# Problem Set 2

*Due*: March 3

**Reading:**

- Chapter 3.6. *Predicate Formulas*,
- Chapter 4. *Mathematical Data Types* through 4.4. *Binary Relations*,
- Chapter 5. *Induction*.

These assigned readings do **not** include the Problem sections. (Many of the problems in the text will appear as class or homework problems.)

**Problem 1.** **(a)** Write predicates that express the following assertions.

- $R$ is a surjection [>= 1 in].
- $R$ is a function [<= 1 out].
- $x = y - z$.
- $x = \{\}$.

**Problem 2.**
Let $A$, $B$ and $C$ be sets. Prove that:

$$A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C). \tag{1}$$

*Hint:* $P$ OR $Q$ OR $R$ is equivalent to

$$(P \text{ AND } \overline{Q}) \text{ OR } (Q \text{ AND } \overline{R}) \text{ OR } (R \text{ AND } \overline{P}) \text{ OR } (P \text{ AND } Q \text{ AND } R).$$

**Problem 3.** **(a)** Prove by induction that a $2^n \times 2^n$ courtyard with a $1 \times 1$ statue of Bill in *a corner* can be covered with L-shaped tiles. (Do not assume or reprove the (stronger) result of Theorem 5.1.2 that Bill can be placed anywhere. The point of this problem is to show a different induction hypothesis that works.)

**(b)** Use the result of part (a) to prove the original claim that there is a tiling with Bill in the middle.

**Problem 4.**
For any binary string $\alpha$ let num $(\alpha)$ be the nonnegative integer it represents in binary notation. For example, num $(\texttt{10}) = 2$, and num $(\texttt{0101}) = 5$.

An $n + 1$-*bit adder* adds two $n + 1$-bit binary numbers. More precisely, an $n + 1$-bit adder takes two length $n + 1$ binary strings

$$\alpha_n ::= a_n \ldots a_1 a_0,$$
$$\beta_n ::= b_n \ldots b_1 b_0,$$

and a binary digit $c_0$ as inputs, and produces a length-$(n + 1)$ binary string

$$\sigma_n ::= s_n \ldots s_1 s_0,$$

and a binary digit $c_{n+1}$ as outputs, and satisfies the specification:

$$\text{num}(\alpha_n) + \text{num}(\beta_n) + c_0 = 2^{n+1} c_{n+1} + \text{num}(\sigma_n). \tag{2}$$

There is a straighforward way to implement an $n+1$-bit adder as a digital circuit: an $n+1$-bit *ripple-carry circuit* has $1 + 2(n + 1)$ binary inputs

$$a_n, \ldots, a_1, a_0, b_n, \ldots, b_1, b_0, c_0,$$

and $n + 2$ binary outputs,

$$c_{n+1}, s_n, \ldots, s_1, s_0.$$

As in Problem 3.6, the ripple-carry circuit is specified by the following formulas:

$$s_i ::= a_i \ \text{XOR} \ b_i \ \text{XOR} \ c_i \tag{3}$$
$$c_{i+1} ::= (a_i \ \text{AND} \ b_i) \ \text{OR} \ (a_i \ \text{AND} \ c_i) \ \text{OR} \ (b_i \ \text{AND} \ c_i),. \tag{4}$$

for $0 \leq i \leq n$.

**(a)** Verify that definitions (3) and (4) imply that

$$a_n + b_n + c_n = 2c_{n+1} + s_n. \tag{5}$$

for all $n \in \mathbb{N}$.

**(b)** Prove by induction on $n$ that an $n + 1$-bit ripple-carry circuit really is an $n + 1$-bit adder, that is, its outputs satisfy (2).

*Hint:* You may assume that, by definition of binary representation of integers,

$$\text{num}(\alpha_{n+1}) = a_{n+1} 2^{n+1} + \text{num}(\alpha_n). \tag{6}$$