

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science
6.042J/18.062J

Propositional Algebra



Albert R Meyer

February 14, 2018

propositional algebra.1

6	9	13	7
12		10	5
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15	8	11	2

Proving Equivalence

Use an algebra of equivalence to prove formulas equivalent.



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propositional algebra.2

6	9	13	7
12		10	5
3	1	4	14
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Strategy: Convert to DNF

Come up with enough equivalence rules to convert any formula to an equivalent Full DNF.

Two formulas are equiv when convert to same Full DNF.



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propositional algebra.3

6	9	13	7
12		10	5
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Algebra for Equivalence

Rules for XOR, IMPLIES

$$P \text{ IMPLIES } Q \equiv \text{NOT}(P) \text{ OR } Q$$

$$P \text{ XOR } Q \equiv (\text{NOT}(P) \text{ AND } Q) \text{ OR } (\text{NOT}(Q) \text{ AND } P)$$

Just leaves AND, OR, NOT



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propositional algebra.4

6	9	13	7
12		10	5
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Algebra for Equivalence

Double Negation

$$\text{NOT}(\text{NOT}(P)) \equiv P$$



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propositional algebra.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Algebra for Equivalence

DeMorgan's law -AND

$$\text{NOT}(P \text{ AND } Q) \equiv \text{NOT}(P) \text{ OR } \text{NOT}(Q)$$



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propositional algebra.6

6	9	13	7
12		10	5
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Algebra for Equivalence

DeMorgan's law -OR

$$\text{NOT}(P \text{ OR } Q) \equiv \text{NOT}(P) \text{ AND } \text{NOT}(Q)$$

rewrite left to right until
NOT's only on variables



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propositional algebra.7

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example

converting to a sum
of products



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propositional algebra.8

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example move NOTs down to literals

$\text{NOT}[\text{NOT}(P \text{ OR } Q) \text{ OR } (R \text{ AND } Q)]$



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propositional algebra.9

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

move NOTs down to literals

$\text{NOT}[\text{NOT}(P \text{ OR } Q) \text{ OR } (R \text{ AND } Q)]$
use DeMorgan



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propositional algebra.10

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

move NOTs down to literals

$\text{NOT}[\text{NOT}(P \text{ OR } Q) \text{ OR } (R \text{ AND } Q)]$

$\text{NOT}[\text{NOT}(P \text{ OR } Q)] \text{ AND } \text{NOT}(R \text{ AND } Q)$



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propositional algebra.11

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

move NOTs down to literals

$\text{NOT}[\text{NOT}(P \text{ OR } Q)] \text{ AND } \text{NOT}(R \text{ AND } Q)$
Double NOT



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propositional algebra.12

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

move **NOT**s down to literals

NOT[**NOT**(P **OR** Q)] **AND** **NOT**(R **AND** Q)
(P **OR** Q) **AND** **NOT**(R **AND** Q)



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propositional algebra.13

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

move **NOT**s down to literals

(P **OR** Q) **AND** **NOT**(R **AND** Q)
use **DeMorgan**



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propositional algebra.14

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

move **NOT**s down to literals

(P **OR** Q) **AND** **NOT**(R **AND** Q)
(P **OR** Q) **AND** (**NOT**R **OR** **NOT**Q)



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propositional algebra.15

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Done move **NOT**s down to literals

(P **OR** Q) **AND** (**NOT**R **OR** **NOT**Q)



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propositional algebra.16

6	9	13	7
12		10	5
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Algebra for Equivalence

Distributive Law

$$P \cdot (Q + R) \equiv (P \cdot Q) + (P \cdot R)$$



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propositional algebra.17

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Algebra for Equivalence

Distributive Law

$$P \text{ AND } (Q \text{ OR } R) \equiv (P \text{ AND } Q) \text{ OR } (P \text{ AND } R)$$

rewrite left to right until
" OR of AND's "



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propositional algebra.18

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Get Sum of Products

$$(P \text{ OR } Q) \text{ AND } (\bar{R} \text{ OR } \bar{Q})$$

Distribute (P OR Q)



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propositional algebra.19

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Get Sum of Products

$$(P \text{ OR } Q) \text{ AND } (\bar{R} \text{ OR } \bar{Q})$$

$$((P \text{ OR } Q) \text{ AND } \bar{R}) \text{ OR } ((P \text{ OR } Q) \text{ AND } \bar{Q})$$



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propositional algebra.20

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Get Sum of Products

$$((P \text{ OR } Q) \text{ AND } \bar{R}) \text{ OR} \\ ((P \text{ OR } Q) \text{ AND } \bar{Q})$$



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propositional algebra.21

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Get Sum of Products

$$((P \text{ OR } Q) \text{ AND } \bar{R}) \text{ OR} \\ ((P \text{ OR } Q) \text{ AND } \bar{Q}) \\ \text{Distribute } \bar{R}$$



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propositional algebra.23

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Get Sum of Products

$$((P \text{ OR } Q) \text{ AND } \bar{R}) \text{ OR} \\ ((P \text{ OR } Q) \text{ AND } \bar{Q}) \\ (P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ ((P \text{ OR } Q) \text{ AND } \bar{Q})$$



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propositional algebra.24

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Get Sum of Products

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ ((P \text{ OR } Q) \text{ AND } \bar{Q}) \\ \text{Distribute } \bar{Q}$$



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propositional algebra.25

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Get Sum of Products

$$\begin{aligned} & (P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ & ((P \text{ OR } Q) \text{ AND } \bar{Q}) \\ & (P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ & (P \text{ AND } \bar{Q}) \text{ OR } (Q \text{ AND } \bar{Q}) \end{aligned}$$



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propositional algebra.26

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Done: Sum of Products

$$\begin{aligned} & (P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ & (P \text{ AND } \bar{Q}) \text{ OR } (Q \text{ AND } \bar{Q}) \end{aligned}$$



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propositional algebra.27

6	9	13	7
12		10	5
3	1	4	14
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Simplification rules

$$\begin{aligned} & (Q \text{ OR } Q) \equiv Q \\ & (Q \text{ AND } Q) \equiv Q \\ & P \text{ OR } (\bar{Q} \text{ AND } Q) \equiv P \\ & P \text{ AND } (\bar{Q} \text{ OR } Q) \equiv P \end{aligned}$$



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propositional algebra.28

6	9	13	7
12		10	5
3	1	4	14
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example

$$\begin{aligned} & (P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ & (P \text{ AND } \bar{Q}) \text{ OR } (Q \text{ AND } \bar{Q}) \end{aligned}$$

Simplify



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propositional algebra.29

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ (P \text{ AND } \bar{Q}) \text{ OR } (Q \text{ AND } \bar{Q})$$

Simplify



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propositional algebra.30

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR} \\ (P \text{ AND } \bar{Q})$$



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propositional algebra.31

6	9	13	7
12		10	5
3	1	4	14
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we have DNF!

$$(P \text{ AND } \bar{R}) \text{ OR } (Q \text{ AND } \bar{R}) \text{ OR } (P \text{ AND } \bar{Q})$$

now to get Full DNF:



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propositional algebra.32

6	9	13	7
12		10	5
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Full DNF for an AND-term

$$(P \text{ AND } \bar{R}) \quad \text{unsimplify}$$

$$(P \text{ AND } \bar{R}) \text{ AND } (Q \text{ OR } \bar{Q})$$



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propositional algebra.33

6	9	13	7
12		10	5
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Full DNF for an AND-term

$$(P \text{ AND } \bar{R})$$

$(P \text{ AND } \bar{R}) \text{ AND } (Q \text{ OR } \bar{Q})$ distribute

$$(P \text{ AND } \bar{R} \text{ AND } Q) \text{ OR}$$

$$(P \text{ AND } \bar{R} \text{ AND } \bar{Q})$$



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propositional algebra.34

6	9	13	7
12		10	5
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Full DNF for an AND-term

$$(P \text{ AND } \bar{R})$$

$$(P \text{ AND } \bar{R} \text{ AND } Q) \text{ OR}$$

$$(P \text{ AND } \bar{R} \text{ AND } \bar{Q})$$

Full!



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propositional algebra.35

6	9	13	7
12		10	5
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Rearrangement rules

$$P \text{ AND } Q \equiv Q \text{ AND } P$$

$$(P \text{ AND } Q) \text{ AND } R \equiv$$

$$(P \text{ AND } Q \text{ AND } R)$$

...likewise for OR



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propositional algebra.36

6	9	13	7
12		10	5
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example

$$(P \text{ AND } \bar{R} \text{ AND } Q) \text{ OR}$$

$$(P \text{ AND } \bar{R} \text{ AND } \bar{Q})$$



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propositional algebra.37

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example

$(P \text{ AND } \bar{R} \text{ AND } Q) \text{ OR}$
 $(P \text{ AND } \bar{R} \text{ AND } \bar{Q})$ sort



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propositional algebra.38

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example

$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR}$
 $(P \text{ AND } \bar{Q} \text{ AND } \bar{R})$



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propositional algebra.39

6	9	13	7
12		10	5
3	1	4	14
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Sorted Full DNF for $(P \text{ AND } \bar{R})$

$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR}$
 $(P \text{ AND } \bar{Q} \text{ AND } \bar{R})$



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propositional algebra.40

6	9	13	7
12		10	5
3	1	4	14
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example

same for each AND-term,
 and OR them together:

$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR}$ $(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR}$
 $(P \text{ AND } \bar{Q} \text{ AND } \bar{R})$ OR $(\bar{P} \text{ AND } Q \text{ AND } \bar{R})$ OR
 $(P \text{ AND } \bar{Q} \text{ AND } R) \text{ OR}$
 $(P \text{ AND } \bar{Q} \text{ AND } \bar{R})$



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propositional algebra.41

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example

simplify (duplicates)

$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR }$
 $(P \text{ AND } \bar{Q} \text{ AND } \bar{R}) \text{ OR } (\bar{P} \text{ AND } Q \text{ AND } \bar{R}) \text{ OR }$
 $(P \text{ AND } \bar{Q} \text{ AND } R) \text{ OR }$
 ~~$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR }$~~



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propositional algebra.43

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

example

also sort the clauses

$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR }$
 $(P \text{ AND } \bar{Q} \text{ AND } \bar{R}) \text{ OR } (\bar{P} \text{ AND } Q \text{ AND } \bar{R}) \text{ OR }$
 $(P \text{ AND } \bar{Q} \text{ AND } R)$



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propositional algebra.44

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12		10	5
3	1	4	14
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Canonical DNF

$(P \text{ AND } Q \text{ AND } \bar{R}) \text{ OR } (P \text{ AND } \bar{Q} \text{ AND } R) \text{ OR }$
 $(P \text{ AND } \bar{Q} \text{ AND } \bar{R}) \text{ OR } (\bar{P} \text{ AND } Q \text{ AND } \bar{R})$

Done Sorted Full DNF
unique for each formula



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propositional algebra.45

6	9	13	7
12		10	5
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Algebra for Equivalence

This set of rules for \equiv
 is complete:
 if two formulas are \equiv ,
 these rules can prove it.



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propositional algebra.46

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12		10	5
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Algebra for Equivalence

Because two formulas are equivalent iff have same truth table iff have same canonical DNF.



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propositional algebra.49

6	9	13	7
12		10	5
3	1	4	14
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Validity Checking still hard

Algebraic proofs in general don't beat truth tables. The canonical DNF is just a copy of the truth table as an algebraic formula.



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propositional algebra.51

6	9	13	7
12		10	5
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Validity Checking still hard

Algebraic proofs in general don't beat truth tables. No efficient method known for equivalence or validity.



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propositional algebra.52