# Problem Set 2

**Due:** Monday, September 19

**Reading Assignment:** Sections 2.5-2.7, 3.0-3.4, & 3.5 (optional)

## Problem 1. [10 points]

**(a)** [5 pts] Prove that the number of subsets of $\{1, 2, \ldots n\}$ for positive integers $n$ is $2^n$ using induction.

**(b)** [5 pts] Prove that for $n \in \mathbb{N}$, $2^{3n} - 1$ is divisible by 7 using induction.

## Problem 2. [15 points]

**(a)** [5 pts] Consider three integers x, y, z written down on a piece of paper. Any of the integers may be replaced by the sum of the other two plus 1. This operation is repeated a number of times until the final result is $17, 1967, 1985$. Is it possible that the initial integers were $2, 4, 6$? Prove it. (*Hint*: Try out a few rounds of operations starting with $2, 4, 6$ and see if a pattern emerges)

**(b)** [10 pts] Prove that there is no way to cover a 6 x 6 board with rectangles of size 1 x 4 such that each square is covered by exactly one rectangle. As a hint, one way to prove that it is not possible involves reasoning about the following assignment of letters $A, B, C, D$ to each square of the board.

| | | | | | |
|---|---|---|---|---|---|
| $A$ | $B$ | $C$ | $D$ | $A$ | $B$ |
| $B$ | $C$ | $D$ | $A$ | $B$ | $C$ |
| $C$ | $D$ | $A$ | $B$ | $C$ | $D$ |
| $D$ | $A$ | $B$ | $C$ | $D$ | $A$ |
| $A$ | $B$ | $C$ | $D$ | $A$ | $B$ |
| $B$ | $C$ | $D$ | $A$ | $B$ | $C$ |

**Problem 3. [20 points]** The following problem is fairly tough until you hear a certain one-word clue. The solution is elegant but is slightly tricky, so don't hesitate to ask for hints!

During 6.042, the students are sitting in an $n \times n$ grid. A sudden outbreak of beaver flu (a rare variant of bird flu that lasts forever; symptoms include year ning for problem sets and craving for ice cream study sessions) causes some students to get infected. Here is an example where $n = 6$ and infected students are marked $\times$.

| | | | | | |
|---|---|---|---|---|---|
| × | | | | × | |
| | × | | | | |
| | | × | × | | |
| | | | | | |
| | | × | | | |
| | | | × | | × |

Now the infection begins to spread every minute (in discrete time-steps). Two students are considered *adjacent* if they share an edge (i.e., front, back, left or right, but NOT diagonal); thus, each student is adjacent to 2, 3 or 4 others. A student is infected in the next time step if either

- the student was previously infected (since beaver flu lasts forever), or
- the student is adjacent to *at least two* already-infected students.

In the example, the infection spreads as shown below.

| | | | | | |
|---|---|---|---|---|---|
| × | | | | × | |
| | × | | | | |
| | | × | × | | |
| | | | | | |
| | | × | | | |
| | | | × | | × |

$\Rightarrow$

| | | | | | |
|---|---|---|---|---|---|
| × | × | | | × | |
| × | × | × | | | |
| | × | × | × | | |
| | | × | | | |
| | | × | × | | |
| | | × | × | × | × |

$\Rightarrow$

| | | | | | |
|---|---|---|---|---|---|
| × | × | × | | × | |
| × | × | × | × | | |
| × | × | × | × | | |
| | × | × | × | | |
| | | × | × | × | |
| | | × | × | × | × |

In this example, over the next few time-steps, all the students in class will become infected.

**Theorem.** *If fewer than $n$ students in class are initially infected, the whole class will never be completely infected.*

Prove this theorem.

*Hint:* To understand how a system such as the above "evolves" over time, it is usually a good strategy to (1) identify an appropriate property of the system at the initial stage, and (2) prove, by induction on the number of time-steps, that the property is preserved at every time-step. So look for a property (of the set of infected students) that remains invariant as time proceeds.

If you are stuck, ask your recitation instructor for the one-word clue and even more hints!

**Problem 4. [10 points]** Can raising an irrational number $a$ to an irrational power $b$ result in a rational number? Provide a proof that it can by considering $\sqrt{5}^{\sqrt{2}}$ and using proof by cases.

**Problem 5. [15 points]** For any nonempty set $C$, let $f(C)$ be the square of the product of the elements in $C$. For example, if $C = \{1, 4, 5\}$, then $f(C) = (1 \cdot 4 \cdot 5)^2 = 400$. Show that the sum of $f(S)$ for all nonempty subsets of $\{1, 2, \ldots, n\}$ containing no consecutive elements is $(n + 1)! - 1$. For example, if we consider $\{1, 2, 3\}$, then we have $f(\{1, 3\}) + f(\{1\}) + f(\{2\}) + f(\{3\}) = 23 = 4! - 1$. (Hint: Use strong induction)

**Problem 6. [15 points]** A group of $n \geq 1$ people can be divided into teams, each containing either 5 or 6 people. What is the largest $n$ for which the group cannot be divided into such teams? Use induction to prove that your answer is correct.

**Problem 7. [15 points]** *The Well Ordering Principle (WOP)* states that "every *nonempty* set of *nonnegative* integers has a *smallest* element." (See Section 3.1 of the text *Mathematics for Computer Science*.) It captures a special property about nonnegative integers and can be extremely useful in proofs.

Prove using the Well Ordering Principle that for all nonnegative integers, $n$:

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{1}$$

(*Hint:* Begin by pretending that the equation is false and consider the nonempty set of nonnegative integers for which the equation does not hold. Try to show that this set has no least element, which would contradict the Well Ordering Principle)