Fractal Dimension Calculation

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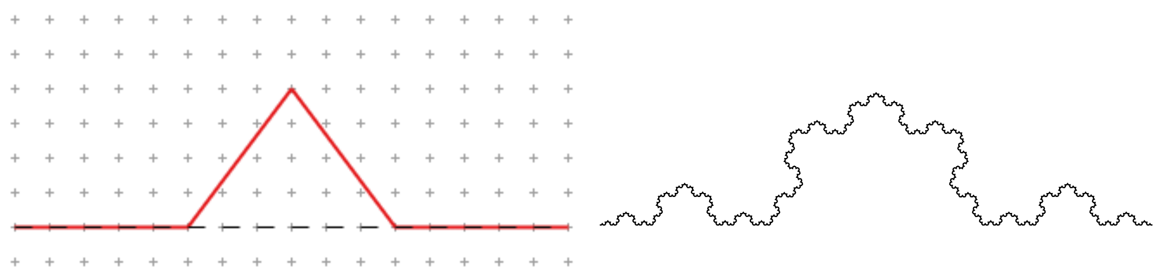
The purpose of this document is to describe how fractal dimensions are calculated in the JavaScript "`fracexpl.js`" fractal explorer. A fractal is defined by a seed shape made up of individual segments, where each segment represents either a scaled version of the fractal or a simple line segment. For example, the seed shape on the left below defines the Koch curve (rendered on the right), where each segment is a replicating segment of length 5, and the length of the baseline (the dashed line) is 16.



We denote the size (either length or area) of a shape with baseline length b as $S(b)$. For a shape with dimension d, the function should satisfy

$$S(b) = k \cdot b^d$$

for some constant k. For example, when the shape is a line segment that is twice the length of the baseline, then $S(b) = 2b$, so the dimension is 1. When the shape is a rectangle that is b units on one side and $3b$ on the other side, then $S(b) = 3b^2$ so the dimension is 2.

For regular fractals, such as the Koch curve, every segment of the seed shape is replicated and of the same length, and we can solve for d analytically. In the Koch seed above, each of the 4 replicating segments has length $\frac{5}{16}b$, so $4S(\frac{5}{16}b) = S(b)$ for all b. We plug into our general form for $S(b)$ to get

$$4k\left(\frac{5}{16}b\right)^d = k \cdot b^d \iff \left(\frac{16}{5}\right)^d = 4 \iff d = \frac{\log 4}{\log 3.2} = 1.191844...$$

Therefore, this Koch curve has dimension approximately 1.19.

Unfortunately, when you get beyond regular seed shapes and when you put in non-replicating line segments, analytic solutions are impossible. For the general case, consider a seed shape in which there are n replicating segments, with lengths $c_1 b, c_2 b, \cdots, c_n b$, and m fixed (non-replicating) segments with lengths $f_1 b, f_2 b, \cdots, f_m b$. The size of the fractal then satisfies

$$S(b) = \sum_{i=1}^{k} S(c_i b) + \sum_{i=1}^{m} f_i b. \tag{1}$$

Consider two cases: when $m = 0$ (i.e., there are no non-replicating segments), and when $m > 0$.

Case 1: $m = 0$**.** In this case, the second sum in (1) is empty, so

$$\sum_{i=1}^{n} S(c_i b) = S(b) \iff k \cdot b^d = \sum_{i=1}^{n} k \cdot (c_i b)^d \iff \sum_{i=1}^{n} c_i^d = 1$$

For a given seed shape (i.e., for given c_i values), the `fracexpl.js` code solves this equation using bisection.

Case 2: $m > 0$**.** In this case we must consider the full formula in (1), so we need

$$\sum_{i=1}^{k} S(c_i b) + \sum_{i=1}^{m} f_i b = S(b) \iff \sum_{i=1}^{n} k \cdot (c_i b)^d + \sum_{i=1}^{m} f_i b = k \cdot b^d \iff b^d \sum_{i=1}^{n} c_i^d + \frac{b}{k} \sum_{i=1}^{m} f_i = b^d.$$

This equation must hold for all values of b (since dimension is scale-invariant), and it is not hard to see that if the sum of the f_i's is non-zero this can only hold when $d = 1$. Fixing $d = 1$, we get

$$\sum_{i=1}^{n} c_i + \frac{1}{k} \sum_{i=1}^{m} f_i = 1 \iff k = \frac{\sum_{i=1}^{m} f_i}{1 - \sum_{i=1}^{n} c_i}.$$

We can solve for a positive k if and only if $\sum_{i=1}^{n} c_i < 1$, so the `fracexpl.js` code tests for this special case and returns dimension 1 if this holds. If there are non-replicating segments and this does *not* hold, then there is no way find a scale-invariant dimension, and so the code indicates that there is unknown dimension in this case.