

Quiz 1

Problem 1. [**10 points**] In problem set 1 you showed that the `nand` operator by itself can be used to write equivalent expressions for all other Boolean logical operators. We call such an operator *universal*. Another universal operator is `nor`, defined such that $P \texttt{nor} Q \Leftrightarrow \neg(P \vee Q)$.

Show how to express $P \wedge Q$ in terms of: `nor`, P, Q, and grouping parentheses.

Solution. $(\neg P)\ \texttt{nor}\ (\neg Q) = (P\ \texttt{nor}\ P)\ \texttt{nor}\ (Q\ \texttt{nor}\ Q)$. ■

Problem 2. [**15 points**] We define the sequence of numbers

$$a_n = \begin{cases} 1 & \text{if } 0 \le n \le 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \ge 4. \end{cases}$$

Prove that $a_n \equiv 1 \pmod 3$ for all $n \ge 0$.

Solution. Proof by strong induction. Let $P(n)$ be the predicate that $a_n \equiv 1 \pmod 3$.

Base case: For $0 \le n \le 3$, $a_n = 1$ and is therefore $\equiv 1 \pmod 3$.

Inductive step: For $n \ge 4$, assume $P(k)$ for $0 \le k \le n$ in order to prove $P(n+1)$.

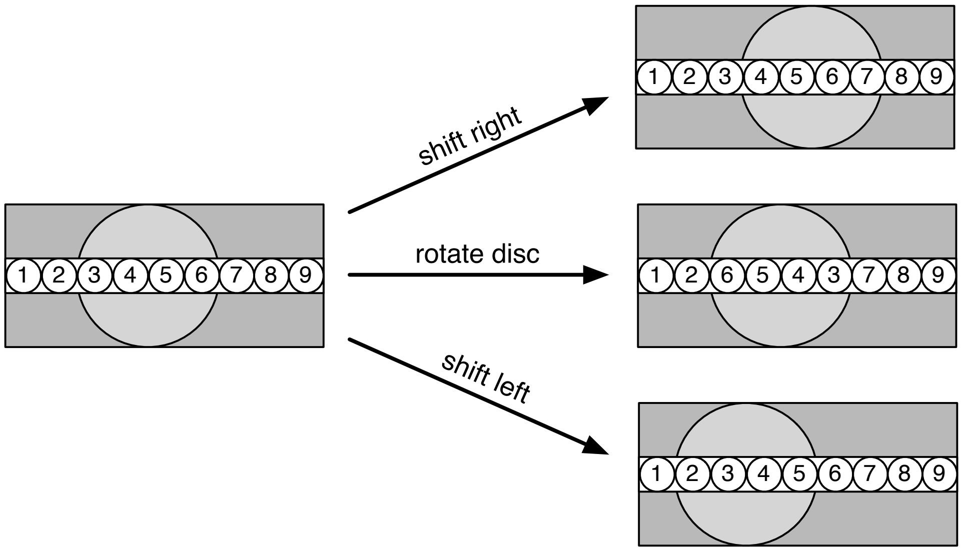
In particular, since each of a_{n-4}, a_{n-3}, a_{n-2} and a_{n-1} is $\equiv 1 \pmod 3$, their sum must be $\equiv 4 \equiv 1 \pmod 3$. Therefore, $a_n \equiv 1 \pmod 3$ and $P(n+1)$ holds. ■

Problem 3. [**20 points**] The Slipped Disc Puzzle™ consists of a track holding 9 circular tiles. In the middle is a disc that can slide left and right and rotate 180° to change the positions of *exactly* four tiles. As shown below, there are three ways to manipulate the puzzle:

Shift Right: The center disc is moved one unit to the right (if there is space)

Rotate Disc: The four tiles in the center disc are reversed

Shift Left: The center disc is moved one unit to the left (if there is space)



Prove that if the puzzle starts in an initial state with all but tiles 1 and 2 in their natural order, then it is impossible to reach a goal state where all the tiles are in their natural order. The initial and goal states are shown below:

Write your proof on the next page...

Solution. Order the tiles from left to right in the puzzle. Define an *inversion* to be a pair of tiles that is out of their natural order (e.g. 4 appearing to the left of 3).

Lemma. *Starting from the initial state there is an odd number of inversions after any number of transitions.*

Proof. The proof is by induction. Let $P(n)$ be the proposition that starting from the initial state there is an odd number of inversions after n transitions.

Base case: After 0 transitions, there is one inversion, so $P(0)$ holds.

Inductive step: Assume $P(n)$ is true. Say we have a configuration that is reachable after $n + 1$ transitions.

1. Case 1: The last transition was a shift left or shift right

   In this case, the left-to-right order of the discs does not change and thus the number of inversions remains the same as in

2. The last transition was a rotate disc.

   In this case, six pairs of disks switch order. If there were x inversions among these pairs after n transitions, there will be $6 - x$ inversions after the reversal. If x is odd, $6 - x$ is odd, so after $n + 1$ transitions the number of inversions is odd.

□

Conclusion: Since all reachable states have an odd number of inversions and the goal state has an even number of inversions (specifically 0), the goal state cannot be reached. ■

Problem 4. [10 points] Find the multiplicative inverse of 17 modulo 72 in the range $\{0, 1, \ldots, 71\}$.

Solution. Since 17 and $72 = 2^3 3^2$ are relatively prime, an inverse exists and can be found by either Euler's theorem or the Pulverizer.

Solution 1: Euler's Theorem

$$\begin{aligned}
\phi(72) &= \phi(2^3 \cdot 3^2) \\
&= \phi(2^3) \cdot \phi(3^2) && \text{(since } 2^3 \text{ and } 3^2 \text{ are rel. prime)} \\
&= (2^3 - 2^2)(3^2 - 3^1) && \text{(since 2 and 3 are prime)} \\
&= 4 \cdot 6 = 24
\end{aligned}$$

Therefore, $17^{\phi(72)-1} = 17^{23}$ is *an* inverse of 17. To find *the* inverse in the range $\{0, 1, \ldots, 71\}$ we take the remainder using the method of repeated squaring:

$$\begin{aligned}
17 &= 17 \\
17^2 &= 289 \\
&\equiv 1 && \text{(since } 289 = 4 \cdot 72 + 1\text{)} \\
17^4 &\equiv 1^2 = 1 \\
17^8 &\equiv 1 \\
&\ldots \textit{etc.}
\end{aligned}$$

Therefore the inverse of 17 in the range $\{0, 1, \ldots, 71\}$ is given by,

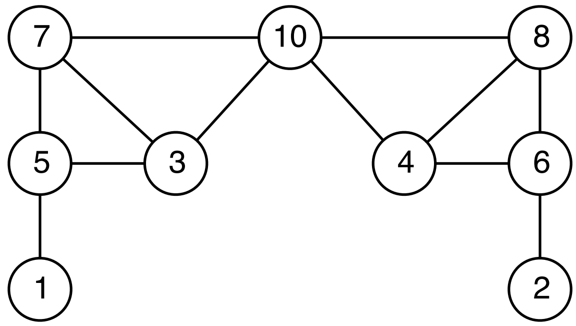
$$\begin{aligned}
17^{23} &= 17^{16} 17^4 17^2 17^1 \\
&\equiv 1 \cdot 1 \cdot 1 \cdot 17 \\
&= 17
\end{aligned}$$

Solution 2: The Pulverizer

x	y	rem xy	$=$	$x - q \cdot y$
72	17	4	$=$	$72 - 4 \cdot 17$
17	4	1	$=$	$17 - 4 \cdot 4$
			$=$	$17 - 4 \cdot (72 - 4 \cdot 17)$
			$=$	$17 \cdot 17 - 4 \cdot 72$
4	1	0		

Since $17^2 - 4 \cdot 72 = 1$, $17^2 \equiv 1 \pmod{72}$ and so 17 is self inverse. ■

Problem 5. [15 points] Consider a graph representing the main campus buildings at MIT.



(a) [5 pts] Give the diameter of this graph:

Solution. The diameter is 6, the length of a shortest path between buildings 1 and 2. ■

(b) [5 pts] Is this graph bipartite? Provide a brief argument for your answer.

Solution. No, there is an odd-length cycle ■

(c) [5 pts] Does this graph have an Euler circuit? Provide a brief argument for your answer.

Solution. This graph does not have an Euler circuit because there are vertices with odd degree ■

Problem 6. [10 points]

A tournament graph $G = (V, E)$ is a directed graph such that there is either an edge from u to v or an edge from v to u for *every* distinct pair of nodes u and v. (The nodes represent players and an edge $u \to v$ indicates that player u beats player v.)

Consider the "beats" relation implied by a tournament graph. Indicate whether or not each of the following relational properties hold *for all* tournament graphs and briefly explain your reasoning. You may assume that a player never plays herself.

1. **transitive**

   Solution. The "beats" relation is not transitive because there could exist a cycle of length 3 where x beats y, y beats z and z beats x. By the definition of a tournament, x cannot then beat y in such a situation. ■

2. **symmetric**

   Solution. The "beats" relation is not symmetric by the definition of a tournament: if x beats y then y does not beat x. ■

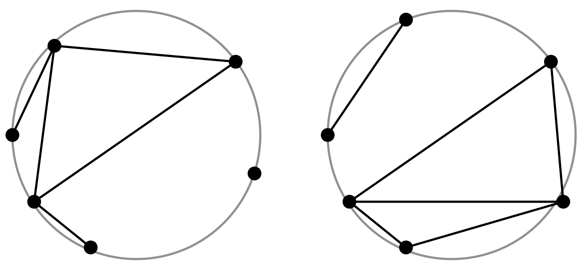
3. **antisymmetric**

   Solution. The "beats" relation is antisymmetric since for any distinct players x and y, if x beats y then y does not beat x. ■

4. **reflexive**

   Solution. The "beats" relation is not reflexive since a tournament graph has no self-loops. ■

Problem 7. [20 points] An outerplanar graph is an undirected graph for which the vertices *can be* placed on a circle in such a way that no edges (drawn as straight lines) cross each other. For example, the complete graph on 4 vertices, K_4, is not outerplanar but any proper subgraph of K_4 with strictly fewer edges is outerplanar. Some examples are provided below:



Prove that any outerplanar graph is 3-colorable. A fact you may use without proof is that any outerplanar graph has a vertex of degree at most 2.

Solution. *Proof.* Proof by induction on the number of nodes n with the induction hypothesis $P(n) =$ "every outerplanar graph with n vertices is 3-colorable."

Base case: For $n = 1$ the single node graph with no edges is trivially outerplanar and 3-colorable.

Inductive step: Assume $P(n)$ holds and let G_{n+1} be an outerplanar graph with $n+1$ vertices. There must exist a vertex v in G_{n+1} with degree at most 2. Removing v and all its incident edges leaves a subgraph G_n with n vertices.

Since G_{n+1} could be drawn with its vertices on a circle and its edges drawn as straight lines without intersections, any subgraph can also be drawn in such a way and so G_n is also an outerplanar graph. $P(n)$ implies G_n is 3-colorable. Therefore we can color all the vertices in G_{n+1} other than v using only 3 colors and since $\deg(v) \leq 2$ we may color it a color different than the vertices adjacent to it using only 3 colors. Therefore, G_{n+1} is 3-colorable and $P(n+1)$ holds.

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