CS 205 Midterm Review

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1 01.01 Statements, Symbolic Representations, and Tautologies

Identifying Statements A **statement** is a sentence that is either true or false. Questions, commands, and exclamations are not statements. We do not have to know the truth value of a statement to determine if it is a statement. We only need to know if it can be true or false.

Example Questions

1. Which of the following are statements?
   (a) The sky is blue.
   (b) What time is it?
   (c) The moon is made of green cheese.
   (d) Hello!
   (e) The sum of 2 and 3 is 5.

Solution:

(a) Statement

(b) Not a statement. This is a question.

(c) Statement

(d) Not a statement. This is an exclamation.

(e) Statement

Truth Values of Compound Statements A **compound statement** is a statement formed by combining two or more statements. The **truth value** of a compound statement is determined by the truth values of the component statements and the logical connective used to combine them. You will need to know the truth values of the component statements (conjunction, disjunction, negation, implication, equivalence) and the order of precedence for evaluating compound statements.

Example Questions

1. Given the truth values A true, B false, and C true, determine the truth value of the following compound statements.
   (a) $A \wedge B$
   (b) $A \vee B$
   (c) $A \wedge (B \vee C)$

(d) $(A \wedge B) \vee C$
(e) $A \wedge \neg B$

Solution:

(a) False. True and false is false.

(b) True. True or false is true.

(c) True. True and (false or true) is true.

(d) True. (True and false) or true is true.

(e) True. True and not false is true.

1.1 Rewrite English Statements as Symbolic Statements

Rewriting English Statements as "If ... Then" English statements have multiple ways for expressing logical conncectives. Implication statements can be challenging, because the order of the component statements in English does not always match the order in the symbolic expression $P \to Q$. Table 1.5 in your textbook includes the following english phrases for $A \to B$.

- IF A, then B
- A implies B
- A only if B
- B follows from A
- A is a sufficient condition for B
- B is a necessary condition for A

Example Questions

1. Rewrite each of the following statements in the form "if ... then".
   (a) The sky is blue only if the grass is green.
   (b) The sky is blue if and only if the grass is green.
   (c) The sky is blue unless the grass is green.

Solution:

(a) If the sky is blue, then the grass is green. P only if Q is equivalent to $P \to Q$.

(b) If the sky is blue, then the grass is green and if the grass is green, then the sky is blue. P if and only if Q is equivalent to $P \leftrightarrow Q$.

(c) If the grass is not green, then the sky is blue. P unless Q is equivalent to $\neg Q \to P$.

Representing the negation of compound statements The negation of conjunction and disjunction statements can be tricky. It is tempting to think that the negation of $A \wedge B$ is $\neg A \wedge \neg B$ and the negation of $A \vee B$ is $\neg A \vee \neg B$. However, the negation of a compound statement

- $A \wedge B$ is $A' \vee B'$
- $A \vee B$ is $A' \wedge B'$

Example Questions

1. Express the following English statements in symbolic notation.
   (a) Peter is tall and thin.
   (b) Peter is tall or thin.
   (c) Peter is not tall and thin.
   (d) Peter is neither tall nor thin.

> **Solution:**
> (a) $T \wedge N$
>
> (b) $T \vee N$
>
> (c) $(T \wedge N)'$, which is equivalent to $\neg T \vee \neg N$
>
> (d) $(T \vee N)'$, which is equivalent to $\neg T \wedge \neg N$

Converting compound English Expressions into well-formed formulas. Refer to the examples in Table 1.5 on page 4 of your textbook for help converting English expressions into well-formed formulas. The order of precedence for logical connectives is as follows:

1. Parentheses
2. Negation
3. Conjunction
4. Disjunction
5. Implication
6. Equivalence

Example Questions

1. Using the letters indicated for the component statements, translate the following compound statement into symbolic notation.
   A: prices go up; B: housing will be plentiful; C: housing will be expensive
   If prices go up, then housing will be plentiful and expensive; but if housing is not expensive, then it will still be plentiful.

> **Solution:** $(A \rightarrow (B \wedge C)) \wedge (C' \rightarrow B)$

Simplifiying Conditional Statements in pseudocode When writing pseudocode, it is important to simplify conditional statements to make them easier to read and understand. Watch for tautologies (statements that are always true) and contradictions (statements that are always false) when simplifying conditional statements. De Morgan's Laws can also be helpful when simplifying conditional statements. When working with the negation of comparison operators, remember that the negation of $<$ is $\geq$, the negation of $\leq$ is $>$, the negation of $=$ is $\neq$, and the negation of $\neq$ is $=$.

Example Questions

1. Rewrite the following statement form with a simplified conditional expression, where the function `odd(n)` returns true if n is odd:

```
if not((Value1 < Value2) or odd(Number))
   or (not(Value1 < Value2) and odd(Number)) then
   statement1
else
   statement2
end if
```

> **Solution:**
>
> ```
> if (Value1 >= Value2) and odd(Number) then
>    statement1
> else
>    statement2
> end if
> ```

2. You want your program to execute `statement1` when A is false, B is false, and C is true, and to execute `statement2` otherwise. You wrote:

```
if not(A and B) and C then
   statement1
else
   statement2
end if
```

Does this do what you want? Justify your answer.

> **Solution:** No. The original statement is equivalent to $\neg A \wedge \neg B \wedge C$. The original code is equivalent to $\neg(A \wedge B) \wedge C$. The original code will execute `statement1` when A is false, B is false, and C is true, but it will also execute `statement1` when A is true, B is true, and C is true.

2 01.02 Propositional Logic

Derivation Rules in Propositional Logic Derivation rules include equivalence rules and inference rules. Equivalence rules are used to derive equivalent statements. Inference rules are used to derive new statements from given hypotheses. Table 1.11 (page 28) defines the following equivalence rules:

- commutative
- associative
- de Morgan's
- implication
- double negation

- definition of equivalence

Table 1.12 (page 29) defines the following inference rules:

- modus ponens
- modus tollens
- conjunction
- simplification
- addition

Example Questions

1. For the following arguments, what inference rule is illustrated?
   (a) If Martina is the author, then the book is fiction. But the book is nonfiction. Therefore, Martina is not the author.
   (b) If the business declares bankruptcy, then all assets must be confiscated. The business declared bankruptcy. Therefore, all assets must be confiscated.

> **Solution:**
> (a) Modus tollens
>
> (b) Modus ponens

Validating Arguments To validate an argument, you must determine if the conclusion can be reached from the given hypotheses. If the conclusion can be reached, the argument is valid. If the conclusion cannot be reached, the argument is invalid. To validate an argument, you must use the inference rules in Table 1.12. You must also know the truth values of the component statements and the order of precedence for evaluating compound statements.

Example Questions

1. For the following arguments, decide what conclusion, if any, can be reached from the given hypotheses and justify your answer.
   (a) If the car was involved in the hit-and-run, then the paint would be chipped. But the paint is not chipped.
   (b) Either the weather will turn bad or we will leave on time. If the weather turns bad, then the flight will be canceled.

> **Solution:**
> (a) From modus tollens, we can conclude that the car was not involved in the hit-and-run.
>
> (b) We cannot conclude anything from the given hypotheses.

Proof Sequence Steps An argument can be represented as $P_1 \wedge P_2 \wedge \ldots \wedge P_n \to Q$. To prove the argument, you must justify each step in the proof sequence. The proof sequence is a series of statements that lead to the conclusion.

The first steps in a proof sequence are the **hypotheses**, the component statements that are given $(P_1, P_2, \ldots, P_n)$. The last step in a proof sequence is the **conclusion**, the statement that is derived from the hypotheses (Q). The steps in between the hypotheses and the conclusion are the **proof steps**, the statements that are derived from the hypotheses using equivalence rules and inference rules. Each proof step must be justified by an equivalence rule or an inference rule.

Example Questions

1. Justify each step in the proof sequence of the following argument.
   $A \wedge (B \to C) \to (B \to (A \wedge C))$

   1. A
   2. $B \to C$
   3. B
   4. C
   5. $A \wedge C$

   Solution:

   1. Hypothesis
   2. Hypothesis
   3. Hypothesis
   4. Modus ponens on 2 and 3
   5. Conjunction of 1 and 4

2. Use propositional logic to prove that the argument is valid.
   $(A \vee B')' \wedge (B \to C) \to (A' \wedge C)$

   Solution:

   1. $(A \vee B')'$ Hypothesis
   2. $(B \to C)$ Hypothesis
   3. $(A' \wedge B)$ De Morgan's Law on 1
   4. A' Simplification on 3
   5. B Simplification on 3
   6. C Modus ponens on 2 and 5
   7. $A' \wedge C$ Conjunction on 4 and 6

3. Write the argument using propositional wffs (use the statement letters shown.) Then, using propositional logic, including the rule in Table 1.14, prove that the argument is valid.
If the program is efficient, it executes quickly. Either the program is efficient, or it has a bug. However, the program does not execute quickly. Therefore, it has a bug. E, Q, B

Solution: Argument: $(E \to Q) \wedge (E \vee B) \to (Q \to B)$

1. $(E \to Q)$ Hypothesis
2. $(E \vee B)$ Hypothesis
3. Q' Hypothesis
4. E' Modus tollens on 1 and 3
5. B Disjunctive syllogism on 2 and 4

3 2.1 Proof Techniques

Counterexamples A **counterexample** is a specific example that shows a statement is false. To prove a statement is false, you can provide a counterexample.

Example Questions

1. Provide a counterexample to the following statement: If an animal lives in the ocean, then it is a fish.

Solution: A whale is an animal that lives in the ocean but is not a fish.

Exhaustive Proof An **exhaustive proof** is a proof that considers all possible cases. To prove a statement is true, you can use an exhaustive proof.

Example Questions

1. Prove the following statement: If n is an even integer, $4 \leq n \leq 12$, then n is a sum of two prime numbers.

Solution:

- $n = 4$: $4 = 2 + 2$
- $n = 6$: $6 = 3 + 3$
- $n = 8$: $8 = 3 + 5$
- $n = 10$: $10 = 3 + 7$
- $n = 12$: $12 = 5 + 7$

Informal Proofs You can use an informal proof to prove a statement is true. An informal proof is a proof that uses words to explain why a statement is true. When writing an informal proof about mathematical statements, you can start with the definitions of the terms in the statement and logically explain why the statement is true. You can also use examples to show that the statement is true for specific cases.

Example Questions

1. Prove the following statement: If n is an even integer, $4 \leq n \leq 12$, then n is a sum of two prime numbers.

> **Solution:**
>
> - $n = 4$: $4 = 2 + 2$
> - $n = 6$: $6 = 3 + 3$
> - $n = 8$: $8 = 3 + 5$
> - $n = 10$: $10 = 3 + 7$
> - $n = 12$: $12 = 5 + 7$

Proof by Contraposition The implication $P \to Q$ has a **converse** $(Q \to P)$, an **inverse** $(P' \to Q')$ and a **contrapositive** $(P \to Q)$. Only the contrapositive is logically equivalent to the original implication. To prove an implication, you can prove the contrapositive instead.

Example Questions

1. Write the converse and contrapositive of the following statement: If it rained, then the grass is wet.

> **Solution:** Converse: If the grass is wet, then it rained.
> Contrapositive: If the grass is not wet, then it did not rain.

2. Prove the following statement: If a number x is positive, then so is $x+1$. (Do a proof by contraposition.)

> **Solution:** Assume $x + 1$ is not positive. Then $x + 1 \leq 0$. Subtracting 1 from both sides, we get $x \leq -1$. Since x is positive, this is a contradiction. Therefore, $x + 1$ is positive.

Proof by Contradiction A **proof by contradiction** is a proof that assumes the statement is false and then shows that this assumption leads to a contradiction. To prove a statement is true, you can use a proof by contradiction.

Example Questions

1. Prove the following statement: The sum of two even integers is even.(Do a proof by contradiction.)

> **Solution:** Let $x = 2m$ and $y = 2n$ be two even integers. Assume that $x + y$ is odd. Then $x + y = 2m + 2n = 2k + 1$ for some integer k or $2m + 2n - 2k = 1$. We can factor out a 2 to get $2(m + n - k) = 1$. Since the left side is even and the right side is odd, this is a contradiction. Therefore, the sum of two even integers is even.

4 3.1 Recursive Definitions

Recursively Defined-Sequences

1. Write the first five values in the sequence:
   1. $a_1 = 1$
   2. $a_n = a_{n-1} + 2$ for $n \geq 2$

Solution:

1. $a_1 = 1$
2. $a_2 = 1 + 2 = 3$
3. $a_3 = 3 + 2 = 5$
4. $a_4 = 5 + 2 = 7$
5. $a_5 = 7 + 2 = 9$

Recursively Defined Sets

1. A set T of numbers is defined recursively by the following rules:
   1. $2 \in T$
   2. If $x \in T$, then $x + 3 \in T$ and $2 * x \in T$

   Which of the following numbers are in T?
   1. 6
   2. 7
   3. 19
   4. 12

Solution:

1. $6 = 2 * 3 \in T$
2. $7 = 2 + 3 \in T$
3. $19 = 2 + 3 + 3 + 3 + 3 + 3 + 2 * 3 \in T$
4. $12 = 2 + 3 + 3 + 2 * 3 \in T$

2. A set S of strings is defined recursively by the following rules:
   1. a and b belong to S.
   2. If x belongs to S, so does xb, the concatenation of x and b.

   Which of the following strings belong to S?
   1. ab

2. abb
3. $abab$
4. $abbbb$

Solution: $abab$ does not belong to S. You can only add a b onto the end of an existing string in S. Creating the string $abab$ would require adding an a to the end of ab.

Recursively-Defined Algorithms The textbook introduces two algorithms defined recursively: SelectionSort and BinarySearch. SelectionSort is a sorting algorithm that selects the maximum element in a list and moves it to the end of the list. The steps for the SelectionSort algorithm are as follows:

1. Find the largest element in the list.
2. Exchange the largest element with the last element in the list.
3. Repeat the process with the remaining elements in the list.

The steps for the BinarySearch algorithm are as follows:

1. If the list is empty, the target element is not in the list.
2. If the target element is in the middle of the list, the search is successful.
3. If the target element is less than the middle element, search the left half of the list.
4. If the target element is greater than the middle element, search the right half of the list.

Ihe list has an odd number of elements, the middle element is the median element. If the list has an even number of elements, the middle element is the element to the left of the median element.

BinarySearch is a search algorithm that divides a list in half and determines which half contains the target element.

1. Simulate the execution of the algorithm *SelectionSort* on the following list, L; write the list after every exchange that changes the list.
   $L = [3, 1, 4, 1, 5, 9, 2, 6, 3]$. Remember that this textbook uses the maximum element selection sort algorithm.

Solution:

1. Start with the original list: $[3, 1, 4, 1, 5, 9, 2, 6, 3]$
2. The largest element in $L[0 : 8]$ is 9. Exchange 9 with 3: $[3, 1, 4, 1, 5, 3, 2, 6, 9]$
3. The largest element in $L[0 : 7]$ is 6, which is already in the correct position: $[3, 1, 4, 1, 5, 3, 2, 6, 9]$
4. The largest element in $L[0 : 6]$ is 5. Exchange 5 with 2, which is at L[6]: $[3, 1, 4, 1, 2, 3, 5, 6, 9]$
5. The largest element in $L[0 : 5]$ is 4. Exchange 4 with 3, which is at L[5]: $[3, 1, 3, 1, 2, 4, 5, 6, 9]$
6. The largest element in $L[0 : 4]$ is 3. Exchange 3 with 2, which is at L[4]: $[2, 1, 3, 1, 3, 4, 5, 6, 9]$
7. The largest element in $L[0 : 3]$ is 3. Exchange 3 with 1, which is at L[3]: $[2, 1, 1, 3, 3, 4, 5, 6, 9]$
8. The largest element in $L[0 : 2]$ is 2. Exchange 2 with 1, which is at L[2]: $[1, 1, 2, 3, 3, 4, 5, 6, 9]$
9. The largest element in $L[0 : 1]$ is 1. The list is already sorted: $[1, 1, 2, 3, 3, 4, 5, 6, 9]$

2. The binary search algorithm is used with the following list; x has the value "flour." Name the elements against which x is compared.
   $L = \{\text{butter}, \text{chocolate}, \text{eggs}, \text{milk}, \text{shortening}, \text{sugar}\}$

> **Solution:**
>
> - The list has six elements. That means there are an even number of elements. Therefore the middle element is the element to the left of the calculated median. The median is the element at index 3, which is "milk."
> - The target element, "flour," is less than "milk." The search continues with the left half of the list.
> - The left half of the list is $\{\text{butter}, \text{chocolate}, \text{eggs}\}$. The median is the element at index 1, which is "chocolate."
> - The target element, "flour," is greater than "chocolate." There are no more sublists to search. The search is unsuccessful.

5 4.1 Sets

Set Definitions Set notation includes the following symbols:

- $\in$ means "is an element of"
- $\subseteq$ means "is a subset of"
- $\emptyset$ is the empty set
- $\wp(S)$ is the power set of S

Example Questions

1. Let $S = \{2, 5, 17, 27\}$. Which of the following expressions are true?
   1. $5 \in S$
   2. $\emptyset \in S$
   3. $S \in S$
   4. $\{5\} \in S$

> **Solution:**
>
> 1. True. 5 is an element of S.
> 2. False. The empty set is a subset of every other set. It is not a subset of S.
> 3. False. S is not an element of itself.
> 4. False. $\{5\}$ is a subset of S, but it is not an element of S.

2. List the unique sets described here.

1. $\{2, 3, 4\}$
2. $\{x | x$is the first letter in cat, bat, or apple$\}$
3. $\{x | x \in \mathbb{N}$ and $2 \leq x \leq 4\}$
4. $\{a, b, c\}$
5. $\emptyset$
6. $\{x | x$ is the first letter in cat, bat, and apple$\}$
7. $\{2, a, 3, b, 4, c\}$
8. $\{3, 4, 2\}$

Solution:

1. $\{2, 3, 4\}$
2. $\{a, b, c\}$
3. $\emptyset$
4. $\{2, a, 3, b, 4, c\}$

3. Describe the following set by listing its elements: $\{x | x \in \mathbb{N}$ and $x^2 < 25\}$

Solution: The set is comprised of all natural numbers whose square is less than 25. The elements are $\{0, 1, 2, 3, 4\}$

4. Let $A = \{x | x \in \mathbb{N}$ and $1 < x < 50\}$, $B = \{x | x \in \mathbb{R}$ and $1 < x < 50\}$, $C = \{x | x \in \mathbb{Z}$ and $|x| \geq 25\}$. Which of the following statements are true?

   1. $A \subseteq B$
   2. $17 \in A$
   3. $A \subseteq C$
   4. $\sqrt{3} \in B$
   5. $\{0, 1, 2\} \subseteq C$
   6. $\emptyset \in B$
   7. $\{x | x \in \mathbb{Z}$ and $x^2 > 625\} \subseteq C$

Solution: $A = \{2, 3, \ldots, 49\}$.
B is the set of real numbers between 1 and 50, exclusively.
$C = \{\ldots, -27, -26, -25, 25, 26, 27, \ldots\}$

1. True.
2. True.
3. False. The number in A less than 25 are not in C.

4. True
5. False. None of the elements in $\{0, 1, 2\}$ are in C.
6. False. The empty set is not an element of B.
7. True

5. Find $\wp(S)$ for $S = \{a, b\}$

Solution: $\wp S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

6. What can be said about A if $\wp(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$?

Solution: $A = \{a, b\}$

7. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Find $A \cup B$, $A \cap B$, $A - B$, and $B - A$.

Solution:

- $A \cup B = \{1, 2, 3, 4\}$
- $A \cap B = \{2, 3\}$
- $A - B = \{1\}$
- $B - A = \{4\}$

8. Let $A = \{2, 4, 5, 6, 8\}$
$B = \{1, 4, 5, 9\}$
$C = \{x | x \in \mathbb{Z} \text{ and } 2 \leq x < 5\}$
be subsets of $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find
   1. $A \cup B$
   2. $A \cap B$
   3. $A \cap C$
   4. $B \cup C$
   5. $A - B$
   6. A'
   7. $A \cap A'$
   8. $B \times C$

Solution:

1. $A \cup B = \{1, 2, 4, 5, 6, 8, 9\}$
2. $A \cap B = \{4, 5\}$
3. $A \cap C = \{2, 4\}$
4. $B \cup C = \{1, 2, 3, 4, 5\}$
5. $A - B = \{2, 6, 8\}$
6. $A' = \{0, 1, 3, 7, 9\}$
7. $A \cap A' = \emptyset$
8. $B \times C =$
   $\{(1, 2), (1, 3), (1, 4), (1, 5),$
   $(4, 2), (4, 3), (4, 4), (4, 5),$
   $(5, 2), (5, 3), (5, 4), (5, 5),$
   $(9, 2), (9, 3), (9, 4), (9, 5)\}$

9. Consider the following subsets of the set of all students:
A: students who are taking calculus
B: students who are taking discrete mathematics
C: students who are taking linear algebra
D: students who are taking calculus and discrete mathematics
E: students who are taking calculus and linear algebra
Using set operations, describe each of the following sets in terms of A, B, C, D, and E.

1. Students who are taking calculus but not discrete mathematics.
2. Students who are taking calculus but not linear algebra.
3. Students who are taking calculus but not discrete mathematics or linear algebra.
4. Students who are taking calculus, discrete mathematics, and linear algebra.
5. Students who are taking exactly two of the three courses.
6. Students who are taking at least one of the three courses.
7. Students who are taking all three courses.
8. Students who are taking at most one of the three courses.

Solution:

1. $A - B$
2. $A - C$
3. $A - (B \cup C)$
4. $D \cap E$
5. $(A \cap B) \cup (A \cap C) \cup (B \cap C)$
6. $A \cup B \cup C$
7. $D \cap E$
8. $(A \cup B \cup C) - (D \cup E)$

6 4.2 Counting

1. A frozen yogurt shop allows you to choose one flavor (vanilla, strawberry, lemon, cherry, or peach), one topping (chocolate shavings, crushed toffee, or crushed peanut brittle), and one condiment (whipped cream or shredded coconut). How many different frozen yogurts can you create?

   Solution: $5 * 3 * 2 = 30$

2. A multiple choice exam has 20 questions, each with 4 possible answer and 10 additional questions each with 5 possible answers. How many different answer sheets are possible?

   Solution: A student can choose from four answers for the first 20 questions. That means there are $4_1 \times 4_2 \ldots \times 4_2 0 = 4^{20}$ possible answer sheets for the first 20 questions. For the next 10 questions, a student can choose from 5 answers. That means there are 5^{10} possible answer sheets for the next 10 questions. Each of the possible answer sheets for the first 20 questions can be paired with each of the possible answer sheets for the next 10 questions. That means there are $4^{20} \times 5^{10}$ possible answer sheets.

3. A, B, C, and D are nodes on a computer network. There are 2 paths between A and C, 2 between B and D, 4 between A and B and 4 between C and D Along how many routes can a message from A to D be sent?

   Solution: The possible paths from A to D are $A \to C \to D$ and $A \to B \to D$. There are 2 paths from A to C and 4 paths from C to D. Along the $A \to C \to D$ path, there are 2 paths from A to C and 4 paths from C to D. Along the $A \to B \to D$ path, there are 4 paths from A to B and 2 paths from B to D.That means there are $2 * 4$ paths along that route. Along the $A \to C \to D$ path, there are $2 * 4 = 8$ paths. Along the $A \to B \to D$ path, there are $4 * 2 = 8$ paths. Therefore, there are $8 + 8 = 16$ paths from A to D.

4. A president and vice president must be chosen for the executive committee of an organization. There are 17 volunteers from the Eastern Division and 24 volunteers from the Western Division. If both officers must come from the same division, in how many ways can the officers be selected?

   Solution: There are 17 ways to choose the president from the Eastern Division and 16 ways to choose the vice president from the Eastern Division. That means there are $17 * 16 = 272$ ways to choose the president and vice president from the Eastern Division. There are 24 ways to choose the president from the Western Division and 23 ways to choose the vice president from the Western Division. That means there are $24 * 23 = 552$ ways to choose the president and vice president from the Western Division. Therefore, there are $272 + 552 = 824$ ways to choose the president and vice president.

5. A new car can be ordered with a choice of 10 exterior colors; 7 interior colors; automatic or three-speed or five-speed transmission; with or without air conditioning; with or without power steering; and with or without the option package that contains the door lock and the rear-window defroster. How many different cars can be ordered?

Solution: $10 * 7 * 3 * 2 * 2 * 2 = 840$

6. A computer password consists of 3 letters followed by 4 digits. How many different passwords are possible?

Solution: $26^3 * 10^4 = 175,760,000$

7. What is the value of `Count` after the following pseudocode has been executed?

```
Count = 0
for i = 1 to 5 do
    for Letter = 'A' to 'C' do
        Count = Count + 1
    end for
end for
```

Solution: $Count = 15$

8. What is the value of `Result` after the following pseudocode has been executed?

```
Result = 0
for Index = 20 down to 10 do
    for Inner = 5 to 10 do
        Result = Result + 2
    end for
end for
```

Solution: $Result = 66$

7 4.3 Principle of Inclusion and Exclusion; Pigeonhole Principle

1. In a group of 42 tourists, everyone speaks English or French; there are 35 English speakers and 25 French speakers. How many tourists speak both languages?

Solution: $35 + 25 - 42 = 18$

2. A survey of 150 college students reveals that
   - 83 own automobiles,
   - 97 own bikes
   - 29 own motorcycles
   - 53 own a car and a bike

- 14 own a car and a motorcycle
- 7 own a bike and a motorcycle,
- 2 own all three.

1. How many students own a car but not a bike or a motorcycle?
2. How many students do not own any of the three?

Solution:

1. $53 - 14 - 2 = 37$
2. $150 - 83 - 97 - 29 + 53 + 14 + 7 - 2 = 5$

3. How many people must be in a group to guarantee that at least two people have the same birthday? Remember that there are 365 days in a year, and 366 days in a leap year.

Solution: 366

4. In a group of 25 people, must there be at least 3 who were born in the same month?

Solution: Yes

8 4.4 Permutations and Combinations

1. Compute the value of the following expressions:
   1. $P(7, 2)$
   2. $P(8, 5)$

Solution:

1. $P(7, 2) = 42$
2. $P(8, 5) = 6720$

2. How many ways can 19 people be seated in a row?

Solution: 19!

3. How many different ways can 11 men and 8 women be seated in a row?

Solution: 19!

4. How many different ways can 11 men and 8 women be seated in a row if all the men sit together and all the women sit together?

Solution: $11! * 8!$

5. Compute the value of the following expressions:
   1. $C(7, 2)$
   2. $C(8, 5)$

Solution:

1. $C(7, 2) = 21$
2. $C(8, 5) = 56$

6. Quality control wants to select 3 items from a production line of 20 items to check for defects. How many different ways can the 3 items be selected?

Solution: $C(20, 3) = 1140$

7. How many different sets of 4 novels and 3 plays can be created from a collection of 21 novels and 11 plays?

Solution: $C(21, 4) * C(11, 3) = 598,752$

8. Of a company's personnel, 7 people work in design, 14 in manufacturing, 4 in testing, 5 in sales, 2 in accounting, and 3 in marketing. A committee of 6 people is to be formed to meet with upper management.
   (a) How many committees with 1 member from each department are possible?
   (b) How many committees with exactly 2 members from manufacturing ar possible?
   (c) How many committees with no representative from accounting and exactly one representative from marketing are possible?
   (d) How many committees with at least 2 members from manufacturing are possible?

Solution:

(a) $C(7, 1) * C(14, 1) * C(4, 1) * C(5, 1) * C(2, 1) * C(3, 1) = 5040$

(b) $C(14, 2) * C(7, 4) = 90090$

(c) $C(7, 1) * C(14, 1) * C(4, 1) * C(5, 1) * C(3, 1) = 2940$

(d) $C(14, 2) * C(7, 4) + C(14, 3) * C(7, 3) + C(14, 4) * C(7, 2) + C(14, 5) * C(7, 1) + C(14, 6) = 3003 + 2002 + 1001 + 364 + 84 = 6454$

9. A computer network has 60 switching nodes.
   (a) THe network is designed to withstand the failure of any 2 nodes. In how many ways can such a failure occur?
   (b) In how many ways can 1 or 2 nodes fail?
   (c) If 1 node has failed, in how many ways can 7 nodes be selected to include exactly 1 failed node?
   (d) If 2 nodes have failed, in how many ways can 7 nodes be selected to include exactly 1 failed node?

Solution:

(a) $C(60, 2) = 1770$

(b) $C(60, 1) + C(60, 2) = 60 + 1770 = 1830$

(c) $C(2, 1) * C(58, 6) = 2 * 386, 206 = 772, 412$

(d) $C(2, 2) * C(58, 5) = 1 * 383, 838 = 383, 838$

10. How many distinct permutations of the characters in the word "MISSISSIPPI" are there?

Solution: $\frac{11!}{4!4!2!} = 34, 650$

11. A florist has a large number of roses, carnations, lilies, and snapdragons in stock. How many different bouquets of one dozen flowers can be made?

Solution: From 4 choose 12 with repetitions. $C(4 + 12 - 1, 12) = C(15, 12) = 455$

12. Arrange the following permutations of the numbers $\{1, \ldots, 6\}$ in lexicographic order.
    163542, 345621, 643125, 634521, 163452, 356421

Solution: 163452, 163542, 345621, 356421, 634521, 643125

13. Use algorithm permutation generator to generate the next permutation after the given permutaion in the set of all permutations of the numbers $\{1, \ldots, 7\}$.
    7,4,3,1,6,5,2

Solution:

1. Find the pivot: 1
2. Find the element to the right of the pivot that is the smallest element greater than the pivot: 2
3. Swap the pivot with the element found in step 2: 7,4,3,2,6,5,1
4. Reverse the order of the elements to the right of the pivot: 7,4,3,2,1,5,6
5. The next permutation is 7,4,3,2,1,5,6

14. In generating all combinations of five items from the set $\{1, \ldots, 9\}$, find the next five values in the list after 24579.

> **Solution:**
> Starting with 24570, 9 is at its maximum value, bump 7 up to 8, giveing 24589. Now 9 and 8 are at their maximum values. Bump 4 up to 6 and us 7, 8 as the last two values, giving 24678. Bump 8 up to 9, giving 24679. Bump 6 up to 7, giving 24789.
> The sequence is 24589, 24678, 24679, 24789.

9 4.6 Probability

1. Three coins are tossed at the same time, each equally likely to come up heads or tails.
   (a) What is the size of the sample space?
   (b) What is the probability of getting 1 head and 2 tails?
   (c) What is the probability of getting all tails?
   (d) What is the probability that no coin comes up heads?
   (e) What is the probability of getting all tails or all heads?
   (f) What is the probability of getting all tails and all heads?

> **Solution:**
> (a) $2^3 = 8$
> (b) $\frac{3}{8}$
> (c) $\frac{1}{8}$
> (d) $\frac{1}{8}$
> (e) $\frac{2}{8}$
> (f) 0

2. Two cards are drawn from a 52-card deck. A standard deck has 13 cards in each of four suits: hearts, diamonds, clubs, and spades. The cards have a face value 2 through 10, jack, queen, king, and ace. The jack, queen, and king are "face cards."
   (a) What is the size of the sample space?
   (b) What is the probability that both cards are in the same suit? What is the probability that exactly one card is a spade?
   (c) What is the probability that both cards are face cards?

> **Solution:**
> (a) $C(52, 2) = 1326$
> (b) $\frac{13}{51}$
> (c) $\frac{13*39}{1326}$
> (d) $\frac{12}{51}$

3. E_1 and E_2 are events from the same sample space; $P(E_1) = 0.37$ and $P(E_2) = 0.45$. $P(E_1 \cap E_2) = 0.14$.
   (a) Find the probability that E_2 does not occur.
   (b) Find the probability that either E_1 or E_2 occurs.
   (c) Find the probability that neither E_1 nor E_2 occurs.

Solution:
(a) $1 - 0.45 = 0.55$

(b) $0.37 + 0.45 - 0.14 = 0.68$

(c) $1 - 0.68 = 0.32$

4. A loaded die has the following probabilities: $P(1) = 0.1$, $P(2) = 0.2$, $P(3) = 0.3$, $P(4) = 0.2$, $P(5) = 0.1$, $P(6) = 0.1$. Whe the die is rolled, let E_1 be the event that the number rolled is odd, let E_2 be the event that the number rolled is 3 or 6, and let E_3 be the event that the number rolled is 4 or more.
   (a) Find $P(E_1)$, $P(E_2)$, and $P(E_3)$.
   (b) Find $P(E_2 \cap E_3)$ and $P(E_1 \cup E_3)$.

Solution: Find the probabilities of events E_1, E_2, and E_3 by adding the probabilities of their outcomes. Find the probabilities of the intersections and unions by using the principle of inclusion and exclusion.
(a) $P(E_1) = 0.1 + 0.3 + 0.1 = 0.5$, $P(E_2) = 0.3 + 0.1 = 0.4$, $P(E_3) = 0.2 + 0.1 + 0.1 = 0.4$

(b) $P(E_2 \cap E_3) = 0.1$, $P(E_1 \cup E_3) = 0.5 + 0.2 + 0.1 = 0.8$

5. On a particular manufacturing job, the probability that there will be a shortage of copper is 0.37. The probability that there will be a shortage of both copper and aluminum is 0.28. Find the probabilitiy that there will be a shortage of aluminum given that there is a shortage of copper.

Solution: $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0.28}{0.37} = 0.76$

6. An online pharmacy sells an over-the-counter drug, medication X, that is ued for a variety of purposes. The pharmacy has data that says 18% of its customers are HIV positive, 9% of its HIV-positive customers buy medication X and 3% of its HIV-negative customers buy medication X. Find the probability that a customer who buys medication X is HIV positive. The pharmacy can use this data to shape its marketing/advertising plan.

Solution: $P(H|X) = \frac{P(H \cap X)}{P(X)} = \frac{0.09}{0.18} = 0.5$

7. A fair six-sided die is rolled once. Let the random variable X equal the value that comes up.
   (a) Find the expected value of X, $E(X)$.
   (b) The die is now "loaded" so that 2 comes up twice as often as any other number. Find the new expected value of X.

Solution:

(a) $E(X) = \frac{1+2+3+4+5+6}{6} = 3.5$

(b) $E(X) = \frac{1+2+2+3+4+5+6}{7} = 3.14$

8. A box contains 43 red balls, 27 green balls, and 8 blue balls. A player marks a game card with the color he or she believes will be picked. The price money for guesssing the correct color is
$3 for a red ball, $6 for a green ball, and $10 for a blue ball.
The price of the game card is $5. Find the expected value of the price money.

Solution: $E(X) = \frac{43}{78} * 3 + \frac{27}{78} * 6 + \frac{8}{78} * 10 - 5 = 3.56$