Counting.

1. Say I have a standard 6-sided dice. I generate a sequence of 5 numbers by tossing this dice 5 times, and writing down the result after each toss.

   (a) How many possible distinct sequences of 5 numbers can I generate this way? Please enter your answer as an integer.

   \# sequences =

   How many options do we have each time? Is repetition allowed? (Review the flipping coins and rolling dice sections of the notes).

   The answer is $6 \times 6 \times 6 \times 6 \times 6 = 7776$ because each time we have 6 options.

   (b) True or False: In the above procedure, the sequence of numbers $6, 6, 6, 6, 6$ is less likely to come up that the sequence $6, 5, 6, 6, 5$.

   How many possible rolling outcomes generate the first outcome? How many possible rolling routines generate the second?

   - True
   - False

   False. Both are equally likely to come up.

   (c) True or False: In the above procedure, the sequence of numbers $6, 6, 6, 6, 6$ is less likely to come up than some sequence of the form x_1, x_2, x_3, x_4, x_5, where each $x_i \in \{5, 6\}$.

   How many possible tosses generate the first outcome? How many possible tosses generate the second?

   - True
   - False

   True. There is one way for the first sequence to come up. But there are many more ways for a sequence of the second form come up (2^5 ways).

   (d) How many sequences generated by the procedure above have the form $6, 6, 6, 6, 6$? Please enter your answer as an integer.

   \# sequences =

   Review the flipping coins and rolling dice sections of the notes.

   There is just one way, since every dice roll is already uniquely determined.

   (e) How many sequences generated by the procedure above are of the form x_1, x_2, x_3, x_4, x_5, where each $x_i \in \{5, 6\}$? Please enter your answer as an integer.

   \# sequences =

   Review the flipping coins and rolling dice sections of the notes.

   32. For each roll we have 2 choices, so the answer is $2^5 = 32$.

   (f) What is the probability of my generating the sequence $6, 6, 6, 6, 6$? Please enter your answer as a decimal with a leading zero and to exactly 5 decimal place precision (round to the nearest 10^{-5}, rounding up on 5).

   probability =

   Recall that the definition of the probability of an event is the number of times this event occurs divided by the number of possible events.

   We divide the number of ways, i.e. 1, by 6^5 which is the total number of possible outcomes to get the answer.

   (g) What is the probability of my generating the sequence $6, 5, 6, 6, 5$? Please enter your answer as a decimal with a leading zero and to exactly 5 decimal place precision (round to the nearest 10^{-5}, rounding up on 5).

   probability =

   Recall that the definition of the probability of an event is the number of times this event occurs divided by the number of possible events.

   As before, the number of ways, i.e. 1, divided by the total number of outcomes, i.e. 6^5, gives us the answer.

   (h) What is the probability of my generating a sequenceof the form x_1, x_2, x_3, x_4, x_5, where each $x_i \in \{5, 6\}$? Please enter your answer as a decimal with a leading zero and to exactly 5 decimal place precision (round to the nearest 10^{-5}, rounding up on 5).

   probability =

   Recall that the definition of the probability of an event is the number of times this event occurs divided by the number of possible events.

   The total number of ways to get such a sequence is 2^5. Dividing by 6^5 gives us the probability.

2. Suppose that my 5 friends are roommates, and that they also share a sock drawer. In this sock drawer they have 10 pairs of socks. Each pair of socks is a different color, but it is folded together so that no roommate is ever wearing two different colors of socks.

   (a) How many distinct roommate-sock combinations are there? Please enter your answer as an integer.

   \# combinations =

   Review the counting sequences section of the notes.

   For the socks of the first roommate, we have 10 choices, for the second one, we have 9 choices and so on. The answer is $10 \times 9 \times 8 \times 7 \times 6 = 30240$.

   (b) Each day, the roommates record the combination of sock colors that they are all wearing in a special scrapbook. How many such combinations are possible? Please enter your answer as an integer.

   \# combinations =

   Notice that order doesn't matter anymore–we only care about which socks are present, not about which roommate is wearing them. Review the counting sets section of the notes.

   The answer is $\binom{10}{5} = 252$ which is the number of ways we can pick 5 out of the 10 socks when the order does not matter.

   (c) Assuming that the roommates randomly choose their socks each morning, what is the probability that the red, teal, orange, pink, and magenta socks are all worn by the roommates on a specific day? Please enter your answer as a decimal with a leading zero and to exactly 5 decimal place precision (round to the nearest 10^{-5}, rounding up on 5).

   probability =

   Recall the definition of probability. What is the total number of events possible here?

   There are $\binom{10}{5}$ possible combinations of socks, all of which are equally likely. Here we are interested in only one of these combinations. The probability of it is simply $1/\binom{10}{5}$.

   (d) Say that one roommate decides he likes the pink, red, and teal socks more than all of the rest, and decides to jealously hoard them so that no one else can ever wear them (but he can still also wear the other 7 colors of socks). Now, how many distinct roommate-sock combinations are there? Please enter your answer as an integer.

   \# combinations =

   Review the counting sequences section of the notes. How many choices does each roommate get if the first roommate chooses a hoarded sock? How many choices does each roommate get if the first roommate chooses a shared sock?

   Let's begin with the other roommates choosing their socks. The first one has 7 choices, the second one has 6, the third one 5, and the fourth one 4. Now the roommate who has hoarded socks can pick any pair from the remaining 3 hoarded socks and 3 not-hoarded ones. So this roommate has 6 choices. The answer is $7 \times 6 \times 5 \times 4 \times 6 = 5040$.

   (e) The other roommates grow frustrated with the hoarding roommate and tell him that if he does not return the pink, red, and teal socks, then he is not allowed to wear the other colors of socks. The hoarding roommate does not relent. Now, how many distinct roommate-sock combinations are there? Please enter your answer as an integer.

   \# combinations =

   Review the counting sequences section of the notes. How many choices does each roommate get?

   The hoarding roommate has 3 choices. The next roommate has 7 (all the not-hoarded socks), the next one 6, the next 5, and the next 4. So the answer is $7 \times 6 \times 4 \times 3 = 2520$.

   (f) How many color combinations of socks are now possible in the scrapbook? Please enter your answer as an integer.

   \# combinations =

   Review the counting sets section of the notes. Which sets are being chosen from?

   We need to pick 4 colors from the 7 not-hoarded socks and one from the hoarded socks. So the answer is $\binom{7}{4} \times \binom{3}{1} = 105$.

3. A combinatorial proof is a proof which shows that two quantities are the same by explaining that each quantity is a different way of counting the same thing. This question is intended to help you see how this technique is applied.

   In the question below, a quantity is given. Select all choices that are a valid different way of counting the given quantity (allowing you to deduce that the formulae given by each way of counting are equal).

   (a) The number of squares in an $n \times n$ grid.

   i. In an $n \times n$ grid, there are n rows of squares, each of which has n squares in it. Thus, there are n^2 squares in an $n \times n$ grid.

   ii. We know there are exactly n squares on the diagonal. Now, when we remove the diagonal, we have two equally sized triangles that have $n-1$ squares on the hypotenuse. When we remove those, we end up with smaller triangles with $n-2$ squares on the hypotenuse. We continue this until we are left with one square on each side, and we've counted all of the squares in the grid. This gives us a total of $n + 2\sum_{k=1}^{n-1} k$ squares in the grid.

   iii. Take the $(n-1) \times (n-1)$ subgrid that is the upper lefthand corner of this grid. This subgrid has $n-1$ rows, each of which has $n-1$ squares, so this part contributes $(n-1)^2$ squares. Now, the squares that we excluded from this subgrid come to a total of $n + n - 1$ squares. Thus, there are $(n-1)^2 + 2n - 1$ squares in an $n \times n$ grid.

   iv. First, we peel off the leftmost column, and topmost row, removing exactly $2n-1$ squares. We then peel off the leftmost column and topmost row remaining, removing exactly $2(n-1)-1$ squares. We continue this process until we are left with a single square, which we also remove. This gives us a total of $(2n-1) + (2n-3) + \cdots + 3 + 1 = \sum_{k=1}^{n} 2k - 1$ squares in the $n \times n$ grid.

   Review combinatorial proofs in the notes. Also, drawing out a couple of examples might help.

   All of these are valid ways of counting the number of squares in an $n \times n$ grid.

Notice you can verify that these quantities are actually equal algebraically. While this example was simple enough to do so, the proof technique also extends to situations where the equality is less obvious.

For example, you can try to prove that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ by combinatorially proving that

$$\binom{n}{k} k! = n(n-1) \cdots (n-k+1),$$

using only the set-counting definition of $\binom{n}{k}$. What is the quantity on the left counting? Can you argue that the quantity on the right is counting the same thing?