# CSC 433/33
# Computer Graphics

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Credit: Joshua Levine

# Matrix Math

Oct. 22, 2020

# What is a Matrix?

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  - Geometric transforms, in particular of shapes represented as vectors
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- Compare with both vectors and scalars?

# Recall: Vector Multiplication

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$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b$$

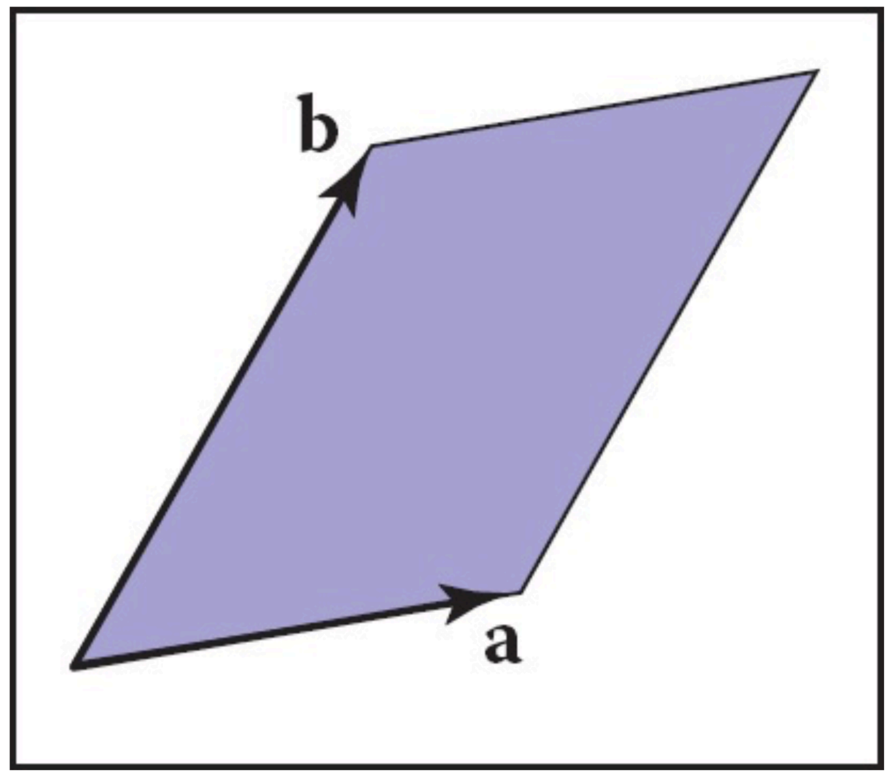
  - Cross product (2 vectors in, 1 vector out)

$$\mathbf{a} \times \mathbf{b} = (y_a z_b - z_a y_b,\ z_a x_b - x_a z_b,\ x_a y_b - y_a x_b)$$

# Determinants as Vector Multiplication

- Usually thought of as an operation on a matrix (similar to vector norms) that produces a scalar, but they can also be considered a multiplication of vectors:

- For 2d vectors **a** and **b**, the **determinant**, det(**ab**), is equal to the *signed* area of the parallelogram formed by **a** and **b**
  - **Signed** here means that det(**ab)** = -det(**ba),**
  - It is positive iff **b** is **CCW** to **a**
  - Related: ||**a** × **b**||
  - Conclusion det(**ab**)=0 iff **ab** are collinear .

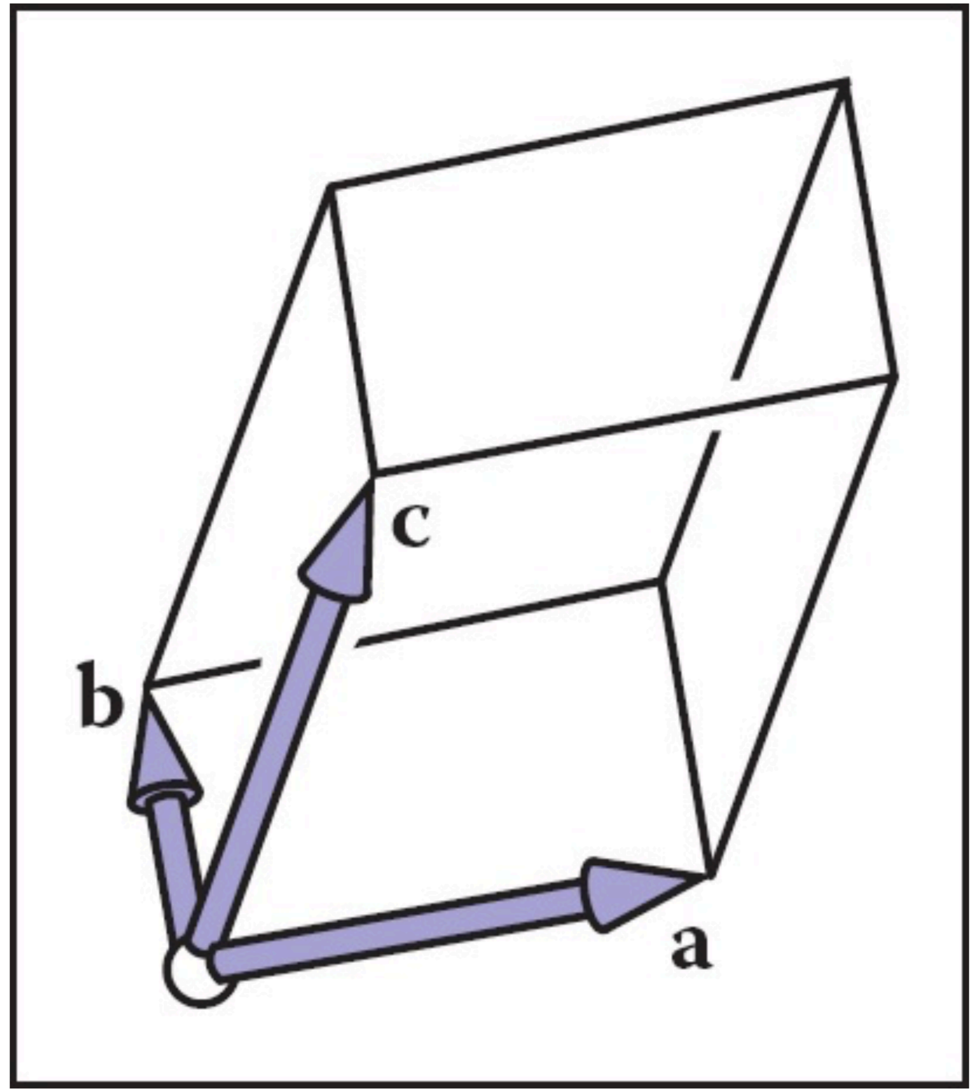


# Determinants as Vector Multiplication

- For 3d vectors **a**, **b**, and **c**, the determinant, |**abc**|, is the *signed* volume of the parallelepiped formed by **a**, **b**, and **c**

- Sign refers to left-handed or right-handed coordinate system

- What could we say about **a,b,c if** |**abc**|=0?



# Properties of Determinants

- Scale one side, scale determinant:

$$\left|(k\mathbf{a})\mathbf{b}\right| = \left|\mathbf{a}(k\mathbf{b})\right| = k\left|\mathbf{a}\mathbf{b}\right|$$

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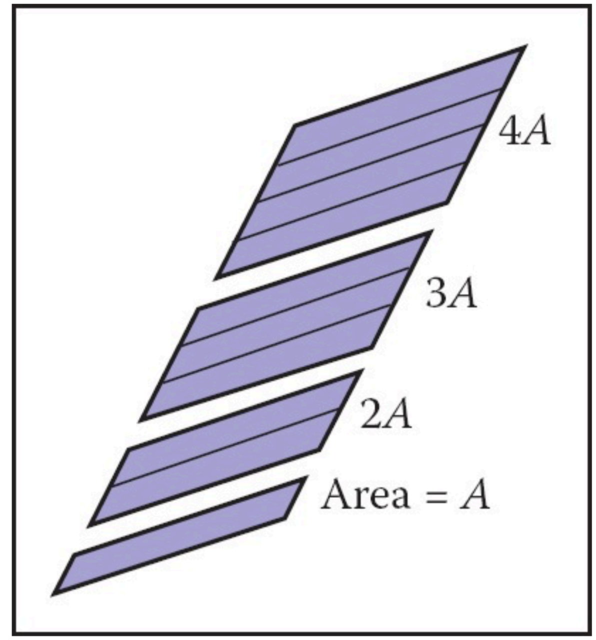


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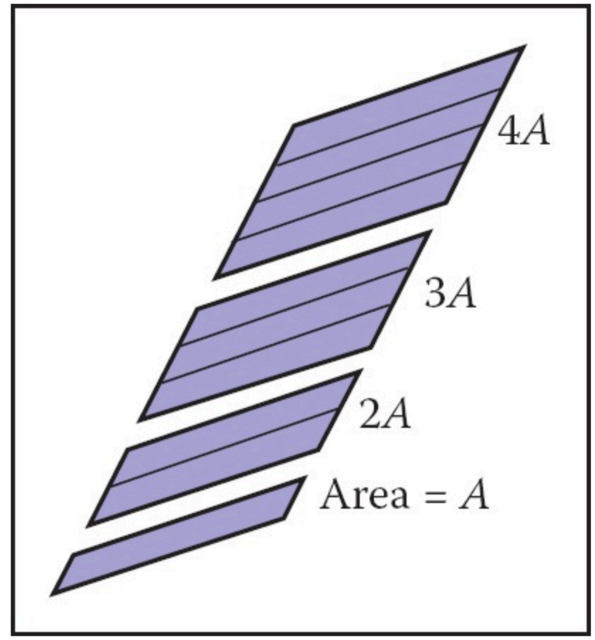


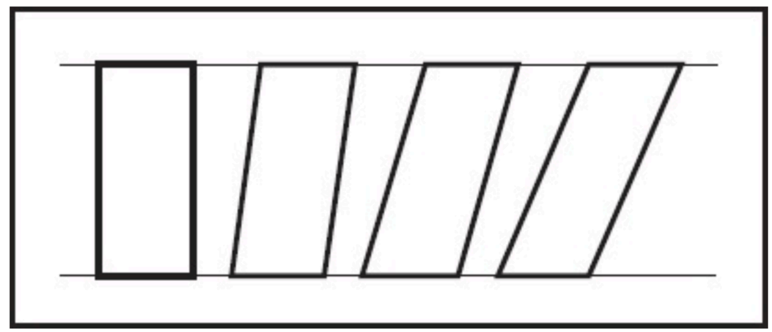

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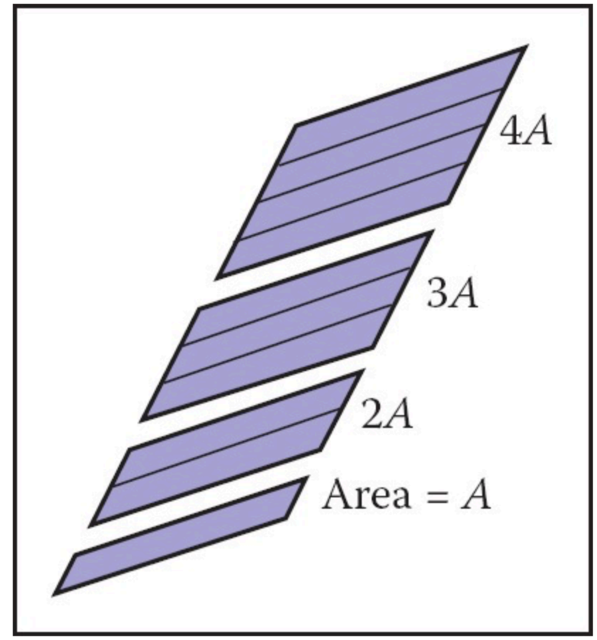
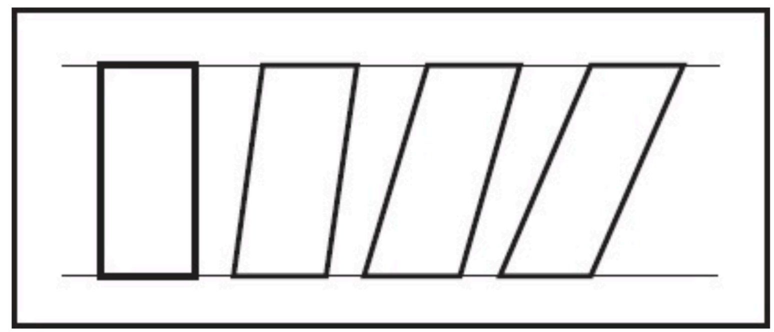
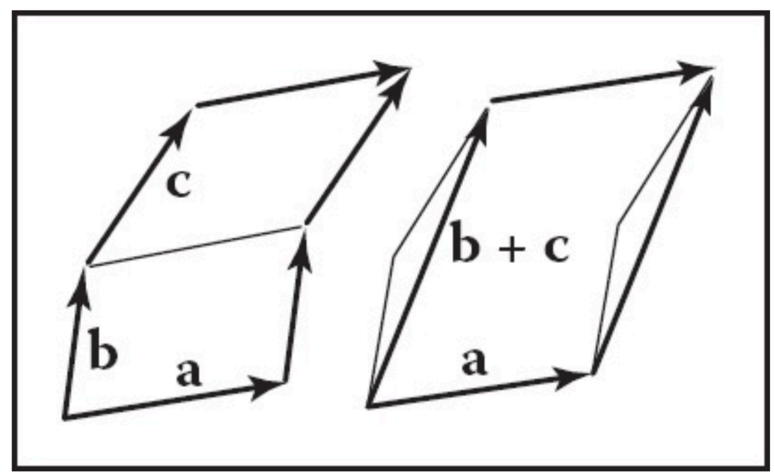
- Distribution:

$$|\mathbf{a}(\mathbf{b} + \mathbf{c})| = |\mathbf{ab}| + |\mathbf{ac}|$$





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- For 2, 2D vectors

$$|\mathbf{ab}| = |(x_a\mathbf{x} + y_a\mathbf{y})(x_b\mathbf{x} + y_b\mathbf{y})|$$

$$= x_a x_b|\mathbf{xx}| + x_a y_b|\mathbf{xy}| + y_a x_b|\mathbf{yx}| + y_a y_b|\mathbf{yy}|$$

$$= x_a x_b(0) + x_a y_b(+1) + y_a x_b(-1) + y_a y_b(0)$$

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- For 3, 3D vectors

$$|\mathbf{abc}| = |(x_a\mathbf{x}+y_a\mathbf{y}+z_a\mathbf{z})(x_b\mathbf{x}+y_b\mathbf{y}+z_b\mathbf{z})(x_c\mathbf{x}+y_c\mathbf{y}+z_c\mathbf{z})|$$
$$= x_a y_b z_c - x_a z_b y_c - y_a x_b z_c + y_a z_b x_c + z_a x_b y_c - z_a y_b x_c$$

# Matrix Operations

# Matrix Arithmetic

- Multiplication by a scalar, element-wise

$$2\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -8 \\ 6 & 4 \end{bmatrix}$$

- Matrix-Matrix addition, element-wise

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$$

# Matrix-Matrix Multiplication

- Can multiply any *r-by-m* matrix with any *m-by-c* matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{im} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rm} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1c} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \cdots & b_{mj} & \cdots & b_{mc} \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1c} \\ \vdots & & \vdots & & \vdots \\ p_{i1} & \cdots & \boxed{p_{ij}} & \cdots & p_{ic} \\ \vdots & & \vdots & & \vdots \\ p_{r1} & \cdots & p_{rj} & \cdots & p_{rc} \end{bmatrix}$$

- Each new term is defined as

$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

# Matrix-Matrix Multiplication

- Multiplication is not commutative, $\mathbf{AB} \neq \mathbf{BA}$
- And if $\mathbf{AB} = \mathbf{AC}$, it does not necessarily follow that $\mathbf{B} = \mathbf{C}$
- But, multiplication is associative and distributive:

$$
\begin{aligned}
(\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC}), \\
\mathbf{A}(\mathbf{B}+\mathbf{C}) &= \mathbf{AB}+\mathbf{AC}, \\
(\mathbf{A}+\mathbf{B})\mathbf{C} &= \mathbf{AC}+\mathbf{BC}.
\end{aligned}
$$

# Matrix Inverses

- For square (*n-by-n*) matrices, we can define the action of undoing multiplication, called the **matrix inverse**
    - To do so, we need a matrix that behaves like "1", called the **identity matrix**, $\mathbf{I}$, e.g.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The **inverse matrix** of **A**, written $\mathbf{A}^{-1}$, is the matrix that ensures $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

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$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} \text{ because } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Matrix Inverses

- If the inverse of $\mathbf{A}$ is $\mathbf{A}^{-1}$, then the inverse of $\mathbf{A}^{-1}$ is $\mathbf{A}$, thus

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- Inverse of the product of two matrices, $\mathbf{A}$ and $\mathbf{B}$, is the product of the inverses, with the order reversed

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

# Matrix Transpose

- The **transpose** of a matrix **A**, written $\mathbf{A}^\mathbf{T}$, swaps the rows and columns, e.g. $a_{ij} = a^\mathbf{T}_{ji}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^\mathrm{T} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

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- Transpose of the product of two matrices is similar to inverse of product:

$$(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$$

# Matrix Determinant

- If the matrix is **square** (has the same number of rows/columns), it can be computed by taking the **columns** of the matrix as vectors and computing their determinant

# Rules for Determinants and Matrix Operations

- Determinant and Matrix-Matrix Multiplication:

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- Determinant and Matrix Transpose:

$$|\mathbf{A}^{T}| = |\mathbf{A}|$$

# Matrix-Vector Multiplication

- Can consider a vector of length *m* as just an *m-by-1* matrix
    - Convention is that vectors are columns
- Multiplying a vector by a matrix produces another vector, in doing so, transforms the vector, e.g.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} -y_a \\ x_a \end{bmatrix}$$

- This matrix has the effect of rotating the vector by 90 degrees

# Vector Operations as Matrix Operations

- Can rewrite dot product, outputting a 1-by-1 matrix

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} \qquad \begin{bmatrix} x_a & y_a & z_a \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} x_a x_b + y_a y_b + z_a z_b \end{bmatrix}$$

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- Likewise, one can also define the **outer product** of a pair of vectors, which takes two vectors and produces a matrix

$$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \begin{bmatrix} x_b & y_b & z_b \end{bmatrix} = \begin{bmatrix} x_a x_b & x_a y_b & x_a z_b \\ y_a x_b & y_a y_b & y_a z_b \\ z_a z_b & z_a y_b & z_a z_b \end{bmatrix}$$

# Matrix Multiplication as Vector Operations

- We can think of matrix-matrix multiplication as a collection of vector operations

$$\begin{bmatrix} | \\ \mathbf{y} \\ | \end{bmatrix} = \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ - & \mathbf{r}_3 & - \end{bmatrix} \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix}$$

- Alternate interpretation is a weighted sum

$$\begin{bmatrix} | \\ \mathbf{y} \\ | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};$$

$$\mathbf{y} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 .$$

- More generally, the matrix-matrix product **AB** is an array containing all pairwise dot products

# Special Matrix Types for Matrix Operations

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- Matrices whose columns are orthogonal vectors *of unit length* are called **orthogonal matrices.**
  - Determinant of an orthogonal matrix is always 1 or -1
  - Orthogonal matrices are almost their own inverse, for an orthogonal matrix $\mathbf{R}$, $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$.

# Basis

- We say that vectors **a,b** form a basis in 2D if any other vector **v** could be written as a linear combination of **a,b** $\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b}$ where $\alpha, \beta$ are scalars.

- A. set of vectors $\mathbf{v_1}\ldots\mathbf{v_n}$ are **linearly dependent** if we can find scalars $\alpha_1 \ldots \alpha_n$ such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + .. + \alpha_n\mathbf{v}_n = 0$

- A basis is any set of linearly independent vectors.

- Examples: any two non-collinear vectors. They define a plane that they span. Any other vector **v** on this plane is a linear combination of these vectors.

- Examples, any 3 linearly independent vectors in 3D. Furthermore, we can create any other vector v as a linear combination of these vectors:
$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + .. + \alpha_n\mathbf{v}_n$

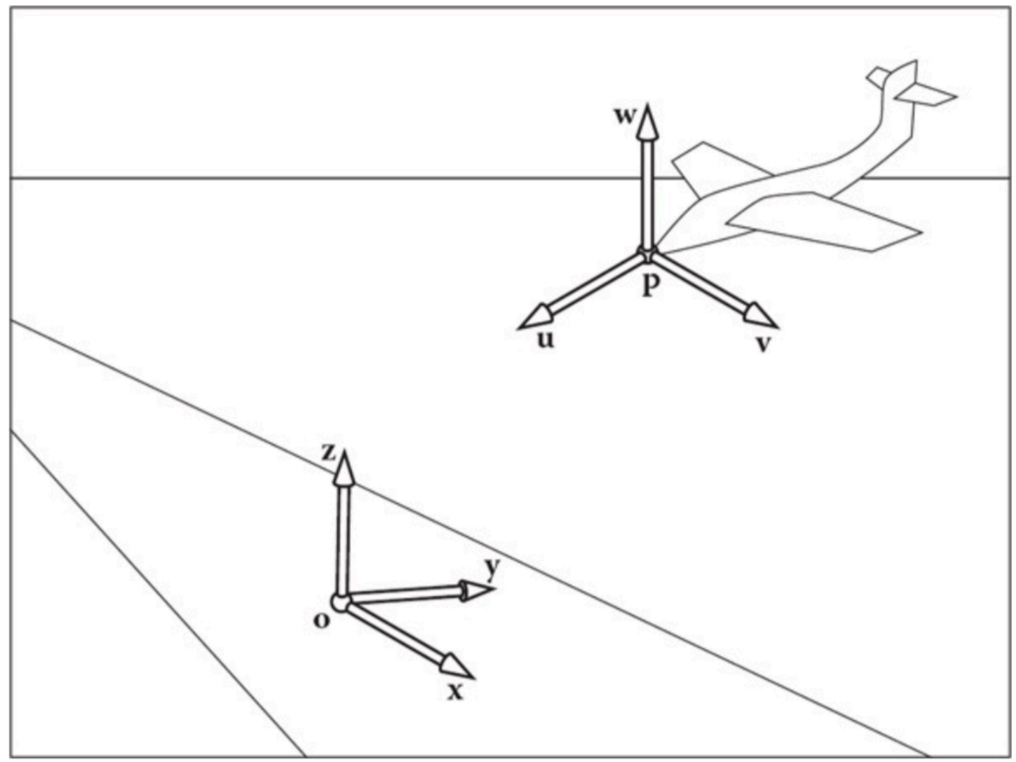
# Orthonormal Bases

- The columns of an orthogonal matrix define an orthonormal basis!

- Vectors **u**, **v**, and **w** define an orthonormal basis if

$$||\mathbf{u}|| = ||\mathbf{v}|| = ||\mathbf{w}|| = 1$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0$$



# Examples of Matrix Types

- Diagonal, Symmetric, but not Orthogonal

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

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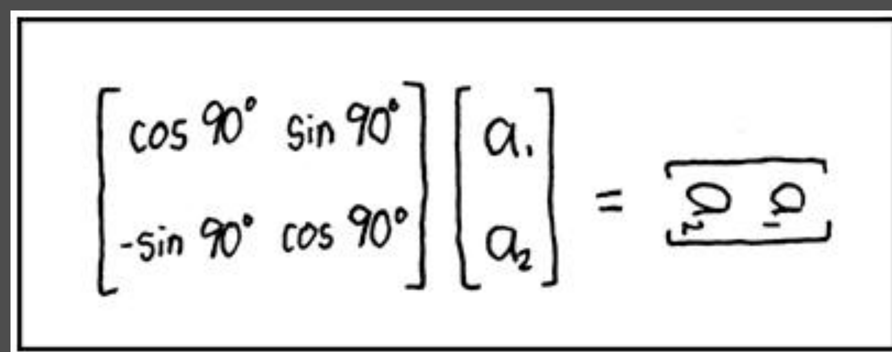
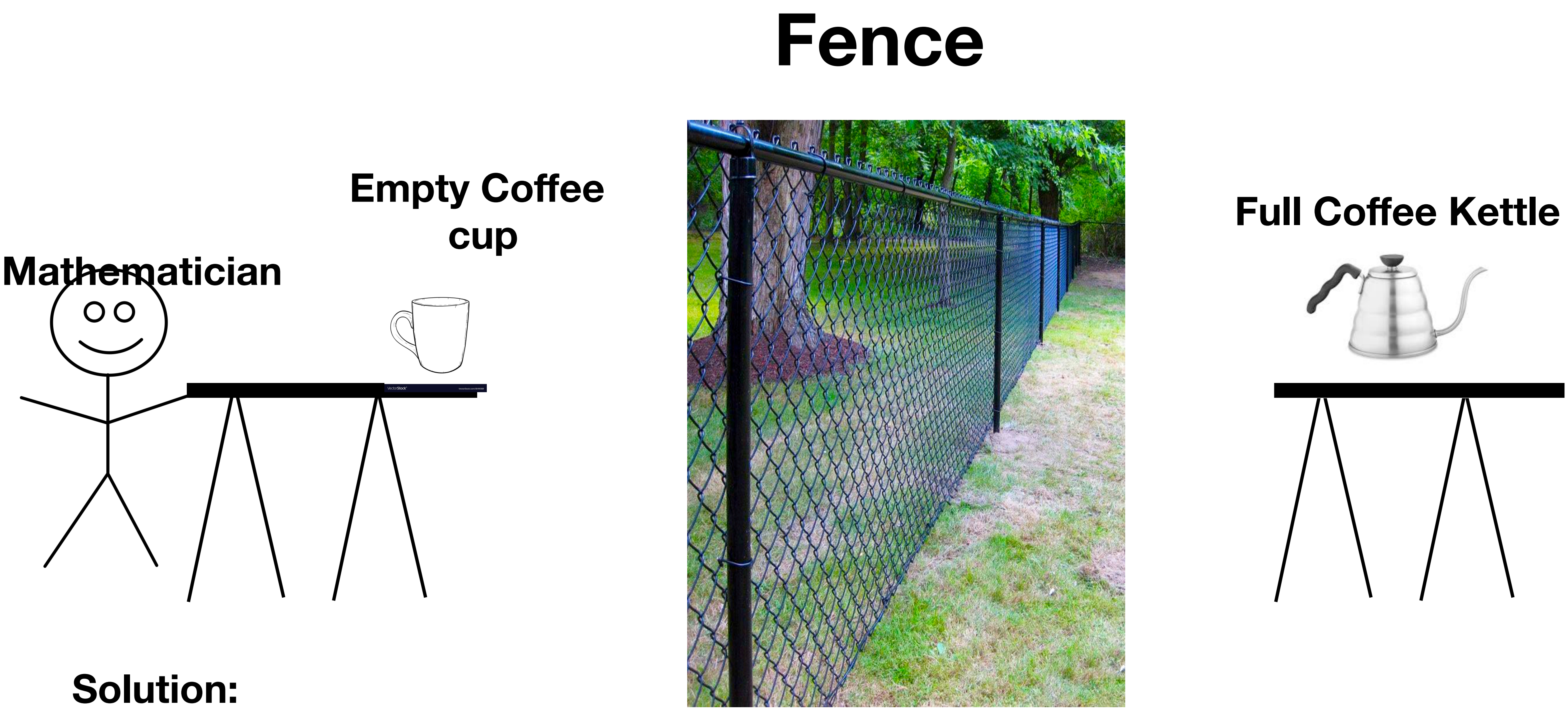
- Orthogonal, but not Symmetric or Diagonal $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

# Transformations



# The mathematician and coffee cup non-funny joke Part 1




**Solution:**

1. **Walk around the fence,**
2. **fetch coffee kettle,**
3. **walk back pure coffee,**
4. **drink**

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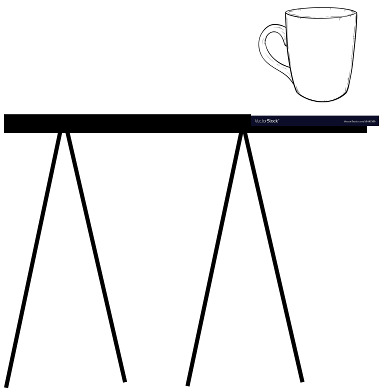
# Part 1

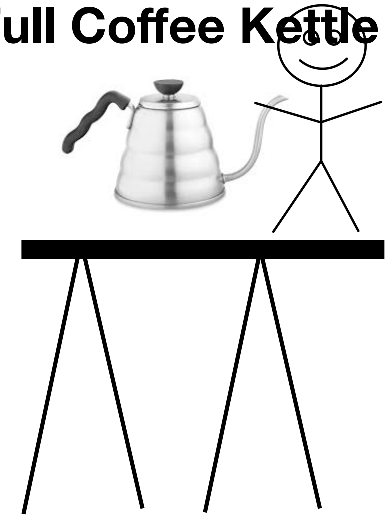
## Fence

**Empty Coffee cup**

**Mathematician**



**Full Coffee Kettle**



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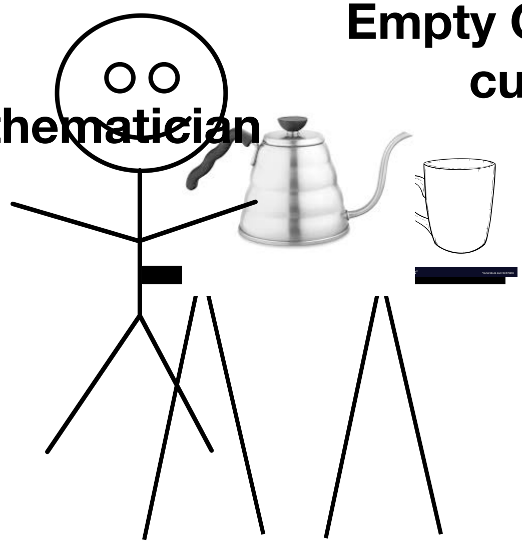

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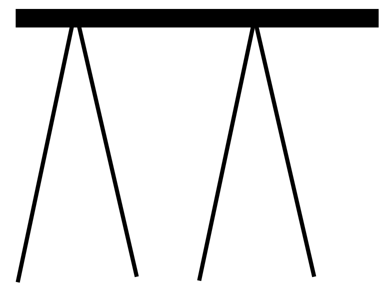

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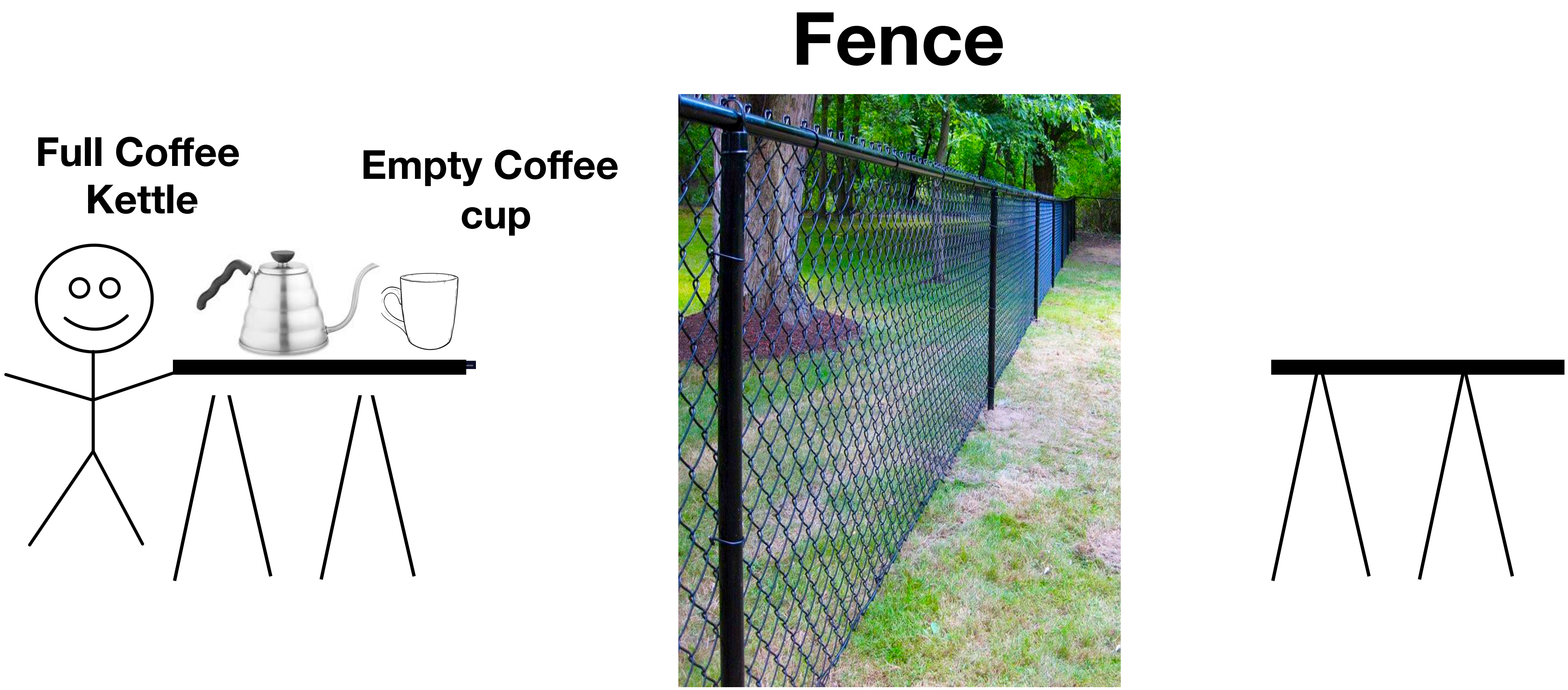


**Solution:**

1. **Bring the coffee Kettle to the other table, and walk to the left table**
2. **Apply the solution from the previous slide**

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# Part 2

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**Full Coffee Kettle**

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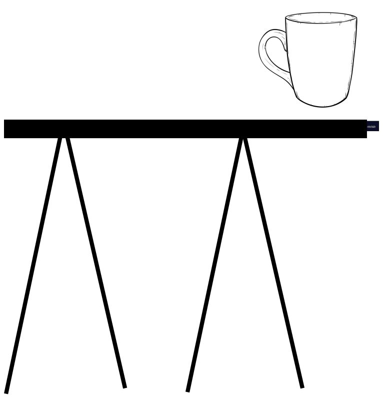

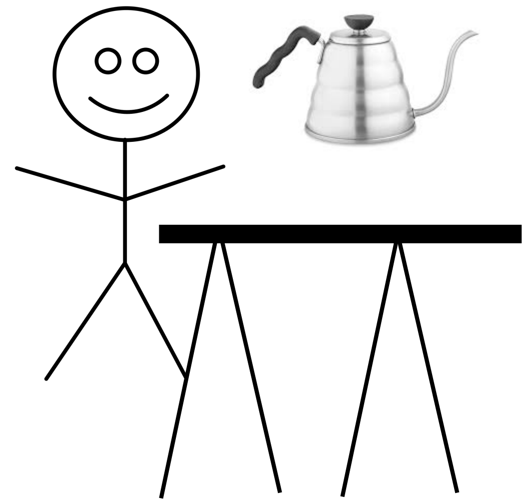

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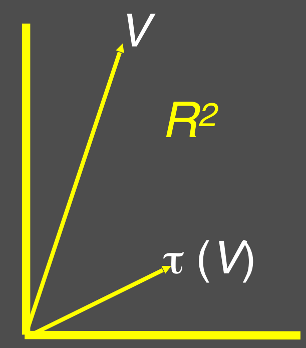
# Transformations

- A transformation τ is a mapping of $R^n$ to itself
  - (not necessarily one-to-one. Many points might be transformed to the same point.)
  - Linear transformation:

    $$\tau(\alpha\mathbf{u} + \beta\mathbf{b}) = \alpha \cdot \tau(\mathbf{u}) + \beta \cdot \tau(\mathbf{v})$$

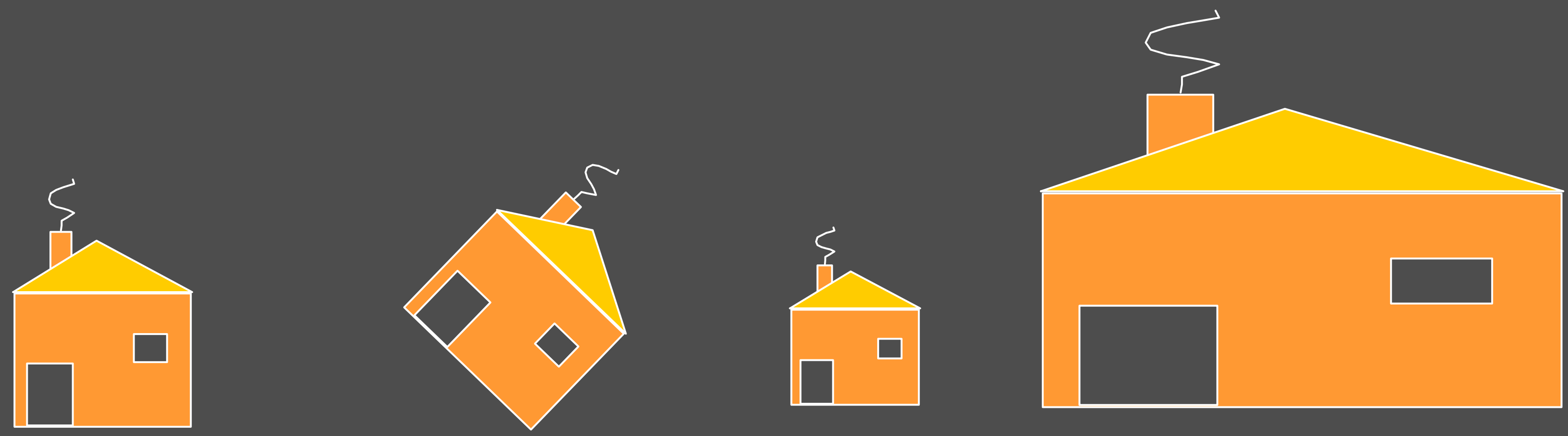
    for any $\mathbf{u}, \mathbf{v}$ vectors, and $\alpha, \beta$ scalars.
    In particular, zero is transformed to zero.




- *Affine* transformation – $\tau(V) = \mathbf{A}V + b$
  - $\mathbf{A}$ – matrix
  - $b$ – vector
- Relevant Example: Holography (project a billboard to the image plane). More about homographs later in the course.

# Transformations

- Transforming an object = transforming all its points
- For a polygonal model = transforming its vertices

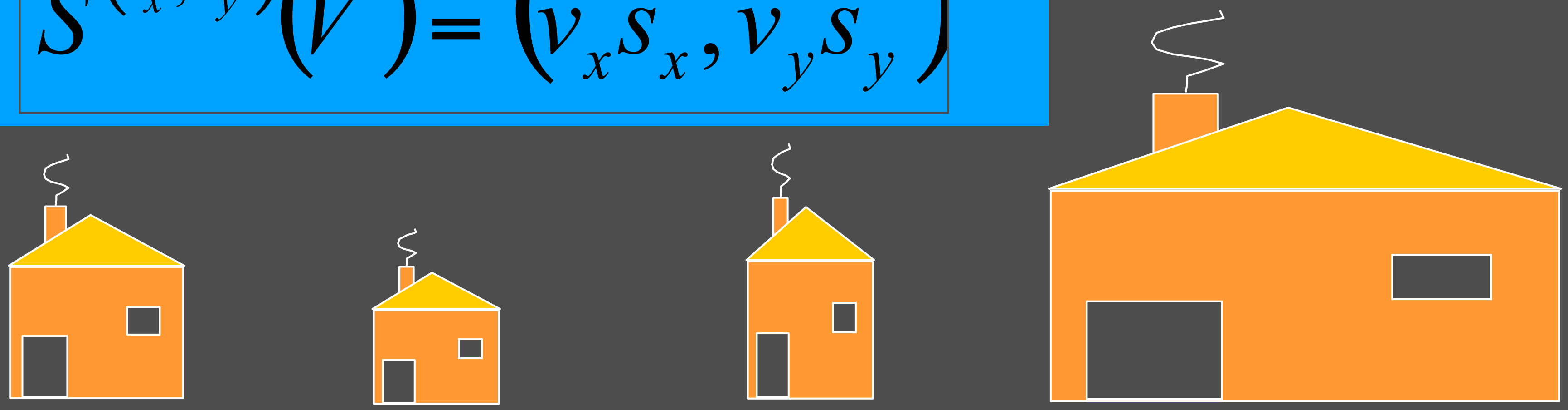


# Scaling

- $V = (v_x, v_y)$ – vector in XY plane

- *Scaling* operator $S$ with parameters $(s_x, s_y)$:

$$S^{(s_x, s_y)}(V) = (v_x s_x, v_y s_y)$$



# Scaling

- Matrix form:

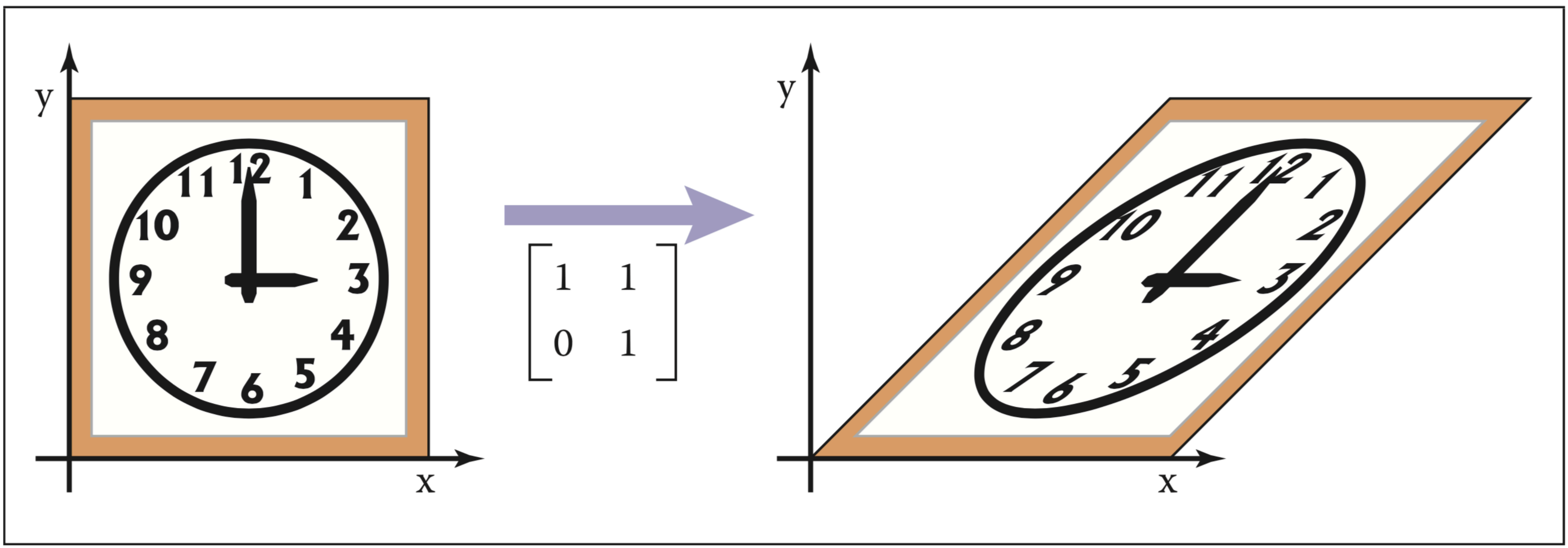
$$S^{(s_x, s_y)}(V) = (v_x, v_y)\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} = (v_x s_x, v_y s_y)$$

- Independent in $x$ and $y$
  - Non-uniform scaling is allowed
- What is the meaning of scaling by zero?

# Shearing

- Horizontal shearing shifts each row based on the y value.

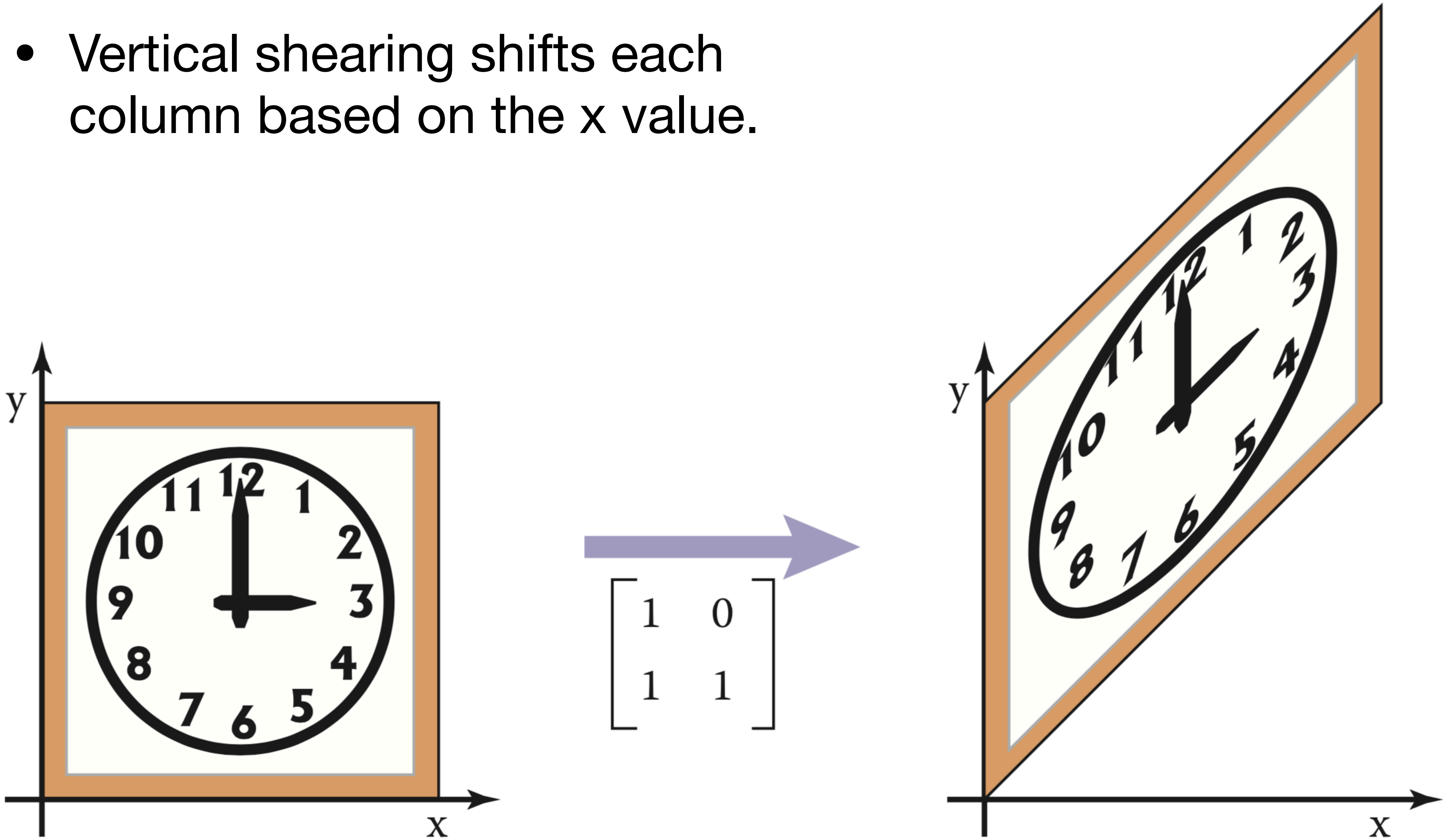
$$\text{shear-x}(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$



# Shearing

$$\text{shear-y}(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

- Vertical shearing shifts each column based on the x value.

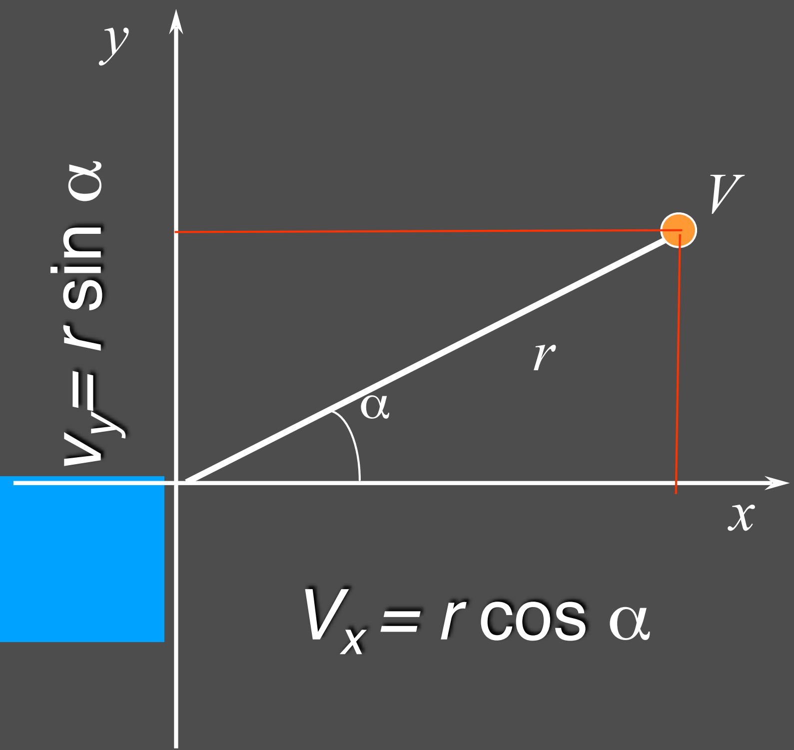



$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

# Polar form of a point

- Polar form:

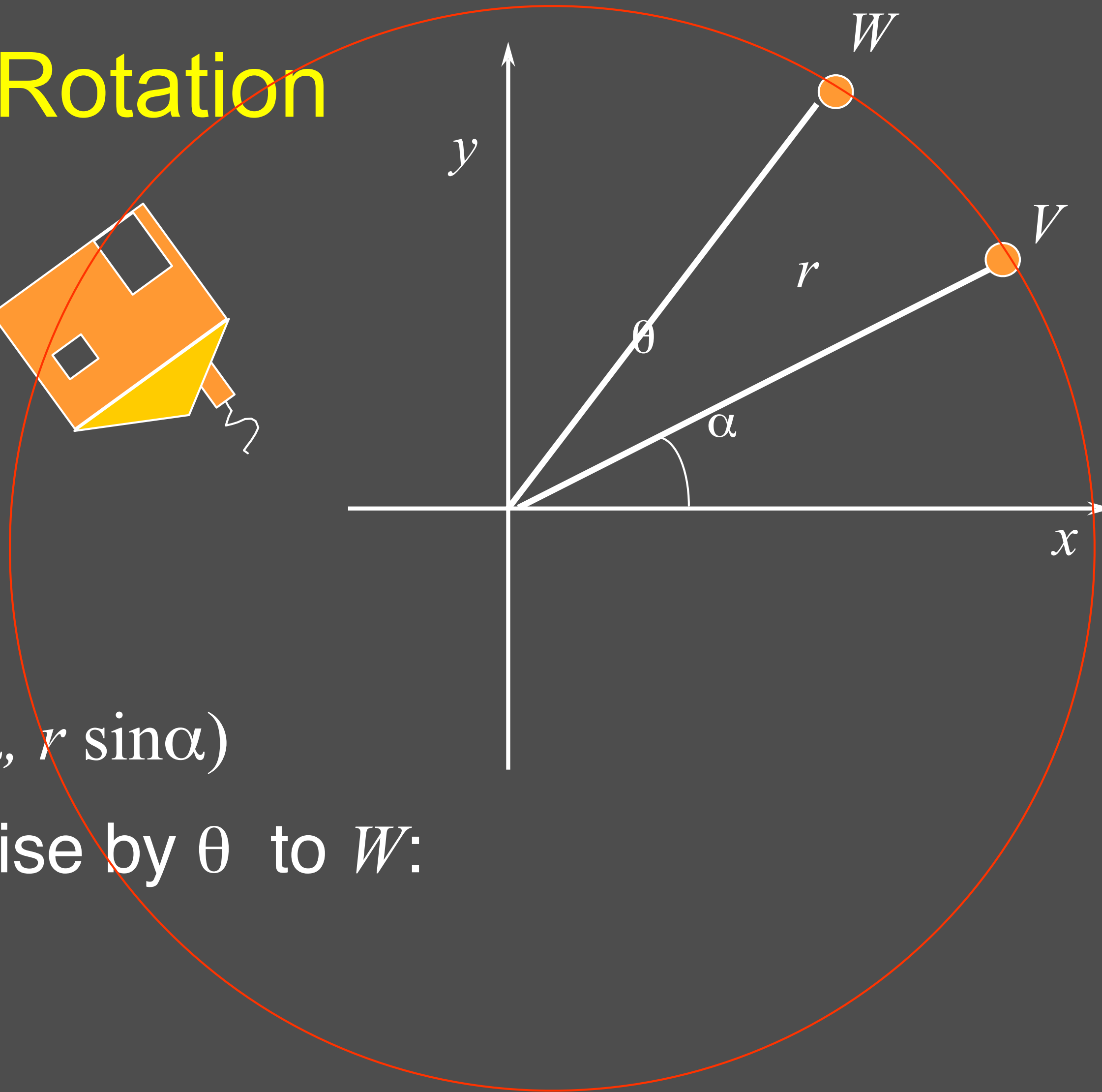
$$V=(v_x, v_y)=(r\cos\alpha,\ r\sin\alpha)$$




$r$ is the distance of $v$ from the origin (0,0)

$r = \text{sqrt}(\ v_x^2 + v_y^2\ )$

# Rotation

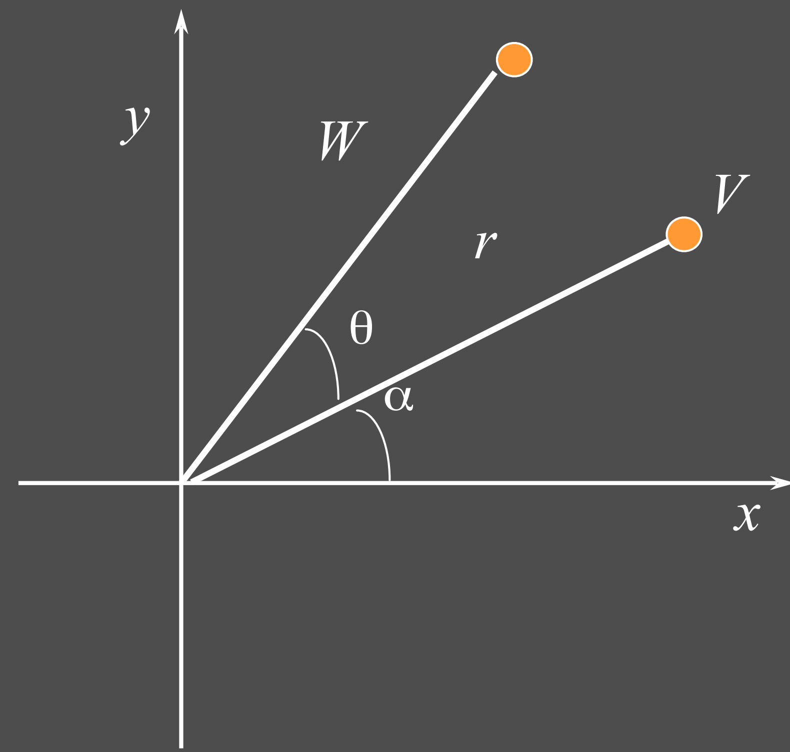



- Polar form:

$$V=(v_x, v_y)=(r\cos\alpha,\ r\sin\alpha)$$

- Rotate $V$ anti-clockwise by $\theta$ to $W$:

# Rotation CCW by $\theta$




- Polar form:

$$V=(v_x,v_y)=(r\cos\alpha,\ r\sin\alpha)$$

- Rotate $V$ counter-clockwise by $\theta$ to $W$

$$W=(r\cos\alpha, r\sin\alpha)\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$=V\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$W=(w_x,w_y)$$
$$=(r\cos(\alpha+\theta), r\sin(\alpha+\theta))$$
$$=(r\cos\alpha\cos\theta-r\sin\alpha\sin\theta,$$
$$r\cos\alpha\sin\theta+r\sin\alpha\cos\theta)$$

$$R^{\theta}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

# Rotation Properties

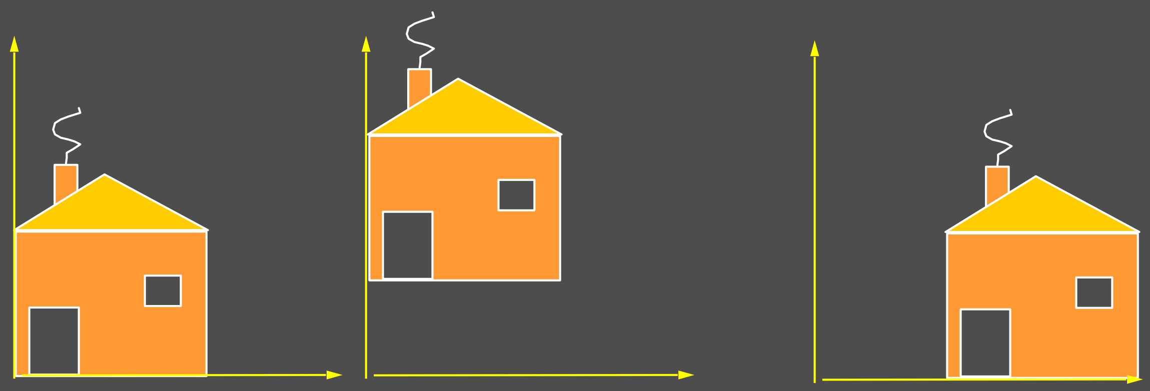
- $R^\theta$ is orthogonal

$$R^{-1} = R^{-\theta} = (R^{\theta})^T$$

- Rotation by $-\theta$ is

$$R^{-\theta}(V) = (v_x, v_y)\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \left(R^{\theta}\right)^{-1}$$

# Translation



- Translation operator $T$ with parameters $(t_x, t_y)$:

$$T^{(t_x, t_y)}(V) = (v_x + t_x, v_y + t_y)$$

- Can we express $T$ in a matrix form?

# Translation - Homogeneous Coordinates

- To represent $T$ in matrix form – use homogeneous coordinates:

$$V^h = \left(v_x^h, v_y^h, v_w^h\right) = \left(v_x^h, v_y^h, 1\right)$$

- Conversion (projection) from homogeneous to Euclidean:

$$V = (v_x, v_y) = \left(\frac{v_x^h}{v_w^h}, \frac{v_y^h}{v_w^h}\right)$$

- In homogeneous coordinates:

$$(2, 2, 1) = (4, 4, 2) = (1, 1, 0.5)$$

# Translation

- Using homogeneous coordinates:

$$T^{(t_x,t_y)}(V^h) = (v_x, v_y, 1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

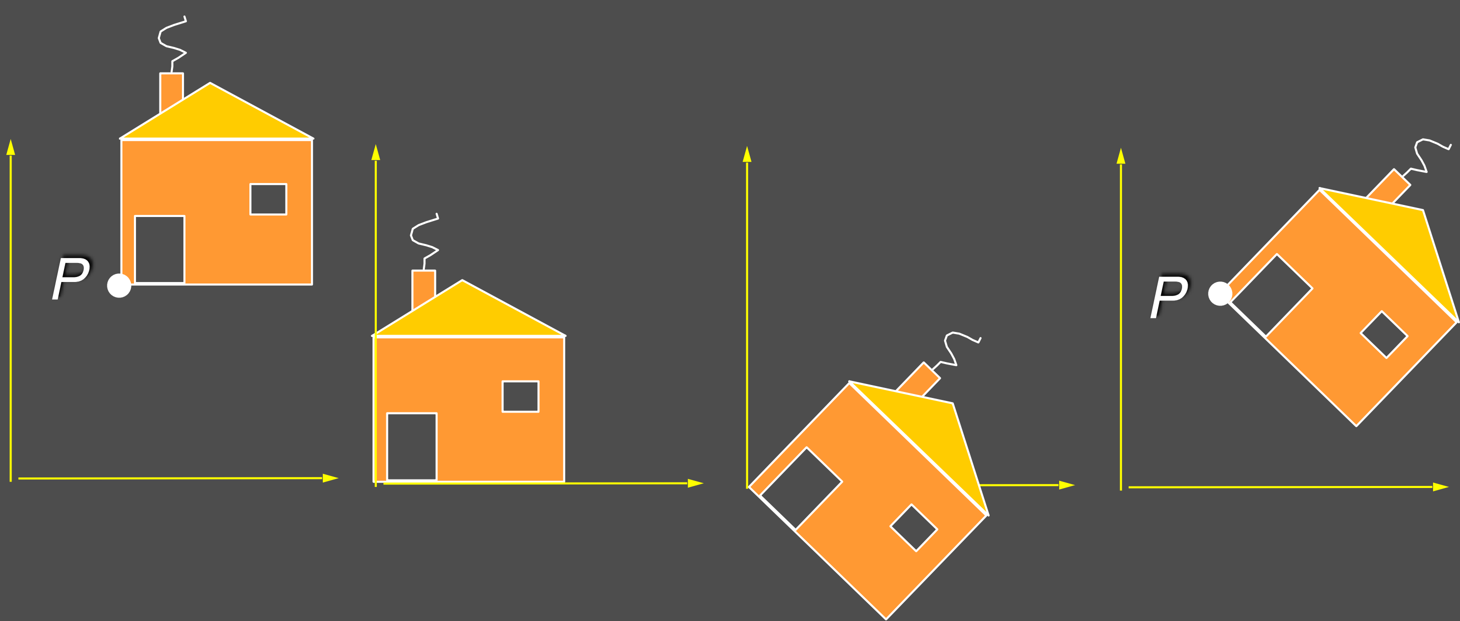
$$= (v_x + t_x, v_y + t_y, 1)$$

# Matrix Form

- Why is it useful to use **Matrix** form to represent the **transformations** ?
- The answer depends if we think about CPU or about GPU

# Transformation Composition

- What operation rotates by $\theta$ around $P = (p_x, p_y)$ ?
  - Translate $P$ to origin
  - Rotate around origin by $\theta$
  - Translate back



# Transformation Composition

$$T^{(-p_x,-p_y)} R^{\theta} T^{(p_x,p_y)} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p_x & -p_y & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_x & p_y & 1 \end{bmatrix}$$

Let *Q* denote this matrix (computed once).

For every point *p* of the many points of the "house", we apply

*p'*=*pQ*

(read: The old corner *p* is transformed to the new corner *p'*)

# Transformations Quiz

- What do these transformations do?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

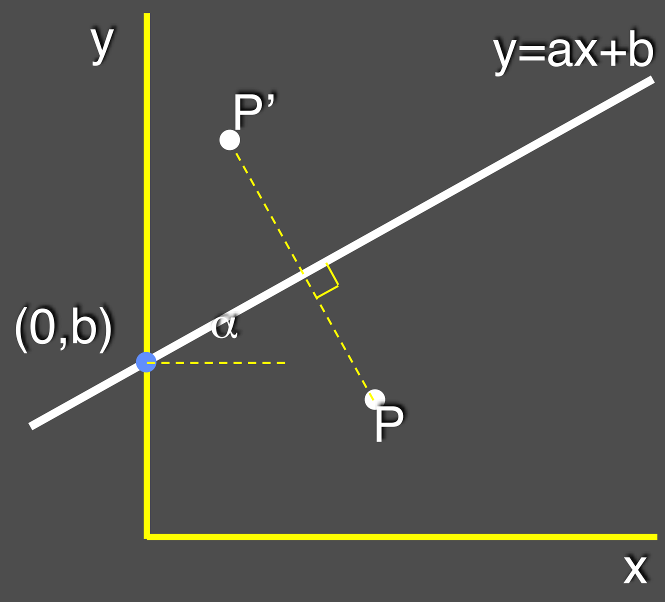
$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

- And these homogeneous ones?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

- How can one reflect a planar object through an arbitrary line in the plane?
- Can one rotate a planar object in the plane by reflection?

# Arbitrary Reflection




Shift by (0,-b)

Rotate by $-\alpha$ $\quad\quad \alpha=\tan^{-1}(a)$

Reflect through x

Rotate by $\alpha$

Shift by (0,b)

$$T^{(0,-b)} R^{-\alpha} \text{ Ref}^{x} R^{\alpha} T^{(0,b)}$$