Quantum Fourier Transform

Basics of Discrete FT:

Primitive roots of Unity

$z^n = 1$ $\quad z$ is $\mathbb{C}$ $\quad$ (z is complex number)

It has n roots - such roots are called primitive roots of Unity. These roots are:

$\{\omega_n^0, \omega_n^1, \omega_n^2, \ldots \omega_n^{n-1}\}$

where $\omega_n^k = e^{2\pi i k/n}$

Property 1:

ω_n^k lie on Unit circle. i.e. $|\omega_n^k| = 1$

$$|\omega_n^k| = \sqrt{\omega_n^{k*} \cdot \omega_n^k}$$

$$= \sqrt{e^{-2\pi i \frac{k}{n}} \, e^{2\pi i \frac{k}{n}}} = \sqrt{e^0} = 1$$

Property 2: ω_n^k is a periodic function

$$\omega_n^k = \omega_n^{k \bmod n}$$

ex: write all roots $z^4 = 1$.

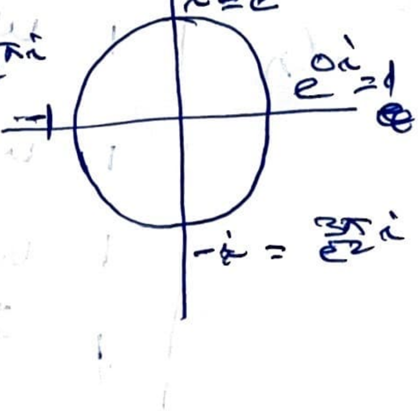
The roots are $\{\omega_4^0, \omega_4^1, \omega_4^2, \omega_4^3\}$

All these Roots lie on Unit circle.

$$= \left\{ e^{\frac{2\pi i 0}{4}}, e^{\frac{2\pi i 1}{4}}, e^{\frac{2\pi i 2}{4}}, e^{\frac{2\pi i 3}{4}} \right\}.$$

$$= \left\{ e^{0i}, e^{\frac{\pi}{2} i}, e^{\pi i}, e^{\frac{3\pi i}{2}} \right\}$$

We know $e^{i\theta}$ lie on Unit circle at angle θ.



Hence roots are $\{1, i, -1, -i\}$

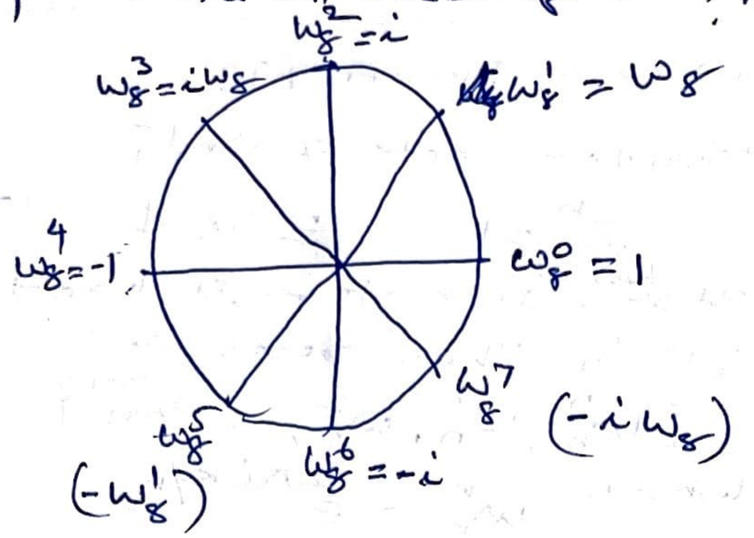
Re arrange them: $\{1, -1, i, -i\}$

Few properties:

$$\omega_4^4 = \omega_4^{4 \bmod 4} = \omega_4^0 = 1$$

$$\omega_4^7 = \omega_4^{7 \bmod 4} = \omega_4^3 = -i$$

Ex: 2 ~~4~~ $z^8 = 1$ Write all roots of $z^8 = 1$.



Hence the roots are.
$(1, \omega_8, i, i\omega_8, -1, -\omega_8, -i, -i\omega_8)$ $\qquad \omega_n^{ij}$

DFT Matrix: $n \times n$ square matrix
i^{th} rows and j^{th} column element $= \omega_n^{ij}$
We starts counting rows and columns from 0 instead of 1

$$
f_n = \frac{1}{\sqrt{n}} \begin{bmatrix}
\omega_n^{0\times 0} & \omega_n^{0\times 1} & \omega_n^{0\times 2} & \cdots & \omega_n^{0\times(n-1)} \\
\vdots & \vdots & \vdots & & \vdots \\
\omega_n^{k\times 0} & \omega_n^{k\times 1} & \omega_n^{k\times 2} & \cdots & \omega_n^{k\times(n-1)} \\
\vdots & \vdots & \vdots & & \vdots \\
\omega_n^{(n-1)\cdot 0} & \omega_n^{(n-1)1} & \omega_n^{(n-1)2} & \cdots & \omega_n^{(n-1)(n-1)}
\end{bmatrix}
$$

$$
= \frac{1}{\sqrt{n}} \left[\begin{array}{cccccc}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{(n-1)} \\
1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\
1 & \omega_n^3 & \omega_n^6 & & \omega_n^{3(n-1)} \\
\vdots & \vdots & & & \\
1 & \cdots & & & \omega_n^{(n-1)(n-1)}
\end{array}\right]
\begin{array}{l}
\text{row \#1} \\
\text{row \#2} \\
\text{row-3} \\
\text{row-} \\
\\
\text{row n.}
\end{array}
$$

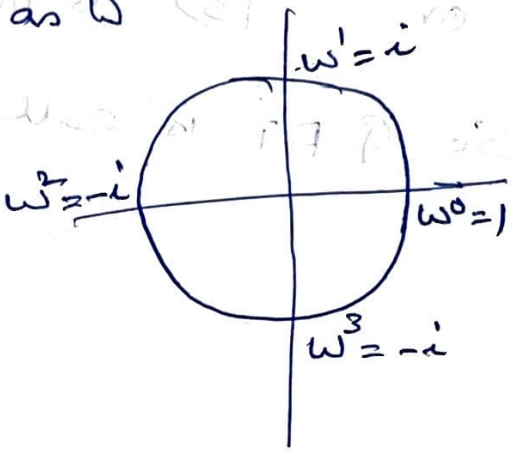
Ex: Transform $|\psi\rangle = \frac{|0\rangle + |3\rangle}{\sqrt{2}}$ using F_4.

$$F_4|\psi\rangle = |\phi\rangle$$

$$F_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix}$$

Note:

$\omega_4^1 \rightarrow$ is shown as ω^1 and ω_4^2 as ω^2



$$F_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^0 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

$$= \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$F_4|\psi\rangle = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ 1-i \\ 0 \\ 1+i \end{bmatrix} = |\phi\rangle$$

$$|\phi\rangle = \frac{\cancel{2}}{\cancel{2}\sqrt{2}}|0\rangle + \frac{(1-i)}{2\sqrt{2}}|1\rangle + \frac{(1+i)}{2\sqrt{2}}|3\rangle$$