Random Variables, and Probability Distributions.

1. **Expectation and Variance.** This problem will give you some practice calculating expectations and variances of random variables.

   (a) Suppose that the random variable X takes on 3 values, $10, 25, 70$. Suppose $\mathbb{P}[X = 10] = 0.5$, $\mathbb{P}[X = 25] = 0.2$, and $\mathbb{P}[X = 70] = 0.3$.

      i. What is $\mathbb{E}[X]$?
      $\mathbb{E}[X] =$
      Review the definition of expectation.

      ii. What is $\mathbb{E}[X^2]$?
      $\mathbb{E}[X^2] =$
      Review the definition of expectation.

      iii. What is $\mathrm{Var}[X]$?
      $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 =$
      You can calculate this with the quantities above.

   (b) Let X, Y be random variables, let a, b, c be constants, and let $\mathbb{E}[X] = x$ and $\mathbb{E}[Y] = y$. What is $\mathbb{E}[aX + bY + c]$?
   Review linearity of expectation

      i. $a^2x + b^2y$
      ii. $ax + by$
      iii. $ax + by + c$
      iv. $aX + bY + c$

   (c) Let X, Y be random variables, let a, b, c be constants. Which of the following statements are always true?
   Review the definitions of expectation and variance, as well as properties of variance given in notes 12 and 13.

      i. $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$.
      ii. $\mathrm{Var}[aX + bY] = a\mathrm{Var}[X] + b\mathrm{Var}[Y]$
      iii. $\mathrm{Var}[aX] = a^2\mathrm{Var}[X]$
      iv. $\mathrm{Var}[aX + bY] = a^2\mathrm{Var}[X] + b^2\mathrm{Var}[Y]$
      v. $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ assuming X, Y are independent.
      vi. $\mathrm{Var}[X] + \mathbb{E}[X]^2 = \mathbb{E}[X^2]$
      vii. $\mathrm{Var}[aX + c] = a^2\mathrm{Var}[X]$ (remember that $\mathrm{Var}[Y] = \mathbb{E}[(Y - \mathbb{E}[Y])^2]$)

2. **A Game–Part I.** This game will give you practice with some expectations.

   (a) Suppose I have a bag full of equal numbers of \$1 bills and \$5 bills. If I let you choose a bill uniformly at random from this bag, let X be the random variable corresponding to your profit. What is your expected profit?
   $\mathbb{E}[X] =$
   Review the definition of expectation.

   (b) Say that I now add a second bag, which has equal numbers of \$10 and \$20 bills. Let the random variable corresponding to your profit from this bag be Y. If I let you choose one bill from the first bag and one bill from the second bag, what is your expected proft?
   $\mathbb{E}[X + Y] =$
   Review the definition of expectation and linearity of expectation.

   (c) Now, suppose I decide to charge you \$2.50 to draw a bill from the first bag, and \$15.50 to draw from the second bag. If you draw from both bags, what is your expected net profit?
   $\mathbb{E}[X + Y - 18] =$
   Review the definition of expectation and linearity of expectation.

   (d) Given the pay scheme above (\$2.50 to draw from the first bag, \$15.50 to draw from the second bag) Which of the following actions gives the maximum expected profit?

      i. Drawing from the first bag only.
      ii. Drawing from the second bag only.
      iii. Drawing from both bags.
      iv. Drawing from neither bag.

   (e) As the designer of this game, I want your expected profit to be net negative. I have told you that the proportions of \$1 and \$5 bills in the bag are the same. However, I could lie to you and rig the game so that if I charge you \$2.50, your expected profit for drawing from the first bag is negative. What percentage of bills in the bag should be \$1 bils in order to make your expected profit $-\$1$ (round up to the nearest percent)?
   % of \$1s? =
   Review the definition of expectation.

3. **A Game–Part II.** This problem will give you some practice with variances.

   (a) Recall the game from Part I: I have a bag full of equal numbers of \$1 bills and \$5 bills. If I let you choose a bill uniformly at random from this bag (without charging you a fee), let X be the random variable corresponding to your profit. What is the variance of X?
   $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 =$
   Review the definition of variance.

   (b) Say now that I have a new game, in which I do not make any claims about the composition of the bag. However, I do advertise that the **average earnings for one round are \$3**, that **the variance is 3**, and that each round is independent of the previous round.
   After playing the game 10 times, your total earnings are \$14–you drew \$1 bills 9 out of 10 times. You suspect me of cheating customers at this game. Can you use expectation and variance to make a case against me? Intuitively the answer is yes–you can show that your earnings are consistently below the advertized expectation, in a way that is more extreme than the advertized variance predicts.
   This intuition can be formalized by *Chebyshev's Inequality*. Chebyshev's Inequality states that for a random variable X,

   $$\Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}.$$

      i. Define $Y = \sum_{i=1}^{10} \frac{1}{10} Y_i$ to be the random variable corresponding to your average earnings, and let Y_i be the random variable corresponding to your earnings the ith time you play. What is $\mathbb{E}[Y]$, if my claims about the expectation and variance are true? Round your answer to the nearest hundredth.
      $\mathbb{E}[Y] =$
      Remember linearity of expectation. Also, this is not for the 10 games you already played–this is the expected earnings for 10 arbitrary games.

      ii. What is $\mathrm{Var}[Y]$, if my claims are correct? Recall your answer from problem 1c, and that each time you play is independent of the others. Enter your answer as a decimal with leading 0.
      $\mathrm{Var}[Y] =$
      Remember how variances of sums of independent random variables behave.

      iii. Let $|1.40 - \mathbb{E}[Y]| = a$. Give an upper bound on the probability that $Y = \$1.40$ (if my claims about the expectation and variance are true), by using Chebyshev's Inequality to upper bound the probability that $|Y - \mathbb{E}[Y]| \geq a$. Give your answer as a decimal, rounding to the nearest hundredth.
      $\Pr[Y = \$1.40] \leq \Pr[|Y - \mathbb{E}[Y]| \geq a] \leq$
      Review the definition of expectation and linearity of expectation.

      iv. Say you decided that this probability is not convincing enough. You decided to repeat the game 100 times, and after these 100 times you find that your average earnings are still \$1.40.
      Define a new random variable corresponding to your average earnings, $Z = \sum_{i=1}^{100} \frac{1}{100} Z_i$, where Z_i is your earnings on the ith round. Repeat the process above (calculate $\mathrm{Var}[Z]$, then use Chebyshev's Inequality) to give the best upper bound you can on the probability that $Z = \$1.40$. Give your answer as a decimal, rounding to the nearest hundredth.
      $\Pr[Z = \$1.40] \leq \Pr[|Z - \mathbb{E}[Z]| \geq a] \leq$
      Review the definition of expectation and linearity of expectation.

      v. Even though the average is the same after 10 and 100 games, do you have a stronger case for my cheating after repeating the experiment more times?

      - Yes
      - No