Quiz 1

Name: ______________________________

Circle the name of your recitation instructor:

David Darren Martyna Nick Oscar Stav

- This quiz is **closed book**, but you may have one $8.5 \times 11''$ sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- The exam ends at 9:30 PM.

Problem	Points	Grade	Grader
1	10		
2	15		
3	20		
4	10		
5	15		
6	10		
7	20		
Total	100		

Problem 1. [20 points] Define a number $S_p = 1^p + 2^p + 3^p + \ldots (p-1)^p$. Use Fermat's theorem to show that if p is an odd prime, then $p|S_p$.

Problem 2. [20 points] Suppose $S(n)$ is a predicate on natural numbers, n, and suppose

$$\forall k \in \mathbb{N}\ S(k) \Rightarrow S(k+2). \tag{1}$$

If (1) holds, some of the assertions below must *always* (**A**) hold, some *can* (**C**) hold but not always, and some can *never* (**N**) hold. Indicate which case applies for each of the assertions by **circling** the correct letter.

(a) [2 pts] **A C N** $\forall n \geq 0\ S(n)$

(b) [2 pts] **A C N** $\neg S(0) \wedge \forall n \geq 1\ S(n)$

(c) [2 pts] **A C N** $\forall n \geq 0\ \neg S(n)$

(d) [2 pts] **A C N** $(\forall n \leq 100\ S(n)) \wedge (\forall n > 100\ \neg S(n))$

(e) [2 pts] **A C N** $(\forall n \leq 100\ \neg S(n)) \wedge (\forall n > 100\ S(n))$

(f) [2 pts] **A C N** $S(0) \Rightarrow \forall n\ S(n+2)$

(g) [2 pts] **A C N** $S(1) \Rightarrow \forall n\ S(2n+1)$

(h) [2 pts] **A C N** $[\exists n\, S(2n)] \Rightarrow \forall n\ S(2n+2)$

(i) [2 pts] **A C N** $\exists n\, \exists m > n\, [S(2n) \wedge \neg S(2m)]$

(j) [2 pts] **A C N** $[\exists n\, S(n)] \Rightarrow \forall n\, \exists m > n\, S(m)$

Problem 3. [20 points]

Let $G_0 = 1$, $G_1 = 2$, $G_2 = 4$, and define

$$G_n = G_{n-1} + 2G_{n-2} + G_{n-3} \tag{2}$$

for $n \geq 3$. Show by induction that $G_n \leq (2.2)^n$ for all $n \geq 0$.

Problem 4. [0 points] Consider a stable marriage problem with 4 boys and 4 girls. Here are their preference rankings:

Alfred:	Grace, Helen, Emily, Fiona
Billy:	Emily, Grace, Fiona, Helen
Calvin:	Helen, Emily, Fiona, Grace
David:	Helen, Grace, Emily, Fiona

Emily:	Calvin, Alfred, David, Billy
Fiona:	Alfred, Billy, Calvin, David
Grace:	Alfred, Calvin, David, Billy
Helen:	Alfred, Billy, David, Calvin

(a) [0 pts] Exhibit a stable matching between the boys and girls.

(b) [0 pts] Explain why this is the only stable matching possible.

(c) [0 pts] Suppose that Harry is one of the boys and Alice is one of the girls when a Mating Ritual is performed. Circle the properties below that must be preserved invariants.

(i) Harry is serenading Alice.

(ii) Alice is crossed off Harry's list.

(iii) Alice likes her favorite better than Harry.

(iv) Alice has at least one suitor.

(v) Harry is serenading a girl he likes better than Alice.

(vi) Harry is serenading a girl he likes less than Alice.

Problem 5. [0 points] In a stable matching between n boys and girls produced by the Mating Ritual, call a person *lucky if they are matched up with one of their $\lceil n/2 \rceil$ top choices. We will prove:

Theorem. *There must be at least one lucky person.*

To prove this, define the following derived variables for the Mating Ritual:

$r(B) = j$, when the boy B is courting the jth girl on his list.

$r(G)$ is the number of boys that girl G has rejected.

(a) [0 pts] Let

$$S \sum_{B \in \text{Boys}} r(B) - \sum_{G \in \text{Girls}} r(G). \tag{3}$$

Show that S remains the same from one day to the next in the Mating Ritual.

(b) [0 pts] Prove the Theorem above. (You may assume for simplicity that n is even.)

(*Hint: A*) girl is sure to be lucky if she has rejected half the boys.

Problem 6. [0 points] Prove: Any graph in which every vertex has deg ≥ 2 contains a cycle.

Problem 7. [0 points] In this problem we consider *edge coloring* of a simple graph. An edge coloring is a graph coloring where we color edges instead of vertices. Recall that in a coloring of vertices, two adjacent vertices may not be the same coloring. Similarly, in an edge coloring, any two edges which are incident on the same vertex must not be the same color.

Prove that a bipartite graph is edge-colorable with a number of colors equal to the maximum degree of any vertex in the graph.

Problem 8. [0 points] Induction: Prove that a sum of consecutive odd numbers (beginning with 1); i.e.

$$\sum_{i=0}^{n} 2i + 1$$

with $n \geq 1$; is a perfect square.

Hint: prove something stronger

Problem 9. [10 points] Taken from: http://www.cut-the-knot.org/ctk/invariant.shtml

(a) [5 pts] In the game of Squares and Circles, the players (you and your computer) start with a sequence of shapes: some circles and some squares. On each move a player selects two shapes. These two are replaced with a single one according to the following rule:

Identical shapes are replaced with a square. Different shapes are replaced with a circle.

At the end of the game, when only one shape remains, you are a winner if the remaining shape is a circle. Otherwise, your computer wins. Prove that you will win if the number of circles initially is odd. Hint: Use an invariant about the parity of the number of circles.

(b) [5 pts] The game extends to a 3-shape puzzle. Let there be three kinds of shapes: squares, circles, and triangles. A move consists in selecting two objects of different shapes which are then replaced with an object of the remaining shape. Prove that we can always achieve a state with all shapes the same (e.g. all squares). Hint: Use an invariant relating to the total number of shapes.

Problem 10. [10 points] Use integration to find upper and lower bounds that differ by at most 0.5 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{i^3}$$