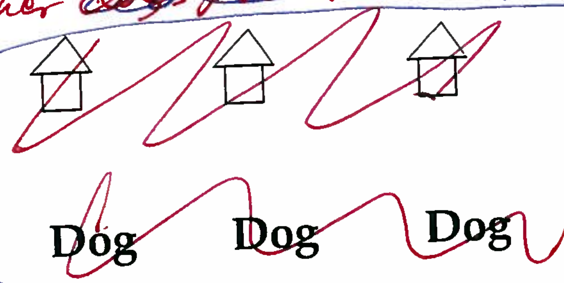
~~Chapter 11~~

5, 8 Planar Graphs

5.8.1 ~~11.1~~ Drawing Graphs in the Plane

Suppose there are three dogs and three houses ~~Here are three dogs and three houses.~~ as shown in Figure DP. Can you find a route ~~a path~~ from each dog to each ~~Is there a way for each dog to walk to~~ ~~the~~ house such that no ~~dog's path~~ route crosses any other ~~dog's path~~ route?



Keep this as is

Figure DP: Three dogs and three houses. Is there a ~~path~~ route from each dog to each house so that no pair of the 9 routes cross each other?

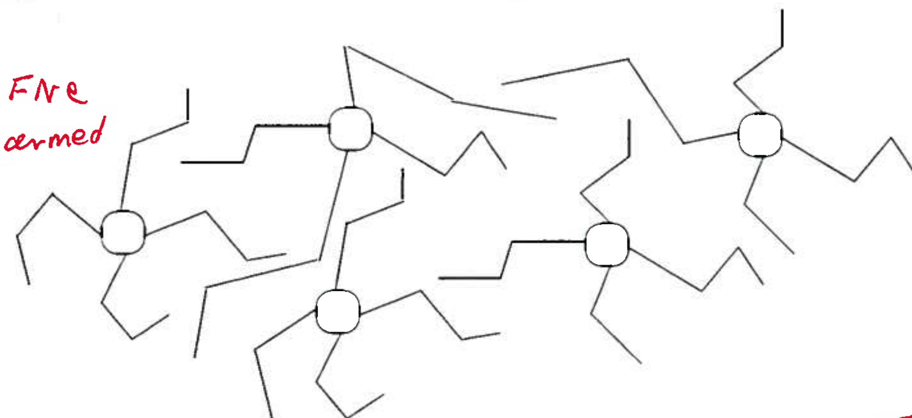
no new ¶

Can you find a path ~~from each~~ dog to each ~~house such that~~ no two ~~paths intersect?~~

A *quadapus* is a little-known animal similar to an octopus, but with four arms.

~~Here~~ Suppose there are five quadapi resting on the seafloor~~:~~, as shown in Figure DA.

Figure DA: Five quadapi (4-armed creatures).



Can each quadapus simultaneously shake hands with every other in such a way that no arms cross? INSERT DB goes here

~~Informally, a~~ A *planar graph* is a graph ~~that can be drawn in~~ the plane so ~~that no~~ nodes or edges overlap ~~and so that no edges cross each other~~ ¶ ~~edges cross, as in a map of showing the borders of countries or states.~~ Thus, these two puzzles are asking whether the graphs ~~below~~ in Figure DB are planar; that is, whether they can be redrawn so that no edges cross. The first graph is called the *complete bipartite graph*, $K_{3,3}$, and the second is K_5.

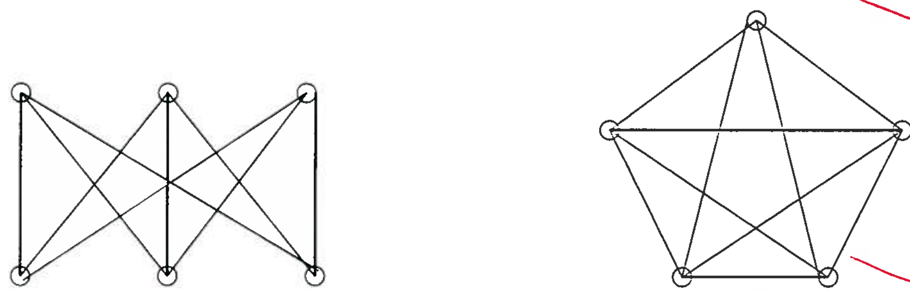
INSERT PB

Definition 5.1 A *planar graph* is a graph that can be drawn in the plane so that no nodes or edges overlap and so that no edges cross each other. ~~By a~~ A *drawing of a graph in the plane* ~~(a.k.a. embedding)~~ ~~are~~ consists of an assignment of vertices ~~vertices to~~ ~~each vertex~~ to ~~a~~ distinct points in the plane ~~and~~ and an assignment of edges to smooth, non-self-intersecting curves in the plane ~~where~~ ~~each curve connect~~ ~~whose~~ ~~whose endpoints~~ ~~each such curve are the nodes incident to the edges~~ ~~where~~ ~~that~~ (whose endpoints are the nodes incident to the edge). ~~a planar embedding~~ The drawing is *planar* (i.e., it is a *planar drawing*) ~~a planar embedding~~ if none of the curves "cross" — i.e., the only points that appear on more than one curve are the vertex points.

~~a.k.a., planar embedding~~

~~Definition 5.1 is a mouthful!~~

we have illustrated planar drawings for each ~~graph~~ resulting graph in Figure DC.

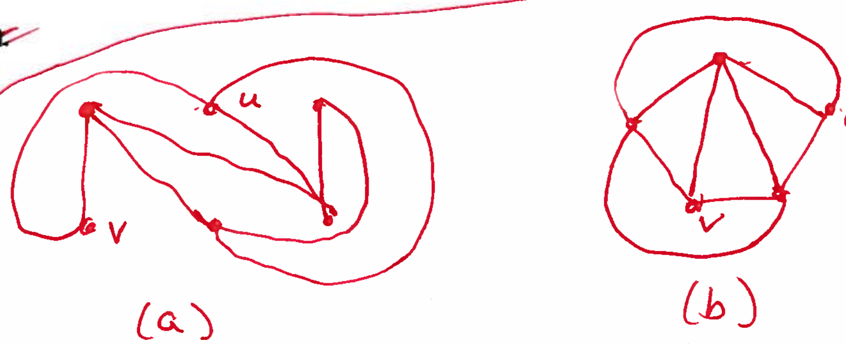


(a) (b)

Figure DB: ~~The graph~~ $K_{3,3}$ (a) and K_5 (b). Can you redraw these graphs so that no pair of edges cross?

In each case, the answer is, "No— but almost!" In fact, ~~each drawing *would* be~~ if you remove an edge from either of them, then the resulting graphs ~~possible if any single edge were removed.~~ can be redrawn in the plane so that no edges cross. For example, ~~see Figure DC. Such drawings are called planar drawings~~.

Planar ~~graphs~~ drawings have applications in circuit layout and are helpful in displaying graphical data, ~~for example,~~ such as program flow charts, organizational charts, and scheduling conflicts. ~~We will treat them as a recursive data type and use structural induction to establish their basic properties. Then we'll be able to describe a simple recursive procedure to color any planar graph with *five* colors, and also prove that there is no uniform way to place n satellites around the globe unless $n = 4, 6, 8, 12,$ or 20.~~ For these applications, the goal is to draw the graph in the plane with as few edge crossings as possible. (See the box on the following page for one such example.)



(a) (b)

Figure DC: Planar drawings of $K_{3,3} - \{u,v\}$ (a) and $K_5 - \{u,v\}$ (b).

When wires are arranged on a surface, like a circuit board or microchip, crossings require troublesome three-dimensional structures. When Steve Wozniak designed the disk drive for the early Apple II computer, he struggled mightily to achieve a nearly planar design:

> For two weeks, he worked late each night to make a satisfactory design. When he was finished, he found that if he moved a connector he could cut down on feedthroughs, making the board more reliable. To make that move, however, he had to start over in his design. This time it only took twenty hours. He then saw another feedthrough that could be eliminated, and again started over on his design. "The final design was generally recognized by computer engineers as brilliant and was by engineering aesthetics beautiful. Woz later said, 'It's something you can only do if you're the engineer and the PC board layout person yourself. That was an artistic layout. The board has virtually no feedthroughs.'"[a]

[a] From apple2history.org which in turn quotes *Fire in the Valley* by Freiberger and Swaine.

— INSERT PD goes here —

~~11.2 Continuous & Discrete Faces~~

~~Planar graphs are graphs that can be drawn in the plane — like familiar maps of countries or states.~~ "Drawing" ~~the~~ a graph in the plane requires ~~means~~ that each vertex of the graph corresponds to a distinct point in the plane, and if two vertices are adjacent, their vertices are connected by a smooth, non-self-intersecting curve. The drawing is planar if none ~~None~~ of the curves ~~may~~ "cross" —i.e., the only points that may appear on more than one curve are the vertex points. ~~These curves are the boundaries of connected regions of the plane called the *continuous faces* of the drawing.~~

~~For example~~, the drawing in Figure 11.1 has four continuous faces. Face IV,

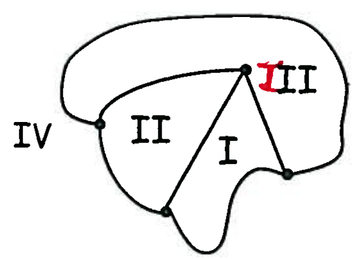


Figure 11.1: A Planar Drawing with Four Faces.

5.8.2 A Recursive Definition for Planar Graphs

Definition 5.1 is perfectly precise but has the challenge ~~that~~ that it requires us to work with concepts such as a "smooth curve" when trying to prove results about planar graphs. The trouble is that ~~to~~ we have not really laid the groundwork ~~to~~ from geometry and topology to ~~be ab~~ be able to reason carefully ~~about~~ about such ~~continuous~~ concepts. For example, we haven't really defined what it means for a curve to be smooth— we just drew a simple picture ~~(Figure~~ (e.g.) Figure DC) and hoped that you would get the idea.

~~Working with smooth curves and continuous regions is~~ possible ~~doable, but is not really something that we have prepared for in this text. For example, we haven't even defined what it really means for a curve to be smooth or what it means for a region to be continuous — we just drew a simple picture and hoped that you would get the idea.~~

Relying on pictures to convey new concepts is generally not a good idea and can sometimes lead to disaster (or, at least, false proofs). Indeed, it is because of this issue that there have been so many false proofs relating to planar graphs over time.[1] ~~The~~ Such proofs usually ~~push~~ rel~~ies~~y way too heavily on pictures and ~~see~~ ~~have~~ have way too many statements like,

David: center the quote

"As you can see from Figure ABC, it must be that property XYZ holds for all planar graphs."

This is not good.

1 XYZ's false proof of the 4-color theorem for planar graphs is not the only example. ~~The literature is filled~~

~~In order to avoid these the difficulties.~~

The good news is that there is another way to define planar graphs ~~that~~ ~~that uses only~~ that uses only discrete mathematics; In particular, we can define planar graphs as a recursive data type. In order to understand how it works, we first need to understand the concept of a face in a planar drawing.

Faces ← subsubsection (no #)

~~When a~~ In ~~any~~ a planar drawing of a graph, the curves corresponding to the edges ~~form the boundaries~~ divide up the plane ~~of the graph~~ into connected regions. These regions are called ~~faces~~ ~~the~~ the continuous faces[1] of the drawing. For example,

1 Most texts drop the word continuous from the definition of a face. We need it to differentiate ~~this definition~~ the connected region in the plane from the closed walk in the graph that bounds the region, which we will call a discrete face.

~~to working with clue to completely, working with smooth curves and continuous regions~~

which extends off to infinity in all directions, is called the *outside face*.

~~This definition~~ of a drawing planar ~~graphs~~ is ~~perfectly precise~~, but also a little ~~completely~~ ~~unsatisfying~~ in that ~~it invokes smooth curves and continuous regions of the plane to define a~~ graph, which is a ~~property of a discrete data type. So the first thing we'd like to find is a discrete data type that represents planar drawings.~~ ~~Given that this is a discrete math book, it is only natural that we would try to find a discrete~~ data type to ~~represent~~ planar drawings. — ~~INSERT DC goes here~~

~~The clue to how to do this is to~~ Notice that the vertices along the boundary of each of the faces in Figure 11.1 form a ~~simple~~ cycle. For example, labeling the vertices as in Figure 11.2, the ~~simple~~ cycles for the face boundaries are

abca abda bcdb acda. → (Eqn DA)

~~These four cycles are correspond to the nicely to the four continuous faces. Since they are a discrete data type,~~ ~~Since every edge in the drawing appears on the boundaries of exactly two continuous faces, every edge of the simple graph appears on exactly two of the simple cycles.~~ we refer to them as the discrete faces of the drawing in Figure 11.1. If it were always this simple, we would ~~be done~~ have ~~found~~ a discrete way to represent a planar drawing.

~~Vertices around the boundaries of states and countries in an ordinary map are always simple cycles, but oceans are slightly messier. The ocean boundary is the set of all boundaries of islands and continents in the ocean; it is a *set* of simple cycles~~

— INSERT DE goes here —

INSERT DE

These four cycles correspond nicely to the four continuous faces in Figure 11.2. So nicely, in fact, that we can identify each of the ~~four~~ faces in Figure 11.2 by its cycle. For example, the cycle abca identifies ~~the~~ face III. ~~So~~ Hence, ~~we can just call~~ we say that the cycles in ~~Eq~~ (Eqn DA) are the <u>discrete faces</u> of the graph in Figure 11.2. We use the term "discrete" since cycles in a graph are a discrete data type (as opposed to a ~~[illegible]~~ ~~subset~~ region in the plane, ~~[illegible]~~ which is a continuous data type).

Unfortunately, continuous faces ~~are~~ in planar drawings are not always bounded by cycles in the graph — things can get a little more complicated. For example, consider the planar drawing in Figure 11.3. This graph has ~~a~~ what we will call a <u>bridge</u> (namely, the edge ~~[illegible]~~ (—e) and the outer face is

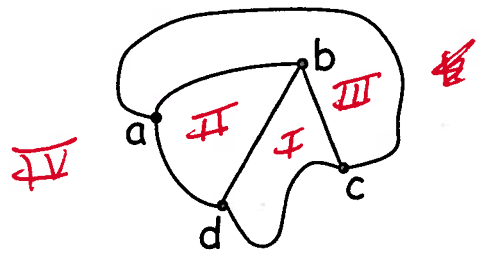


Figure 11.2: The Drawing with Labelled Vertices.

(this can happen ~~for countries too —like Bangladesh). But this happens~~ because islands (and the two parts of Bangladesh) are not connected to each other. So we can dispose of ~~this complication by treating each connected component~~ separately.

But general planar graphs, even when they are connected, may be a bit more complicated than ~~maps~~. For example a planar graph may have a "bridge," as in ~~Figure 11~~ the graph in Figure 16.1. ~~Figure 11.3.~~ Now ~~the cycle around~~ the outer face is closed walk ~~there is no cycle~~

abcefgecda.

This is not a ~~simple~~ cycle, since it has to traverse the bridge c—e twice, but it is a closed walk.

~~Planar graphs may~~ also have ~~"dongles," as in Figure 11.4. Now the cycle~~

INSERT DF goes here

INSERT DF

As another example, consider the ~~graph~~ planar drawing in Figure 11.4. This graph has what we will call a <u>dangle</u> (namely the nodes ~~x,~~ v, x, y, z and w, and the edges incident to them) and the inner face is

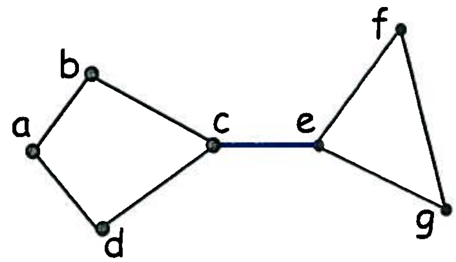


Figure 11.3: A Planar Drawing with a ~~B~~ b*ridge*; ~~Se~~ namely the edge {c,e}.

~~around the inner face is~~

`rstvxyxvwvtur.`

This is not a cycle because it has to traverse *every* edge of the dongle twice —once "coming" and once "going," but once again, it is a closed walk.

~~The good news is that~~
~~But bridges and dongles are really the only complications,~~ ~~(at least for connected graphs),~~ ~~which leads us to~~

INSERT DG goes here

the discrete data type of *planar embeddings* that we can use in place of continuous planar drawings. Namely, we'll define a planar embedding recursively to be the set of boundary-tracing ~~cycles~~ closed walks we could get by drawing one edge after another.

INSERT DG

~~The good news is th~~

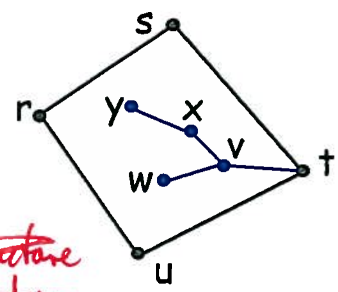
It turns out that bridges and dongles are ~~really~~ the only complications, at least for connected graphs. In particular, every ~~face~~ continuous face in a planar drawing corresponds ~~nicely~~ to a closed walk in the graph. ~~This means that we can define the plan~~ ~~association will allow us to~~ ~~characterize (and hence define) the a planar drawing in terms of its the closed walks bounding the continuous faces.~~

We ~~call~~ refer to such closed walks as the discrete faces of the drawing. ~~The association between the continuous faces of a draw planar drawing and closed walks will allows us to~~

~~The~~ A Recursive Definition for Planar Embeddings

subsubsection (no #)

The association between the continuous faces of a planar drawing and closed walks ~~will~~ will allow us to characterize a planar drawing in terms of the closed walks that bound the continuous faces. In particular, it leads us to



~~a Recursive Data Structure~~ ~~Using Structural Induction~~ ~~to Define Planar embeddings~~

Figure 11.4: A Planar Drawing with a ~~B~~dongle; namely the nodes v, x, y, z and the edges incident to them.

~~8.8.7~~ ~~11.3 Planar Embeddings~~ ~~as a Recursive~~

~~By thinking of the process of drawing a planar graph edge by edge, we can give a useful recursive definition of planar embeddings.~~

Definition 11.3.1. A *planar embedding* of a *connected* graph consists of a nonempty set of ~~cycles~~ closed walks of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

- **Base case:** If G is a graph consisting of a single vertex, v, then a planar embedding of G has one discrete face, namely the length zero ~~cycle~~ closed walk, v.

- **Constructor Case:** (split a face) Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face, γ, of the planar embedding. That is, γ is a ~~cycle~~ closed walk of the form

$$a \dots b \cdots a.$$

Then the graph obtained by adding the edge a—b to the edges of G has a planar embedding with the same discrete faces as G, except that face γ is replaced by the two discrete faces[1]

$$a \dots ba \quad \text{and} \quad ab \cdots a,$$

as illustrated in Figure 11.5.

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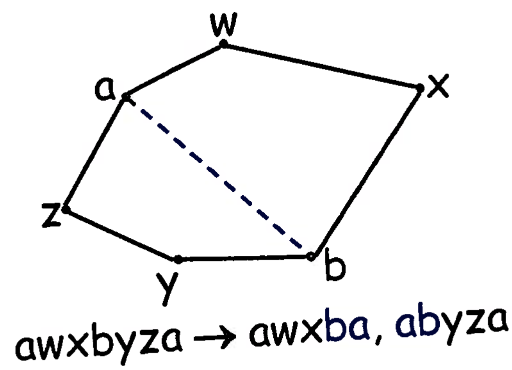
[1] There is ~~one exception to~~ a special case of this rule. If G is a line graph beginning with a and ending with b, then the cycles into which γ splits are actually the same. That's because adding edge a—b creates a simple cycle graph, C_n, that divides the plane into an "inner" and an "outer" region with the same border. In order to maintain the correspondence between continuous faces and discrete faces, we have to allow two "copies" of this same cycle to count as discrete faces. ~~But since this is the only situation in which two faces are actually the same cycle, this exception is better explained in a footnote than mentioned explicitly in the definition.~~



"split a face" case.

Figure 11.5: The ~~Split a Face Case.~~

- **Constructor Case:** (add a bridge) Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Let a be a vertex on a discrete face, γ, in the embedding of G. That is, γ is of the form

$$a \ldots a.$$

Similarly, let b be a vertex on a discrete face, δ, in the embedding of H, so δ is of the form

$$b \cdots b.$$

Then the graph obtained by connecting G and H with a new edge, a—b, has

INSERT D~~E~~H

Does it Work? ← sub ~~se~~ subsection

Yes! ~~Ce~~ In general, a graph is planar if and only if each of its connected components has a planar embedding as defined in Definition 11.3.1. Unfortunately, proving this fact requires a bunch of mathematics that we ~~won~~ don't cover in this text — stuff like geometry and topology. Of course, that is why we went to the trouble of ~~dealin~~ ~~a defining planar~~ ~~embed~~ including Definition 11.3.1 — we don't want to deal with that stuff ~~here~~ ~~to~~ in this text and ~~by~~ now that we have a recursive definition for planar graphs, we won't need to. That's the good news.

The bad news is that Definition 11.3.1 looks alot more complicated than the intuitively simple notion of a ~~planar gr~~ drawing where edges don't cross. It seems like it would be easier to stick to the simple notion and give proofs using pictures. Perhaps ~~this is~~ so, but ~~it will~~ ~~also~~ your proofs are more likely to be complete a planar embedding whose discrete faces are the union of the discrete faces of G and H, except that faces γ and δ are replaced by one new face

$$a \ldots ab \cdots ba.$$

This is illustrated in Figure 11.6, where the faces of G and H are:

$$G : \{\texttt{axyza, axya, ayza}\} \qquad H : \{\texttt{btuvwb, btvwb, tuvt}\},$$

and after adding the bridge a—b, there is a single connected graph with faces

$$\{\texttt{axyzabtuvwba, axya, ayza, btvwb, tuvt}\}.$$

INSERT DH goes here

A: ~~In general a~~

~~An arbitrary graph is *planar* iff each of its connected components has a planar embedding.~~ ~~as defined as described in Definition 11.3.1.~~

~~Proving that Definition 11.3.1 produces exactly any and all graphs that can be drawn in the plane without crossing edges, so we won't include it here.~~ However

~~11.4 What outer face?~~

Notice that the recursive definition of a planar embedding does not distinguish an "outer" face. There really isn't any need to distinguish one.

~~Since the some topology is required since we need to formally tie the discrete notion of a face to the continuous notion of a face.~~

and correct if you work from the discrete ~~d~~ Definition 11.3.1 instead of the continuous Definition 5.1.

Where Did the Outer Face Go? ← subsubsection

Every planar drawing has an immediately-recognizable outer face — it's the one that goes to infinity in all directions. But where is the outer face in ~~the~~ a planar embedding?

There isn't one! That's because there really isn't any need to distinguish one. In fact,

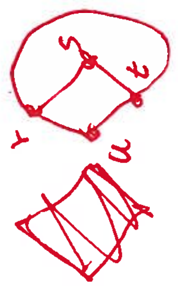
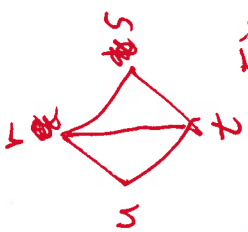



Figure DE: Two illustrations of the same embedding.

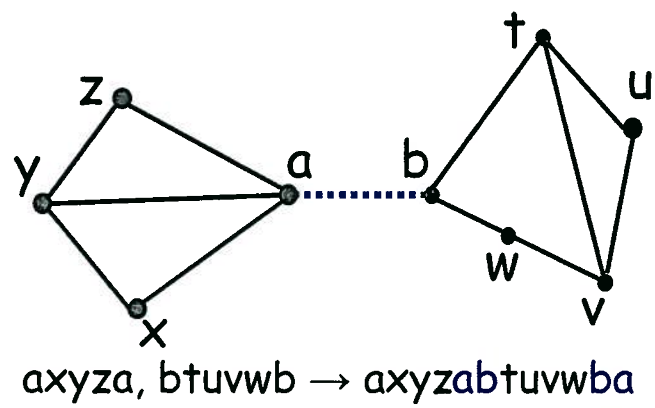


Figure 11.6: The ~~Add Bridge Case.~~ "add a bridge" case.

~~In fact~~, a planar embedding could be drawn with any given face on the outside. An intuitive explanation of this is to think of drawing the embedding on a *sphere* instead of the plane. Then any face can be made the outside face by "puncturing" that face of the sphere, stretching the puncture hole to a circle around the rest of the faces, and flattening the circular drawing onto the plane.

So pictures that show different "outside" boundaries may actually be illustrations of the same planar embedding. For example, the two embeddings shown in Figure DE are really the same.

This is what justifies the "add bridge" case in a planar embedding: whatever face is chosen in the embeddings of each of the disjoint planar graphs, we can draw a bridge between them without needing to cross any other edges in the drawing, because we can assume the bridge connects two "outer" faces.

5.8.3 ~~11.5~~ Euler's Formula

The value of the recursive definition is that it provides a powerful technique for proving properties of planar graphs, namely, structural induction. ~~The~~ For example, we ~~can~~ will now use Definition 11.3.1 and structural induction to establish one ~~One~~ of the most basic properties of a connected planar graph, ~~is that its num-~~ ~~ber~~ namely that the number of vertices and edges completely determines the number of faces in every possible planar embedding~~s~~ of the graph.

Theorem 11.5.1 (Euler's Formula). *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges, and f is the number of faces.

For example, in Figure 11.1, $|V| = 4$, $|E| = 6$, and $f = 4$. Sure enough, $4 - 6 + 4 =$

2, as Euler's Formula claims.

Proof. The proof is by structural induction on the definition of planar embeddings. Let $P(\mathcal{E})$ be the proposition that $v - e + f = 2$ for an embedding, $\mathcal{E}$.

Base case: ($\mathcal{E}$ is the one vertex planar embedding). By definition, $v = 1$, $e = 0$, and $f = 1$, so $P(\mathcal{E})$ indeed holds.

Constructor case: (split a face) Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face, $\gamma = a \ldots b \cdots a$, of the planar embedding.

Then the graph obtained by adding the edge a—b to the edges of G has a planar embedding with one more face and one more edge than G. So the quantity $v - e + f$ will remain the same for both graphs, and since by structural induction this quantity is 2 for G's embedding, it's also 2 for the embedding of G with the added edge. So P holds for the constructed embedding.

Constructor case: (add bridge) Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Then connecting these two graphs with a bridge merges the two bridged faces into a single face, and leaves all other faces unchanged. So the bridge operation yields a planar embedding of a connected graph with $v_G + v_H$ vertices, $e_G + e_H + 1$ edges, and $f_G + f_H - 1$ faces.

since

$$
\begin{aligned}
&(v_G + v_H) - (e_G + e_H + 1) + (f_G + f_H - 1) \\
&= (v_G - e_G + f_G) + (v_H - e_H + f_H) - 2 \\
&= (2) + (2) - 2 && \text{(by structural induction hypothesis)} \\
&= 2.
\end{aligned}
$$

So $v - e + f$ remains equal to 2 for the constructed embedding. That is, P also holds in this case.

This completes the proof of the constructor cases, and the theorem follows by structural induction. ■

Bounding the Number of Edges in a Planar Graph

~~More Identities~~

5.8.~~6~~4 11.6 ~~Number of Edges versus Vertices~~

~~Bounding the Number of Edges in a Planar Graph~~

Like Euler's formula, the following lemmas follow by structural induction directly from the ~~definition of planar embedding~~ Definition 11.3.1.

Lemma 11.6.1. *In a planar embedding of a connected graph, each edge is traversed once by each of two different faces, or is traversed exactly twice by one face.*

Lemma 11.6.2. *In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.*

Combining Lemmas 11.6.1 and 11.6.2 with Euler's Formula, we can now prove that planar graphs ~~cannot have too many more edges than vertices~~ have a limited number of edges:

~~Corollary~~ Theorem 11.6.3. *Suppose a connected planar graph has $v \geq 3$ vertices and e edges. Then*

$$e \leq 3v - 6.$$

Proof. By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with v vertices and e edges has a planar embedding with f faces. By Lemma 11.6.1, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Lemma 11.6.2, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{11.1}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (11.1) gives

$$3(e - v + 2) \leq 2e$$

$$e - 3v + 6 \leq 0$$

$$e \leq 3v - 6$$

■

5.8.5 Returning to K_5 and $K_{3,3}$

#

Theorem ~~Corollary~~ 11.6.3 lets us prove that the quadapi can't all shake hands without crossing. Representing quadapi by vertices and the necessary handshakes by edges, we get the complete graph, K_5. Shaking hands without crossing amounts to showing that K_5 is planar. But K_5 is connected, has 5 vertices and 10 edges, and $10 > 3 \cdot 5 - 6$. This violates the condition of Theorem ~~Corollary~~ 11.6.3 required for K_5 to be planar, which proves

Corollary ~~Lemma~~ 11.6.4. *K_5 is not planar.*

— INSERT DI goes here —

~~5.8.5 Returning to K_5 and $K_{3,3}$~~

~~We can also~~

We can also use Euler's Formula to show that $K_{3,3}$ is not planar. The proof is similar to that of Theorem 11.6.3 except that we use the additional fact that $K_{3,3}$ is a bipartite graph.

Lemma D5: ~~In a bip~~ Every closed walk in a bipartite graph ~~has~~ has even length.

Proof: ~~A~~ A bipartite graph $G=(V,E)$ is defined by the property that the nodes are partitioned into two sets L and R where every edge connects ~~a~~ a node in L to a node in R. Hence, any closed walk in G must alternate between a node in L followed by a node in R. Since a closed walk ends on the same node it started with, it must ~~an an~~ visit ~~an edge~~ nodes in L equally as often as it visits nodes in R. Hence it must have even length. ■

Corollary D6: In a planar embedding of a connected bipartite ~~If G is a bipartite~~ graph with at least 3 vertices, ~~nodes~~ each ~~then every~~ face of ~~G~~ has length at least 4.

Proof: By Lemma 11.6.2, every face has length 3. Since ~~the~~ the graph is bipartite and since each face is a closed walk, Lemma D5 implies that ~~the~~ no faces ~~cannot have~~ can have length 3. Hence, every face must have length at least 4. ∎

Theorem D6: $K_{3,3}$ is not planar.

Proof: By contradiction. Assume $K_{3,3}$ is planar. ~~By~~ and consider any planar embedding of $K_{3,3}$. ~~By Euler's Formula~~ with f faces. Arguing as in the proof of Lemma 11.6.3, where ~~we derived equation (1)~~ ~~we know~~ (but using Lemma D6 in place of Lemma 11.6.2 since $K_{3,3}$ is bipartite), we find that the sum of the lengths of the face boundaries is exactly $2e$ and ~~that the sum~~ is at least $4f$. ~~$4f \le 2e$~~

Hence,

$$4f \le 2e$$

for any bipartite graph. Plugging in $e = 9$ and $v = 6$ for $K_{3,3}$ in Euler's formula, we find that

$$f = 2 + e - v$$
$$= 5.$$

But ~~&~~

$$4 \cdot 5 \not\leq 2 \cdot 9,$$

and so we have a contradition. Hence $K_{3,3}$ must not be planar.

5.8.6 Another ~~Characterization~~ Characterization for ~~of~~ Planar Graphs

~~$K_{3,3}$ and K_5~~

We did not choose to pick on K_5 and $K_{3,3}$ because of ~~the~~ ~~well-known~~ ~~well-known challenges~~ their applications to ~~get~~ dogs getting ~~to~~ ~~get the~~ ~~dog house~~ home or quadrapi shaking hands. Rather, we ~~s~~ selected these graphs as examples because they provide another way to characterize ~~all~~ the set of planar graphs, ~~In particular, if a~~ as follows.

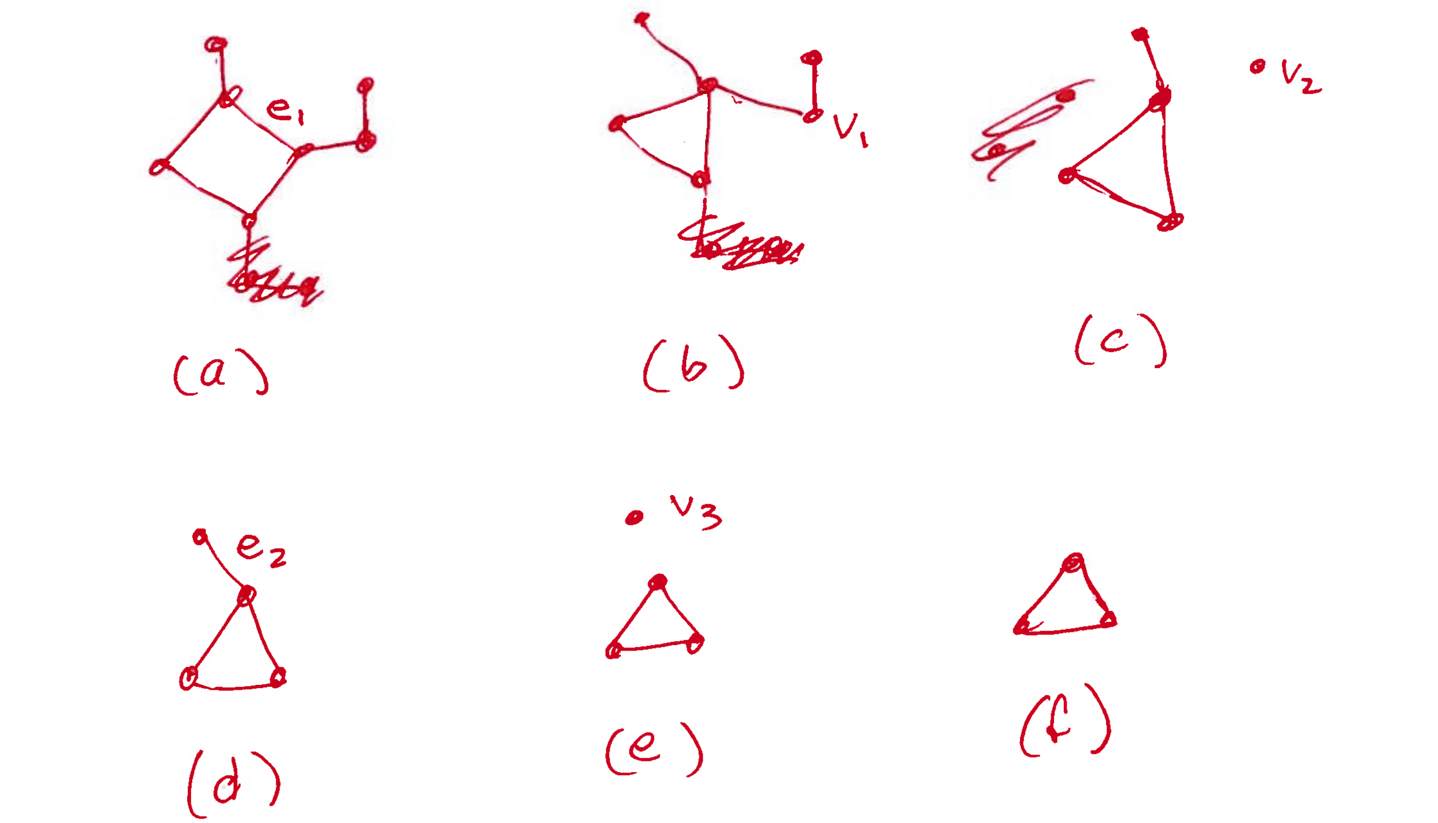
~~graph is not~~

Theorem 11.8.4 (Kuratowski) A graph is not planar if and only if it contains K_5 or $K_{3,3}$ as a minor.

Definition: A minor of a graph G is a graph that can be obtained by repeatedly[1] deleting vertices, deleting edges, and merging adjacent vertices of G. — INSERT DJ goes here — (text on p 514)

1 The three operations can be performed in any order and ~~as we do not~~ in any quantities, ~~including 0 times~~, or not at all.

For example, Figure DL illustrates why ~~K_3~~ C_3 is a minor of the graph in Figure DL(a). In fact C_3 ~~K_3~~ is a minor of a connected graph G if and only if G is not a tree.



<u>Figure DL</u>: ~~The process~~ one method by which the graph in ~~Figure~~ (a) ~~be~~ can be reduced to ~~K_3~~ C_3 (f), thereby showing that C_3 is a minor of the graph. The steps are: merging the nodes incident to e_1 (b), deleting v_1 and all edges incident to it (c), deleting v_2 (d), deleting e_2 (e), and deleting v_3 (f).

¶ ~~We will not prove Theorem 11.8.4 here~~

¶ We will not prove Theorem 11.8.4 here, nor will we prove the following ~~useful~~ handy facts, ~~some~~ ~~of~~ which are obvious given the definition of a planar drawing from section 5.1, and ~~and all of~~ which can be proved using ~~from~~ the recursive definition of a planar embeddings from section 5.2.

~~Theorem 11.7.5~~

This is INSERT DZ and goes into INSERT DL

~~Another consequence is~~

~~Corollary~~ Lemma

~~Lemma~~ 11.6.5. *Every planar graph has a vertex of degree at most five.*

Proof. By contradiction. If every vertex had degree at least 6, then the sum of the vertex degrees is at least $6v$, but since the sum of the vertex degrees equals $2e$, by the Handshake Lemma (Lemma ??), we have $e \geq 3v$ contradicting the fact that $e \leq 3v - 6 < 3v$ by ~~Corollary~~ Theorem 11.6.3. ■

INSERT DI goes here (it is old section 11.9) on pp 518-521

~~11.7 Planar Subgraphs~~

~~If you draw a graph in the plane by repeatedly adding edges that don't cross, you clearly could add the edges in any other order and still wind up with the same drawing. This is so basic that we might presume that our recursively defined planar embeddings have this property. But that wouldn't be fair: we really need to prove it. After all, the recursive definition of planar embedding was pretty technical —maybe we got it a little bit wrong, with the result that our embeddings don't have this basic draw-in-any-order property.~~

~~Now any ordering of edges can be obtained just by repeatedly switching the~~

order of successive edges, and if you think about the recursive definition of embedding for a minute, you should realize that you can switch *any* pair of successive edges if you can just switch the last two. So it all comes down to the following lemma.

Lemma 11.7.1. *Suppose that, starting from some embeddings of planar graphs with disjoint sets of vertices, it is possible by two successive applications of constructor operations to add edges e and then f to obtain a planar embedding, $\mathcal{F}$. Then starting from the same embeddings, it is also possible to obtain $\mathcal{F}$ by adding f and then e with two successive applications of constructor operations.*

We'll leave the proof of Lemma 11.7.1 to Problem ??.

Corollary 11.7.2. *Suppose that, starting from some embeddings of planar graphs with disjoint sets of vertices, it is possible to add a sequence of edges $e_0, e_1, \ldots, e_n$ by successive applications of constructor operations to obtain a planar embedding, $\mathcal{F}$. Then starting from the same embeddings, it is also possible to obtain $\mathcal{F}$ by applications of constructor* *operations that successively add any permutation*[2] *of the edges* $e_0, e_1, \ldots, e_n$.

Lemma

~~Fact~~

Keep **~~Corollary~~ 11.7.3.** *Deleting an edge from a planar graph leaves a planar graph.*

Proof. By Corollary 11.7.2, we may assume the deleted edge was the last one added in constructing an embedding of the graph. So the embedding to which this last edge was added must be an embedding of the graph without that edge. ■

Since we can delete a vertex by deleting all its incident edges, Corollary 11.7.3 immediately implies

~~Fact~~ Corollary

~~Corollary~~ 11.7.4. *Deleting a vertex from a planar graph, along with all its incident edges ~~of course~~, leaves another planar graph.*

~~A *subgraph* of a graph, G, is any graph whose set of vertices is a subset of the vertices of G and whose set of edges is a subset of the set of edges of G. So we can summarize Corollaries 11.7.3 and 11.7.4 and their consequences in a Theorem.~~

Theorem 11.7.5. *Any subgraph of a planar graph is planar.*

~~[2] If $\pi : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ is a bijection, then the sequence $e_{\pi(0)}, e_{\pi(1)}, \ldots, e_{\pi(n)}$ is called a *permutation* of the sequence $e_0, e_1, \ldots, e_n$.~~

~~11.8 Planar 5-Colorability~~

~~We need to know one more property of planar graphs in order to prove that planar graphs are 5-colorable.~~

Theorem

~~Lemma~~ 11.8.1. *Merging two adjacent vertices of a planar graph leaves another planar graph.*

This is text Insert DJ and goes to page DI-3.

Here merging two adjacent vertices, n_1 and n_2 of a graph means deleting the two vertices and then replacing them by a new "merged" vertex, m, adjacent to all the vertices that were adjacent to either of n_1 or n_2, as illustrated in Figure 11.7.

~~Lemma 11.8.1 can be proved by structural induction, but the proof is kind of boring, and we hope you'll be relieved that we're going to omit it. (If you insist, we can add it to the next problem set.)~~

INSERT DL goes here

~~Now we've got all the simple facts we need to prove 5-colorability.~~

Theorem 11.8.2. *Every planar graph is five-colorable.*

Proof. The proof will be by strong induction on the number, v, of vertices, with induction hypothesis:

INSERT D2

5.8.7 Coloring Planer Graphs

~~We~~ We've ~~done~~ covered alot of ground with planer graphs, but not nearly enough to prove the famous 4-color theorem. ~~namely~~ But we can get awefully close. Indeed, we have done ~~just~~ almost enough work to prove that every planer graph can be colored using only 5 colors. We need only one more lemma:

— INSERT D3 goes here —
(it is the text in the box on p511)

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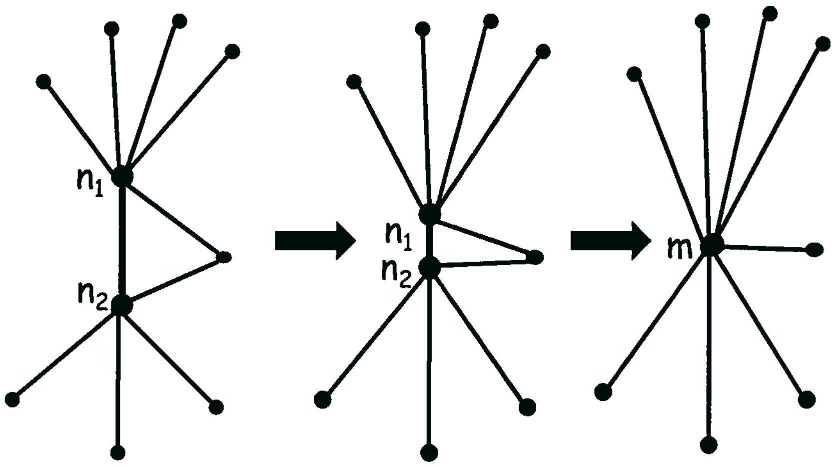

Figure 11.7: Merging adjacent vertices n_1 and n_2 into new vertex, m.

Every planar graph with v vertices is five-colorable.

Base cases ($v \leq 5$): immediate.

Inductive case: Suppose G is a planar graph with $v + 1$ vertices. We will describe a five-coloring of G.

First, choose a vertex, g, of G with degree at most 5; Lemma 11.6.5 guarantees there will be such a vertex.

Case 1 ($\deg(g) < 5$): Deleting g from G leaves a graph, H, that is planar by ~~Lemma~~ Corollary 11.7.4, and, since H has v vertices, it is five-colorable by induction hypothesis. Now define a five coloring of G as follows: use the five-coloring of H for all the vertices besides g, and assign one of the five colors to g that is not the same as the color assigned to any of its neighbors. Since there are fewer than 5 neighbors, there will always be such a color available for g.

Case 2 ($\deg(g) = 5$): If the five neighbors of g in G were all adjacent to each other, then these five vertices would form a nonplanar subgraph isomorphic to K_5, contradicting Theorem 11.7.5 (since K_5 is not planar). So there must be two neighbors, n_1 and n_2, of g that are not adjacent. Now merge n_1 and g into a new vertex, m. ~~as in Figure 11.7~~. In this new graph, n_2 is adjacent to m, and the graph is planar by ~~Lemma~~ Theorem 11.8.1. So we can then merge m and n_2 into a another new vertex, m', resulting in a new graph, G', which by ~~Lemma~~ Theorem 11.8.1 is also planar. ~~Now~~ Since G' has $v - 1$ vertices, ~~and so~~ it is five-colorable by the induction hypothesis.

~~Now d~~ Define a five coloring of G as follows: use the five-coloring of G' for all the vertices besides g, n_1 and n_2. Next assign the color of m' in G' to be the color of the neighbors n_1 and n_2. Since n_1 and n_2 are not adjacent in G, this defines a proper five-coloring of G except for vertex g. But since these two neighbors of g have the same color, the neighbors of g have been colored using fewer than five colors altogether. So complete the five-coloring of G by assigning one of the five colors to g that is not the same as any of the colors assigned to its neighbors.

■

A graph obtained from a graph, G, be repeatedly deleting vertices, deleting edges, and merging adjacent vertices is called a *minor* of G. Since K_5 and $K_{3,3}$ are

X

not planar, Lemmas 11.7.3, 11.7.4, and 11.8.1 immediately imply:

Corollary 11.8.3. *A graph which has K_5 or $K_{3,3}$ as a minor is not planar.*

We don't have time to prove it, but the converse of Corollary 11.8.3 is also true. This gives the following famous, very elegant, and purely discrete characterization of planar graphs:

Theorem 11.8.4 (Kuratowksi). *A graph is not planar iff it has K_5 or $K_{3,3}$ as a minor.*

~~11.9~~ 5.8.8 Classifying Polyhedra

The Pythagoreans had two great mathematical secrets, the irrationality of $\sqrt{2}$ and a geometric construct that we're about to rediscover!

A *polyhedron* is a convex, three-dimensional region bounded by a finite number of polygonal faces. If the faces are identical regular polygons and an equal number of polygons meet at each corner, then the polyhedron is *regular*. Three examples of regular polyhedra ~~are~~ shown ~~below~~ in Figure DP: the tetrahedron, the cube, and the octahedron.

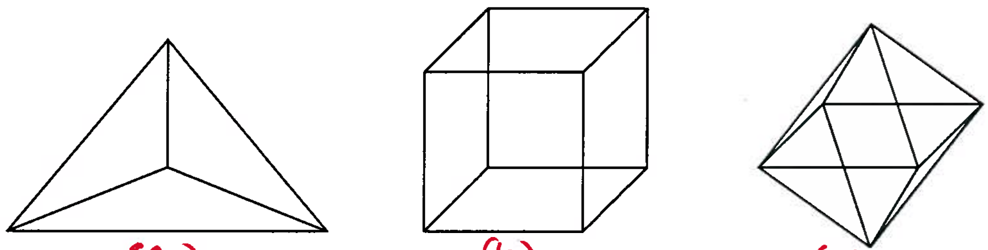

Figure DP: The tetrahedron (a), cube (b), and octahedron (c).

We can determine how many more regular polyhedra there are by thinking about planarity. Suppose we took *any* polyhedron and placed a sphere inside it. Then we could project the polyhedron face boundaries onto the sphere, which would give an image that was a planar graph embedded on the sphere, with the images of the corners of the polyhedron corresponding to vertices of the graph. X ~~But~~ We've already observed that embeddings on a sphere are the same as embeddings on the plane, so Euler's formula for planar graphs can help guide our search for regular polyhedra.

For example, planar embeddings of the three polyhedra ~~above look like this:~~ in Figure DP are shown in Figure DQ.

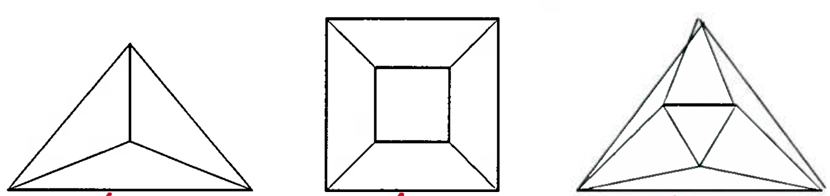

Figure DQ: Planar embeddings of the tetrahedron (a), cube (b), and octahedron (c).

Let m be the number of faces that meet at each corner of a polyhedron, and let

n be the number of ~~sides~~ edges on each face. In the corresponding planar graph, there are m edges incident to each of the v vertices. ~~Since each edge is incident to two vertices~~ By the Handshake Lemma ??, we know:

$$mv = 2e$$

Also, each face is bounded by n edges. Since each edge is on the boundary of two faces, we have:

$$nf = 2e$$

Solving for v and f in these equations and then substituting into Euler's formula gives:

$$\frac{2e}{m} - e + \frac{2e}{n} = 2$$

which simplifies to

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2} \tag{11.2}$$

~~This last e~~Equation (11.2) places strong restrictions on the structure of a polyhedron. Every nondegenerate polygon has at least 3 sides, so $n \geq 3$. And at least 3 polygons must meet to form a corner, so $m \geq 3$. On the other hand, if either n or m were 6 or more, then the left side of the equation could be at most $1/3 + 1/6 = 1/2$, which is less than the right side. Checking the finitely-many cases that remain turns up

as shown in Figure DR.

only five solutions. For each valid combination of n and m, we can compute the associated number of vertices v, edges e, and faces f. And polyhedra with these properties do actually exist.

n	m	v	e	f	polyhedron
3	3	4	6	4	tetrahedron
4	3	8	12	6	cube
3	4	6	12	8	octahedron
3	5	12	30	20	icosahedron
5	3	20	30	12	dodecahedron

Figure DR: The only possible regular polyhedra.

largest

The ~~last~~ polyhedron ~~in this list~~, the dodecahedron, was the other great mathematical secret of the Pythagorean sect. ~~These five, then, are the only possible regular polyhedra.~~

The 5 polyhedra in Figure DR are the only possible regular polyhedra.

So if you want to put more than 20 geocentric satellites in orbit so that they *uniformly* blanket the globe —tough luck!

5.9 ~~11.9.1~~ Problems

~~Exam Problems~~

~~Class Problems~~

~~Homework Problems~~