

## Problem Set 9

**Due:** Monday, November 8

### Problem 1. [10 points]

(a) [5 pts] Show that of any  $n + 1$  distinct numbers chosen from the set  $\{1, 2, \dots, 2n\}$ , at least 2 must be relatively prime. (*Hint:*  $\gcd(k, k + 1) = 1$ .)

(b) [5 pts] Show that any finite connected undirected graph with  $n \geq 2$  vertices must have 2 vertices with the same degree.

### Problem 2. [10 points] Under Siege!

Fearing retribution for the many long hours his students spent completing problem sets, Prof. Leighton decides to convert his office into a reinforced bunker. His only remaining task is to set the 10-digit numeric password on his door. Knowing the students are a clever bunch, he is not going to pick any passwords containing the forbidden consecutive sequences "18062", "6042" or "35876" (his MIT extension).

How many 10-digit passwords can he pick that don't contain forbidden sequences if each number  $0, 1, \dots, 9$  can only be chosen once (i.e. without replacement)?

**Problem 3. [50 points]** Be sure to show your work to receive full credit. In this problem we assume a standard card deck of 52 cards.

(a) [4 pts] How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

(b) [4 pts] For fixed positive integers  $n$  and  $k$ , how many nonnegative integer solutions  $x_0, x_1, \dots, x_k$  are there to the following equation?

$$\sum_{i=0}^k x_i = n$$

(c) [4 pts] For fixed positive integers  $n$  and  $k$ , how many nonnegative integer solutions  $x_0, x_1, \dots, x_k$  are there to the following equation?

$$\sum_{i=0}^k x_i \leq n$$

- (d) [4 pts] How many simple undirected graphs are there with  $n$  vertices?
- (e) [4 pts] How many directed graphs are there with  $n$  vertices (self loops allowed)?
- (f) [4 pts] How many tournament graphs are there with  $n$  vertices?
- (g) [4 pts] How many acyclic tournament graphs are there with  $n$  vertices?
- (h) [4 pts] How many numbers are there that are in the range  $[1..700]$  which are divisible by 2, 5 or 7?
- (i) [9 pts] In how many ways can you arrange  $n$  books on  $k$  bookshelf (assuming the order of books on a shelf matters?)
- (j) [9 pts] How about if there has to be at least 1 book at each bookshelf?

**Problem 4. [15 points]** Give a combinatorial proof of the following theorem:

$$n2^{n-1} = \sum_{k=1}^n k \binom{n}{k}$$

(Hint: Consider the set of all length- $n$  sequences of 0's, 1's and a single \*.)

**Problem 5. [15 points]** At a congressional hearing, there are  $2n$  members present. Exactly  $n$  of them are Democrats and  $n$  of them are Republicans. The members want to select a smaller subcommittee of size  $n$  from within those present at the hearing. However, since the Democrats currently hold majority, they want there to be more Democrats than Republicans in the committee. In how many ways can you select such a committee? (Hint: Consider two cases:  $n$  odd and  $n$  even.)