CSC2541: Introduction to Causality

Lecture 2 - Causal Models

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Outline

Lecture 1 Recap

- **What is Causal Inference:** It is the study of statistical methods to identify the effect of interventions.
- **Fundamental Problem Of Causal Inference:** We never observe both **potential outcomes** $(Y_1(u), Y_0(u))$ simultaneously.
- **Estimands of interest:**
  1. Individual Treatment Effect (ITE): What is the effect of an intervention on this individual: $\text{ITE}(u) := Y_1(u) - Y_0(u)$.
  2. Average Treatment Effect (ATE): What is the effect of an intervention on a population: $\text{ATE} := \mathbb{E}_{u \sim P(u)}\left[Y_1(u) - Y_0(u)\right]$.
  3. Conditional Average Treatement Effect: What is the effect of an intervention on a group summarized by covariates that can be conditioned on: $\mathbb{E}[Y_1|X] - \mathbb{E}[Y_0|X]$.

Lecture 1 Recap

Problem: The fundamental problem of causal inference makes it challenging to find these estimands without access to an oracle.

Strategy:

1. Write down the estimate of interest,
2. Make assumptions about the behavior of random variables in the problem,
3. Assumptions enable us to write down causal effects using quantities we can estimate from data.

We'll see this strategy arise time and again in this class.

Lecture 1 Recap

Assumptions we covered:

1. SUTVA: $Y_{0,1}(u_1) \perp\!\!\!\perp Y_{0,1}(u_k)\ \forall k \neq 1$
2. Consistency: Factual matches the observed outcome
3. Ignorability/Exchangeability: Potential outcomes are independent given treatment
4. Conditional Ignorability/Exchangeability: Potential outcomes are independent given treatment conditional on covariates [adjustment set]
5. Positivity/Overlap: The non-parameteric estimator for ATE requires us to have a positive probability of being assigned treatment or control for each configuration of patient

Positivity Unconfoundedness tradeoff: Including more variables means we're likely to have a valid adjustment set. Comes at the cost of satisfying overlap due to high-dimensionality

Modeling the joint distribution

$P(x_1, x_2, \ldots, x_n)$ needs $2^n - 1$ parameters to store for binary x_i

$$\textbf{chain rule:}\ P(x_1, x_2, \ldots, x_n) = \prod_i P(x_i | x_{i-1}, \ldots, x_1)$$

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pa_i: a minimal subset of $\{x_1, \ldots, x_{i-1}\}$ that $P(x_i | x_{i-1}, \ldots, x_1) = P(x_i | pa_i)$

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$$P(x_1, x_2, x_3, x_4) = P(x_1)P(x_2|x_1)P(x_3|x_2, x_1)P(x_4|x_3, x_2, x_1)$$

(Bayesian network factorization = compact representations)

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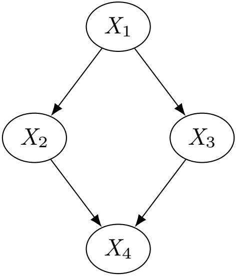
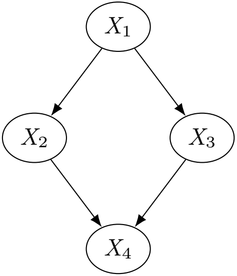
(Bayesian network factorization = compact representations)

Needs $2^0 + 2^1 + 2^1 + 2^2 = 9$ parameters $< 2^4 - 1 = 15$

Graphical models of probabilities

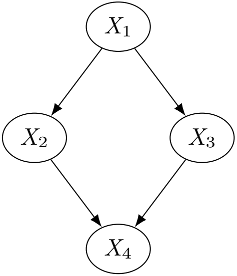


Graphical models of probabilities



- Directed Acyclic Graph (DAG) $\mathcal{G}$ (Bayesian network)

Graphical models of probabilities



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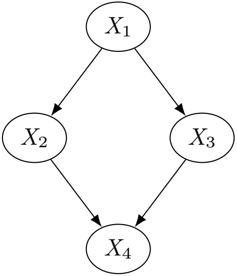

- Directed Acyclic Graph (DAG) $\mathcal{G}$ (Bayesian network)
- P is Markov compatible with $\mathcal{G}$ if our joint distribution admits a factorization compatible with the graph.
- $\mathcal{G}$ describes the conditional independence (CI) structure of distribution P

Conditional Independencies (CI)

What are they?

- Describe structure among the random variables: e.g. what edges do not exist.
- Provide insight into how information flows within the graph.

Why should we care about CI?

- We can use this to reduce the storage complexity of joint distribution.
- Identifying what we should *adjust for* to extract causal effects.

Conditional independence in DAGs

Question

What are the conditional independencies among random variables in a given graph $\mathcal{G}$?

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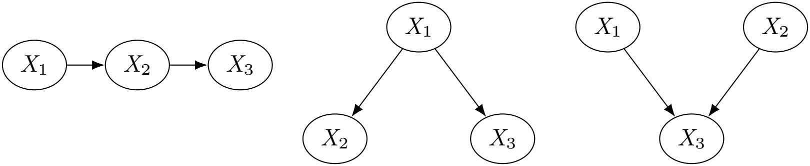
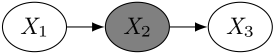
We first consider the building blocks of DAGs



Chain   Fork   Collider (v-structure)

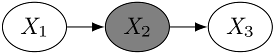
Conditional independence - Chain and v-structure



$$
\begin{aligned}
&P(x_1, x_3 | x_2) \\
&= \frac{P(x_1, x_2, x_3)}{P(x_2)} \\
&= \frac{P(x_1) P(x_2 | x_1) P(x_3 | x_2)}{P(x_2)} \\
&= P(x_1 | x_2) P(x_3 | x_2) \quad \text{(Bayes rule)}
\end{aligned}
$$

$$X_1 \perp\!\!\!\perp X_3 | X_2$$

Conditional independence - Chain and v-structure



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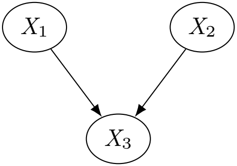
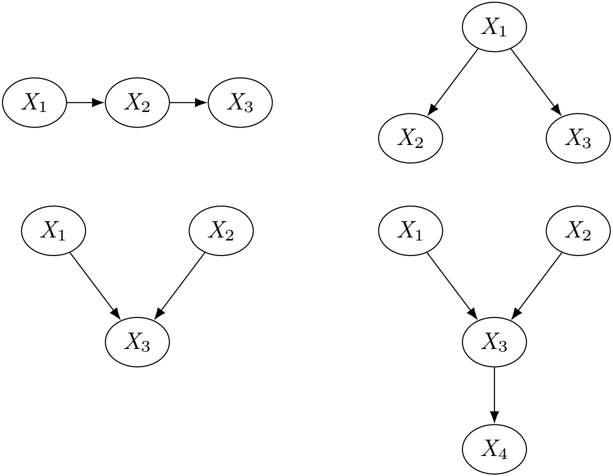


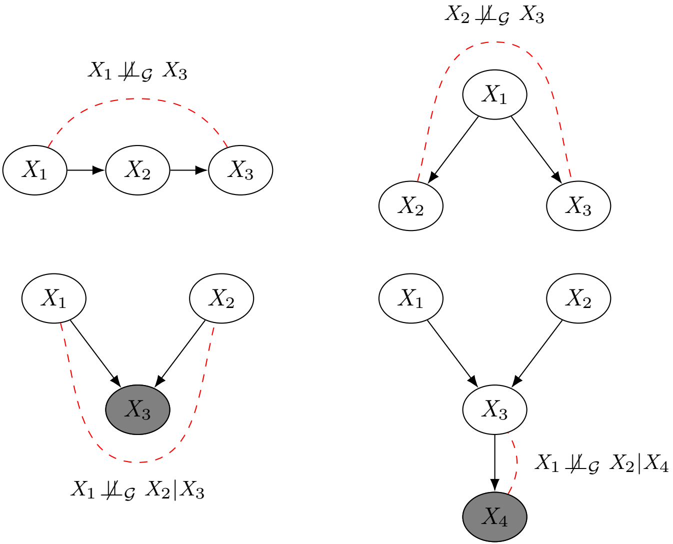
$$
\begin{aligned}
P(x_1, x_2) &= \sum_{x_3} P(x_1, x_2, x_3) \\
&= \sum_{x_3} P(x_1)P(x_2)P(x_3|x_1, x_2) \\
&= P(x_1)P(x_2) \sum_{x_3} P(x_3|x_1, x_2) \\
&= P(x_1)P(x_2)
\end{aligned}
$$

$$X_1 \perp\!\!\!\perp X_3$$

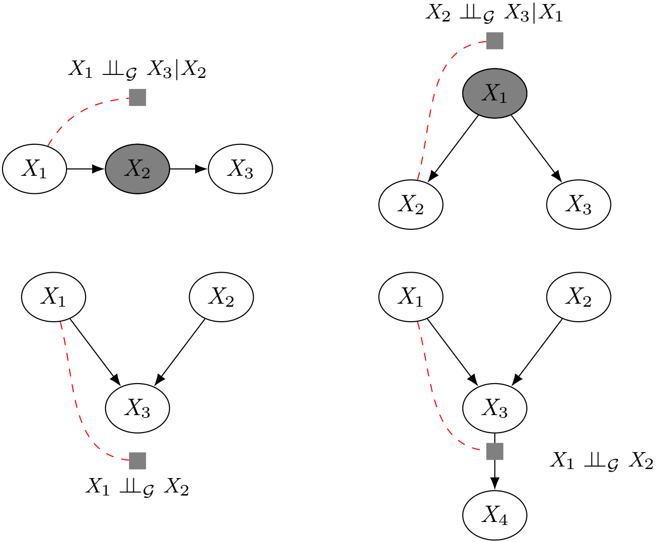
Conditional independence - Analyzing paths in a graph



Conditional independence - Unblocked paths



Conditional independence - Blocked paths



D-separation

Blocked path

Given a DAG $\mathcal{G}$, a (undirected) path between nodes X and Y is blocked by a set Z iff

- There is a **chain** $U \to W \to V$ or a **fork** $U \leftarrow W \to V$ on the path, where $W \in Z$, *or*

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D-separation ($X \perp\!\!\!\perp_{\mathcal{G}} Y | Z$)

Given a DAG $\mathcal{G}$, two sets of nodes X and Y are d-separated by a set Z iff all the paths between nodes of X and Y are blocked by Z

Global and local Markov properties

Idea

Given $\mathcal{G}$, we can use the Bayes Ball algorithm (Shachter, 1998) to find the conditional independencies in a graph.

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Global Markov property

A distribution P satisfies the *global Markov property* w.r.t. a DAG $\mathcal{G}$ if $X \perp\!\!\!\perp_{\mathcal{G}} Y|Z \implies X \perp\!\!\!\perp Y|Z$ for all disjoint sets of nodes X, Y, Z.

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Local Markov property

A distribution P satisfies the *local Markov property* w.r.t. a DAG $\mathcal{G}$ if each variable is independent of its nondescendants (in $\mathcal{G}$) conditioned on its parents.

Global and local Markov properties

Idea

Given $\mathcal{G}$, we can use the Bayes Ball algorithm (Shachter, 1998) to find the conditional independencies in a graph.

Theorem - Equivalence of Markov properties

Given a distribution P and a DAG $\mathcal{G}$, if P has a density function, then the followings are equivalent

1. P is Markov compatible w.r.t. $\mathcal{G}$
2. P satisfies the global Markov property w.r.t. $\mathcal{G}$
3. P satisfies the local Markov property w.r.t. $\mathcal{G}$

In Markov Random Fields, these properties are shown by the Hammersley-Clifford Theorem.

Observational equivalence

Question

Markov properties relate graphical separation to conditional independencies. Is it possible to have multiple graphs with the same CI structure?

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Markov equivalence of graphs

Let $\mathcal{M}(\mathcal{G}) := \{P; P \text{ is Markov compatible with } \mathcal{G}\}$. Then, $\mathcal{G}_1$ and $\mathcal{G}_2$ are called Markov equivalent if $\mathcal{M}(\mathcal{G}_1) = \mathcal{M}(\mathcal{G}_2)$.

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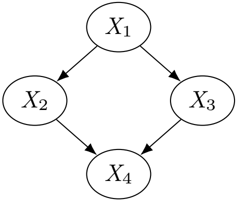

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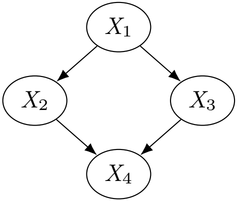

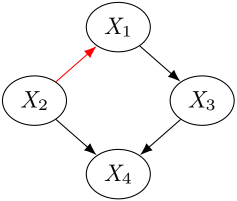

$$P(x_1)P(x_2|x_1)P(x_3|x_1)P(x_4|x_2,x_3) \quad = \quad P(x_2)P(x_1|x_2)P(x_3|x_1)P(x_4|x_2,x_3)$$

All these DAGs are observationally valid - They capture the same CI structure

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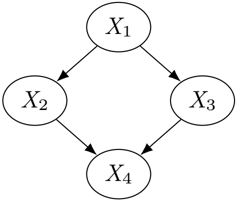

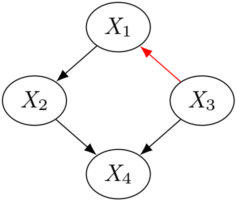

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Questions?

Question

Any questions on Bayesian Networks?

Observation and intervention in the kidney stone data - The *do*-operator

Stone size S, treatment T, and recovery rate R. Each item shows $P(R=1|S=s, T=t)$, i.e., # recovered / #total with $T=t, S=s$

	Small stone ($S=s$)	Large stone ($S=l$)
$T=a$ Open surgery	81/87	192/263
$T=b$ Percutaneous nephrolithotomy	234/270	55/80

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Observing a patient with open surgery treatment, what can we infer about their stone size?

$$P(S=\text{ small}|T=a) = \frac{P(S=\text{ small}, T=a)}{P(T=a)} = \frac{87/700}{(87+263)/700} \approx 0.25$$

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Question

Now, assume we *intervene* on all patients with open surgery treatment. What can we infer about their stone size?

Intuitively, we expect that changes to treatment assignment has no effect on the stone size

$$P(S = \text{ small}|\underbrace{do(T = a)}_{\text{intervention}})=P(S = \text{ small}) = \frac{87 + 270}{700} = 0.51$$

Can we use potential outcomes to compute interventional distributions?

- $P(S|T=a)$: We *see* (observe) $T=a$ and infer the stone size
- $P(S|do(T=a))$: We *do* (intervene) $T=a$ and infer the stone size
- Generally, $P(Y|do(X=x)) \neq P(Y|X=x)$. In the kidney stone data:

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What about $P(R=1|do(T=a))$ and $P(R=1|do(T=b))$?

$P(R=1|do(T=a))$
$= P(R_a = 1)$ (Potential outcome)
$= \mathbb{E}[R_a]$

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$$\begin{aligned}
&P(R=1|do(T=a)) \\
&= P(R_a=1) \quad \text{(Potential outcome)} \\
&= \mathbb{E}[R_a] \\
&= \mathbb{E}_S[\mathbb{E}[R|S,T=a]] \quad \text{(G-formula)} \\
&= P(S=s)P(R=1|S=s,T=a) + P(S=l)P(R=1|S=l,T=a) \\
&= \frac{87+270}{700}\cdot\frac{81}{87} + \frac{263+80}{700}\cdot\frac{192}{263} \approx 0.832
\end{aligned}$$

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$$\begin{aligned}
&P(R=1|do(T=b))\\
&= P(R_b=1)\\
&= \mathbb{E}[R_b] \quad \text{(Potential outcome)}\\
&= \mathbb{E}_S[\mathbb{E}[R|S,T=b]] \quad \text{(G-formula)}\\
&= P(S=s)P(R=1|S=s,T=b) + P(S=l)P(R=1|S=l,T=b)\\
&= \frac{87+270}{700}\cdot\frac{234}{270} + \frac{263+80}{700}\cdot\frac{55}{80} \approx 0.779
\end{aligned}$$

Can we use potential outcomes to compute interventional distributions?

- Using the potential outcome framework (G-formula), we saw that treatment a is, on average, a better choice

$$P(R = 1|do(T = a)) > P(R = 1|do(T = b))$$

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- Can we use the same G-formula for the following data?

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	Normal BP	High/low BP
$T = a$	81/87	192/263
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- Since the data is the same, using the same formula will choose treatment a again. But, we saw in lecture 1 that treatment b is better in this case. Why?

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Bayesian networks as data generating processes (DGPs)

Bayesian networks model the conditional independence structure of distribution

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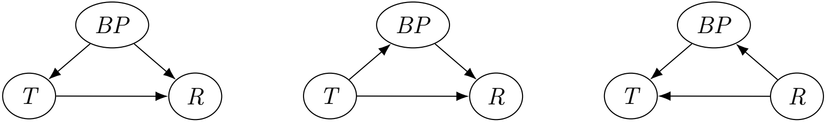

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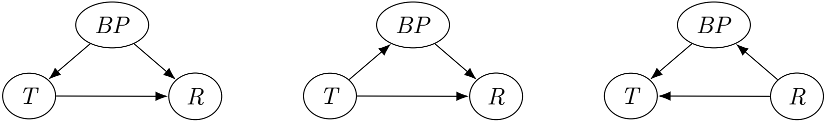


Order: (BP, T, R) (T, BP, R) (R, BP, T)

All three graphs are plausible based on data (*observationally equivalent*)

We are interested in the "real" graph, i.e., the one that describes the real/physical data generating process (Causal order)

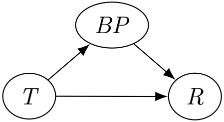
Questions?

Question

Any questions on Observations v.s. Interventions?

Bayesian networks as data generating processes (DGP)

Assume the nature generates the data with ordering (T, BP, R)



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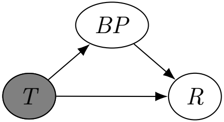

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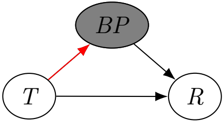

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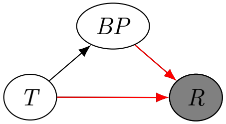

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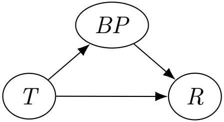

For each node X_i in the data generating Bayesian network, $P(x_i|pa_i)$ is called the *mechanism* that generates X_i

How to characterize these "causal" mechanisms?

Characterizing causal mechanisms

Suppose we know the joint distribution $P(A, T)$ of altitude of cities A and their average temperature T. Which one is the cause?

Characterizing causal mechanisms

Case 1: Suppose the causal DGP is $T \rightarrow A$ with mechanisms $P(T)$ and $P(A|T)$

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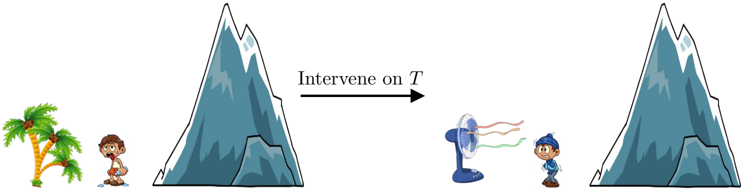

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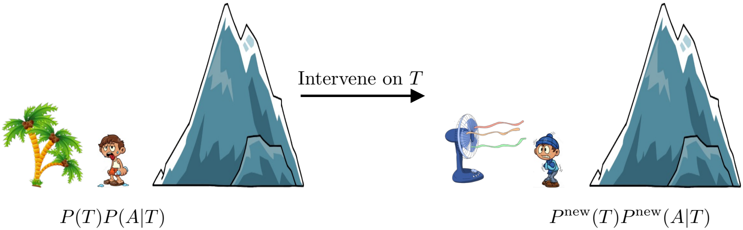

Intervention on T does **not** affect the value of A but **both** mechanisms $P(T)$ and $P(A|T)$ change.

Characterizing causal mechanisms

Case 2: Suppose the causal DGP is $A \to T$ with mechanisms $P(A)$ and $P(T|A)$

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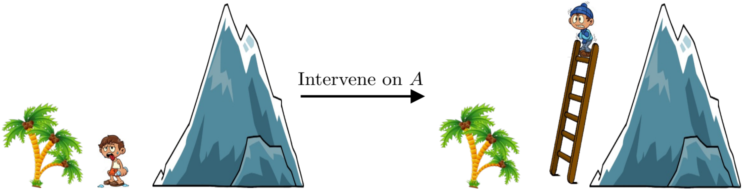

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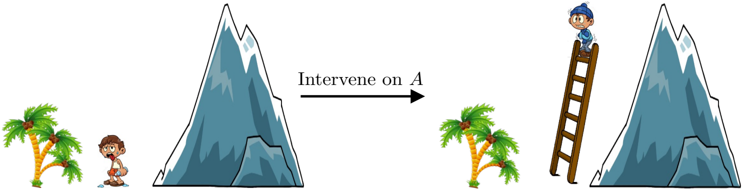

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Intervention on A **does** affect the value of T. Also, only **one** mechanism changes ($P(A)$).

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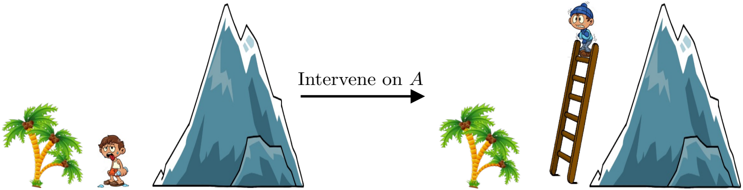

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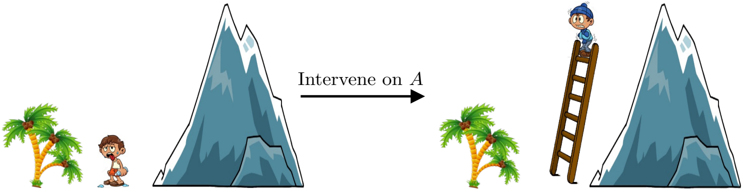

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Intervention on A **does** affect the value of T. Also, only **one** mechanism changes ($P(A)$).

A is the cause since intervention in A changes T. Moreover, interventions can only change **one** mechanism in the causal DGP $A \rightarrow T$.

Modularity assumption

Modularity assumption (Independent mechanisms / Autonomy)

A (data generating) Bayesian network has modular mechanisms if intervention on a node X_i **only** changes the mechanism $P(x_i|pa_i)$

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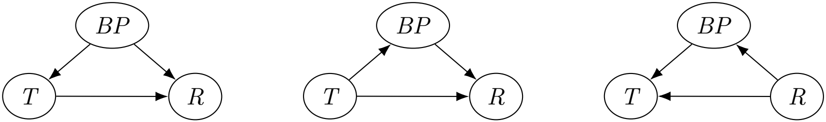

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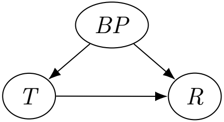

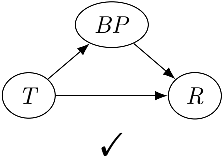

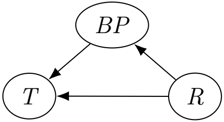

Formal definition of causal Bayesian networks

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How to calculate interventional distributions? - Truncated factorization

Bayesian network factorization

$$P(x_1, x_2, \cdots, x_n) = \prod_i P(x_i|pa_i)$$

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&= \prod_{x_i \notin Z} P(x_i | pa_i) && \text{Modularity (property 3)}
\end{aligned}
$$

All of these assumes x consistent with z, otherwise it will be zero

$$\text{E.g.,} \quad P(X_1 = 1, X_2 = 2 | do(X_1 = 0)) = 0$$

Which treatment should we choose?

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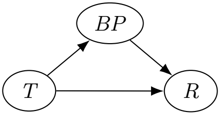

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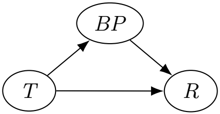

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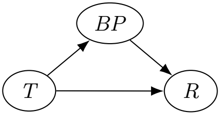

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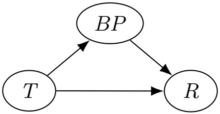

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$$P(R=1|do(T=\color{red}{a})) = \sum_{t,x} P(R=1, T=t, BP=x|do(T=\color{red}{a})) \quad \text{marginalization}$$

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truncated factorization

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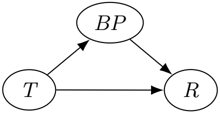

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$$= P(R=1|T=a) = \frac{81+192}{87+263} = 0.78$$

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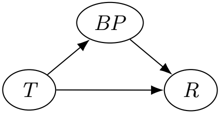

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$$\begin{aligned} P(R = 1|do(T = b)) &= \sum_{t,x} P(R = 1, T = t, BP = x|do(T = b)) \quad \text{marginalization} \\ &= \sum_{x} P(R = 1, T = b, BP = x|do(T = b)) \\ &\qquad T = a \text{ is inconsistent} \\ &= \sum_{x} P(BP = x|T = b)P(R = 1|BP = x, T = b) \\ &\qquad \text{truncated factorization} \\ &= P(R = 1|T = b) = \frac{234 + 55}{270 + 80} \approx 0.826 \end{aligned}$$

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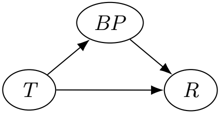

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Here, treatment b is better (and association *is* causation)

Unobserved variables can change everything!

What if, in the previous dataset, the stone size had also influenced both T and R but we didn't observe it?

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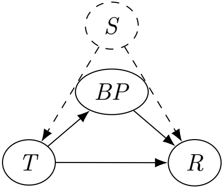

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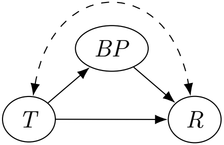

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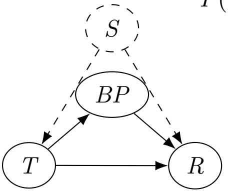

$P(R = 1|do(T = a))$

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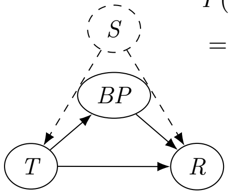

$$P(R=1|do(T=a))$$
$$= \sum_{x,y} P(S=x)P(BP=y|T=a)P(R=1|T=a,S=x,BP=y)$$

(truncated factorization - Try this as HW)

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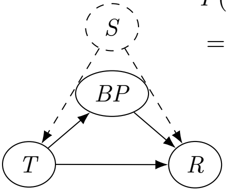

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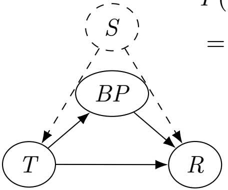

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What to do in the presence of unobserved variable? $\rightarrow$ **Lecture 3**

When does a node has a causal effect on another one?

Causal effect

A variable (set) Z has causal effect on a (disjoint) variable (set) X if at least for two z, z'

$$P(X|do(Z=z)) \neq P(X|do(Z=z'))$$

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Proof by induction: (x_i is a root node)

$$\begin{aligned} P(x_i|do(Z=z)) &= P(x_i|\emptyset, do(Z=z)) \\ &= P(x_i) && \text{(Modularity)} \\ &= P(x_i|do(Z=z')) \end{aligned}$$

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$$P(x_i|do(Z=z)) = \sum_{pa_i} P(x_i, pa_i|do(Z=z))$$
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&= P(x_i|do(Z=z'))
\end{aligned}$$

Flow of causation v.s. statistical association

A node can only influence its descendents in a causal graph $\mathcal{G}$,

1. If X is a child of Z in $\mathcal{G}$: *direct* cause
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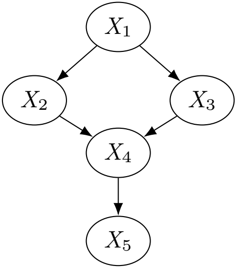

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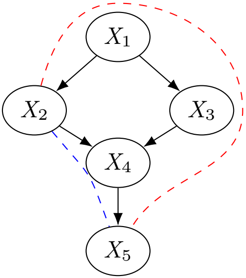

Remember d-separation

- Unblocked paths between X_2 and X_5 are (potential) dependencies between X_2 and X_5

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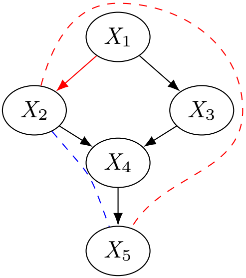

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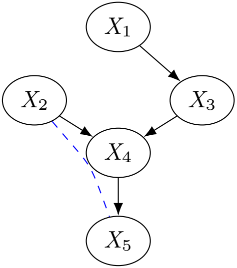

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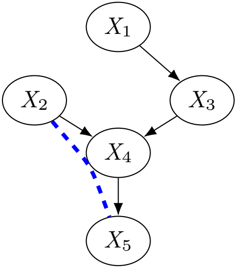

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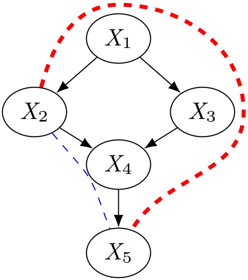

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- Every unblocked path from X_2 to X_5 in $\mathcal{G}_{\overline{X_2}}$ is a *causal* path (directed paths)
- Other unblocked paths in the original graph are *backdoor* paths

Questions?

Question

Any questions on Causal Bayesian Networks?

Modeling data generating processes with equations

- We assumed the data is being generated using (independent) mechanisms $P(x_i|pa_i)$.

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$$R \sim P(R|T = t, BP = x)$$

$$R = f_{x,t}(U) \text{ where } U \sim \texttt{Unif}[0, 1] \text{ and } f_{x,t}(u) = P^{-1}(u|T = t, BP = x)$$

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- Any causal mechanism $P(x_i|pa_i)$ can be written as a deterministic function f_i of its direct causes pa_i and some exogenous noise U_i
- We call f_i the law (process) that generates X_i

Structural causal models - A mathematical framework to define causal effects

Structural causal model (SCM)

A structural causal model is a tuple $\mathcal{M} = (X, U, F, P_U)$ of

1. Endogenous set of variables X, (observed variables)
2. Exogenous set of noises U, (unobserved noise)
3. Set of functions F, (data generating rules/processes)
4. *Product* distribution P_U over variables in U, i.e., $P_U(u_1, \ldots, u_d) = \prod_{i=1}^{d} P_U(u_i)$ (noises are independent)

such that for any variable $X_i \in X$, we have an assignment

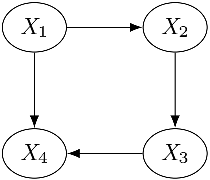
$$X_i := f_i(PA_i, U_i) \text{ But not } f_i(PA_i, U_i) := X_i$$

for some $PA_i \subseteq X \backslash \{X_i\}$, $U_i \in U$, and $f_i \in F$. We call the elements of PA_i *direct causes* of X_i.

SCMs induce causal graphs

- For each SCM $\mathcal{M}$, we can construct a (unique) graph $\mathcal{G}$ by drawing edges from each direct cause in PA_i to X_i

$$X_1 := f_1(U_1)$$
$$X_2 := f_2(X_1, U_2)$$
$$X_3 := f_3(X_2, U_3)$$
$$X_4 := f_4(X_1, X_3, U_4)$$
$U_1, \ldots, U_4$ are jointly independent



causal graph

- We assume $\mathcal{G}$ is acyclic (no feedback loop in assignments)

$$X_1 := f_1(X_2, U_1)$$
$$X_2 := f_2(X_1, U_2)$$

Generating "observational" distribution with SCMs

How to generate data from an SCM?

1. Consider a topological order of endogenous variables $X_1, \ldots, X_d$ (since the assignments are acyclic)
2. Sample from exogenous noises $u_1, \ldots, u_d \sim P_U$
3. Generate samples $x_1, \ldots, x_d$ by assignments $x_i = f_i(pa_i, u_i)$

Each X_i can be written as a unique function of noises $(U_k)_{k \in An_i}$ that belong to ancestors of X_i, i.e.,

$$X_i = g_i((U_k)_{k \in An_i})$$

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Observational distribution

An SCM $\mathcal{M}$ induces a unique distribution over endogenous variables $X_1, \dots, X_d$, which we call the observational distribution of $\mathcal{M}$ and denote it by $P_X^{\mathcal{M}}$, or simply P.

Generating "interventional" distribution with SCMs

Remember the independent mechanisms assumption: intervention on a variable X_i can only change the mechanism $P(x_i|pa_i)$

We can use SCMs to formally define interventions

Interventional distribution

Consider an SCM $\mathcal{M}$. An intervention on a variable X_i (or multiple variables) is replacing the assignment $X_i := f_i(PA_i, U_i)$ with a new assignment

$$X_i := \hat{f}(\overline{PA}_i, \hat{U}_i)$$

We call the induced distribution of the new SCM an interventional distribution and denote it by $P_X^{\mathcal{M}}\left(\cdot|do\left(X_i := \hat{f}(\overline{PA}_i, \hat{U}_i)\right)\right)$.

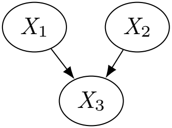
If $\hat{f}(\overline{PA}_i, \hat{U}_i)$ is a constant value c, we simply write it as $P(\cdot|do\,(X_i = c))$.

Types of intervention - soft intervention

- Soft interventions: $\overline{PA}_i = PA_i$, i.e., only the mechanism changes but direct causes remain active

$$X_1 := \mathcal{N}(0,1)$$
$$X_2 := \mathcal{N}(0,1)$$
$$X_3 := X_1 + X_2 + \mathcal{N}(0,1)$$

before intervention



Types of intervention - soft intervention

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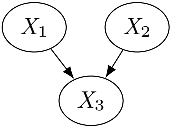
after intervention on X_3

$$X_1 := \mathcal{N}(0,1)$$
$$X_2 := \mathcal{N}(0,1)$$
$$X_3 := X_1^2 + X_2^2 + \texttt{Unif}(0,1)$$



Types of intervention - hard intervention

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$$
\begin{aligned}
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$$

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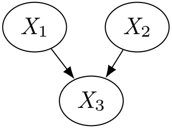

Types of intervention - hard intervention

- Hard intervention: $\overline{PA}_i \neq PA_i$

$$
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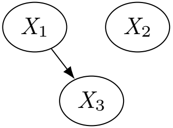
X_1 &:= \mathcal{N}(0,1)\\
X_2 &:= \mathcal{N}(0,1)\\
X_3 &:= 2X_1 + \mathcal{N}(0,1)
\end{aligned}
$$

after intervention on X_3



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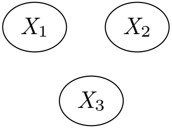

$$X_1 := \mathcal{N}(0,1)$$
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- Atomic intervention is a type of hard intervention, where $X_i := c$ for some constant value $c \rightarrow$ mutilated graph $\mathcal{G}_{\overline{X_3}}$

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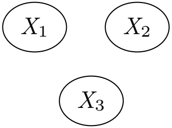

- Atomic intervention is a type of hard intervention, where $X_i := c$ for some constant value $c \rightarrow$ mutilated graph $\mathcal{G}_{\overline{X_3}}$
- We previously defined causal Bayesian networks only using atomic interventions (see slide 25). SCMs give us more flexibility in defining interventions
- The causal graph corresponding to an SCM $\mathcal{M}$ is a causal Bayesian network compatible with $P^{\mathcal{M}}$

Example - good predictors are not always causes

Consider the following SCM:

$$X_1 := U_{X_1}$$
$$Y := X_1 + U_Y$$
$$X_2 := Y + U_{X_2}$$

$$U_{X_1}, U_Y \sim \mathcal{N}(0, 1)$$
$$U_{X_2} \sim \mathcal{N}(0, \mathbf{0.1})$$

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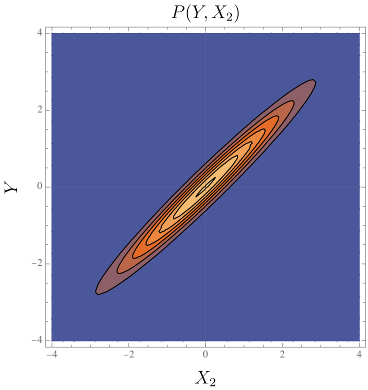

Observational distribution for X_2 and Y

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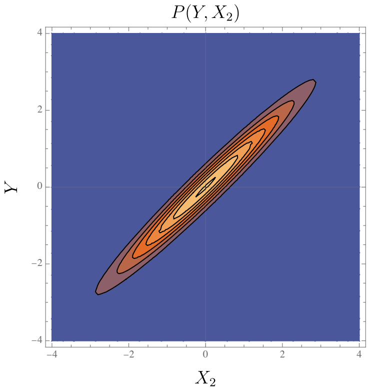
We train two linear models to predict Y:

1. $\hat{Y}_1 = \theta_1 X_1 : \quad \mathbb{E}\left[\|\hat{Y}_1 - Y\|_2^2\right] \approx 1$
2. $\hat{Y}_2 = \theta_2 X_2 : \quad \mathbb{E}\left[\|\hat{Y}_2 - Y\|_2^2\right] \approx \mathbf{0.1}$



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Example - good predictors are not always causes

Now, we intervene on X_2:

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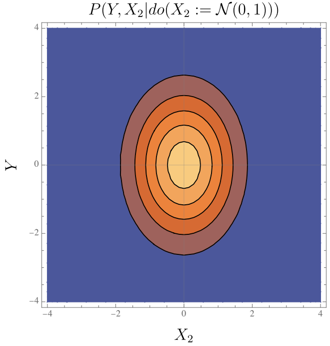
X_2 is not a good predictor for Y anymore (independent of Y)



Interventional distribution for X_2 and Y

Questions?

Question

Any questions on Structural Causal Models?