Problem Set 5

Due: Wednesday, October 12

Reading Assignment: Sections 5.2, 6.3

Problem 1. [20 points] The following series of problems deals with the notion of Hamiltonian cycles and Eulerian circuits in undirected graphs. A Hamiltonian cycle is a path that visits each vertex of a graph exactly once and ends at its starting vertex. An Eulerian circuit is a path that traverses all edges of a graph exactly once and ends at its starting vertex. A useful fact for this series of problems is that a connected graph has an Eulerian circuit if and only if all vertices of the graph have even degree (you may use this fact without proof throughout the problems).

(a) [5 pts] Is it true that if a simple graph has an Eulerian circuit, then it has a Hamiltonian cycle? If yes, then provide a proof. Otherwise, provide a counterexample.

(b) [5 pts] Is it true that if a simple graph has an Hamiltonian cycle, then it has a Eulerian circuit? If yes, then provide a proof. Otherwise, provide a counterexample.

(c) [10 pts]

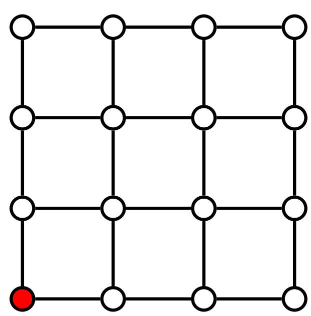

Figure 1: 4 x 4 grid graph

Bruce missed out on trick-or-treating as a kid. However, at the ripe age of 22, Bruce decides to dress up as Batman and try out trick-or-treating for the first time. Suppose that the above graph represents Bruce's neighborhood with each edge representing some street that is lined with houses and Bruce's home being the lower left corner vertex (colored red). Bruce wants to visit all the streets to get as much candy as possible, but doesn't get any candy from re-visiting a street. What is the fewest number of streets Bruce has to revisit in order to visit all the streets in his neighborhood and return home?

Problem 2. [15 points]

In the cycle C_{2n} of length $2n$, we'll call two vertices *opposite* if they are on opposite sides of the cycle, that is that are distance n apart in C_n. Let G be the graph formed from C_{2n} by adding an edge, which we'll call a *crossing edge*, between each pair of opposite vertices. So G has n crossing edges.

(a) [5 pts] Give a simple description of the shortest path between any two vertices of G.

Hint: Argue that a shortest path between two vertices in G uses at most one crossing edge.

(b) [3 pts] What is the *diameter* of G, that is, the largest distance between two vertices?

(c) [3 pts] We say that a graph is *k-edge connected* if removing $(k-1)$ edges can not disconnect the graph. Prove that the graph above is not 4-edge connected.

(d) [4 pts] Prove that the graph is 3-edge connected.

Problem 3. [10 points] Consider a Hunger Games scenario in which a tribute must be selected from a set of boys and a second tribute must be selected from a set of girls. However, suppose that the Hunger Games committee wants to first match each boy with a girl and then simply select a pair from the possible matchings. Suppose that the committee lets each boy rank each of the girls he wants to be paired with and vice-versa. Prove or disprove the following claim: for some $n \geq 3$ (n boys and n girls, for a total of $2n$ people), there exists a set of boys' and girls' rankings such that any matching made by the committee is stable.

Problem 4. [10 points]

Show that the congestion of the N-input butterfly is $\sqrt{N}$ if N is an even power of 2.

Problem 5. [20 points]

In a *perfect shuffle*, a deck of N cards is cut exactly in half and then perfectly interlaced. Thus, for N cards, we would obtain the resulting cards in the following order:

$$1,\ \left(\frac{N}{2}+1\right),\ 2,\ \left(\frac{N}{2}+2\right),\ \ldots,\ \left(\frac{N}{2}-1\right),\ (N-1),\ \frac{N}{2},\ N.$$

(a) [10 pts] Show that m perfect shuffles will return a deck of N cards to its original order provided that $2^m = 1 \pmod{(N-1)}$.

(b) [4 pts] Show that 8 perfect shuffles are necessary and sufficient to return a deck of 52 cards to their original order.

(c) [5 pts] How many perfect shuffles are necessary and sufficient to return a deck of cards to its original order if there are two jokers added to the deck (so that it has 54 cards)?

Problem 6. [10 points] Construct a 16-bit de Bruijn sequence, by considering an Eulerian tour of the de Bruijn graph.