

Problem Set 6

Due: Thursday, October 20

Reading Assignment: Sections 5.4.3, 6.1, 6.2, 7.1-7.9

Problem 1. [20 points]

This problem involves trees, which are a special class of graphs. Suppose we consider a simple connected graph G on n nodes. Then, G is a tree if and only if G is minimally connected (that is removing an edge disconnects G).

(a) [5 pts] Show that our definition of a tree implies that a simple connected graph G is a tree if and only if G contains no cycles.

(b) [5 pts] Prove that a simple connected graph with n nodes and $n - 1$ edges is a tree.

(c) [10 pts] Prove that in any tree, any two longest paths cross each other (that is any two longest paths share a vertex).

Problem 2. [10 points] The adjacency matrix A of a graph G with n vertices as defined in lecture is an $n \times n$ matrix in which $A_{i,j}$ is 1 if there is an edge from i to j and 0 if there is not. In lecture we saw how the smallest k where $A_{i,j}^k \neq 0$ describes the length of the shortest path from i to j . Give a combinatorial interpretation of the following statements about the adjacency matrix in terms of connectivity properties of G . For example, the smallest k such that $A_{i,j}^k$ is non-zero means that the distance from i to j is at most k .

(a) [3 pts] The trace of A^k (the trace of a matrix is the sum of its diagonal entries).

(b) [2 pts] $\forall k. A_{i,j}^k = 0$

(c) [2 pts] $\forall i \forall k. A_{i,i}^k = 0$

(d) [3 pts] $\forall k$, we can write A^k as $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$

Problem 3. [15 points] A set of PageRank values is stationary if the amount of PageRank going into each vertex is the same as the amount leaving the vertex on every update. In this problem we will prove that a strongly connected graph has at most one set of PageRank values that are stationary. There always is a set of PageRank values that are stationary, but we are not asking you prove this.

(a) [7 pts] For two sets of values of PageRank, d_1 and d_2 , let γ be defined as $\gamma : \max_{x \in V} \frac{d_1(x)}{d_2(x)}$, the maximum ratio of a value in d_1 over the corresponding value in d_2 . Show that there exists a directed edge from y to z such that $d_1(y)/d_2(y) < \gamma$ and $d_1(z)/d_2(z) = \gamma$.

(b) [8 pts] Prove that a strongly connected graph has at most one set of PageRank values that are stationary by deriving a contradiction using the edge found in part a.

Problem 4. [15 points]

(a) [5 pts] For the graph in Figure 1, compute the first two iterations of PageRank, starting from uniform PageRank values across all vertices.

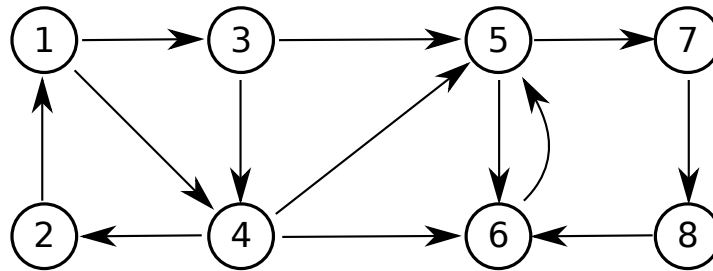


Figure 1: A graph of web pages and links.

(b) [10 pts] A strongly connected component of a directed graph is a subgraph which has the property that for every pair of vertices u and v , there exists a path from u to v and one from v to u . Also, every vertex which can be reached by a path starting in the strongly connected component is also in the strongly connected component. Suppose that a graph G consists of exactly two strongly connected components C_1 and C_2 , and that there exist edges from C_1 to C_2 (but not from C_2 to C_1). There is always a stationary set of values which is non-negative. Prove that the stationary PageRank values of this graph are entirely concentrated in C_2 , i.e. that the PageRank values are all zero on C_1 .

Problem 5. [10 points] For each of the following, either prove that it is an equivalence relation and state its equivalence classes, or give an example of why it is not an equivalence relation.

(a) [2 pts] $R_n := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } x \equiv y \pmod{n}\}$

(b) [2 pts] $R := \{(x, y) \in P \times P \text{ s.t. } x \text{ is taller than } y\}$ where P is the set of all people in the world today.

(c) [3 pts] $R := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } \gcd(x, y) = 1\}$

(d) [3 pts] $R_G :=$ the set of $(x, y) \in V \times V$ such that V is the set of vertices of a graph G , and there is a path x, v_1, \dots, v_k, y from x to y along the edges of G .

Problem 6. [10 points] Let R_1 and R_2 be two equivalence relations on a set, A . Prove or give a counterexample to the claims that the following are also equivalence relations:

(a) [5 pts] $R_1 \cap R_2$.

(b) [5 pts] $R_1 \cup R_2$.

Problem 7. [20 points] In this problem we study partial orders (posets). Recall that a weak partial order \preceq on a set X is reflexive ($x \preceq x$), anti-symmetric ($x \preceq y \wedge y \preceq x \rightarrow x = y$), and transitive ($x \preceq y \wedge y \preceq z \rightarrow x \preceq z$). Note that it may be the case that neither $x \preceq y$ nor $y \preceq x$. A chain is a list of *distinct* elements x_1, \dots, x_i in X for which $x_1 \preceq x_2 \preceq \dots \preceq x_i$. An antichain is a subset S of X such that for all distinct $x, y \in S$, neither $x \preceq y$ nor $y \preceq x$.

(a) [10 pts] A chain in a poset is called maximal if it cannot be extended (that is no other element of the poset can be added to it). Let B_n (the Boolean algebra) be the set of subsets of $\{1, 2, \dots, n\}$ where the partial order is defined by the contains operation (that is for subsets S_1, S_2 , $S_1 < S_2$ if $S_1 \subset S_2$ and the number of elements in B_n is 2^n). How many maximal chains does B_n have?

(b) [10 pts] Let π be a permutation, and let d be the smallest integer so that π can be written as a union of d increasing sequences. Prove that the longest decreasing subsequence of π consists of d elements. (Hint: use Dilworth's theorem)