Problem Set 9 Solutions

Due: Monday, November 14

Reading Assignment: Sections 7.2, 11.1-11.10

Problem 1. [10 points]

(a) [5 pts] Show that of any $n + 1$ distinct numbers chosen from the set $\{1, 2, \ldots, 2n\}$, at least 2 must be relatively prime. (*Hint:* $\gcd(k, k + 1) = 1$.)

Solution. Treat the $n + 1$ numbers as the pigeons and the n disjoint subsets of the form $\{2j - 1, 2j\}$ as the pigeonholes. The pigeonhole principle implies that there must exist a pair of consecutive integers among the $n + 1$ chosen which, as suggested in the hint, must be relatively prime. ■

(b) [5 pts] Show that any finite connected undirected graph with $n \geq 2$ vertices must have 2 vertices with the same degree.

Solution. In a finite connected graph with $n \geq 2$ vertices, the domain for the vertex degrees is the set $\{1, 2, \ldots, n - 1\}$ since each vertex can be adjacent to at most all of the remaining $n-1$ vertices and the existence of a degree 0 vertex would violate the assumption that the graph be connected. Therefore, treating the n vertices as the pigeons and the $n-1$ possible degrees as the pigeonholes, the pigeonhole principle implies that there must exist a pair of vertices with the same degree. ■

Problem 2. [15 points]

Consider the 40 most popular cities on Earth. Use the pigeonhole principle to show that there are two subsets of these cities that have exactly the same number of people. Assume that there are 10^{10} people on the Earth.

Solution. Let the pigeonholes be the positive integers mod 10^{10}. Let the pigeons be the number of people in each subset of the 40 cities. There are 2^{40} subsets of our cities. Now 2^{40} is much larger than 10^{10}, and so at least two of the subsets of cities must fall into the same pigeonhole (that is they have the same number of people mod 10^{10}). But, note that the number of people in each subset must necessarily be less than the number of people in the world. Hence the number of people in each subset is exactly one of the positive integers mod 10^{10}. Hence, we have that at least two subsets of the cities have the same number of people. ■

Problem 3. [15 points] In this problem, we will use the principle of inclusion-exclusion (PIE) to solve the problem of derangements.

Suppose you attend a Halloween party with n guests, where each guest dressed up as Harley-Quinn. Each Harley-Quinn brought a bat as an accessory to the party, but didn't want to carry it through the night, and so all the bats were left stacked together in a single room. Unfortunately, later on in the party, one of the party-goers bumped into the stack of bats causing them all to fall in a pile. Thus, at the end of the party, each Harley-Quinn grabbed a random bat and left. It turns out none of them got his/her own bat back. We will count the number of ways this can happen.

(a) [2 pts] How many ways could the guests have picked up the bats at random?

Solution. $n!$ as we are simply assigning one of the n bats to each guests. ■

(b) [3 pts] How many ways there are to choose 1 person to get their bat back and randomly assign everyone else?

Solution. $\binom{n}{1} \cdot (n-1)! = \frac{n!}{1!}$ as we have $\binom{n}{1}$ choices for the person to get their bat back, and $(n-1)!$ ways to arrange the other guests and bats. ■

(c) [10 pts] Use the principle of inclusion-exclusion to determine the number of ways in which no guest got their hat back.

Solution. Following the reasoning in part b, we can use PIE to conclude that the number of ways in which at least one guest got their hat back will be:

$$\binom{n}{1} \cdot (n-1)! - \binom{n}{2} \cdot (n-2)! + \ldots (-1)^{n-1}\binom{n}{n} \cdot (n-n)! = n! - \frac{n!}{2!} + \frac{n!}{3!} - \ldots + (-1)^{n-1}\frac{n!}{n!}$$

Therefore by subtracting this value from part a as we are counting the complement, the answer will be $n! - (n! - \frac{n!}{2!} + \frac{n!}{3!} - \ldots + (-1)^{n-1}\frac{n!}{n!})$. ■

Problem 4. [45 points] Be sure to show your work to receive full credit. In this problem we assume a standard card deck of 52 cards.

(a) [4 pts] How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

Solution. There is a bijection with sequence of the form:

$$(\text{value of pair}, \text{suits of pair}, \text{value of other three cards}, \text{suits of other three cards})$$

Thus, the number of hands with a single pair is:

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,098,240$$

Alternatively, there is also a 3!-to-1 mapping to the tuple:

(value of pair, suits of pair,
value 3rd card, suit 3rd card, value 4th card, suit 4th card, value 5th card, suit 5th card)

Thus, the number of hands with a single pair is:

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot 4 \cdot 11 \cdot 4 \cdot 10 \cdot 4}{3!} = 1,098,240$$

∎

(b) [4 pts] For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i = n$$

Solution. There is a bijection from the solutions of the equation to the binary strings containing n zeros and k ones where x_0 is the number of 0s preceding the first 1, x_k is the number of 0s following the last 1 and x_i is the number of 0s between the ith and $(i+1)^{th}$ 1 for $0 < i < k$.

$$\binom{n+k}{k}$$

∎

(c) [4 pts] For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i \leq n$$

Solution. There is a bijection from the solutions of

$$\sum_{i=0}^{k} x_i \leq n$$
$$= n - x_{k+1} \qquad\qquad \text{(for some } x_{k+1} \geq 0)$$

and the solutions of

$$\sum_{i=0}^{k+1} x_i = n.$$

$$\binom{n+k+1}{k+1}$$

∎

(d) [4 pts] How many simple undirected graphs are there with n vertices?

Solution. There are $\binom{n}{2}$ potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{\binom{n}{2}}$$

∎

(e) [4 pts] How many directed graphs are there with n vertices (self loops allowed)?

Solution. There are n^2 potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{n^2}$$

∎

(f) [4 pts] How many tournament graphs are there with n vertices?

Solution. There are no self-loops in a tournament graph and for each of the $\binom{n}{2}$ pairs of distinct vertices a and b, either $a \to b$ or $b \to a$ but not both. Therefore, the number of tournament graphs is:

$$2^{\binom{n}{2}}$$

∎

(g) [4 pts] How many acyclic tournament graphs are there with n vertices?

Solution. For any path from x to y in a tournament graph, an edge $y \to x$ would create a cycle. Therefore in any acyclic tournament graph, the existence of a path between distinct vertices x and y would require the edge $x \to y$ also be in the graph. That is, the "beats" relation for such a graph would be transitive. Since each pair of distinct players are comparable (either $x \to y$ or $y \to x$) we can uniquely rank the players $x_1 < x_2 < \cdots < x_n$. There are $n!$ such rankings. ∎

(h) [4 pts] How many numbers are there that are in the range [1..700] which are divisible by $2, 5$ or 7?

Solution. Let S be the set of all numbers in range $[1..700]$. Let S_2 be the numbers in this range divisible by 2, S_5 be the numbers in this range divisible by 5 and S_7 be the numbers in this range divisible by 7. By inclusion-exclusion, the number of elements in S divisible by 2, 5 or 7 is

$$\begin{aligned} n &= |S_2| + |S_5| + |S_7| - |S_2 S_5| - |S_2 S_7| - |S_5 S_7| + |S_2 S_5 S_7| \\ &= \frac{700}{2} + \frac{700}{5} + \frac{700}{7} - \frac{700}{2 \cdot 5} - \frac{700}{2 \cdot 7} - \frac{700}{5 \cdot 7} + \frac{700}{2 \cdot 5 \cdot 7} \\ &= 350 + 140 + 100 - 70 - 50 - 20 + 10 \\ &= 460. \end{aligned}$$

∎

(i) [4 pts] How many ways are there to list the digits $\{1, 1, 2, 2, 3, 4, 5\}$ so that the 1's are always consecutive?

Solution. Just treat the ones as a block B. Then we just list the permutations of $\{B, 2, 2, 3, 4, 5\}$, which is $\frac{6!}{2!}$. ∎

(j) [4 pts] What is the coefficient of $x^3 y^2 z$ in the expansion of $(x + y + z)^6$?

Solution. This is just $\frac{6!}{2! \cdot 3! \cdot 1!}$ by considering the number of ways of selecting 3 x's, 2 y's, and 1 z from the 6 factors $(x + y + z)$. ∎

(k) [5 pts] How many unique terms are there in the expansion of $(x + y + z)^6$?

Solution. We want to essentially find non-negative integers a, b, c such that $x^a y^b z^c$ is a term and so $a + b + c = 6$. Hence we have a bijection from this problem with the problem of finding non-negative integers a, b, c such that $a + b + c = 6$. Hence, by following the solution in part b, there are $\binom{8}{2}$ terms. ∎

Problem 5. [**15 points**] Give a combinatorial proof of the following theorem:

$$n 2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$

Solution. Here is the **first solution**:

Consider the ways of forming a committee out of n members and picking 1 leaders.

The left hand side indicates picking a leader first out of the n members ($\binom{n}{1}$ ways), and then picking a subset of the remaining $n-1$ members to form the committee (2^{n-1} ways).

The right hand side indicates picking a committee first, and then selecting a leader from the committee. The committee can have any size from 1 to n, since we have to have at least one member in the committee to be the leader. Hence we sum from $k = 1$ to n of the number of ways to select a k person committee from n people ($\binom{n}{k}$ ways), and then selecting a leader from the committee in $\binom{k}{1}$ ways.

Here is a **second solution**:

Let $P = \{0, \ldots, n-1\} \times \{0, 1\}^{n-1}$. On the one hand, there is a bijection from P to S by mapping (k, x) to the word obtained by inserting a * just after the kth bit in the length-$n-1$ binary word, x. So

$$|S| = |P| = n2^{n-1} \tag{1}$$

by the Product Rule.

On the other hand, every sequence in S contains between 1 and n nonzero entries since the $*$, at least, is nonzero. The mapping from a sequence in S with exactly k nonzero entries to a pair consisting of the set of positions of the nonzero entries and the position of the $*$ among these entries is a bijection, and the number of such pairs is $\binom{n}{k}k$ by the Generalized Product Rule. Thus, by the Sum Rule:

$$|S| = \sum_{k=1}^{n} k\binom{n}{k}$$

Equating this expression and the expression (1) for $|S|$ proves the theorem.   ■