Problem Set 11

Due: Monday, November 28, 7:30pm

Problem 1. **[20 points]** You are organizing a neighborhood census and instruct your census takers to knock on doors and note the sex of any child that answers the knock. Assume that there are two children in a household, that children are equally likely to be girls and boys, and that girls and boys are equally likely to open the door.

A sample space for this experiment has outcomes that are triples whose first element is either B or G for the sex of the elder child, likewise for the second element and the sex of the younger child, and whose third coordinate is E or Y indicating whether the *e*lder child or *y*ounger child opened the door. For example, (B, G, Y) is the outcome that the elder child is a boy, the younger child is a girl, and the girl opened the door.

(a) [5 pts] Let T be the event that the household has two girls, and O be the event that a girl opened the door. List the outcomes in T and O.

(b) [5 pts] What is the probability $\Pr(T \mid O)$, that both children are girls, given that a girl opened the door?

(c) [10 pts] Where is the mistake in the following argument for computing $\Pr(T \mid O)$?

> If a girl opens the door, then we know that there is at least one girl in the household. The probability that there is at least one girl is
>
> $$1 - \Pr(\text{both children are boys}) = 1 - (1/2 \times 1/2) = 3/4.$$
>
> So,
>
> $$\begin{aligned} &\Pr(T \mid \text{there is at least one girl in the household}) \\ &= \frac{\Pr(T \cap \text{there is at least one girl in the household})}{\Pr\{\text{there is at least one girl in the household}\}} \\ &= \frac{\Pr(T)}{\Pr\{\text{there is at least one girl in the household}\}} \\ &= (1/4)/(3/4) = 1/3. \end{aligned}$$
>
> Therefore, given that a girl opened the door, the probability that there are two girls in the household is 1/3.

Problem 2. [15 points] In lecture we discussed the Birthday Paradox. Namely, we found that in a group of m people with N possible birthdays, if $m \ll N$, then:

$$\Pr\{\text{all } m \text{ birthdays are different}\} \sim e^{-\frac{m(m-1)}{2N}}$$

To find the number of people, m, necessary for a half chance of a match, we set the probability to 1/2 to get:

$$m \sim \sqrt{(2\ln 2)N} \approx 1.18\sqrt{N}$$

For $N = 365$ days we found m to be 23.

We could also run a different experiment. As we put on the board the birthdays of the people surveyed, we could ask the class if anyone has the same birthday. In this case, before we reached a match amongst the surveyed people, we would already have found other people in the rest of the class who have the same birthday as someone already surveyed. Let's investigate why this is.

(a) [5 pts] Consider a group of m people with N possible birthdays amongst a larger class of k people, such that $m \leq k$. Define $\Pr\{A\}$ to be the probability that m people all have different birthdays *and* none of the other $k - m$ people have the same birthday as one of the m.

Show that, if $m \ll N$, then $\Pr\{A\} \sim e^{\frac{m(m-2k)}{2N}}$. (Notice that the probability of no match is $e^{-\frac{m^2}{2N}}$ when k is m, and it gets smaller as k gets larger.)

Hints: For $m \ll N$: $\frac{N!}{(N-m)!N^m} \sim e^{-\frac{m^2}{2N}}$, and $(1 - \frac{m}{N}) \sim e^{-\frac{m}{N}}$.

(b) [10 pts] Find the approximate number of people in the group, m, necessary for a half chance of a match (your answer will be in the form of a quadratic). Then simplify your answer to show that, as k gets large (such that $\sqrt{N} \ll k$), then $m \sim \frac{N \ln 2}{k}$.

Hint: For $x \ll 1$: $\sqrt{1-x} \sim (1 - \frac{x}{2})$.

Problem 3. [20 points]

(a) [7 pts] Suppose you repeatedly flip a fair coin until you see the sequence `HHT` or the sequence `TTH`. What is the probability you will see `HHT` first?

Hint: Use a bijection argument.

(b) [7 pts] What is the probability you see the sequence `HTT` before you see the sequence `HHT`?

Hint: Try to find the probability that `HHT` comes before `HTT` conditioning on whether you first toss an `H` or a `T`. Somewhat surprisingly, the answer is not 1/2.

(c) [6 pts] Suppose you flip three fair, mutually independent coins. Define the following events:

- Let A be the event that *the first* coin is heads.
- Let B be the event that *the second* coin is heads.
- Let C be the event that *the third* coin is heads.
- Let D be the event that *an even number of* coins are heads.

Use the four step method to determine the probability of each of A, B, C, D.

Problem 4. [20 points]

Professor Moitra has a deck of 52 randomly shuffled playing cards, 26 red, 26 black. He proposes the following game: he will continually draw a card off the top of the deck, turn it face up so that you can see it and then put it aside. At any point while there are still cards left in the deck, you may say "stop" and he will flip over one last card. If that next card turns up black you win and otherwise you lose. Either way, the game ends.

(a) [4 pts] Show that if you say "stop" before you have seen any cards, you then have probability $1/2$ of winning the game.

(b) [4 pts] Suppose you don't say "stop" before the first card is flipped and it turns up red. Show that you then have a probability of winning the game that is greater than $1/2$.

(c) [4 pts] If there are r red cards left in the deck and b black cards, show that the probability of winning if you say "stop" before the next card is flipped is $b/(r + b)$.

(d) [8 pts] Either,

1. come up with a strategy for this game that gives you a probability of winning strictly greater than $1/2$ and prove that the strategy works, or,
2. come up with a proof that no such strategy can exist.

Problem 5. [20 points] Suppose you have seven standard dice with faces numbered 1 to 6. Each die has a label corresponding to a letter of the alphabet (A through G). A *roll* is a sequence specifying a value for each die in alphabet order. For example, one roll is $(6, 1, 4, 1, 3, 5, 2)$ indicating that die A showed a 6, die B showed 1, die C showed 4, ...

(a) [5 pts] What is the probability of a roll where *exactly* two dice have the value 3 and the remaining five dice all have different values?

Example: $(3, 2, 3, 1, 6, 4, 5)$ is a roll of this type, but $(1, 1, 2, 6, 3, 4, 5)$ and $(3, 3, 1, 2, 4, 6, 4)$ are not.

(b) [5 pts] What is the probability of a roll where two dice have an even value and the remaining five dice all have different values?

Example: $(4, 2, 4, 1, 3, 6, 5)$ is a roll of this type, but $(1, 1, 2, 6, 1, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

(c) [10 pts] What is the probability of a roll where two dice have one value, two different dice have a second value, and the remaining three dice a third value?

Example: $(6,1,2,1,2,6,6)$ is a roll of this type, but $(4,4,4,4,1,3,5)$ and $(5,5,5,6,6,1,2)$ are not.