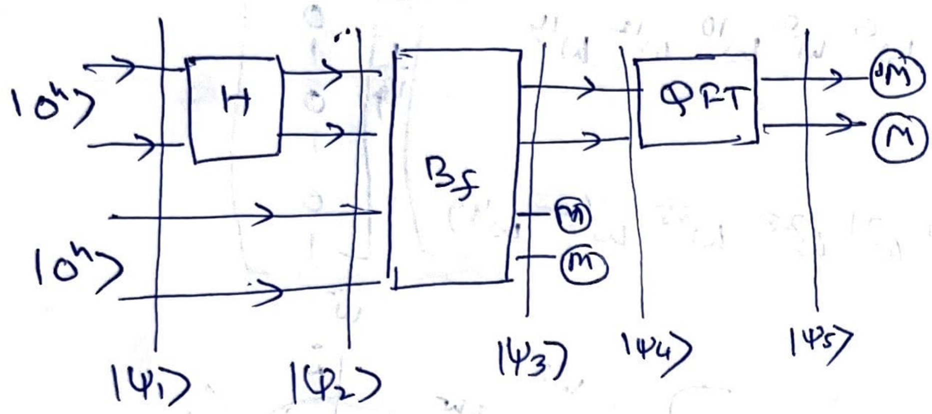
Example of Period Finding Algorithm

Find period r

given $f : \{0,1\}^3 \rightarrow \{0,1\}^3$

For use in B_f $\quad f(x) = x \bmod 2$



$$|\psi_1\rangle = |0^n\rangle|0^n\rangle = |000\rangle|000\rangle$$

$$|\psi_2\rangle = H^{\otimes 3}|0^3\rangle|0^3\rangle = \frac{1}{\sqrt{2^3}} \sum_{x=0}^{7} |x\rangle|000\rangle$$

$$= \frac{|0\rangle + |1\rangle + \cdots + |7\rangle}{\sqrt{2^3}} |000\rangle$$

$$|\psi_3\rangle = B_f|\psi_2\rangle \qquad \text{we know } B_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle$$

$$= \frac{1}{\sqrt{2^3}}\Big(|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|0\rangle + |3\rangle|1\rangle + |4\rangle|0\rangle + |5\rangle|1\rangle + |6\rangle|0\rangle + |7\rangle|1\rangle\Big).$$

Assume we measure $|1\rangle$ in second register,

$$|\psi_4\rangle = \frac{1}{\sqrt{2^3}} \frac{(|1\rangle + |3\rangle + |5\rangle + |7\rangle)}{\sqrt{4}} |1\rangle \qquad // \sqrt{4} \text{ for normalization}$$

Similarly if we measure $|0\rangle$ in second register

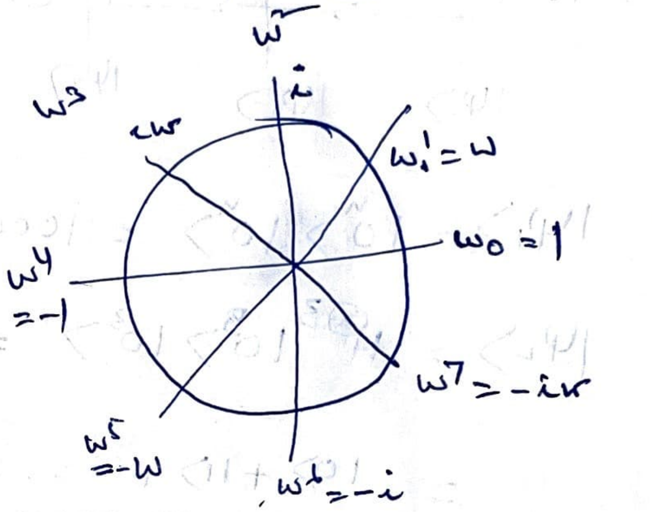
$$|\psi_4\rangle = \frac{1}{\sqrt{2^3}} \frac{|0\rangle + |2\rangle + |4\rangle + |6\rangle}{\sqrt{4}} |0\rangle$$

Now we remove linear shift by applying QFT

$$|\psi_5\rangle = QFT_8 \, |\psi_4\rangle$$

$$|\psi_5\rangle = \frac{1}{\sqrt{2^3}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} \\ & & & & & & & \\ 1 & w^7 & w^{14} & w^{21} & w^{28} & w^{35} & w^{42} & w^{49} \end{bmatrix} \frac{1}{\sqrt{4}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{2}} \begin{bmatrix} 1+1+1+1 \\ w + w^3 + w^5 + w^7 \\ w^2 + w^6 + w^{10} + w^{14} \\ w^3 + w^9 + w^{15} + w^{21} \\ w^4 + w^{12} + w^{20} + w^{28} \\ \vdots \\ w^7 + w^{21} + w^{35} + w^{49} \end{bmatrix}$$



$$= \frac{1}{4\sqrt{2}} \begin{bmatrix} 4 \\ w + iw - w - iw \\ i - i + i - i \\ iw + w - iw - iw \\ -1 -1 -1 -1 \\ \\ \\ \\ \end{bmatrix} = \frac{1}{4\sqrt{2}} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4\sqrt{2}} \cdot 4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{|0\rangle - |4\rangle}{\sqrt{2}}$$

Measure $|\psi_5\rangle$

$|0\rangle$ $\quad$ $|1\rangle$

$\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$ Prob $\quad \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$.

Conclusion:
we measure '0' with $\frac{1}{2}$ Probability and we measure 4 with $\frac{1}{2}$ Probability

Let $|\psi_4\rangle = \dfrac{(|0\rangle + |2\rangle + |4\rangle + |6\rangle)\,|0\rangle}{\sqrt{4}}$

then $|\psi_5\rangle = \dfrac{|0\rangle + |4\rangle}{\sqrt{2}} \longrightarrow$ $|0\rangle$ with $1/2$ Prob, $|1\rangle$ with $1/2$ Prob we have to rerun this circuit $\log N$ times (15)

Rerun $O(\log N) = O(\log 2^3)$ times.

Find GCD of all our measurement (Let it be 4).

$4 \leftarrow GCD(0, 4)$

~~Find~~ $\cancel{\frac{N}{r} = 4}$

Find r, Given $\frac{N}{r} = 4$

$N = 8 \therefore \frac{N}{r} = \frac{8}{r} = 4$

$r = 2$

$\Rightarrow r = 2$

Hence period $= 2$.