In-Class Problems Week 12, Mon.

Problem 1.
Find the coefficients of

(a) x^5 in $(1+x)^{11}$

(b) x^8y^9 in $(3x+2y)^{17}$

(c) a^6b^6 in $(a^2+b^3)^5$

Problem 2.
According to the Multinomial theorem, $(w+x+y+z)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, r_3, r_4} w^{r_1} x^{r_2} y^{r_3} z^{r_4}.$$

(a) How many terms are there in the sum?

(b) The sum of these multinomial coefficients has an easily expressed value. What is it?

$$\sum_{\substack{r_1+r_2+r_3+r_4=n,\\ r_i \in \mathbb{N}}} \binom{n}{r_1,\ r_2,\ r_3,\ r_4} = ? \tag{1}$$

Hint: How many terms are there when $(w+x+y+z)^n$ is expressed as a sum of monomials in w, x, y, z *before* terms with like powers of these variables are collected together under a single coefficient?

Problem 3. **(a)** Use the Multinomial Theorem **??** to prove that

$$(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p} \tag{2}$$

for all primes p. (Do not prove it using Fermat's "little" Theorem. The point of this problem is to offer an independent proof of Fermat's theorem.)

Hint: Explain why $\binom{p}{k_1, k_2, \ldots, k_n}$ is divisible by p if all the k_i's are positive integers less than p.

(b) Explain how (2) immediately proves Fermat's Little Theorem **??**: $n^{p-1} \equiv 1 \pmod{p}$ when n is not a multiple of p.

Problem 4.
You want to choose a team of m people for your startup company from a pool of n applicants, and from

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these m people you want to choose k to be the team managers. You took a Math for Computer Science subject, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n-k}{m-k}.$$

Before doing the reasonable thing—dump on your CFO or Harvard Business School—you decide to check his answer against yours.

(a) Give a *combinatorial proof* that your CFO's formula agrees with yours.

(b) Verify this combinatorial proof by giving an *algebraic* proof of this same fact.

Problem 5.

(a) Give a combinatorial proof of the following identity by letting S be the set of all length-n sequences of letters a, b and a single c and counting $|S|$ is two different ways.

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} \tag{3}$$

(b) Now prove (3) algebraically by applying the Binomial Theorem to $(1+x)^n$ and taking derivatives.

Problem 6.

What do the following expressions equal? Give both algebraic and combinatorial proofs for your answers.

(a)

$$\sum_{i=0}^{n}\binom{n}{i}$$

(b)

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i$$

Hint: Consider the bit strings with an even number of ones and an odd number of ones.