

CSC 433/533

Computer Graphics

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Credit: Joshua Levine

Vector Math + Coding

Today's Agenda

- Reminders:
 - A02 questions?
- Goals for today:
 - Introduce some mathematics and connect it to code

Vectors

What is a Vector?

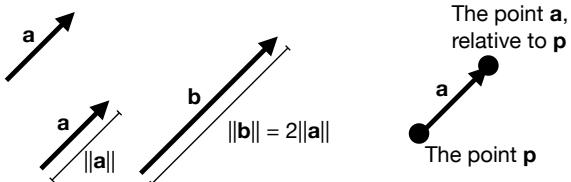
- A **vector** describes a length and a direction
- A vector is also a tuple of numbers
 - But, it often makes more sense to think in terms of the length/direction than the coordinates/numbers
- And, especially in code, we want to manipulate vectors as objects and abstract the low-level operations
- Compare with a **scalar**, or just a single number

Properties

- Two vectors, **a** and **b**, are the same (written $\mathbf{a} = \mathbf{b}$) if they have the same length and direction. (other notation: \bar{a}, \vec{a}')
- A vector's **length** is denoted with $\|\cdot\|$, (sometimes we just denote \cdot). When $\mathbf{a} = (x,y)$, then $|\mathbf{a}| = \sqrt{a.x^2 + a.y^2}$
 - e.g. the length of **a** is $\|\mathbf{a}\|$
- A **unit vector** has length one
- The **zero vector** has length zero, and undefined direction

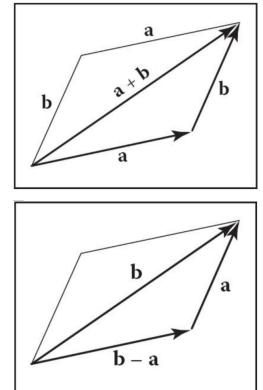
Vectors in Pictures

- We often use an arrow to represent a vector
- The length of the arrow indicates the length of the vector, the direction of the arrow indicates the direction of the vector.
- The position of the arrow is irrelevant!
- However, we can use vectors to represent positions by describing displacements from a common point



Vector Operations

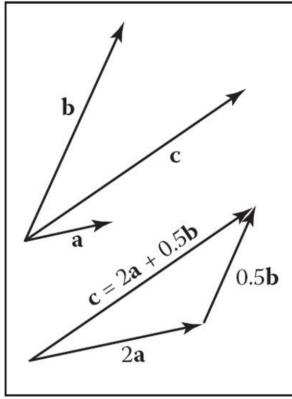
- Vectors can be added, e.g. for vectors \mathbf{a}, \mathbf{b} , there exists a vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$
 $\mathbf{a} + \mathbf{b} = (a.x + b.x, a.y + b.y)$
- Defined using the parallelogram rule: idea is to trace out the displacements and produce the combined effect
- Vectors can be negated (flip tail and head), and thus can be subtracted
- Vectors can be multiplied by a scalar, which scales the length but not the direction
 $\beta\mathbf{a} = (\beta a.x, \beta a.y)$



Vectors Decomposition

- By linear independence, any 2D vector can be written as a combination of any two nonzero, nonparallel vectors
- Such a pair of vectors is called a **2D basis**

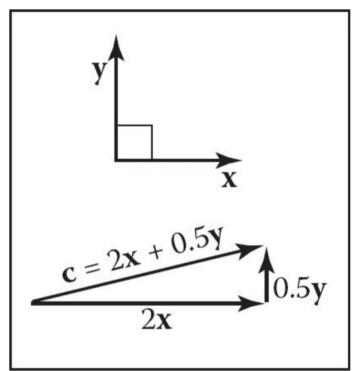
$$\mathbf{c} = a_c \mathbf{a} + b_c \mathbf{b}$$



Canonical (Cartesian) Basis

- Often, we pick two perpendicular vectors, \mathbf{x} and \mathbf{y} , to define a common **basis**
- Notationally the same,
 $\mathbf{a} = x_a \mathbf{x} + y_a \mathbf{y}$
- But we often don't bother to mention the basis vectors, and write the vector as $\mathbf{a} = (x_a, y_a)$, or

$$\mathbf{a} = \begin{bmatrix} x_a \\ y_a \end{bmatrix}$$

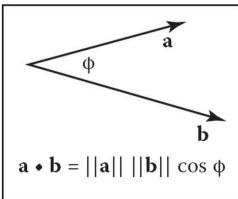


Vector Multiplication: Dot Products

- Given two vectors \mathbf{a} and \mathbf{b} , the **dot product**, relates the lengths of \mathbf{a} and \mathbf{b} with the angle ϕ between them:

$$\mathbf{a} \cdot \mathbf{b} = (a.x \cdot b.x + a.y \cdot b.y)$$

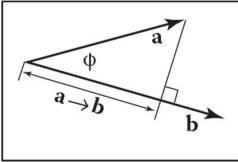
$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos \phi$$



- Sometimes called the scalar product, as it produces a scalar value

- Also can be used to produce the **projection**, $\mathbf{a} \rightarrow \mathbf{b}$, of \mathbf{a} onto \mathbf{b}

$$\mathbf{a} \rightarrow \mathbf{b} = ||\mathbf{a}|| \cos \phi = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||}$$



Dot Products are Associative and Distributive

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$,
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$,
- $(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k\mathbf{a} \cdot \mathbf{b}$
- And, we can also define them directly if \mathbf{a} and \mathbf{b} are expressed in Cartesian coordinates:

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b$$

3D Vectors

- Same idea as 2D, except these vectors are defined typically with a basis of three vectors
- Still just a direction and a magnitude
- But, useful for describing objects in three-dimensional space
- Most operations exactly the same, e.g. dot products:

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b$$

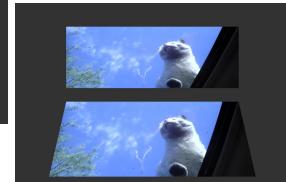
Assignment 3. Balls and Billboards

Input: JSON file describing locations of billboards and spheres.

Images placed on the billboards.

Output: scene showing what a viewer could see, and

A video showing camera movement



Billboards are extremely important for interactive computer graphics

- They could use as texture
- They could use as "imposer" of a very detailed huge geometric scene (e.g. the mountains at the background)
- The user could move (slightly) and not notice that the background mountains don't move properly. Very small errors.



Each tree is its own billboard



- But if we render a tree on a billboard, why are the billboard not occluding each other?
- We store at the data base a set of 2D images. Each shows the tree from a different directions.
- If the camera moves slightly, Small errors are not noticeable. Sometimes we need to switch with image with another

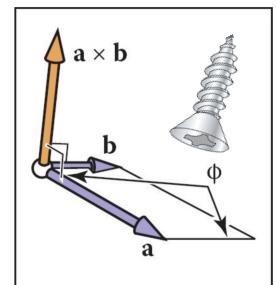
Cross Products

In 3D, another way to "multiply" two vectors is the **cross product**, $\mathbf{a} \times \mathbf{b}$:

- $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| ||\mathbf{b}|| \sin \phi$
- $||\mathbf{a} \times \mathbf{b}||$ is always the area of the parallelogram formed by \mathbf{a} and \mathbf{b} , and $\mathbf{a} \times \mathbf{b}$ is always in the direction perpendicular (two possible answers).
- A screw turned from \mathbf{a} to \mathbf{b} will progress in the direction $\mathbf{a} \times \mathbf{b}$
- Cross products distribute, but order matters:

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$\mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b}) \quad \mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$$
- $$\mathbf{a} \times \mathbf{b} = \left(\begin{array}{c} y_a z_b - z_a y_b \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{array} \right)$$
 x component z component



Cross Products

- Since the cross product is always orthogonal to the pair of vectors, we can define our 3D Cartesian coordinate space with it:
- In practice though (and the book derives this), we use the following to compute cross products:

$$\mathbf{x} = (1, 0, 0)$$

$$\mathbf{y} = (0, 1, 0)$$

$$\mathbf{z} = (0, 0, 1)$$

$$\mathbf{x} \times \mathbf{y} = +\mathbf{z},$$

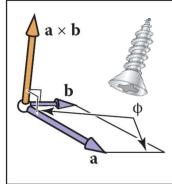
$$\mathbf{y} \times \mathbf{x} = -\mathbf{z},$$

$$\mathbf{y} \times \mathbf{z} = +\mathbf{x},$$

$$\mathbf{z} \times \mathbf{y} = -\mathbf{x},$$

$$\mathbf{z} \times \mathbf{x} = +\mathbf{y},$$

$$\mathbf{x} \times \mathbf{z} = -\mathbf{y}.$$



$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$$

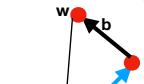
Checking orientation

Assume \mathbf{a}, \mathbf{b} are in 2D ($z=0$). There are 3 possible scenarios.

\mathbf{a} might be counter-clockwise (ccw) of \mathbf{b}

\mathbf{a} might be clockwise (cw) of \mathbf{b}

\mathbf{a} is collinear with \mathbf{b}



$$x_a y_b - y_a x_b > 0$$

$$x_a y_b - y_a x_b < 0$$

$$x_a y_b - y_a x_b = 0$$

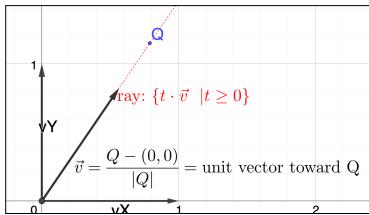
\mathbf{a} is counter-clockwise (ccw) of \mathbf{b}

\mathbf{a} is clockwise (cw) of \mathbf{b}

\mathbf{a}, \mathbf{b} collinear

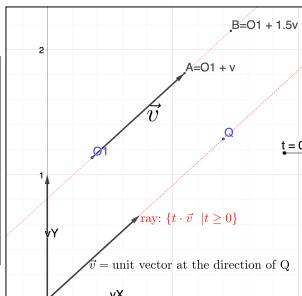
This will provide a convenient way to check if a triangle with vertices u, v, w (when vertices are given to us in this order) is CCW or CW

Rays, lines, Orthogonal Projections



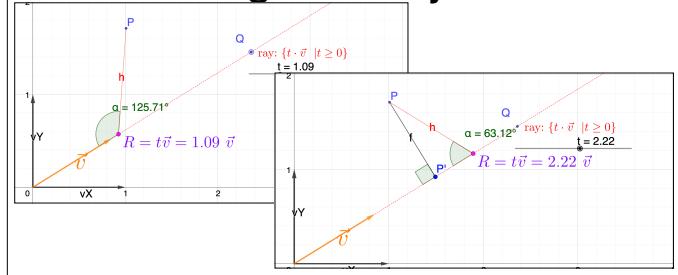
The ray $\{t \cdot \vec{v} \mid t \geq 0\}$

The line that \vec{v} defines is
 $\ell = \{t \cdot \vec{v} \mid v \in \mathbb{R}\}$
 (that is, t is any real value)



The ray $\{O1 + t \cdot \vec{v} \mid t \geq 0\}$
 This is the same ray, shifted by O1
 That is, the ray emerges from O1

Orthogonal Projections



- Let P be a point not on the ray
- Need to find: The point P' which is the orthogonal projection of P on $\ell = \{t \vec{v} \mid t \in \mathbb{R}\}$
- P' is the closest point on ℓ to P
- Assume t start at zero, and slowly increases. Let $R = t \cdot \vec{v}$. Monitor the angle $\angle(O, R, P')$. At some time t_0 , this angle is 0, and R and P' coincide, and $\angle(O, R, P') = 0$. This means:

$$(t_0 \cdot \vec{v}) \perp (P - t_0 \vec{v}) \Rightarrow (t_0 \cdot \vec{v}) \cdot (P - t_0 \vec{v}) = 0 \\ t_0 \vec{v} \cdot P = t_0^2 (\vec{v} \cdot \vec{v}) \\ \Leftrightarrow t_0 = P \cdot \vec{v} \\ \Rightarrow P' = (P \cdot \vec{v}) \vec{v}$$

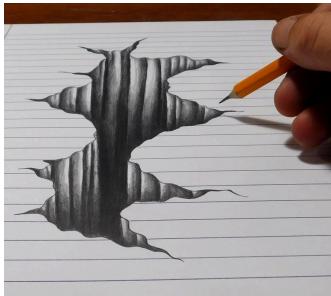
Rendering

What is Rendering?

Rendering is the task of taking three-dimensional objects and producing a 2D image that shows the objects as viewed from a particular viewpoint

Two Ways to Think About How We Make Images

- Drawing



- Photography



Two Ways to Think About Rendering

- Object-Ordered

- Decide, for every object in the scene, its contribution to the image

- Image-Ordered

- Decide, for every pixel in the image, its contribution from every object

Two Ways to Think About Rendering

- Object-Ordered or Rasterization

```
for each object {  
    for each image pixel {  
        if (object affects pixel)  
        {  
            do something  
        }  
    }  
}
```

- Image-Ordered or Ray Tracing

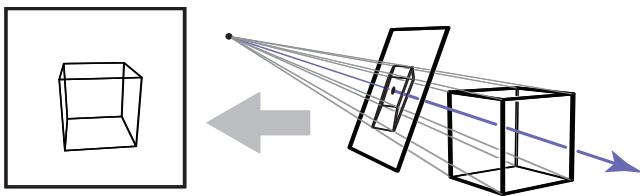
```
for each image pixel {  
    for each object {  
        if (object affects pixel)  
        {  
            do something  
        }  
    }  
}
```

TODAY

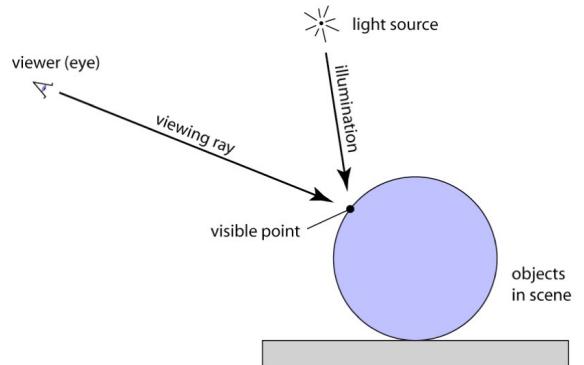
Basics of Ray Tracing

Idea of Ray Tracing

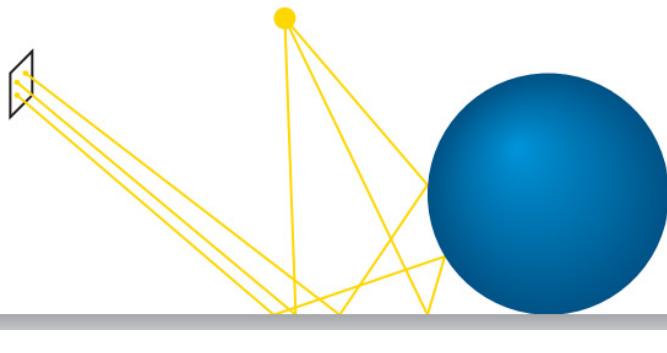
- Ask first, for each pixel: what belongs at that pixel?
- Answer: The set of objects that are visible if we were standing on one side of the image looking into the scene



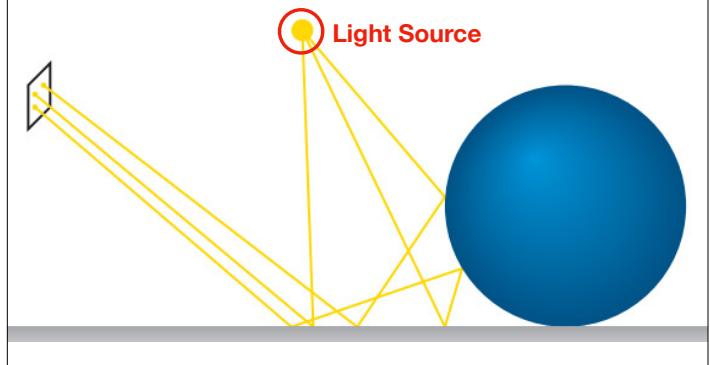
Key Concepts, in Diagram



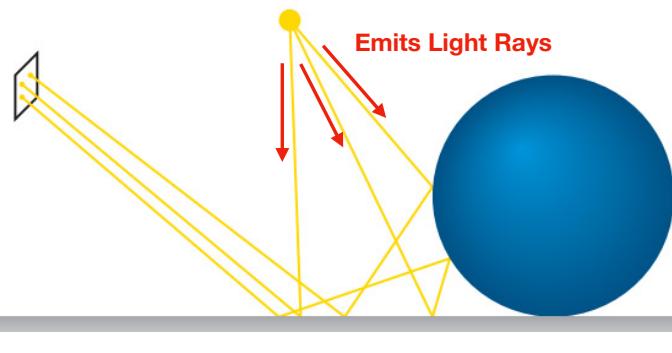
Idea: Using Paths of Light to Model Visibility



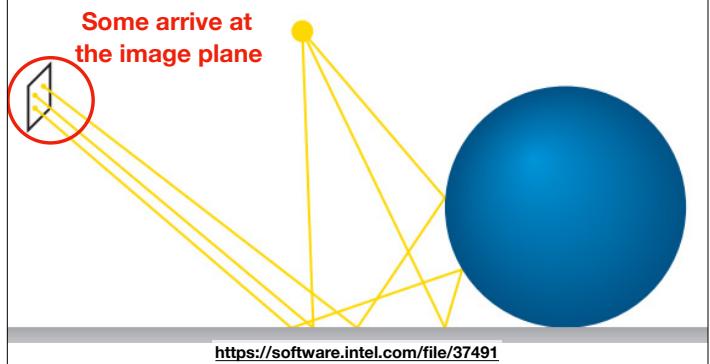
Using Paths of Light to Model Visibility



Using Paths of Light to Model Visibility



Using Paths of Light to Model Visibility



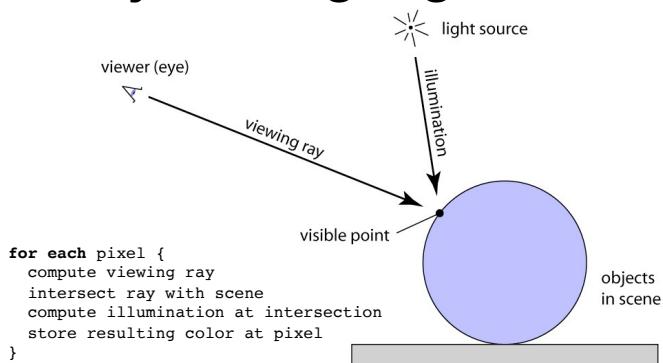
Using Paths of Light to Model Visibility



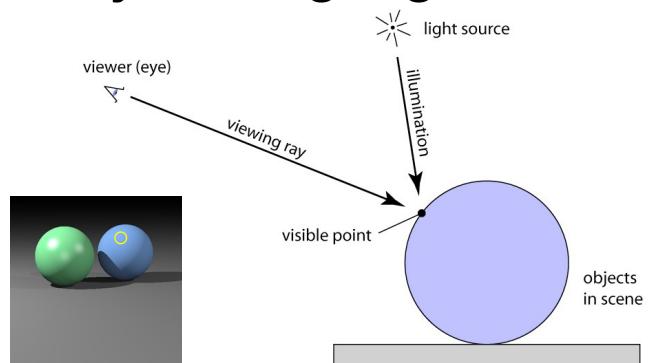
Forwarding vs Backward Tracing

- Idea: Trace rays from light source to image
 - This is slow!
- Better idea: Trace rays from image to light source

Ray Tracing Algorithm



Ray Tracing Algorithm



Cameras and Perspective

```
If illumination is uniform and directional-free (ambient light):
for each pixel {
    compute viewing ray
    intersect ray with scene
    copy the color of the object at this point to this pixel.
}
```

Commonly, we need slightly more involved

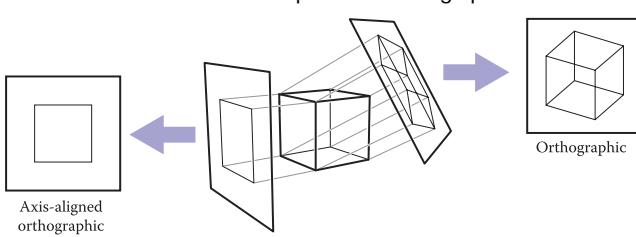
```
for each pixel {
    compute viewing ray
    intersect ray with scene
    compute illumination at intersection
    store resulting color at pixel
}
```

Linear Perspective

- Standard approach is to project objects to an image plane so that straight lines in the scene stay straight lines on the image
- Two approaches:
 - Parallel projection: Results in **orthographic** views
 - Perspective projection: Results in **perspective** views

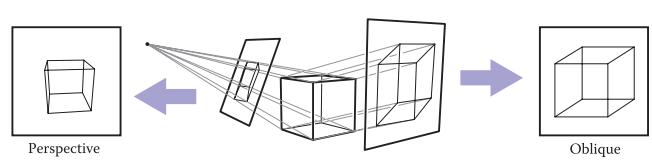
Orthographic Views

- Points in 3D are moved along parallel lines to the image plane.
- Resulting view determined solely by choice of projection direction and orientation/position of image plane



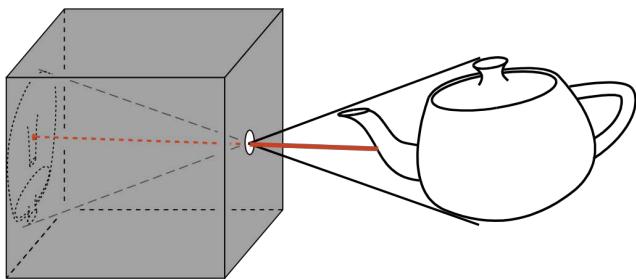
Perspective Views

- But, objects that are further away should look smaller!
- Instead, we can project objects through a single viewpoint and record where they hit the plane.
- Lines which are parallel in 3D might be non-parallel in the view



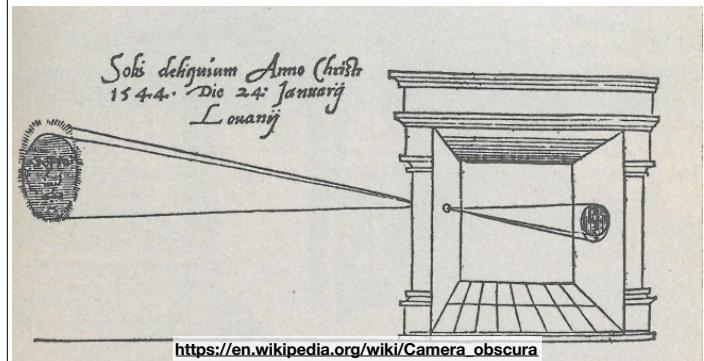
Pinhole Cameras

- Idea: Consider a box with a tiny hole. All light that passes through this hole will hit the opposite side
- Produced image inverts



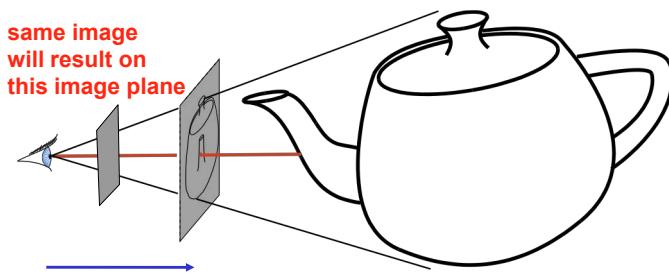
Camera Obscura

- Gemma Frisius, 16th century



Simplified Pinhole Cameras

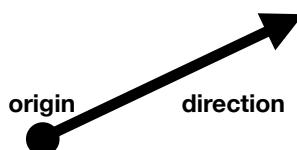
- Instead, we can place the eye at the pinhole and consider the eye-image pyramid (sometimes called **view frustum**)



Defining Rays

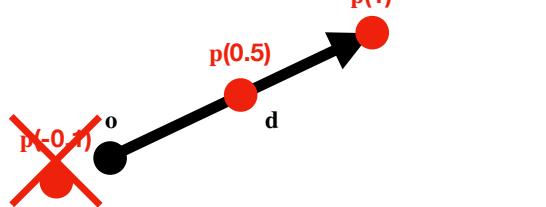
Mathematical Description of a Ray

- Two components:
 - An **origin**, or a position that the ray starts from
 - A **direction**, or a vector pointing in the direction the ray travels
 - Not necessarily unit length, but it's sometimes helpful to think of these as normalized

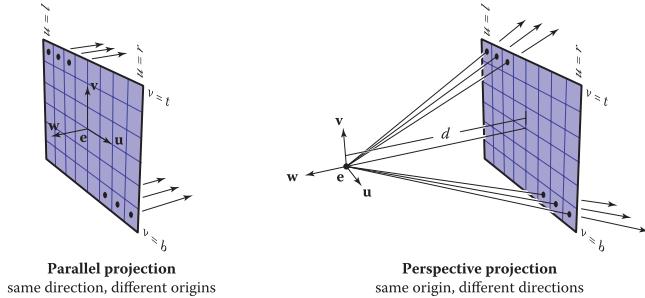


Mathematical Description of a Ray

- Rays define a family of points, $p(t)$, using a **parametric** definition
- $p(t) = \mathbf{o} + t\mathbf{d}$, \mathbf{o} is the origin and \mathbf{d} the direction
- Typically, $t \geq 0$ is a non-negative number

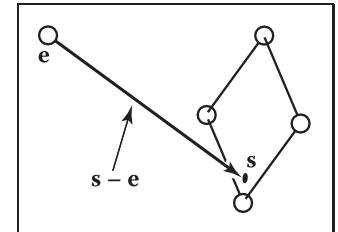


Orthographic vs. Perspective Rays



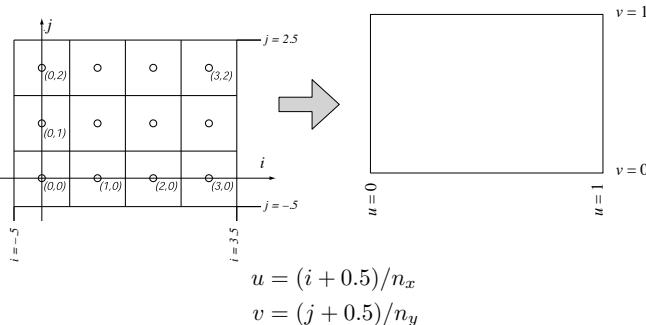
Defining o and d in Perspective Projection

- Given a viewpoint, e , and a position on the image plane, s
- $$\mathbf{o} = \mathbf{e}$$
- $$\mathbf{d} = \mathbf{s} - \mathbf{e}$$
- And thus $\mathbf{p}(t) = \mathbf{e} + t(\mathbf{s} - \mathbf{e})$



Pixel-to-Image Mapping

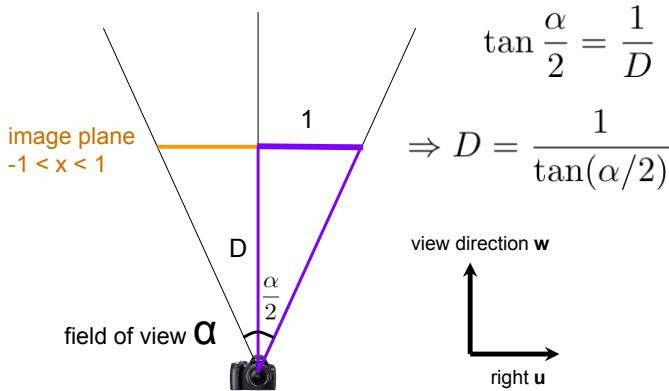
- Exactly where are pixels located? Must convert from pixel coordinates (i,j) to positions in 3D space (u,v,w)
- What should w be?



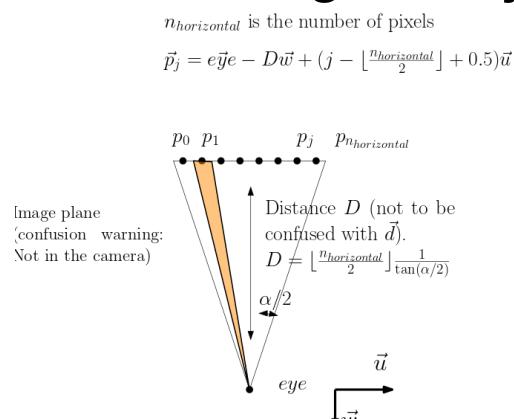
Camera Components

- Definition of an image plane
 - Both in terms of pixel resolution AND position in 3D space or more frequently in **field of view** and/or **distance**
- Viewpoint
- View direction
- Up vector (note that is not necessarily the "up" of the geometric scene)

Ray Generation in 2D

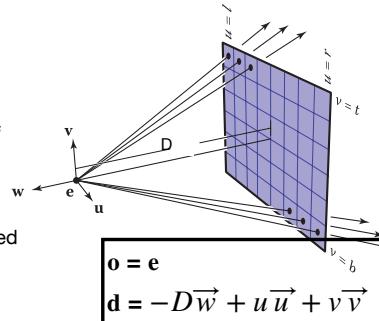


Generating the rays



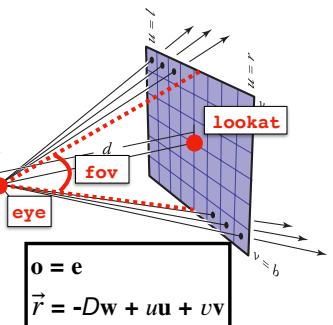
From 2D to 3D

- Moving from 2D to 3D is essentially the same thing once you can define the positions of pixels in uvw space
- Following the convention of the book, w is the negated viewpoint vector
- The up vector, v , can be used to define a local coordinate space by computing u



In the Assignment

- An eye position
- A position to lookat, which is centered in the image
- w can be defined using eye and lookat as well as the distance D , together with the up vector.
- A fov_angle for the vertical FOV
 - The FOV defines the **height** of the image plane in world space
 - You can then use this to compute the **width** of the image plane in world space using the aspect ratio (rows/columns) of the image
- Using the number of rows/columns you can then sample u, v

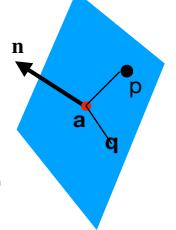


Intersecting Objects

```
for each pixel {
  compute viewing ray
  intersect ray with scene
  compute illumination at intersection
  store resulting color at pixel
}
```

Defining a Plane

- Let h be a plane with normal n , and containing a point a . Let p be some other point. Then p is on this plane if and only if (iff) $\mathbf{p} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
- Proof. Consider the segment $p-a$. p is on the plane iff $p-a$ is orthogonal to n . Using the property of dot product $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{n} = |\mathbf{p} - \mathbf{a}| |\mathbf{n}| \cos \alpha$
- Here α is the angle between them. Now $\cos(90^\circ) = 0$. So if p on this plane then $\mathbf{p} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ implying
- If $\mathbf{p} \cdot \mathbf{n} > \mathbf{a} \cdot \mathbf{n}$ then p lies on the "front" side of the plane (in the direction pointed to by the normal)
- $\mathbf{p} \cdot \mathbf{n} - \mathbf{a} \cdot \mathbf{n} < 0$ means that p lies on the "back" side.
- Sometimes used as $f(p)=0$ iff " p on the plane". So the function $f(p)$ is $f(p)=(\mathbf{p}-\mathbf{a})\mathbf{n}$
- If we have 3 points a, p, q all on the plane, then we can compute a normal $\mathbf{n} = (\mathbf{p} - \mathbf{a}) \times (\mathbf{q} - \mathbf{a})$. (cross product).
- Warning: The term "normal" does not mean that it was normalized.

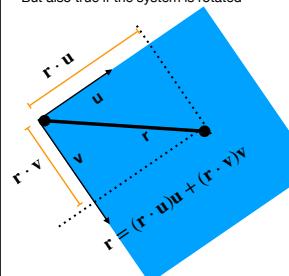


Ray-Plane Intersection

- A ray $\mathbf{p}(t) = \mathbf{o} + t\mathbf{d}$
 - Two conditions must be satisfied:
 - Must be on a ray: $\mathbf{p}(t) = \mathbf{o} + t\mathbf{d}$
 - Must be on the plane: $f(\mathbf{p}) = (\mathbf{p} - \mathbf{a}) \cdot \mathbf{n} = 0$
 - Can substitute the equations and solve for t in $f(\mathbf{p}(t))$:
- $$(\mathbf{o} + t\mathbf{d} - \mathbf{a}) \cdot \mathbf{n} = 0$$
- This means that $t_{hit} = ((\mathbf{a} - \mathbf{o}) \cdot \mathbf{n}) / (\mathbf{d} \cdot \mathbf{n})$. The intersection point is $\mathbf{o} + t_{hit}\mathbf{d}$

Revisiting dot product: projections

Let \mathbf{u}, \mathbf{v} be orthonormal vectors (orthogonal and unit length). \mathbf{r} is another vector
 $\mathbf{r} \cdot \mathbf{u} = |\mathbf{r}| |\mathbf{u}| \cos \alpha = |\mathbf{r}| \cos \alpha$
 is the projection of \mathbf{r} on the direction of \mathbf{u}
 the length of the "shadow" that \mathbf{r} cast on the line containing \mathbf{u}
 Sounds obvious when the coordinate system is xyz
 But also true if the system is rotated

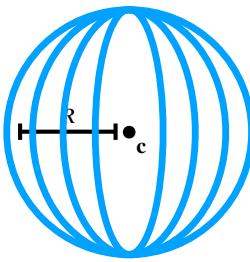


Defining a Sphere

- We can define a sphere of radius R , centered at position c , using the implicit form

$$f(\mathbf{p}) = (\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) - R^2 = 0$$

- Any point \mathbf{p} that satisfies the above lives on the sphere



Ray-Sphere Intersection

- Two conditions must be satisfied:
 - Must be on a ray: $\mathbf{p}(t) = \mathbf{o} + t\mathbf{d}$
 - Must be on a sphere: $f(\mathbf{p}) = (\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) - R^2 = 0$
- Can substitute the equations and solve for t in $f(\mathbf{p}(t))$:

$$(\mathbf{o} + t\mathbf{d} - \mathbf{c}) \cdot (\mathbf{o} + t\mathbf{d} - \mathbf{c}) - R^2 = 0$$

- Solving for t is a quadratic equation

Ray-Sphere Intersection

- Solve $(\mathbf{o} + t\mathbf{d} - \mathbf{c}) \cdot (\mathbf{o} + t\mathbf{d} - \mathbf{c}) - R^2 = 0$ for t :

- Rearrange terms:

$$(\mathbf{d} \cdot \mathbf{d})t^2 + (2\mathbf{d} \cdot (\mathbf{o} - \mathbf{c}))t + (\mathbf{o} - \mathbf{c}) \cdot (\mathbf{o} - \mathbf{c}) - R^2 = 0$$

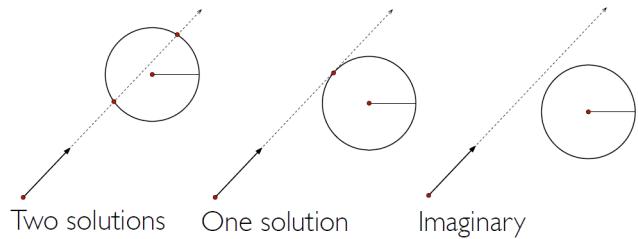
- Solve the quadratic equation $At^2 + Bt + C = 0$ where

- $A = (\mathbf{d} \cdot \mathbf{d})$
- $B = 2\mathbf{d} \cdot (\mathbf{o} - \mathbf{c})$
- $C = (\mathbf{o} - \mathbf{c}) \cdot (\mathbf{o} - \mathbf{c}) - R^2$

Discriminant, $D = B^2 - 4A^2C$
Solutions must satisfy:
 $t = (-B \pm \sqrt{|D|}) / 2A$

Ray-Sphere Intersection

- Number of intersections dictated by the discriminant
- In the case of two solutions, prefer the one with lower t



Geometric Method (instead of Algebraic)

Ray: $\mathbf{P} = \mathbf{P}_0 + t\mathbf{V}$
Sphere: $\|\mathbf{P} - \mathbf{O}\|^2 - r^2 = 0$

$$\mathbf{L} = \mathbf{O} - \mathbf{P}_0$$

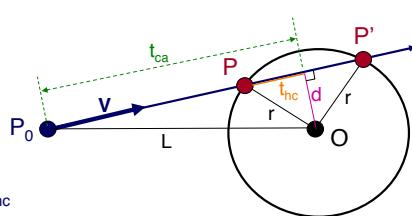
$t_{ca} = \mathbf{L} \cdot \mathbf{V}$
if $(t_{ca} < 0)$ return 0

$d^2 = \mathbf{L} \cdot \mathbf{L} - t_{ca}^2$
if $(d^2 > r^2)$ return 0

$t_{hc} = \sqrt{r^2 - d^2}$
 $t = t_{ca} - t_{hc}$ and $t_{ca} + t_{hc}$

$$\mathbf{P} = \mathbf{P}_0 + t\mathbf{V}$$

Geometric Method



Constructing Orthonormal Bases from a Pair of Vectors

- Given two vectors \mathbf{a} and \mathbf{b} , which might not be orthonormal to begin with:

$$\begin{aligned} \mathbf{w} &= \frac{\mathbf{a}}{\|\mathbf{a}\|}, \\ \mathbf{u} &= \frac{\mathbf{b} \times \mathbf{w}}{\|\mathbf{b} \times \mathbf{w}\|}, \\ \mathbf{v} &= \mathbf{w} \times \mathbf{u}. \end{aligned}$$

- In this case, \mathbf{w} will align with \mathbf{a} and \mathbf{v} will be the closest vector to \mathbf{b} that is perpendicular to \mathbf{w}