Notes for Recitation 1

1 Team Problem: Contrapositive

Prove by truth table that an implication is equivalent to its contrapositive.

Solution.

x	y	x → y	¬y	¬x	¬y → ¬x	(x → y) ↔ (¬y → ¬x)
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

In every row, x → y is T precisely when ¬y → ¬x is T. Thus, we conclude that an implication is equivalent to its contrapositive. ■

2 Team Problem: A Mystery

A certain cabal within the 6.042 course staff is plotting to make the final exam *ridiculously hard*. ("Problem 1. Prove that the axioms of mathematics are complete and consistent. Express your answer in Mayan hieroglyphics.") The only way to stop their evil plan is to determine exactly who is in the cabal. The course staff consists of nine people:

$$\{\text{Madalina}, \text{Catherine}, \text{Jeffrey}, \text{Brando}, \text{Nirvan}, \text{Ashley}, \text{Chennah}, \text{Ankur}, \text{Tom}\}$$

The cabal is a subset of these nine. A membership roster has been found and appears below, but it is deviously encrypted in logic notation. The predicate *incabal* indicates who is in the cabal; that is, $incabal(x)$ is true if and only if x is a member. Translate each statement below into English and deduce who is in the cabal.

(i) $\exists x\ \exists y\ \exists z\ (x \neq y \land x \neq z \land y \neq z \land incabal(x) \land incabal(y) \land incabal(z))$

Solution. A direct English paraphrase would be "There exist people we'll call x, y, and z, who are all different, such that x, y and z are each in the cabal." A better version would use the fact that there's no need in this case to give names to the people. Namely, a better paraphrase is, "There are 3 different people in the cabal." Perhaps a simpler way to say this is, "The cabal is of size at least 3." ■

(ii) $\neg(incabal(\text{Chennah}) \land incabal(\text{Nirvan}))$

Solution. Chennah and Nirvan are not both in the cabal. Equivalently: at least one of Chennah and Nirvan is not in the cabal. ■

(iii) $(incabal(\text{Ashley}) \lor incabal(\text{Jeffrey})) \to \forall x\ incabal(x)$

Solution. If either Ashley or Jeffrey is in the cabal, then everyone is. ■

(iv) $incabal(\text{Chennah}) \to incabal(\text{Nirvan})$

Solution. If Chennah is in the cabal, then Nirvan is also. ■

(v) $incabal(\text{Catherine}) \to incabal(\text{Ashley})$

Solution. If Catherine is in the cabal, then Ashley is also. ■

(vi) $(incabal(\text{Madalina}) \lor incabal(\text{Brando})) \to \neg incabal(\text{Tom})$

Solution. If either of Madalina or Brando is in the cabal, then Tom is not. Equivalently, if Tom *is* in the cabal, then neither Madalina nor Brando is. ■

(vii) $(incabal(\text{Madalina}) \lor incabal(\text{Nirvan})) \to \neg incabal(\text{Ankur})$

Solution. If either of Madalina or Nirvan is in the cabal, then Ankur is not. Equivalently, if Ankur *is* in the cabal, then neither Madalina nor Nirvan is. ■

So much for the translations. We now argue that the only cabal satisfying all seven propositions above is one whose members are exactly Madalina, Nirvan, and Brando.

We first observe that by (ii), there must be someone — either Chennah or Nirvan — who is not in the cabal. But if either Ashley or Jeffrey were in the cabal, then by (iii), everyone would be. So we conclude by contradiction that

$$\text{Ashley and Jeffrey are not in the cabal.} \tag{1}$$

Now consider that (v) implies its contrapositive: if Ashley is not in the cabal, then neither is Catherine. Therefore, since Ashley is not in the cabal,

$$\text{Catherine is not in the cabal.} \tag{2}$$

Next observe that if Chennah were in the cabal, then by (iv), Nirvan would be too, contradicting (ii). So by again contradiction, we conclude that

$$\text{Chennah is not in the cabal.} \tag{3}$$

Now suppose Tom is in the cabal. Then by (vi), Madalina and Brando are not. We already know Ashley, Jeffrey, Catherine, and Chennah are not in the cabal, leaving only three who could be — Tom, Ankur, and Nirvan. But by (i) the cabal must have at least three members, so it follows that the cabal must consist of exactly these three. This proves:

Lemma 1. *If Tom is in the cabal, then Ankur and Nirvan are in the cabal.*

But by (vii), if Nirvan is the cabal, then Ankur is not. That is,

Lemma 2. *Nirvan and Ankur cannot both be in the cabal.*

Now from Lemma 2 we conclude that the conclusion of Lemma 1 is false. So by contrapositive, the hypothesis of Lemma 1 must also be false, namely,

$$\text{Tom is not in the cabal.} \tag{4}$$

Finally, suppose Ankur is in the cabal. Then by (vii), Madalina and Nirvan are not, and we already know Ashley, Jeffrey, Catherine, Chennah, and Tom are not. So the cabal must consist of at most two people (Ankur and Brando). This contradicts (i), and we conclude by contradiction that

$$\text{Ankur is not in the cabal.} \tag{5}$$

So the only remaining people who could be in the cabal are Madalina, Nirvan, and Brando. Since the cabal must have at least three members, we conclude that

Lemma 3. *The only possible cabal consists of Madalina, Nirvan, and Brando.*

But we're not done yet: we haven't shown that a cabal consisting of Madalina, Nirvan, and Brando actually does satisfy all seven conditions. So let $\mathcal{A} = \{\text{Madalina}, \text{Nirvan}, \text{Brando}\}$, and let's quickly check that $\mathcal{A}$ satisfies (i)–(vii):

- $|A| = 3$, so A satisfies (i).
- Chennah is not in A, so A satisfies (ii) and (iv).
- Neither Ashley nor Jeffrey is in A, so the hypothesis of (iii) is false, which means that A satisfies (iii).
- Catherine is not in A, so A satisfies (v).
- Finally, Tom and Ankur are not in A, so the conclusions of both (vi) and (vii) are true, and so A satisfies (vi) and (vii).

So now we have proved

Proposition. $\{$*Madalina*, *Nirvan*, *Brando*$\}$ *is the* unique *cabal satisfying conditions* (i)–(vii).