Preface

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CS 6840 Advanced Complexity Theory

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Date: Jan 10, 2012

LECTURE

4

Quest for Structure in Counting Problems

In the previous lecture, we saw that the counting problem can be as hard as (or harder than) the decision problem as given an algorithm for counting problem the decision problem reduces to just checking the count to be zero or not. We also saw an easy decision problem CYCLE whose counting version $\#\mathsf{CYCLE}$ is NP-hard (by reduction from $\mathsf{HAMCYCLE}$) implying that easy decision problems can also have corresponding counting problems hard. We also argued that talking about counting problems still makes sense as the count value, though exponential, can still be represented in polynomial number of bits.

In this lecture, we will study counting problems and understand their structural complexity. We shall also make attempts to develop the theory of complexity classes capturing the counting problems (especially $\#\mathsf{P}$). We shall also discuss their basic containments.

4.1 Preliminaries

Firstly, we fix our computation model where we have a Turing machine with an input tape, work tape and an output tape. We are interested in the resources used by the Turing machine - space (considering only the work tape) and time.

We want to capture the notion of counting formally. One such way is to see it as computing a function $f : \Sigma^* \rightarrow \mathbb{N}$ where $\Sigma = \{0, 1\}$ which gives an integer value. So, how can we capture the notion of computing a function? There are two possible ways of capturing function computation.

Variant 1 We say that a function f is computable if each bit of the output can be computed in some decision complexity class $\mathcal{C}$.

Variant 2 f is computable if the value of the function computation can be written down within the resource bounds.

Analogous to the decision problems, we define complexity classes for function computation problems. A natural extension of P is FP which is defined as

$$\mathsf{FP} = \{f \mid f : \Sigma^* \rightarrow \mathbb{N}, f(x) \text{ for any } x \in \Sigma^* \text{ can be written down in } poly(|x|) \text{ time}\}$$

Now, we shall plugging in the two variants of function computation and see which of them is more appropriate.

4.2 Comparing the variants

We quickly observe that the first and the second variant really coincides when we are talking about deterministic computations. Let us do this analysis by attempting the definition of FP. Following the first variant $f \in \mathsf{FP}$ iff there exists an algorithm that can compute each bit of f in class P. But since the algorithm is deterministic, this is equivalent to saying that $\forall\ i$, the language defined by the i^{th} bit

$$L_{f_i} = \{x : (f(x))_i = 1\} \in \mathsf{P}$$

On the other hand, if each bit can be computed in polynomial time and since the count value can be represented in polynomial number of bits, for evaluation, we just run a polynomial time algorithm polynomial times which is still a polynomial. Hence we have an algorithm that satisfies the second variant.

Hence for deterministic polynomial time computation both variants are equivalent. We can define other classes for deterministic computation like FL(log space bounded function computation), $\mathsf{FPSPACE}$ (polynomial space bounded function computation). Containments of these classes are analogous to their decision versions. We leave the proof as an exercise.

Lemma 4.1. $\mathsf{FL} \subseteq \mathsf{FP} \subseteq \mathsf{FPSPACE}$

Now we turn into the non-deterministic world. Following variant 1 of definition of function computation, we must have each bit computable in class NP. But variant 2 is not useful because an non-deterministic poly time Turing machine by our model is not set to output a value. How can we capture the function computation for a non-deterministic machine for decision problems which works by guess-verify mechanism?

Now, consider the non-deterministic algorithm we had for SAT, which does guessing of an assignment and verifying it. We can observe the following additional property.

Observation 4.2. *Number of satisfying assignments is exactly equal to the number of accepting paths.*

This leads to the question as to whether this is accidental or is there some hidden structure? It also assigns a function value to the non-deterministic Turing machine. This motivates us to give a new model, for us to call f is computable by a non-deterministic polynomial time Turing machine.

Definition 4.3. f is $\#\mathsf{P}$ if there exists a non-deterministic Turing machine M running in time $p(n)$ such that $\forall x \in \Sigma^*$, $f(x) = \left|\{y \in \{0,1\}^{p(n)} : M \text{ accepts on path } y\ \}\right|$.

Remark 4.4. The RHS is also the number of accepting paths of M on x if the lengths of all paths are equal to $p(n)$. We remark that this can be achieved without loss of generality. That is, from an arbitrary TM M, we can get to a new TM M' which has the same number of accepting paths, such that the number of accepting paths on any input x remains the same. We recall our observation that for length of all non-deterministic paths of an NP machine on any input can be made equal without changing the accepted language[1] . But this construction makes the language accepted the same, and need not keep the number of

[1] In particular, we showed that a language $A \in \mathsf{NP}$ if and only if there is a language $B \in \mathsf{P}$ and a polynomial $p(n)$ such that $x \in A \iff \exists y \in \{0,1\}^{p(n)} : (x, y) \in B$

accepting paths the same. We modify it slightly to achieve our goal. Indeed, if a path is shorter than $p(n)$ bits and decided A/R, we extend it to the required length using a binary tree of paths rooted at that node and make the left most path (in this binary tree) report A/R respectively and make all other paths reject. The number of accepting paths does not change due to this construction.

Remark 4.5. Try this as an exercise. Initiate the thought process on : how does this definition compare with variant 1? What does computing/testing each bit to be 0/1 mean?

Counting version of SAT denoted as $\#\mathsf{SAT}$ can be defined as

$$\#\mathsf{SAT}(\phi) = |\{\sigma|\phi(\sigma) = 1, \sigma \text{ is a boolean assignment to variables in } \phi\}|$$

It follows from our observation that $\#\mathsf{SAT} \in \#\mathsf{P}$, since we can give a non-deterministic machine (i,e. a machine for SAT) where number of accepting paths equals to the number of satisfying truth assignments. It can also be shown that $\#\mathsf{CYCLE} \in \#\mathsf{P}$.

Claim 4.6. $\#\mathsf{CYCLE} \in \#\mathsf{P}$

Proof. Following is a non-deterministic Turing machine N, such that number of cycles equals number of accepting paths.

$N =$ " On input G,

1. Guess subsets $V' \subseteq V(G)$ and $E' \subseteq E(G)$ non-deterministically.
2. Accept iff V', E' form a simple cycle. "

Now it follows that $\#\mathsf{CYCLE}\ (G) = |\{\#$ of accepting paths of N on $G\}|$ since any cycle can uniquely be characterised by an edge set and a vertex set. ∎

4.3 Basic Containments

In the functional world, we have the following scenario.

$$\begin{array}{c} \mathsf{FPSPACE} \\ | \\ \mathsf{FP} \\ | \\ \mathsf{FL} \end{array}$$

So where does set of functions, $\#\mathsf{P}$ lie? We will argue that $\#\mathsf{P}$ lies between $\mathsf{FP}$and $\mathsf{FPSPACE}$ thus replicating the picture in the decision world.

Lemma 4.7. $\#\mathsf{P} \subseteq \mathsf{FPSPACE}$

Proof. Given a non-deterministic poly time Turing machine M computing function f, we just need to do a simulation in deterministic poly space. This can be done by simulating M over all non deterministic paths while reusing space across the paths. Since length of any path is polynomially bounded, space used will also be polynomial. In the process we need to keep the count of accepting paths which can be $\leq 2^{p(n)}$ but still representable with $p(n)$ bits in binary. Hence space requirement is only polynomial in input length. ∎

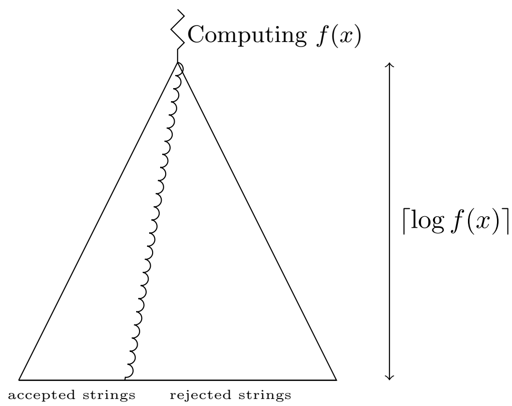
Lemma 4.8. $\mathsf{FP} \subseteq \#\mathsf{P}$

Proof. Given an $\mathcal{L} \in \mathsf{FP}$, there exists a deterministic Turing machine M which $\forall x \in \mathcal{L}$ writes $f(x)$ in $p(n)$ time where p is a polynomial and $n = |x|$. To show that $\mathcal{L} \in \#\mathsf{P}$, we need to construct a non-deterministic such that

$$\text{No of accepting paths} = f(x).$$

N can compute $f(x)$ by simulating M. Let the value obtained be k. Now, N must have exactly k accepting paths. This can be ensured by guessing $\lceil \log k \rceil$ bits and accepting all the paths whose address have binary representations $\leq k$ and rejecting the remaining paths.

N runs in poly time since $f(x)$ computation (simulation of M) takes only polynomial time. Even though $f(x)$ is exponential, number of bits guessed is $\lceil \log f(x) \rceil$ which will be polynomial in n. Hence depth is polynomially bounded. Also number of accepting paths equals $f(x)$ by construction. Thus N' is a $\#\mathsf{P}$machine accepting $\mathcal{L}$. Hence $\mathcal{L} = L(N') \in \#\mathsf{P}$. ∎



It can be observed that if function computation can be done in polynomial time then $\mathsf{P} = \mathsf{NP}$. This is because solving decision problem amounts to checking if the corresponding counting function gives a non zero value or not hence making decision problem easy if function computation is in P.

Lemma 4.9. $\#\mathsf{P} = \mathsf{FP} \implies \mathsf{P} = \mathsf{NP}$

An interesting question would be to ask if the converse it true? That is

$$\text{Does } \mathsf{P} = \mathsf{NP} \text{ imply } \#\mathsf{P} = \mathsf{FP}?$$

We will address this and show a weaker implication (that is, based on slightly stronger LHS) in the next lecture.

CS 6840 Advanced Complexity Theory

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LECTURE
5

FP vs #P question

We know (#P = FP) $\implies$ (P = NP). In the last lecture we left the question about the converse. i.e., is it true that, (P = NP) $\implies$ (#P = FP)?

In this lecture, we will show a weaker version of the above containment. We will define a new class of languages PP (probabilistic polynomial time) in decision world, that contains NP, and will show that if we make a stronger assumption PP = P (seemingly stronger than NP = P) we get the required implication. We will show in particular that PP = P $\iff$ #P = FP.

5.1 Complexity Class PP

For exploring the complexity class, we revisit the definition of NP. With respect to a "branching" Turing machine(NTM) (which can branch at each computation by using a guess bit), we have two different classes of languages as follows, which differs in their acceptance conditions. We denote for a machine M, by $\#acc_M(x)$ as the number of accepting paths. Let the machine run for $p(n)$ time on each path.

- NP = $\{$L : $\exists$ branching TM M_1, x $\in$ L $\iff$ $\#acc_{M_1}(x) \geq 1\}$.
- coNP = $\{$L : $\exists$ branching TM M_2, x $\in$ L $\iff$ $\#acc_{M_2}(x) \geq 2^{p(n)}\}$.

In terms of number of accepting paths on input x, the above classes represent the extremes. On one side, in NP we talk of atleast one accepting path, and on the other side, in coNP we talk of all accepting paths. To understand the structure of these classes we can ask the variant of this as : what different class of languages do we get if we set the accepting condition as the number of accepting paths being more than a fraction of the total number of paths on input x. A simpler situation is to consider the fraction to be half and the corresponding class of languages is called PP(probabilistic polynomial) which is defined as follows. We will come across other variants in the later lectures.

DEFINITION 5.10. PP =$\{$L : $\exists$ NTM M, x $\in$ L $\iff$ $\#acc_{M_1}(x) > 2^{p(n)-1}\}$

We first understand this complexity class with respect to the other ones we have already seen. We start with the following proposition.

PROPOSITION 5.11. $\mathsf{NP} \subseteq \mathsf{PP}$.

Proof. Let $L \in \mathsf{NP}$ via a nondeterministic turing machine M, $x \in L \iff M$ has atleast one accepting path on x. Our aim is to give another turing machine M' for the same language L such that $x \in L \iff M'$ has more than half of the total number of paths as accepting paths on x.
Description of M' (on input x):

1. Simulate M on x.
2. If M accepts then choose one bit nondeterministically and accept in both branches.
3. If M rejects then choose one bit nondeterministically,accept in one branch and reject in other.

As we remarked in the last lecture, we can assume without loss of generality that the height of computation tree of M on x is exactly $p(n)$ for some polynomial p. Since the total number of paths for M' on input x is $2^{p(n)+1}$, it suffices to prove that $x \in L \iff \#acc_M(x) > 2^{p(n)}$. Let $x \in L$. Since M has at least one accepting path on x. Since M', by construction, creates an imbalance (in count) between the number of accepting and rejecting paths precisely when M accepts, the number of accepting paths in M' will be more than the number of rejecting paths. Thus $\#acc_M(x) > 2^{p(n)}$.
If $x \notin L$, all paths reject in M on input x, and thus number of accepting and rejecting paths in M' on input x are exactly equal and each of themsame is equal to $2^{p(n)}$. ∎

A similar proof will also show that coNP is contained in PP. Thus we have the following containment relationship among different classes of languages

$$\begin{array}{c} \mathsf{PSPACE} \\ | \\ \mathsf{PP} \\ | \\ \mathsf{NP} \\ | \\ \mathsf{P} \end{array}$$

Now we will prove the main theorem of the lecture which characterizes the FP vs $\#\mathsf{P}$ question in the decision world.

THEOREM 5.12. $(\#\mathsf{P} = \mathsf{FP}) \iff (\mathsf{PP} = \mathsf{P})$

Proof. $(\Rightarrow)$ Assume $\#\mathsf{P} = \mathsf{FP}$, our aim is to prove $\mathsf{PP} = \mathsf{P}$. The reverse containment follows since we know $\mathsf{NP} \subseteq \mathsf{PP}$ by proposition5.11. Now we show that $\mathsf{PP} \subseteq \mathsf{P}$. Let $L \in \mathsf{PP}$ via a machine M running in time $p(n)$ for some polynomial p, such that $x \in L \iff \#acc_M(x) > 2^{p(n)-1}$. Define the function, $f(x) = \#acc_M(x)$. By definition $f \in \#\mathsf{P}$ and hence $f \in \mathsf{FP}$. There is a deterministic polynomial time Turing machine N which on input x outputs the value of $f(x)$. Given an x, to test whether it is in L or not, it suffices to test whether whether the MSB of the binary representation of $f(x)$ is 1 or not. This can be done by simply running the machine N on x and testing the MSB of the output. Hence $L \in P$.

$(\Leftarrow)$ Assume $\mathsf{PP} = \mathsf{P}$, our aim is to prove $\#\mathsf{P} = \mathsf{FP}$. The reverse direction is easy. $\mathsf{FP} \subseteq \#\mathsf{P}$ as we argued in the last lecture. To show the forward direction, let $f \in \#\mathsf{P}$ via M

such that $\forall x \in \Sigma^*,\ f(x) = \#accept_M(x)$. Note that the naive approach to compute $f(x)$ is to compute the number of accepting paths in computation tree of height $p(n)$ will take exponential time. But it suffices to find the minimum $0 \le k \le 2^{p(n)}$ such that :

$$k + \#acc_M(x) > 2^{p(n)} \tag{5.1}$$

Indeed, the minimum is achieved when $k = k_{min} = 2^{p(n)} - \#acc_M(x) + 1$. Thus $\#acc_M(x) = 2^{p(n)} - k_{min} + 1$. This is an indirect way of finding out $\#acc_M(x)$ and we are moving towards using our assumption that $\mathsf{PP} = \mathsf{P}$.

We have search problem in hand; to search for the minimum $k(k_{min})$ satisfying equation (5.1). We do this by binary search over the range $0 \le k \le 2^{p(n)}$. We solve the decision problem first. Given x and k, check if $k + \#acc_M(x) > 2^{p(n)}$.

For this, we construct an another Turing machine N such that the number of accepting paths is exactly $k + \#acc_M(x)$ and the total number of paths is $2^{p(n)+1}$. Assume such a construction exists. Define a language $A \subset \Sigma^*$ such that $x \in A \iff k + \#acc_M(x) > 2^{p(n)}$. Thus $x \in A \iff \#acc_N(x) > 2^{p(n)}$. This implies that $A \in PP$ (via the machine N !) and hence $A \in P$ by assumption. Given $x \in \Sigma^*$ we can test if $k + \#acc_M(x) > 2^{p(n)}$ by testing if $x \in A$ or not, which can be done in polynomial time. Now we can do this construction and simulation through a binary search in order to find the minimum value of k that satisfies our inequality.

To complete the proof, we give the construction of N (for a given k). We first construct a machine M_k that runs in time $p(n)$ and has exactly k accepting paths. We slightly modify the idea in the previous lecture to do this. Let $\ell = \lceil \log k \rceil$. The machine M_k guesses a $y \in \{0,1\}^{p(n)}$ and *accepts* if the first ℓ bits of y in binary represents a number less than k and the last $p(n) - \ell$ bits is all-zero (lexicographically first path) and *reject* otherwise.

We combine M_k and N (both using exactly $p(n)$ non-deterministic bits), to get the machine N. The machine N on input x guesses 1 bit and on the 0-branch it simulates M_k on x and on the 1-branch it simulates M on x. Clearly $\#acc_N(x) = \#acc_{M_k}(x) + \#acc_M(x) = k + \#acc_M(x)$. The length of each path is exactly $p(n) + 1$ and hence total number of paths is $2^{p(n)+1}$. ∎

CS 6840 Advanced Complexity Theory

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LECTURE
6

#P-Completeness

Thus we motivated to answer the questions. We saw (#P = FP) $\iff$ (P = PP) in decision world. This motivates understanding structure in the counting world. We start with the notion of reductions.

6.1 Reduction in Counting World

Our aim is to come up with a notion of #P-completeness and show structural classification of counting problems. Informally, we have seen counting problems that are hard to solve. We want our notion of hardness to capture this as closely as possible. We consider some natural options for such a definition and proceed from there.

Attempt 1: Let A,B be two languages. In decision world $A \leq_m^p B$ if and only if

1. $x \in A \Longleftrightarrow \sigma \in B$.
2. $\sigma : \Sigma^* \to \Sigma^*$ is polynomial time computable.

Translating the above statements into counting scenario.

1. σ is polynomial time computable.
2. $f(x) = g(\sigma(x))$.

Two lectures back we showed SAT can be decided if $\#CYCLE$ can be computed. But following the above attempted notion of a reduction, we get the following.

$$\#\mathsf{SAT}(\phi) = \#CYCLE(\sigma(\phi)) \tag{6.2}$$

But if such a reduction exists, then we can solve SAT by checking if $\#SAT(\phi) = \#CYCLE(\sigma(\phi))$, which in turn can be done in polynomial time. Thus, if we impose this strictness then in #P only counting versions of NP-complete problems will be hard. Notice that the issue here is that the reduction is set to preserve the number of certificates !.

Let us make some preliminary observations. What if we allow factors in equation6.2? Could this save us? No, still if such a reduction exists, we can decide SAT by counting the value of $\#CYCLE$. Indeed, there would not have been a problem if there was a "+1" on the right hand side of the above expression, such that the zeroness of $\#\mathsf{SAT}$ does not carry over to the zeroness of $\#CYCLE(\sigma(\phi))$. This shows that may be we should allow non-trivial computations after the query to the function $\#CYCLE$. Since we would also like structural composibility and transitivity of reductions, this naturally leads to the attempt 2, in its generality.

Attempt 2: We attempt now a generalization of the notion of Turing reductions.

DEFINITION 6.13. We say that a function $f \leq g$ if $f \in FP^g$. That is, if there exists a functional oracle Turing machine[2] with queries to $g : \Sigma^* \to \mathbb{N}$ that given any $x \in \Sigma^*$ can compute $f(x)$.
A function g is $\#\mathsf{P}$-hard if $\forall f \in \#\mathsf{P}, f \leq_m^p g$. g is complete for $\#\mathsf{P}$ if $g \in \#\mathsf{P}$ and g is $\#\mathsf{P}$-hard.

THEOREM 6.14. $\#\mathsf{SAT}$ *is* $\#\mathsf{P}$*-complete.*

1. $\#\mathsf{SAT} \in \#\mathsf{P}$: $\exists$ a non-deterministic turing machine polynomial time which guesses assignments (bit by bit), verifies the same, accepts if it satisfies the formula, and rejects otherwise. No assignment is guessed by two different non-deterministic computation paths. Hence the number of accepting paths will precisely be equal.

2. $\forall f \in \#\mathsf{P}, f \leq_m^p \#\mathsf{SAT}$: Let $f \in \#\mathsf{P}$ via a machine M such that $\forall x \in \Sigma^*, f(x) = \#acc_M(x)$.

   Cook-Levin Theorem - Revisited.

   $$L \in \mathsf{NP} \implies L \leq_m^p \mathsf{SAT}.$$

   Let $L \in \mathsf{NP}$, via machine M such that $x \in L \implies M$ has an accepting path. We construct $\phi_{(M,x)}$ from computation history such that M has an accepting path implies $\phi_{(M,x)}$ is satisfiable. For every accepting path, we have a satisfying assignment and for every satisfying assignment there is a corresponding accepting path too. Moreover, this map is There is a one-to one mapping between the set of accepting paths and the set of satisfying assignments. The certificates corresponds to the first set is the choice of non-deterministic bits and for the second set is the satisfying assignments. Such reductions are called *Parsimonious reductions*. An additional point is that this certificate bijection is polynomial time computable. That is given an accepting path of the machine M the reduction also gives you a way to transform it into a satisfying assignment of the formula and vice versa. Reductions satisfying the latter property are called *Levin reductions*. Note that, in order to prove NP-completeness, the reduction neither need to be parsimonious nor it should be Levin reduction. It is an additional structural property that the reductions seems to satisfy. Interestingly, the NP-complete reductions that we have seen in the last course (like the SAT to 3SAT, 3SAT to

[2] A functional oracle TM is similar to the normal oracle Turing machine, but since the output of the oracle query is not a 1-bit, instead of q_{yes}/q_{no} as the two states to which the machine moves after the oracle answers, the machine has has an oracle output tape for the oracle to write the function value.

independent set, Independent set to Vertex Cover), has this additional structural property. Thus all of them are #P-complete by our definition.

We have already talked out one example of a decision problem which is in P, but the counting version seems to be as hard as NP. We will consider another similar problem and show that the counting version can be shown to be #P-complete as well. Indeed $\#CYCLE$ is also #P-complete. We introduce the problem now:

6.2 Perfect Matching in a graph

Definition 6.15. : Perfect Matching: Let $G = (X, Y, E)$ be a bipartite graph. A subset $S \subseteq E$ is said to be a perfect matching iff

$$\forall u \in X \cup Y : \exists! v : (u, v) \in S$$

In words, each vertex has exactly one edge from S incident on it.

We will show a connection that the perfect matching has, with the following combinatorial parameter of the bipartite adjacency matrix. We state this parameter for an arbitrary matrix first.

Definition 6.16. : Permanent of a Matrix For an $n \times n$ matrix A,
Determinant of A, $Det(A) = \displaystyle\sum_{\sigma \text{ is a permutation} \in S_n} (-1)^{sign(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$
Permanent of A, $Per(A) = \displaystyle\sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i,\sigma(i)}$

Notice that the parameter is very similar to the determinant of the matrix A which is written as $Det(A) = \displaystyle\sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$ where $sgn(\sigma)$ assigns a sign to each term based on the sign of the permutation. Observing the differences, permanent of the 0-1 matrix can never be negative, where as that of a determinant of a 0-1 matrix can be negative. Determinant can be computed in FP, whereas we will show the permanent cannot be computed efficiently unless #P = FP.

For a bipartite graph G, let A be the bipartite adjacency matrix in which rows are indexed by x and columns are indexed by y. Note that here $|x| = |y|$, otherwise graph does not have a perfect matching. Now we will show the following connection:

Lemma 6.17. *Let G be the bipartite graph G, and let A be the bipartite adjacency matrix of G. Then,*

$$Per(A) = \#PM(G)$$

Proof. We show that, for any σ, $\displaystyle\prod_{i=1}^{n} A_{i,\sigma(i)} = 1$ if and only if σ gives a perfect matching in G. By definition, $A_{i,\sigma(i)} = 1 \iff (i, \sigma(i)) \in E$. Since the entries are Boolean, and the product is 1, it must be the case that if $\prod_{i=1}^{n} A_{i,\sigma(i)} = 1$ then all edges $A_{i,\sigma(i)}$ are present in the graph. Since σ is a permutation, it must provide an edge $(i, \sigma(i)$ for each vertex $i \in [n]$. Conversely, let S be a perfect matching. It must provide an edge for every $i \in [n]$, and since the perfect matching does not allow any vertex to be covered for more than once, and covers every vertex, we can define a permutation σ such that $\sigma(i) = j$ if $(i, j) \in E$. By our observation, $A_{i,\sigma(i)} = 1$ as well. Hence the proof. ∎

Note that the problem of testing whether there exist a perfect matching or not can be done in polynomial time, by using Floyd-Warshall flow algorithm. Thus testing whether perfect matching is zero or not can be done in polynomial time. But what about computing the value of the permanent function exactly? Is this in $\#\mathsf{P}$ at least? We design a non-deterministic Turing machine: given the matrix A, guess the permutation σ, and check if $\prod_{i=1}^{n} A_{i,\sigma(i)} = 1$, and if so accept and reject otherwise. The number of accepting paths is precisely the number of permutations which contributes 1 to the $Per(A)$. Hence $Per \in \#\mathsf{P}$. Indeed, one can also see this by observing that counting the number of perfect matchings in a bipartite graph is in $\#\mathsf{P}$ (guess a subset of edges and check if it is a perfect matching). We will show soon that Per is $\#\mathsf{P}$-complete too.

Before we end this lecture, let us talk about matrices with non-Boolean entries. Let us say $A \in \mathbb{Z}^{n \times n}$. Is the permanent function in $\#\mathsf{P}$? An obvious difficulty seems to be that the value of the function can be negative, since the entries could be negative, and it does not make sense to ask for a machine to have negative number of accepting paths. Thus our condition is too stringent, we should relax and ask for can we compute permanent with an oracle access to $\#\mathsf{P}$. We will address these in detail in the next class where we introduce a similar combinatorial interpretation of permanent and use that to argue the $\mathsf{FP}^{\#\mathsf{P}}$ upper bound for computing permanent of integer matrices.

CS 6840 Advanced Complexity Theory

Instructor: Jayalal Sarma
Scribe: Dinesh K.
Date: Jan 16, 2012

LECTURE
7

A Combinatorial Interpretation of Permanent

We consider the problem of computing the permanent of a square matrix $A_{n\times n}$ defined as

$$\mathsf{perm}(A) = \sum_{\sigma\in S_n}\prod_{i=n}^{n} A_{i,\sigma(i)}$$

where S_n denotes the set of all permutations of $\{1,2,\ldots,n\}$. Note that this formula is very similar to that of the determinant computation which can be expressed as,

$$|A| = \sum_{\sigma\in S_n} sign(\sigma)\prod_{i=n}^{n} A_{i,\sigma(i)}$$

where *sign* of a permutation is $-1/+1$ depending on if the number of inversions in the permutations is odd or even. Though the definitions are similar "syntactically", the computation problems are very different in the level of hardness. In this lecture, we try to understand more on permanents and will come up with equivalent combinatorial characterisations.

7.1 Characterisation of Permanents of Binary Square Matrices

Our aim is to how characterise the problem of computing permanent as a combinatorial problem thereby gives us a handle for attacking the problem.

Consider a bipartite graph $G(X,Y,E)$ where $|X| = |Y| = n$. Consider the adjacency matrix $A = (A_{i,j})_{n\times n}$ where,

$$A_{i,j} = \begin{cases} 1 & \text{if } (x_i,y_j)\in E, x_i\in X, y_j\in Y \\ 0 & \text{otherwise}\end{cases}$$

We give the following characterisation for permanent of A.

LEMMA 7.18. $\mathsf{perm}(A) = |\#$ *of perfect matchings in* $G|$

Proof. We show that every permutation $\sigma \in S_n$ corresponds to a unique perfect matching $\mathcal{M}$.

($\Rightarrow$) Note that every permutation σ of S_n with the property $\prod_{i=1}^{n} A_{i,\sigma(i)} = 1$ means that there is an edge connecting vertices x_i and $y_{\sigma(i)}$. Since σ is a permutation, the map is bijective and $y_{\sigma(i)}$ exists for every x_i and is unique. Hence by definition $\{(x_i, y_{\sigma(i)})|i \in \{1, 2, \ldots, n\}\}$ forms a perfect matching.

($\Leftarrow$) Similarly, given a perfect matching $\mathcal{M}$, one can define a permutation from X to Y as $\sigma(i) = j$ if $(x_i, y_j) \in \mathcal{M}$. It can be seen that the map is bijective and $\prod_{i=1}^{n} A_{i,\sigma(i)}$ will be 1.

Hence for every $\sigma \in S_n$,

$$\prod_{i=1}^{n} A_{i,\sigma(i)} = 1 \iff \big\{(x_i, y_{\sigma(i)})|i \in \{1, 2, \ldots, n\} \text{ is a matching}\big\} \tag{7.3}$$

Now taking sum over all permutations (on both sides) would give the lemma. ∎

Hence for a $\{0, 1\}$ square matrix, checking if perm is larger than 0 can be solved by showing a perfect matching in the corresponding graph. Standard flow algorithms like the *Fork-Fulkerson algorithm* can be used to find the matching in polynomial time.

This characterisation also gives us a $\#\mathsf{P}$ machine for perm, thereby showing that perm is in $\#\mathsf{P}$.

Lemma 7.19. *For* $A \in \{0, 1\}^{n\times n}$, $\mathsf{perm}(A) \in \#\mathsf{P}$

Proof. Following machine M computes perm(A).
M = "On input $A_{n\times n}$

1. Construct a bipartite graph $G(X, Y, E)$ with $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}, (x_i, y_j) \in E \iff A_{i,j} = 1$

2. Non deterministically guess a subset of edges.

3. Check if they form a perfect matching and accept iff the edges form a perfect matching."

Note that M runs in polynomial time and by the equivalence (7.3), each perfect matching contributes a count of one to the permanent. Hence the number of accepting paths of M will exactly be equal to the perm(A). ∎

7.2 Characterisation of Permanents of Integer matrices

Now we get to matrices with integer entries. Since the entries could also be negative, the permanent of the matrix can be negative. Hence this function cannot be in $\#\mathsf{P}$, but we will show that it is in $\mathsf{FP}^{\#\mathsf{P}}$. To do this, we start with the combinatorial characterisation of the permanent, for which we need the notion of cycle covers.

Definition 7.20. (Cycle Cover) Consider a directed weighted graph $G(V, E, w)$ with $w : E \to \mathbb{N}$. A *cycle cover* of G is a subset of edges that forms a set of vertex disjoint directed cycles in G such that each vertex is a part of at least one cycle in the cycle cover.

Weight of a cycle cover $\mathcal{C}$ is defined as the product of weights of edges in $\mathcal{C}$.

$$wt(\mathcal{C}) = \prod_{e \in \mathcal{C}} w(e)$$

Consider the following examples. Note the effect of adding self loops.

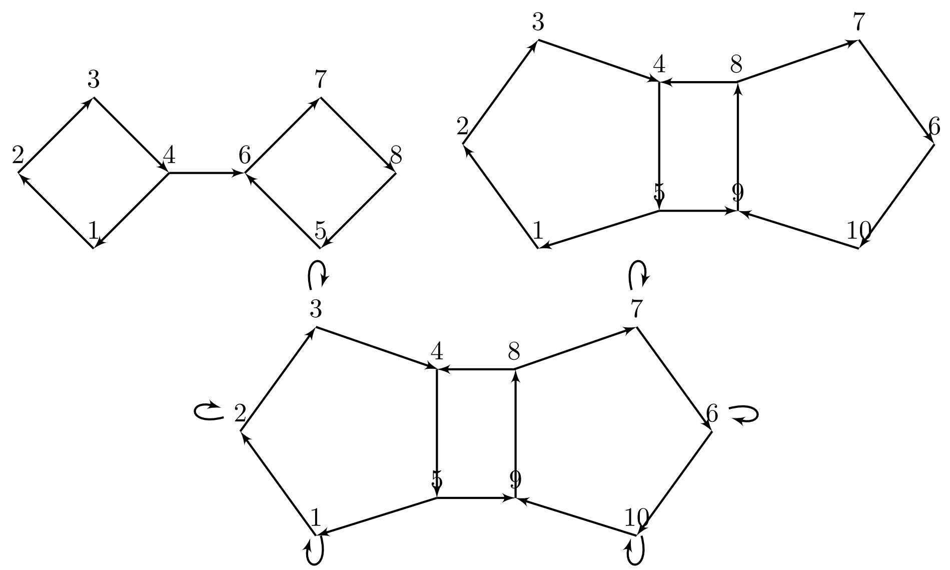

FIGURE 7.1: In the first example, we can see that the only cycle cover is `(1,2,3,4),(5,6,7,8)`. For the second one also there is only one cycle cover, namely `(1,2,3,4,5),(10,9,8,7,6)`. The third is a modified version of the second example. Now there are *two* cycle covers namely, `{(1,2,3,4,5),(10,9,8,7,6)}, {(1),(2),(3),(6),(7),(10),(4,5,9,8)}`.

We are now set to give a characterisation of permanent over matrices in $\mathbb{Z}^{n \times n}$, which we will be crucially using in the next lecture.

LEMMA 7.21. *Let $A \in \mathbb{Z}^{n \times n}$. Let G be the directed weighted graph obtained by interpreting A as the weighted adjacency matrix of G, i,e.*

$$|V(G)| = n,$$

$$\forall 1 \leq i, j \leq n, wt(i,j) = w \iff A_{i,j} = w$$

then

$$\mathsf{perm}(A) = \sum_{\mathcal{C} \in Cyclecover(G)} wt(\mathcal{C})$$

Proof. We show that corresponding to every permutation $\sigma \in S_n$, there is a unique cycle cover $\mathcal{C}$. That is,

$$\prod_{i=1}^{n} A_{i,\sigma(i)} = k \iff \exists\, \mathcal{C} \text{ such that } wt(\mathcal{C}) = k$$

($\Leftarrow$) Suppose there is a cycle cover $\mathcal{C}$ such that $wt(\mathcal{C}) = k$. Without loss of generality assume $k \neq 0$. (If $k = 0$, then one of the edges is not present and hence no cycle cover is possible using the edges in $\mathcal{C}$). Now we can define a permutation to be $\sigma(i) = j \ \forall (i,j) \in \mathcal{C}$. Since $\mathcal{C}$ covers all vertices, σ is defined for all $\{1, 2, \ldots, n\}$ and since $\mathcal{C}$ is composed of cycles, every (i,j) will be unique. Hence the map σ will be bijective and is a valid permutation in S_n. Also $\prod_{i=1}^{n} A_{i,\sigma(i)} = \prod_{(i,j)\in\mathcal{C}} wt(i,j) = wt(\mathcal{C}) = k$.

($\Rightarrow$) Similarly, given a permutation σ such that $\prod_{i=1}^{n} A_{i,\sigma(i)} = k$, we construct a cycle cover as follows.

If $k = 0$, then there is no cycle cover possible with the given permutation since a zero weight edge is appearing. Hence let $k \neq 0$. Fix an $i \in \{1, 2, \ldots, n\}$. Consider the sequence $(i, \sigma(i), \sigma^2(i), \ldots, \sigma^n(i))$ where σ^i for $i \geq 0$ is obtained by applying σ function i times. It can be seen by pigeon hole principle that atleast one values in the sequence must repeat (since, σ is defined on n terms and there are $n+1$ terms in the sequence). Without loss of generality, let $\sigma^l(i)$ and $\sigma^m(i)$ repeat with $0 \leq l < m \leq n$. Since all the weights are non-zero, the vertices $\sigma^l(i), \sigma^{l+1}(i), \ldots, \sigma^m(i) = \sigma^l(i)$ have edges between them and clearly forms a cycle. We can find all the cycles by repeating this process for various values of i. Note that since σ is a permutation, the cycles obtained will be disjoint and will cover the entire graph.

Now taking sum over all permutations on both sides proves the claim. ∎

Using this result the following observation (which is left as an exercise) can also be made.

Observation 7.22. *For $A \in \mathbb{Z}^{n\times n}$ computing permanent of A is in $\mathsf{FP}^{\#\mathsf{P}}$.*

CS 6840 Advanced Complexity Theory

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Date: Jan 17, 2012

LECTURE
8

Permanent Computation is #P-Complete

In this lecture, we define the permanent of a matrix and study the complexity of computing the permanent for various classes of matrices. Finally we prove a theorem by Valiant that $\mathsf{Per}(A)$ is #P-Complete.

8.1 Definitions

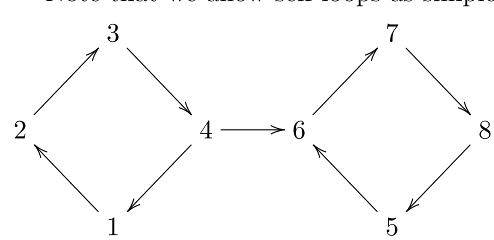
Definition 8.23. **Permanent** Given a matrix $A_{n\times n}$, Let S_n be the set of all permutations of $\{1, 2, \ldots, n\} = [n]$. Then

$$\mathsf{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i,\sigma(i)} \tag{8.4}$$

Earlier, we have seen a characterization of the permanent as the number of perfect matchings in a graph G. To prove the result, we characterize the permanent using **Cycle Covers**.

Definition 8.24. **Cycle Cover** A Cycle Cover C of a directed graph G is a set of pairwise disjoint simple cycles such that each vertex lies in exactly one cycle in C.

Note that we allow self-loops as simple cycles.



Has a cycle cover $\big\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\big\}$.

Definition 8.25. **Weight of a Cycle Cover** Given a Graph $G(V, E)$, let $C = C_1, C_2, \ldots C_l$ be any cycle cover

$$\textbf{Weight of Cycle -}\ \mathsf{Weight}(C_i) = \prod_{e \in E} w(e) \tag{8.5}$$

$$\textbf{Weight of Cycle Cover - } \mathsf{Weight}(C_i) = \prod_{c \in C} \mathsf{Weight}(c) \tag{8.6}$$

LEMMA 8.26. *Let $G(V, E)$ be a graph with adjacency matrix A. $A \in \mathbb{Z}^{n\times n}$. Then*

$$\mathsf{Per}(A) = \sum_{\substack{C \text{ is a} \\ \text{Cycle Cover}}} \mathsf{Weight}(C) \tag{8.7}$$

Proof. Consider a single term from the permanent - $\sum_{i=1}^{n} A_{i,\sigma(i)}$.

We can view σ as a cycle cover as follows.

First we decompose σ into cycles of the form $a, \sigma(a), \sigma^2(a) \dots \sigma^k(a) = a$. Now each cycle in the permutation can be viewed as a cycle in the graph G, the edges being

$$\big\{(1, \sigma(1)), (2, \sigma(2)), \dots (n, \sigma(n))\big\}$$

This is the required cycle cover.

For the other direction, some of the cycle covers generated from permutation might not be valid as some edges are absent. In that case, we just see that their weight is 0 as $A_{i,j} = 0$ for the missing edge (i, j). And in the case that it is a valid cover, $\mathsf{Weight}(C) = \prod_{i\in[n]} A_{i,\sigma(i)}$, because both of them are exactly the product of the corresponding edges.

This shows a bijective correspondence between the permutations and the cycle covers which completes the proof. ∎

EXERCISE 8.27. Let $\mathbb{Z}_+$ denote the non negative integers. Using the previous construction involving cycle covers, show that for $A \in \mathbb{Z}_+^{n\times n}, \mathsf{Per}(A) \in \mathsf{FP}^{\#\mathsf{P}}$.

EXERCISE 8.28. Is the reduction from SAT to 3SAT parsimonious? If yes, show that #3SAT is #P-complete.

8.2 Proof of harndess of Permanent computation

In this section, we show that computing permanant of a $0 - 1$ matrix is #P complete. The result is due to Valiant [2]. The proof presented here is from Dell. et.al [1].

We will show this by using a gadget construction.

THEOREM 8.29. $\#\mathsf{3SAT} \in \mathsf{FP}^{\mathsf{Per}}$

Proof. Consider a formula ϕ in 3SAT. $\phi = C_1 \wedge C_2 \dots C_m$ where $C_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$.

We will construct a directed graph G such that A_G is the adjacency matrix of G, such that $A_G \in \{-1, 0, 1\}^{n\times n}$ and also, $\mathsf{Per}(A_G) = (-2)^k(\#\phi)$ where $(\#\phi)$ is the number of satisfying assignments of ϕ and k is a quantity that we specify later.

LEMMA 8.30. *Let $\#x$ be the number of times a variable x occurs in ϕ.*

Then ϕ can be converted to ϕ' such that $\forall x$, $k = \#x = \#\overline{x}$ and $\#phi = \#phi'$

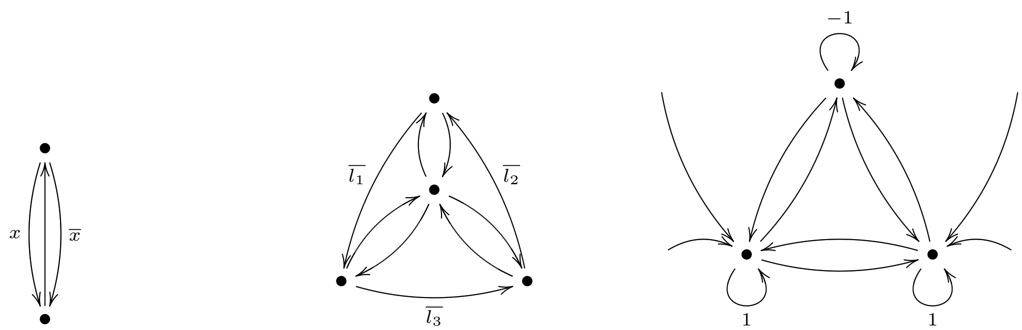
Proof. We first observe that if $\#x \neq \#\overline{x}$. Then this imbalance can be removed by adding terms of the form $(x \vee x \vee \overline{x})$ and/or $(x \vee \overline{x} \vee \overline{x})$ because they add or decrease the relative number of x compared to $\overline{x}$.

Now, we assume that $\forall x, \#x = \#\overline{x}$. Now to compensate for relative differences between x, y we add terms of the form $(x \vee \overline{x} \vee \overline{y}) \wedge (x \vee \overline{x} \vee \overline{y})$ to increase the number of x and the other way to decrease.

And we just note that the number of solutions is invariant because each of the added terms are always true.

This completes the lemma. $\square$

We will be constructing the graph from three gadgets - Variable Gadget, Clause Gadget and the Equality Gadget.



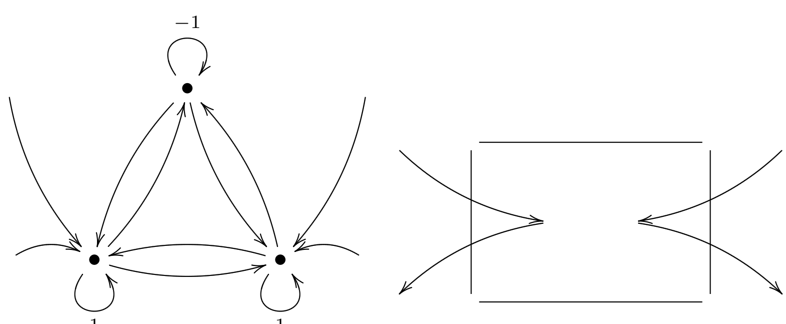
(a) Variable Gadget (b) Clause Gadget (c) Equality Gadget

Figure 8.2: The Gadgets

Now, we consider the construction of the graph.

For each variable pair $(x, \overline{x})$ we have one variable gadget. Each clause has a Clause Gadget and the equality gadget is used to join each variable with all the clauses the variable is in.

The equality gadget is represented as a black box as follows -



(a) Equality Gadget (b) Equality Gadget Representation

Figure 8.3: Equality Gadget Blackbox representation

The way we connect a variable and a clause is shown in the next figure. Here $\overline{x}$ is the literal l_1 in the clause.

When the same variable appears in multiple clauses, we split the edge representing the variable and attach multiple equality gadgets. The next figure shows the split.

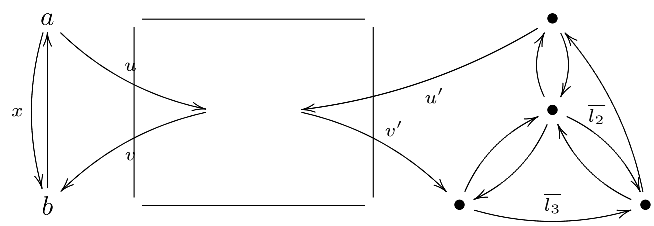


FIGURE 8.4: Connection between variables and clauses.

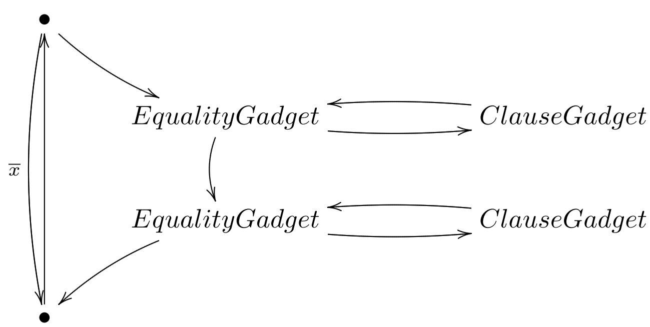


FIGURE 8.5: One variable occurring in multiple clauses.

This essentially completes the construction. We will prove the correctness of the construction in a series of claims.

CLAIM 8.31. *Any cycle cover can either use the edge x or $\overline{x}$ in the variable gadget, but not both.*

Proof. If a cycle cover used both the edges, then the vertices would be covered twice. ∎

CLAIM 8.32. *Each assignment corresponds to atleast one cycle cover.*

Proof. For each variable, choose x or $\overline{x}$ based on the assignment. And for each clause choose the cycle $l_1 \to l_2 \to l_3$ and choose self-loops everywhere else. ∎

We will derive a much precise correspondence in the remaining proof.

CLAIM 8.33. *In any cycle cover C, either both u, v are used, or neither u, v are used.*

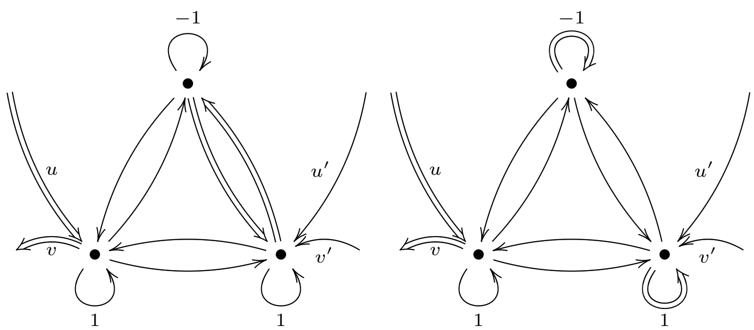
Proof. The proof is just a verification, We see in 8.4 that if the edge u is used, edge (b, a) will have to be used, and to complete a cycle, we will need v to complete the cycle as the edge x cannot be used.

The proof holds unmodified for edges u', v' too. ∎

CLAIM 8.34. *If both u, v and u', v' edges are used in the cycle cover, then the -1 valued self-loop has to be chosen in the cycle cover.*

CLAIM 8.35. *If edges u, v are used while u', v' are not used, the corresponding Cycle Covers* *contribute weight 0.*

Proof. We just observe that there are two components one contributing $+1$ and the other as -1 in the cycle weight. Cycle covers are marked in double lines.



(a) Weight +1 (b) Weight -1

FIGURE 8.6: Only one of the two sets of edges are present

$\square$

CLAIM 8.36. *If both u, v and u', v' are not used, then the Cycle Covers have a contribution of 2 from this gadget.*

Proof. The figure 8.7 contains all the possible cycle covers of the gadget. Their contributions sum upto 2. $\square$

CLAIM 8.37. *Weight of each cycle cover is* $(-2)^{kn}$

Proof. We want to claim that if a variable x is assigned value 1, then all the equality gadgets for x will contribute 1 to the weight because u, v and u', v' will both be a part of the cycle cover for each of the gadgets, if just one pair is in the cycle cover, we have seen that those covers would contribute 0 to the weight.

Also, the gadgets that correspond to $\overline{x}$ will not have either u, v or u', v' being used - because, u, v cannot be used as x edge will be used, hence $\overline{x}$ cannot be used. Now, if u', v' are used, those cycles will have 0 weight as seen in the observation.

So, the only possibility there is both u, v and u', v' are not used. In that case, the contribution would be 2 for each gadget. As there are k such gadgets, we will have a contribution of 2^k from these.

So, multiplying them would give us, that each variable pair $x, \overline{x}$ contribute exactly $(-2)^k$ to the weight. Hence the cycle cover would have a weight of exactly $(-2)^{kn}$.

So, We sum them up over all the possible assignments to get the required result. This completes the proof.

$\square$

So, we have the result -

$$\sum_{\text{C is a Cycle Cover}} \mathsf{Weight}(C) = \mathsf{Per}_{-1,0,1}(A) \tag{8.8}$$

Hence computing $\#\mathsf{SAT}$ reduces to computing $\mathsf{Per}_{-1,0,1}$. This completes the proof. ∎

Now we will first show that $\mathsf{Per}_{-1,0,1}$ reduces to $\mathsf{Per}_{0,1,\dots n}$ and finally show that $\mathsf{Per}_{0,1,\dots n}$ reduces to $\mathsf{Per}_{0,1}$ and hence completing the theorem.

THEOREM 8.38. $\mathsf{Per}_{-1,0,1} \in \mathsf{FP}^{\mathsf{Per}_{0,1,\dots n}}$.

Proof. The first thing we observe is that all the -1 terms in the adjacency matrix A represent self-loops because in the construction, -1 was the edge weight of only one self-loop.

Consider $\mathsf{Per}(A)$ as a polynomial in x where each -1 is replaced by x denoted by $p(x)$

Now we just observe that, using $\mathsf{Per}_{0,1,\dots n}$ as an oracle, we can find $p(0), p(1), ...p(n)$. Also, degree of p $\leq n$ because x occurs only in the diagonal entries, hence only n x can be present.

So, now we just use Lagrange Interpolation to find the polynomial p. This can be done in poly time. Once this is done, $\mathsf{Per}_{-1,0,1}(A) = p(-1)$, which can be computed easily.

This completes the reduction. ∎

In the final reduction, we show that $\mathsf{Per}_{0,1,\dots n} \in \mathsf{FP}^{\mathsf{Per}_{0,1}}$, and that $\mathsf{Per}_{0,1}$ is as hard as the other Per computations.

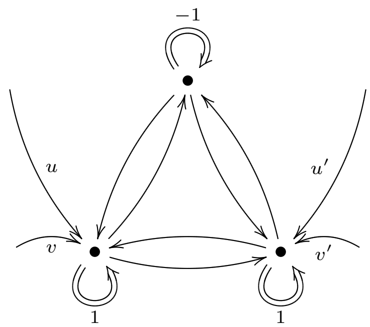
THEOREM 8.39. $\mathsf{Per}_{0,1,\dots n} \in \mathsf{FP}^{\mathsf{Per}_{0,1}}$

Proof. This proof involves substituting the -1 self-loop with a gadget so that we can compute the values of the polynomial $p(x)$ at points $x = 0, x = 1, \dots x = n$.

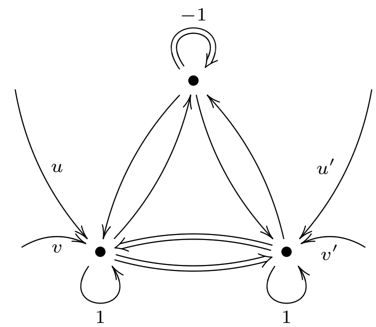
The gadget we use is - Consider any $a = (a_k, a_{k-1} \dots a_0)_2$ in base 2 where $a \in \{0, 1, \dots n\}$. Now we want to replace the self-loop of weight a with the gadget, so that the gadget contributes exactly k weight to the Cycle cover.

The gadget has precisely n cycles of weight 1 each. This gadget can be used to replace the self loop and then query the $\mathsf{Per}_{0,1}$ oracle.

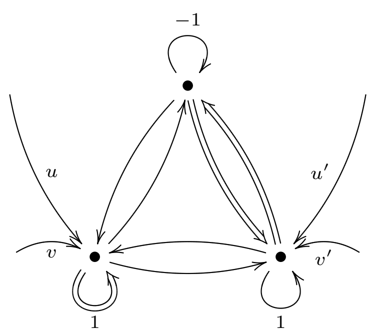
This completes the reduction. Hence proved. ∎



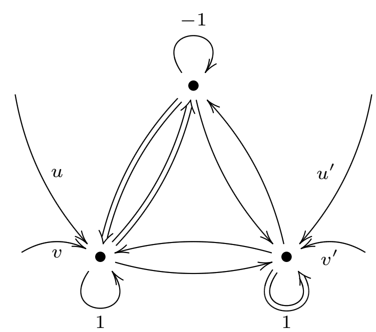
(a) Weight -1



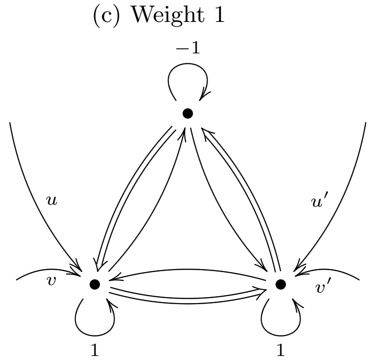
(b) Weight -1



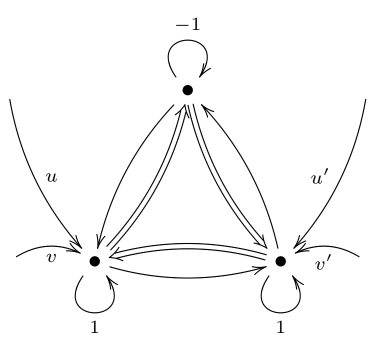
(c) Weight 1



(d) Weight 1

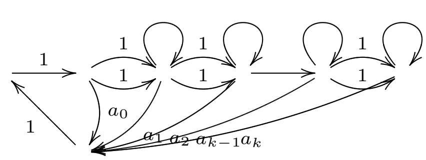


(e) Weight 1



(f) Weight 1

FIGURE 8.7: The weights sum upto 2



CS 6840 Advanced Complexity Theory

LECTURE

9

Instructor: Jayalal Sarma
Scribe: Balagopal
Date: 23 Jan, 2012

Probabilistic TMs and Randomized Algorithms

9.1 Review of Branching machines

A branching machine is a machine that is allowed to make non-deterministic guesses while computation. This is a generalization of NTMs where the branching machine accepts iff at least one path accepts. Similarly we can think of various definitions of acceptance each of them leading to a (possibly) new complexity class. For example, PP is defined as the class which has a branching machine where more than half of the paths accept. $\oplus\mathsf{P}$ is defined as the class where each language has a branching machine in which an odd number of paths accept if $x \in L$. Similarly we can think of every branching machine as computing a function $f(x)$ where $f(x)$ is defined as the number of accepting paths of M on x.

Question. How is $\oplus\mathsf{P}$ related to P , NP , PSPACE , and PP ?

We will come back to answer this question, and show that NP can be *almost* solved in P if $\oplus\mathsf{P}$ can be solved in P. The *almost* here will refer to possibility of error by the algorithm in the decision. We will now do a systematic formal study of randomized algorithms (algorithms which may make an error but with low probability). We introduce these from the branching machine perspective that we have seen so far.

9.2 Characterizing Randomized Algorithms

We are going to characterize randomized algorithms using branching machines that we defined previously.

Consider a branching machine M for the language L. If $x \in L$ call all paths of M that reject as erroneous. If $x \notin L$ call all paths of M that accept as erroneous. Intuitively, we want the erroneous paths to be as small as possible. We use the following notations for counting paths with specific properties for a branching machine M.

$\#paths_M(x)$ Total number of paths of M on x

$\#acc_M(x)$ Total number of accepting paths of M on x

$\#rej_M(x)$ Total number of rejecting paths of M on x

$\#err_M(x)$ Total number of erroneous paths of M on x

We defined class PP as the set of languages L with branching machines satisfying the following property.

$$\begin{aligned} x \in L &\implies P(A \text{ accepts } x) > 1/2 \\ x \notin L &\implies P(A \text{ rejects } x) \geq 1/2 \end{aligned}$$

To connect the notion of a branching program and a randomized algorithm, we have to make sure that all paths of the branching machine are of the same length (i.e., the computation tree is a full binary tree). This can be done by a construction similar to the one used to do this while defining #P. However, when a path is extended by branching it yields multiple paths. We must ensure that the error probabilities are not increased during this process. Note that this is different from what we did while defining #P. The goal there was to preserve the count (Number of accepting paths). The goal here is to keep the error probability (which depends on the relative number of accepting and rejecting paths) the same.

Now consider a randomized algorithm A that chooses one path of M uniformly at random and executes it (This shows that there is a randomized algorithm corresponding to every branching machine). Clearly

$$x \in L \iff \#err_M(x) \leq \frac{1}{2}\#paths_M(x) \tag{9.9}$$

We see that the probability of A making an error is at most 1/2. But this can be achieved by a trivial randomized algorithm that flips a coin and determines the result according to the outcome of the coin flip. So the class PP is not a good candidate for formally capturing "good" randomized algorithms.

What we want is the error probability to be bounded away from 1/2. If $\#err_M(x) < \frac{1}{4}\#paths_M(x)$, then certainly the corresponding randomized algorithm is better than a trivial one.

We will now work towards defining a problem for which there is an efficient (poly time) randomized algorithm but for which no poly time deterministic algorithm is known. This gives reason to study randomized algorithms formally.

9.3 Polynomial Identity Testing

This problem has its roots in the simple high school arithmetic. Suppose we are given a polynomial in a complicated form where the monomials may repeat with arbitrary coefficients etc. We want to find out if the coefficient of the monomials cancel out to zero. This in effect is testing whether the polynomial is the zero polynomial, and equivalenty it is testing if the polynomials evaluates to zero on all substitutions of the variable from the underlying field $\mathbb{F}$.

How are we given the polynomial? This indeed is going to have effect on the complexity of the problem. Let us start with the high school arithmetic again. Suppose we are given it in the monomial form (though some monomials may repeat) along with their coefficients.

To solve the problem, it suffices to check, for each monomial whether the coefficient in its various appearences is adding up to zero. Given the explicit representation at the input, this is very easy to do by simply going over the input for each monomial. Hence this can be done in time polynomial in the input.

What if the polynomial is not given that explicity. What is the most implicit form that we can think of? A black box which evaluates the polynomial. That is, we have an oracle p when given input a returns $p(a)$, the value of polynomial at a.

Assume that we are also given an upper bound on the degree of the polynomial $deg(p) \leq d$. Indeed, we do not have access to the actual polynomial except through the blackbox. We have to use some property of the degree d polynomials. The most obvious one is the number of points in which they can evaluate to zero. Based on this thought, the following deterministic algorithm solves the problem.

1. Choose $d+1$ different points $a_1, \ldots, a_{d+1}$.
2. Call the oracle $d+1$ times to evaluate $p(a_1), \ldots, p(a_{d+1})$.
3. If all calls returned 0 accept else reject.

FIGURE 9.8: A deterministic algorithm for univariate polynomial identity testing

If p were really 0 then all calls will return 0 and we will definitely accept. If p were not 0, then at most d calls can return 0 since a polynomial with degree at most d has at most d roots. Hence if $p \neq 0$, then our algorithm will definitely reject.

Now let us think about the problem when p is a multivariate polynomial. The previous assertion that a degree d polynomial has at most d roots no longer holds. To see this, consider the degree 2 polynomial $p(x_1, x_2) = x_1 x_2$. This has an infinite number of roots $x_1 = 0, x_2 \in \mathbb{F}$, where $\mathbb{F}$ is the (possibly infinite) field over which p is defined. We can work around this problem by considering a finite subset of the field, say $S = \{0, \ldots, 10\}$. The polynomial p has 19 zeroes. So if x_1, x_2 is chosen uniformly at random from S there is at most 19/100 chance that we will get a false result. As can be seen from the above example, by making the size of S arbitrarily large, we can make the error probability arbitrarily small. But then the disadvantage is that we will need more random bits in order to choose an element at random from the set $|S|$, and the running time of our algorithm will also increase.

In the next lecture, we will show that this intuition is correct by exhibiting a low error polynomial time randomized algorithm for the multivariate case. The question of finding a deterministic algorithm for this problem is open. Although it looks like a simple algorithmic problem from algebra which only mathematicians might be interested in, there are several computational problems that can be encoded into this form and hence can be solved efficiently if this algorithmic problem can be solved efficiently.

CS 6840 Advanced Complexity Theory

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Date: Jan 24, 2012

LECTURE
10

Polynomial Identity Testing

Towards the end of last lecture, we introduced the following problem : *Given a polynomial p, test if it is identically zero.* That is, do all the terms cancel out and become the zero polynomial. Described as a language :

$$\mathrm{PIT} = \{p \mid p \equiv 0\}$$

We also saw some easy cases of the problem:

1. When it is given as a sum of monomials: Given p, run over the input to figure out the coefficient of each monomial, and if all of them turn out to be zero, then report that p is in PIT. This algorithm runs in $O(n^2)$ time.
2. When it is given as a Black Box: In uni-variate case, check $p(x)$ for $d+1$ different points where d is the degree bound. If the polynomial is not equivalent to zero, then at-least one of the steps gives a non-zero value. Indeed, if the polynomial is zero, then all the $(d+1)$ evaluations will result in a zero value. Thus the algorithm is correct and runs in time $O(d)$ where d is the degree of the polynomial.

As we observed, this strategy could not be generalized in multi-variate case. We took an example as $p(x_1, x_2) = x_1 x_2$. For the assignment $x_1 = 0$, whatever x_2 chose, the value will always be 0. However, if $p \equiv 0$, no matter what we choose as the substitution for x_1, and x_2, the polynomial will be identically zero.

The strategy that we will follow is as follows: If the total degree of the polynomial is $\leq d$, and if $S \subseteq \mathbb{F}$, such that $|S| \geq 2d$, instead of picking elements arbitrarily, we pick elements uniformly at random from S. Indeed, there may be many choices for the values which may lead to zero. But how many?

Lemma 10.40 (Schwartz-Zippel Lemma). *Let $p(x_1, x_2, \cdots, x_n)$ be a non-zero polynomial over a field $\mathbb{F}$. Let $S \subseteq \mathbb{F}$*

$$Pr[p(\bar{a} = 0] \leq \frac{d}{|S|}$$

Proof. (By induction on n) For $n = 1$: For a univariate polynomial p of degree d, there are $\leq d$ roots. Now in the worst case the set S that we picked has all d roots. Thus for a random choice of substitution for the variable from S, the probability that it is a zero of the polynomial p is at most $\frac{d}{|S|}$.

For $n > 1$, write the polynomial p as a univariate polynomial in x_1 with coefficients as polynomials in the variables $p(x_2, \dots, x_n)$.

$$\sum_{j=0}^{d} x_1^j p_j(x_2, x_3, \dots, x_n)$$

For example: $x_1x_2^2 + x_1^2x_2x_3 + x_3^2 = (x_2x_3)x_1^2 + (x_2^2)x_1 + x_1^0(x_3^2)$.

To analyse the probability that we will choose a zero of the polynomial (even though the polynomial is not identically zero). For a choice of the variables as $(a_1, a_2, \cdots, a_n) \in S^n$, we ask the question : how can $p(a_1, a_2, \cdots, a_n)$ be zero? It could be because of two reasons:

1. $\forall j \ : \ 1 \leq j \leq n, \ \ p_j(a_2, a_3, \dots, a_n) = 0$.
2. Some coefficients $p_j(a_2, a_3, \dots, a_n) = 0$ are non-zero, but the resulting univariate polynomial in x_1 evaluates to zero upon substituting $x_1 = a_1$.

Now we are ready to calculate $Pr[p(a_1, a_2, \dots, a_n) = 0]$. For a random choice of $(a_1, \dots, a_n)$. Let A denote the event that the polynomial $p(a_1, \dots, a_n) = 0$. Let B denote the event that $\forall j \ : \ 1 \leq j \leq n, \ \ p_j(a_2, a_3, \cdots, a_n) = 0$. Now we simply write : $Pr[A] = Pr[A \wedge B] + Pr[A \wedge \bar{B}]$.

We calculate both the terms separately: $Pr[A \wedge B] = Pr[B].Pr[A|B] = Pr[B]$ where the last equality is because $B \implies A$. Let ℓ be the highest power of x_1 in $p(x)$. That is $p_\ell \neq 0$. Since the event B insists that for all j, $p_j(a_2, a_3, \dots, a_n) = 0$, we have that $Pr[B] \leq Pr[p_\ell(a_1, a_2, \dots, a_n) \neq 0]$. By induction hypothesis, since this polynomial has only $n-1$ variables and has degree at most $\frac{d-\ell}{S}$. Thus, $Pr[B] \leq \frac{d-\ell}{S}$.

To calculate the other term,

$$Pr[A \cap \bar{B}] \ \ = \ \ Pr[\bar{B}].Pr[A|\bar{B}] \leq Pr[A|\bar{B}] \leq \frac{\ell}{|S|}$$

where the last inequality holds because the degree of the non-zero univariate polynomial after substituting for $a_2, \dots, a_n$ is at most ℓ and hence the base case applies. ∎

This suggests the following efficient algorithm for solving PIT. Given d and a blackbox evaluating the polynomial p of degree at most d.

```
1.  Choose S ⊆ 𝔽 of size ≥ 4d.
1.  Choose (a_1, a_2, ..., a_n) ∈_R S^n.
2.  Evaluate p(a_1, a_2, ... a_n) by querying the blackbox.
3.  If it evaluates to 0 accept else reject.
```

FIGURE 10.9: A randomized algorithm for multivariate polynomial identity testing

The algorithm is clearly running in polynomial time. The following Lemma states the error probability and follows from the Schwartz-Zippel Lemma that we saw before.

LEMMA 10.41. *There is a randomized polynomial time algorithm A, which, given a black box access to a polynomial p of degree d (d is also given in unary), answers whether the polynomial is identically zero or not, with probability at least $\frac{3}{4}$.*

Notice that in fact the lemma is weak in the sense that it ignores the fact that when the polynomial is identically zero then the success probability of the algorithm is actually 1 !.

Now we connect to where we left out from Branching machines, by observing that this randomized algorithm is indeed a branching machine. Let $\chi_L(x)$ denote the characterestic function of the language. That $\chi_L(x) = 1$ if $x \in L$ and 0 otherwise. Let us call a computation path to be *erroneous* if the decision (1 for accept and 0 for reject) reported in that path is not $\chi_L(x)$. Let $\#err_M(x)$ denote the number of erroneous paths. Thus the braching machine has some guarantees about $\#err_M(x)$.

Corollary 10.42. *Let L be the language PIT, then there exists a branching machine M, running in $p(n)$ time (hence using at most $p(n)$ branching bits).*

$$\#err_M(x) \leq \frac{1}{4}2^{p(n)}$$

Is there anything special about $\frac{1}{4}$? As we can go back an observe, this number can be reduced to say $\frac{1}{5}$ by easily choosing the size of the set S to be larger than $5d$ where d is the degree of the polynomial. We get better success probability then, but what do we lose? We lose on the running time, since we have to spend more time and random bits now in order to choose the elements from S^n as $|S|$ has gone up.

But more seriously, this seems to be an adhoc method which applies only to this problem. In general, if we have a randomized algorithm that achieves a success probability of $\frac{3}{4}$, can we boost it to another constant?

Based on the discussion so far, we can make the following definition of a set of languages. For a fixed ϵ, define the class BPP_ϵ as follows. $L \in BPP_\epsilon$, for some $0\epsilon < \frac{1}{2}$, if there is a branching machine M running in time $p(n)$, such that $\#err_M(x) \leq \epsilon 2^{p(n)}$.

Notice that all these sets of classes are contained in PSPACE. Let $L \in BPP_\epsilon$ via a machien M. By just brute force run over all the choice bits of the machine M (reusing space across different paths) we can exactly calculate how many paths are accepting. This information will be sufficient to decide whether $x \in L$ or not..

All of them contain P since there is a trivial choice machine which achieves any success probability (of 1 !).

How do they compare, for different ϵ and ϵ'? Could they be incomparable with each other (and hence form an antichain in the poset of languages)? In the next lecture we will show a lemma which will imply that for any constants $0 < \epsilon \neq \epsilon' < \frac{1}{2}$, $\mathsf{BPP}_\epsilon = \mathsf{BPP}_{\epsilon'}$. This eliminates the possibility of an antichain in the poset and makes the definition of the following complexity class.

Definition 10.43 (BPP). A language L is said to be in BPP if there is an ϵ such that $0 < \epsilon \leq \frac{1}{2}$, and a branching machine M running in time $p(n)$ such that: $\#err_M(x) \leq \epsilon 2^{p(n)}$

We begin the thoughts on proving $\mathsf{BPP}_\epsilon = \mathsf{BPP}_{\epsilon'}$ for $0 < \epsilon \neq \epsilon' < \frac{1}{2}$. Without loss of generality, assume that $\epsilon < \epsilon'$. Note that $\mathsf{BPP}_\epsilon \subseteq \mathsf{BPP}_{\epsilon'}$. To show the other direction we need to improve the success probability of the algorithm. Viewing the success of the algorithm as a favourable probability event, a natural strategy is to repeat the process independently again, so that the probability of error goes down multiplicatively. Thus it improves the success probability.

CS 6840 Advanced Complexity Theory

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LECTURE
11

Amplification Lemma

In the last lecture, we saw the polynomial identity testing problem and a randomized algorithm for it. We also discussed how branching machines with guarantees on the number of erroneous paths characterize randomized algorithms. We ended the last lecture with a question about how two sets of languages compare. BPP_ϵ and $\mathsf{BPP}_{\epsilon'}$. for different ϵ and ϵ'?

11.1 Amplification of Success Probability

We showed that if $\epsilon < \epsilon'$ then $\mathsf{BPP}_\epsilon \subseteq \mathsf{BPP}_{\epsilon'}$. A strategy to prove the other direction was the following : Repeat the randomized algorithm (experiment) multiple times (say k), and then take the majority of the outcomes in order to improve our success probability. One remark is that the repetition is sequential and happens on each branch. Thus we are essentially producing a new branching machine with many deeper computation paths.

Why would this improve the success probability? and if so, how does it depend on k?. The following lemma answers these.

Lemma 11.44. *If $\mathcal{E}$ is an event that $Pr(\mathcal{E}) \geq \frac{1}{2} + \epsilon$, then the probability the $\mathcal{E}$ occurs atleast $\frac{k}{2}$ times on k independent trials is at least* $1 - \frac{1}{2}(1-4\epsilon^2)^{\frac{k}{2}}$

Proof. Let q denote the probability the $\mathcal{E}$ occurs atleast $\frac{k}{2}$ times on k independent trials. Let $q_i = \Pr(\mathcal{E}$ occurs exatly i times in k trials), $0 \leq i \leq k$. Thus, $q = 1 - \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} q_i$. We will analyse the complementary event: $\Pr(\mathcal{E}$ occurs atmost $\frac{k}{2}$ times) $= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} q_i$.

We show an upper bound on each q_i and thus show an lower bound on q.

$$\begin{aligned}
q_i &= \binom{k}{i}(\frac{1}{2}+\epsilon)^i(\frac{1}{2}-\epsilon)^{k-i} \\
&\leq \binom{k}{i}\left(\frac{1}{2}+\epsilon\right)^i\left(\frac{1}{2}-\epsilon\right)^{k-i}\left(\frac{\frac{1}{2}+\epsilon}{\frac{1}{2}-\epsilon}\right)^{\frac{k}{2}-i} (because \epsilon \leq \frac{1}{2}) \\
&= \binom{k}{i}\left(\frac{1}{2}+\epsilon\right)^{\frac{k}{2}}\left(\frac{1}{2}-\epsilon\right)^{\frac{k}{2}} \\
&= \binom{k}{i}\left(\frac{1}{4}-\epsilon^2\right)^{\frac{k}{2}}
\end{aligned}$$

Now we analyse the sum:

$$\begin{aligned}
\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor} q_i &\leq \sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{i}\left(\frac{1}{4}-\epsilon^2\right)^{\frac{k}{2}} \\
q = 1-\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor} q_i &\geq \sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{i}\left(\frac{1}{4}-\epsilon^2\right)^{\frac{k}{2}} \\
&= 1-\left(\frac{1}{4}-\epsilon^2\right)^{\frac{k}{2}} 2^{k-1} \\
&= 1-\frac{1}{2}\left(1-4\epsilon^2\right)^{\frac{k}{2}} \\
\text{Thus, } q &\geq 1-\frac{1}{2}\left(1-4\epsilon^2\right)^{\frac{k}{2}}
\end{aligned}$$

∎

In the last lecture, we defined the class BPP_ϵ (Bounded Error Probabilistic Polynimial Time) and now we can use the above amplification lemma to prove that

$$\mathsf{BPP}_\epsilon = \mathsf{BPP}_{\epsilon'} \quad \forall 0 \leq \epsilon, \epsilon' < \frac{1}{2}$$

We want to calculate In general, the above lemma can be used to prove that , at the cost of running time, the the error probability of a language L , $L \in \mathsf{BPP}$ can be reduced to $\frac{1}{2^{q(n)}}$ where $q(n)$ is a polynomial in n.

Lemma 11.45. *$L \in \mathsf{BPP}$ if an only if for any polynomial $q(n)$ there is a machine M that runs for time $p(n)$ (which depends on $q(n)$) such that*

$$Pr(M \text{ errs on input } x)) \leq 2^{-q(n)}$$

In terms of number of paths,

$$\#err_M(x) \leq 2^{p(n)-q(n)}$$

Proof. Given a language $L \in \mathsf{BPP}_\epsilon$ with PTM M, we design a PTM N such that $L(N) \in \mathsf{BPP}$ and $L(n) = L$ as follows:

- Run the machine M on input x k times independently where choice of k is such that

$$\frac{1}{2}\left(1-4\epsilon^2\right)^{\frac{k}{2}} \leq 2^{-q(n)} \tag{11.10}$$

The above equation (11.10) yields a value of k polynomial in n and hence N runs a polynomial number of times. The amplification lemma ensures that the error probability reduces to the LHS of the equation (11.10). ∎

The Structure of BPP

We explore some interesting structural properties about the class BPP.

PROPOSITION 11.46. BPP *is closed under complementation.*

Proof. Let $L \in BPP$ via PTM M with error probability $\epsilon < \frac{1}{2}$. We show that $\bar{L}$ is in BPP. We design a new machine $\bar{M}$ by switching the accept and reject states of M.

$$\begin{aligned}
x \in L &\implies \#acc_M(x) \geq (1-\epsilon)\#path_M(x) \\
&\implies \#rej_M(x) \leq \epsilon\#path_M(x). \\
&\implies \#acc_{\bar{M}} \leq \epsilon\#path_{\bar{M}}(x). \\
&\implies x \notin L(\bar{M}). \\
x \notin L &\implies \#acc_M(x) \leq \epsilon.\#path_M(x). \\
&\implies \#rej_M(x) \geq (1-\epsilon)\#path_M(x). \\
&\implies \#acc_{\bar{M}} \geq (1-\epsilon)\#path_{\bar{M}}(x). \\
&\implies x \in L(\bar{M}).
\end{aligned}$$

Hence, we have,

$$x \in L \iff x \notin L(\bar{M})$$

Therefore, $L(\bar{M}) = \bar{L}$ and $L \in BPP$ via machine $\bar{M}$. Hence, BPP is closed under complementation. ∎

One-sided Error Randomized Algorithms

Consider the language , PIT that is, Polynomial Identity Testing,

$$PIT = \{p | p \equiv 0\}$$

where p is a polynomial. From, the last lecture, we make the following observation about PIT, if $p \in PIT$, PTM makes no error, if $p \notin PIT$, PTM makes some error (less than half the number of paths).

We now explore how complexity theory can be extended to these kind of algorithms too.

DEFINITION 11.47. (RP) A language L is said to be in RP if there is an ϵ such that $0 < \epsilon < \frac{1}{2}$, and a randomized algorithm A such that :

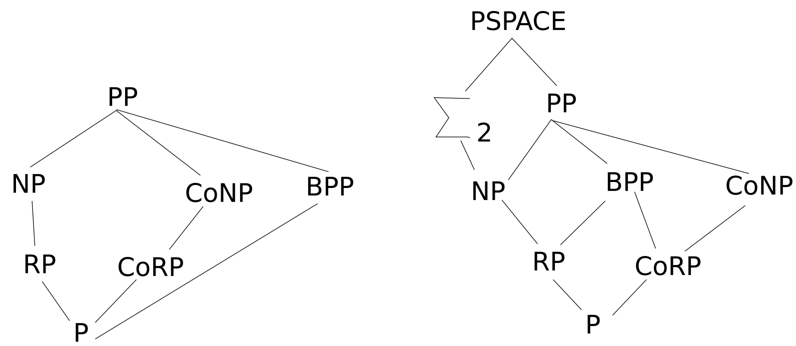
$$\begin{aligned}
x \in A &\implies Pr[\text{ } A \text{ accepts }] \geq \frac{1}{2} + \epsilon \\
x \notin A &\implies Pr[\text{ } A \text{ accepts }] = 0
\end{aligned}$$

Hence we have the following proposition:

$$PIT \in \mathsf{coRP} \tag{11.11}$$

PROPOSITION 11.48. $\mathsf{RP} \subseteq \mathsf{NP}$

Proof. Consider language $L \in \mathsf{RP}$ via machine M such that $x \in L \implies M$ accepts x with some error $\epsilon < \frac{1}{2} \implies M$ accepts x on atleast 1 path.
$x \notin L \implies M$ accepts x with probability $0 \implies M$ rejects on all paths. Thus $L \in \mathsf{NP}$. Moreover, even with the acceptance condition of a non-deterministic machine, the branching machine corresponding to the RP algorithm accepts the language L itself. ∎



11.2 Derandomization of BPP

There are several questions connected to the new class BPP that contains several natural problems. We saw one example of multivariate polynomial identity testing problem. Is $BPP \subseteq P$? This would amount to showing that in the world of efficient computations, randomization does not add any power. There are reasons to remotely believe this to be the case, but till date there is no proof.

A question of slightly different flavour is, if problems in BPP are contained in NP? That is, can we trade non-determinism with randomness? We already know that if the randomness causes only one-sided error, then it can be replaced by simple non-deterministm ($\mathsf{RP} \subseteq \mathsf{NP}$). But extending this to two-sided error version is an interesting open problem in the area.

We show a relaxed containment which can be seen to be an improved upper bound for problems in BPP compared to the trivial upper bound of PSPACE.

THEOREM 11.49. $BPP \in \Sigma_2$

Proof. Let $L \in \mathsf{BPP}$. By using amplification lemma for $q(n) = n$ we can state: there is a probabilistic Turing machine M and polynomial p(n) such that,

$$\#err_M(x) \leq 2^{-n}2^{p(n)}$$

Let us recall the definition and a characterization of the class Σ^2 Σ_2 is defined as follows: $L \in \Sigma_2$ iff $\exists B \in P$ such that:

$$x \in L \iff \exists y, \forall z, (x, y, z) \in B$$

There is a clear mindblock here. How do we tradeoff quantifiers to randomness?
Let us define a set A(x) as follows:

$$A(x) = \{y \in \{0,1\}^{p(n)} | M \text{ accepts } x \text{ on path } y\}$$

Observe that,

$$x \in L \Rightarrow |A(x)| \geq (1 - 2^{-n})2^{p(n)}$$

that is, no. of y's such that $M(x, y) = 1$ is large.

$$x \notin L \Rightarrow |A(x)| \leq 2^{-n}2^{p(n)})$$

that is, no, of y's such that $M(x, y) = 1$ is small.

Parity Map: For two strings $y, z \in \{0,1\}^{p(n)}$, let $y \oplus z$ denote the bit-wise parity of the two strings. We can extend the parity map to operate on subsets of $\{0,1\}^{p(n)}$ as follows:

$$S \oplus z = \{y \oplus z \mid y \in S\, z \in \{0,1\}^{p(n)}\}$$

OBSERVATION 11.50. *For a fixed z, $\oplus_z$ is a bijection from $\{0,1\}^{p(n)} \to \{0,1\}^{p(n)}$. That is, for any $z \in \{0,1\}^{p(n)}$, and $S \subseteq \{0,1\}^{p(n)}$, $|S| = |S \oplus z|$.*

Ask the question : how many z's do we need to cover $\{0,1\}^{p(n)}$ entirely? That is, how large do we need m to be, such that there exists strings $z_1, z_2, \ldots, z_m$ such that:

$$\bigcup_{i=1}^{m} (A(x) \oplus z_i) = \{0,1\}^{p(n)}$$

Intuitively, we expect the answer to be *small* when the size of $A(x)$ is large, and *large* when the size of $A(x)$ is small. Now we formalize this.

Case 1: Small $|A(x)| \leq 2^{-n}2^{p(n)}$ In the best case, let each z_i maps $A(x)$ to non-intersecting sets.

$$\forall i, j (A(x) \oplus z_i) \cap (A(x) \oplus z_j) = \phi, i \neq j$$

$$|\bigcup_{i=1}^{m} (A(x) \oplus z_i)| \geq |\{0,1\}^{p(n)}|$$

$$\Rightarrow m(2^{-n})2^{p(n)} \geq 2^{p(n)}$$

$$\Rightarrow m \geq 2^n$$

Hence, the no. of z's required is exponential in n when $A(x)$ is small, that is, when $x \notin L$.

Case 2: Large $|A(x)| \geq (1 - 2^{-n})2^{p(n)}$ We prove that $\exists z_1, z_2, z_3 ... z_m$ for a small m such that

$$|\bigcup_{i=1}^{m} (A(x) \oplus z_i)| = |\{0,1\}^{p(n)}|$$

We call the m-tuple $z_1, z_2, z_3 ... z_m$ *bad*, if ,

$$|\bigcup_{i=1}^{m} (A(x) \oplus z_i)| \neq |\{0,1\}^{p(n)}|$$

$$\Rightarrow \exists w \in \{0,1\}^{p(n)}, z_i \oplus y \neq w, \forall y \in A(x), \forall i$$

$$\Rightarrow \{z_i \oplus w | 1 \leq i \leq m\} \subset R(x)$$

where $R(x) = \bar{A}(x)$. $|R(x)| = 2^{p(n)} - |A(x)| \Rightarrow |R(x)| \leq 2^{p(n)-n}$

For a given w and a given subset of $R(x)$ of size m, we get a *bad* m-tuple $z_1, z_2, z_3...z_m$. Hence,

Number of *bad* $z_1, z_2, z_3...z_m \leq$ Number of of w's $\times$ Number of subsets of $R(x)$ of size m.

$\Rightarrow$ Number of *bad* $z_1, z_2, z_3...z_m \leq 2^{p(n)}(2^{p(n)-n})^m$

Total number of $z_1, z_2, z_3...z_m = (2^{p(n)})^m$. m should be such that,

$$2^{p(n)}(2^{p(n)-n})^m < (2^{p(n)})^m$$

$$p(n) + (p(n) - n)m < p(n)m$$

$$p(n) - nm < 0$$

$$m > \frac{p(n)}{n}$$

This goes well with our intuition. If m is allowed to be very small, then we should not be able to cover the entire set $\{0,1\}^n$. For, $m > \frac{p(n)}{n}$ we are guaranteed to have atleast one *good* m-tuple, that is,

$$z_i \oplus w = y, y \in A(x)$$

Hence we conclude that,

$$\exists z_1, z_2, z_3...z_m, \forall w \in \{0,1\}^{p(n)} \left(\bigwedge_{i=1}^{m} [\ z_i \oplus w \in A(x)\ ] \right)$$

$$\exists z_1, z_2, z_3...z_m, \forall w \in \{0,1\}^{p(n)}, \left(\bigwedge_{i=1}^{m} [\ M(x, z_i \oplus w) = 1\ ] \right)$$

Checking if M accepts x on a given path is a polynomial time operation, and repeating it for each z_i where the number of z_i's is polynomial in n is also a polynomial time operation. Fix $m = p(n)$, Thus we have a $B \in \mathsf{P}$ such that

$$x \in L \iff \exists \overline{z} \in \{0,1\}^{p(n)^2}, \forall w \in \{0,1\}^{p(n)} (x, \overline{z}, w) \in B$$

Hence, the above language $L \in \Sigma_2$. $\square$

CS 6840 Advanced Complexity Theory

Instructor: Jayalal Sarma
Scribe: Sajin Koroth
Date: January 28, 2012

LECTURE
12

One random string for all

Today we will be showing an interesting consequence of amplification of BPP introduced earlier. We will show that for an amplified BPP algorithm there is a good string of random bits for each input length n such that the algorithm run with these random bits is correct for all inputs x of length n. Hence if you could get this good random string some how then you can decide a language in BPP in polynomial time without any randomness. But there is a catch, although we prove the existence of such a random string we do not know how to compute such a string efficiently. Hence we will introduce a new model of computation where you are given such advice strings for free, but the advice for all inputs of length n has to be the same. We will introduce an advice string based class called $P/poly$, and will discuss its connection to BPP

12.1 One random string for all

Recall that amplification allows to transform in polynomial time any BPP algorithm to a BPP algorithm with error bound 2^{-2n}(i.e. at most 2^{-2n} fraction of random strings are "bad") using $p(n)$ randomness. We will show that for a BPP algorithm with the above mentioned error bound there is one random string for every length n such that for any input of that length n, the BPP algorithm outputs correctly on that random string. For the rest of the lecture we will work with sufficiently amplified success probability BPP machines, where the notion of sufficient success probability is defined as given below :

$$x \in L \implies \Pr_y [M(x, y) \text{ accepts }] \geq 1 - 2^{-2n} \tag{12.12}$$

$$x \notin L \implies \Pr_y [M(x, y) \text{ accepts }] \geq 2^{-2n} \tag{12.13}$$

That is in such a machine the number of random strings y which lead the machine to output a wrong answer is bounded by 2^{-2n}. Now let us consider a matrix A whose rows are indexed by inputs of length n and columns are indexed by random strings of length $p(n)$, and the (i, j)th entry is 1 if on fixing the random bits to be j the machine M on input i outputs correctly and it is 0 otherwise. That is $A(i, j) = 1$ if and only if $M(i, j) = \chi_L(i)$, where $\chi_L(i)$ is the membership function of the language L (i.e. $\chi_L(i) = 1$ if and only if $i \in L$).

By the amplification we are guaranteed that for a given input i at most 2^{-2n} fraction of the random strings can have $A(i, j) = 0$. Hence the total number of zeros in the A matrix is at most the number of rows times the maximum number of zeros in a row, which is equal to

$$\begin{aligned} \#\ 0\text{'s in matrix } A \quad & \leq 2^n \times 2^{-2n} \times 2^{p(n)} \\ & \leq 2^{p(n)-n} \end{aligned}$$

But the total number of zeros, $2^{p(n)-n}$ is strictly less than the number of columns in the matrix A. Hence there must be at least one column with no zeros in it. If a column in the A matrix has no zeros then by the definition of A matrix, the random string represented by this column when fed as random bits to machine M would output correctly $\chi_L(x)$ for every $x \in \{0, 1\}^n$.

12.2 Class P/poly

Even though we have proved the existence of a fixing of random bits for an arbitrary input length n of an amplified BPP machine M such that the M on these random bits decides all inputs x of a given length correctly for $L(M)$, we do not know how to compute such a string efficiently (deterministically or using a randomized algorithm) for arbitrary amplified BPP machines. Also note that the good random string can vary with the input length. But if we can get this random string for each input length n for **free** then we can decide a language in BPP in P. That if there is a function $h : N \to \{0, 1\}^*$ such that $h(n)$ is at most polynomial in n and is the correct random string for the given BPP machine M, for all inputs of length n, for all n then we can construct a machine M' such that it on input $(x, h(|x|))$ will simulate M on x using $h(|x|)$ as the random bits tape.

We will generalize the above ideas to define a class such that every language in BPP is also in this class.

DEFINITION 12.51 (P/poly). A language L is in P/poly if there exists a polynomial $p(n)$, an advice function $h : N \to \{0, 1\}^*$ and a language $B \in \mathsf{P}$ such that $\forall n, |h(n)| \leq p(n)$ and

$$x \in L \iff (x, h(|x|)) \in B$$

where $|x|$ denotes the length of the string x.

12.2.1 BPP ⊂ P/poly

This is a straight forward corollary of the existence of a good random string for any BPP machine, which works correctly for all inputs of a given length. To show that for any $L \in \mathsf{BPP}$ it is also true that $L \in \mathsf{P/poly}$ we will use the fact that there a BPP machine M_L accepting L with error at most 2^{-2n} using at most $p(n)$ random bits. We have already shown that for such a machine for every input length n at least one of $2^{p(n)}$ possible random strings is good for all inputs of length n. We define the advice function $h(n)$ to be a good random string which works for all inputs of length n. Hence $|h(n)| = p(n)$ is at most polynomial in input length. Note that definition of *P/poly* doesn't have any requirements on the computability of such a function, but needs the guarantee that such a function exists. We will construct a machine M_B running in deterministic polynomial time which would accept the language $B \in \mathsf{P}$ which accepts $(x, h(|x|))$ for all $x \in L$. The machine M_B on input $(x, h(|x|))$ starts simulating M_L on input x using $h(|x|)$ as the random bits. By the definition of $h(|x|)$, $M_L(x)$ using random bits $h(|x|)$ accepts if and only if $x \in L$. Hence the proof.

12.2.2 With advice comes the undecidable

A consequence of the above definition of class $\mathsf{P/poly}$ is that it not only contains BPP, but it also contains some undecidable languages as we do not insist on computability of advice function h. One such undecidable language is **Unary Halting Problem** defined as

$$\text{UHP} = \{1^n \mid \text{ Turing machine encoded by bin}(n) \text{ halts on all inputs}\}$$

It is easy to note that the general halting problem reduces to the unary halting problem. Hence UHP is undecidable because HP is.

We can also show that UHP is in $\mathsf{P/poly}$. This is very straight forward because the language is a unary language and there is exactly one input of length n. Hence the advice function is simply a bit representing the answer to the UHP on input 1^n. Since we just need a single bit of advice note that UHP is also in $\mathsf{P}/\theta(1)$. Also from the above argument we can deduce that complement of UHP is also in $\mathsf{P/poly}$ because by modifying the $\mathsf{P/poly}$ machine for UHP to accept when $h(n) = 0$ and reject otherwise where $h()$ is the advice function for UHP, we get a $\mathsf{P/poly}$ machine for $\overline{\text{UHP}}$. Hence $\mathsf{P/poly}$ not only contains complete problems for semi-decidable languages, like UHP also contains languages which are complete for co-semi-decidable languages, like $\overline{\text{UHP}}$.

CS 6840 Advanced Complexity Theory

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Date: January 28, 2012

LECTURE
13

Self Reducibility of SAT, Complete problem for Σ_k^{P}

Recall that we introduced the advice based class $\mathsf{P/poly}$ in the last lecture. We also saw that $\mathsf{BPP} \subsetneq \mathsf{P/poly}$, and by definition $\mathsf{P} \subsetneq \mathsf{P/poly}$. But we don't know whether $\mathsf{NP} \subsetneq \mathsf{P/poly}$ or not. Hence if we could prove that $\mathsf{NP} \not\subset \mathsf{P/poly}$ then we would essentially be separating P from NP. The reason why most of the complexity theorists believe $\mathsf{NP} \not\subset \mathsf{P/poly}$ is, if $\mathsf{NP} \subset \mathsf{P/poly}$ then we would be able to prove that $\mathsf{PH} = \Sigma_2^{\mathsf{P}}$, contrary to the common belief that PH does not collapse. In today's lecture we will detail two key ingredients needed for showing the above mentioned conditional collapse of PH, **a complete problem for** Σ_k^{P} and **self reducibility property of** SAT

13.1 Complete problem for the hierarchy

We will first show a complete problem for the kth level of polynomial hierarchy. Later on we will use the self-reducibility nature of this problem to show the conditional collapse mentioned earlier. Recall that a language L is said to be in Σ_k^{P} if there exists polynomials $p_1, \ldots, p_k$ and a machine M running in deterministic polynomial time such that

$$x \in L \iff \exists y_1 \forall y_2 \exists y_3 \ldots Q_k y_k \left[M(x, y_1, y_2, y_3, \ldots, y_k) = 1 \right], \forall i, |y_i| \le p_i(|x|)$$

Cook-Levin theorem guarantees that machine M on input x can be converted into formula ϕ_x in polynomial time on variables $y_1, y_2, \ldots, y_k$ such that $\phi_x(y_1, y_2, y_3, \ldots, y_k)$ is satisfiable if and only if $M(x, y_1, y_2, y_3, \ldots, y_k)$ accepts. Hence we can say that the following problem is complete for Σ_k^{P},

Definition 13.52 ($\Sigma_k - \mathsf{SAT}$). $\Sigma_k - \mathsf{SAT}$ is the set of all quantified Boolean formulas with at most k alternations (starting with an existential quantifier) which are true. That is

$$\Sigma_k - \mathsf{SAT} = \{\exists y_1 \forall y_2 \exists y_3 \ldots Q_k y_k \phi(y_1, \ldots, y_k) \mid \exists y_1 \forall y_2 \exists y_3 \ldots Q_k y_k \phi(y_1, \ldots, y_k) \text{ is true}\}$$

The above problem is clearly in Σ_k^{P} as you can in polynomial time construct from a formula, a machine in P for checking if the formula is satisfiable or not given an assignment of all the variables as input. The problem is Σ_k^{P} hard because of Cook-Levin reduction from any machine in P to an equivalent formula.

13.2 Self reducibility of SAT

Suppose we are given that $\mathsf{NP} \subset \mathsf{P/poly}$ then we know that there is a polynomial time deterministic Turing machine and a polynomial length advice string for each input length such that the machine decides a given language in NP. We will sketch how this can cause a collapse in the Polynomial Hierarchy, without giving the details but exposing some difficulties which we have to overcome before getting to the proof. To prove that PH collapses to Σ_2^{P} it suffices to show that $\Sigma_3^{\mathsf{P}} = \Sigma_2^{\mathsf{P}}$. Recall that Σ_3^{P} is the set of true quantified Boolean formulas which are of the form $\exists y_1 \forall y_2 \exists y_3 M(x, y_1, y_2, y_3)$, and Σ_2^{P} are true quantified Boolean formulas which are of the form $\exists y_1 \forall y_2 M(x, y_1, y_2)$. Also we are given that $\mathsf{NP} \subset \mathsf{P/poly}$ hence for any $L \in \mathsf{NP}$ there exists $h : N \to \{0,1\}^*$ and an $M \in \mathsf{P}$ such that $x \in L$ if and only if $(x, h(|x|))$ is accepted by M. The idea to place Σ_3^{P} in Σ_2^{P} is the following, the third there exists y_3 and $M(x, y_1, y_2, y_3)$ can be combined to a machine in NP, where it first guesses a string y_3 of size $p_3(|y_3|)$ and then runs M on (x, y_1, y_2, y_3). We have assumed that equivalent to this NP machine there is a P/poly machine, and even though we don't know the advice string we know there exists a good advice string, and given the advice string the last "there exists" quantifier in Σ_3^{P} can be eliminated by replacing it with the polynomial time machine which is given the advice string, hence we would a get a language in Σ_2^{P}. But unfortunately we don't know the advice string, hence the next best thing to do is to guess the advice string using the first "there exists" quantifier in Σ_2^{P}. We are guaranteed that at least one guess is the correct advice string. But there is a catch here, we could have guessed the advice string incorrectly in some branch which in turn could have led the machine M to accept incorrectly thus falsely accepting a string outside the language L in Σ_3^{P}. To get around this problem we will use the first part to reduce the problem in Σ_3^{P} to $\Sigma_3 - \mathsf{SAT}$ and then use an algorithm for SAT which given a sub-routine which tells a formula is satisfiable or not, which uses a crucial property of the SAT problem, self-reducibility to construct a satisfying assignment for the given formula. And in the case it cannot construct a satisfying assignment we would be able to guarantee that the advice string guessed is bad.

Self reducibility of SAT refers to the property of the SAT problem that checking a formula on n variables is satisfiable reduces to checking the satisfiability of two formulas on $n-1$ variables. This property leads to a polynomial time algorithm for constructing a satisfying assignment given a polynomial time sub-routine deciding the decision version of SAT problem correctly. Algorithm **??** constructs a satisfying assignment given a sub-routine which correctly solves SAT instances of up to n variables. Notice that one important property of Algorithm **??** is that even if the sub-routine which checks the satisfiability of a formula is wrong, the algorithm would not be accepting an un-satisfiable formula as a satisfiable formula. Because at the end of the algorithm we are checking whether the assignment constructed by the algorithm is satisfiable or not, so even if the sub-routine for SAT, **SATISFIABLE** is erroneous we would not be able to construct a satisfying assignment for an un-satisfiable formula. But it might fail to construct a satisfying assignment for a satisfiable formula if the sub-routine is erroneous.

CS 6840 Advanced Complexity Theory

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Date: Jan 30, 2012

LECTURE
14

Karp-Lipton-Sipser Collapse Theorem

We showed that $\mathsf{BPP} \subseteq \mathsf{P/poly}$, and as we argued $\mathsf{P/poly}$ seems to be a huge class containing P and BPP, and even some undecidable languages. A natural question is whether NP is also contained in $\mathsf{P/poly}$. We show that both answers to this question has interesting consequences.

Suppose we are able to prove that $\mathsf{NP} \not\subseteq \mathsf{P/poly}$, then we are indeed are proving that $\mathsf{NP} \not\subseteq \mathsf{P}$. That is big !.

Suppose we are able to prove that $\mathsf{NP} \subseteq \mathsf{P/poly}$. Does it have any consequences? In this lecture, we will prove the Karp-Lipton-Sipser theorem, which says that if NP is contained in $\mathsf{P}/poly$, then the polynomial hierarchy collapses to Σ_2. It is believed that the polynomial hierarchy does not collapse, since the flavour of the question about each level of the hierarchy is about elimination of a quantifier, and is of a similar difficulty to to P vs NP question.

Summarising this discussion; we believe that $\mathsf{NP} \not\subseteq \mathsf{P}$, but we do not know how to prove it. But then, since $\mathsf{P} \subseteq \mathsf{P/poly}$ is this not a harder problem to solve that $\mathsf{P}vs\mathsf{NP}$? Yes, but why do we even bother to address it when we do not know how to attack the easier question? As we will see later in the course (when we do circuit complexity) this class $\mathsf{P/poly}$ provides this nice escape from the "combinatorics of a Turing machine" and helps us to prove theorems which we do not know how to prove otherwise. It was for precisely this reason that, in the definition of $\mathsf{P/poly}$ we did not make the advice function even computable (to avoid references to Turing machines).

We state the theorem.

Theorem 14.53 (**Karp-Lipton-Sipser, 1980**). *If* $\mathsf{NP} \subseteq \mathsf{P/poly}$*, then* PH *collapses to* Σ_2.

We prove the theorem by proving two lemmas. We first show that our assumption implies something much stronger. That is if $\mathsf{NP} \subseteq \mathsf{P/poly}$ then not only NP, but the entire PH will be in $\mathsf{P/poly}$.

Lemma 14.54. *If* $\mathsf{NP} \subseteq \mathsf{P/poly}$*, then* $\mathsf{PH} \subseteq \mathsf{P/poly}$.

Proof. It suffices to show that $\Sigma_k \subseteq \mathsf{P}/poly$ for any k. We prove this by induction on k. For $k = 1$, it is trivially true, since $\Sigma_1 = \mathsf{NP}$. Hence, the base case is true. Consider an $L \in \Sigma_2$, then $L \in \mathsf{NP}^B$ for some $B \in \mathsf{NP}$. But $\mathsf{NP} \subseteq \mathsf{P}/poly$ (by the induction hypothesis), hence, $\exists h : \mathbb{N} \to \{0,1\}^*$ and $C \in P$, such that, $y \in B \leftrightarrow (y, h(y)) \in C$. Now, membership in B is decidable in polynomial time with the help of the advice function. Hence, we do not need to make oracle query to B to resolve membership questions in L, we can embed the polynomial time computation of the oracle with advice function in the NTM for L itself. So we can say that, $L \in \mathsf{NP}$ with the advice function $h : \mathbb{N} \to \{0,1\}^*$. We can rewrite it as $\exists h : \mathbb{N} \to \{0,1\}^* \wedge C' \in \mathsf{NP}$ such that $x \in L \leftrightarrow (x, h(|x|)) \in C'$.

Now, what can we say about C'? We know that $C' \in \mathsf{NP}$, hence $C' \in \mathsf{P}/poly$. Hence there is an advice function for C' also, so that membership in C' is computable in polynomial time with the help of that advice function. That is, $\exists g : \mathbb{N} \to \{0,1\}^* \wedge D \in \mathsf{P}$ such that $y \in C' \leftrightarrow (y, g(|y|)) \in D$. Rewriting it we get:

$$(x, h(|x|)) \in C' \leftrightarrow (x, h(|x|), g(p(x))) \in D$$

Hence from the above argument we see that $L \in \mathsf{P/poly}$. ∎

Now we show that if any level of PH is in $\mathsf{P}poly$, then it essentially gives a way to express the acceptance condition using only two quantifiers. This is done in the following lemma.

LEMMA 14.55. *For any $k > 2$, if $\Sigma_k \subseteq P/poly$, then $\Sigma_k \subseteq \Sigma_2$.*

Proof. It suffices to show that $L \in \Sigma_k \implies L \in \Sigma_2$. For this, we take the language SAT_k which is a quantified boolean formula with at most k alternating quantifiers. Since SAT_k is Σ_k-complete for any k, the lemma follows.

Let us assume that for any $k > 2$, $\Sigma_k \subseteq \mathsf{P/poly}$. Let $L \in \Sigma_k$, and since SAT_k is Σ_k-complete, then by our assumption, $\mathsf{SAT}_k \subseteq P/poly$. By the definition of $\mathsf{P/poly}$, $\exists h : \mathbb{N} \to \{0,1\}^* \wedge B \in \mathsf{P}$ such that $\phi \in \mathsf{SAT}_k \leftrightarrow (\phi, h(|\phi|)) \in B$. If $|\phi| = n$, then $h(n) \in \{0,1\}^{p(n)}$. Let us define a new function w in the following way:

$$w = g(n) = (h(0), h(1), ..., h(n))$$

For any $\phi \in \Sigma_k$, such that $|\phi| \leq n$, the string w has the following properties:

1. $(0, w) \notin B \wedge (1, w) \in B$
2. If $\phi = \exists y, \psi \wedge \phi \in \mathsf{SAT}_k$, then $(\psi|_{y=0}, w) \in B \vee (\psi|_{y=1}, w) \in B$.
3. If $\phi = \forall y, \psi \wedge \phi \in \mathsf{SAT}_k$, then $(\psi|_{y=0}, w) \in B \wedge (\psi|_{y=1}, w) \in B$.

Hence, $(\phi, w) \in B \implies 1, 2$, and 3 are satisfied.

It is also true in the other direction. That is, if $1, 2$, and 3 are satisfied, then in order to check if ϕ is true, it suffices to check if $(\phi, w) \in B$. We can show this inductively. When ϕ is either 0 or 1, then by 1, the above claim is true. Suppose, $\phi = (\exists x)\psi$, then we can apply 2 on ψ to evaluate it. Otherwise, if $\phi = (\forall x)\psi$, then we can apply 3 on ψ to evaluate it. Thus, by recursively evaluating, we can reach upto the leaf where we apply 1. Hence by a consistency check we can make sure that the advice does not give us a wrong answer, and indeed, if it gives a wrong answer to us, we can detect it at some point in the recursive evaluation. Note that we are using the self-reducibility property of SAT_k here. Thus we have proved that,

$$\phi \in \mathsf{SAT}_k \leftrightarrow (\exists w, |w| \leq p(n))(\forall \psi, |\psi| \leq n)(1 \wedge 2 \wedge 3 \wedge (\phi, w) \in B).$$

∎

CS 6840 Advanced Complexity Theory

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Date: Jan 31, 2012

LECTURE
15

Introduction to Toda's Theorem

There are two different way to generalize hard problems like optimization problems.

1. Non-determinism or quantification
2. Counting

It was an important open question in 1980's about relative power of those two approach, alternation and counting. There are various classes are defined independently in those two different approach. But how are those class comparable? In 1991, Toda came up with the following relation among the two different world.

THEOREM 15.56. $PH \subseteq \mathsf{P}^{\#\mathsf{P}}$

This theorem take the whole quantification world to counting world. That is we can solve any problem in PH in polynomial with an oracle access to a #P-Complete problem. Before going to the proof of Toda's theorem lets define the following.

DEFINITION 15.57. $\exists$. Operator.
Let us consider the definition of NP.

$$L \in \mathsf{NP} \text{ if } \exists B \in \mathsf{P} \text{ such that}$$

$$x \in L \Longleftrightarrow \exists y \text{ such that } (x, y) \in B$$

So we can think $\exists$ as an operator and let us define it as follows
Let C be any class of languages. Define,

$$L \in \exists.C \text{ if } \exists B \in C \text{ such that } x \in L \Longleftrightarrow \exists y \text{ such that } (x, y) \in B$$

So $\mathsf{NP} = \exists.\mathsf{P}$

DEFINITION 15.58. BP. Operator
Let us consider definition of the class BPP

$L \in \mathsf{BPP}$ if $\exists B \in \mathsf{P}$ such that

$$x \in L \Longrightarrow \mathsf{Pr_y}[(x,y) \in B] \geq \frac{3}{4}$$
$$x \notin L \Longrightarrow \mathsf{Pr_y}[(x,y) \in B] \leq \frac{1}{4}$$

So similarly we can define BP. operator as

Let C be a class of languages. $L \in BP.C$ if $\exists B \in$ C such that

$$x \in L \Longrightarrow \mathsf{Pr}[(x,y) \in B] \geq \frac{3}{4}$$
$$x \notin L \Longrightarrow \mathsf{Pr}[(x,y) \in B] \leq \frac{1}{4}$$

So $\mathsf{BPP} = BP.\mathsf{P}$

Now lets define the class $\oplus\mathsf{P}$:

Definition 15.59. L $\in \oplus\mathsf{P}$ if there is a branching machine M such that

$$x \in L \Longrightarrow \#Acc_M(x) \text{ is odd}$$

$\oplus\mathsf{P}$ can be considered as the class of decision problems corresponding to the least significant bit of a $\#\mathsf{P}$-problem. Now the natural question is "Is $\mathsf{NP} \subseteq \oplus\mathsf{P}$?".

We will see that NP problems are reduced to $\oplus\mathsf{P}$ in some extend. Lets now define Randomize Reduction as follows

Definition 15.60. $A \leq B$ via a randomize reduction function σ if

1. σ is computable by choosing a random string $y \in \{0,1\}^{p(n)}$.
2. $x \in L \iff \sigma_y(x) \in B$ with high probability.

We will see that NP problems are reduced to some $\oplus\mathsf{P}$ problem via randomize reduction. More specifically $\mathsf{NP} \subseteq BP.(\oplus\mathsf{P})$. Now we will develop the proof strategy for Toda's theorem. We will proof the theorem in the following steps ...

$$\mathsf{NP} \subseteq BP.(\oplus\mathsf{P}) \subseteq \mathsf{BPP}^{\oplus\mathsf{P}} \subseteq \mathsf{P}^{\#\mathsf{P}} = \mathsf{P}^{\mathsf{PP}}$$

Then by induction on the number of quantifiers we will show that the entire $PH \subseteq \mathsf{P}^{\#\mathsf{P}}$.

So our first step is to show $\mathsf{NP} \subseteq BP.(\oplus\mathsf{P})$.

To prove this claim let us define the following languages ....

Definition 15.61.

$$\oplus\mathsf{SAT} = \{\phi : \#\text{ of satisfying assignment of } \phi \text{ is odd}\}$$

Let us define the following promise problem USATas,

Definition 15.62. For all boolean formulas ϕ,

$$\phi \in \mathsf{USAT} \Longrightarrow \#SAT(\phi) = 1$$
$$\phi \notin \mathsf{USAT} \Longrightarrow \#SAT(\phi) = 0$$

This is a promise problem as we will consider only those ϕ which falls into above two cases. Now we introduce a very important result called *Valiant-Vazirani lemma* proved by Valiant and Vazirani in 1986.

LEMMA 15.63. *(Valiant-Vazirani Lemma)*
There exists a randomize reduction σ_y such that

$$\phi \in \mathsf{SAT} \iff Pr_y[\sigma_y(\phi) \in \mathsf{USAT}] \geq \frac{3}{4}$$
$$\phi \notin \mathsf{SAT} \iff Pr_y[\sigma_y(\phi) \in \mathsf{USAT}] \leq \frac{1}{4}$$

We will prove this lemma in the next lecture and using this lemma we will prove our first step $\mathsf{NP} \subseteq BP.(\oplus\mathsf{P})$.

CS 6840 Advanced Complexity Theory

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Date: Feb 2, 2012

LECTURE
16

Valiant-Vazirani Lemma

In the last lecture, we outlined an approach to prove Toda's theorem. One of the key ingredients was a partial answer to the question that we had in one of the earlier lectures. Is NP contained in $\oplus\mathsf{P}$? In the last lecture we viewed $\exists$, $\oplus$, BP as operators on complexity classes and stated that NP is contained in $\mathsf{BP}.(\oplus\mathsf{P})$. We also interpreted this in the following way; there is a randomized reduction from SAT to a language in $\oplus\mathsf{P}$.

16.1 Randomized Reduction from SAT to ⊕SAT

Define languages,

USAT=$\{\phi \mid \#\phi = 1\}$
where $\#\phi$ is the number of satisfying assignments to the formula ϕ

$\oplus$SAT=$\{\phi \mid \exists k \in \mathbb{N}, \#\phi = 2k+1\}$

The following Lemma, famously known as the Valiant-Vazirani Lemma, was proved by Valiant and Vazirani in 1986. It proved instrumental for many results later, including Toda's theorem which we will be taking up in the next lecture.

The lemma states the following reduction from SAT to $USAT$.

Lemma 16.64 (**Valiant-Vazirani Lemma**). *There exists a randomized polynomial time algorithm that takes input ϕ and produces a formula $\psi_{\phi,y}$ (say ψ) such that,*

$$\begin{aligned} \phi \in \mathsf{SAT} &\implies Pr(\psi \in USAT) &\geq& \frac{1}{8n} \\ \phi \notin \mathsf{SAT} &\implies Pr(\psi' \notin USAT) &=& 1 \end{aligned}$$

Clearly, $\psi \in$ USAT $\Rightarrow \psi \in \oplus$ SAT. In the other case since $\psi \notin \mathsf{SAT}$, it is also case that $\psi \notin \oplus\mathsf{SAT}$. Thus as a corollary to the lemma, we get:

Corollary 16.65. *There exists a randomized polynomial time algorithm that takes input ϕ and produces a formula ψ such that,*

$$\begin{aligned} \phi \in \mathsf{SAT} &\implies Pr(\psi \in \oplus\mathsf{SAT}) &\geq& \frac{1}{8n} \\ \phi \notin \mathsf{SAT} &\implies Pr(\psi \notin \oplus\mathsf{SAT}) &=& 1 \end{aligned}$$

Proof. We present a high-level idea first. Given ϕ, a natural approach to produce a formula with unique satisfying assignment is to add another conjunction to produce $\psi = \phi \wedge (\omega)$ such that ω filters out the satisfying assignments using the clause ω such that only one of them will satisfy the resulting formula. Clearly, if ϕ is not satisfiable, then by construction, ψ is also not satisfiable, no matter what ω is. Consider the case when ϕ is satisfiable. Thus, we want ω to state a property which only one of the satisfying assignments have.

We use hashing to achieve this. That is, we will make ω state that $h(x) = 0^k$ where 0^k is in range of the hash family and x is an assignment. The probability that there is a collision at 0^k for two xs that satisfy ϕ (that is, probability that two assignments that satisfy ϕ also satisfies ω) can be controlled by choosing a nice hash family. Thus our filter, with high probability, filters out a unique satisfying assignment for ψ from the set of satisfying assignments of ϕ.

Usually, we design hash families with a size parameter k; the size up to which we want to guarantee collision-free property. But here we do not know the size of the subset (the set of satisfying assignments) a priori for which we are trying to achieve unique mapping(collision-free). The smaller the subset the smaller the range (of the hash functions) that we can work with. Let us say we choose randomly the number k such that the number of satisfying assignments is between 2^k and 2^{k+1} and then decide hash function for all subsets of that size. Since the number of satisfying assignments could be any number from 0 to 2^n, we have already lost out a bit on the probability but by a factor of at most $\frac{1}{n}$.

Now we will formally address this intuition. Let $T \subseteq \{0,1\}^n$ be the set of satisfying assignments of ϕ. Select $k \in \{0, 1, ..., n-1\}$ such that $2^k \leq |T| \leq 2^{k+1}$. Let $H_{n,k}$ be a collection of functions $h : \{0,1\}^n \mapsto \{0,1\}^k$. $H_{n,k}$ is said to be pairwise independent if for every x,$x' \in \{0,1\}^n$ with $x \neq x'$ and for every y,$y' \in \{0,1\}^k$,

$$\Pr_{h \in_R H_{n,k}} [h(x) = y \wedge h(x') = y'] = \frac{1}{2^k} . \frac{1}{2^k} = 2^{-2k}$$

Construct a family of pairwise independent hash functions $H_{n,k+2}$. Hence,

$$\Pr_{h \in H, x \in T} [h(x) = 0^{k+2}] = \frac{1}{2^{k+2}}$$

We calculate the probability of existence of a hash function $h \in H$ such that h maps exactly one satisfying assignment $x \in T$ to 0^{k+2}. This parameter is given by,

$$\Pr_{h \in H} [|\{x \mid x \in T, h(x) = 0^{k+2}\}| = 1]$$

Once we have an h satisfying the above condition, the number of satisfying assignments for ϕ will be exactly 1. That is, in the described transformation, if ω is $h(x) = 0^{k+2}$, the number of satisfying assignments for ψ will be exactly 1, if it is satisfiable. Note that we can consider the computation sequence of the hash function h on a turing machine and convert that to a SAT formula using Cook-Levin reduction. We claim that,

$$\Pr_{h \in H} [|\{x \mid x \in T, h(x) = 0^{k+2}\}| = 1] \geq \frac{1}{8}$$

Consider,

$$\Pr_{h \in H} [\exists x \in T, h(x) = 0^{k+2} \wedge \forall_{x' \in T, x' \neq x} h(x') \neq 0^{k+2}]$$

$$
\begin{aligned}
&= \Pr_h[\forall_{x' \in T, x' \neq x} h(x') \neq 0^{k+2} \mid h(x) = 0^{k+2}]. \Pr_h[h(x) = 0^{k+2}] \\
&= (1 - \Pr_h[\exists_{x' \in T, x' \neq x} h(x') = 0^{k+2} | h(x) = 0^{k+2}]). \Pr_h[h(x) = 0^{k+2}] \\
&= (1 - \sum_{\substack{x' \in T \\ x' \neq x}} \Pr_h[h(x') = 0^{k+2} \mid h(x) = 0^{k+2}]). \Pr_h[h(x) = 0^{k+2}]
\end{aligned}
$$

Let us calculate the first part of the expression, $(1 - \sum_{\substack{x' \in T \\ x' \neq x}} \Pr_h[h(x') = 0^{k+2} \mid h(x) = 0^{k+2}])$. Since h is sampled from a family of pairwise independent hash functions, h satisfies the condition of simple uniform hashing. That is, the events $h(x) = 0^{k+2}$ and $h(x') = 0^{k+2}$ are independent. Therefore,

$$
\sum_{\substack{x' \in T \\ x' \neq x}} \Pr_h[h(x') = 0^{k+2} \mid h(x) = 0^{k+2}]) = \sum_{\substack{x' \in T \\ x' \neq x}} \frac{1}{2^{k+2}} = |T - \{x\}|.\frac{1}{2^{k+2}}
$$

We bound the expression from above by bounding $|T - \{x\}|$ from above. That is,

$$
\sum_{\substack{x' \in T \\ x' \neq x}} \Pr_h[h(x') = 0^{k+2} \mid h(x) = 0^{k+2}]) \leq (2^{k+1} - 1)\frac{1}{2^{k+2}}
$$

Hence,

$$
\begin{aligned}
(1 - \sum_{\substack{x' \in T \\ x' \neq x}} \Pr_h[h(x') = 0^{k+2} \mid h(x) = 0^{k+2}]) &\geq 1 - (2^{k+1} - 1)\frac{1}{2^{k+2}} \\
&= 1 - \frac{1}{2}[\frac{(2^{k+1} - 1)}{2^{k+2}}] \\
&\geq \frac{1}{2}
\end{aligned}
$$

Therefore,

$$
(1 - \sum_{\substack{x' \in T \\ x' \neq x}} \Pr_h[h(x') = 0^{k+2} | h(x) = 0^{k+2}]). \Pr_h[h(x) = 0^{k+2}] \geq \frac{1}{2} \cdot \frac{1}{2^{k+2}}
$$

Thus,

$$
|T|.(1 - \sum_{\substack{x' \in T \\ x' \neq x}} \Pr_h[h(x') = 0^{k+2} \mid h(x) = 0^{k+2}]). \Pr_h[h(x) = 0^{k+2}] \geq |T|.\frac{1}{2}\frac{1}{2^{k+2}}
$$

By bounding $|T|$ from below, the probability that there is a $x \in T$ which uniquely gets mapped to 0^{k+2} is given by,

$$
|T|.(1 - \sum_{\substack{x' \in T \\ x' \neq x}} \Pr_h[h(x') = 0^{k+2} \mid h(x) = 0^{k+2}]). \Pr_h[h(x) = 0^{k+2}] \geq \frac{1}{2^k}\frac{1}{2}\frac{1}{2^{k+2}} \geq \frac{1}{8}
$$

Considering the fact that $k \in \{0, 1, ..., n-1\}$ satisfies $2^k \leq |T| \leq 2^{k+1}$ with probability $\frac{1}{n}$ we get,

$$
\phi \in \mathsf{SAT} \implies \Pr_x[\psi_{\phi,x} \in USAT] \geq \frac{1}{8n}
$$
$$
\phi \notin \mathsf{SAT} \implies \Pr_x[\psi_{\phi,x} \notin USAT] = 1
$$

∎

Now we make comments about amplification of success probability. Since the algorithm is one-sided error, an approach will be to simply repeat the experiment some ℓ times, and take the $\vee$ of the results. This however, causes loss of uniqueness of the assignment since different h may make different xs to go to 0^k. However, we can still preserve the parity with high probability that results in the following amplification result which we state as a Lemma.

LEMMA 16.66. *There exists a randomized polynomial time algorithm that takes input ϕ and produces a formula ψ' such that,*

$$\begin{aligned} \phi \in \mathsf{SAT} \implies Pr(\psi' \in \oplus\mathsf{SAT}) &\geq 1 - \frac{1}{2^{q(n)}} \\ \phi \notin \mathsf{SAT} \implies Pr(\psi' \notin \oplus\mathsf{SAT}] &= 1 \end{aligned}$$

The OR amplification says, that we report that $\phi \in \oplus\mathsf{SAT}$ if and only if at least one of the trials produces a formula in $\oplus\mathsf{SAT}$. But then, can we produce a single formula ψ' such that if ϕ is in SAT, then ψ' is in $\oplus\mathsf{SAT}$ with high probability. We will address such a reduction in the next lecture.

Note that it is an open problem to boost the probability of $\frac{1}{8n}$ in the Valiant-Vazirani reduction to USAT.

CS 6840 Advanced Complexity Theory

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Date: Feb 6, 2012

LECTURE
17

Reductions to $\oplus$-SAT: Amplified version

The last few lectures focus on the Toda's theorem which states that $\mathsf{PH} \subseteq \mathsf{P}^{\#\mathsf{P}}$. The first half of the proof of the theorem shows a randomized reduction from PH to $\oplus$SAT.

We proved the Valiant-Vazirani lemma which stated a randomized polynomial time algorithm that takes in a formula ϕ and produces a new formula ψ such that it gives a *weak form* of a randomized reduction from SAT to USAT. We have the following

$$\begin{aligned} \phi \in \mathsf{SAT} \quad &\Longrightarrow \quad Pr\left(\psi \in \mathrm{U}\mathsf{SAT}\right) \geq \frac{1}{8n} \\ \phi \notin \mathsf{SAT} \quad &\Longrightarrow \quad Pr\left(\psi \notin U\mathsf{SAT}\right) = 1 \end{aligned}$$

We also concluded that this gives a weak[3] randomized reduction from NP to $\oplus$SAT. Now we show how to amplify the success probability in the case of the reduction to $\oplus$SAT. [4]

Indeed, something special about $\oplus$ is going to help us. In this lecture, we explore some properties of the $\oplus$ quantifier which are used to come up with a formula ϕ'' from a given formula $\phi \in \mathsf{SAT}$ such that $\phi'' \in \oplus\mathsf{SAT}$ with high probability. We thereby deduce that $\mathsf{NP} <_r^m \oplus\mathsf{SAT}$.

17.1 Parity Addition, Complementation and Multiplication

Given two boolean formulae, ϕ and ϕ', their parity can be added as follows:

$$\oplus(\phi + \phi')(\bar{z}) = \oplus(z_0 = 0 \wedge \phi) \vee \oplus(z_1 = 0 \wedge \phi')$$

Similarly, the parity can be complemented as:

$$(z = 0) \wedge (\sim x_1 \wedge \sim x_2 \wedge \ldots \wedge \sim x_n) \vee (\phi \wedge z = 1)$$

which is represented as $\phi + 1$. Multiplication of parity is obtained as:

$$\oplus\phi(\bar{z}) \times \oplus\phi'(\bar{z}) = \oplus(\phi(\bar{z}) \wedge \phi'(\bar{z}))$$

[3]The reduction is weak because the error probability$(1 - \frac{1}{8n})$ is much more than what is allowed in a randomized reduction $(\frac{1}{2})$

[4]Such an amplification is not known for the case of USAT.

17.2 Randomized Reduction

In our construction for the Valiant-Vazirani Lemma , we defined a formula $\psi_i = \phi \wedge (h_i(x) = 0^k)$ where x is an assignment of ϕ.Now consider a formula $\phi' = \overset{l-1}{\underset{i=0}{\wedge}} (\oplus \psi_i)$. if $\Pr[\psi_i \in \texttt{USAT}] \geq \frac{1}{8n}$, then $\Pr[\phi' \in \mathsf{SAT}] = 1 - \left(1 - \frac{1}{8n}\right)^l]$. We now come up with a formula ϕ'' equivalent to ϕ' such that the parity of ϕ'' is odd. that is, $\phi'' \in \oplus\mathsf{SAT}$ conditioned on the probability that atleast one $\psi_i \in \oplus\mathsf{SAT}$.
For simplicity, consider ψ_1 and ψ_2, one of which is in $\oplus\mathsf{SAT}$.We observe that both , $\psi_1 + 1$ and $\psi_2 + 1$ cannot have an odd parity. Therefore,we have,

$$\begin{aligned} \oplus(\psi_1 + 1).(\psi_2 + 1) =& 0 \\ \oplus((\psi_1 + 1).(\psi_2 + 1) + 1) =& 1 \end{aligned}$$

Hence, $((\psi_1 + 1).(\psi_2 + 1) + 1) \in \oplus\mathsf{SAT}$.In general , $\overset{l-1}{\underset{i=0}{\wedge}} ((\psi_i + 1) + 1) \in \oplus\mathsf{SAT}$ which is our new formula ϕ'' equivalent to ϕ. Hence , $\Pr[\phi'' \in \oplus\mathsf{SAT}]$ is the same as that of atleast one ψ_i having an odd parity.

$$\phi \in \mathsf{SAT} \Rightarrow Pr\left[\phi'' \in \oplus\mathsf{SAT}\right] = 1 - \left(1 - \frac{1}{8n}\right)^l$$

To amplify this probabilty to $1 - 2^{-m}$ we choose an appropriate value of l such that,

$$1 - \left(1 - \frac{1}{8n}\right)^l \geq 1 - 2^{-m}$$

Hence, we have

$$\begin{aligned} \phi \in \mathsf{SAT} \quad &\Rightarrow \quad Pr\left[\phi'' \in \oplus\mathsf{SAT}\right] \geq 1 - 2^{-m} \\ \phi \notin \mathsf{SAT} \quad &\Rightarrow \quad Pr\left[\phi'' \in \oplus\mathsf{SAT}\right] = 0 \end{aligned}$$

Hence, $\mathsf{NP} <_r^m \oplus\mathsf{SAT}$.

CS 6840 Advanced Complexity Theory

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Date: Feb 07, 2012

LECTURE
18

Proof of Toda's Theorem

18.1 Valiant-Vazirani Lemma

We have already saw Valiant-Vazirani theorem which stated that :

LEMMA 18.67. *Valiant-Vazirani Theorem: There exists a probabilistic polynomial-time algorithm f such that for every n-variable boolean formula φ*

$$\begin{aligned} \varphi \in SAT \Rightarrow Pr[f(\chi) \in \oplus SAT] &\geq \frac{1}{8n} \\ \varphi \notin SAT \Rightarrow Pr[f(\chi) \in \oplus SAT] &= 0 \end{aligned}$$

But to prove that $\mathsf{PH} \subseteq \mathsf{BP}(\oplus\mathsf{P})$ we will need amplified version of this lemma.

LEMMA 18.68. *Valiant-Vazirani lemma (Amplified version): There exists a randomized algorithm which produce φ from a given boolean formula ϕ such that for a polynomial $q(n)$:*

$$\begin{aligned} x \in L \Rightarrow Pr[\varphi \in \oplus SAT] &\geq \left(1 - \frac{1}{2^{q(n)}}\right) \\ x \notin L \Rightarrow Pr[\varphi \in \oplus SAT] &\leq \left(\frac{1}{2^{q(n)}}\right) \end{aligned}$$

18.2 Toda's Theorem: $\mathsf{PH} \subseteq \mathsf{P}^{\#\mathsf{P}}$

Any problem in PH can be solved by a P machine by making queries to a $\#\mathsf{P}$ oracle.

Proof
This theorem will be proved in two parts

- $\mathsf{PH} \subseteq \mathsf{BP}(\oplus\mathsf{P})$
- $\mathsf{BP}(\oplus\mathsf{P}) \subseteq \mathsf{P}^{\#P}$

Claim: $\mathsf{PH} \subseteq \mathsf{BP}(\oplus\mathsf{P})$

Proof. Proof by induction on the number of alterations, k.
Basis: for $k = 0$, we already have the result $\mathsf{NP} \subseteq \mathsf{BP}(\oplus\mathsf{P})$.
Assume the result for k. ie; Σ_k or $SAT_k \in \mathsf{BP}(\oplus\mathsf{P})$.
We need to prove that $SAT_{k+1} \in \mathsf{BP}(\oplus\mathsf{P})$.

Consider a language L $\in \Sigma_{k+1}^p$,

$$x \in L \Leftrightarrow \exists y_1 \forall y_2 (y_1, y_2) \in B$$

where $B \in \Sigma_k^p$

From inductive hypothesis, there exists a formula $\phi(x)$ such that B can be reduced to $\oplus\mathsf{SAT}$.
Consider the formula ,

$$\phi_1(x, y_1, y_2) = \phi(x) \wedge (h_1(y_2) = 0^{k_2}) \wedge (h_2(y_1) = 0^{k_1})$$

where $h_i(x)$ is chosen from a pairwise independent hash family.

Now, using the techniques seen in last lecture, reduce this formula $\phi_1(x, y_1, y_2)$ to a formula $\psi(x)$ in $\oplus\mathsf{SAT}$.
Hence,

$$x \in L \Leftrightarrow \psi(x) \in \oplus\mathsf{SAT}$$

Therefore, from induction, $\mathsf{PH} \leq_r \oplus\mathsf{P}$.
$\mathsf{PH} \subseteq \mathsf{BP}(\oplus\mathsf{P})$

Comment 18.69.

$$\phi \in SAT_k \Rightarrow Pr[\oplus_z \phi \wedge \tau(x, z)] \geq (1 - \frac{1}{2^{q(n)}})$$

$$\phi \notin SAT_k \Rightarrow Pr[\oplus_z \phi \wedge \tau(x, z)] \leq \frac{1}{2^{q(n)}}$$

Now, any $\phi' \in SAT_{k+1}$ can be written as $\phi' = \exists\sigma$ where $\sigma \in SAT_k$

$$\phi' \in SAT_{k+1} \Rightarrow \exists\sigma \in SAT_k \text{ with } \sigma \in \Pi_k$$

Let $\sigma' = \exists(\neg\varphi)$ where $\sigma = \neg\varphi$

$$\neg(\forall\varphi) \Rightarrow \exists\sigma \Rightarrow Pr[\sigma' \in \oplus SAT] \geq (1 - \frac{1}{2^{q(n)}})$$

$$\forall\varphi \Rightarrow \neg\exists\sigma \Rightarrow Pr[\sigma' \notin \oplus SAT] \geq (1 - \frac{1}{2^{q(n)}})$$

From inductive hypothesis,

$$\forall\varphi \Rightarrow Pr[\sigma'' \in \oplus SAT] \geq (1 - \frac{1}{2^{q(n)}})$$

$$\neg\forall\varphi \Rightarrow Pr[\sigma'' \notin \oplus SAT] \geq (1 - \frac{1}{2^{q(n)}})$$

$$\varphi \rightarrow \varphi \wedge (h(y) = 0^k) \rightarrow \varphi \wedge (h(y) = 0^{k_1} \wedge h(x) = 0^{k_2})$$

$$\phi = \exists\forall\varphi$$

$\square$

Claim: $\mathsf{BP}(\oplus\mathsf{P}) \subseteq \mathsf{P}^{\#\mathsf{P}}$
Before proving the above claim, let us set up a following lemma.

Let $A \in \oplus\mathsf{P}$, then by definition,

$$x \in A \implies \#acc_M(x) = -1 \ (mod\ 2)$$
$$x \notin A \implies \#acc_M(x) = 0 \ (mod\ 2)$$

But what Toda argued was that, not only does this hold true under modulo 2 operation, but under any modulo $2^{q(n)}$ operation where q(n) is a polynomial.

Lemma 18.70. *Let $A \in \oplus\mathsf{P}$, then ,*

$$x \in A \implies \#acc_M(x) = -1 \ (mod\ 2^{q(n)})$$
$$x \notin A \implies \#acc_M(x) = 0 \ (mod\ 2^{q(n)})$$

Proof. Let y = $\#acc_M(x)$,where M is the NDTM corresponding to the $\oplus\mathsf{P}$ function f, such that f(x) = y.

$$x \in A \implies y = -1 \ (mod\ 2^{q(n)})$$
$$x \notin A \implies y = 0 \ (mod\ 2^{q(n)})$$

Now construct a polynomial m = $3y^4 \ + \ 4y^3$. Using the NDTM M(where $\#acc_M(x) = m$), construct a new NDTM M' such that,

$$\#acc_{M'}(x) = m$$

To construct such a NDTM, let us look at the following two constructions

- Construction of a machine M such that , $\#acc_M(x) = \#acc_{M_1}(x) + \#acc_{M_2}(x)$
  Describe the NDTM M as its choice tree as follows :
  1) Create a new root node such that the machine can take two choices on it.
  2) On one of the choices, simulate the machine M_1.
  3) On the other choice, simulate machine M_2.
  4) Extend the leaves of the corresponding choice tree to balance the height.

  This new choice tree will have $\#acc_M(x) = \#acc_{M_1}(x) + \#acc_{M_2}(x)$.

- Construction of a machine M such that , $\#acc_M(x) = \#acc_{M_1}(x) \times \#acc_{M_2}(x)$
  Describe the NDTM M as its choice tree as follows :
  1) Create a choice tree of either machine M_1 or M_2. W.L.G., lets assume the choice tree of machine M_1.
  2) On each of the accept node of machine M_1, extend the tree downwards by simulating the machine M_2.
  3) On the reject nodes, extend the tree downwards to balance the height of the choice tree.

  This new choice tree will have $\#acc_M(x) = \#acc_{M_1}(x) \times \#acc_{M_2}(x)$.

Using the above two constructions, a NDTM M' can be constructed from machine M.

Observe that,

$$x \in A \implies y = -1 \ (mod \ 2^2)$$
$$x \notin A \implies y = 0 \ (mod \ 2^2)$$

via the NDTM M'

Repeating the above procedure at the i^{th} iterations, we will have,

$$x \in A \implies y = -1 \ (mod \ 2^{2^i})$$

$$x \notin A \implies y = 0 \ (mod \ 2^{2^i})$$

Hence, repeating the procedure for O(log($q(n)$)) iterations, we will have ,

$$x \in A \implies y = -1 \ (mod \ 2^{q(n)})$$

$$x \notin A \implies y = 0 \ (mod \ 2^{q(n)})$$

Note:
An important observation in the proof is that, the height of the choice tree increases by a factor of 4 in every iteration. Hence, the height of the choice tree in the final machine is $4^{\log(q(n))}$, which is a polynomial. Hence, the running time of the finally obtained choice machine is also polynomial and hence the function corresponding to it, still remains in $\oplus$P. ∎

Now, using the above lemma, lets prove $\mathsf{BP}(\oplus\mathsf{P}) \subseteq \mathsf{P}^{\#\mathsf{P}}$ which will complete the proof of Toda's Theorem.

Consider $L \in \mathsf{BP}(\oplus\mathsf{P})$
By definition,

$$Pr_{y \in \{0,1\}^{p(n)}}(x \in L \Leftrightarrow (x, y) \in A) \geq \frac{3}{4}$$

for some A $\in \oplus$P

Consider h(x) = $|\{y : (x, y) \in A\}|$

If we calculate the value of h(x) and estimate it to be greater than a fraction of $\frac{3}{4}$, then accept x, else reject.
Hence, L $\in \mathsf{P}^{\#\mathsf{P}}$.

To estimate this value consider the function,

$$l(x) = \Sigma_{y \in \{0,1\}^{p(n)}} acc_M(x, y)$$

This can be written as,

$$l(x) = \Sigma_{(x,y) \in A} acc_M(x, y) + \Sigma_{(x,y) \notin A} acc_M(x, y)$$

$$l(x) = (-1) \times h(x) + 0(mod 2^{q(n)})$$

$$l(x) = -h(x)(mod 2^{q(n)})$$

If we choose a large enough value of $2^{q(n)}$, then we can determine value of h(x) uniquely from l(x). Formally, h(x) $< 2^{q(n)}$

We can easily see that function l(x) $\in$ #P. i.e. Create an NDTM M' , which first guesses nondeterministically a string $y \in \{0,1\}^{p(n)}$, and simulates M on (x, y). The number of accepting paths of the machine M' is precisely equal to l(x).

COMMENT 18.71.

LEMMA 18.72. *If $\mathcal{A} \in \oplus\mathsf{P}$ then $\exists B$ such that for any polynomial q and input x of length n,*

$$x \in \mathcal{A} \Rightarrow (\#(x,y) \in B) \equiv -1 \mod 2^{q(n)}$$

$$x \notin \mathcal{A} \Rightarrow (\#(x,y) \in B) \equiv 0 \mod 2^{q(n)}$$

Now we can state the lemma as, Let $\mathcal{A} \in \oplus\mathsf{P}$. Then for any polynomial q, there exists a polynomial-time NTM M such that for any input x of length n,

$$x \in \mathcal{A} \Rightarrow (\chi_M(x)) \equiv -1 \mod 2^{q(n)}$$

$$x \notin \mathcal{A} \Rightarrow (\chi_M(x)) \equiv 0 \mod 2^{q(n)}$$

Here $\chi_M(x)$ denotes the number of accepting computations of M on x.

Proof. Let M_1 be a polynomial time NTM such that

$$\chi_{M_1} \equiv 1 \mod 2 \text{ if } x \in \mathcal{A}$$

$$\chi_{M_1} \equiv 0 \mod 2 \text{ if } x \notin \mathcal{A}$$

In other words,

$$x \in \mathcal{A} \Rightarrow (\#acc_M(x)) \text{ is odd}$$

$$x \notin \mathcal{A} \Rightarrow (\#acc_M(x)) \text{ is even}$$

Define another polynomial time NTM M_2 that repeats M_1 on x a number of times such that

$$x \in \mathcal{A} \Rightarrow (\#acc_{M_2}(x)) \text{ is odd}$$

$$x \notin \mathcal{A} \Rightarrow (\#acc_{M_2}(x)) \text{ is even}$$

Note: Given two NDTMs M_1 and M_2, we know how to get an M_3 with:

- $\#acc_{M_3}(x) = \#acc_{M_1}(x) + \#acc_{M_2}(x)$: By running M_1 and M_2 in parallel.
- $\#acc_{M_3}(x) = \#acc_{M_1}(x) \times \#acc_{M_2}(x)$: By rnning M_1 and M_2 one after other.

Let $f(x,i) = \chi_{M_2}(< x, i >)$. It if clear that f satisfies the recurrence relation given below:

$$f(x, i+1) = 3f(x,i)^4 + 4f(x,i)^3, i \geq 0 \tag{18.14}$$

From equation 18.14

$$f(x,0) \text{ is even } \Rightarrow f(x,i) \equiv 0 \mod 2^{2^i}$$

$$f(x,0) \text{ is odd } \Rightarrow f(x,i) \equiv -1 \mod 2^{2^i}$$

Now from the above analysis,

$$x \in \mathcal{A} \Rightarrow (\chi_M(x) \equiv -1 \mod 2^{2^{\log q(n)}} \equiv -1 \mod 2^{q(n)}$$

$$x \notin \mathcal{A} \Rightarrow (\chi_M(x) \equiv 0 \mod 2^{2^{\log q(n)}} \equiv 0 \mod 2^{q(n)}$$

∎

THEOREM 18.73. $\mathsf{BP}(\oplus\mathsf{P}) \subseteq \mathsf{P}^{\#\mathsf{P}}$

Proof. $L \in \mathsf{BP}(\oplus\mathsf{P})$ means there exists a set $\mathcal{A} \in \oplus\mathsf{P}$ and a polynomial p such that for all x,

$$x \in L \Rightarrow Pr_y[(x,y) \in \mathcal{A}] \geq \frac{2}{3}$$

$$x \notin L \Rightarrow Pr_y[(x,y) \in \mathcal{A}] \leq \frac{1}{3}$$

where y ranges over all strings of length $p(|x|)$.

By lemma 18.72, if $\mathcal{A} \in \oplus\mathsf{P}$, there exists a polynomial-time NTM M such that for all (x,y), with $|x| = n$ and $|y| = p(n)$

$$\chi_M(< x,y >) \equiv -1 \mod 2^{p(n)}, \text{ if } < x,y > \in \mathcal{A}$$

$$\chi_M(< x,y >) \equiv 0 \mod 2^{p(n)}, \text{ if } < x,y > \notin \mathcal{A}$$

Let $g(x)$ and $h(x)$ are two functions defined as,

$$g(x) = |\{y : |y| = p(|x|), (x,y) \in \mathcal{A}\}|$$
$$h(x) = \sum_{|y|=p(|x|)} \chi_M(< x,y >)$$

Then for any x of length n,

$$\begin{aligned}
h(x) &= \sum_{<x,y> \in \mathcal{A}} \chi_M(< x,y >) + \sum_{(<x,y> \notin \mathcal{A})} \chi_M(< x,y >) \\
&= (\sum_{<x,y> \in \mathcal{A}} (-1) + \sum_{(<x,y> \notin \mathcal{A})} 0) \mod 2^{q(n)} \\
&\equiv (g(x).(-1) + (2^{p(n)} - g(x)).0) \mod 2^{p(n)} \\
&\equiv (-g(x)) \mod 2^{p(n)}
\end{aligned}$$

Construct a machine N that has $h(x)$ accepting path (Just guess a y and run N). Now make a query to a $\#\mathsf{P}$ machine to compute $h(x)$. By having $q(n)$ sufficiently large, ie., $2^{q(n)} > 2p(n)$ we can compute $g(x) = (2^{p(n)} - h(x))$.

We know $x \in L$ if and only if $g(x) > 2^{p(n)-1}$. Since $g(x)$ can be computed from $h(x)$ and $p(n)$ it is clear that we can decide whether $x \in L$ from $h(x)$. The function h is in $\#\mathsf{P}$, we can define an NTM M_1 that on input x first nondeterministically guesses a string y of length $p(|x|)$ and then simulate M on $< x,y >$. Hence, $L \in P^{\#\mathsf{P}}$. ∎

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