CSC2541: Introduction to Causality

Lecture 5 - Estimation (cont.) and Instrumental Variables

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Recap - Lecture 4

- Identification
  - Backdoor criteria: Identical to adjustment via the G-formula,
  - Frontdoor criteria: Using mediators to identify causal effect on outcomes.
- Do-Calculus: Three rules to identify causal effects:
  1. Insertion or deletion of observations : Generalization of d-separation,
  2. Interchanging actions with observations : Generalization of the backdoor criteria,
  3. Insertion or deletion of actions
- Parametric Estimation:
  - Conditional outcome models
  - Grouped conditional outcome models
  - TAR-Net

Matching

1. For each observation in the treatment group, find "statistical twins" in the control group with similar covariates X (and vice versa), where X is a valid adjustment set
2. Use the Y values of the matched observations as the counterfactual outcomes for one at hand
3. Estimate average treatment effect as the difference between observed and imputed counterfactual values

Slide credit to David Sontag

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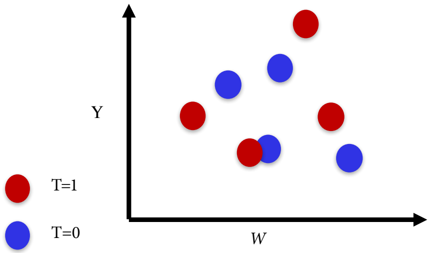

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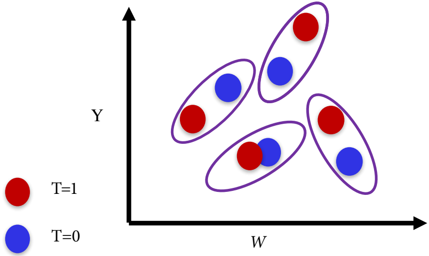

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Matching - Formal definition

Let the data $\mathcal{D} = \{(T^i, X^i, Y^i)\}_{i=1}^N$. To estimate the counterfactual Y_0^i for a sample i in the treatment group, we use (similar) samples from the control group ($T = 0$):

$$\hat{Y}_0^i = \sum_{j \text{ s.t. } T^j=0} w_{ij} Y^j$$

Similarly, to estimate the counterfactual Y_1^i for a sample i in the control group, we use samples from the treatment group:

$$\hat{Y}_1^i = \sum_{j \text{ s.t. } T^j=1} w_{ij} Y^j$$

An estimation of ATE will be

$$\widehat{\text{ATE}} = \frac{1}{N} \sum_i Y_1^i - Y_0^i = \frac{1}{N} \left[\sum_{i;T^i=1} \left(Y^i - \hat{Y}_0^i \right) + \sum_{i;T^i=0} \left(\hat{Y}_1^i - Y^i \right) \right]$$

Different matching algorithms use different definitions of w_{ij}

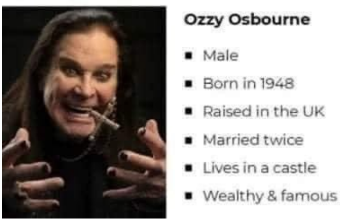
Types of matching

- **Exact matching**: $w_{ij} = \begin{cases} \frac{1}{k_i} & \text{if } X^i = X^j \\ 0 & o.w. \end{cases}$ with k_i as the number of samples j with $X^i = X^j$
  - Problem: For high-dimensional X, it will be less likely to find an exact match
- **Multivariate distance matching (MDM)**: Use (Euclidean) distance metric to find "close" observations as potential matches
  - We can use KNN algorithm to find the k closes observations in the control (treatment) group for each treated (controlled) sample, i.e.,

$$w_{ij} = \begin{cases} \frac{1}{k} & \text{if } X^j \in \text{KNN}(X^i) \\ 0 & o.w. \end{cases}$$

Matching - Pros and Cons

+ Interpretable, especially in small samples
+ Non-parametric
- KNN-matching can be biased since $X^i \approx X^j \implies Y_0^i \approx Y_0^j, Y_1^i \approx Y_1^j$ (See Abadie and Imbens, 2011 for bias-correction for matching estimators)
- Curse of dimensionality - it gets harder to find good matches as dimension grows



Source: `https://mobile.twitter.com/HallaMartin/status/1569311697717927937`

Propensity scores

- Matching can suffer from curse of dimensionality of X
- Let's look at probability of treatment assignment given X

$$e(X) := P(T = 1|X)$$

Propensity scores

- Matching can suffer from curse of dimensionality of X
- Let's look at probability of treatment assignment given X

$$e(X) := P(T = 1|X)$$

- $e(X)$ summarizes high-dimensional variables X into one dimension!

Theorem - Propensity Score

Assume X satisfies the backdoor criterion (conditional ignorability) w.r.t. T, Y. Given positivity, $e(X)$ will also satisfy conditional ignorability, i.e.,

$$Y_0, Y_1 \perp\!\!\!\perp T | e(X)$$

- Helpful for matching!

Propensity score theorem - Intuition

$$Y_0, Y_1 \perp\!\!\!\perp T | X \implies Y_0, Y_1 \perp\!\!\!\perp T | e(X)$$

$$\begin{tikzcd} X \arrow[d] \arrow[dr] & \\ T \arrow[r] & Y \end{tikzcd}$$

For the formal proof, see Rosenbaum and Rubin, 1983.

Slide credit to Brady Neal

Propensity score theorem - Intuition

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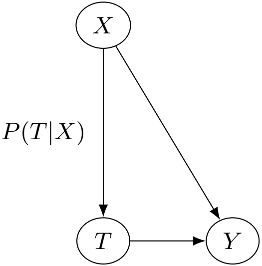

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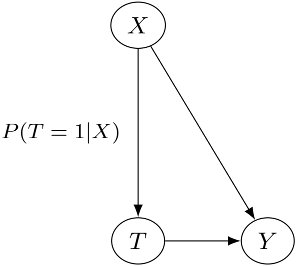

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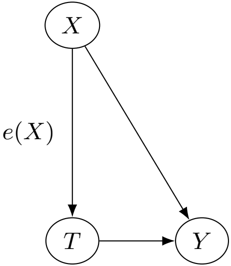

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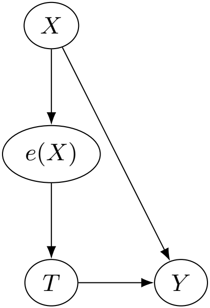

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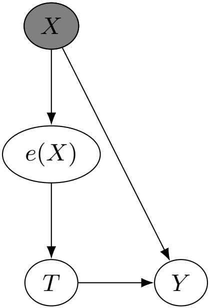

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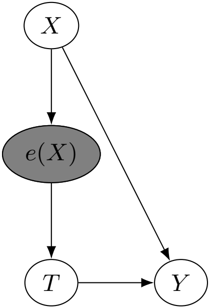

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Propensity score matching

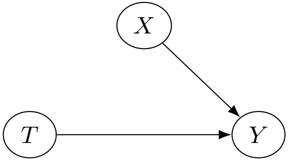
- Instead of computing multivariate distances, we can match the one-dimensional propensity score:
- Step 1: Estimate $e(X)$ using a **parametric** method
- Step 2: Apply a matching algorithm (KNN) with distance $|e(X_i) - e(X_j)|$

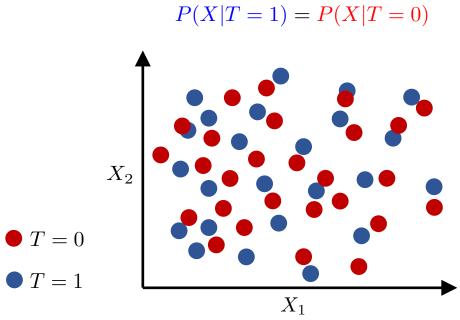
Propensity score matching

- Instead of computing multivariate distances, we can match the one-dimensional propensity score:
- Step 1: Estimate $e(X)$ using a **parametric** method
- Step 2: Apply a matching algorithm (KNN) with distance $|e(X_i) - e(X_j)|$
- This is not a magic, we still need to estimate $P(T = 1|X)$!
- A perfect predictor of T is not always good - we can include more variables as X to get better treatment assignment predictions
  - Can increase variance,
  - See "Why Propensity Scores Should Not Be Used for Matching" by King and Nielsen, 2019.

Inverse probability weighting (IPW)

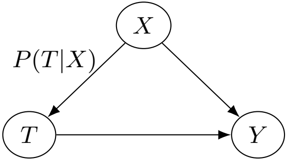
- Causal estimation in RCTs is easier (control and treatment groups are similar)



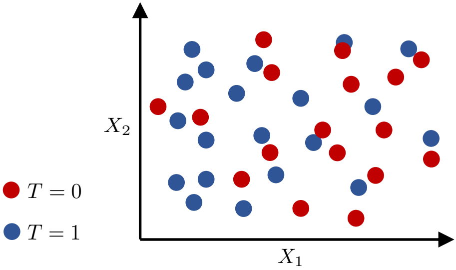


Inverse probability weighting (IPW)

- In observational studies, however, the treatment and control groups are not comparable.

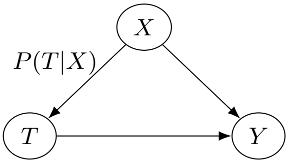


$P(X|T=1) \neq P(X|T=0)$



Inverse probability weighting (IPW)

- In observational studies, however, the treatment and control groups are not comparable. Can we make a pseudo-RCT by re-weighting samples?



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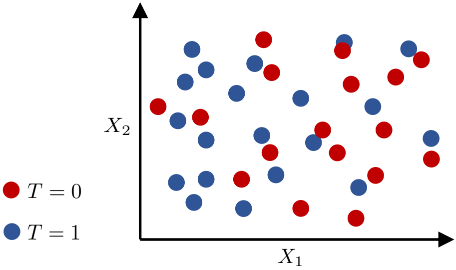

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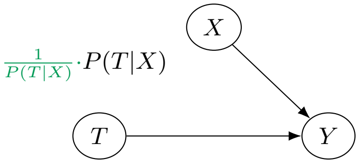
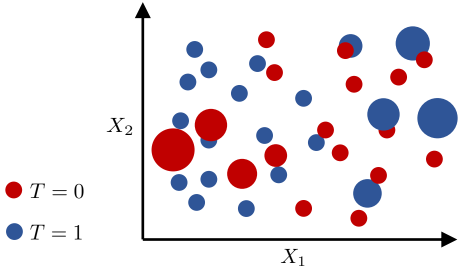


$$w_1(X) \cdot P(X|T=1) \approx w_0(X) \cdot P(X|T=0)$$



Samples re-weighted by the inverse propensity score of the treatment they received

Inverse probability weighting (IPW) - Formal

$$\mathbb{E}[Y_t] = \mathbb{E}_X\left[\mathbb{E}[Y|X, T=t]\right] \qquad \text{(conditional ignorability)}$$

Inverse probability weighting (IPW) - Formal

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\begin{aligned}
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&= \sum_x \sum_y y P(y|x,t) P(x) \frac{P(t|x)}{P(t|x)} \\
&= \sum_{x,y} \frac{1}{P(t|x)} y P(x,y,t) && (P(y|x,t)P(x)P(t|x) = P(x,y,t))
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&= \sum_{x,y,t'} f(x,y,t') P(x,y,t') \\
&= \mathbb{E}\left[f(X,Y,T)\right] = \mathbb{E}\left[\frac{\mathbb{I}(T=t)Y}{P(t|X)}\right]
\end{aligned}
$$

Inverse probability weighting (IPW) - Formal

- Hence,

$$\begin{aligned}\text{ATE} = \mathbb{E}\left[Y_1 - Y_0\right] &= \mathbb{E}\left[\frac{\mathbb{I}(T=1)Y}{P(T=1|X)}\right] - \mathbb{E}\left[\frac{\mathbb{I}(T=0)Y}{P(T=0|X)}\right] \\ &= \mathbb{E}\left[\frac{\mathbb{I}(T=1)Y}{e(X)}\right] - \mathbb{E}\left[\frac{\mathbb{I}(T=0)Y}{1-e(X)}\right]\end{aligned}$$

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- For a given dataset $\mathcal{D} = \{(x^i, t^i, y^i)\}_{i=1}^N$, an estimate of ATE will be

$$\widehat{\text{ATE}} = \frac{1}{N_1}\sum_{i;t^i=1}\frac{y^i}{\hat{e}(x^i)} - \frac{1}{N_0}\sum_{i;t^i=0}\frac{y^i}{1-\hat{e}(x^i)}$$

for $N_1 = |\{i; t^i = 1\}|, N_0 = N - N_1$.

Inverse probability weighting (IPW) - Formal

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  for $N_1 = |\{i; t^i = 1\}|, N_0 = N - N_1$.
- Still we need to estimate $e(X)$. If positivity is violated, propensity scores become non-informative and miscalibrated
- Small propensity scores can create large variance/errors

Questions?

Question

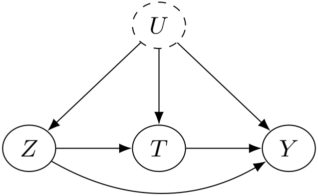
Any questions on weighting based estimators?

Instrumental Variables

- Unobserved confounding (variables that we know exist, but do not observe) is a real concern when attempting to identify causal effects in practical scenarios,
- In such scenarios, we might be able to rely on the use of *instruments* to help us,
- Instruments can be thought of as random variables in a causal Bayesian network that:
  - Are independent of unobserved confounding and,
  - Are related to the outcome *only through* the treatment,

Motivation via causal Bayesian networks

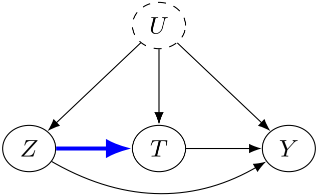
- Consider the following complete graph with unobserved U and observed Z (which as we'll see is the instrument variable),
- We care about estimating the causal effect of T on Y,
- The causal effect of T on Y is non-identifiable (why?). We'll make *assumptions* to make causal inference feasible:



Presentation credit to Brady Neal

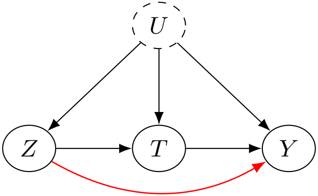
Assumption 1: Relevance

- First, we'll need to assume that there exists an edge from Z to T,
- This is saying the instrument has an effect on treatment.



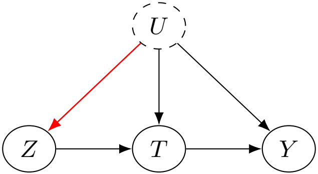
Assumption 2: Exclusion Restriction

- Next, we'll need to assume that there is no edge from Z to Y,
- This is equivalent to saying that the only effect that Z can have on Y is through T.



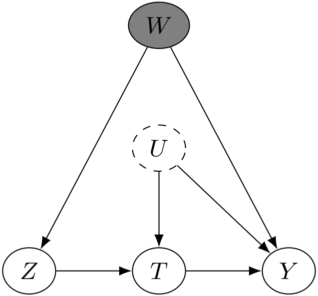
Assumption 3: Instrumental Unconfoundedness

- Finally, we'll need to assume that there is no edge from U to Z,
- This is equivalent to saying that the instrument is independent of the confounder.



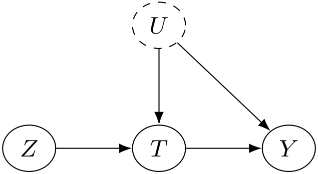
Assumption 3: (Conditional) Instrumental Unconfoundedness

- If there exists a W that couples Z and Y, we can still obtain a valid instrument by conditioning on W.



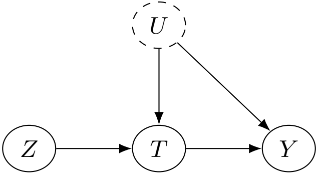
Instrumental variables - Intuition

- How to estimate ATE (or CATE) with unobserved U?
- *Intuition:*
  - Changes in the instrument Z lead to changes in the treatment T, and consequently the outcome Y,
  - If we modify Z, then T, Y will co-vary based on the relationship induced by U,
  - If we can modify Z in different ways, we can see how T, Y co-vary and subtract off the influence of U.



Intuition - Partial derivatives and differences in conditional expectations

- Note that $\frac{\partial y}{\partial z}$ represents the effect on the outcome by perturbation of the instrument,
- In the (implicit) SCM for the figure below, what we really want is to assess $\frac{\partial y}{\partial t}$,
- We have $\frac{\partial y}{\partial t} = \frac{\partial y}{\partial z}\frac{\partial z}{\partial t} = \frac{\frac{\partial y}{\partial z}}{\frac{\partial t}{\partial z}}$,
- In the binary setting, $\frac{\partial y}{\partial z}$ can be seen as $\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0]$, and $\frac{\partial y}{\partial t}$ as $\mathbb{E}[Y|T=1] - \mathbb{E}[Y|T=0]$.



Binary Linear Model

Assume $Y = \delta T + \alpha U + \epsilon$:

$$
\begin{aligned}
&\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0] \\
&= \mathbb{E}[\delta T + \alpha U + \epsilon|Z=1] - \mathbb{E}[\delta T + \alpha U + \epsilon|Z=0] \\
&= \mathbb{E}[\delta T + \alpha U|Z=1] - \mathbb{E}[\delta T + \alpha U|Z=0] + \cancel{\mathbb{E}[\epsilon|Z=1] - \mathbb{E}[\epsilon|Z=0]}
\end{aligned}
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&= \delta(\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0]) + \alpha \underbrace{(\mathbb{E}[U|Z=1] - \mathbb{E}[U|Z=0])}_{U \perp\!\!\!\perp Z}
\end{aligned}
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&= \mathbb{E}[\delta T + \alpha U + \epsilon|Z=1] - \mathbb{E}[\delta T + \alpha U + \epsilon|Z=0] \\
&= \mathbb{E}[\delta T + \alpha U|Z=1] - \mathbb{E}[\delta T + \alpha U|Z=0] + \cancel{\mathbb{E}[\epsilon|Z=1] - \mathbb{E}[\epsilon|Z=0]} \\
&= \delta(\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0]) + \alpha\underbrace{(\mathbb{E}[U|Z=1] - \mathbb{E}[U|Z=0])}_{U \perp\!\!\!\perp Z} \\
&= \delta(\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0]) + \alpha\cancel{(\mathbb{E}[U] - \mathbb{E}[U])} \\
&= \delta(\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0])
\end{aligned}$$

Binary Linear Model

Assume $Y = \delta T + \alpha U + \epsilon$:

$$\begin{aligned}
&\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0] \\
&= \mathbb{E}[\delta T + \alpha U + \epsilon|Z=1] - \mathbb{E}[\delta T + \alpha U + \epsilon|Z=0] \\
&= \mathbb{E}[\delta T + \alpha U|Z=1] - \mathbb{E}[\delta T + \alpha U|Z=0] + \cancel{\mathbb{E}[\epsilon|Z=1] - \mathbb{E}[\epsilon|Z=0]} \\
&= \delta(\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0]) + \alpha \underbrace{(\mathbb{E}[U|Z=1] - \mathbb{E}[U|Z=0])}_{U \perp\!\!\!\perp Z} \\
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&= \delta(\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0])
\end{aligned}$$

Simplifying gives us the Wald Estimand:

$$\delta = \frac{\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0]}{\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0]}$$

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&= \delta(\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0])
\end{aligned}
$$

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$$\delta = \frac{\mathbb{E}[Y|Z=1] - \mathbb{E}[Y|Z=0]}{\mathbb{E}[T|Z=1] - \mathbb{E}[T|Z=0]}$$

We can estimate this from data via the Wald Estimator:

$$\hat{\delta} = \frac{\frac{1}{n_1}\sum_{i:z_i=1} Y_i - \frac{1}{n_1}\sum_{i:z_i=0} Y_i}{\frac{1}{n_1}\sum_{i:z_i=1} T_i - \frac{1}{n_1}\sum_{i:z_i=0} T_i}$$

Continuous Linear Model

In the continuous case, we use a similar intuition but instead of differences in conditional expectations, we look at $\text{Cov}(Y, Z)$.

$$\begin{aligned}
&\text{Cov}(Y, Z) \\
&= \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] \\
&= \mathbb{E}[(\delta T + \alpha U + \epsilon)Z] - \mathbb{E}[(\delta T + \alpha U + \epsilon)]\mathbb{E}[Z]
\end{aligned}$$

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&= \delta\mathbb{E}[TZ] + \alpha\mathbb{E}[UZ] - \delta\mathbb{E}[T]\mathbb{E}[Z] - \alpha\mathbb{E}[U]\mathbb{E}[Z]
\end{aligned}$$

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&= \delta\mathbb{E}[TZ] + \alpha\mathbb{E}[UZ] - \delta\mathbb{E}[T]\mathbb{E}[Z] - \alpha\mathbb{E}[U]\mathbb{E}[Z] \\
&= \delta(\mathbb{E}[TZ] - \mathbb{E}[T]\mathbb{E}[Z]) + \alpha \underbrace{(E[UZ] - \mathbb{E}[U]\mathbb{E}[Z])}_{\mathrm{Cov}(U,Z)=0 \quad U \perp\!\!\!\perp Z} \\
&= \delta\mathrm{Cov}(T, Z)
\end{aligned}$$

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&= \delta\mathbb{E}[TZ] + \alpha\mathbb{E}[UZ] - \delta\mathbb{E}[T]\mathbb{E}[Z] - \alpha\mathbb{E}[U]\mathbb{E}[Z] \\
&= \delta(\mathbb{E}[TZ] - \mathbb{E}[T]\mathbb{E}[Z]) + \alpha\underbrace{(E[UZ] - \mathbb{E}[U]\mathbb{E}[Z])}_{\mathrm{Cov}(U,Z)=0 \quad U \perp\!\!\!\perp Z} \\
&= \delta\mathrm{Cov}(T, Z)
\end{aligned}
$$

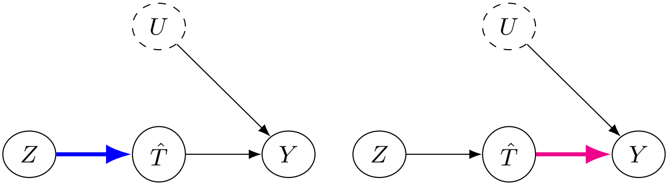
Simplifying gives us

$$\delta = \frac{\mathrm{Cov}(Y, Z)}{\mathrm{Cov}(T, Z)}$$

We can estimate this from data via the empirical covariances.

Another approach - Two-stage estimator

1. Estimate (via linear regression) $\mathbb{E}[T|Z]$. The model then gives us $\hat{T}$,
2. Estimate (via linear regression) $\mathbb{E}[Y|\hat{T}]$. The coefficient in front of $\hat{T}$ is our estimate $\hat{\delta}$.



For one-dimensional variables, this method matches the previous one:

$$\hat{T} = \frac{\mathrm{Cov}(T,Z)}{\mathrm{Var}(Z)} Z$$

$$\hat{\delta} = \frac{\mathrm{Cov}(\hat{T},Y)}{\mathrm{Var}(\hat{T})} = \frac{\frac{\mathrm{Cov}(T,Z)}{\mathrm{Var}(Z)}\mathrm{Cov}(Z,Y)}{\left(\frac{\mathrm{Cov}(T,Z)}{\mathrm{Var}(Z)}\right)^2 \mathrm{Var}(Z)} = \frac{\mathrm{Cov}(Z,Y)}{\mathrm{Cov}(T,Z)}$$

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