Notes for Recitation 7

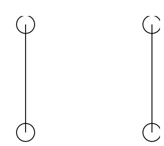
1 Build-up error

Recall a graph is ***connected*** iff there is a path between every pair of its vertices.

False Claim. *If every vertex in a graph has positive degree, then the graph is connected.*

(a) Prove that this Claim is indeed false by providing a counterexample.

Solution. There are many counterexamples; here is one:



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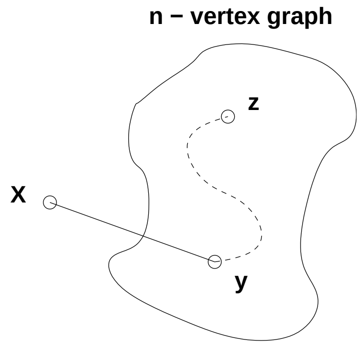
(b) Since the Claim is false, there must be a logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

Proof. We prove the Claim above by induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has positive degree, then the graph is connected.

Base cases: ($n \leq 2$). In a graph with 1 vertex, that vertex cannot have positive degree, so $P(1)$ holds vacuously.

$P(2)$ holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 2$. Consider an n-vertex graph in which every vertex has positive degree. By the assumption $P(n)$, this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to obtain an $(n+1)$-vertex graph:



All that remains is to check that there is a path from x to every other vertex z. Since x has positive degree, there is an edge from x to some other vertex, y. Thus, we can obtain a path from x to z by going from x to y and then following the path from y to z. This proves $P(n+1)$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, which proves the Claim.

□

Solution. This one is tricky: the proof is actually a good proof of something else. The first error in the proof is only in the final statement of the inductive step: "This proves $P(n+1)$".

What we have actually shown (above) is that there are graphs on n+1 vertices where each vertex has positive degree, and are connected. But if we want to show that every graph on n+1 vertices where each vertex has positive degree, is necessarily connected, when we start with G and try to remove a node and all edges incident to it we do not necessarily get a graph on n vertices that satisfies the conditions we want. To put it more succinctly, in the counterexample to part (a) every vertex we choose to delete results in a graph where some node has degree zero afterwards.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$-vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh!

The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck— and properly so, since the claim is false! ■

2 Euler tours

The statement of (a) in the original version was incorrect! This has been corrected below.

(a) Prove that a graph G has an Euler tour if and only if every vertex of G has even degree.

Note that there are two directions to prove!

Solution. The proof is the same as in the proof of Theorem 5.6.3 in the book (pp. 159–160). ■

(b) Suppose that G is strongly connected. (A strongly connected graph is one in which any vertex can be reached from any other). Come up with a necessary and sufficient condition for the existence of an Euler tour in a *directed* graph. Adapt your proof above to prove that your condition is the right one.

Solution. The condition is: an Euler tour exists if and only if for every vertex, the indegree equals the outdegree.

The proof is basically the same. Any Euler tour must enter and exit a vertex the same number of times; so the condition is certainly necessary.

Now suppose the condition holds, and let $W = w_0, w_1, \ldots, w_k$ be a longest walk in G using every directed edge at most once. Then W must be a closed walk; for suppose that $w_k \neq w_0$. Then we must have entered w_k one more time than we left it, which means that there is some outgoing directed edge that we have not used. This would allow us to extend the walk, contradicting that W was as long as possible.

Suppose that W is not an Euler tour. Either W contains all vertices or it does not. First consider the case that all vertices are in W. Because it is not an Euler tour, there is at least one edge not used. Given this edge, there must be an unused edge out of the end of this edge. If there weren't, that would contradict the assumption that the indegree equals the outdegree and W is a closed walk. We can continue to follow this chain of edges. The only thing that can stop this is getting to the start of the first edge since this would allow the chain to stop. This chain is a closed walk all not in W. We can combine the closed walks which makes a longer walk, contradicting the assumption that W was the longest.

Now, suppose that not all vertices are in W. There must be an unused edge directed away from some vertex in the walk W; for if not, there would be no path from any vertex on W to a vertex not in W, contradicting the assumption that G is strongly connected. Let $w_i \to u$ be this edge. Construct a walk W' beginning with this edge and traversing only unused edges, stopping when we cannot make a move. Again by the condition that indegree equals outdegree, this walk will end at w_i. We thus obtain a longer walk

$$W' = w_0, w_1, \ldots, w_i, u, \ldots, w_i, w_{i+1}, \ldots, w_k.$$

This is again a contradiction.

■

3 Connectivity

Prove that any simple graph with n nodes and strictly more than $\frac{1}{2}(n-1)(n-2)$ edges is connected.

Solution. We'll show the equivalent statement that any disconnected graph on n nodes has at most $(n-1)(n-2)/2$ edges.

Let $G = (V, E)$ be any graph on n nodes that is not connected. Then there must be more than one connected component; let $G_1 = (V_1, E_1)$ be any connected component, and let $G_2 = (V_2, E_2)$ be the graph induced on $V_2 := V - V_1$. Note that there are no edges going between G_1 and G_2, and so $E_1 \cup E_2 = E$.

How many edges can G_1 have? At most $\binom{|V_1|}{2}$ edges (one for each pair of nodes). Similarly, G_2 can have at most $\binom{|V_2|}{2}$ edges.

Write $t := |V_1|$; then $|V_2| = |V| - |V_1| = n - t$. So the total number of edges in G is at most

$$\frac{t(t-1)}{2} + \frac{(n-t)(n-t-1)}{2}.$$

If we simplify this, we get

$$|E| \leq \frac{n(n-1)}{2} - t(n-t).$$

But since $1 \leq t \leq n-1$, $t(n-t) \geq n-1$. (You can confirm this with some calculus; it might help to draw $t(n-t)$ as a function of t; it's just a parabola.) So

$$|E| \leq \frac{n(n-1)}{2} - (n-1) = \frac{(n-2)(n-1)}{2}.$$

■