Final

Name: ____

Circle the name of your recitation instructor:

David Darren Martyna Nick Oscar Stav

- This quiz is **closed book**, but you may have one 8.5 × 11″ sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- The exam ends at 4:30 PM.

Problem	Points	Grade	Grader
1	10		
2	10		
3	20		
4	15		
5	20		
6	25		
7	10		
8	10		
Total	120		

Problem 1. [10 points]

Consider the recursion

$$T_{n+2} = T_{n+1} + 2T_n, T_0 = T_1 = 1$$

(a) [5 pts] Show that T_n is odd

(b) [5 pts] Using part (a), prove that $\gcd(T_{n+1}, T_n) = 1$. Note that this implies that T_{n+1} and T_n are relatively prime.

Problem 2. [10 points] Let R be a positive random variable with $E[R^3] = k$. Show $Pr(R \geq x) \leq k/x^3$

Problem 3. [20 points] In the far off land of Spain two soccer teams: Barcelona and Madrid have been battling each other for centuries to see who is more awesome. They play two games a year: one in the Spring and one in the Fall. The Madrid team has a tendency to fire their coaches very often to try to improve their results. The outcomes of the games are as follows:

1. If Madrid has not just fired their coach:
   - Barcelona wins with probability $\frac{2}{5}$
   - Madrid wins with probability $\frac{2}{5}$
   - They tie with probability $\frac{1}{5}$
2. If Madrid has just fired their coach (they haven't yet realized that it's a bad idea):
   - Barcelona wins with probability $\frac{3}{5}$
   - Madrid wins with probability $\frac{1}{5}$
   - They tie with probability $\frac{1}{5}$

Now, Madrid does not fire their coach if they win or tie, but if Barcelona wins, they will fire their coach with 90% probability. For the following two questions, assume that Madrid did not fire their coach before the Spring 2010 game.

(a) [10 pts] Given that Madrid lost the Fall 2010 game, what is the probability that they fired their coach between the Spring and Fall games this year.

(b) [10 pts] What is the probability that Madrid fired their coach between the Spring and Fall games this year given that they lost BOTH of these games?

Problem 4. [25 points]

(a) [5 pts] You roll two fair dice and look at the sum. What is the expected value of that sum?

(b) [5 pts] What is the variance?

(c) [5 pts] You repeatedly roll two fair dice and look at the sum. What is the probability that you will roll a sum of 4 before you roll a sum of 7?

(d) [5 pts] What is the expected number of rolls before you get a sum of 4 or a sum of 7?

(e) [5 pts] You roll 10 dice. Using the Chernoff Bound, give an upper bound for the probability that 8 or more of them rolled a 1 or a 2?

(f) [5 pts] Recall that we can approximate the CDF of the binomial distribution by $F_{n,p}(\alpha)$ whenever $\alpha < p$. In terms of $F_{n,p}(\alpha)$, give an upper bound for the probability that 8 or more of them rolled a 1 or a 2?

Problem 5. [15 points] Show that for any m, n, k such that $k < m < n - k$, the following identity holds:

$$\binom{n}{k} = \sum_{i=0}^{k} \binom{m}{i} \cdot \binom{n-m}{k-i}$$

Problem 6. [10 points]

(a) [5 pts] Three sets of twins are sharing a bucket of 24 pieces of chicken from Olive Oyl's. If each person eats at least one piece of chicken, how many ways can the chicken pieces be distributed amongst this family? You do not have to calculate a specific number, but please give your answer in a closed form (no summations). Also, for the purposes of this problem, chicken pieces are indistinguishable.

(b) [5 pts]

Across town, a family of seven folk eat a meal that comes with unlimited sides of biscuits (the restaurant owner has deep pockets). Being a caring family, some number of biscuits are shared between the family members. However, the family obeys the following two rules:

- If a biscuit is shared, it must be shard between **exactly two** people
- If two people share a biscuit, they cannot share another biscuit.

Prove that there exist two members of this family who shared the same number of biscuits as each other.

Problem 7. [10 points]

Find a closed-form solution to the following recurrence:

$x_n = 11x_{n-1} - 30x_{n-2}$ where $(x_0 = 4, x_1 = 23)$

Problem 8. [20 points]

(a) [10 pts] Show that in any direct acyclic graph there exists a node with no outgoing edges.

(b) [10 pts] Show that any direct acyclic graph with maximum in-degree k is $k+1$-colorable.

Problem 9. [10 points] Let G be a bipartite graph with n nodes and k components. You independently color each of the nodes of G red or black with equal probabilities. What is the probability that your coloring is a valid 2-coloring of G?

Problem 10. [[10 points] points] There is a complete deck of 52 cards, and half of another deck (for a total of 52 + 26 cards) left over. The decks have been mixed together, and repeated cards cannot be told apart. A hand of 5 cards is picked randoly from the shuffled deck and a half.

[[5 points] pts] What is the probability that the hand has no identical cards (i.e., cards with the same suit and value. For example, the hand $\langle Q\heartsuit, 5\spadesuit, 6\spadesuit, 8\clubsuit, Q\heartsuit\rangle$ *has* identical cards.)? (you can leave your answer in terms of binomial coefficients)

We can calculate this probability by calculating

$$\frac{\text{hands with no identical cards}}{\text{total possible hands}}$$

Note there are essentially two families of types (the type of a card is its (suit, number) pair) of cards: 26 types are repeated in the set and 26 are unique in the set. We can count the number of possible hands with no identical cards by considering the following cases: the hand is made out of 5 cards of non-repeated types, the hand is made out of 4 cards of non-repeated types in the set and 1 of a repeated type, ... up to the case of all cards are of repeated types. If all cards are from the non repeated kind, there are only $\binom{52}{5}$ possibilities. If all cards are from the repeated types, then we have $\binom{52}{5} \cdot 2^5$ hands that could be created. If the hand has k cards from the non-repeated types and $5 - k$ from the repeated types, then there are $\binom{52}{k} \cdot \binom{52}{5-k} \cdot 2^{5-k}$ possible hands.

So we get a total count of:

$$\sum_{i=0}^{5} \binom{52}{k} \cdot \binom{52}{5-k} \cdot 2^{k-5}$$

Therefore the probability of drawing a hand with no identical cards is:

$$\frac{\sum_{i=0}^{5} \binom{52}{k} \cdot \binom{52}{5-k} \cdot 2^{k-5}}{\binom{78}{5}}$$

[[5 points] pts] What is the probability that the hand has exactly two pairs of identical cards?

If we have two pairs of identical cards, both pairs will be of the repeated kinds. All possible hands of this type can be concisely described by a tuple $(\{T_1, T_2\}, t)$ where $\{T_1, T_2\}$ is a set of the two distinct types that appear twice, and t is any leftover card. There are $\binom{26}{2}$ options for the set, and $52 + 26 - 4 = 74$ options for the third card.

Hence the probability we pick such a hand is:

$$\frac{\binom{26}{2} \cdot 74}{\binom{78}{5}}$$

Problem 11. [10 points]

Consider a length n vector of integers, $x = (x_1, x_2, \ldots, x_n)$ where the entries of the vector are integers in the set $\{1, 2, \ldots, n\}$. Let A be the number of entries x_i in the vector where $x_i \leq i$.

(a) [5 pts] Calculate $E[A]$.

(b) [5 pts] Calculaate the variance of A.

Problem 12. [20 points] In the card game of bridge, you are dealt a hand of 13 cards from the standard 52-card deck.

(a) [5 pts] A balanced hand is one in which a player has roughly the same number of cards in each suit. How many different hands are there where the player has 4 cards in one suit and 3 cards in each of the other suits?

(b) [5 pts] Not surprisingly, a non-balanced hand is one in which a player has more cards in some suits than others. Hands that are very disired are ones where over half the cards are in one suit. How many different hands are there where there is exactly 7 cards in one suit?

(Notes)