CS 581 – Advanced Artificial Intelligence

Topic: Bayesian Networks

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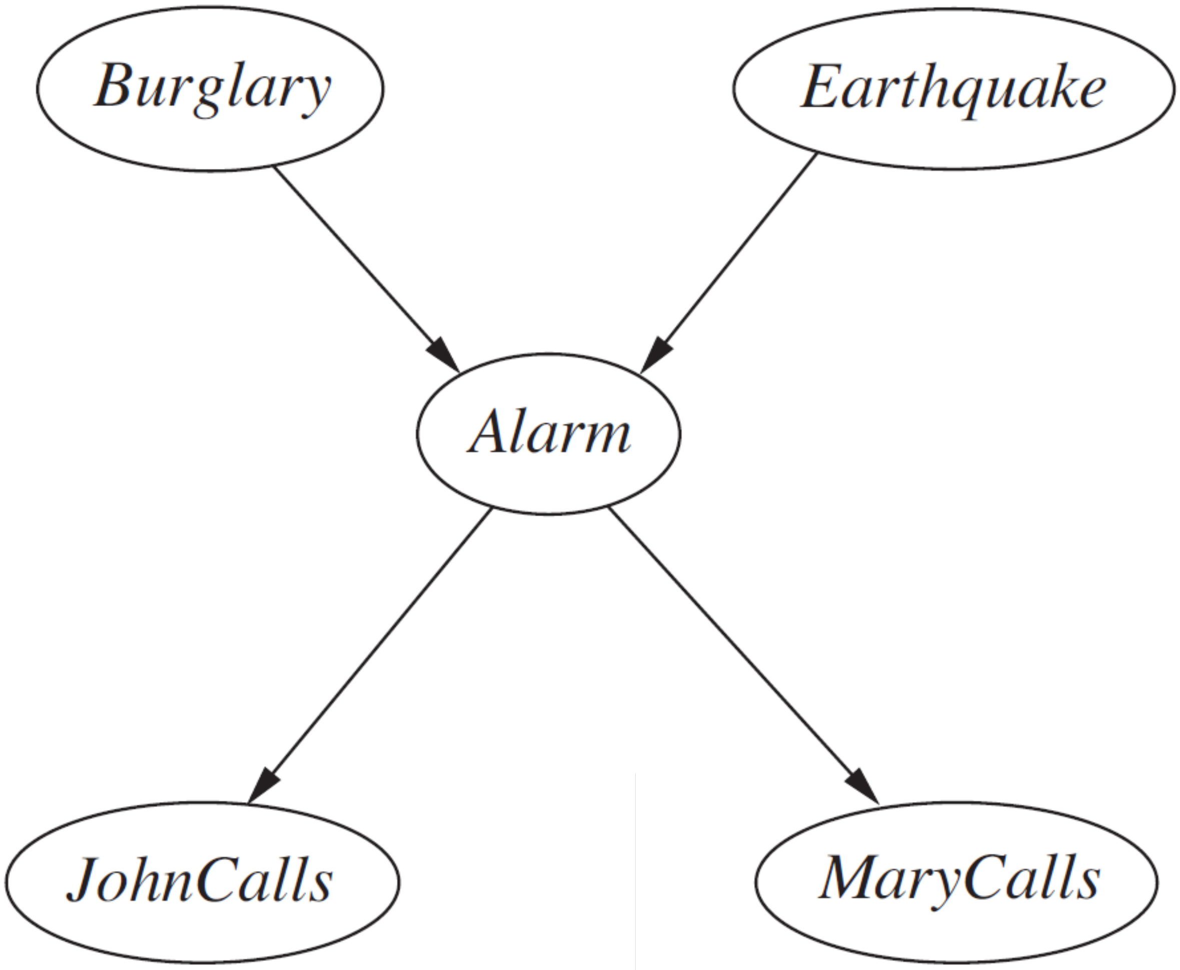
Motivation

- Efficient, intuitive, and modular representation of probability distributions
  - Represent joint and conditional distributions
- Structured and efficient inference
  - Answer probability and MAP queries
- BN structure represents correlation but can be used to answer causality questions under certain conditions

An Example

- Five binary variables
  - Earthquake, Burglary, Alarm, MaryCalls, JohnCalls
- Assume the following
  - E and B are uncorrelated
  - E and M are related only through A; similarly, E, J, and A
  - B and M are related only through A; similarly, B, J, and A
  - M and J are directly related through A; undirectly related through E and B; otherwise, M and J are unrelated
- One approach
  - Represent and estimate the full joint $P(E, B, A, M, J)$
    - How many independent parameters?
    - What can you tell about the relationships between the variables?
- Alternative approach
  - Bayesian network (next slide)

Burglary Example



Possible Queries

- $P(B \mid J = true)$
- $P(B \mid M = true, J = true)$
- $P(M \mid B = true)$
- $P(M \mid B = false)$
- $P(M, J \mid B = true)$
- $P(M \mid J = true)$
- …

We'll Cover

- Bayesian networks (in detail)
  - https://en.wikipedia.org/wiki/Bayesian_network
- Hidden Markov Models (in detail)
  - https://en.wikipedia.org/wiki/Hidden_Markov_model
- Dynamic Bayesian networks (brief)
  - https://en.wikipedia.org/wiki/Dynamic_Bayesian_network
- Influence diagrams (in detail)
  - https://en.wikipedia.org/wiki/Influence_diagram
- Causal networks (brief)

Bayesian Networks

- Random variables = nodes
- Direct relationships = directed edges
- BNs capture independencies
  - More compact than full joint representation
- Graphs provide
  - Graph theory / efficient reasoning
  - Intuition

Directed Graphs

- A **graph** consists of **nodes** and **edges**
- **Nodes:** $\mathcal{X}=\{X_1, X_2, \ldots, X_n\}$
- **Undirected Edge:** $X_i - X_j$
- **Directed Edge:** $X_i \rightarrow X_j$
- A graph is **directed** if its *all* edges are directed

Relationships

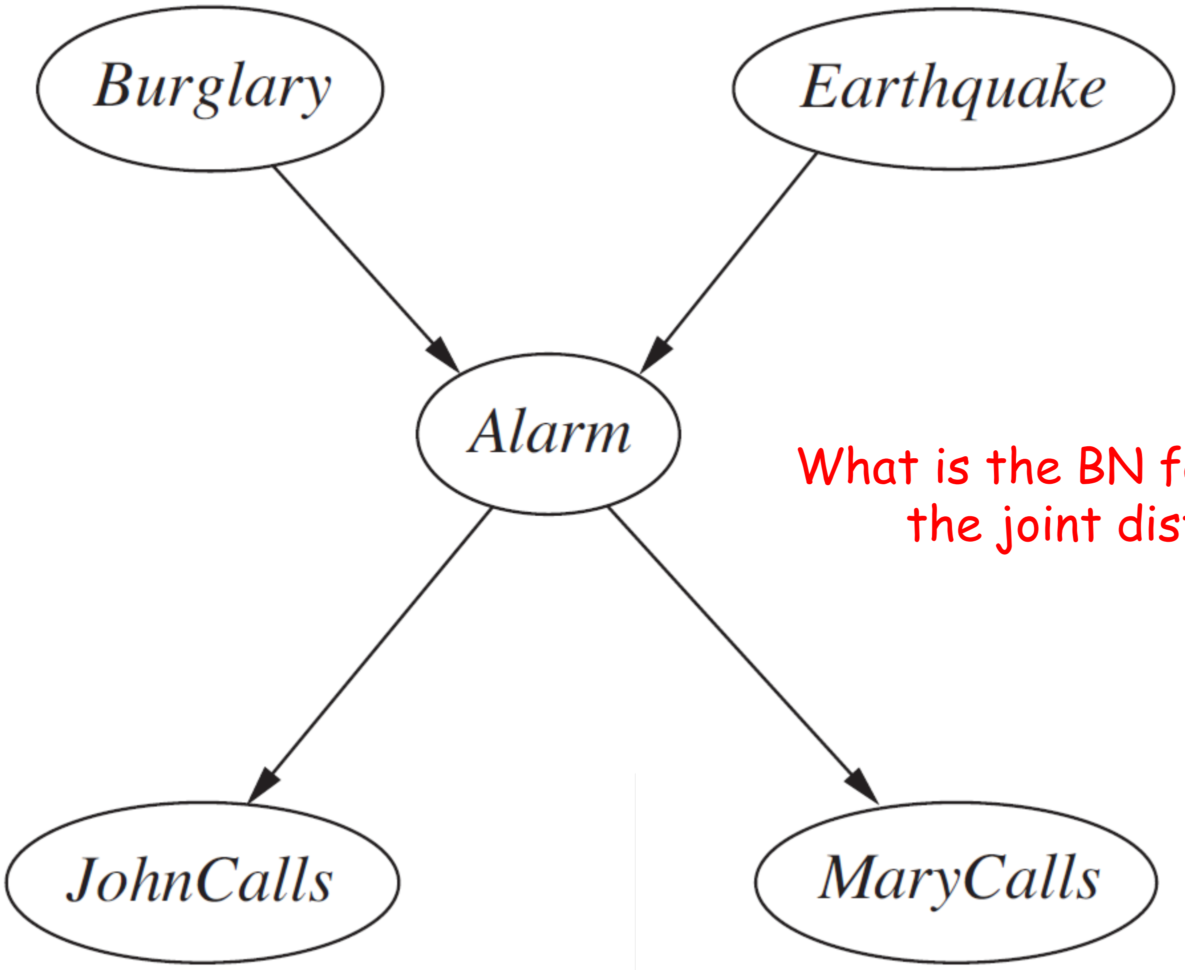
- $X_i \rightarrow X_j$
  - X_i is the **parent**
  - X_j is the **child**
- X_i is an **ancestor** of X_j if there is a directed path from X_i to X_j
- X_i is a **descendant** of X_j if there is a directed path from X_j to X_i
- **Nondescendants**$(X_i) \equiv \mathcal{X} \setminus$ Descendants(X_i)

Bayesian network Factorization

$$P\left(X_1,\ldots,X_n\right)=\prod_i P\left(X_i \mid Pa\left(X_i\right)\right)$$

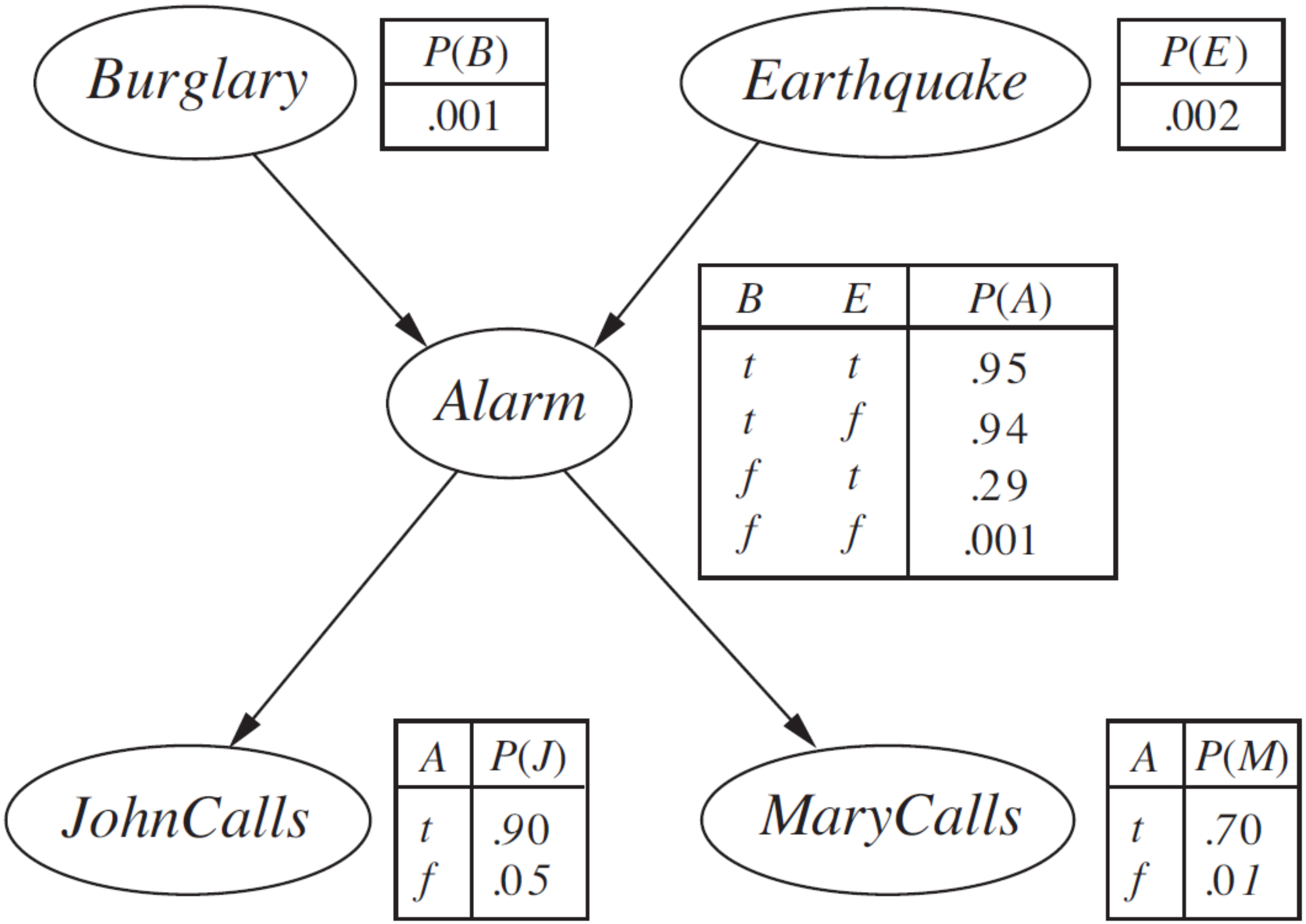
Burglary Example

$$P(B,E,A,J,M) = P(B) \cdot P(E) \cdot P(A|B,E) \cdot P(M|A) \cdot P(J|A)$$



What is the BN factorization of the joint distribution?

Burglary Example



P(B)
.001

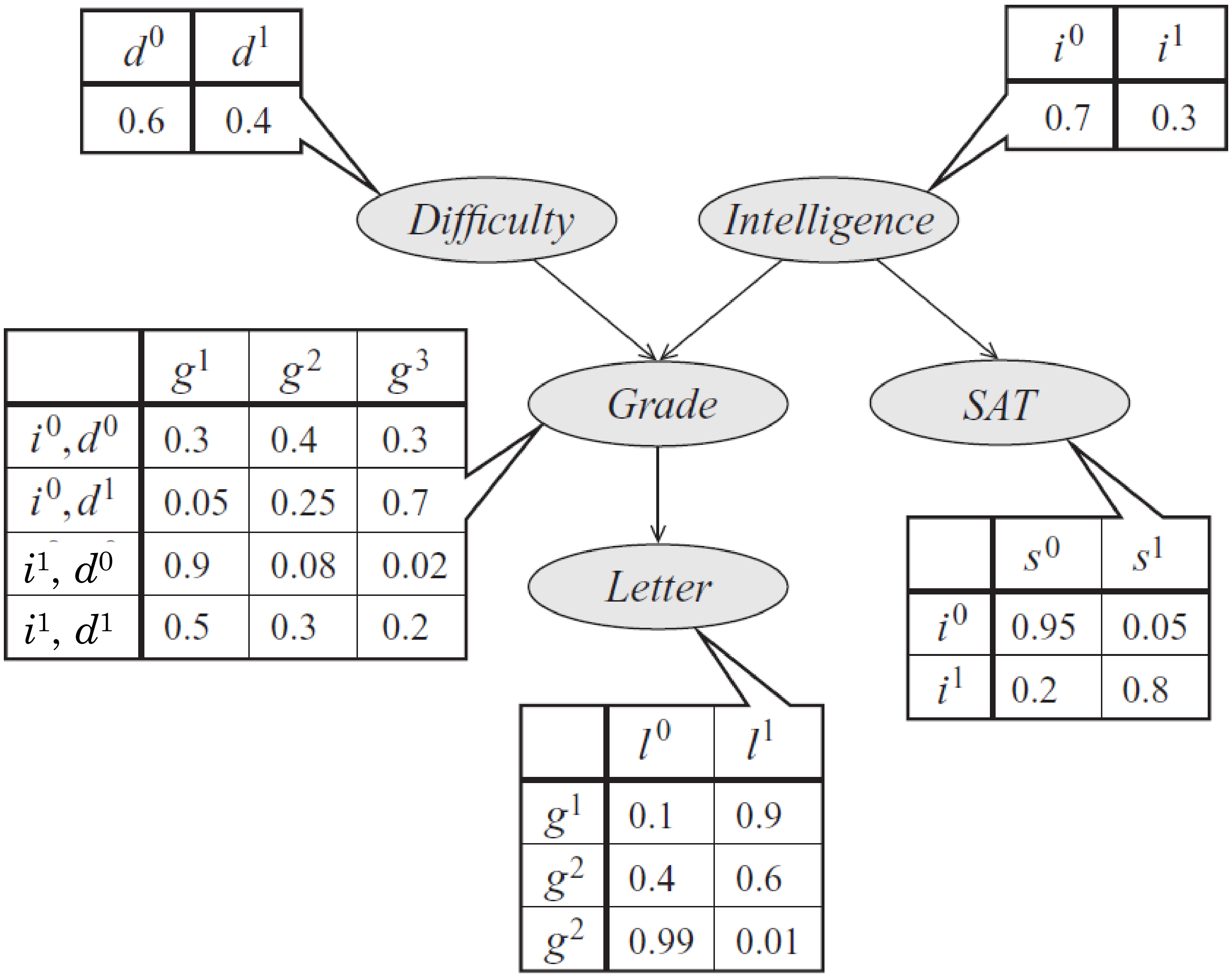
P(E)
.002

B	E	P(A)
t	t	.95
t	f	.94
f	t	.29
f	f	.001

A	P(J)
t	.90
f	.05

A	P(M)
t	.70
f	.01

STUDENT EXAMPLE



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^2	0.99	0.01

Independencies

- X is independent of its non-descendants given its parents
  - $X \perp \text{Non-descendants}(X) \mid \text{Parents}(X)$
- D-separation

Independencies – D-separation

- Definition: Observed ≡ Its value is known
- Causal trail
  - $X \rightarrow Y \rightarrow Z$; E.g., Burglary → Alarm → MaryCalls
  - X and Z are independent if Y is observed
- Evidential trail
  - $X \leftarrow Y \leftarrow Z$; E.g., MaryCalls ← Alarm ← Burglary
  - X and Z are independent if Y is observed
- Common cause
  - $X \leftarrow Y \rightarrow Z$; E.g., JohnCalls ← Alarm → MaryCalls
  - X and Z are independent if Y is observed
- Common effect
  - $X \rightarrow Y \leftarrow Z$; E.g., Burglary → Alarm ← Earthquake
  - X and Z are marginally independent, but they become dependent if Y or any of Y's descendants are observed

EXAMPLES

- X causes Y and Y causes Z; no direct relationship between X and Z
  - $X \rightarrow Y \rightarrow Z$
  - Nothing is marginally independent of each other
  - $Z \perp X \mid Y$
- Y causes both X and Z; no direct relationship between X and Z
  - $X \leftarrow Y \rightarrow Z$
  - Nothing is marginally independent of each other
  - $Z \perp X \mid Y$
- Both X and Z cause Y; no direct relationship between X and Z
  - $X \rightarrow Y \leftarrow Z$
  - X and Z are marginally independent
  - X and Z become dependent when the value of Y is known

Independence ⇔ Factorization

- Independence ⇒ Factorization
- Factorization ⇒ Independence

Reasoning Patterns

- **Causal reasoning**
  - From causes to effects
    - E.g., Burglary to Alarm to MaryCalls
    - E.g., Intelligence to Grade to Letter
- **Evidential reasoning**
  - From effects to the causes
    - E.g., JohnCalls to Alarm to Earthquake
    - E.g, Letter to Grade to Difficulty
- **Explaining away/inter-causal reasoning**
  - Causes of a common effect interact
    - E.g., Earthquake, Burglary, and Alarm (and Alarm's descendants)
    - E.g., Difficulty, Intelligence, and Grade (and Grade's descendants)

Inference in Bayesian Networks

- There are several methods, some are exact and some are approximate
- We will study two in this class
  - *Variable elimination*
  - *Message passing on linear chains*

QUERYING A DISTRIBUTION

- **Evidence (E=e)**: what is known, **Query (Y)**: variables of interest, **X** is the set of all variables that include **E**, **Y**, and potentially others

1. **Probability query**
   - $P(\mathbf{Y} \mid \mathbf{e}) = ?$
2. **MAP query**
   - $\mathbf{W} = \mathbf{X} \setminus \mathbf{E}$ (i.e., all the non-evidence variables)
   - $\mathrm{MAP}(\mathbf{W} \mid \mathbf{e}) = \mathrm{argmax}_{\mathbf{w}} P(\mathbf{w}, \mathbf{e})$
   - Important: We *cannot* find **w** by finding the maximum likely value for each variable individually
3. **Marginal MAP query**
   - $\mathrm{MAP}(\mathbf{Y} \mid \mathbf{e}) = \mathrm{argmax}_{\mathbf{y}} P(\mathbf{y} \mid \mathbf{e})$
   - Let $\mathbf{Z} = \mathbf{X} \setminus \mathbf{E} \cup \mathbf{Y}$
   - $\mathrm{MAP}(\mathbf{Y} \mid \mathbf{e}) = \mathrm{argmax}_{\mathbf{y}} \sum_{\mathbf{z}} P(\mathbf{z}, \mathbf{y} \mid \mathbf{e})$

Variable Elimination

- Let
  - **V** be the set of all variables, **Q** be the set of query variables, **E** be the set of evidence variables
  - P(**Q** | **E**) be the query

1. Write down the joint dist. using the Bayesian network structure
2. Set the variables in **E** to their respective values
3. Sum over all variables in $\mathbf{V} \setminus (\mathbf{Q} \cup \mathbf{E})$
   a) Pick an order for variables in $\mathbf{V} \setminus (\mathbf{Q} \cup \mathbf{E})$
   b) For each variable V_i in $\mathbf{V} \setminus (\mathbf{Q} \cup \mathbf{E})$, create a new factor by
      - Multiplying all the factors that contains V_i, and
      - Summing over possible values of V_i
4. Normalize the last remaining factor (this step is unnecessary if **E** is empty)

IRRELEVANT

- Let
  - **V** be the set of all variables, **Q** be the set of query variables, **E** be the set of evidence variables
  - P(**Q** | **E**) be the query
- $Y \in \boldsymbol{V} \setminus \{\boldsymbol{Q} \cup \boldsymbol{E}\}$ is irrelevant iff
  - $Y \notin Ancestors\ of \{\boldsymbol{Q} \cup \boldsymbol{E}\}$
    - or
  - $Y \perp \boldsymbol{Q} \mid \boldsymbol{E}$
- Examples

Variable elimination for MAP

- Variable elimination works by multiplying and summing
- It's also called *sum-product* algorithm
- We can use the same technique for MAP, if we replace the sum operator with a max operator
- The algorithm is called *max-product*
- A few differences
  - Sum is replaced with max
  - All variables (except evidence) are eliminated
  - We need to trace back our steps to find the MAP assignment

Marginal MAP

- **Marginal MAP query**
  - $\mathrm{MAP}(\boldsymbol{Y} \mid \boldsymbol{e}) = \mathrm{argmax}_{\mathrm{y}} \mathrm{P}(\boldsymbol{y} \mid \boldsymbol{e})$
  - Let $\boldsymbol{Z} = \mathcal{X} \setminus \boldsymbol{E} \cup \boldsymbol{Y}$
  - $\mathrm{MAP}(\boldsymbol{Y} \mid \boldsymbol{e}) = \mathrm{argmax}_{\boldsymbol{y}} \sum_{\boldsymbol{z}} \mathrm{P}(\boldsymbol{z}, \boldsymbol{y} \mid \boldsymbol{e})$
- We need to sum out $\boldsymbol{Z}$
- Max-sum-product algorithm
  - First, eliminate $\boldsymbol{Z}$ using sum-product algorithm
  - Then, find MAP for $\boldsymbol{Y}$ using max-product algorithm
- Unfortunately, max and sum cannot be interleaved
- The variables are partitioned into three disjoint sets: $\boldsymbol{E}$, $\boldsymbol{Z}$, $\boldsymbol{Y}$
- This partitioning limits which orders we can choose, as we can order only within $\boldsymbol{Z}$ and within $\boldsymbol{Y}$

Variable Elimination Examples

- See OneNote

Message Passing

- We will cover this for HMMs

Linear Gaussian Exercise

- Given $X \rightarrow Y$
  - $p(X) \sim N(\mu_X; \sigma_X^2)$
  - $p(Y \mid X) \sim N(\beta_0 + \beta_1 \mu_X; \sigma_Y^2)$
- Calculate
  - $p(Y)$ $\leftarrow$ $\int p(x) p(y \mid x) dx$
  - $p(Y \mid X = 5)$
  - $p(X \mid Y = 5)$
  - $p(X, Y)$

General Discussion

- Applications
- Other topics
  - Approximate inference
  - Learning
    - Parameters
    - Structure