Problem Set 10

Due: Monday, November 21, 7:30pm

Reading Assignment: Sections 12.1-12.6, 14.1-14.5, 16.5

Problem 1. [30 points] Generating functions are very useful for turning difficult combinatorial problems into simple algebra. Solve the following combinatorial problems by using generating functions:

(a) [4 pts] Consider rolling a pair of normal six-sided dice. How many ways are there for the dice to sum to 7?

(b) [6 pts] Bob has a basket with 4 apples and 5 bananas. Suppose the apples are all distinguishable, and the bananas are all distinguishable. How many ways are there for Bob to select 6 pieces of fruit such that he picks an even number of apples and at least two bananas?

(c) [4 pts] Bob has a basket with 4 apples and 5 bananas. Suppose the apples are now indistinguishable, and the bananas are indistinguishable. How many ways are there for Bob to select 6 pieces of fruit such that he picks an even number of apples and at least two bananas?

(d) [6 pts] Find the number of ways to collect 15 dollars from 20 people if each of the first 19 people can give a dollar or nothing, and the twentieth person can give either 1 dollar, 5 dollars, or nothing.

(e) [6 pts] We have three pennies, four nickels, and two quarters. Find the generating function of the number of ways we can make change for n cents. Assume the coins are indistinguishable.

(f) [4 pts] We have three pennies, four nickels, and two quarters. Find the generating function of the number of ways we can make change for n cents. Assume the coins are distinguishable.

Problem 2. [10 points] Let $\mathcal{C}$ be the set of sequences formed by $\{a, b, c, d, 1, 2, 3\}$ such that the letters $\{a, b, c, d\}$ appear before the numbers $\{1, 2, 3\}$. As an example, $abba12$ and $cdab321$ are sequences in $\mathcal{C}$ but $a3b2c1$ is not a sequence in $\mathcal{C}$. Let c_n be the number of sequences in $\mathcal{C}$ of length n. Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$.

(a) [6 pts] Determine an expression for $C(x)$.

(b) [4 pts] Determine an explicit expression for c_n (*Hint*: use partial fraction decomposition on the generating function you find in part a).

Problem 3. [10 points] Let $a_n = a_{n-1} + 2a_{n-2}$ for $n \in \mathbb{N}$ with $a_0 = 1, a_1 = 1$. Use generating functions to find an explicit expression for a_n.

Problem 4. [20 points] For a given n, let p_n be the number of ways of writing n as a sum of 3 positive integers, where the order matters. For example, we can write 5 as:

$$1+1+3 \qquad 1+3+1 \qquad 3+1+1 \qquad 1+2+2 \qquad 2+1+2 \qquad 2+2+1$$

(a) [8 pts] What is a formula for the generating function $P(x) = \sum_{n=0}^{\infty} p_n x^n$?

Now again, for a given n, consider the number of ways of writing n as a sum of 3 positive integers as describe above. Let f_n be the sum of the product of each of the triples of numbers. For example, for $n = 5$ again, we get

$$f_5 = 1\cdot 1\cdot 3 + 1\cdot 3\cdot 1 + 3\cdot 1\cdot 1 + 1\cdot 2\cdot 2 + 2\cdot 1\cdot 2 + 2\cdot 2\cdot 1$$

Let $f_0 = 0$.

(b) [12 pts] What is the formula for the generating function $F(x) = \sum_{n=0}^{\infty} f_n x^n$? (Your answer should be a closed formula, not a Taylor Series)

Problem 5. [20 points] In lecture we discussed the Birthday Paradox. Namely, we found that in a group of m people with N possible birthdays, if $m \ll N$, then:

$$\Pr\{\text{all } m \text{ birthdays are different}\} \sim e^{-\frac{m(m-1)}{2N}}$$

To find the number of people, m, necessary for a half chance of a match, we set the probability to 1/2 to get:

$$m \sim \sqrt{(2\ln 2)N} \approx 1.18\sqrt{N}$$

For $N = 365$ days we found m to be 23.

We could also run a different experiment. As we put on the board the birthdays of the people surveyed, we could ask the class if anyone has the same birthday. In this case, before we reached a match amongst the surveyed people, we would already have found other people in the rest of the class who have the same birthday as someone already surveyed. Let's investigate why this is.

(a) [10 pts] Consider a group of m people with N possible birthdays amongst a larger class of k people, such that $m \leq k$. Define $\Pr\{A\}$ to be the probability that m people all have different birthdays *and* none of the other $k - m$ people have the same birthday as one of the m.

Show that, if $m \ll N$, then $\Pr\{A\} \sim e^{\frac{m(m-2k)}{2N}}$. (Notice that the probability of no match is $e^{-\frac{m^2}{2N}}$ when k is m, and it gets smaller as k gets larger.)

Hints: For $m \ll N$: $\frac{N!}{(N-m)!N^m} \sim e^{-\frac{m^2}{2N}}$, and $(1 - \frac{m}{N}) \sim e^{-\frac{m}{N}}$.

(b) [10 pts] Find the approximate number of people in the group, m, necessary for a half chance of a match (your answer will be in the form of a quadratic). Then simplify your answer to show that, as k gets large (such that $\sqrt{N} \ll k$), then $m \sim \frac{N \ln 2}{k}$.

Hint: For $x \ll 1$: $\sqrt{1-x} \sim (1 - \frac{x}{2})$.

Problem 6. [10 points] We're covering probability in 6.042 lecture one day, and you volunteer for one of Professor Leighton's demonstrations. He shows you a coin and says he'll bet you \$1 that the coin will come up heads. Now, you've been to lecture before and therefore suspect the coin is biased, such that the probability of a flip coming up heads, $\Pr\{H\}$, is p for $1/2 < p \leq 1$.

You call him out on this, and Professor Leighton offers you a deal. He'll allow you to come up with an algorithm using the biased coin to *simulate* a fair coin, such that the probability you win and he loses, $\Pr\{W\}$, is equal to the probability that he wins and you lose, $\Pr\{L\}$. You come up with the following algorithm:

1. Flip the coin twice.
2. Based on the results:
   - $TH \Rightarrow$ you win $[W]$, and the game terminates.
   - $HT \Rightarrow$ Professor Leighton wins $[L]$, and the game terminates.
   - $(HH \vee TT) \Rightarrow$ discard the result and flip again.
3. If at the end of N rounds nobody has won, declare a tie.

As an example, for $N = 3$, an outcome of HT would mean the game ends early and you lose, $HHTH$ would mean the game ends early and you win, and $HHTTTT$ would mean you play the full N rounds and result in a tie.

(a) [5 pts] Assume the flips are mutually independent. Show that $\Pr\{W\} = \Pr\{L\}$.

(b) [5 pts] Show that, if $p < 1$, the probability of a tie goes to 0 as N goes to infinity.