

### Table Matchings.

Say that we want to pair up the men (1,2,3) with the women (A,B,C); such a pairing is called a *matching*. The table below gives the ranked preferences of the people, with most preferred on the left and least preferred on the right.

man	ranking	woman	ranking
1	B, A, C	A	3, 2, 1
2	A, B, C	B	1, 2, 3
3	B, A, C	C	2, 1, 3

Given a matching, we say that two people form a *rouge couple* if both prefer each other to the partner they are currently matched to.

1. Select all of the rouge couples for the matching (1A, 2B, 3C) with respect to the preferences in the table above.

Review the definition of a matching and a rouge couple above. For more details read course note 4. Remember that a rouge couple requires the man and the woman to *mutually* prefer each other!

- (a) (1, B)
- (b) (1, C)
- (c) (2, A)
- (d) (2, C)
- (e) (3, A)
- (f) (3, B)

Let's consider the choices in turn.

- a) Man 1 ranks woman *B* above his current partner *A*, and woman *B* likewise ranks man 1 above her current partner 2, so (1, *B*) *do* form a rouge couple.
- b) Woman *C* prefers man 1 to her current partner 3, *but* man 1 prefers his partner *A* over *C*, so (1, *C*) *do not* form a rouge couple.
- c) Man 2 prefers *A* over his current partner, and *A* prefers him over her current partner, so (2, *A*) *do* form a rouge couple.
- d) While woman *C* prefers man 2 over her current partner 3, man 2 prefers his current partner over *C*, so (2, *C*) *do not* form a rouge couple.
- e) Man 3 prefers *A* over his current partner *C*, and *A* prefers him over her current partner 1, so (3, *A*) *do* form a rouge couple.
- f) Man 3 prefers *B* over his current partner, but *B* does not prefer him over her current partner, so (3, *B*) *do not* form a rouge couple.

2. Again given the preferences above, select all of the rouge couples for the matching (3A, 2B, 1C).

Review the definition of a matching and a rouge couple above. For more details read course note 4. Remember that a rouge couple requires the man and the woman to *mutually* prefer each other!

- (a) (1, A)
- (b) (1, B)
- (c) (2, A)
- (d) (2, C)
- (e) (3, B)
- (f) (3, C)

Let's consider each choice in turn.

- a) Man 1 prefers woman *A* to his current partner *C*, but *A* prefers her current partner 3 over 1, so (1, *A*) *do not* form a rouge couple.
- b) Man 1 and woman *B* both prefer each to their current partners, so (1, *B*) *do* form a rouge couple.
- c) Man 2 prefers woman *A* to his current partner, but woman *A* does not prefer man 2, so (2, *A*) *do not* form a rouge couple.
- d) Woman *C* prefers man 2 over her current partner, but man 2 does not prefer woman *C*, so (2, *C*) *do not* form a rouge couple.
- e) Woman *A* prefers man 3 to her current partner, but man 3 does not prefer woman *A*, so (3, *A*) *do not* form a rouge couple.
- f) Man 3 prefers woman *B* over his current partner, but woman *B* does not prefer 3, so (3, *B*) *do not* form a rouge couple.

A matching is considered *stable* if it contains no rouge couples. This is because no one can convince anyone else to leave their partner in order to form a new pair. The stable marriage algorithm allows us to find stable matchings. This algorithm is described in course note 4. The following problems are designed to give you practice with this algorithm.

3. Consider the preference ranks given below for the men/suitors (1,2,3,4) and women/suitees (A,B,C,D).

man	ranking	woman	ranking
1	A, B, D, C	A	3, 1, 4, 2
2	A, C, B, D	B	2, 1, 3, 4
3	B, C, D, A	C	4, 1, 3, 2
4	B, A, D, C	D	1, 2, 3, 4

Simulate the stable marriage algorithm (with male suitors), and answer the following questions about the algorithm. We consider the first offers to have taken place on day 1.

- (a) Who did suitor 1 propose to on day 1?  
His first choice, A.
- (b) Was he rejected?
  - Yes
  - No  
No, A received proposals from 1 and 2 and she preferred 1.
- (c) Who did suitor 4 propose to on day 1?  
His top choice, B.
- (d) Was he rejected?
  - Yes
  - No  
Yes, B received proposals from 3 and 4 and preferred 3.
- (e) Who proposed (with a new proposal) to A on day 2?  
4, because he was rejected by B on day 1 and A is the next woman on his list.
- (f) Did he get rejected?
  - Yes
  - No  
Yes, A received proposals from 4 and 1 and preferred 1.
- (g) Who was 1 matched to when the algorithm terminated?  
Woman A.
- (h) Who was 2 matched to when the algorithm terminated?  
Woman C.
- (i) Who was 3 matched to when the algorithm terminated?  
Woman B.
- (j) Who was 4 matched to when the algorithm terminated?  
Woman D.
- (k) Now let us simulate the algorithm with women proposing (being the suitors). Before finding the outcome, what do you think is going to happen to men's matches (compared to when they were proposing)?
  - Each man will either get the same match as before, or prefer his new match.
  - Each man will either get the same match as before, or prefer his old match.
  - Some men will get the same match as before, some will prefer their new matches, and some will prefer their old matches.

The algorithm run with men proposing produces a male-optimal pairing. Therefore the original matches are as high on their list as each man can hope for. Another way to arrive at the same conclusion is to note that the algorithm produces male-pessimal pairings if run with women proposing. Therefore men get partners that are as low as it can be on their list in the second run.

- (l) Who will woman A propose to on the first day?  
Her top choice, 3.
- (m) Who gets rejected on the first day?
  - A
  - B
  - C
  - D
  - No one  
No one gets rejected, because all men receive exactly one proposal.
- (n) In a male-optimal stable matching, who would be paired with man 4?  
The algorithm with male suitors returns a male-optimal pairing. We have already seen that the algorithm run with men proposing matches 4 with D.
- (o) In a female-optimal stable matching, who would be paired with man 4?  
The algorithm with female suitors returns a female-optimal pairing. In that matching, 4 is matched with C.
- (p) In a male-pessimal stable matching, who would be paired with woman A?  
We get the male-pessimal pairing by running the algorithm with women proposing. In that case A is matched with 3.
- (q) In a female-pessimal stable matching, who would be paired with woman A?  
We get the female pessimal pairing by running the algorithm with men proposing. In that case A is matched with 1 as we have already seen.