Midterm

Name: ______________________________

Circle the name of your recitation instructor:

David Darren Martyna Nick Oscar Stav

- This quiz is **closed book**, but you may have one $8.5 \times 11''$ sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- The exam ends at 9:30 PM.

Problem	Points	Grade	Grader
1	10		
2	15		
3	20		
4	15		
5	18		
6	20		
7	10		
8	12		
Total	120		

Problem 1. [10 points]

Consider these two propositions:

$$\texttt{P:}\quad (A \vee B) \Rightarrow C$$

$$\texttt{Q:}\quad (\neg C \Rightarrow \neg A) \vee (\neg C \Rightarrow \neg B)$$

Use the truth table below to show whether P and Q are equivalent, $P \Rightarrow Q$, $Q \Rightarrow P$, or none of the above.

A	B	C					

Problem 2. [15 points]

Let $G_0 = 1$, $G_1 = 3$, $G_2 = 9$, and define

$$G_n = G_{n-1} + 3G_{n-2} + 3G_{n-3} \tag{1}$$

for $n \geq 3$. Show by induction that $G_n \leq 3^n$ for all $n \geq 0$.

Problem 3. [20 points]

In the game of Squares and Circles, the players (you and your computer) start with a sequence of shapes: some circles and some squares. On each move a player chooses any two shapes from the sequence. These two are replaced with a single one according to the following rule:

Identical shapes are replaced with a square. Different shapes are replaced with a circle.

At the end of the game, when only one shape remains, you are a winner if the remaining shape is a circle. Otherwise, your computer wins.

(a) [5 pts] Prove that the game will end.

(b) [15 pts] Prove that you will win if the number of circles initially is odd. Hint: Use an invariant about the number of circles.

Problem 4. [15 points]

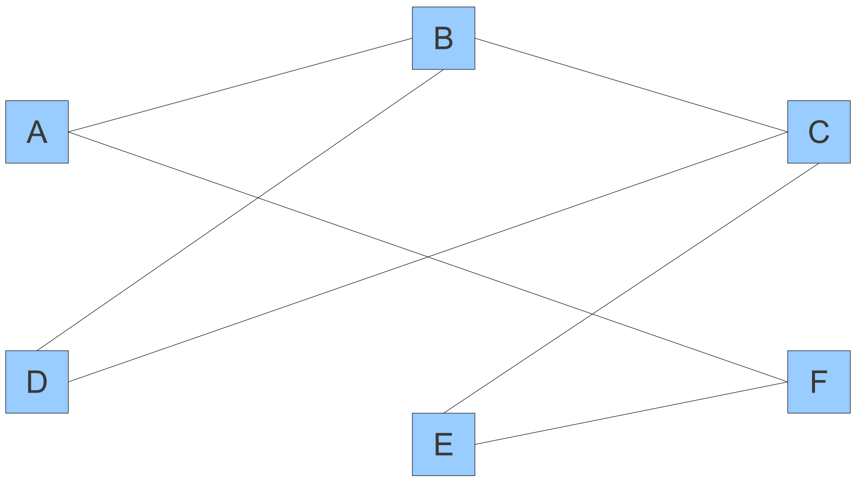
(a) [8 pts] Find a number $x \in \{0, 1, \ldots, 112\}$ such that $18x \equiv 1 \pmod{113}$.

(b) [7 pts] Find a number $y \in \{0, 1, \ldots, 112\}$ such that $18^{112111} \equiv y \pmod{113}$ (*Hint: x is congruent to which power of* 18*, modulo* 113*?*)

Problem 5. [18 points]

Consider the simple graph G given in figure 1.

Figure 1: Simple graph G



(a) [3 pts] Give the diameter of G.

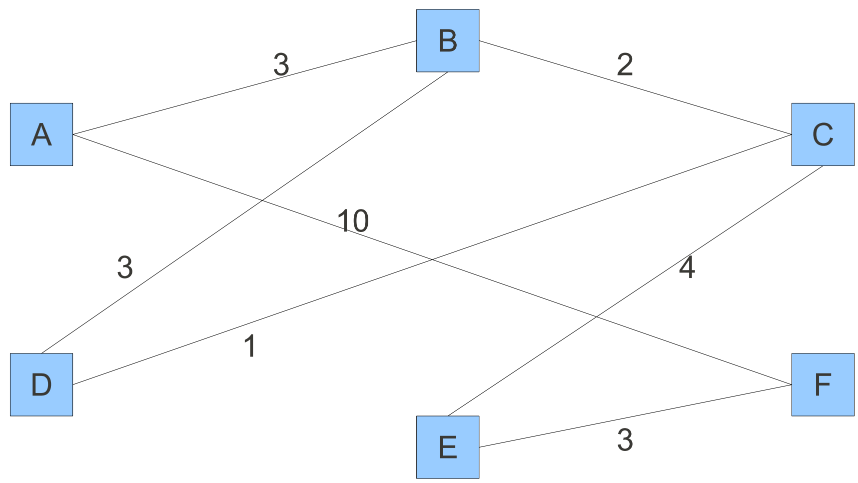
(b) [3 pts] Give a longest path on G.

(c) [3 pts] Give a coloring on G and show that it uses the smallest possible number of colors.

(d) [3 pts] Does G have an Eulerian cycle? Justify your answer.

Now consider graph H, which is like G but with weighted edges, in figure 2:

Figure 2: Weighted graph H



(e) [3 pts] Draw a minimum spanning tree on H.

(f) [3 pts] Give a list of edges reflecting the order in which a greedy algorithm would choose edges when finding an MST on H.

Problem 6. [20 points] Let G be a graph with n vertices, m edges and k components. Prove that G contains at least $m + k - n = c$ cycles. (Hint: Prove this by induction on the number of edges, m)

Problem 7. [10 points] Use integration to find upper and lower bounds that differ by at most 1 for the following sum.

$$\sum_{i=1}^{\infty} \frac{1}{i^4}$$

Problem 8. [12 points] Give a proof of the following propositions.

(a) [3 pts] x is $O(x \ln x)$

(b) [3 pts] $x / \ln x$ is $o(x)$

(c) [3 pts] x^{n+1} is $\Omega(x^n)$

(d) [3 pts] $n!$ is $\Theta\left(\frac{2\pi n! \ln n}{H_n}\right)$ Where H_n is the n^{th} Harmonic number.

(Notes)