CS 581 – Advanced Artificial Intelligence

Topic: Logistic Regression

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Logistic Regression

P(y|x)

- Learns $P(Y|\boldsymbol{X})$ directly, without going through $P(\boldsymbol{X}|Y)$ and $P(Y)$
- Assumes $P(Y|\boldsymbol{X})$ follows the logistic function

$$P(Y = false \mid X_1, X_2, \cdots, X_n) = \frac{1}{1 + e^{w_0 + \sum_{i=1}^{n} w_i X_i}}$$

$$\frac{1}{1+e^s}$$

$$P(Y = true \mid X_1, X_2, \cdots, X_n) = \frac{e^{w_0 + \sum_{i=1}^{n} w_i X_i}}{1 + e^{w_0 + \sum_{i=1}^{n} w_i X_i}}$$

$$\frac{e^s}{1+e^s}$$

- Learning: estimate the weights $w_0, w_1, \ldots, w_n$

Learning – Parameter Estimation

- Maximize (conditional) log-likelihood

$$W \leftarrow \underset{W}{\operatorname{argmax}} \prod P(Y[d] \mid \boldsymbol{X}[d])$$

$$W \leftarrow \underset{W}{\operatorname{argmax}} \sum \ln P(Y[d] \mid \boldsymbol{X}[d])$$

Take derivative of CLL wrt W

- See OneNote

OPTIMIZATION

- No closed-form solution for W
- One solution: gradient ascent
- Good news: log-likelihood for logistic regression is concave

Gradient Optimization

MOTIVATION

- Maximize / minimize a function *f(x)*
- Typical approach
  - Take gradient of *f(x)* wrt *x* and set it to 0
  - That is, solve $\nabla f(x) = 0$
- What if there is no analytical solution to $\nabla f(x) = 0$?
- One approach is gradient ascent (for maximization of *f*) and gradient descent (for minimization of *f*)

Two Simple Examples

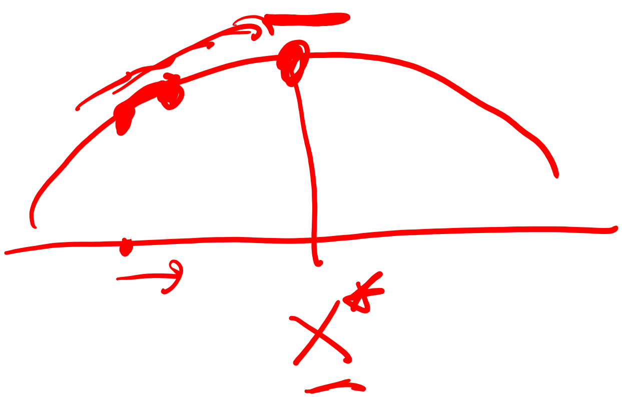
1. Maximize $f(x) = -2x^2 + 8x + 10$
2. Maximize $f(x) = -x^5 - 2x^4 + 13x^3 + 14x^2 - 24x$

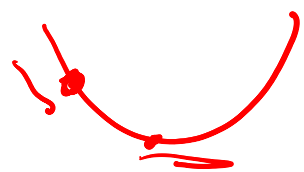
Gradient Ascent

- Using the Taylor expansion, a function *f(x)* can be approximated at around a as
  - $f(x) \approx f(a) + \nabla f(a) * (x - a)$

$x = a + \varepsilon$

$f(a+\varepsilon) \approx f(a) + f'(a) \cdot \varepsilon$





Gradient Ascent

- Find maximum of $f(x)$ where there is no closed form solution
- Start with some initial guess $\vec{x}_0$
- While change is not much
  - $\vec{x}_{i+1} = \vec{x}_i + \eta * \nabla f(x_i)$
- η is called the learning rate and it is a user specified parameter

Gradient Ascent

- Find maximum of $f(x)$ where there is no closed form solution
- Start with some initial guess x_0
- While change is not much
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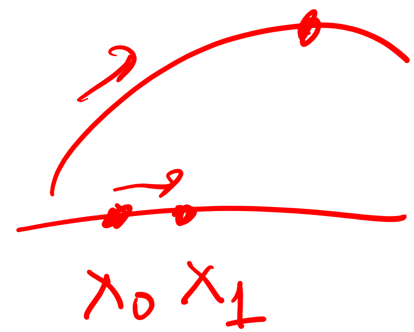

$x_1 = \underline{x_0} + \eta \cdot f'(x_0)$

$x_2 = x_1 + \eta f'(x_1)$

Let's See a Simple Example

- Maximize $f(x) = -2x^2 + 8x + 10$
- Note that $\nabla f(x) = 0$ has a closed from solution for this example. We'll use this simple example just for illustration purposes
- See OneNote and Jupyter Notebook

Logistic Regression Gradient Example

- See OneNote

The meaning of Intercept, w_0

- Can we interpret w_0 similar to $P(Y)$ in naïve Bayes?
  - No!
  - w_0 depends on both the class distribution and the scale of the features
  - Please see OneNote

Categorical Features

- Logistic regression's parameters are feature weights
  - Hence, features need to have values that can be multiplied by a weight
- What if you have a binary feature?
  - Two choices: 0/1, or -1/+1.
- What if you have a categorical features that has more than two possible values, such as R, G, B?
  - Incorrect way: R=1, G=2, B=3. Why?
  - How should we handle these features?

REGULARIZATION

Regularization

- Prefer smaller weights
  - Why?

L_2 Regularization

- Objective function
  - $W \leftarrow \underset{W}{\operatorname{argmax}}\left(\sum \ln P(Y[d] \mid \boldsymbol{X}[d]) - \frac{\lambda}{2}\|W\|^2\right)$
  - Trade-off between fit to the data vs model complexity
- Assuming n features
  - $W \leftarrow \underset{W}{\operatorname{argmax}}\left(\sum \ln P(Y[d] \mid \boldsymbol{X}[d]) - \frac{\lambda}{2}\sum_{i=1}^{n} w_i^2\right)$
- Take derivate of the objective function with respect to w_i.

$\frac{\partial CLL}{\partial w_i} - \frac{\lambda}{2} \cdot 2 \cdot w_i = \frac{\partial CLL}{\partial w_i} - \lambda w_i$

$p(\theta)$ $p(\theta|D)$

$p(X_{next}|D)$

L_2 Regularization & Bayesian Estimation

- Unregularized version corresponds to maximum likelihood estimate of the parameters
- Bayesian means we put a prior on what we do not know. Remember $p(\theta)$, $p(\theta|D), P(X|D)$.
- In this case, we put a prior on w. That is, we have a prior distribution $p(w)$. $\leftarrow N(0, \frac{1}{\lambda})$
- L_2 regularization corresponds to
  - $p(w)$ is a Gaussian distribution with zero mean and variance related to $1/\lambda$, and
  - Taking the maximum of the posterior $\underset{w}{\operatorname{argmax}}\, P(w|D)$

L_1 REGULARIZATION

- Instead of a quadratic penalty, absolute value is used
- Assuming n features
  - $W \leftarrow \underset{W}{\operatorname{argmax}}(\sum \ln P(Y[d] \mid \boldsymbol{X}[d] - \beta \sum_{i=1}^{n} |w_i|)$
- In the Bayesian case, $p(w)$ is assumed to be not a Gaussian distribution but instead a Laplace distribution

L_2 VS L_1

- L_2 forces the large weights to get closer to zero and places an emphasis on the large weights
  - Even though the weights get closer to zero, they are often not zero
- L_1 also penalizes large weights but the emphasis is not necessarily on the large weights
  - Some of the weights become zero
  - Leads to sparser representation

References

- http://www.cs.cmu.edu/~tom/mlbook/NBayesLogReg.pdf
- https://scikit-learn.org/stable/modules/linear_model.html#logistic-regression