CS 581 – Advanced Artificial Intelligence

Topic: Bayesian Networks

Mustafa Bilgic

http://www.cs.iit.edu/~mbilgic

https://twitter.com/bilgicm

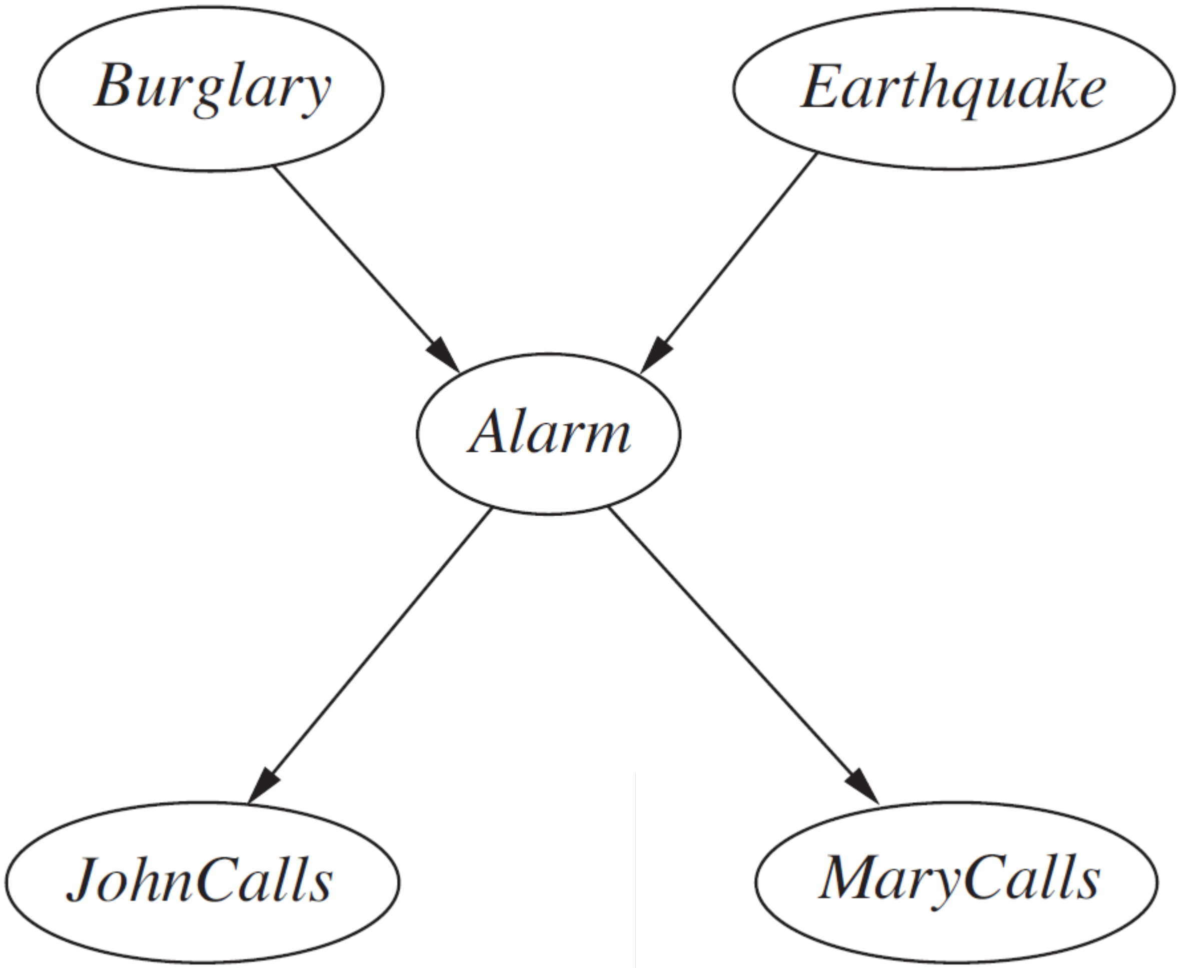
Motivation

- Efficient, intuitive, and modular representation of probability distributions
  - Represent joint and conditional distributions
- Structured and efficient inference
  - Answer probability and MAP queries
- BN structure represents correlation but can be used to answer causality questions under certain conditions

An Example

- Five binary variables
  - Earthquake, Burglary, Alarm, MaryCalls, JohnCalls
- Assume the following
  - E and B are uncorrelated
  - E and M are related only through A; similarly, E, J, and A
  - B and M are related only through A; similarly, B, J, and A
  - M and J are directly related through A; undirectly related through E and B; otherwise, M and J are unrelated
- One approach
  - Represent and estimate the full joint $P(E, B, A, M, J)$
    - How many independent parameters?
    - What can you tell about the relationships between the variables?
- Alternative approach
  - Bayesian network (next slide)

Burglary Example



Possible Queries

- $P(B \mid J = true)$
- $P(B \mid M = true, J = true)$
- $P(M \mid B = true)$
- $P(M \mid B = false)$
- $P(M, J \mid B = true)$
- $P(M \mid J = true)$
- …

We'll Cover

- Bayesian networks (in detail)
  - https://en.wikipedia.org/wiki/Bayesian_network
- Hidden Markov Models (in detail)
  - https://en.wikipedia.org/wiki/Hidden_Markov_model
- Dynamic Bayesian networks (brief)
  - https://en.wikipedia.org/wiki/Dynamic_Bayesian_network
- Influence diagrams (in detail)
  - https://en.wikipedia.org/wiki/Influence_diagram
- Causal networks (brief)

Bayesian Networks

- Random variables = nodes
- Direct relationships = directed edges
- BNs capture independencies
  - More compact than full joint representation
- Graphs provide
  - Graph theory / efficient reasoning
  - Intuition

Directed Graphs

- A **graph** consists of **nodes** and **edges**
- **Nodes:** $\mathcal{X}=\{X_1, X_2, \ldots, X_n\}$
- **Undirected Edge:** $X_i - X_j$
- **Directed Edge:** $X_i \rightarrow X_j$
- A graph is **directed** if its *all* edges are directed

Relationships

- $X_i \rightarrow X_j$
  - X_i is the **parent**
  - X_j is the **child**
- X_i is an **ancestor** of X_j if there is a directed path from X_i to X_j
- X_i is a **descendant** of X_j if there is a directed path from X_j to X_i
- **Nondescendants**$(X_i) \equiv \mathcal{X} \setminus \text{Descendants}(X_i)$

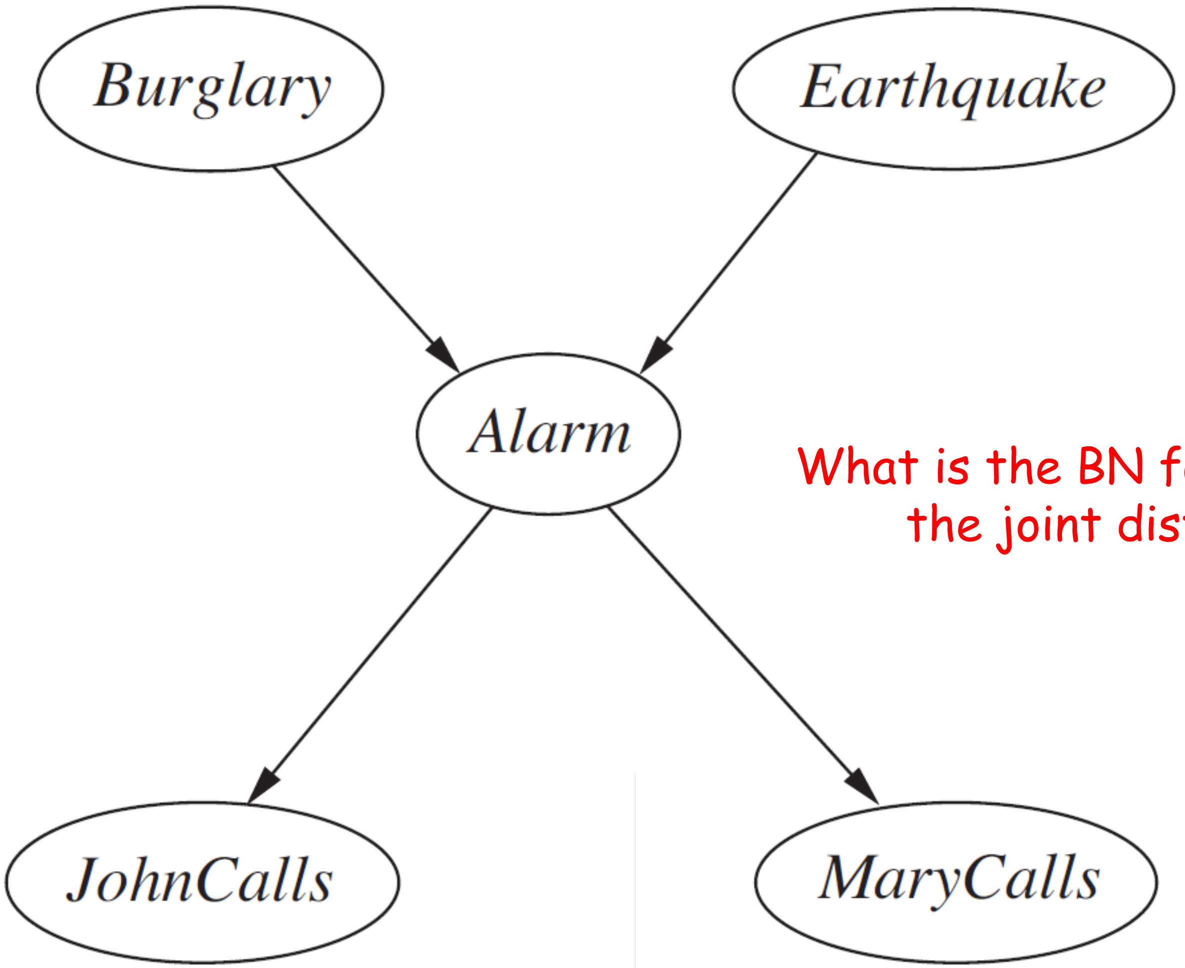
Bayesian network Factorization

$$P\left(X_1,\ldots,X_n\right)=\prod_i P\left(X_i \mid Pa\left(X_i\right)\right)$$

BURGLARY EXAMPLE

$P(B,E,A,J,M)$

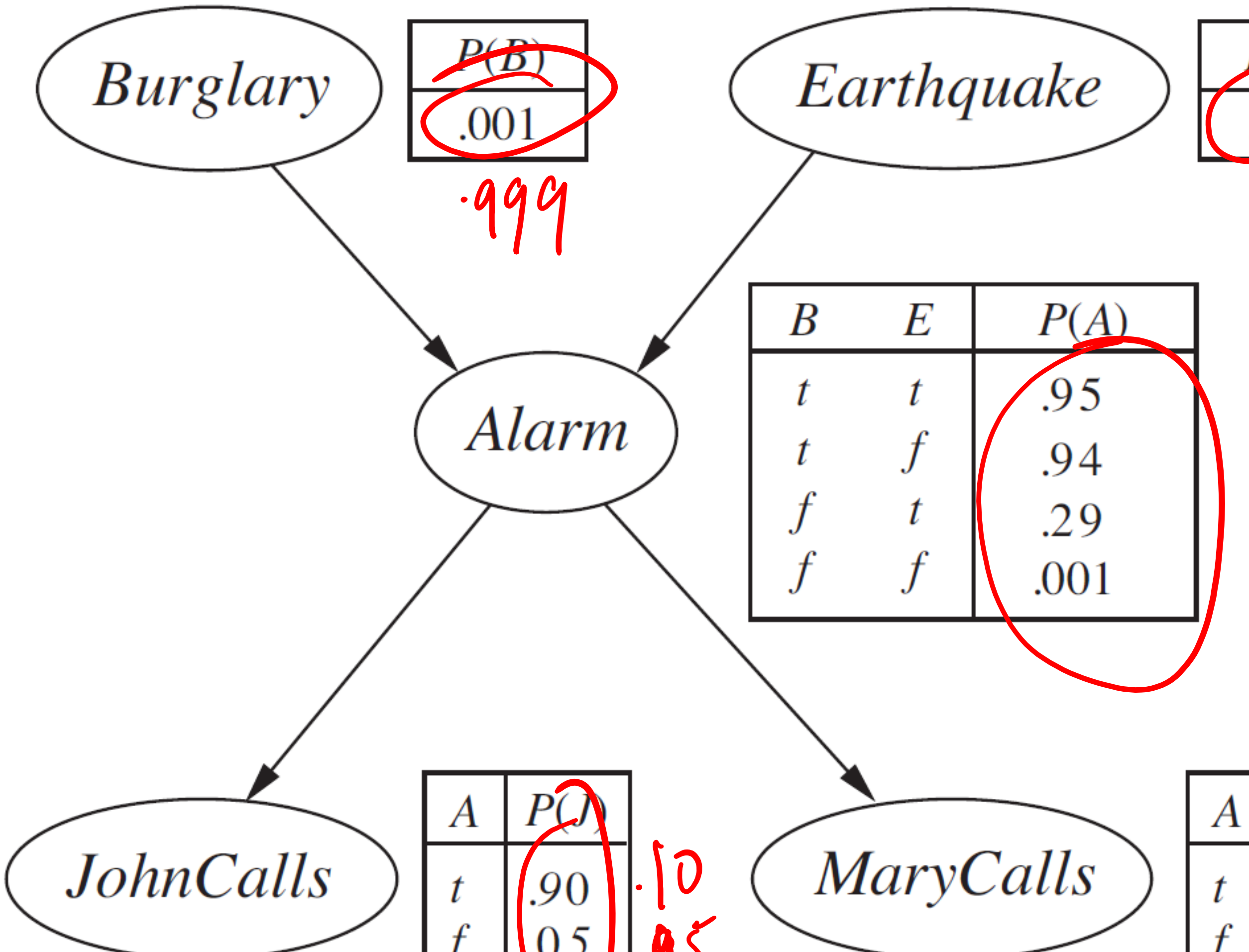
$= P(B) \cdot P(E) \cdot P(A|B,E) \cdot P(M|A) \cdot P(J|A)$



What is the BN factorization of the joint distribution?

BURGLARY EXAMPLE

10 ind vs 31



P(B)
.001

.999

P(E)
.002

.998

B	E	P(A)
t	t	.95
t	f	.94
f	t	.29
f	f	.001

.05

.999

A	P(J)
t	.90
f	.05

.10

.95

A	P(M)
t	.70
f	.01

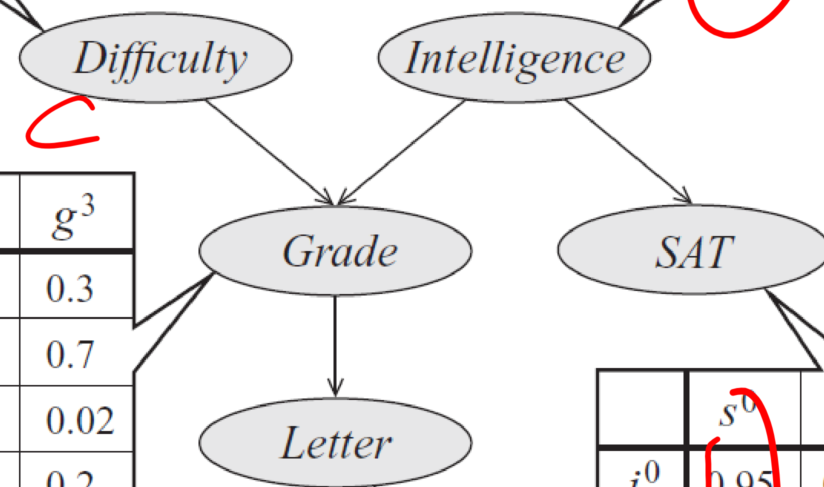
Student Example

$P(D, I, G, S, L)$

$2 \times 2 \times 3 \times 2 \times 2 - 1 = 47$

d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3



A B C

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^2	0.99	0.01

15 ind

Independencies

- X is independent of its non-descendants given its parents
  - $X \perp \text{Non-descendants}(X) \mid \text{Parents}(X)$
- D-separation

INDEPENDENCIES – D-SEPARATION

- Definition: Observed ≡ Its value is known
- Causal trail
  - $X \rightarrow Y \rightarrow Z$; E.g., Burglary → Alarm → MaryCalls
  - X and Z are independent if Y is observed
- Evidential trail
  - $X \leftarrow Y \leftarrow Z$; E.g., MaryCalls ← Alarm ← Burglary
  - X and Z are independent if Y is observed
- Common cause
  - $X \leftarrow Y \rightarrow Z$; E.g., JohnCalls ← Alarm → MaryCalls
  - X and Z are independent if Y is observed
- Common effect
  - $X \rightarrow Y \leftarrow Z$; E.g., Burglary → Alarm ← Earthquake
  - X and Z are marginally independent, but they become dependent if Y or any of Y's descendants are observed

EXAMPLES

- X causes Y and Y causes Z; no direct relationship between X and Z
  - $X \rightarrow Y \rightarrow Z$
  - Nothing is marginally independent of each other
  - $Z \perp X \mid Y$
- Y causes both X and Z; no direct relationship between X and Z
  - $X \leftarrow Y \rightarrow Z$
  - Nothing is marginally independent of each other
  - $Z \perp X \mid Y$
- Both X and Z cause Y; no direct relationship between X and Z
  - $X \rightarrow Y \leftarrow Z$
  - X and Z are marginally independent
  - X and Z become dependent when the value of Y is known

INDEPENDENCE ⇔ FACTORIZATION

- Independence ⇒ Factorization
- Factorization ⇒ Independence

Reasoning Patterns

- **Causal reasoning**
  - From causes to effects
    - E.g., Burglary to Alarm to MaryCalls
    - E.g., Intelligence to Grade to Letter
- **Evidential reasoning**
  - From effects to the causes
    - E.g., JohnCalls to Alarm to Earthquake
    - E.g, Letter to Grade to Difficulty
- **Explaining away/inter-causal reasoning**
  - Causes of a common effect interact
    - E.g., Earthquake, Burglary, and Alarm (and Alarm's descendants)
    - E.g., Difficulty, Intelligence, and Grade (and Grade's descendants)

Inference in Bayesian Networks

- There are several methods, some are exact and some are approximate
- We will study two in this class
  - *Variable elimination*
  - *Message passing*

Variable Elimination

- Let
  - **V** be the set of all variables, **Q** be the set of query variables, **E** be the set of evidence variables
  - P(**Q** | **E**) be the query

1. Write down the joint dist. using the Bayesian network structure
2. Set the variables in **E** to their respective values
3. Sum over all variables in $\mathbf{V} \backslash (\mathbf{Q} \cup \mathbf{E})$
   a) Pick an order for variables in $\mathbf{V} \backslash (\mathbf{Q} \cup \mathbf{E})$
   b) For each variable V_i in $\mathbf{V} \backslash (\mathbf{Q} \cup \mathbf{E})$**,** create a new factor by
      - Multiplying all the factors that contains V_i, and
      - Summing over possible values of V_i
4. Normalize the last remaining factor (this step is unnecessary if **E** is empty)

IRRELEVANT

- Let
  - **V** be the set of all variables, **Q** be the set of query variables, **E** be the set of evidence variables
  - P(**Q** | **E**) be the query
- $Y \in \boldsymbol{V} \setminus \{\boldsymbol{Q} \cup \boldsymbol{E}\}$ is irrelevant iff
  - $Y \notin Ancestors\ of \{\boldsymbol{Q} \cup \boldsymbol{E}\}$
    - or
  - $Y \perp \boldsymbol{Q} \mid \boldsymbol{E}$
- Examples

VARIABLE ELIMINATION EXAMPLES

- See OneNote

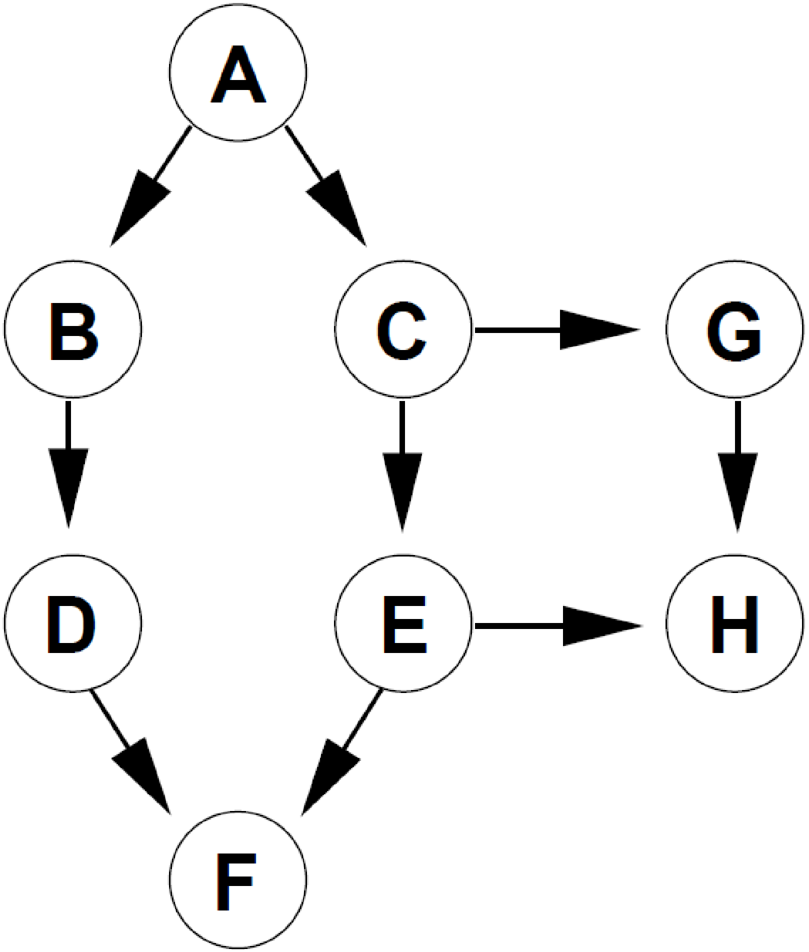
Message Passing

- Junction tree algorithm
- See OneNote for an example

Message Passing - motivation

- We are interested in multiple marginal/conditional probabilities
- In variable elimination, we define our target upfront and then eliminate the others
- If we need probabilities for other variables, there is no apparent way of reusing shared computations
- In the student example, assume that I'm interested in P(G) and P(L). What are some of the shared computations?

EXAMPLE



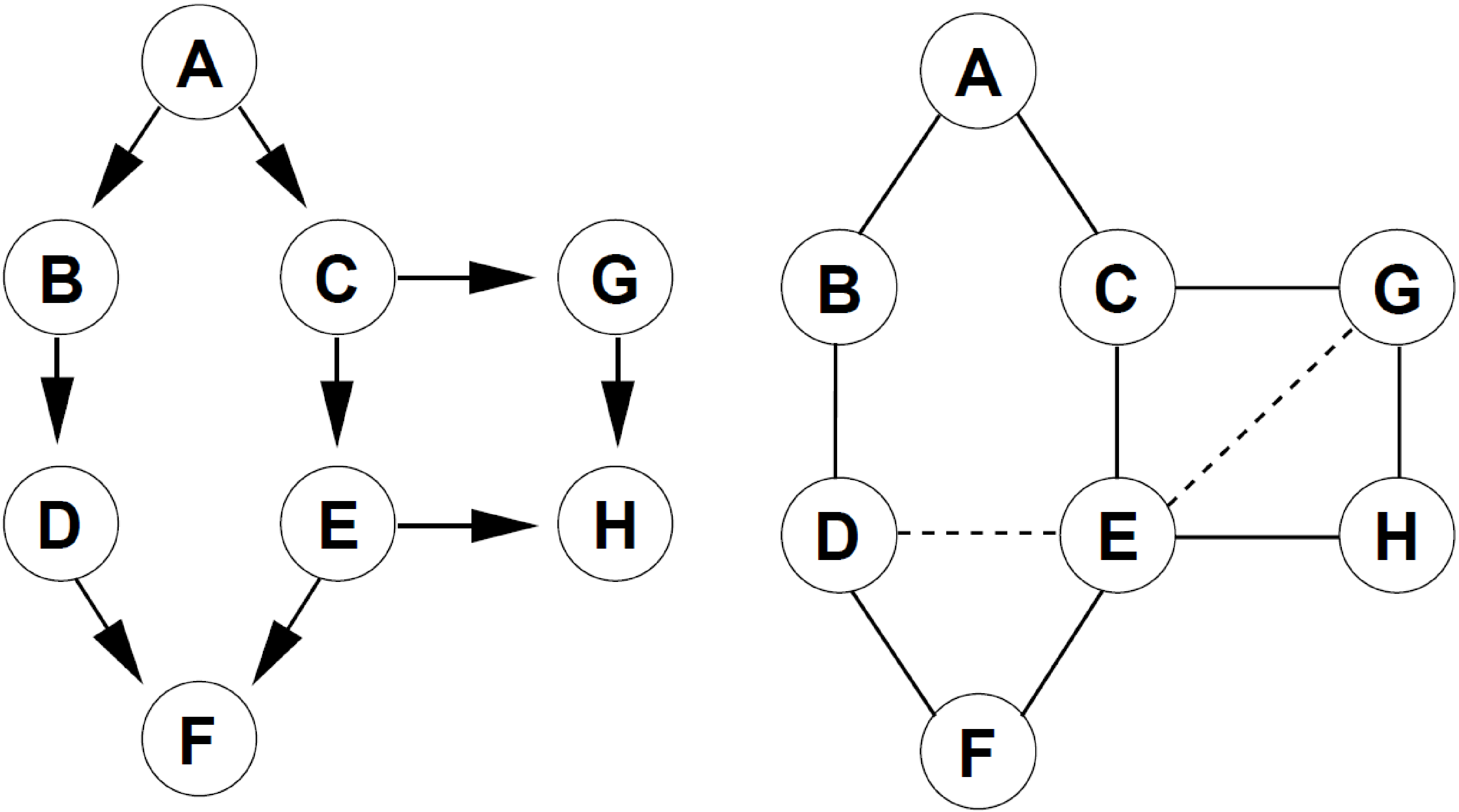
Calculate P(H) using variable elimination

Now, calculate P(G) using variable elimination

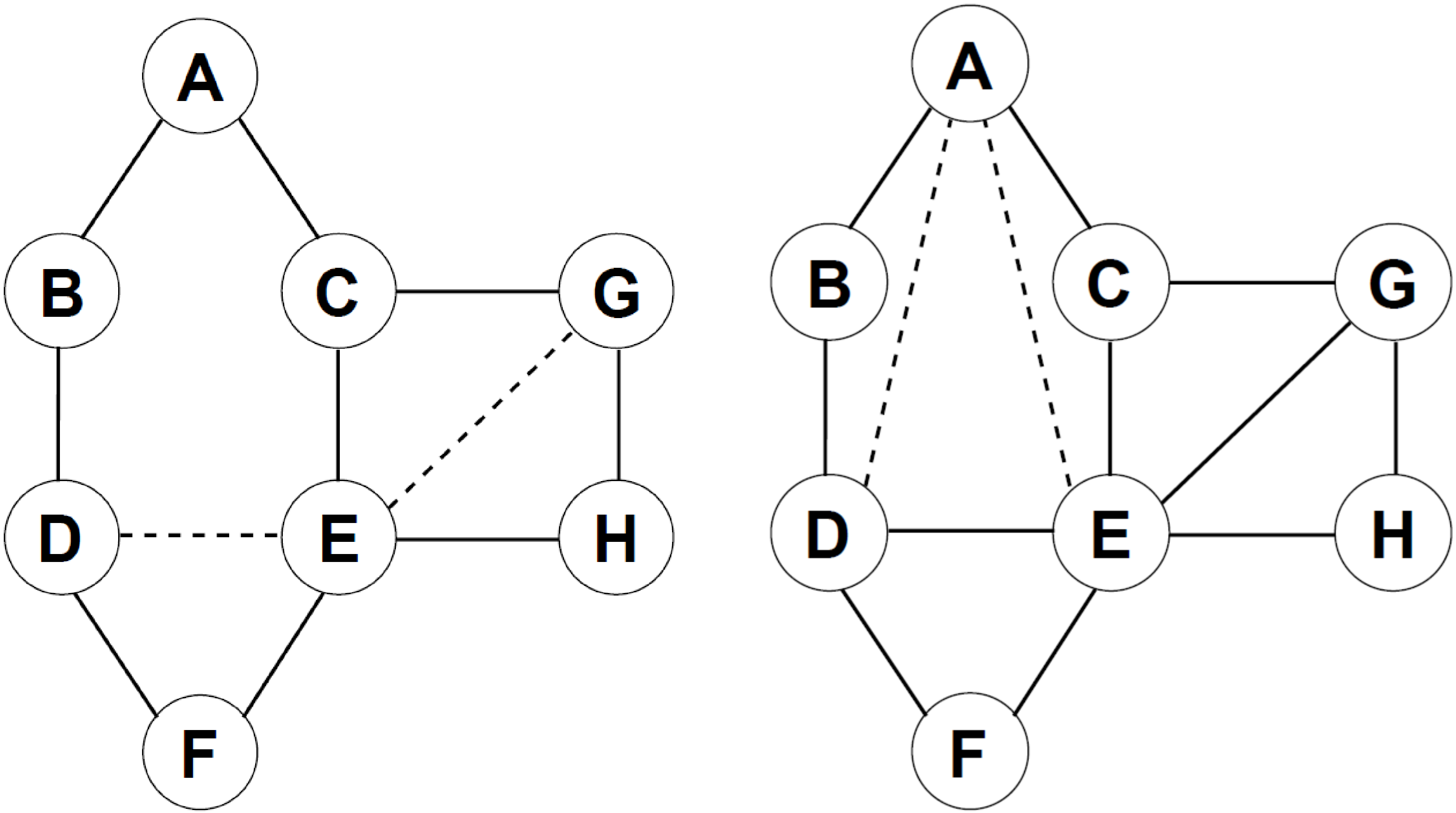
Variable elimination as graph transformation

- First, construct the moral graph
- Then, eliminate variables so that each elimination introduces the fewest number of edges
- Take note of the factors

EXAMPLE - MORALIZE



ELIMINATION ORDER: H, G, F, C, B, D, E, A



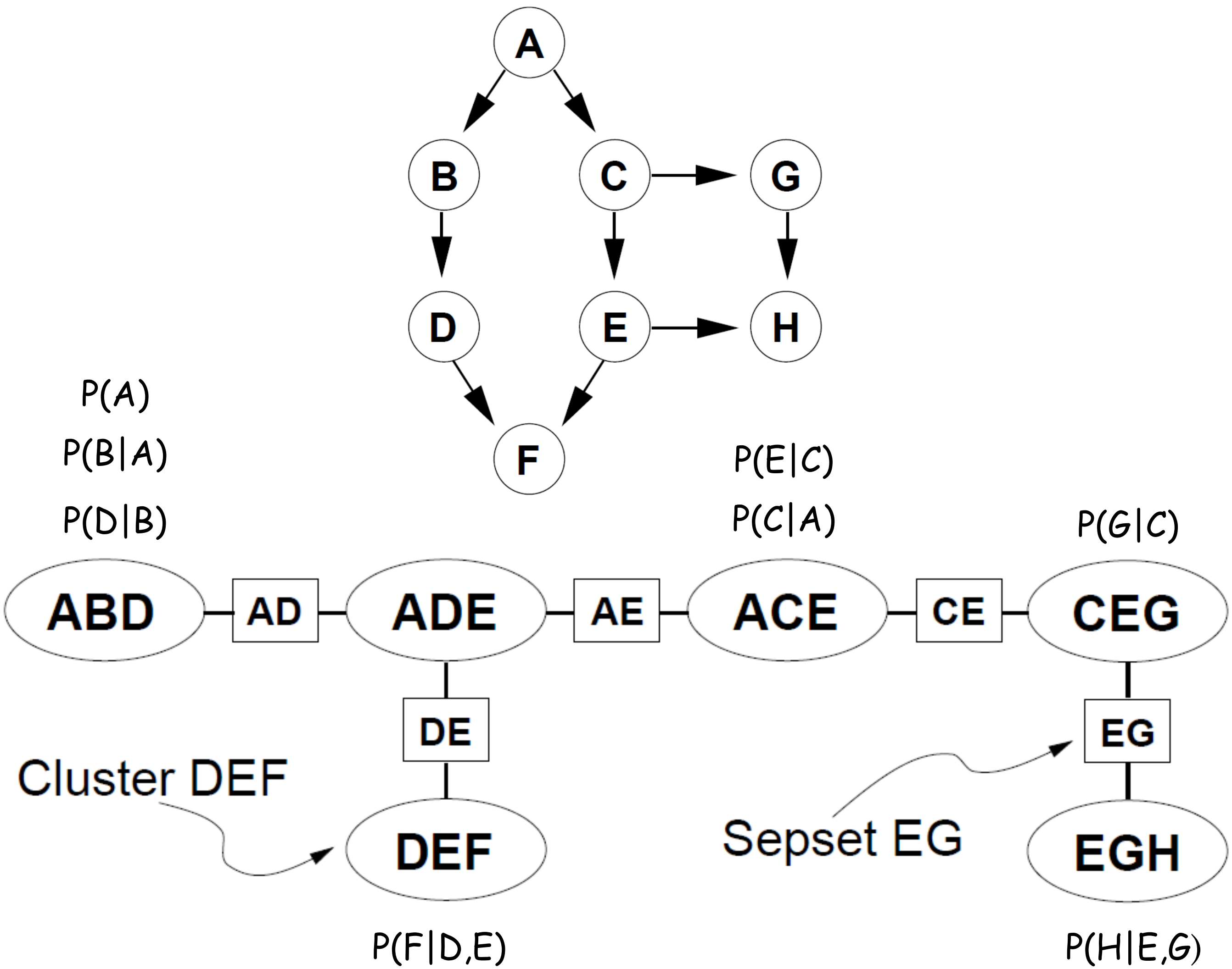
Cluster graph

- A *cluster graph* $\mathcal{U}$ for a set of factors Φ over $\mathcal{X}$ is an undirected graph, each of whose nodes are associated with a cluster $\boldsymbol{C}_i \subseteq \mathcal{X}$.
- A cluster graph must be *family preserving* – each factor $\phi \in \Phi$ must be associated with a cluster $\boldsymbol{C}_i$, denoted as $\alpha(\phi)$, such that Scope$[\phi] \subseteq \boldsymbol{C}_i$.
- Each edge between a pair of clusters $\boldsymbol{C}_i$ and $\boldsymbol{C}_j$ is associated with a *sepset* $\boldsymbol{S}_{ij} \subseteq \boldsymbol{C}_i \cap \boldsymbol{C}_j$

Running intersection property

- Let $\mathcal{T}$ be a cluster tree. $\mathcal{T}$ has *running intersection property* if, whenever there is a variable X such that $X \in \boldsymbol{C}_i$ and $X \in \boldsymbol{C}_j$, then X is also in every cluster in the unique path in $\mathcal{T}$ between $\boldsymbol{C}_i$ and $\boldsymbol{C}_j$.
- A cluster tree that satisfies the running intersection property is also called the *join/clique/junction tree*.
- **Theorem**: A cluster tree obtained through a run of variable elimination satisfies the running intersection property; that is, it is a clique tree.

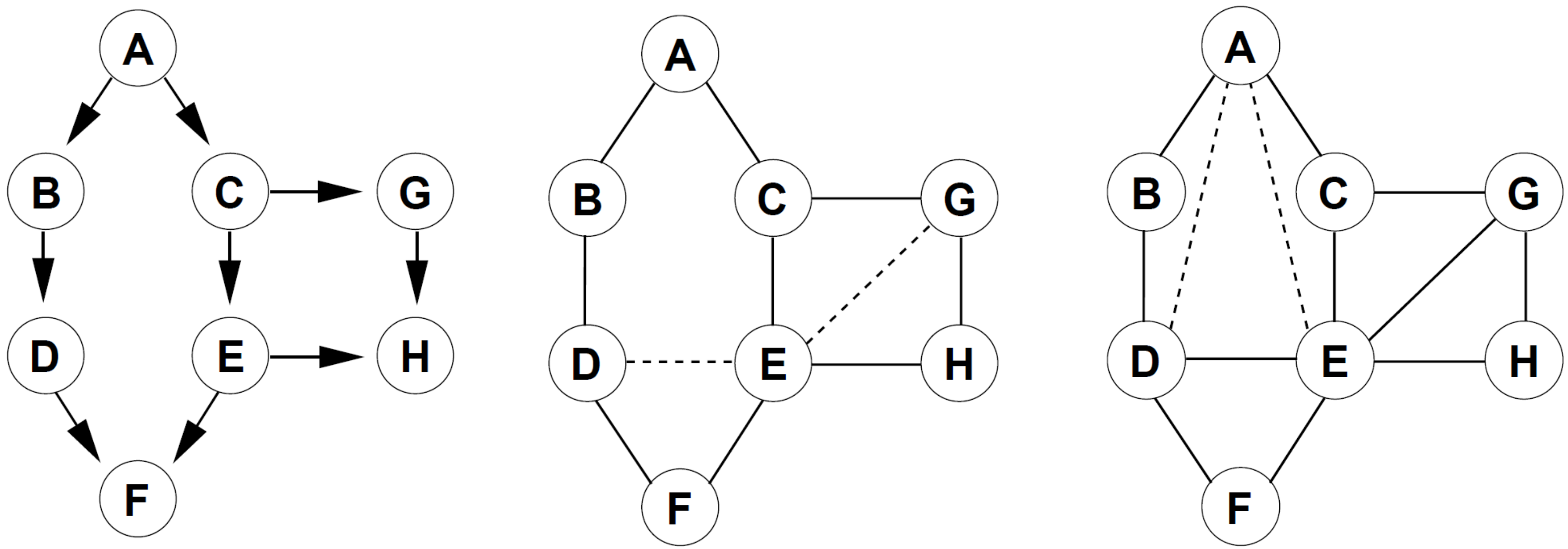
EXAMPLE CLIQUE TREE

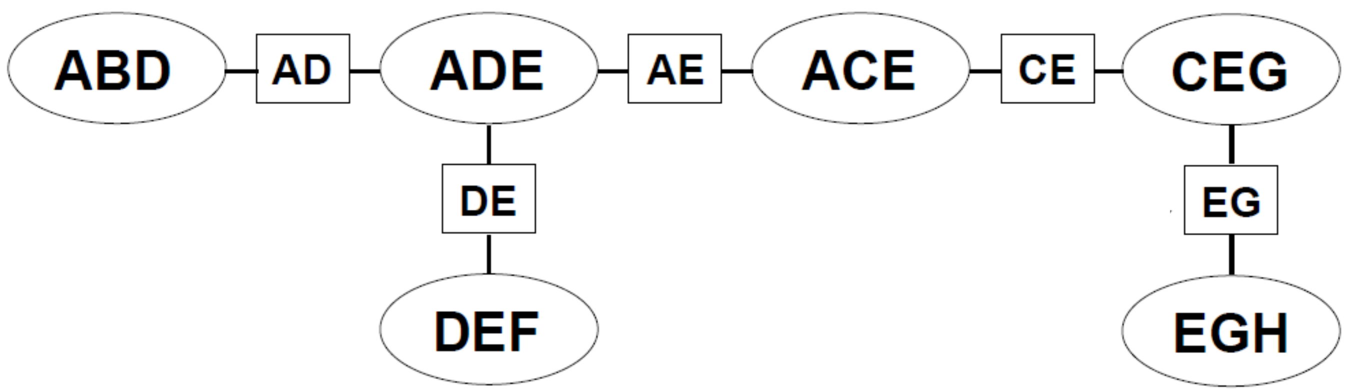


Construct a clique tree

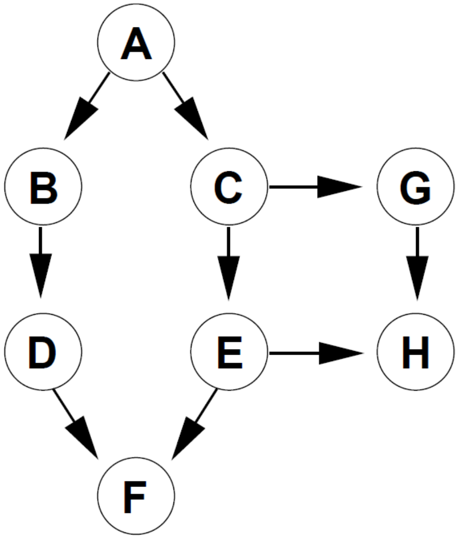
1. Moralize the graph
2. Pick a variable elimination order
3. Eliminate the variables, noting the maximal cliques
4. The cliques are the nodes of the tree
5. Until a tree is formed (i.e., n-1 edges are added)
   a) Connect two disconnected components by a maximal size sepset

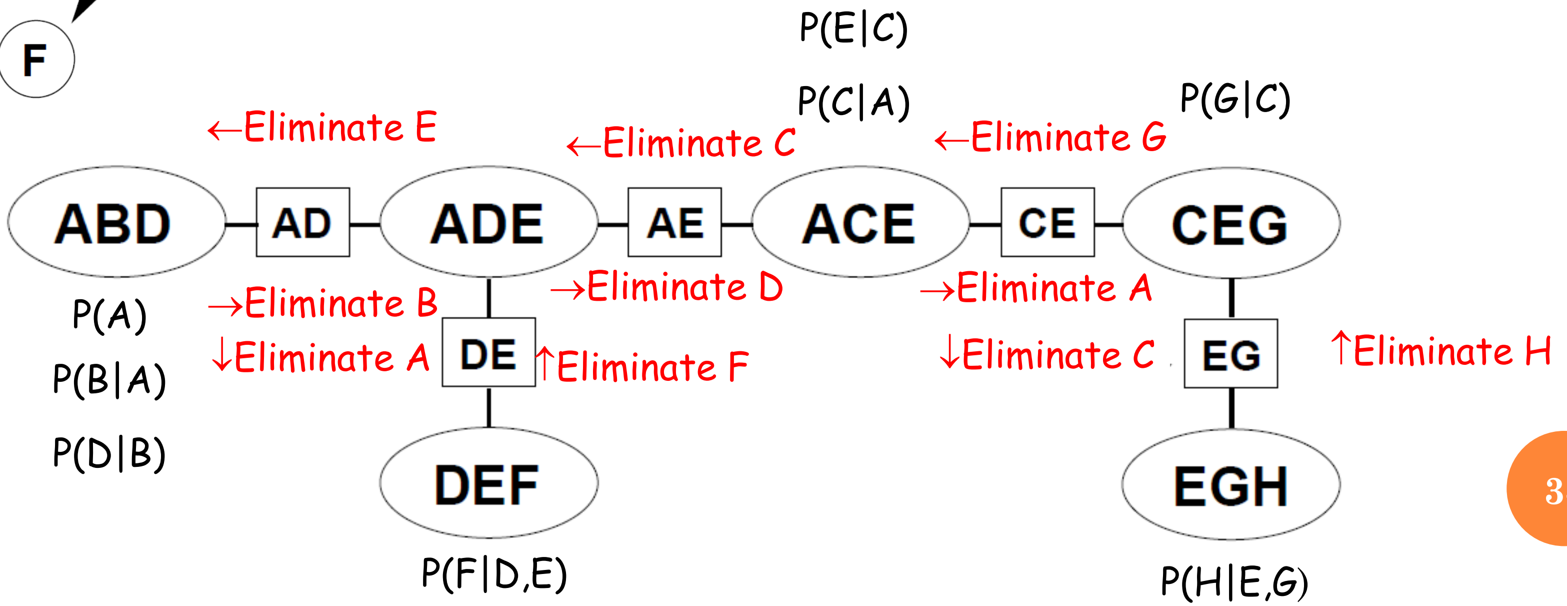
Elimination order: H, G, F, C, B, D, E, A





Variable elimination on junction tree





MESSAGE PASSING ON JUNCTION TREE

- Clusters receive from and send messages to its neighbors
- Each message pass consists of elimination of one or more variables
- A cluster $\boldsymbol{C}_i$ is ready to send a message to its neighbor $\boldsymbol{C}_j$, when it receives messages from its *all other* neighbors
- A message from $\boldsymbol{C}_i$ to $\boldsymbol{C}_j$ is computed as follows
  - $\boldsymbol{C}_i$ multiples all the factors assigned to it, and all the messages it received from its *other* neighbors
  - It sums out $\boldsymbol{C}_i \setminus \boldsymbol{S}_{ij}$

A MESSAGE

$$\delta_{i\to j} = \sum_{C_i \setminus S_{ij}} \left(\left(\prod_{\phi:\alpha(\phi)=i} \phi \right) \times \left(\prod_{k\in(Nb_i - \{j\})} \delta_{k\to i} \right) \right)$$

Belief

$$\beta_i = \left(\prod_{\phi:\alpha(\phi)=i} \phi \right) \times \left(\prod_{k \in Nb_i} \delta_{k \to i} \right)$$

Linear Gaussian Exercise

- Given
  - $p(X) \sim N\left(\mu_X;\ \sigma_X^2\right)$
  - $p(Y \mid X) \sim N\left(\beta_0 + \beta_1 \mu_X;\ \sigma_Y^2\right)$
- Calculate
  - $p(Y)$
  - $p(Y \mid X = 5)$
  - $p(X \mid Y = 5)$
  - $p(X, Y)$