# Problems for Recitation 9

## 1 Getting around a graph

**Definition 1.** *A* ***walk***[1] *in a graph, $G$, is a sequence of vertices*

$$v_0, v_1, \ldots, v_k$$

*and edges*

$$\{v_0,v_1\}, \{v_1,v_2\}, \ldots, \{v_{k-1},v_k\}$$

*such that $\{v_i,v_{i+1}\}$ is an edge of $G$ for all $i$ where $0 \leq i < k$ . The walk is said to* start *at $v_0$ and to* end *at $v_k$, and the* length *of the walk is defined to be $k$. An edge, $\{u,v\}$, is* ***traversed*** *$n$ times by the walk if there are $n$ different values of $i$ such that $\{v_i,v_{i+1}\} = \{u,v\}$.*

**Definition 2.** *A* ***path*** *is a walk where all the $v_i$'s are different, that is, $i \neq j$ implies $v_i \neq v_j$. For simplicity, we will refer to paths and walks by the sequence of vertices.*[2]

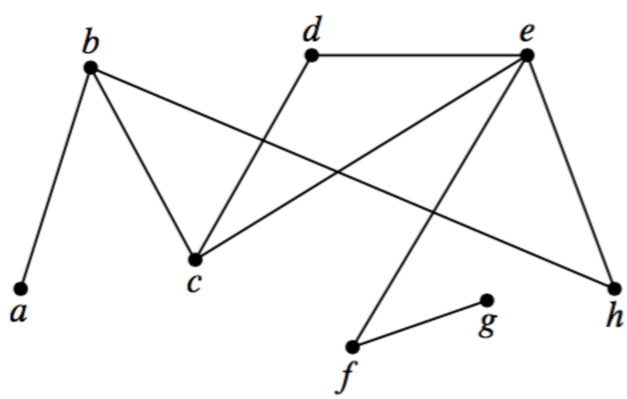


Figure 1: A graph containing a path $a$, $b$, $c$, $d$, $e$, $f$, $g$ of length 6.

[1]Some texts use the word *path* for our definition of walk and the term *simple path* for our definition of path.

[2]This works fine for simple graphs since the edges in a walk are completely determined by the sequence of vertices and there is no ambiguity. For graphs with multiple edges, we would need to specify the edges as well as the nodes.

For example, the graph in Figure 1 has a length 6 path $a$, $b$, $c$, $d$, $e$, $f$, $g$. This is the longest path in the graph. Of course, the graph has walks with arbitrarily large lengths; for example, $a$, $b$, $a$, $b$, $a$, $b$, ....

The length of a walk or path is the total number of times it traverses edges, which is *one less* than its length as a sequence of vertices. For example, the length 6 path $a$, $b$, $c$, $d$, $e$, $f$, $g$ contains a sequence of 7 vertices.

# Problem: Finding a Path

Use the Well Ordering Principle to prove the following lemma.

**Lemma 1.** *If there is a walk from a vertex $u$ to a vertex $v$ in a graph, then there is a path from $u$ to $v$.*

# 2 Matrix Multiplication

Although this may be a review for some, it will be an important part of tomorrow's lecture when we describe graphs using an *adjacency matrix*.

Recall the definition of matrix multiplication. Given two matrices $A$ and $B$, where $A$ is $m \times n$ and $B$ is $n \times k$ their product is given by:

$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{bmatrix} = C$$

where

$$c_{ij} = \sum_{p=1}^{n} a_{ip} b_{pj}$$

Note that $C$ is a $m \times k$ matrix.

It is also important to note that in general $A \cdot B \neq B \cdot A$. In fact, this operation may not even be defined, since the row dimension of the first matrix must equal the column dimension of the second (above, $A$ has $n$ rows and $B$ has $n$ columns).

Multiplying a matrix and a vector is identical, noting that a column vector is simply a $n \times 1$ matrix.

Let's do some examples.

# Problems: Matrix Multiplication Practice

1. Evaluate the following expressions.

   (a)
   $$\begin{bmatrix} \alpha & \rho \\ \beta & \sigma \\ \gamma & \tau \end{bmatrix} \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$$

   (b)
   $$\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} \begin{bmatrix} \alpha & \rho \\ \beta & \sigma \\ \gamma & \tau \end{bmatrix}$$

   (c)
   $$\begin{bmatrix} a & b & c & d \\ w & x & y & z \end{bmatrix} \begin{bmatrix} \alpha & \rho \\ \beta & \sigma \\ \gamma & \tau \end{bmatrix}$$

2. Prove the following lemma:

   **Lemma 2.** *Let $b$ be a $m \times 1$ vector whose entries are nonnegative and sum to 1. Let $A$ be a $n \times m$ matrix whose entries are nonnegative and each column sums to one. Then, the product $c = Ab$ is a $n \times 1$ vector whose entries are nonnegative and sum to one.*

# 3 Problem: Connectivity

Prove that any simple graph with $n$ nodes and strictly more than $\frac{1}{2}(n-1)(n-2)$ edges is connected.