Problem Set 7 Solutions

Due: Thursday, October 23

Problem 1. [15 points] Express

$$\sum_{i=0}^{n} i^2 x^i$$

as a closed-form function of n.

Solution. We use the derivative method. Let us start with the following formula, derived in lecture (for $x \neq 1$):

$$\sum_{i=0}^{n} i x^i = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

Differentiating both sides:

$$\begin{aligned}
x^{-1}\sum_{i=0}^{n} i^2 x^i =& \frac{(1-(n+1)^2x^n + n(n+2)x^{n+1})(1-x)^2 - (x-(n+1)x^{n+1}+nx^{n+2})(2(1-x)(-1))}{(1-x)^4} \\
=& \frac{(1-(n+1)^2x^n + n(n+2)x^{n+1})(1-x) + 2(x-(n+1)x^{n+1}+nx^{n+2})}{(1-x)^3} \\
=& \frac{1-(n+1)^2x^n + n(n+2)x^{n+1} - x + (n+1)^2x^{n+1} - n(n+2)x^{n+2}}{(1-x)^3} \\
&+ \frac{2x - 2(n+1)x^{n+1} + 2nx^{n+2}}{(1-x)^3} \\
=& \frac{1+x-(n+1)^2x^n + (n(n+2)+(n+1)^2-2(n+1))x^{n+1} + (2n - n(n+2))x^{n+2}}{(1-x)^3} \\
=& \frac{1+x-(n+1)^2x^n + (2n^2+2n-1)x^{n+1} - n^2x^{n+2}}{(1-x)^3}.
\end{aligned}$$

Multiplying both sides by x, we get

$$\sum_{i=0}^{n} i^2 x^i = \frac{x(1+x-(n+1)^2x^n + (2n^2+2n-1)x^{n+1} - n^2x^{n+2})}{(1-x)^3}.$$

■

Problem 2. [20 points]

(a) [5 pts] What is the product of the first n odd powers of two: $\prod_{k=1}^{n} 2^{2k-1}$?

Solution.

$$\Pi_{k=1}^{n} 2^{2k-1} = 2^{\sum_{k=1}^{n} 2k-1} = 2^{2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1} = 2^{n(n+1)-n} = 2^{n^2}$$

■

(b) [5 pts] Find a closed expression for

$$\sum_{i=0}^{n}\sum_{j=0}^{m} 3^{i+j}$$

Solution.

$$\begin{aligned}
\sum_{i=0}^{n}\sum_{j=0}^{m} 3^{i+j} &= \sum_{i=0}^{n}\left(3^i \cdot \sum_{j=0}^{m} 3^j\right) \\
&= \left(\sum_{j=0}^{m} 3^j\right) \cdot \left(\sum_{i=0}^{n} 3^i\right) \\
&= \left(\frac{3^{m+1}-1}{2}\right) \cdot \left(\frac{3^{n+1}-1}{2}\right)
\end{aligned}$$

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(c) [5 pts] Find a closed expression for

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)$$

Solution.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j) &= \left(\sum_{i=1}^{n}\sum_{j=1}^{n} i\right) + \left(\sum_{i=1}^{n}\sum_{j=1}^{n} j\right) \\
&= \left(\sum_{i=1}^{n} ni\right) + \left(\sum_{i=1}^{n} \frac{n(n+1)}{2}\right) \\
&= \frac{2n^2(n+1)}{2} \\
&= n^2(n+1)
\end{aligned}$$

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(d) [5 pts] Find a closed expression for

$$\prod_{i=1}^{n}\prod_{j=1}^{n} 2^i \cdot 3^j$$

Solution.

$$\begin{aligned}
\prod_{i=1}^{n}\prod_{j=1}^{n} 2^i \cdot 3^j &= \left(\prod_{i=1}^{n} 2^{ni}\right)\left(\prod_{j=1}^{n} 3^{nj}\right) \\
&= 2^{n\sum_{i=1}^{n} i} 3^{n\sum_{j=1}^{n} j} \\
&= 2^{n^2(n+1)/2} 3^{n^2(n+1)/2}
\end{aligned}$$

■

Problem 3. [10 points]

(a) [6 pts] Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

Solution. Let's first try standard bounds:

$$\int_1^{\infty} \frac{1}{(2x+1)^2}\,dx \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad f(1) + \int_1^{\infty} \frac{1}{(2x+1)^2}\,dx$$

Evaluating the integrals gives:

$$-\frac{1}{2(2x+1)}\Bigg|_1^{\infty} \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + -\frac{1}{2(2x+1)}\Bigg|_1^{\infty}$$

$$\frac{1}{6} \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{9} + \frac{1}{6}$$

These bounds are too far apart, so let's sum the first couple terms explicitly and bound the rest with integrals.

$$\frac{1}{3^2} + \int_2^{\infty} \frac{1}{(2x+1)^2}\,dx \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + f(2) + \int_2^{\infty} \frac{1}{(2x+1)^2}\,dx$$

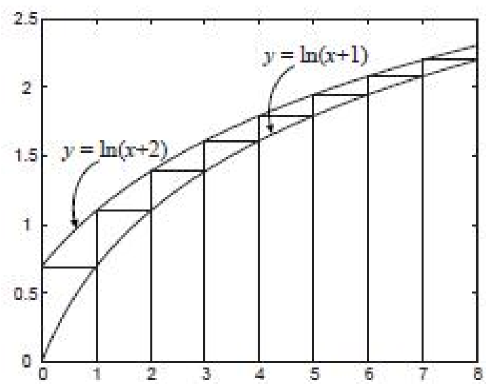
Integration now gives:

$$\frac{1}{3^2} + \left(-\frac{1}{2(2x+1)}\Bigg|_2^{\infty}\right) \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + \frac{1}{5^2} + \left(-\frac{1}{2(2x+1)}\Bigg|_2^{\infty}\right)$$

$$\frac{1}{3^2} + \frac{1}{10} \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{10}$$

Now we have bounds that differ by $1/5^2 = 0.04$. ■

(b) [4 pts] Assume n is an integer larger than 1. Which of the following inequalities, if any, hold. You may find the graph helpful.



1. $\displaystyle\sum_{i=1}^{n} \ln(i+1) \leq \int_0^n \ln(x+2)dx$
2. $\displaystyle\sum_{i=1}^{n} \ln(i+1) \leq \ln 2 + \int_1^n \ln(x+1)dx$

Solution. The 1st inequality holds. ■

Problem 4. [15 points] We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of n times.

(a) [10 pts] Describe a closed form formula for the amount of wine in the first glass after n back-and-forth pourings.

Solution. The state of the system of glasses/wine/water at the beginning of a round of pouring and pouring back is determined by the total amount of wine in the first glass. Suppose at the beginning of some round, the first glass contains w pints of wine, $0 \leq w \leq 1$ and $1 - w$ pints of water. The second glass contains the rest of the wine and water.

Pouring 1/3 pint from the first glass to the second leaves 2/3 pints of liquid and $(2/3)w$ wine in the first glass, and 4/3 pints of liquid and $1 - (2/3)w$ wine in the second glass.

Pouring 1/3 pint back from the second into the first transfers a proportion of $(1/3)/(4/3)$ of the wine in the second glass into the first. So the round completes with both glasses containing a pint of liquid, and the first glass containing

$$(2/3)w + (1/4)(1 - (2/3)w) = 1/4 + w/2$$

pints of wine. After one more round, the first glass contains

$$1/4 + (1/4 + w/2)/2 = 1/4 + 1/8 + w/2^2$$

pints of wine, and after n more rounds

$$\begin{aligned} w/2^n + \Sigma_{i=1}^n (1/2)^{i+1} &= w/2^n + (1/2)\Sigma_{i=1}^n (1/2)^i \\ &= w/2^n + (1/2)(-1 + \Sigma_{i=0}^n (1/2)^i) \\ &= w/2^n + (1/2)(-1 + (1 - (1/2)^{n+1})/(1 - 1/2)) \\ &= w/2^n - 1/2 + 1 - (1/2)^{n+1} \\ &= w/2^n + 1/2 - (1/2)^{n+1}. \end{aligned}$$

Since $w = 0$ initially, the pints of wine in the first glass after n rounds is

$$1/2 - (1/2)^{n+1}.$$

■

(b) [5 pts] What is the limit of the amount of wine in each glass as n approaches infinity?

Solution. The limiting amount of wine in the first glass approaches 1/2 from below as n approaches infinity. In fact, it approaches 1/2 no matter how the wine was initially distributed. This of course is what you would expect: after a thorough mixing the glasses should contain essentially the same amount of wine.

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Problem 5. [20 points] For each of the following six pairs of functions f and g (parts (a) through (f)), state which of these order-of-growth relations hold (more than one may hold, or none may hold):

$$f = o(g) \qquad f = O(g) \qquad f = \omega(g) \qquad f = \Omega(g) \qquad f = \Theta(g) \qquad f \sim g$$

(a)	$f(n) = \log_2 n$	$g(n) = \log_{10} n$
(b)	$f(n) = 2^n$	$g(n) = 10^n$
(c)	$f(n) = 0$	$g(n) = 17$
(d)	$f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$	$g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$
(e)	$f(n) = 1.0000000001^n$	$g(n) = n^{10000000000}$

Solution.

- $f(n) = \log_2 n$ and $g(n) = \log_{10} n$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{\ln n/\ln 2}{\ln n/\ln 10}\\ &= \frac{\ln 10}{\ln 2}\end{aligned}$$

So $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$ and $f(n) = \Theta(g(n))$.

- $f(n) = 2^n$ and $g(n) = 10^n$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{2^n}{10^n}\\ &= \lim_{n\to\infty}(1/5)^n\\ &= 0\end{aligned}$$

So $f(n) = o(g(n))$ and $f(n) = O(g(n))$.

- $f(n) = 0$ and $g(n) = 17$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \frac{0}{17}\\ &= 0\end{aligned}$$

So $f(n) = o(g(n))$ and $f(n) = O(g(n))$.

- $f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$ and $g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$:

  For all $n \equiv 1 \pmod 4$, $f(n)/g(n) = 0$, so $f(n) \neq \Omega(g(n))$. Likewise, for all $n \equiv 0 \pmod 4$, $g(n)/f(n) = 0$, so $f(n) \neq O(g(n))$. The quotient never converges to some particular limit, so no relation holds.

- $f(n) = 1.0000000001^n$ and $g(n) = n^{10000000000}$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{1.0000000001^n}{n^{10000000000}}\\ &= \lim_{n\to\infty}\frac{1.0000000001^n\ln 1.0000000001}{10000000000n^{9999999999}}\\ &= \lim_{n\to\infty}\frac{1.0000000001^n(\ln 1.0000000001)^{10000000000}}{10000000000!}\\ &= \infty\end{aligned}$$

So $f(n) = \omega(g(n))$ and $f(n) = \Omega(g(n))$.

■

Problem 6. [20 points] This problem continues the study of the asymptotics of factorials.

(a) [5 pts]

Either prove or disprove each of the following statements.

- $n! = O((n+1)!)$
- $n! = \Omega((n+1)!)$
- $n! = \Theta((n+1)!)$
- $n! = \omega((n+1)!)$
- $n! = o((n+1)!)$

Solution. Observe that $n! = (n+1)!/(n+1)$, and thus $n! = o((n+1)!)$. Thus, $n! = O((n+1)!)$ as well, but the remaining statements are false. ■

(b) [5 pts] Show that $n! = \omega\left(\left(\frac{n}{3}\right)^{n+e}\right)$.

Solution. By Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

On the other hand, note that $\left(\frac{n}{3}\right)^{n+e} = \left(\frac{n}{3}\right)^e \left(\frac{n}{3}\right)^n$. Dividing $n!$ by this quantity,

$$\frac{3^e\sqrt{2\pi}}{n^{e-1/2}} \cdot \left(\frac{3}{e}\right)^n,$$

we see that since $3 > e$, this expression goes to ∞. Thus, $n! = \omega\left(\frac{n}{3}\right)^{n+e}$. ■

(c) [5 pts] Show that $n! = \Omega(2^n)$

Solution. We can proceed straight from the definition. Recall $n!$ is $\Omega\left(2^n\right)$ if and only if

$$\lim_{n\to\infty} \frac{n!}{2^n} > 0$$

By multiplying and dividing by the same factor, we get

$$\lim_{n\to\infty} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{2^n}$$

And using Stirling's approximation, we know the left part tends to 1. So we only need to worry about

$$\lim_{n\to\infty} \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{2^n}$$

The expression in the limit can be manipulated to be

$$\left(\frac{n}{2e}\right)^n \sqrt{2\pi n}$$

Since n^n is strictly larger than 10^n for $n > 10$, then

$$\lim_{n\to\infty} \left(\frac{n}{2e}\right)^n \sqrt{2\pi n} > \lim_{n\to\infty} \left(\frac{10}{2e}\right)^n \sqrt{2\pi n} = \infty$$

So the original limit must also be ∞. This also shows that in fact $n! = \omega\left(2^n\right)$ And the same argument can be used to show that $n! = \omega\left(10^n\right)$ or any other constant base.

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(d) [5 pts] Show that

$$\sum_{k=1}^{n} k^6 = \Theta(n^7).$$

Solution. Let $S_n = \sum_{k=1}^{n} k^6$.

One approach is to use the Integral Method:

$$\frac{n^7}{7} = \int_0^n x^6 \, dx \le S_n \le \int_0^n (x+1)^6 \, dx = \frac{(n+1)^7}{7} - \frac{1}{7}.$$

So we have $n^7 \le 7S_n$, and so $n^7 = O(S_n)$. Also $(n+1)^7/7 - 1/7 = O(n^7)$, and so $S_n = O(n^7)$. Hence, $S_n = \Theta(n^7)$.

An alternative approach not using the Integral Method goes as follows. There are n terms in S_n and each term is at most n^6, so $S_n \le n \cdot n^6 = n^7 = O(n^7)$. So $S_n = O(n^7)$.

On the other hand, at least $(n-1)/2$ of the terms are as large as $[(n-1)/2]^6$, so

$$\begin{aligned} S_n &\ge ((n-1)/2) \cdot [(n-1)/2]^6 \\ &= [(n-1)/2]^7 \\ &\ge (n/3)^7 \end{aligned}$$

for $n > 3$, so $n^7 \le 3^7 \cdot S_n$. In other words, $n^7 = O(S_n)$. ■