19 ~~15~~

Random ~~Processes~~ Walks

Random Walks are used to model situations in which an object moves in a sequence of steps in randomly chosen directions. ~~For example in Physics, three-dimensional random walks are used to model Brownian motion and gas diffusion. In this chapter we'll examine two examples of random walks. First, we'll model gambling as a simple 1-dimensional random walk—a walk along a straight line. Then we'll explain how the~~ many phenomena can be modeled as a random walk and we will see several examples in this chapter. Among other things, we'll see why it is rare that you leave the casino with more money that you entered with and we'll see how the Google search engine uses random walks through the graph of world-wide web links to determine the relative importance of websites.

~~INSERT A goes here~~

David – we are not going to use the rest of the original text in Ch 15. The first part of the new CH 19 is based on lecture notes from 12/9/08 and Eric is writing the last part from scratch.

P-2

1-dimensional

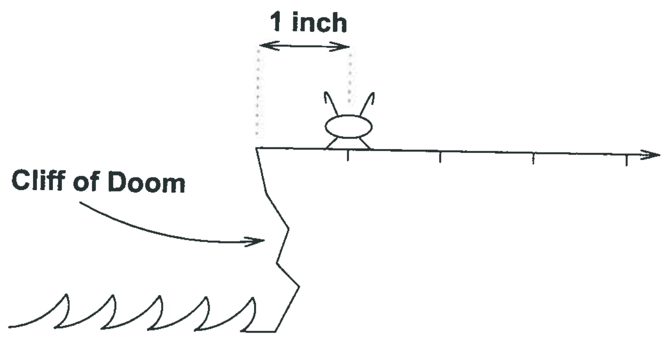
~~Random Walks~~

unbiased

19.1 ~~Unbiased~~ Random Walks ~~one~~

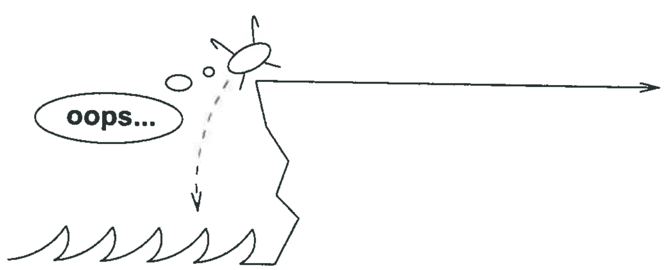
19.1.1 ~~1~~ A Bug's Life

There is a small flea named Stencil. To his right, there is an endless flat plateau. One inch to his left is the Cliff of Doom, which drops to a raging sea filled with flea-eating monsters.



boundary condition or absorbing barrier

Each second, Stencil hops 1 inch to the right or 1 inch to the left with equal probability, independent of the direction of all previous hops. If he ever lands on the very edge of the cliff, then he teeters over and falls into the sea.



So, for example, if Stencil's first hop is to the left, he's fishbait. On the other hand, if his first few hops are to the right, then he may bounce around happily on the plateau for quite some time.

Our job is to analyze the life of Stencil. Does he have any chance of avoiding a fatal plunge? If not, how long will he hop around before he takes the plunge?

Stencil's movement is an example of a ***random walk***. A typical one-dimensional random walk involves some value that randomly wavers up and down over time. Many natural phenomena are

If the walk ends when a certain ~~va~~ value is reached, then that ~~value~~ ~~threshold~~ value is called ~~an~~ a ~~absorbing barrier~~. For example, the Cliff of Doom is ~~an~~ ~~absorbing barrier~~ a boundary condition in the example above.

The walk is said to be unbiased if the value is equally likely to move up or down.

next

P-3

nicely modeled by random walks. However, for some reason, they are traditionally discussed in the context of some social vice. For example, the value is often regarded as the position of a drunkard who randomly staggers left, staggers right, or just wobbles in place during each time step. Or the value is the wealth of a gambler who is continually winning and losing bets. So discussing random walks in terms of fleas is actually sort of elevating the discourse.

19.1.2 ~~1.1~~ A Simpler Problem

Let's begin with a simpler problem. Suppose that Stencil is on a small island; now, not only is the Cliff of Doom 1 inch to his left, but also there is ~~a Cliff~~ another Pit of Disaster, 2 inches to his right! For example, see Figure P1.

the boundary conditions; ~~absorbing barrier, the~~

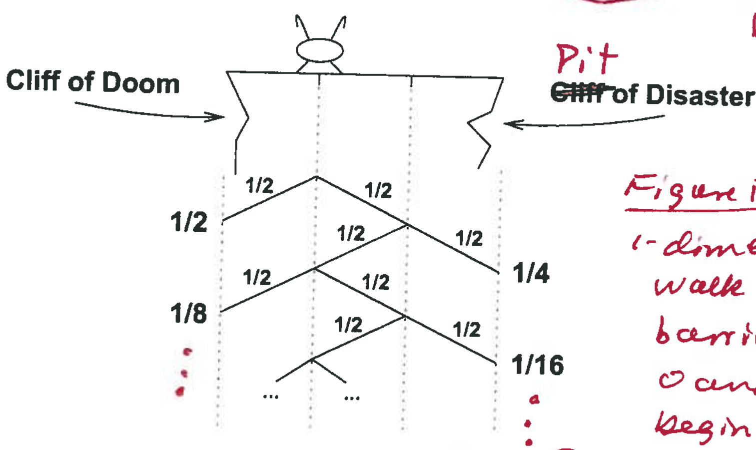


Figure P1: An unbiased, 1-dimensional random walk with absorbing barriers at positions 0 and 3. The walk begins at position 1. The tree diagram shows the probabilities of hitting each barrier.

~~Below~~ In the figure, we've worked out a tree diagram for ~~his~~ Stencil's possible fates. In particular, he falls off the Cliff of Doom on the left side with probability:

$$\begin{aligned}\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots &= \frac{1}{2}\left(1+\frac{1}{4}+\frac{1}{16}+\cdots\right)\\ &= \frac{1}{2}\cdot\frac{1}{1-1/4}\\ &= \frac{2}{3}.\end{aligned}$$

Similarly, he falls ~~off~~ into the ~~Cliff~~ Pit of Disaster on the right side with probability:

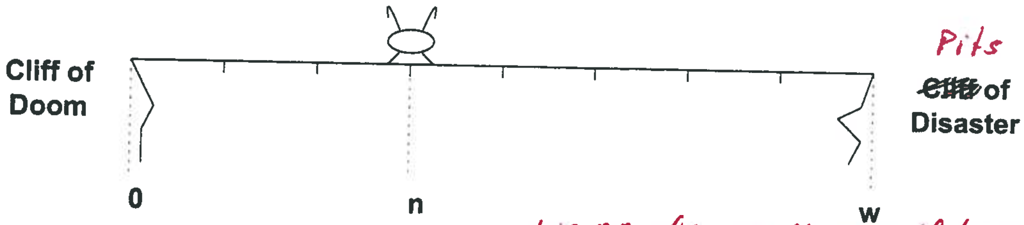
$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots = \frac{1}{3}.$$

There is a remaining possibility: ~~he~~ Stencil *could* hop back and forth in the middle of the ~~table~~ island forever. However, we've already identified two disjoint events with probabilities 2/3 and 1/3, so this happy alternative must have probability 0.

P-4

19.1.3 ~~1.2~~ A Big Island

Putting Stencil on such a tiny island was sort of cruel. Sure, he's probably carrying bubonic plague, but there's no reason to pick on the little fella. So suppose that we instead place him n inches from the left side of an island w inches across:



In other words, Stencil starts at position n and ~~there are cliffs at~~ his random walk ends if he ever reaches positions 0 ~~and~~ or w.

Now he has three possible fates: he could fall off the Cliff of Doom, fall ~~off~~ into the ~~Cliff~~ Pit of Disaster, or hop around on the island forever. We could compute the probabilities of these three events with a horrific summation, but fortunately there's a far easier method: we can use a linear recurrence.

Let R_n be the probability that Stencil falls to the right ~~off~~ into the ~~Cliff~~ Pit of Disaster, given that he starts at position n. In a couple special cases, the value of R_n is easy to determine. If he starts at position w, he falls ~~from~~ into the ~~Cliff~~ Pit of Disaster immediately, so $R_w = 1$. On the other hand, if he starts at position 0, then he falls from the Cliff of Doom immediately, so $R_0 = 0$.

Now suppose that our frolicking friend starts somewhere in the middle of the island; that is, $0 < n < w$. Then we can break the analysis of his fate into two cases based on the direction of his first hop:

- If his first hop is to the left, then he lands at position $n - 1$ and eventually falls ~~off~~ into the ~~Cliff~~ Pit of Disaster with probability R_{n-1}.
- On the other hand, if his first hop is to the right, then he lands at position $n + 1$ and eventually falls ~~off~~ into the ~~Cliff~~ Pit of Disaster with probability R_{n+1}.

Therefore, by the Total Probability Theorem, we have:

$$R_n = \frac{1}{2} R_{n-1} + \frac{1}{2} R_{n+1} .$$

Solving the Recurrence

~~A Recurrence Solution~~ ← subsubsection

Let's assemble all our observations about R_n, the probability that Stencil falls ~~from~~ into the ~~Cliff~~ Pit of Disaster if he starts at position n:

$$\begin{aligned} R_0 &= 1 \\ R_w &= 0 \\ R_n &= \tfrac{1}{2} R_{n-1} + \tfrac{1}{2} R_{n+1} \qquad (0 < n < w) . \end{aligned}$$

P-5

This is just a linear recurrence— and we know how to solve those! Uh, right? ~~(We've attached a quick reference guide to be on the safe side.)~~ Rember Chapter 10, or Chapter 12.

There is one unusual complication: in a normal recurrence, R_n is written a function of preceding terms. In this recurrence equation, however, R_n is a function of both a preceding term (R_{n-1}) and a *following* term (R_{n+1}). This is no big deal, however, since we can just rearrange the terms in the recurrence equation:

$$R_{n+1} = 2R_n - R_{n-1}.$$

Now we're back on familiar territory.

Let's solve the recurrence. The characteristic equation is:

$$x^2 - 2x + 1 = 0.$$

This equation has a double root at $x = 1$. There is no inhomogenous part, so the general solution has the form:

$$R_n = a \cdot 1^n + b \cdot n1^n = a + bn.$$

Substituting in the boundary conditions $R_0 = 0$ and $R_w = 1$ gives two linear equations:

$$0 = a,$$
$$1 = a + bw.$$

The solution to this system is $a = 0$, $b = 1/w$. Therefore, the solution to the recurrence is:

$$R_n = n/w.$$

19.1.4 Death is Certain

~~Interpreting the Answer~~

Our analysis shows that if we place Stencil n inches from the left side of an island w inches across, then he falls off the right side with probability n/w. For example, if Stencil is $n = 4$ inches from the left side of an island $w = 12$ inches across, then he falls off the right side with probability $n/w = 4/12 = 1/3$.

We can compute the probability that he falls off the *left* side by exploiting the symmetry of the problem: the probability the falls off the *left* side starting at position n is the same as the probability that he falls of the *right* side starting at position $w - n$, which is $(w - n)/n$.

This is bad news. The probability that Stencil eventually falls off one ~~cliff~~ side or the other is:

$$\frac{n}{w} + \frac{w-n}{w} = 1.$$

There's no hope! The probability that ~~he~~ Stencil hops around on the island forever is zero.

And there's even worse news. Let's go back to the original problem where Stencil is 1 inch from the left edge of an infinite plateau. In this case, the probability that he eventually falls into the sea is:

$$\lim_{w\to\infty} \frac{w-1}{w} = 1.$$

So even if there were no Pit of Disaster, Stencil still falls ~~into~~ off the Cliff of Doom with probability 1. And since

$\lim_{w \to \infty} \frac{w-n}{w} = 1$ P-6

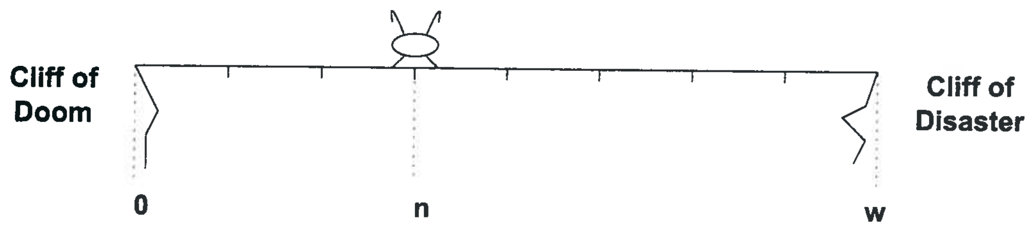
for any finite n, this is true no matter where Stencil starts.)

Our little friend is doomed!

Hey, you know how in the movies they often make it look like the hero dies, but then he comes back in the end and everything turns out okay? Well, ~~I'm~~ we're not sayin' anything, just pointing that out.

~~1.3~~ 19.1.5 Life Expectancy

On the bright side, Stencil may get to hop around for a while before he ~~sinks beneath the waves~~ goes over ~~the~~ an edge. Let's use the same setup as before, where he starts out n inches from the left side of an island w inches across:



What is the expected number of hops he takes before falling off ~~a cliff?~~ an edge?

Let X_n be ~~his~~ Stencil's expected lifespan, measured in hops. If he starts at either edge of the island, then he dies immediately:

$$X_0 = 0,$$
$$X_w = 0.$$

If he starts somewhere in the middle of the island ($0 < n < w$), then we can again break down the analysis into two cases based on his first hop:

- If his first hop is to the left, then he lands at position $n-1$ and can expect to live for another X_{n-1} steps.
- If his first hop is to the right, then he lands at position $n+1$ and ~~his expected lifespan is X_{n+1}.~~ can expect to live for another X_{n+1} steps,

Thus, by the Law of Total Expectation ~~Theorem~~ and ~~linearity, his~~ Linearity of Expectation, Stencil's expected lifespan is:

$$X_n = 1 + \frac{1}{2}X_{n-1} + \frac{1}{2}X_{n+1}.$$

The leading 1 accounts for his first hop.

Solving the Recurrence ← subsubsection

Once again, Stencil's fate hinges on a recurrence equation:

$$
\begin{aligned}
X_0 &= 0 \\
X_w &= 0 \\
X_n &= 1 + \tfrac{1}{2} X_{n-1} + \tfrac{1}{2} X_{n+1} \qquad (0 < n < w).
\end{aligned}
$$

We can rewrite the last line as:

$$X_{n+1} = 2X_n - X_{n-1} - 2. \qquad \text{(eqn P1)}$$

As before, the characteristic equation is:

$$x^2 - 2x + 1 = 0.$$

There is a double-root at 1, so the homogenous solution has the form:

$$X_n = a + bn.$$

But this time, there's ~~There's~~ an inhomogenous term, so we also need to find a particular solution. Since this term is a constant, we should try a particular solution of the form $X_n = c$ and then try $X_n = c + dn$ and then $X_n = c + dn + en^2$ and so forth. As it turns out, the first two possibilities don't work, but the third does. Substituting ~~in this guess gives:~~ ~~into~~ $X_n = c + dn + en^2$ into Equation P1 gives

$$\cancel{X_{n+1} = 2X_n - X_{n-1} - 2}$$

$$c + d(n+1) + e(n+1)^2 = 2(c + dn + en^2) - (c + d(n-1) + e(n-1)^2) - 2,$$

which simplifies to $\cancel{e = -1}$ $e = -1$. Since all

~~All~~ the c and d terms cancel, ~~so~~ $X_n = c + dn - n^2$ is a particular solution for all c and d. For simplicity, let's take $c = d = 0$. Thus, our particular solution is $X_n = -n^2$.

Adding the homogenous and particular solutions gives the general form of the solution:

$$X_n = a + bn - n^2.$$

Substituting in the boundary conditions $X_0 = 0$ and $X_w = 0$ gives two linear equations:

$$
\begin{aligned}
0 &= a, \\
0 &= a + bw - w^2.
\end{aligned}
$$

The solution to this system is $a = 0$ and $b = w$. Therefore, the solution to the recurrence equation is:

$$X_n = wn - n^2 = n(w - n).$$

Interpreting the Solution ← subsubsection

Stencil's expected lifespan is $X_n = n(w - n)$, which is the *product* of the distances to the two ~~cliffs~~ edges. Thus, for example, if he's 4 inches from the left ~~cliff~~ edge and 8 inches from the right edge ~~cliff~~, then his expected lifespan is $4 \cdot 8 = 32$.

Let's return to the original problem where Stencil has the Cliff of Doom 1 inch to his left and an infinite plateau to this right. (Also, cue the "hero returns" theme music.) In this case, his expected lifespan is:

$$\lim_{w \to \infty} 1(w - 1) = \infty.$$

Yes, Stencil is certain to eventually fall off ~~*the cliff into the sea*~~ the Cliff of Doom — *but his expected lifespan is infinite!* This sounds almost like a contradiction, but both answers are correct!

Here's an informal explanation. ~~The~~ It turns out that the probability that Stencil falls from the Cliff of Doom on the k-th step is ~~approximately $1/k^{3/2}$. Thus, the probability that he falls eventually is:~~ p_k where $p_k = \Theta(1/k^{3/2})$ and $\sum_{k=1}^{\infty} p_k = 1$.

$$\cancel{\Pr[\text{falls off cliff}] \approx \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}}$$

~~You can verify by integration that this sum converges. The exact sum actually converges to 1. On the other hand, the expected time until he falls is:~~

$$\cancel{\operatorname{Ex}[\text{hops until fall}] \approx \sum_{k=1}^{\infty} k \cdot \frac{1}{k^{3/2}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}}$$

INSERT A goes here

~~And you can verify by integration that this sum diverges. So our answers are compatible!~~

2 The Gambler's Ruin

We took the high road for a while, but now let's discuss random walks in more conventional terms. A gambler goes to Las Vegas with n dollars in her pocket. Her plan is to make only \$1 bets on red or black in roulette, each of which she'll win with probability $9/19 \approx 0.473$. She'll play until she's either broke or up \$100. What's the probability that she goes home a winner?

This is similar to the flea problem. The gambler's wealth goes up and down randomly, just like the Stencil's position. Going broke is analogous to falling off the Cliff of Doom and winning \$100 corresponds to falling off the Cliff of Disaster. In fact, the only substantive difference is that the gambler's wealth is slightly more likely to go down than up, whereas Stencil was equally likely to hop left or right.

We determined the flea usually falls off the nearest cliff. So we might expect that the gambler can improve her odds of going up \$100 before going bankrupt by bringing more

$$\cancel{\lim_{n\to\infty} \sum_{k=1}^{\infty} k p_k = \infty}$$

p_k where $p_k = \Theta\left(\frac{1}{k^{3/2}}\right)$ and $\sum_{k=1}^{\infty} p_k = 1$. You can verify by the integration bound that $\sum_{k=1}^{\infty} 1/k^{3/2}$ ~~is~~ converges.

On the other hand, the expected time until stencil falls over the edge is ~~at least~~

$$\sum_{k=1}^{\infty} k p_k \geq c \sum_{k=1}^{\infty} k/k^{3/2}$$

$$= c \sum_{k=1}^{\infty} 1/\sqrt{k}$$

$$= \infty,$$

where c is a constant that comes from the Θ notation. So our answers are ~~not~~ compatible.

19.1.6 Application to Fair Gambling

We took the high road for a while, but let's now discuss random walks in a more conventional setting — gambling. ~~Start~~ ~~Suppose you start the evening with \$n in your wallet and that you wager every wager you make is~~

~~Suppose for the sake of an example that~~

~~you~~ A gambler goes to Las Vegas with \$$n$ in her ~~wallet~~ pocket. Her plan is to make only \$1 bets and somehow she has ~~found~~ a casino that will offer her truly even odds[1]; namely, she will win or lose \$1 on each bet with probability 1/2. She'll play until she is broke or she has ~~a total of \$~~ ~~for som~~ won \$$m$. In the latter case, she will go home with ~~\$~~

$$w = n + m$$

dollars. ~~But~~ what's the probability that she goes ~~this~~ home a winner?

This is identical to the flea problem that we just analyzed. Going broke is analogous to falling off the Cliff of Doom. ~~When~~ Going home a winner is analogous to falling into the Pit of Disaster, just a lot more fun. ~~So we can~~

---

[1] Don't worry, we'll get to the more realistic scenario where she is more likely to lose than win in a moment, but's lets just fantasize about the fair scenario for a ~~little~~ bit.

Our analysis of Stencil's life ~~answers~~ ~~reveals~~ ~~that~~ tell's us most everything we want to know about the gambler's prospects:

- the gambler goes home broke with probability

$$\frac{n}{w} = \frac{n}{n+m},$$

- the gambler goes home a winner with probability

$$\frac{w-n}{w} = \frac{m}{n+m},$$

- the gambler goes home with probability

$$\frac{n}{n+m} + \frac{m}{n+m} = 1.$$

~~Playing Time & Subscribers~~

- the ~~gambler expects to make~~ number of bets before the gambler goes home is expected to be

$$n(w-n) = nm.$$

If the gambler gets greedy and ~~plays~~ ~~forever~~ ~~until~~ ~~too~~ keeps playing ~~she goes broke~~ unless or until she goes broke, then

- the gambler eventually goes broke

with probability 1,

- the ~~gambler~~ number of bets before the gambler goes broke is expected to be infinite.

The bottom line here is clear: ~~set modest~~ when gambling, quit ~~when~~ while you are ahead — if you play until you ~~are~~ go broke, you will certainly go broke.

And that's the good news! Matters get much worse for the more typical scenario where the odds are against you. ~~Let's see~~ Let's see why.

19.2 Gambler's Ruin

So far, we have considered unbiased random walks, where the probability of moving up or down (or left or right) is 1/2. ~~Now we'll consider~~ ~~Unfortunately~~ ~~In casinos, it never quite works~~ ~~In casinos out that way.~~
Unfortunately, things are never quite this simple (or fair) in real casinos.

For example, suppose ~~our~~ the gambler goes to Las Vegas and ~~bets~~ makes \$1 bets on red or black in roulette. In this case, she will win \$1 with probability ~~$\frac{18}{38} = \frac{9}{19}$~~

$$\frac{18}{38} \approx 0.473$$

and she will lose \$ with probability

$$\frac{20}{38} \approx 0.527.$$

That's because the casinos add those bothersome green 0 and 00 to give the house a slight advantage.

At first glance (or after a few drinks), $\frac{18}{38}$ seems awfully close to 1/2 and so our intuition tells us that the game is "almost fair". So we might expect the analysis we just did for the fair game to be "almost right" for the real game. For example, if the gambler starts with \$100 ~~in the~~ and quits when she gets ahead by \$100 in the fair game, then she goes home a winner with probability

$$\frac{100}{200} = .5 .$$

~~At~~ And, if ~~If~~ she~~s~~ wants to improve her chances of going home a winner, she could bring more money. If she brings \$1000 and quits when she gets ahead by \$100 in the fair game, then she goes home a winner with probability

$$\frac{1000}{1100} \approx .91 .$$

So, given that the real game is "almost fair," we might expect the probabilities of going home a winner ~~to~~ in these two scenarios to be "almost 50% and 91%," respectively.

~~Nothing could be further from the truth.~~

Unfortunately for the gambler, all this "almost ~~to~~ ~~reas~~ reasoning" will almost surely lead to disaster. Here ~~is the~~ are the grim facts ~~of~~ for the real game where the gambler wins \$1 with probability $\frac{18}{38}$.

p-15

~~money to Vegas. But here's some actual data:~~

n = starting wealth	probability she reaches $n + \$100$ before $0
$100	1 in 37649.619496...
$1000	1 in 37648.619496...
$1,000,000,000	1 in 37648.619496...

She is almost certain to go broke before winning $100.

Except on the very low end, the amount of money she brings makes almost no difference! ~~The fact that only one digit changes from the first case to the second is a peripheral bit of bizarreness that we'll leave in your hands.~~

Let's see why.

19.2.1 ~~2.1~~ Finding a Recurrence

We can approach the gambling problem the same way we studied the life of Stencil. Supose that the gambler starts with n dollars. She wins each bet with probability p and plays until she either goes bankrupt or has ~~$w \geq n$~~ $w = n + m$ dollars in her pocket. (To be clear, w is the total amount of money she wants to end up with, not the amount by which she wants to increase her wealth, which is m.) Our objective is to compute R_n, the probability that she goes home a winner.

As usual, we begin by identifying some boundary conditions. If she starts with no money, then she's bankrupt immediately so $R_0 = 0$. On the other hand, if she starts with w dollars, then she's an instant winner, so $R_w = 1$.

Now we divide the analysis of the general situation into two cases based on the outcome of her first bet:

- She wins her first bet with probabilty p. She then has $n + 1$ dollars and probability R_{n+1} of reaching her goal of w dollars.
- She loses her first bet with probability $1 - p$. This leaves her with $n - 1$ dollars and probability R_{n-1} of reaching her goal.

Plugging these facts into the Total Probability Theorem gives the equation:

$$R_n = pR_{n+1} + (1-p)R_{n-1}.$$ (eqn P3)

19.2.2 ~~2.2~~ Solving the Recurrence

~~We now have~~ Rearranging terms in Equation P3 gives us a recurrence for R_n, the probability that the gambler reaches her goal of w dollars if she starts with n:

$$R_0 = 0$$
$$R_w = 1$$
$$\cancel{R_n = pR_{n+1} + (1-p)R_{n-1}} \qquad (0 < n < w).$$

$$pR_{n+1} - R_n + (1-p)R_{n-1} = 0$$

1

P-16

The characteristic equation is:

$$px^2 - x + (1-p) = 0$$

The quadratic formula gives the roots:

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} \\ &= \frac{1 \pm \sqrt{(1-2p)^2}}{2p} \\ &= \frac{1 \pm (1-2p)}{2p} \\ &= \frac{1-p}{p} \text{ or } 1 \end{aligned}$$

There's an important point lurking here. If the gambler is equally likely to win or lose each bet, then $p = 1/2$, and the characteristic equation has a double root at $x = 1$. This is the situation we considered in the flea problem. The double root led to a general solution of the form:

$$R_n = a + bn$$

Now suppose that the gambler is *not* equally likely to win or lose each bet; that is, $p \neq 1/2$. Then the two roots of the characteristic equation are different, which means that the solution has a completely different form:

$$R_n = a \cdot \left(\frac{1-p}{p}\right)^n + b \cdot 1^n$$

In mathematical terms, this is where the ~~flea problem~~ fair game and the ~~gambler problem~~ "almost fair" game take off in completely different directions: in one case we get a linear solution and in the other we get an exponenetial solution! This is going to be bad news for ~~flea~~ anyone playing the "almost fair" game.

Anyway, substituting the boundary conditions into the general form of the solution gives a system of linear equations:

$$\begin{aligned} 0 &= a + b \\ 1 &= a \cdot \left(\frac{1-p}{p}\right)^w + b \end{aligned}$$

Solving this system, gives:

$$a = \frac{1}{\left(\frac{1-p}{p}\right)^w - 1} \qquad b = -\frac{1}{\left(\frac{1-p}{p}\right)^w - 1}$$

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Substituting these values back into the general solution gives:

$$R_n = \left(\frac{1}{\left(\frac{1-p}{p}\right)^w - 1}\right) \cdot \left(\frac{1-p}{p}\right)^n - \frac{1}{\left(\frac{1-p}{p}\right)^w - 1}$$
$$= \frac{\left(\frac{1-p}{p}\right)^n - 1}{\left(\frac{1-p}{p}\right)^w - 1} .$$

(Suddenly, Stencil's life doesn't seem so bad, huh?)

19.2.3 ~~You Lose~~ Bad News!

~~2.3 Interpreting the Solution~~

But it's not good news.

We have an answer! If the gambler starts with n dollars and wins each bet with probability p, then the probability she reaches w dollars before going broke is:

$$\frac{\left(\frac{1-p}{p}\right)^n - 1}{\left(\frac{1-p}{p}\right)^w - 1} .$$

Let's try to make sense of this expression. If the game is biased against her, as with roulette, then $1 - p$ (the probability she loses) is greater than p (the probability she wins). If n, her starting wealth, is also reasonably large, then both exponentiated fractions are big numbers and the -1's don't make much difference. Thus, her probability of reaching w dollars is very close to:

$$\left(\frac{1-p}{p}\right)^{n-w} = \left(\frac{1-p}{p}\right)^m .$$

In particular, if she is hoping to come out \$100 ahead in roulette, then $p = \cancel{9/19}$ 18/38 and ~~$w = n + 100$, so~~ her probability of success is: $m =$

$$\left(\frac{10}{9}\right)^{-100} = 1 \text{ in } 37648.619496 .$$

This explains the strange number we arrived at earlier! In fact, this number does not change no matter how large n gets so even if the gambler starts with a trillion dollars, she is still not likely to get ahead by even \$100.

19.2.4 But why?

~~2.4 Some Intuition~~

Why does the gambler's starting wealth have so little impact on her probability of coming out ahead? Intuitively, there are two forces at work. First, the gambler's wealth has random upward and downward *swings* due to runs of good and bad luck. Second, her wealth has a steady, downward *drift* because she has a small expected loss on every bet. The situation is illustrated ~~below:~~ in Figure P3.

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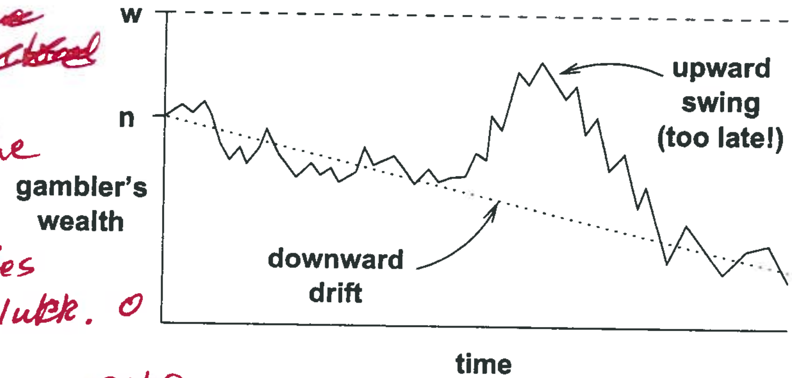
Figure P3: In a ~~predicted~~ ~~explana~~ biased random walk, the downward drift usually dominates swings of good luck.



$1 \cdot \frac{18}{38} + (-1)\frac{20}{38} = \frac{-2}{38} = -1/19.$

~~18/38~~

For example, in roulette, the gambler wins a dollar with probability ~~9/19~~ 18/38 and loses a dollar with probability ~~10/19~~ 20/38. Therefore, her expected return on each bet is 1 ~~$\cdot 9/19 + (-1) \cdot 10/19 = -1/19$~~. Thus, her expected wealth drifts downward by a little over 5 cents per bet.

One might think that if the gambler starts with a billion dollars, then she will play for a long time, so at some point she should have a lucky, upward swing that puts her \$100 ahead. The problem is that her capital is steadily drifting downward. And after her capital drifts down a few hundred dollars, she needs a huge upward swing to save herself. And such a huge swing is extremely improbable. So if she does not have a lucky, upward swing early on, she's doomed forever. As a rule of thumb, *drift* dominates *swings* over the long term.

—— INSERT B goes here ——

3 Pass the Broccoli

~~Here's a game that involves a random walk. There are $n+1$ people, numbered $0, 1, \ldots, n$ sitting in a circle:~~

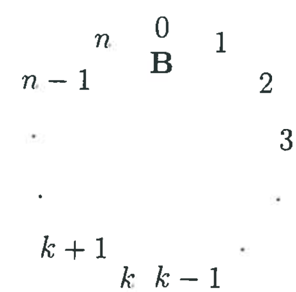


Figure P6: ~~The~~ n+1 people sitting in a circle. The B indicates the person with the broccoli — in this case, person 0.

The B in Figure P6 indicates that person 0 has a big stalk of nutritious broccoli, which provides 250% of the US recommended daily allowance of vitamin C and is also a good source of vitamin A and iron. (Typical for a random walk problem, this game orginally involved a pitcher of beer instead of a broccoli. We're taking the high road again.)

Person 0 passes the broccoli either to the person on his left or the person on his right with equal probability. Then, that person also passes the broccoli left or right at random and so

INSERT B

19.2.5 Expected Playing Time

Even though casino gamblers are destined to lose, some of them enjoy the process. So let's figure out how long their enjoyment is expected to last.

Let x_n be the expected number of bets before going home ~~broke or (one way or the other)~~ broke or a winner. Reasoning as in Section 19.1.5, we can set up a reccurence for x_n:

$$x_0 = 0,$$

$$x_w = 0,$$

$$x_n = 1 + (1-p)x_{n-1} + p x_{n+1} . \qquad \text{(eqn P5)}$$

This is the same as the reccurence for R_n in Equation P3 except for the inhomogeneous part.

To find the particular solution, we try $x_n = c$. ~~(which doesn't work) and then $x_n = c \cdot dn$, and (which does work). Plugging $x_n = c \cdot dn$ into Equation P5 yields~~

(which doesn't work) and then $x_n = c + dn$ (which does work as long as $p \neq 1/2$). Plugging $x_n = c + dn$ into Equation P5a yields:

$$c + dn = 1 + (1-p)(c + d(n-1)) + p(c + d(n+1))$$

$$= 1 + c + dn - (1-p)d + pd$$

and thus that ~~c is anything and~~

$$d = \frac{1}{1-2p}.$$

Since c is arbitrary, we will set $c = 0$ and our particular solution is

$$x_n = \frac{n}{1-2p}.$$

The characteristic equation for Equation P5 is

$$px^2 - x + (1-p) = 0.$$

We have already determined that the roots for this equation are

$$\frac{1-p}{p} \text{ and } 1.$$

Hence, the general solution to the recurrence is

$$X_n = a\left(\frac{1-p}{p}\right)^n + b + \frac{n}{1-2p}.$$

Plugging in the boundary conditions, we find that

$$0 = a + b,$$

$$0 = a\left(\frac{1-p}{p}\right)^w + b + \frac{w}{1-2p}.$$

Hence

$$a = \frac{-\left(\frac{w}{1-2p}\right)}{\left(\frac{1-p}{p}\right)^w - 1} \text{ and } b = \frac{\left(\frac{w}{1-2p}\right)}{\left(\frac{1-p}{p}\right)^w - 1}.$$

~~We~~ The final solution to the recurrence is then

$$X_n = \frac{-\left(\frac{w}{1-2p}\right)\left(\frac{1-p}{p}\right)^n}{\left(\frac{1-p}{p}\right)^w - 1} + \frac{\left(\frac{w}{1-2p}\right)}{\left(\frac{1-p}{p}\right)^w - 1} + \frac{n}{1-2p}.$$

~~Yikes! The gambler won't have any fun at all if she is thinking about this equation. Let's~~

~~see if we can make it simpler~~

$$= \frac{n}{1-2p} - \left(\frac{w}{1-2p}\right)\left[\frac{\left(\frac{1-p}{p}\right)^n - 1}{\left(\frac{1-p}{p}\right)^w - 1}\right].$$

Yikes! The gambler won't have any fun at all if she is thinking about this equation. Let's see if we can make it simpler in the case when ~~n and~~ $m = w - n$ ~~are both~~ is large.

Since $p < 1/2$, $\frac{1-p}{p} > 1$ and for large m,

$$\lim_{m\to\infty} \left(\frac{w}{1-2p}\right)\left[\frac{\left(\frac{1-p}{p}\right)^n - 1}{\left(\frac{1-p}{p}\right)^w - 1}\right] = \lim_{m\to\infty} \left(\frac{w}{1-2p}\right)\left(\frac{1-p}{p}\right)^{-m}$$

$$= 0.$$

This means that as m gets large

$$X_n \sim \frac{n}{1-2p},$$

which is much simpler. It says that if the gambler starts with $n, she will expect to make about $\frac{n}{1-2p}$ bets before she goes home broke. This seems to makes sense since ~~But~~ she expects to lose

$$1\cdot(1-p) + (-1)p = 1-2p$$

dollars on every bet and she started with n dollars.[1]

---

1 ~~However~~ Be careful, it is tempting to use such a direct and simple argument instead of all those nasty recurrences, ~~but~~ ~~is~~ ~~that~~ ~~not~~ ~~ok~~ ~~correct.~~ ~~There~~ but such an argument is not correct. There are ~~are many~~ examples where the expected duration of a ~~walk~~ process is not ~~even~~ close to the starting point divided by the expected decrease at each step.

19.3 ~~Random~~ Walks ~~on~~ Around a Circle ~~Pass the Brocool Broccol~~.

So far, we have considered ~~random~~ random walks on a line. ~~What about?~~ Now we'll look at a problem where the random walk is on a circle. Going from a line to a circle

~~Suppose the~~

may not seem like such a big change, but as we have seen so often with probability,

~~things are not always the wa~~

~~things do not always turn out like our intuition~~

small changes can have large ~~ones~~ consequences

~~that ou escape~~

~~that could our intuition.~~

that are often beyond the grasp of our ~~intut~~ intuition.

19.3.1 Pass the Broccoli

Suppose there are $n+1$ people, numbered $0, 1, \ldots, n$, sitting in a circle as shown in Figure P6. For this example

19.3.2 There is no Escape

P-25

, which would be

on. After a while, everyone in an arc of the circle has touched the broccoli and everyone outside that arc has not. Eventually, the arc grows until all but one person has touched the broccoli. That final person is declared the winner and gets to keep the broccoli!

Suppose that you allowed to position yourself anywhere in the circle. Where should you stand in order to maxmize the probability that you win? You shouldn't be person 0; you can't win in that position. The answer is "intuitively obvious": you should ~~stand~~ sit as far as possible from person 0 ~~at~~ position $n/2$ or $\frac{n+1}{2}$ depending on whether n is even or odd.

Let's try to verify this intuition. Suppose that you ~~stand~~ sit at position $k \neq 0$. At some point, the broccoli is going to end up in the hands of one of your neighbors. This has to happen eventually; the game can't end until at least one of them touches it. Let's say that person $k-1$ gets the broccoli first. Now let's cut the circle between yourself and your other neighbor, person $k+1$:

$$k \quad (k-1) \quad \ldots \quad 3 \quad 2 \quad 1 \quad 0 \quad n \quad (n-1) \quad \ldots \quad (k+1).$$

B

~~Now~~ There are two possibilities. If the broccoli reaches you before it reaches person $k+1$, then you lose. But if the broccoli reaches person $k+1$ before it reaches you, then every other person has touched the broccoli and you wins. ~~This is just the flea problem all over again: The probability that the broccoli hops $n-1$ people to the right (reaching person $k+1$) before it hops 1 person to the left (reaching you) is $1/n$. Therefore, our intuition was compeletely wrong: *your probability of winning is $1/n$ regardless of where you're standing!*~~

— INSERT C goes here —

~~Well, that's it for 6.042. Good luck on the final exam and have a fun IAP!~~

19.4 Random Walks on Graphs

— material to be supplied by Eric

19.5 Problems

INSERT C

So we need to compute the probability that the broccoli hops $n-1$ people to the right before it takes 1 hop to the left. This will be the probability that you win.

But this is just the flea problem all over again. From the analysis in Section 19.1.3, we know that the probability of moving $n-1$ steps rightward before moving one step leftward is ~~just~~ simply $1/n$. This means that wherever you sit ~~except if you~~ (aside from position 0, of course), your probability of getting the brocolli last is $1/n$.

So our intuition was completely wrong (again)! It doesn't matter where you sit. Being close to the brocooli or far away at the start makes no difference; there is no escape; you still get the brocolli last with probability $1/n$.

~~Enough with the bad news~~

Enough with the bad news: Stencil's doomed, you ~~loses~~ go home broke from the casino, and you can't escape the brocolli. Let's see how to use probability to *make* some money.