Propositions and Proofs

A proposition is a statement which is either true or false.
Are the following propositions?

1. $2+2=4$
   (a) Yes, it is a proposition.
   (b) No, it is not a proposition.
   This is a proposition and it is true.

2. $x+2=4$
   (a) Yes, it is a proposition.
   (b) No, it is not a proposition.
   This is not a proposition. It is a predicate: whether it is true or false is predicated on the value of x.

3. All photos are taken by some human.
   (a) Yes, it is a proposition.
   (b) No, it is not a proposition.
   This is a proposition and it is false. Search for "macaque selfie" online for a counterexample.

4. How is your new semester so far?
   (a) Yes, it is a proposition.
   (b) No, it is not a proposition.
   This is not a proposition. A question is not a statement.

5. Let $\mathbb{X} = \{\text{photos}\}$ and $\mathbb{Y} = \{\text{humans}\}$, which one of the following is equivalent to "All photos are taken by some human"?
   (a) $(\forall x \in \mathbb{X})(\forall y \in \mathbb{Y})(x \text{ is taken by } y)$
   (b) $(\forall x \in \mathbb{X})(\exists y \in \mathbb{Y})(x \text{ is taken by } y)$
   (c) $(\exists x \in \mathbb{X})(\forall y \in \mathbb{Y})(x \text{ is taken by } y)$
   (d) $(\exists x \in \mathbb{X})(\exists y \in \mathbb{Y})(x \text{ is taken by } y)$
   Let's break down this proposition. $(\forall x \in \mathbb{X})(\exists y \in \mathbb{Y})(x \text{ is taken by } y)$ translates to "For all photos x, there exists some human y such that the human y took the photo x."

6. Let $\mathbb{Z}$ denote the set of all integers, and let $P(x)$ denote the proposition formula $x \geq 0$, which ones of the following are equivalent to "For every pair of integers, at least one of them is negative"?
   (a) $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\neg P(x) \vee \neg P(y))$
   (b) $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})\neg(P(x) \vee P(y))$
   (c) $\neg((\exists x \in \mathbb{Z})(\exists y \in \mathbb{Z})(P(x) \wedge P(y))$
   (d) $(\forall x \in \mathbb{Z})\neg((\exists y \in \mathbb{Z})(P(x) \wedge P(y))$
   a), c), and d) are all correct. Let's consider them one by one. Choice a) translates to "For all integers x and y, either x is less than 0 or y is less than 0 or both are less than 0". Choice c) translates to "It is NOT true that there exist integers x and y such that both x and y are greater than or equal to 0." Choice d) translates to "For every integer x, there does not exist an integer y such that both x and y are greater than or equal to 0". Note that d) can be easily derived from c) using the rules for distributing $\neg$ across quantifiers.

7. Select the correct truth table for the boolean function

$$Y = (A \implies \neg B) \wedge (C \implies B).$$

Note that $P \implies Q$ is logically equivalent to $\neg P \vee Q$. Try converting $(A \implies \neg B)$ and $(C \implies B)$ to their equivalent disjuction forms first.

(a)

A	B	C	Y
0	0	0	1
0	0	1	0

(b)

A	B	C	Y
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

(c)

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

(d)

A	B	C	Y
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Let's build up the truth table in stages, by first finding truth tables for $A \implies \neg B$ and $C \implies B$. Recall that $P \implies Q$ is logically equivalent to $\neg P \vee Q$. This gives us the following truth tables for $A \implies \neg B$ and $C \implies B$

A	B	$A \implies \neg B$
0	0	1
0	1	1
1	0	1
1	1	0

B	C	$C \implies B$
0	0	1
0	1	0
1	0	1
1	1	1

We can put these two truth tables together while conveying the same information

A	B	C	$A \implies \neg B$	$C \implies B$
0	0	0	1	1
0	0	1	1	0
0	1	0	1	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	0
1	1	0	0	1
1	1	1	0	1

The truth table for $(A \implies \neg B) \wedge (C \implies B)$ is only 1 when *both* of the right columns are 1, so it'll look like this

A	B	C	$A \implies \neg B$	$C \implies B$	$(A \implies \neg B) \wedge (C \implies B)$
0	0	0	1	1	1
0	0	1	1	0	0
0	1	0	1	1	1
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	0	0
1	1	0	0	1	0
1	1	1	0	1	0

The following questions have a proposition and corresponding proof. For each question: decide whether the proof is correct, and if not, identify the proof's flaw.

8. We call integer n an even number if and only if there exists an integer k, such that $n = 2k$.
   Proposition: The negative of any even integer n is even.
   Proof: The proposition is true:
   (1) By definition of even number, there exists some integer k, such that $n = 2k$.
   (2) Multiply both sides by -1, we get
$$\begin{aligned} -n &= -(2k) \\ &= 2 \times (-k) \end{aligned}$$
   (3) Now let $r = -k$. We have $-n = 2r$ for some integer r.
   (4) Hence, by definition of even number, $-n$ is even.

   (a) The proof is correct.
   (b) There is an error in line (1).
   (c) There is an error in line (2).
   (d) There is an error in line (3).
   (e) There is an error in line (4).
   This proof is correct; it is a form of direct proof.

9. **Proposition:** For any positive integer k, if $2^k = 0 \mod (3)$ then $8^k = 1 \mod 3$.
   Proof: The proposition is false. Proof by counterexample: let $k = 1$, then $8^1 = 8 \neq 1 \mod 3$.
   (a) The proof is correct.
   (b) The proof is incorrect because the proof only mentions the case of $k = 1$, while the original proposition contains a universal quantifier for any positive integer k.
   (c) The proof is incorrect because for any positive integer k, $2^k = 0 \mod 3$ will always be false.
   (d) The proof is incorrect because the 'counterexample' provided is not a counterexample for the overall proposition. Letting $k = 1$ does not make the implication false.
   The proof is incorrect because the provided counterexample is not actually a counterexample. The hypothesis of the proposition is always false, so the implication is always true. Thus, the statement is vacuously true and cannot be a counterexample.

10. **Proposition:** For any integer n, if $n^2 = 0 \mod (4)$ then $n = 0 \mod 4$.
    Proof: The proposition is false. Proof by counterexample: let $n = 6$, then $n^2 = 36 = 0 \mod 4$, but $n \neq 0 \mod 4$.
    (a) The proof is correct.
    (b) The proof is incorrect because the proof only mentions the case of $n = 6$, while the original proposition contains a universal quantifier for any positive integer n.
    (c) The proof is incorrect because the proposition is actually true.
    The proof is correct; it is an example of proof by contradiction.

11. **Proposition:** Let x and y be two positive integers. If $x \times y < 36$ then $x < 6$ or $y < 6$.
    Proof:
    (1) Suppose $x \geq 6$ and $y \geq 6$.
    (2) Since $x \geq 6$, multiply both sides by y, we get
$$x \times y \geq 6 \times y$$
    (3) and since $y \geq 6$, multiply both sides by 6, we get
$$6 \times y \geq 6 \times 6 \text{ i.e. } 6 \times y \geq 36$$
    (4) So $x \times y \geq 36$
    (5) Therefore, if $x \times y < 36$ then $x < 6$ or $y < 6$.

    (a) The proof is correct.
    (b) The proof is incorrect because (1) is not a negation of the original implication's conclusion.
    (c) The proof is incorrect because there is an error at (2).
    (d) The proof is incorrect because from (1) - (4) we cannot get (5).

    The proof is correct. It is a proof by contraposition, i.e., to prove "$x \times y < 36 \implies (x < 6 \vee y < 6)$", we first prove "$(x \geq 6 \wedge y \geq 6) \implies x \times y \geq 36$"

12. **Proposition:** The negative of any irrational number is irrational.
    Proof:
    (1) Assume (for the sake of contradiction) that there exists an irrational number x, such that $-x$ is rational.
    (2) By definition of rational, $\exists a, b \in \mathbb{Z}$, $b \neq 0$, s.t.
$$-x = a/b$$
    (3) multiply both sides by -1, we get
$$x = -(a/b) = (-a)/b$$
    (4) Since $-a$ and b are integers and $b \neq 0$, and by definition of rational, x is rational, which is a contradiction.
    (5) Therefore, the negative of any irrational number is irrational.

    (a) The proof is correct.
    (b) The proof is incorrect because (1) is not a negation of the original proposition.
    (c) The proof is incorrect because there is an error at (4).
    (d) The proof is incorrect because from (1) - (4) we cannot get (5).

    The proof is correct. It is a proof by contradiction, i.e., to prove "$\forall$ irrational number x,$-x$ is irrational." we show that "$\exists$ irrational number x, $-x$ is rational." leads to a contradiction.