IITM-CS6840: Advanced Complexity Theory March 02 2012

Lecture 31 : Expander Graphs, Degree reduction

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THEME: Probabilistic Proof Systems
LECTURE PLAN: In the last class we have given the outline of Irit Dinur's proof of PCP Theorem in which we viewed SAT as qCSP Problem .We then transformed the q-query CSP to 2-query CSP Ψ .In general, after applying query reduction, the constraint graph constructed from 2-query CSP may have very large degree.Now the goal is to make the degree of the graph constant independent of the instance size , for which we will be using a special type of graphs called expander graphs.

Expander Graphs A family $G = \{G_1, G_2, \ldots\}$ of d-regular undirected graphs is an "edge expander family" if there is a constant $\alpha > 0$ such that $\forall G' \in G$, $\forall S \subseteq V(G'), |S| \leqslant \frac{1}{2}.|V|$ $E(S, S^c) \geqslant \alpha.d.|S|$
Algebrically expander graphs have intereresting properties.Let $A' = \frac{1}{d}.A$ where A is the adjacency matrix of the expander graph.Because A' is symmetric, the spectral theorem implies that A has n real-valued eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ between -1 and 1. Because G is regular, $\lambda_1 = 1$. In some contexts, the spectral gap of G is defined to be $1 - \lambda_2$. In other contexts, the spectral gap is $1 - \lambda$, where

$\lambda = \max\{|\lambda_2|, |\lambda_n|\}$.

Theorem Constant degree expander graphs exist and can be constrcuted explicitly.
We will use this theorem in our analysis without proof.

Random walks on Graphs A random walk of length k on a graph G with a root 0 is a stochastic process with random variables $X_1, X_2, \ldots, X_k$ such that $X_1 = 0$ and X_{i+1} is a vertex chosen uniformly at random from the neighbors of X_i.Let P_i be a random vector of dimension n ,where n is the number of vertices in the graph , whose j^{th} component gives the probabilities of the walk being at that j^{th} vertex after i steps of Random walk.We can see that

$P_{i+1} = A' P_i$

The larger the spectral gap ,the smaller the number of walks required to be equally probable of ending at any vertex in graph.

Degree Reduction Let G be a constraint graph of Ψ.We create a new CSP Ψ' by replacing every vertex i in G of degree n_i by a cloud of n_i vertices. So each variable x_i of Ψ will give rise to n_i variables $x'_{i1}, \ldots, x'_{in_i}$ in Ψ' . Each constraint in Ψ gives rise to d/2 parallel constraints in Ψ'between unique vertices in the corresponding clouds. Within each cloud,

we interconnect the vertices by a 1/4-edge expander and make each expander constraint an equality constraint (i.e. requiring that variables get the same value). Notice that if Ψ has m constraints, then $\Psi^{'}$ will have m variables and dm constraints.
Clearly, if Ψ has a satisfying assignment x, we can obtain a satisfying assignment for $\Psi^{'}$ by setting $x^{'}_{i1} = \ldots = x^{'}_{in_i} = x_i$ for every i.

Now we prove soundness.

Theorem If some assignment $x^{'}$ the violates at most an ε -fraction of contraints in $\Psi^{'}$, then there exists an assignment x that violates at most a 18ε fraction of constraints in Ψ.

Proof The assignment x is obtained from $x^{'}$ as follows: Within each cloud, let x_i be the plurality value (i.e., the most representative value) among $x^{'}_{i1}, \ldots, x^{'}_{in_i}$. Let ε_i be the fraction of constraints violated in cloud i. Then $\Sigma_{i=1}^{n} \varepsilon_i (dn_i/4) \varepsilon (dm/2)$, the total number of violated constraints.

Lets fix a cloud i. Let S_i be the set of vertices j within this cloud where $x^{'}_{ij}$ agrees with the plurality assignment. We will argue that, because of the expansion in the cloud, the assignment within the cloud must largely agree with the plurality assignment. We split the analysis into three cases:

i) If $|S_i| > n_i/2$, then by edge expansion $|E(S_i, S_i^c)| \geq d|S_i^c|/4$. Since all the constraints in the cut (S_i, S_i^c) are violated by the assignment, $|E(S_i, S_i^c)| \leq \varepsilon_i (dn_i/4)$, so $|S_i^c| \leq 4_i n_i$.

ii) If $n_i/4 \leq |S_i| \leq n_i/2$, then by edge expansion $|E(S_i, S_i^c)| \geq d|S_i|/4 \geq dn_i/16$. Since all the constraints in the cut are violated, it follows that $\varepsilon_i \geq 1/4$, so $|S_i^c| \leq n_i \leq 4\varepsilon_i n_i$.

iii) If $|S_i| < n_i/4$, then no value in Σ is taken more than $\frac{1}{4}$-fraction of the time inside the cloud, so there must exist some partition of the values within the cloud so that the smaller side of the partition has between $\frac{n_i}{4}$ and $\frac{n_i}{2}$ vertices. Just like in the previous case, we get that $|S_i^c| \leq n_i \leq 4\varepsilon_i n_i$. In all cases above, $|S_i^c| \leq 4\varepsilon_i n_i$ for every i.

Now consider what happens in $\Psi^{'}$ when we replace the assignment $x^{'}$ with the plurality assignment $x^{'}_{plur}$ (i.e. one that equals the plurality of x on every cloud). For each cloud, this may cause the violation of at most $(d/2)|S^{'}_i|$ additional constraints that go out of the cloud. So if $x^{'}$ violates εdm constraints in $\Psi^{'}$, $x^{'}_{plur}$ will violate at most $\varepsilon(dm)/2 + \Sigma_{i=1}^{n} (d/2)|S_i^c|$ $\leq \varepsilon(dm)/2 + \Sigma_{i=1}^{n} (d/2) 4\varepsilon_i n_i \leq \varepsilon(dm)/2 + 8(\varepsilon(dm)/2) = 9(\varepsilon(dm)/2)$. constraints of $\Psi^{'}$. This is a 9ε-fraction of all the constraints in $\Psi^{'}$. So the assignment x can violate at most a 18ε fraction of constraints in Ψ.
So the soundness gap goes down by at most a constant factor in this transformation .
Acknowledgements:
–Lecture notes of Prof.Jayalal Sharma
–Lecture 13 , Theory of Computational Complexity , The Chinese University of Hong Kong Scribe.
–The PCP Theorem by Gap Amplification by Irit Dinur.