Midterm Practice Problems

Name: ____

- This quiz is **closed book**, but you may have one 8.5×11" sheet with notes in your own handwriting on both sides.
- Calculators and electronic devices are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

Problem 1. [12 points] Let X be the set of students in 6.042. Let Y be the set of problems on this quiz. For $x \in X$ and $y \in Y$, let $P(x, y)$ be the statement "Student x got full points on problem y". Let $Q(x)$ be the statement "Student x drops 6.042".

(a) [6 pts] Convert the following statements into English.

1. $(\exists x \in X, Q(x)) \Rightarrow (\forall x \in X, Q(x))$.
2. $\forall x \in X((\exists y \in Y \neg P(x, y)) \Rightarrow Q(x))$
3. $\exists x \in X(\neg Q(x))$.

(b) [6 pts] Assuming 1,2 and 3 are true, what can you say about your score on this quiz?

Problem 2. [20 points] 6 people are sitting in a circle. Each person has a number. They do a ritual during each round that makes their numbers update. Each person, to get his number for round $n+1$, takes his number from round n, then adds his right neighbor's number from round n to it, and then subtracts his left neighbor's number from round n from it.

For example if on the current round there is a person A whose number is 5, and A's right neighbor's number is 4, and A's left neighbor's number is 6, then in the following round, A's number will be $5 + 4 - 6 = 3$.

Initially the people are given the numbers $1, 2, 3, 4, 5, 6$ (in clockwise order, and the person with 6 is sitting next to the person with 1).

Thus, after 1 round, their numbers will be $5, 0, 1, 2, 3, 10$.

Prove that there will never be a round when all their numbers are equal.

BIG HINT: Consider what happens to the sum of the numbers.

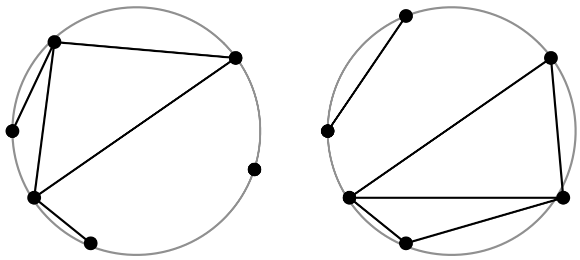
Problem 3. [15 points] We define the sequence of numbers

$$a_n = \begin{cases} 1 & \text{if } 0 \leq n \leq 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \geq 4. \end{cases}$$

Prove that $a_n \equiv 1 \pmod 3$ for all $n \geq 0$.

Problem 4. [10 points] Find the multiplicative inverse of 17 modulo 72 in the range $\{0, 1, \ldots, 71\}$.

Problem 5. [20 points] An outerplanar graph is an undirected graph for which the vertices *can be* placed on a circle in such a way that no edges (drawn as straight lines) cross each other. For example, the complete graph on 4 vertices, K_4, is not outerplanar but any proper subgraph of K_4 with strictly fewer edges is outerplanar. Some examples are provided below:



Prove that any outerplanar graph is 3-colorable. A fact you may use without proof is that any outerplanar graph has a vertex of degree at most 2.

Problem 6. [16 points] Consider a stable marriage problem with 4 boys and 4 girls. Here are their preference rankings:

Alfred:	Grace, Helen, Emily, Fiona
Billy:	Emily, Grace, Fiona, Helen
Calvin:	Helen, Emily, Fiona, Grace
David:	Helen, Grace, Emily, Fiona

Emily:	Calvin, Alfred, David, Billy
Fiona:	Alfred, Billy, Calvin, David
Grace:	Alfred, Calvin, David, Billy
Helen:	Alfred, Billy, David, Calvin

(a) [5 pts] Exhibit a stable matching between the boys and girls.

(b) [5 pts] Explain why this is the only stable matching possible.

(c) [6 pts] Suppose that Harry is one of the boys and Alice is one of the girls when a Mating Ritual is performed. Circle the properties below that must be preserved invariants.

(i) Harry is serenading Alice.

(ii) Alice is crossed off Harry's list.

(iii) Alice likes her favorite better than Harry.

(iv) Alice has at least one suitor.

(v) Harry is serenading a girl he likes better than Alice.

(vi) Harry is serenading a girl he likes less than Alice.

Problem 7. [20 points]

Consider the simple graph G given in Figure 1.

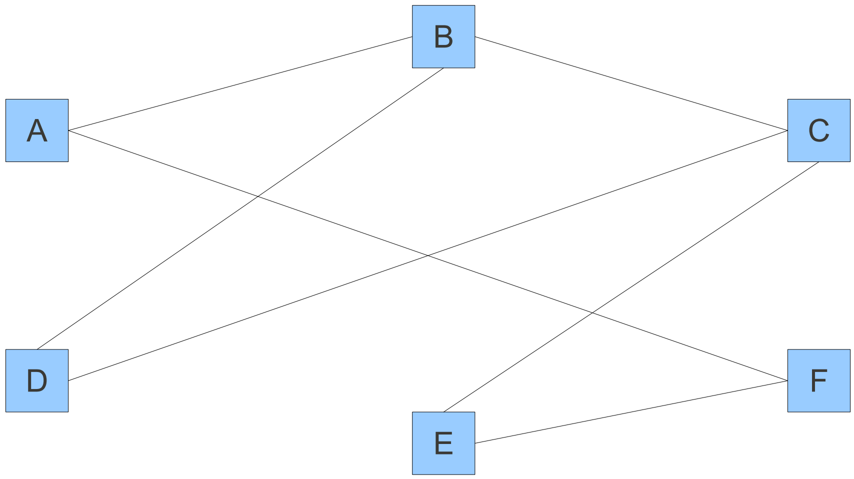


Figure 1: Simple graph G

(a) [4 pts] Give the diameter of G.

(b) [4 pts] Give a Hamiltonian Cycle on G.

(c) [4 pts] Give a coloring on G and show that it uses the smallest possible number of colors.

(d) [4 pts] Does G have an Eulerian cycle? Justify your answer.

Now consider graph H, which is like G but with weighted edges, in Figure 2:

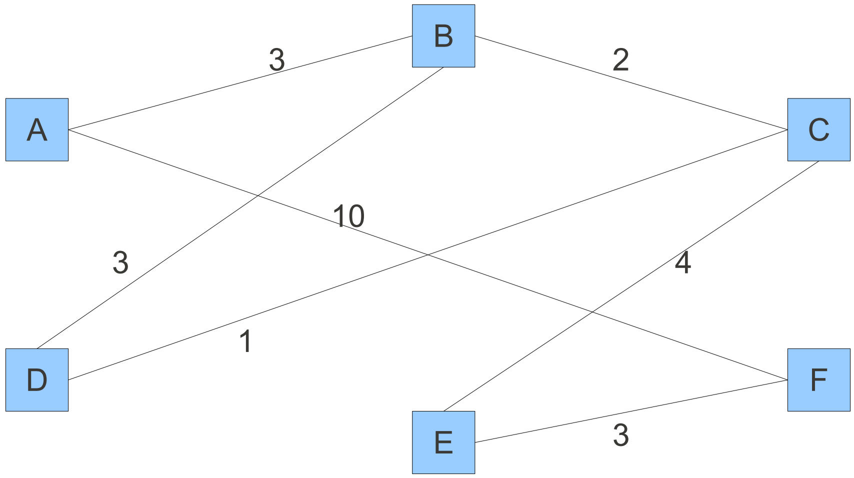


Figure 2: Weighted graph H

(e) [4 pts] Give a list of edges reflecting the order in which one of the greedy algorithms presented in class (i.e. in lecture, recitation, or the course text) would choose edges when finding an MST on H.

Problem 8. [18 points]

In this problem, we say that an $M \times N$ *grid network* is a routing network consisting of an undirected grid of M rows and N columns, with M inputs on the left and M outputs on the right, as depicted in Figure 3. (Note: this is not the same as the "2-D array" in the notes, which has outputs on the bottom.)

(a) [6 pts] What is the congestion of the 3×3 grid in Figure 3? (Feel free to use the grids drawn on the next page in expressing your answer.)

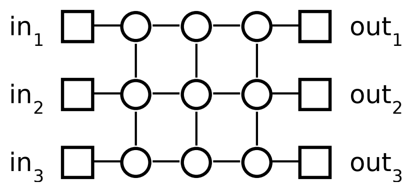
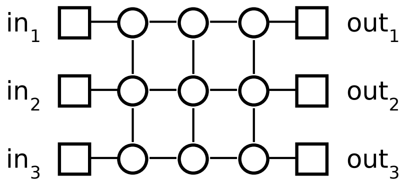


Figure 3: A 3×3 grid network.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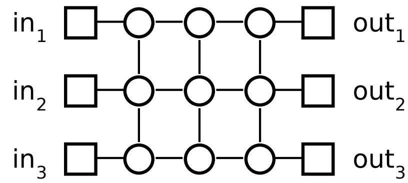
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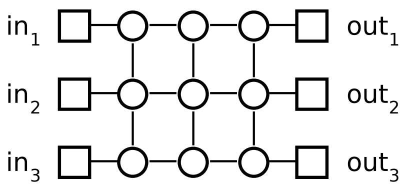
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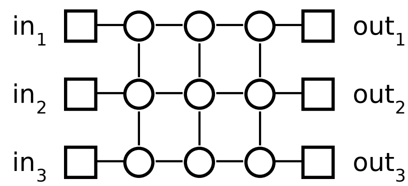
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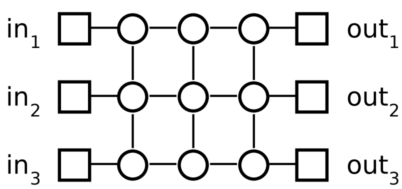
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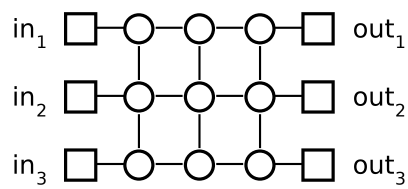
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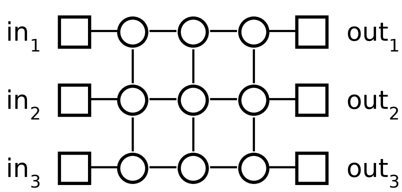
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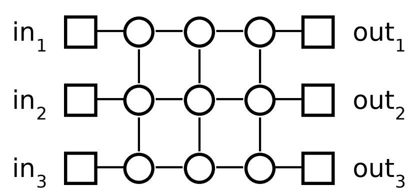
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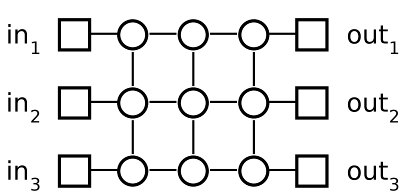
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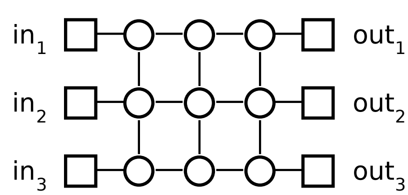
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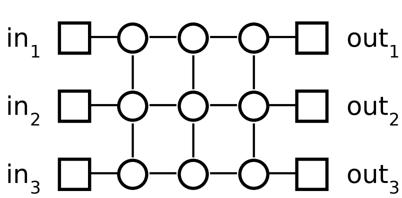
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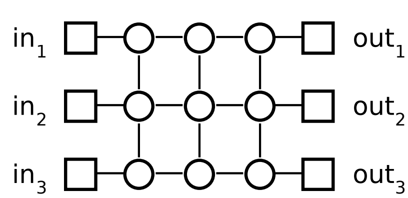
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(b) [6 pts] For the $M \times N$ grid network with M inputs on the left and M outputs on the right, prove carefully that the congestion is always strictly greater than 1, so long as $M > 1$.

(c) [6 pts] For the $M \times N$ grid, as in part (b), argue that for any $M > 1$, it is possible to choose N large enough so that the grid has congestion 2. (A fully-fledged proof is not required, but give justification.)

Hint: imagine routes traveling along rows and swapping in various places.

Problem 9. [15 points]

Throughout this problem, PageRank is understood to mean *unscaled* PageRank.

(a) [5 pts] Given any strongly connected web graph G on at least 2 vertices, argue that no single vertex can have equilibrium PageRank value greater than $\frac{1}{2}$.

(b) [5 pts] Let $N \geq 2$ be given. Exhibit a web graph on N vertices in which one vertex has equilibrium PageRank value equal to $\frac{1}{2}$.

(c) [5 pts] Argue that for each N, there is only one solution to part (b) that is strongly connected.

Problem 10. [10 points] Consider the following relation on the set of natural numbers:

$$R = \{(x, y) : x \le y^2 \text{ for } x, y \in \mathbb{N}\}.$$

Which of the following properties holds for R? If it has the property, prove it. If not, provide a counterexample.

(a) [2 pts] Reflexivity.

(b) [2 pts] Symmetry.

(c) [2 pts] Transitivity.

(d) [2 pts] Antisymmetry.

(e) [2 pts] The property of being an equivalence relation.