

Problem Set 1

Due: Monday, September 8

Problem 1. [20 points] Suppose that $w^2 + x^2 + y^2 = z^2$, where w, x, y , and z always denote positive integers. (*Hint:* It may be helpful to represent even integers as $2i$ and odd integers as $2j + 1$, where i and j are integers.)

Prove the proposition: z is even if and only if w, x , and y are even. Do this by considering all the cases of w, x, y being odd or even.

Problem 2. [14 points] The three logical expressions below are nearly equivalent. However, for one particular setting of x, y , and z , one of the expressions is not like the others. Write out *complete truth tables for all three*, and determine which one of these expressions just doesn't belong.

1. $(x \rightarrow y) \wedge (y \rightarrow z) \wedge (z \rightarrow x)$
2. $(\neg x) \wedge (\neg y) \wedge (\neg z)$
3. $(\neg x \vee (y \wedge z)) \wedge (x \vee (\neg y \wedge \neg z))$

Problem 3. [14 points] A student is trying to prove that propositions p, q , and r are all true. She proceeds as follows. First, she proves three facts: $p \rightarrow q$, $q \rightarrow r$, and $r \rightarrow p$. Then she concludes, “thus obviously p, q , and r are all true.” Let's first formalize her deduction as a logical statement and then evaluate whether or not it is correct.

(a) [5 pts] Using logic notation and the symbols p, q , and r , write down the logical implication that she uses in her final step.

(b) [5 pts] Use a truth table to determine whether this logical implication is a tautology (i.e., a universal truth in logic).

(c) [4 pts] Is her proof that propositions p, q , and r are all true correct? Briefly explain.

Problem 4. [24 points] Translate the following statements into predicate logic. For each, specify the domain. In addition to logic symbols, you may build predicates using arithmetic, relational symbols, and constants. For example, the statement “ n is an odd number” could be translated into $\exists m.(2m + 1 = n)$, where the domain is \mathbb{Z} , the set of integers. Another example, “ p is a prime number,” could be translated to

$$(p > 1) \text{ AND } \text{NOT}(\exists m.\exists n.(m > 1 \text{ AND } n > 1 \text{ AND } mn = p))$$

Let $\text{prime}(p)$ be an abbreviation that you could use to denote the above formula in this problem.

(a) [4 pts] (Lagrange's Four-Square Theorem) Every nonnegative integer is expressible as the sum of four perfect squares.

(b) [4 pts] (Goldbach Conjecture) Every even integer greater than two is the sum of two primes.

(c) [4 pts] The function $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous.

(d) [4 pts] (Fermat's Last Theorem) There are no nontrivial solutions to the equation:

$$x^n + y^n = z^n$$

over the nonnegative integers when $n > 2$.

(e) [4 pts] There is no largest prime number.

(f) [4 pts] (Bertrand's Postulate) If $n > 1$, then there is always at least one prime p such that $n < p < 2n$.

Problem 5. [16 points] You have a balance; by putting objects on each side of the balance, you can determine which side is heavier (or if both sides have the same weight).

You also have a large collection of stones of known weight; there are stones that weigh 1 ounce, 2 ounces, and so on all the way up to 40 ounces (and you have many stones of each weight).

You are given an object of unknown weight; all you know is that it weighs an integer number of ounces between 1 and 40, inclusive. Your goal is to determine the weight of this object, by using as small a set of stones from your collection as possible (you may do as many weighings as you like with the stones you select). Give a set of possible stones, and summarize the procedure that you will use to identify the weight of the object. (You will get 8/16 points for using 6 stones; 12/16 points for using just 5 stones; and 16/16 points for using only 4 stones).

Hint: To identify the unknown weight, you do not need to make every weight from 1 through 40 (e.g., making only the even weights also helps you identify the unknown weight should it be odd.)

Additional hint: When using the balance, you can place some stones on the same side as the object of unknown weight.

Problem 6. [12 points] A *triangle* is a set of three people such that either every pair has shaken hands or no pair has shaken hands.

(a) [8 pts] Prove that among every six people there is a triangle.

(Suggestion: Initially, break the problem into two cases: for any person X in a given group of six people,

1. there exist at least three other people who shook hands with X ;
2. there exist at least three other people who didn't shake hands with X .

Why must exactly one of these conditions hold?)

(b) [4 pts] Also, is there always a triangle among every five people?