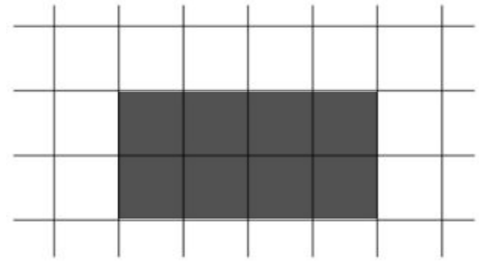
Transformations in 2D
Short version

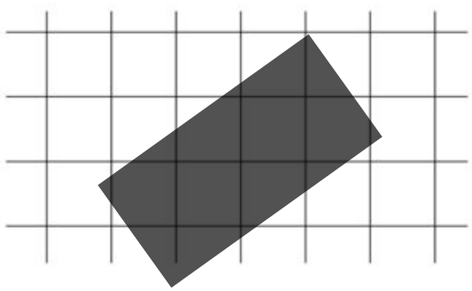
We will discuss transformation in 3D, and with full details, later in the course
(will need Matrix Multiplication and Homogenous coordinates)

About hw1
Aliasing and Anti-Aliasing



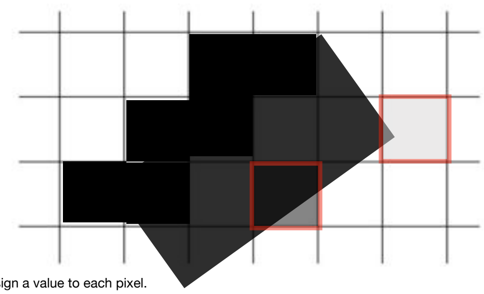
- This about an image where each pixels is fully black or fully white

What if we rotate the rectangle



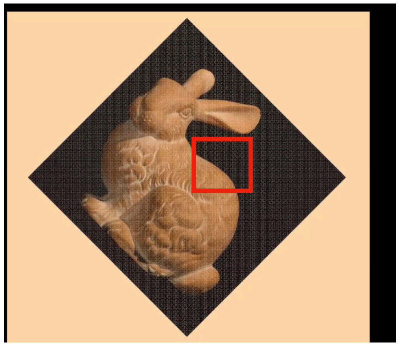
- Some pixels are **partially** covered by the rectangle. Show they be rendered as black, white, or some shade of grey ?

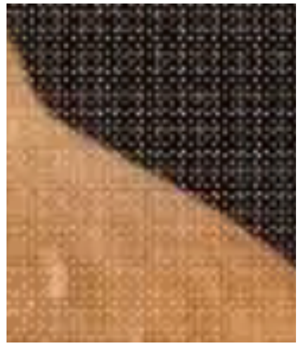
What if we rotate the rectangle



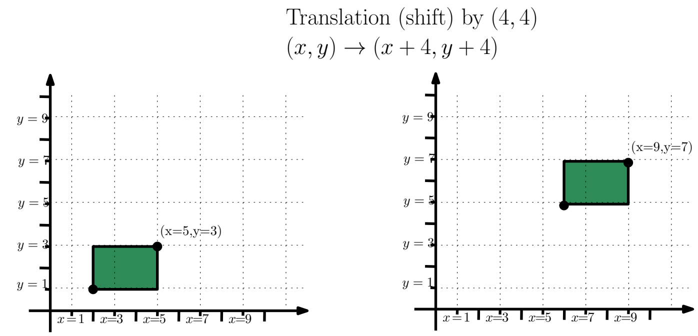
- IWe still need to assign a value to each pixel.
- If we draw each **partially covered** pixel as black, we will obtain a very pixelated shape. This is an example of **aliasing.**
- A possible solution is to render some pixels as gray. For example, based on the portion of its area which is covered. This technique is call antialiasing. Essentially, the color of a pixel might be determined using input from several neighboring pixels.
- We will study much much more about it. Do not worry about it in hw1.
- In hw1, each rendered pixel has the (rgb) value of one (single) input pixel. No averaging or mixing.

Something to be careful about with hw1





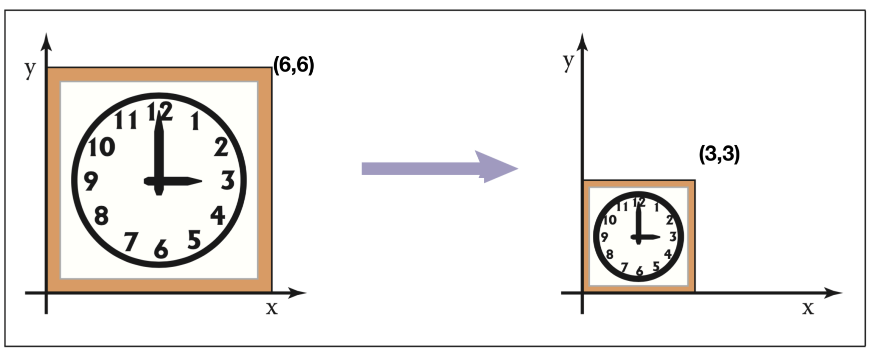
Translations (shift) by(α, β)



- Adding a constant α to the x-coordinate of every point
- Adding a constant β to the y-coordinate of every point
- $(x, y) \rightarrow (x+\alpha, y+\beta)$

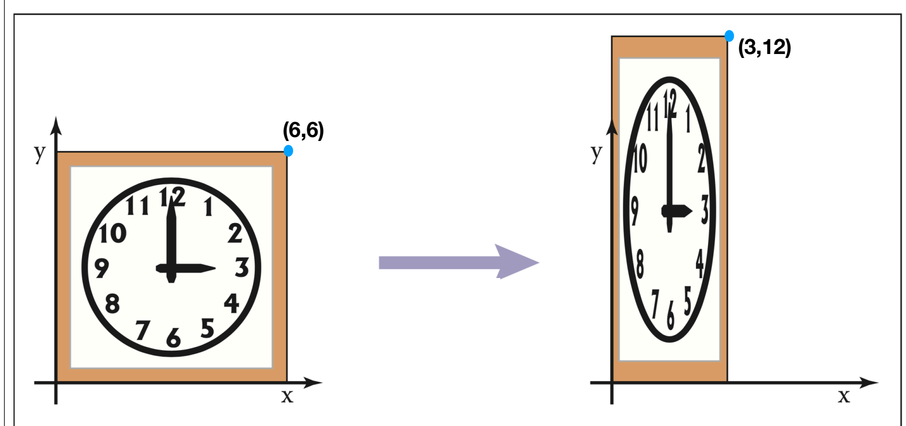
Scaling

- We can use two constants *(s_x,s_y)* for the x-axis and the y-axis. Then we shift each point *(x,y)* into the point $(s_x \cdot x, \ \ s_y \cdot y)$
- $(x, y) \rightarrow (s_x \cdot x, \ \ s_y \cdot y)$
- Example $(x, y) \rightarrow (x/2, \ y/2)$

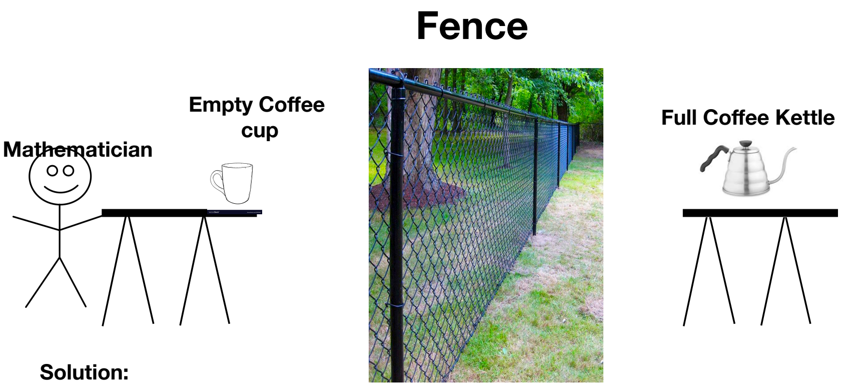


Scaling

- Example: $(x, y) \rightarrow (0.5x, \ \ 2y)$

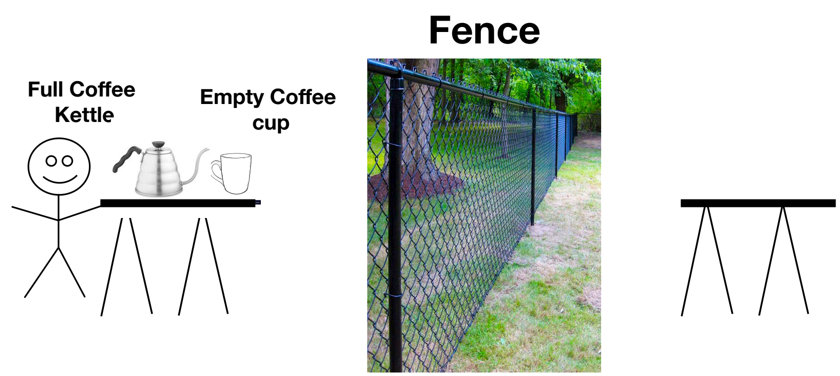


The mathematician and coffee cup non-funny joke Part 1



Solution:
1. **Walk around the fence,**
2. **fetch coffee kettle,**
3. **walk back pure coffee,**
4. **drink**

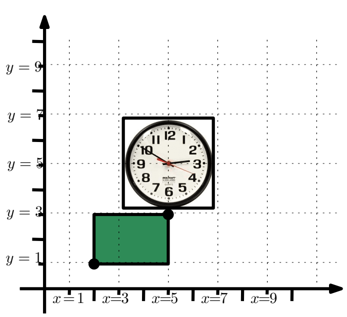
The mathematician and coffee cup non-funny joke Part 2

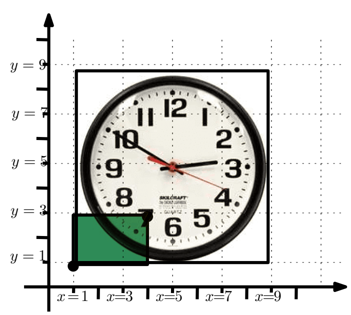


Solution:
1. **Bring the coffee Kettle to the other table, and walk to the left table**
2. **Apply the solution from the previous slide**

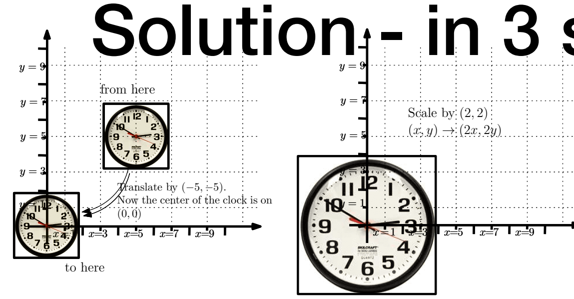
Resize the clock, without changing its center

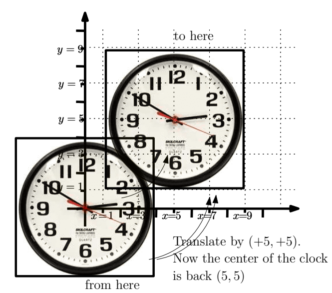
Problem: scale the clock, but without changing its center and without effecting the green rectangle





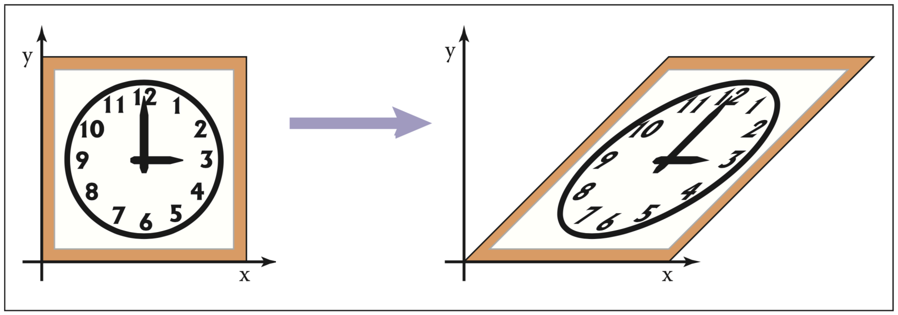
Solution - in 3 steps





Shearing

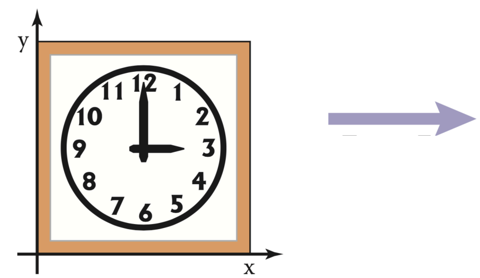
- If we move each point (x,y) into the point
  $(x, y) \rightarrow (x + y,\ y)$

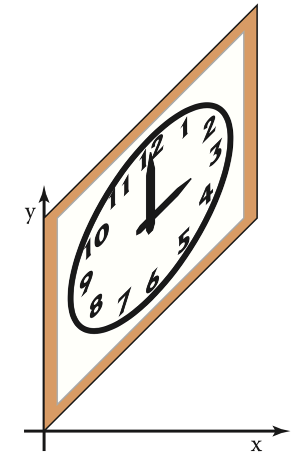


Shearing

- Vertical shearing shifts each column based on the x value.

$$(x, y) \rightarrow (x,\ x + y)$$

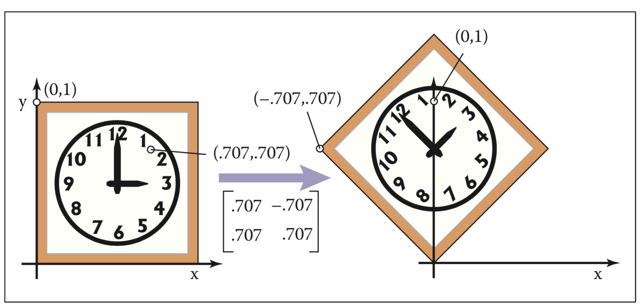




Rotation

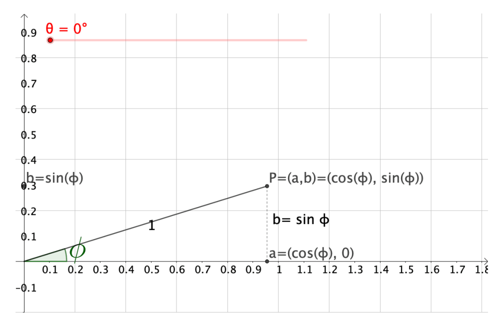
- Rotate counterclockwise by an angle θ about the origin.

$$(x, y) \rightarrow (\underbrace{x\cos\theta - y\sin\theta}_{\textbf{New } x},\ \ \underbrace{x\sin\theta + y\cos\theta}_{\textbf{New } y})$$



Assume we rotate p by an angle θ CCW

Starting from a point $P = (a, b)$, where will this point find itself after rotation by θ in the CounterClockwise direction ?
Let $p' = (x', y')$ denote the new location of this point. Lets compute this location:
For simplicity, assume $a^2 + b^2 = 1$



$$x' = \cos(\phi + \theta) = \underbrace{\cos(\phi)}_{=a}\cos(\theta) - \underbrace{\sin(\phi)}_{=b}\sin(\theta) = a\cos(\theta) - b\sin(\theta)$$

$$y' = \sin(\phi + \theta) = \underbrace{\sin(\phi)}_{=b}\cos(\theta) + \underbrace{\cos(\phi)}_{=a}\sin(\theta) \\ a\sin(\theta) + y\sin(\theta)$$

Assume we rotate p by an angle θ CCW

Starting from a point $P=(a,b)$, where will this point find itself after rotation by θ in the CounterClockwise direction ?
Let $p'=(x',y')$ denote the new location of this point. Lets compute this location:
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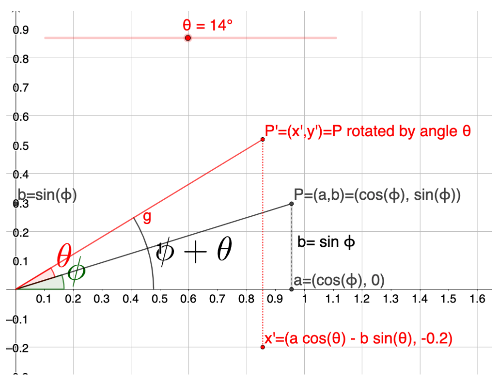
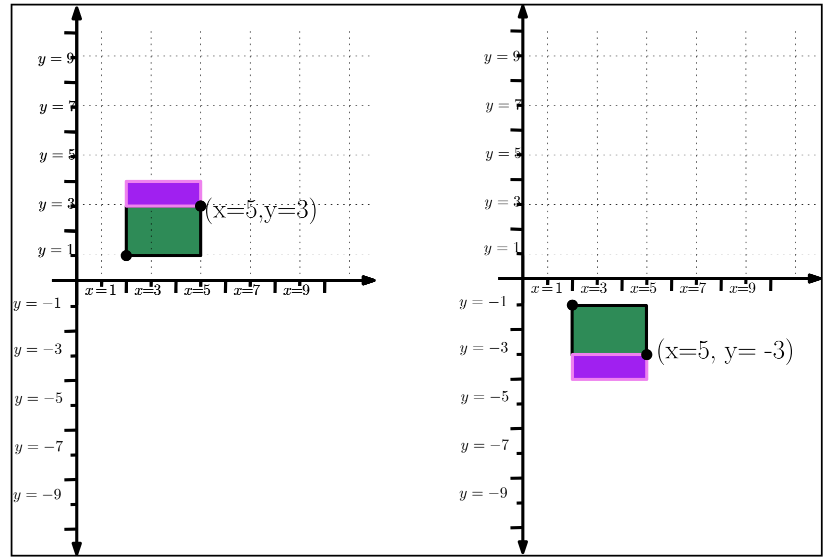


$$x' = \cos(\phi+\theta) = \underbrace{\cos(\phi)}_{=a}\cos(\theta) - \underbrace{\sin(\phi)}_{=b}\sin(\theta) = a\cos(\theta) - b\sin(\theta)$$

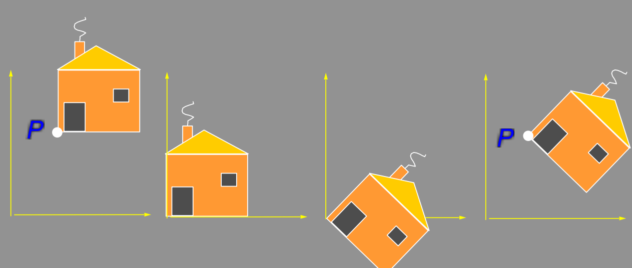
$$y' = \sin(\phi+\theta) = \underbrace{\sin(\phi)}_{=b}\cos(\theta) + \underbrace{\cos(\phi)}_{=a}\sin(\theta)\; a\sin(\theta) + y\sin(\theta)$$

Reflection on the x-axes: $(x,y) \rightarrow (x,\ -y)$



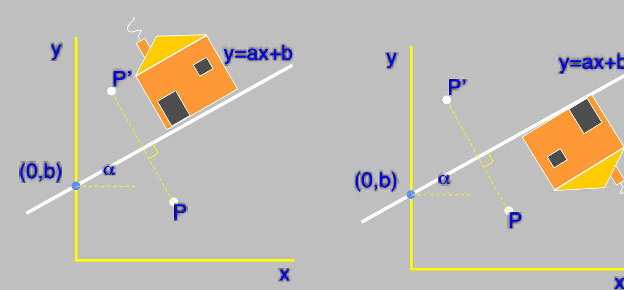
Transformation Composition

- What operation rotates by θ around $P=(p_x,p_y)$?
  - Translate P to origin
  - Rotate around origin by θ
  - Translate back



Arbitrary Reflection - promo
We will get back to it later in the semester

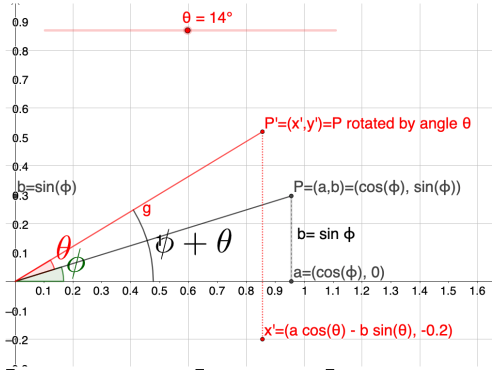
α=tan^{-1}(a)



1. Compute b.
2. Shift by (0,-b)
3. Rotate by –α CCW
4. Reflect through x
5. Rotate by α
6. Shift by (0,b)

Very scarrrrry….
Unless we represent transformation by matrices
And then it is trivial

Expressing rotations with matrices



- $$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix} = \begin{bmatrix} a' \\ y' \end{bmatrix}$$

What about other operaions

- Scaling ?
- Reflection by the x-axis ?
- Sheering ?
- Translation is problematic ?

Homogeneous coordinates

- We represent a point $p = (x, y)$ using 3 numbers $p = (x, y, w)_h$
- What are the coordinates of this point in Euclidean Cartesian representation ? $p = (x/w,\ y/w) = (x, y, w)_h$
- So $(4,2)_{Cartesian} = (4,2,1)_{homog} = (8,\ 4,\ 2)_h = (2,\ 1,\ 0.5)_{homog}$

Homogenous transformations are extremely useful in multiple graphics settings - including translations

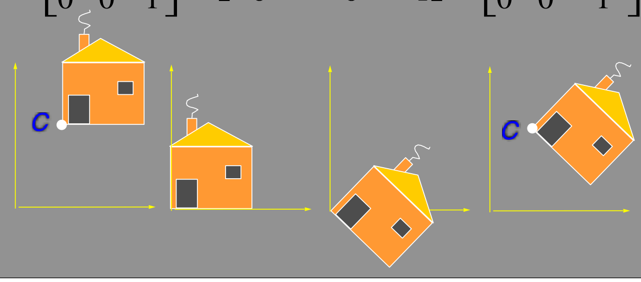
$$\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix}_h = \begin{bmatrix} x_0 + \alpha \cdot 1 \\ y_0 + \beta \cdot 1 \\ 1 \cdot 1 \end{bmatrix} =$$

- $(x, y) \to (x + \alpha, y + \beta)$

Transformation Composition

- What operation rotates by -θ around $C = (x_0, y_0)$
  - Translate by $-(x_0, y_0)$. It translate C to the origin
  - Rotate around origin by -θ
  - Translate back
  - $M = \Big(Trans(P) \cdot Rotate(\theta) \cdot Trans(-P)\Big)p$

$$M = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}$$



Identity Matrix and Inverse matrix

- The matrix I= $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called the identity matrix.
- Note that for every matrix M, it holds that $M \cdot I = I \cdot M = 1$
- For a matrix M, we denote by M^{-1} a matrix M such that $M \cdot M^{-1} = I$
- Question: What is the inverse of $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$??

Rotations - more perspective

- If z_1, z_2 are complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$
- Then $z_1 \cdot z_2$ is a new complex number, whose length is $|z_1 \cdot z_2| = |z_2| \cdot |z_2|$, and whose angle is the sum of angles of z_1, z_2

  $\arg(z_1 \cdot z_2) = \arg(z_1) \cdot \arg(z_2)$

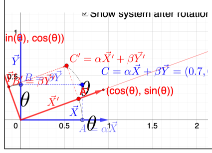
  We also know that $z_1 \cdot z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$
- Now, if $z_2 = \cos\theta + i\sin\theta$,
- then $z_1 \cdot z_2 = \underbrace{x_1\cos\theta - y_1\sin\theta}_{real\ part=x'} + i\underbrace{(x_1\sin\theta + y_1\cos\theta)}_{=y'}$

Rotations - more perspectives
Transforming from one coordinate system to another

- From Linear algebra: A **basis** $\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_d\}$ is a set of vectors such that every point p in a space(plane/space…) could be expressed as a linear combination. $p = \alpha_1 \cdot \vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_d \cdot \vec{v}_d \ldots.$
- and in addition, we could not drop any of these vectors.
- The space is **spanned** by this basis.
- Multiplication by a matrix M is a linear operation: That is
  - $M \cdot \vec{0} = \vec{0}$
  - $M \cdot (\vec{u} + \vec{v}) = M\vec{u} + M\vec{v}$
  - $M(\alpha\vec{u}) = \alpha(M\vec{u})$

We can express rotation by creating a new basis of $\mathbb{R}^2$

- This is going to be extremely useful when discussing rotations in 3D
- To specify a rotation, it is sufficient to create a new coordinates system, and specify what is the correspondence between the old and new basis.
- To be precise, create M, such that the it's **column** of M is the $i'th$ vector (represented as a linear combination of the old basis)
- The text above is probably very cryptic without multiple examples
- "Tricky" way to find rotation matrix. If $\overrightarrow{X}, \overrightarrow{Y}$ are unit vectors in the old coordinate system, then we could think about the rotation as rotating the coordinates systems as well, and after the rotation we expect
- $\overrightarrow{X} \xrightarrow[by\ \theta]{Rotation} \overrightarrow{X'}$ and $\overrightarrow{Y} \xrightarrow[by\ \theta]{Rotation} \overrightarrow{Y'}$. Let R_θ be the rotation matrix. This means $R_\theta \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \overrightarrow{X'}$ and $R_\theta \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \overrightarrow{Y'}$.
- But $R_\theta \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \overrightarrow{X'}$ is just the first column of R_θ. And $R_\theta \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the second column.
- Lets try: Write $R_\theta = \begin{bmatrix} \vdots & \vdots \\ \overrightarrow{X'} & \overrightarrow{Y'} \\ \vdots & \vdots \end{bmatrix}$.
- then for every data point $C = (\alpha, \beta)$, we could write (in a somehow obnoxious way) $C = \alpha\overrightarrow{X} + \beta\overrightarrow{Y}$.
- $R_\theta \cdot C = \begin{bmatrix} \vdots & \vdots \\ \overrightarrow{X'} & \overrightarrow{Y'} \\ \vdots & \vdots \end{bmatrix} \left(\alpha\overrightarrow{X} + \beta\overrightarrow{Y}\right) \overset{\text{linearity}}{=} \alpha \begin{bmatrix} \vdots & \vdots \\ \overrightarrow{X'} & \overrightarrow{Y'} \\ \vdots & \vdots \end{bmatrix} \overrightarrow{X} + \beta \begin{bmatrix} \vdots & \vdots \\ \overrightarrow{X'} & \overrightarrow{Y'} \\ \vdots & \vdots \end{bmatrix} \overrightarrow{Y} = \alpha\overrightarrow{X'} + \beta\overrightarrow{C'}$



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- $R_\theta \cdot C = \begin{bmatrix} \vdots & \vdots \\ \overrightarrow{X'} & \overrightarrow{Y'} \\ \vdots & \vdots \end{bmatrix} \left(\alpha\overrightarrow{X} + \beta\overrightarrow{Y}\right) \overset{\text{linearity}}{=} \alpha \begin{bmatrix} \vdots & \vdots \\ \overrightarrow{X'} & \overrightarrow{Y'} \\ \vdots & \vdots \end{bmatrix} \overrightarrow{X} + \beta \begin{bmatrix} \vdots & \vdots \\ \overrightarrow{X'} & \overrightarrow{Y'} \\ \vdots & \vdots \end{bmatrix} \overrightarrow{Y} = \alpha\overrightarrow{X'} + \beta\overrightarrow{C'} = C'$
- Important take home message: To find the rotation matrix, just create a matrix where each column is one of the new basis vector (written using coordinates of the old coordinate system)