CS 581 – Advanced Artificial Intelligence

Topic: Probability Theory

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Motivation

- The agent needs reason in an uncertain world
- Uncertainty can be due to
  - Noisy sensors (e.g., temperature, GPS, camera, etc.)
  - Imperfect data (e.g., low resolution image)
  - Missing data (e.g., lab tests)
  - Imperfect knowledge (e.g., medical diagnosis)
  - Exceptions (e.g., all birds fly except ostriches, penguins, birds whose wings are broken, dead birds, …)
  - Changing data (e.g., flu seasons, traffic conditions during rush hour, etc.)
  - Maybe others
- The agent still must act (e.g., step on the breaks, diagnose a patient, order a lab test, …)

We'll Cover

- Background in probability theory
- Bayesian networks
- Hidden Markov Models
- Kalman filters
- Influence diagrams
- Markov decision processes
- Probabilistic classifiers

Some exercises

- In a class, 70% of the grad students got an A. John got an A. What is the probability that John is a grad student?
- You design a covid test with the following behavior
  - $P(+ \mid covid) = 0.95$; $P(- \mid covid) = 0.05$
  - $P(+ \mid \sim covid) = 0.10$; $P(- \mid \sim covid) = 0.90$
  - John takes the test, and the result is +. What is the probability that John has covid?
- $P(toothache \mid cavity) = 0.75$. What is $P(cavity \mid toothache)$?

Random Variables

- Pick variables of interest
  - Medical diagnosis
    - Age, gender, weight, temperature, LT1, LT2, …
  - Loan application
    - Income, savings, payment history, …
  - Exercises
    - Grad student, Grade, Covid, Test result, Toothache, Cavity
- Every variable has a domain
  - Binary (True/False)
  - Categorical
  - Real-valued
- Possible world
  - An assignment to all variables of interest

PROBABILITY MODEL

- A **probability model** associates a numerical probability P(w) with each possible world w
  - P(w) sums to 1 over all possible worlds
- An **event** is the set of possible worlds where a given predicate is true
  - Roll two dice
    - The possible worlds are (1,1), (1,2), …, (6,6); 36 possible worlds
    - Predicate = two dice sum to 10
    - Event = {(4,6), (5,5), (6,4)}
  - Toothache and cavity
    - Four possible worlds: $(t, c), (t, \sim c), (\sim t, c), (\sim t, \sim c)$
    - Some worlds are more likely than others
    - Predicate can be anything about these variables: $t \wedge c, t, t \vee \sim c,$

Axioms of Probability

1. The probability $P(a)$ of a proposition a is a real number between 0 and 1

2. $P(true) = 1$, $P(false) = 0$

3. $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$

$P(\neg a)$

- $P(a \vee \neg a) = P(a) + P(\neg a) – P(a \wedge \neg a)$
- $P(true) = P(a) + P(\neg a) – P(false)$
- $1 = P(a) + P(\neg a) – 0$
- $P(\neg a) = 1 – P(a)$
- Intuitive explanation:
  - The probability of all possible worlds is 1
  - Either a or $\neg a$ holds in one world
  - The worlds that a holds and the worlds that $\neg a$ holds are mutually exclusive and exhaustive

Random Variables – Notation

- Capital: X: variable
- Lowercase: x: a particular value of X
- Val(X): the set of values X can take
- Bold Capital: **X**: a set of variables
- Bold lowercase: **x**: an assignment to all variables in **X**
- P(X=x) will be shortened as P(x)
- $P(X=x \cap Y=y)$ will be shortened as P(x,y)

Joint Distribution

- We have n random variables, $V_1, V_2, \ldots, V_n$
- We are interested in the probability of a possible world, where
  - $V_1=v_1, V_2=v_2, \ldots, V_n = v_n$
- $P(V_1, V_2, \ldots, V_n)$ associates a probability for each possible world ≡ the **joint distribution**
- How many entries are there, if we assume the variables are all binary?

Toothache example

Toothache	Cavity	P(T,C)
toothache	cavity	0.15
toothache	¬cavity	0.10
¬toothache	cavity	0.05
¬toothache	¬cavity	0.70

Prior and Posterior

- Prior probability
  - Probability of a proposition in the absence of any other information
  - E.g., $P(V_1, V_3, V_5)$
- Conditional/posterior probability
  - Probability of a proposition given another piece of information
  - E.g., $P(V_2, V_3 \mid V_5 = T, V_7 = F)$
  - $P(A \mid B) = P(A \wedge B) / P(B)$

Marginalization

- Given $P(V_1, V_2, \ldots, V_n \mid V_{n+1}, V_{n+2}, \ldots, V_{n+m})$, where $n>0$ and $m \geq 0$, we can find, for example
  - $P(V_i, V_j, V_k \mid V_{n+1}, V_{n+2}, \ldots, V_{n+m})$ where $i, j, k < n$ by summing out all the irrelevant variables
- Examples

Let's Answer a few queries

Toothache	Cavity	P(T,C)
toothache	cavity	0.15
toothache	¬cavity	0.10
¬toothache	cavity	0.05
¬toothache	¬cavity	0.70

- P(cavity) = ?
- P(¬cavity) = ?
- P(toothache) = ?
- P(¬toothache) = ?

Let's Answer a few queries

Toothache	Cavity	P(T,C)
toothache	cavity	0.15
toothache	¬cavity	0.10
¬toothache	cavity	0.05
¬toothache	¬cavity	0.70

- $P(cavity \mid toothache) = ?$
- $P(cavity \mid \neg toothache) = ?$
- $P(\neg cavity \mid toothache) = ?$
- $P(\neg cavity \mid \neg toothache) = ?$
- $P(toothache \mid cavity) = ?$
- $P(\neg toothache \mid cavity) = ?$
- $P(toothache \mid \neg cavity) = ?$
- $P(\neg toothache \mid \neg cavity) = ?$

Bayes' Rule

- $P(B|A) = P(A|B)*P(B) / P(A)$
- Example use
  - $P(\text{cause}|\text{effect}) = P(\text{effect}|\text{cause})*P(\text{cause}) / P(\text{effect})$
- Why is this useful?
  - Because in practice it is easier to get probabilities for P(effect|cause) and P(cause) than for P(cause|effect)
    - E.g., P(disease|symptoms) = P(symptoms|disease)*P(disease) / P(symptoms)
    - It is easier to know what symptoms diseases cause. It is harder to diagnose a disease given symptoms

Bayes Rule

- Can we compute $P(\alpha|\beta)$ from $P(\beta|\alpha)$?

Let's Revisit Some of the Exercises

- See OneNote

Chain Rule

- $P(X_1, X_2, X_3, \ldots, X_k) =$
  - $P(X_1)\ P(X_2 | X_1)\ P(X_3 | X_1,X_2) \ldots P(X_k | X_1, X_2, X_3, \ldots, X_{k-1})$
    - or
  - $P(X_2)\ P(X_1 | X_2)\ P(X_3 | X_1,X_2) \ldots P(X_k | X_1, X_2, X_3, \ldots, X_{k-1})$
    - or
  - $P(X_2)\ P(X_3 | X_2)\ P(X_1 | X_3,X_2) \ldots P(X_k | X_1, X_2, X_3, \ldots, X_{k-1})$
    - or
  - Pick an order, then
    - P(first)P(second | first)P(third | first,second)…P(last | all_previous)

Marginal Independence

- An event α is **independent** of event β in P, denoted as $P \models \alpha \perp \beta$, if
  - $P(\alpha \mid \beta) = P(\alpha)$, or
  - $P(\beta) = 0$
- Proposition: A distribution P satisfies $\alpha \perp \beta$ if and only if
  - $P(\alpha, \beta) = P(\alpha)\, P(\beta)$
  - Can you prove it?
- Corollary: $\alpha \perp \beta$ implies $\beta \perp \alpha$

MARGINAL INDEPENDENCE

X	Y	P(X, Y)
t	t	0.18
t	f	0.42
f	t	0.12
f	f	0.28

Is $X \perp Y$?

Conditional Independence

- Two events are independent given another event
- An event α is **independent** of event β given event γ in P, denoted as $P \models (\alpha \perp \beta \mid \gamma)$, if
  - $P(\alpha \mid \beta, \gamma) = P(\alpha \mid \gamma)$, or
  - $P(\beta, \gamma) = 0$
- Proposition: A distribution P satisfies $\alpha \perp \beta \mid \gamma$ if and only if
  - $P(\alpha, \beta \mid \gamma) = P(\alpha \mid \gamma)\, P(\beta \mid \gamma)$

Querying a distribution

- **Evidence ($\mathbf{E}=\mathbf{e}$)**: what is known, **Query ($\mathbf{Y}$)**: variables of interest, $\mathbf{X}$ is the set of all variables that include $\mathbf{E}$, $\mathbf{Y}$, and potentially others

1. **Probability query**
   - $P(\mathbf{Y} \mid \mathbf{e}) = ?$
2. **MAP query**
   - $\mathbf{W} = \mathbf{X} \setminus \mathbf{E}$ (i.e., all the non-evidence variables)
   - $\mathrm{MAP}(\mathbf{W} \mid \mathbf{e}) = \mathrm{argmax}_{\mathbf{w}} P(\mathbf{w}, \mathbf{e})$
   - Important: We *cannot* find $\mathbf{w}$ by finding the maximum likely value for each variable individually
3. **Marginal MAP query**
   - $\mathrm{MAP}(\mathbf{Y} \mid \mathbf{e}) = \mathrm{argmax}_{\mathbf{y}} P(\mathbf{y} \mid \mathbf{e})$
   - Let $\mathbf{Z} = \mathbf{X} \setminus \mathbf{E} \cup \mathbf{Y}$
   - $\mathrm{MAP}(\mathbf{Y} \mid \mathbf{e}) = \mathrm{argmax}_{\mathbf{y}} \sum_{\mathbf{z}} P(\mathbf{z}, \mathbf{y} \mid \mathbf{e})$

MAP EXAMPLE

A	B	P(A, B)
t	t	0.10
t	f	0.25
f	t	0.35
f	f	0.30

Maximum likely assignment for A = f

Maximum likely assignment for B = f

MAP (A,B) = <A=f, B=t>

Number of Parameters

- Assuming everything is binary
- $P(V_1)$ requires
  - 1 independent parameter
- $P(V_1, V_2, \ldots, V_n)$ requires
  - 2^n-1 independent parameters
- $P(V_1 | V_2)$ requires
  - 2 independent parameters
- $P(V_1, V_2, \ldots, V_n \mid V_{n+1}, V_{n+2}, \ldots, V_{n+m})$ requires
  - $2^m \times (2^n\text{-}1)$ independent parameters

Continuous Spaces

- Assume X is continuous and Val(X) = [0,1]
- If you would like to assign the same probability to all real numbers in [0, 1], what is, for e.g., P(X=0.5) = ?
- Answer: P(X=0.5) = 0.

Probability Density Function

- We define **probability density function**, p(x), a non-negative integrable function, such that $\int_{Val(X)} p(x)dx = 1$

$$P(X \le a) = \int_{-\infty}^{a} p(x)dx$$

$$P(a \le X \le b) = \int_{a}^{b} p(x)dx$$

UNIFORM DISTRIBUTION

- A variable X has a uniform distribution over [a,b] if it has the PDF

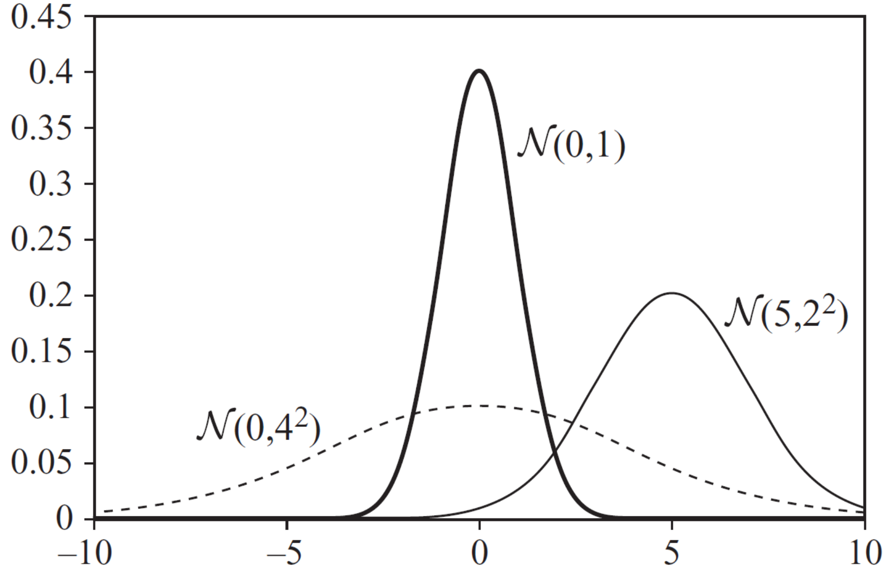
$$p(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & otherwise \end{cases}$$

Check and make sure that p(x) integrates to 1.

Gaussian Distribution

- A variable X has a Gaussian distribution with mean μ and variance σ^2, if it has the PDF

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Can p(x) be ever greater than 1?

Conditional Probability

- We want P(Y | X=x) where X is continuous, Y is discrete
- P(Y | X=x) = P(Y,X=x) / P(X=x)
  - What's wrong with this expression?
- Instead, we use the following expression

$$P(Y \mid X = x) = \lim_{\varepsilon \to 0} P(Y \mid x - \varepsilon \leq X \leq x + \varepsilon)$$

Conditional Probability

- We want p(Y|X) where X is discrete, Y is continuous
- How would you represent it?

Expectation

$$E_P[X] = \sum_x xP(x)$$

$$E_P[X] = \int_x xp(x)dx$$

$$E_P[aX + b] = aE_P[X] + b$$

$$E_P[X + Y] = E_P[X] + E_P[Y]$$

$$E_P[X \mid y] = \sum_x xP(x \mid y)$$

What about E[X*Y]?

VARIANCE

$$Var_P[X] = E_P\left[\left(X - E_P[X]\right)^2\right]$$

$$Var_P[X] = E_P\left[X^2\right] - \left(E_P[X]\right)^2$$

Can you derive the second expression using the first expression?

$$Var_P[aX + b] = a^2 Var_P[X]$$

What is Var[X+Y]?

Uniform and Gaussian Distribution

- If $X \sim N(\mu,\sigma^2)$, then $E[X] = \mu$, $Var[X] = \sigma^2$
- What about the expectation and variance of a uniform distribution?

SUMMARY

- Definition of probability
- Conditional rule
- Summation rule
- Bayes rule
- Chain rule
- Marginal independence
- Conditional independence
- Querying a distribution
- Number of parameters
- Continuous spaces
- Uniform distribution
- Gaussian distribution
- Expectation
- Variance