

## Notes for Recitation 7

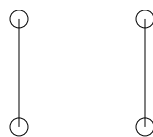
### 1 Build-up error

Recall a graph is **connected** iff there is a path between every pair of its vertices.

**False Claim.** *If every vertex in a graph has positive degree, then the graph is connected.*

- (a) Prove that this Claim is indeed false by providing a counterexample.

**Solution.** There are many counterexamples; here is one:



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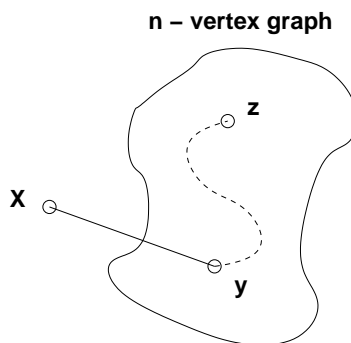
- (b) Since the Claim is false, there must be a logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

*Proof.* We prove the Claim above by induction. Let  $P(n)$  be the proposition that if every vertex in an  $n$ -vertex graph has positive degree, then the graph is connected.

**Base cases:** ( $n \leq 2$ ). In a graph with 1 vertex, that vertex cannot have positive degree, so  $P(1)$  holds vacuously.

$P(2)$  holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

**Inductive step:** We must show that  $P(n)$  implies  $P(n+1)$  for all  $n \geq 2$ . Consider an  $n$ -vertex graph in which every vertex has positive degree. By the assumption  $P(n)$ , this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex  $x$  to obtain an  $(n+1)$ -vertex graph:



All that remains is to check that there is a path from  $x$  to every other vertex  $z$ . Since  $x$  has positive degree, there is an edge from  $x$  to some other vertex,  $y$ . Thus, we can obtain a path from  $x$  to  $z$  by going from  $x$  to  $y$  and then following the path from  $y$  to  $z$ . This proves  $P(n + 1)$ .

By the principle of induction,  $P(n)$  is true for all  $n \geq 0$ , which proves the Claim. □

**Solution.** This one is tricky: the proof is actually a good proof of something else. The first error in the proof is only in the final statement of the inductive step: “This proves  $P(n + 1)$ ”.

The issue is that to prove  $P(n + 1)$ , *every*  $(n + 1)$ -vertex positive-degree graph must be shown to be connected. But the proof doesn’t show this. Instead, it shows that every  $(n + 1)$ -vertex positive-degree graph *that can be built up by adding a vertex of positive degree to an  $n$ -vertex connected graph*, is connected.

The problem is that *not every*  $(n + 1)$ -vertex positive-degree graph can be built up in this way. The counterexample above illustrates this: there is no way to build that 4-vertex positive-degree graph from a 3-vertex positive-degree graph.

More generally, this is an example of “buildup error”. This error arises from a faulty assumption that every size  $n + 1$  graph with some property can be “built up” in some particular way from a size  $n$  graph with the same property. (This assumption is correct for some properties, but incorrect for others— such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a “shrink down, grow back” process in the inductive step: start with a size  $n + 1$  graph, remove a vertex (or edge), apply the inductive hypothesis  $P(n)$  to the smaller graph, and then add back the vertex (or edge) and argue that  $P(n + 1)$  holds. Let’s see what would have happened if we’d tried to prove the claim above by this method:

*Inductive step:* We must show that  $P(n)$  implies  $P(n + 1)$  for all  $n \geq 1$ . Consider an  $(n + 1)$ -vertex graph  $G$  in which every vertex has degree at least 1. Remove an arbitrary vertex  $v$ , leaving an  $n$ -vertex graph  $G'$  in which every vertex has degree... uh-oh!

The reduced graph  $G'$  might contain a vertex of degree 0, making the inductive hypothesis  $P(n)$  inapplicable! We are stuck— and properly so, since the claim is false!

■

## 2 Euler tours

*The statement of (a) in the original version was incorrect! This has been corrected below.*

- (a) Prove that a graph  $G$  has an Euler tour if and only if: i) every vertex of  $G$  has even degree, and ii) the subgraph obtained after removing all isolated vertices is connected. (An *isolated vertex* is a vertex of degree 0.)

Note that there are two directions to prove!

**Solution.** Let  $G'$  be the subgraph induced by the vertices that are not isolated vertices. Note that  $G'$  has an Euler tour if and only if  $G$  has. Any Euler tour must visit every vertex in  $G'$ , since all edges must be visited. Thus ii) is certainly a necessary condition for the existence of an Euler tour.

The rest of the proof is as in the proof of Theorem 5.6.3 in the book (pp. 159–160). ■

- (b) Come up with a necessary and sufficient condition for the existence of an Euler tour in a *directed* graph. Adapt your proof above to prove that your condition is the right one.

**Solution.** The condition is: an Euler tour exists if and only if i) for every vertex, the indegree equals the outdegree, and ii) the subgraph obtained after removing all isolated vertices is strongly connected.

The proof is basically the same. Again, let  $G'$  be the subgraph of  $G$  induced on the non-isolated vertices of  $G$ ;  $G'$  has an Euler tour if and only if  $G$  does. Any Euler tour provides a directed path between any two vertices of  $G'$  (since we need to visit every arc), and must enter and exit a vertex the same number of times; so the condition is certainly necessary.

Now suppose the condition holds, and let  $W = w_0, w_1, \dots, w_k$  be a longest walk in  $G'$  using every directed edge at most once. Then  $W$  must be a closed walk; for suppose that  $w_k \neq w_0$ . Then we must have entered  $w_k$  one more time than we left it, which means that there is some outgoing directed edge that we have not used. This would allow us to extend the walk, contradicting that  $W$  was as long as possible.

Suppose that  $W$  is not an Euler tour. There must be an unused edge directed away from some vertex in the walk  $W$ ; for if not, there would be no path from any vertex on  $W$  to a vertex not in  $W$ , contradicting the assumption that  $G'$  is strongly connected. Let  $w_i \rightarrow u$  be this edge. Construct a walk  $W'$  beginning with this edge and traversing only unused edges, stopping when we cannot make a move. Again by the condition that indegree equals outdegree, this walk will end at  $w_i$ . We thus obtain a longer walk

$$W' = w_0, w_1, \dots, w_i, u, \dots, w_i, w_{i+1}, \dots, w_k.$$

This is again a contradiction. ■

### 3 Connectivity

Prove that any simple graph with  $n$  nodes and strictly more than  $\frac{1}{2}(n-1)(n-2)$  edges is connected.

**Solution.** We'll show the equivalent statement that any disconnected graph on  $n$  nodes has at most  $(n-1)(n-2)/2$  edges.

Let  $G = (V, E)$  be any graph on  $n$  nodes that is not connected. Then there must be more than one connected component; let  $G_1 = (V_1, E_1)$  be any connected component, and let  $G_2 = (V_2, E_2)$  be the graph induced on  $V_2 := V - V_1$ . Note that there are no edges going between  $G_1$  and  $G_2$ , and so  $E_1 \cup E_2 = E$ .

How many edges can  $G_1$  have? At most  $\binom{|V_1|}{2}$  edges (one for each pair of nodes). Similarly,  $G_2$  can have at most  $\binom{|V_2|}{2}$  edges.

Write  $t := |V_1|$ ; then  $|V_2| = |V| - |V_1| = n - t$ . So the total number of edges in  $G$  is at most

$$\frac{t(t-1)}{2} + \frac{(n-t)(n-t-1)}{2}.$$

If we simplify this, we get

$$|E| \leq \frac{n(n-1)}{2} - t(n-t).$$

But since  $1 \leq t \leq n-1$ ,  $t(n-t) \geq n-1$ . (You can confirm this with some calculus; it might help to draw  $t(n-t)$  as a function of  $t$ ; it's just a parabola.) So

$$|E| \leq \frac{n(n-1)}{2} - (n-1) = \frac{(n-2)(n-1)}{2}.$$

■