Martin Gardner's Mistake

Tanya Khovanova

Tanya Khovanova received a Ph.D. from Moscow State University in 1988. Research in representation theory, integrable systems, super-string theory, and quantum groups was interrupted by employment in industry, where she became interested in algorithms, complexity theory, cryptography, and networks. Her current interests (now as a research affiliate at MIT and freelance mathematician) lie in combinatorics, number theory, recreational mathematics, and probability theory. Check out her website (tanyakhovanova.com) or her highly popular math blog (blog.tanyakhovanova.com) or her Number Gossip website (numbergossip.com).

Martin Gardner was amazingly accurate and reliable. That he made a mistake is simply testimonial to the difficulty of this particular problem, which appeared in 1959 and was republished in [**3**]:

> Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?
>
> Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?

Mr. Jones has failed to stir any controversy, so we ignore him and his two children [**5**]. Instead, we concentrate on Mr. Smith. Here is the solution that Martin Gardner published with the problem:

> If Smith has two children, at least one of which is a boy, we have three equally probable cases: boy-boy, boy-girl, girl-boy. In only one case are both children boys, so the probability that both are boys is $1/3$.

The corrected solution

Later Martin Gardner wrote a column titled "Probability and Ambiguity," which was also republished (in [**4**]). In this column Gardner corrects himself, writing "...the answer depends on the procedure by which the information "at least one is a boy" is obtained."

He suggested two potential procedures.

(i) Pick all the families with two children, one of which is a boy. If Mr. Smith is chosen randomly from this list, then the answer is $1/3$.

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(ii) Pick a random family with two children; suppose the father is Mr. Smith. Then if the family has two boys, Mr. Smith says, "At least one of them is a boy." If he has two girls, he says, "At least one of them is a girl." If he has a boy and a girl he flips a coin to choose one or another of those two sentences. In this case the probability that both children are the same sex is $1/2$.

Thus, the original problem without a specified procedure is ambiguous.

More procedures

Let us call procedure (i) "boy-centered," because from the start we know that we are talking about boys. Correspondingly, we call (ii) "gender neutral."

Gardner wanted to emphasize that the problem is ambiguous. For him it was enough to show two different procedures leading to different answers. However, there are many procedures. Here are two more that demonstrate the full range of ambiguity.

(iii) Suppose that Mr. Smith wants to brag about his sons and will always mention as many as he can. In this case the procedure might be the following: If he has two boys, he says, "I have two boys." If he has one son, he says "At least one of them is a boy." In this case the answer to the problem is 0.

(iv) Suppose Mr. Smith dislikes boys, and wants to de-emphasize the number of boys he has. In this case the procedure might be the following: If he has two boys, he says, "At least one of them is a boy." If he has a boy and a girl, he says, "I am the proud father of a girl." In this case the answer is 1.

We challenge the reader to invent a procedure for any given answer between 0 and 1.

Tuesday-child problem

Fast-forward to 2010. In a talk at the 9th Gathering for Gardner, Gary Foshee posed the following problem:

> I have two children. One is a boy born on a Tuesday. What is the probability that I have two boys?

This is Martin Gardner's Two-Children problem with an extra twist. Before discussing the solution, let us agree on some basic assumptions:

- Sons and daughters are equally probable—not exactly true, but a reasonable approximation.
- Twins do not exist. Not only is the proportion of twins in the population small, but, because they are usually born on the same day, twins might complicate the calculation.
- All days of the week are equally probable birthdays. While this isn't actually true—for example, assisted labors are unlikely to be scheduled for weekends—it is a reasonable approximation.

History repeats itself

Just as occurred with the classical Two-Children problem, many mathematicians have been fighting for the wrong solution. This is their argument:

Each child can be one of two genders and can be born on one of seven days of the week. Thus, gender and day present 14 equally probable cases for each child. That, in turn, makes each two-children family belong to one of 196 equally probable cases. When we restrict all possible cases to the given information that one child is a boy born on a Tuesday, we get 27 equally probable cases. We can divide these cases into several groups, as follows:

- 7 cases, where the first child is a son born on a Tuesday and the second child is a daughter.
- 7 cases, where the second child is a son born on a Tuesday and the first child is a daughter.
- 6 cases, where the first child is a son born on a Tuesday and the second child is a son not born on a Tuesday.
- 6 cases, where the second child is a son born on a Tuesday and the first child is a son not born on a Tuesday.
- 1 case, where both children are sons born on a Tuesday.

These are a total of 27 equally probable cases, 13 of which correspond to two sons. Thus the probability must be $13/27$.

This incorrect solution was widely published, for example, it appeared in Devlin's Angle [**1**]. But, like Martin Gardner, Devlin corrected himself in his next column [**2**].

Procedures

The ambiguity that Martin Gardner found in the Two-Children problem is also present in the Tuesday-Child problem. To resolve the ambiguity, we need to specify the procedure by which the information was obtained. Here we present just four procedures. The calculations are based on 196 equally probable cases for different combinations of gender and the day of the week.

Gender-neutral, day-of-the-week-neutral procedure. In this scenario, a father of two children is picked at random. He is instructed to choose a child by flipping a coin. Then he has to provide true information about the chosen child in the following format: "I have a son/daughter born on a Mon/Tue/Wed/Thu/Fri/Sat/Sun." If his statement is, "I have a son born on a Tuesday," what is the probability that the other child is also a son?

The solution is the following: A father has two daughters in 49 cases. Such a father will make the above statement with probability zero. A father has a son and a daughter in 98 cases, and will produce the above statement with probability $1/14$: with probability $1/2$ the son is chosen over the daughter and with probability $1/7$ Tuesday is the birthday. A father has two sons in 49 cases, and he will make the statement with probability $1/7$. The father of two sons is twice as likely to make the statement as the father of a son and a daughter, but there are half as many such fathers. Thus the probability is $1/2$ that the other child is also a son. This is the most symmetric scenario and produces the most symmetric answer.

Boy-centered, day-of-the-week-neutral procedure. A father of two children is picked at random. If he has two daughters, he is sent home and another father is picked until one is found with at least one son. If he has just one son, he is instructed to provide true information on his son's day of birth. If he has two sons, he has to choose one at random. His statement will be, “I have a son born on a Mon/Tue/Wed/Thu/Fri/Sat/Sun.” If his statement is, “I have a son born on a Tuesday,” what is the probability that the other child is also a son?

The solution is as follows. A father has a son and a daughter in 98 cases, and will produce the statement above with probability $1/7$. If he has two sons (49 cases), the probability of the statement above will likewise be $1/7$. The father of two sons is exactly as likely to make the statement as the father of a son and a daughter, but there are half as many such fathers. Thus the probability is $1/3$ that the other child is also a son.

This scenario corresponds to the original procedure leading to the first solution of Martin Gardner’s Two-Children problem. Unsurprisingly, the answer is the same.

Boy-centered, Tuesday-centered procedure. Now let us consider the third scenario. A father of two children is picked at random. If he doesn’t have a son born on a Tuesday, he is sent home and another father is picked at random until one who has a son born on a Tuesday is found. He is instructed to tell you, “I have a son born on a Tuesday.” What is the probability that the other child is also a son?

Here is the solution. A father of a boy and a girl has a son born on a Tuesday in 14 cases. He will make the statement in question with probability 1. A father of two sons has a son born on a Tuesday in 13 cases. He too is guaranteed to make the statement. Thus the probability is $13/27$ that the other child is a son.

This procedure corresponds to the procedure many mathematicians assume while solving the Tuesday-Child problem. This implicit assumption is the source of the erroneous solution.

Gender-neutral, Tuesday-centered procedure. For completeness let us consider a fourth scenario. A father of two children is picked at random. If he doesn’t have a child who is born on a Tuesday, he is sent home and another father is picked at random until one who has a child born on a Tuesday is found. He is instructed to tell you, “I have a son/daughter born on a Tuesday.” If both of his children were born on Tuesdays, he has to pick one at random. If his statement is, “I have a son born on a Tuesday,” what is the probability that the other child is also a son?

Here is the solution. A father of two daughters will have a child born on a Tuesday in 13 cases. He makes the statement in question with probability 0. A father of a boy and a girl has a child born on a Tuesday in 26 cases. The probability that he makes the statement is $1/2$. A father of two sons will have a son born on a Tuesday in 13 cases. The probability that he makes the statement is 1. The father of two sons is twice as likely to make the statement as the father of a son and a daughter, but there are half as many such fathers. Thus the probability is $1/2$ that the other child is also a son.

In this procedure we added an additional constraint on the families with two children—that a child was born on a Tuesday—that’s independent of gender. Not surprisingly, the answer is $1/2$.

Back to Gardner

Many people I argue with don’t want to listen to me. They refer to Martin Gardner as the final authority in support of their wrong solution. Gardner was a great thinker; he corrected his mistake. I urge those who do not agree with me to trust Martin Gardner. Revisit and rethink this problem together with him.

Acknowledgment. I am grateful to all my friends and colleagues who discussed the problem with me and supported me in writing about it for my blog [**6, 7, 8**]. I am especially grateful to

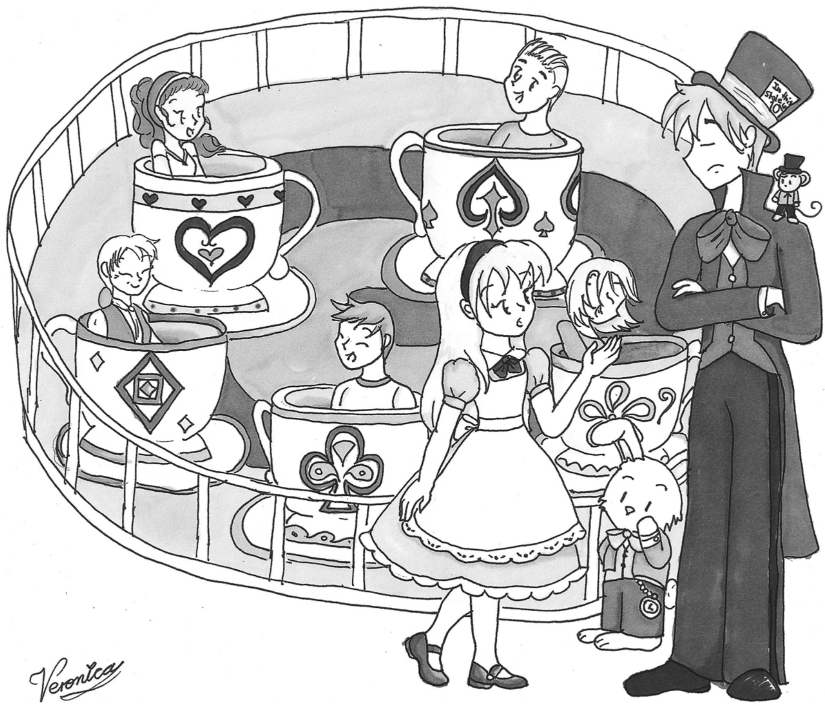
Alexey Radul [9] and Peter Winkler [10] who contributed essays on the subject to my blog. I also thank Sue Katz and Julie Sussman, P.P.A., for editing.

Summary. When Martin Gardner first presented the Two-Children Problem, he made a mistake in its solution. Later he corrected the error, but unfortunately the incorrect solution is more widely known than his correction. In fact, a Tuesday-Child variation of this problem went viral in 2010, and the same flaw keeps reappearing in proposed solutions of that problem too. In this article, we re-visit Martin Gardner's correction and discuss the new problem in detail.

References

1. K. Devlin, Probability can bite, (2010); available at `http://www.maa.org/devlin/devlin_04_10.html`.
2. K. Devlin, The problem with word problems, (2010); available at `http://www.maa.org/devlin/devlin_05_10.html`.
3. M. Gardner, "Nine more problems" in *The Second Scientific American Book of Mathematical Puzzles and Diversions*, The Chicago University Press, 1987, 152–164.
4. ———, "Probability and ambiguity" in *The Second Scientific American Book of Mathematical Puzzles and Diversions*, The Chicago University Press, 1987, 220–232.
5. T. Khovanova, Mr. Jones, (2010); available at `http://blog.tanya.khovanova.com/?p=313`.
6. ———, A son born on Tuesday, (2010); available at `http://blog.tanyakhovanova.com/?p=221`.
7. ———, Sons and Tuesdays, (2010); available at `http://blog.tanyakhovanova.com/?p=233`.
8. ———, A Tuesday quiz, (2010); available at `http://blog.tanyakhovanova.com/?p=247`.
9. A. Radul, Shannon entropy rescues the Tuesday child, (2010); available at `http://blog.tanyakhovanova.com/?p=254`.
10. P. Winkler, Conditional probability and "He said, she said", (2010); available at `http://blog.tanyakhovanova.com/?p=234`.

The Mad Tea Party Ride



Alice and friends begin a mathematical adventure (see next page).