Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on a separate sheet of paper.

Counting by complement

1. Give numerical answers for the questions below.

   (a) There are 5 kids on the math team. Two kids will be selected from the team to compete in the state competition. How many ways are there to select the 2 competitors?

   Solution: $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = \frac{5\cdot4}{2} = 10$

   (b) The math team has 3 girls and 2 boys. How many ways are there to select the two competitors if they are both girls?

   Solution: $\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3\cdot2\cdot1}{2\cdot1\cdot1} = \frac{6}{2} = 3$

   (c) The math team has 3 girls and 2 boys. How many ways are there to select the two competitors so that at least one boy is chosen?

   Solution: Counting directly Two choices for first boy. For each of those choices, there are four choices (1 boy, 3 girls) for the other competitor, which gives $2 \cdot 4 = 8$. However, counting this way counts the team of two boys twice, once for boy 1 and once for boy 2, thus must be subtracted away, so $2 \cdot 4 - 1 = 7$.

   Counting by complement From Part (a), there are 10 total ways to make a two person team, three of which have no boys. Thus, the remaining $10 - 3 = 7$ teams must have at least one boy on them.

Inclusion-exclusion principle

2. You are contracted by the FBI to unlock the phone of a suspected criminal. Your forensics team informs you that the material on the glass where the 8-key would appear during passcode entry has more oil-residue, left behind by fingers, than other places on the screen. This is understood to be a strong indication that the passcode contains at least one 8. How much does knowing that the passcode includes the digit 8 narrow the search space, considering that the passcode is 4 digits?

   (a) Use the inclusion-exclusion principle to count the number of 2 digit passcode that include an 8.

   > **Solution:** There are 10 passcodes of the form '`8[0-9]`' and 10 passcodes of the form '`[0-9]8`'. However these two sets are not disjoint. In fact their intersection has exactly one element, the passcode '`88`'. So to count the total number of 2 digit passcodes that include an 8, we have $10 + 10 - 1 = 19$.

   (b) Use the inclusion-exclusion principle to count the number of 3 digit passcode that include an 8.

   > **Solution:** There are 100 passcodes of the form '`8[0-9][0-9]`', 100 passcodes of the form '`[0-9]8[0-9]`' and 100 passcodes of the form '`[0-9][0-9]8`'. However these three sets are not disjoint. Furthermore, no pair of these sets is disjoint. In fact, given any pair of these sets, their intersection has exactly 10 elements and the intersection of all three sets has exactly one element. So to count the total number of 3 digit passcodes that include an 8, we have $100 + 100 + 100 - 10 - 10 - 10 + 1 = 271$.

   (c) Use the inclusion-exclusion principle to count the number of 4 digit passcode that include an 8.

   > **Solution:** There are 1000 passcodes of the form '`8[0-9][0-9][0-9]`', likewise for an 8 in the second, third, or fourth position. As before, the sets are not disjoint, so we subtract away elements that are shared by any two sets, 100 per pair. Then we add back elements shared by any three sets, 10 per triple, than subtract away the single passcode that is in all four sets, namely '`8888`'. This gives us $1000+1000+1000+1000-\binom{4}{2}\cdot 100+\binom{4}{3}\cdot 10-1 = 4000-600+40-1 = 3439$.

3. Solve the previous three sub-problems using counting by complement rather than the inclusion-exclusion principle, to verify that you have the correct answers, i.e., the total number of passcodes should be the same.

   (a)

   > **Solution:** The total number of 2 digit passcodes is 10^2. The total number of 2 digit passcodes that do not include an 8 is 9^2. So the total number of 2 digit passcodes that include at least one 8 is $10^2 - 9^2 = 19$.

   (b)

   > **Solution:** The total number of 3 digit passcodes is 10^3. The total number of 3 digit passcodes that do not include an 8 is 9^3. So the total number of 3 digit passcodes that include at least one 8 is $10^3 - 9^3 = 271$.

(c)

Solution: The total number of 4 digit passcodes is 10^4. The total number of 4 digit passcodes that do not include an 8 is 9^4. So the total number of 4 digit passcodes that include at least one 8 is $10^4 - 9^4 = 3439$.