Probability Spaces and Conditional Probability.

1. Suppose I have a biased coin, with outcomes H and T, with the probability of heads $\mathbb{P}[H] = \frac{3}{4}$ and the probability of tails $\mathbb{P}[T] = \frac{1}{4}$. Suppose I perform an experiment in which I toss the coin 3 times–an outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

   (a) What is the *sample space* for my experiment?
   i. $\{H, T\}$
   ii. $\{\frac{3}{4}, \frac{1}{4}\}$
   iii. $\{(X_1, X_2, X_3)\ s.t.\ X_1, X_2, X_3 \in \{H, T\}\}$

   Review the definition of an sample space in the notes.

   We have three flips, therefore our sample space needs to capture all sequences of length 3 of possible outcomes.

   (b) Which of the following are examples of *events*? Select all that apply.
   i. $\{(H,H,T),(H,H),(T)\}$
   ii. $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
   iii. $\{(T,T,T)\}$
   iv. $\{(T,T,T),(H,H,H)\}$
   v. $\{(T,H,T),(H,H,T)\}$

   Review the definition of an event in the notes.

   Any subset of the sample space is an event. All choices but the first one are valid subsets.

   (c) What is the probability of the outcome H, H, T? Enter your answer as a fully reduced fraction (i.e. your answer should have the format x/y, where $x, y \in \mathbb{N}$ are the smallest possible integers they can be).

   $\mathbb{P}[(H,H,T)] =$

   Note that the coin tosses are independent, and you know $\mathbb{P}[H], \mathbb{P}[T]$.

   We need to multiply the probabilities together. We get $\frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{9}{64}$.

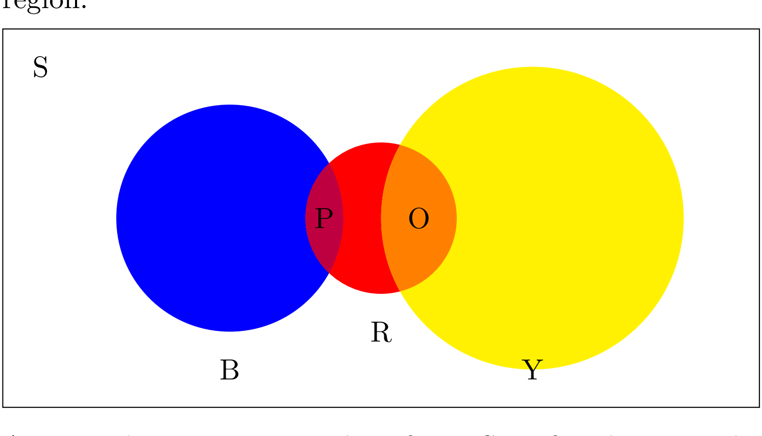
   (d) What is the probability of the event that my outcome has exactly two heads? Enter your answer as a fully reduced fraction (i.e. your answer should have the format x/y, where $x, y \in \mathbb{N}$ are the smallest possible integers they can be).

   $\mathbb{P}[\text{ exactly 2 heads}] =$

   Note that the coin tosses are independent, and you know $\mathbb{P}[H], \mathbb{P}[T]$. How many such outcomes are there, and what is each one's probability? Try a combinatorial approach.

   There are three such outcomes (three places where the single tails can occur). Each such outcome has probability $\frac{9}{64}$ like before. So the total probability is $3 \times \frac{9}{64} = \frac{27}{64}$.

2. Consider the drawing of the probability space S below. Here, the blue/purple region is the set of events B, the red/purple/orange region is the set of events R, and the yellow/orange region is the set of events Y. The set of events P is the set of events in both B and R, and is represented by the purple region. The set of events O is the set of events in both R and Y, and is represented by the orange region.



   Assume that we are sampling from S uniformly at random. Please answer the following multiple choice questions about this space, selecting all that apply.

   (a) What is $\mathbb{P}[R]$, the probability that an element from S is in R?

   Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes.
   i. $\frac{|R|-|P|-|O|}{|S|}$
   ii. $\frac{|R|}{|S|}$
   iii. $|R|$

   Since we are sampling uniformly at random, the probabilities are computed by dividing the number of elements in the event of interest (R) by the total number of elements (S).

   (b) What is $\mathbb{P}[R|Y]$, the probability that an element from S is in R given that it is also in Y?

   Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes. When we condition on an event Z, we are restricting the total outcomes to be outcomes for which Z occurs.
   i. $\frac{|O|}{|S|}$
   ii. $\frac{|O|}{|Y|}$
   iii. $\frac{|R\cap Y|}{|Y|}$

   Conditioning on Y is like considering everything outside of Y as not even being there. So we need to restrict R to Y which gives us $Y \cap R = O$. Then we need to divide the number of elements in that region by $|Y|$ which is our new sample space.

   (c) What is $\mathbb{P}[R|O]$, the probability that an element from S is in R given that it is in O?

   Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes. When we condition on an event Z, we are restricting the total outcomes to be outcomes for which Z occurs.
   i. 1
   ii. $\frac{|R\cap O|}{|O|}$
   iii. $\frac{|R\cap O|}{|S|}$

   Like before this probability is equal to $\frac{|R\cap O|}{|O|}$. But note that $O \cap R = O$, so this fraction is just 1.

   (d) What is $\mathbb{P}[P|B]$, the probability that an element from S is in P given it is in B?

   Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes. When we condition on an event Z, we are restricting the total outcomes to be outcomes for which Z occurs.
   i. $\frac{|P|}{|S|}$
   ii. $\frac{|P|}{|B|}$
   iii. $\frac{|B\cap P|}{|B|}$

   Similar to the previous questions, the answer is $\frac{|B\cap P|}{|B|}$. Note that $P \subseteq B$, so $B \cap P = P$.

   (e) What is $\mathbb{P}[B \cup R \cup Y]$, the probability that an element of S is in B or R or Y?

   Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes. Be careful not to double count!
   i. $\frac{|B|+|R|+|Y|-|P|-|O|}{|S|}$
   ii. $\frac{|B\setminus P|+|R|+|Y\setminus O|}{|S|}$ $\qquad (X \setminus Z = \{x\ s.t.\ x \in X \text{ and } x \notin Z\})$
   iii. $\frac{|B|+|R|+|O|}{|S|}$

   We just need to count and make sure that each region inside the union is counted exactly once. This happens in the first two choices. In the last choice, O is counted twice (once from R and once from O) as is P. So the last choice is not right.

   (f) What is $\mathbb{P}[O|R \cup Y]$, the probability that an element of S is in O given that it is also in R or Y?

   Recall the definition of probability–the probability of an event X is the number of outcomes for which X occurs divided by the total number of outcomes. When we condition on an event Z, we are restricting the total outcomes to be outcomes for which Z occurs. Be careful not to double count!
   i. $\frac{|O|}{|S|-|B|}$
   ii. $\frac{|O|}{|Y|+|R|-|O|}$
   iii. $\frac{|O|}{|Y|+|R|}$

   O is a subset of $R \cup Y$. So in the numerator we just need $|O|$. In the denominator we need $\mathbb{P}[R \cup Y]$. Only the second choice gives us that in the denominator.

3. Suppose I have a bag of candy containing 10 chocolate bars, 5 lollipops and 5 toffees.

   (a) If I randomly select a piece of candy to eat, what is the probability that it will be a chocolate bar? Please enter your answer as a decimal. Enter the answer as an exact decimal with a leading zero.

   $\mathbb{P}[\text{chocolate}] =$

   Recall the definition of probability.

   Exactly half of the candies are chocolate bars. So the probability is 1/2.

   (b) Suppose that I am trying to randomly select a candy for a friend who does not like chocolate, so that every time I choose a chocolate I return it to the bag, and I stop when I draw a candy that is not chocolate. What is the probability that I choose a toffee? Enter the answer as an exact decimal with a leading zero.

   Probability =

   Recall the definition of probability–what are we conditioning on?

   Conditioning on the candy not being a chocolate bar, we have a sample space consisting of 5 lollipops and 5 toffees. In this space, exactly half of the outcomes are toffees. So the probability is again 1/2.

   (c) Say that I have eaten one chocolate, one toffee, and one lollipop. True or false: now that I have eaten one of each candy, my probability of choosing a chocolate has decreased.

   Calculate the probability of drawing a chocolate in the updated bag.
   - True
   - False

   In the new space we have 9 chocolate bars, 4 toffees and 4 lollipops. Therefore the new probability of picking a chocolate bar is 9/17 which is slightly above 1/2.

4. **Bayesian Inference.** In this problem, we will work through an example of Bayesian Inference.

   Suppose you would like to decide whether to go to on a picnic tomorrow. You have some data about the weather in the area. You know the following:
   - The probability of rain, $\mathbb{P}[R] = 0.2$
   - The probability you see clouds the day before it rains, $\mathbb{P}[C|R] = 0.75$
   - The probability you see clouds the day before a clear day, $\mathbb{P}[C|\overline{R}] = 0.1$

   You notice heavy clouds in the sky. If you could calculate the probability that it will rain tomorrow conditioned on the clouds in the sky tonight, then you can make a more informed decision about tomorrow's plans.

   You want to calculate $\mathbb{P}[R|C]$, the probability that it rains tomorrow if it is cloudy tonight. You remember that by Bayes' formula,

$$\mathbb{P}[R|C] = \frac{\mathbb{P}[C|R]\mathbb{P}[R]}{\mathbb{P}[C]}.$$

   You do not know $\mathbb{P}[C]$. But, you know that

$$\mathbb{P}[C] = \mathbb{P}[C \cap \overline{R}] + \mathbb{P}[C \cap R].$$

   (a) You can compute $\mathbb{P}[C \cap R]$ from the known quantities above. Recall that

$$\mathbb{P}[C \cap R] = \mathbb{P}[C|R]\mathbb{P}[R]$$

   What is $\mathbb{P}[C \cap R]$? Enter your answer as an exact decimal with a leading zero.

   $\mathbb{P}[C \cap R] =$

   Do you know each of these quantites? Refer to your data above.

   We know that $\mathbb{P}[C \cap R] = \mathbb{P}[C|R]\mathbb{P}[R] = 0.75 \times 0.2 = 0.15$.

   (b) Now, you can compute $\mathbb{P}[C \cap \overline{R}]$:

$$\mathbb{P}[C \cap \overline{R}] = \mathbb{P}[C|\overline{R}]\mathbb{P}[\overline{R}]$$

   What is $\mathbb{P}[C \cap \overline{R}]$? Enter your answer as an exact decimal with a leading zero.

   $\mathbb{P}[C \cap \overline{R}] =$

   Do you know each of these quantites? Refer to your data above.

   We have $\mathbb{P}[C \cap \overline{R}] = \mathbb{P}[C|\overline{R}]\mathbb{P}[\overline{R}] = 0.1 \times (1 - 0.2) = 0.08$.

   (c) Finally, calculate $\mathbb{P}[R|C] = \frac{\mathbb{P}[C|R]\mathbb{P}[R]}{\mathbb{P}[C\cap\overline{R}]+\mathbb{P}[C\cap R]}$. Enter your answer as a decimal with a leading 0 and 2 decimal place's percision (i.e. $0.xy$ where x, y are digits).

   $\mathbb{P}[R|C] =$

   Refer to your Bayes' formula above, and plug in your known quantities.

   We have calculated the numerator already. It is 0.15. The denominator is the sum of two terms, each of which has already been computed. So the denominator is $0.15 + 0.08 = 0.23$. Therefore the probability $\mathbb{P}[R|C]$ is simply $0.15/0.23 \simeq 0.65$.

   (d) Your prior was $\mathbb{P}[R]$, the probability that it rains tomorrow. Assuming you do not want to get rained on, will you be more or less likely to go on a picnic tomorrow considering your posterior probability $\mathbb{P}[R|C]$?

   Is the posterior probability that it will rain greater than 0.5? Was the prior probability greater than 0.5?
   - More
   - Less

   Since the posterior is greater, it will be more likely to rain tomorrow, and therefore we should be less likely to go on a picnic.

5. **Which is the biased coin?** In this problem, we will use Bayesian inference to try and distinguish a biased coin from a fair coin.

   Suppose I have 2 coins. Let's call them A and B. Coin A is unbiased, *i.e.*, when you flip it, it comes up with a heads with probability exactly 0.5, or about half the time. Coin B is biased. When you flip it, it comes up with a heads with probablility 0.75, or about three fourths of the time.

   The problem is that outwardly, the two coins are identical and indistinguishable. I can't tell which is which just by looking at them, or weighing them, or subjecting them to any physical test. All I can do is flip the coins many times, and hope to tell them apart based on the outcomes of these flips.

   (a) So I pick up one of these coins at random and flip it a 100 times, and I get 50 heads.

   Given the outcome above, what is the probability that the coin I picked is unbiased?
   - 0.0
   - 0.5
   - 1.0
   - $\frac{1}{1+\left(\frac{3}{4}\right)^{50}}$
   - $\frac{1}{1+\left(\frac{1}{2}\right)^{100}}$
   - $\frac{\binom{100}{50}\frac{1}{2^{100}}}{\binom{100}{75}\left(\frac{3}{4}\right)^{100}+\binom{100}{50}\left(\frac{1}{2}\right)^{100}}$
   - $\frac{1}{1+\left(\frac{1}{3}\right)^{25}}$

   Let A be the event that I get 50 flips out of the 100, and let B be the event that I pick the unbiased coin. Then $\mathbb{P}[B] = 0.5$. We also have $\mathbb{P}[A|B] = \binom{100}{50}(1/2)^{100}$, since there are $\binom{100}{50}$ ways to pick the 50 heads and each such configuration happens with probability $(1/2)^{100}$.

   We also have $\mathbb{P}[A|\overline{B}] = \binom{100}{50}(3/4)^{50}(1/4)^{50}$ since if we pick the biased coin, there are again $\binom{100}{50}$ configurations with 50 heads and each occurs with probability $(3/4)^{50}(1/4)^{50}$.

   Now using the Bayes' formula we have

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A|B]\mathbb{P}[B]}{\mathbb{P}[A|B]\mathbb{P}[B] + \mathbb{P}[A|\overline{B}]\mathbb{P}[\overline{B}]} = \frac{\mathbb{P}[A|B] \times 0.5}{\mathbb{P}[A|B] \times 0.5 + \mathbb{P}[A|\overline{B}] \times 0.5}$$

   And the last part is equal to

$$\frac{(1/2)^{100}}{(1/2)^{100} + (3/4)^{50}(1/4)^{50}}$$

   which can be further simplified into $\frac{1}{1+(3/4)^{50}}$.

   (b) Now I pick up the other coin and flip *it* a 100 times, and I get 75 heads.

   Given the two outcomes above (*i.e.*, the heads counts from both sets of 100 flips), what is the probability that the first coin that I picked is unbiased?
   - 0.0
   - 0.5
   - 1.0
   - $\frac{1}{1+\left(\frac{3}{4}\right)^{50}}$
   - $\frac{1}{1+\left(\frac{1}{2}\right)^{100}}$
   - $\frac{\binom{100}{50}\frac{1}{2^{100}}}{\binom{100}{75}\left(\frac{3}{4}\right)^{100}+\binom{100}{50}\left(\frac{1}{2}\right)^{100}}$
   - $\frac{1}{1+\left(\frac{1}{3}\right)^{25}}$

   Same as before, we just need to use the Bayes' formula. If C is the event of seeing 75 heads for the second coin, then $\mathbb{P}[C|\overline{B}] = \binom{100}{75}(1/2)^{100}$ and $\mathbb{P}[C|B] = \binom{100}{75}(3/4)^{75}(1/4)^{25}$.

   Note that A and C are independent conditioned on B, meaning that once we fix whether we picked the biased coin or the unbiased coin, the two events A and C become independent (since they involve two series of unrelated coin tosses). So we have $\mathbb{P}[A \cap C|B] = \mathbb{P}[A|B]\mathbb{P}[C|B]$ and similarly $\mathbb{P}[A \cap C|\overline{B}] = \mathbb{P}[A|\overline{B}]\mathbb{P}[C|\overline{B}]$. We need $\mathbb{P}[B|A \cap C]$. We can again use the Bayes' formula and we end up with

$$\mathbb{P}[B|A \cap C] = \frac{(1/2)^{100} \times (3/4)^{75}(1/4)^{25}}{(1/2)^{100} \times (3/4)^{75}(1/4)^{25} + (3/4)^{50}(1/4)^{50} \times (1/2)^{100}}$$

   This further simplifies into $\frac{1}{1+(1/3)^{25}}$.

6. **Accounting fraud.** There are 3 ways a CEO can commit accounting fraud: (a) he can overstate revenue by booking fake orders, (b) he can overstate profits by recording near-term expenses as long-term capital expenses, and (c) he can understate liabilities by making overly optimistic assumptions about the growth prospects of the company's pension funds. In the United States, 50% of CEOs commit fraud (a), 20% of CEOs commit fraud (b), 15% of CEOs commit both fraud (a) and fraud (b), and all CEOs commit fraud (c).

   The probability of getting caught if a CEO does (a) is 0.01, independent of whether or not he commits other kinds of fraud. Similarly, the probability of getting caught if he does (b) is 0.05, and of getting caught if he does (c) is 0.001.

   (a) What are the chances (rounded to 2 decimal places) that a CEO who perpetrates all 3 kinds of fraud will get caught?
   - 0.21
   - 0.76
   - 0.06
   - 0.01
   - 0.00

   If A, B, C are the events of getting caught for each of the three frauds, then we are interested in $\mathbb{P}[A \cup B \cup C]$. But note that this is simply $1 - \mathbb{P}[\overline{A} \cap \overline{B} \cap \overline{C}]$. Since $\overline{A}, \overline{B}, \overline{C}$ are independent, we have $\mathbb{P}[\overline{A} \cap \overline{B} \cap \overline{C}] = (1-0.01)(1-0.05)(1-0.001) \simeq 0.94$. Therefore $\mathbb{P}[A \cup B \cup C] \simeq 0.06$.

   (b) You read on the front page of the Wall Street Journal that the CEO of TrustMe, Inc. (for all practical purposes, a randomly chosen CEO) has been arrested for accounting fraud. What are the chances (rounded to 2 decimal places) that this CEO perpetrated at least 2 kinds of fraud?
   - 0.00
   - 0.95
   - 1.00
   - 0.33
   - 0.97

   We can partition the sample space into 4 parts. Let's call the CEOs who perpetrate all frauds D, the ones who perpetrate only type (a) and (c) as E, the ones who perpetrate only type (b) and (c) as F, and the ones who perpetrate only type (c) as G. Also let A, B, C be the events of being arrested for frauds of type (a), (b), (c) respectively. Then we are interested in $\mathbb{P}[D \cup E \cup F|A \cup B \cup C]$. To use the Bayes' formula we need compute $\mathbb{P}[A \cup B \cup C|x]\mathbb{P}[x]$ where x ranges over D, E, F, G.

   We have $\mathbb{P}[A \cup B \cup C|D] = (1-(1-0.01)(1-0.05)(1-0.001))$ and $\mathbb{P}[D] = 0.15$. Therefore $\mathbb{P}[A \cup B \cup C|D]\mathbb{P}[D] \simeq 0.009$.

   We have $\mathbb{P}[A \cup B \cup C|E] = (1-(1-0.01)(1-0.001))$ and $\mathbb{P}[E] = 0.5 - 0.15 = 0.35$. Therefore $\mathbb{P}[A \cup B \cup C|E]\mathbb{P}[E] \simeq 0.00385$.

   We have $\mathbb{P}[A \cup B \cup C|F] = (1-(1-0.05)(1-0.001))$ and $\mathbb{P}[F] = 0.2 - 0.15 = 0.05$. Therefore $\mathbb{P}[A \cup B \cup C|F]\mathbb{P}[F] \simeq 0.00255$.

   We have $\mathbb{P}[A \cup B \cup C|G] = 0.001$ and $\mathbb{P}[G] = 1 - \mathbb{P}[D] - \mathbb{P}[E] - \mathbb{P}[F] = 0.45$. Therfore $\mathbb{P}[A \cup B \cup C|G]\mathbb{P}[G] = 0.00045$.

   Now using the Bayes' formula we get

$$\mathbb{P}[D \cup E \cup F|A \cup B \cup C] = \frac{0.009 + 0.00385 + 0.00255}{0.009 + 0.00385 + 0.00255 + 0.00045} \simeq 0.97.$$

   So the chances are roughly 97%.

   (c) What are the chances (rounded to 2 decimal places) that the above CEO perpetrated exactly 2 kinds of fraud?
   - 0.40
   - 0.60
   - 0.20
   - 0.10
   - 1.00

   This time we are interested in $\mathbb{P}[E \cup F|A \cup B \cup C]$. We can reuse the computations from last time and the Bayes' formula to get

$$\mathbb{P}[E \cup F|A \cup B \cup C] = \frac{0.00385 + 0.00255}{0.009 + 0.00385 + 0.00255 + 0.00045} \simeq 0.40.$$

   So the chances are roughly 40%.