Notes for Recitation 11

1. Give a description of the equivalence classes associated with each of the following equivalence relations.

   (a) Integers x and y are equivalent if $x \equiv y \pmod 3$.

   Solution.
$$\{\ldots, -6, -3, 0, 3, 6, \ldots\}$$
$$\{\ldots, -5, -2, 1, 4, 7, \ldots\}$$
$$\{\ldots, -4, -1, 2, 5, 8, \ldots\}$$
■

   (b) Real numbers x and y are equivalent if $\lceil x \rceil = \lceil y \rceil$, where $\lceil z \rceil$ denotes the smallest integer greater than or equal to z.

   Solution. For each integer n, all the real numbers r such that $n - 1 < r \leq n$ form an equivalence class. ■

2. Show that neither of the following relations is an equivalence relation by identifying a missing property (reflexivity, symmetry, or transitivity).

   (a) The "divides" relation on the positive integers.

   Solution. This relation is reflexive (since $a \mid a$) and transitive (since $a \mid b$ and $b \mid c$ implies $a \mid c$), but not symmetric (since $3 \mid 6$, but not $6 \mid 3$). ■

   (b) The "implies" relation on propositional formulas.

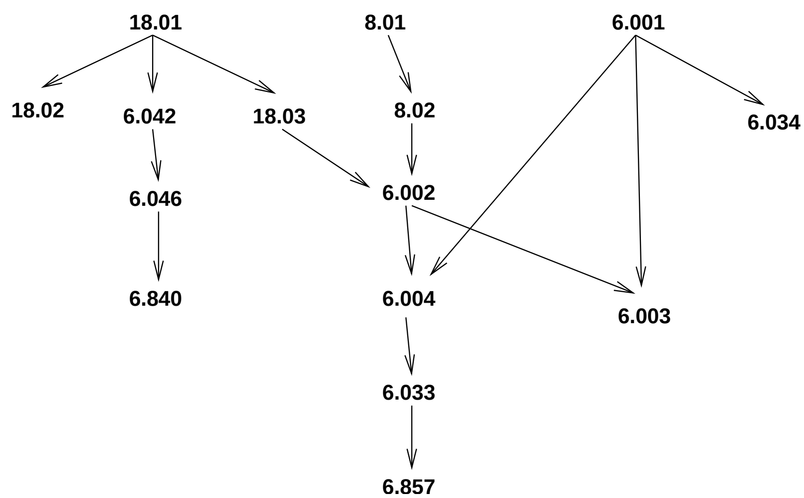
   Solution. This relation is reflexive since $p \Rightarrow p$. It is also transitive, since if $p \Rightarrow q$ and $q \Rightarrow r$, then $p \Rightarrow r$. However, it isn't symmetric since, for example, `false` $\Rightarrow$ `true`, but not `true` $\Rightarrow$ `false`. ■

3. Here is prerequistite information for some MIT courses:

$$18.01 \to 6.042 \qquad 18.01 \to 18.02$$
$$18.01 \to 18.03 \qquad 6.046 \to 6.840$$
$$8.01 \to 8.02 \qquad 6.01 \to 6.034$$
$$6.042 \to 6.046 \qquad 18.03, 8.02 \to 6.02$$
$$6.01, 6.02 \to 6.003 \qquad 6.01, 6.02 \to 6.004$$
$$6.004 \to 6.033 \qquad 6.033 \to 6.857$$

(a) Draw a Hasse diagram for the corresponding partially-ordered set. (A ***Hasse diagram*** is a way of representing a poset $(A, \preceq)$ as a directed acyclic graph. The vertices are the element of A, and there is generally an edge $u \to v$ if $u \preceq v$. However, self-loops and edges implied by transitivity are omitted.) You'll need this diagram for all the subsequent problem parts, so be neat!

Solution.



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(b) Identify a largest chain. (A ***chain*** in a poset $(S, \preceq)$ is a subset $C \subseteq S$ such that for all $x, y \in C$, either $x \preceq y$ or $y \preceq x$.)

Solution. There are two largest chains:

$$8.01 \preceq 8.02 \preceq 6.02 \preceq 6.004 \preceq 6.033 \preceq 6.857$$

and

$$18.01 \preceq 18.03 \preceq 6.02 \preceq 6.004 \preceq 6.033 \preceq 6.857$$

■

(c) Suppose that you want to take all the courses. What is the minimum number of terms required to graduate, if you can take as many courses as you want per term?

Solution. Six terms are necessary, because at most one course in the longest chain can be taken each term. Six terms are sufficient by a theorem proved in lecture. ■

(d) Identify a largest **antichain**. (An ***antichain*** in a poset $(S, \preceq)$ is a subset $A \subseteq S$ such that for all $x, y \in A$ with $x \neq y$, neither $x \preceq y$ nor $y \preceq x$.)

Solution. There are several five-element antichains. One is:

$$\{18.02, 6.042, 18.03, 8.01, 6.01\}$$

■

(e) What is the maximum number of classes that you could possibly take at once?

Solution. Classes you are taking simultaneously must form an antichain, so you can take at most five at once. ■

(f) Identify a topological sort of the classes. (A ***topological sort*** of a poset $(A, \preceq)$ is a total order of all the elements such that if $a_i \preceq a_j$ in the partial order, then a_i precedes a_j in the total order.)

Solution. Many answers are possible. One is 18.01, 8.01, 6.01, 18.02, 6.042, 18.03, 8.02, 6.034, 6.046, 6.02, 6.840, 6.004, 6.003, 6.033, 6.857. ■

(g) Suppose that you want to take all of the courses, but can handle only two per term. How many terms are required to graduate?

Solution. There are 15 courses, so at least 8 terms are necessary. The schedule below shows that 8 terms are sufficient as well:

1:	18.01	8.01
2:	6.01	18.02
3:	6.042	18.03
4:	8.02	6.034
5:	6.046	6.02
6:	6.840	6.004
7:	6.003	6.033
8:	6.857	

■

(h) What if you could take three courses per term?

Solution. In part (c) we argued that six terms are required even if there is no limit on the number of courses per term. Six terms are also sufficient, as the following schedule shows:

1:	18.01	8.01	6.01
2:	6.042	18.03	8.02
3:	18.02	6.046	6.02
4:	6.004	6.003	6.034
5:	6.840	6.033	
6:	6.857		

■

(i) Stanford's computer science department offers n courses, limits students to at most k classes per term, and has its own complicated prerequisite structure. Describe two different lower bounds on the number of terms required to complete all the courses. One should be based on your answers to parts (b) and (c) and a second should be based on your answer to part (g).

Solution. One lower bound is the length of the longest chain and another is $\lceil n/k \rceil$. ■