Deutsch - Jozsa Algorithm

$f : \{0,1\}^n \to \{0,1\}$ f is constant or balanced ?.

Problem

Input : a function as above

Promise : f is either constant or balanced only

Task : Determine whether f is const or bal.

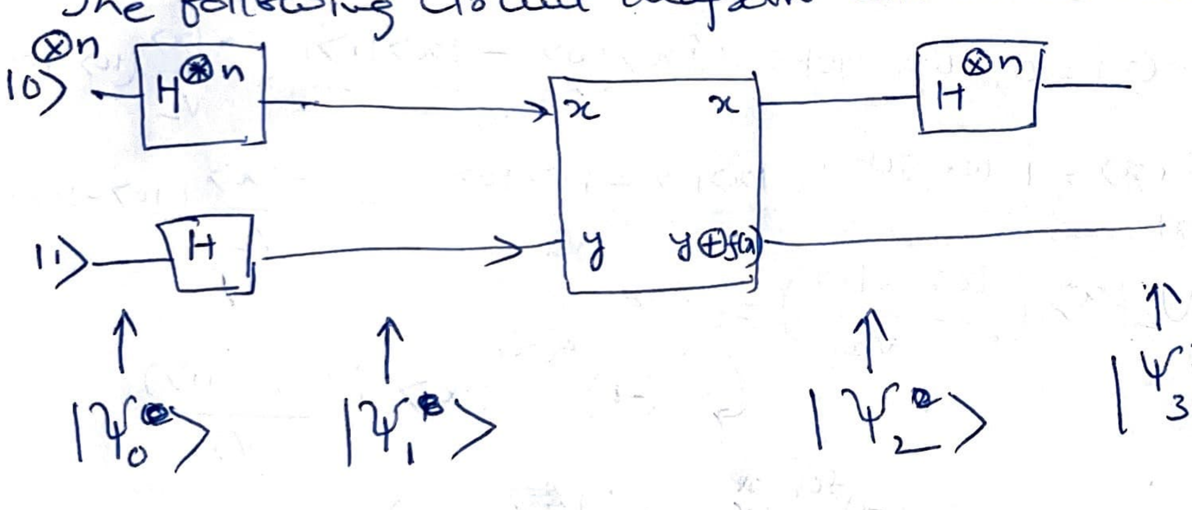
Let U_f be the oracle that performs the transformt

$$U_f|x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle \quad -(1)$$

for $x \in \{0, 1, \ldots 2^n-1\}$ and $y \in \{0,1\}$

$f(x) \in \{0,1\}$

The following circuit diagram can describe the alg.



$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

In short $H|x\rangle = \sum_{z \in \{0,1\}} \frac{(-1)^{xz}|z\rangle}{\sqrt{2}}$ for $x \in \{0,1\}$ $-(2)$

$$H^{\otimes n} = \underbrace{H \otimes H \otimes \cdots \otimes H}_{n \text{ times}}$$

$$H^{\otimes n}|0\rangle^{\otimes n} = H|0\rangle \otimes H|0\rangle \otimes \cdots \otimes H|0\rangle$$

$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \cdots \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \sum_{x \in \{0,1\}^n} \frac{|x\rangle}{\sqrt{2^n}}$$

where $x = \underbrace{\{0000..0, 0000..1, \ldots 1111..1\}}_{2^n-1}$

$-(3)$ $-\cancel{(3)}$

The initial state is $|\Psi_0\rangle = |0\rangle^{\otimes n} |1\rangle$

ie we start with $n+1$ qubits, n in $|0\rangle$ and 1 in $|1\rangle$

$$|\Psi_1\rangle = H^{\otimes n}|0\rangle H|1\rangle \quad \text{(from ③)}$$

$$= \sum_{x \in \{0,1\}^n} \frac{|x\rangle}{\sqrt{2^n}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \quad - ④$$

from ①

Now $U_f|x\rangle|0\rangle = |x\rangle|0 \oplus f(x)\rangle = |x\rangle|f(x)\rangle$

$$U_f|x\rangle|1\rangle = |x\rangle|1 \oplus f(x)\rangle$$

$$\therefore U_f|x\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{|x\rangle|f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle}{\sqrt{2}}$$

If $f(x) = 0$, we get $\dfrac{(|x\rangle|0\rangle - |x\rangle|1\rangle)}{\sqrt{2}} = \dfrac{|x\rangle}{\sqrt{2}}(|0\rangle - |1\rangle)$

If $f(x) = 1$, we get $\dfrac{|x\rangle|1\rangle - |x\rangle|0\rangle}{\sqrt{2}} = \dfrac{-|x\rangle}{\sqrt{2}}(|0\rangle - |1\rangle)$

In Short

$$U_f|x\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = (-1)^{f(x)}|x\rangle\frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

$$\therefore |\Psi_2\rangle = \underbrace{\sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}}}_{\text{Apply } H^{\otimes n}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \quad - ⑤$$

Note that $H|x\rangle = \sum_{z \in \{0,1\}} \dfrac{(-1)^{xz}|z\rangle}{\sqrt{2}}$ where $x \in \{0,1\}$ [From ②]

For $x \in \{0,1\}^n$

$$H^{\otimes n}|x\rangle = \sum_{z \in \{0,1\}^n} \frac{(-1)^{x \cdot z}|z\rangle}{\sqrt{2^n}} \quad \text{[From ③]}$$

where $x \cdot z = x_1 z_1 \oplus x_2 z_2 \oplus \dots \oplus x_n z_n$

$-$ ④

③

$$H^{\otimes n}\left(\sum \frac{(-1)^{f(x)}}{\sqrt{2^n}}|x\rangle\right) = \sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}}{\sqrt{2^n}} H^{\otimes n}|x\rangle$$

③ applied to H gate

$$= \sum_{x} \frac{(-1)^{f(x)}}{\sqrt{2^n}}\left(\sum_{z\in\{0,1\}^n} \frac{(-1)^{x\cdot z}}{\sqrt{2^n}}|z\rangle\right)$$

[Sub. Eq ④] ④

$$= \sum_{z}\sum_{x} \frac{(-1)^{x\cdot z + f(x)}}{2^n}|z\rangle$$

$$|\psi_3\rangle = H^{\otimes n}\left(\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}}{\sqrt{2^n}}|x\rangle\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \sum_{z}\sum_{x} \frac{(-1)^{x\cdot z + f(x)}}{2^n}|z\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Now look at $|\psi_3\rangle$ when f is either constant or balanced.

~~we~~ when f is constant, we get

$$|\psi_3\rangle = \sum_{z}\sum_{x} \frac{(-1)^{xz+c}}{2^n}|z\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \quad \text{where } c\in\{0,1\}$$

In this case, except ~~of~~ for $z = 0^{\otimes n}$, coefficients for all $|z\rangle$ vanish. Since $x\cdot z = 0$ and so $x\cdot z + c = c$.

ie $$|\psi_3\rangle = (-1)^c |0\rangle^{\otimes n}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

When f is balanced, then for $z = 0^{\otimes n}$, the amplitude $\sum_x (-1)^{f(x)} = 0$

$\therefore |\psi_3\rangle$ does not contain $|z = 0^{\otimes n}\rangle = |0\rangle^{\otimes n}$

Hence when we measure the first register, a result in $|0\rangle^{\otimes n}$ implies f is constant, otherwise f is balanced.