Merge Sort / Recursion

Lecture Question

Restriction: No state is allowed in this question. Specifically, the keyword "var" is banned. (ie. You are expected to use a recursive solution)

Question: In a package named "functions" create an **object** named Numbers with a method named fib that:

- Takes an Int as a parameter and returns the n^{th} fibonacci number
- Fibonacci numbers are equal to the sum of the previous two fibonacci numbers starting with 1 and 1 as the first two numbers in the sequence
- Fibonacci numbers: 1, 1, 2, 3, 5, 8...
- fib(1) == 1
- fib(2) == 1
- fib(3) == 2
- fib(4) == 3
- You method will not be tested with inputs < 1

Runtime Analysis

- Last time we said Selection sort is inefficient
- Let's be more specific
- We'll measure the asymptotic runtime of the algorithm
  - Often use big-O notation
- Count the number of "steps" the algorithm take
  - A step is typically a basic operation (+, -, &&, etc)

Runtime Analysis

- Asymptotic runtime
  - Measures the order of magnitude of the runtime in relation to the size of the input
  - Name the input size **n**
  - For sorting - Size of the input is the number of values in the data structure
  - Ignore constants
- Ex. Runtime of O(n) grows linearly with the size of the input

Selection Sort - Runtime

- Abridged runtime analysis

Outer loop runs once for each index

Runs O(n) times

```
def selectionSort[T](inputData: List[T], comparator: (T, T) => Boolean): List[T] = {
  var data: List[T] = inputData
  for (i <- data.indices) {
    var minFound = data.apply(i)
    var minIndex = i
    for (j <- i until data.size) {
      val currentValue = data.apply(j)
      if (comparator(currentValue, minFound)) {
        minFound = currentValue
        minIndex = j
      }
    }
    data = data.updated(minIndex, data.apply(i))
    data = data.updated(i, minFound)
  }
  data
}
```

Selection Sort - Runtime

- Abridged runtime analysis

Inner loop runs once for each index from i to the end of the list

Runs for each iteration of the outer loop with a worst case of O(n)

```
def selectionSort[T](inputData: List[T], comparator: (T, T) => Boolean): List[T] = {
  var data: List[T] = inputData
  for (i <- data.indices) {
    var minFound = data.apply(i)
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    for (j <- i until data.size) {
      val currentValue = data.apply(j)
      if (comparator(currentValue, minFound)) {
        minFound = currentValue
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    }
    data = data.updated(minIndex, data.apply(i))
    data = data.updated(i, minFound)
  }
  data
}
```

Selection Sort - Runtime

- Abridged runtime analysis

Run O(n) iterations O(n) times results in an O(n^2) total runtime

```
def selectionSort[T](inputData: List[T], comparator: (T, T) => Boolean): List[T] = {
  var data: List[T] = inputData
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        minIndex = j
      }
    }
    data = data.updated(minIndex, data.apply(i))
    data = data.updated(i, minFound)
  }
  data
}
```

Selection Sort - Runtime

- Abridged runtime analysis

We reach $O(n^3)$ since apply takes $O(n)$

More details next week

```
def selectionSort[T](inputData: List[T], comparator: (T, T) => Boolean): List[T] = {
  var data: List[T] = inputData
  for (i <- data.indices) {
    var minFound = data.apply(i)
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      val currentValue = data.apply(j)
      if (comparator(currentValue, minFound)) {
        minFound = currentValue
        minIndex = j
      }
    }
    data = data.updated(minIndex, data.apply(i))
    data = data.updated(i, minFound)
  }
  data
}
```

Selection Sort - Runtime

- More mathematical analysis
  - Inner loop runs Σ i times where i ranges from n to 1
  - $n + n\text{-}1 + n\text{-}2 + \ldots + 2 + 1 = n^2/2 + n/2$
  - For asymptotic we only consider the highest order term and ignore constant multipliers
  - Therefore $n^2/2 + n/2$ is $O(n^2)$
  - Selection Sort has $O(n^2)$ runtime

Merge Sort

- We briefly saw in CSE115 that we can do better by using merge sort and reaching O(n log(n)) runtime
- Let's analyze this in more depth

Merge Sort

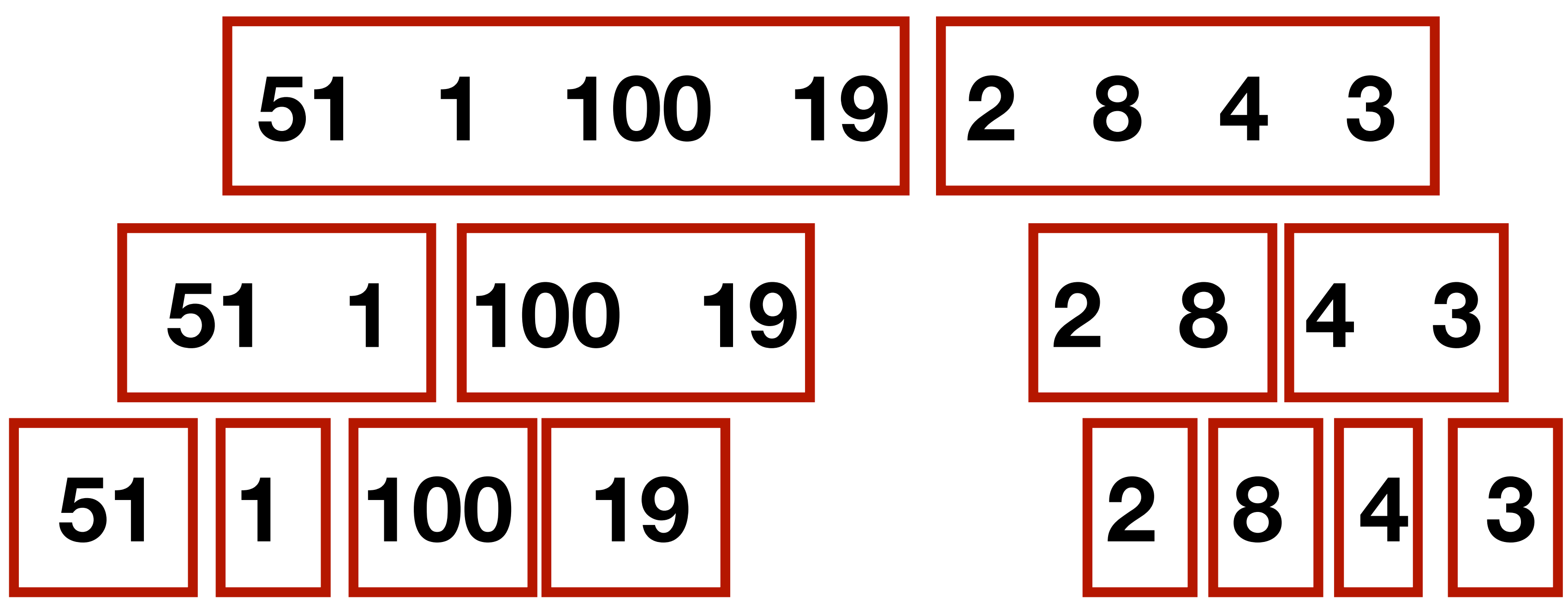
- The algorithm
  - If the input list has < 2 elements
    - Return it (It's already sorted)
  - Else
    - Divide the input list in two halves
    - Recursively call merge sort on each half (Repeats until the lists are size 1)
    - Merge the two sorted lists together into a single sorted list

Merge Sort

- Given an input

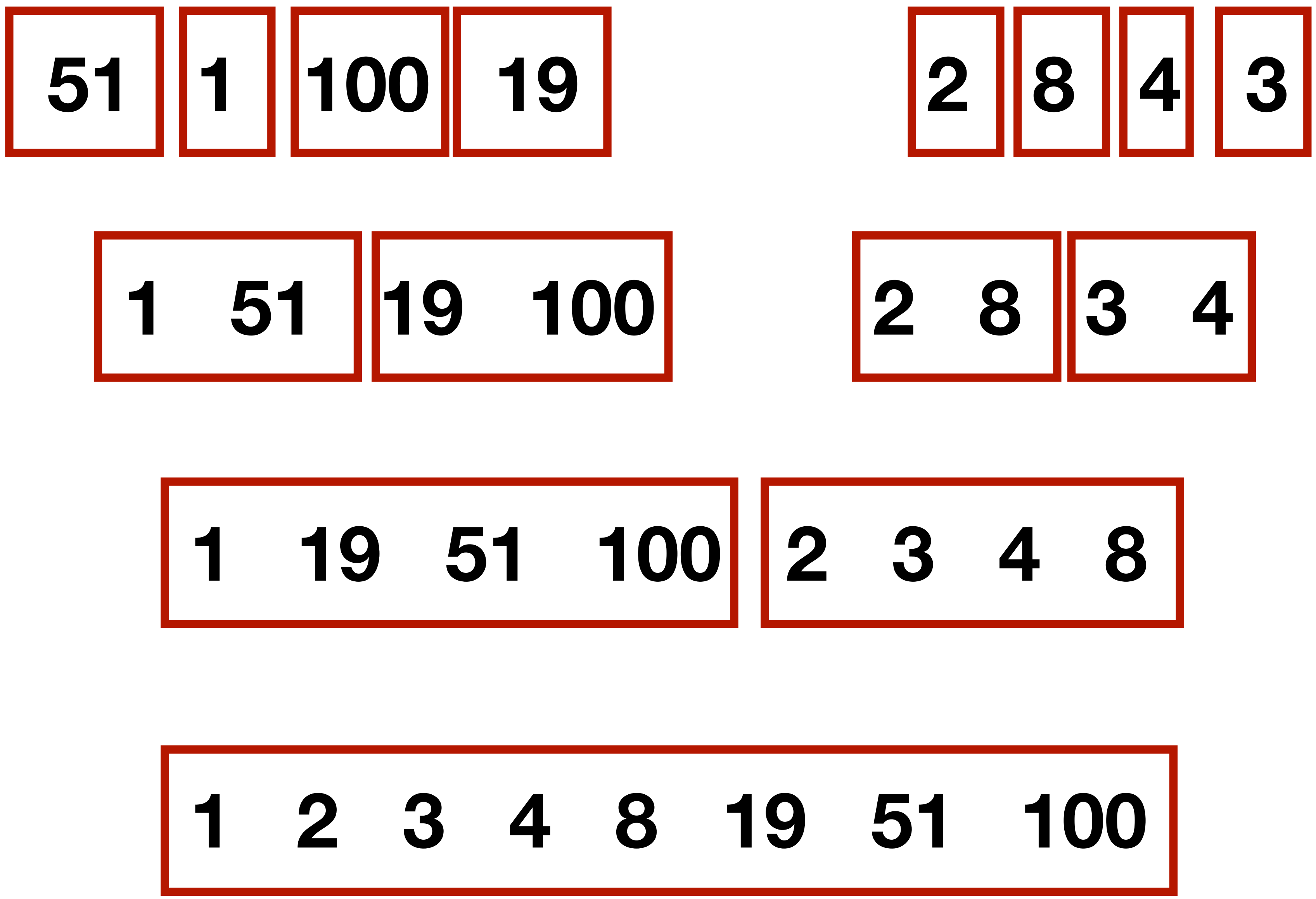
51 1 100 19 2 8 4 3

- Divide into two lists recursively until n=1



Merge Sort

- Merge lists until the original input is sorted



Merge Sort - Merge

```
def mergeSort[T](inputData: List[T], comparator: (T, T) => Boolean): List[T] = {
  if (inputData.length < 2) {
    inputData
  } else {
    val mid: Int = inputData.length / 2
    val (left, right) = inputData.splitAt(mid)
    val leftSorted = mergeSort(left, comparator)
    val rightSorted = mergeSort(right, comparator)

    merge(leftSorted, rightSorted, comparator)
  }
}
```

Recursion!

Merge Sort - Runtime

- Each level of the recursion has 2^i lists of size $n/2^i$
- Recursion ends when is $n/2^i == 1$
  - $i = \log(n)$
  - $\log(n)$ levels of recursion
- Each level needs to merge a total of n elements across all sub-lists
- If we can merge in $O(n)$ time we'll have $O(n \log(n))$ total runtime

Merge Sort - Merge

- Merge two sorted lists in O(n) time
- Take advantage of each list being sorted
- Start with pointers at the beginning of each list
- Compare the two values at the pointers and find which come first based on the comparator
    - Append it to a new list and advance that pointer
- When a pointer reaches the end of a list copy the rest of the contents

Merge Sort - Merge

1 19 51 100 2 3 4 8

Merge Sort - Merge

1 19 51 100

2 3 4 8

1

Merge Sort - Merge

1 19 51 100

2 3 4 8

1 2

Merge Sort - Merge

1 19 51 100

2 3 4 8

1 2 3

Merge Sort - Merge

1 19 51 100 2 3 4 8

1 2 3 4

Merge Sort - Merge

1 19 51 100

2 3 4 8

When a pointer reaches the end of a list,
copy the rest of the other list to the result

1 2 3 4 8

Merge Sort - Merge

1 19 51 100

2 3 4 8

When a pointer reaches the end of a list, copy the rest of the other list to the result

1 2 3 4 8 19 51 100

Merge Sort - Merge

```scala
def merge[T](left: List[T], right: List[T], comparator: (T, T) => Boolean): List[T] = {
  var leftPointer = 0
  var rightPointer = 0

  var sortedList: List[T] = List()

  while (leftPointer < left.length && rightPointer < right.length) {
    if (comparator(left.apply(leftPointer), right.apply(rightPointer))) {
      sortedList = sortedList :+ left.apply(leftPointer)
      leftPointer += 1
    } else {
      sortedList = sortedList :+ right.apply(rightPointer)
      rightPointer += 1
    }
  }

  while (leftPointer < left.length) {
    sortedList = sortedList :+ left.apply(leftPointer)
    leftPointer += 1
  }
  while (rightPointer < right.length) {
    sortedList = sortedList :+ right.apply(rightPointer)
    rightPointer += 1
  }

  sortedList
}
```

Use the comparator to make ordering decisions

Writing Recursive Methods

- Suggested approach:
  - Assume your recursive calls return the correct values
  - Write your method based on this assumption
  - Add a base case(s) for an input that has a trivial return value
  - Only write recursive calls that get closer to the base case

Writing Recursive Methods

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Writing Recursive Methods

- Assume your recursive calls return the correct values

  -and-

- Write your method based on this assumption

- The primary benefit of writing recursive methods/functions is that we can assume that the recursive calls are correct
- If these calls are not correct, we have work to do elsewhere
  - While writing the top level functionality, assume they are correct and fix the other issues if they are not

Writing Recursive Methods

- Add a base case(s) for an input that has a trivial return value
  - A simple input where the return value is trivial
  - Ex. An empty list, an empty String, 0, 1
- Add a conditional to your method to check for the base case(s)
  - If the input is a base case, return the trivial solution
  - Else, run your code that makes the recursive call(s)

Writing Recursive Methods

- Ensure your recursive calls always get closer to your base case
  - Base case is eventually reached and returned
  - Ex. Base case is 0, each recursive call decreases the input
  - Ex. Base case is the empty String and an each recursive call removes a character from the input
- If your recursive calls don't reach a base case
  - Infinite recursion
  - Stack overflow

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