Problem Set 8

Due: Tuesday, November 8

Problem 1. [15 points] This problem continues the study of the asymptotics of factorials.

(a) [5 pts]

Either prove or disprove each of the following statements.

- $n! = O((n+1)!)$
- $n! = \Omega((n+1)!)$
- $n! = \Theta((n+1)!)$
- $n! = \omega((n+1)!)$
- $n! = o((n+1)!)$

(b) [5 pts] Is $n! = o\left(\left(\frac{n}{2}\right)^{n+e}\right)$ or is $n! = \omega\left(\left(\frac{n}{2}\right)^{n+e}\right)$? (Hint: Use Stirling's formula)

(c) [5 pts] Show that $n! = \Omega(3^n)$

Problem 2. [25 points] Find Θ bounds for the following divide-and-conquer recurrences. Assume $T(1) = 1$ in all cases. Show your work.

(a) [5 pts] $T(n) = 8T(\lfloor n/2 \rfloor) + n$

(b) [5 pts] $T(n) = 2T(\lfloor n/8 \rfloor + 1/n) + n$

(c) [5 pts] $T(n) = 7T(\lfloor n/20 \rfloor) + 2T(\lfloor n/8 \rfloor) + n$

(d) [5 pts] $T(n) = 2T(\lfloor n/4 \rfloor + 1) + n^{1/2}$

(e) [5 pts] $T(n) = 3T(\lfloor n/9 + n^{1/9} \rfloor) + 1$

Problem 3. [20 points] It is easy to misuse induction when working with asymptotic notation.

False Claim If

$$T(1) = 1 \text{ and}$$
$$T(n) = 4T(n/2) + n$$

Then T(n) = O(n).

False Proof We show this by induction. Let $P(n)$ be the proposition that $T(n) = O(n)$.

Base Case: $P(1)$ is true because $T(1) = 1 = O(1)$.

Inductive Case: For $n \geq 1$, assume that $P(n-1), \ldots, P(1)$ are true. We then have that

$$T(n) = 4T(n/2) + n = 4O(n/2) + n = O(n)$$

And we are done.

(a) [5 pts] Identify the flaw in the above proof.

(b) [5 pts] Using Akra-Bazzi theorem, find the correct asymptotic behavior of this recurrence.

(c) [10 pts] We have now seen several recurrences of the form $T(n) = aT(\lfloor n/b \rfloor) + n$. Some of them give a runtime that is $O(n)$, and some don't. Find the relationship between a and b that yields $T(n) = O(n)$, and prove that this is sufficient.

Problem 4. [15 points] Define the sequence of numbers A_i by

$A_0 = 2$

$A_{n+1} = A_n/2 + 1/A_n$ (for $n \geq 1$)

Prove that $A_n \leq \sqrt{2} + 1/2^n$ for all $n \geq 0$.

Problem 5. [25 points] Find closed-form solutions to the following linear recurrences.

(a) [5 pts] $x_n = 5x_{n-1} - 6x_{n-2} \quad (x_0 = 0, x_1 = 1)$

(b) [10 pts] $x_n = 4x_{n-1} - 4x_{n-2} \quad (x_0 = 0, x_1 = 2)$

(c) [10 pts] $x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3} \quad (x_0 = 3, x_1 = 4, x_2 = 14)$

Problem 6. [25 points] In this problem, we will solve inhomogeneous linear recurrences. For the following problems, use the technique learned in class. That is, first solve the homogeneous linear recurrence, and guess the form of the particular solution. Then add the particular solution and the homogeneous solution to get the general solution, and then use the boundary case to determine remaining constants.

(a) [10 pts] Find the solution to $x_n = 3x_{n-1} + n \quad (x_0 = 2)$.

(b) [15 pts] Find the solution to $x_n = -x_{n-1} + 2x_{n-2} + n \quad (x_0 = 5, x_1 = -4/9)$.