# Problem Set 2

**Due:** Monday, September 15

**Reading Assignment:** Sections 2.5-2.7, 3.0-3.4, & 3.5 (optional)

**Problem 1. [6 points]** Can raising an irrational number $a$ to an irrational power $b$ result in a rational number? Provide a proof that it can by considering $\sqrt{3}^{\sqrt{2}}$ and using proof by cases.

**Problem 2. [18 points]** The following problem is fairly tough until you hear a certain one-word clue. The solution is elegant but is slightly tricky, so don't hesitate to ask for hints!

During 6.042, the students are sitting in an $n \times n$ grid. A sudden outbreak of beaver flu (a rare variant of bird flu that lasts forever; symptoms include yearning for problem sets and craving for ice cream study sessions) causes some students to get infected. Here is an example where $n = 6$ and infected students are marked $\times$.



Now the infection begins to spread every minute (in discrete time-steps). Two students are considered *adjacent* if they share an edge (i.e., front, back, left or right, but NOT diagonal); thus, each student is adjacent to 2, 3 or 4 others. A student is infected in the next time step if either

- the student was previously infected (since beaver flu lasts forever), or

- the student is adjacent to *at least two* already-infected students.

In the example, the infection spreads as shown below.

In this example, over the next few time-steps, all the students in class will become infected.

**Theorem.** *If fewer than n students in class are initially infected, the whole class will never be completely infected.*

Prove this theorem.

*Hint:* To understand how a system such as the above "evolves" over time, it is usually a good strategy to (1) identify an appropriate property of the system at the initial stage, and (2) prove, by induction on the number of time-steps, that the property is preserved at every time-step. So look for a property (of the set of infected students) that remains invariant as time proceeds.

If you are stuck, ask your recitation instructor for the one-word clue and even more hints!

## Problem 3. [20 points]

In the 15-puzzle, there are 15 lettered tiles and a blank square arranged in a $4 \times 4$ grid. Any lettered tile adjacent to the blank square can be slid into the blank. For example, a sequence of two moves is illustrated below:

| A | B | C | D |
|---|---|---|---|
| E | F | G | H |
| I | J | K | L |
| M | O | N |   |

$\rightarrow$

| A | B | C | D |
|---|---|---|---|
| E | F | G | H |
| I | J | K | L |
| M | O |   | N |

$\rightarrow$

| A | B | C | D |
|---|---|---|---|
| E | F | G | H |
| I | J |   | L |
| M | O | **K** | N |

In the leftmost configuration shown above, the O and N tiles are out of order. Using only legal moves, is it possible to swap the N and the O, while leaving all the other tiles in their original position and the blank in the bottom right corner? In this problem, you will prove that the answer is "no".

**Theorem.** *No sequence of moves transforms the board below on the left into the board below on the right.*

| A | B | C | D |
|---|---|---|---|
| E | F | G | H |
| I | J | K | L |
| M | **O** | **N** |   |

| A | B | C | D |
|---|---|---|---|
| E | F | G | H |
| I | J | K | L |
| M | **N** | **O** |   |

**(a)** [2 pts] We define the "order" of the tiles in a board to be the sequence of tiles on the board reading from the top row to the bottom row and from left to right within a row. For example, in the right board depicted in the above theorem, the order of the tiles is $A$, $B$, $C$, $D$, $E$, etc.

Can a row move change the order of the tiles? Prove your answer.

**(b)** [2 pts] How many pairs of tiles will have their relative order changed by a column move? More formally, for how many pairs of letters $L_1$ and $L_2$ will $L_1$ appear earlier in the order of the tiles than $L_2$ before the column move and later in the order after the column move? Prove your answer correct.

**(c)** [2 pts] We define an *inversion* to be a pair of letters $L_1$ and $L_2$ for which $L_1$ precedes $L_2$ in the alphabet, but $L_1$ appears after $L_2$ in the order of the tiles. For example, consider the following configuration:

| A | B | C | E |
|---|---|---|---|
| D | H | G | F |
| I | J | K | L |
| M | N | O |   |

There are exactly four inversions in the above configuration: $E$ and $D$, $H$ and $G$, $H$ and $F$, and $G$ and $F$.

What effect does a row move have on the parity of the number of inversions? Prove your answer.

**(d)** [4 pts] What effect does a column move have on the parity of the number of inversions? Prove your answer.

**(e)** [8 pts] The previous problem part implies that we must make an *odd* number of column moves in order to exchange just one pair of tiles (N and O, say). But this is problematic, because each column move also knocks the blank square up or down one row. So after an *odd* number of column moves, the blank can not possibly be back in the last row, where it belongs! Now we can bundle up all these observations and state an *invariant*, a property of the puzzle that never changes, no matter how you slide the tiles around.

**Lemma.** *In every configuration reachable from the position shown below, the parity of the number of inversions is different from the parity of the row containing the blank square.*

*row 1*
*row 2*
*row 3*
*row 4*

| A | B | C | D |
|---|---|---|---|
| E | F | G | H |
| I | J | K | L |
| M | O | N |   |

Prove this lemma.

**(f)** [2 pts] Prove the theorem that we originally set out to prove.

**Problem 4. [14 points]** *The Well Ordering Principle (WOP)* states that "every *nonempty* set of *nonnegative* integers has a *smallest* element." (See Section 3.1 of the text *Mathematics for Computer Science*.) It captures a special property about nonnegative integers and can be extremely useful in proofs.

**(a)** [4 pts] Show that WOP can be proved when the principle of mathematical induction is taken as an axiom. (*Hint:* Begin by pretending that the well ordering principle were false and carry out induction on a nonempty set of nonnegative integers that has no least element. The induction should lead to the conclusion that the set is empty.)

**(b)** [5 pts] Prove using the Well Ordering Principle that for all nonnegative integers, $n$:

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{1}$$

**(c)** [5 pts] Prove equation (1) by induction.

**Problem 5. [8 points]** *Euler's Conjecture* in 1769 was that there are no positive integer solutions to the equation

$$a^4 + b^4 + c^4 = d^4. \tag{2}$$

Integer values for $a, b, c, d$ that do satisfy this equation were first discovered in 1986. So Euler guessed wrong, but it took more than two hundred years to prove it.

Now let's consider Moitra's equation, similar to Euler's but with some coefficients:

$$27a^4 + 9b^4 + 3c^4 = d^4 \tag{3}$$

Prove that Moitra's equation (3) really does not have any positive integer solutions.

*Hint:* Consider the minimum value of $a$ among all possible solutions to (3).

**Problem 6. [18 points]** Nim is a game played between two players with three piles of stones. Players alternate removing stones. A player picks a pile and removes any positive number of stones. The goal is to be the last player to take a stone.

**(a)** [5 pts] The winning strategy in Nim requires computing a Nim sum. A Nim sum is defined to be the binary xor of the number of stones in each pile. Prove that if the Nim sum is zero, then any move will result in Nim sum that is not zero.

**(b)** [5 pts] Prove that if the Nim sum is not zero that it is always possible to make the Nim sum zero with one move.

**(c)** [8 pts] Using parts a and b, prove that if the game begins with a non-zero Nim sum, then the first player has a winning strategy.

**Problem 7. [8 points]** A group of $n \geq 1$ people can be divided into teams, each containing either 4 or 7 people. What are all the possible values of $n$? Use induction to prove that your answer is correct.

**Problem 8. [8 points]** Three pirates are considering an attack on the *Piñata*, a Spanish galleon laden with $n \geq 20$ pieces of gold. Redbeard insists that his share of the treasure be a multiple of 5 gold pieces, Bluebeard insists that his share be a multiple of 7 gold pieces, and Blackbeard demands a multiple of 9 gold pieces.

Furthermore, no pirate can tolerate treasure going to waste; if that is unavoidable, the pirates will surely have a fatal quarrel. For example, if there were 12 gold pieces aboard, then Redbeard could take 5, Bluebeard 7, and Blackbeard 0. However, if there were 13 gold pieces aboard, then the pirates would kill each other.

Can the pirates safely attack the *Piñata*? Use strong induction to prove that your answer is correct.