Problem Set 8

Due: May 6

Reading:

- Sections 14.5. *Products*, 14.7. *Asymptotic Notation* (omit 14.6).
- Chapter 15. *Counting* through Section 15.7. *Counting Practice*

Problem 1.
The New York Yankees and the Boston Red Sox are playing a two-out-of-three series. In other words, they play until one team has won two games. Then that team is declared the overall winner and the series ends. Assume that the Red Sox win each game with probability $3/5$, regardless of the outcomes of previous games.

Answer the questions below using the four step method. You can use the same tree diagram for all three problems.

(a) What is the probability that a total of 3 games are played?

(b) What is the probability that the winner of the series loses the first game?

(c) What is the probability that the *correct* team wins the series?

Problem 2.
Suppose you have three cards: A♡, A♠, and a Jack. From these, you choose a random hand (that is, each card is equally likely to be chosen) of two cards, and let n be the number of Aces in your hand. You then randomly pick one of the cards in the hand and reveal it.

(a) Describe a simple probability space (that is, outcomes and their probabilities) for this scenario, and list the outcomes in each of the following events:

1. $[n \geq 1]$, (that is, your hand has an Ace in it),
2. A♡ is in your hand,
3. the revealed card is an A♡,
4. the revealed card is an Ace.

(b) Then calculate $\Pr\left[n = 2 \mid E\right]$ for E equal to each of the four events in part (a). Notice that most, but *not all*, of these probabilities are equal.

Now suppose you have a deck with d distinct cards, a different kinds of Aces (including an A♡), you draw a random hand with h cards, and then reveal a random card from your hand.

(c) Prove that Pr[A♡ is in your hand] $= h/d$.

(d) Prove that

$$\Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right] = \Pr[n = 2] \cdot \frac{2d}{ah}. \tag{1}$$

(e) Conclude that

$$\Pr\left[n = 2 \mid \text{the revealed card is an Ace}\right] = \Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right].$$

Problem 3.
Suppose $\Pr[\cdot] : \mathcal{S} \to [0, 1]$ is a probability function on a sample space, $\mathcal{S}$, and let B be an event such that $\Pr[B] > 0$. Define a function $\Pr_B[\cdot]$ on outcomes $\omega \in \mathcal{S}$ by the rule:

$$\Pr_B[\omega] ::= \begin{cases} \Pr[\omega]/\Pr[B] & \text{if } \omega \in B, \\ 0 & \text{if } \omega \notin B. \end{cases} \tag{2}$$

(a) Prove that $\Pr_B[\cdot]$ is also a probability function on $\mathcal{S}$ according to Definition 17.5.2.

(b) Prove that

$$\Pr_B[A] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

for all $A \subseteq \mathcal{S}$.

(c) Explain why the Disjoint Sum Rule carries over for conditional probabilities, namely,

$$\Pr\left[C \cup D \mid B\right] = \Pr\left[C \mid B\right] + \Pr\left[D \mid B\right] \qquad (C, D \text{ disjoint}).$$

Give examples of several further such rules.