Problems for Recitation 11

Chains and antichains

Recall some definitions and theorems from lecture.

Definition 1. *A* ***chain*** *in a poset* $(S, \preceq)$ *is a subset* $C \subseteq S$ *such that for all* $x, y \in C$*, either* $x \preceq y$ *or* $y \preceq x$*. In other words, it is a sequence* $a_1 \preceq a_2 \preceq a_3 \cdots \preceq a_t$*, where* $a_i \neq a_j$ *for all* $i \neq j$*, such that each item is comparable to the next in the chain, and it is smaller with respect to* $\preceq$*.*

Theorem 1. *Given any finite poset* $(A, \preceq)$ *for which the longest chain has length* t*, it is possible to partition* A *into* t *subsets* $A_1, A_2, \ldots, A_t$ *such that for all* $i \in \{1, 2, \ldots, t\}$ *and for all* $a \in A_i$*, we have that all* $b \preceq a$ *appear in the set* $A_1 \cup \cdots \cup A_{i-1}$*.*

Corollary 2. *The total amount of parallel time needed to complete the tasks is the same as the length of the longest chain.*

Definition 2. *An* ***antichain*** *in a poset* $(S, \preceq)$ *is a subset* $A \subseteq S$ *such that for all* $x, y \in A$ *with* $x \neq y$*, neither* $x \preceq y$ *nor* $y \preceq x$*.). In other words, it is a set of incomparable items, e.g., things that can be scheduled at the same time.*

1 Problem: Antichains

The above theorem can be recast in the language of antichains. Prove the following corollary.

Corollary 3. *If* i *is the length of the longest chain in a poset* $(A, \preceq)$*, then* A *can be partitioned into* t *antichains.*

Note: It turns out that the dual of this corollary, which states that if the longest antichain has size t, then A can be partitioned into t chains, is also true! However, this is much harder to prove. It is known as ***Dilworth's theorem***.

2 Problem: Taking classes

Here is prerequstitite information for some MIT courses:

$18.01 \to 6.042$	$18.01 \to 18.02$
$18.01 \to 18.03$	$6.046 \to 6.840$
$8.01 \to 8.02$	$6.01 \to 6.034$
$6.042 \to 6.046$	$18.03, 8.02 \to 6.02$
$6.01, 6.02 \to 6.003$	$6.01, 6.02 \to 6.004$
$6.004 \to 6.033$	$6.033 \to 6.857$

1. Draw a Hasse diagram for the corresponding partially-ordered set. (Recall: A ***Hasse diagram*** is a way of representing a poset $(A, \preceq)$ as a directed acyclic graph. The vertices are the element of A, and there is generally an edge $u \to v$ if $u \preceq v$. However, self-loops and edges implied by transitivity are omitted.) You'll need this diagram for all the subsequent problem parts, so be neat!

2. Identify a largest chain.

3. Suppose that you want to take all the courses. What is the minimum number of terms required to graduate, if you can take as many courses as you want per term?

4. Identify a largest antichain.

5. What is the maximum number of classes that you could possibly take at once?

6. Identify a topological sort of the classes. (A ***topological sort*** of a poset $(A, \preceq)$ is a total order of all the elements such that if $a_i \preceq a_j$ in the partial order, then a_i precedes a_j in the total order.)

7. Suppose that you want to take all of the courses, but can handle only two per term. How many terms are required to graduate?

8. What if you could take three courses per term?

9. Stanford's computer science department offers n courses, limits students to at most k classes per term, and has its own complicated prerequisite structure. Describe two different lower bounds on the number of terms required to complete all the courses. One should be based on your answers to parts (b) and (c) and a second should be based on your answer to part (g).

10. We now complement these lower bounds with an upper bound. This upper bound is known as ***Brent's Theorem***, and it implies that the *sum* of these two lower bounds is an *upper bound* on the number of terms required to complete all courses.

    Suppose the length of the longest prerequisite chain is c. Show that the maximum number of terms required to complete all the courses is

    $$M(n, c, k) := (c - 1) + \left\lceil \frac{n - (c - 1)}{k} \right\rceil .$$

    Hint: Try induction on c. You may find it helpful to use the fact that if $a \geq b \geq 0$, then

    $$\lceil a - b \rceil \leq 1 + \lceil a \rceil - \lceil b \rceil \tag{1}$$

    for all real numbers a, b.

3 Problem: Relations

1. Give a description of the equivalence classes associated with each of the following equivalence relations.

   (a) Integers x and y are equivalent if $x \equiv y \pmod 3$.

   (b) Real numbers x and y are equivalent if $\lceil x \rceil = \lceil y \rceil$, where $\lceil z \rceil$ denotes the smallest integer greater than or equal to z.

   (c) Vertices x and y in the graph G are equivalent if there is a path from x to y.

2. Show that neither of the following relations is an equivalence relation by identifying a missing property (reflexivity, symmetry, or transitivity).

   (a) The "divides" relation on the positive integers.

   (b) The "implies" relation on propositional formulas.

   (c) The "intersects" relation on nonempty subsets of $\mathbb{N}$.