SP10 Final Exam Problem 1

Problem 1.
A *literal* is a propositional variable or its negation. A k-*clause* is an OR of k literals, with no variable occurring more than once in the clause. For example,

$$P \text{ OR } \overline{Q} \text{ OR } \overline{R} \text{ OR } V,$$

is a 4-clause, but

$$\overline{V} \text{ OR } \overline{Q} \text{ OR } \overline{X} \text{ OR } V,$$

is not, since V appears twice.

Let $\mathcal{S}$ be a set of n distinct k-clauses involving v variables. The variables in different k-clauses may overlap or be completely different, so $k \leq v \leq nk$.

A random assignment of true/false values will be made independently to each of the v variables, with true and false assignments equally likely. Write formulas in n, k, and v in answer to the first two parts below.

(a) What is the probability that the last k-clause in $\mathcal{S}$ is true under the random assignment?

(b) What is the expected number of true k-clauses in $\mathcal{S}$?

(c) A set of propositions is *satisfiable* iff there is an assignment to the variables that makes all of the propositions true. Use your answer to part (b) to prove that if $n < 2^k$, then $\mathcal{S}$ is satisfiable.

SP10 Final Exam Problem 2

Problem 2.

For each of the relations below, indicate whether it is *transitive* but not a partial order (**Tr**), a *total order* (**Tot**), a *strict partial order* that is not total (**S**), a *weak partial order* that is not total (**W**), or *none* of the above (**N**).

- the "is a subgraph of" relation on graphs. ________
  (Note that every graph is considered a subgraph of itself.)

Let f, g be nonnegative functions on the real numbers.

- the "Big Oh" relation, $f = O(g)$, ________
- the "Little Oh" relation, $f = o(g)$, ________
- the "asymptotically equal" relation, $f \sim g$. ________

SP10 Final Exam Problem 3

Problem 3.

False Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If G has a vertex of degree strictly less than k, then G is k-colorable.*

(a) Give a counterexample to the False Claim when $k = 2$.

(b) Underline the exact sentence or part of a sentence that is the first unjustified step in the following bogus proof of the False Claim.

Bogus proof. Proof by induction on the number n of vertices:

Induction hypothesis:

$P(n)$::= "Let G be an n-vertex graph whose vertex degrees are all $\leq k$. If G also has a vertex of degree strictly less than k, then G is k-colorable."

Base case: $(n = 1)$ G has one vertex, the degree of which is 0. Since G is 1-colorable, $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices whose vertex degrees are all k or less. Also, suppose G_{n+1} has a vertex, v, of degree strictly less than k. Now we only need to prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Let u be a vertex that is adjacent to v in G_{n+1}. Removing v reduces the degree of u by 1. So in G_n, vertex u has degree strictly less than k. Since no edges were added, the vertex degrees of G_n remain $\leq k$. So G_n satisfies the conditions of the induction hypothesis, $P(n)$, and so we conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. ■

(c) With a slightly strengthened condition, the preceding proof of the False Claim could be revised into a sound proof of the following Claim:

Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If* ⟨statement inserted from below⟩ *has a vertex of degree strictly less than k, then G is k-colorable.*

Circle each of the statements below that could be inserted to make the proof correct.

- G is connected and
- G has no vertex of degree zero and
- G does not contain a complete graph on k vertices and
- every connected component of G
- some connected component of G

SP10 Final Exam Problem 4

Problem 4.

Definition 12.2.2 of planar graph embeddings applied only to connected planar graphs. The definition can be extended to planar graphs that are not necessarily connected by adding the following additional constructor case to the definition:

- **Constructor Case:** (collect disjoint graphs) Suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are planar embeddings with no vertices in common. Then $\mathcal{E}_1 \cup \mathcal{E}_2$ is a planar embedding.

Euler's Planar Graph Theorem now generalizes to unconnected graphs as follows: if a planar embedding, $\mathcal{E}$, has v vertices, e edges, f faces, and c connected components, then

$$v - e + f - 2c = 0. \tag{1}$$

This can be proved by structural induction on the definition of planar embedding.

(a) State and prove the base case of the structural induction.

(b) Let v_i, e_i, f_i, and c_i be the number of vertices, edges, faces, and connected components in embedding $\mathcal{E}_i$ and let v, e, f, c be the numbers for the embedding from the (collect disjoint graphs) constructor case. Express v, e, f, c in terms of v_i, e_i, f_i, c_i.

(c) Prove the (collect disjoint graphs) case of the structural induction.

SP10 Final Exam Problem 5

Problem 5.

(a) What is the value of $\phi(175)$, where ϕ is Euler's function?

(b) Call a number from 0 to 174 *powerful* iff some positive power of the number is congruent to 1 modulo 175. What is the probability that a random number from 0 to 174 is powerful?

(c) What is the remainder of $(-12)^{482}$ divided by 175?

SP10 Final Exam Problem 6

Problem 6.
In this problem we consider the famous Math for Computer Science magic trick. Unlike the one performed in class by the TAs, this time the Assistant will be choosing 4 cards and revealing 3 of them to the Magician (in some particular order) instead of choosing 5 and revealing 4.

(a) Show that the Magician could not pull off this trick with a deck larger than 27 cards.

(b) Show that, in principle, the Magician could pull off the Card Trick with a deck of exactly 27 cards. (You do not need to describe the actual method.)

SP10 Final Exam Problem 7

Problem 7.

(a) Let S be a set with i elements. How many ways are there to divide S into a pair of subsets?

(b) Here is a combinatorial proof of an equation giving a closed form for a certain summation $\sum_{i=0}^{n}$:

> There are n marbles, each of which is to be painted red, green, blue, or yellow. One way to assign colors is to choose red, green, blue, or yellow successively for each marble.
>
> An alternative way to assign colors to the marbles is to
>
> - choose a number, i, between 0 and n,
> - choose a set, S, of i marbles,
> - divide S into two subsets; paint the first subset red and the other subset green.
> - divide the set of all the marbles not in S into two subsets; paint the first subset blue and the other subset yellow.

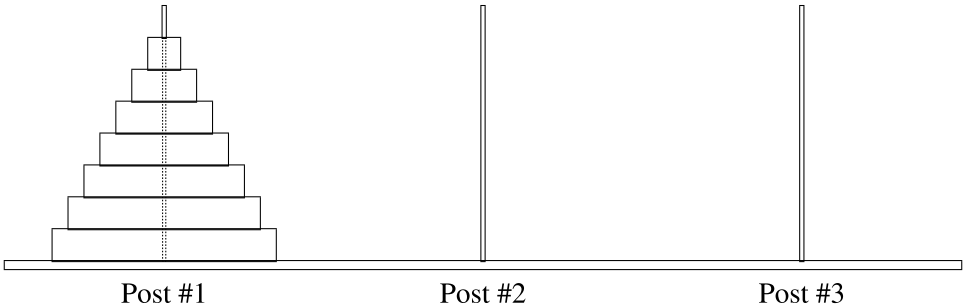
What is the equation?

(c) Now use the binomial theorem to prove the same equation.

SP10 Final Exam Problem 8

Problem 8.

Less well-known than the Towers of Hanoi—but no less fascinating—are the Towers of Sheboygan. As in Hanoi, the puzzle in Sheboygan involves 3 posts and n disks of different sizes. Initially, all the disks are on post #1:



The objective is to transfer all n disks to post #2 via a sequence of moves. A move consists of removing the top disk from one post and dropping it onto another post with the restriction that a larger disk can never lie above a smaller disk. Furthermore, a local ordinance requires that *a disk can be moved only from a post to the next post on its right—or from post #3 to post #1.* Thus, for example, moving a disk directly from post #1 to post #3 is not permitted.

(a) One procedure that solves the Sheboygan puzzle is defined recursively: to move an initial stack of n disks to the next post, move the top stack of $n-1$ disks to the furthest post by moving it to the next post two times, then move the big, nth disk to the next post, and finally move the top stack another two times to land on top of the big disk. Let s_n be the number of moves that this procedure uses. Write a simple linear recurrence for s_n.

(b) Let $S(x)$ be the generating function for the sequence $\langle s_0, s_1, s_2, \ldots \rangle$. Carefully show that

$$S(x) = \frac{x}{(1-x)(1-4x)}.$$

(c) Give a simple formula for s_n.

SP10 Final Exam Problem 9

Problem 9.
The hat-check staff has had a long day serving at a party, and at the end of the party they simply return the n checked hats in a random way such that the probability that any particular person gets their own hat back is $1/n$.

Let X_i be the indicator variable for the ith person getting their own hat back. Let S_n be the total number of people who get their own hat back.

(a) What is the expected number of people who get their own hat back?

(b) Write a simple formula for $\text{Ex}[X_i X_j]$ for $i \neq j$.
Hint: What is $\Pr\left[X_j = 1 \mid X_i = 1\right]$?

(c) Explain why you cannot use the variance of sums formula to calculate $\text{Var}[S_n]$.

(d) Show that $\text{Ex}[S_n^2] = 2$. *Hint:* $X_i^2 = X_i$.

(e) What is the variance of S_n?

(f) Show that there is at most a 1% chance that more than 10 people get their own hat back. Try to give an intuitive explanation of why the chance remains this small regardless of n.

SP10 Final Exam Problem 10

Problem 10.
Yesterday, the bakers at a local cake factory baked a huge number of cakes. To estimate the fraction, b, of cakes in this program that are improperly prepared, the cake-testers will take a small sample of cakes chosen randomly and independently (so it is possible, though unlikely, that the same cake might be chosen more than once). For each cake chosen, they perform a variety of non-destructive tests to determine if the cake is

improperly prepared, after which they will use the fraction of bad cakes in their sample as their estimate of the fraction b.

The factory statistician can use estimates of a binomial distribution to calculate a value, s, for a number of cakes to sample which ensures that with 97% confidence, the fraction of bad cakes in the sample will be within 0.006 of the actual fraction, b, of bad cakes in the back.

Mathematically, the *batch* is an actual outcome that already happened. The *sample* is a random variable defined by the process for randomly choosing s cakes from the batch. The justification for the statistician's confidence depends on some properties of the batch and how the sample of s cakes from the batch are chosen. These properties are described in some of the statements below. Mark each of these statements as **T** (true) or **F** (false), and then briefly explain your answer.

1. The probability that the ninth cake in the *batch* is bad is b. _______
2. All cakes in the batch are equally likely to be the third cake chosen in the *sample*. _______
3. The probability that the ninth cake chosen for the *sample* is bad, is b. _______
4. Given that the first cake chosen for the *sample* is bad, the probability that the second cake chosen will also be bad is greater than b.

   ____
5. Given that the last cake in the *batch* is bad, the probability that the next-to-last cake in the batch will also be bad is greater than b.

   ____
6. Given that the first two cakes selected in the *sample* are the same kind of cake —they might both be chocoloate, or both be angel food cakes,...—the probability that the first cake is bad may be greater than b.

   ____
7. The expectation of the indicator variable for the last cake in the *sample* being bad is b. _______
8. There is zero probability that all the cakes in the *sample* will be different. _______

FA11 CP3f Problem 4

Problem 11.
This problem provides a proof of the [Schröder-Bernstein] Theorem:

$$\text{If } A \text{ surj } B \text{ and } B \text{ surj } A, \text{ then } A \text{ bij } B. \tag{2}$$

(a) It is OK to assume that A and B are disjoint. Why?

(b) Explain why there are total injective functions $f : A \to B$, and $g : B \to A$.

Picturing the diagrams for f and g, there is *exactly one* arrow *out* of each element —a left-to-right f-arrow if the element is in A and a right-to-left g-arrow if the element is in B. This is because f and g are total functions. Also, there is *at most one* arrow *into* any element, because f and g are injections.

So starting at any element, there is a unique, and unending path of arrows going forwards. There is also a unique path of arrows going backwards, which might be unending, or might end at an element that has no arrow into it. These paths are completely separate: if two ran into each other, there would be two arrows into the element where they ran together.

This divides all the elements into separate paths of four kinds:

i. paths that are infinite in both directions,

ii. paths that are infinite going forwards starting from some element of A.

iii. paths that are infinite going forwards starting from some element of B.

iv. paths that are unending but finite.

(c) What do the paths of the last type (iv) look like?

(d) Show that for each type of path, either

- the f-arrows define a bijection between the A and B elements on the path, or
- the g-arrows define a bijection between B and A elements on the path, or
- both sets of arrows define bijections.

For which kinds of paths do both sets of arrows define bijections?

(e) Explain how to piece these bijections together to prove that A and B are the same size.

FA11 CP11w Problem 3

Problem 12.
Here are the solutions to the next 10 short answer questions, in no particular order. Enter the solution number after each question.

1. $\dfrac{n!}{(n-m)!}$ 2. $\dbinom{n+m}{m}$ 3. $(n-m)!$ 4. m^n

5. $\dbinom{n-1+m}{m}$ 6. $\dbinom{n-1+m}{n}$ 7. 2^{mn} 8. n^m

(a) How many solutions over the nonnegative integers are there to the inequality ________

$$x_1 + x_2 + \cdots + x_n \leq m\ ?$$

(b) How many length m words can be formed from an n-letter alphabet, if no letter is used more than once? ________

(c) How many length m words can be formed from an n-letter alphabet, if letters can be reused? ________

(d) How many binary relations are there from set A to set B when $|A| = m$ and $|B| = n$? ________

(e) How many total injective functions are there from set A to set B, where $|A| = m$ and $|B| = n \geq m$? ________

(f) How many ways are there to place a total of m distinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls? ________

(g) How many ways are there to place a total of m indistinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls? ________

(h) How many ways are there to put a total of m distinguishable balls into n distinguishable urns with at most one ball in each urn? ________

FA11 CP11f Problem 1

Problem 13.
Section 15.11.3 explained why it is not possible to perform a four-card variant of the hidden-card magic trick with one card hidden. But the Magician and her Assistant are determined to find a way to make a trick like this work. They decide to change the rules slightly: instead of the Assistant lining up the three unhidden cards for the Magician to see, he will line up all four cards with one card face down and the other three visible. We'll call this the *face-down four-card trick*.

For example, suppose the audience members had selected the cards $9\heartsuit$, $10\diamondsuit$, $A\clubsuit$, $5\clubsuit$. Then the Assistant could choose to arrange the 4 cards in any order so long as one is face down and the others are visible. Two possibilities are:

$A\clubsuit$	?	$10\diamondsuit$	$5\clubsuit$
?	$5\clubsuit$	$9\heartsuit$	$10\diamondsuit$

(a) Explain how to model this the face-down four-card trick as a matching problem, and show that there must be a bipartite matching which theoretically will allow the Magician and Assistant to perform the trick.

(b) There is actually a simple way to perform the face-down four-card trick.[1]

Case 1. *there are two cards with the same suit*: Say there are two $\spadesuit$ cards. The Assistant proceeds as in the original card trick: he puts one of the $\spadesuit$ cards *face up as the first card*. He will place the second $\spadesuit$ card *face down*. He then uses a permutation of the face down card and the remaining two face up cards to code the offset of the face down card from the first card.

Case 2. *all four cards have different suits*: Assign numbers $0, 1, 2, 3$ to the four suits in some agreed upon way. The Assistant computes, s, the sum modulo 4 of the ranks of the four cards, and chooses the card with suit s to be placed *face down as the first card*. He then uses a permutation of the remaining three face-up cards to code the rank of the face down card.

Explain how in Case 2. the Magician can determine the face down card from the cards the Assistant shows her.

(c) Explain how any method for performing the face-down four-card trick can be adapted to perform the regular (5-card hand, show 4 cards) with a 52-card deck consisting of the usual 52 cards along with a 53rd card call the *joker*.

FA11 MQ6-(both) Problem 1

Problem 14.

(a) Each row in the following table starts with a partial order. The second row starts with the divides relation on $\mathbb{N}$ where $a \mid b$ iff $b = ak$ for some $k \in \mathbb{N}$. The third row starts with the less than relation on $\mathbb{N}$ and the complex number, i, where $a < b$ iff $a, b \in \mathbb{N}$ and $a < b$.

Fill in the remaining entries in each row.

[1] This elegant method was devised in Fall '09 by student Katie E Everett.

partial order	**path-total order** (YES, NO)	**minimal(s)**	**maximal(s)**
$\subseteq$ on $\mathcal{P}(\{1,2,3\})$			
divides on $\mathbb{N}$			
$<$ on $\mathbb{N} \cup \{i\}$			

(b) What is the longest *chain* on the subset relation, $\subseteq$, on $\mathcal{P}(\{1,2,3\})$? (If there is more than one, provide **one** of them.)

(c) What is the longest *anti-chain* on the subset relation, $\subseteq$, on $\mathcal{P}(\{1,2,3\})$? (If there is more than one, provide **one** of them.)

FA11 MQ11-morning Problem 1

Problem 15.
Alyssa Hacker sends out a video that spreads like wildfire over the UToob network. On the day of the release (call it zero-day) and the very next day the video doesn't receive any hits. However, starting with day 2, the number of hits, r_n, can be expressed as seven times the number of hits on the previous day, four times the number of hits the day before that, and the number of days that has passed since the release of the video plus one. So, for example on day 2, there will be $7 \times 0 + 4 \times 0 + 3 = 3$ hits.

Provide a recurrence that models the number of hits that the video receives on day n.

Express the generating function of this sequence as a quotient of polynomials or products of polynomials. You do *not* have to find a closed form for r_n.

SP05 Final Exam Problem 2 (problem not shown)
SP05 Final Exam Problem 6 (problem not shown)