Notes for Recitation 16

1 Combinatorial Proof

A ***combinatorial proof*** is an argument that establishes an algebraic fact by relying on counting principles. Many such proofs follow the same basic outline:

1. Define a set S.
2. Show that $|S| = n$ by counting one way.
3. Show that $|S| = m$ by counting another way.
4. Conclude that $n = m$.

Consider the following theorem:

Theorem.

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

We can prove it with a combinatorial approach:

Proof. We give a combinatorial proof. Let S be the set of all binary sequences with exactly n zeroes and $k+1$ ones.

On the one hand, we know from a previous recitation that the number of such sequences is equal to $\binom{k+n+1}{k+1}$.

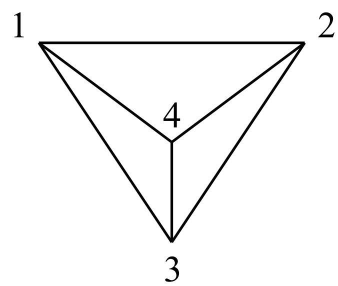
On the other hand, the number of zeroes i to the left of the rightmost one ranges from 0 to n. For a fixed value of i, there are $\binom{k+i}{k}$ possible choices for the sequence of bits before the rightmost one. If we sum over all possible i, we find that $|S| = \sum_{i=0}^{n} \binom{k+i}{k}$.

Equating these two expressions for $|S|$ proves the theorem. $\square$

Triangles

Let $T = \{X_1, \ldots, X_t\}$ be a set whose elements X_i are themselves sets such that each X_i has size 3 and is $\subseteq \{1, 2, \ldots, n\}$. We call the elements of T "triangles". Suppose that for all "edges" $E \subseteq \{1, 2, \ldots, n\}$ with $|E| = 2$ there are exactly λ triangles $X \in T$ with $E \subseteq X$.

For example, if we might have the triangles depicted in the following diagram, which has $\lambda = 2$, $n = 4$, and $t = 4$:



In this example, each edge appears in exactly two of the following triangles:

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$$

Prove

$$\lambda \cdot \frac{n(n-1)}{2} = 3t$$

by counting the set

$$C = \{(E, X) : X \in T, E \subseteq X, |E| = 2\}$$

in two different ways.

Solution. We give a combinatorial proof. Let C be $\{(E, X) : X \in T, E \subseteq X, |E| = 2\}$.

On the one hand, there are $\binom{n}{2}$ sets $E \subseteq \{1, \ldots, n\}$ of size $|E| = 2$. For each such E there are exactly λ triangles $X \in T$ with $E \subseteq X$. So, $|C| = \lambda\binom{n}{2} = \lambda \cdot \frac{n(n-1)}{2}$.

On the other hand, there are t triangles. Each triangle has exactly $\binom{3}{2} = 3$ subsets E of size 2. So, $|C| = 3t$.

Equating these two expressions for $|C|$ proves the theorem. ■

2 Generating Functions

The ***(ordinary) generating function*** for a sequence $\langle a_0, a_1, a_2, a_3, \ldots \rangle$ is the power series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Problem 1

Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) $\langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle$

Solution.

$$2 + 3x + 5x^2$$

■

(b) $\langle 1, 1, 1, 1, 1, 1, 1, \ldots \rangle$

Solution.

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

■

(c) $\langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle$

Solution.

$$\begin{aligned} 1 + 2x + 4x^2 + 8x^3 + \ldots &= (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots \\ &= \frac{1}{1-2x} \end{aligned}$$

■

(d) $\langle 1, 0, 1, 0, 1, 0, 1, 0, \ldots \rangle$

Solution.

$$1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1-x^2}$$

■

(e) $\langle 0, 0, 0, 1, 1, 1, 1, 1, \ldots \rangle$

Solution.

$$x^3 + x^4 + x^5 + x^6 + \ldots = x^3(1 + x + x^2 + x^3 + \ldots) \qquad = \frac{x^3}{1-x}$$

■

(f) $\langle 1, 3, 5, 7, 9, 11, \ldots \rangle$

Solution.

$$\begin{aligned}
1 + x + x^2 + x^3 + \ldots &= \frac{1}{1-x} \\
\frac{d}{dx}\ 1 + x + x^2 + x^3 + \ldots &= \frac{d}{dx}\ \frac{1}{1-x} \\
1 + 2x + 3x^2 + 4x^2 + \ldots &= \frac{1}{(1-x)^2} \\
2 + 4x + 6x^2 + 8x^2 + \ldots &= \frac{2}{(1-x)^2} \\
1 + 3x + 5x^2 + 7x^3 + \ldots &= \frac{2}{(1-x)^2} - \frac{1}{1-x} \\
&= \frac{1+x}{(1-x)^2}
\end{aligned}$$

■

Problem 2

Suppose that:

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ g(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots \end{aligned}$$

What sequences do the following functions generate?

(a) $f(x) + g(x)$

Solution.

$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + \ldots$$

■

(b) $f(x) \cdot g(x)$

Solution.

$$a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \ldots + \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n + \ldots$$

■

(c) $f(x)/(1 - x)$

Solution. This is a special case of the preceding problem part where:

$$\begin{aligned} g(x) &= \frac{1}{1-x} \\ &= 1 + x + x^2 + x^3 + x^4 + \cdots \end{aligned}$$

and so $b_0 = b_1 = b_2 = \ldots = 1$. In this case, we have:

$$f(x) \cdot g(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \ldots + \left(\sum_{k=0}^{n} a_k \right) x^n + \ldots$$

Thus, $f(x)/(1 - x)$ is the generating function for sums of prefixes of the sequence generated by f. ■

Problem 3

There is a jar containing n different flavors of candy (and lots of each kind). I'd like to pick out a set of k candies.

(a) In how many different ways can this be done?

Solution. There is a bijection with sequences containing k zeroes (representing candies) and $n-1$ ones (separating the different varieties). The number of such sequences is:

$$\binom{n+k-1}{k}$$

■

(b) Now let's approach the same problem using generating functions. Give a closed-form generating function for the sequence $\langle s_0, s_1, s_2, s_3, \ldots \rangle$ where s_k is the number of ways to select k candies when there is only $n = 1$ flavor available.

Solution.

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

■

(c) Give a closed-form generating function for the sequence $\langle t_0, t_1, t_2, t_3, \ldots \rangle$ where t_k is the number of ways to select k candies when there are $n = 2$ flavors.

Solution.

$$(1 + x + x^2 + x^3 + \ldots)^2 = \frac{1}{(1-x)^2}$$

■

(d) Give a closed-form generating function for the sequence $\langle u_0, u_1, u_2, u_3, \ldots \rangle$ where u_k is the number of ways to select k candies when there are n flavors.

Solution.

$$\frac{1}{(1-x)^n}$$

■