We see ① How many Grover rotations are required
② Complete circuit.

Problem: Given a function $f: \{0,1\}^n \to \{0,1\}$
find x such that $f(x) = 1$

Definitions:

$$A: \{x \in \{0,1\}^n : f(x) = 1\}$$

x	$f(x)$
000	0
001	0
010	1
011	0
100	1
101	0
110	0
111	0

$A = \{010, 100\}$

$B = \{000, 001, 011, 101, 110, 111\}$

$A = \{x \in \{0,1\}^n : f(x) = 1\}$

$B = \{x \in \{0,1\}^n : f(x) = 0\}$

Let $|A| = a = 2$ $|B| = b = 6$

$$|A\rangle = \frac{1}{\sqrt{a}} \sum_{x \in A} |x\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |100\rangle)$$

$$|B\rangle = \frac{1}{\sqrt{b}} \sum_{x \in B} |x\rangle = \frac{1}{\sqrt{6}}(|000\rangle + |001\rangle + |011\rangle + |101\rangle + |110\rangle + |111\rangle).$$

$$|h\rangle = H|0\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle \qquad N = 2^n$$

↳ Equal superposition of all the inputs i.e. Hadamard gate

We can rewrite $|h\rangle$ as

$$|h\rangle = \sqrt{\frac{a}{N}}|A\rangle + \sqrt{\frac{b}{N}}|B\rangle$$

In our example

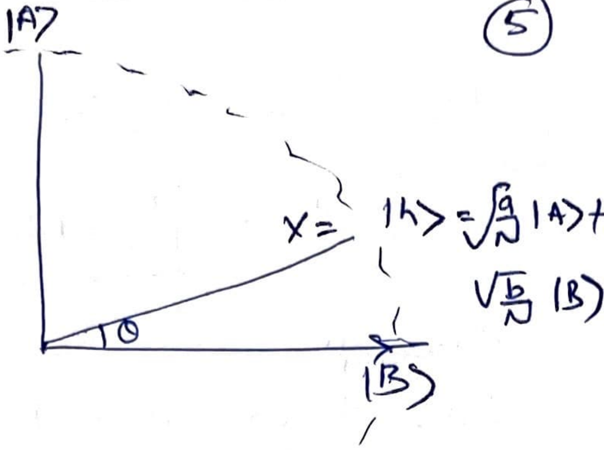
$$|h\rangle = \frac{1}{\sqrt{8}} \sum_{x=0}^{7} |x\rangle = \sqrt{\frac{2}{8}}|A\rangle + \sqrt{\frac{6}{8}}|B\rangle$$

One qubit version of Hadamard gate
$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

// n-qubit version is
$$H|0\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle$$

R_B : Reflect x about $|B\rangle$

R_h : Reflect x about $|h\rangle$



θ after 1 Grover Rotation

$\theta + 2\theta$

θ after 2 Grover Rotations

$\theta + 2\cdot 2\theta = \theta + 4\theta$

θ after k Grover Rot.

$\theta + k\cdot 2\theta$

$= (1+2k)\theta$

$$|h\rangle = \sqrt{\frac{a}{N}}|A\rangle + \sqrt{\frac{b}{N}}|B\rangle$$

In polar coordinates

$$|h\rangle = \sin\theta\,|A\rangle + \cos\theta\,|B\rangle$$

With above two eqns we can deduce

$\sin\theta = \sqrt{\frac{a}{N}}$ $\quad$ $\sin\theta \approx \theta$ for small θ.

$\therefore \theta = \sqrt{\frac{a}{N}}$ — (1)

We can write x as

$$x = \sin(1+2k)\theta\,|A\rangle + \cos(1+2k)\theta\,|B\rangle$$

Our Aim : Prob. of measure $|A\rangle \approx 1$

$(\sin(1+2k)\theta)^2 \approx 1$

$\sin(1+2k)\theta = 1$

$(1+2k)\theta = \sin^{-1} 1 = \frac{\pi}{2}$

Sub. $\theta = \sqrt{\frac{a}{N}}$ and find k

$(1+2k)\sqrt{\frac{a}{N}} = \frac{\pi}{2} \Rightarrow$

$(1+2k) = \frac{\pi}{2} \times \frac{\sqrt{N}}{\sqrt{a}}$

$2k = \frac{\pi}{2}\sqrt{\frac{N}{a}} - 1$ $\quad$ $\therefore k = \frac{\pi}{4}\sqrt{\frac{N}{a}} - \frac{1}{2}$

i.e. $\quad$ Here $a = 1$

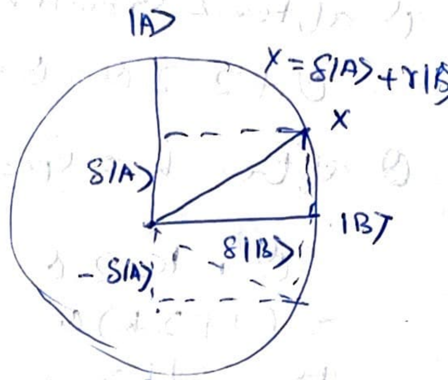
K can be rewritten as

$$K = \left\lfloor \frac{\pi}{4}\sqrt{N} \right\rfloor$$

~~Circuit~~

Quantum circuit for reflection

Reflect X about $|B\rangle$



$X = \delta|A\rangle + \gamma|B\rangle$

Rule of Parallelogram:

Reflection Rule:

General Rule: Reflect $|M\rangle$ about $|\pi\rangle$ by changing signs of component orthogonal to $|\pi\rangle$

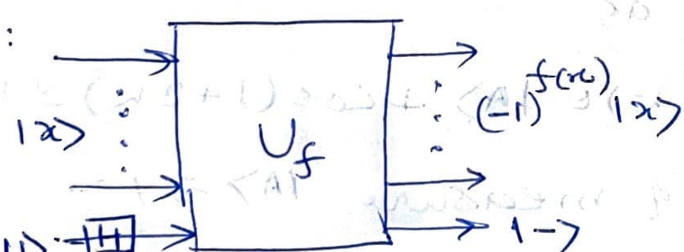
Reflect X about $|B\rangle$

$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}|0\rangle = -|0\rangle \qquad \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}|1\rangle = -|1\rangle$$

Similarly $|i\rangle \rightarrow -|i\rangle$, $|x\rangle \rightarrow -|x\rangle$

Goal $|x\rangle = (-1)^{f(x)}|x\rangle$

~~Equation~~ :



$U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$

$U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$ Proved in Deustch Jos Alg

$= (-1)^{f(x)}|x\rangle|-\rangle \quad \because H|1\rangle = |-\rangle$