Final

- The exam is **closed book**, but you may have four 8.5″ × 11″ sheet with notes (either printed or in your own handwriting) on both sides.
- Calculators and electronic devices (including cell phones) are not allowed.
- You may assume all of the results presented in class. This does **not** include results demonstrated in practice quiz material.
- Write your name on each page of the exam
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

NAME: ____________________

TA: ____________________

Problem	**Value**	**Score**	**Grader**
1	10		
2	10		
3	15		
4	20		
5	10		
6	20		
7	10		
8	15		
9	10		
10	10		
11	10		
Total	140		

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Problem 1. [10 points] Find a closed form for $\sum_{i=1}^{n}\sum_{j=i}^{m}\frac{i}{j}$. Leave your answer in terms of n, m.

Hint: Use H_k to represent the kth harmonic number

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Problem 2. [10 points] We define the sequence of numbers

$$a_n = \begin{cases} 1, & \text{for } n \leq 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}, & \text{for } n > 3. \end{cases}$$

Use *strong induction* to prove that remainder$(a_n, 3) = 1$ for all $n \geq 0$.

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Problem 3. [15 points]

(a) [10 pts] Find a solution to $x_n = 4x_{n-1} + n + 1$ with $x_0 = 2$.

(b) [5 pts] Give an asymptotic expression for the following recurrence, in Θ notation:

$$T(n) = 2T(\frac{n}{4}) + 3T(\frac{n}{6}) + n^3, \qquad T(1) = 0$$

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Problem 4. [20 points]

(a) [10 pts] Suppose that we are flipping a fair coin n times. What is the probability that there are exactly k heads, where the heads must be separated by at least 2 tails?

(b) [10 pts] Give a combinatorial proof for this identity:

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

Hint: Let S_i be the set of binary sequences with exactly n zeroes, $k+1$ ones, and a total of exactly i occurrences of zeroes appearing before the rightmost occurrence of a one.

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Problem 5. [10 points] Is there a bipartite graph with ordered degree sequence $3, 3, 3, 3, 3, 3, 4, 4, 4$?
Hint: The vertices of a bipartite graph can be divided into two subsets. Consider the sum of degrees of the vertices in each subset.

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Problem 6. [20 points] The hat-check staff has had a long day serving at a party, and at the end of the party they simply return the n checked hats uniformly at random, such that the probability that any particular person gets their own hat back is $1/n$.

Let X_i be the *indicator variable* for the ith person getting their own hat back. Let S_n be the total number of people who get their own hat back.

(a) [2 pts] What is the expected number of people who get their own hat back?

(b) [3 pts] Write a simple formula for $\mathrm{E}[X_i \cdot X_j]$ for $i \neq j$.

Hint: What is $\Pr\{X_j = 1 | X_i = 1\}$?

(c) [5 pts] Show that $\mathrm{E}[(S_n)^2] = 2$.

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(d) [5 pts] What is the variance of S_n?

(e) [5 pts] Use the *Chebyshev bound* to show that there is at most a 1% chance that more than 10 people get their own hat back.

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Problem 7. [10 points] Find the generating function for the number of ways to pay any amount of money using only pennies, nickels, dimes, and quarters.

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Problem 8. [15 points] Vlad has been sick for the past few days and is curious to know which disease he is suffering from. He knows that he has the flu with probability .3 and the common cold with probability .7.

If he has the flu, the conditional probability that the flu medicine will work and cure all symptoms is .4. Similarly, if he has the common cold, then he will be free of symptoms by taking the common cold medicine with conditional probability .15. Vlad can only take medicine for one disease each day.

You don't have to reduce your answer for the following problems.

(a) [5 pts] Which disease should Vlad take medicine for on the first day in order to maximize the probability of curing his disease?

(b) [5 pts] Vlad took medicine for the flu on the first day but is still sick. What is the probability that he has the flu?

(c) [5 pts] Vlad flips a fair coin to determine which medicine to take the first day and gets better on the first day. What is the probability that he took the flu medicine?

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Problem 9. [10 points] In a permutation of n elements, a pair (i, j) is called an inversion if and only if $i < j$ and i comes after j. For example, the permutation 31542 in the case of $n = 5$ has five inversions: $(3, 1), (3, 2), (5, 4), (5, 2)$ and $(4, 2)$. What is the expected number of inversions in a uniform random permutation of the number $1, 2, \ldots n$?

Hint: Use appropriate indicator variables and linearity of expectation.

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Problem 10. [10 points] Consider the following three random variables:

1. Let A be a binary random variable that is 1 if a coin C_1 comes up heads and 0 otherwise.
2. Let B be a binary random variable that is 1 if a coin C_2 comes up heads and 0 otherwise.
3. Let C be a binary random variable that is 1 if both A and B are different values and 0 otherwise.

Assume that C_1 and C_2 are independent coins.

(a) [5 pts] Are A, B, C mutually independent?

(b) [5 pts] Are A, B, C pairwise independent?

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Problem 11. [10 points] Consider tossing a fair coin C until one throws a heads. Tossing C results in heads with probability $\frac{1}{2}$. Let X be a random variable corresponding to the number of tosses needed until one throws a heads (so $X \geq 1$).

(a) [5 pts] Calculate $\mathbb{E}(X)$.

(b) [10 pts] Calculate $\mathbb{E}(X^3)$.

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