Problem Set 2 Solutions

Due: Monday, September 20 (7pm)

Problem 1. [12 points] Define a *3-chain* to be a (not necessarily contiguous) subsequence of three integers, which is either monotonically increasing or monotonically decreasing. We will show here that any sequence of five distinct integers will contain a *3-chain*. Write the sequence as a_1, a_2, a_3, a_4, a_5. Note that a monotonically increasing sequences is one in which each term is greater than or equal to the previous term. Similarly, a monotonically decreasing sequence is one in which each term is less than or equal to the previous term. Lastly, a subsequence is a sequence derived from the original sequence by deleting some elements without changing the location of the remaining elements.

(a) [4 pts] Assume that $a_1 < a_2$. Show that if there is no *3-chain* in our sequence, then a_3 must be less than a_1. (Hint: consider a_4!)

Solution. We first assume that $a_1 < a_2$. Now consider where a_3 must be placed. If $a_3 > a_2$, then $a_1 < a_2 < a_3$, so a *3-chain* is created. So $a_3 < a_2$. Now, consider the case where $a_3 > a_1$ as well. In that case, there is no place for a_4 to go without creating a *3-chain*. If $a_4 < a_3$, then $a_4 < a_3 < a_2$, so a *3-chain* is made. If $a_4 > a_3$, then $a_1 < a_3 < a_4$, so a *3-chain is formed as well. So a_3 must be less than a_1.* ■

(b) [2 pts] Using the previous part, show that if $a_1 < a_2$ and there is no *3-chain* in our sequence, then $a_3 < a_4 < a_2$.

Solution. From the previous part, we have that $a_3 < a_1 < a_2$. In this case, a_4 cannot be less than a_3, because then $a_4 < a_3 < a_2$. Similarly, a_4 cannot be greater than a_2, because then $a_1 < a_2 < a_4$. So we conclude that $a_3 < a_4 < a_2$. ■

(c) [2 pts] Assuming that $a_1 < a_2$ and $a_3 < a_4 < a_2$, show that any value of a_5 must result in a *3-chain*.

Solution. Consider how a_5 compares to the other numbers in the sequence. If $a_5 < a_4$, then $a_5 < a_4 < a_2$, and a *3-chain* is formed. If $a_5 > a_4$, then $a_3 < a_4 < a_5$, so a *3-chain* is formed again. And finally, a_5 cannot equal a_4, because it was stipulated that the numbers were all distinct. So any value of a_5 must result in a *3-chain*. ■

(d) [4 pts] Using the previous parts, prove by contradiction that any sequence of five distinct integers must contain a *3-chain*.

Solution. From the previous parts, we see that assuming that $a_1 < a_2$, any choice of a_5 must result in a *3-chain*. However, the other case, that $a_1 > a_2$, results in the same argument; one only needs to reverse the $>$ and $<$ signs. So any choice of a_5 results in a *3-chain* in that case as well. What we have been doing is assuming that we could construct such a five-integer sequence, and going through the steps to construct it. Because the only results from such a construction contain *3-chains*, we reach a contradiction, and conclude that all sequences of five distinct integers must contain a *3-chain*. ∎

Problem 2. [8 points]

Prove by either the Well Ordering Principle or induction that for all nonnegative integers, n:

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{1}$$

Solution. Proof by Well Ordering Principle

Proof. The proof is by contradiction and use of the Well Ordering Principle. Assume that the theorem is false. Then, some nonnegative integers serve as counterexamples to it. Lets collect these counterexamples in a set: $C ::= \{n \in \mathbb{N} | \sum_{i=0}^{n} i^3 \neq \left(\frac{n(n+1)}{2}\right)^2\}$.

By our assumption that the theorem admits counterexamples, C is a nonempty set of nonnegative integers. So, by the Well Ordering Principle, C has a minimum element, call it c. That is, c is the smallest counterexample to the theorem.

Since c is the smallest counterexample, we know that equation (1) is false for $n = c$ but true for all nonnegative integers $n < c$. But equation (1) is true for $n = 0$ since $\sum_{i=0}^{0} i^3 = 0 = \left(\frac{0(0+1)}{2}\right)^2$. Hence $c > 0$. This means $c - 1$ is a nonnegative integer, and since it is less than c, equation (1) is true for $c - 1$. That is,

$$\sum_{i=0}^{c-1} i^3 = \left(\frac{(c-1)c}{2}\right)^2. \tag{2}$$

But then, adding c^3 to both sides of equation (2) gives us

$$\sum_{i=0}^{c} i^3$$

on the left hand side. And the right hand side now equals

$$
\begin{aligned}
\left(\frac{(c-1)c}{2}\right)^2 + c^3 &= \frac{(c-1)^2 c^2 + 4c^3}{2^2} \\
&= \frac{c^2\left((c-1)^2 + 4c\right)}{2^2} \\
&= \frac{c^2\left(c^2 - 2c + 1 + 4c\right)}{2^2} \\
&= \frac{c^2\left(c^2 + 2c + 1\right)}{2^2} \\
&= \frac{c^2(c+1)^2}{2^2} \\
&= \left(\frac{c(c+1)}{2}\right)^2.
\end{aligned}
$$

That is,

$$
\sum_{i=0}^{c} i^3 = \left(\frac{c(c+1)}{2}\right)^2,
$$

which means that equation 1 does hold for c, after all! This is a contradiction, and we are done.

$\square$

Proof by Induction

Proof. The proof is by induction on n. Let $P(n)$ be the proposition that equation 1 holds.

Base case: $P(0)$ is true because

$$
\sum_{i=0}^{0} i^3 = 0 = \left(\frac{0(0+1)}{2}\right)^2.
$$

Inductive step: Assume $P(n)$ is true, that is $\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Then we can prove

$P(n+1)$ is also true as follows:

$$\sum_{i=0}^{n+1} i^3 = \sum_{i=0}^{n} i^3 + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{2^2}$$

$$= \frac{(n+1)^2 \left(n^2 + 4(n+1)\right)}{2^2}$$

$$= \frac{(n+1)^2 \left(n^2 + 4n + 4\right)}{2^2}$$

$$= \frac{(n+1)^2(n+2)^2}{2^2}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2$$

The first step breaks up the sum. The second step uses the assumption $P(n)$. The rest of the steps are algebraic simplifications.

Thus, $P(0)$ is true and $P(n)$ implies $P(n+1)$ for all nonnegative integers. Therefore, $P(n)$ is true for all nonegative integers by the principle of induction.

$\square$

■

Problem 3. [20 points] The following problem is fairly tough until you hear a certain one-word clue. The solution is elegant but is slightly tricky, so don't hesitate to ask for hints!

During 6.042, the students are sitting in an $n \times n$ grid. A sudden outbreak of beaver flu (a rare variant of bird flu that lasts forever; symptoms include yearning for problem sets and craving for ice cream study sessions) causes some students to get infected. Here is an example where $n = 6$ and infected students are marked ×.
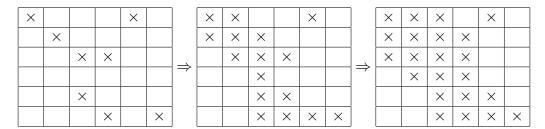
×				×	
	×				
		×	×		
		×			
			×		×

Now the infection begins to spread every minute (in discrete time-steps). Two students are considered *adjacent* if they share an edge (i.e., front, back, left or right, but NOT diagonal); thus, each student is adjacent to 2, 3 or 4 others. A student is infected in the next time step if either

- the student was previously infected (since beaver flu lasts forever), or

- the student is adjacent to *at least two* already-infected students.

In the example, the infection spreads as shown below.

×				×	
	×				
		×	×		
		×			
			×		×

$\Rightarrow$

×	×			×	
×	×	×			
	×	×	×		
			×		
		×	×		
		×	×	×	×

$\Rightarrow$

×	×	×		×	
×	×	×	×		
×	×	×	×		
	×	×	×		
		×	×	×	
		×	×	×	×

In this example, over the next few time-steps, all the students in class become infected.

Theorem. *If fewer than n students in class are initially infected, the whole class will never be completely infected.*

Prove this theorem.

Hint: When one wants to understand how a system such as the above "evolves" over time, it is usually a good strategy to (1) identify an appropriate property of the system at the initial stage, and (2) prove, by induction on the number of time-steps, that the property is preserved at every time-step. So look for a property (of the set of infected students) that remains invariant as time proceeds.

If you are stuck, ask your recitation instructor for the one-word clue and even more hints!

Solution. *Proof.* Define the *perimeter* of an infected set of students to be the number of edges with infection on exactly one side. Let I denote the perimeter of the initially-infected set of students.

Now we use induction on the number of time steps to prove that the perimeter of the infected region never increases. Let $P(k)$ be the proposition that after k time steps, the perimeter of the infected region is at most I.

Base case: $P(0)$ is true by definition; the perimeter of the infected region is at most I after 0 time steps, because I is defined to be the perimeter of the initially-infected region.

Inductive step: Now we must show that $P(k)$ implies $P(k+1)$ for all $k \geq 0$. So assume that $P(k)$ is true, where $k \geq 0$; that is, the perimeter of the infected region is at most I after k steps. The perimeter can only change at step $k+1$ because some squares are newly infected. By the rules above, each newly-infected square is adjacent to at least two previously-infected squares. Thus, for each newly-infected square, at least two edges are removed from the perimeter of the infected region, and at most two edges are added to the perimeter. Therefore, the perimeter of the infected region can not increase and is at most I after $k+1$ steps as well. This proves that $P(k)$ implies $P(k+1)$ for all $k \geq 0$.

By the principle of induction, $P(k)$ is true for all $k \geq 0$.

If an $n \times n$ grid is completely infected, then the perimeter of the infected region is $4n$. Thus, the whole grid can become infected only if the perimeter is initially at least $4n$. Since each square has perimeter 4, at least n squares must be infected initially for the whole grid to be infected. □

The above proof shows that if initially k students are infected, then the perimeter of the infected region will never exceed $4k$. The largest number of students that can be contained within a region with perimeter $\leq 4k$ is equal to k^2, therefore, if k students in class are initially infected, then at most k^2 students will eventually be infected. This feels intuitively true after having done the previous proof. However, to give a formal proof requires some case analysis (try it!). ■

Problem 4. [10 points] Find the flaw in the following *bogus* proof that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Proof. The *bogus* proof is by induction on n, with hypothesis

$$P(n) ::= \forall k \leq n. \, a^k = 1,$$

where k is a nonnegative integer valued variable.

Base Case: $P(0)$ is equivalent to $a^0 = 1$, which is true by definition of a^0. (By convention, this holds even if $a = 0$.)

Inductive Step: By induction hypothesis, $a^k = 1$ for all $k \in \mathbb{N}$ such that $k \leq n$. But then

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1,$$

which implies that $P(n+1)$ holds. It follows by induction that $P(n)$ holds for all $n \in \mathbb{N}$, and in particular, $a^n = 1$ holds for all $n \in \mathbb{N}$. □

Solution. *NOTE TO GRADERS* A student may decide the flaw is in the base case because by extending the basecase to include $P(0)$ and $P(1)$ we could get a valid base case/induction step combination. If this is their argument, then mark it as correct.

The more solid argument though is the following, since the inductive hypothesis was given in 'strong form':

The flaw comes in the *inductive step*, where we implicitly assume $n \geq 1$ in order to talk about a^{n-1} in the denominator (otherwise the exponent is not a nonnegative integer, so we cannot apply the inductive hypothesis). The inductive step must work for all n but in this case it does not. We checked the base case only for $n = 0$, so we are not justified in assuming that $n \geq 1$ when we try to prove the statement for $n + 1$ in the inductive step. And of course the proposition first breaks precisely at $n = 1$. ■

Problem 5. [10 points] Let the sequence $G_0, G_1, G_2, \ldots$ be defined recursively as follows: $G_0 = 0$, $G_1 = 1$, and $G_n = 5G_{n-1} - 6G_{n-2}$, for every $n \in \mathbb{N}, n \geq 2$.

Prove that for all $n \in \mathbb{N}$, $G_n = 3^n - 2^n$.

Solution. *Proof.* The proof is by strong induction on n. Let $P(n)$ be the proposition that $G_n = 3^n - 2^n$.

Base case: $P(0)$ is true because $G_0 = 0$ and $3^0 - 2^0 = 1 - 1 = 0$. $P(1)$ is true because $G_1 = 1$ and $3^1 - 2^1 = 3 - 2 = 1$.

Inductive step: Let $n \geq 2$ and suppose that $P(k)$ is true for $0 \leq k \leq n$. That is, assume that for every k, $1 \leq k \leq n$, $G_k = 3^k - 2^k$. Then by definition of the sequence $G_{n+1} = 5G_n - 6G_{n-1}$. By our induction hypothesis, $G_n = 3^n - 2^n$ and $G_{n-1} = 3^{n-1} - 2^{n-1}$. Thus, $G_{n+1} = 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1})$. Simplifying we get,

$$
\begin{aligned}
G_{n+1} &= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) \\
&= 5(3 \cdot 3^{n-1} - 2 \cdot 2^{n-1}) - 6(3^{n-1} - 2^{n-1}) \\
&= 15 \cdot 3^{n-1} - 10 \cdot 2^{n-1}) - 6 \cdot 3^{n-1} + 6 \cdot 2^{n-1} \\
&= 9 \cdot 3^{n-1} - 4 \cdot 2^{n-1} \\
&= 3^{n+1} - 2^{n+1}
\end{aligned}
$$

So, $G_{n+1} = 3^{n+1} - 2^{n+1}$, as needed for the inductive step.

This completes the proof.

$\square$

$\blacksquare$

Problem 6. [25 points]

In the 15-puzzle, there are 15 lettered tiles and a blank square arranged in a 4×4 grid. Any lettered tile adjacent to the blank square can be slid into the blank. For example, a sequence of two moves is illustrated below:

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

$\rightarrow$

A	B	C	D
E	F	G	H
I	J	K	L
M	O		**N**

$\rightarrow$

A	B	C	D
E	F	G	H
I	J		L
M	O	**K**	N

In the leftmost configuration shown above, the O and N tiles are out of order. Using only legal moves, is it possible to swap the N and the O, while leaving all the other tiles in their original position and the blank in the bottom right corner? In this problem, you will prove the answer is "no".

Theorem. *No sequence of moves transforms the board below on the left into the board below on the right.*

A	B	C	D
E	F	G	H
I	J	K	L
M	**O**	**N**	

A	B	C	D
E	F	G	H
I	J	K	L
M	**N**	**O**	

(a) [3 pts] We define the "order" of the tiles in a board to be the sequence of tiles on the board reading from the top row to the bottom row and from left to right within a row. For example, in the right board depicted in the above theorem, the order of the tiles is A, B, C, D, E, etc.

Can a row move change the order of the tiles? Prove your answer.

> **Solution.** No. A row move moves a tile from cell i to cell $i+1$ or vice versa. This tile does not change its order with respect to any other tile. Since no other tile moves, there is no change in the order of any of the other pairs of tiles. ∎

(b) [3 pts] How many pairs of tiles will have their relative order changed by a column move? More formally, for how many pairs of letters L_1 and L_2 will L_1 appear earlier in the order of the tiles than L_2 before the column move and later in the order after the column move? Prove your answer correct.

> **Solution.** A column move changes the relative order of exactly three pairs of tiles. Sliding a tile down moves it after the next three tiles in the order. Sliding a tile up moves it before the previous three tiles in the order. Either way, the relative order changes between the moved tile and each of the three it crosses. ∎

(c) [3 pts] We define an *inversion* to be a pair of letters L_1 and L_2 for which L_1 precedes L_2 in the alphabet, but L_1 appears after L_2 in the order of the tiles. For example, consider the following configuration:

A	B	C	E
D	H	G	F
I	J	K	L
M	N	O	

There are exactly four inversions in the above configuration: E and D, H and G, H and F, and G and F.

What effect does a row move have on the parity of the number of inversions? Prove your answer.

> **Solution.** A row move never changes the parity of the number of inversions. A row move does not change the order of the tiles, so it does not affect the total number of inversions. ∎

(d) [5 pts] What effect does a column move have on the parity of the number of inversions? Prove your answer.

> **Solution.** A column move always changes the parity of the number of inversions. A column move changes the relative order of exactly three pairs of tiles. An inverted pair becomes uninverted and vice versa. Thus, one exchange flips the total number of inversions to the opposite parity, a second exhange flips it back to the original parity, and a third exchange flips it to the opposite parity again. ∎

(e) [8 pts] The previous problem part implies that we must make an *odd* number of column moves in order to exchange just one pair of tiles (N and O, say). But this is problematic, because each column move also knocks the blank square up or down one row. So after an *odd* number of column moves, the blank can not possibly be back in the last row, where it belongs! Now we can bundle up all these observations and state an *invariant*, a property of the puzzle that never changes, no matter how you slide the tiles around.

Lemma. *In every configuration reachable from the position shown below, the parity of the number of inversions is different from the parity of the row containing the blank square.*

<table>
<tr><td>row 1</td><td>A</td><td>B</td><td>C</td><td>D</td></tr>
<tr><td>row 2</td><td>E</td><td>F</td><td>G</td><td>H</td></tr>
<tr><td>row 3</td><td>I</td><td>J</td><td>K</td><td>L</td></tr>
<tr><td>row 4</td><td>M</td><td>O</td><td>N</td><td></td></tr>
</table>

Prove this lemma.

Solution. *Proof.* The proof is by induction. Let $P(n)$ be the proposition that after n moves, the parity of the number of inversions is different from the parity of the row containing the blank square.

Base case: After zero moves, exactly one pair of tiles is inverted (O and N), which is an odd number. And the blank square is in row 4, which is an even number. Therefore, $P(0)$ is true.

Inductive step: Now we must prove that $P(n)$ implies $P(n + 1)$ for all $n \geq 0$. So assume that $P(n)$ is true; that is, after n moves the parity of the number of inversions is different from the parity of the row containing the blank square. There are two cases:

1. Suppose move $n + 1$ is a row move. Then the parity of the total number of inversions does not change. The parity of the row containing the blank square does not change either, since the blank remains in the same row. Therefore, these two parities are different after $n + 1$ moves as well, so $P(n + 1)$ is true.

2. Suppose move $n+1$ is a column move. Then the parity of the total number of inversions changes. However, the parity of the row containing the blank square also changes, since the blank moves up or down one row. Thus, the parities remain different after $n + 1$ moves, and so $P(n + 1)$ is again true.

Thus, $P(n)$ implies $P(n + 1)$ for all $n \geq 0$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$. $\square$

■

(f) [3 pts] Prove the theorem that we originally set out to prove.

Solution. In the target configuration on the right, the total number of inversions is zero, which is even, and the blank square is in row 4, which is also even. Therefore, by the lemma, the target configuartion is unreachable. ■

Problem 7. [15 points] There are two types of creature on planet Char, Z-lings and B-lings. Furthermore, every creature belongs to a particular generation. The creatures in each generation reproduce according to certain rules and then die off. The subsequent generation consists entirely of their offspring.

The creatures of Char pair with a mate in order to reproduce. First, as many Z-B pairs as possible are formed. The remaining creatures form Z-Z pairs or B-B pairs, depending on whether there is an excess of Z-lings or of B-lings. If there are an odd number of creatures, then one in the majority species dies without reproducing. The number and type of offspring is determined by the types of the parents

- If both parents are Z-lings, then they have three Z-ling offspring.

- If both parents are B-lings, then they have two B-ling offspring and one Z-ling offspring.

- If there is one parent of each type, then they have one offspring of each type.

There are 200 Z-lings and 800 B-lings in the first generation. Use induction to prove that the number of Z-lings will always be at most twice the number of B-lings.

Hint: You may want to use a stronger hypothesis for the induction.

Solution. An induction proof with the hypothesis that the number of Z-lings is always at most twice the number of B-lings will not go through. In fact, you can check that if the number of Z-lings is larger than the one of B-lings, then the fraction of B-lings will decrease at every step. A stronger induction hypothesis is required.

Proof. We will prove that there are always at least as many B-lings as Z-lings; the claim that the number of Z-lings is always at most twice the number of B-lings is weaker and thus follows.

The proof is by induction on the generation number. Let $P(n)$ be the proposition that there are at least as many B-lings as Z-lings in generation n. In the base case, $P(1)$ is true because there are 800 B-lings and only 200 Z-lings in the first generation.

In the inductive step, for $n \geq 1$ assume that there are at least as many B-lings as Z-lings in generation n to prove that there are at least as many B-lings as Z-lings in generation $n + 1$. Let a be the number of Z-lings in generation n, and let b be the number of B-lings. Note that $b \geq a$ by the induction assumption. Then there are no Z-Z pairs formed, there are a Z-B pairs formed, and there are $\lfloor \frac{b-a}{2} \rfloor$ $B - B$ pairs formed. As a result, the number of Z-lings in generation $n + 1$ is $a + \lfloor \frac{b-a}{2} \rfloor$, and the number of B-lings is $a + 2\lfloor \frac{b-a}{2} \rfloor$. Therefore, there are at least as many B-lings as Z-lings in generation $n + 1$. Thus, for all $n \geq 1$, $P(n)$ implies $P(n + 1)$, and the claim is proved by induction. □

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