ELSEVIER

Discrete Mathematics 159 (1996) 119–130

DISCRETE MATHEMATICS

The complexity of planar graph choosability [1]

Shai Gutner

Department of Computer Science, School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel

Received 25 July 1994; revised 12 January 1995

Abstract

A graph G is *k-choosable* if for every assignment of a set $S(v)$ of k colors to every vertex v of G, there is a proper coloring of G that assigns to each vertex v a color from $S(v)$. We consider the complexity of deciding whether a given graph is k-choosable for some constant k. In particular, it is shown that deciding whether a given planar graph is 4-choosable is NP-hard, and so is the problem of deciding whether a given planar triangle-free graph is 3-choosable. We also obtain simple constructions of a planar graph which is not 4-choosable and a planar triangle-free graph which is not 3-choosable.

1. Introduction

All graphs considered here are finite, undirected and simple (i.e., have no loops and no parallel edges). If $G=(V,E)$ is a graph, and f is a function that assigns to each vertex v of G a positive integer $f(v)$, we say that G is f-choosable if for every assignment of sets of integers $S(v) \subseteq Z$ for all vertices $v \in V$, where $|S(v)| = f(v)$ for all v, there is a proper vertex coloring $c\colon V \mapsto Z$ so that $c(v) \in S(v)$ for all $v \in V$. The graph G is *k-choosable* if it is f-choosable for the constant function $f(v) \equiv k$. The *choice number* of G, denoted $\mathrm{ch}(G)$, is the minimum integer k so that G is k-choosable.

The study of choice numbers of graphs was initiated by Vizing in [11] and by Erdős et al. in [2]. A characterization of all 2-choosable graphs is given in [2]. If G is a connected graph, the *core* of G is the graph obtained from G by repeatedly deleting vertices of degree 1 until there is no such vertex.

Theorem 1.1 (Erdos et al. [2]). *A simple graph is 2-choosable if and only if the core of each connected component of it is either a single vertex, or an even cycle, or*

[1] This paper forms part of a Ph.D. thesis written by the author under the supervision of Prof. N. Alon and Prof. M. Tarsi in Tel Aviv University.

SSDI 0012-365X(95)00104-2

a graph consisting of two vertices with three even internally disjoint paths between them, where the length of at least two of the paths is exactly 2.

In the present paper we consider the complexity of deciding whether a given graph is k-choosable for some constant k. It is shown in [2] that the following problem is Π_2^p-complete: (for terminology see [3])

BIPARTITE GRAPH (2,3)-CHOOSABILITY (BG (2,3)-CH)
Instance: A bipartite graph $G=(V,E)$ and a function $f\colon V \mapsto \{2,3\}$.
Question: Is G f-choosable?

Consider the following decision problem:

BIPARTITE GRAPH k-CHOOSABILITY (BG k-CH)
Instance: A bipartite graph G.
Question: Is G k-choosable?

It is proved in [4] that this problem is Π_2^p-complete for every constant $k \geqslant 3$. It follows easily from Theorem 1.1 that the case $k=2$ is solvable in polynomial time.

The following results are known concerning the choice numbers of planar graphs:

Theorem 1.2 (Thomassen [9]). *Every planar graph is* 5-*choosable.*

Theorem 1.3 (Voigt [12]). *There exists a planar graph* (*with* 238 *vertices*) *which is not* 4-*choosable.*

Theorem 1.4 (Alon and Tarsi [1]). *Every bipartite planar graph is* 3-*choosable.*

Theorem 1.5 (Voigt [13]). *There exists a planar triangle-free graph* (*with* 166 *vertices*) *which is not* 3-*choosable.*

Theorem 1.6 (Thomassen [10]). *Every planar graph with girth* 5 *is* 3-*choosable.*

The following two theorems improve upon Theorems 1.3 and 1.5 and use much simpler constructions.

Theorem 1.7. *There exists a planar graph with* 75 *vertices which is not* 4-*choosable.*

Theorem 1.8. *There exists a planar triangle-free graph with* 164 *vertices which is not* 3-*choosable.*

It follows easily from Theorems 1.1 and 1.4 that the choice number of a given bipartite planar graph can be determined in polynomial time. Consider the following decision problems:

BIPARTITE PLANAR GRAPH (2,3)**-CHOOSABILITY (BPG** (2,3)**-CH)**
Instance: A bipartite planar graph $G = (V, E)$ and a function $f : V \mapsto \{2, 3\}$.
Question: Is G f-choosable?

PLANAR TRIANGLE-FREE GRAPH 3**-CHOOSABILITY (PTFG** 3**-CH)**
Instance: A planar triangle-free graph G.
Question: Is G 3-choosable?

PLANAR GRAPH 4**-CHOOSABILITY (PG** 4**-CH)**
Instance: A planar graph G.
Question: Is G 4-choosable?

UNION OF TWO FORESTS 3**-CHOOSABILITY (U2F** 3**-CH)**
Instance: Two forests F_1 and F_2 with $V(F_1) = V(F_2)$.
Question: Is the union of F_1 and F_2 3-choosable?

We prove the following results:

Theorem 1.9. BIPARTITE PLANAR GRAPH (2,3)-CHOOSABILITY *is Π_2^p-complete*

Theorem 1.10. PLANAR TRIANGLE-FREE GRAPH 3**-CHOOSABILITY** *is Π_2^p-complete.*

Theorem 1.11. PLANAR GRAPH 4**-CHOOSABILITY** *is Π_2^p-complete.*

The decision problem **U2F** 3**-CH** was formulated by Stiebitz [8] in light of the fact that every planar triangle-free graph is the union of two forests. The following theorem can be derived easily from the constructions used in the proofs of Theorems 1.9 and 1.10.

Theorem 1.12. UNION OF TWO FORESTS 3**-CHOOSABILITY** *is Π_2^p-complete.*

The rest of the paper is organized as follows. In Section 2 we prove Theorems 1.7 and 1.8. The Π_2^p-completeness proof of the decision problem **BG (2,3)-CH** taken from [2] forms the basis for the proof of Theorem 1.9 given in Section 3. Section 4 contains the proofs of Theorems 1.10 and 1.11.

2. Two planar graphs

In this section we construct two planar graphs in order to prove Theorems 1.7 and 1.8.

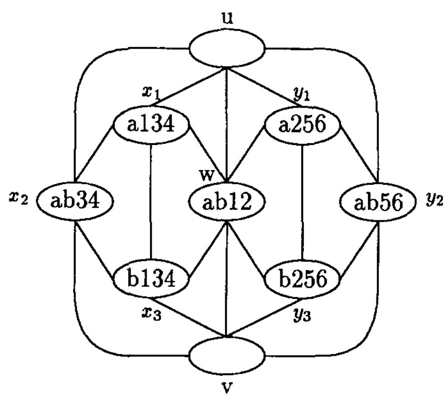

Fig. 1. The graph W_1.

Proof of Theorem 1.7. The graph H is constructed as follows: We take the disjoint union of the graphs $\{G_i\colon 1 \leqslant i \leqslant 12\}$, where each G_i is a copy of the graph W_1 in Fig. 1. All the 12 vertices named u are identified, as well as all the 12 vertices named v. The edge (u, v) is added to obtain the graph H, which is obviously planar. We claim that the graph H is not 4-choosable. To prove this, take $S(u) = S(v) = \{7, 8, 9, 10\}$. Denote $A = \{(a, b) \in S(u) \times S(v) | a \neq b\}$, then surely $|A| = 12$. With every i, $1 \leqslant i \leqslant 12$, we associate a different element $p_i = (a, b) \in A$, and define the sets of every vertex of G_i except for u and v to be as in Fig. 1. It can be easily verified that there is no proper vertex coloring for this assignment, and therefore H is not 4-choosable. To see this, suppose the vertex u is colored with the color a and the vertex v is colored with the color b, where $(a, b) = p_i \in A$. The vertex w in the graph G_i can be colored with either the color 1 or the color 2, and in both cases the coloring in the graph G_i cannot be completed.

We now construct a planar graph H' which is not 4-choosable and has fewer vertices than H. The graph H' is obtained from H by identifying the vertex y_2 of G_i with the vertex x_2 of G_{i+1} for every i, $1 \leqslant i < 12$. We claim that H' is not 4-choosable. The previous definitions of $S(u)$, $S(v)$, A and p_i are used. For every i, $1 \leqslant i < 12$, we do the following: Denote $p_i = (a, b)$ and $p_{i+1} = (c, d)$. The set of the vertex y_2 of G_i (which is the same as the set of the vertex x_2 of G_{i+1}) is chosen so that it contains the colors a, b, c and d (and maybe other colors if p_i and p_{i+1} are not disjoint). In the same manner as before, we conclude that H' is not 4-choosable. The graph H' is planar and has $2 + 12 * 7 - 11 = 75$ vertices. □

Proof of Theorem 1.8. The graph H is constructed as follows: We take the disjoint union of the graphs $\{G_i\colon 1 \leqslant i \leqslant 9\}$, where each G_i is a copy of the graph W_2 in Fig. 2. All the 9 vertices named u are identified, as well as all the 9 vertices named v, to obtain the planar triangle-free graph H. We claim that the graph H is not 3-choosable. To prove this, take $S(u) = \{10, 11, 12\}$ and $S(v) = \{13, 14, 15\}$. With every i, $1 \leqslant i \leqslant 9$, we associate a different element $(a, b) \in S(u) \times S(v)$, and define the sets of every vertex of

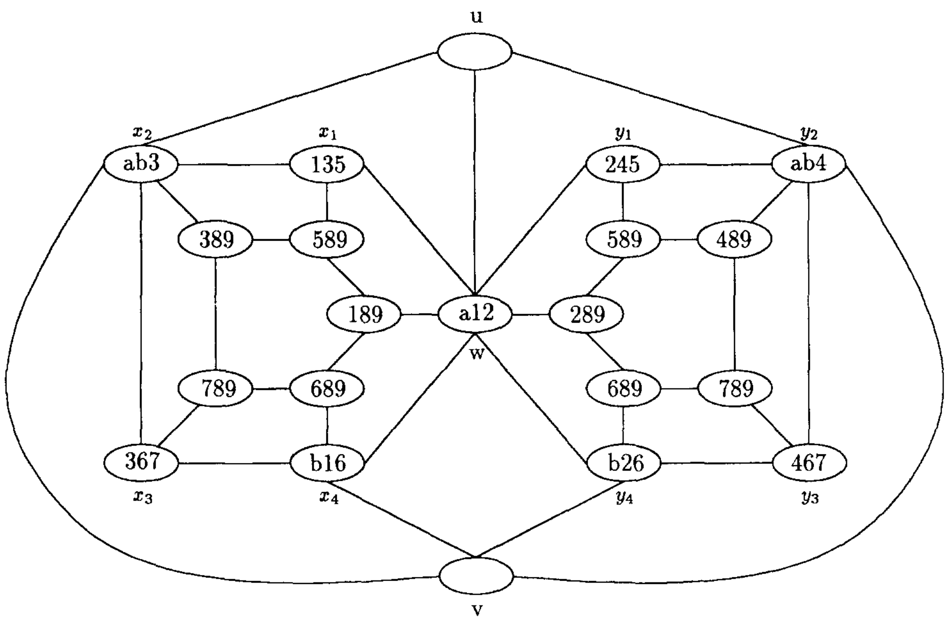

Fig. 2. The graph W_2.

G_i except for u and v to be as in Fig. 2. As in the proof of Theorem 1.7, we conclude that H is not 3-choosable.

We now construct a planar triangle-free graph H' which is not 3-choosable and has fewer vertices than H. The graph H' is obtained from H by identifying the vertex y_2 of G_i with the vertex x_2 of G_{i+1} for every i, $1 \leqslant i \leqslant 9$ (indices taken modulo 9). We claim that H' is not 3-choosable. The previous definitions of $S(u)$ and $S(v)$ are used. Consider the following ordering of the elements of $S(u) \times S(v)$:

$$\{p_i\}_{i=1}^{9} = (10,13),\ (10,14),\ (10,15),\ (11,15),\ (11,13),\ (11,14),\ (12,14),$$
$$(12,15),\ (12,13).$$

For every i, $1 \leqslant i \leqslant 9$, we do the following: Denote $p_i = (a,b)$ and $p_{i+1} = (c,d)$. The set of the vertex y_2 of G_i (which is the same as the set of the vertex x_2 of G_{i+1}) is defined as $\{a,b,c,d\}$ (this is a set of size 3). In the same manner as before, we conclude that H' is not 3-choosable. H' is a planar triangle-free graph and has $2 + 9 * 18 = 164$ vertices. $\square$

3. The choosability of bipartite planar graphs

The Π_2^p-completeness proof of the decision problem **BG (2,3)-CH** taken from [2] forms the basis for the proof of Theorem 1.9 given in this section. The ordinary Planar

Satisfiability problem is well known to be NP-complete [3,6]. We use a reduction from the following problem:

RESTRICTED PLANAR SATISFIABILITY (RPS)

Instance: An expression of the form $(\forall U_1)\cdots(\forall U_k)(\exists V_1)\cdots(\exists V_r)\Phi$ such that (1) Φ is a formula in conjunctive normal form with a set C of clauses over the set $X = \{U_1,\ldots,U_k,V_1,\ldots,V_r\}$ of variables, (2) each clause involves exactly three distinct variables, (3) every variable occurs in at most three clauses, and (4) the graph $G_\Phi = (X \cup C, \{xc \mid x \in c \in C \text{ or } \bar{x} \in c \in C\})$ is planar.

Question: Is this expression true?

A similar problem is used in [5] for proving results concerning the complexity of list colorings. The same transformation used in [6] for proving that the decision problem Planar Quantified Boolean Formula is P-space-complete can be used for proving that the following problem is Π_2^p-complete:

ORDINARY PLANAR SATISFIABILITY (OPS)

Instance: An expression of the form $(\forall U_1)\cdots(\forall U_k)(\exists V_1)\cdots(\exists V_r)\Phi$ such that (1) Φ is a formula in conjunctive normal form with a set C of clauses over the set $X = \{U_1,\ldots,U_k,V_1,\ldots,V_r\}$ of variables, (2) each clause involves at most three distinct variables, (3) the graph $G_\Phi = (X \cup C, \{xc \mid x \in c \in C \text{ or } \bar{x} \in c \in C\})$ is planar.

Question: Is this expression true?

We apply ideas from [7] for proving the following lemma:

Lemma 3.1. RESTRICTED PLANAR SATISFIABILITY *is Π_2^p-complete.*

Proof. It is easy to see that **RPS** $\in \Pi_2^p$. We transform **OPS** to **RPS**. Let the expression B be an instance of **OPS**, and suppose that B has the form $(\forall U_1)\cdots(\forall U_k)(\exists U_{k+1})\cdots(\exists U_{k+r})\Phi$. Take a planar embedding of G_Φ. For every variable V we do the following: Let $(V,C_1),\ldots,(V,C_n)$ be the edges adjacent to the variable V in the graph G_Φ in a clockwise order according to the planar embedding. Now introduce new variables $V_1,\ldots,V_n$ and clauses $V_i \vee \overline{V}_{i+1}$, $i = 1,\ldots,n$ (indices taken modulo n), and replace the literals $V,\overline{V}$ in clauses C_i by the literals $V_i,\overline{V}_i$, respectively, for $i=1,\ldots,n$. The quantified variable V is replaced with the variable V_1 quantified with the same quantifier. A new quantifier block existentially quantifying the variables $V_2,\ldots,V_n$ is appended to the list of quantifiers.

To every clause which involves exactly two variables we add a new variable V and insert the quantified variable $(\forall V)$ in the beginning of the expression. In a similar manner we handle clauses with only one variable. It is easily seen that the modified formula has the desired properties and that it is true if and only if B is true. $\square$

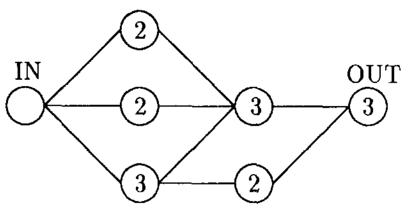

Fig. 3. Half-propagator.

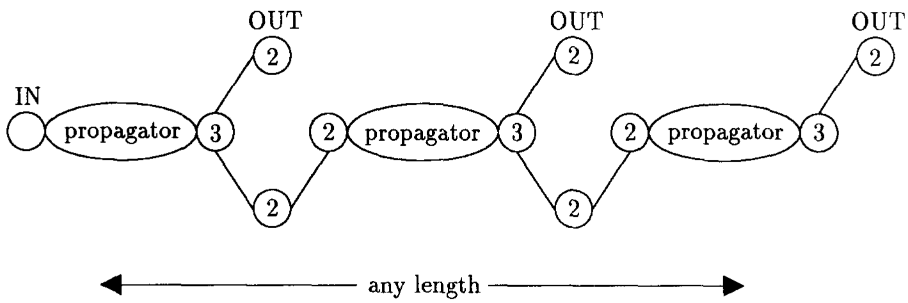

Fig. 4. Multioutput propagator.

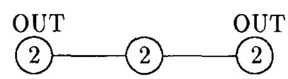

Fig. 5. A '$\exists$-graph'.

Proof of Theorem 1.9. It is easy to see that **BPG** $(2,3)$**-CH** $\in \Pi_2^p$. We transform **RPS** to **BPG** $(2,3)$**-CH**. Let the expression $(\forall U_1)\cdots(\forall U_k)(\exists U_{k+1})\cdots(\exists U_{k+r})\Phi$, denoted as B, be an instance of **RPS**. We shall construct a bipartite planar graph $G=(V,E)$ and a function $f: V \mapsto \{2,3\}$ such that G is f-choosable if and only if B is true. Suppose that Φ has the following form: $C_1 \wedge C_2 \wedge \cdots \wedge C_m$ where each C_i is of the form $(X_{i1} \vee X_{i2} \vee X_{i3})$ and each X_{ij} is U_s or $\overline{U}_s$.

The basic ideas of constructs for the graph involve 'propagators', 'half-propagators', 'multioutput propagators', and 'initial graphs', with some nodes designated as input nodes, and some nodes designated as output nodes. In the following figures a number on a node will be the value f takes on that node when G is formed. The value on an *in* node will be acquired when it gets merged with an *out* node. A half-propagator is the graph is Fig. 3. A propagator can be made by merging the *out* node of any half-propagator with the *in* node of any other half-propagator. A multioutput propagator is shown in Fig. 4. The initial graphs are the graphs in Figs. 5 and 6.

The graph G consists of the following. For each i from 1 to k, we have a $\forall$-graph, with the *out* nodes named U_i and $\overline{U}_i$. For each i from $k+1$ to $k+r$, we have a $\exists$-graph, with the *out* nodes names U_i and $\overline{U}_i$. We think of the C_i's as clauses, and think of

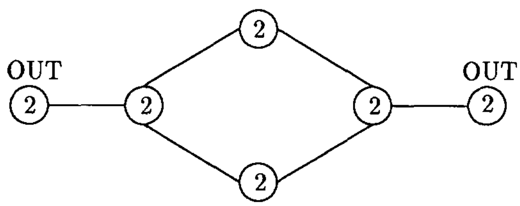

Fig. 6. A '∀-graph'.

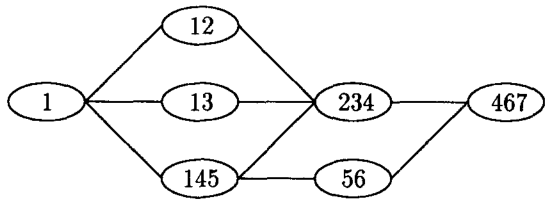

Fig. 7. An assignment for the half-propagator.

U_s and $\overline{U}_s$ as literals. For each literal V we connect a multioutput propagator to the node named V, identifying the *in* node of the propagator with V. All the multioutput propagators look alike having $3m$ output nodes, one for each ij where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant 3$.

Now we add m new nodes (each with $f(C_i) = 3$) named $C_1, C_2, \ldots, C_m$. For each i from 1 to m, and each j from 1 to 3, connect C_i to the ij node of the multioutput propagator attached to the node named X_{ij}.

That describes the graph G, which is obviously bipartite. Every variable occurs in at most three clauses, and therefore it occurs either at most once positive or at most once negative. Combining this with the fact that G_Φ is planar, we conclude that G is planar.

We use here a different half-propagator from the one used in [2], and therefore the following properties needed for the proof should be verified for our half-propagator.

1. A 2-coloration will give the *out* node opposite color to that of the *in* node.
2. For any choice of a letter from the *in* node, and no matter what letters are put on nodes other than the *in* node, there is a compatible choice of letters from the remaining nodes of the half-propagator.
3. For any assignment of letters to nodes other than the *in* node, for any choice of a letter from the *out* node, there is at most one choice of letter incompatible with it on the *in* node. (This is a direct consequence of $K_{2,3}$ being 2-choosable)
4. There is an assignment of letters, and a choice of *in* letter, such that only one choice of a letter from the *out* node is compatible with it. (See Fig. 7)

The proof which appears in [2] can be used to conclude that G is f-choosable iff B is true. □

4. The choosability of planar graphs

In this section we prove Theorems 1.10 and 1.11.

Lemma 4.1. *Let $G=(V,E)$ be an odd cycle, and suppose we have an assignment of sets of integers $S(v) \subseteq Z$ for all vertices $v \in V$, where $S(v)=2$ for all v. There exists a proper coloring $c: V \mapsto Z$ so that $c(v) \in S(v)$ for all $v \in V$ if and only if not all the sets $S(v)$ are equal.*

Proof. Suppose first that not all the sets $S(v)$ are equal. Let x_1 and x_k be adjacent vertices for which $S(x_1) \neq S(x_k)$, where G is the cycle $x_1 - \cdots - x_k - x_1$. Choose a color $c_1 \in S(x_1) - S(x_k)$, and go in a sequence choosing $c_2 \in S(x_2) - \{c_1\}$, $c_3 \in S(x_3) - \{c_2\}, \ldots$ until $c_k \in S(x_k) - \{c_{k-1}\}$. We have obtained a proper coloring of G, as needed.

If the sets $S(v)$ are equal there is no coloring as $\chi(G)=3$. □

Lemma 4.2. *Suppose that C_1 and C_2 are two disjoint copies of the odd cycle of length k, which we denote by $C_1 = x_1 - \cdots - x_k - x_1$ and $C_2 = y_1 - \cdots - y_k - y_1$. Let G be composed of C_1 and C_2 together with the edges (x_i, y_i), $i=1,\ldots,k$. Suppose we have an assignment of sets of integers $S(v) \subseteq Z$ for all vertices $v \in C_2$, where $S(v)=3$ for all $v \in C_2$. Then there is at most one proper coloring of C_1 which cannot be completed to a proper coloring of G by assigning to each vertex $v \in C_2$ a color from $S(v)$.*

Proof. Suppose that c is a proper coloring of C_1 which cannot be completed to a proper coloring of G. Denote $c_i = c(x_i)$, $i=1,\ldots,k$. If follows from Lemma 4.1 that there exist two colors a and b so that $S(y_i)=\{a,b,c_i\}$, $i=1,\ldots,k$. Since c is a proper coloring, surely $\bigcap_{i=1}^{k} S(y_i)=\{a,b\}$. By applying Lemma 4.1 again, we conclude that c is the only proper coloring with the required properties for the considered assignment of sets $S(y_i)=\{a,b,c_i\}$. □

Definition 4.3. A graph $G=(V,E)$ is *k-restrictly-choosable* if G is f_v-choosable for every $v \in V$, where the function f_v is defined as $f_v(v)=k-1$ and $f_v(w)=k$ for every $w \in V-\{v\}$.

Definition 4.4. A graph G is *k-choice-critical* if G is k-choosable but not k-restrictly-choosable.

Definition 4.5. Let $G=(V,E)$ be a graph, and suppose that u and v are two distinct vertices of G. Let S be an assignment of sets of integers $S(w) \subseteq Z$ for all vertices $w \in V$. We denote by *incomp*(G,u,v,S) the set $\{(a,b) \in S(u) \times S(v) |$ there is no proper vertex coloring $c: V \mapsto Z$ so that $c(u)=a$, $c(v)=b$ and $c(w) \in S(w)$ for all $w \in V\}$.

Lemma 4.6. *Let $W_2 = (V,E)$ be the graph in Fig. 2. If S is an assignment of sets of integers $S(w) \subseteq Z$ for all vertices $w \in V$, where $S(w) = 3$ for all w, then* $|\text{incomp}(W_2, u, v, S)| \leqslant 1$.

Proof. Suppose that $(a,b) \in \text{incomp}(W_2, u, v, S)$. It is easy to verify, by applying Lemma 4.2, that $a \neq b$, $a \in S(w)$, $b \in S(x_4) \cap S(y_4)$ and $\{a,b\} \subseteq S(x_2) \cap S(y_2)$. Combining Lemma 4.2 with the fact that $(a,b) \in \text{incomp}(W_2, u, v, S)$, we obtain that there exist a coloring of the vertices $x_1, \ldots, x_4, w$ with the colors $c_1, \ldots, c_5$, respectively, and a coloring of the vertices $y_1, \ldots, y_4, w$ with the colors $d_1, \ldots, d_5$, respectively, which have the properties stated in the lemma. It follows easily that $S(w) = \{a, c_5, d_5\}$ and $S(x_2) = \{a, b, c_2\}$.

In the same manner we can prove that if $(g,h) \in \text{incomp}(W_2, u, v, S)$, then $g \neq h$, $S(w) = \{g, c_5, d_5\}$ and $S(x_2) = \{g, h, c_2\}$, which implies that $g = a$ and $h = b$. This proves that $|\text{incomp}(W_2, u, v, S)| \leqslant 1$, as needed. □

We construct the graph H_1 as follows: We take the disjoint union of the graphs $\{G_i\colon 1 \leqslant i \leqslant 6\}$, where each G_i is a copy of the graph W_2 in Fig. 2. All the 6 vertices named u are identified, as well as all the 6 vertices named v, to obtain the planar triangle-free graph H_1.

Lemma 4.7. *The graph H_1 is 3-choosable.*

Proof. Let S be an assignment of sets of integers $S(w) \subseteq Z$ for all vertices $w \in V$, where $S(w) = 3$ for all w. Suppose first that there exists a color $c \in S(u) \cap S(v)$. It follows immediately that by coloring u and v with the color c we can find a proper coloring.

Suppose next that $S(u) \cap S(v) = \emptyset$. It follows from Lemma 4.6 that $|\text{incomp}(G_i, u, v, S)| \leqslant 1$ for $i=1,\ldots,6$, and therefore $|\text{incomp}(H_1, u, v, S)| \leqslant 6$. Since $|\text{incomp}(H_1, u, v, S)| < |S(u) \times S(v)| = 9$, we conclude that a coloring in possible. □

Lemma 4.8. *The graph H_1 is not 3-restrictly-choosable.*

Proof. Take $S(u) = \{10, 11\}$ and $S(v) = \{12, 13, 14\}$. Proceed as in the proof of Theorem 1.8. □

Lemma 4.9. *There exists a planar triangle-free graph which is 3-choice-critical.*

Proof. Combine Lemmas 4.7 and 4.8. □

Proof of Theorem 1.10. It is easy to see that **PTFG 3-CH** $\in \Pi_2^p$. We transform **BPG (2,3)-CH** to **PTFG 3-CH**. Let the graph $G = (V,E)$ and the function $f\colon V \mapsto \{2,3\}$ be an instance of **BPG (2,3)-CH**. We shall construct a planar triangle-free graph

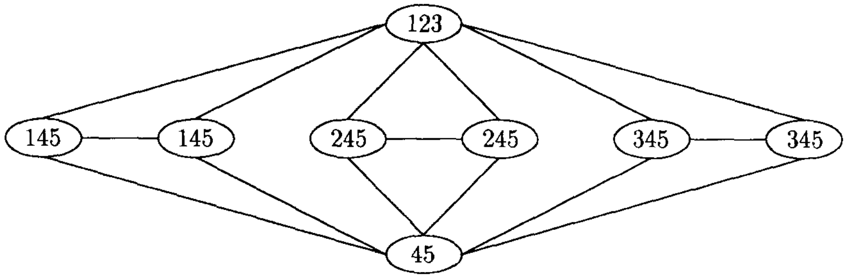

Fig. 8. The graph W_3.

$G' = (V', E')$ such that G' is 3-choosable if and only if G is f-choosable. If follows from Lemma 4.9 that there exists a planar triangle-free graph W which is 3-choice-critical. Let u be a vertex of W for which W is not g_u-choosable, where the function g is defined as $g_u(u) = 2$ and $g_u(w) = 3$ otherwise. The graph G' is obtained from G by adding a disjoint copy of W for every $v \in V(G)$ for which $f(v) = 2$, and connecting v to the vertex u of this copy.

Since both G and W are planar triangle-free graphs, it is easy to see that G' is also a planar triangle-free graph (recall that W has an embedding in the plane so that u appears on the exterior face.) We first prove that if G is f-choosable, then G' is 3-choosable. Take an assignment of sets of integers $S(w) \subseteq Z$ for all vertices $w \in V'$, where $S(w) = 3$ for all w. The graph W is 3-choosable, and so we find a proper coloring in each copy of W in the graph G'. For each copy of W, the color chosen in the vertex u is removed from the vertex of G adjacent to u. The coloring can be completed, since G is f-choosable.

We now prove that if G' is 3-choosable, then G is f-choosable. Suppose we have an assignment of sets of integers $S(w) \subseteq Z$ for all vertices $w \in V(G)$, where $|S(w)| = f(w)$ for all w. Take an assignment which proves that W is not g_u-choosable, and put it in each copy of W in the graph G'. Let d be a new color. For each copy W, we add the color d to the vertex u of this copy and to its neighbor in G. Since G' is 3-choosable, we can find a proper coloring c of G' assigning to each vertex a color from its set. The coloring c restricted to G implies that G is f-choosable. $\square$

In order to prove that deciding whether a given planar graph is 3-choosable is Π_2^p-complete (a weaker version of Theorem 1.10), it is possible to use the planar graph W_3 in Fig. 8. In a similar manner to the previous proofs, one can prove that W_3 is 3-choice-critical. The assignment given in Fig. 8 proves that W_3 is not 3-restrictly-choosable.

Lemma 4.10. *Let $W_1 = (V, E)$ be the graph in Fig. 1. If S is an assignment of sets of integers $S(w) \subseteq Z$ for all vertices $w \in V$, where $S(w) = 4$ for all w, then* $|\mathrm{incomp}(W_1, u, v, S)| \leqslant 1$.

Proof. Suppose that $\{a,b\} \in \text{incomp}(W_1,u,v,S)$. It is easy to verify, by applying Lemma 4.1, that $a \neq b$, $a \in S(x_1) \cap S(y_1)$, $b \in S(x_3) \cap S(y_3)$ and $\{a,b\} \subseteq S(w) \cap S(x_2) \cap S(y_2)$. Combining Lemma 4.1 with the fact that $\{a,b\} \in \text{incomp}(W_1,u,v,S)$, we obtain that there exist three distinct colors c, d and e so that $S(x_2) = \{a,b,c,d\}$, $S(x_1) = \{a,c,d,e\}$ and $S(x_3) = \{b,c,d,e\}$.

In the same manner we can prove that if $(g,h) \in \text{incomp}(W_1,u,v,S)$, then $g \neq h$, $S(x_1) = \{g,c,d,e\}$ and $S(x_3) = \{h,c,d,e\}$, which implies that $g = a$ and $h = b$. This proves that $|\text{incomp}(W_1,u,v,S)| \leqslant 1$, as needed. □

Lemma 4.11. *There exists a planar graph which is 4-choice-critical.*

Proof. Take 12 pairwise disjoint copies of the graph W_1 in Fig. 1 and identify all the 12 vertices named u as well as all the 12 vertices named v. Use Lemma 4.10 and proceed as in the proofs of Lemmas 4.7 and 4.8. □

Proof of Theorem 1.11. Apply Lemma 4.11 as in the proof of Theorem 1.10. □

Acknowledgements

I would like to thank Noga Alon and Michael Tarsi for helpful discussions.

References

[1] N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992) 125–134.
[2] P. Erdős, A.L. Rubin and H. Taylor, Choosability in graphs, in: Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congr. Numer. XXVI (1979) 125–157.
[3] M.R. Garey and D.S. Johnson, Computers and Intractability, A Guide to the Theory of NP-Completeness (W.H. Freeman and Company, New York, 1979).
[4] S. Gutner and M. Tarsi, Some results on $(a:b)$-choosability, submitted.
[5] J. Kratochvil and Zs. Tuza, Algorithmic complexity of list colorings, Discrete Appl. Math. in press.
[6] D. Lichtenstein, Planar formulae and their uses, SIAM J. Comput. 11 (1982) 329–343.
[7] M. Middendorf and F. Pfeiffer, On the complexity of the disjoint paths problem, Combinatorica 13 (1993) 97–107.
[8] M. Stiebitz, private communication.
[9] C. Thomassen, Every planar graph is 5-choosable, Technical Report, The Technical University of Denmark, 1993.
[10] C. Thomassen, 3-list-coloring planar graphs of girth 5, Technical Report, The Technical University of Denmark, February 1994.
[11] V.G. Vizing, Coloring the vertices of a graph in prescribed colors (in Russian), Diskret. Analiz. No. 29, Metody Diskret. Anal. v. Teorii Kodov i Shem 101 (1976) 3–10.
[12] M. Voigt, List colourings of planar graphs, Discrete Math. 120 (1993) 215–219.
[13] M. Voigt, A not 3-choosable planar graph without 3-cycles, Discrete Math., in press.