Final Exam

YOUR NAME: ___________________________

Circle the name of your recitation instructor:

Albert Claire Edmond Florent Nick

- You may use **two** $8.5 \times 11''$ sheets with notes in your own handwriting on both sides, but no other reference materials. Calculators are not allowed.
- You may assume all results presented in class.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.

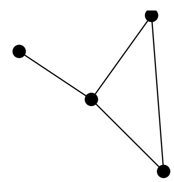
GOOD LUCK!

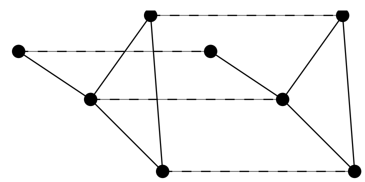
Problem	Points	Grade	Grader
1	13		
2	15		
3	12		
4	12		
5	12		
6	12		
7	12		
8	12		
Total	100		

Problem 1. [13 points] Give an inductive proof that the Fibonacci numbers F_n and F_{n+1} are relatively prime for all $n \geq 0$. The Fibonacci numbers are defined as follows:

$$F_0 = 0 \qquad F_1 = 1 \qquad F_n = F_{n-1} + F_{n-2} \quad \text{(for } n \geq 2\text{)}$$

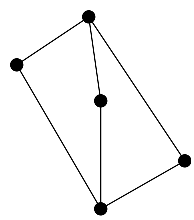
Problem 2. [15 points] The *double* of a graph G consists of two copies of G with edges joining corresponding vertices. For example, a graph appears below on the left and its double appears on the right.





Some edges in the graph on the right are dashed to clarify its structure.

(a) Draw the double of the graph shown below.



(b) Suppose that G_1 is a bipartite graph, G_2 is the double of G_1, G_3 is the double of G_2, and so forth. Use induction on n to prove that G_n is bipartite for all $n \geq 1$.

Problem 3. [12 points] *Finalphobia* is a rare disease in which the victim has the delusion that he or she is being subjected to an intense mathematical examination.

- A person selected uniformly at random has finalphobia with probability $1/100$.
- A person with finalphobia has shaky hands with probability $9/10$.
- A person without finalphobia has shaky hands with probability $1/20$.

What is the probablility that a person selected uniformly at random has finalphobia, given that he or she has shaky hands?

Problem 4. [12 points] Suppose that you roll five 6-sided dice that are fair and mutually independent. For the problems below, answers alone are sufficient, but we can award partial credit only if you show your work. Also, you do not need to simplify your answers; you may leave factorials, binomial coefficients, and arithmetic expressions unevaluated.

(a) What is the probability that all five dice show different values?

Example: $(1, 2, 3, 4, 5)$ is a roll of this type, but $(1, 1, 2, 3, 4)$ is not.

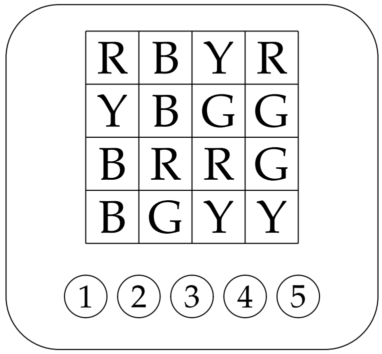
(b) What is the probability that two dice show the same value and the remaining three dice all show different values?

Example: $(6, 1, 6, 2, 3)$ is a roll of this type, but $(1, 1, 2, 2, 3)$ and $(4, 4, 4, 5, 6)$ are not.

(c) What is the probability that two dice show one value, two different dice show a second value, and the remaining die shows a third value?

Example: $(6, 1, 2, 1, 2)$ is a roll of this type, but $(4, 4, 4, 4, 5)$ and $(5, 5, 5, 6, 6)$ are not.

Problem 5. [12 points] An electronic toy displays a 4×4 grid of colored squares. At all times, four are red, four are green, four are blue, and four are yellow. For example, here is one possible configuration:



For parts (a) and (b) below, you need not simplify your answers.

(a) How many such configurations are possible?

(b) Below the display, there are five buttons numbered 1, 2, 3, 4, and 5. The player may press a sequence of buttons; however, the same button can not be pressed twice in a row. How many different sequences of n button-presses are possible?

(c) Each button press scrambles the colored squares in a complicated, but nonrandom way. Prove that there exist two *different* sequences of 32 button presses that both produce the *same* configuration, if the puzzle is initially in the state shown above. (Hint: $4^{32} = 16^{16} > 16!$)

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Problem 6. [12 points] MIT students sometimes delay laundry for a few days. Assume all random values described below are mutually independent.

(a) A ***busy*** student must complete 3 problem sets before doing laundry. Each problem set requires 1 day with probability $2/3$ and 2 days with probability $1/3$. Let B be the number of days a busy student delays laundry. What is $\text{Ex}(B)$?

Example: If the first problem set requires 1 day and the second and third problem sets each require 2 days, then the student delays for $B = 5$ days.

(b) A ***relaxed*** student rolls a fair, 6-sided die in the morning. If he rolls a 1, then he does his laundry immediately (with zero days of delay). Otherwise, he delays for one day and repeats the experiment the following morning. Let R be the number of days a relaxed student delays laundry. What is $\text{Ex}(R)$?

Example: If the student rolls a 2 the first morning, a 5 the second morning, and a 1 the third morning, then he delays for $R = 2$ days.

(c) Before doing laundry, an ***unlucky*** student must recover from illness for a number of days equal to the product of the numbers rolled on two fair, 6-sided dice. Let U be the expected number of days an unlucky student delays laundry. What is $\text{Ex}(U)$?

Example: If the rolls are 5 and 3, then the student delays for $U = 15$ days.

(d) A student is ***busy*** with probability $1/2$, ***relaxed*** with probability $1/3$, and ***unlucky*** with probability $1/6$. Let D be the number of days the student delays laundry. What is $\text{Ex}(D)$? *Leave your answer in terms of* $\text{Ex}(B)$, $\text{Ex}(R)$, *and* $\text{Ex}(U)$.

Problem 7. [12 points] I have twelve cards:

$$\boxed{1}\ \boxed{1}\quad \boxed{2}\ \boxed{2}\quad \boxed{3}\ \boxed{3}\quad \boxed{4}\ \boxed{4}\quad \boxed{5}\ \boxed{5}\quad \boxed{6}\ \boxed{6}$$

I shuffle them and deal them in a row. For example, I might get:

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{3}\ \boxed{4}\ \boxed{6}\ \boxed{1}\ \boxed{4}\ \boxed{5}\ \boxed{5}\ \boxed{2}\ \boxed{6}$$

What is the expected number of adjacent pairs with the same value? In the example, there are two adjacent pairs with the same value, the 3's and the 5's.

We can award partial credit only if you show your work.

Problem 8. [12 points] Each time a baseball player bats, he hits the ball with some probability. The table below gives the hit probability and number of chances to bat next season for five players.

player	prob. of hit	# chances to bat
Player A	$1/3$	300
Player B	$1/4$	200
Player C	$1/4$	400
Player D	$1/5$	250
Player E	$2/5$	500

(a) Let X be the total number times these five players hit the ball next season. What is $\mathrm{Ex}(X)$?

(b) Give a nontrivial upper bound on $\Pr(X \geq 1500)$ and justify your answer. *Do not* assume that hits happen mutually independently.

(c) Using a Chernoff inequality, give a nontrivial upper bound on $\Pr(X \leq 400)$. For this part, you *may* assume that all hits happen mutually independently.