CS 583: Probabilistic Graphical Models

Topic: Foundations

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This Slide Deck

- Foundations in
  - Probability
  - Graphs

PROBABILITY

Probability Distribution

- **Ω**: **Space** of possible outcomes
  - E.g., Rolling a die $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **S**: Measurable **events**
  - E.g., An odd roll of die $S = \{1, 3, 5\}$
- A **probability distribution P** over (Ω, S) is a mapping from events in S to real values that satisfies
  - $P(\alpha) \geq 0$ for all $\alpha \in S$
  - $P(\Omega) = 1$
  - If $\alpha, \beta \in S$ and $\alpha \cap \beta = \emptyset$, then $P(\alpha \cup \beta) = P(\alpha) + P(\beta)$

Random Variables

- A problem is represented through variables
  - Age, fever, lab tests, …
  - Industrious (student), Difficulty (of a class), Grade (of a student in that class), …
- A variable takes on values from its domain
  - Fever takes on True, False
  - Grade takes on A, B, C
- Can be either discrete or continuous
  - Grade is discrete, Age is continuous
- In an uncertain world, a variable takes on values from its domain probabilistically
  - For example, Grade can be A, B, or C probabilistically
  - P(Grade = A), P(Grade = B), P(Grade = C)

Random Variables – Notation

G

- Capital: X: variable

b

- Lowercase: x: a particular value of X
- Val(X): the set of values X can take
- Bold Capital: **X**: a set of variables
- Bold lowercase: **x**: an assignment to all variables in **X**

P(G = a) P(a)

- P(X=x) will be shortened as P(x)
- P(X=x ∩ Y=y) will be shortened as P(x,y)

Table – the most basic representation

Industrious	P(Industrious)
~industrious	0.7
industrious	0.3

Grade	P(Grade)
a	0.25
b	0.37
c	0.38

Joint Distribution

- Several random variables
  - $\mathbf{X}=\{X_1, X_2, \ldots, X_n\}$
- Joint Distribution
  - $P(\mathbf{X}) = P(X_1, X_2, \ldots, X_n)$
  - Specifies a probability value to all possible assignments

Joint Distribution

Industrious	Grade	P(Industrious, Grade)
~industrious	a	0.07
~industrious	b	0.28
~industrious	c	0.35
industrious	a	0.18
industrious	b	0.09
industrious	c	0.03

Conditional Probability

$\langle a|i, b|i, c|i \rangle$

- $P(X \mid Y) = P(X, Y) / P(Y)$
- What do the following mean?
  - P(Grade) $\langle a, b, c \rangle$
  - P(Grade | Industrious)

    $\langle a|\neg i, b|\neg i, c|\neg i \rangle$
  - P(Grade | Industrious = industrious)
  - P(Grade = a | Industrious = ~industrious)

Summation Rule

- Given P(X, Y), P(X) can be computed using
  - $P(X) = \Sigma_y P(X,y)$ where y ranges over Val(Y)
- Answer the following
  - $\Sigma_x P(X) = ?$ $\quad P(a)+P(b)+P(c) = ? = 1$
  - $\Sigma_x P(X \mid y) = ?$ $\quad P(a \mid \cdot)+P(b \mid \cdot)+P(c \mid \cdot) = 1$
  - $\Sigma_y P(X \mid Y) = ?$

$\neq 1$

$\sum_x P(X \mid Y) \neq 1 \quad ?$

Chain Rule

- $P(X_1, X_2, X_3, \ldots, X_k) =$
  - $P(X_1)\, P(X_2 | X_1)\, P(X_3 | X_1,X_2) \ldots P(X_k | X_1, X_2, X_3, \ldots, X_{k-1})$
    - or
  - $P(X_2)\, P(X_1 | X_2)\, P(X_3 | X_1,X_2) \ldots P(X_k | X_1, X_2, X_3, \ldots, X_{k-1})$
    - or
  - $P(X_2)\, P(X_3 | X_2)\, P(X_1 | X_3,X_2) \ldots P(X_k | X_1, X_2, X_3, \ldots, X_{k-1})$
    - or
  - Pick an order, then
    - P(first)P(second | first)P(third | first,second)…P(last | all_previous)

Bayes Rule

- Bayes Rule
  - $P(X | Y) = P(Y | X)P(X) / P(Y)$
- Conditional Bayes Rule
  - $P(X | Y, Z) = P(Y | X, Z)P(X | Z) / P(Y | Z)$

Bayes Rule

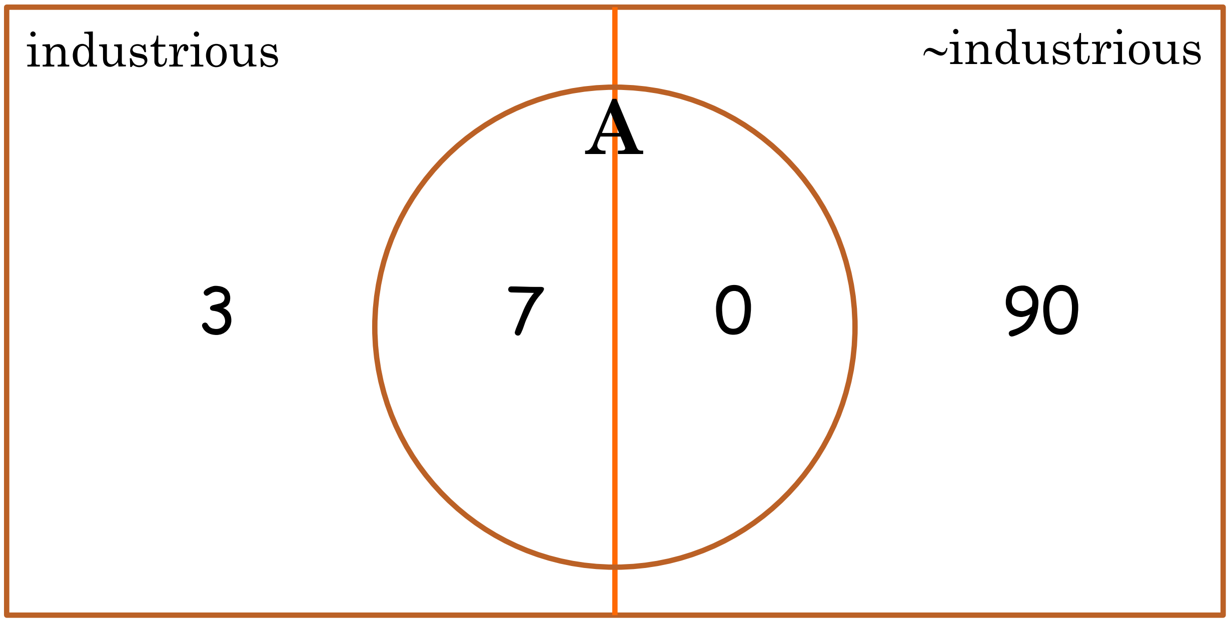
- Can we compute $P(\alpha|\beta)$ from $P(\beta|\alpha)$?
- E.g.,
  - In a class, 70% of the industrious students got an A.
    - P(a | industrious) = 0.7
  - John got an A. What is the probability of John being industrious given he got an A?
    - P(industrious | a) = ?

Note: these numbers have nothing to do with the previous tables and numbers.

Class Example

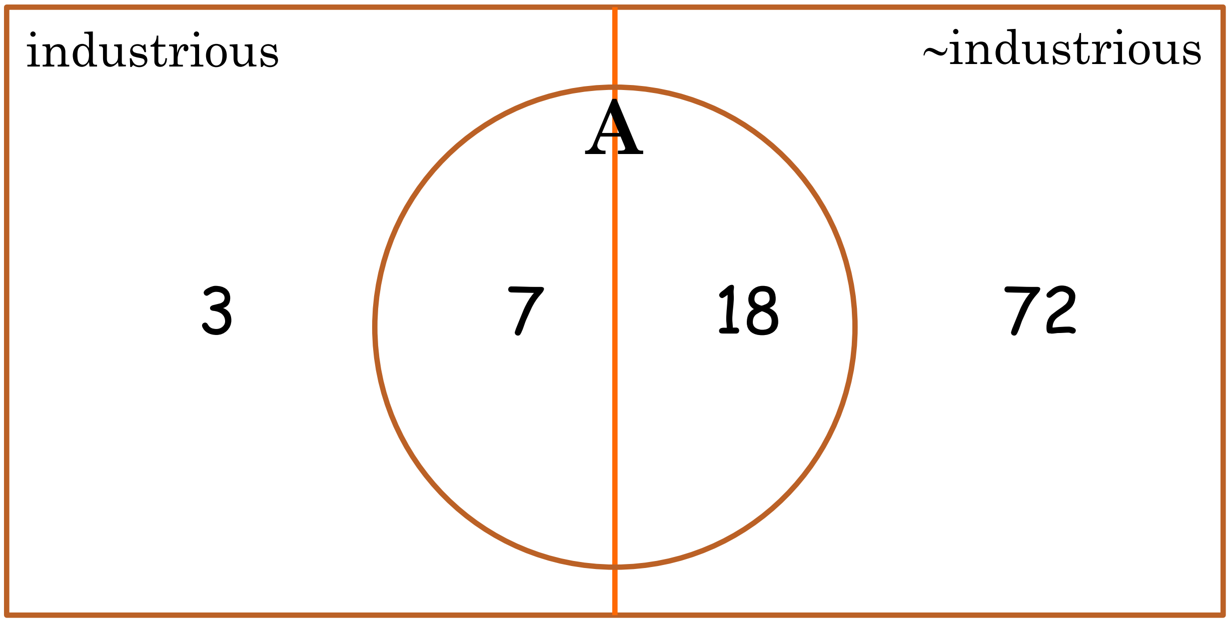
- Let's say there are 100 students in the class
- Let's say 10 of them are industrious, 90 are ~industrious
- Probability of a randomly picked student being industrious
  - P(industrious) = 0.1
- We know that 70% of the industrious students got an A.
  - P(a | industrious) = 0.7
  - 7 industrious students got an A; 3 did not get an A.
- What is P(industrious|a) = ?
  - Depends on P(a)

VERY HARD CLASS



$P(i \mid a) = ?$ $\frac{7}{7}$

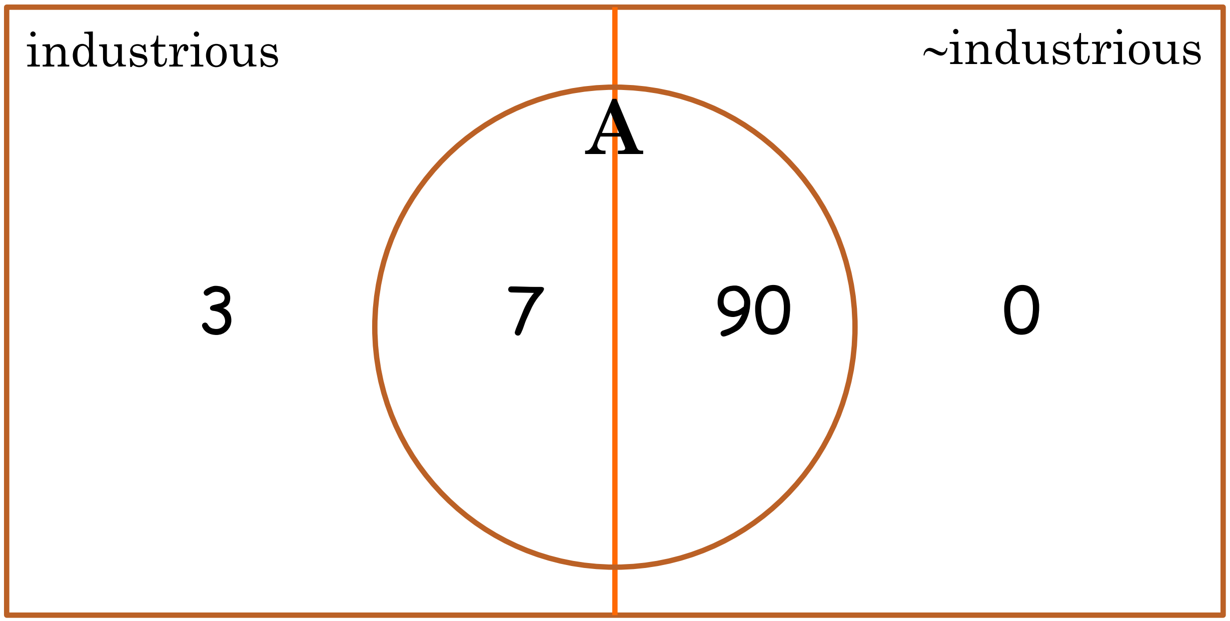
Medium Hard Class



P(i | a) = ? $\frac{7}{25} = 0.28$

$P(i) = 0.10$

Weird Class



P(i | a) = ? $\frac{7}{97} \approx 0.07$

$P(i) = 0.10$

EXERCISE

- In a state, 60% of the hospitalized patients are vaccinated.
  - $P(v \mid h) = 0.6$

$P(h)$ $P(v)$

- What does this number tell you about the effectiveness of the vaccines in preventing hospitalizations?

So Far

- Definition of probability distributions
- Random variables
- Joint distribution $P(X_1, X_2, \ldots X_n)$
- Conditional distribution $P(A|B) = \frac{P(A,B)}{P(B)}$
- Chain rule
- Summation rule $P(A,C) = \sum_{B,D} P(A,B,C,D)$
- Bayes rule $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Marginal Independence

- An event α is **independent** of event β in P, denoted as $P \models \alpha \perp \beta$, if
  - $P(\alpha \mid \beta) = P(\alpha)$, or — $P(\alpha \mid \beta) = \frac{P(\alpha, \beta)}{P(\beta)} = P(\alpha)$
  - $P(\beta) = 0$
- Proposition: A distribution P satisfies $\alpha \perp \beta$ if and only if
  - $P(\alpha, \beta) = P(\alpha)\, P(\beta)$
  - Can you prove it? Cond. Def $\perp$.
- Corollary: $\alpha \perp \beta$ implies $\beta \perp \alpha$

Marginal Independence

X	Y	P(X, Y)
t	t	0.18
t	f	0.42
f	t	0.12
f	f	0.28

Is $X \perp Y$?

$$P(X,Y) \stackrel{?}{=} P(X)\,P(Y)$$

Conditional Independence

- Two events are independent given another event
- An event α is **independent** of event β given event γ in P, denoted as $P \models (\alpha \perp \beta \mid \gamma)$, if
  - $P(\alpha \mid \beta, \gamma) = P(\alpha \mid \gamma)$, or
  - $P(\beta, \gamma) = 0$
- Proposition: A distribution P satisfies $\alpha \perp \beta \mid \gamma$ if and only if
  - $P(\alpha, \beta \mid \gamma) = P(\alpha \mid \gamma)\, P(\beta \mid \gamma)$

QUERYING A DISTRIBUTION

- **Evidence (E=e)**: what is known, **Query (Y)**: variables of interest, **X** is the set of all variables that include **E**, **Y**, and potentially others

1. **Probability query**
   - $P(\mathbf{Y} \mid \mathbf{e}) = ?$
2. **MAP query**
   - $\mathbf{W} = \mathbf{X} \setminus \mathbf{E}$ (i.e., all the non-evidence variables)
   - $MAP(\mathbf{W} \mid \mathbf{e}) = \text{argmax}_{\mathbf{w}} P(\mathbf{w},\mathbf{e})$
   - Important: We *cannot* find **w** by finding the maximum likely value for each variable individually
3. **Marginal MAP query**
   - $MAP(\mathbf{Y} \mid \mathbf{e}) = \text{argmax}_{\mathbf{y}} P(\mathbf{y} \mid \mathbf{e})$
   - Let $\mathbf{Z} = \mathbf{X} \setminus \mathbf{E} \cup \mathbf{Y}$
   - $MAP(\mathbf{Y} \mid \mathbf{e}) = \text{argmax}_{\mathbf{y}} \sum_{\mathbf{z}} P(\mathbf{z}, \mathbf{y} \mid \mathbf{e})$

MAP Example

A	B	P(A, B)
t	t	0.10
t	f	0.25
f	t	0.35
f	f	0.30

Maximum likely assignment for A = f

Maximum likely assignment for B = f

MAP (A,B) = <A=f, B=t>

Continuous Spaces

- Assume X is continuous and Val(X) = [0,1]
- If you would like to assign the same probability to all real numbers in [0, 1], what is, for e.g., P(X=0.5) = ?
- Answer: P(X=0.5) = 0.

Probability Density Function

- We define **probability density function**, p(x), a non-negative integrable function, such that $\int_{Val(X)} p(x)dx = 1$

$$P(X \leq a) = \int_{-\infty}^{a} p(x)dx$$

$$P(a \leq X \leq b) = \int_{a}^{b} p(x)dx$$

UNIFORM DISTRIBUTION

- A variable X has a uniform distribution over [a,b] if it has the PDF

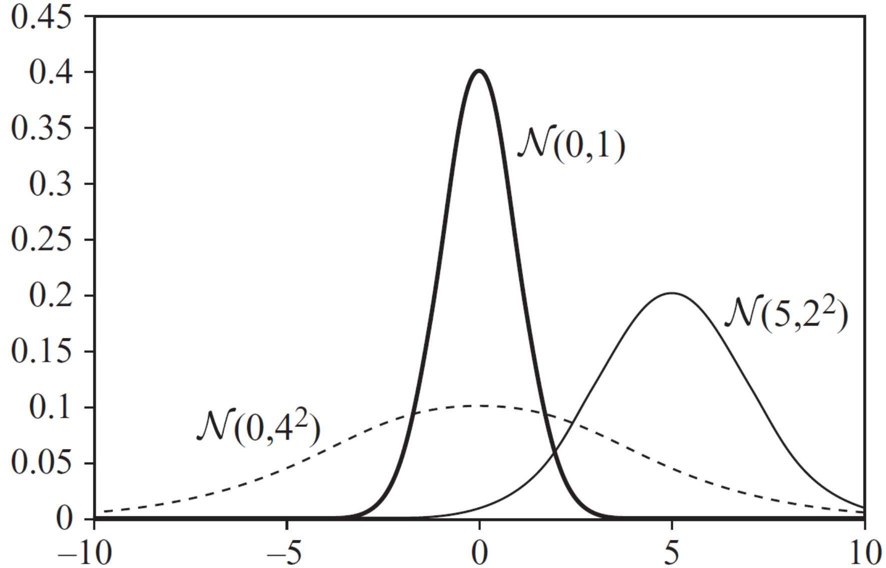
$$p(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & otherwise \end{cases}$$

Check and make sure that p(x) integrates to 1.

GAUSSIAN DISTRIBUTION

- A variable X has a Gaussian distribution with mean μ and variance σ^2, if it has the PDF

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Can p(x) be ever greater than 1?

Conditional Probability

- We want P(Y | X=x) where X is continuous, Y is discrete
- P(Y | X=x) = P(Y,X=x) / P(X=x)
  - What's wrong with this expression?
- Instead, we use the following expression

$$P(Y \mid X = x) = \lim_{\varepsilon \to 0} P(Y \mid x - \varepsilon \le X \le x + \varepsilon)$$

Conditional Probability

- We want $p(Y|X)$ where X is discrete, Y is continuous
- How would you represent it?

EXPECTATION

$$E_P[X] = \sum_x xP(x)$$

$$E_P[X] = \int_x xp(x)dx$$

$$E_P[aX + b] = aE_P[X] + b$$

$$E_P[X + Y] = E_P[X] + E_P[Y]$$

$$E_P[X \mid y] = \sum_x xP(x \mid y)$$

What about E[X*Y]?

VARIANCE

$$Var_P[X] = E_P\left[\left(X - E_P[X]\right)^2\right]$$

$$Var_P[X] = E_P\left[X^2\right] - \left(E_P[X]\right)^2$$

Can you derive the second expression using the first expression?

$$Var_P[aX + b] = a^2 Var_P[X]$$

What is Var[X+Y]?

Uniform and Gaussian Distribution

- If $X \sim N(\mu,\sigma^2)$, then $E[X] = \mu$, $Var[X] = \sigma^2$
- What about the expectation and variance of a uniform distribution?

GRAPHS

Graphs

- A **graph** consists of **nodes** and **edges**
- **Nodes:** $\mathcal{X}=\{X_1, X_2, \ldots, X_n\}$
- **Undirected Edge:** $X_i - X_j$
- **Directed Edge:** $X_i \rightarrow X_j$
- Between a pair of nodes, at most one type of edge exists
  - We cannot have $X_i \rightarrow X_j$ and $X_j \rightarrow X_i$ at the same time, and
  - We cannot have $X_i \rightarrow X_j$ and $X_i - X_j$ at the same time
- Some edge: $X_i \leftrightarrows X_j$

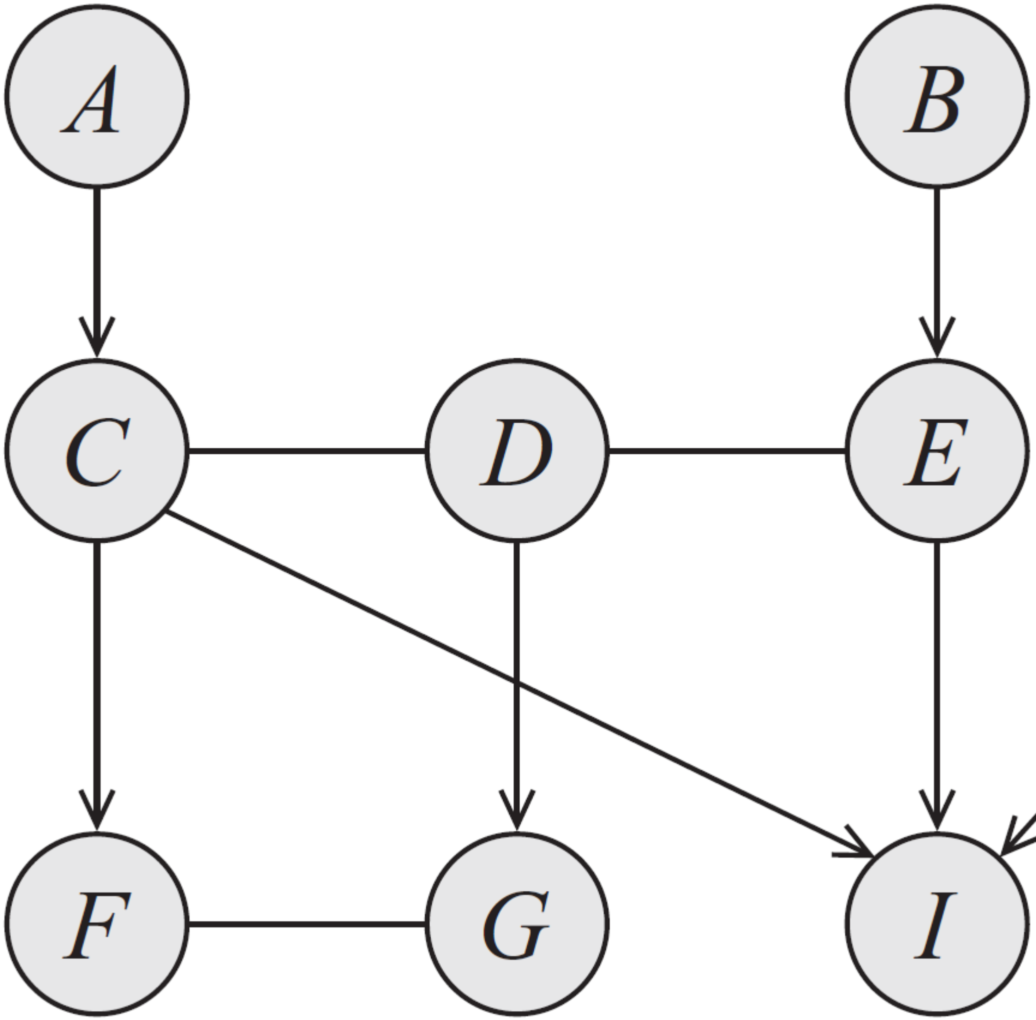
Directed and undirected

- A graph is **directed** if its *all* edges are directed
- A graph is **undirected** if its *all* edges are undirected

Relationships

- $X_i \rightarrow X_j$
  - X_i is the **parent**
  - X_j is the **child**
- $X_i - X_j$
  - X_i and X_j are **neighbors**
- $X_i \leftrightarrows X_j$
  - X_i and X_j are **adjacent**
- **Degree** of X_i: The number of edges X_i is part of
- **Indegree** of X_i: The number of directed edges pointing to X_i
- **Degree** of a graph: The maximal degree of a node in the graph

EXAMPLE



Examples of:

Parents
Children
Neighbors
Adjacent nodes
Degree
Indegree

Complete Graphs and Cliques

- A subgraph over $\mathbf{X} \subseteq \mathcal{X}$ is **complete** if *every* two nodes in $\mathbf{X}$ are connected by some edge
- Such a set $\mathbf{X}$ is also called a **clique**
- A clique is maximal if for any superset of nodes $\mathbf{Y} \supset \mathbf{X}$, $\mathbf{Y}$ is not a clique

Paths and Trails

- $X_1, X_2, \ldots, X_k$ forms a **path** if, for every i =1, 2, …, k-1, we have that either $X_i - X_{i+1}$ or $X_i \rightarrow X_{i+1}$.
- A path is directed if, for *at least one* i, $X_i \rightarrow X_{i+1}$.
- $X_1, X_2, \ldots, X_k$ forms a **trail** if, for every i =1, 2, …, k-1, we have $X_i \leftrightarrows X_{i+1}$.
- What is the difference between a path and a trail? Is every path also a trail? Is every trail also a path?

Ancestors and descendants

- X_i is an **ancestor** of X_j if there is a directed path from X_i to X_j
- X_i is a **descendant** of X_j if there is a directed path from X_j to X_i
- **Nondescendants**$(X_i) \equiv \mathcal{X} \setminus \text{Descendants}(X_i)$

Cycles and Loops

- A **cycle** is a directed path from a node to itself
- A graph is **acyclic** if it contains no cycles
- A directed acyclic graph is the one where all edges are directed and there are no cycles
- A **loop** is a trail from a node to itself
- A graph is **singly-connected** if it contains no loops

Next

- Bayesian networks