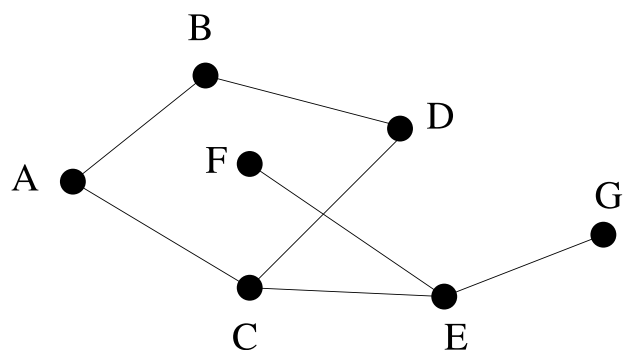
Notes for Recitation 6

1 Graph Basics

Let $G = (V, E)$ be a graph. Here is a picture of a graph.



Recall that the elements of V are called vertices, and those of E are called edges. In this example the vertices are $\{A, B, C, D, E, F, G\}$ and the edges are

$$\{A—B, B—D, C—D, A—C, E—F, C—E, E—G\}.$$

Deleting some vertices or edges from a graph leaves a *subgraph*. Formally, a subgraph of $G = (V, E)$ is a graph $G' = (V', E')$ where V' is a nonempty subset of V and E' is a subset of E. Since a subgraph is itself a graph, the endpoints of every edge in E' must be vertices in V'. For example, $V' = \{A, B, C, D\}$ and $E' = \{A—B, B—D, C—D, A—C\}$ forms a subgraph of G.

In the special case where we only remove edges incident to removed nodes, we say that G' is the *subgraph induced on* V' if $E' = \{(x—y | x, y \in V' \text{ and } x—y \in E\}$. In other words, we keep all edges unless they are incident to a node not in V'. For instance, for a new set of vertices $V' = \{A, B, C, D\}$, the induced subgraph G' has the set of edges $E' = \{A—B, B—D, C—D, A—C\}$.

2 Problem 1

An undirected graph G has ***width*** w if the vertices can be arranged in a sequence

$$v_1,\ v_2,\ v_3,\ \ldots,\ v_n$$

such that each vertex v_i is joined by an edge to at most w preceding vertices. (Vertex v_j *precedes* v_i if $j < i$.) Use induction to prove that every graph with width at most w is $(w+1)$-colorable.

(Recall that a graph is *k-colorable* iff every vertex can be assigned one of k colors so that adjacent vertices get different colors.)

Solution. We use induction on n, the number of vertices. Let $P(n)$ be the proposition that every graph with width w is $(w+1)$ colorable.

Base case: Every graph with $n = 1$ vertex has width 0 and is $0+1 = 1$ colorable. Therefore, $P(1)$ is true.

Inductive step: Now we assume $P(n)$ in order to prove $P(n+1)$. Let G be an $(n+1)$-vertex graph with width w. This means that the vertices can be arranged in a sequence

$$v_1, v_2, v_3, \ldots, v_n, v_{n+1}$$

such that each vertex v_i is connected to at most w preceding vertices. Removing vertex v_{n+1} and all incident edges gives a graph G' with n vertices and width at most w. (If original sequence is retained, then removing v_{n+1} does not increase the number of edges from a vertex v_i to a preceding vertex.) Thus, G' is $(w+1)$-colorable by the assumption $P(n)$. Now replace vertex v_{n+1} and its incident edges. Since v_{n+1} is joined by an edge to at most w preceding vertices, we can color v_{n+1} differently from all of these. Therefore, $P(n+1)$ is true.

The theorem follows by the principle of induction. ■

3 Problem 2

A **planar graph** is a graph that can be drawn without any edges crossing.

1. First, show that any subgraph of a planar graph is planar.

   Solution. The edge set of any subgraph will be a subset of the set of edges in the original planar graph. This means that since edges in the original graphs do not cross, edges in a subset of the original set of edges also do not cross. ■

2. Also, any planar graph has a node of degree at most 5. Now, prove by induction that any graph can be colored in at most 6 colors.

   Solution. We prove by induction. First, let n be the number of nodes in the graph. Then define

   $$P(n) = \text{Any planar graph with } n \text{ nodes is 6-colorable.}$$

   Base case, $P(1)$: Every graph with $n = 1$ vertex is 6-colorable. Clearly true since it's actually 1-colorable.

Inductive step, $P(n) \to P(n+1)$: Take a graph G with $n+1$ nodes. Then take a node v with degree at most 5 (which we know exists because we know any planar graph has a node of degree ≤ 5), and remove it. We know that the induced subgraph G' formed in this way has n nodes, so by our inductive hypothesis, G' is 6-colorable. But v is adjacent to at most 5 other nodes, which can have at most 5 different colors between them. We then choose v to have an unused color (from the 6 colors), and as we have constructed a 6-coloring for G, we are done with the inductive step.

Because we have shown the base case and the inductive step, we have proved

$$\forall n \in \mathbb{Z}_+ : P(n)$$

(Note: $\mathbb{Z}_+$ refers to the set of positive integers.) ■