CSCI 239—Discrete Structures of Computer Science
Lab 6—Vectors and Matrices

This lab consists of exercises on real-valued vectors and matrices. Most of the exercises will required pencil and paper. Put your answers either on paper or in a Word file.

Objectives:

- to understand real-valued vectors and matrices and their operations
- to learn to calculate vector and matrix operations by hand

Part 1: Real-Valued Vectors

A *real-valued vector* is a finite sequence of values from *R*. (In a program, we usually use an array of floating-point values to represent a vector; in a functional language like Haskell, we use a list.) We'll use the notation of a list of values in parentheses, separated by commas, to express a vector; we'll use boldface variable names for vectors:

$\mathbf{u}$ = (1.0, 2.0, 0.0, −3.0)
$\mathbf{v}$ = (7.6, −9.2, −45.1)
$\mathbf{w}$ = (0.0, 9.0, 1.0, −7.0)

In the examples above, **u** and **w** are vectors of length four; **v** is a vector of length three. We define *scalar multiplication* of a single real value (a *scalar*) with a vector as the vector resulting from multiplying each element of the vector by the scalar:

$3.0\mathbf{u}$ = (3.0, 6.0, 0.0, −9.0)
$0.0\mathbf{v}$ = (0.0, 0.0, 0.0)
$-1.0\mathbf{w}$ = (0.0, −9.0, −1.0, 7.)

We define *Vector addition* for vectors of the same length as the pairwise addition of their elements:

$\mathbf{u} + \mathbf{w}$ = (1.0 + 0.0, 2.0 + 9.0, 0.0 + 1.0, -3.0 + (−7.0))
= (1.0, 11.0, 1.0, −10.0)

Exercise 1.1

Given the following vector definitions, compute the results of the following vector operations:

a = (1, 2, 3, 4)
b = (2, 1, 0, -1)
c = (−4, −3, −2, −1)
d = (1, 0, -1, 0)

i. 2**a**
ii. -4**b**
iii. **a** + **b**
iv. **a** + **c**
v. **b** + **d**
vi. 2**a** + (−5)**c**

The *dot product* of two same-length vectors **u** and **v**, denoted **u**·**v**, is the sum of the pairwise products of the elements of **u** and **v**. For example, the dot product of the two vectors **a** and **b** defined in Exercise 1.1 is given as follows:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (1, 2, 3, 4) \cdot (2, 1, 0, -1) \\ &= 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot (-1) \\ &= 2 + 2 + 0 + (-4) \\ &= 0\end{aligned}$$

When the dot product of two vectors is 0, we say that the two vectors are *orthogonal* or *perpendicular*. In Euclidean space (ordinary geometry), this means that the angle between the two vectors is π/2 radians or 90°.

Exercise 1.2

Using the vectors **a**, **b**, **c**, and **d** from Exercise 1.1, compute the following:

i. **a** · **c**
ii. **c** · **d**
iii. (**a** · **d**)**b** *Note that this is a dot product, then a scalar multiplication*
iv. (**d** · **b**)**a**
v. (**a** + **d**) · **b**

The *norm* or *magnitude* of a vector **v**, denoted $|\mathbf{v}|$ or sometimes $\|\mathbf{v}\|$, is the square root of the dot product of the vector with itself. That is, $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. For example, the magnitude of vector **a** from Exercise 1.1 is:

$$
\begin{aligned}
|\mathbf{a}| &= \sqrt{\mathbf{a} \cdot \mathbf{a}} \\
&= \sqrt{(1, 2, 3, 4) \cdot (1, 2, 3, 4)} \\
&= \sqrt{1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4} \\
&= \sqrt{1 + 4 + 9 + 16} \\
&= \sqrt{30}
\end{aligned}
$$

Exercise 1.3

Compute the norm of vectors **b**, **c**, and **d** from Exercise 1.1. Leave the answer in radical form unless the dot product is a perfect square.

Note that for any two non-zero vectors **u** and **v**, the following equality holds, where θ is the angle between the two vectors:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|}$$

If the cosine is one, the two vectors are parallel, pointing in the same direction; if it is negative one, they are parallel pointing in opposite directions; if the cosine is zero, as we noted before, the vectors are orthogonal.

Part 2: Matrices

A *real-valued matrix* is a collection of values from **R**, arranged into a rectangle of rows and columns. (In a program, we usually use a two-dimensional array, an array of arrays, or a list of lists to represent a matrix, depending on the language.) We notate an array as a rectangular arrangement of values enclosed in large parentheses; we use italic uppercase letters, with or without subscripts, for vector variable names:

$$M_1 = \begin{pmatrix} 1.0 & 0.0 \\ 1.0 & 2.0 \\ 0.0 & 3.0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1.0 & 2.0 & 3.0 \\ 2.0 & 4.0 & 6.0 \\ 3.0 & 6.0 & 9.0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}$$

The plural of matrix is *matrices* (pronounced **may**-tri-seez).

The *dimension* or *size* of a matrix is the number of rows and columns it contains. In the examples above, M_1 is a 3×2 matrix and M_2 is a 3×3 matrix; note that the number of rows comes first, then the number of columns. We refer to a matrix with the same number of rows and columns, like M_2, as a *square matrix*.

We define *scalar multiplication* of a single real value with a matrix in the same way we did for vectors—we just multiply each element of the matrix by the scalar:

$$3.0M_1 = \begin{pmatrix} 3.0 & 0.0 \\ 3.0 & 6.0 \\ 0.0 & 9.0 \end{pmatrix}$$

$$-0.5M_2 = \begin{pmatrix} -0.5 & -1.0 & -1.5 \\ -1.0 & -2.0 & -3.0 \\ -1.5 & -3.0 & -4.5 \end{pmatrix}$$

We define *matrix addition* on two matrices of the same size as the pairwise addition of their element. Matrix addition is undefined for matrices with different sizes; matrix addition is commutative ($A + B = B + A$) and associative ($A + (B + C) = (A + B) + C$).

$$\begin{aligned} M_2 + M_3 &= \begin{pmatrix} 1.0 & 2.0 & 3.0 \\ 2.0 & 4.0 & 6.0 \\ 3.0 & 6.0 & 9.0 \end{pmatrix} + \begin{pmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{pmatrix} \\ &= \begin{pmatrix} 2.0 & 2.0 & 3.0 \\ 2.0 & 5.0 & 6.0 \\ 3.0 & 6.0 & 10.0 \end{pmatrix} \end{aligned}$$

Exercise 2.1

Given the following matrix definitions and the definition of M_1, M_2, and M_3 above, compute the results of the following matrix operations:

$$M_4 = \begin{pmatrix} 1.0 & -1.0 \\ -2.0 & 2.0 \\ 3.0 & -3.0 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 1.0 & 2.0 & 1.0 \\ 1.0 & 3.0 & 2.0 \\ 0.0 & 1.0 & 2.0 \end{pmatrix}$$

i. $2.0M_4$
ii. $-M_5$
iii. $M_1 + M_4$
iv. $M_2 + M_5$
v. $3.0M_3 + 2.0M_5$
vi. $M_2 + M_3 + M_5$

The *product* of two matrices (*matrix multiplication*) is a bit more complicated. To add two matrices, we need matrices with the same size; to multiply two matrices, we need matrices with compatible dimensions. Two matrices are compatible for multiplication if the first matrix has the same number of columns as the second matrix has rows. That is, if A has size $r_1 \times c_1$ and B has size $r_2 \times c_2$, then we can compute $A \times B$ if and only if $c_1 = r_2$. In the matrices we defined above, M_1 has size 3×2 and M_2 has size 3×3, so we can't multiply them in that order ($M_1 \times M_2$) since M_1 has two columns and M_2 has three rows. On the other hand, we can compute $M_2 \times M_1$ since M_2 has three columns and M_1 has three rows. When we multiply a $k \times m$ matrix times an $m \times n$ matrix, the product will be a $k \times n$ matrix.

Let's consider an example to illustrate this situation. We said we could compute $M_2 \times M_1$ because M_2 has three columns and M_1 has three rows. Since M_2 has three rows and M_1 has two columns, the resulting matrix will have a size of 3×2:

$$M_2 \times M_1 = \begin{pmatrix} 1.0 & 2.0 & 3.0 \\ 2.0 & 4.0 & 6.0 \\ 3.0 & 6.0 & 9.0 \end{pmatrix} \times \begin{pmatrix} 1.0 & 0.0 \\ 1.0 & 2.0 \\ 0.0 & 3.0 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \\ ? & ? \end{pmatrix}$$

Now we just have to describe how to compute each value in the new matrix. There's a mathematical formula involving a big sigma and a whole bunch of subscripts, but we can make things easier to describe by using vectors and the dot product. In English, the value in row i, column j of the product is the dot product of row i in the first matrix with column j in the second matrix, considering the rows and columns of matrices as if they were vectors. So the example above becomes:

$$
\begin{aligned}
M_2 \times M_1 &= \begin{pmatrix} 1.0 & 2.0 & 3.0 \\ 2.0 & 4.0 & 6.0 \\ 3.0 & 6.0 & 9.0 \end{pmatrix} \times \begin{pmatrix} 1.0 & 0.0 \\ 1.0 & 2.0 \\ 0.0 & 3.0 \end{pmatrix} \\
&= \begin{pmatrix} row\ 1(M_2) \cdot col\ 1(M_1) & row\ 1(M_2) \cdot col\ 2(M_1) \\ row\ 2(M_2) \cdot col\ 1(M_1) & row\ 2(M_2) \cdot col\ 2(M_1) \\ row\ 3(M_2) \cdot col\ 1(M_1) & row\ 3(M_2) \cdot col\ 2(M_1) \end{pmatrix} \\
&= \begin{pmatrix} 1.0 \cdot 1.0 + 2.0 \cdot 1.0 + 3.0 \cdot 0.0 & 1.0 \cdot 0.0 + 2.0 \cdot 2.0 + 3.0 \cdot 3.0 \\ 2.0 \cdot 1.0 + 4.0 \cdot 1.0 + 6.0 \cdot 0.0 & 2.0 \cdot 0.0 + 4.0 \cdot 2.0 + 6.0 \cdot 3.0 \\ 3.0 \cdot 1.0 + 6.0 \cdot 1.0 + 9.0 \cdot 0.0 & 3.0 \cdot 0.0 + 6.0 \cdot 2.0 + 9.0 \cdot 3.0 \end{pmatrix} \\
&= \begin{pmatrix} 1.0 + 2.0 + 0.0 & 0.0 + 4.0 + 9.0 \\ 2.0 + 4.0 + 0.0 & 0.0 + 8.0 + 18.0 \\ 3.0 + 6.0 + 0.0 & 0.0 + 12.0 + 27.0 \end{pmatrix} \\
&= \begin{pmatrix} 3.0 & 13.0 \\ 6.0 & 26.0 \\ 9.0 & 39.0 \end{pmatrix}
\end{aligned}
$$

This example illustrates why we need the number of columns in the first matrix to be the same as the number of rows in the second—the dot product is only defined if the two vectors are the same length.

In general, matrix multiplication is associative ($A \times (B \times C) = (A \times B) \times C$) but not commutative ($A \times B \neq B \times A$).

Exercise 2.2

Using the matrices defined above; compute the results of the following matrix multiplications:

i. $M_5 \times M_1$
ii. $M_2 \times M_4$
iii. $M_5 \times M_4$
iv. $M_2 \times M_3$
v. $M_2 \times M_5$
vi. $M_2 \times M_2$

Note the result of item *iv* in the exercise. The result is just M_2! What would happen if you multiplied $M_3 \times M_2$, $M_5 \times M_3$, or $M_3 \times M_5$? M_3 is a square matrix with the special property that if you multiply it with any other 3 × 3 matrix, the result is the other matrix. Any square matrix with ones in the main diagonal (upper left to lower right) and zeros everywhere else has this property, and we refer to these matrices as *identity matrices*. We denote the identity matrices with the capital letter I and a subscript indicating the number of rows and columns, so M_3 is better known by the proper name I_3.

Part 3: Matrix representation of graphs

We've already seen that we can use matrices to represent graphs and other relations. To do this we usually use a Boolean matrix, that is, a matrix with only zeros and ones. (We could use *true*s and *false*s, but we want to be able to multiply these matrices, so we use zero for *false* and one for *true*.) The matrix has a one in row i column j if there is an edge from vertex i to vertex j in the graph; otherwise, it has a zero.

Consider the graph $G = (V, E)$ where $V = \{0, 1, 2, 3\}$, and $E = \{(0, 1), (0, 3), (1, 0), (1, 2), (2, 0), (2, 3), (3, 1)\}$. You may wish to draw a dot and arrow diagram of the graph. The matrix representation of this graph is:

$$G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The matrix implicitly gives the whole definition of the graph. The rows and columns are numbered starting at zero, giving the set V of vertex labels; the ones in the matrix correspond to the edges in E. Note there are zeros in the main diagonal because this graph as no self-edges.

We can now use the matrix representation to compute the transitive closure of a graph. We note first that we can multiply any square matrix by itself; in particular, we can multiply the matrix representation of a graph by itself. Thus, we can denote $G \times G$ as G^2, $G \times G \times G$ as G^3, and so on. We assert that there is an edge between two vertices in G^2 if there is a path between those vertices of length two in G, and further that there is an edge between two vertices in G^i if there is a path between those vertices of length i in G. We will not prove this assertion here, but exercise 3.1 gives it support.

Exercise 3.1

Compute the value of G^2, G^3, and G^4. Draw dot and arrow diagrams for all of these graphs (including G, if you have not already done so).

The *transitive closure* of a graph is a graph with the same vertex set and an edge set that contains the edge (i, j) if and only if there is a path in the original graph from vertex i to vertex j. If a path exists between two vertices, then there must be a path no longer than the number of vertices, since otherwise, there would be at least one repeated vertex, creating a cycle, and we could remove that cycle to eliminate the repetition. Thus, the transitive closure of a graph GN with n vertices has an edge set that is the union of the edge set of GN, GN^2, … GN^n.

We can compute the transitive closure from the power graphs by computing the *or* (Boolean sum) of the graphs (including the original graph itself.)

Exercise 3.2

Use your results from exercise 3.1 to compute the transitive closure of G. Draw a dot and arrow diagram for the result. Verify that for every edge in the transitive closure graph there is a path between the edge vertices in G.