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Notes for Recitation 1

1 Logic

How can one discuss mathematics with logical precision, when the English language is itself riddled with ambiguities? For example, imagine that you ask a friend what kind of dessert was offered at the party you couldn't make it to last week, and your friend says,

You could have cake or ice cream.

Does this mean that you could have *both* cake and ice cream? Or does it mean you had to choose either one or the other?

To cope with such ambiguities, mathematicians have defined precise meanings for some key words and phrases. Furthermore, they have devised symbols to represent those words. For example, if P is a proposition, then "not P" is a new proposition that is true whenever P is false and vice versa. The symbolic representation for "not P" is $\neg P$ or $\overline{P}$.

Two propositions, P and Q, can be joined by "and", "or", "implies", or "if and only if" to form a new proposition. The truth of this new proposition is determined by the truth of P and Q according to the table below. Symbolic equivalents are given in parentheses.

P	Q	"P and Q" ($P \wedge Q$)	"P or Q" ($P \vee Q$)	"P implies Q" or "if P, then Q" ($P \Rightarrow Q$)	"P if and only if Q" or "P iff Q" ($P \Leftrightarrow Q$)
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

There are a couple notable features hidden in this table:

- The phrase "P or Q" is true if P is true, Q is true, or both. Thus, you can have your cake and ice cream too.
- The phrase "P implies Q" (equivalently, "if P, then Q") is true when P *is false or* Q *is true*. Thus, "if the moon is made of green cheese, then there will be no final in 6.042" is a true statement.

There are two more important phrases in mathematical writing: "for all" (symbolized by $\forall$) and "there exists" (symbolized by $\exists$). These are called ***quantifiers***. A quantifier is always followed by a variable (and perhaps an indication of the range of that variable) and then a predicate, which typically involves that variable. Here are two examples:

$$\begin{aligned} \forall\, x \in \mathbb{R}^+ &\quad e^x < (1+x)^{1+x} \\ \exists\, n \in \mathbb{N} &\quad 2^n > (100n)^{100} \end{aligned}$$

The first statement says that e^x is less than $(1+x)^{1+x}$ for every positive real number x. The second statement says that there exists a natural number n such that $2^n > (100n)^{100}$.

The special symbols such as $\forall$, $\exists$, $\neg$, and $\vee$ are useful to logicians trying to express mathematical ideas without resorting to English at all. And other mathematicians often use these symbols as a shorthand. We recommend using them sparingly, however, because decrypting statements written in this symbolic language can be challenging!

2 Proving an Implication

Let's try to prove the following theorem.

Theorem 1. *Let $P(a,b)$ be any predicate defined for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then:*

$$\big(\exists\, a \in \mathcal{A} \quad \forall\, b \in \mathcal{B} \quad P(a,b)\big) \quad \Rightarrow \quad \big(\forall\, b \in \mathcal{B} \quad \exists\, a \in \mathcal{A} \quad P(a,b)\big)$$

Yuck! Now you *know* you're in a math class! Let's impose a specific interpretation in order to give concrete meaning to this claim. Define:

$$\begin{aligned} \mathcal{A} &= \{6.042 \text{ students}\} \\ \mathcal{B} &= \{6.042 \text{ lectures}\} \\ P(a,b) &= \text{"student } a \text{ falls asleep during lecture } b\text{"} \end{aligned}$$

Interpreting the left side in these terms gives:

$$\exists\, a \in \mathcal{A} \quad \forall\, b \in \mathcal{B} \quad P(a,b) = \text{"there exists a student that falls asleep in every lecture"}$$

So this side asserts that some particular student — let's call him Snoozer — always falls asleep. Now on the right side, we have:

$$\forall\, b \in \mathcal{B} \quad \exists\, a \in \mathcal{A} \quad P(a,b) = \text{"in every lecture, some student falls asleep"}$$

This is a slightly different assertion, because there might be a different sleeper in each lecture. Intuitively, the left side should imply the right; if Snoozer sleeps in every lecture, then in every lecture some student is surely asleep.

The implication in Theorem 1 is actually true for *every* predicate P and choice of sets $\mathcal{A}$ and $\mathcal{B}$. A universally-true statement, like this one, is called a ***validity***. (Every tautology (*cf.* Lecture Notes 9/4, p.6) is a validity, but validities may also involve quantifiers.) The ***converse*** of an implication $P \Rightarrow Q$ is the reverse implication $Q \Rightarrow P$. In this case, the converse is:

$$\big(\forall\ b \in \mathcal{B} \quad \exists\ a \in \mathcal{A} \quad P(a,b)\big) \quad \Rightarrow \quad \big(\exists\ a \in \mathcal{A} \quad \forall\ b \in \mathcal{B} \quad P(a,b)\big)$$

Under our interpretation, this says, "If in every lecture some student falls asleep, then there is some student who falls asleep in every lecture." This is not *necessarily* true, although it might be true for certain choices of predicate and sets. But since the truth of this converse proposition depends on the particular choice of predicate and sets, it is *not* a validity.

Anyway, let's prove the theorem.

Proof. We consider two cases.

Case 1: Suppose that the left side of the implication is false. Then the claim as a whole is true by default.

Case 2: Suppose that the left side of the implication is true. Then there exists some element $a_0 \in \mathcal{A}$ such that $P(a_0, b)$ is true for all $b \in \mathcal{B}$. Thus, for all $b \in \mathcal{B}$ there exists an $a \in \mathcal{A}$ (namely, a_0) such that $P(a, b)$ is true. Therefore, the right side of the implication is also true.

In both cases, the left side implies the right side, and so the theorem holds. □

Broadly speaking, we just proved that $P \Rightarrow Q$ for some nasty-looking propositions P and Q. When P was false (case 1), the implication held trivially. When P was true (case 2), we had to do some work to show that Q was also true. Every implication proof has this same structure: all the substance is in case 2. Thus, ordinarily no one even bothers to write down case 1 or even to identify two cases! Instead, when proving an implication, you may dispense with everything except for the body of case 2; the boxed text alone is considered a valid proof of the theorem. In summary, *in order to prove that P implies Q, you should assume that P is true and prove that Q is also true subject to that assumption.*

Notes for Recitation 1

1 Team Problem: Contrapositive

Prove by truth table that an implication is equivalent to its contrapositive.

Solution.

x	y	$x\rightarrow$ y	¬y	¬x	¬y → ¬x	($x\rightarrow$ y) ⟷ (¬y → ¬x)
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

In every row, x → y is T precisely when ¬y → ¬x is T. Thus, we conclude that an implication is equivalent to its contrapositive. ■

2 Team Problem: A Mystery

A certain cabal within the 6.042 course staff is plotting to make the final exam *ridiculously hard*. ("Problem 1. Prove that the axioms of mathematics are complete and consistent. Express your answer in Mayan hieroglyphics.") The only way to stop their evil plan is to determine exactly who is in the cabal. The course staff consists of nine people:

$$\{\text{Devin}, \text{Elizabeth}, \text{Emanuele}, \text{Hao}, \text{Henry}, \text{Hyungie}, \text{Michael}, \text{Patrick}, \text{Rachel}\}$$

The cabal is a subset of these nine. A membership roster has been found and appears below, but it is deviously encrypted in logic notation. The predicate *incabal* indicates who is in the cabal; that is, *incabal*(x) is true if and only if x is a member. Translate each statement below into English and deduce who is in the cabal.

(i) $\exists x\ \exists y\ \exists z\ (x \neq y \land x \neq z \land y \neq z \land \mathit{incabal}(x) \land \mathit{incabal}(y) \land \mathit{incabal}(z))$

Solution. A direct English paraphrase would be "There exist people we'll call x, y, and z, who are all different, such that x, y and z are each in the cabal." A better version would use the fact that there's no need in this case to give names to the people. Namely, a better paraphrase is, "There are 3 different people in the cabal." Perhaps a simpler way to say this is, "The cabal is of size at least 3." ■

(ii) $\neg(\mathit{incabal}(\text{Michael}) \land \mathit{incabal}(\text{Henry}))$

Solution. Michael and Henry are not both in the cabal. Equivalently: at least one of Michael and Henry is not in the cabal. ■

(iii) $(\mathit{incabal}(\text{Hyungie}) \lor \mathit{incabal}(\text{Emanuele})) \to \forall x\ \mathit{incabal}(x)$

Solution. If either Hyungie or Emanuele is in the cabal, then everyone is. ■

(iv) $\mathit{incabal}(\text{Michael}) \to \mathit{incabal}(\text{Henry})$

Solution. If Michael is in the cabal, then Henry is also. ■

(v) $\mathit{incabal}(\text{Elizabeth}) \to \mathit{incabal}(\text{Hyungie})$

Solution. If Elizabeth is in the cabal, then Hyungie is also. ■

(vi) $(\mathit{incabal}(\text{Devin}) \lor \mathit{incabal}(\text{Hao})) \to \neg\mathit{incabal}(\text{Rachel})$

Solution. If either of Devin or Hao is in the cabal, then Rachel is not. Equivalently, if Rachel *is* in the cabal, then neither Devin nor Hao is. ■

(vii) $(\mathit{incabal}(\text{Devin}) \lor \mathit{incabal}(\text{Henry})) \to \neg\mathit{incabal}(\text{Patrick})$

> **Solution.** If either of Devin or Henry is in the cabal, then Patrick is not. Equivalently, if Patrick *is* in the cabal, then neither Devin nor Henry is. ■

So much for the translations. We now argue that the only cabal satisfying all seven propositions above is one whose members are exactly Devin, Henry, and Hao.

We first observe that by (ii), there must be someone — either Michael or Henry — who is not in the cabal. But if either Hyungie or Emanuele were in the cabal, then by (iii), everyone would be. So we conclude by contradiction that

$$\text{Hyungie and Emanuele are not in the cabal.} \tag{1}$$

Now consider that (v) implies its contrapositive: if Hyungie is not in the cabal, then neither is Elizabeth. Therefore, since Hyungie is not in the cabal,

$$\text{Elizabeth is not in the cabal.} \tag{2}$$

Next observe that if Michael were in the cabal, then by (iv), Henry would be too, contradicting (ii). So by again contradiction, we conclude that

$$\text{Michael is not in the cabal.} \tag{3}$$

Now suppose Rachel is in the cabal. Then by (vi), Devin and Hao are not. We already know Hyungie, Emanuele, Elizabeth, and Michael are not in the cabal, leaving only three who could be — Rachel, Patrick, and Henry. But by (i) the cabal must have at least three members, so it follows that the cabal must consist of exactly these three. This proves:

Lemma 1. *If Rachel is in the cabal, then Patrick and Henry are in the cabal.*

But by (vii), if Henry is the cabal, then Patrick is not. That is,

Lemma 2. *Henry and Patrick cannot both be in the cabal.*

Now from Lemma 2 we conclude that the conclusion of Lemma 1 is false. So by contrapositive, the hypothesis of Lemma 1 must also be false, namely,

$$\text{Rachel is not in the cabal.} \tag{4}$$

Finally, suppose Patrick is in the cabal. Then by (vii), Devin and Henry are not, and we already know Hyungie, Emanuele, Elizabeth, Michael, and Rachel are not. So the cabal must consist of at most two people (Patrick and Hao). This contradicts (i), and we conclude by contradiction that

$$\text{Patrick is not in the cabal.} \tag{5}$$

So the only remaining people who could be in the cabal are Devin, Henry, and Hao. Since the cabal must have at least three members, we conclude that

Lemma 3. *The only possible cabal consists of Devin, Henry, and Hao.*

But we're not done yet: we haven't shown that a cabal consisting of Devin, Henry, and Hao actually does satisfy all seven conditions. So let $\mathcal{A} = \{\text{Devin}, \text{Henry}, \text{Hao}\}$, and let's quickly check that $\mathcal{A}$ satisfies (i)–(vii):

- $|A| = 3$, so A satisfies (i).
- Michael is not in A, so A satisfies (ii) and (iv).
- Neither Hyungie nor Emanuele is in A, so the hypothesis of (iii) is false, which means that A satisfies (iii).
- Elizabeth is not in A, so A satisfies (v).
- Finally, Rachel and Patrick are not in A, so the conclusions of both (vi) and (vii) are true, and so A satisfies (vi) and (vii).

So now we have proved

Proposition. $\{\mathit{Devin}, \mathit{Henry}, \mathit{Hao}\}$ *is the* unique *cabal satisfying conditions* (i)–(vii).