CSC 1201 Probability and Statistics

Discrete Random Variables and Distributions

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Learning Outcomes

After careful study of this chapter, you should be able to:

1. Determine probabilities from probability mass functions and the reverse.
2. Determine probabilities and probability mass functions from cumulative distribution functions and the reverse.
3. Calculate means and variances for discrete random variables.
4. Understand the assumptions for some common discrete probability distributions.
5. Select an appropriate discrete probability distribution to calculate probabilities in specific applications.
6. Calculate probabilities and determine means and variances for some common discrete probability distributions.

Introduction to Random Variables

- In many random experiments, elements of the sample space are not necessarily numbers.
- Example: Tossing a coin gives outcomes 'Head' or 'Tail'.

$$S = \{Head, Tail\}$$

- A random variable quantifies outcomes of the sample space.
- Example: X could be the temperature of a room at time t.

Therefore, a random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.

Random Variables

Definition

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

Remark on Notation

We will use capital letters (e.g. X) to denote random variables. After the random experiment is performed, the observed value of the random variable is denoted by a lower-case letter (e.g. x).

Random Variables

Space of a Random Variable

The space of the random variable X will be denoted by $\mathcal{R}_X$. In fact, the space of X is just the *range* of the function $X : S \to \mathbb{R}$.

Example

- X is the random variable corresponding to the temperature of the room at time t.
- x is the measured temperature of the room at time t.

Discrete Random Variables

Definition

A **discrete random variable** is a random variable with a finite or countable range. For example:

- Number of scratches in the surface of a disk
- Number of TCP connections to a server
- Number of errors in a transmission

Many situations can be adequately modeled using discrete random variables:

- Number of failed hard-drives in a RAID system
- Number of customers attempting to access a webserver
- Number of times you need to call an airline company until you reach a competent agent

Probability Distributions

- The probability distribution of a random variable X is a list of all the possible outcomes of X together with their corresponding probability values.
- For a discrete random variable X, this is quite straightforward because X can only take a finite or countable number of possible values.
- The **probability mass function (pmf)** is the probability distribution of a discrete random variable, giving the possible values and their associated probabilities.

Definition: Probability Mass Function (pmf)

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n$, the **probability mass function** is a function f with domain $\{x_1, \ldots, x_n\}$ s.t.:

1. $f(x_i) \geq 0$ for each i,
2. $\sum_{i=1}^{n} f(x_i) = 1$,
3. $P(X = x_i) = f(x_i)$ for each i.

Probability Distributions (Countably Infinite Case)

If the space of possible outcomes is countably infinite, the previous definition of the pmf can be generalized:

Definition: Probability Mass Function (pmf)

For a discrete random variable X with possible values $x_1, x_2, \ldots$, the **probability mass function** is a function

$$f : \{x_1, x_2, \ldots\} \to [0, 1]$$

satisfying:

1. $f(x_i) \geq 0$ for each i,
2. $\sum_{i=1}^{\infty} f(x_i) = 1$,
3. $P(X = x_i) = f(x_i)$ for each i.

Example 1

Situation: A user can make one of three possible requests to a GUI. We can view this as a random experiment with sample space $\{\text{Print}, \text{Save}, \text{Cancel}\}$.

Mapping to Numerical Values:

- Print $\rightarrow 0$
- Save $\rightarrow 1$
- Cancel $\rightarrow 2$

Let X be the random variable corresponding to the request made.

Constructing a pmf for X:

$$P(X = 0) = 0.2, \quad P(X = 1) = 0.5, \quad P(X = 2) = 0.3.$$

Example 1

Thus, the set of possible values is $\{0, 1, 2\}$, and the probability mass function f can be written as

$$f(x) = P(X = x) = \begin{cases} 0.2 & \text{if } x = 0, \\ 0.5 & \text{if } x = 1, \\ 0.3 & \text{if } x = 2. \end{cases}$$

It is straightforward to check that this is a valid p.m.f. because all probabilities are non-negative and sum to 1.

Probability Distributions (Sample Questions)

Question 2: A pair of dice consisting of a six-sided die and a four-sided die is rolled, and the sum is determined. Let the random variable X denote this sum.
Task: Find the sample space, the space of X, and the probability mass function of X.

Question 3: A fair coin is tossed 3 times. Let the random variable X denote the number of heads in the 3 tosses.
Task: Find the sample space, the space of X, and the probability mass function of X.

Question 4: If the probability of a random variable X with space $\mathcal{R}_X = \{1, 2, 3, \ldots, 12\}$ is given by

$$f(x) = k(2x - 1),$$

Task: Determine the value of the constant k.

Solution to Question 3

Sample Space:

$$S = \{\text{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}\}.$$

Each element of S is one outcome of tossing a fair coin three times.

Definition of the Random Variable X: Let X be the number of heads in the three tosses.

$$\mathcal{R}_X = \{0, 1, 2, 3\}.$$

Probability Mass Function (pmf): Because each of the 8 outcomes is equally likely with probability $\frac{1}{8}$,

$$f(x) = P(X = x) = \begin{cases} \frac{1}{8}, & x = 0, \\ \frac{3}{8}, & x = 1, \\ \frac{3}{8}, & x = 2, \\ \frac{1}{8}, & x = 3. \end{cases}$$

Cumulative Distribution Function

Motivation: For any random variable, we can describe its distribution using the *cumulative distribution function* (cdf) rather than the probability mass function (pmf).

Definition: Cumulative Distribution Function (cdf)

The **cdf** of a random variable X is the function $F : \mathbb{R} \to [0,1]$ defined by

$$F(x) \ = \ P\big(X \leq x\big),$$

for all real x.

Cumulative Distribution Function

Discrete Case: If X is a discrete random variable with pmf $f(\cdot)$, then

$$F(x) = P(X \leq x) = \sum_{i: x_i \leq x} f(x_i),$$

where the sum runs over all x_i that are less than or equal to x.

Properties of $F(x)$:

1. $0 \leq F(x) \leq 1$.
2. If $x \leq y$, then $F(x) \leq F(y)$ (non-decreasing).

Cumulative Distribution Function

Theorem

If the space $\mathcal{R}_X$ of the random variable X is given by

$$\mathcal{R}_X = \{x_1 < x_2 < x_3 < \cdots < x_n\},$$

then the corresponding pmf f can be recovered from the cdf F as follows:

$$f(x_1) = F(x_1), \quad f(x_2) = F(x_2) - F(x_1), \quad f(x_3) = F(x_3) - F(x_2), \quad \ldots, \quad f(x_n) = F(x_n) - F(x_{n-1}).$$

Important: Recall that the cdf $F(x)$ is defined *for all real numbers* x, not just the specific values that X can take in $\mathcal{R}_X$.

Cumulative Distribution Function — Example 1

Given: A random variable X has the cdf

$$F(x) = \begin{cases} 0.00, & x < -1, \\ 0.25, & -1 \leq x < 1, \\ 0.50, & 1 \leq x < 3, \\ 0.75, & 3 \leq x < 5, \\ 1.00, & x \geq 5. \end{cases}$$

Find the pmf: For a discrete variable, $P(X = a) = F(a) - \lim_{x \to a^-} F(x)$. Thus,

$$P(X = -1) = 0.25 - 0.00 = 0.25, \quad P(X = 1) = 0.50 - 0.25 = 0.25,$$

$$P(X = 3) = 0.75 - 0.50 = 0.25, \quad P(X = 5) = 1.00 - 0.75 = 0.25.$$

Cumulative Distribution Function — Example 1

Hence, the random variable X takes values $\{-1, 1, 3, 5\}$, each with probability 0.25.

Compute:

(a) $P(X \leq 3) = 0.25 + 0.25 + 0.25 = 0.75,$

(b) $P(X = 3) = 0.25,$

(c) $P(X < 3) = 0.25 + 0.25 = 0.50.$

CDF — Example

Let X be a discrete random variable taking values $\{-1, 0, 1, 3, 5\}$ with probability mass function (pmf)

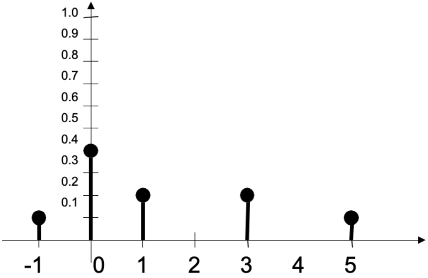
$$f(x) = \begin{cases} 0.1 & x = -1, \\ 0.4 & x = 0, \\ 0.2 & x = 1, \\ 0.2 & x = 3, \\ 0.1 & x = 5, \\ 0 & \text{otherwise.} \end{cases}$$

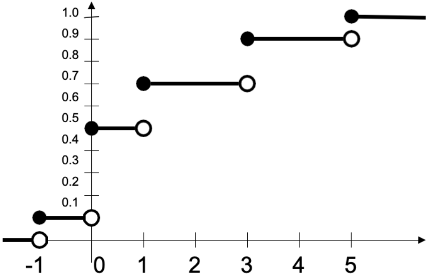
The corresponding cumulative distribution function (cdf) is

$$F(x) = \begin{cases} 0.0 & x < -1, \\ 0.1 & -1 \leq x < 0, \\ 0.5 & 0 \leq x < 1, \\ 0.7 & 1 \leq x < 3, \\ 0.9 & 3 \leq x < 5, \\ 1.0 & x \geq 5. \end{cases}$$

Sketches of the PMF and CDF

- The *pmf sketch* shows discrete probabilities at the points $-1, 0, 1, 3, 5$.
- The *cdf sketch* is a step function that jumps at each point in the support of X.





CDF — Example

Given: A discrete random variable X with cdf:

$$F(x) = \begin{cases} 0.0, & x < -1, \\ 0.1, & -1 \le x < 0, \\ 0.5, & 0 \le x < 1, \\ 0.7, & 1 \le x < 3, \\ 0.9, & 3 \le x < 5, \\ 1.0, & x \ge 5. \end{cases}$$

What is $P(1 \le X \le 4)$?

- A) 0.3
- B) 0.4
- C) 0.5
- D) 0.6

CDF Example - Solution

Example Probability Calculations:

$$P(X \leq 4) \; = \; 0.9.$$

$$P(1 < X \leq 4) = P(X \leq 4) \; - \; P(X \leq 1) = 0.9 \; - \; 0.7 = 0.2.$$

$$P(1 \leq X \leq 4) = P(X \leq 4) \; - \; P(X < 1) = 0.9 \; - \; 0.5 = 0.4.$$

The correct choice is $\boxed{0.4}$ *(refer to previous slide).*

Warning!

Note that $P(1 < X \leq 4)$ is correctly computed as

$$P(1 < X \leq 4) = P(X \leq 4) - P(X \leq 1),$$

rather than

$$P(X \leq 4) - P(X < 1).$$

The reason is that $P(X \leq 1)$ includes the probability mass at $X = 1$, so subtracting it excludes $X = 1$ from the interval $(1, 4]$ correctly.

1. We want $1 < X \leq 4$:
   - This interval does *not* include $X = 1$.
   - So we must subtract everything up to *and including* 1 from everything up to 4.
2. $P(X \leq 1)$ *includes* $X = 1$:
   - Therefore, $P(X \leq 4) - P(X \leq 1)$ removes the probability of $X = 1$.
   - This precisely leaves us with probabilities where X is *greater* than 1 but still ≤ 4.
3. $P(X < 1)$ *does not* include $X = 1$:
   - If you only subtract $P(X < 1)$, you have $P(X \leq 4) - P(X < 1)$.
   - That still includes the probability mass at $X = 1$, effectively giving $P(1 \leq X \leq 4)$, not $P(1 < X \leq 4)$.

Hence, to exclude $X = 1$ properly, we must subtract $P(X \leq 1)$.

Mean and Variance

Two important summary measures for a random variable X:

- **Mean (or Expected Value)**: measures the "center" of the distribution.
- **Variance**: measures the dispersion or "spread" of the distribution.

Definition: Mean and Variance (Discrete X)

Let X be a discrete random variable taking values in $\{x_1, x_2, \dots\} \subset \mathbb{R}$.

- **Mean (Expected Value)** of X:

$$\mathbb{E}[X] = \mu_X = \sum_{x \in \{x_i\}} x\, f(x).$$

- **Variance** of X, denoted σ_X^2 or $\mathrm{V}(X)$:

$$\sigma_X^2 = \mathrm{V}(X) = \sum_{x \in \{x_i\}} (x - \mu_X)^2 f(x).$$

Mean and Variance

Useful Alternative Formula for Variance:

$$\mathrm{V}(X) = \sum_{x \in \{x_i\}} x^2 f(x) - \mu_X^2.$$

Mean and Variance

Mean:

- The mean of X is a weighted average of the possible values, where the weights are the corresponding probabilities.
- It describes the "center" of the distribution, that is, on average, where X tends to lie.

Variance:

- The variance of X measures the dispersion of X around its mean.
- A larger variance indicates X is more spread out.

Important

The variance is ALWAYS non-negative:

$$\mathrm{V}(X) \geq 0.$$

Mean and Variance

Definition: Standard Deviation

Let X be a discrete random variable. The standard deviation of X is simply

$$\sigma = \sqrt{\mathrm{V}(X)}.$$

Example 1

Scenario: A certain manufacturer produces CD-RWs, each of which may have a random number of surface defects. This number is modeled by a discrete random variable X with the following pmf:

$$f(x) = P(X = x) = \begin{cases} 0.90, & x = 0, \\ 0.08, & x = 1, \\ 0.02, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Question (Q1): What is $\mu_x = \mathbb{E}[X]$?

Solution:

$$\mathbb{E}[X] = 0 \cdot 0.90 + 1 \cdot 0.08 + 2 \cdot 0.02 = 0.00 + 0.08 + 0.04 = 0.12.$$

Q2: What is $\sigma_x^2 = \mathrm{V}[X]$?

First Method (Definition):

$$\mathrm{V}(X) = \sum_{i=0}^{2} (\, i - \mu_x \,)^2 \, P(X = i).$$

We already have $\mu_x = 0.12$. Therefore:

$$\mathrm{V}(X) = (0 - 0.12)^2 \times 0.9 + (1 - 0.12)^2 \times 0.08 + (2 - 0.12)^2 \times 0.02.$$

Numerically:

$$(0 - 0.12)^2 \; = \; 0.0144, \quad (1 - 0.12)^2 \; = \; 0.7744, \quad (2 - 0.12)^2 \; = \; 3.5344.$$

Hence,

$$\mathrm{V}(X) = 0.0144 \times 0.9 + 0.7744 \times 0.08 + 3.5344 \times 0.02 = 0.1456.$$

Q2: What is $\sigma_x^2 = \mathrm{V}[X]$?

Second Method (Shortcut Formula):

$$\mathrm{V}(X) = \sum_{i=0}^{2} i^2\, P(X = i) \; - \; \mu_x^2.$$

Compute:

$$\sum_{i=0}^{2} i^2 P(X = i) = 0^2 \times 0.9 + 1^2 \times 0.08 + 2^2 \times 0.02 = 0 + 0.08 + 0.08 = 0.16.$$

Then

$$\mathrm{V}(X) = 0.16 \; - \; (0.12)^2 = 0.16 - 0.0144 = 0.1456.$$

Answer: $\mathrm{V}(X) = 0.1456.$

Example 2

Problem Setup:

- A couple decides to have 3 children.
- If none of these 3 is a girl, they will have a 4th child.
- If that 4th child is also not a girl, they will have one more child (the 5th).
- Then they stop, regardless of whether the 5th child is a girl or not.

Let X be the total number of children the couple ends up having.

Example 2

Possible Values of X:

- $X = 3$ if at least one of the first 3 children is a girl.

$$P(X = 3) \;=\; 1 - P(\text{all 3 are boys}) \;=\; 1 - \left(\tfrac{1}{2}\right)^3 = 1 - \tfrac{1}{8} = \tfrac{7}{8}.$$

- $X = 4$ if the first 3 are boys, but the 4th is a girl.

$$P(X = 4) \;=\; \left(\tfrac{1}{2}\right)^3 \times \tfrac{1}{2} = \tfrac{1}{8} \times \tfrac{1}{2} = \tfrac{1}{16}.$$

- $X = 5$ if the first 3 are boys, and the 4th is also a boy (the 5th then happens anyway, regardless of gender).

$$P(X = 5) \;=\; \left(\tfrac{1}{2}\right)^3 \times \tfrac{1}{2} = \tfrac{1}{8} \times \tfrac{1}{2} = \tfrac{1}{16}.$$

Expected Value:

$$\mathbb{E}[X] = 3\times\tfrac{7}{8}+4\times\tfrac{1}{16}+5\times\tfrac{1}{16} = \tfrac{21}{8}+\tfrac{4}{16}+\tfrac{5}{16} = 2.625+0.25+0.3125 = 3.1875$$

Functions of Random Variables

Key Point: If X is a random variable and $h : \mathbb{R} \to \mathbb{R}$ is any function, then $Y = h(X)$ is also a random variable.

Example: Let $X \in \{0, 1, 2\}$ have pmf

$$f(x) = P(X = x) = \begin{cases} 0.90, & x = 0, \\ 0.08, & x = 1, \\ 0.02, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Define $Y = (X - 1)^2$. Then Y takes values

$$Y = \begin{cases} 0, & \text{if } X = 1, \\ 1, & \text{if } X = 0 \text{ or } X = 2. \end{cases}$$

So $\{0, 1\}$ is the support of Y.

Functions of Random Variables

What is the pmf of Y?

$$P(Y = 0) = P(X = 1) = 0.08,$$

$$P(Y = 1) = P(X = 0 \text{ or } X = 2) = 0.90 + 0.02 = 0.92.$$

What is the expected Value of Y?

$$\mathbb{E}[Y] = 0 \times 0.08 \, + \, 1 \times 0.92 = 0.92.$$

Note: There is a quicker way to compute $\mathbb{E}[Y]$ using $\mathbb{E}[h(X)] = \sum_x h(x)\, f(x)$ directly!

Law of the Unconscious Statistician

Proposition

Let X be a random variable, and let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then $Y = h(X)$ is also a random variable. If X is discrete and takes values $\{x_1, x_2, \ldots\}$ with pmf $f(x)$, then

$$\mathbb{E}[Y] \;=\; \mathbb{E}[h(X)] \;=\; \sum_i h(x_i)\, f(x_i).$$

Law of the Unconscious Statistician

Example

Let $X \in \{0, 1, 2\}$ have the pmf

$$f(x) = P(X = x) = \begin{cases} 0.90, & x = 0, \\ 0.08, & x = 1, \\ 0.02, & x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Define a new random variable $Y = (X - 1)^2$. What is $\mathbb{E}[Y]$?

$$\mathbb{E}[Y] = \sum_{x \in \{0,1,2\}} (x - 1)^2 f(x) = (0 - 1)^2 \cdot 0.90 + (1 - 1)^2 \cdot 0.08 + (2 - 1)^2 \cdot 0.02 = 0.92.$$

Variance as an Expected Value

Key Idea: From the Law of the Unconscious Statistician, we see that

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mu_X)^2\right] = \sum_x (x - \mu_X)^2 f(x),$$

i.e., the variance is the expected value of the random variable $Y = (X - \mu_X)^2$.

In Other Words

$$\text{Let } h(X) = (X - \mu_X)^2. \quad \text{Then } \mathbb{E}[h(X)] = \sum_x h(x) f(x) = \sum_x (x - \mu_X)^2 f(x).$$

Note: The name "Law of the Unconscious Statistician" comes from how intuitive this result seems—yet rigorously proving it for both discrete and continuous random variables can be non-trivial.

Example: Missing TeslaCoil Toy

Setup:

- You collect Happy Meal toys. There are 5 total toys, and you're missing only the "TeslaCoil" toy.
- Each meal independently contains the TeslaCoil with probability 0.2. Otherwise, you get another toy (with probability 0.8).
- You buy exactly 2 meals.

Define: X = number of TeslaCoils obtained in those 2 meals.

Possible Values of X:

$$X \in \{0, 1, 2\}.$$

$$P(X = 0) = 0.8 \times 0.8 = 0.64,$$

$$P(X = 1) = 2 \times 0.2 \times 0.8 = 0.32, \quad P(X = 2) = 0.2 \times 0.2 = 0.04.$$

Expected Value:

$$\mathbb{E}[X] = 0 \cdot 0.64 + 1 \cdot 0.32 + 2 \cdot 0.04 = 0.4.$$

Example: Missing TeslaCoil Toy

Variance: Using $\mathrm{V}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$:

$$\mathbb{E}[X^2] = 0^2 \cdot 0.64 + 1^2 \cdot 0.32 + 2^2 \cdot 0.04 = 0.00 + 0.32 + 0.16 = 0.48,$$

$$\mathrm{V}(X) = 0.48 - (0.4)^2 = 0.48 - 0.16 = 0.32.$$

Properties of the Mean and Variance

Properties

Let X be a random variable, and let $a, b \in \mathbb{R}$. Then:

1. $\mathbb{E}[\,aX + b\,] \;=\; a\,\mathbb{E}[X] \;+\; b.$
2. $\mathrm{Var}(X) \;=\; \mathbb{E}\big[(\,X - \mu_X\,)^2\big] \;=\; \mathbb{E}[X^2] \;-\; \mu_X^2.$
3. $\mathrm{Var}(aX + b) \;=\; a^2\,\mathrm{Var}(X).$
4. $\sqrt{\mathrm{Var}(aX + b)} \;=\; |a|\,\sqrt{\mathrm{Var}(X)}.$

Homework: Prove the above properties.

Important Warnings:

$$\mathbb{E}[X^2] \;\neq\; (\mathbb{E}[X])^2 \quad \text{and} \quad \mathbb{E}[\sqrt{X}] \;\neq\; \sqrt{\mathbb{E}[X]}\,.$$

Expectation is linear, but not *generally* compatible with non-linear operations.

Example: Slot-Machine Game

Scenario:

- You play a slot machine at a casino that charges €5 per play.
- With probability 0.02, you win €100; otherwise, you win nothing.
- These events are modeled by a Bernoulli random variable X, where

$$P(X = 1) = 0.02, \quad P(X = 0) = 0.98.$$

Profit Random Variable:

Let Y be your net profit from one play:

$$Y = 100X \; - \; 5.$$

What is the mean and variance of the profit?

Compute the Mean:

$$\mathbb{E}[X] = 1 \cdot 0.02 + 0 \cdot 0.98 = 0.02 \quad \Longrightarrow \quad \mathbb{E}[Y] = \mathbb{E}[\,100X - 5\,] = 100\,\mathbb{E}[X] - 5 = 100 \cdot 0.02 - 5 = -3\,\text{(Euros)}.$$

Note: You may ignore the units for now.

Example: Slot-Machine Game

Compute the Variance:

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 0.02 \cdot 1^2 - (0.02)^2 = 0.02 - 0.0004 = 0.0196.$

Then,

$$\mathrm{Var}(Y) = \mathrm{Var}(100X - 5) = 100^2\,\mathrm{Var}(X) = 10{,}000 \times 0.0196 = 196$$

$$\Rightarrow\ \sigma_Y = \sqrt{\mathrm{Var}(Y)} = 14.$$

Result:

$$\mathbb{E}[Y] = -3$$

,

$$\mathrm{Var}(Y) = 196,$$

$$\sigma_Y = 14.$$

Independence of Random Variables

Earlier we discussed *independent events*. The concept of *independence* extends naturally to random variables and plays a crucial role in both probability and statistics.

Definition: Independent Random Variables

Let X and Y be two random variables. They are said to be **independent** if, for any sets A and B,

$$P(\{X \in A\} \cap \{Y \in B\}) = P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B).$$

Independence of Random Variables

Definition: Independence of Multiple Random Variables

Let $X_1, X_2, \ldots, X_n$ be n random variables. They are said to be **jointly independent** if, for any sets $A_1, A_2, \ldots, A_n$, we have

$$P(X_1 \in A_1,\, X_2 \in A_2, \ldots, X_n \in A_n) \;=\; P(X_1 \in A_1)\; P(X_2 \in A_2)\; \cdots\; P(X_n \in A_n).$$

Independence of Random Variables: Examples

Example 1

Setup:

- Let X be the number of students in the classroom today.
- Let Y be the outcome of flipping a fair coin ($Y = 0$ for tails, $Y = 1$ for heads).

Question: Are X and Y independent?

Example 2

Setup:

- Let X be as before, the number of students in the classroom.
- Let Y be a random variable taking the value 1 if today is sunny, and 0 otherwise.

Question: Are X and Y independent?

Example 3

Setup:

- Let X again be the number of students in the classroom.
- Let Y be the number of *female* students in the classroom.

Question: Are X and Y independent?

Properties of Independent Random Variables

Proposition

Let $X_1, X_2, \ldots, X_n$ be random variables. Then

$$\mathbb{E}[\,X_1 + X_2 + \cdots + X_n\,] \;=\; \mathbb{E}[X_1] \;+\; \mathbb{E}[X_2] \;+\; \cdots + \;\mathbb{E}[X_n].$$

Furthermore, if $X_1, \ldots, X_n$ are *jointly independent*, then

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) \;=\; \mathrm{Var}(X_1) \;+\; \mathrm{Var}(X_2) \;+\; \cdots + \;\mathrm{Var}(X_n).$$

Importance:

- Summation of random variables arises often (e.g., sums of independent trial outcomes, sums of independent error terms).
- Many well-known distributions (e.g., Binomial, Poisson, Normal) can be viewed as sums of independent r.v.s.

Revision Questions — Q1

Question 1:
A fair coin is tossed 3 times. The player receives \$10 if all three turn up heads and pays \$3 if there is one or no heads. No gain or loss is incurred otherwise.
Let Y be the gain of the player.

What is $\mathbb{E}[Y]$?

Revision Questions — Q2

Question 2:
A discrete random variable X has a probability mass function of the form

$$f(x) = \begin{cases} c(8-x), & x = 0,1,2,3,4,5, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the constant c.
2. Find $P(X > 2)$.
3. Find the expected value $\mathbb{E}[X]$.

Revision Questions — Q3

Question 3:
A discrete random variable X has CDF

$$F(x) = \begin{cases} 0, & x < 1, \\ 0.30, & 1 \leq x < 2, \\ 0.50, & 2 \leq x < 4, \\ 0.65, & 4 \leq x < 7, \\ 0.80, & 7 \leq x < 20, \\ 1.0, & x \geq 20. \end{cases}$$

1. What is $P(1.5 \leq X \leq 5)$? *(Use the CDF only.)*
2. What is $P(X > 7)$? *(Use the CDF only.)*
3. Determine the probability mass function (pmf) of X.

Solution to Q1

Question 1 Recap: A fair coin is tossed 3 times. Gain:

- \$10 if 3 heads occur,
- $-$\$3 if 1 or 0 heads occur,
- \$0 otherwise (if exactly 2 heads).

Solution:

$$P(3 \text{ heads}) = \tfrac{1}{8}, \quad P(2 \text{ heads}) = \tfrac{3}{8}, \quad P(1 \text{ or } 0 \text{ heads}) = \tfrac{4}{8}.$$

Hence

$$\mathbb{E}[Y] = 10 \cdot \tfrac{1}{8} + 0 \cdot \tfrac{3}{8} - 3 \cdot \tfrac{4}{8} = \tfrac{10}{8} - \tfrac{12}{8} = -\tfrac{2}{8} = -0.25.$$

$$\boxed{\text{The expected gain is } -\$0.25.}$$

Solution to Q2

Question 2 Recap:

$$f(x) = \begin{cases} c(8-x), & x = 0,1,2,3,4,5, \\ 0, & \text{otherwise.} \end{cases}$$

1. **Find c:**

$$1 = \sum_{x=0}^{5} f(x) = \sum_{x=0}^{5} c(8-x) = c\sum_{x=0}^{5}(8-x) = c\,(8+7+6+5+4+3) = c \cdot 33.$$

Hence $c = \frac{1}{33}$.

2. **Find** $P(X > 2)$:

$$P(X > 2) = \sum_{x=3}^{5} c(8-x) = \frac{1}{33}\left[(8-3)+(8-4)+(8-5)\right] = \frac{1}{33}(5+4+3) = \frac{12}{33} = \frac{4}{11}.$$

3. **Find** $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \sum_{x=0}^{5} x\,f(x) = \frac{1}{33}\sum_{x=0}^{5} x(8-x).$$

Compute

$$\sum_{x=0}^{5} x(8-x) = 8\sum_{x=0}^{5} x - \sum_{x=0}^{5} x^2 = 8 \cdot 15 - 55 = 120 - 55 = 65.$$

Thus

$$\mathbb{E}[X] = \frac{1}{33} \cdot 65 = \frac{65}{33} \approx 1.97.$$

Solution to Q3

Question 3 Recap: The CDF $F(x)$ is:

$$F(x) = \begin{cases} 0, & x < 1, \\ 0.30, & 1 \leq x < 2, \\ 0.50, & 2 \leq x < 4, \\ 0.65, & 4 \leq x < 7, \\ 0.80, & 7 \leq x < 20, \\ 1.0, & x \geq 20. \end{cases}$$

1. $P\big(1.5 \leq X \leq 5\big) = F(5) - F(1.5^{-}) = 0.65 - 0.30 = 0.35.$
2. $P(X > 7) = 1 - F(7) = 1 - 0.80 = 0.20.$
3. To get the pmf, look at the jumps of F:

$$P(X = 1) = F(1) - \lim_{x \to 1^{-}} F(x) = 0.30 - 0.0 = 0.30,$$

$$P(X = 2) = 0.50 - 0.30 = 0.20,$$

$$P(X = 4) = 0.65 - 0.50 = 0.15,$$

$$P(X = 7) = 0.80 - 0.65 = 0.15,$$

$$P(X = 20) = 1.0 - 0.80 = 0.20.$$

All other values have probability 0.

Next Lecture