Induction

The following questions have a proposition and corresponding inductive proof. For each question: decide whether the proof is correct, and if not, identify the proof's flaw.

1. **Proposition:** For every integer $n \geq 0$,

$$0^2 + 1^2 + 2^2 + \cdots + n^2 = \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof:

- **Base Case.** For $n = 0$, this clearly holds since $\sum_{i=0}^{0} i^2 = 0 = \frac{0(1)(1)}{6}$.
- **Inductive Hypothesis.** Assume $\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.
- **Inductive Step.** Consider $\sum_{i=0}^{k+1} i^2$. Notice that

$$\sum_{i=0}^{k+1} i^2 = \left(\sum_{i=0}^{k} i^2\right) + (k+1)^2. \tag{1}$$

Apply the inductive hypothesis to the right-hand side to obtain

$$\begin{aligned} \sum_{i=0}^{k+1} i^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && (2)\\ &= (k+1)\frac{(2k^2+k)+6k+6}{6} && (3)\\ &= (k+1)\frac{(k+2)(2k+3)}{6}. && (4) \end{aligned}$$

Thus, the statement holds for $k+1$, and the proposition follows by the principle of induction.

(a) The proof is correct.
(b) The wrong base case was used; we should start from $n = 1$.
(c) This proof only holds for some of the integers; it is not general enough.
(d) The inductive hypothesis was not applied correctly.
(e) Equation (1) is incorrect.
(f) There is an error in line (2).
(g) There is an error in line (3).
(h) There is an error in line (4).

The proof is correct.

2. **Proposition:** $(\forall n \in \mathbb{N})(n^2 \leq n)$.
Proof:

- **Base Case.** When $n = 1$, the statement is $1^2 \leq 1$ which is true.
- **Inductive Hypothesis.** Assume that $k^2 \leq k$.
- **Inductive Step.** (1)We need to show that

$$(k+1)^2 \leq k+1$$

Working backwards we see that:
(2)

$$k^2 \leq (k+1)^2 - 1 \leq (k+1) - 1 = k$$

(3)So we get back to our original hypothesis which is assumed to be true.
(4)Hence, for every $n \in \mathbb{N}$ we know that $n^2 \leq n$. ♠

(a) The proof is correct.
(b) The proof of the base case is incorrect.
(c) The inductive hypothesis does not hold.
(d) The goal in (1) is incorrect.
(e) The application of the inductive hypothesis in (2) is incorrect.
(f) Step (3) is incorrect.
(g) Step (4) is incorrect.

The application of the inductive hypothesis in (2) is incorrect.

The goal of the inductive hypothesis is to let you prove that if the proposition is true for $n = k$, then it will be true for $n = k+1$. In other words, to prove that $P(k) \implies P(k+1)$. However, what we actually showed here was that $P(k+1) \implies P(k)$. Remember that implications are not bidirectional!

3. **Proposition:** All students love homework equally.
Proof:

- **Base Case.** For $n = 1$, a single student clearly loves homework exactly as much as him- or herself, so the base case holds.
- **Inductive Hypothesis.** Assume any k students love homework equally.
- **Inductive Step.** Consider $k+1$ students.
(1) By the inductive hypothesis, the first k students all love homework equally.
(2) By the inductive hypothesis, the last k students all love homework equally.
(3) Therefore, all $k+1$ students love homework equally, and the proposition holds by the principle of induction.

Why are we separately considering the first k students and the last k students in a group of $k+1$? What assumptions are we making about the two sets that allow us to draw conclusions in (3)?

(a) The proof is correct.
(b) The proof of the base case is incorrect.
(c) The inductive hypothesis does not hold.
(d) The application of the inductive hypothesis in (1) is incorrect.
(e) The application of the inductive hypothesis in (2) is incorrect.
(f) The inductive step logic in step (3) is flawed for at least one value of k.

The inductive step logic in step (3) is flawed for at least one value of k - namely, $k = 2$.

The implicit assumption we make in step (3) is that there will be some overlap between the first k students and the last k students in a set of $k+1$.

For example, in a set of three students $\{A\ B\ C\}$, the first two students are $\{A\ B\}$ and the last two students are $\{B\ C\}$. By the inductive hypothesis, A and B both like homework equally, and B and C both like homework equaly, so by transitivity A and C must like homework equally, so the entire set of three must like homework equally. Similar logic will hold for all larger k.

However, this assumption is *not* valid for $k = 2$. In a set of two students $\{A\ B\}$, $k - 1 = 1$. The set of the first one student is A and the last one students is B, and while the inductive hypothesis would hold for both sets, we cannot make any arguments by transitivity because there is no overlap.

4. **Proposition:** Every integer $n \geq 2$ can be written as a product of prime numbers.
Proof:

- **Base Case.** The base case is $k = 2$. This is prime, so the base case holds.
- **Inductive Hypothesis.** Assume for some integer $k \geq 2$ that it can be expressed as a product of prime numbers.
- **Inductive Step.** Consider $k+1$. If $k+1$ is prime, then we are done. (1) Otherwise, it must have a smallest integer divisor $a > 1$ such that $k+1 = a \cdot b$ for some integer b. (2) Applying the inductive hypothesis, we know that a and b can be written as products of primes, and therefore $k+1$ can be written as a product of those primes in turn. Thus, the proposition holds by the principle of induction.

(a) The proof is correct.
(b) The logic of the inductive step does not apply to every integer.
(c) The claim in (1) is incorrect, because we are not guaranteed that $k+1$ has a smallest integer divisor.
(d) The inductive hypothesis was too weak to apply in (2); strong induction should have been used.
(e) We cannot apply our inductive hypothesis to more than one subproblem at a time in (2).

The inductive hypothesis was too weak to apply in (2).

Our inductive hypothesis only allowed us to assume that a single number $k \in \mathbb{N}$ was the product of prime numbers. However, in (2) we assume that a and b can be written as the product of prime numbers, with $a, b \leq k$.

Simply assuming that k is the product of primes does not give us any information about all the natural numbers up to k. Our inductive hypothesis should have been "Assume that for some integer $k \geq 2$, all $i \leq k$ can be written as the product of prime numbers".

5. **Proposition:** Consider the function f defined as:

$$f(x) = \begin{cases} x & x = 1, 2, 3 \\ f(x-1) + f(x-2) + f(x-3) & x \in \mathbb{N} \text{ and } x > 3 \end{cases}$$

Show that $\forall x \in \mathbb{N}, f(x) < 2^x$.
Proof:

- **Base Case.** $f(1) = 1 < 2^1$,$f(2) = 2 < 2^2$, $f(3) = 3 < 2^3$, and $f(4) = 1 + 2 + 3 < 2^4$.
- **Inductive Hypothesis.** Assume for $n \geq 3, \forall x \leq n, f(x) < 2^x$.
- **Inductive Step.** Consider $x = n+1$. We get

$$\begin{aligned} f(n+1) = f(n) + f(n-1) + f(n-2) &< 2^n + 2^{(n-1)} + 2^{(n-2)} && (5)\\ &= (2^2 + 2 + 1) \times 2^{(n-2)} && (6)\\ &= 7 \times 2^{(n-2)} && (7)\\ &< 8 \times 2^{(n-2)} = 2^{(n+1)} && (8) \end{aligned}$$

Therefore, $\forall x \in \mathbb{N}, f(x) < 2^x$.

(a) The proof is correct.
(b) The proof is incorrect because the base case does not need to verify $f(4)$.
(c) The proof is incorrect because the inductive hypothesis does not include the cases when $n = 1, 2$.
(d) The proof is incorrect because the inductive step should consider $x = n$ instead of $x = n+1$.

The proof is correct.

6. **Proposition:** Any shape which can be drawn using n squares of side length ℓ is an $\ell \times n\ell$ rectangle.
Proof:

- **Base Case.** For $n = 1$, a single square is an $\ell \times 1\ell$ rectangle, and so the base case holds.
- **Inductive Hypothesis.** Any shape which can be drawn using k squares is an $\ell \times k\ell$ rectangle.
- **Inductive Step.** Consider a shape drawn using k squares. (1) By the inductive hypothesis, this shape must be an $\ell \times k \cdot \ell$ rectangle. (2) Now, we add one more square to the end of the rectangle, and we have a $\ell \times (k+1)\ell$ rectangle. Therefore, the statement holds for $k+1$, and we have proven the proposition by induction.

(a) The proof is correct.
(b) The base case is incorrect.
(c) The inductive hypothesis is applied too early in the proof.
(d) The inductive hypothesis does not hold as applied in (1).
(e) The way in which the argument is extended to $k+1$ in (2) does not hold for all cases.
(f) The inductive hypothesis does not actually hold for the subproblem to which it was applied.

The way in which the argument is extended to $k+1$ in (2) doesn't hold.

It is true that if we added another square to the end of the rectangle, extending the kl side by one unit, we would have an $l \times (k+1)l$ rectangle. However, this is *not* the only possible way to place the square.

For example, a square could be placed on a 1×2 rectangle extending the short side to form an L shape. What this actually proves is that it is always *possible* to construct an $l \times nl$ rectangle out of n squares of length l, which is a much less extraordinary claim.

7. Suppose you are trying to prove the following proposition by induction on n:
Proposition: For $n \geq 1$, every set of n numbers whose elements sum to 0 must contain at least one non-positive number.
Which of the following is a suitable induction hypothesis:

(a) Assume that for all $k \geq 1$, some set of k numbers whose elements sum to 0 must contain at least one non-positive number.
(b) Assume that there is a set of $k \geq 1$ numbers whose elements sum to 0 and which contains at least one non-positive number.
(c) Assume that for every $k \geq 1$, any set of k numbers whose elements sum to 0 must contain at least one non-positive number.
(d) Assume that for some $k \geq 1$, any set of k numbers whose elements sum to 0 must contain at least one non-positive number.

The correct proposition is "Assume that for some $k \geq 1$, any set of k numbers whose elements sum to 0 must contain at least one positive number." This is because we are inducting over the number of elements in the sets, so our hypothesis must assert that the proposition is true for sets of size k.

8. The following questions will help you figure out sufficient assumptions for induction to work. Assume that in all cases our goal is to prove $(\forall n \in \mathbb{N})P(n)$. In each case determine whether this statement can be proven using the given assumptions or not.

(a) We know $(\forall n)(P(2n) \implies P(2n+2))$ and $P(0), P(1)$ are true.
   - Induction proves $\forall n P(n)$.
   - The assumptions are not sufficient.
   No. For example consider the statement $P(n)$ which says either n is even or $n = 1$. Then the assumptions are satisfied, but not all n satisfy $P(n)$. The inductive step only works on even numbers, so we can never use it to derive $P(n)$ for odd n.

(b) We know $(\forall n)(P(n) \implies P(2n))$ and $\forall n(P(n) \implies P(2n+1))$ and $P(0)$ are true.
   - Induction proves $\forall n P(n)$.
   - The assumptions are not sufficient.
   Yes. From $P(0)$, we can derive $P(0), P(1)$. From $P(1)$ we can derive $P(2), P(3)$. In general if we manage to prove $P(0), \ldots, P(k-1)$, then $P(\lfloor \frac{k}{2} \rfloor)$ implies $P(2\lfloor \frac{k}{2} \rfloor)$ and $P(2\lfloor \frac{k}{2} \rfloor + 1)$, one of which is simply $P(k)$. And since $\lfloor \frac{k}{2} \rfloor$ is amongst $0, \ldots, k-1$, we have already proven it.

9. Assuming that we know $P(0), P(1)$ are true, which of the following assumptions imply that $(\forall n \in \mathbb{N})P(n)$?

- $\forall n(P(n) \implies P(2n))$.
- $\forall n((P(n) \wedge P(n+1)) \implies P(n+2))$.
- $\forall n((P(n) \wedge P(n+2)) \implies P(n+1))$.
- $\forall n(P(n) \implies (P(n+1) \implies P(n+2)))$.

For the first choice, we can only use the rule to deduce $P(n)$ for even n. So it will never help us prove $P(9)$ for example.

For the second choice, using $P(0), P(1)$ we can prove $P(2)$, then using $P(1), P(2)$ we can prove $P(3)$ and so on.

For the third choice, note that to prove $P(n+1)$ we must already know $P(n+2)$. So if we know $P(0), \ldots, P(k)$, the rule will never help us prove $P(k+1)$.

For the fourth choice, note that $P(n) \implies (P(n+1) \implies P(n+2))$ is logically equivalent to $(P(n) \wedge P(n+1)) \implies P(n+2)$. So it is equivalent to the second choice.

10. We wish to prove the following proposition: Let $r \neq 1$ be a real number and let $n \geq 0$ be an integer. Then we have

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

- What should we induct on?
  (a) r
  (b) i
  (c) n
  (d) Σ
  We cannot induct on r, because it is a real number and not a natural number. i is an internal variable, and the statement is not defined in terms of it (i.e. you can't say this is the statement for $i = 2$). So the only remaining choice is to induct on n.
- **Base Case:** What should our base case be?
  (a) $n = 0$
  (b) $n = 1$
  (c) $i = 0$
  (d) $i = 1$
  (e) $r = 0$
  (f) $r = 1$
  (g) $r = -1$
  Since we are inducting on n, the only two logical choices are $n = 0$ and $n = 1$. But since we want to prove the statement for all $n \geq 0$, $n = 0$ is the logical base case. Otherwise we would have to provide a separate proof for the $n = 0$ case.
- **Inductive Hypothesis:** What should our inductive hypothesis be? We want the simplest statement that will still prove our proposition.
  (a) Suppose that our proposition holds for any integer $k < n+1$.
  (b) Suppose that our proposition holds for some even integer n.
  (c) Suppose that our proposition holds for some n.
  (d) Suppose that for some n, we have $\sum_{i=0}^{n} r^i \leq \frac{r^{n+1}-1}{r-1}$.
  The difference between the sum on the left hand side of the goal statement, for n and $n+1$, is just one single term. So it looks like the sum for $n+1$ should be easily expressible in terms of the sum for n. Therefore assuming that the statement holds for some n seems to be enough.

  Now, we consider the case of $n+1$, and examine the sum $\sum_{i=0}^{n+1} r^i$. Note that this sum has $n+2$ terms, since it starts from 0.
- **Inductive Step:** How do we apply our hypothesis?
  (a) We apply our hypothesis to the last $n+1$ terms of the sum.
  (b) We apply our hypothesis to the first $n+1$ terms of the sum.
  (c) We first apply our hypothesis to the first $n+1$ terms, then apply it to the last $n+1$ terms, then subtract the expression for the middle n terms.
  It is only the first $n+1$ terms that resemble a sum of the same format. So the logical thing to do is to apply the inductive hypothesis to the first $n+1$ terms.