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Notes for Recitation 17

The ***(ordinary) generating function*** for a sequence $\langle a_0, a_1, a_2, a_3, \ldots \rangle$ is the power series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) $\langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle$

Solution.

$$2 + 3x + 5x^2$$

■

(b) $\langle 1, 1, 1, 1, 1, 1, 1, \ldots \rangle$

Solution.

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

■

(c) $\langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle$

Solution.

$$\begin{aligned} 1 + 2x + 4x^2 + 8x^3 + \ldots &= (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots \\ &= \frac{1}{1-2x} \end{aligned}$$

■

(d) $\langle 1, 0, 1, 0, 1, 0, 1, 0, \ldots \rangle$

Solution.

$$1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1-x^2}$$

■

(e) $\langle 0, 0, 0, 1, 1, 1, 1, 1, \ldots \rangle$

Solution.

$$x^3 + x^4 + x^5 + x^6 + \ldots = x^3(1 + x + x^2 + x^3 + \ldots) \qquad = \frac{x^3}{1-x}$$

■

(f) $\langle 1, 3, 5, 7, 9, 11, \ldots \rangle$

Solution.

$$\begin{aligned}
1 + x + x^2 + x^3 + \ldots &= \frac{1}{1-x} \\
\frac{d}{dx}\ 1 + x + x^2 + x^3 + \ldots &= \frac{d}{dx}\ \frac{1}{1-x} \\
1 + 2x + 3x^2 + 4x^2 + \ldots &= \frac{1}{(1-x)^2} \\
2 + 4x + 6x^2 + 8x^2 + \ldots &= \frac{2}{(1-x)^2} \\
1 + 3x + 5x^2 + 7x^3 + \ldots &= \frac{2}{(1-x)^2} - \frac{1}{1-x} \\
&= \frac{1+x}{(1-x)^2}
\end{aligned}$$

■

Problem 2

Suppose that:

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ g(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots \end{aligned}$$

What sequences do the following functions generate?

(a) $f(x) + g(x)$

Solution.

$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + \ldots$$

■

(b) $f(x) \cdot g(x)$

Solution.

$$a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \ldots + \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n + \ldots$$

■

(c) $f(x)/(1-x)$

Solution. This is a special case of the preceding problem part where:

$$\begin{aligned} g(x) &= \frac{1}{1-x} \\ &= 1 + x + x^2 + x^3 + x^4 + \cdots \end{aligned}$$

and so $b_0 = b_1 = b_2 = \ldots = 1$. In this case, we have:

$$f(x) \cdot g(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \ldots + \left(\sum_{k=0}^{n} a_k \right) x^n + \ldots$$

Thus, $f(x)/(1-x)$ is the generating function for sums of prefixes of the sequence generated by f. ■

Problem 3

There is a jar containing n different flavors of candy (and lots of each kind). I'd like to pick out a set of k candies.

(a) In how many different ways can this be done?

Solution. There is a bijection with sequences containing k zeroes (representing candies) and $n-1$ ones (separating the different varieties). The number of such sequences is:

$$\binom{n+k-1}{k}$$

■

(b) Now let's approach the same problem using generating functions. Give a closed-form generating function for the sequence $\langle s_0, s_1, s_2, s_3, \ldots \rangle$ where s_k is the number of ways to select k candies when there is only $n = 1$ flavor available.

Solution.

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

■

(c) Give a closed-form generating function for the sequence $\langle t_0, t_1, t_2, t_3, \ldots \rangle$ where t_k is the number of ways to select k candies when there are $n = 2$ flavors.

Solution.

$$(1 + x + x^2 + x^3 + \ldots)^2 = \frac{1}{(1-x)^2}$$

■

(d) Give a closed-form generating function for the sequence $\langle u_0, u_1, u_2, u_3, \ldots \rangle$ where u_k is the number of ways to select k candies when there are n flavors.

Solution.

$$\frac{1}{(1-x)^n}$$

■