Graph products, prime factorization, unique roots, and cancellation

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First Part

Prime factorizations of finite and infinite connected graphs
with respect to the
hierarchical and the generalized hierarchical product of graphs.

Second Part

Unique root and cancellations properties
of finite and infinite disconnected graphs
with respect to associative products of graphs.

Acknowledgement

First Part

Most results about finite graphs are joint work with
Rafał Kalinowski, and Monika Pilśniak,
AGH Cracow, Poland,
and the results on infinite homogeneous trees were obtained together with
Juliana Palmen, Gabriela Makar and Piotr Zajac,
students AGH Cracow, Poland.

Second Part

This part is joint work with
Daniel Smertnig and Igor Klep,
University of Ljubljana, Slovenia.

What is a product of graphs?

Well known products are
the Cartesian, the strong, the direct and the lexicographic product.

Speaking of a product, we may mean the operation of forming a new graph, or the result of such an operation.

The mentioned products are binary operations.
If $*$ denotes such an operation, then, given two graphs G, H, their product $H * G$ is a new graph formed by certain rules.

But, there are also multiary products, by which a new graph is formed from several factors. Examples are the join and the X-join of graphs.

Reasons to study graph products

Many well known graphs are either products of graphs or subgraphs thereof. Hypercubes, Hamming graphs, lattices, and grid graphs are products.

Benzenoid graphs are isometric subgraphs of hypercubes. Median graphs not only subgraphs, but even retracts of hypercubes, that is, images of hypercubes under idempotent endomorphims. These facts help in studying them.

Many products have interesting properties with respect to their automorphism groups, spectra, chromatic numbers and other graph parameters.

Products are used to construct graphs with given properties.

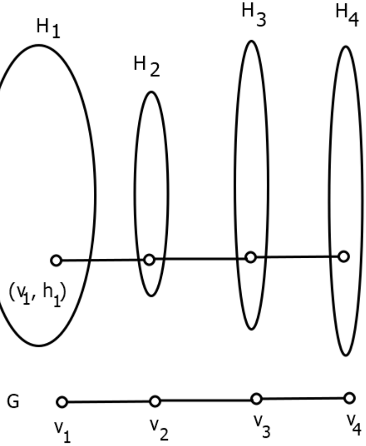
There are many open problems pertaining to products, for example Vizing's conjecture on the domination number of Cartesian products.

Example 1: A multiary product of Godsil and McKay

Given a graph G on n vertices $v_1, \ldots, v_n$, and rooted graphs $H_1[h_1], \ldots, H_n[h_n]$, then the product

$$G \times \bigcup_{i=1}^{n} H_i[h_i]$$

of Godsil and McKay is formed by identifying each v_i with h_i.



Example 2: The hierarchical product

If all $H_i[h_i]$ are isomorphic to $H[h]$, then we obtain the *hierarchical product*, or simply *h-product*

$$G \sqcap H[h],$$

of Barriere, Comellas, Dalfó, and Fiol from 2009.

The h-product is a special case of the product of Godsil and McKay.

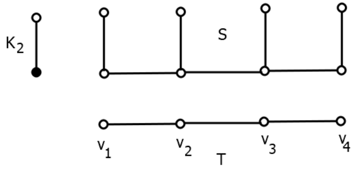
Spectral characterization of $T \sqcap K_2$

Godsil & McKay call a polynomial $p(\lambda)$ of order $2n$ symmetric if

$$p(\lambda) = (-1)^n \lambda^{2n} p\left(\frac{1}{\lambda}\right), \text{ e.g. } p(\lambda) = \lambda^8 - 9\lambda^6 + 16\lambda^4 - 9\lambda^2 + 1.$$

They proved: To any finite tree S, there exists a tree T such that $S \cong T \sqcap K_2[0]$

iff the characteristic polynomial of S symmetric.



Characteristic Polynomial

If G is a graph and $A(G)$ its adjacency matrix, recall that

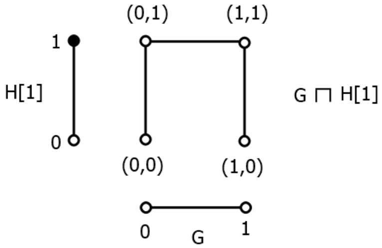
$$p(\lambda) = \mathrm{Det}(\lambda I - A(G))$$

is the characteristic polynomial of G.

The roots of $p(\lambda)$ are eigenvalues of G, and the set of eigenvalues is the spectrum of G.

The first papers on the h-product concentrated on its spectral properties. We study its prime factorizations.

Properties of the h-product



The product $G \sqcap H[1]$ and G are unrooted, but $H[1]$ has a root.

$$G \sqcap K_1[1] \cong G; \quad K_1[1] \text{ is a right unit,}$$
$$K_1 \sqcap G[v] \cong G; \quad K_1 \text{ is a left unit.}$$

The h-product is neither commutative nor associative.

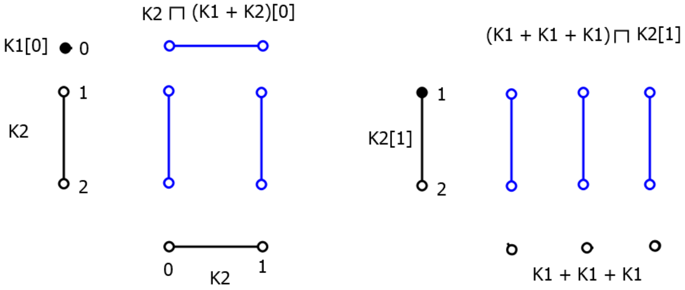
$G \neq K_1$ is called prime if it is not a product of two graphs A, B with $|A|, |B| \geq 2$.

Properties of the h-product

Non-unique prime factorization for disconnected graphs[1]

Let $V(K_1) = \{0\}$, $V(K_2) = \{1, 2\}$. Then

$$K_2 \sqcap (K_1 + K_2)[0] \cong (K_1 + K_1 + K_1) \sqcap K_2[1].$$



[1] Anderson, Guo, Tenney, and Wash, 2017

Properties of the h-product

Unique first prime factors for finite connected graphs.

Theorem *To any finite connected graph $X \neq K_1$ there exist a unique prime graph G and a unique rooted graph $H[v]$ such that*

$$X = G \sqcap H[v].$$

The subgraph of X induced by $\{(g, v) \mid g \in V(G)\}$, which is isomorphic to G, is invariant under all automorphisms of X.

Corollary $\mathrm{Aut}(G) \ltimes \mathrm{Aut}(H[v])$ *is the full automorphisms group of X.*

h-products of several factors

Recall: $X = G \sqcap H[v]$
We rewrite it as $X = G_1 \sqcap X_2[x_2]$

Theorem *Each finite, connected graph $X \neq K_1$ has a unique standard prime factorization of the form*

$$X = G_1 \sqcap (\, G_2 \sqcap \, (\, G_3 \sqcap (\cdots \sqcap \, G_k[x_k]\,)[x_{k-1}] \cdots)[x_3])[x_2],$$

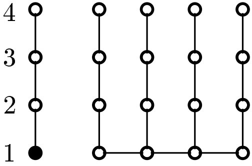
where the prime factors $G_1, \ldots, G_k$ and the roots $x_2, \ldots, x_k$, are uniquely determined up to isomorphisms.

There may be other presentations as products of prime graphs: prime factorizations. For example

$$K_2 \sqcap \Big(K_2 \sqcap \, (K_2 \sqcap K_2[1])[v]\Big)[w] = (K_2 \sqcap K_2[1]) \sqcap \, (K_2 \sqcap K_2[1])[v].$$

Example

$P_4[1]$

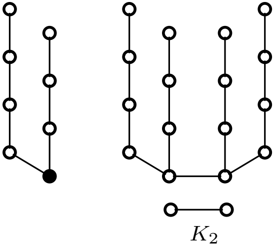


$(K_2 \sqcap K_2[1]) \sqcap P_4[1]$

$P_4 = K_2 \sqcap K_2[1]$

Non-standard prime factorization

$(K_2 \sqcap (K_2 \sqcap K_2[1])\,[0,0])\,[0,0]$

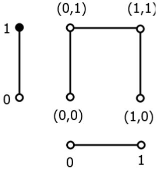


Standard prime factorization

Remark on non-associativity

One might think, given a product, that one can always reset parentheses by choosing appropriate roots. This is not true:

Let $P_4 = 1234$ and $K_2 = 01$. Recall $P_4 \cong K_2 \sqcap K_2[1]$.



Then

$$K_2 \sqcap P_4[1] = K_2 \sqcap (K_2 \sqcap K_2[1])[0,0] \ncong (K_2 \sqcap K_2[1]) \sqcap K_2[1] = P_4 \sqcap K_2[1],$$

because $K_2 \sqcap P_4[1]$ has two vertices of degree 1, but $P_4 \sqcap K_2[1]$ has four.

Condition for associativity

We have

$$(A \sqcap B[b]) \sqcap C[c] = A \sqcap (B \sqcap C[c])[(b, c],$$

but

$$(A \sqcap B[b]) \sqcap C[c] \neq A \sqcap (B \sqcap C[c'])[(b', c'']$$

unless $b = b'$ and $c = c' = c''$.

Furthermore, given a graph X with

$$X = (G_1 \sqcap G_2[g]) \sqcap H_2[h],$$

then

$$X = G_1 \sqcap (G_2 \sqcap H_2[h])[g, h],$$

and if G_1 and G_2 are prime, then they are unique.

Theorem *In each non-standard prime factorization of a connected graph with respect to the h-product, the prime factors are, considered as unrooted graphs, the same as in the standard representation. Only the roots will be different.*

Admissible bracketings

Suppose we are given a prime factorization of a connected graph G in standard form, and a prime factorization of G in non-standard form.

By the previous observations, one can show that the prime factors are the same as unrooted graphs, and appear in the same order.

We can also determine whether a given bracketing is admissible.

The number of binary bracketings of n letters are the Catalan numbers C_{n-1}, where $C_n = \frac{1}{n+1}\binom{2n}{n}$.

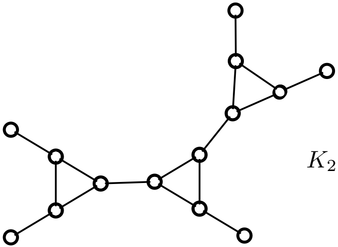
Algorithm for finite trees

Let T be a finite tree of order n. Then, the standard prime factorization of T with respect to the hierarchical product can be computed in $O(|T|^{3/2})$ time.

For the proof it helps that the center of T is always contained in the first factor, and that isomorphism testing of trees is linear in their size.

Problem *Can one reduce the complexity of finding the standard prime factorization of trees? Is it linear?*

Infinite graphs as hierarchical products



$K_2 * C_3$

Free product of K_2 by C_3

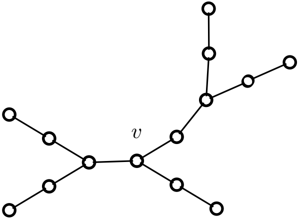
It is an h-product of the form $K_3 \sqcap H[1]$, where 1 is the vertex of degree 1 in the factor H.
If one removes the pendant edge from H one obtains a graph H', and clearly

$$K_3 \sqcap H[1] = K_2 \sqcap H'[2],$$

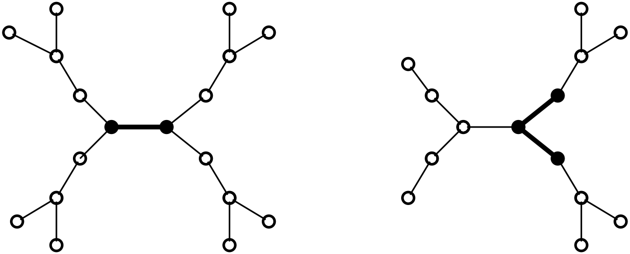
where 2 is the vertex of degree 2 in H'.

Hence, the first prime factors need not be unique.

Nonunique first prime factors for trees



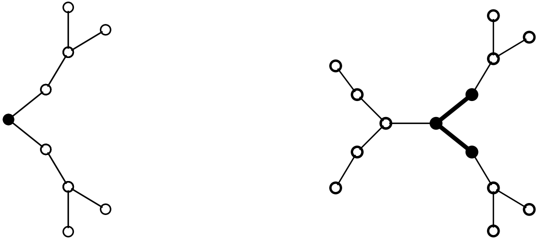
The tree Z



Two different first prime factors for Z: K_2 and P_3

Condition for unique first prime factors

Let $Z = K_2 \sqcap H[2]$ and $Z = P_3 \sqcap H'[1]$.



$H[2]$ is shown above, and $H'[1]$ is an isomorphic proper subgraph of $H[2]$.

But, $H[2]$ is also an isomorphic proper subgraph of $H'[1]$.

Theorem *Let*

$$G_1 \sqcap H_1[h_1] \cong G_2 \sqcap H_2[h_2]$$

be connected, locally finite graphs, where G_1 and G_2 are prime. If neither of the graphs H_1 or H_2 is a proper isomorphic subgraph of the other, then $G_1 \cong G_2$.

Homogeneous trees of finite degree have unique prime factorizations

Let T_n be the infinite tree where each vertex has degree n, and
T_n^1 be the tree where one vertex has degree 1 and all others degree n.

Note that $T_1^1 = K_2$, and that
T_2^1 is a one-sided infinite path, also called a ray, often denoted by R.

Theorem[2]. *For each $n \in \mathbb{N}$, the regular infinite tree T_n has a unique presentation as a hierarchical product in the standard form. It is given by the formula*

$$T_n = T_1^1 \sqcap (T_2^1 \sqcap (\cdots \sqcap T_n^1[v_n])[v_{n-1}]) \cdots)[v_2],$$

where each root v_i, $2 \le i \le n$, is the unique vertex of degree $n - i + 1$ in the product of the last $n - i + 1$ factors.

The proof is by induction, beginning with $T_2 = K_2 \sqcap R[1]$.

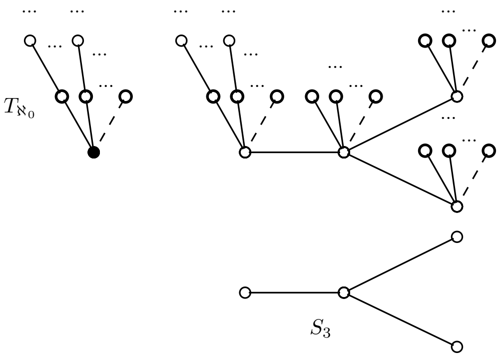
Note that the embedded K_2 in T_n is not invariant under automorphisms, contrary to the finite case.

[2] Result with Juliana Palmen, Piotr Zajac, and Gabriela Makar

What happens if we admit infinite degrees?

$$T_{\aleph_0} = K_2 \sqcap T_{\aleph_0}[0].$$

In fact $T_{\aleph_0} = T \sqcap T_{\aleph_0}[0]$ for any countable tree T.

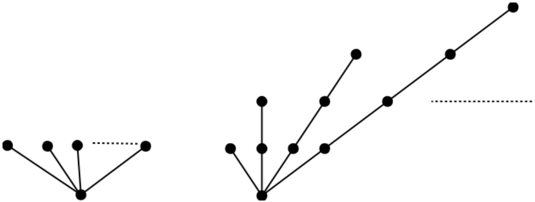


$$S_3 \sqcap T_{\aleph_0} \cong T_{\aleph_0}$$

Rayless graphs

By König's Lemma each infinite connected locally finite graph has a ray. Hence, a rayless graph must have a vertex of infinite degree.

Below are examples of infinite rayless trees.



Infinite rayless trees also have a unique center consisting of a single edge or a single vertex, as shown in 1994 by Polat and Sabidussi.
But, it also follows from a result of Schmidt 1983, who showed that rayless graphs have a uniquely defined finite core, and this core is invariant under automorphisms.
In many ways rayless graphs are similar to finite graphs.

Theorem *Each connected rayless graph G has a standard prime factorization of the form*

$$X = G_1 \sqcap (G_2 \sqcap (G_3 \sqcap (\cdots \sqcap G_k[x_k])[x_{k-1}] \cdots)[x_3])[x_2],$$

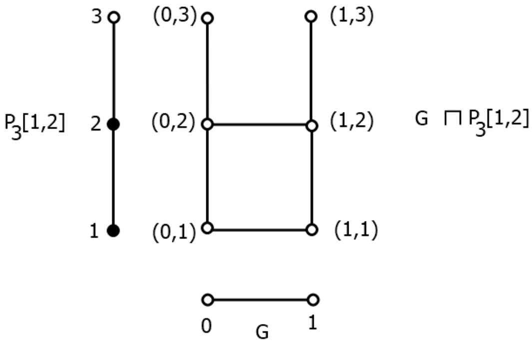
where the prime factors $G_1, \dots, G_k$ and the roots $x_2, \dots, x_k$, are uniquely determined up to isomorphisms.

As in the finite case, $\mathrm{Aut}(X)$ *is the wreath product of the groups of the prime factors.*

The generalized hierarchical product

Given graphs G and H and $U \subseteq V(H)$, the generalized hierarchical product[3] $G \sqcap H[U]$ is an unrooted graph defined on $V(G) \times V(H)$ by

$$G \sqcap H[U] = (G \times U) \cup (V(G) \times H).$$



[3]Barrière, Dalfó, Fiol and Mitjana, 2009

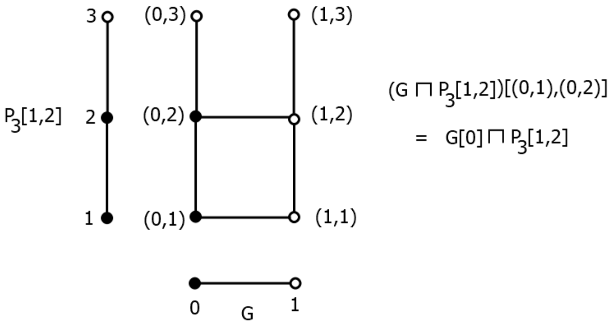
Properties of the generalized hierarchical product

The generalized hierarchical product is a variant of the Cartesian product. It is neither commutative, nor associative.

Special cases are $|U| = 1$, which yields the hierarchical product, and $U = V(H)$, which yields the Cartesian product.
There are examples when the generalized hierarchical product does not have unique prime factorizations.

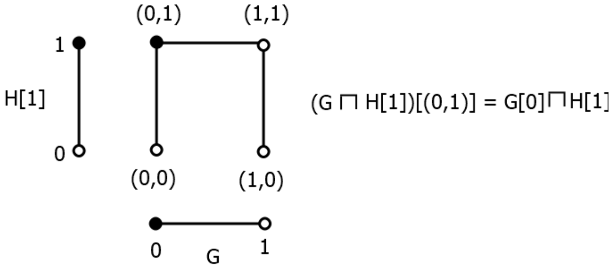
The rooted generalized h-product

Let G, H be graphs, and $U_G \subseteq V(G)$, $U_H \subseteq V(H)$. Then the rooted generalized hierarchical product $G[U_G] \sqcap H[U_H]$ is defined as $(G \sqcap H[U_H])\,[U_G \times U_H]$. It is associative.



Rooted hierarchical product

If all root sets have cardinality 1 we obtain the rooted hierarchical product.



$P_4 = K_2 \sqcap K_2[0]$

$P_4[(0,1)] = G[0] \sqcap H[1]$ is a rooted h-product, but

$P_4[(0,0)]$ is prime with respect to the rooted h-product.

Prime factorization via the rooted generalized hierarchical product

Theorem *Each finite connected rooted graph $G[U]$ is uniquely representable as a rooted generalized hierarchical product*

$$G_1[U_1] \sqcap G_2[U_2] \sqcap \cdots \sqcap G_k[U_k]$$

of rooted prime graphs, where two factors

$$G_i[U_i], G_{i+1}[U_{i+1}], \;\; 1 \leq i < k,$$

commute if they are isomorphic as rooted graphs, or if

$$U_i = V(G_i) \text{ and } U_{i+1} = V(G_{i+1}).$$

Rayless graphs, and extensions to infinitely many factors

We conjecture that infinite connected rayless graphs also have unique prime factorizations with respect to the rooted generalized hierarchical product.

One can also define the product for infinitely many factors. Recall:

The vertices of $X = G_1[U_1] \sqcap \cdots \sqcap G_k[U_k]$ are vectors $x = (x_1, \ldots x_k)$, and two vertices x, y are adjacent if there exists an i, for $1 \leq i \leq k$, such that the following two conditions are satisfied:

(i) x, y differ only in coordinate i, for which $x_i \sim y_i$, i.e. x_i is adjacent to y_i in G_i.

(ii) $x_j = y_j \in U_j$ for all j with $i < j \leq k$.

Hence, given an infinite index set I that is well ordered, and connected graphs $G_\iota, \iota \in I$, one defines $\prod_{\iota \in I} G_\iota$ on the vectors x, with $x_\iota \in V(G_\iota)$, and $x \sim y$ are adjacent when (i) and (ii) are satisfied.

Note that for the special case of the Cartesian product condition (ii) is void, and that I need not be ordered.

The weak rooted generalized hierarchical product

Under no further conditions $\prod_{\iota \in I} G_\iota$ will be disconnected.

But, if one chooses a vertex $a \in V(G_\iota, \iota \in I)$ and requires that each vertex in $\prod_{\iota \in I} G_\iota$ differs from a in only finitely many coordinates, then one obtains a connected component of $\prod_{\iota \in I} G_\iota$, denoted

$$\prod_{\iota \in I}^{a} G_\iota,$$

called the weak rooted generalized hierarchical product.

Of the special case of the weak Cartesian product one knows that it has unique prime factorization. Is this also case now?

What happens if one just requires a linear order?

The monoid of rooted prime graphs

Let $\mathcal{G}$ be the class of rooted finite graphs.

It is closed with respect to the rooted h-product, which is associative, non-commutative, and has a left and a right unit.

Let $X_1, X_2, \ldots$ be the countably many non-trivial, pairwise non-isomorphic, connected, rooted graphs that are prime with respect to rooted hierarchical product.

Each non-trivial, connected $G \in \mathcal{G}$ is a unique product of the X_i.

$$G = X_{g_1} \sqcap X_{g_2} \sqcap \cdots \sqcap X_{g_k}.$$

Recall $X_i \neq X_j$, where $i, j > 1$, and then $X_i \sqcap X_j \ncong X_j \sqcap X_i$.

Free monoids

If we set $\varphi(X_i) = x_i$ we see that $\mathcal{G}$ is isomorphic to the free monoid $\langle X \rangle$, consisting of finite words over a countable alphabet $X = \{x_1, \ldots\}$. That is, of words,

$$x_{i_1} x_{i_2} \cdots x_{i_k},$$

for which multiplication is defined by juxtaposition.

Note that $x_i x_j \neq x_j x_i$ unless $i = j$.

The empty word, denoted by 1, plays the role of the unit element, and corresponds to $K_1[1]$.

Extension to disconnected graphs

Now we extend $\mathcal{G}$ to all finite graphs, where each connected component is rooted. That is, to all finite graphs G of the form

$$G = G_1 + G_2 + \cdots + G_\ell,$$

where the G_i are connected rooted graphs, and where $+$ denotes the disjoint union.

We admit that $G_i = K_1[1]$. If we group isomorphic components together we can present each G, up to the ordering of the factors, uniquely in the form

$$G = a_1 G_1 + a_2 G_2 + \cdots + a_k G_k,$$

with pairwise nonisomorphic connected graphs G_i, and $a_i \geq 1$.

The free algebra $\mathbb{Z}\langle X\rangle$

Recall that $\varphi(X_i) = x_i$, and that each connected rooted graph G corresponds to a unique product of elements of X, say

$$x_{101}^2 x_1 x_7 x_1^3.$$

For finite, connected rooted graphs A, B, C we have the distributive laws

$$(A+B)\sqcap C = A\sqcap C + B\sqcap C \quad \text{and} \quad C\sqcap(A+B) = C\sqcap A + C\sqcap B.$$

Hence, φ extends to a mapping of all graphs G into the free algebra $\mathbb{Z}\langle X\rangle$ over $\langle X\rangle$.

For example, if $G = 3X_6 \sqcap X_3^2 + X_7 + 12X_9 \sqcap X_4^5$, then

$$\varphi(G) = 3x_6x_3^2 + x_7 + 12x_9x_4^5.$$

$\mathbb{Z}^+\langle X\rangle$

We only have positive coefficients and the rooted K_1 corresponds to the empty word, hence φ extends to a bijection between all finite graphs and the elements $\mathbb{Z}^+\langle X\rangle$ of the free algebra $\mathbb{Z}\langle X\rangle$.

The empty graph corresponds to 0 in $\mathbb{Z}\langle X\rangle$.

Unique vs. non-unique factorizations in $\mathbb{Z}\langle X\rangle$ and $\mathbb{Z}^+\langle X\rangle$

It is well known that unique prime factorization holds in $\mathbb{Z}\langle X\rangle$
For example, if $x \in X$, then

$$x^4 + 2x^3 + x^2 + 4x + 4 = (x+1)(x^2 - x + 2)(x+2).$$

But, polynomials that correspond to graphs have no negative coefficients, and so we only obtain the factorizations

$$(x^3 + x + 2)(x+2) = (x+1)(x^3 + x^2 + 4)$$

which are prime factorizations in $\mathbb{Z}^+\langle X\rangle$.
Hence, prime factorization in $\mathbb{Z}^+\langle X\rangle$is not unique.

Unique nth roots and cancellation

But, one still has unique nth roots.
In other words, if G and H are disconnected graphs in which each connected component has a root, then

$$G^n \cong H^n \text{ implies that } G \cong H,$$

where powers are taken with respect to the rooted hierarchical product.
Furthermore, cancellation holds too. That is, if A is a disconnected graph in which each connected component has a root, then

$$G \cong H \text{ if } A \sqcap G \cong A \sqcap H, \text{ or } G \sqcap A \cong H \sqcap A.$$

Other products

The nth root and cancellation properties also hold for:

The generalized rooted hierarchical product

The Cartesian product[4]

The Cartesian product of graphs with loops

The strong product

The direct product of non-bipartite graphs, where loops are allowed[5]

The lexicographic product, despite the fact that prime factorization is not unique for connected graphs[6]

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[4] Fernández, Leighton, and José Luis López-Presa, 2007

[5] Lovász, 1971, showed the n-th root property without restriction to bipartite graphs, and cancelation if there are homomorphisms from G and H to A.

[6] Imrich, 1972, direct proof.

Power series rings

The new results about nth roots and cancellation are joint work with Daniel Smertnig and Igor Klep.

With the exception of the lexicographic product, we can extend these results to power series rings.

In other words, we can admit infinitely many connected components, where each component has finite multiplicity.

The semistrong product of graphs

The rooted hierarchical products and the other products considered are associative.

But a few days ago a non-associative product called my attention that also might have the nth root and the cancellation property.

It is the semistrong product $G \overline{\underline{\times}} H$. It was already defined 1976 by Ringeisen et al., but not investigated with respect to prime factorization.

It is defined as the strong product $G \boxtimes H$, from which all layers with respect to the second factor are removed.

But it can also be defined as

$$G \times H^o,$$

where $\times$ denotes the direct product, and where H^o is formed from H by adding loops to all vertices.

The fact that non-bipartite graphs have the unique prime factorization property with respect to the direct product might help in further investigations.

Orphan product

The semistrong product was called an orphan product, but there are many products of its kind.

Let G and H be graphs. To define a product $G * H$ on $V(G) \times V(H)$, we have to define whether two vertices $(x_g, x_h), (y_g, y_h) \in V(G * H)$ are adjacent or not.

When vertices x, y in a graph are adjacent, we write $x\mathrm{E}y$, when $x = y$ we write $x\Delta y$, otherwise $x\mathrm{C}y$.

The relation between $(x_g, x_h), (y_g, y_h)$ is then defined by the relation between x_g, y_g, and the relation between x_h, y_h.

$G * H$	E	Δ	C
E	$\bullet$	$\bullet$	$\bullet$
Δ	$\bullet$	Δ	$\bullet$
C	$\bullet$	$\bullet$	$\bullet$

Each $\bullet$ is either E or C.

Orphan products

For the strong and the semistrong product we have the following tables

$$\begin{pmatrix} E & E & C \\ E & \Delta & C \\ C & C & C \end{pmatrix}$$

Strong product

$$\begin{pmatrix} E & C & C \\ E & \Delta & C \\ C & C & C \end{pmatrix}$$

Semistrong product

Clearly there are 256 such products. 20 are associative, 10 of them depend on the structure of both factors. The remaining products are not associative.
Note that a product depends on the structure of both factors if the first and the third column are different, as well as the first and the third row.

There are around 150 non-associative products that depend on the structure of both factors, just like the semistrong product. It may be an orphan, but it is not alone.

Many of them may be worth studying with respect to graph parameters of the product in dependence of the values for the factors, and for their algebraic properties.

More possibly interesting weak products

The 10 associative products that depend on the structure of both factors are:

the lexicographic product $A \circ B$, which is not commutative, and $B \circ A$.

Then there are 8 commutative ones, of which 4, namely the Cartesian $\square$, the direct $\times$, the strong $\boxtimes$ and the modular $\bigtriangledown$ are well known.
The other four are their complementary products. If $*$ is a product, then its complementary product $\bar{*}$ is defined by

$$A \bar{*} B = \overline{\bar{A} * \bar{B}},$$

where $\bar{G}$ denotes the complementary graph of G.

For these complementary products, and the lexicographic product, one can define "weak" versions, some of which may be worth studying.

Thank you for your attention!