Final Exam Solutions

- The exam is **closed book**, but you may have two 8.5″ × 11″ sheet with notes (either printed or in your own handwriting) on both sides.
- Calculators and electronic devices (including cell phones) are not allowed.
- You may assume all of the results presented in class. This does **not** include results demonstrated in practice quiz material.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

NAME: ____________________

TA: ____________________

Problem	Value	Score	Grader
1	10		
2	13		
3	18		
4	10		
5	24		
6	15		
7	10		
8	15		
9	15		
10	15		
11	15		
12	10		
13	10		
Total	180		

Problem 1. [10 points] *Finalphobia* is a rare disease in which the victim has the delusion that he or she is being subjected to an intense mathematical examination.

- A person selected uniformly at random has finalphobia with probability $\frac{1}{50}$.
- A person with finalphobia has shaky hands with probability $\frac{9}{10}$.
- A person without finalphobia has shaky hands with probability $\frac{1}{20}$.

What is the probability that a person selected uniformly at random has finalphobia, given that he or she has shaky hands?

Your answer should be expressed as a ratio of integers.

Solution. We first compute:

$$\begin{aligned}
\Pr[\text{shaky hands}] &= \Pr[\text{shaky hands} \mid \text{finalphobia}] \cdot \Pr[\text{finalphobia}] \\
&\quad + \Pr[\text{shaky hands} \mid \text{no finalphobia}] \cdot \Pr[\text{no finalphobia}] \\
&= \frac{9}{10} \cdot \frac{1}{50} + \frac{1}{20} \cdot \frac{49}{50} \\
&= \frac{67}{1000}.
\end{aligned}$$

By Bayes's theorem,

$$\begin{aligned}
\Pr[\text{finalphobia} \mid \text{shaky hands}] &= \frac{\Pr[\text{shaky hands} \mid \text{finalphobia}] \Pr[\text{finalphobia}]}{\Pr[\text{shaky hands}]} \\
&= \frac{\frac{9}{10} \cdot \frac{1}{50}}{\frac{67}{1000}} \\
&= \frac{18}{67}.
\end{aligned}$$

■

Problem 2. [13 points]

(a) [3 pts] Give a closed form for

$$\sum_{i=1}^{n}\sum_{j=1}^{n} i \cdot j$$

Solution.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} i \cdot j &= \sum_{i=1}^{n}(i \cdot \sum_{j=1}^{n} j) \\
&= \sum_{i=1}^{n} i \cdot \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2} \cdot \sum_{i=1}^{n} i \\
&= \left(\frac{n(n+1)}{2}\right)^2.
\end{aligned}$$

■

(b) [5 pts] Give a closed form, as a ratio of polynomials in n, for

$$\sum_{j=1}^{n}\sum_{i=j}^{n} j.$$

For convenience, we provide the identity: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution.

$$\begin{aligned}
\sum_{j=1}^{n}\sum_{i=j}^{n} j &= \sum_{j=1}^{n} j \sum_{i=j}^{n} 1 \\
&= \sum_{j=1}^{n} j(n-j+1) \\
&= n\sum_{j=1}^{n} j - \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} j \\
&= (n+1)\sum_{j=1}^{n} j - \sum_{j=1}^{n} j^2 \\
&= (n+1) \cdot \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n(n+1)(3n+3) - n(n+1)(2n+1)}{6} \\
&= \frac{n(n+1)(n+2)}{6}.
\end{aligned}$$

■

(c) [5 pts] Let $f(n) = \prod_{i=1}^{n} 2i$. Give a function g in closed form such that $f(n) = \Theta(g(n))$. You may not include factorials in your answer.

Solution. We can write

$$f(n) = 2^n \prod_{i=1}^{n} i = 2^n \cdot n!$$

Using Stirling's approximation, we have that

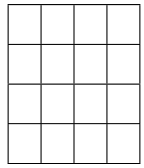
$$f(n) \sim 2^n \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

This asymptotic on the right is an adequate answer. Another expression (which equals the same thing up to a constant) is

$$\sqrt{n} \left(\frac{2n}{e}\right)^n .$$

■

Problem 3. [18 points] We play a simplified game of battleship. We are given a 4x4 board



on which you have placed two pieces. Your destroyer is 1 square $\times$ 2 squares and your submarine is 1 square $\times$ 3 squares. The pieces lie entirely on the board, cannot overlap, and are arranged either vertically or horizontally.

Your opponent picks 8 of the 16 squares uniformly at random and then shoots at those 8 squares. A ship is sunk if all the squares it occupies are shot at.

For this problem, you may leave your answer as the sum of expressions that are products or ratios of integers, exponentials, factorials and/or choose expressions.

(a) [8 pts] What is the probability that both of your ships are sunk?

Solution. There are $\binom{16}{8}$ total choices for your opponent. There are $\binom{16-5}{8-5}$ choices that pick all of your squares. So the probability that both of your ships are sunk is $\frac{\binom{11}{3}}{\binom{16}{8}}$. ■

(b) [10 pts] What is the probability of sinking the submarine but not the destroyer?

Solution. To sink the submarine (and perhaps also the destroyer), your opponent must choose the 3 submarine squares and any 5 of the 13 others; so the probability of this event is

$$\frac{\binom{13}{5}}{\binom{16}{8}}.$$

To exclude the possibility of also sinking the destroyer, we subtract off our answer to part (a), and conclude with

$$\frac{\binom{13}{5} - \binom{11}{3}}{\binom{16}{8}}.$$

■

Problem 4. [10 points] You get 5 cards at random from a standard 52 card deck. What is the probability that you have exactly one pair? (This means that exactly 2 cards share the same rank, so two pairs or three of a kind would not count.)

You may express your answer as the product or ratio of integers, factorials, exponentials, and/or choose expressions.

Solution. There are 13 different ranks that could have a pair. For each rank, you need 2 out of the 4 cards. There are 12 remaining ranks for the last three cards and each has four possible suits, for a final answer of:

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}}$$

∎

Problem 5. [24 points] Alice decides to play the lottery. She bought 10,000 tickets, each with a probability of $\dfrac{1}{1,000,000}$ of winning a payout of \$1,000,000, probability $\dfrac{1}{10}$ of paying out \$10, and probability $1 - \dfrac{1}{10} - \dfrac{1}{1,000,000}$ of being worth nothing. If multiple tickets win, the payouts remain as above for each ticket. The tickets are mutually independent.

(a) [5 pts] What is Alice's expected return? Express your answer as an integer.

Solution. Each ticket has an expected return of $\frac{1}{1,000,000} \cdot 1,000,000 + \frac{1}{10} \cdot 10 = 2$ dollars. By the linearity of expectation, Alice's expected return is the sum of the expected returns of 10,000 such tickets, which gives a total of \$20,000. ■

(b) [2 pts] Does your answer to part (a) depend on the tickets being mutually independent?

Solution. Part (a) uses only the linearity of expectation, so independence is not required. ■

(c) [5 pts] What is the variance of Alice's return? Express your answer as an integer.

Solution. Because the payoffs X_i of individual tickets are mutually independent, the variance of the sum is the sum of the variances. So we compute the variance of an individual ticket:

$$\begin{aligned}
\mathrm{Var}(X_i) &= \mathrm{E}\left[(X_i - \mathrm{E}[X_i])^2\right] \\
&= (-2)^2 \cdot \left(1 - \frac{1}{10} - \frac{1}{1,000,000}\right) + 999,998^2 \cdot \frac{1}{1,000,000} + 8^2 \cdot \frac{1}{10} \\
&= 1,000,006.
\end{aligned}$$

An arithmetically easier way to compute this is as follows:

$$\mathrm{E}\left[X_i^2\right] = 1,000,000^2 \cdot \frac{1}{1,000,000} + 10^2 \cdot \frac{1}{10} = 1,000,010,$$

$$\mathrm{Var}(X_i) = \mathrm{E}\left[X_i^2\right] - (\mathrm{E}[X_i])^2 = 1,000,010 - 4 = 1,000,006.$$

The total variance of $R = \sum_{i=1}^{100} X_i$ is then 10,000 times this, for a final answer of 10,000,060,000 (in units of dollars squared). ■

(d) [2 pts] Does your answer to part (c) depend on the tickets being mutually independent?

Solution. Part (c) uses the additivity of variance, so independence is required. ■

(e) [4 pts] Give a Markov upper bound for the probability that Alice wins at least \$100,000,000. Your answer should be expressed as a ratio of integers.

Solution. Markov's Theorem states that, since R is non-negative,

$$\Pr[R \geq x] \leq \frac{\mathrm{E}[R]}{x}.$$

In this case,

$$\Pr[R \geq 100{,}000{,}000] \leq \frac{20{,}000}{100{,}000{,}000} = \frac{1}{5{,}000}.$$

■

(f) [6 pts] Give a Chebyshev upper bound for the probability that Alice wins at least \$100,000,000. Your answer should be expressed in terms of basic arithmetic operations on integers, but need not be simplified further.

Solution.

$$\begin{aligned}
\Pr[R \geq 100{,}000{,}000] &= \Pr[|R - 20{,}000| \geq 99{,}980{,}000] \\
&= \Pr[|R - \mathrm{E}[R]\,| \geq 99{,}980{,}000] \\
&\leq \frac{\operatorname{Var} R}{99{,}980{,}000^2} \\
&= \frac{10{,}000{,}060{,}000}{99{,}980{,}000^2} \\
&= \frac{1{,}000{,}006}{999{,}800^2} \\
&= \frac{1{,}000{,}006}{999{,}600{,}040{,}000} \\
&= \frac{500{,}003}{499{,}800{,}020{,}000}.
\end{aligned}$$

■

Problem 6. [15 points]

How many of the numbers $1, 2, \ldots, 3000$ are divisible by one or more of 4, 5, or 6?

Solution. We will apply the inclusion–exclusion principle.

Let S_i be the set of numbers $1, 2, \ldots, 3000$ that are divisible by i; and let T denote the set of numbers $1, 2, \ldots, 3000$ divisible by one or more of 4, 5, or 6; so we are trying to count T. Notice that $T = S_4 \cup S_5 \cup S_6$. Then

$$\begin{aligned}
|T| &= |(S_4 \cup S_5) \cup S_6| \\
&= |S_4 \cup S_5| + |S_6| - |(S_4 \cup S_5) \cap S_6| && \text{(by inclusion–exclusion)} \\
&= |S_4| + |S_5| + |S_6| - |S_4 \cap S_5| - |(S_4 \cup S_5) \cap S_6| && \text{(by inclusion–exclusion)} \\
&= |S_4| + |S_5| + |S_6| - |S_4 \cap S_5| - |(S_4 \cap S_6) \cup (S_5 \cap S_6)| \\
&= |S_4| + |S_5| + |S_6| - |S_4 \cap S_5| - |S_4 \cap S_6| \\
&\quad - |S_5 \cap S_6| + |(S_4 \cap S_6) \cap (S_5 \cap S_6)| && \text{(by inclusion–exclusion)} \\
&= |S_4| + |S_5| + |S_6| - |S_4 \cap S_5| - |S_4 \cap S_6| \\
&\quad - |S_5 \cap S_6| + |S_4 \cap S_5 \cap S_6| \\
&= |S_4| + |S_5| + |S_6| - |S_{20}| - |S_{12}| - |S_{30}| + |S_{60}| && \text{(by taking LCMs)} \\
&= \lfloor \frac{3000}{4} \rfloor + \lfloor \frac{3000}{5} \rfloor + \lfloor \frac{3000}{6} \rfloor \\
&\quad - \lfloor \frac{3000}{20} \rfloor - \lfloor \frac{3000}{12} \rfloor - \lfloor \frac{3000}{30} \rfloor + \lfloor \frac{3000}{60} \rfloor \\
&= 750 + 600 + 500 - 150 - 250 - 100 + 50 \\
&= 1400.
\end{aligned}$$

A more concise solution might directly apply the general inclusion–exclusion formula for three sets, whereas here we deal with unions of two sets at a time. ■

Problem 7. [10 points]

Prove the following identity for all natural numbers n:

$$\sum_{i=1}^{n}(i \cdot i!) = (n+1)! - 1.$$

Solution. We proceed by induction. The base case $n = 1$ is clear:

$$1 \cdot 1! = 2! - 1.$$

For the inductive step, suppose that the identity holds for n; we prove it for $n+1$:

$$\begin{aligned}
\sum_{i=1}^{n+1}(i \cdot i!) &= (n+1)! - 1 + (n+1)(n+1)! && \text{(by inductive hypothesis)} \\
&= (n+2)(n+1)! - 1 && \text{(factoring)} \\
&= (n+2)! - 1.
\end{aligned}$$

This completes the inductive step. By induction, the identity holds for all natural numbers n. ∎

Problem 8. [15 points]

(a) [8 pts] Give an exact solution to the following recurrence:

$$T(n) = 5T(n-1) - 6T(n-2), \quad T(0) = 0,\ T(1) = 1.$$

Solution. This is a linear recurrence with characteristic equation $x^2 = 5x - 6$, which factors as

$$(x-2)(x-3) = 0.$$

We therefore expect a solution of the form $T(n) = a \cdot 2^n + b \cdot 3^n$. From the initial conditions, we have $a + b = 0$ and $2a + 3b = 1$, from which $b = 1$ and $a = -1$. So the exact solution is:

$$T(n) = 3^n - 2^n.$$

■

(b) [7 pts] Give an asymptotic expression for the following recurrence, in Θ notation:

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2, \quad T(1) = 0.$$

Solution. We apply the Akra–Bazzi method with $g(n) = n^2$, $a = 4$, $b = \frac{1}{2}$. Solving $ab^p = 1$ for p, we find $p = 2$. Then

$$\begin{aligned} T(n) &= \Theta\left(n^2\left(1 + \int_1^n \frac{u^2}{u^3}\,du\right)\right) \\ &= \Theta\left(n^2\left(1 + \log n\right)\right) \\ &= \Theta(n^2 \log n). \end{aligned}$$

An alternative is to apply the Master Theorem. ■

Problem 9. [15 points] Suppose that we build a graph on N vertices as follows: for each (unordered) pair of distinct vertices, we independently toss a fair coin, and draw an edge between that pair of vertices if the coin lands 'heads'.

Justify your responses for all parts below.

(a) [6 pts] Given a vertex, what is the probability that the vertex has degree exactly 3?

Solution. This vertex is a member of $N-1$ vertex pairs, and this vertex has degree 3 precisely when exactly 3 of these vertex pairs are connected by edges. Considering the 2^{N-1} equally probable outcomes for the edge status of these $N-1$ vertex pairs, there are $\binom{N-1}{3}$ outcomes for which the vertex has degree 3. Thus the probability is

$$2^{-N+1}\binom{N-1}{3}.$$

■

(b) [3 pts] Let D_i denote the degree of vertex i as a random variable. Which standard family of probability distributions does D_i belong to?

Solution. The degree follows a binomial distribution with parameters $n = N-1$ and $p = \frac{1}{2}$. The argument of the part above essentially shows this; we could replace '3' with any other value in part (a) and recover the pdf for a binomial distribution.

For a more formal argument, let E_{ij} denote the indicator random variable taking on value 1 if (i, j) is an edge, which is true with probability $\frac{1}{2}$; this is a Bernoulli random variable with parameter $\frac{1}{2}$, and the random variables E_{ij} are independent. Fixing a vertex i, the degree of i is $\sum_j E_{ij}$, and a sum of $N-1$ independent Bernoulli variables with parameter $\frac{1}{2}$ is a binomial distribution as described. ■

(c) [6 pts] What is the expected number of vertices in the graph with degree exactly 3?

Partial credit will be given for answers written in terms of p, where p represents the correct answer to part (a).

Solution. Let X_i be the indicator that vertex i has degree 3; then the expectation of X_i is the same as the probability that vertex i has degree 3, which we computed in part (a). The number of degree 3 vertices in the graph is $X = \sum_{i=1}^{N} X_i$. Its expectation is

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} \mathrm{E}[X_i] = N \cdot p = N \cdot 2^{-N+1}\binom{N-1}{3},$$

by the linearity of expectation. ■

Problem 10. [15 points]

(a) [8 pts] Let $d \geq 1$ be a given constant, and consider the sequence $(a_k)_{k\geq 0}$ defined by

$$a_k = \begin{cases} 1 & \text{if } 1 \leq k \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Give a closed form expression for the generating function $f_d(x)$ of this sequence, expressed as a ratio of polynomials in the variable x. The answer will depend on d.

Hint: Be careful of the first term, $a_0 = 0$.

Solution. If we were to shift the sequence left by one, so that $a_0 = 1$ but $a_d = 0$, the generating function would be a finite geometric series with initial term 1 and closed form

$$\frac{1 - x^d}{1 - x}.$$

Shifting the sequence right corresponds to multiplying by x, so the actual generating function is

$$\frac{x(1 - x^d)}{1 - x}.$$

■

(b) [7 pts] Suppose we have two dice, one with values $1, 2, \ldots, d_1$ on its sides, and the other with values $1, 2, \ldots, d_2$. Define the sequence $(b_k)_{k\geq 0}$ as the number of outcomes for the dice that yield a total sum of k. What is the generating function of this sequence?

Hint: Your answer to part (a) might be helpful.

Partial credit will be given for answers written in terms of $f_d(x)$, the generating function that correctly answers part (a).

Solution. The sequence (b_k) is a convolution of two copies of (a_k), where the two copies have d_1 and d_2 respectively substituted for d. Convolution of sequences corresponds to multiplication of generating functions, so the generating function is:

$$\begin{aligned} f_{d_1}(x) \cdot f_{d_2}(x) &= \frac{x(1 - x^{d_1})}{1 - x} \cdot \frac{x(1 - x^{d_2})}{1 - x} \\ &= \left(\frac{x}{1 - x}\right)^2 (1 - x^{d_1})(1 - x^{d_2}). \end{aligned}$$

■

Problem 11. [15 points] Suppose we choose a random 9-digit MIT ID number, taken uniformly from the numbers 000000000 to 999999999.

Each answer should be expressed as a ratio of integers or in scientific notation.

(a) [8 pts] What is the expected number of total occurrences of the digit sequence '6042' in the ID number?

Solution. For $1 \leq i \leq 6$, let X_i be the indicator random variable with value 1 precisely when the ith digit of the ID is a 6, the $(i+1)$th digit is a 0, the $(i+2)$th is a 4, **and** the $(i+3)$th is a 2.

Each X_i is 1 with probability $\frac{1}{10000}$. By the linearity of expectation, the expected number of occurrences of '6042' is

$$\mathrm{E}[X_1 + \ldots + X_6] = \mathrm{E}[X_1] + \ldots + \mathrm{E}[X_6] = \frac{6}{10000}.$$

■

(b) [7 pts] What is the expected total number of occurrences of '6042' or '6041'? That is,

$$\mathrm{E}[\#\text{ occurrences of '6042'} + \#\text{ occurrences of '6041'}].$$

Solution. Again we use the linearity of expectation to rewrite this as

$$\mathrm{E}[\#\text{ occurrences of '6042'}] + \mathrm{E}[\#\text{ occurrences of '6041'}].$$

The first term is the answer to part (a); the second term is the same, since we would not have answered part (a) differently for the sequence '6041'. Thus the answer is $\frac{12}{10000}$. ■

Problem 12. [10 points]

Suppose that we randomly scramble (uniformly) the letters of the word 'ABRACADABRA'; what is the probability that the word remains 'ABRACADABRA' afterwards? Express your answer as a ratio of integers, exponentials, factorials, and/or choose expressions.

Solution. We first count the letters in the given word: five As, two Bs, two Rs, a C and a D; 11 letters in total. In order to shuffle letters around and still obtain the word 'ABRACADABRA', we can only permute the five As among themselves, permute the two Bs, and permute the two Rs, for which there are $5! \cdot 2! \cdot 2! = 480$ possible outcomes. On the other hand, there are 11! total ways to shuffle the letters. So the final probability is

$$\frac{5! \cdot 2! \cdot 2!}{11!}.$$

■

Problem 13. [10 points]

Identify each of the following asymptotic statements as true or false, with brief justification.

(a) [2 pts] $4^x = O(3^x)$.

Solution. False. The ratio

$$\frac{4^x}{3^x} = \left(\frac{4}{3}\right)^x$$

tends to ∞ as $x \to \infty$. ■

(b) [2 pts] $x^2 = O(x^3)$.

Solution. True. We can multiply x^3 by a constant (say, 1) such that it eventually forms an upper bound for x^2 (say, for all $x \geq 1$). ■

(c) [2 pts] $x^3 - x^2 = o(5x^3)$.

Solution. False. The ratio $\dfrac{5x^3}{x^3 - x^2}$ tends to 5, not to ∞, as $x \to \infty$. ■

(d) [2 pts] $\frac{1}{x} = \Theta(1)$.

Solution. False. $\frac{1}{x}$ is not bounded below by any *positive* constant times 1. ■

(e) [2 pts] $e^{\ln^2 x} = \Omega(x^{100})$.

Solution. True. We can write $e^{\ln^2 x} = x^{\ln x}$, which exceeds x^{100} for all sufficiently (very!) large x. ■