In-Class Problems Week 12, Wed.

Problem 1. **(a)** Use the Multinomial Theorem **??** to prove that

$$(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p} \tag{1}$$

for all primes p. (Do not prove it using Fermat's "little" Theorem. The point of this problem is to offer an independent proof of Fermat's theorem.)

Hint: Explain why $\binom{p}{k_1, k_2, \ldots, k_n}$ is divisible by p if all the k_i's are positive integers less than p.

(b) Explain how (1) immediately proves Fermat's Little Theorem **??**: $n^{p-1} \equiv 1 \pmod{p}$ when n is not a multiple of p.

Problem 2.
We are interested in generating functions for the number of different ways to compose a bag of n donuts subject to various restrictions. For each of the restrictions in (a)-(e) below, find a closed form for the corresponding generating function.

(a) All the donuts are chocolate and there are at least 3.

(b) All the donuts are glazed and there are at most 2.

(c) All the donuts are coconut and there are exactly 2 or there are none.

(d) All the donuts are plain and their number is a multiple of 4.

(e) The donuts must be chocolate, glazed, coconut, or plain and:

- there must be at least 3 chocolate donuts, and
- there must be at most 2 glazed, and
- there must be exactly 0 or 2 coconut, and
- there must be a multiple of 4 plain.

(f) Find a closed form for the number of ways to select n donuts subject to the constraints of the previous part.

Problem 3. **(a)** Let

$$S(x) ::= \frac{x^2 + x}{(1-x)^3}.$$

What is the coefficient of x^n in the generating function series for $S(x)$?

(b) Explain why $S(x)/(1-x)$ is the generating function for the sums of squares. That is, the coefficient of x^n in the series for $S(x)/(1-x)$ is $\sum_{k=1}^{n} k^2$.

Creative Commons 2011, Eric Lehman, F Tom Leighton, Albert R Meyer .

(c) Use the previous parts to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Appendix

Let $[x^n]F(x)$ denote the coefficient of x^n in the power series for $F(x)$. Then,

$$[x^n]\left(\frac{1}{(1-\alpha x)^k}\right) = \binom{n+k-1}{k-1}\alpha^n. \tag{2}$$

Partial Fractions

Here's a particular case of the Partial Fraction Rule that should be enough to illustrate the general Rule. Let

$$r(x) ::= \frac{p(x)}{(1-\alpha x)^2(1-\beta x)(1-\gamma x)^3}$$

where α, β, γ are distinct complex numbers, and $p(x)$ is a polynomial of degree less than the demoninator, namely, less than 6. Then there are unique numbers $a_1, a_2, b, c_1, c_2, c_3 \in \mathbb{C}$ such that

$$r(x) = \frac{a_1}{1-\alpha x} + \frac{a_2}{(1-\alpha x)^2} + \frac{b}{1-\beta x} + \frac{c_1}{1-\gamma x} + \frac{c_2}{(1-\gamma x)^2} + \frac{c_3}{(1-\gamma x)^3}$$