

Problem Set 1 Solutions

Due: Monday, September 8

Reading Assignment: Chapter 1 & Sections 2.0-2.4

Problem 1. [20 points] Suppose that $w^2 + x^2 + y^2 = z^2$, where w, x, y , and z always denote positive integers. (*Hint:* It may be helpful to represent even integers as $2i$ and odd integers as $2j + 1$, where i and j are integers.)

Prove the proposition: z is even if and only if w, x , and y are even. Do this by considering all the cases of w, x, y being odd or even.

Solution. Proof. As the problem suggests, we will build a truth table to figure out what z is in each case of x, y, z being even.

- w, x, y are all even. In this case, we can write w, x, y as $2i, 2j, 2k$, and the sum of squares is $4i^2 + 4j^2 + 4k^2 = 4(i^2 + j^2 + k^2)$. In this case, z can be any even integer when $i^2 + j^2 + k^2 = l^2$ for some l .
- w, x, y are all odd. In this case, each of their squares are odd, and the sum of three odd numbers is odd. However, if z is even, then z^2 is also even. So it cannot be that $z^2 = w^2 + x^2 + y^2$.
- One of w, x, y is odd. This case is the same as above. Assume x is odd. The x^2 is odd, but w^2 and y^2 are even. So the sum is once again odd, which cannot equal the square of an even integer. So z cannot be even in that case either. The same argument can be repeated for when w and y are odd as well.
- Two of w, x, y are odd. Assume that w, x are odd and y is even. Then we can write the sum as $w^2 + x^2 + y^2 = (2i + 1)^2 + (2j + 1)^2 + (2k)^2$. This can be rewritten as $4i^2 + 4i + 1 + 4j^2 + 4j + 1 + 4k^2 = 4(i^2 + i + j^2 + j) + 2$. However, if z is even, then z^2 is a multiple of 4. So z cannot be an even integer in this case. The same argument can be used when w, y are odd or when x, z are odd.

w even	x even	y even	z even
<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>
<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>
<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>
<i>F</i>	<i>F</i>	<i>F</i>	<i>F</i>

□

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Problem 2. [14 points] The three logical expressions below are nearly equivalent. However, for one particular setting of x , y , and z , one of the expressions is not like the others. Write out *complete truth tables for all three*, and determine which one of these expressions just doesn't belong.

1. $(x \rightarrow y) \wedge (y \rightarrow z) \wedge (z \rightarrow x)$
2. $(\neg x) \wedge (\neg y) \wedge (\neg z)$
3. $(\neg x \vee (y \wedge z)) \wedge (x \vee (\neg y \wedge \neg z))$

Solution.

x	y	z	(1)	(2)	(3)
<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>
<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>F</i>
<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>
<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>F</i>
<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	F	<i>T</i>

Expression 2 just doesn't belong.

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Problem 3. [14 points] A student is trying to prove that propositions p , q , and r are all true. She proceeds as follows. First, she proves three facts: $p \rightarrow q$, $q \rightarrow r$, and $r \rightarrow p$. Then she concludes, “thus obviously p , q , and r are all true.” Let's first formalize her deduction as a logical statement and then evaluate whether or not it is correct.

(a) [5 pts] Using logic notation and the symbols p , q , and r , write down the logical implication that she uses in her final step.

Solution.

$$((p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)) \rightarrow (p \wedge q \wedge r)$$

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(b) [5 pts] Use a truth table to determine whether this logical implication is a tautology (i.e., a universal truth in logic).

Solution.

p	q	r	$((p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p))$	$(p \wedge q \wedge r)$	complete expression
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	T	F	$\rightarrow F \leftarrow$

The truth table indicates that the implication she uses is *not* a tautology. ■

(c) [4 pts] Is her proof that propositions p , q , and r are all true correct? Briefly explain.

Solution. Her proof is incorrect; she makes use of a false proposition. (However, the first part of her argument is sufficient to show that either all three statements are true *or* all three statements are false.) ■

Problem 4. [24 points] Translate the following statements into predicate logic. For each, specify the domain. In addition to logic symbols, you may build predicates using arithmetic, relational symbols, and constants. For example, the statement “ n is an odd number” could be translated into $\exists m.(2m + 1 = n)$, where the domain is \mathbb{Z} , the set of integers. Another example, “ p is a prime number,” could be translated to

$$(p > 1) \text{ AND } \text{NOT} (\exists m. \exists n. (m > 1 \text{ AND } n > 1 \text{ AND } mn = p))$$

Let $\text{prime}(p)$ be an abbreviation that you could use to denote the above formula in this problem.

(a) [4 pts] (Lagrange’s Four-Square Theorem) Every nonnegative integer is expressible as the sum of four perfect squares.

Solution. The domain is \mathbb{N} , the natural numbers.

$$\forall n. \exists w \exists x \exists y \exists z. (n = w^2 + x^2 + y^2 + z^2).$$
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(b) [4 pts] (Goldbach's Conjecture) Every even integer greater than two is the sum of two primes.

Solution. The domain is \mathbb{N} , the natural numbers. The statement could be translated as $\forall n. ((n > 2) \text{ AND } \exists m(n = 2m)) \text{ IMPLIES } \exists p \exists q(\text{prime}(p) \text{ AND } \text{prime}(q) \text{ AND } (n = p + q))$.

Another translation is

$$\forall n. ((n > 2) \text{ AND } \exists m(n = 2m)). \exists p. \text{prime}(p). \exists q. \text{prime}(q). n = p + q.$$

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(c) [4 pts] The function $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous.

Solution. The domain is \mathbb{R} , the real numbers.

$$\forall a \forall x. \exists b. \forall y. ((a > 0 \text{ AND } b > 0 \text{ AND } |x - y| < b) \text{ IMPLIES } |f(x) - f(y)| < a).$$

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(d) [4 pts] (Fermat's Last Theorem) There are no nontrivial solutions to the equation:

$$x^n + y^n = z^n$$

over the nonnegative integers when $n > 2$.

Solution. The domain is \mathbb{N} .

$$\forall x \forall y \forall z \forall n. ((x > 0 \text{ AND } y > 0 \text{ AND } z > 0 \text{ AND } n > 2) \text{ IMPLIES NOT}(x^n + y^n = z^n)).$$

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(e) [4 pts] There is no largest prime number.

Solution. The domain is \mathbb{Z} , the integers.

$$\text{NOT } (\exists p(\text{Prime}(p) \text{ AND } (\forall q(\text{Prime}(q) \text{ IMPLIES } p \geq q)))).$$

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(f) [4 pts] (Bertrand's Postulate) If $n > 1$, then there is always at least one prime p such that $n < p < 2n$.

Solution. The domain is \mathbb{Z} .

$$\forall n. ((n > 1) \text{ IMPLIES } (\exists p(\text{Prime}(p) \text{ AND } (n < p) \text{ AND } (p < 2n))))).$$

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Problem 5. [16 points] You have a balance; by putting objects on each side of the balance, you can determine which side is heavier (or if both sides have the same weight).

You also have a large collection of stones of known weight; there are stones that weigh 1 ounce, 2 ounces, and so on all the way up to 40 ounces (and you have many stones of each weight).

You are given an object of unknown weight; all you know is that it weighs an integer number of ounces between 1 and 40, inclusive. Your goal is to determine the weight of this object, by using as small a set of stones from your collection as possible (you may do as many weighings as you like with the stones you select). Give a set of possible stones, and summarize the procedure that you will use to identify the weight of the object. (You will get 8/16 points for using 6 stones; 12/16 points for using just 5 stones; and 16/16 points for using only 4 stones).

Hint: To identify the unknown weight, you do not need to make every weight from 1 through 40 (e.g., making only the even weights also helps you identify the unknown weight should it be odd.)

Additional hint: When using the balance, you can place some stones on the same side as the object of unknown weight.

Solution. There are many possible sets of size 4, 5, and 6 that work. As long as you explained your solution, you will receive credit commensurate to the number of stones you use.

We will show two possible ways to approach this problem, and then optimize these approaches with the same key insights to get from 6 stones to 4 stones.

Approach 1 involves building our set of stones "bottom up": that is, first ensuring that you can identify an unknown stone of weight 1, then ensuring that you can identify an unknown stone of weight 2, then ensuring that you can identify an unknown stone of weight 3 . . . , and finally ensuring that you can identify an unknown stone of weight 40. Using this approach, one set of stones we could pick is 1, 2, 4, 8, 16, 32, which is equivalent to the set $\{2^0, 2^1, 2^2, 2^3, 2^4, 2^5\}$. Using this set of stones, we can "make" every weight from 1 to 40 (in fact, from 1 to 63): that is, to make a particular weight, we look at the binary representation of the weight, and use the stones that correspond to the binary representation. For example, to make 13, whose binary representation is 1101, we should use the stones 1, 4, and 8.

Approach 2 starts in the middle of the weight range for the unknown object, and we will refer to it as the "binary search" approach: first, ensure that you can identify a stone in the middle, which will divide the weight range of unidentifiable object into two groups. Then, we will pick a stone that can be used to identify a weight in the middle of all resulting groups, and repeat the process until we have no possible weights that cannot be identified. Using this approach, the set of stones that we pick is $\{20, 10, 5, 3, 2, 1\}$. Using this set of stones, we can again make every weight from 1 through 40: we can make 1 through 10 using the stones $\{5, 3, 2, 1\}$, and to make larger weights, we add either the 10oz stone or the 20oz stone (or both) to the stones we use to make 1 through 10.

Since both approaches can make every possible weight of the unknown object, we can incrementally test every possible weight to see what the unknown weight is. Using the hint, we can modify the bottom up approach to only include the set of stones $\{20, 4, 8, 16, 32\}$. Using these even-ounce stones, we can make every even weight from 1 through 40. Likewise, we can use the hint to modify the binary search approach to only include the set of stones $\{20, 10, 6, 4, 2\}$: using the stones 2, 4, and 6, we can make every even number from 1 through 10, and by also using the 10oz and 20oz stones, we can make every even number between 1 through 40.

Using the additional hint, we can improve both approaches even more. We can modify the (original) bottom up approach to only include the stones $\{1, 3, 9, 37\}$ (or $\{3^0, 3^1, 3^2, 3^3\}$), which is a set of stones that can create every number from 1 through 40. For example, to create 2, we can put 3 on the side opposite the unknown weight, and 1 on the same side as the unknown weight. We can use the stones 1 and 3 to create the numbers 1 through 4; then by adding the stone 9, create the numbers 5-9 (by putting 9 and “subtracting 0-4”); and create 10-13 (by putting 9 and adding 1-4). Finally, by adding the stone 27, we can create the numbers 14-27 (by putting 27 and subtracting 0-13) or create 28-40 (by putting 27 and adding 1-13). Since we can create every weight from 1 through 40, we can incrementally check through the possible weight range. We can also modify the binary search approach using the additional hint to get $\{20, 12, 6, 2\}$, which is the set that can create every even weight from 1 through 40. ■

Problem 6. [12 points] A *triangle* is a set of three people such that either every pair has shaken hands or no pair has shaken hands.

(a) [8 pts] Prove that among every six people there is a triangle.

(Suggestion: Initially, break the problem into two cases: for any person X in a given group of six people,

1. there exist at least three other people who shook hands with X ;
2. there exist at least three other people who didn't shake hands with X .

Why must exactly one of these conditions hold?)

Solution. Proof. We first prove the statement given in the suggestion, which we state as a lemma:

Lemma. For any person X in a given group of six people, exactly one of these conditions holds:

1. there exist at least three other people who shook hands with X ;
2. there exist at least three other people who didn't shake hands with X .

Proof. We prove the lemma by case analysis. There are three possible cases: neither condition holds; both conditions hold; or exactly one of the two conditions holds.

- Neither condition holds: that is, at most two other people have shaken hands with X , and at most two other people have not shaken hands with X . So at most four other people have shaken hands or have not shaken hands with X . However, since the set of people who are distinct from X consists of five individuals and each individual must either have shaken hands with X or have not shaken hands with X , we have a contradiction.
- Both conditions hold: that is, at least three other people have shaken hands with X , and at least three other people have not shaken hands with X . Since no individual can be counted in both groups, the union of these two sets has at least six members. This contradicts the fact that there are only five individuals distinct from X .

Since two of the three cases lead to contradiction, we conclude that the only remaining possibility—that exactly one of the two conditions holds—must be correct. \square

We now prove the proposition directly using case analysis.

Let X denote one of the six people. By our lemma, there are two possibilities.

1. There exist three other people who shook hands with X . Now there are two further possibilities:
 - (a) Among these three, some pair shook hands. Then these two and X form a triangle.
 - (b) Among these three, no pair shook hands. Then these three form a triangle.
2. Otherwise, there are exactly three people who did not shake hands with X . Again, there are two further possibilities:
 - (a) Among these three, every pair shook hands. Then these three form a triangle.
 - (b) Among these three, some pair did not shake hands. Then these two and X form a triangle.

\square

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(b) [4pts] Also, is there always a triangle among every five people?

Solution. No. One counter example is for persons A,B,C,D,E where

- A has shaken hands with B and E;
- B has shaken hands with A and C;
- C has shaken hands with B and D;
- D has shaken hands with C and E;
- E has shaken hands with D and A.

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