Solutions to Problem Set 6

Reading: Notes Ch. **??**– **??**; Ch. **??**

Problem 1.
An edge is said to *leave* a set of vertices if one end of the edge is in the set and the other end is not.

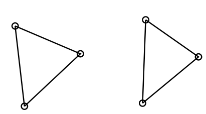
(a) An n-node graph is said to be *mangled* if there is an edge leaving every set of $\lfloor n/2 \rfloor$ or fewer vertices. Prove the following claim.
Claim. *Every mangled graph is connected.*

Solution. The proof is by contradiction. Assume for the purpose of contradiction that these exists an n-node graph that is mangled, but not connected. Then the graph must have at least two connected components. However, there can be at most one connected component of size more than $\lfloor n/2 \rfloor$, since $2(\lfloor n/2 \rfloor + 1) > n$. Therefore, there exists a connected component with $\lfloor n/2 \rfloor$ or fewer vertices. Since the graph is mangled, there is an edge leaving this component. But this contradicts the definition of a connected component. ■

An n-node graph is said to be *tangled* if there is an edge leaving every set of $\lceil n/3 \rceil$ or fewer vertices.

(b) Draw a tangled graph that is not connected.

Solution. A tangled but non-connected graph is the following:



■

(c) Find the error in the proof of the following
False Claim. *Every tangled graph is connected.*

False proof. The proof is by strong induction on the number of vertices in the graph. Let $P(n)$ be the proposition that if an n-node graph is tangled, then it is connected. In the base case, $P(1)$ is true because the graph consisting of a single node is trivially connected.

For the inductive case, assume $n \geq 1$ and $P(1), \ldots, P(n)$ hold. We must prove $P(n+1)$, namely, that if an $(n+1)$-node graph is tangled, then it is connected.

So let G be a tangled, $(n+1)$-node graph. Choose $\lceil n/3 \rceil$ of the vertices and let G_1 be the tangled subgraph of G with these vertices and G_2 be the tangled subgraph with the rest of the vertices. Note that since $n \geq 1$, the graph G has a least two vertices, and so both G_1 and G_2 contain at least one vertex. Since G_1 and G_2 are tangled, we may assume by strong induction that both are connected. Also, since G is tangled, there is an edge leaving the vertices of G_1 which necessarily connects to a vertex of G_2. This means there is a path between any two vertices of G: a path within one subgraph if both vertices are in the same subgraph, and a path traversing the connecting edge if the vertices are in separate subgraphs. Therefore, the entire graph, G, is connected. This completes the proof of the inductive case, and the Claim follows by strong induction.

□

Solution. The error is in the statement, "Let G_1 be the *tangled* subgraph ...." This makes the implicit assumption that a tangled graph can be split into tangled subgraphs, one of which is of size at most $\lceil n/3 \rceil$. This assumption is false. To see why, consider the counterexample given in part (b). That graph is tangled and $\lceil n/3 \rceil = 2$. But, no matter how we split the graph into two subgraphs of sizes either 2 and 4 or 1 and 5, one of the two is not tangled.

It's a common blunder to assume that a property of a graph is "inherited" by a subgraph. This is true for many familiar properties such as being k-colorable, having degrees bounded by a constant d, being planar, being no larger than the whole graph. But many other familiar are not inherited, for example, being a tree, being a simple cycle, requiring more than k colors, being nonplanar. ■

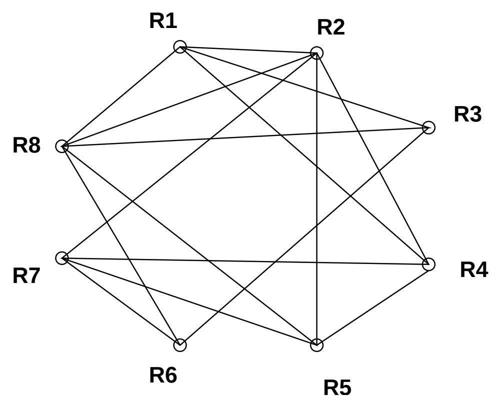
Problem 2.
6.042 is often taught using recitations. Suppose it happened that 8 recitations were needed, with two or three staff members running each recitation. The assignment of staff to recitation sections is as follows:

- R1: Eli, Jodyann, Rich
- R2: Eli, Megumi, Albert
- R3: Jodyann, Justin
- R4: Rajeev, Megumi, Steven
- R5: Rajeev, Tom, Albert
- R6: Tom, Justin
- R7: Tom, Megumi
- R8: Jodyann, Justin, Albert

Two recitations can not be held in the same 90-minute time slot if some staff member is assigned to both recitations. The problem is to determine the minimum number of time slots required to complete all the recitations.

(a) Recast this problem as a question about coloring the vertices of a particular graph. Draw the graph and explain what the vertices, edges, and colors represent.

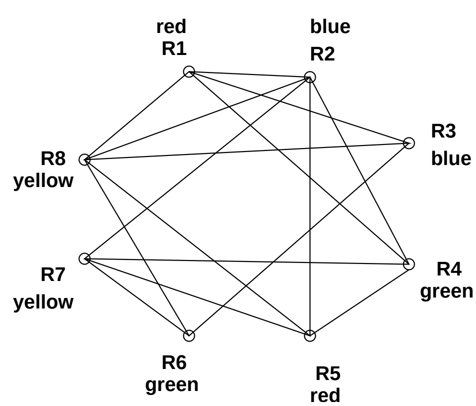
Solution. Each vertex in the graph below represents a recitation section. An edge connects two vertices if the corresponding recitation sections share a staff member and thus can not be scheduled at the same time. The color of a vertex indicates the time slot of the corresponding recitation.



■

(b) Show a coloring of this graph using the fewest possible colors. What schedule of recitations does this imply?

Solution. Four colors are necessary and sufficient. To see why they are *sufficient*, consider the coloring:



This corresponds to the following assignment of recitations to four time slots:

1. R1, R5
2. R2, R3
3. R4, R6
4. R7, R8

Other schedules are also possible.

To see why 4 colors are *necessary*, look at the subgraph defined by the vertices for R2, R4, R5, and R7. This is the complete graph on 4 vertices, and it obviously needs 4 colors. ■

Problem 3.
In this problem you will prove:

Theorem. *A graph G is 2-colorable iff it contains no odd length cycle.*

As usual with "iff" assertions, the proof splits into two proofs: part (a) asks you to prove that the left side of the "iff" implies the right side. The other problem parts prove that the right side implies the left.

(a) Assume the left side and prove the right side. Three to five sentences should suffice.

Solution. Assume G is 2-colorable and select a 2-coloring of G. Consider an arbitrary cycle with successive vertices $v_1, v_2, \ldots, v_k, v_1$. Then the vertices v_i must be one color for all even i and the other color for all odd i. (This is obvious, but could of course, be proved by induction.) Since v_1 and v_k must be colored differently, k must be even. Thus, the cycle has even length. ■

(b) Now assume the right side. As a first step toward proving the left side, explain why we can focus on a single connected component H within G.

Solution. If we can 2-color every connected component of G, then we can 2-color all of G. Thus, it suffices to show that an arbitrary connected component H of G is 2-colorable. ■

(c) As a second step, explain how to 2-color any tree.

Solution. A 2-coloring of a tree can be defined by selecting any fixed vertex v, and coloring a vertex one color if the (unique) path to it from v has odd length, and coloring it with the other color if the path has even length.

To verify that adjacent vertices in the tree get different colors, let $e ::= x\text{—}y$ be an edge in the tree. There is a unique path from v to x. If this path traverses e, it must consist of a path from v to y followed by the e traversal to x. If this path does not traverse e, then it can be extended to a path to y by adding a final traversal of e. In either case, the paths to these vertices from v differ by a single traversal of e, and so the lengths of the paths differ by 1; in particular, one is of odd length and the other is of even length, so x and y are differently-colored. ■

(d) Choose any 2-coloring of a spanning tree, T, of H. Prove that H is 2-colorable by showing that any edge *not* in T must also connect different-colored vertices.

Solution. Let $x\text{—}y$ be an edge not in T, and consider the unique paths from v to x and from v to y in T. Exactly one of these two paths must have odd length; otherwise, these two paths together with the edge $x\text{—}y$ would form an odd length cycle. But this means x and y are colored differently. ■

Problem 4.
Scholars through the ages have identified *twenty* fundamental human virtues: honesty, generosity, loyalty, prudence, completing the weekly 6.042 reading-response email, etc. At the beginning of the term, every student in 6.042 possessed exactly *eight* of these virtues. Furthermore, every student was unique; that is, no two students possessed exactly the same set of virtues. The 6.042 course staff must select *one* additional virtue to impart to each student by the end of the term. Prove that there is a way to select an additional virtue for each student so that every student is unique at the end of the term as well.

Suggestion: Use Hall's theorem. Try various interpretations for the vertices on the left and right sides of your bipartite graph.

Solution. Construct a bipartite graph G as follows. The vertices on on the left are all students and the virtues on the right are all subset of nine virtues. There is an edge between a student and a set of 9 virtues if the student already has 8 of those virtues.

Each vertex on the left has degree 12, since each student can learn one of 12 additional virtues. The vertices on the right have degree at most 9, since each set of 9 virtues has only 9 subsets of size 8. So this bipartite graph is degree-constrained, and therefore, by Lemma **??**, there is a matching for the students. Thus, if each student is taught the additional virtue in the set of 9 virtues with whom he or she is matched, then every student is unique at the end of the term. ■

Problem 5.
Consider a n-player round-robin tournament where every pair of distinct players compete in a single game that doesn't allow for a tie. We can model the results of such a tournament using a *tournament digraph* where the vertices correspond to players and there is an edge $x \rightarrow y$ if x beat y.

(a) Explain why there can be no cycles of length 1 or 2.

Solution. There are no self-loops in a tournament graph since no player plays himself, so no length 1 cycles. Also, it cannot be that x beats y and y beats x for $x \neq y$, since every pair competes exactly once and there are no ties. This means there are no length 2 cycles.

■

(b) Is the "beats" relation for a tournament graph is always/sometimes/never:

- asymmetric?
- reflexive?
- irreflexive?
- transitive?

Explain.

Solution. No self-loops implies the relation is irreflexive. It is also asymmetric since it is irreflexive and for every pair of distinct players, exactly one game is played and results in a win for one of the players. Some tournament graphs represent transitive relations and others don't. ■

(c) Show that a tournament graph represents a total order iff there are no cycles of length 3.

Solution. As observed in the previous part, the "beats" relation whose graph is a tournament is asymmetric and irreflexive. Since every pair of players is comparable, the relation is a total order iff it is transitive.

"Beats" is transitive iff for any players x, y and z, $x \to y$ and $y \to z$ implies that $x \to z$ (and consequently that there is no edge $z \to x$). Therefore, "beats" is transitive iff there are no cycles of length 3. ■