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Problem Set 2

Due: Monday, September 19

Reading Assignment: Sections 2.5-2.7, 3.0-3.4, & 3.5 (optional)

Problem 1. [15 points]

(a) [5 pts] Consider three integers x, y, z written down on a piece of paper. Any of the integers may be replaced by the sum of the other two plus 1. This operation is repeated a number of times until the final result is $11031, 19871, 16343$. Is it possible that the initial integers were $2, 4, 6$? Prove it.

(b) [10 pts] Is it possible to cover a 6 x 6 board with rectangles of size 1 x 4?

Problem 2. [10 points] Show that in the past 1000 years, you have had an ancestor P such that there was a person Q who was an ancestor of both the father and mother of P. You may use the following assumptions:

1. Assume that it takes 25 years for one generation to produce an offspring.
2. There have not been more than 10^{10} people on this planet over any period of 25 years.

Problem 3. [12 points] The following problem is fairly tough until you hear a certain one-word clue. The solution is elegant but is slightly tricky, so don't hesitate to ask for hints!

During 6.042, the students are sitting in an $n \times n$ grid. A sudden outbreak of beaver flu (a rare variant of bird flu that lasts forever; symptoms include year ning for problem sets and craving for ice cream study sessions) causes some students to get infected. Here is an example where $n = 6$ and infected students are marked $\times$.

$\times$				$\times$	
	$\times$				
		$\times$	$\times$		
		$\times$			
			$\times$		$\times$

Now the infection begins to spread every minute (in discrete time-steps). Two students are considered *adjacent* if they share an edge (i.e., front, back, left or right, but NOT diagonal); thus, each student is adjacent to 2, 3 or 4 others. A student is infected in the next time step if either

- the student was previously infected (since beaver flu lasts forever), or
- the student is adjacent to *at least two* already-infected students.

In the example, the infection spreads as shown below.

×				×	
	×				
		×	×		
		×			
			×		×

$\Rightarrow$

×	×			×	
×	×	×			
	×	×	×		
		×			
		×	×		
		×	×	×	×

$\Rightarrow$

×	×	×		×	
×	×	×	×		
×	×	×	×		
	×	×	×		
		×	×	×	
		×	×	×	×

In this example, over the next few time-steps, all the students in class will become infected.

Theorem. *If fewer than n students in class are initially infected, the whole class will never be completely infected.*

Prove this theorem.

Hint: To understand how a system such as the above "evolves" over time, it is usually a good strategy to (1) identify an appropriate property of the system at the initial stage, and (2) prove, by induction on the number of time-steps, that the property is preserved at every time-step. So look for a property (of the set of infected students) that remains invariant as time proceeds.

If you are stuck, ask your recitation instructor for the one-word clue and even more hints!

Problem 4. [8 points] Can raising an irrational number a to an irrational power b result in a rational number? Provide a proof that it can by considering $\sqrt{5}^{\sqrt{2}}$ and using proof by cases.

Problem 5. [10 points] For any nonempty set C, let $f(C)$ be the square of the product of the elements in C. For example, if $C = \{1, 4, 5\}$, then $f(C) = (1 \cdot 4 \cdot 5)^2 = 400$. Show that the sum of $f(S)$ for all nonempty subsets of $\{1, 2, \ldots, n\}$ containing no consecutive elements is $(n + 1)! - 1$. For example, if we consider $\{1, 2, 3\}$, then we have $f(\{1, 3\}) + f(\{1\}) + f(\{2\}) + f(\{3\}) = 23 = 4! - 1$.

Problem 6. [10 points] A group of $n \geq 1$ people can be divided into teams, each containing either 5 or 6 people. What is the largest n for which the group cannot be divided into such teams? Use induction to prove that your answer is correct.

Problem 7. [10 points] The Fibonacci sequence is a sequence of integers following the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ with $F_1 = 1, F_2 = 1$. Show that any positive integer S can be written as a sum of unique Fibonacci numbers as follows:

$$S = F_{a_1} + F_{a_2} + \ldots F_{a_m}$$

for $2 \leq a_1 < a_2 < \ldots < a_m$.

Problem 8. [25 points] *The Well Ordering Principle (WOP)* states that "every *nonempty* set of *nonnegative* integers has a *smallest* element." (See Section 3.1 of the text *Mathematics for Computer Science*.) It captures a special property about nonnegative integers and can be extremely useful in proofs.

(a) [5 pts] For practice, prove using the Well Ordering Principle as in class that for all nonnegative integers, n:

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

We refer to integers of the form $\frac{n(n+1)}{2}$ as **triangular numbers** (Just imagine making a triangle of dots where the first row has 1 dot, the second row has 2 dots, ..., and the the n^{th} row has n dots).

(b) [20 pts] A function $f(x)$ is defined as follows:

$$\begin{aligned} f(1) &= 6042 \\ f(1) + f(2) + \ldots + f(n) &= n^2 f(n) \end{aligned}$$

Find $f(6042)$.

This is a tougher problem so here are a few hints:

1. Examine the first few terms of the sequence, and try to guess $f(n)$ in general. Use induction to verify your guess.
2. You may use the fact that $\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.