Disk Graphs and Transmission Graphs—Recent Developments

Wolfgang Mulzer

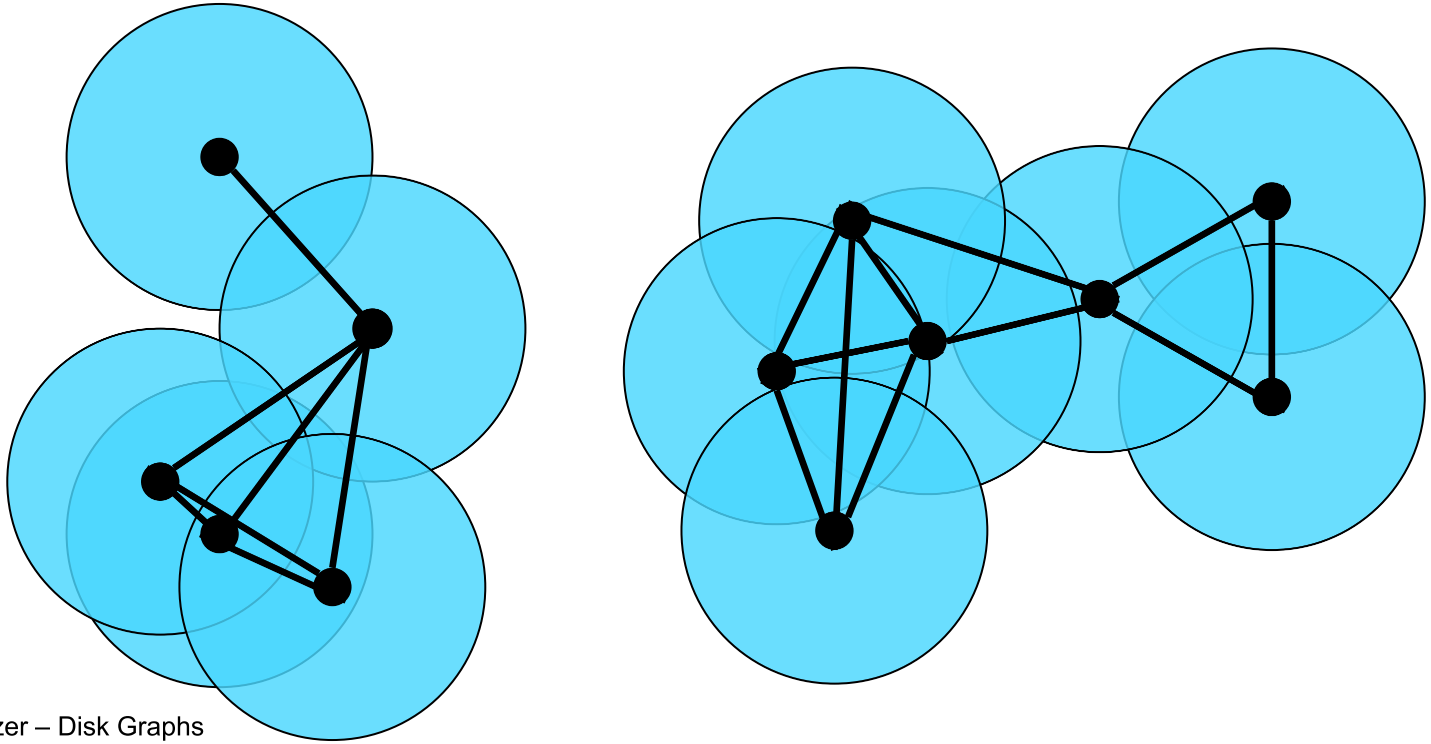
Disk Graphs

n sites in the plane

each site p has an associated radius r_p

sites p, q are adjacent iff $|pq| \leq r_p + r_q$

undirected



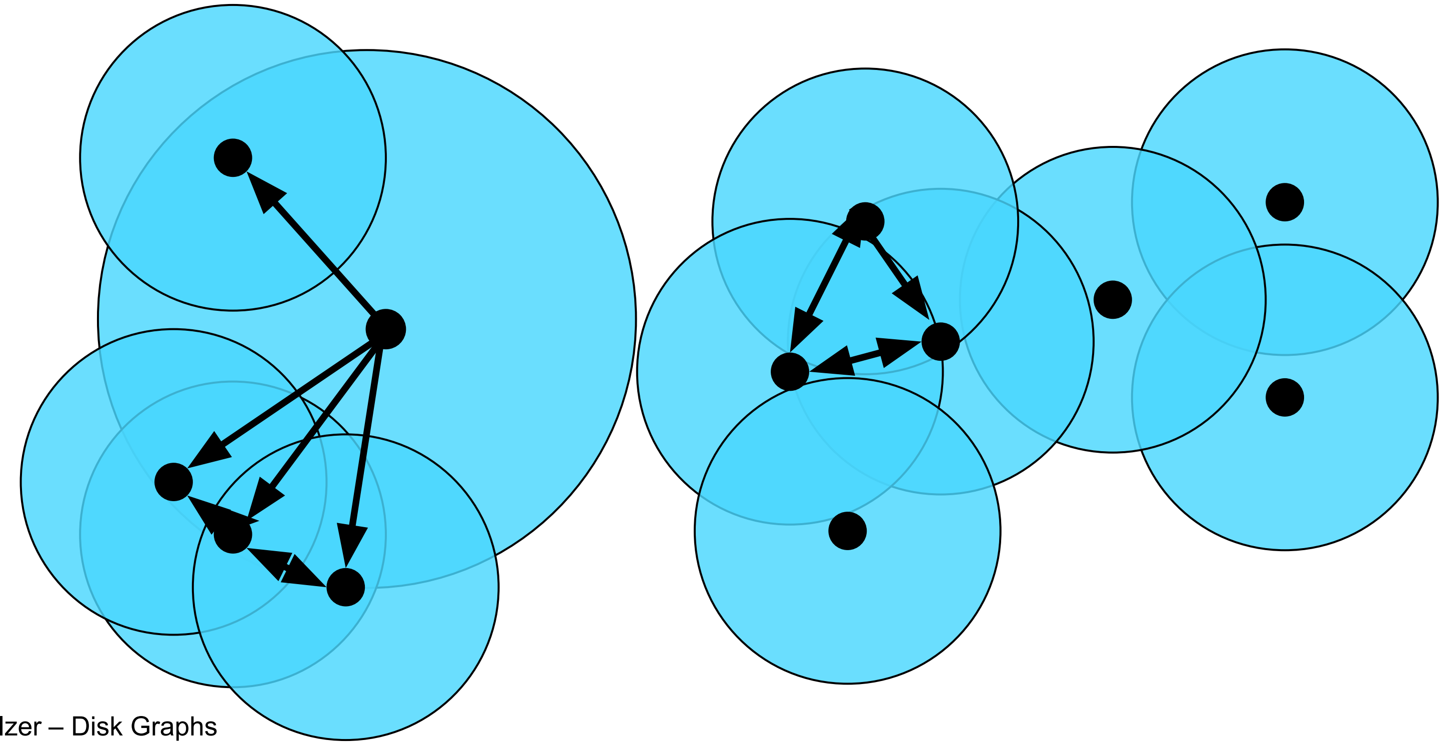
Transmission Graphs

n sites in the plane

each site p has an associated radius r_p

edge from p to q iff $|pq| \le r_p$

directed



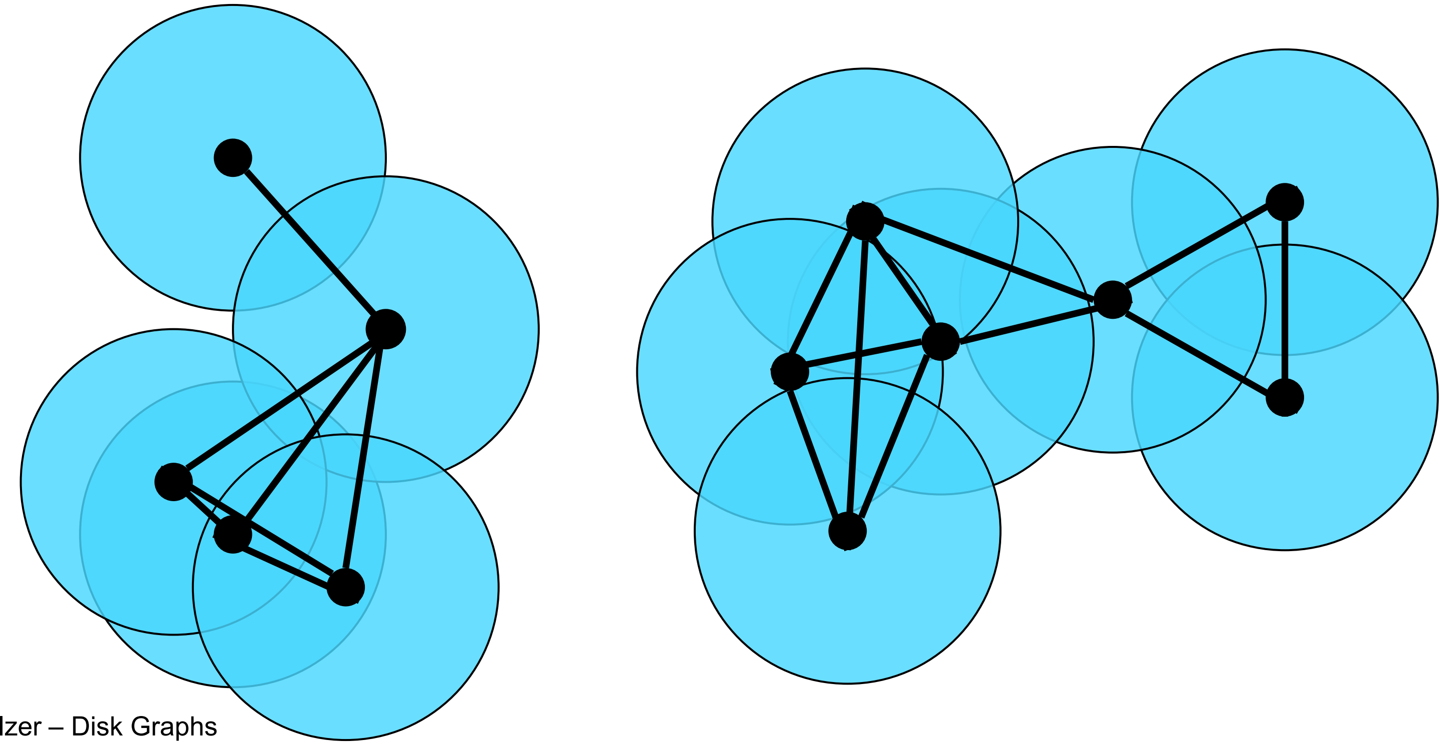
Disk Graphs and Transmission Graphs

natural model for geometrically defined graphs

can be dense and contain large cliques

However: description is sparse

Question: Do geometry and sparse description help?

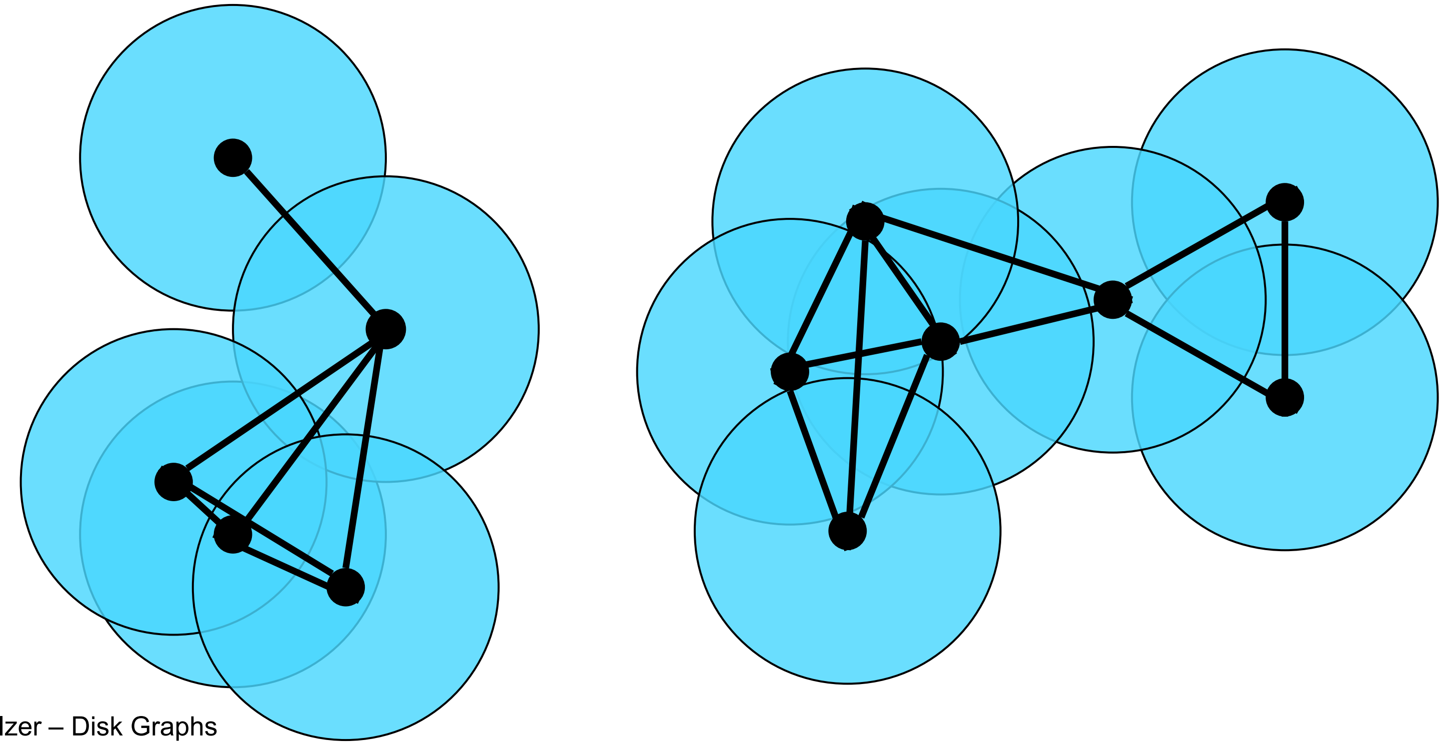


Three Examples

dynamic connectivity

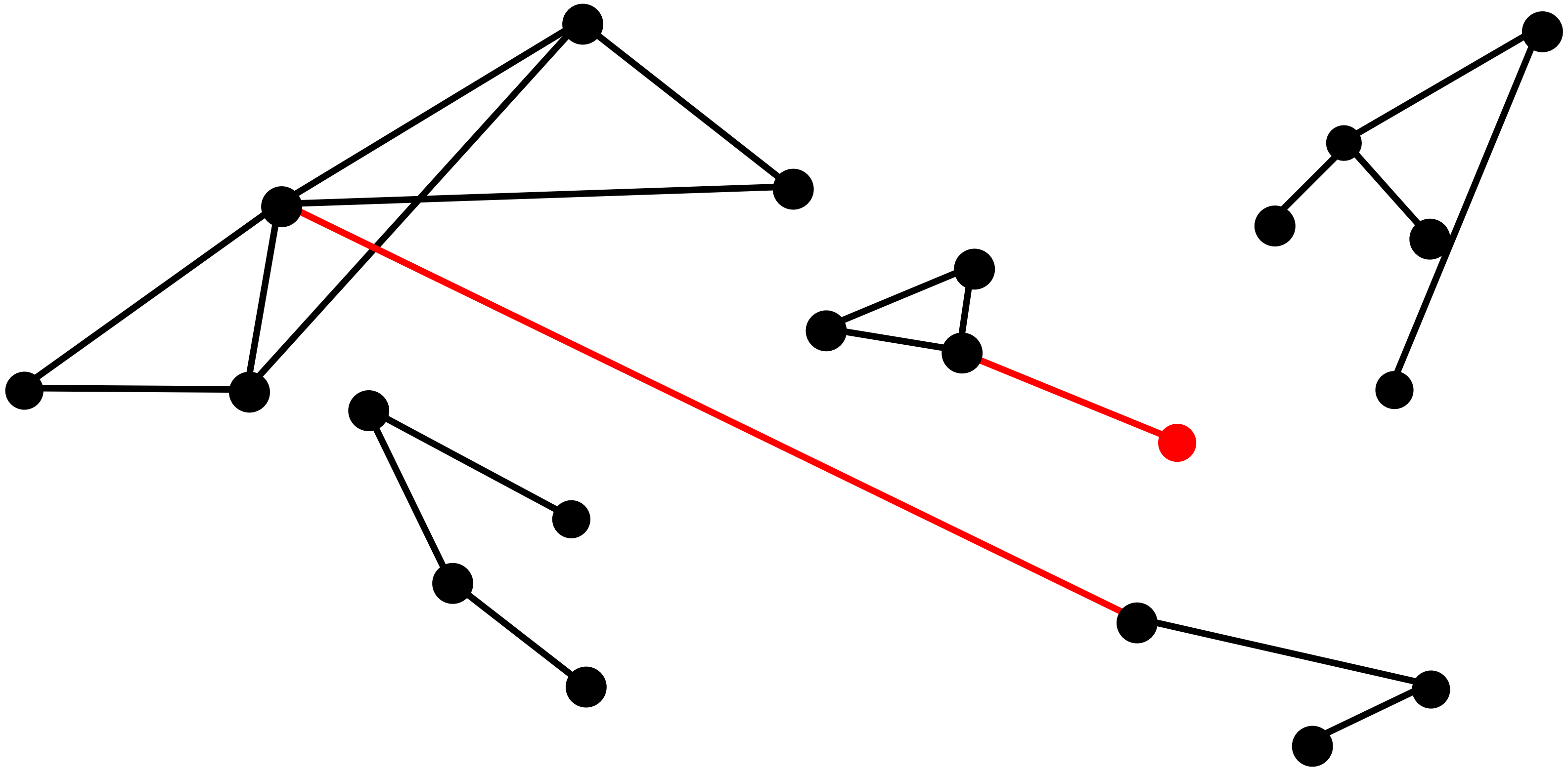
girth

maximum matching



Dynamic Connectivity

dynamic graph: insert, delete, connected?



Dynamic Connectivity

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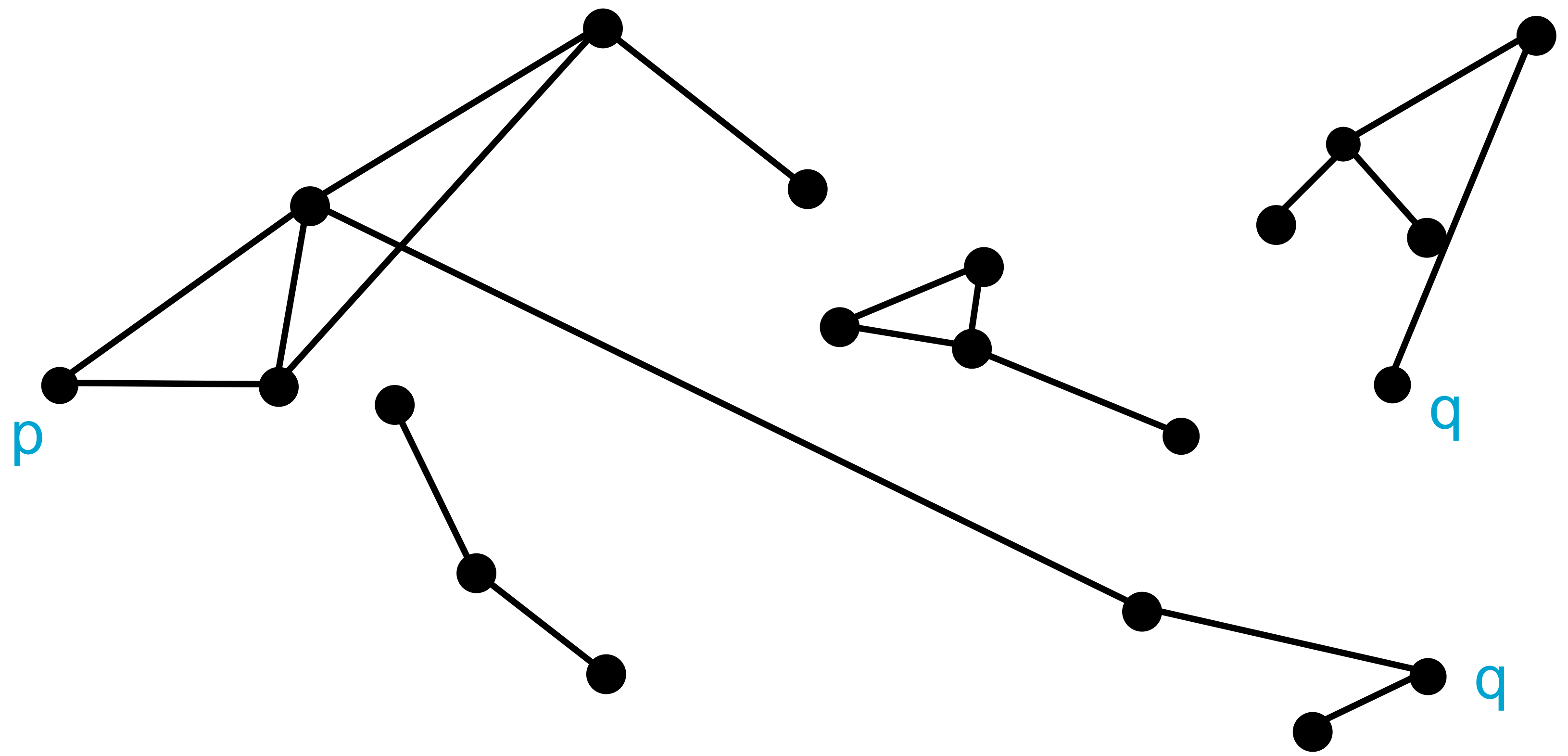
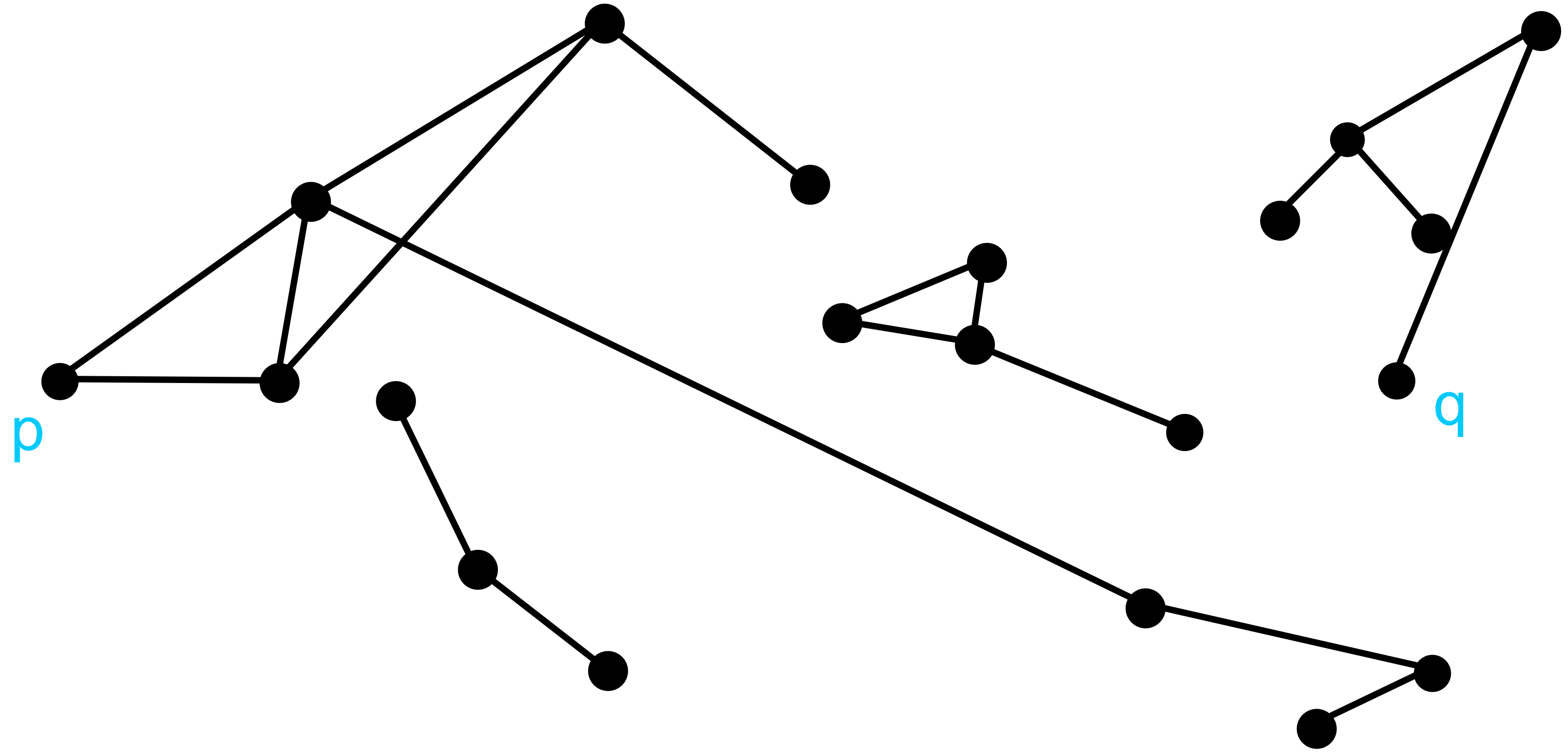


Dynamic Connectivity

general graphs: $O(\log^2 n)$ update, $O(\log n / \log\log n)$ query [Holm et al.]

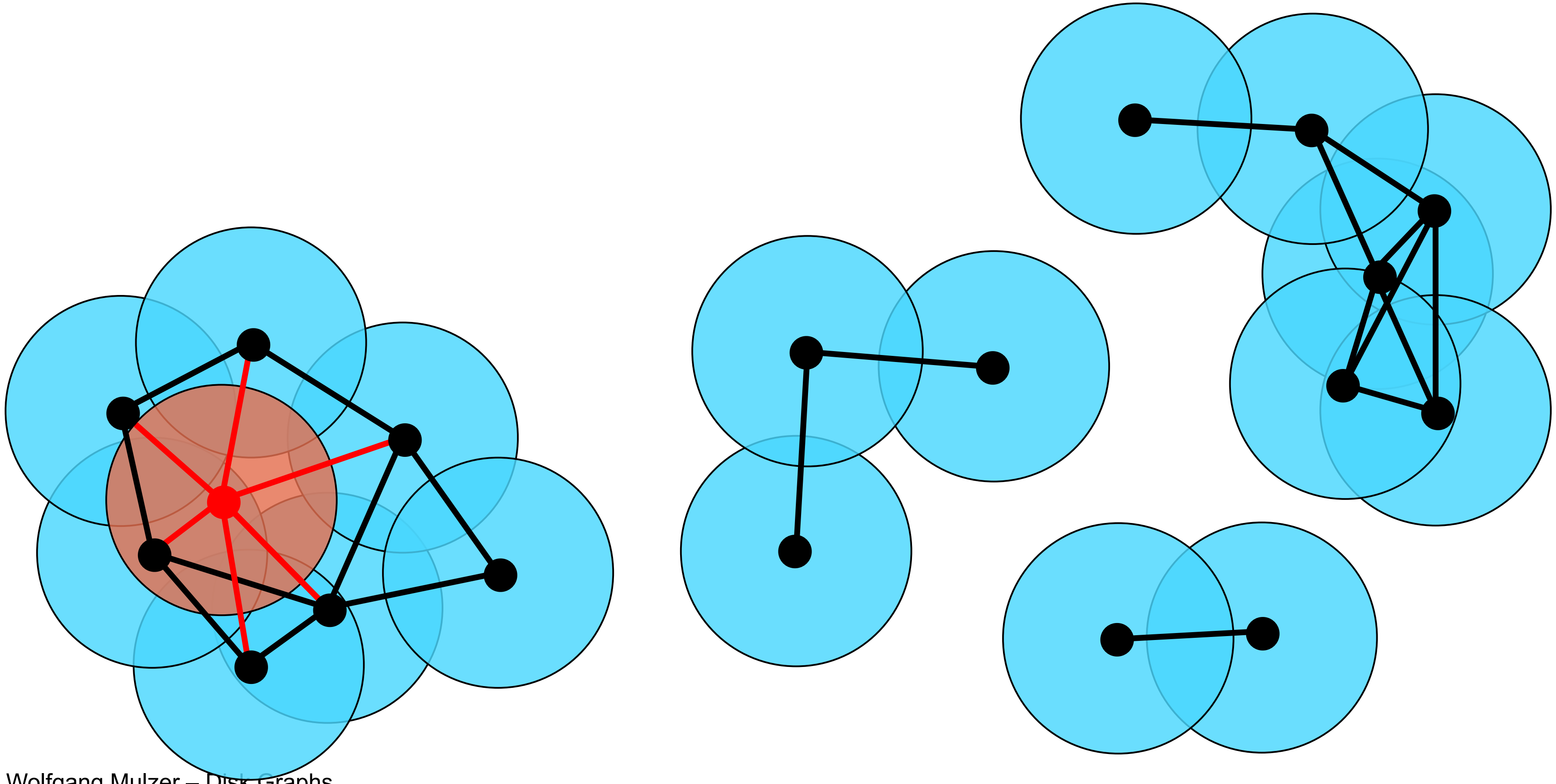
planar graphs: $O(\log n)$ update, $O(\log n)$ query [Eppstein et al.]

unit disk graphs?



Dynamic Connectivity in Unit Disk Graphs

dynamic set $P \subset \mathbb{R}^2$, $|P| = n$



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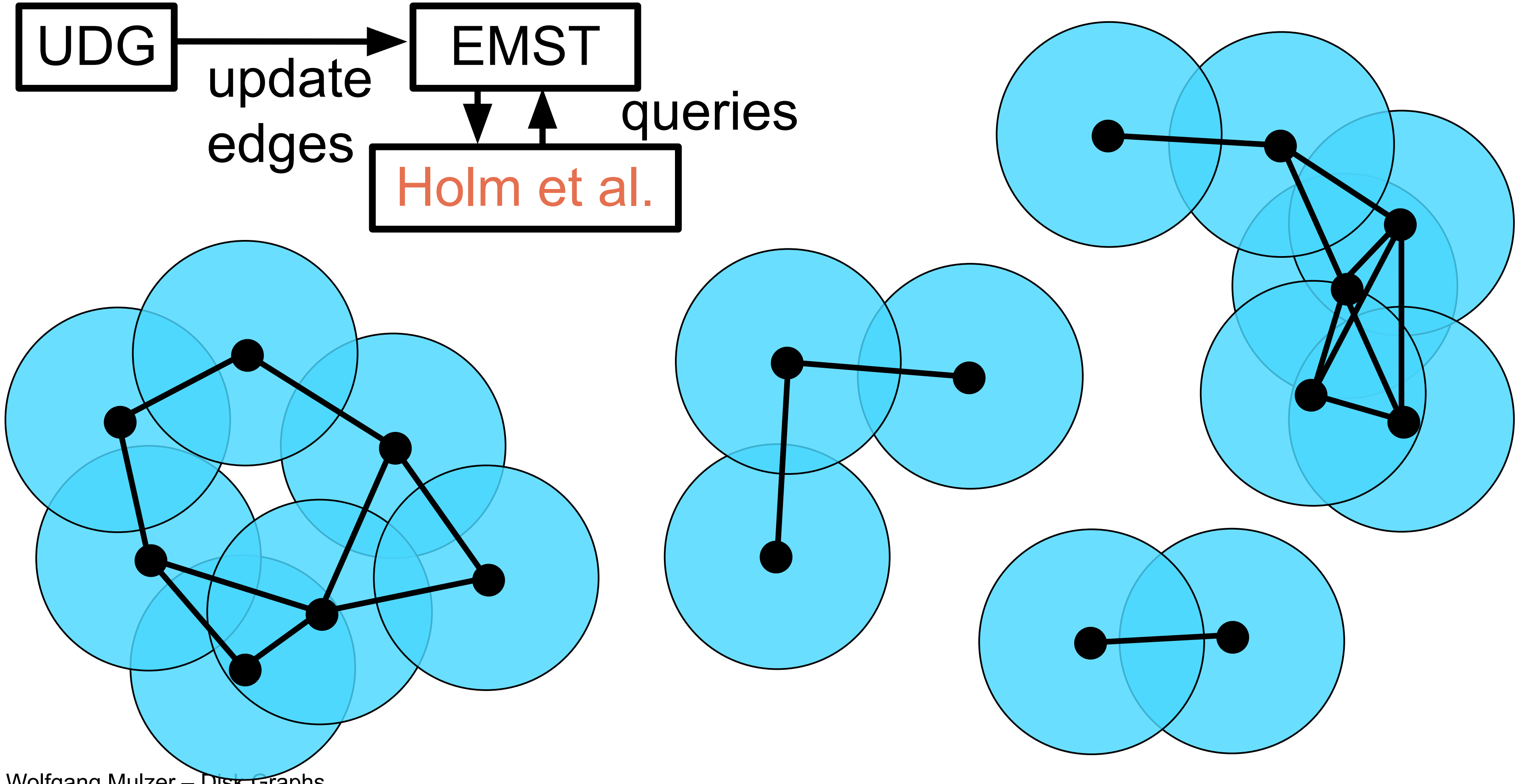
Chan et al: $O(\log^{10} n)$ updates, $O(\log n / \log\log n)$ queries



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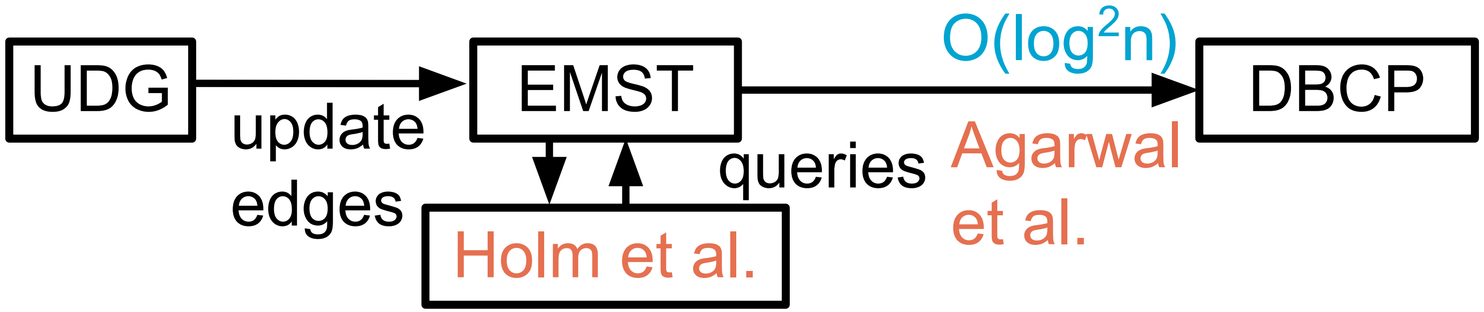
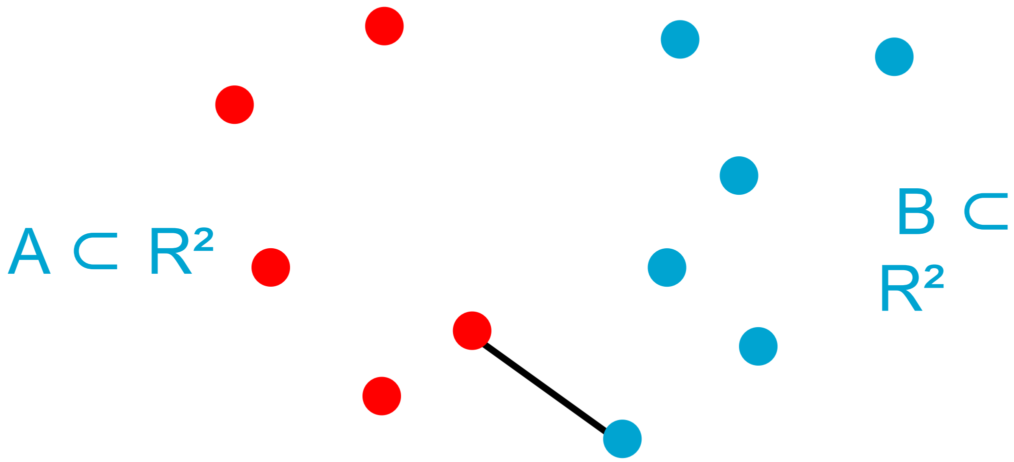




Dynamic Connectivity in Unit Disk Graphs

dynamic set $P \subset R^2$, $|P| = n$

Chan et al: $O(\log^{8} m)$ updates, $O(\log m/\log\log m)$ queries

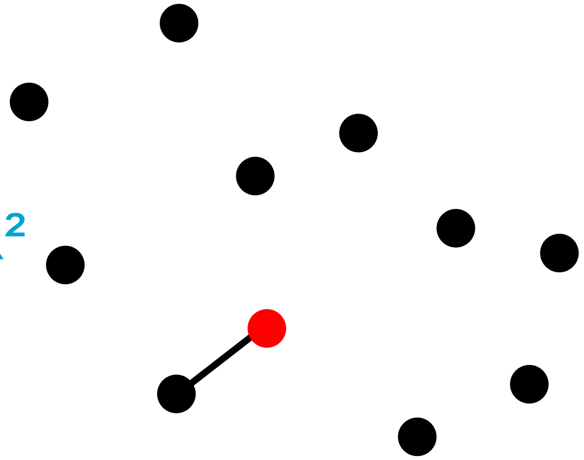
Chan et al: $O(\log^{10} m)$ updates, $O(\log m/\log\log m)$ queries

Chan: $O(\log^6 n)$

Kaplan et al.: $O(\log^5 n)$

Chan: $O(\log^4 n)$

$P \subset R^2$



Dynamic Connectivity in Unit Disk Graphs

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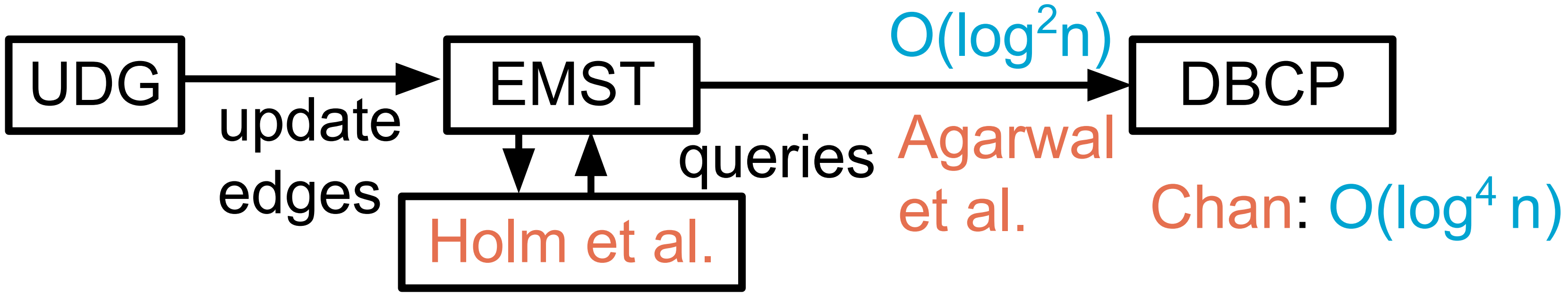
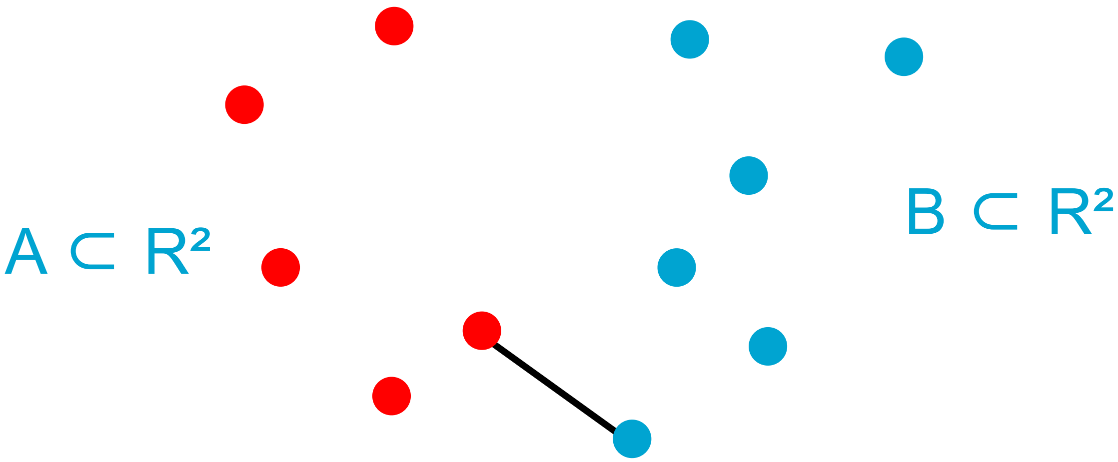
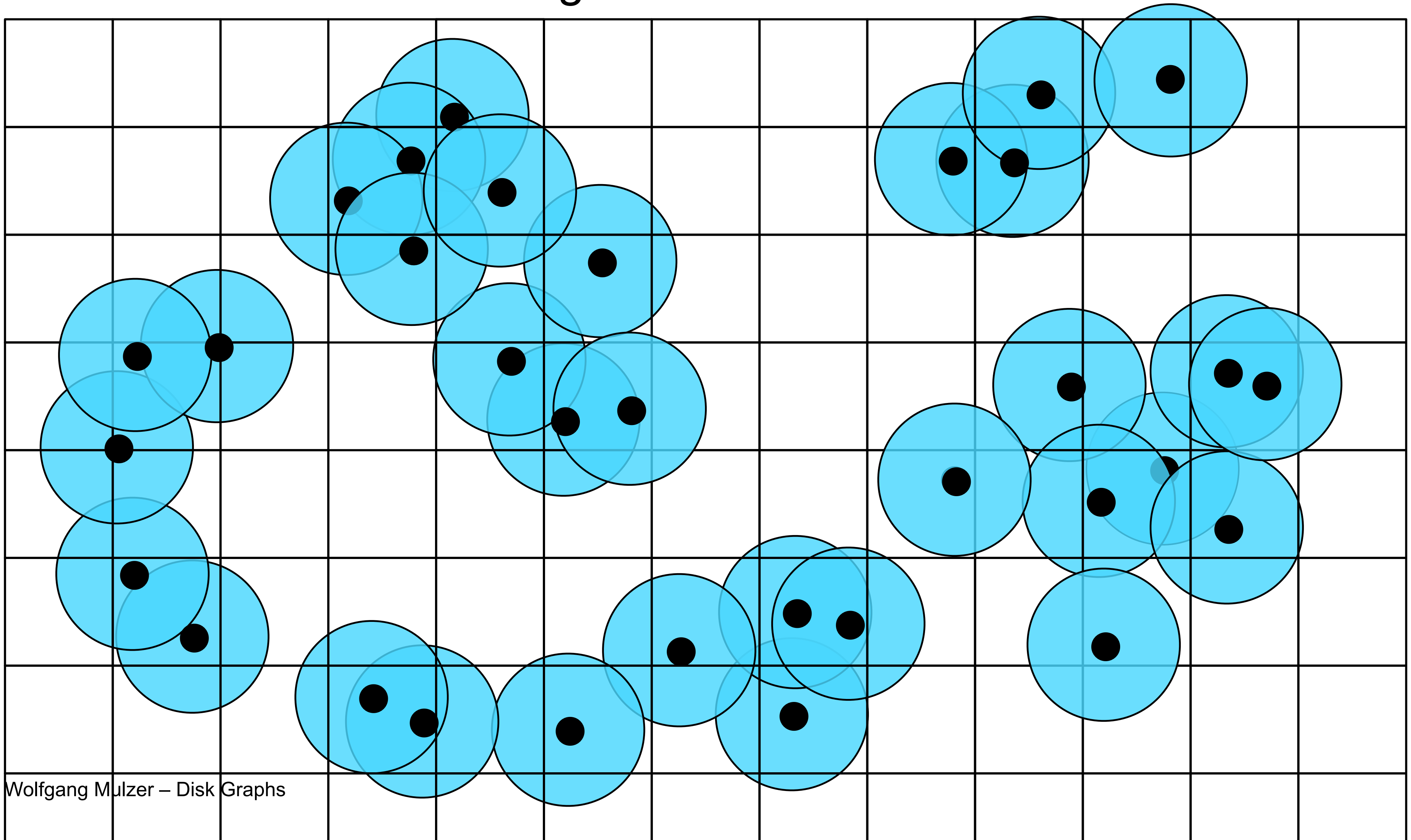
Chan et al: $O(\log^6 n)$ updates, $O(\log n / \log\log n)$ queries





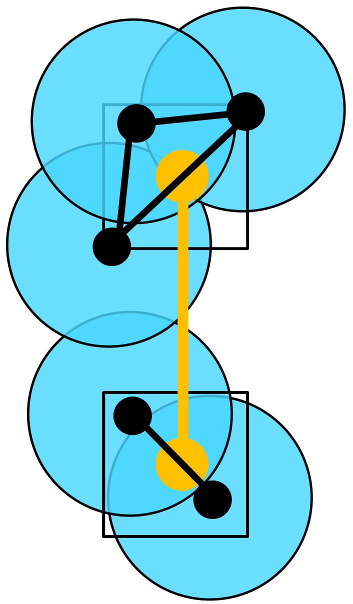
The Grid Graph

grid of diameter 1



The Grid Graph

connect nonempty adjacent cells



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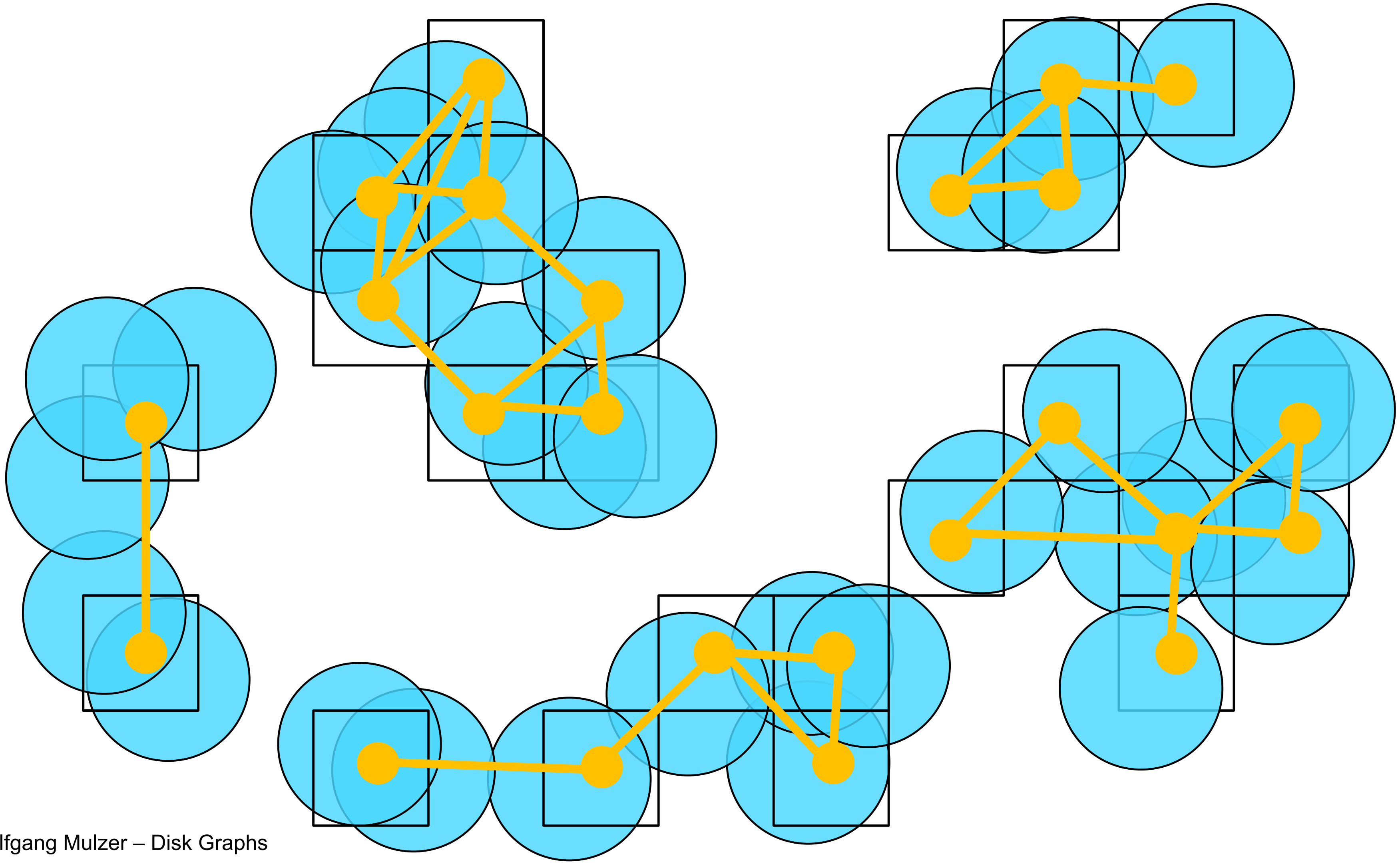
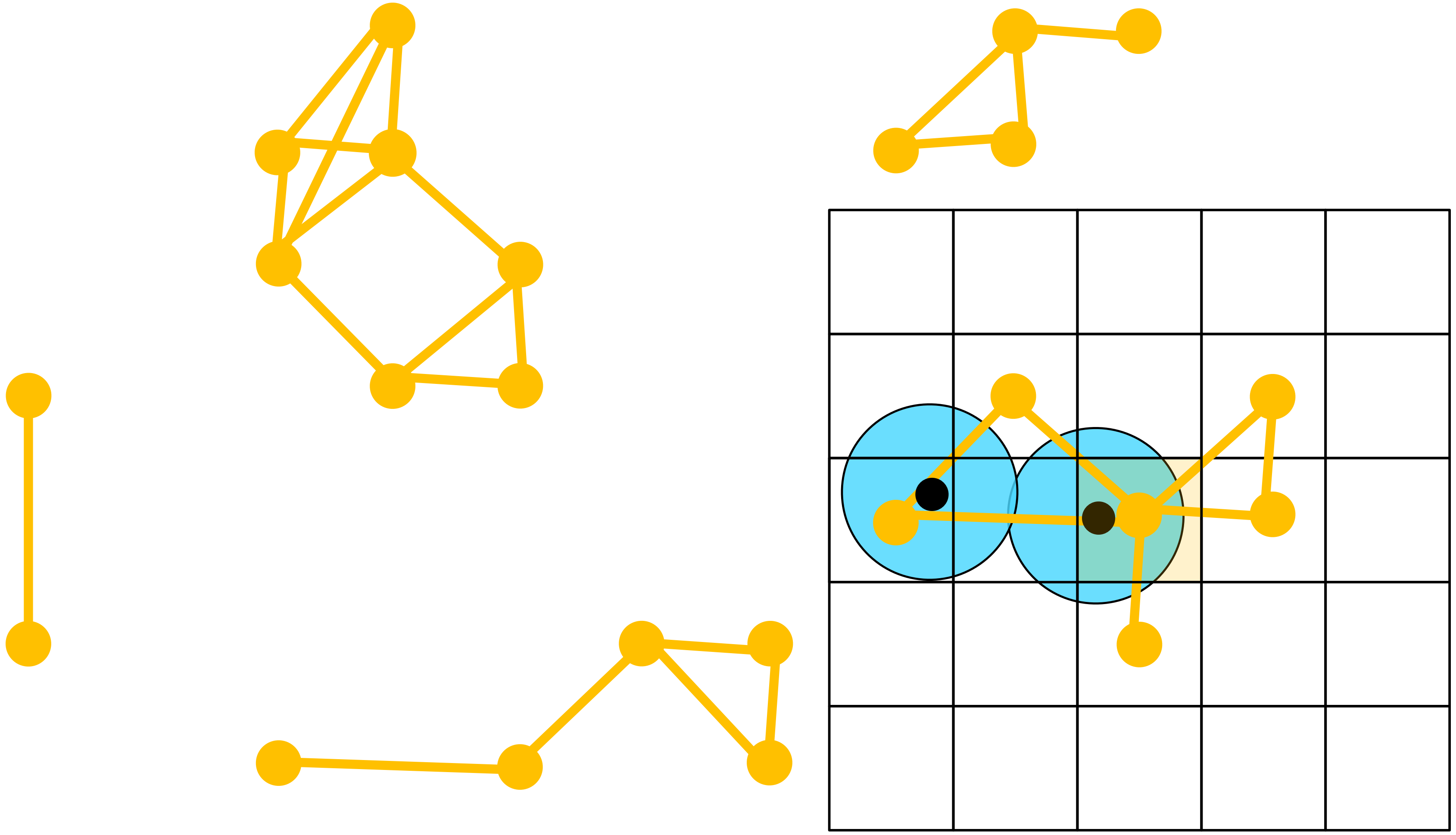


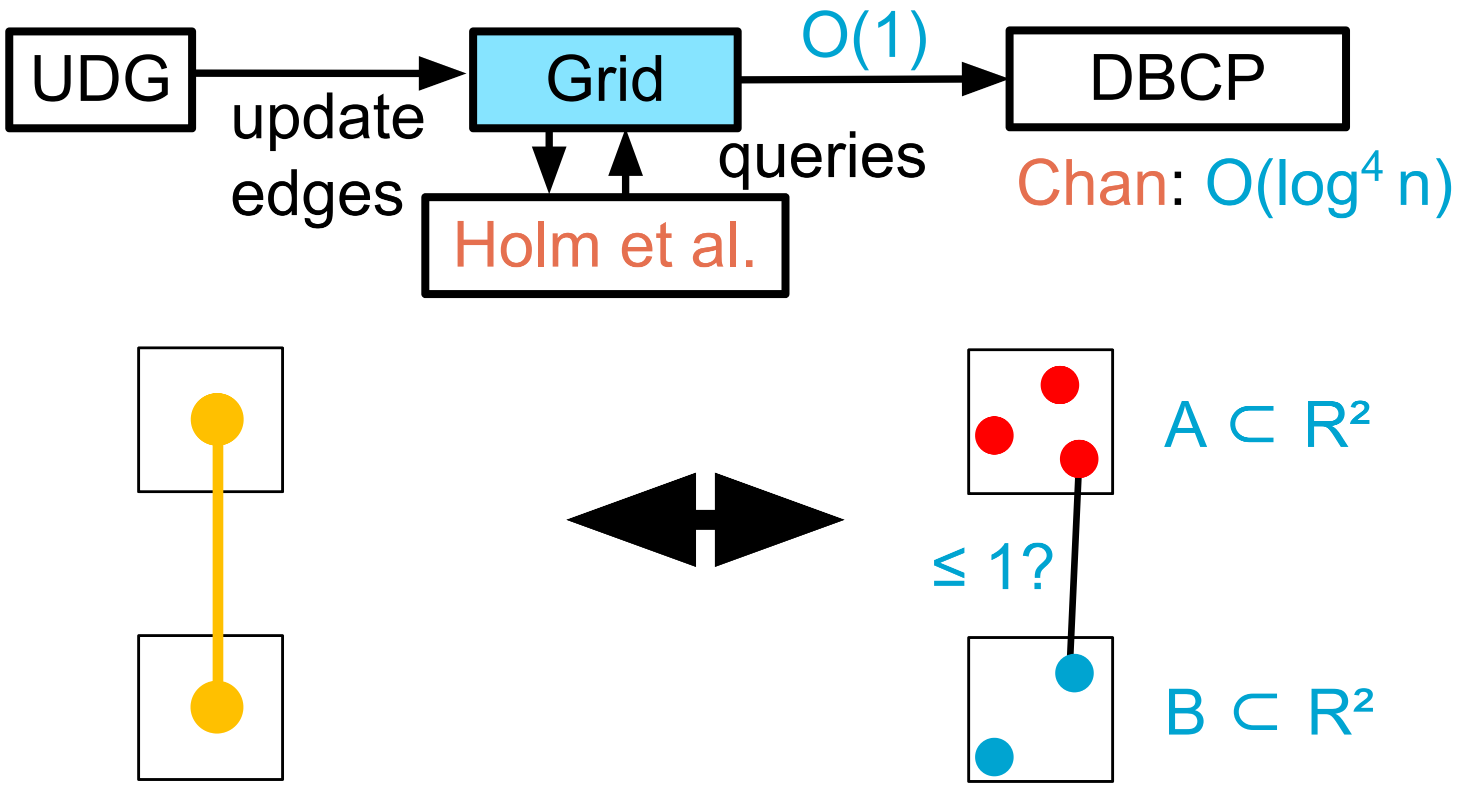
The Grid Graph

consider neighborhoods



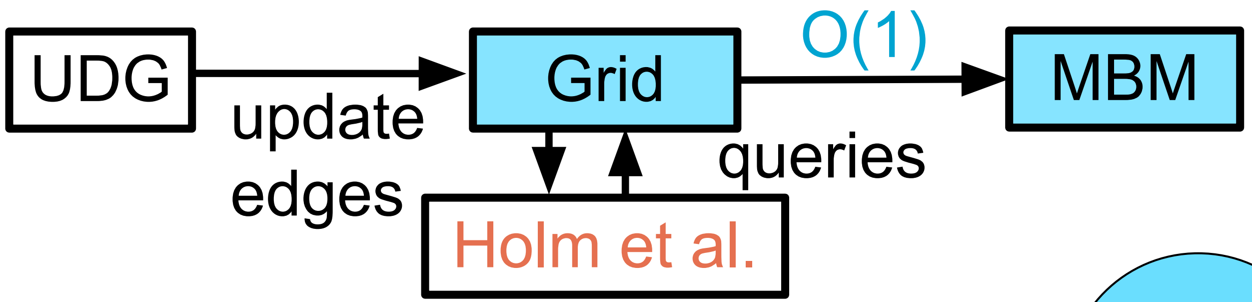
Dynamic Connectivity in Unit Disk Graphs

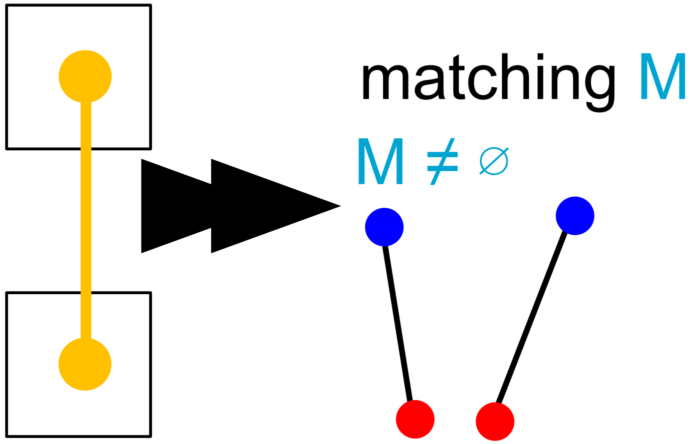
update time: $O(\log^6 n)$

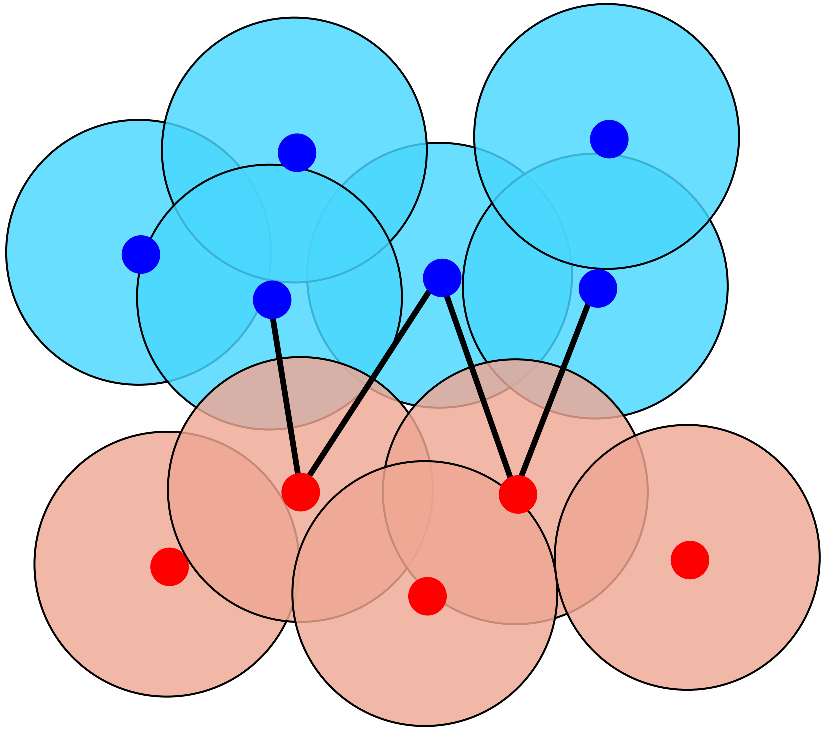


Dynamic Connectivity in Unit Disk Graphs

update time: $O(\log^4 n)$







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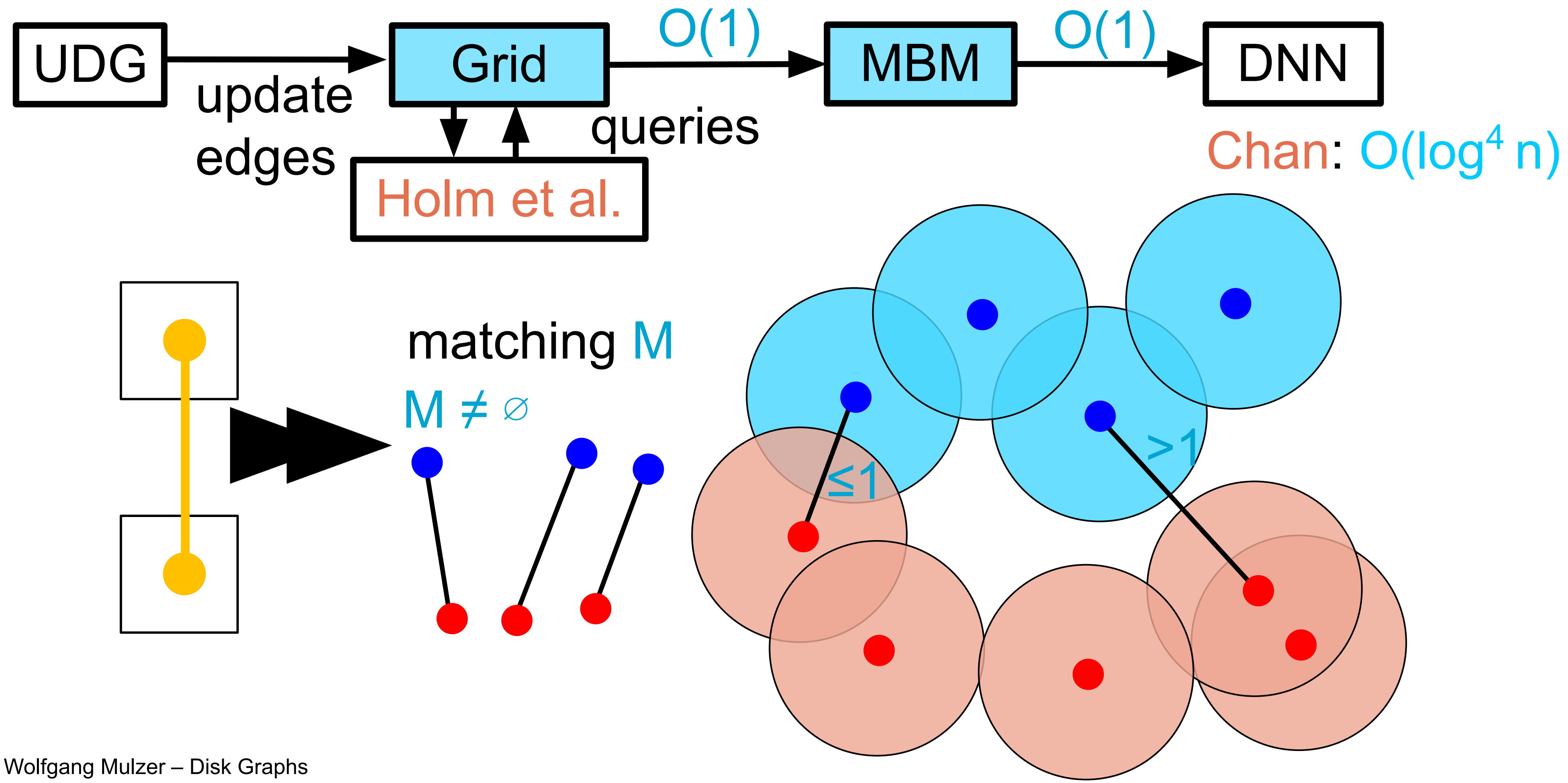

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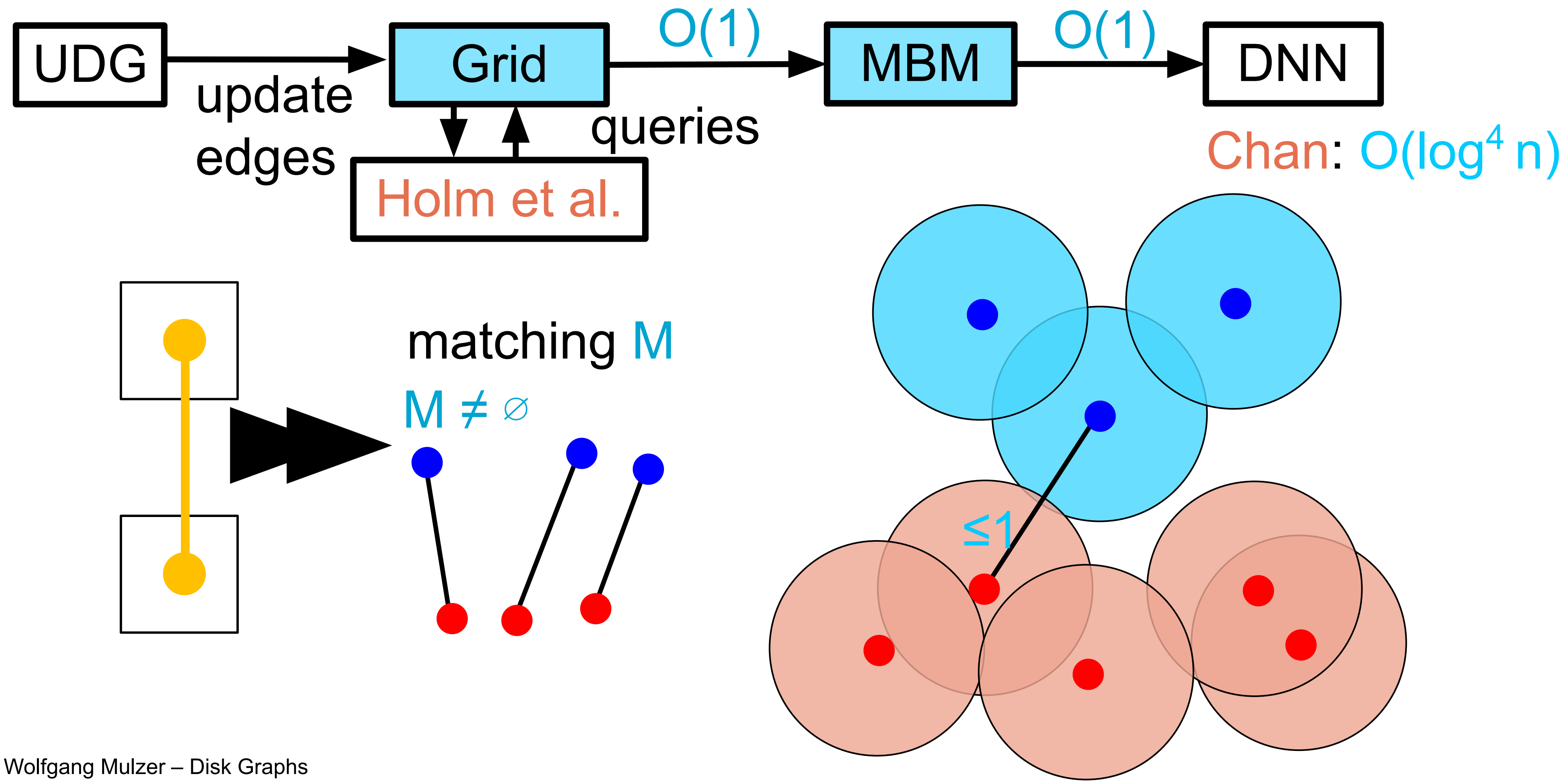
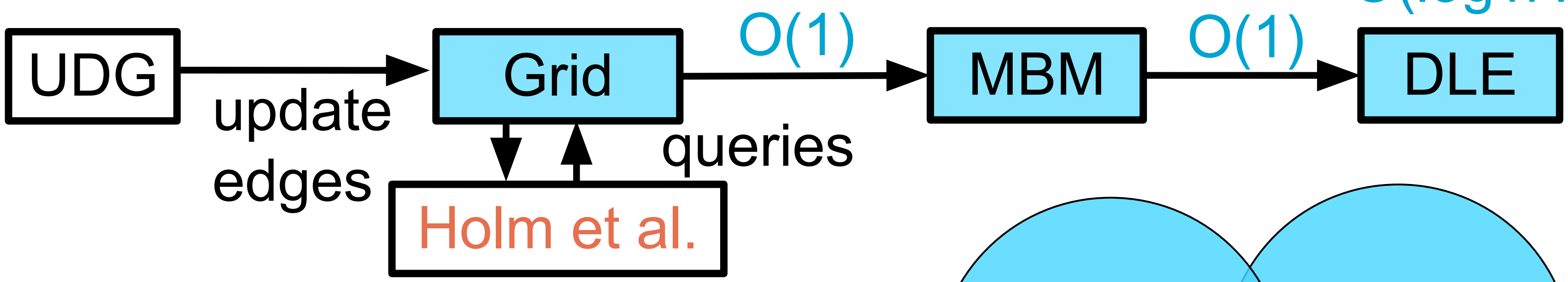
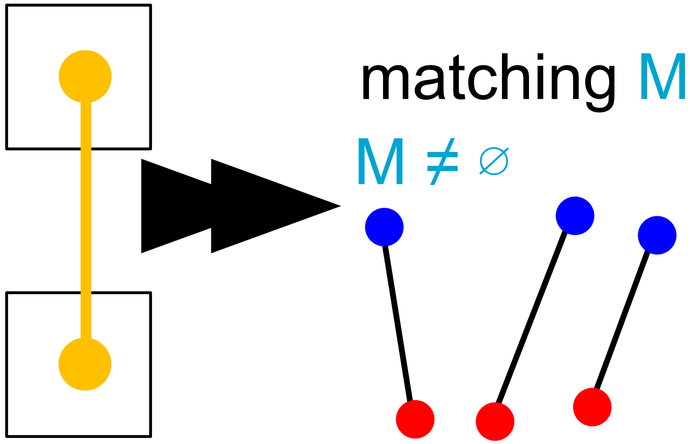


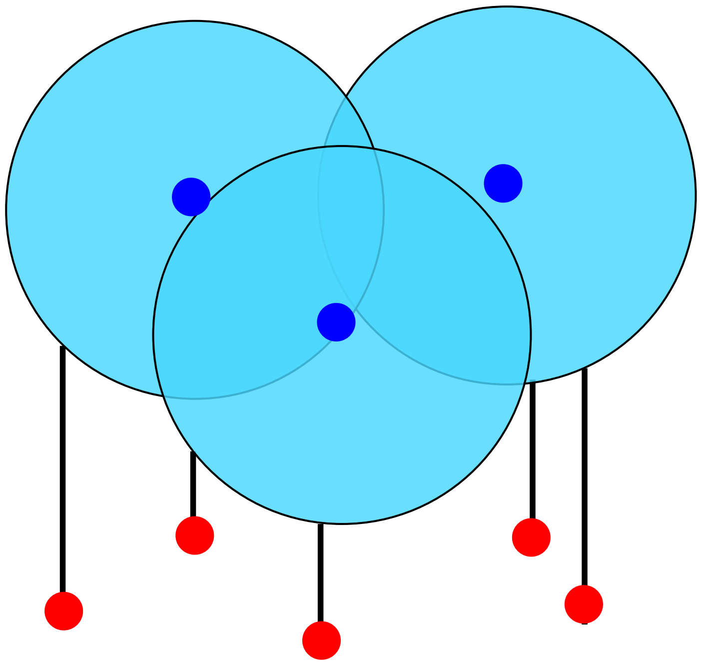
Dynamic Connectivity in Unit Disk Graphs

update time: $O(\log^2 n)$

Kaplan et al.,
Agarwal et al.:
$O(\log n \log\log n)$







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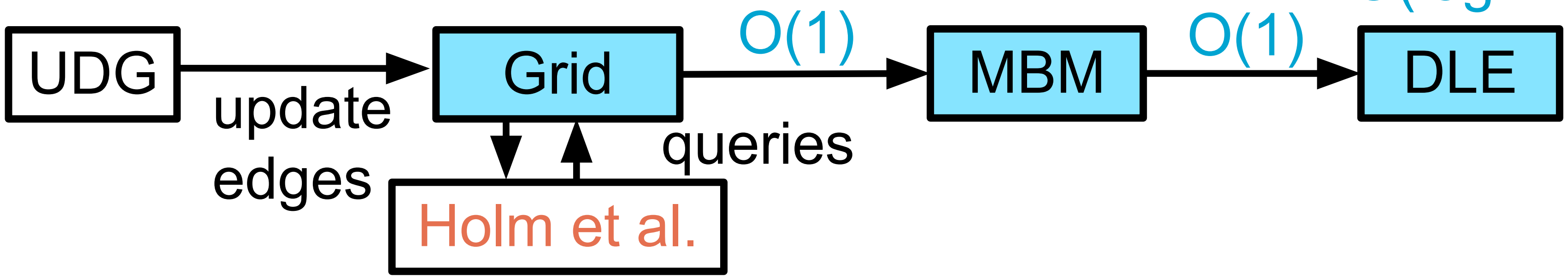
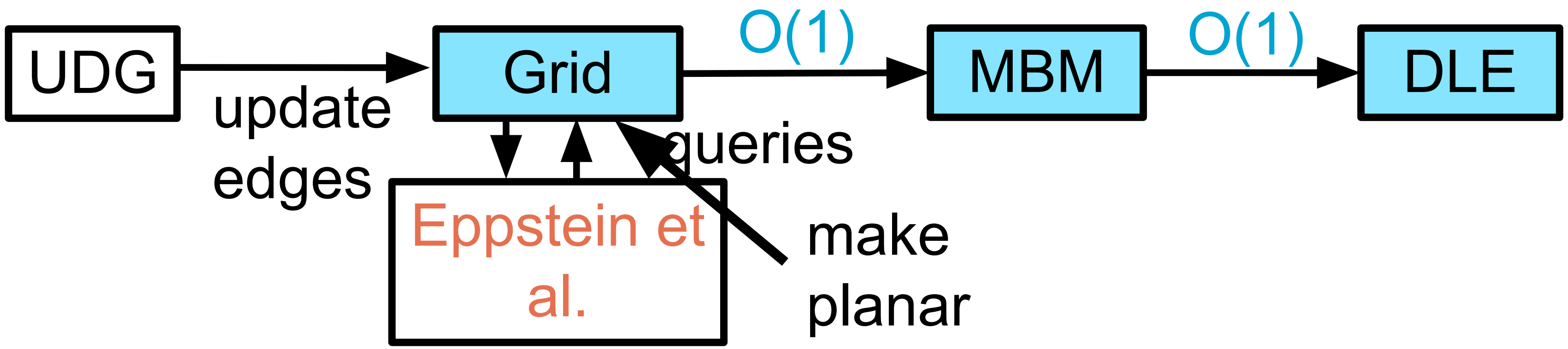


Thm 1 [w/ HK, LR, PS]:
Can maintain UD(P) with
$O(\log^2 n)$ updates and
$O(\log n/\log \log n)$ queries.

Dynamic Connectivity in Unit Disk Graphs

update time: $O(\log^2 n)$



Thm 1 [w/ HK, LR, PS]: Can maintain UD(P) with $O(\log^2 n)$ updates and $O(\log n/\log \log n)$ queries.

Thm 2 [w/ HK, LR, PS]: Can maintain UD(P) with $O(\log n \log\log n)$ updates and $O(\log n)$ queries.

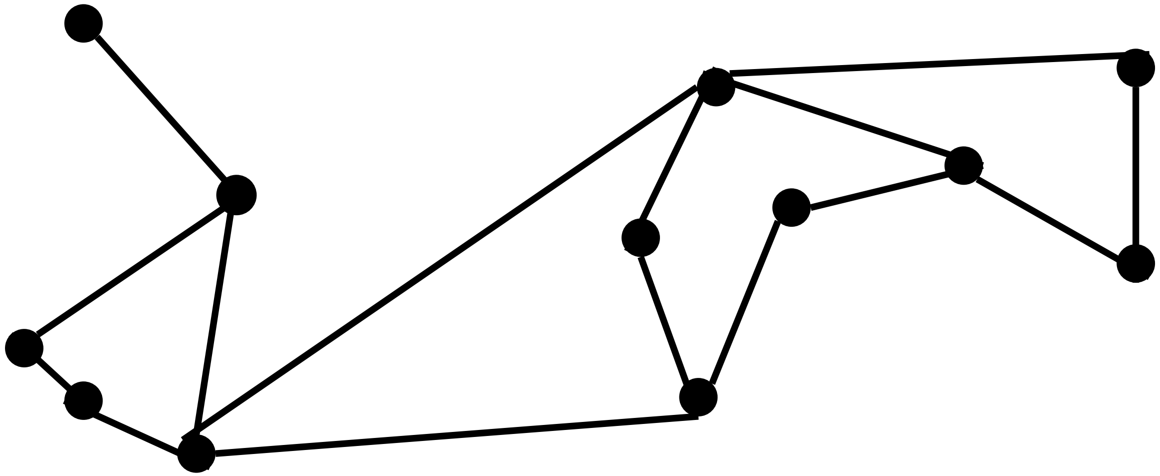
Open Questions:

- arbitrary radii
- general shapes

Computing the Girth

Given: simple graph $G = (V, E)$

Girth: shortest cycle in G (minimum number of edges)



Computing the Girth

Given: simple graph $G = (V, E)$

Girth: shortest cycle in G (minimum number of edges)

General results:

$O(n^{\omega}) = O(n^{2.371339})$ with fast MM
[Itai, Rode 1978][Alman et al. 2024]

$O(n^3 \text{polyloglog } n / \log^4 n)$ „combinatorial" algorithm
[Yu 2015]

Question: What about disk graphs?

$O(n^2)$ additive $+1$ approximation
[Itai, Rode 1978]

further approximation and hardness results
[Vassilevska Williams,Williams 2010]
[Roditty, Vassilevska Williams 2012]

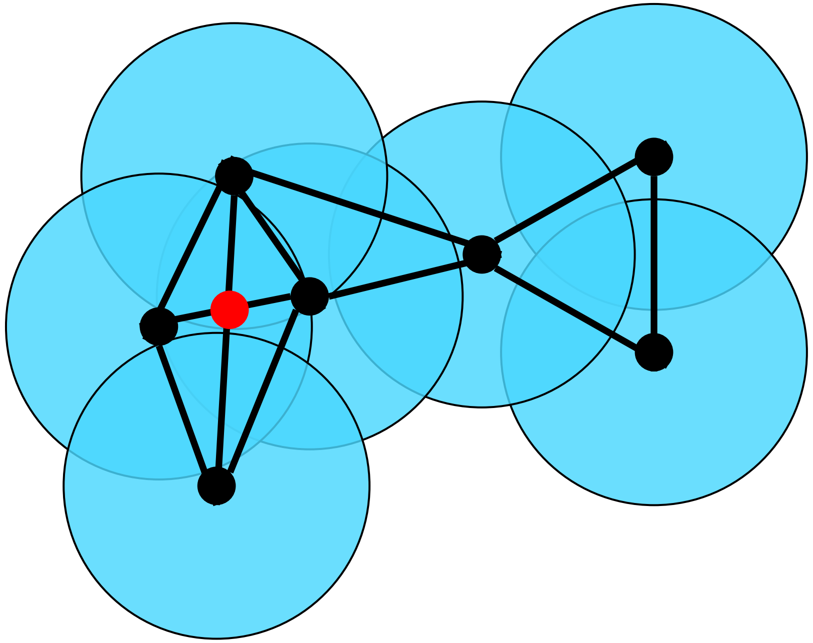
Planar graphs:

$O(n)$ time
[Chang, Lu 2013]

A Useful Fact about Disk Graphs

Lemma: Let G be a disk graph that is not plane. Then, there are three sites whose disks intersect in a common point.

[Evans, van Gardern, Löffler, Polishchuk 2016, and earlier]

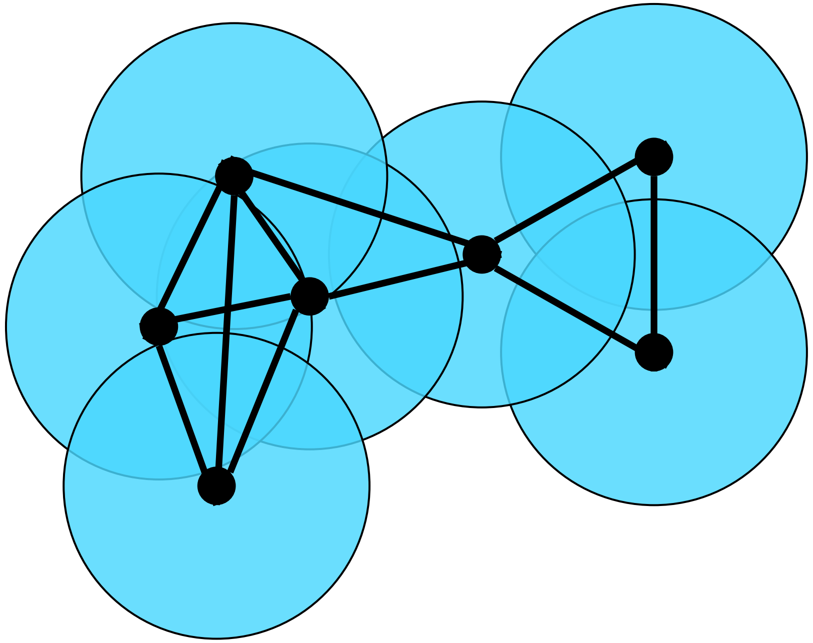


A Useful Fact about Disk Graphs

Lemma: Let G be a disk graph that is not plane. Then, there are three sites whose disks intersect in a common point.

[Evans, van Gardern, Löffler, Polishchuk 2016, and earlier]

Consequence: a disk graph is either plane, or it has girth 3



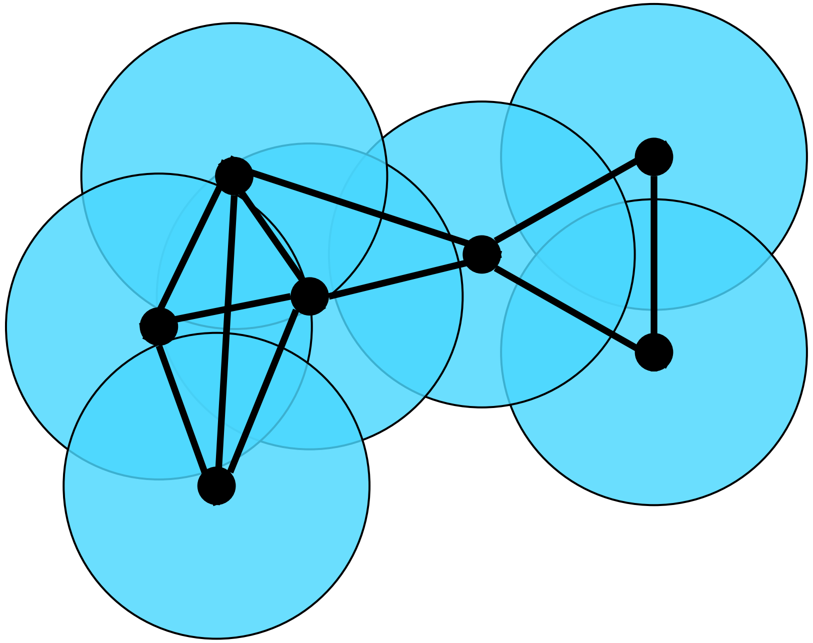
Algorithm

Construct disk graph using a sweep line algorithm

If two edges cross → report girth 3

Otherwise: use algorithm for planar graphs

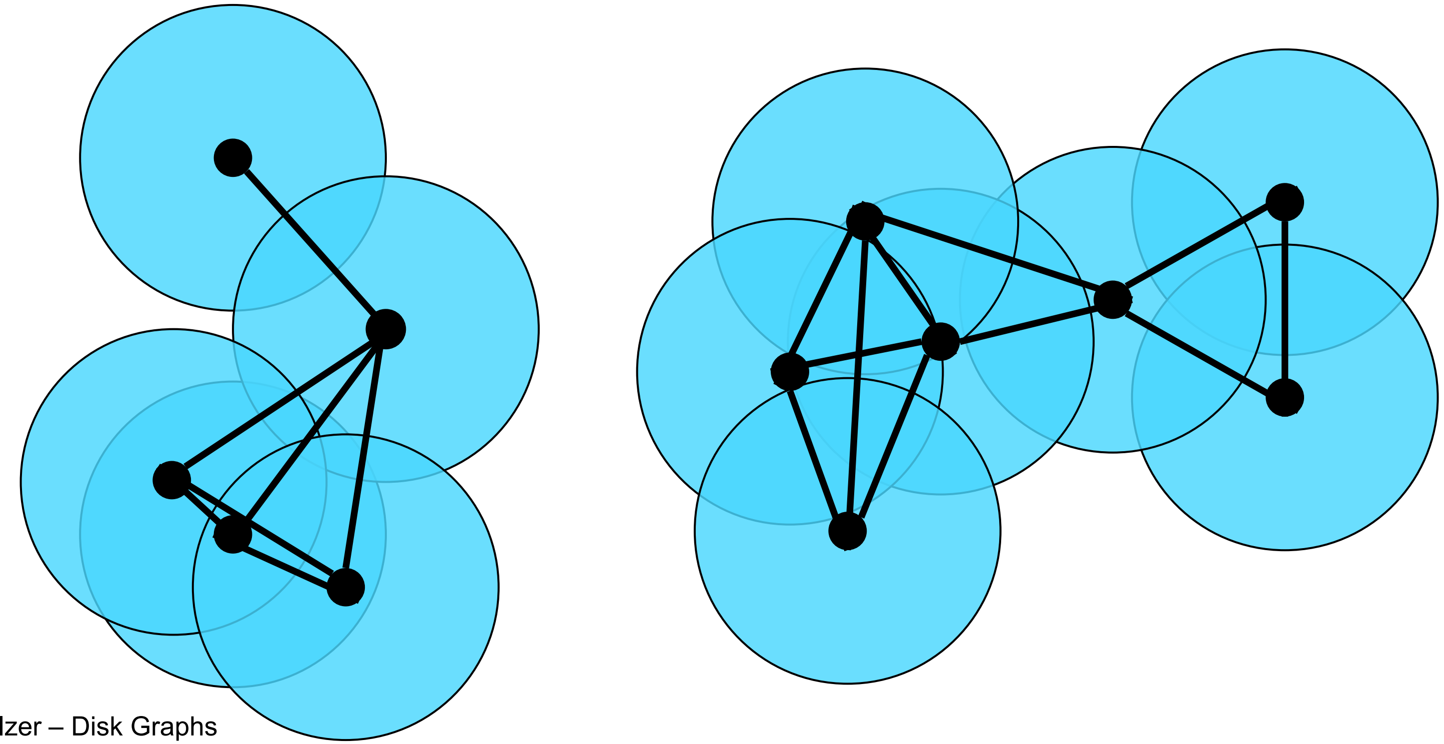
Running time: $O(n \log n)$



Finding the Shortest Triangle

Given: disk graph G, edges weighted by Euclidean length

Want: triangle that minimizes the total edge length

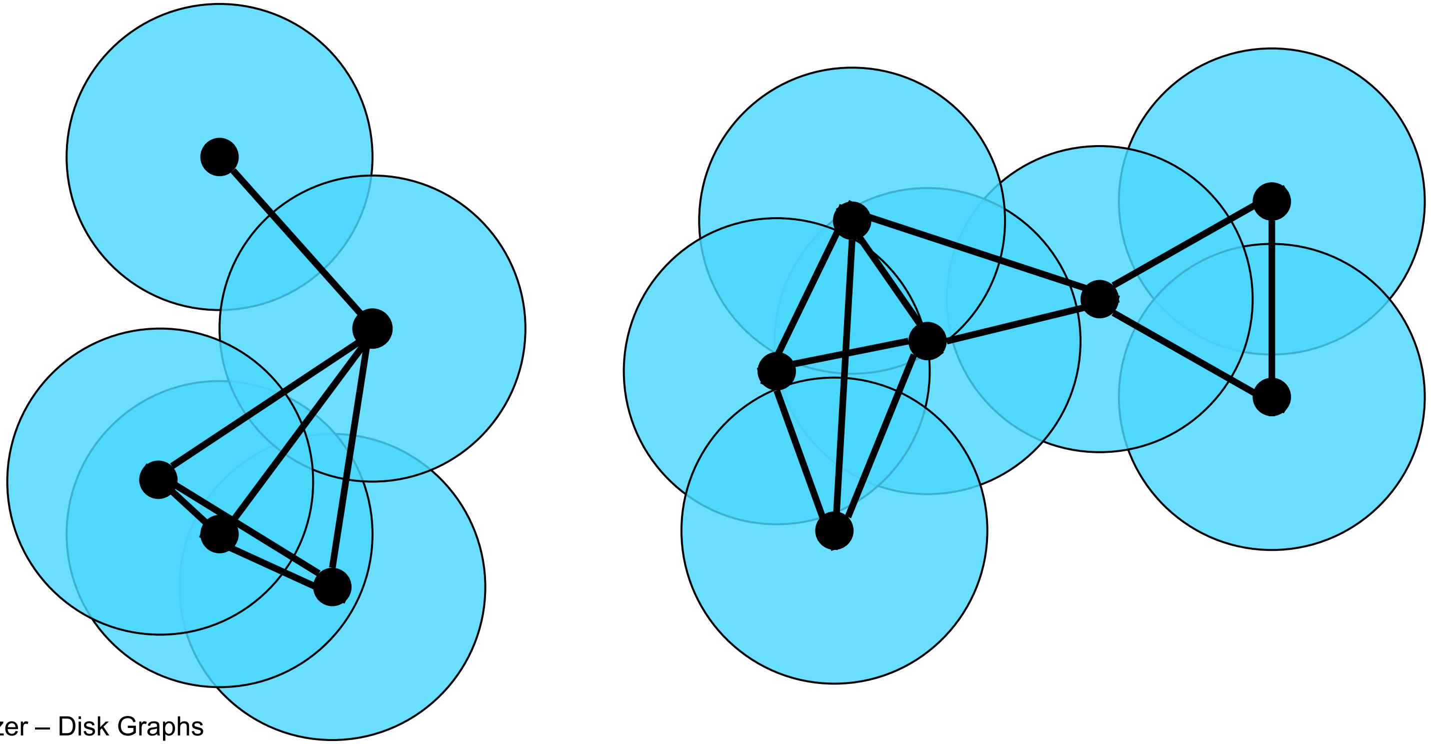


Finding the Shortest Triangle

Given: disk graph G, edges weighted by Euclidean length

Want: triangle that minimizes the total edge length

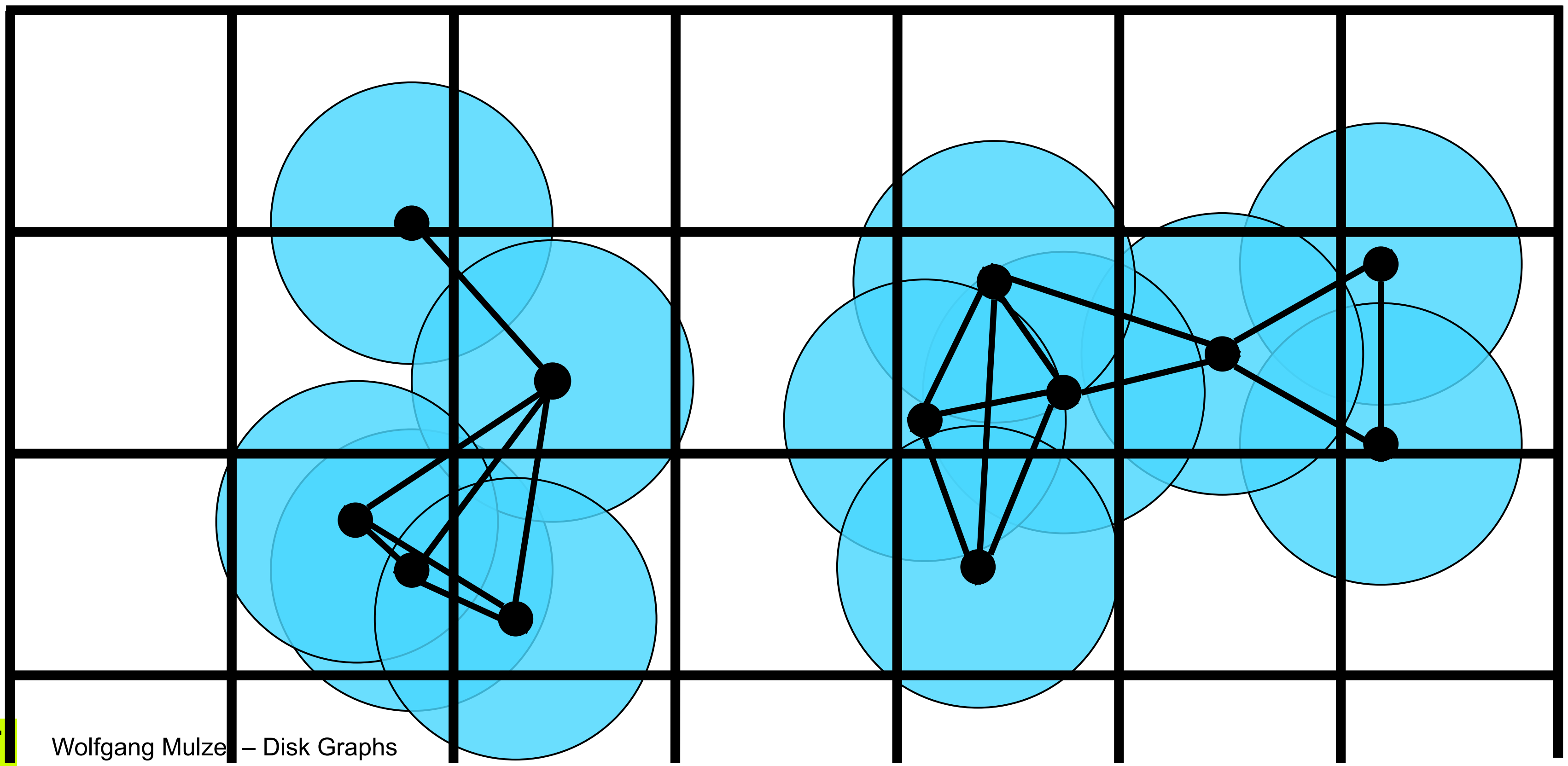
First: given $W > 0$, is there a triangle of length W?



Shortest Triangle: Decision Version

Impose grid of diameter W/3

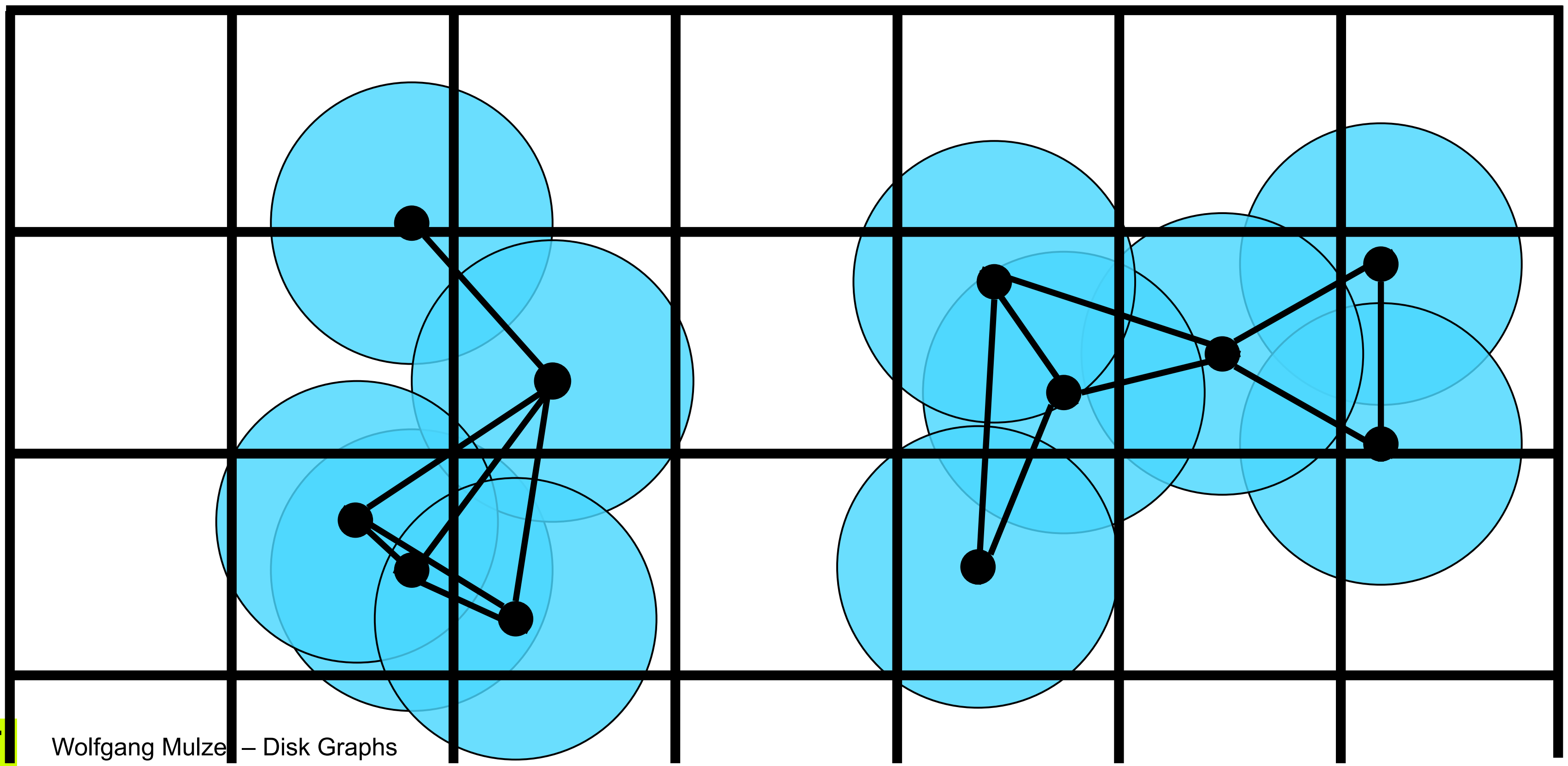
If a grid cell contains a triangle → done (check in O(n log n) time)



Shortest Triangle: Decision Version

Impose grid of diameter W/3

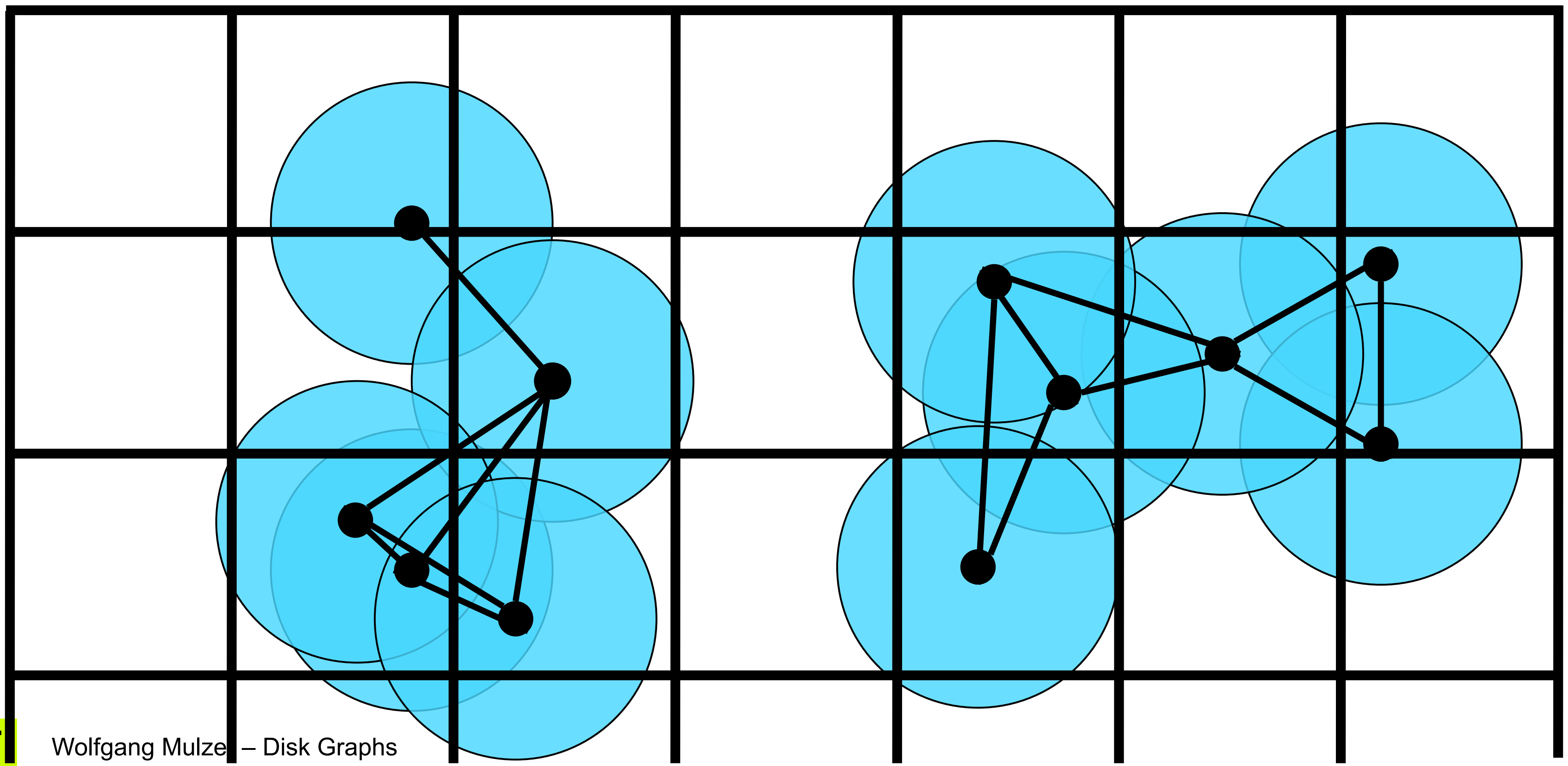
Otherwise: triangle goes between neighboring grid cells; induced graph in each cell is plane; few “long” edges → check explicitly



Shortest Triangle: Decision Version

Result: The decision version can be solved in O(n log n) time.

Then: Plug into Chan's randomized framework for geometric optimization problems → O(n log n) expected time.

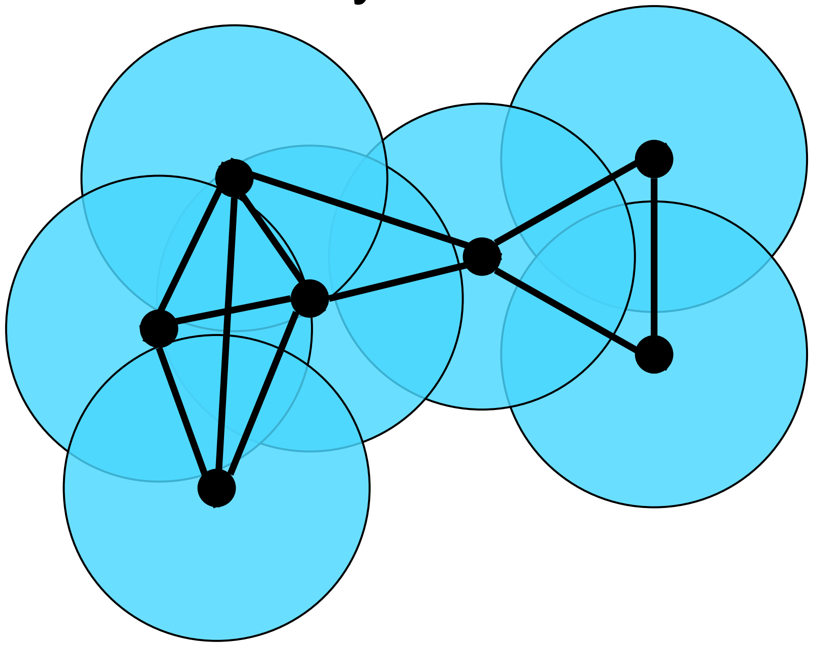


Extension: Shortest Cycle in a Disk Graph

Theorem [w/ HK, KK, LR, MS, PS]: The shortest (weighted) cycle in a disk graph can be found in $O(n \log n)$ expected time.

Similar strategy: first solve decision version, then plug into Chan's randomized framework.

Interesting subproblem: Given weighted graph $G = (V, E)$, vertex v in V, find shortest cycle in G that contains v.

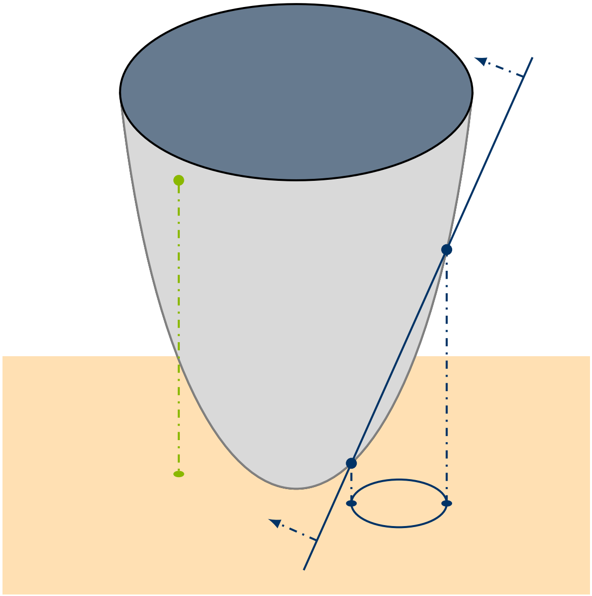


Extension: Triangle in a Transmission Graph

Theorem [w/ HK, KK, LR, MS, PS]: A directed triangle in a transmission graph can be found in $O(n \log n)$ expected time.

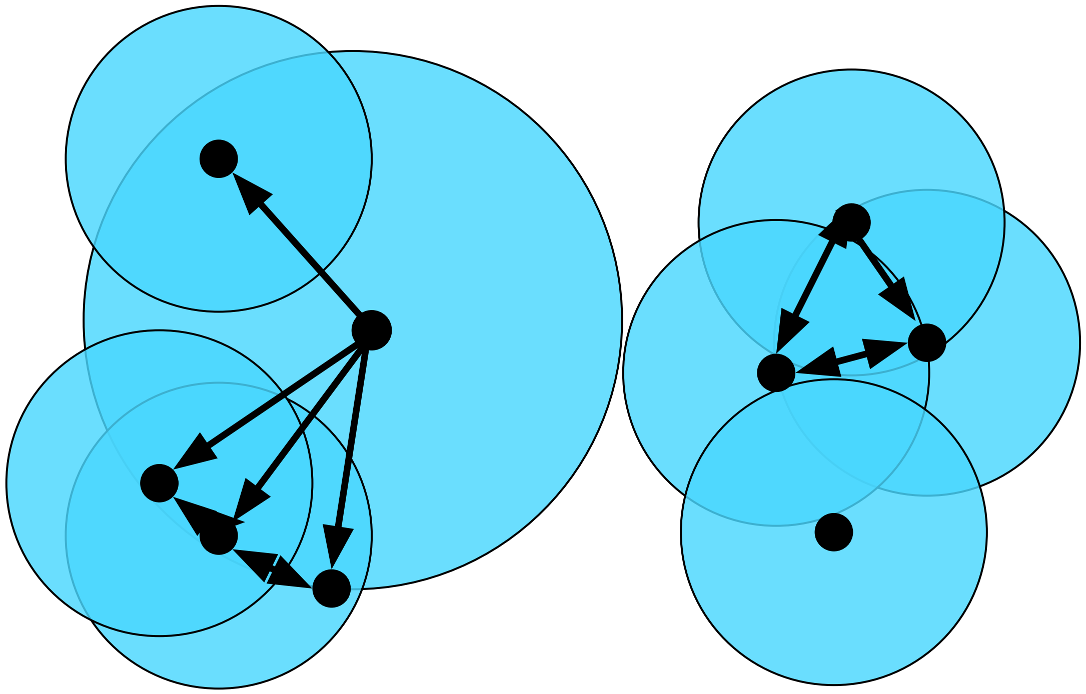
Requires additional range searching techniques.

Extends to finding a shortest triangle.



Extension: k-Cycle in a Transmission Graph

Theorem [w/ HK, KK, LR, MS, PS]: A directed k-cycle a transmission graph can be found in $O(n \log^2 n) + n2^{O(k)}$ time.



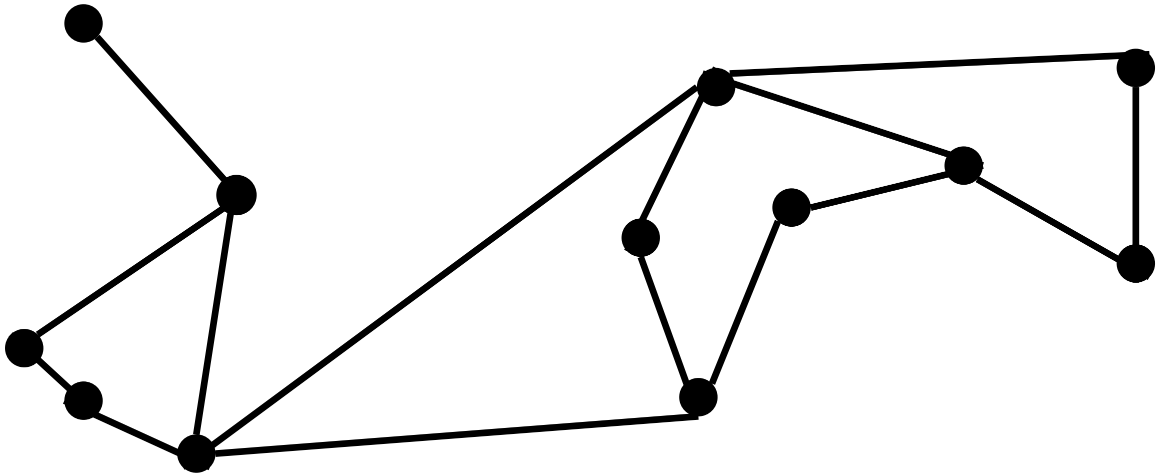
Open Questions: Find a shortest cycle, derandomize

Maximum Matching

Given: simple graph $G = (V, E)$.

Matching: set of edges with pairwise distinct endpoints

Goal: find matching of maximum cardinality



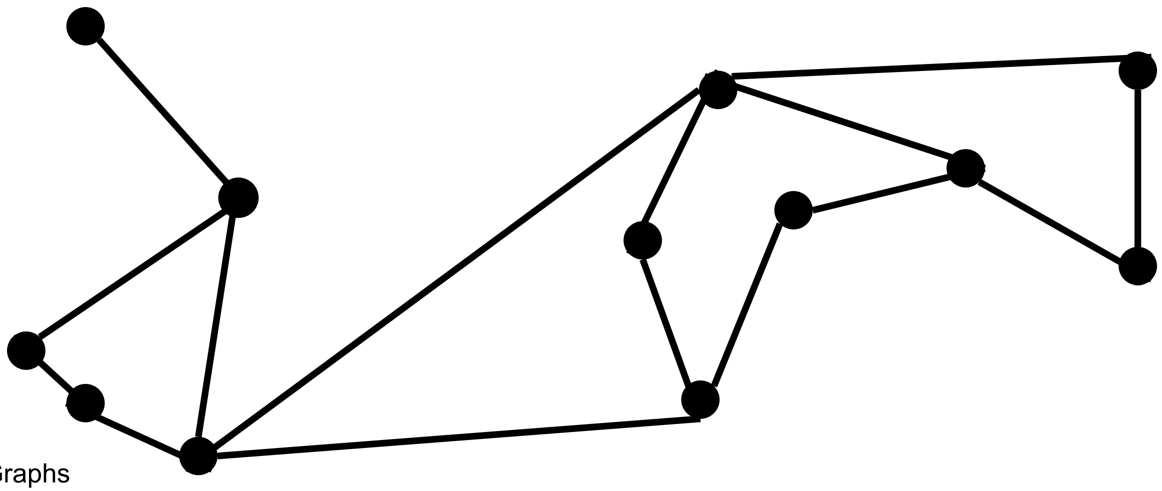
Maximum Matching

Goal: find matching of maximum cardinality

Extremely classic problem

Fastest algorithms: $O(\sqrt{n}\, m)$ [Micali, Vazirani]
$O(n^{\omega})$ [Mucha, Sankowski]
$O(m^{10/7})$ [Mądry]

Bipartite unit disk graphs: $O(n^{3/2} \log n)$ [Efrat, Itai, Katz]
$O(n^{4/3+\varepsilon} \log n)$ [Cabello et al.]

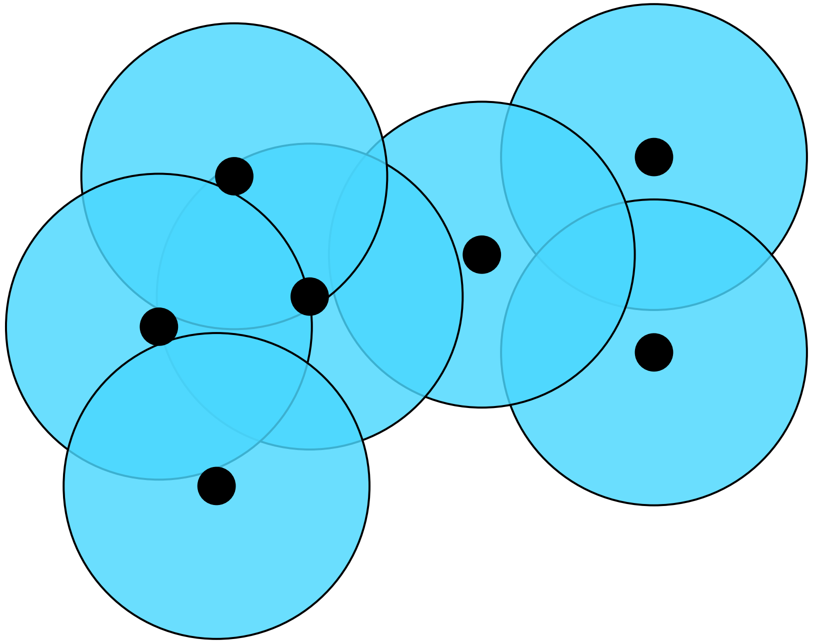


Maximum Matching – Bounded Depth

Depth: maximum number ρ of disks that cover any single point

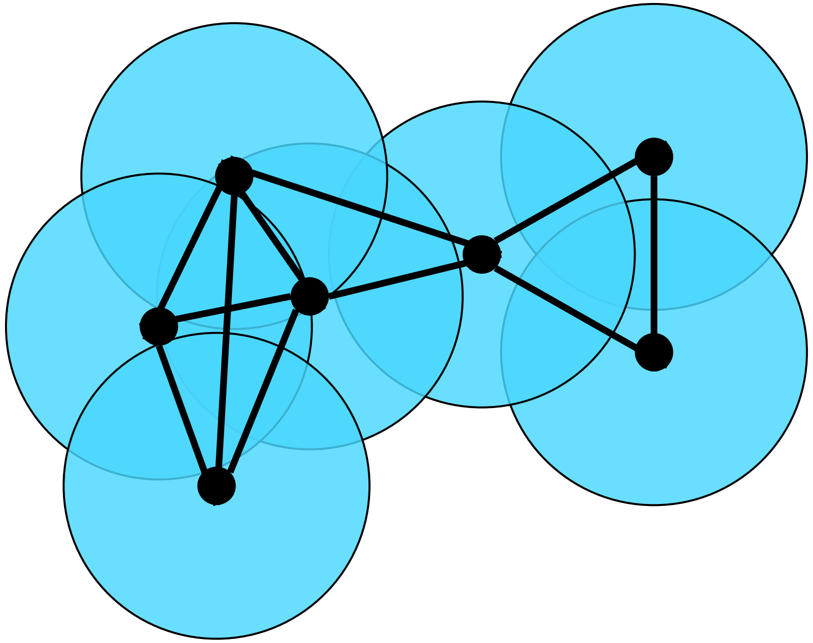
Easy: bounded depth implies bounded average degree

[Mucha, Sankowski], [Yuster, Zwick], [Alon, Yuster]: Maximum matching in hereditary graph families with bounded average degree and small separators can be found quickly



Maximum Matching – Bounded Depth

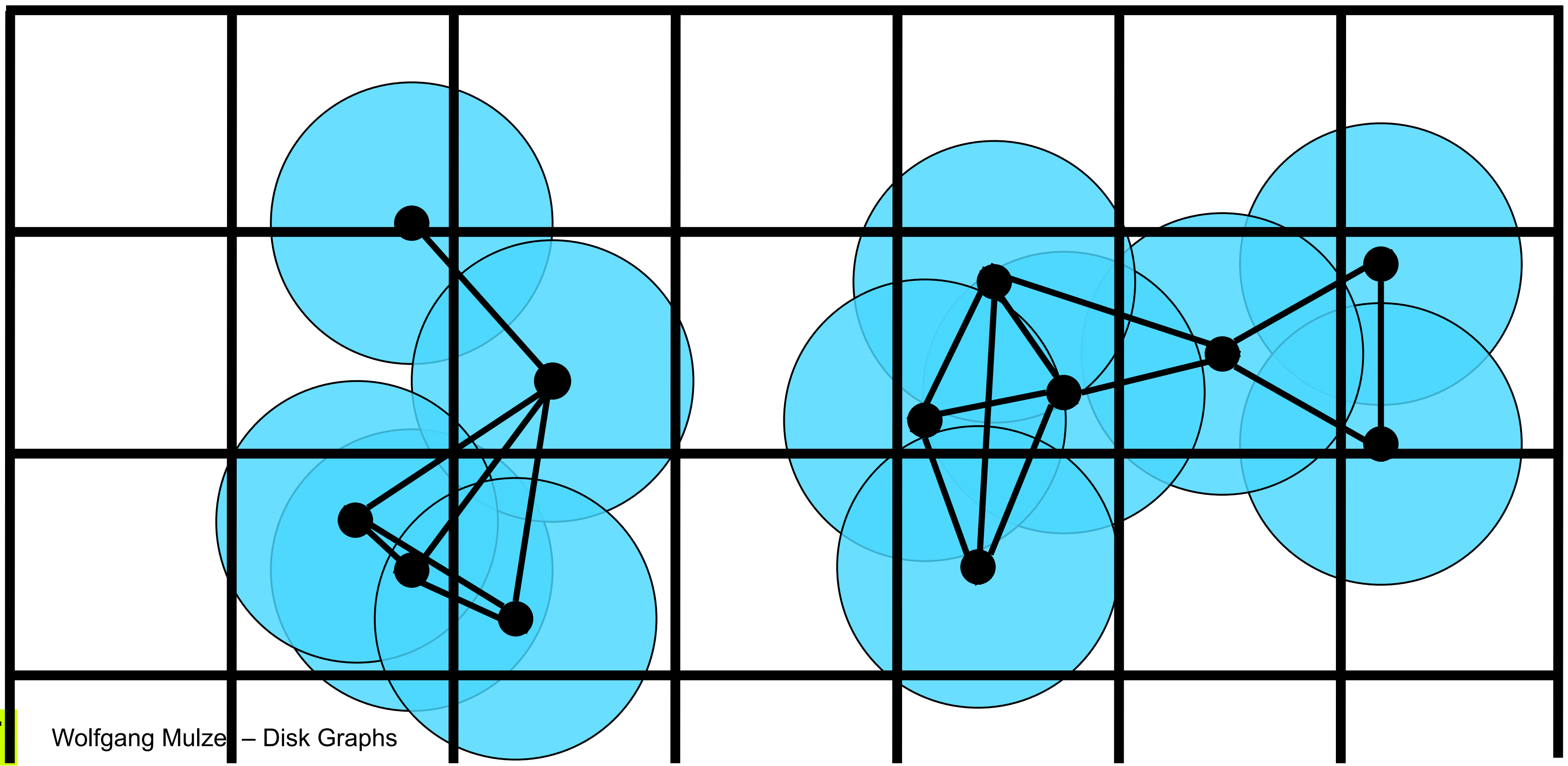
Theorem [w/ E. Bonnet and S. Cabello]: A maximum matching in a unit disk graph of depth ρ can be found in expected time $O(\rho^{3\omega/2} n^{\omega/2})$.



Maximum Matching – Sparsification

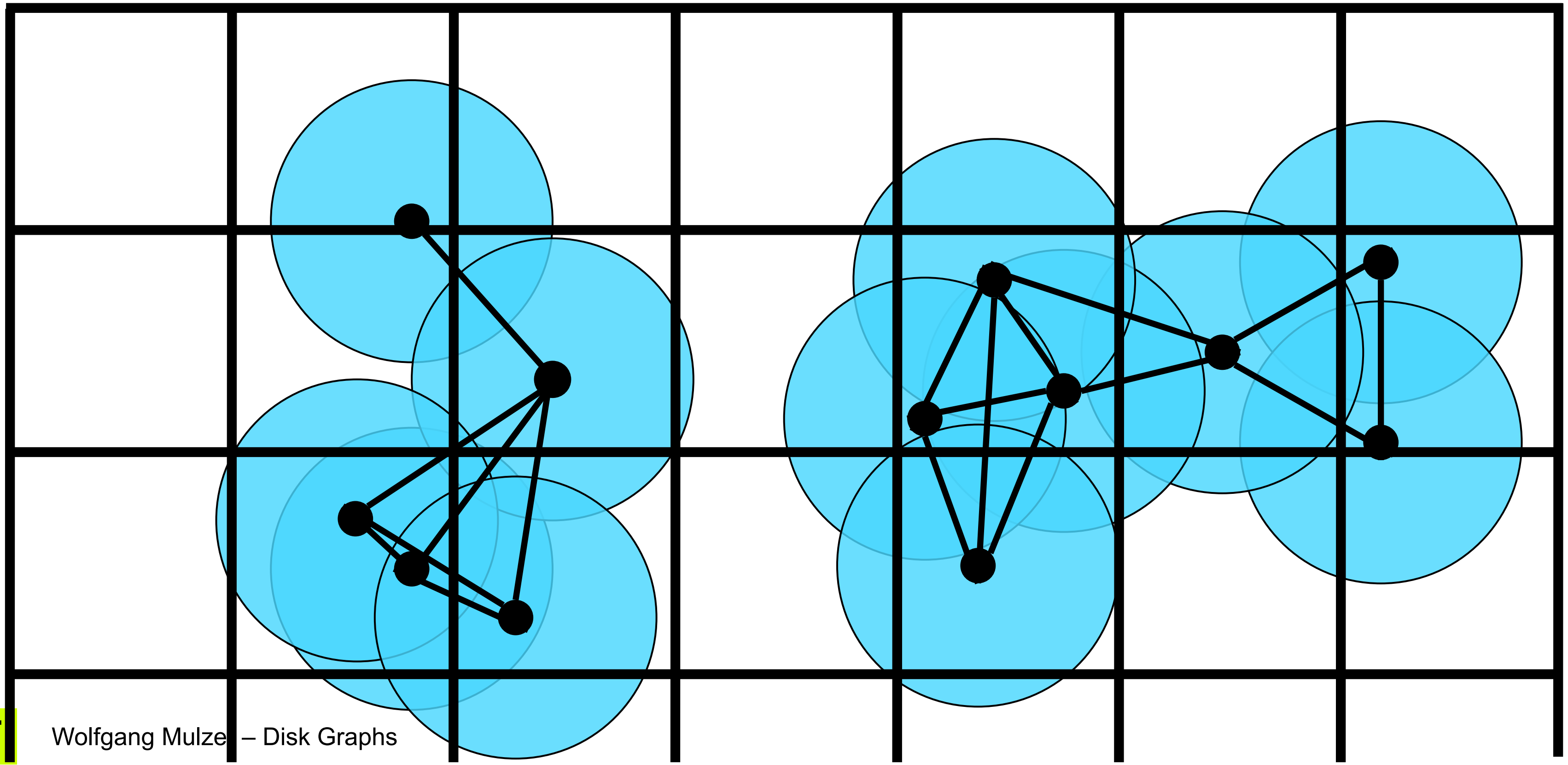
Can reduce the general case to bounded depth case.

Strategy: use grid (again); maximum matching in each cell is easy; define auxiliary graph between cells, of bounded depth



Maximum Matching – Sparsification

Theorem [w/ E. Bonnet and S. Cabello]: A maximum matching in a unit disk graph can be found in expected time $O(n^{\omega/2})$.

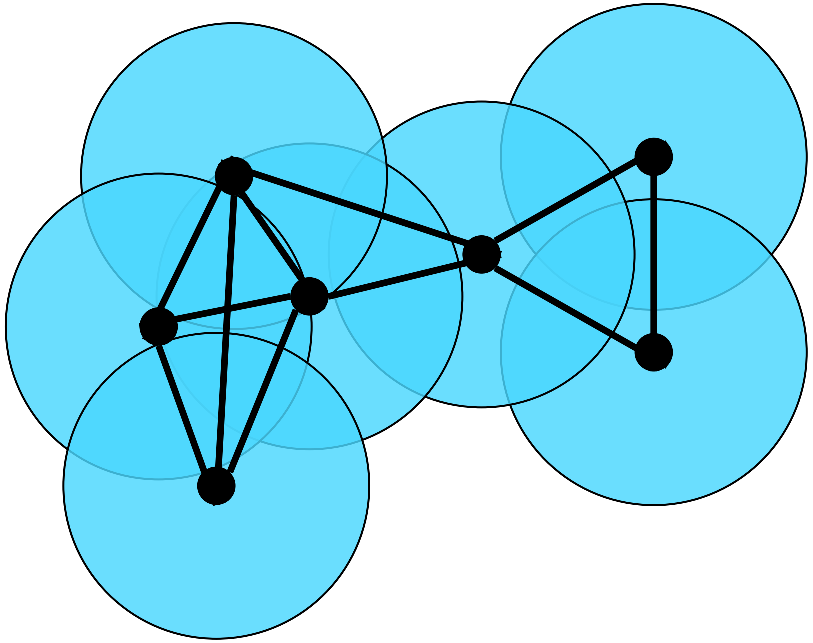


Conclusion

Disk graphs are useful and interesting

Many possible directions, many results, many open problems

Not mentioned: routing, reachability oracles, shortest paths, recognition …



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Questions?