On weight-equitable partitions of graphs

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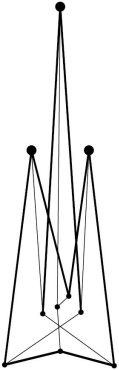
Outline

Introduction

Graph spectrum



$$\Longleftrightarrow \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

spectrum (eigenvalues): $\lambda_1 \geq \cdots \geq \lambda_n$

Graph spectrum

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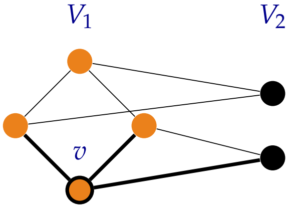
Equitable partitions

$$\mathcal{P} = \{V_1, V_2\}$$



$b_{11}(v) = 2, b_{12}(v) = 1$

$$B = (b_{ij}) = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$$

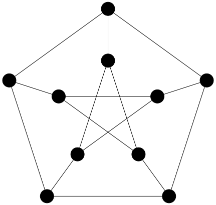
Equitable if b_{ij} only depends on i and j.

Representing partitions

$$\mathcal{P} = \{V_1, V_2, \ldots, V_m\}$$

$$S = \begin{bmatrix} | & & & \\ e_1 & & & \\ | & & & \\ & | & & \\ & e_2 & & \\ & | & & \\ & & \vdots & \\ & & & | \\ & & & e_m \\ & & & | \end{bmatrix}$$

Shrinking graphs while preserving (part) of the spectrum



$\{3, 1^5, -2^4\}$

$$\longrightarrow \qquad B = (b_{ij}) = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$\{3, 1, -2\}$

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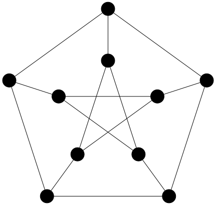

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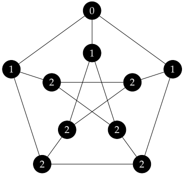
$\{3, 1^5, -2^4\}$ $\{3, 1, -2\}$

Theorem (e.g. Cvetković, Doob, Sachs 1980)
Every eigenvalue of B is also an eigenvalue of $A(G)$.

Equitable partitions in algebraic combinatorics

Equitable partitions in algebraic combinatorics

- Naturally occur in graphs with rich algebraic structures:



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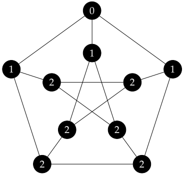
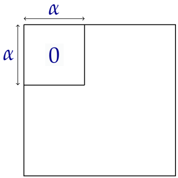


- Useful for proving eigenvalue bounds on graph parameters like the k-independence number (Cvetković 1972), (Haemers 1995), (A., Coutinho, Fiol 2019)



Extending equitable partitions

Equitable: every neighbor contributes to b_{ij} equally.

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What if we assign weights to the vertices?

Extending equitable partitions

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Use weights which 'regularize' the graph.

Perron eigenvector

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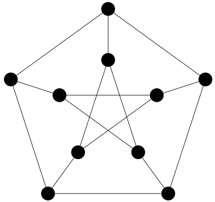

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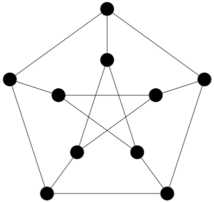

$\lambda_1 = 3$, $v = \mathbf{1}$

We call v the *Perron eigenvector*.

Weight partitions

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Vertex weights: Perron eigenvector ν, scale such that $\min \nu_i = 1$.

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Weight quotient matrix $B^* = (b^*_{ij})$ with entries (*weight-intersection numbers*):

$$b^*_{ij}(u) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v \qquad\qquad u \in V_i$$

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Note that the sum of the weight-intersection numbers for all $1 \le j \le m$ gives the weight-degree of each vertex $u \in V_i$:

$$\sum_{j=1}^{m} b^*_{ij}(u) = \frac{1}{\nu_u} \sum_{v \in G(u)} \nu_v = \delta^*_u = \lambda_1$$

Weight-equitable partitions

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$$b_{ij}^*(\cancel{u}) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v \qquad \forall u \in V_i$$

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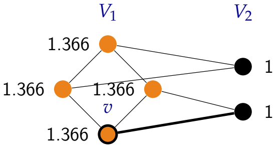


$$b^*_{12}(v) = \frac{1}{\nu_v} \sum_{w \in G(v) \cap V_2} \nu_w = \frac{1}{1.366} \cdot 1 = 0.732$$

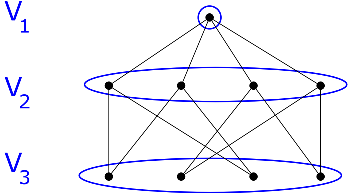
$$B^* = (b^*_{ij}) = \begin{pmatrix} 2 & 0.732 \\ 2.732 & 0 \end{pmatrix}$$

Weight-equitable if b^*_{ij} only depends on i and j.

Note: $\sum_j b^*_{ij} = \lambda_1$.

Example weight-equitable partition

$$\nu = (\ 2j \mid \sqrt{2}j \mid 1j\ )$$



$$b^*_{ij}(u) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v$$

$$b^*_{12}(1) = \tfrac{1}{2}(\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2})$$

$$b^*_{21}(2) = \tfrac{1}{\sqrt{2}} 2$$

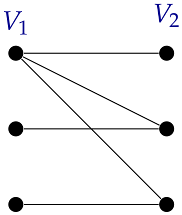
$$b^*_{21}(3) = \tfrac{1}{\sqrt{2}} 2$$

$$b^*_{21}(4) = \tfrac{1}{\sqrt{2}} 2$$

$$b^*_{21}(5) = \tfrac{1}{\sqrt{2}} 2$$

...

Example weight-equitable partition but not equitable



$$\nu = (2.732, 1, 1, 1.414, 1.932, 1.932)$$

equitable $\quad \not\Longleftarrow\!\!\Rightarrow \quad$ weight-equitable

Origin of weight-equitable partitions

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Ratio bound (Hoffman 1970)

If G is regular with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

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Origin weight partitions

number $\chi(G)$ of G, is bounded below by $n/(\alpha(G))$. Thus upper bounds for $\alpha(G)$ give lower bounds for $\chi(G)$. For instance, if G is regular, Theorem 3.2 implies that $\chi(G) \geqslant 1 - \lambda_1/\lambda_n$. This bound, however, remains valid for nonregular graphs (but note that it does not follow from Theorem 3.3).

THEOREM 4.1.

(i) *If G is not the empty graph, then* $\chi(G) \geqslant 1 - (\lambda_1/\lambda_n)$.
(ii) *If* $\lambda_2 > 0$, *then* $\chi(G) \geqslant 1 - (\lambda_{n-\chi(G)+1}/\lambda_2)$.

Proof. Let $X_1, \dots, X_\chi$ $[\chi = \chi(G)]$ denote the color classes of G and let $u_1, \dots, u_n$ be an orthonormal set of eigenvectors of A (where u_i corresponds to λ_i). For $i = 1, \dots, \chi$, let s_i denote the restriction of u_1 to X_i, that is,

$$(s_i)_j = \begin{cases} (u_1)_j, & \text{if } j \in X_i, \\ 0, & \text{otherwise,} \end{cases}$$

and put $\tilde{S} = [s_1 \cdots s_\chi]$ (if some $s_i = 0$, we delete it from $\tilde{S}$ and proceed similarly) and $D = \tilde{S}^\mathsf{T}\tilde{S}$, $S = \tilde{S}D^{-1/2}$, and $B = S^\mathsf{T}AS$. Then B has zero diagonal (since each color class corresponds to a zero submatrix of A) and an eigenvalue λ_1 ($d = D^{1/2}\underline{1}$ is a λ_1-eigenvector of B). Moreover, interlacing Theorem 2.1 gives that the remaining eigenvalues of B are at least λ_n. Hence

$$0 = \operatorname{tr}(B) = \mu_1 + \cdots + \mu_\chi \geqslant \lambda_1 + (\chi - 1)\lambda_n,$$

which proves (i), since $\lambda_n < 0$. The proof of (ii) is similar, but a bit more

Origin weight-equitable partitions

Formally defined and used by
(Garriga, Fiol 1999)

Origin weight-equitable partitions

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Theory of eigenvalue interlacing extended (Fiol 1999)

Motivation

Why using weight-equitable partitions?

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Powerful tool used to extend several spectral bounds known for regular graphs also for **non-regular graphs**.

Applications of weight-equitable partitions

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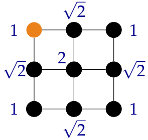
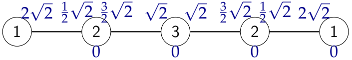
- (Fiol, Garriga, Yebra 1996) (Locally) pseudo-distance-regular graphs.





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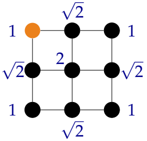

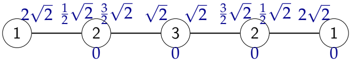

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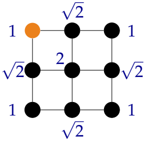

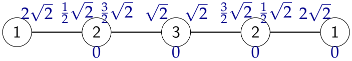

- (Fiol, Garriga 1999) Spectral characterizations of distance-regularity around a set and spectral characterizations of completely regular codes.

- (Lee, Weng 2012) Spectral excess theorem for irregular graphs.

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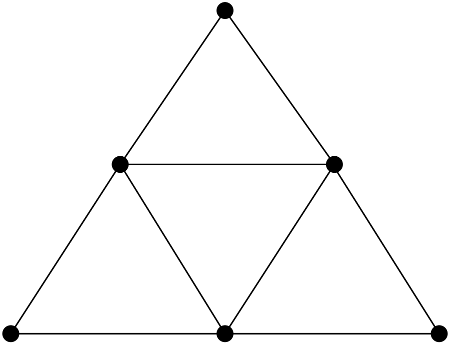
Equitable vs Weight-equitable partitions



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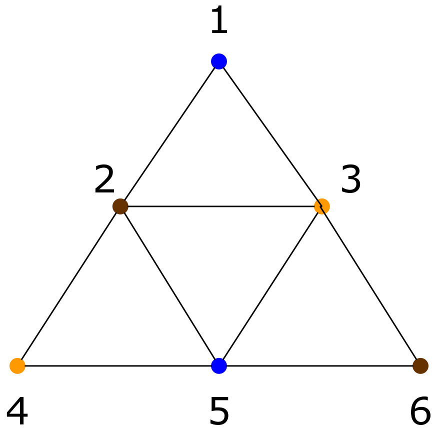

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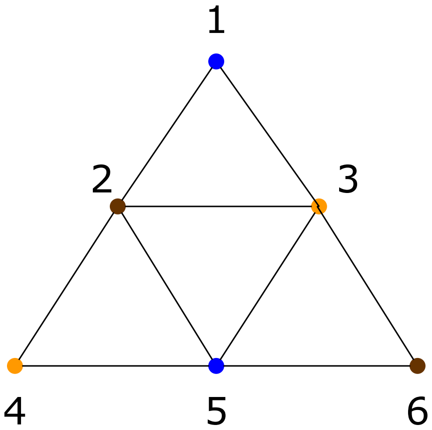

weight-equitable BUT NOT equitable

Equitable vs Weight-equitable partitions

$$\text{Equitable} \Longrightarrow \text{Weight-equitable}$$

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Converse not true!

$$\text{Equitable} \not\Longleftarrow \text{Weight-equitable}$$

Relation between (weight-)equitable partitions

	graph class admitting . . . partition with m cells	
number of cells m	equitable	weight-equitable
1	$\Longleftrightarrow$ regular	all
2	biregular	bipartite
n	all	all

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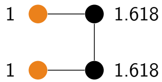
Proposition

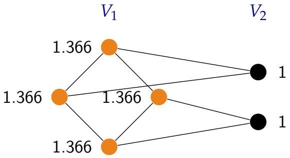
Weight-equitable and ν constant over all cells $\quad\Leftrightarrow\quad$ equitable



Spectral properties

Notation

Let $\rho : U \mapsto \sum_{u \in U} v_u e_u$.



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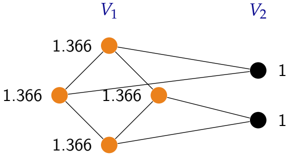
Let $\rho : U \mapsto \sum_{u \in U} \nu_u e_u$.



$$\bar{S}^* = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Normalized weight-characteristic matrix: $\bar{s}^*_{ui} = \begin{cases} \frac{\nu_u}{\|\rho(V_i)\|} & \text{if } u \in V_i, \\ 0 & \text{otherwise.} \end{cases}$

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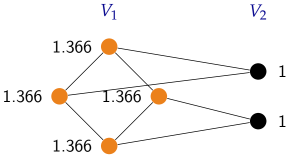

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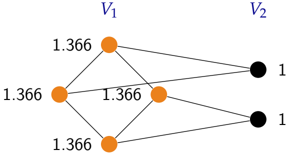

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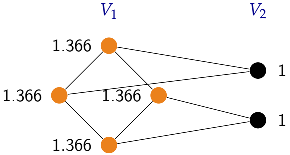

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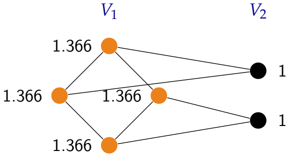

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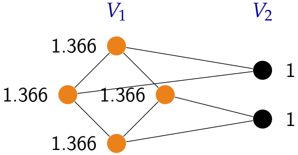
Notation

Let $\rho : U \mapsto \sum_{u \in U} v_u e_u$.



$\bar{S}^* = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ $\bar{B}^* = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$

$$(\bar{S}^*)^\top \bar{S}^* = I \qquad \bar{B}^* = (\bar{S}^*)^\top A \bar{S}^*$$

Theorem

- $\bar{B}^*$ has largest eigenvalue λ_1;
- All eigenvalues of $\bar{B}^*$ are eigenvalues of G.

Motivation

It is often useful (why, in next section) to know whether a graph admits a weight-equitable partition :

Motivation

It is often useful (why, in next section) to know whether a graph admits a weight-equitable partition :

$\rightarrow$ Find characterizations and conditions.

Characterization I:

generalized double stochastic matrices and weight-regularity

Known characterizations

Theorem (Fiol 1999)
$AS^* = S^*B^* \iff \mathcal{P}$ weight-equitable partition

Double stochastic matrices

A matrix is *double stochastic* if it is nonnegative and each of its rows and each of its columns sums up to one.

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Note: $\Omega(A)$ is a convex polytope since it consists of all matrices X such that

$$XA = AX, \quad X\mathbf{1} = \mathbf{1}X = 1, \quad X \geq 0.$$

Double stochastic matrices and equitable partitions

Lemma (Godsil 1997)

Let A be the adjacency matrix of a graph G, and let $\mathcal{P}$ be a partition of the vertex set with normalized characteristic matrix S. Then, $\mathcal{P}$ is equitable if and only if A and $SS^\top$ commute.

Double stochastic matrices and equitable partitions

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Question: Can we extend this result to weight-equitable partitions?

Generalized double stochastic matrices

A matrix is *generalized double stochastic* if it is nonnegative and each of its rows and each of its columns sums up to the same constant.

Generalized double stochastic matrices

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Note: $\Omega^*(A)$ is also a convex polytope since it consists of all matrices X such that

$$XA = AX, \quad X\mathbf{1} = \mathbf{1}X, \quad X \geq 0.$$

Generalized double stochastic matrices and weight-equitable partitions

Lemma (A. 2019)
Let A be the adjacency matrix of a graph G, and let $\mathcal{P}$ be a weight partition of the vertex set with normalized weight-characteristic matrix $\overline{S}^*$. Then, $\mathcal{P}$ is weight-equitable if and only if A and $\overline{S}^*\overline{S}^{*\top}$ commute.

Corollary (A. 2019)
Let $\mathcal{P}$ be a weight partition of the vertex set of a graph with normalized weight-characteristic matrix $\overline{S}^*$. Then $\mathcal{P}$ is weight-equitable if and only if $\overline{S}^*\overline{S}^{*\top} \in \Omega^*(A)$.

Characterization II:

Fractional automorphisms and weight-regularity

Fractional automorphisms

A adjacency matrix of a graph

Fractional automorphisms

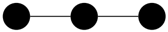
A adjacency matrix of a graph

Graph automorphism:

Permutation matrix P
s.t. $PA = AP$

Fractional automorphism:

Doubly stochastic matrix X
s.t. $XA = AX$



$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

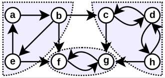
$$X = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Fractional automorphisms

Let $X = (x_{ij})$ be doubly stochastic and define the directed graph G_A with adjacency matrix

$$A = (a_{ij}) = \begin{cases} 1 & \text{if } x_{ij} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\mathcal{P}_X$ be the partition of $[n]$ into the strongly connected components of G_A.



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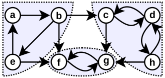

Proposition (A., Hojny, Zeijlemaker 2022)

If X commutes with $A(G)$, then $\mathcal{P}_X$ is weight-equitable.

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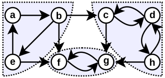

Proposition (A., Hojny, Zeijlemaker 2022)
If X commutes with $A(G)$, then $\mathcal{P}_X$ is weight-equitable.

Unfortunately no hope for an iff result ...

Fractional automorphisms

Proposition (A., Hojny, Zeijlemaker 2022)

Given a partition $\mathcal{P}$, let $X_{\mathcal{P}}$ be a matrix with entries

$$x_{vw} = \begin{cases} \frac{\nu_v \nu_w}{\|\rho(P)\|^2} & \text{if } v, w \in P \text{ for some } P \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{P}$ is a weight-equitable partition, then $X_{\mathcal{P}} A = A X_{\mathcal{P}}$.

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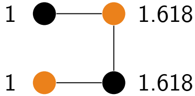

$$X_{\mathcal{P}} = \begin{pmatrix} 0.276 & 0 & 0.447 & 0 \\ 0 & 0.724 & 0 & 0.447 \\ 0.447 & 0 & 0.724 & 0 \\ 0 & 0.447 & 0 & 0.276 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$X_{\mathcal{P}}$ not a double stochastic, but quite symmetric ...

Characterization III:

Hoffman-type polynomial and weight-regularity

Related results

(Hoffman 1963)
Characterization of regular graphs in terms of the Hoffman polynomial.

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Hoffman's-like characterization for biregular graphs.

Related results

Theorem (A., Dalfó, Fiol 2013)
A bipartite graph $G = (V_1 \cup V_2, E)$, with $n = n_1 + n_2$ vertices in (δ_1, δ_2)-biregular if and only if the odd part of its preHoffman polynomial satisfies

$$H_1(A) = \alpha \begin{pmatrix} \boldsymbol{O} & J \\ J & \boldsymbol{O} \end{pmatrix}$$

with $\alpha = \frac{n_1+n+2}{2\sqrt{n_1 n_2}} = \frac{\delta_1+\delta_2}{2\sqrt{\delta_1\delta_2}}$.

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Question: Can we find a Hoffman-like polynomial to characterize weight-regularity?

Polynomials and weight-regularity

Theorem (A. 2019)

Let G be a connected graph with a partition of its vertices into m sets, $\mathcal{P} = \{V_1, \ldots, V_m\}$, such that $n = n_1 + \cdots + n_m$ and such that the map on V, denoted by $\boldsymbol{\rho} : u \to \nu_u$, is constant over each V_k. Then there exists a polynomial $H \in \mathbb{R}_d[x]$ such that

$$H(A) = \begin{pmatrix} b^*_{11}J & b^*_{12}J & \cdots & b^*_{1m}J \\ b^*_{21}J & b^*_{22}J & \cdots & b^*_{2m}J \\ \vdots & & \ddots & \\ b^*_{m1}J & b^*_{m2}J & \cdots & b^*_{mm}J \end{pmatrix}$$

if and only if $\mathcal{P}$ is a weight-equitable partition of G.

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Weight-equitable partitions maintain the structure of the Perron eigenvector $\boldsymbol{\nu}$.

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if and only if $\mathcal{P}$ is a weight-equitable partition of G.

As a corollary, for a regular graph $\nu = 1$: (Hoffman 1963)

Intermezzo: last work visit to India

International Conference on Algebraic Geometry, Coding Theory and Combinatorics
Indian Institute of Technology Hyderabad, India

in honour of prof. Sudhir Ghorpade.


An application to graph theory: improvement of Hoffman's bound

Hoffman's ratio bound

Theorem (Hoffman 1970)

If G has at least one edge, then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

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Theorem (Hoffman 1970)
If G has at least one edge, then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

When equality holds we call the coloring a *Hoffman coloring*.

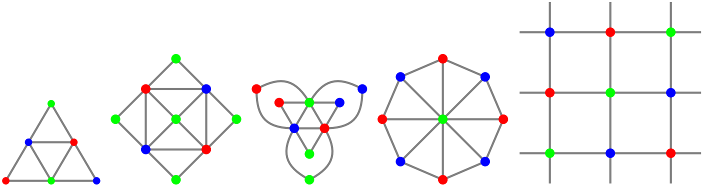
Examples of Hoffman colorable graphs

Trivially Hoffman colorable graphs:

- Bipartite graphs;
- Regular complete multipartite graphs (e.g. $K_{3,3,3}$), including complete graphs.

BUT not many non-trivial infinite families of Hoffman colorable graphs are known!

Some irregular Hoffman colorable graphs:



Hoffman colorings in the literature

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- (A., Bosma, Van Veluw 2025) Structural properties of Hoffman colorings of irregular graphs.

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$h(G) = 1 - \frac{\lambda_1}{\lambda_n} \leq \chi(G)$

- For Hoffman colorable graphs, h is optimal,
- Improving Hoffman's bound,
- Sandwiching: if $h(G) = \chi(G)$ then

$$h(G)=\chi_v(G)=\chi_{sv}(G)=\chi_q(G)=\chi(G)$$

Exact values of chromatic parameters for free, even some that are not known to be computable like $\chi_q(G)$!

Improving Hoffman's bound

Theorem (A. 2019)

If G has chromatic number $\chi(G)$ and a Hoffman coloring, then the partition defined by the color classes is weight-equitable.

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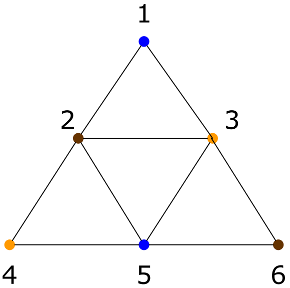

$$\mathcal{P} = \{V_1, V_2, V_3\} = \{\{2,6\},\{1,5\},\{3,4\}\}$$

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What can we do with this theorem?

Corollary (A. 2019)

If G has at least one edge and the vertex partition defined by the χ color classes is not weight-equitable, then

$$\chi(G) \geq 2 - \frac{\lambda_1}{\lambda_n}.$$

Improving Hoffman's bound

Theorem (A. 2019)

If G has chromatic number $\chi(G)$ and a Hoffman coloring, then the partition defined by the color classes is weight-equitable.

What can we do with this theorem?

If G does not have a weight-regular partition
$\Longrightarrow$ G cannot have a Hoffman coloring
$\Longrightarrow$ useful for finding families of non-regular Hoffman colorable graphs (A., Bosma, Van Veluw 2025)

Computing weight-equitable partitions

Complexity equitable partitions

https://mathoverflow.net/questions/96858/complexity-of-equitable-partitions

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Complexity of equitable partitions

Asked 11 years, 5 months ago Modified 6 years, 5 months ago Viewed 2k times

18

We are talking about undirected simple graphs and *partitions* of their vertex sets into disjoint non-empty *cells*. Such a partition is *equitable* if for any two vertices v, w in the same cell, and any cell C, it holds that v, w have the same number of neighbours in C. The *trivial* partition (with only one vertex per cell) is always equitable.

Given any partition π, there is a unique coarsest equitable partition $\bar{\pi}$ finer than π. (The concepts *finer* and *coarser* include equality). This is a very old result, as also are polynomial-time algorithms for computing $\bar{\pi}$ from π.

Another fact is that it is NP-complete to determine if a graph has an equitable partition with every cell of size 2. (This follows from Lubiw, SIAM J Comput 10, 1981, 11–21 on noting that such a partition corresponds to a fixed-point-free automorphism of order 2.)

My question is: **what else?** Are any other complexity results known? In particular:

1. What is the complexity of: Given a regular graph, does it have any non-trivial equitable partition other than the partition with just one cell?
2. What is the complexity of: Given a regular graph, does it have an equitable partition with exactly two cells?
3. What is the complexity of: Given a graph and two vertices v, w, is there a non-trivial equitable partition which has v, w in different cells?
4. Is there any problem on equitable partitions with complexity equal to graph isomorphism?

graph-theory computational-complexity

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edited May 14, 2012 at 9:09

asked May 13, 2012 at 23:34

Brendan McKay

Applications and coarseness

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Depending on the application, different levels of coarseness are required:

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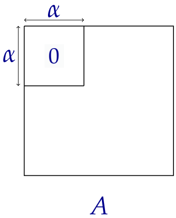
- Bounds on α: two cells;
- Pseudo-distance-regular graphs: $\#$cells $=$ diameter $+ 1$.



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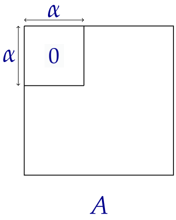
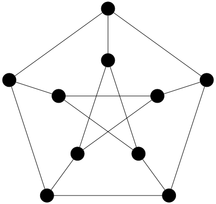


In general, coarser means more shrinkage.

Shrinking graphs while preserving (part) of the spectrum



$$\longrightarrow \quad B = (b_{ij}) = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$\{3, 1^5, -2^4\}$ $\{3, 1, -2\}$

Our focus

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Coarse(st) weight-equitable partitions

- Largest size reduction;
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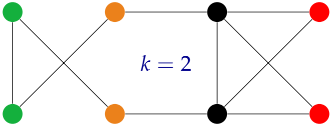
k-homogeneous weight-equitable partitions



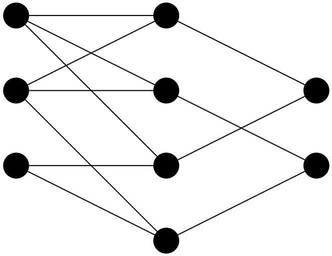
- Finding general complexity results is hard, so start with a regular case.

Computational results for equitable partitions: coarsest

Theorem (Bastert 1999)

The coarsest equitable partition of a graph can be found in polynomial time.

Color splitting:



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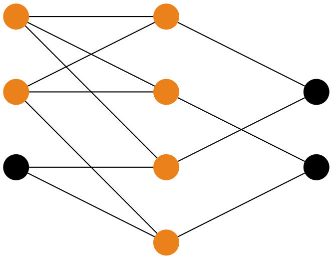

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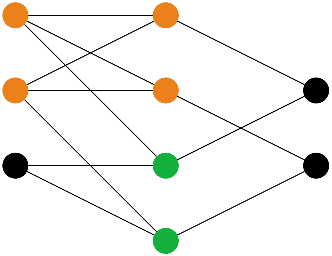

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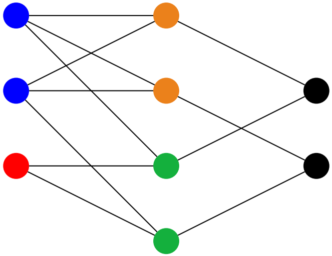

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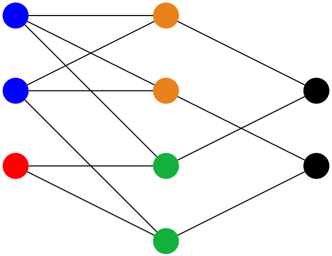

Based on algorithm for minimising finite automata (Hopcroft 1971).

Can be computed in $O(m \log n)$ time (Cardon, Crochemore 1982).

Extension to weight-equitable partitions?

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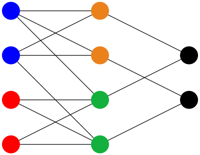
	graph class admitting ... partition with m cells	
number of cells m	equitable	weight-equitable
1	$\iff$ regular	**all**
2	biregular	bipartite
n	all	all

Computational results for equitable partitions: 2-homogeneous

Lemma

2-homogeneous equitable partition $\Leftrightarrow$ the graph has an automorphism being an *involution without fixed points* (autom of the graph where every vertex is in a pair).



Computational results for equitable partitions: 2-homogeneous

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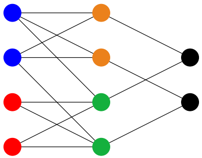

(Lubiw 1981)
Deciding whether a given graph has a fixed-point-free automorphism of order two is NP-complete.

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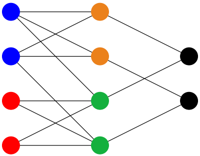

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Corollary

Finding 2-homogeneous equitable partitions is NP-complete.

Overview computational results equitable partitions

Coarsest:

Theorem (Bastert 1999)

The coarsest equitable partition of a graph can be found in polynomial time.

Overview computational results equitable partitions

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Finding 2-homogeneous equitable partitions is NP-complete.

Extension to weight-equitable partitions?

2-homogeneous equitable partition NP-completeness
Cannot extend proof unless the partition also happens to be equitable.

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Question: When does this happen?

Maybe a polynomial algorithm for some graph class?

We just saw that in general, we cannot decide in polynomial time whether a graph admits an equitable partition with cells of size two.

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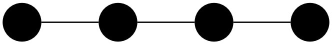
Question: Are there graph classes for which we can obtain an efficient algorithm to compute such 2-homogeneous equitable partitions?

A small example



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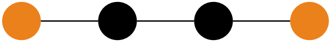
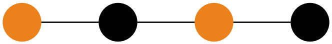


Equitable
(and hence weight-equitable)

A small example



Not equitable
(but weight-equitable)

Graphs without P_4?

Cographs

Graphs without <u>induced</u> P_4

Cographs

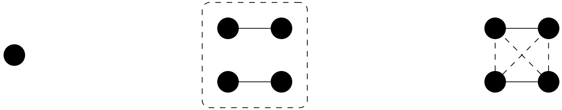
Graphs without <u>induced</u> P_4

(Corneil, Lerchs, Burlingham 1981)

(i) K_1 is a cograph

(ii) If $G_1, \ldots, G_k$ cographs, then $G_1 \cup \cdots \cup G_k$ cograph

(iii) The join of cographs is a cograph



(i) (ii) (iii)

Cographs

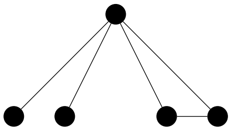
Graphs without <u>induced</u> P_4

Proposition (A., Hojny, Zeijlemaker 2022)
In cographs, all 2-homogeneous weight-equitable partitions are equitable.

Goal: devise algorithm to find 2-homogeneous weight-equitable partitions of cographs.

Representing cographs with trees

Cotree: Represent union with 0, join with 1



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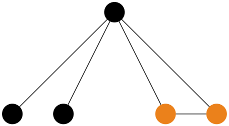

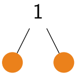

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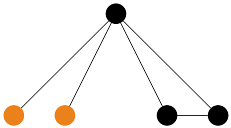

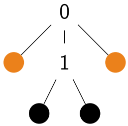

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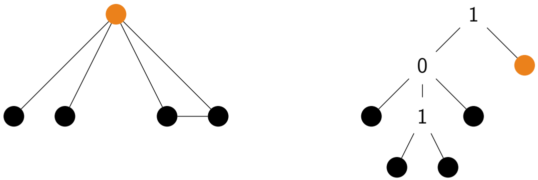

Note that leaves of the cotree are the vertices in the cograph.

(Corneil, Lerchs, Burlingham 1981)
If 0 and 1 alternate, this tree is unique.

Representing cographs with trees

Lemma (A., Hojny, Zeijlemaker 2022)
Automorphism ϕ of cograph $G \quad \Leftrightarrow \quad$ automorphism of the cotree which:

- acts as the original ϕ on the leaves,
- respects the $0/1$-labeling.

Representing cographs with trees

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Automorphism ϕ of cograph G $\Leftrightarrow$ automorphism of the cotree which:

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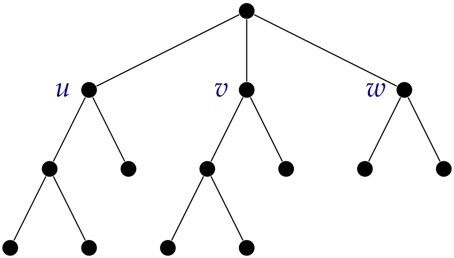
This allows us to translate the problem of finding 2-homogeneous partitions to the problem of finding automorphisms in a tree ...

Computing 2-homogeneous equitable partitions: intuition

2-homogeneous equitable partition
$\xLeftrightarrow{\text{Lubiw}}$ involution of the graph without fixed points (automorphisms of order 2 in the graph)
$\xLeftrightarrow{\text{Lemma}}$ automorphism of cotree which is a fixed-point-free involution on the leaves.



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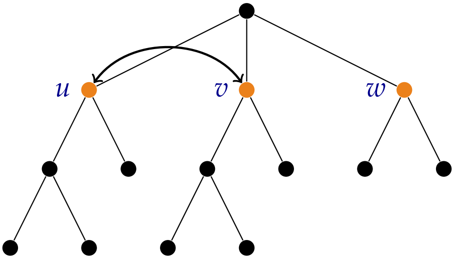

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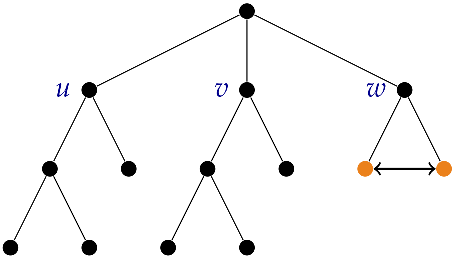

Computing 2-homogeneous equitable partitions: algorithm

Input : Labeled (co)tree T, root vertex r

for *each child of* r *with distinct subtree* **do**
    if *an odd number of children have the same subtree* **then**
        if *the child is a leaf* **then**
            return false
        else
            recurse
return true

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Edmonds' algorithm

Computing 2-homogeneous equitable partitions: algorithm

Edmonds' algorithm (Busacker, Saaty 1965)
Algorithm for detecting isomorphic subtrees.

(Colbourn and Booth 1981)
Linear time extension of Edmonds' algorithm.

Theorem (A., Hojny, Zeijlemaker 2022)
Let G be a cograph. The problem of deciding whether G admits a (weight-) equitable partition with $\frac{n}{2}$ cells of size 2 can be solved in $O(n^2)$ time.

Closing remarks

Open problems: algebraic flavour

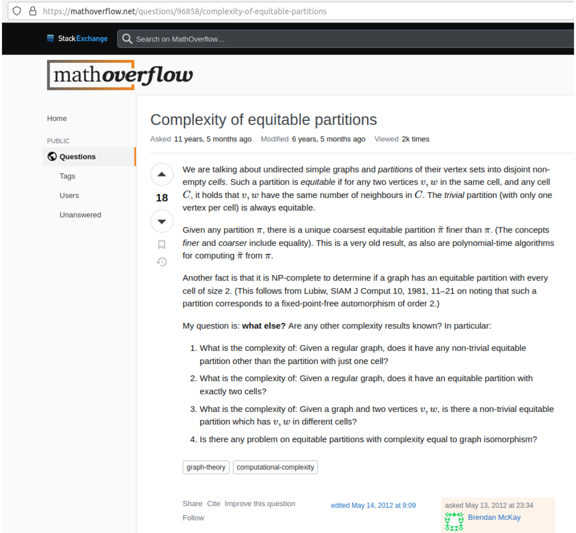
- New characterizations of weight-equitable partitions.
- Find new applications of weight(-equitable) partitions.
- Characterize graphs meeting Hoffman's bound improvement:

  (A. 2019) *If the vertex partition defined by the χ color classes is not weight-equitable, then*

$$\chi(G) \geq 2 - \frac{\lambda_1}{\lambda_n}.$$

Open problems: algorithmic flavour

- Complexity of (weight-)equitable partitions?



https://mathoverflow.net/questions/96858/complexity-of-equitable-partitions

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Complexity of equitable partitions

Asked 11 years, 5 months ago Modified 6 years, 5 months ago Viewed 2k times

18

We are talking about undirected simple graphs and *partitions* of their vertex sets into disjoint non-empty *cells*. Such a partition is *equitable* if for any two vertices v, w in the same cell, and any cell C, it holds that v, w have the same number of neighbours in C. The *trivial* partition (with only one vertex per cell) is always equitable.

Given any partition π, there is a unique coarsest equitable partition $\bar{\pi}$ finer than π. (The concepts *finer* and *coarser* include equality). This is a very old result, as also are polynomial-time algorithms for computing $\bar{\pi}$ from π.

Another fact is that it is NP-complete to determine if a graph has an equitable partition with every cell of size 2. (This follows from Lubiw, SIAM J Comput 10, 1981, 11–21 on noting that such a partition corresponds to a fixed-point-free automorphism of order 2.)

My question is: **what else?** Are any other complexity results known? In particular:

1. What is the complexity of: Given a regular graph, does it have any non-trivial equitable partition other than the partition with just one cell?
2. What is the complexity of: Given a regular graph, does it have an equitable partition with exactly two cells?
3. What is the complexity of: Given a graph and two vertices v, w, is there a non-trivial equitable partition which has v, w in different cells?
4. Is there any problem on equitable partitions with complexity equal to graph isomorphism?

graph-theory computational-complexity

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edited May 14, 2012 at 9:09

asked May 13, 2012 at 23:34

Brendan McKay

Open problems: algorithmic flavour

- Algorithms for more general graph classes (A., Coutinho, Silva, Zeijlemaker 2025++).

- Algorithm to compute the coarsest weight-equitable partition? (extending Bastert)

Thank you for your attention!

a.abiad.monge@tue.nl

Further reading:

A. Abiad
A characterization and an application of weight-regular partitions of graphs
Linear Algebra and Appl. 569 (2019).

A. Abiad, C. Hojny, S. Zeijlemaker.
Characterizing and computing weight-equitable partitions of graphs
Linear Algebra and Appl. 645 (2022).