

## Notes for Recitation 15

### 1 The Tao of BOOKKEEPER

In this problem, we seek enlightenment through contemplation of the word *BOOKKEEPER*.

1. In how many ways can you arrange the letters in the word *POKE*?

**Solution.** There are  $4!$  arrangements corresponding to the  $4!$  permutations of the set  $\{P, O, K, E\}$ . ■

2. In how many ways can you arrange the letters in the word  $BO_1O_2K$ ? Observe that we have subscripted the O's to make them distinct symbols.

**Solution.** There are  $4!$  arrangements corresponding to the  $4!$  permutations of the set  $\{B, O_1, O_2, K\}$ . ■

3. Suppose we map arrangements of the letters in  $BO_1O_2K$  to arrangements of the letters in *BOOK* by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

$O_2BO_1K$	<i>BOOK</i> <i>OBOK</i> <i>KOBO</i> <i>...</i>
$KO_2BO_1$	
$O_1BO_2K$	
$KO_1BO_2$	
$BO_1O_2K$	
$BO_2O_1K$	
...	

4. What kind of mapping is this, young grasshopper?

**Solution.** 2-to-1 ■

5. In light of the Division Rule, how many arrangements are there of *BOOK*?

**Solution.**  $4!/2$  ■

6. Very good, young master! How many arrangements are there of the letters in  $KE_1E_2PE_3R$ ?

**Solution.**  $6!$  ■

7. Suppose we map each arrangement of  $KE_1E_2PE_3R$  to an arrangement of  $KEEPER$  by erasing subscripts. List all the different arrangements of  $KE_1E_2PE_3R$  that are mapped to  $REPEEK$  in this way.

**Solution.**  $RE_1PE_2E_3K, RE_1PE_3E_2K, RE_2PE_1E_3K, RE_2PE_3E_1K, RE_3PE_1E_2K, RE_3PE_2E_1K$  ■

8. What kind of mapping is this?

**Solution.**  $3!$ -to-1 ■

9. So how many arrangements are there of the letters in  $KEEPER$ ?

**Solution.**  $6!/3!$  ■

10. *Now you are ready to face the BOOKKEEPER!*

How many arrangements of  $BO_1O_2K_1K_2E_1E_2PE_3R$  are there?

**Solution.**  $10!$  ■

11. How many arrangements of  $BOOK_1K_2E_1E_2PE_3R$  are there?

**Solution.**  $10!/2!$  ■

12. How many arrangements of  $BOOKKE_1E_2PE_3R$  are there?

**Solution.**  $10!/(2! \cdot 2!)$  ■

13. How many arrangements of  $BOOKKEEPER$  are there?

**Solution.**  $10!/(2! \cdot 2! \cdot 3!)$  ■

14. How many arrangements of  $VOODOODOLL$  are there?

**Solution.**  $10!/(2! \cdot 2! \cdot 5!)$  ■

15. **(IMPORTANT)** How many  $n$ -bit sequences contain  $k$  zeros and  $(n - k)$  ones?

**Solution.**  $n!/(k! \cdot (n - k)!)$  ■

This quantity is denoted  $\binom{n}{k}$  and read “ $n$  choose  $k$ ”. You will see it almost every day in 6.042 from now until the end of the term.

*Remember well what you have learned: subscripts on, subscripts off.*

*This is the Tao of Bookkeeper.*

## 2 Pigeonhole Principle

Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

1. In a room of 500 people, there exist two who share a birthday.

**Solution.** The pigeons are the 500 people. The pigeonholes are 366 possible birthdays. Map each person to his or her own birthday. Since there 500 people and 366 birthdays, some two people must have the same birthday by the Pigeonhole Principle. ■

2. Suppose that each of the 115 students in 6.042 sums the nine digits of his or her ID number. Must two people arrive at the same sum?

**Solution.** Yes. The students are the pigeons, the possible sums are the pigeonholes, and we map each student to the sum of the digits in his or her MIT ID number. Every sum is in the range from 0 to  $9 \cdot 9 = 81$ , which means that there are 82 pigeonholes. Since there are more pigeons than pigeonholes, there must be two pigeons in the same pigeonhole; in other words, there must be two students with the same sum. ■

3. In every set of 100 integers, there exist two whose difference is a multiple of 37.

**Solution.** The pigeons are the 100 integers. The pigeonholes are the numbers 0 to 36. Map integer  $k$  to  $k \bmod 37$ . Since there are 100 pigeons and only 37 pigeonholes, two pigeons must go in the same pigeonhole. This means  $k_1 \bmod 37 = k_2 \bmod 37$ , which implies that  $k_1 - k_2$  is a multiple of 37. ■

### 3 More Counting Problems

Solve the following counting problems. Define an appropriate mapping (bijective or  $k$ -to-1) between a set whose size you know and the set in question.

1. **(IMPORTANT)** In how many ways can  $k$  elements be chosen from an  $n$ -element set  $\{x_1, x_2, \dots, x_n\}$ ?

**Solution.** There is a bijection from  $n$ -bit sequences with  $k$  ones and  $n - k$  zeros. The sequence  $(b_1, \dots, b_n)$  maps to the subset that contains  $x_i$  if and only if  $b_i = 1$ . Therefore, the number of such subsets is  $\binom{n}{k}$ . ■

2. How many different ways are there to select a dozen donuts if five varieties are available? (We discussed a bijection for this set in Recitation 15. Now use that bijection to give a count.)

**Solution.** There is a bijection from selections of a dozen donuts to 16-bit sequences with exactly 4 ones. In particular, suppose that the varieties are glazed, chocolate, lemon, sugar, and Boston creme. Then a selection of  $g$  glazed,  $c$  chocolate,  $l$  lemon,  $s$  sugar, and  $b$  Boston creme maps to the sequence:

$$(g \text{ 0's}) \text{ 1 } (c \text{ 0's}) \text{ 1 } (l \text{ 0's}) \text{ 1 } (s \text{ 0's}) \text{ 1 } (b \text{ 0's})$$

Therefore, the number of selections is equal to the number of 16-bit sequences with exactly 4 ones, which is:

$$\frac{16!}{4! 12!} = \binom{16}{4}$$

■

3. An independent living group is hosting eight pre-frosh, affectionately known as  $P_1, \dots, P_8$  by the permanent residents. Each pre-frosh is assigned a task: 2 must wash pots, 2 must clean the kitchen, 1 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. In how many ways can  $P_1, \dots, P_8$  be put to productive use?

**Solution.** There is a bijection from sequences containing two  $P$ 's, two  $K$ 's, a  $B$ , a  $C$ , and two  $D$ 's. In particular, the sequence  $(t_1, \dots, t_8)$  corresponds to assigning  $P_i$  to washing pots if  $t_i = P$ , to cleaning the kitchen if  $t_i = K$ , to cleaning the bathrooms if  $t_i = B$ , etc. Therefore, the number of possible assignments is:

$$\frac{8!}{2! 2! 1! 1! 2!}$$

■

4. Suppose that two identical 52-card decks of are mixed together. In how many ways can the cards in this double-size deck be arranged?

**Solution.** The number of sequences of the 104 cards containing 2 of each card is  $104!/(2!)^{52}$ . ■

## 4 Fun with Phonology: Hawaiian

The Hawaiian language is rich in vowels: it contains 8 consonants and 25 vowels<sup>1</sup>! In addition, every word in Hawaiian must end in a vowel and must not contain two consonants in a row. Let's assume that all combinations of vowels and consonants that satisfy these constraints are valid.

We'd like to know how many  $n$ -phoneme words there are in Hawaiian. (A *phoneme* is either a single vowel or a single consonant. Assume no phoneme can be both a vowel and a consonant.) For simplicity, let's assume  $n$  is even.

1. Before tackling the general problem, work out how many different words there are with exactly 4 phonemes. (Which distributions of vowels and consonants are possible?)

**Solution.** Since a consonant cannot go at the end of a word and no consonant can directly follow another (or equivalently, each consonant must be followed by a vowel), we have these possibilities for vowel/consonant distributions:

$$VVVV \quad VVCV \quad VCVV \quad CVVV \quad CVCV$$

Since these are mutually exclusive, we can find the number of words for each of the five types and sum them together. Using the product rule for each type, we find that the total number of  $n$ -phoneme words is

$$\begin{aligned} 25^4 + 25^2 \cdot 8 \cdot 25 + 25 \cdot 8 \cdot 25^2 + 8 \cdot 25^3 + 8 \cdot 25 \cdot 8 \cdot 25 &= 25^4 + 3 \cdot 25^3 \cdot 8 + 8^2 \cdot 25^2 \\ &= 805625 \end{aligned}$$

■

2. Now for the general case. Let  $A$  be the set of all  $n$ -phoneme words, and let  $A_k$  be the set of all  $n$ -phoneme words with exactly  $k$  consonants. Express  $|A|$  in terms of  $|A_k|$  for all possible  $k$ .

**Solution.**  $k$  can range from 0 to  $n/2$  since every consonant is followed by a vowel. Since the set of words with  $k$  consonants and the set of words with  $j$  consonants where  $j \neq k$  are disjoint, we can use the sum rule to compute  $|A|$ :

$$|A| = \sum_{k=0}^{n/2} |A_k|$$

■

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<sup>1</sup>Counting long vowels and diphthongs. For this problem, treat each of the 25 vowels as a unique single vowel.

3. Now let's find  $|A_k|$  for an arbitrary  $k$ . For simplicity's sake, assume Hawaiian has only one consonant and only one vowel. Find a bijection between  $A_k$  and a set of arbitrary sequences of 0 and 1 of length  $p$ . What is  $p$ ?

**Solution.** Since every consonant must be followed by a vowel, we can group each consonant and the vowel after it into a cluster. If there are  $k$  consonants, then there are  $k$  clusters. Since there are no further constraints on the distribution of these clusters, we can map each cluster to 0 and each remaining vowel to 1. Since the clustering removes  $k$  vowels from consideration, and the number of consonants is equal to the number of clusters, the resulting sequences of 0 and 1 have length  $n - k$ .

■

4. Using this bijection, compute  $|A_k|$ .

**Solution.** The number of sequences of  $k$  0's and  $n - 2k$  1's is

$$\binom{n - k}{k}$$

■

5. How would you change your expression for  $|A_k|$  to allow for 8 consonants and 25 vowels, not just one of each?

**Solution.** Each word in  $A_k$  is a sequence of  $V$ 's and  $C$ 's, where each  $V$  can represent any vowel and each  $C$  can represent any consonant. The total number of these sequences is  $\binom{n - k}{k}$ , as derived in the previous part.

Since each sequence has  $k$   $C$ 's and  $n - k$   $V$ 's, there are  $8^k \cdot 25^{n - k}$  distinct words that map to the same sequence of  $V$ 's and  $C$ 's. In other words, this mapping is  $(8^k \cdot 25^{n - k})$ -to-1, so

$$|A_k| = \binom{n - k}{k} \cdot 8^k \cdot 25^{n - k}$$

■

6. How many  $n$ -phoneme words are there in Hawaiian? (You don't have to find a closed form for your expression.)

**Solution.** Plugging this into the summation, we get

$$\begin{aligned} |A| &= \sum_{k=0}^{n/2} |A_k| \\ &= \sum_{k=0}^{n/2} \binom{n - k}{k} \cdot 8^k \cdot 25^{n - k} \end{aligned}$$

■

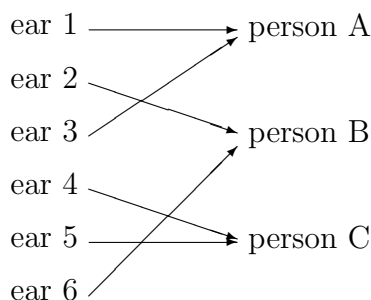
## Appendix: Basic Counting Notions

**Rule 1** (Bijection Rule). *If there exists a bijection  $f : A \rightarrow B$ , then  $|A| = |B|$ .*

**Rule 2** (Generalized Pigeonhole Principle). *If  $|X| > k \cdot |Y|$ , then for every function  $f : X \rightarrow Y$  there exist  $k + 1$  different elements of  $X$  that are mapped to the same element of  $Y$ .*

“If more than  $n$  pigeons are assigned to  $n$  holes, then there must exist two pigeons assigned to the same hole.”

A ***k-to-1 function*** maps exactly  $k$  elements of the domain to every element of the range. For example, the function mapping each ear to its owner is 2-to-1:



**Rule 3** (Division Rule). *If  $f : A \rightarrow B$  is  $k$ -to-1, then  $|A| = k \cdot |B|$ .*

**Rule 4** (Product Rule). *If  $P_1, P_2, \dots, P_n$  are sets, then:*

$$|P_1 \times P_2 \times \dots \times P_n| = |P_1| \cdot |P_2| \cdot \dots \cdot |P_n|$$

**Rule 5** (Generalized Product Rule). *Let  $S$  be a set of length- $k$  sequences. If there are:*

- $n_1$  possible first entries,
- $n_2$  possible second entries for each first entry,
- $n_3$  possible third entries for each combination of first and second entries, etc.

*then:*

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdot \dots \cdot n_k$$

**Rule 6** (Sum Rule). *If  $A_1, \dots, A_n$  are disjoint sets, then:*

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n |A_k|$$