Notes for Recitation 18

Problem 1

Write a formula for the generating function whose successive coefficients are given by the sequence:

1. $0, 0, 1, 1, 1, \ldots$

   Solution.
$$\frac{x^2}{1-x}$$
■

2. $1, 1, 0, 0, 0, \ldots$

   Solution.
$$1+x$$
■

3. $1, 0, 1, 0, 1, 0, 1, \ldots$

   Solution.
$$\frac{1}{1-x^2}$$
■

4. $1, 4, 6, 4, 1, 0, 0, 0, \ldots$

   Solution.
$$(1+x)^4$$
■

5. $1, 2, 3, 4, 5, \ldots$

   Solution. $1/(1-x)^2$, the derivative of $1/(1-x)$. ■

6. $1, 4, 9, 16, 25, \ldots$

   Solution. $(1+x)/(1-x)^3$, the derivative of $x/(1-x)^2$. ■

7. $1, 1, 1/2, 1/6, 1/24, 1/120, \ldots$

   Solution.
$$e^x$$
■

Problem 2

T-Pain is planning an epic boat trip and he needs to decide what to bring with him.

- He must bring some burgers, but they only come in packs of 6.
- He and his two friends can't decide whether they want to dress formally or casually. He'll either bring 0 pairs of flip flops or 3 pairs.
- He doesn't have very much room in his suitcase for towels, so he can bring at most 2.
- In order for the boat trip to be truly epic, he has to bring at least 1 nautical-themed pashmina afghan.

1. Let $B(x)$ be the generating function for the number of ways to bring n burgers, $F(x)$ for the number of ways to bring n pairs of flip flops, $T(x)$ for towels, and $A(x)$ for Afghans. Write simple formulas for each of these.

   Solution.

$$\begin{aligned}
B(x) &= \frac{x^6}{1-x^6}, \\
F(x) &= (1+x^3), \\
T(x) &= 1+x+x^2 = \frac{1-x^3}{1-x} \\
A(x) &= \frac{x}{1-x}.
\end{aligned}$$

■

2. Let g_n be the the number of different ways for T-Pain to bring n items (burgers, pairs of flip flops, towels, and/or afghans) on his boat trip. Let $G(x)$ be the generating function $\sum_{n=0}^{\infty} g_n x^n$. Verify that

$$G(x) = \frac{x^7}{(1-x)^2}.$$

   Solution. By the Convolution Rule,

$$\begin{aligned}
G(x) &= B(x)F(x)T(x)A(x) \\
&= \frac{x^6}{1-x^6}(1+x^3)\frac{1-x^3}{1-x}\frac{x}{1-x} \\
&= \frac{x^6(1+x^3)(1-x^3)x}{(1-x^6)(1-x)^2} \\
&= \frac{x^7}{(1-x)^2}
\end{aligned}$$

■

3. Find a simple formula for g_n.

Solution.

$$g_n = \begin{cases} 0 & \text{for } n < 7 \\ n - 6 & \text{for } n \geq 7. \end{cases} \tag{1}$$

Let

$$H(x) ::= \frac{1}{(1-x)^2},$$

so $G(x) = x^7 H(x)$. We know that the coefficient, h_n, of x^n in the series for $H(x)$ is, by the Convolution Rule, the number of ways to select n items of two different kinds, namely, $h_n = \binom{n+1}{1} = n + 1$. So we conclude that for $n \geq 7$, the nth coefficient in the series for $G(x)$ is h_{n-7} namely (1). ■

Problem 3

Let a_n be the number of ways to make change for \$$n$ using \$2 and \$3 coins. For example, $a_5 = 1$ because the only way to make change for \$5 is with one \$2 coin and one \$3 coin, but $a_6 = 2$ because there are two ways to make change for \$6, namely using three \$2 coins or using two \$3 coins.

Express the generating function for the sequence of a_n's as a rational function (quotient of products of polynomials). You need not simplify your formula or solve for a_n.

Solution.

$$1/(1-x^2)(1-x^3)$$

Using \$2 coins, there is only one way to make change for \$$n$ when n is even, and no way to do it when n is odd. So the generating function for the number of ways to make change for \$$n$ using only \$2 coins is

$$1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

Similarly, the generating function for the number of ways to make change for \$$n$ using only \$3 coins is

$$\frac{1}{1-x^3}$$

The generating function for the number of ways to make change using both kinds of coins is the product of the generating functions for each kind of coin. ■

Problem 4

You would like to buy a bouquet of flowers. You find an online service that will make bouquets of **lilies**, **roses** and **tulips**, subject to the following constraints:

- there must be at most 1 lily,
- there must be an odd number of tulips,
- there must be at least two roses.

Example: A bouquet of no lilies, 3 tulips, and 5 roses satisfies the constraints.

Express $B(x)$, the generating function for the number of ways to select a bouquet of n flowers, as a quotient of polynomials (or products of polynomials). You do not need to simplify this expression.

Solution. Generating function for the number of ways to choose lilies:

$$L(x) = 1 + x$$

Generating function for the number of ways to choose tulips:

$$T(x) = x + x^3 + x^5 + \cdots = \frac{x}{1 - x^2}$$

Generating function for the number of ways to choose roses:

$$R(x) = x^2 + x^3 + x^4 + \cdots = \frac{x^2}{1 - x}$$

By the Convolution Property, the generating function $B(x)$ is the product of these functions, namely,

$$\begin{aligned} B(x) &= L(x)R(x)T(x) \\ &= (1 + x)\frac{x}{1 - x^2}\frac{x^2}{1 - x} \\ &= \frac{x^3(1 + x)}{(1 + x)(1 - x)^2} \\ &= \frac{x^3}{(1 - x)^2} \end{aligned}$$

■