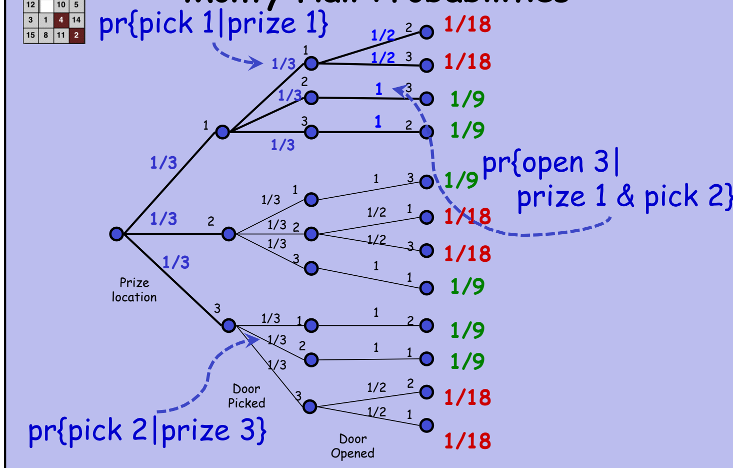
Mathematics for Computer Science

MIT 6.042J/18.062J

Conditional Probability

Monty Hall Probabilities



Conditional Probability

We were reasoning about conditional probability in the way we defined are probability space in the first place.

We were using:

Product Rule

$$\Pr\{A \cap B\} = \Pr\{A\} \cdot \Pr\{B \mid A\}$$

Conditional Probability

In fact, we use this reasoning to *define* conditional probability:

Conditional Probability

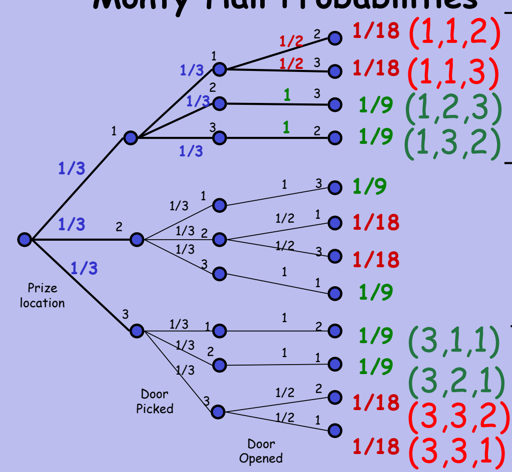
$\Pr\{B \mid A\}$ is the probability of event B, *given* that event A has occurred:

$$\Pr\{B \mid A\} ::= \frac{\Pr\{A \cap B\}}{\Pr\{A\}}$$

Product Rule

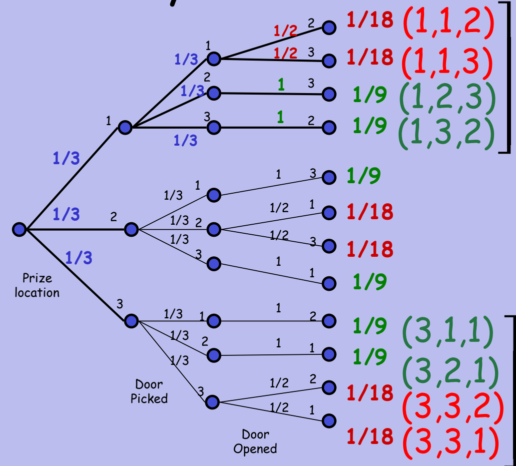
$$\Pr\{A \cap B \cap C\} = \Pr\{A\} \cdot \Pr\{B \mid A\} \cdot \Pr\{C \mid A \cap B\}$$

Monty Hall Probabilities



goat at 2

Monty Hall Probabilities



prize at 1

goat at 2

Conditional Probability: Monty Hall

Pr{ prize at 1 | goat at 2} = 1/2

Outcomes: Really!

(Prize Door, Picked Door, Carol door)

[goat at 2] =

{ (1,1,2),(1,1,3), (1,2,3),(1,3,2)
(3,3,1),(3,3,2),(3,1,2),(3,2,1)}

Conditional Probability: Monty Hall

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Conditional Probability: Monty Hall

Pr{ prize at 1 | Carol opens 2} = 1/2

Outcomes:

(Prize Door, Picked Door, Carol door)

[Carol opens 2] =

{ (1,1,2), (1,3,2),
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Conditional Probability: Monty Hall

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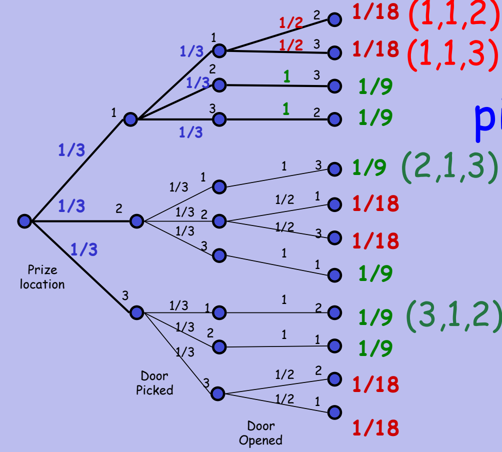
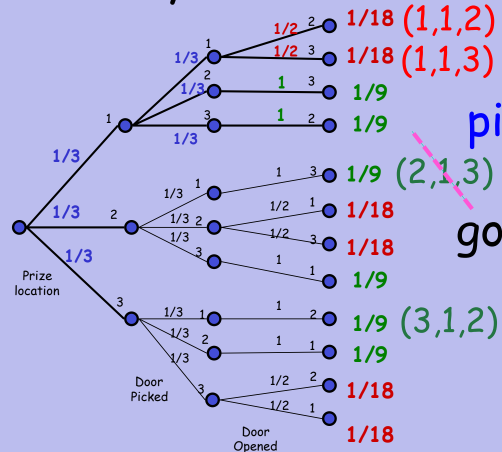
Monty Hall Probabilities



picked 1

Monty Hall Probabilities



picked 1 & goat at 2

Conditional Probability: Monty Hall

Pr{ prize at 1 | picked 1 & goat at 2} $= \frac{1}{2}$ Really!

[picked 1 & goat at 2] =

{ (1,1,2), (1,1,3), (3,1,2)}

Conditional Probability: Monty Hall

Pr{ prize at 1 | picked 1 & goat at 2} $= \frac{1}{2}$

[picked 1 & goat at 2] =

$\{ \underbrace{(1,1,2), (1,1,3)}_{\text{prize at 1}}, (3,1,2)\}$

Conditional Probability: Monty Hall

Pr{ prize at 1 | picked 1 & goat at 2} $= \frac{1}{2}$

[picked 1 & goat at 2] =

$\{ \underbrace{(1,1,2)}_{pr=1/18}, \underbrace{(1,1,3)}_{pr=1/18}, \underbrace{(3,1,2)}_{pr=1/9}\}$

Conditional Probability: Monty Hall

Seems the contestant may as well **stick**, since the probability is 1/2 given what he knows when he chooses. But wait, contestant knows more than goat at 2: he knows

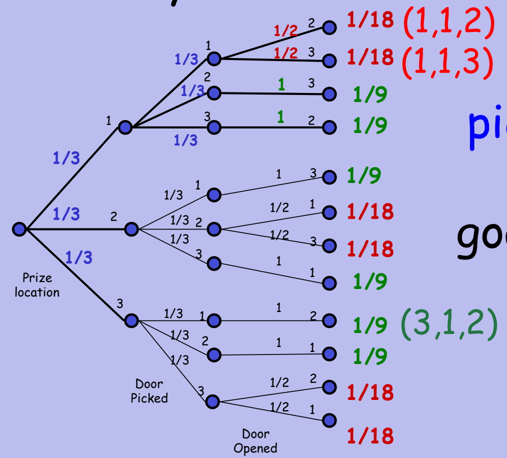
Carol opened door 2!

Conditional Probability: Monty Hall

So until now, we have been conditioning on the wrong events —a common blunder.

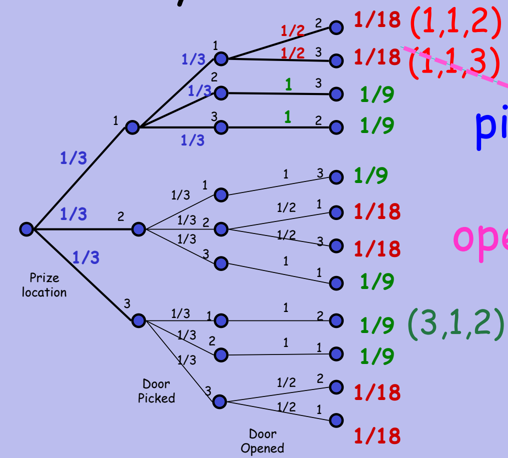
Using the correct one:

Monty Hall Probabilities



picked 1
&
goat at 2

Monty Hall Probabilities



picked 1
&
opened 2

Conditional Probability: Monty Hall

Pr{ | picked 1 &
Carol opened 2}

[picked 1 & Carol opened 2] =
{ (1,1,2),(3,1,2) }

Conditional Probability: Monty Hall

Pr{ prize at 1 | picked 1 &
Carol opened 2}

[picked 1 & Carol opened 2] =
{ $\underbrace{(1,1,2)}_{\text{Pr}=1/18}$,(3,1,2) }

Conditional Probability: Monty Hall

$$\Pr\{\text{prize at } 1 \mid \text{picked } 1 \ \& \ \text{Carol opened } 2\} = \boxed{1/3}$$

[picked 1 & Carol opened 2] =

$$\{ \underbrace{(1,1,2)}_{\Pr=1/18}, \underbrace{(3,1,2)}_{\Pr=1/9} \}$$

$$\frac{1/18}{1/18 + 1/9} =$$

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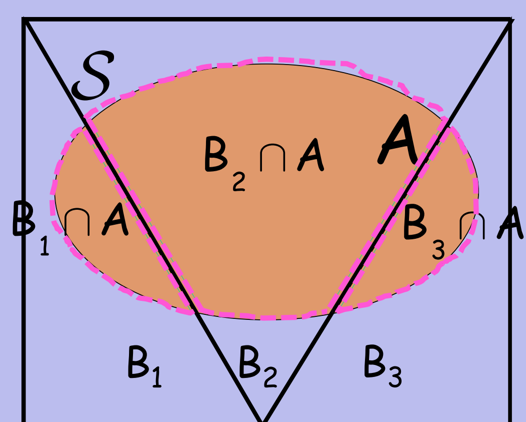
Law of Total Probability

Law of Total Probability

Law for reasoning about probability by cases

Law of Total Probability



Law of Total Probability

$$A = (B_1 \cap A) \cup (B_2 \cap A) \cup (B_3 \cap A)$$

$$\boxed{\begin{aligned}\Pr\{A\} = \Pr\{B_1 \cap A\} &+ \\ \Pr\{B_2 \cap A\} &+ \\ \Pr\{B_3 \cap A\}&\end{aligned}}$$

Law of Total Probability

$$A = (B_1 \cap A) \cup (B_2 \cap A) \cup (B_3 \cap A)$$

$$\begin{aligned}\Pr\{A\} = &\Pr\{A \mid B_1\}\Pr\{B_1\} \\ &+ \Pr\{A \mid B_2\}\Pr\{B_2\} \\ &+ \Pr\{A \mid B_3\}\Pr\{B_3\}\end{aligned}$$

Law of Total Probability

If $\mathcal{S}$ is disjoint union of $B_0, B_1, \ldots$

$$\Pr\{A\} = \sum_i \Pr\{A \cap B_i\}$$

$$= \sum_i \Pr\{A \mid B_i\} \cdot \Pr\{B_i\}$$

TB testing by cases

$$\Pr[\text{test positive} \mid \text{TB}] = 1$$

$$\Pr[\text{test negative} \mid \text{not TB}] = \frac{99}{100}$$

$$\Pr[\text{TB}] = \frac{1}{10{,}000}$$

TB testing by cases

$$\Pr[\text{test negative} \mid \text{not TB}] = \frac{99}{100}$$

$$\Pr\left[\text{false positive}\right] = \frac{1}{100}$$

TB testing by cases

$$\Pr[+] = \Pr[+ \mid TB] \cdot \Pr[TB] + \Pr[+ \mid \text{not } TB] \cdot \Pr[\text{not } TB]$$

$$\Pr[+] = 1 \cdot \frac{1}{10,000} + \frac{1}{100} \cdot \frac{9,999}{10,000}$$

$$\approx \frac{1}{100}$$ —dominated by false positive rate

TB testing by cases

$$\Pr[TB \mid +] = \frac{\Pr[TB \text{ AND } +]}{\Pr[+]}$$

TB testing by cases

$$\Pr[TB \mid +] = \frac{\Pr[+ \mid TB] \cdot \Pr[TB]}{\Pr[+]}$$

TB testing by cases

$$\Pr[TB \mid +] = \frac{\Pr[+ \mid TB] \cdot \Pr[TB]}{\Pr[+]}$$

$$\approx \frac{1 \cdot \frac{1}{10,000}}{\frac{1}{100}} = \frac{1}{100}$$

TB testing by cases

So, because of relatively high false positive rate compared to TB rate,

TB testing by cases

So, because of relatively high false positive rate compared to TB rate, chance of having TB even when a 99% accurate test says so remains small (1%)!

Bayes Rule

$$\Pr[TB \mid +] = \frac{\Pr[+ \mid TB] \cdot \Pr[TB]}{\Pr[+]}$$

$$\Pr[B \mid A] = \frac{\Pr[A \mid B] \cdot \Pr[B]}{\Pr[A]}$$

Independent Events

Independent Events

Definition 1:

Events A and B are independent iff

$$\Pr\{A\} = \Pr\{A \mid B\}.$$

Definition 2:

Events A and B are independent iff

$$\Pr\{A\} \cdot \Pr\{B\} = \Pr\{A \cap B\}.$$

Definitions of Independence

proof of equivalence:

$$\Pr\{A\} = \Pr\{A \mid B\} \quad \text{iff}$$

$$\Pr\{A\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}} \quad \text{iff}$$

$$\Pr\{A\} \cdot \Pr\{B\} = \Pr\{A \cap B\}.$$

Definitions of Independence

need $\Pr\{B\} \neq 0$ for Def. 1.

Def. 2 always works:

$$\boxed{\Pr\{A\} \cdot \Pr\{B\} = \Pr\{A \cap B\}}$$

Independence

$Pr\{A\} \cdot Pr\{B\} = Pr\{A \cap B\}$

symmetric in A and B so,

A independent of B iff

B independent of A.

Independence

Corollary: If $Pr\{B\} = 0$, then B is independent of every event – even itself.

Independence

Quickies:

Reflexive?

Transitive?

Intuition for Symmetry?

Independence

A independent of B means A is independent of whether or not B occurs:

Independence

Lemma:

A independent of B iff

A independent of $\overline{B}$.

Simple proof using:

$$\Pr\{A-B\} = \Pr\{A\}-\Pr\{A\cap B\}.$$

Independence

Lemma:

$$\Pr\{A \cap \overline{B}\} = \Pr\{A - B\} =$$
$$\Pr\{A\} - \Pr\{A \cap B\} =$$
$$\Pr\{A\} - \Pr\{A\} \cdot \Pr\{B\} =$$
$$\Pr\{A\} \cdot (1 - \Pr\{B\}) =$$
$$\Pr\{A\} \cdot \Pr\{\overline{B}\}$$

Independent Events?

B: Baby born at Mass General Hospital between 1:00AM and 1:01AM.

F: Jupiter's moon IO is full.

Independent Events?

Does event B (baby born) have anything to do with event F (IO is full)?

Independent Events?

Does event B (baby is born) have anything to do with event F (IO is full)?

of course not!

Babies & Full Moons

The events are

independent:

IO phase has *no effect* on birth frequency.

Babies & Full Moons

My sweet Aunt Daisy believed in Astrology. She thought celestial events could influence babies.

We might say "nonsense," there's no effect.

But Daisy might be right (for wrong reasons)

Babies & Full Moons

But there *is* an effect:

IO full and IO "new" are different distances from Earth.

C:\42\pub\jup-radio_070115.htm

** INFORMATION FOR AMATEUR RADIO ASTRONOMERS ** JUPITER DECAMETRIC EMISSIONS ** JUPITER EPHEMERIS 01 Jul 1994, 0000UTC, Julian Day: 2449534.5, GMT Sidereal Time: 18h35m17s ....

C:\42\pub\jup-radio_070115.htm

SUMMARY: Jupiter's HF emissions are ...heard on earth when Jupiter's magnetic field "sweeps" the earth every 9h55m27s and at other times when Io's geometric position influences activity.

Babies & Full Moons

influence of IO's magnetic field changes with phases! --might affect radios in ambulances, for example

Babies & Full Moons

So independence of B and F is actually unclear.

Deciding whether to treat them as independent is a matter of experiment, not Mathematics.

Babies & Full Moons

have to compare

- all daily birth statistics
- daily birth statistics when IO was full,

to see if different

Babies & Full Moons

if baby frequency 1:00--01AM is same when IO is full:

Pr{Baby born 1AM | IO is full}

= Pr{Baby born 1AM},

then B and F are independent

2-way Independence

Events $E_1, E_2, \ldots$ are

2-way independent

iff every pair of them are independent

Independent Product Rule

Sets $A_1, A_2, \ldots, A_n$ satisfy the independent product rule

when

$$\Pr\{A_1 \cap A_2 \cap \cdots \cap A_n\} = \Pr\{A_1\} \cdot \Pr\{A_2\} \cdot \cdots \cdot \Pr\{A_n\}$$

k-way Independence

Events $A_1, A_2, \ldots$ are

k-way independent

iff every collection of $\leq k$

of them satisfies the

independent product rule.

k-way Independence

Events $E_1, E_2, \ldots$ are

k-way independent

iff every they are

(k-1)-way independent

and $\Pr\{E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\} =$

$\Pr\{E_{i_1}\} \cdot \Pr\{E_{i_2}\} \cdots \Pr\{E_{i_k}\}$

2-way vs 3-way independence

make independent flips of

3 fair coins.

A ::= [coin 1 matches coin 2]

B ::= [coin 1 matches coin 3]

C ::= [coin 2 matches coin 3]

A,B,C are 2-way independent

but not 3-way

2-way vs 3-way Independence

choose values $v_1, v_2, \ldots$ independently with = probability.

for events $[v_m = v_n]$:

any 2 are independent

but $[v_1 = v_2]$, $[v_2 = v_3]$, $[v_1 = v_3]$

not 3-way indep.

Mutual Independence

events $A_1, A_2, \ldots, A_n$ are

mutually independent

when they are n-way independent

$\left(\begin{array}{l}2^n-(n+1) \text{ equations} \\ \text{to check!}\end{array}\right)$

Mutual Independence

events $A_1, A_2, \ldots, A_n$ are

mutually independent

iff $\Pr\{A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\} =$

$\Pr\{A_{i_1}\} \cdot \Pr\{A_{i_2}\} \cdots \Pr\{A_{i_k}\}$

for all A_{i_j} $\left(\begin{array}{l}2^n-(n+1) \text{ equations} \\ \text{to check!}\end{array}\right)$