Problem Set 9 Solutions

Due: Wednesday, November 12

Reading assignment: 11.8–11.11, 12.1–12.3, 12.4.1, 12.6

Problem 1. [10 points]

(a) [5 pts] Show that of any $n + 1$ distinct numbers chosen from the set $\{1, 2, \ldots, 2n\}$, at least 2 must be relatively prime. (*Hint:* $\gcd(k, k + 1) = 1$.)

Solution. Treat the $n + 1$ numbers as the pigeons and the n disjoint subsets of the form $\{2j - 1, 2j\}$ as the pigeonholes. The pigeonhole principle implies that there must exist a pair of consecutive integers among the $n + 1$ chosen which, as suggested in the hint, must be relatively prime. ∎

(b) [5 pts] Show that any finite connected undirected graph with $n \geq 2$ vertices must have 2 vertices with the same degree.

Solution. In a finite connected graph with $n \geq 2$ vertices, the domain for the vertex degrees is the set $\{1, 2, \ldots, n - 1\}$ since each vertex can be adjacent to at most all of the remaining $n-1$ vertices and the existence of a degree 0 vertex would violate the assumption that the graph be connected. Therefore, treating the n vertices as the pigeons and the $n-1$ possible degrees as the pigeonholes, the pigeonhole principle implies that there must exist a pair of vertices with the same degree. ∎

Problem 2. [15 points] Under Siege!

Fearing retribution for the many long hours his students spent completing problem sets, Prof. Leighton decides to convert his office into a reinforced bunker. His only remaining task is to set the 10-digit numeric password on his door. Knowing the students are a clever bunch, he is not going to pick any passwords containing the forbidden consecutive sequences "18062", "6042" or "35876" (his MIT extension).

How many 10-digit passwords can he pick that don't contain forbidden sequences if each number $0, 1, \ldots, 9$ can only be chosen once (i.e. without replacement)?

Solution. The number of passwords he can choose is the number of permutations of the 10 digits minus the number of passwords containing one or more of the forbidden words, which we will find using inclusion-exclusion.

There are 6 positions 18062 could appear and the remaining digits could be any permutation of the remaining 5 digits. Therefore, there are $6 \cdot 5!$ passwords containing 18062. Similarly, there are $7 \cdot 6!$ passwords containing 6042 and $6 \cdot 5!$ passwords containing 35876.

Each of the forbidden words contain the digit 6 and since he must choose each number exactly once, the only way two forbidden words can appear in the same password is if they overlap at 6. The only case where this can happen is if the password contains 35876042 and there are $3 \cdot 2!$ such passwords.

By inclusion-exclusion the total number of passwords not containing any of the forbidden words is

$$10! - (6 \cdot 5! + 7 \cdot 6! + 6 \cdot 5!) + 3 \cdot 2! = 3622326$$

∎

Problem 3. [15 points] Give a combinatorial proof of the following theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$

*(Hint: Consider the set of all length-n sequences of 0's, 1's and a single *.)*

Solution. Let $P = \{0, \ldots, n-1\} \times \{0,1\}^{n-1}$. On the one hand, there is a bijection from P to S by mapping (k, x) to the word obtained by inserting a * just after the kth bit in the length-$n-1$ binary word, x. So

$$|S| = |P| = n2^{n-1} \tag{1}$$

by the Product Rule.

On the other hand, every sequence in S contains between 1 and n nonzero entries since the *, at least, is nonzero. The mapping from a sequence in S with exactly k nonzero entries to a pair consisting of the set of positions of the nonzero entries and the position of the * among these entries is a bijection, and the number of such pairs is $\binom{n}{k}k$ by the Generalized Product Rule. Thus, by the Sum Rule:

$$|S| = \sum_{k=1}^{n} k \binom{n}{k}$$

Equating this expression and the expression (1) for $|S|$ proves the theorem.   ∎

Problem 4. [20 points] Calculate the following. Make sure to explain your reasoning.

(a) [5 pts] How many n-digit PIN numbers are there where no 2 consecutive digits are the same?

Solution. There are 10 choices for the first digit, and 9 choices for each of the remaining $n - 1$ digits, since you can choose any digit that is not the same as the one right in front of it, so there are

$$10 \cdot 9^{n-1}$$

such PIN numbers. ∎

(b) [5 pts] How many numbers are there that are in the range $[1..700]$ which are divisible by $2, 5$ or 7?

Solution. Let S be the set of all numbers in range $[1..700]$. Let S_2 be the numbers in this range divisible by 2, S_5 be the numbers in this range divisible by 5 and S_7 be the numbers in this range divisible by 7. By inclusion-exclusion, the number of elements in S divisible by 2, 5 or 7 is

$$
\begin{aligned}
n &= |S_2| + |S_5| + |S_7| - |S_2 S_5| - |S_2 S_7| - |S_5 S_7| + |S_2 S_5 S_7| \\
&= \frac{700}{2} + \frac{700}{5} + \frac{700}{7} - \frac{700}{2 \cdot 5} - \frac{700}{2 \cdot 7} - \frac{700}{5 \cdot 7} + \frac{700}{2 \cdot 5 \cdot 7} \\
&= 350 + 140 + 100 - 70 - 50 - 20 + 10 \\
&= 460.
\end{aligned}
$$

∎

(c) [10 pts] How many 10 digit numbers are there in which there are exactly 5 occurances of the digit 9 and the first two digits are not the same?

Solution. Consider 2 cases

1. The first digit is a 9.
   Then the remaining nines have to be in the last 8 slots and the remaining 5 slots can be filled with any of the remaining 9 digits. There are

   $$\binom{9}{4} 9^5$$

   such numbers.

2. The first digit is not a 9, but the second one is.
   Then the remaining nines have to be in the last 8 slots and the remaining 5 slots can again be filled with any of the remaining 9 digits. There are again

   $$\binom{9}{4} 9^5$$

   such numbers.

3. The first two digits are not nines. In this case, there are $9 \cdot 8$ ways to select the first two digits, the 5 nines have to be in the last 8 slots and the remaining 3 slots can be filled with any digit other than 9. This gives

$$9 \cdot 8 \binom{8}{5} 9^3$$

such numbers.

In the end, the final answer is

$$2 \cdot \binom{9}{4} \cdot 9^5 + 8 \cdot \binom{8}{5} \cdot 9^4.$$

∎

Problem 5. [10 points] In the card game of bridge, you are dealt a hand of 13 cards from the standard 52-card deck.

(a) [5 pts] A balanced hand is one in which a player has roughly the same number of cards in each suit. How many different hands are there where the player has 4 cards in one suit and 3 cards in each of the other suits?

Solution. There are 4 suits to pick from for the longest suit, 4 cards out of 13 to choose from in that suit, and 3 cards out of 13 to choose from in each of the remaining 3 suits. This gives

$$4 \cdot \binom{13}{4} \cdot \binom{13}{3} \cdot \binom{13}{3} \cdot \binom{13}{3}$$

such hands.   ∎

(b) [5 pts] Not surprisingly, a non-balanced hand is one in which a player has more cards in some suits than others. Hands that are very disired are ones where over half the cards are in one suit. How many different hands are there where there is exactly 7 cards in one suit?

Solution. There are 4 suits to choose from for the long suit, 7 cards out of 13 to choose in that suit, and 6 cards out of the remaining 39 in the other 3 suits. This gives

$$4 \cdot \binom{13}{7} \cdot \binom{39}{6}$$

such hands.   ∎

Problem 6. [30 points]

(a) [5 pts] How many ways are there to distibute n dollars among k people assuming each person has to get an integer amount of dollars?

Solution. We can encode a solution as a string of $n + k - 1$ bits with $k - 1$ of these as ones and n of these as dollars: a zero indicates to give a dollar to the 'current' person, and a one indicates to move to the next person. By the bookkeeper rule, the number of such strings is:

$$\binom{n + k - 1}{k - 1}$$

∎

(b) [5 pts] How about if everyone has to get at least one dollar?

Solution. We can modify the approach of (a) as follows: we are looking for a string consisting of n tokens '0' or '10', and necessarily starting with 0. This ensures that, if we interpret zeroes and ones as in the solution to part (a), each person gets at least one dollar. Applying the bookkeeper rule to this scenario (excluding the guaranteed leading 0), we obtain:

$$\binom{n - 1}{k - 1}$$

∎

(c) [10 pts] In how many ways can you arrange n books on k bookshelves (assuming the books are distinguishable, so that order matters)?

Solution. There are $\binom{n+k-1}{k-1}$ ways to distribute n books to k bookshelves, if we ignore order, using the same zeroes-and-ones bookkeeper method as in part (a). Given such a distribution, there are now $n!$ permutations of the particular books. By the (generalized) product rule, the result is:

$$n! \cdot \binom{n + k - 1}{k - 1}$$

∎

(d) [10 pts] How about if there has to be at least 1 book at each bookshelf?

Solution. Following the approach of part (b), there are $\binom{n-1}{k-1}$ ways to distribute n books to k shelves, ignoring order, if each shelf must have at least one book. Again there are $n!$ permutations of the particular books in question, so that the total count of arrangements is

$$n! \binom{n - 1}{k - 1} = \frac{n!(n - 1)!}{(k - 1)!(n - k)!}$$

∎

Problem 7. [25 points] We will use generating functions to determine how many ways there are to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

(a) [4 pts] Write the generating function $P(x)$ for for the number of ways to use only pennies to make n cents.

Solution. Since there is only one way to change any given amount with only pennies, so the generating function for the sequence is

$$P(x) := 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

∎

(b) [4 pts] Write the generating function $N(x)$ for the number of ways to use only nickels to make n cents.

Solution. There is no way to change amounts that are not multiples of five with only nickels. So the generating function for the sequence is

$$
\begin{aligned}
N(x) &\\
&:= 1 + 0x + 0x^2 + 0x^3 + 0x^4 + 1x^5 + 0x^6 + 0x^7 + 0x^8 + 0x^9 + x^{10} + 0x^{11} + \cdots\\
&= 1 + x^5 + x^{10} + x^{15} + \cdots\\
&= \frac{1}{1-x^5}.
\end{aligned}
$$

∎

(c) [8 pts] Write the generating function for the number of ways to use only nickels and pennies to change n cents.

Solution. Since $P(x)$ and $N(x)$ are generating functions for the number of ways to choose pennies and nickels separately, by the Convolution Rule for generating functions, the generating function for the number of ways to choose pennies and nickels together, is the their product

$$N(x) \cdot P(x) = \frac{1}{(1-x)(1-x^5)}.$$

∎

(d) [4 pts] Write the generating function for the number of ways to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

Solution. Generalizing our method gives;

$$C(x) := \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}.$$

∎

(e) [5 pts] Explain how to use this function to find out how many ways are there to change 50 cents; you do *not* have to provide the answer or actually carry out the process.

Solution. The answer is the coefficient to x^{50} of the power series for $C(x)$, which happens to be 50. This coefficient could be extracted by taking the 50th derivative of $C(x)$. This approach would not be appropriate for hand calculation, but the answer would be easy to get using a symbolic mathematics program such as Mathematica. ∎

Problem 8. [20 points] Recall the operation of taking derivatives of generating functions. This is done termwise, that is, if

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots ,$$

then

$$F'(x) := f_1 + 2 f_2 x + 3 f_3 x^2 + \cdots .$$

For example,

$$\frac{1}{(1-x)^2} = \left(\frac{1}{(1-x)} \right)' = 1 + 2x + 3x^2 + \cdots$$

so

$$H(x) := \frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + \cdots$$

is the generating function for the sequence of nonnegative integers. Therefore

$$\frac{1+x}{(1-x)^3} = H'(x) = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots ,$$

so

$$\frac{x^2 + x}{(1-x)^3} = x H'(x) = 0 + 1x + 2^2 x^2 + 3^2 x^3 + \cdots + n^2 x^n + \cdots$$

is the generating function for the nonnegative integer squares.

(a) [10 pts] Prove that for all $k \in \mathbb{N}$, the generating function for the nonnegative integer kth powers is a quotient of polynomials in x. That is, for all $k \in \mathbb{N}$ there are polynomials $R_k(x)$ and $S_k(x)$ such that

$$[x^n] \left(\frac{R_k(x)}{S_k(x)} \right) = n^k. \tag{2}$$

(Hint: Observe that the derivative of a quotient of polynomials is also a quotient of polynomials. It is not necessary work out explicit formulas for R_k and S_k to prove this part.)

Solution. The proof is by induction on k with the hypothesis that there are polynomials $R_k(x)$ and $S_k(x)$ satisfying (2).

Base case $k = 0$: Let $R_0(x) := 1$, $S_0(x) := (1-x)$.

Inductive step: Assuming by induction that we have R_k, S_k satisfying (2), the generating function for the $k+1$st powers will be

$$x \left(\frac{R_k(x)}{S_k(x)} \right)' = \frac{x R_k' S_k - x R_k S_k'}{S_k^2} \tag{3}$$

But the derivative of a polynomial is a polynomial, so the right hand side of (3) is a sum of quotients of polynomials, which can always be simplified into a quotient of two polynomials. Let R_{k+1} and S_{k+1} be these two polynomials. ∎

(b) [10 pts] Conclude that if $f(n)$ is a function on the nonnegative integers defined recursively in the form

$$f(n) = af(n-1) + bf(n-2) + cf(n-3) + p(n)\alpha^n$$

where the $a, b, c, \alpha \in \mathbb{C}$ and p is a polynomial with complex coefficients, then the generating function for the sequence $f(0), f(1), f(2), \ldots$ will be a quotient of polynomials in x.

(*Hint: Consider* $R_k(\alpha x)/S_k(\alpha x)$.)

Solution. By part (a), $R_k(\alpha x)/S_k(\alpha x)$ is the generating function for the sequence whose nth term is $n^k \alpha^n$. So for any function of n of the form $p(n)\alpha^n$, a linear combination of such expressions R_k/S_k for $k = 0, 1, \ldots, \deg(p)$ will yield a quotient, $Q(x)$, of polynomials that is a generating function, for $p(n)\alpha^n$, that is $[x^n]Q(x) = p(n)\alpha^n$.

Letting $F(x)$ be the generating function with $[x^n]F(x) = f(n)$, the usual argument shows that

$$F(x) - axF(x) - bx^2F(x) - cx^3F(x) - Q(x)$$
$$= rx^2 + sx + t,$$

where

$$r = f(2) - af(1) - bf(0) - p(2)\alpha^2$$
$$s = f(1) - af(0) - p(1)\alpha$$
$$t = f(0) - p(0).$$

This shows that $F(x)$ is a quotient of polynomials, namely,

$$F(x) = \frac{Q(x) + rx^2 + sx + t}{1 - ax - bx^2 - cx^3}.$$

∎

Problem 9. [15 points]

Prove, using generating functions, that

$$\binom{i+j}{k} = \sum_{\ell=0}^{k} \binom{i}{l}\binom{j}{k-\ell}.$$

(*Hint: Think of both expressions as kth terms of some sequences whose generating functions you might want to determine. The right-hand side looks like it could be a convolution.*)

Solution. The left-hand side is the kth coefficient of the generating function

$$\sum_{n=0}^{\infty} \binom{i+j}{n}x^n = (x+1)^n.$$

The right-hand side is the kth coefficient of the convolution of the following sequences:

$$a_n = \binom{i}{n}, \qquad b_n = \binom{j}{n},$$

which have generating functions

$$(x+1)^i \quad \text{and} \quad (x+1)^j.$$

The result now follows from the convolution law: the convolution on the right-hand side has generating function

$$(x+1)^i \cdot (x+1)^j = (x+1)^{i+j},$$

which is precisely the generating function for the left-hand side; so the left and right are equal. ∎