Problem Set 1

Due: Monday, September 12

Problem 1. [**20 points**]

(a) [12 pts] Use a truth table to verify that $(P \oplus (P \oplus Q))$ is equivalent to Q.

(b) [8 pts] Find a predicate $P(x)$ that is a counterexample to the following:

$$\exists x.P(x) \text{ IMPLIES } \forall x.P(x)$$

Problem 2. [**20 points**] A student is trying to prove that propositions p, q, and r are all true. She proceeds as follows. First, she proves three facts: $p \to q$, $q \to r$, and $r \to p$. Then she concludes, "thus obviously p, q, and r are all true." Let's first formalize her deduction as a logical statement and then evaluate whether or not it is correct.

(a) [6 pts] Using logic notation and the symbols p, q, and r, write down the logical implication that she uses in her final step.

(b) [8 pts] Use a truth table to determine whether this logical implication is a tautology (i.e., a universal truth in logic).

(c) [6 pts] Is her proof that propositions p, q, and r are all true correct? Briefly explain.

Problem 3. [**24 points**] Translate the following statements into predicate logic. For each, specify the domain. In addition to logic symbols, you may build predicates using arithmetic, relational symbols, and constants. For example, the statement "n is an odd number" could be translated into $\exists m.(2m + 1 = n)$, where the domain is $\mathbb{Z}$, the set of integers. Another example, "p is a prime number," could be translated to

$$(p > 1) \text{ AND NOT } (\exists m.\exists n.(m > 1 \text{ AND } n > 1 \text{ AND } mn = p))$$

Let prime(p) be an abbreviation that you could use to denote the above formula in this problem.

(a) [4 pts] (Lagrange's Four-Square Theorem) Every nonnegative integer is expressible as the sum of four perfect squares.

(b) [4 pts] (Goldbach Conjecture) Every even integer greater than two is the sum of two primes.

(c) [4 pts] Every finite integer set has a maximum element.

(d) [4 pts] (Fermat's Last Theorem) There are no nontrivial solutions to the equation:

$$x^n + y^n = z^n$$

over the nonnegative integers when $n > 2$.

(e) [4 pts] There are infinitely many primes.

(f) [4 pts] If integers a and b are coprime, then there exist integers x and y such that $ax + by = 1$.

Problem 4. [16 points] Translate the following sentences from English to predicate logic. Let S denote the set of all students and let T denote the set of all TAs. You may use the functions $R(x, y)$, meaning that "x is in y's recitation," $P(x)$, meaning that "x will pass 6.042," $H(x)$, meaning that "x does their homework regularly," and $U(x)$, meaning that "x is an undergraduate."

(a) [4 pts] All the students in at least one TA's recitation will pass 6.042.

(b) [4 pts] All the students in 6.042 who do their homework will pass 6.042.

(c) [4 pts] Every TA will have a student in their recitation pass 6.042.

(d) [4 pts] There are at least three undergraduate TAs.

Problem 5. [20 points] Suppose that $w^2 + x^2 + y^2 = z^2$, where w, x, y, and z always denote positive integers. (*Hint:* It may be helpful to represent even integers as $2i$ and odd integers as $2j + 1$, where i and j are integers.)

Prove the proposition: z is even if and only if w, x, and y are even. Do this by considering all the cases of w, x, y being odd or even.