Tom Leighton and Marten van Dijk

Problems for Recitation 15

1 The Tao of BOOKKEEPER

In this problem, we seek enlightenment through contemplation of the word $BOOKKEEPER$.

1. In how many ways can you arrange the letters in the word $POKE$?

2. In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

3. Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	$\ldots$
BO_2O_1K	
$\ldots$	

4. What kind of mapping is this, young grasshopper?

5. In light of the Division Rule, how many arrangements are there of $BOOK$?

6. Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

7. Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

8. What kind of mapping is this?

9. So how many arrangements are there of the letters in $KEEPER$?

10. *Now you are ready to face the BOOKKEEPER!*

    How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

11. How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

12. How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

13. How many arrangements of $BOOKKEEPER$ are there?

14. How many arrangements of $VOODOODOLL$ are there?

15. **(IMPORTANT)** How many n-bit sequences contain k zeros and $(n - k)$ ones?

This quantity is denoted $\binom{n}{k}$ and read "n choose k". You will see it almost every day in 6.042 from now until the end of the term.

Remember well what you have learned: subscripts on, subscripts off.

This is the Tao of Bookkeeper.

2 Pigeonhole Principle

Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

1. In a room of 500 people, there exist two who share a birthday.

2. Suppose that each of the 115 students in 6.042 sums the nine digits of his or her ID number. Must two people arrive at the same sum?

3. In every set of 100 integers, there exist two whose difference is a multiple of 37.

3 More Counting Problems

Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

1. **(IMPORTANT)** In how many ways can k elements be chosen from an n-element set $\{x_1, x_2, \ldots, x_n\}$?

2. How many different ways are there to select a dozen donuts if five varieties are available? (We discussed a bijection for this set in Recitation 15. Now use that bijection to give a count.)

3. An independent living group is hosting eight pre-frosh, affectionately known as $P_1, \ldots, P_8$ by the permanent residents. Each pre-frosh is assigned a task: 2 must wash pots, 2 must clean the kitchen, 1 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. In how many ways can $P_1, \ldots, P_8$ be put to productive use?

4. Suppose that two identical 52-card decks of are mixed together. In how many ways can the cards in this double-size deck be arranged?

4 Fun with Phonology: Hawaiian

The Hawaiian language is rich in vowels: it contains 8 consonants and 25 vowels[1]! In addition, every word in Hawaiian must end in a vowel and must not contain two consonants in a row. Let's assume that all combinations of vowels and consonants that satisfy these constraints are valid.

We'd like to know how many n-phoneme words there are in Hawaiian. (A *phoneme* is either a single vowel or a single consonant. Assume no phoneme can be both a vowel and a consonant.) For simplicity, let's assume n is even.

1. Before tackling the general problem, work out how many different words there are with exactly 4 phonemes. (Which distributions of vowels and consonants are possible?)

2. Now for the general case. Let A be the set of all n-phoneme words, and let A_k be the set of all n-phoneme words with exactly k consonants. Express $|A|$ in terms of $|A_k|$ for all possible k.

3. Now let's find $|A_k|$ for an arbitrary k. For simplicity's sake, assume Hawaiian has only one consonant and only one vowel. Find a bijection between A_k and a set of arbitrary sequences of `0` and `1` of length p. What is p?

4. Using this bijection, compute $|A_k|$.

5. How would you change your expression for $|A_k|$ to allow for 8 consonants and 25 vowels, not just one of each?

[1]Counting long vowels and diphthongs. For this problem, treat each of the 25 vowels as a unique single vowel.

6. How many n-phoneme words are there in Hawaiian? (You don't have to find a closed form for your expression.)

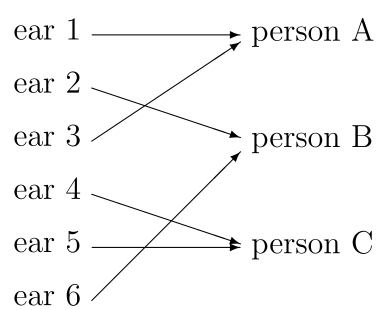
Appendix: Basic Counting Notions

Rule 1 (Bijection Rule). *If there exists a bijection $f : A \to B$, then $|A| = |B|$.*

Rule 2 (Generalized Pigeonhole Principle). *If $|X| > k \cdot |Y|$, then for every function $f : X \to Y$ there exist $k + 1$ different elements of X that are mapped to the same element of Y.*

> "If more than n pigeons are assigned to n holes, then there must exist two pigeons assigned to the same hole."

A ***k-to-1 function*** maps exactly k elements of the domain to every element of the range. For example, the function mapping each ear to its owner is 2-to-1:



Rule 3 (Division Rule). *If $f : A \to B$ is k-to-1, then $|A| = k \cdot |B|$.*

Rule 4 (Product Rule). *If $P_1, P_2, \ldots P_n$ are sets, then:*

$$|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdots |P_n|$$

Rule 5 (Generalized Product Rule). *Let S be a set of length-k sequences. If there are:*

- *n_1 possible first entries,*
- *n_2 possible second entries for each first entry,*
- *n_3 possible third entries for each combination of first and second entries, etc.*

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

Rule 6 (Sum Rule). *If $A_1, \ldots, A_n$ are disjoint sets, then:*

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} |A_k|$$