Notes for Recitation 22

1 Expected Value Rule for Functions of Random Variables

In lecture, we have computed the expectation of a function of a random variable without explicitly discussing the general rule. For example, yesterday we saw that the expectation of the square of the roll of a die is not equal to the square of the expectation of the roll. That is, if R is the outcome of a single roll of a die. Then $\text{Ex}(R^2) \neq \text{Ex}(R)^2$. We will now explicitly present the rule for computing the expection of a function of a random variable R.

Rule (Expected Value for the Function of a Random Variable). *Let R be a random variable, and let $f(R)$ be a function of R. Then, the expected value of the random variable $f(R)$ is given by*

$$\text{E}[f(R)] = \sum_{x \in Range(R)} f(x) \cdot \Pr\{R = x\}$$

2 Properties of Variance

In this problem we will study some properties of the variance and the standard deviation of random variables.

a. Show that for any random variable R, $\text{Var}[R] = \text{E}[R^2] - \text{E}^2[R]$.

Solution. Let $\mu = \text{E}[R]$. Then

$$
\begin{aligned}
\text{Var}[R] &= \text{E}\left[(R - \text{E}[R])^2\right] && \text{(Definition of variance)} \\
&= \text{E}\left[(R - \mu)^2\right] && \text{(def. of } \mu\text{)} \\
&= \text{E}\left[R^2 - 2\mu R + \mu^2\right] \\
&= \text{E}\left[R^2\right] - 2\mu\text{E}[R] + \mu^2 && \text{(linearity of expectation)} \\
&= \text{E}\left[R^2\right] - 2\mu^2 + \mu^2 && \text{(def. of } \mu\text{)} \\
&= \text{E}\left[R^2\right] - \mu^2 \\
&= \text{E}\left[R^2\right] - \text{E}^2[R]. && \text{(def. of } \mu\text{)}
\end{aligned}
$$

■

b. Show that for any random variable R and constants a and b, $\mathrm{Var}\,[aR + b] = a^2\,\mathrm{Var}\,[R]$. Conclude that the standard deviation of $aR+b$ is a times the standard deviation of R.

Solution. We will transform the left side into the right side. The first step is to expand $\mathrm{Var}\,[aR + b]$ using the alternate definition of variance.

$$\mathrm{Var}\,[aR + b] = \mathrm{E}\left[(aR + b)^2\right] - \mathrm{E}^2\,[aR + b].$$

We will work on the first term and then the second term. For the first term, note that by linearity of expectation,

$$\mathrm{E}\left[(aR + b)^2\right] = \mathrm{E}\left[a^2R^2 + 2abR + b^2\right] = a^2\,\mathrm{E}\left[R^2\right] + 2ab\,\mathrm{E}\,[R] + b^2. \qquad (1)$$

Similarly for the second term:

$$\mathrm{E}^2\,[aR + b] = (a\,\mathrm{E}\,[R] + b)^2 = a^2\mathrm{E}^2\,[R] + 2ab\,\mathrm{E}\,[R] + b^2. \qquad (2)$$

Finally, we subtract the expanded second term from the first.

$$
\begin{aligned}
\mathrm{Var}\,[aR + b] &= \mathrm{E}\left[(aR + b)^2\right] - \mathrm{E}^2\,[aR + b] &\text{(previous part)}\\
&= a^2\,\mathrm{E}\left[R^2\right] + 2ab\,\mathrm{E}\,[R] + b^2 -\\
&\quad (a^2\mathrm{E}^2\,[R] + 2ab\,\mathrm{E}\,[R] + b^2) &\text{(by (\textbf{??}) and (\textbf{??}))}\\
&= a^2\,\mathrm{E}\left[R^2\right] - a^2\mathrm{E}^2\,[R]\\
&= a^2(\mathrm{E}\left[R^2\right] - \mathrm{E}^2\,[R])\\
&= a^2\,\mathrm{Var}\,[R] &\text{(previous part)}
\end{aligned}
$$

Since the standard deviation of a random variable is the square root of the variance, the standard deviation of $aR + b$ is $\sqrt{a^2\,\mathrm{Var}\,[R]}$ which is just a times the standard deviation of R. ∎

c. Show that if R_1 and R_2 are independent random variables, then

$$\mathrm{Var}\,[R_1 + R_2] = \mathrm{Var}\,[R_1] + \mathrm{Var}\,[R_2].$$

Solution. We will transform the left side into the right side. We begin by applying the alternate definition of variance.

$$\mathrm{Var}\,[R_1 + R_2] = \mathrm{E}\left[(R_1 + R_2)^2\right] - \mathrm{E}^2\,[R_1 + R_2].$$

We will work on the first term and then the second term separately. For the first term, note

$$
\begin{aligned}
\mathrm{E}\left[(R_1 + R_2)^2\right] &= \mathrm{E}\left[R_1^2 + 2R_1R_2 + R_2^2\right]\\
&= \mathrm{E}\left[R_1^2\right] + \mathrm{E}\,[2R_1R_2] + \mathrm{E}\left[R_2^2\right]\\
&= \mathrm{E}\left[R_1^2\right] + 2\,\mathrm{E}\,[R_1]\,\mathrm{E}\,[R_2] + \mathrm{E}\left[R_2^2\right].
\end{aligned}
$$

First, we multiply out the squared expression. The second step uses linearity of expectation. In the last step, we break the expectation of the product $R_1 R_2$ into a product of expectations; this is where we use the fact that R_1 and R_2 are independent. Now we work on the second term.

$$
\begin{aligned}
\mathrm{E}^2 \left[R_1 + R_2 \right] &= \left(\mathrm{E} \left[R_1 \right] + \mathrm{E} \left[R_2 \right] \right)^2 \\
&= \mathrm{E}^2 \left[R_1 \right] + 2 \mathrm{E} \left[R_1 \right] \mathrm{E} \left[R_2 \right] + \mathrm{E}^2 \left[R_2 \right] .
\end{aligned}
$$

The first step uses linearity of expectation, and in the second step we multiply out the squared expression. Now we subtract the (expanded) second term from the first. Cancelling and rearranging terms, we find that

$$
\begin{aligned}
\mathrm{Var} \left[R_1 + R_2 \right] &= \left(\mathrm{E} \left[R_1^2 \right] - \mathrm{E}^2 \left[R_1 \right] \right) + \left(\mathrm{E} \left[R_2^2 \right] \right) - \mathrm{E}^2 \left[R_2 \right]) \\
&= \mathrm{Var} \left[R_1 \right] + \mathrm{Var} \left[R_2 \right] .
\end{aligned}
$$

∎

d. Give an example of random variables R_1 and R_2 for which

$$
\mathrm{Var} \left[R_1 + R_2 \right] \neq \mathrm{Var} \left[R_1 \right] + \mathrm{Var} \left[R_2 \right] .
$$

Solution. Suppose $R = R_1 = R_2$. If linearity of variance held, then $\mathrm{Var} \left[R + R \right] = \mathrm{Var} \left[R \right] + \mathrm{Var} \left[R \right]$. However, by part b, $\mathrm{Var} \left[R + R \right] = \mathrm{Var} \left[2R \right] = 4 \mathrm{Var} \left[R \right]$. This is only possible if $\mathrm{Var} \left[R \right] = 0$. If, say, we choose R to be the outcome of a fair coin flip, $\mathrm{Var} \left[R \right] \neq 0$. In fact, any R which holds at least 2 distinct values each with positive probability will do. ∎

e. Compute the variance and standard deviation of the Binomial distribution $H_{n,p}$ with parameters n and p.

Solution. We know that $H_{n,p} = \sum_{k=1}^{n} I_k$ where the I_k are mutually independent 0-1-valued variables with $\Pr \{ I_k = 1 \} = p$. The variance of I_k is $\mathrm{E} \left[I_k^2 \right] - \mathrm{E} \left[I_k \right]^2 = \mathrm{E} \left[I_k \right] - \mathrm{E} \left[I_k \right]^2 = \mathrm{E} \left[I_k \right] \left(1 - \mathrm{E} \left[I_k \right] \right) = p(1 - p)$. Thus, by linearity of variance, we have $\mathrm{Var} \left[H_{n,p} \right] = n \mathrm{Var} \left[I_k \right] = np(1 - p)$. Thus, the standard deviation of $H_{n,p}$ is $\sqrt{np(1 - p)}$. ∎

f. Let's say we have a random variable T such that $T = \sum_{j=1}^{n} T_j$, where all of the T_j's are mutually independent and take values in the range $[0, 1]$. Prove that $\mathrm{Var}(\mathrm{T}) \leq \mathrm{Ex}(\mathrm{T})$. We'll use this result in lecture tomorrow. *Hint: Upper bound* $\mathrm{Var} \left[T_j \right]$ *with* $\mathrm{E} \left[T_j \right]$ *using the definition of variance in part (a) and the rule for computing the expectation of a function of a random variable.*

Solution. We know by linearity of variance for mutually-independent random variables that

$$\mathrm{Var}\,[T] = \mathrm{Var}\,[T_1 + \ldots + T_n]$$
$$= \mathrm{Var}\,[T_1] + \ldots + \mathrm{Var}\,[T_n]$$

Now we evaluate the variance of T_j for any j. Using the definition of variance from part (a) above, we have

$$\mathrm{Var}\,[T_j] = \mathrm{E}\,[T_j^2] - \mathrm{E}\,[T_j]^2$$

By the rule for computing the expectation of a function of a random variable, we also know that

$$\mathrm{E}\,[T_j^2] = \sum_{x \in Range(T_j)} x^2 \Pr\{T_j = x\}$$

Now we can use the fact that T_j is in the range $[0, 1]$ to say that $x^2 \leq x$, and that therefore

$$\sum_{x \in Range(T_j)} x^2 \cdot \Pr\{T_j = x\} \leq \sum_{x \in Range(T_j)} x \cdot \Pr\{T_j = x\}$$

Thus, $\mathrm{E}\,[T_j^2] \leq \mathrm{E}\,[T_j]$ and

$$\mathrm{Var}\,[T_j] \leq \mathrm{E}\,[T_j] - \mathrm{E}\,[T_j]^2$$
$$\leq \mathrm{E}\,[T_j]$$

Since this holds for any j, we can now conclude that

$$\mathrm{Var}\,[T] = \mathrm{Var}\,[T_1] + \ldots + \mathrm{Var}\,[T_n]$$
$$\leq \mathrm{E}\,[T_1] + \ldots + \mathrm{E}\,[T_n]$$
$$= \mathrm{E}\,[T]$$

$\blacksquare$

3   Gambler's Ruin

A gambler is placing $1 bets on the "1st dozen" in roulette. This bet wins when a number from one to twelve comes in, and then the gambler gets his $1 back plus $2 more. Recall that there are 38 numbers on the roulette wheel.

The gambler's initial stake in n and his target is T. He will keep betting until he runs out of money ("goes broke") or reaches his target. Let w_n be the probability of the gambler winning, that is, reaching target T before going broke.

a. Write a linear recurrence with boundary conditions for w_n. You need *not* solve the recurrence.

Solution. The probability of winning a bet is 12/38. Thus, by the Law of Total Probability,

$$
\begin{aligned}
w_n &= \Pr\{\text{win with } \$n \text{ start} \mid \text{won 1st bet}\} \cdot \Pr\{\text{won 1st bet}\} \\
&\quad + \Pr\{\text{win with } \$n \text{ start} \mid \text{lost 1st bet}\} \cdot \Pr\{\text{lost 1st bet}\} \\
&= \Pr\{\text{win with } \$n+2 \text{ start}\} \cdot \Pr\{\text{won 1st bet}\} \\
&\quad + \Pr\{\text{win with } \$n-1 \text{ start}\} \cdot \Pr\{\text{lost 1st bet}\},
\end{aligned}
$$

so

$$
w_n = \frac{12}{38} w_{n+2} + \frac{26}{38} w_{n-1}.
$$

Letting $mn+2$ we get

$$
w_m = \frac{38}{12} w_{m-2} - \frac{26}{12} w_{m-3}.
$$

As boundary conditions, we have

$$
w_0 = 0, w_T = 1, w_{T+1} = 1.
$$

Notice that for this degree 3 recurrence, three boundary values are necessary to determine the w_n uniquely. ∎

b. Let e_n be the expected number of bets until the game ends. Write a linear recurrence with boundary conditions for e_n. You need *not* solve the recurrence.

Solution. By the Law of Total Expectation, Theorem,

$$
\begin{aligned}
e_n &= (1 + \mathrm{E}\,[\#\text{bets with } \$n \text{ start} \mid \text{won 1st bet}]) \cdot \Pr\{\text{won 1st bet}\} \\
&\quad + (1 + \mathrm{E}\,[\#\text{bets with } \$n \text{ start} \mid \text{lost 1st bet}]) \cdot \Pr\{\text{lost 1st bet}\} \\
&= (1 + \mathrm{E}\,[\#\text{bets with } \$n+2 \text{ start}]) \cdot \Pr\{\text{won 1st bet}\} \\
&\quad + (1 + \mathrm{E}\,[\text{number of bets starting with } \$n-1]) \cdot \Pr\{\text{lost 1st bet}\},
\end{aligned}
$$

so

$$e_n = (e_{n+2} + 1)\frac{12}{38} + (1 + e_{n-1})\frac{26}{38}$$

Letting $m = n + 2$ we get

$$e_m = \frac{38}{12}e_{m-2} - \frac{26}{12}e_{m-3} - \frac{38}{12}$$

As boundary conditions, we have

$$e_0 = e_T = e_{T+1} = 0.$$

Again, for this degree 3 recurrence three boundary values are necessary to determine the e_n uniquely. ∎