

Problems for Recitation 8

1 Build-up error

Recall a graph is **connected** iff there is a path between every pair of its vertices.

False Claim. *If every vertex in a graph has positive degree, then the graph is connected.*

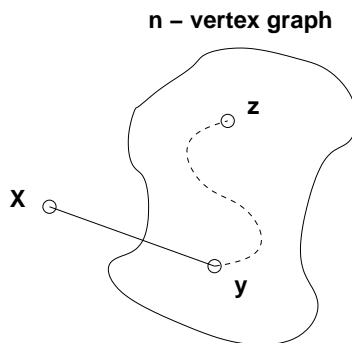
1. Prove that this Claim is indeed false by providing a counterexample.
2. Since the Claim is false, there must be a logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

Proof. We prove the Claim above by induction. Let $P(n)$ be the proposition that if every vertex in an n -vertex graph has positive degree, then the graph is connected.

Base cases: ($n \leq 2$). In a graph with 1 vertex, that vertex cannot have positive degree, so $P(1)$ holds vacuously.

$P(2)$ holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 2$. Consider an n -vertex graph in which every vertex has positive degree. By the assumption $P(n)$, this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to obtain an $(n+1)$ -vertex graph:



All that remains is to check that there is a path from x to every other vertex z . Since x has positive degree, there is an edge from x to some other vertex, y . Thus, we can

obtain a path from x to z by going from x to y and then following the path from y to z . This proves $P(n+1)$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, which proves the Claim.

□

2 The Grow Algorithm

Yesterday in lecture, we saw the following algorithm for constructing a minimum-weight spanning tree (MST) from an edge-weighted N -vertex graph G .

ALG-GROW:

1. Label the edges of the graph e_1, e_2, \dots, e_t so that $wt(e_1) \leq wt(e_2) \leq \dots \leq wt(e_t)$.
2. Let S be the empty set.
3. For $i = 1 \dots t$, if $S \cup \{e_i\}$ does not contain a cycle, then extend S with the edge e_i .
4. Output S .

3 Analysis of ALG-GROW

In this problem you may assume the following lemma from the problem set:

Lemma 1. *Suppose that $T = (V, E)$ is a simple, connected graph. Then T is a tree iff $|E| = |V| - 1$.*

In this exercise you will prove the following theorem.

Theorem. *For any connected, weighted graph G , ALG-GROW produces an MST of G .*

- (a) Prove the following lemma.

Lemma 2. *Let $T = (V, E)$ be a tree and let e be an edge not in E . Then, $G = (V, E \cup \{e\})$ contains a cycle.*

(Hint: Suppose G does *not* contain a cycle. Is G a tree?)

- (b) Prove the following lemma.

Lemma 3. *Let $T = (V, E)$ be a spanning tree of G and let e be an edge not in E . Then there exists an edge $e' \neq e$ in E such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of G .*

(Hint: Adding e to E introduces a cycle in $(V, E \cup \{e\})$.)

(c) Prove the following lemma.

Lemma 4. *Let $T = (V, E)$ be a spanning tree of G , let e be an edge not in E and let $S \subseteq E$ such that $S \cup \{e\}$ does not contain a cycle. Then there exists an edge $e' \neq e$ in $E - S$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of G .*

(Hint: Modify your proof to part (b). Of all possible edges $e' \neq e$ that can be removed to construct T^* , at least one is not in S .)

(d) Prove the following lemma.

Lemma 5. *Define S_m to be the set consisting of the first m edges selected by ALG-GROW from a connected graph G . Let $P(m)$ be the predicate that if $m \leq |V|$ then $S_m \subseteq E$ for some MST $T = (V, E)$ of G . Then $\forall m. P(m)$.*

(Hint: Use induction. There are two cases: $m + 1 > |V|$ and $m + 1 \leq |V|$. In the second case, there are two subcases.)

(e) Prove the theorem. (Hint: Lemma 5 says there exists an MST $T = (V, E)$ for G such that $S \subseteq E$. Use contradiction to rule out the case in which S is a proper subset of E .)