Problem Set 6

Due: Monday, October 13

Reading Assignment: Sections 7.1-7.9

Problem 1. [**20 points**] The adjacency matrix A of a graph G with n vertices as defined in lecture is an $n \times n$ matrix in which $A_{i,j}$ is 1 if there is an edge from i to j and 0 if there is not. In lecture we saw how the smallest k where $A_{i,j}^k \neq 0$ describes the length of the shortest path from i to j. Given a combinatorial interpretation of the following statements about the adjacency matrix in terms of connectivity properties of G. For example, the smallest k such that $A_{i,j}^k$ is non-zero means that the distance from i to j is at most k.

(a) [5 pts] The smallest k such that for every pair (i, j) at least one of $A_{i,j}, A_{i,j}^2, \ldots, A_{i,j}^k$ is non-zero.

(b) [5 pts] $\forall k. A_{i,j}^k = 0$

(c) [5 pts] $\forall i \forall k. A_{i,i}^k = 0$

(d) [5 pts] $\forall k$, we can write A^k as $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$

Problem 2. [**15 points**] A set of PageRank values is stationary if the amount of PageRank going into each vertex is the same as the amount leaving the vertex on every update. Prove that a strongly connected graph has at most one set of PageRank values that are stationary. There always is a set of PageRank values that are stationary, but we are not asking you prove this.

(a) [7 pts] For two sets of values of PageRank, d_1 and d_2, let γ be defined as $\gamma \equiv \max_{x \in V} \frac{d_1(x)}{d_2(x)}$, the maximum ratio of a value in d_1 over the corresponding value in d_2. Show that there exists a directed edge from y to z such that $d_1(y)/d_2(y) < \gamma$ and $d_1(z)/d_2(z) = \gamma$.

(b) [8 pts] Prove that a strongly connected graph has at most one set of PageRank values that are stationary by deriving a contradiction using the edge found in part a.

Problem 3. [**15 points**]

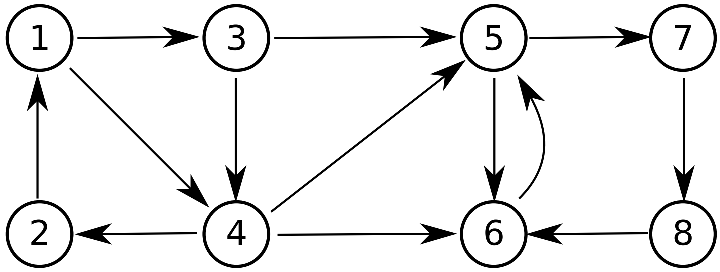

Figure 1: A graph of web pages and links.

(a) [5 pts] For the graph in Figure 1, compute the first two iterations of PageRank, starting from uniform PageRank values across all vertices.

(b) [10 pts] A strongly connected component of a directed graph is a subgraph which has the property that for every pair of vertices u and v, there exists a path from u to v and one from v to u. Also, every vertex which can be reached by a path starting in the strongly connected component is also in the strongly connected component. Suppose that a graph G consists of exactly two strongly connected components C_1 and C_2, and that there exist edges from C_1 to C_2 (but not from C_2 to C_1). There is always a stationary set of values which is non-negative. Prove that the stationary PageRank values of this graph are entirely concentrated in C_2, i.e. that the PageRank values are all zero on C_1.

Problem 4. [20 points] For each of the following, either prove that it is an equivalence relation and state its equivalence classes, or give an example of why it is not an equivalence relation.

(a) [5 pts] $R_n := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } x \equiv y \pmod{n}\}$

(b) [5 pts] $R := \{(x, y) \in P \times P \text{ s.t. } x \text{ is taller than } y\}$ where P is the set of all people in the world today.

(c) [5 pts] $R := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } gcd(x, y) = 1\}$

(d) [5 pts] $R_G :=$ the set of $(x, y) \in V \times V$ such that V is the set of vertices of a graph G, and there is a path $x, v_1, \ldots, v_k, y$ from x to y along the edges of G.

Problem 5. [10 points] Let R_1 and R_2 be two equivalence relations on a set, A. Prove or give a counterexample to the claims that the following are also equivalence relations:

(a) [5 pts] $R_1 \cap R_2$.

(b) [5 pts] $R_1 \cup R_2$.

Problem 6. [15 points] In this problem we study partial orders (posets). Recall that a weak partial order $\preceq$ on a set X is reflexive ($x \preceq x$), anti-symmetric ($x \preceq y \wedge y \preceq x \rightarrow x = y$), and transitive ($x \preceq y \wedge y \preceq z \rightarrow x \preceq z$). Note that it may be the case that neither $x \preceq y$ nor $y \preceq x$. A chain is a list of *distinct* elements $x_1, \ldots, x_i$ in X for which $x_1 \preceq x_2 \preceq \cdots \preceq x_i$. An antichain is a subset S of X such that for all distinct $x, y \in S$, neither $x \preceq y$ nor $y \preceq x$.

The aim of this problem is to show that any sequence of $(n-1)(m-1)+1$ integers either contains a non-decreasing subsequence of length n or a decreasing subsequence of length m. Note that the given sequence may be out of order, so, for instance, it may have the form $1, 5, 3, 2, 4$ if $n = m = 3$. In this case the longest non-decreasing and longest decreasing subsequences have length 3 (for instance, consider $1, 2, 4$ and $5, 3, 2$).

(a) [5 pts] Label the given sequence of $(n-1)(m-1)+1$ integers $a_1, a_2, \ldots, a_{(n-1)(m-1)+1}$. Show the following relation $\preceq$ on $\{1, 2, 3, \ldots, (n-1)(m-1)+1\}$ is a weak poset: $i \preceq j$ if and only if $i \leq j$ and $a_i \leq a_j$ (as integers).

For the next part, we will need to use Dilworth's theorem. Recall that Dilworth's theorem states that if $(X, \preceq)$ is any poset whose longest chain has length n, then X can be partitioned into n disjoint antichains.

(b) [5 pts] Show that in any sequence of $(n-1)(m-1)+1$ integers, either there is a non-decreasing subsequence of length n or a decreasing subsequence of length m.

(c) [5 pts] Construct a sequence of $(n-1)(m-1)$ integers, for arbitrary n and m, that has no non-decreasing subsequence of length n and no decreasing subsequence of length m. Thus in general, the result you obtained in the previous part is best-possible.

Problem 7. [10 points] Let the transitive closure of a graph G be the digraph $G^+ = (V, E^+)$, where:

$$E^+ = \{u \rightarrow v \mid \text{there is a directed path of positive length from } u \text{ to } v \text{ in } G\}.$$

Prove that if the graph G is a directed acyclic graph, then the transitive closure of G is a strong partial order.