

Midterm Practice Problems

Problem 1. [10 points] In problem set 1 you showed that the **nand** operator by itself can be used to write equivalent expressions for all other Boolean logical operators. We call such an operator *universal*. Another universal operator is **nor**, defined such that $P \text{ nor } Q \Leftrightarrow \neg(P \vee Q)$. Show how to express $P \wedge Q$ in terms of: **nor**, P , Q , and grouping parentheses.

Solution. $(\neg P) \text{ nor } (\neg Q) = (P \text{ nor } P) \text{ nor } (Q \text{ nor } Q)$. ■

Problem 2. [15 points] We define the sequence of numbers

$$a_n = \begin{cases} 1 & \text{if } 0 \leq n \leq 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \geq 4. \end{cases}$$

Prove that $a_n \equiv 1 \pmod{3}$ for all $n \geq 0$.

Solution. Proof by strong induction. Let $P(n)$ be the predicate that $a_n \equiv 1 \pmod{3}$.

Base case: For $0 \leq n \leq 3$, $a_n = 1$ and is therefore $\equiv 1 \pmod{3}$.

Inductive step: For $n \geq 4$, assume $P(k)$ for $0 \leq k \leq n$ in order to prove $P(n+1)$.

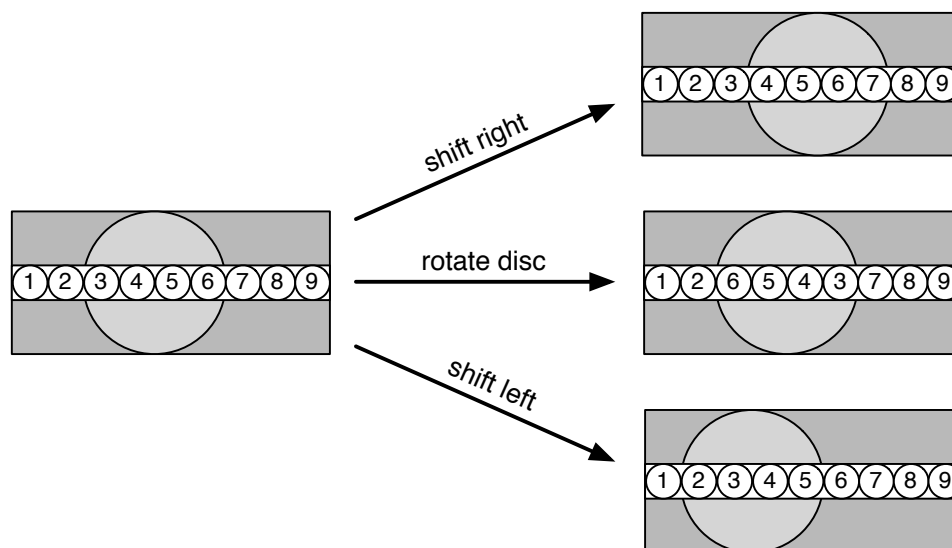
In particular, since each of a_{n-4} , a_{n-3} , a_{n-2} and a_{n-1} is $\equiv 1 \pmod{3}$, their sum must be $\equiv 4 \equiv 1 \pmod{3}$. Therefore, $a_n \equiv 1 \pmod{3}$ and $P(n+1)$ holds. ■

Problem 3. [20 points] The Slipped Disc Puzzle™ consists of a track holding 9 circular tiles. In the middle is a disc that can slide left and right and rotate 180° to change the positions of *exactly* four tiles. As shown below, there are three ways to manipulate the puzzle:

Shift Right: The center disc is moved one unit to the right (if there is space)

Rotate Disc: The four tiles in the center disc are reversed

Shift Left: The center disc is moved one unit to the left (if there is space)



Prove that if the puzzle starts in an initial state with all but tiles 1 and 2 in their natural order, then it is impossible to reach a goal state where all the tiles are in their natural order. The initial and goal states are shown below:



Write your proof on the next page...

Solution. Order the tiles from left to right in the puzzle. Define an *inversion* to be a pair of tiles that is out of their natural order (e.g. 4 appearing to the left of 3).

Lemma. Starting from the initial state there is an odd number of inversions after any number of transitions.

Proof. The proof is by induction. Let $P(n)$ be the proposition that starting from the initial state there is an odd number of inversions after n transitions.

Base case: After 0 transitions, there is one inversion, so $P(0)$ holds.

Inductive step: Assume $P(n)$ is true. Say we have a configuration that is reachable after $n + 1$ transitions.

1. Case 1: The last transition was a shift left or shift right

In this case, the left-to-right order of the discs does not change and thus the number of inversions remains the same as in

2. The last transition was a rotate disc.

In this case, six pairs of disks switch order. If there were x inversions among these pairs after n transitions, there will be $6 - x$ inversions after the reversal. If x is odd, $6 - x$ is odd, so after $n + 1$ transitions the number of inversions is odd.

□

Conclusion: Since all reachable states have an odd number of inversions and the goal state has an even number of inversions (specifically 0), the goal state cannot be reached. ■

Problem 4. [10 points] Find the multiplicative inverse of 17 modulo 72 in the range $\{0, 1, \dots, 71\}$.

Solution. Since 17 and $72 = 2^3 3^2$ are relatively prime, an inverse exists and can be found by either Euler's theorem or the Pulverizer.

Solution 1: Euler's Theorem

$$\begin{aligned}
 \phi(72) &= \phi(2^3 \cdot 3^2) \\
 &= \phi(2^3) \cdot \phi(3^2) && \text{(since } 2^3 \text{ and } 3^2 \text{ are rel. prime)} \\
 &= (2^3 - 2^2)(3^2 - 3^1) && \text{(since 2 and 3 are prime)} \\
 &= 4 \cdot 6 = 24
 \end{aligned}$$

Therefore, $17^{\phi(72)-1} = 17^{23}$ is *an* inverse of 17. To find *the* inverse in the range $\{0, 1, \dots, 71\}$ we take the remainder using the method of repeated squaring:

$$\begin{aligned}
 17 &= 17 \\
 17^2 &= 289 \\
 &\equiv 1 && \text{(since } 289 = 4 \cdot 72 + 1) \\
 17^4 &\equiv 1^2 = 1 \\
 17^8 &\equiv 1 \\
 &\dots etc.
 \end{aligned}$$

Therefore the inverse of 17 in the range $\{0, 1, \dots, 71\}$ is given by,

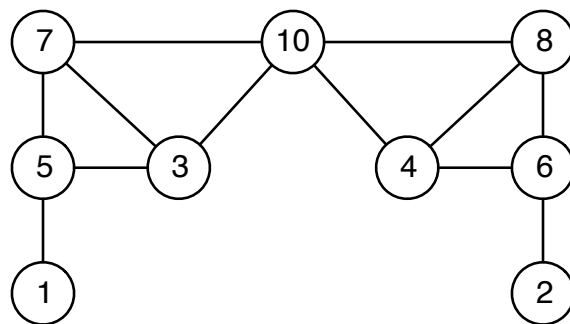
$$\begin{aligned}
 17^{23} &= 17^{16} 17^4 17^2 17^1 \\
 &\equiv 1 \cdot 1 \cdot 1 \cdot 17 \\
 &= 17
 \end{aligned}$$

Solution 2: The Pulverizer

x	y	rem xy	$=$	$x - q \cdot y$
72	17	4	$=$	$72 - 4 \cdot 17$
17	4	1	$=$	$17 - 4 \cdot 4$
			$=$	$17 - 4 \cdot (72 - 4 \cdot 17)$
			$=$	$17 \cdot 17 - 4 \cdot 72$
4	1	0		

Since $17^2 - 4 \cdot 72 = 1$, $17^2 \equiv 1 \pmod{72}$ and so 17 is self inverse. ■

Problem 5. [15 points] Consider a graph representing the main campus buildings at MIT.



(a) [3 pts] Is this graph bipartite? Provide a brief argument for your answer.

Solution. No, there is an odd-length cycle

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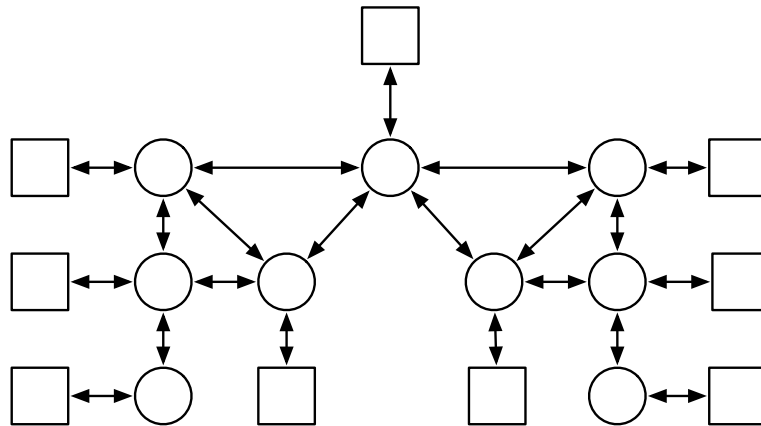
(b) [4 pts] Does this graph have an Euler circuit? Provide a brief argument for your answer.

Solution. This graph does not have an Euler circuit because there are vertices with odd degree

■

Problem 5 continued...

Now suppose each building has separate mail collection and drop-off boxes and each collection box has a single package destined for a unique drop-off box (i.e. a permutation). We can model this as a permutation routing problem by treating the buildings as switches, attaching an input and output terminal to each of the nine buildings, and treating the existing edges as bidirectional as in the graph below:



(c) [4 pts] Give the diameter of this graph:

Solution. The diameter is 8, the length of a shortest path between the terminals for buildings 1 and 2. ■

(d) [4 pts] What is the max congestion of this graph? That is, in the worst case permutation, how many packages would need to pass through a single building? Provide a brief argument for your answer.

Solution. The maximum congestion is 9. Consider a permutation where all the packages on the left side are destined for drop-offs on the right side and vice versa with the building 10 package destined for building 10. In this case, all 9 packages must pass through building 10, giving a congestion of 9, which is the maximum for a graph with nine input-output pairs. ■

Problem 6. [10 points]

A tournament graph $G = (V, E)$ is a directed graph such that there is either an edge from u to v or an edge from v to u for *every* distinct pair of nodes u and v . (The nodes represent players and an edge $u \rightarrow v$ indicates that player u beats player v .)

Consider the “beats” relation implied by a tournament graph. Indicate whether or not each of the following relational properties hold *for all* tournament graphs and briefly explain your reasoning. You may assume that a player never plays herself.

1. transitive

Solution. The “beats” relation is not transitive because there could exist a cycle of length 3 where x beats y , y beats z and z beats x . By the definition of a tournament, x cannot then beat y in such a situation. ■

2. symmetric

Solution. The “beats” relation is not symmetric by the definition of a tournament: if x beats y then y does not beat x . ■

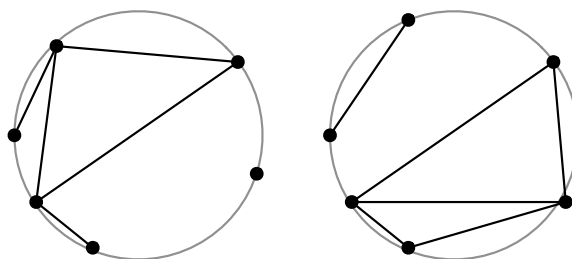
3. antisymmetric

Solution. The “beats” relation is antisymmetric since for any distinct players x and y , if x beats y then y does not beat x . ■

4. reflexive

Solution. The “beats” relation is not reflexive since a tournament graph has no self-loops. ■

Problem 7. [20 points] An outerplanar graph is an undirected graph for which the vertices *can be* placed on a circle in such a way that no edges (drawn as straight lines) cross each other. For example, the complete graph on 4 vertices, K_4 , is not outerplanar but any proper subgraph of K_4 with strictly fewer edges is outerplanar. Some examples are provided below:



Prove that any outerplanar graph is 3-colorable. A fact you may use without proof is that any outerplanar graph has a vertex of degree at most 2.

Solution. *Proof.* Proof by induction on the number of nodes n with the induction hypothesis $P(n)$ = "every outerplanar graph with n vertices is 3-colorable."

Base case: For $n = 1$ the single node graph with no edges is trivially outerplanar and 3-colorable.

Inductive step: Assume $P(n)$ holds and let G_{n+1} be an outerplanar graph with $n+1$ vertices. There must exist a vertex v in G_{n+1} with degree at most 2. Removing v and all its incident edges leaves a subgraph G_n with n vertices.

Since G_{n+1} could be drawn with its vertices on a circle and its edges drawn as straight lines without intersections, any subgraph can also be drawn in such a way and so G_n is also an outerplanar graph. $P(n)$ implies G_n is 3-colorable. Therefore we can color all the vertices in G_{n+1} other than v using only 3 colors and since $\deg(v) \leq 2$ we may color it a color different than the vertices adjacent to it using only 3 colors. Therefore, G_{n+1} is 3-colorable and $P(n+1)$ holds.

□

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Problem 8. [10 points] Give upper and lower bounds for the following expression which differ by at most 1.

$$\sum_{i=1}^n \frac{1}{i^3}$$

Solution. To find upper and lower bounds, we use the integral method:

$$\begin{aligned} \sum_{i=1}^n \frac{1}{i^3} &\leq 1 + \int_1^n \frac{1}{x^3} dx \\ &= 1 - \frac{1}{2} x^{-2} \Big|_1^n \\ &= 1 - \frac{1}{2} \left(\frac{1}{n^2} - 1 \right) = \frac{3}{2} - \frac{1}{2n^2} \\ \sum_{i=1}^n \frac{1}{i^3} &\geq \frac{1}{n^3} + \int_1^n \frac{1}{x^3} dx \\ &= \frac{1}{n^3} - \frac{1}{2} x^{-2} \Big|_1^n \\ &= \frac{1}{n^3} - \frac{1}{2} \left(\frac{1}{n^2} - 1 \right) = \frac{1}{2} + \frac{1}{n^3} - \frac{1}{2n^2} \end{aligned}$$

■

Problem 9. [15 points] Circle every symbol on the left that could correctly appear in the box to its right. For each of the six parts you may need to circle any number of symbols.

(a) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$6n^2 + 7n - 10 = \boxed{O, \Omega, \Theta} (n^2)$$

(b) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$6^n = \boxed{\Omega, \omega} (n^6)$$

(c) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$n! = \boxed{O, o} (n^n)$$

(d) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$\sum_{j=1}^n \frac{1}{j} = \boxed{O, \Omega, \Theta, \sim} (\ln n)$$

(e) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$\ln(n^3) = \boxed{O, \Omega, \Theta} (\ln n)$$