CSC 1201 Probability and Statistics

Moments of Random Variables and Chebychev's Inequality

Denish Azamuke

Makerere University

March 10, 2025

Moments of Random Variables

Introduction.

Definition (nth Moment About the Origin)

Let X be a random variable, and let $f(x)$ denote its probability mass (density) function. The nth moment about the origin of X, denoted $E(X^n)$, is defined as

$$E(X^n) = \begin{cases} \displaystyle\sum_{x \in R_X} x^n f(x), & \text{if } X \text{ is discrete,} \\ \displaystyle\int_{-\infty}^{\infty} x^n f(x)\, dx, & \text{if } X \text{ is continuous.} \end{cases}$$

This definition holds for $n = 0, 1, 2, 3, \ldots$, provided the sum or integral converges absolutely.

Remarks.

- If $n = 1$, then $E(X)$ is called the *first moment about the origin*.
- If $n = 2$, then $E(X^2)$ is called the *second moment about the origin*.
- A random variable may fail to have certain moments if those sums or integrals do not converge absolutely.
- Two important characteristics of a random variable defined via these moments are the expected value and the variance.

Expected Value of Random Variables

Definition. Let X be a random variable with space R_X and probability density function $f(x)$. The mean μ_X of the random variable X is defined as

$$\mu_X = \begin{cases} \displaystyle\sum_{x \in R_X} x\, f(x), & \text{if } X \text{ is discrete,} \\ \displaystyle\int_{-\infty}^{\infty} x\, f(x)\, dx, & \text{if } X \text{ is continuous,} \end{cases}$$

whenever the right-hand side exists.

Interpretation.

- The mean of a random variable is the *average* or *expected* value of X.
- It is a measure of central tendency, reflecting the "balance point" of the distribution.
- The operator $E(\cdot)$ is often used in place of μ_X, i.e. $\mu_X = E(X)$.

Example 1

Problem. If X is a uniform random variable on the interval $(2, 7)$, find the mean of X.

Solution. Since X is uniformly distributed on $(2, 7)$, its probability density function is

$$f(x) = \begin{cases} \dfrac{1}{7-2} = \dfrac{1}{5}, & 2 < x < 7, \\ 0, & \text{otherwise.} \end{cases}$$

Then the mean μ_X is:

$$\mu_X = E(X) = \int_2^7 x\left(\tfrac{1}{5}\right) dx = \tfrac{1}{5}\int_2^7 x\, dx = \tfrac{1}{5}\left[\tfrac{x^2}{2}\right]_2^7 = 4.5.$$

Hence, the mean of X is 4.5.

Cauchy Distribution

Problem. Let X be a Cauchy random variable with location parameter θ, denoted by $X \sim \mathrm{Cauchy}(\theta)$. The probability density function (pdf) of X is

$$f(x) = \frac{1}{\pi\left[1 + (x - \theta)^2\right]}, \quad -\infty < x < \infty.$$

Determine $E(X)$, the expected value of X, if it exists.

Cauchy Distribution

Analysis. The expected value $E(X)$ exists only if $\displaystyle\int_{-\infty}^{\infty} \left| x\, f(x) \right| dx < \infty$.

Hence, we consider

$$\int_{-\infty}^{\infty} \left| x\, f(x) \right| \, dx \; = \; \int_{-\infty}^{\infty} \left| x \, \frac{1}{\pi \left[1 + (x - \theta)^2 \right]} \right| \, dx.$$

Use the substitution $z = x - \theta$, which gives $x = z + \theta$. Then

$$\int_{-\infty}^{\infty} \left| (z + \theta) \, \frac{1}{\pi \left[1 + z^2 \right]} \right| \, dz.$$

Splitting at $z = 0$ and evaluating shows the integral diverges to infinity (the logarithmic term grows without bound).

Conclusion. Since $\displaystyle\int_{-\infty}^{\infty} \left| x\, f(x) \right| \, dx = \infty$, the expected value of a Cauchy (θ) random variable does *not* exist.

Example 2

Problem. Suppose X is a discrete random variable whose probability mass function is given by

$$f(x) = \begin{cases} (1-p)^{x-1}\,p, & x = 1, 2, 3, \ldots, \\ 0, & \text{otherwise}, \end{cases}$$

for some $0 < p < 1$. Determine the expected value $E(X)$.

Example 2

Solution. Observe that this is the pmf of a *geometric* distribution (with the support starting at $x = 1$). We compute the expectation as

$$E(X) = \sum_{x=1}^{\infty} x\, f(x) = \sum_{x=1}^{\infty} x \left[(1-p)^{x-1} p\right].$$

Factor out the constant p:

$$E(X) = p \sum_{x=1}^{\infty} x\, (1-p)^{x-1}.$$

Using the known series identity $\sum_{x=1}^{\infty} x\, r^{x-1} = \frac{1}{(1-r)^2}$ for $|r| < 1$, we have $r = (1-p)$ and thus

$$E(X) = p \cdot \frac{1}{\left(1-(1-p)\right)^2} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Conclusion. The expected value of X is $\frac{1}{p}$ which is the reciprocal of the parameter p.

Linearity of Expectation

Statement. Let X be a random variable with probability density function $f(x)$. For any real numbers a and b,

$$E\big(aX + b\big) \;=\; a\,E(X) \;+\; b.$$

Proof (Continuous Case).

$$\begin{aligned}
E\big(aX + b\big) &= \int_{-\infty}^{\infty} \big(a\,x + b\big)\, f(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} a\,x\,f(x)\,\mathrm{d}x \;+\; \int_{-\infty}^{\infty} b\,f(x)\,\mathrm{d}x \\
&= a\int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x \;+\; b\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \\
&= a\,E(X) \;+\; b,
\end{aligned}$$

where we used $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$.

Linearity of Expectation

Statement. Let X be a random variable with probability density function $f(x)$. For any real numbers a and b,

$$E\big(aX + b\big) \;=\; a\,E(X) \;+\; b.$$

Note (Discrete Case). To prove the discrete case, replace the integral by a sum:

$$E\big(aX+b\big) \;=\; \sum_{x}\big(a\,x+b\big)f(x) \;=\; a\sum_{x}x\,f(x)+b\sum_{x}f(x) \;=\; a\,E(X)+b.$$

Variance of Random Variables

Definition. Let X be a random variable with mean μ_X. The variance of X, denoted $\mathrm{Var}(X)$, is defined as

$$\mathrm{Var}(X) = E\big([X - \mu_X]^2\big).$$

- Often written as σ_X^2.
- The positive square root of the variance, σ_X, is the standard deviation.
- Variance (and standard deviation) measures the *spread* of the distribution of X.

Variance of Random Variables

Statement. If X is a random variable with mean μ_X and variance σ_X^2, then

$$\sigma_X^2 \;=\; E(X^2) \;-\; \big[\mu_X\big]^2.$$

Proof.

$$\begin{aligned}
\sigma_X^2 &= \mathrm{Var}(X) = E\big([X-\mu_X]^2\big) \\
&= E\big(X^2 - 2\,\mu_X\,X + \mu_X^2\big) \\
&= E(X^2) \;-\; 2\,\mu_X\,E(X) \;+\; \mu_X^2 \\
&= E(X^2) \;-\; 2\,\mu_X\,\mu_X \;+\; \mu_X^2 \\
&= E(X^2) \;-\; \mu_X^2.
\end{aligned}$$

Variance of Random Variables

Statement. If X is a random variable with mean μ_X and variance σ_X^2, and a and b are real constants, then

$$\mathrm{Var}(aX+b) \;=\; a^2\,\mathrm{Var}(X).$$

Proof.

$$\begin{aligned}
\mathrm{Var}(aX+b) &= E\Big(\,\big[(aX+b)-\mu_{\,aX+b}\big]^2\Big)\\
&= E\Big(\,\big[aX+b-\big(a\,\mu_X+b\big)\big]^2\Big) \quad \big(\text{since } E(aX+b)=a\mu_X+b\big)\\
&= E\Big(\,\big[a(X-\mu_X)\big]^2\Big)\\
&= a^2\,E\Big(\big[\,X-\mu_X\,\big]^2\Big)\\
&= a^2\,\mathrm{Var}(X).
\end{aligned}$$

Example 1

Given: A random variable X has the density function

$$f(x) = \begin{cases} \dfrac{2x}{k^2}, & 0 \le x \le k, \\ 0, & \text{otherwise}, \end{cases}$$

where $k > 0$.
Question: For what value of k does $\mathrm{Var}(X) = 2$?

Solution

Step 1: Verify that $f(x)$ is a valid pdf.

$$\int_0^k \frac{2x}{k^2}\,dx = \frac{2}{k^2}\int_0^k x\,dx = \frac{2}{k^2}\left[\frac{x^2}{2}\right]_0^k = \frac{2}{k^2}\cdot\frac{k^2}{2} = 1.$$

Step 2: Compute $E(X)$.

$$E(X) = \int_0^k x\,\frac{2x}{k^2}\,dx = \frac{2}{k^2}\int_0^k x^2\,dx = \frac{2}{k^2}\left[\frac{x^3}{3}\right]_0^k = \frac{2}{k^2}\cdot\frac{k^3}{3} = \frac{2k}{3}.$$

Step 3: Compute $E(X^2)$, the second moment.

$$E(X^2) = \int_0^k x^2\,\frac{2x}{k^2}\,dx = \frac{2}{k^2}\int_0^k x^3\,dx = \frac{2}{k^2}\left[\frac{x^4}{4}\right]_0^k = \frac{2}{k^2}\cdot\frac{k^4}{4} = \frac{k^2}{2}.$$

Step 4: Compute $\mathrm{Var}(X)$.

$$\mathrm{Var}(X) = E(X^2) - \left[E(X)\right]^2 = \frac{k^2}{2} - \left(\tfrac{2k}{3}\right)^2 = \frac{k^2}{2} - \frac{4k^2}{9} = \frac{9k^2}{18} - \frac{8k^2}{18} = \frac{k^2}{18}.$$

Solution

Step 5: Set $\mathrm{Var}(X) = 2$**.**

$$\frac{k^2}{18} = 2 \implies k^2 = 36 \implies k = 6 \quad (\text{taking } k > 0).$$

Answer: $k = 6$.

Example 2

Setup. A random variable X has mean μ and variance $\sigma^2 > 0$. We wish to find constants a and b so that the new random variable

$$Y = a + bX$$

has mean 0 and variance 1.
Question. Find such a and b in terms of μ and σ^2.

Solution

1. Enforce mean 0**.**

$$0 = E(Y) = E(a + bX) = a + b\,E(X) = a + b\,\mu.$$

Hence,

$$a = -\,b\,\mu.$$

2. Enforce variance 1**.**

$$1 = \mathrm{Var}(Y) = \mathrm{Var}(a + bX) = b^2\,\mathrm{Var}(X) = b^2\,\sigma^2.$$

Thus,

$$b^2 = \frac{1}{\sigma^2} \implies b = \pm\frac{1}{\sigma}.$$

3. Combine results. Since $a = -b\mu$, we have two solutions:

$$b = \frac{1}{\sigma}, \quad a = -\frac{\mu}{\sigma} \quad \text{or} \quad b = -\frac{1}{\sigma}, \quad a = \frac{\mu}{\sigma}.$$

Solution

Common Choice. Usually, we take $b = \frac{1}{\sigma}$, so

$$a = -\frac{\mu}{\sigma}.$$

Hence $Y = \dfrac{X - \mu}{\sigma}$ has mean 0 and variance 1.

Chebychev's Inequality

Standard Deviation as a Measure of Spread

- The standard deviation σ of a random variable X measures how spread out its values are around its mean μ.
- For a *standard normal distribution* (mean $\mu = 0$ and $\sigma = 1$):
  - About 68% of the area under the pdf lies between $\mu - \sigma$ and $\mu + \sigma$.
  - About 95% lies between $\mu - 2\sigma$ and $\mu + 2\sigma$.
- In general, for a random variable with mean μ and standard deviation σ, the values $\mu \pm k\,\sigma$ describe "spread" for a chosen k.

Motivation

- We might ask: without knowing the exact pdf, can we still estimate the probability that X lies in the interval $[\mu - k\,\sigma,\ \mu + k\,\sigma]$?
- *Chebychev's Inequality* (proved by the Russian mathematician Pafnuty Chebyshev) provides such an estimate:

$$P\big(\,|X - \mu| < k\,\sigma\big) \;\geq\; 1 - \frac{1}{k^2}.$$

Chebychev's Inequality — Statement

Theorem. Let X be a random variable with mean μ and standard deviation $\sigma > 0$. Then, for any positive real k,

$$P\big(\,|X - \mu| < k\,\sigma\big) \;\geq\; 1 \;-\; \frac{1}{k^2}.$$

Interpretation. Without knowing the exact shape of the distribution, we can still guarantee that the probability of being within k standard deviations of the mean is at least $1 - \frac{1}{k^2}$.

Chebychev's Inequality — Proof (Continuous Case)

Step 1. Express σ^2 in three regions:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \underbrace{\int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx}_{\text{Left tail}} + \underbrace{\int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx}_{\text{Middle}} + \underbrace{\int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx}_{\text{Right tail}}.$$

Because $\int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx \geq 0$, we have

$$\sigma^2 \;\geq\; \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx \;+\; \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx.$$

Step 2. Bound $(x-\mu)^2$ outside $[\mu-k\sigma, \mu+k\sigma]$.

$$\text{If } x < \mu - k\sigma, \quad \text{then } (\mu - x) \geq k\sigma \implies (\mu-x)^2 \geq k^2\sigma^2.$$

Similarly, if $x > \mu + k\sigma$, then $(x-\mu)^2 \geq k^2\sigma^2$.
Thus, outside $[\mu-k\sigma,\ \mu+k\sigma]$, $(x-\mu)^2 \geq k^2\,\sigma^2$.

Chebychev's Inequality — Proof (Continuous Case)

Step 3. Plug this lower bound back into σ^2.

$$\sigma^2 \;\geq\; \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx \;+\; \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx \;\geq\; k^2\,\sigma^2 \Big[\underbrace{\int_{-\infty}^{\mu-k\sigma} f(x)\,dx}_{P(X \leq \mu-k\sigma)} \;+\; \underbrace{\int_{\mu+k\sigma}^{\infty} f(x)\,dx}_{P(X \geq \mu+k\sigma)} \Big].$$

Hence,

$$1 \;\geq\; k^2 \Big[P\big(X \leq \mu - k\sigma\big) \;+\; P\big(X \geq \mu + k\sigma\big) \Big] \;\Longrightarrow\; \frac{1}{k^2} \;\geq\; P\big(\,|X-\mu| \,\geq k\sigma\big).$$

Therefore,

$$P\big(\,|X-\mu| < k\sigma\big) \;=\; 1 \;-\; P\big(\,|X-\mu| \,\geq k\sigma\big) \;\geq\; 1 \;-\; \frac{1}{k^2}.$$

Conclusion. This completes the proof of Chebychev's Inequality.

Example 1

Given:

$$X \sim \begin{cases} 630\, x^4\, (1-x)^4, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

which is precisely a $\mathrm{Beta}(5, 5)$ distribution.

Questions:

- What is the exact value of $P\big(|X - \mu| \leq 2\sigma\big)$?
- What does $P\big(|X - \mu| \leq 2\sigma\big)$ become under the Chebychev inequality's estimate?

Solution

Step 1: Mean and Variance.
Recognize X as a $\mathrm{Beta}(5,5)$ distribution. A $\mathrm{Beta}(\alpha,\beta)$ variable has

$$\mu = \frac{\alpha}{\alpha+\beta}, \quad \sigma^2 = \frac{\alpha\,\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Hence,

$$\mu = \frac{5}{5+5} = 0.5, \quad \sigma^2 = \frac{5\cdot 5}{10^2\cdot 11} = \frac{25}{1100} = \frac{1}{44}.$$

Thus $\sigma = \sqrt{\frac{1}{44}} \approx 0.15$.

Step 2: Exact Probability.
We want $P\big(|X-0.5| \le 2\sigma\big)$. Since $2\sigma \approx 0.30$, this is

$$P\big(0.2 \le X \le 0.8\big) \;=\; \int_{0.2}^{0.8} 630\,x^4(1-x)^4\,dx \;\approx\; 0.96.$$

Solution

Step 3: Chebychev's Inequality.
Generally, $P\left(|X-\mu| \leq 2\sigma\right) \geq 1 - \frac{1}{(2)^2} = 0.75.$

Comparison: Exact value ≈ 0.96, while Chebychev's lower bound is 0.75. Hence, Chebychev's result is more conservative but applies to any distribution.

Moment Generating Functions

Motivation.

- Some distributions (e.g., geometric) may have moments that are cumbersome to compute directly.
- A *moment generating function* (mgf) can simplify the computation of moments if it exists.

Definition 4.5. Let X be a random variable with probability density (or mass) function $f(x)$. The *moment generating function (mgf)* of X is the real-valued function

$$M(t) = E\left(e^{tX}\right),$$

provided this expectation exists for t in some interval $(-h, h)$.

Moment Generating Functions

Remarks.

- If the mgf exists, it is given explicitly by

$$M(t) = \begin{cases} \sum_{x \in R_X} e^{tx} f(x), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, & \text{if } X \text{ is continuous.} \end{cases}$$

- Not every random variable has an mgf, but if it does, that mgf is unique.

Derivatives of the MGF

Statement. Let X be a random variable with moment generating function $M(t) = E\left(e^{tX}\right)$. For a positive integer n, show that
$\left.\frac{d^n}{dt^n}M(t)\right|_{t=0} = E\left(X^n\right)$.

Computation.

$$\frac{d}{dt}M(t) \;=\; \frac{d}{dt}E\left(e^{tX}\right) \;=\; E\left(\frac{d}{dt}e^{tX}\right) \;=\; E\left(X\, e^{tX}\right).$$

Derivatives of the MGF

Similarly,

$$\frac{d^2}{dt^2}M(t) = E\left(\frac{d^2}{dt^2}e^{tX}\right) = E\left(X^2\, e^{tX}\right),$$

and in general

$$\frac{d^n}{dt^n}M(t) = E\left(\frac{d^n}{dt^n}e^{tX}\right) = E\left(X^n e^{tX}\right).$$

Conclusion. Evaluating at $t = 0$, we get

$$\left.\frac{d^n}{dt^n}M(t)\right|_{t=0} = E\left(X^n e^0\right) = E\left(X^n\right).$$

Hence the nth derivative of $M(t)$ at $t = 0$ gives the nth moment of X (about the origin).

MGF Example 1

Given:

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find:

- The moment generating function $M(t)$ of X.
- The mean $E(X)$.
- The variance $\mathrm{Var}(X)$.

Solution

Step 1: Moment Generating Function.

$$M(t) \;=\; E\left(e^{tX}\right) \;=\; \int_0^\infty e^{tx}\, e^{-x}\, \mathrm{d}x \;=\; \int_0^\infty e^{(t-1)x}\, \mathrm{d}x, \quad \text{valid for } t < 1.$$

This integral converges to

$$M(t) \;=\; \left[\frac{e^{(t-1)x}}{t-1}\right]_0^\infty \;=\; \frac{1}{1-t}, \quad (t < 1).$$

Step 2: Mean and Variance. We recognize X as an $\mathrm{Exponential}(1)$ random variable, thus

$$E(X) = 1 \quad \text{and} \quad \mathrm{Var}(X) = 1.$$

Alternatively, we can use $M'(t)\big|_{t=0} = E(X)$ and $M''(t)\big|_{t=0} = E(X^2)$ to derive these from the mgf.

Solution — Alternatively

Deriving Mean & Variance from MGF Directly

Given the MGF $M(t) = \dfrac{1}{1-t}, \;\; t < 1$, we can find moments by taking derivatives at $t = 0$.

Mean:

$$E(X) = \frac{d}{dt}M(t)\Big|_{t=0} = \frac{d}{dt}(1-t)^{-1}\Big|_{t=0} = (1-t)^{-2}\big|_{t=0} = 1.$$

Second Moment:

$$E(X^2) = \frac{d^2}{dt^2}M(t)\Big|_{t=0} = \frac{d^2}{dt^2}(1-t)^{-1}\Big|_{t=0} = 2\,(1-t)^{-3}\big|_{t=0} = 2.$$

Hence,

$$\mathrm{Var}(X) = E(X^2) - [\,E(X)\,]^2 = 2 - 1^2 = 1.$$

Example 2

Setup. Let X have the probability mass function

$$f(x) = \begin{cases} \frac{1}{9}\left(\frac{8}{9}\right)^x & \text{for } x = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Question. What is the moment generating function (MGF) of the random variable X?

Solution

Step 1: Definition of the MGF.

The MGF of a discrete random variable X is given by

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} f(x).$$

Step 2: Substitute the given PMF.

Here, $f(x) = \frac{1}{9}\left(\frac{8}{9}\right)^x$ for $x = 0, 1, 2, \ldots$, so

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{1}{9}\left(\frac{8}{9}\right)^x = \frac{1}{9} \sum_{x=0}^{\infty}\left(\frac{8}{9}e^t\right)^x.$$

Step 3: Recognize the geometric series.

A geometric series $\sum_{x=0}^{\infty} r^x$ converges to $\frac{1}{1-r}$ if $|r| < 1$. In this case,

$$r = \frac{8}{9}e^t,$$

Solution

and we require $\frac{8}{9}e^t < 1 \implies t < \ln\left(\frac{9}{8}\right)$. Thus,

$$\sum_{x=0}^{\infty} \left(\frac{8}{9}e^t\right)^x = \frac{1}{1 - \frac{8}{9}e^t} = \frac{9}{9 - 8\,e^t}.$$

Step 4: Simplify to get the MGF.

$$M_X(t) = \frac{1}{9} \cdot \frac{9}{9 - 8\,e^t} = \frac{1}{9 - 8\,e^t}, \quad t < \ln\left(\frac{9}{8}\right).$$

Answer:

$$\boxed{M_X(t) = \frac{1}{9 - 8\,e^t}, \quad t < \ln\left(\frac{9}{8}\right).}$$

Example 3

Given: A continuous random variable X with density

$$f(x) = \begin{cases} b\,e^{-bx}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $b > 0$.

If $M(t)$ is the MGF of X, find $M(-6b)$.

Solution

Step 1: Find $M(t)$.

$$M(t) = \mathbb{E}\left[e^{tX}\right] = \int_0^{\infty} b\, e^{tx}\, e^{-bx}\,\mathrm{d}x = b\int_0^{\infty} e^{-(b-t)x}\,\mathrm{d}x.$$

This integral converges if $b - t > 0$, i.e. $t < b$. In that case,

$$M(t) = b\,\frac{1}{b-t} = \frac{b}{b-t}.$$

Step 2: Evaluate $M(-6b)$.

Substitute $t = -6b$ into $M(t)$:

$$M(-6b) = \frac{b}{b-(-6b)} = \frac{b}{b+6b} = \frac{b}{7b} = \frac{1}{7}.$$

Answer:

$$\boxed{M(-6b) = \tfrac{1}{7}.}$$

Example 4

Given: A random variable X whose MGF is

$$M(t) = (1-t)^{-2}, \quad \text{for } t < 1.$$

Find: The third moment of X about the origin, i.e. $\mathbb{E}[X^3]$.

Solution

Recall: The n-th moment about the origin is given by

$$\mathbb{E}[X^n] \;=\; \frac{d^n}{dt^n} M(t)\Big|_{t=0}.$$

Step 1: Compute successive derivatives of $M(t)$.

$$M(t) = (1-t)^{-2}.$$

$$M'(t) \;=\; 2(1-t)^{-3},$$

$$M''(t) \;=\; 2\cdot(-3)(1-t)^{-4}\cdot(-1) \;=\; 6(1-t)^{-4},$$

$$M^{(3)}(t) \;=\; 6\cdot(-4)(1-t)^{-5}\cdot(-1) \;=\; 24(1-t)^{-5}.$$

Step 2: Evaluate at $t=0$.

$$\mathbb{E}[X^3] \;=\; M^{(3)}(t)\Big|_{t=0} \;=\; 24\cdot(1-0)^{-5} \;=\; 24.$$

Answer:

$$\boxed{\mathbb{E}[X^3] = 24.}$$

Theorem — Statement

Theorem. Let $M(t)$ be the moment generating function (MGF) of the random variable X. If

$$M(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$$

is the Taylor series expansion of $M(t)$ about $t = 0$, then

$$E(X^n) = (n!) a_n$$

for all natural numbers n.

Proof

- Let $M(t)$ be the MGF of X. Its Taylor series expansion about $t = 0$ is

$$M(t) \;=\; M(0) + \frac{M'(0)}{1!}\,t + \frac{M''(0)}{2!}\,t^2 + \frac{M'''(0)}{3!}\,t^3 + \cdots + \frac{M^{(n)}(0)}{n!}\,t^n + \cdots.$$

- Since $M(0) = 1$ and $E\left(X^n\right) = M^{(n)}(0)$ for $n \geq 1$, we also have

$$M(t) \;=\; 1 \;+\; \frac{E(X)}{1!}\,t \;+\; \frac{E(X^2)}{2!}\,t^2 \;+\; \frac{E(X^3)}{3!}\,t^3 \;+\; \cdots \;+\; \frac{E(X^n)}{n!}\,t^n \;+\; \cdots.$$

- Comparing this with the general form

$$M(t) \;=\; a_0 + a_1\,t + a_2\,t^2 + \cdots + a_n\,t^n + \cdots,$$

  we identify

$$a_n \;=\; \frac{E(X^n)}{n!}.$$

  Hence

$$E(X^n) \;=\; (n!)\,a_n,$$

  which completes the proof.

Example 1

Problem: Suppose a random variable X has the moment generating function

$$M(t) = \frac{1}{1+t}.$$

Find the 479th moment of X about the origin, i.e. $\mathbb{E}[X^{479}]$.

Solution

Step 1: Expand $M(t)$ in a power series.

$$M(t) = \frac{1}{1+t} = \frac{1}{1-(-t)} = 1+(-t)+(-t)^2+(-t)^3+\cdots+(-t)^n+\cdots$$

$$= 1 - t + t^2 - t^3 + t^4 \mp \cdots + (-1)^n t^n + \cdots .$$

Hence the coefficient of t^n in this expansion is

$$a_n = (-1)^n.$$

Solution

Step 2: Use the Theorem.

We know that if

$$M(t) = \sum_{n=0}^{\infty} a_n t^n,$$

then

$$\mathbb{E}[X^n] = n!\, a_n.$$

Hence

$$\mathbb{E}[X^{479}] = (479)!\, a_{479} = (479)! \left[(-1)^{479}\right].$$

Since 479 is odd,

$$(-1)^{479} = -1.$$

Therefore,

$$\mathbb{E}[X^{479}] = (479)! \left(-1\right) = -479!.$$

Answer:

$$\boxed{\mathbb{E}[X^{479}] = -479!.}$$