CSC 1201 Probability and Statistics

Bivariate and Conditional Random Variables

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Bivariate Discrete Random Variables

One reason an experimenter may study the joint behavior of more than one random variable is that many real problems involve multiple variables of interest. For example, an educator might involve both the grade and time devoted to study in a single investigation; a physician may study the joint behavior of blood pressure and weight; an economist may study the joint behavior of business volume and profit.

In fact, most real problems we come across will have more than one underlying random variable of interest.

By definition: A random variable (X, Y) is an *ordered pair* of discrete random variables.

Bivariate Discrete Random Variables

Definition

Let (X, Y) be a bivariate random variable and let R_X and R_Y be the range spaces of X and Y, respectively. A real-valued function

$$f : R_X \times R_Y \to \mathbb{R}$$

is called a *joint probability density function* for X and Y if for all $(x, y) \in R_X \times R_Y$,

$$f(x, y) = P(X = x, Y = y),$$

which is the probability of the intersection of the events $\{X = x\}$ and $\{Y = y\}$.

Example 1

Question: Roll a pair of unbiased dice. If X denotes the smaller and Y denotes the larger outcome on the dice, then what is the joint probability density function of X and Y?

Solution to Example 1

Solution Outline:

1. Each die has outcomes $\{1, 2, 3, 4, 5, 6\}$. Thus, there are $6 \times 6 = 36$ equally likely outcomes in total when rolling two dice.
2. The pair (X, Y) must satisfy $1 \leq X \leq Y \leq 6$.
3. For $X = Y$, there is exactly one outcome (both dice showing X). Thus, the probability is

$$P(X = x, Y = x) = \frac{1}{36}, \quad x \in \{1, 2, 3, 4, 5, 6\}.$$

4. For $X < Y$, there are two outcomes leading to (X, Y): (first die $=$ X, second die $= Y$) or (first die $= Y$, second die $= X$). Hence,

$$P(X = x, Y = y) = \frac{2}{36} = \frac{1}{18}, \quad 1 \leq x < y \leq 6.$$

Solution to Example 1

Therefore, the joint PDF $f_{X,Y}(x,y)$ is given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{36}, & \text{if } x = y, \\ \frac{1}{18}, & \text{if } x < y, \\ 0, & \text{otherwise.} \end{cases}$$

Marginal Probability Distributions

Definition

Let (X, Y) be a discrete random variable. Let R_X and R_Y be the range spaces of X and Y, respectively. Let $f(x, y)$ be the joint probability density function of X and Y.

The function

$$f_X(x) = \sum_{y \in R_Y} f(x, y)$$

is called the *marginal probability density function* of X.

Similarly, the function

$$f_Y(y) = \sum_{x \in R_X} f(x, y)$$

is called the *marginal probability density function* of Y.

Example 1: Marginals of X and Y

Problem Statement:

Suppose the joint probability density function of the discrete random variables X and Y is given by

$$f(x, y) = \begin{cases} \frac{1}{36}, & \text{if } 1 \leq x = y \leq 6, \\ \frac{2}{36}, & \text{if } 1 \leq x < y \leq 6, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

Solution

Marginal of X:

$$f_X(x) \;=\; \sum_{y \in R_Y} f(x,y).$$

Here, $R_X = R_Y = \{1,2,3,4,5,6\}$. For a fixed x:

- If $y = x$, then $f(x,x) = \frac{1}{36}$.
- If $y > x$, then $f(x,y) = \frac{2}{36}$ for each $y \in \{x+1, \ldots, 6\}$.

Hence, the number of y values greater than x is $(6-x)$. Therefore,

$$f_X(x) = \frac{1}{36} + (6-x)\,\frac{2}{36} = \frac{1 + 2(6-x)}{36} = \frac{13-2x}{36}, \quad x = 1,2,3,4,5,6.$$

Solution

Marginal of Y:

$$f_Y(y) \;=\; \sum_{x \in R_X} f(x,y).$$

For a fixed y:

- If $x = y$, then $f(y,y) = \frac{1}{36}$.
- If $x < y$, then $f(x,y) = \frac{2}{36}$ for each $x \in \{1, \ldots, y-1\}$.

Hence, the number of x values less than y is $(y-1)$. Therefore,

$$f_Y(y) = (y-1)\,\frac{2}{36} + \frac{1}{36} = \frac{2(y-1)+1}{36} = \frac{2y-1}{36}, \quad y = 1, 2, 3, 4, 5, 6.$$

Example 2: Marginal Distributions

Problem Statement:

Let X and Y be discrete random variables with joint probability density function

$$f(x, y) = \begin{cases} \dfrac{x+y}{21}, & \text{if } x \in \{1, 2\} \text{ and } y \in \{1, 2, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

Solution

Marginal of X:

$$f_X(x) = \sum_{y \in \{1,2,3\}} f(x, y).$$

- For $x = 1$:

$$f_X(1) = f(1,1) + f(1,2) + f(1,3) = \frac{1+1}{21} + \frac{1+2}{21} + \frac{1+3}{21} = \frac{2+3+4}{21} = \frac{9}{21} = \frac{3}{7}.$$

- For $x = 2$:

$$f_X(2) = f(2,1) + f(2,2) + f(2,3) = \frac{2+1}{21} + \frac{2+2}{21} + \frac{2+3}{21} = \frac{3+4+5}{21} = \frac{12}{21} = \frac{4}{7}.$$

Solution

Marginal of Y:

$$f_Y(y) = \sum_{x\in\{1,2\}} f(x,y).$$

- For $y = 1$:

$$f_Y(1) = f(1,1) + f(2,1) = \frac{1+1}{21} + \frac{2+1}{21} = \frac{2+3}{21} = \frac{5}{21}.$$

- For $y = 2$:

$$f_Y(2) = f(1,2) + f(2,2) = \frac{1+2}{21} + \frac{2+2}{21} = \frac{3+4}{21} = \frac{7}{21} = \frac{1}{3}.$$

- For $y = 3$:

$$f_Y(3) = f(1,3) + f(2,3) = \frac{1+3}{21} + \frac{2+3}{21} = \frac{4+5}{21} = \frac{9}{21} = \frac{3}{7}.$$

Thus,

$$f_X(1) = \frac{3}{7}, \quad f_X(2) = \frac{4}{7}, \qquad f_Y(1) = \frac{5}{21}, \quad f_Y(2) = \frac{1}{3}, \quad f_Y(3) = \frac{3}{7}.$$

Note

Checking that all probabilities sum to 1:

- $\displaystyle\sum_{x=1}^{2} f_X(x) \;=\; f_X(1) + f_X(2) = \frac{3}{7} + \frac{4}{7} = 1.$
- $\displaystyle\sum_{y=1}^{3} f_Y(y) \;=\; f_Y(1) + f_Y(2) + f_Y(3) = \frac{5}{21} + \frac{7}{21} + \frac{9}{21} = \frac{21}{21} = 1.$
- Also, $\displaystyle\sum_{x=1}^{2}\sum_{y=1}^{3} f(x,y) = \frac{1}{21}(1+1) + \frac{1}{21}(1+2) + \cdots + \frac{1}{21}(2+3) = 1.$

Therefore, the total probability is indeed 1, confirming that our marginal distributions and the joint distribution are consistent.

Properties

Theorem. A real-valued function f of two variables is a *joint probability density function* of a pair of discrete random variables X and Y (with range spaces R_X and R_Y, respectively) if and only if the following conditions hold:

1. $f(x, y) \geq 0$ for all $(x, y) \in R_X \times R_Y$.
2. $\sum_{x \in R_X} \sum_{y \in R_Y} f(x, y) = 1.$

Example 1

Problem Statement:
For what value of the constant k is the function

$$f(x,y) = \begin{cases} k\,x\,y, & \text{if } x \in \{1,2,3\} \text{ and } y \in \{1,2,3\}, \\ 0, & \text{otherwise,} \end{cases}$$

a joint probability density function of some random variables X and Y?

Solution

Step 1: Nonnegativity Condition.
Since x and y take values in $\{1, 2, 3\}$, and k is presumably nonnegative, we have $f(x, y) \geq 0$ for all (x, y). Hence, the nonnegativity condition is satisfied as long as $k \geq 0$.

Step 2: Total Probability Must Equal 1.
We require:

$$\sum_{x=1}^{3}\sum_{y=1}^{3} f(x, y) = \sum_{x=1}^{3}\sum_{y=1}^{3} kxy = k\sum_{x=1}^{3}\sum_{y=1}^{3} xy = 1.$$

Next, compute the double sum:

$$\sum_{x=1}^{3} x \ = \ 1 + 2 + 3 = 6, \quad \sum_{y=1}^{3} y \ = \ 1 + 2 + 3 = 6.$$

Solution

Hence,

$$\sum_{x=1}^{3}\sum_{y=1}^{3} xy = \left(\sum_{x=1}^{3} x\right)\left(\sum_{y=1}^{3} y\right) = 6 \times 6 = 36.$$

Therefore,

$$k \times 36 = 1 \quad \Longrightarrow \quad k = \frac{1}{36}.$$

Conclusion: The value of k that makes $f(x, y)$ a valid joint probability density function is $\boxed{\frac{1}{36}}$.

Solution

Thus

$$f(x,y) = \begin{cases} \dfrac{1}{36}\, x\, y, & \text{if } x \in \{1,2,3\},\ y \in \{1,2,3\}, \\ 0, & \text{otherwise.} \end{cases}$$

Verification:

$$\sum_{x=1}^{3}\sum_{y=1}^{3}\frac{1}{36}xy = \frac{1}{36}\sum_{x=1}^{3}\sum_{y=1}^{3}xy = \frac{1}{36}\times 36 = 1.$$

Thus, all conditions for a valid joint probability density function are satisfied.

Joint Probability Distribution Function

Definition. Let X and Y be any two discrete random variables. A real-valued function $F : \mathbb{R}^2 \to \mathbb{R}$ is called the *joint cumulative probability distribution function* of X and Y if and only if

$$F(x, y) = P\big(X \leq x,\ Y \leq y\big)$$

for all $(x, y) \in \mathbb{R}^2$. Here, the event $\{X \leq x,\ Y \leq y\}$ means $\{X \leq x\} \cap \{Y \leq y\}$.

From this definition, one can show that for any real numbers a and b:

$$F(a, b) = P(X \leq a,\ Y \leq b).$$

Furthermore, it follows that if $f(x, y)$ is the joint probability mass (density) function of X and Y, then

$$F(x, y) = \sum_{\substack{s \leq x \\ s \in R_X}} \sum_{\substack{t \leq y \\ t \in R_Y}} f(s, t),$$

where R_X and R_Y are the range spaces of X and Y, respectively.

Bivariate Continuous Random Variables

Context: We shall extend the idea of probability density functions of one random variable to that of two random variables.

Definition

Let X and Y be random variables taking values in $\mathbb{R}$. The *joint probability density function* of X and Y is an integrable function $f(x, y)$ such that

1. $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$,
2. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1.$

Note: Probabilities from Joint PDFs

If we know the joint probability density function f of the random variables X and Y, then we can compute the probability of an event A by integrating f over the region A:

$$P(A) = \iint_A f(x, y)\, dx\, dy.$$

This is a direct extension of the single-variable case to two dimensions, where $A \subseteq \mathbb{R}^2$ is the region of interest for (X, Y).

Example 1

Let the joint density function of X and Y be given by

$$f(x,y) = \begin{cases} k\,x\,y^2, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the value of the constant k that makes this a valid joint probability density function?

Solution

To be a valid joint density function, $f(x, y)$ must satisfy

$$\iint f(x, y)\, dx\, dy = 1$$

over its support. Since $0 < x < y < 1$, we integrate in the region where x goes from 0 to y, and y goes from 0 to 1:

$$\int_{y=0}^{1} \int_{x=0}^{y} k\, x\, y^2\, dx\, dy = 1.$$

Step 1: Inner Integral

$$\int_{x=0}^{y} k\, x\, y^2\, dx = k\, y^2 \int_0^y x\, dx = k\, y^2 \left[\frac{x^2}{2}\right]_0^y = k\, y^2 \left(\frac{y^2}{2}\right) = \frac{k\, y^4}{2}.$$

Solution

Step 2: Outer Integral

$$\int_{y=0}^{1} \frac{k\,y^4}{2}\,dy = \frac{k}{2}\int_0^1 y^4\,dy = \frac{k}{2}\left[\frac{y^5}{5}\right]_0^1 = \frac{k}{2}\cdot\frac{1}{5} = \frac{k}{10}.$$

For this to equal 1, we must have

$$\frac{k}{10} = 1 \quad \Longrightarrow \quad k = 10.$$

Conclusion:

$$\boxed{k = 10.}$$

Example 2

Let the joint density of the continuous random variables X and Y be

$$f(x,y) = \begin{cases} \frac{6}{5}\left(x^2 + 2xy\right), & 0 < x \le 1,\ 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability of the event $\{X \le Y\}$.

Solution

To find $P(X \leq Y)$, we integrate $f(x,y)$ over the region $\{(x,y) : 0 < x \leq y < 1\}$. Thus,

$$P(X \leq Y) = \int_{y=0}^{1} \int_{x=0}^{y} \frac{6}{5}\left(x^2 + 2xy\right) dx\, dy.$$

Step 1: Inner Integral

$$\int_{x=0}^{y} \frac{6}{5}(x^2 + 2xy)\, dx = \frac{6}{5}\int_0^y (x^2 + 2xy)\, dx = \frac{6}{5}\left[\frac{x^3}{3} + x^2 y\right]_0^y = \frac{6}{5}\left(\frac{y^3}{3} + y^3\right) = \frac{6}{5}\cdot\frac{4y^3}{3} = \frac{24}{15}\, y^3 = \frac{8}{5}\, y^3.$$

Step 2: Outer Integral

$$P(X \leq Y) = \int_{y=0}^{1} \frac{8}{5}\, y^3\, dy = \frac{8}{5}\int_0^1 y^3\, dy = \frac{8}{5}\left[\frac{y^4}{4}\right]_0^1 = \frac{8}{5}\cdot\frac{1}{4} = \frac{2}{5}.$$

Conclusion:

$$\boxed{P(X \leq Y) = \frac{2}{5}.}$$

Marginal Probability Densities

Definition

Let (X, Y) be a continuous bivariate random variable with joint probability density function $f(x, y)$. Then the function

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

is called the *marginal probability density function* of X. Similarly, the function

$$f_2(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

is called the *marginal probability density function* of Y.

Example 1

If the joint density function for X and Y is given by

$$f(x, y) = \begin{cases} \frac{3}{4}, & \text{if } 0 < y^2 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

find the marginal density function of X for $0 < x < 1$.

Solution

Marginal PDF of X:

$$f_X(x) \;=\; \int_{-\infty}^{\infty} f(x,y)\,dy.$$

From the condition $0 < y^2 < x < 1$, for a fixed x in $(0,1)$, we must have $y^2 < x$. Hence,

$$-\sqrt{x} < y < \sqrt{x}.$$

Therefore,

$$f_X(x) = \int_{y=-\sqrt{x}}^{\sqrt{x}} \frac{3}{4}\,dy = \frac{3}{4}\,[y]_{-\sqrt{x}}^{\sqrt{x}} = \frac{3}{4}\left(\sqrt{x} - (-\sqrt{x})\right) = \frac{3}{4}\cdot 2\sqrt{x} = \frac{3}{2}\,\sqrt{x},$$

Outside this interval, $f_X(x) = 0$.

Hence,

$$f_X(x) = \begin{cases} \dfrac{3}{2}\,\sqrt{x}, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2

Let X and Y have the joint density function

$$f(x, y) = \begin{cases} 2\, e^{-x-y}, & 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

What is the marginal density of X where it is non-zero?

Solution

Marginal PDF of X:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy.$$

Because $0 < x < y < \infty$, for a fixed x, y must range from $y = x$ to $y = \infty$. Hence, for $x > 0$,

$$f_X(x) = \int_{y=x}^{\infty} 2\, e^{-x-y}\, dy.$$

Factor out the terms that do not depend on y:

$$f_X(x) = 2\, e^{-x} \int_{y=x}^{\infty} e^{-y}\, dy.$$

Next, evaluate the integral:

$$\int_{y=x}^{\infty} e^{-y}\, dy = \left[- e^{-y}\right]_{y=x}^{\infty} = 0 - \left(-e^{-x}\right) = e^{-x}.$$

Solution

Thus,

$$f_X(x) = 2\,e^{-x} \cdot e^{-x} = 2\,e^{-2x}, \quad \text{for } x > 0.$$

Outside this region, $f_X(x) = 0$. Therefore,

$$f_X(x) = \begin{cases} 2\,e^{-2x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Joint Probability Distribution Function

Definition

Let X and Y be continuous random variables with joint probability density function $f(x, y)$. The joint cumulative distribution function $F(x, y)$ of X and Y is defined as

$$F(x, y) = P(X \leq x,\ Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, dv\, du$$

for all $(x, y) \in \mathbb{R}^2$.
From the fundamental theorem of calculus, we then obtain

$$f(x, y) = \frac{\partial^2 F}{\partial x\, \partial y}.$$

Example 1

Suppose the joint cumulative distribution function of X and Y is given by

$$F(x, y) = \begin{cases} \frac{1}{5}\left(2x^3 y + 3x^2 y^2\right), & \text{for } 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the joint probability density function $f(x, y)$.

Solution

Step 1: Recall that the joint PDF is obtained by taking the partial derivatives of the CDF with respect to x and y:

$$f(x,y) \;=\; \frac{\partial^2}{\partial x\,\partial y} F(x,y).$$

Step 2: First, compute the partial derivative with respect to x:

$$\frac{\partial}{\partial x}\left[\tfrac{1}{5}\left(2x^3y + 3x^2y^2\right)\right] = \tfrac{1}{5}\left(6x^2y + 6x\,y^2\right) = \tfrac{6}{5}\,x^2\,y \;+\; \tfrac{6}{5}\,x\,y^2.$$

Step 3: Next, take the partial derivative of that result with respect to y:

$$\frac{\partial}{\partial y}\left[\tfrac{6}{5}\,x^2\,y \;+\; \tfrac{6}{5}\,x\,y^2\right] = \tfrac{6}{5}\,x^2 \;+\; \tfrac{12}{5}\,x\,y.$$

Step 4: Thus, the joint PDF is:

$$f(x,y) = \begin{cases} \dfrac{6}{5}\,x^2 \;+\; \dfrac{12}{5}\,x\,y, & 0 < x < 1,\; 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2

Let X and Y have the joint density function

$$f(x,y) = \begin{cases} 2x, & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find

$$P\left(X + Y \le 1 \,\middle|\, X \le \tfrac{1}{2}\right).$$

Solution

Step 1: We want

$$P(X+Y \le 1 \mid X \le \tfrac{1}{2}) = \frac{P(X+Y \le 1,\ X \le \tfrac{1}{2})}{P\big(X \le \tfrac{1}{2}\big)}.$$

Step 2: Compute $P\big(X \le \tfrac{1}{2}\big)$. Integrate the joint density:

$$P\big(X \le \tfrac{1}{2}\big) = \int_{x=0}^{1/2} \int_{y=0}^{1} 2x \, dy \, dx = \int_0^{1/2} 2x \big[y\big]_0^1 \, dx = \int_0^{1/2} 2x \, dx = \Big[x^2\Big]_0^{1/2} = \frac{1}{4}.$$

Step 3: Compute $P(X+Y \le 1,\ X \le \tfrac{1}{2})$. For $0 < x < 1/2$, the condition $X+Y \le 1$ implies $0 < y \le 1-x$:

$$P(X+Y \le 1,\ X \le \tfrac{1}{2}) = \int_{x=0}^{1/2} \int_{y=0}^{1-x} 2x \, dy \, dx = \int_0^{1/2} 2x \Big[y\Big]_0^{1-x} dx = \int_0^{1/2} 2x(1-x) \, dx.$$

$$= \int_0^{1/2} (2x - 2x^2) \, dx = \Big[x^2 - \tfrac{2}{3}x^3\Big]_0^{1/2} = \big(\tfrac{1}{4} - \tfrac{2}{3} \cdot \tfrac{1}{8}\big) = \tfrac{1}{4} - \tfrac{1}{12} = \tfrac{1}{6}.$$

Solution

Step 4: Combine these results:

$$P\big(X + Y \leq 1 \mid X \leq \tfrac{1}{2}\big) = \frac{\frac{1}{6}}{\frac{1}{4}} = \frac{1}{6} \times \frac{4}{1} = \frac{2}{3}.$$

Hence,

$$\boxed{P\big(X + Y \leq 1 \mid X \leq \tfrac{1}{2}\big) = \tfrac{2}{3}.}$$

Conditional Distributions

Setup: Let X and Y be two discrete random variables with joint probability mass function (pmf) $f(x, y) = P(X = x,\ Y = y)$.

Marginal pmfs:

$$f_X(x) = \sum_y f(x, y), \quad f_Y(y) = \sum_x f(x, y).$$

Conditional pmf:

$$f_{X|Y}(x \mid y) \ = \ P(X = x \mid Y = y) \ = \ \frac{f(x, y)}{f_Y(y)}, \quad \text{provided } f_Y(y) > 0.$$

Alternatively, if we denote $P(X = x \mid Y = y)$ by $g(x \mid y)$, then

$$g(x \mid y) \ = \ \frac{f(x, y)}{f_Y(y)}.$$

Conditional Probability Density Function

Definition

Let X and Y be any two random variables with joint density $f(x, y)$ and marginals $f_1(x)$ and $f_2(y)$.

The *conditional probability density function* of X, given the event $Y = y$, is defined as

$$g(x \mid y) \;=\; \frac{f(x, y)}{f_2(y)}, \quad \text{with } f_2(y) > 0.$$

Similarly, the *conditional probability density function* of Y, given the event $X = x$, is defined as

$$h(y \mid x) \;=\; \frac{f(x, y)}{f_1(x)}, \quad \text{with } f_1(x) > 0.$$

Example 1

Let X and Y be discrete random variables with joint probability function

$$f(x,y) = \begin{cases} \dfrac{1}{21}(x+y), & x = 1,2,3;\ y = 1,2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional probability density function of X, given $Y = 2$.

Solution

Step 1: Compute the marginal distribution of Y at $y = 2$:

$$f_Y(2) = \sum_{x=1}^{3} f(x,2) = \sum_{x=1}^{3} \frac{1}{21}(x+2) = \frac{1}{21}\left[(1+2)+(2+2)+(3+2)\right] = \frac{1}{21}(3+4+5) = \frac{12}{21} = \frac{4}{7}.$$

Step 2: The conditional pmf of X given $Y = 2$ is

$$f_{X|Y}(x \mid 2) = \frac{f(x,2)}{f_Y(2)} = \frac{\frac{1}{21}(x+2)}{\frac{4}{7}} = \frac{\frac{1}{21}(x+2)}{\frac{4}{7}} = \frac{1}{21}(x+2) \times \frac{7}{4} = \frac{(x+2)}{12}.$$

Step 3: List the values for $x = 1, 2, 3$:

$$f_{X|Y}(1 \mid 2) = \frac{1+2}{12} = \frac{3}{12} = \frac{1}{4}, \quad f_{X|Y}(2 \mid 2) = \frac{2+2}{12} = \frac{4}{12} = \frac{1}{3}, \quad f_{X|Y}(3 \mid 2) = \frac{3+2}{12} = \frac{5}{12}.$$

Check: $\frac{1}{4} + \frac{1}{3} + \frac{5}{12} = 1.$

Thus, the conditional pmf of X given $Y = 2$ is

$$f_{X|Y}(x \mid 2) = \begin{cases} \dfrac{x+2}{12}, & x = 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2

Let X and Y be discrete random variables with joint probability function

$$f(x, y) = \begin{cases} \dfrac{x+y}{32}, & x = 1, 2;\ y = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional probability of Y given $X = x$, that is, $f_{Y|X}(y \mid x)$.

Solution

Step 1: Find the marginal distribution of X:

$$f_X(x) = \sum_{y=1}^{4} f(x, y).$$

<u>Case $x = 1$:</u>

$$f_X(1) = \sum_{y=1}^{4} \frac{1+y}{32} = \frac{1}{32}\sum_{y=1}^{4}(1+y) = \frac{1}{32}\left[(1+1)+(1+2)+(1+3)+(1+4)\right] = \frac{1}{32}(2+3+4+5) = \frac{14}{32} = \frac{7}{16}.$$

<u>Case $x = 2$:</u>

$$f_X(2) = \sum_{y=1}^{4} \frac{2+y}{32} = \frac{1}{32}\sum_{y=1}^{4}(2+y) = \frac{1}{32}\left[(2+1)+(2+2)+(2+3)+(2+4)\right] = \frac{1}{32}(3+4+5+6) = \frac{18}{32} = \frac{9}{16}.$$

Step 2: The conditional pmf is given by

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)}.$$

<u>Case $x = 1$:</u>

$$f_{Y|X}(y \mid 1) = \frac{\frac{1}{32}(1+y)}{\frac{7}{16}} = \frac{1+y}{32} \times \frac{16}{7} = \frac{1+y}{2} \times \frac{1}{7} = \frac{1+y}{14}, \quad y = 1, 2, 3, 4.$$

<u>Case $x = 2$:</u>

$$f_{Y|X}(y \mid 2) = \frac{\frac{1}{32}(2+y)}{\frac{9}{16}} = \frac{2+y}{32} \times \frac{16}{9} = \frac{2+y}{2} \times \frac{1}{9} = \frac{2+y}{18}, \quad y = 1, 2, 3, 4.$$

Solution

Step 3: Summaries of the conditional distributions:

$$f_{Y|X}(y \mid 1) = \begin{cases} \dfrac{1+y}{14}, & y = 1,2,3,4, \\ 0, & \text{otherwise}, \end{cases} \qquad f_{Y|X}(y \mid 2) = \begin{cases} \dfrac{2+y}{18}, & y = 1,2,3,4, \\ 0, & \text{otherwise}. \end{cases}$$

Independence of Random Variables

Definition

Let X and Y be any two random variables with joint density $f(x, y)$ and marginals $f_1(x)$ and $f_2(y)$. The random variables X and Y are *(stochastically) independent* if and only if

$$f(x, y) = f_1(x) \, f_2(y)$$

for all (x, y) in $R_X \times R_Y$.

Example 1

Let X and Y be discrete random variables with joint probability function

$$f(x,y) = \begin{cases} \frac{1}{36}, & \text{for } 1 \leq x = y \leq 6, \\ \frac{2}{36}, & \text{for } 1 \leq x < y \leq 6, \\ 0, & \text{otherwise.} \end{cases}$$

Are X and Y stochastically independent?

Solution

Step 1: Compute the marginal pmf of X.
For each $x \in \{1, 2, \ldots, 6\}$,

$$f_X(x) \;=\; \sum_{y=1}^{6} f(x, y).$$

Observing the definition:

$$f(x, x) = \frac{1}{36}, \quad f(x, y) = \frac{2}{36} \text{ if } y > x, \quad f(x, y) = 0 \text{ if } y < x.$$

Hence,

$$f_X(x) = \underbrace{\frac{1}{36}}_{\text{when } y=x} + \sum_{y=x+1}^{6} \frac{2}{36} = \frac{1}{36} + (6-x)\,\frac{2}{36} = \frac{1 + 2(6 - x)}{36} = \frac{13 - 2x}{36}.$$

Solution

Step 2: Compute the marginal pmf of Y.
For each $y \in \{1, 2, \dots, 6\}$,

$$f_Y(y) \;=\; \sum_{x=1}^{6} f(x, y).$$

Here,

$$f(y, y) = \frac{1}{36}, \quad f(x, y) = \frac{2}{36} \;\text{ if } x < y, \quad f(x, y) = 0 \;\text{ if } x > y.$$

Hence,

$$f_Y(y) = \underbrace{\frac{1}{36}}_{\text{when } x=y} + \sum_{x=1}^{y-1} \frac{2}{36} = \frac{1}{36} + (y-1)\,\frac{2}{36} = \frac{1 + 2(y-1)}{36} = \frac{2y-1}{36}.$$

Solution

Step 3: Check for independence.
If X and Y were independent, then

$$f(x,y) \;=\; f_X(x)\,f_Y(y) \quad \text{for all } x,y.$$

However, take $(x,y) = (1,1)$:

$$f(1,1) = \frac{1}{36}, \quad f_X(1)\,f_Y(1) = \left(\frac{13-2\cdot 1}{36}\right)\left(\frac{2\cdot 1-1}{36}\right) = \frac{11}{36}\cdot\frac{1}{36} = \frac{11}{1296} \;\neq\; \frac{1}{36}.$$

Thus, $f(x,y) \neq f_X(x)\,f_Y(y)$, and so X and Y are **not** stochastically independent.

Example 2

Let X and Y have the joint density

$$f(x,y) = \begin{cases} e^{-(x+y)}, & 0 < x < \infty,\ 0 < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Are X and Y stochastically independent?

Solution

Step 1: Find the marginal density of X:

$$f_X(x) = \int_0^\infty f(x, y)\, dy = \int_0^\infty e^{-(x+y)}\, dy = e^{-x} \int_0^\infty e^{-y}\, dy = e^{-x} \left[-e^{-y}\right]_0^\infty = e^{-x}, \quad x > 0.$$

Step 2: Find the marginal density of Y:

$$f_Y(y) = \int_0^\infty f(x, y)\, dx = \int_0^\infty e^{-(x+y)}\, dx = e^{-y} \int_0^\infty e^{-x}\, dx = e^{-y} \left[-e^{-x}\right]_0^\infty = e^{-y}, \quad y > 0.$$

Step 3: Check for independence. If X and Y were independent, then

$$f(x, y) = f_X(x)\, f_Y(y).$$

We have

$$f_X(x)\, f_Y(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)} = f(x, y).$$

Therefore, $f(x, y) = f_X(x)\, f_Y(y)$ for all $x > 0$, $y > 0$, and we conclude that X **and Y are stochastically independent**.

Transformations of Independent Random Variables & IID

Observation: If X and Y are independent, then the random variables

$$U = \phi(X) \quad \text{and} \quad V = \psi(Y)$$

are also independent for any real-valued functions ϕ, ψ.

Hence, if X and Y are independent, then e^X and $Y^3 + Y^2 + 1$ are also independent.

Definition

The random variables X and Y are said to be *independent and identically distributed (IID)* if and only if they are independent and have the same distribution.

Independent Random Variables & IID

Example

Let X and Y be two independent random variables with identical probability density function

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability density function of $W = \min(X, Y)$.

Solution

Step 1: Recall that X and Y are i.i.d. exponential(1).
For $w > 0$, we compute the CDF of W:

$$F_W(w) = P(W \leq w) = 1 - P(W > w).$$

But $W > w$ means both $X > w$ and $Y > w$. Hence,

$$P(W > w) = P(X > w,\, Y > w) = P(X > w)\, P(Y > w) \quad \text{(by independence)}.$$

Since $P(X > w) = e^{-w}$ for an Exponential(1) random variable,

$$P(W > w) = e^{-w}\, e^{-w} = e^{-2w}.$$

Thus,

$$F_W(w) = 1 - e^{-2w}.$$

Solution

Step 2: Differentiate $F_W(w)$ to obtain the PDF of W:

$$f_W(w) = \frac{d}{dw}\left(1 - e^{-2w}\right) = 2\,e^{-2w}, \quad w > 0.$$

Step 3: Conclude that $W = \min(X, Y)$ has an exponential distribution with parameter 2:

$$f_W(w) = \begin{cases} 2e^{-2w}, & w > 0, \\ 0, & \text{otherwise.} \end{cases}$$