

## Notes for Recitation 14

### Guessing a Particular Solution

A general linear recurrence has the form:

$$f(n) = b_1f(n-1) + b_2f(n-2) + \dots + b_df(n-d) + g(n)$$

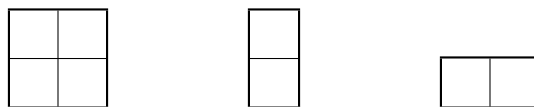
One step in solving this recurrence is finding a *particular solution*. This is a function  $f(n)$  that satisfies the recurrence equation, but may not be consistent with the boundary conditions. Here's a recipe to help you guess a particular solution:

- If  $g(n)$  is a constant, guess that  $f(n)$  is some constant  $c$ . Plug this into the recurrence equation and see if any constant actually works. If not, try  $f(n) = bn + c$ , then  $f(n) = an^2 + bn + c$ , etc.
- More generally, if  $g(n)$  is a polynomial, try a polynomial of the same degree. If that fails, try a polynomial of degree one higher, then two higher, etc. For example, if  $g(n) = n$ , then try  $f(n) = bn + c$  and then  $f(n) = an^2 + bn + c$ .
- If  $g(n)$  is an exponential, such as  $3^n$ , then first guess that  $f(n) = c3^n$ . Failing that, try  $f(n) = bn3^n + c3^n$  and then  $an^23^n + bn3^n + c3^n$ , etc.

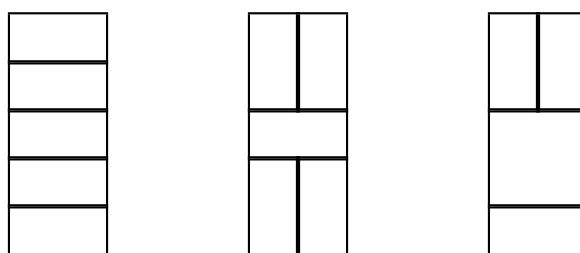
In practice, your first or second guess will almost always work.

## Mini-Tetris

A *winning configuration* in the game of Mini-Tetris is a complete tiling of a  $2 \times n$  board using only the three shapes shown below:



For example, here are several possible winning configurations on a  $2 \times 5$  board:

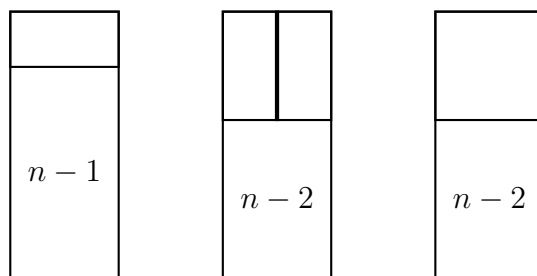


1. Let  $T_n$  denote the number of different winning configurations on a  $2 \times n$  board. Determine the values of  $T_1$ ,  $T_2$ , and  $T_3$ .

**Solution.**  $T_1 = 1$ ,  $T_2 = 3$ , and  $T_3 = 5$ . ■

2. Find a recurrence equation that expresses  $T_n$  in terms of  $T_{n-1}$  and  $T_{n-2}$ .

**Solution.** Every winning configuration on a  $2 \times n$  board is of one three types, distinguished by the arrangement of pieces at the top of the board.



There are  $T_{n-1}$  winning configurations of the first type, and there are  $T_{n-2}$  winning configurations of each of the second and third types. Overall, the number of winning configurations on a  $2 \times n$  board is:

$$T_n = T_{n-1} + 2T_{n-2}$$

■

3. Find a closed-form expression for the number of winning configurations on a  $2 \times n$  Mini-Tetris board.

**Solution.** The characteristic polynomial is  $r^2 - r - 2 = (r - 2)(r + 1)$ , so the solution is of the form  $A2^n + B(-1)^n$ . Setting  $n = 1$ , we have  $1 = T_1 = 2A - B$ . Setting  $n = 2$ , we have  $3 = T_2 = A2^2 + B(-1)^2 = 4A + B$ . Solving these two equations, we conclude  $A = 2/3$  and  $B = 1/3$ . That is,

$$T_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n = \frac{2^{n+1} + (-1)^n}{3}.$$

■

## Linear Recurrences

Find closed-form solutions to the following linear recurrences.

1.  $T_0 = 0$   
 $T_1 = 1$   
 $T_n = T_{n-1} + T_{n-2} + 1$

**Solution.** Following the guide to solving linear recurrences:

- (a) First, we find the general solution to the homogeneous recurrence. The characteristic equation is  $r^2 - r - 1 = 0$ . The roots of this equation are:

$$\begin{aligned} r_1 &= \frac{1 + \sqrt{5}}{2} \\ r_2 &= \frac{1 - \sqrt{5}}{2} \end{aligned}$$

- (b) Using the roots, we write down the homogeneous solution in the form

$$T_n = A \left( \frac{1 + \sqrt{5}}{2} \right)^n + B \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$

- (c) Next, we need a particular solution to the inhomogeneous recurrence. Since the inhomogeneous term is constant, we guess a constant solution,  $c$ . So replacing the  $T(n)$  terms in  $T_n = T_{n-1} + T_{n-2} + 1$  by  $c$ , we require

$$c = c + c + 1,$$

namely,  $c = -1$ . That is,  $T_n = -1$  is a particular solution to the equation.

- (d) Putting it together, the complete solution to the recurrence is the homogenous solution plus the particular solution:

$$T_n = A \left( \frac{1 + \sqrt{5}}{2} \right)^n + B \left( \frac{1 - \sqrt{5}}{2} \right)^n - 1$$

- (e) All that remains is to find the constants  $A$  and  $B$ . Substituting the initial conditions gives a system of linear equations.

$$\begin{aligned} 0 &= A + B - 1 \\ 1 &= A \left( \frac{1 + \sqrt{5}}{2} \right) + B \left( \frac{1 - \sqrt{5}}{2} \right) - 1 \end{aligned}$$

The solution to this linear system is:

$$\begin{aligned} A &= \frac{5 + 3\sqrt{5}}{10} \\ B &= \frac{5 - 3\sqrt{5}}{10} \end{aligned}$$

Therefore, the complete solution to the recurrence is

$$T_n = \left( \frac{5 + 3\sqrt{5}}{10} \right) \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{5 - 3\sqrt{5}}{10} \right) \cdot \left( \frac{1 - \sqrt{5}}{2} \right)^n - 1.$$

■

$$\begin{aligned} 2. \quad S_0 &= 0 \\ S_1 &= 1 \\ S_n &= 6S_{n-1} - 9S_{n-2} \end{aligned}$$

**Solution.** The characteristic polynomial is  $r^2 - 6r + 9 = (r - 3)^2$ , so the solution is of the form  $A3^n + Bn3^n$  for some constants  $A$  and  $B$ . Setting  $n = 0$ , we have  $0 = S_0 = A3^0 + B \cdot 0 \cdot 3^0 = A$ . Setting  $n = 1$ , we have  $1 = S_1 = A3^1 + B \cdot 1 \cdot 3^1 = 3B$ , so  $B = 1/3$ . That is,

$$S_n = 0 \cdot 3^n + \frac{1}{3} \cdot n3^n = n3^{n-1}.$$

■

## Short Guide to Solving Linear Recurrences

A *linear recurrence* is an equation

$$\underbrace{f(n) = a_1 f(n-1) + a_2 f(n-2) + \dots + a_d f(n-d)}_{\text{homogeneous part}} \quad \underbrace{+ g(n)}_{\text{inhomogeneous part}}$$

together with boundary conditions such as  $f(0) = b_0$ ,  $f(1) = b_1$ , etc.

1. Find the roots of the *characteristic equation*:

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_k$$

2. Write down the *homogeneous solution*. Each root generates one term and the homogeneous solution is the sum of these terms. A nonrepeated root  $r$  generates the term  $c_r r^n$ , where  $c_r$  is a constant to be determined later. A root  $r$  with multiplicity  $k$  generates the terms:

$$c_{r_1} r^n, \quad c_{r_2} n r^n, \quad c_{r_3} n^2 r^n, \quad \dots, \quad c_{r_k} n^{k-1} r^n$$

where  $c_{r_1}, \dots, c_{r_k}$  are constants to be determined later.

3. Find a *particular solution*. This is a solution to the full recurrence that need not be consistent with the boundary conditions. Use guess and verify. If  $g(n)$  is a polynomial, try a polynomial of the same degree, then a polynomial of degree one higher, then two higher, etc. For example, if  $g(n) = n$ , then try  $f(n) = bn + c$  and then  $f(n) = an^2 + bn + c$ . If  $g(n)$  is an exponential, such as  $3^n$ , then first guess that  $f(n) = c3^n$ . Failing that, try  $f(n) = bn3^n + c3^n$  and then  $an^2 3^n + bn3^n + c3^n$ , etc.
4. Form the *general solution*, which is the sum of the homogeneous solution and the particular solution. Here is a typical general solution:

$$f(n) = \underbrace{c2^n + d(-1)^n}_{\text{homogeneous solution}} + \underbrace{3n + 1}_{\text{particular solution}}$$

5. Substitute the boundary conditions into the general solution. Each boundary condition gives a linear equation in the unknown constants. For example, substituting  $f(1) = 2$  into the general solution above gives:

$$\begin{aligned} 2 &= c \cdot 2^1 + d \cdot (-1)^1 + 3 \cdot 1 + 1 \\ \Rightarrow -2 &= 2c - d \end{aligned}$$

Determine the values of these constants by solving the resulting system of linear equations.