Staff Solutions to Problem Set 8

Reading:

- Chapter **??**. *Conditional Probability*
- Chapter **??**. *Random Variables & Expectation*

STAFF NOTE: `Conditional Probabillity` `Ch.18.1-18.7`

Problem 1.
There were n Immortal Warriors born into our world, but in the end there can be *only one*. The Immortals' original plan was to stalk the world for centuries, dueling one another with ancient swords in dramatic landscapes until only one survivor remained. However, after a thought-provoking discussion probability, they opt to give the following protocol a try:

(i) The Immortals forge a coin that comes up heads with probability p.

(ii) Each Immortal flips the coin once.

(iii) If *exactly one* Immortal flips heads, then they are declared The One. Otherwise, the protocol is declared a failure, and they all go back to hacking each other up with swords.

One of the Immortals (Kurgan from the Russian steppe) argues that as n grows large, the probability that this protocol succeeds must tend to zero. Another (McLeod from the Scottish highlands) argues that this need not be the case, provided p is chosen carefully.

(a) A natural sample space to use to model this problem is $\{H, T\}^n$ of length-n sequences of H and T's, where the successive H's and T's in an outcome correspond to the Head or Tail flipped on each one of the n successive flips. Explain how a tree diagram approach leads to assigning a probability to each outcome that depends only on p, n and the number h of H's in the outcome.

COMMENTS:

- PS_immortal_probability
- overlaps of PS_ethernet
- F12.rec22

keywords = [*probability sample_space tie_breaking*]

Solution. The tree would have depth n, with each vertex having a branch assigned probability p to a child labelled H and a branch assigned probability $1 - p$ to a child labelled T. An outcome with h H's must have $n - h$ T's and would therefore be assigned a probability of

$$p^h(1-p)^{n-h}.$$

■

(b) What is the probability that the experiment succeeds as a function of p and n?

Solution. Let E be the event that the experiment successfully selects The One. Then E consists of the n outcomes which contain a single head. Each of these has probability $p(1-p)^{n-1}$, so the probability that the procedure succeeds is

$$\Pr[E] = n\left(p(1-p)^{n-1}\right). \tag{1}$$

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(c) How should p, the bias of the coin, be chosen in order to maximize the probability that the experiment succeeds?

STAFF NOTE: You're going to have to compute a derivative! ■

Solution. We compute the derivative of the success probability:

$$\frac{d}{dp}\, np(1-p)^{n-1} \quad = \quad n(1-p)^{n-1} - np(n-1)(1-p)^{n-2}$$

Now we set the right side equal to zero to find the best probability p:

$$\begin{aligned} n(1-p)^{n-1} &= np(n-1)(1-p)^{n-2} \\ (1-p) &= p(n-1) \\ p &= \frac{1}{n}. \end{aligned}$$

This answer makes some intuitive sense, since we want the coin to come up heads exactly 1 time in n. ■

(d) What is the probability of success if p is chosen in this way? What quantity does this approach when n, the number of Immortal Warriors, grows large?

Solution. Setting $p = 1/n$ in the formula (1) for the probability that the experiment succeeds gives:

$$\Pr[E] = \left(1 - \frac{1}{n}\right)^{n-1}$$

In the limit, this tends to $1/e$. McLeod is right. ■

Problem 2.
We're interested in the probability that a randomly chosen poker hand (5 cards from a standard 52-card deck) contains cards from at most two suits.

(a) What is an appropriate sample space to use for this problem? What are the outcomes in the event $\mathcal{E}$ we are interested in? What are the probabilities of the individual outcomes in this sample space?

COMMENTS:

- PS_random_poker_hand
- from: F06.ps11 (ported by Rich)

keywords = [*probability playing cards*]

Solution. The natural sample space to use consists of the $\binom{52}{5}$ possible poker hands. The sample space is *uniform*: Each hand is equally likely and comes up with probability $1/\binom{52}{5}$.

We define $\mathcal{E}$ to be the subset of outcomes in which the 5 cards on the outcome come from at most two suits. ■

(b) What is $\Pr[\mathcal{E}]$?

Solution. Since the sample space is uniform,

$$\Pr[\mathcal{E}] = \frac{|\mathcal{E}|}{\binom{52}{5}}.$$

So we just need to determine the number of outcomes in $\mathcal{E}$. For this, we resort to our usual counting techniques. Doing it by cases works well. There are three cases: 5 cards of one suit, 4 cards of one suit and 1 of another suit, 3 cards of one suit and 2 of another suit.

For 5 of one suit, there are 4 ways to choose the suit and then $\binom{13}{5}$ ways to choose 5 cards of that suit.

For 4 of one suit and 1 of another, there are 4 ways to choose the suit of the 4 and $\binom{13}{4}$ ways to choose 4 cards of that suit, and there are 3 remaining suits to choose for the 1, and 13 choices for the 1 card of that suit.

Finally, for 3 of one suit and 2 of another, there are 4 ways to choose the suit of the 3 and $\binom{13}{3}$ ways to choose 3 cards of that suit, and there are 3 remaining suits to choose for the 2 cards, and $\binom{13}{2}$ choices for the 2 cards of that suit. So the total is

$$4 \cdot \binom{13}{5} + 4 \cdot \binom{13}{4} \cdot 3 \cdot 13 + 4 \cdot \binom{13}{3} \cdot 3 \cdot \binom{13}{2},$$

and the probability of at most two suits is

$$\frac{4 \cdot \binom{13}{5} + 4 \cdot \binom{13}{4} \cdot 3 \cdot 13 + 4 \cdot \binom{13}{3} \cdot 3 \cdot \binom{13}{2}}{\binom{52}{5}} = 88/595 \approx 0.15.$$

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Problem 3.

Suppose you have three cards: A♡, A♠ and a jack. From these, you choose a random hand (that is, each card is equally likely to be chosen) of two cards, and let n be the number of aces in your hand. You then randomly pick one of the cards in the hand and reveal it.

(a) Describe a simple probability space (that is, outcomes and their probabilities) for this scenario, and list the outcomes in each of the following events:

1. $[n \geq 1]$, (that is, your hand has an ace in it),
2. A♡ is in your hand,
3. the revealed card is an A♡,
4. the revealed card is an ace.

COMMENTS:

- PS_conditional_aces
- from S07.ps11
- soln edited ARM 10/23/15

keywords = [*probability probability_space conditional_probability outcomes*]

Solution. Consider each outcome as a pair of cards, the first of which is the revealed card. Each outcome is equally likely (probability $1/6$).

The sets of outcomes are then as follows:

1. $[n \geq 1]$: all pairs:

$$\{(\text{A}\heartsuit, \text{A}\spadesuit), (\text{A}\heartsuit, \text{jack}), (\text{A}\spadesuit, \text{A}\heartsuit), \\ (\text{A}\spadesuit, \text{jack}), (\text{jack}, \text{A}\heartsuit), (\text{jack}, \text{A}\spadesuit)\},$$

2. A$\heartsuit$ is in your hand: $\{(\text{A}\heartsuit, \text{A}\spadesuit), (\text{A}\heartsuit, \text{jack}), (\text{A}\spadesuit, \text{A}\heartsuit), (\text{jack}, \text{A}\heartsuit)\}$,
3. the revealed card is an A$\heartsuit$: $\{(\text{A}\heartsuit, \text{A}\spadesuit), (\text{A}\heartsuit, \text{jack})\}$,
4. the revealed card is an ace: $\{(\text{A}\heartsuit, \text{A}\spadesuit), (\text{A}\heartsuit, \text{jack}), (\text{A}\spadesuit, \text{A}\heartsuit), (\text{A}\spadesuit, \text{jack})\}$.

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(b) Then calculate $\Pr[n = 2 \mid E]$ for E equal to each of the four events in part (a). Notice that most, but *not all*, of these probabilities are equal.

Solution. First, note that $\Pr[n = 2] = 1/3$.

1. $\Pr[n = 2 \mid n \geq 1] = \Pr[n = 2]/1 = 1/3$,
2. $\Pr[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}] = \Pr[n = 2]/(2/3) = 1/2$,
3. $\Pr[n = 2 \mid \text{the revealed card is an A}\heartsuit] = \Pr[(\text{A}\heartsuit, \text{A}\spadesuit)]/(1/3) = 1/2$,
4. $\Pr[n = 2 \mid \text{the revealed card is an ace}] = \Pr[n = 2]/(2/3) = 1/2$.

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Now suppose you have a deck with d distinct cards, a different kinds of aces (including an A$\heartsuit$), you draw a random hand with h cards, and then reveal a random card from your hand.

(c) Prove that $\Pr[\text{A}\heartsuit \text{ is in your hand}] = h/d$.

Solution. The number N of hands is

$$N = \binom{d}{h}.$$

$$
\begin{aligned}
&\Pr[\text{A}\heartsuit \text{ is in your hand}] \\
&= \frac{\#\text{ hands with A}\heartsuit}{N} \\
&= \frac{\#\ (h-1)\text{-card hands from deck w/o A}\heartsuit}{N} \\
&= \frac{\binom{d-1}{h-1}}{N} \\
&= \frac{(d-1)!h!(d-h)!}{(h-1)!(d-h)!d!} && (\text{def. of } \tbinom{m}{n}) \\
&= h/d. && (\text{simplification})
\end{aligned}
$$

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(d) Prove that

$$\Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right] = \Pr[n = 2] \cdot \frac{2d}{ah}. \tag{2}$$

Solution.

$$
\begin{aligned}
&\Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right] \\
&= \frac{\Pr[n = 2 \text{ and A}\heartsuit \text{ is in your hand}]}{\Pr[\text{A}\heartsuit \text{ is in your hand}]} \\
&= \frac{\Pr[n = 2 \text{ and A}\heartsuit \text{ is in your hand}]}{h/d} && (\text{part (c)}) \\
&= \frac{\Pr[n = 2] \cdot \Pr\left[\text{A}\heartsuit \text{ is in your hand} \mid n = 2\right]}{h/d} \\
&= \frac{\Pr[n = 2] \cdot 2/a}{h/d} && (\text{see below}) \\
&= \Pr[n = 2] \cdot \frac{2d}{ah}
\end{aligned}
$$

The $\frac{2}{a}$ substitution above is justified by observing that in a hand with two aces, each of the $\binom{a}{2}$ pairs of aces is equally likely to be the one in the hand. Of these pairs, $a - 1$ include the A♡. So

$$\Pr\left[\text{A}\heartsuit \text{ is in your hand} \mid n = 2\right] = \frac{a-1}{\binom{a}{2}} = \frac{2}{a}.$$

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(e) Conclude that

$$\Pr\left[n = 2 \mid \text{the revealed card is an ace}\right] = \Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right].$$

Solution. Note that

$$\Pr[\text{the revealed card is an ace}] = \frac{a}{d}, \tag{3}$$

since the probability of revealing an ace from the random hand is simply the probability that a random card is an ace. Now,

$$\begin{aligned}
&\Pr\left[n = 2 \mid \text{ the revealed card is an ace}\right] \\
&= \frac{\Pr[n = 2 \text{ and the revealed card is an ace}]}{\Pr[\text{the revealed card is an ace}]} \\
&= \frac{\Pr[n = 2 \text{ and the revealed card is an ace}]}{a/d} && \text{(by (3))} \\
&= \frac{\Pr[n = 2] \Pr\left[\text{the revealed card is an ace} \mid n = 2\right]}{a/d} \\
&= \frac{\Pr[n = 2](2/h)}{a/d} && \text{(see below)} \\
&= \frac{2d}{ah} \cdot \Pr[n = 2] \\
&= \Pr\left[n = 2 \mid \text{ A}\heartsuit \text{ is in your hand}\right]. && \text{(by (2))}
\end{aligned}$$

The $\frac{2}{h}$ substitution for $\Pr\left[\text{the revealed card is an ace} \mid n = 2\right]$ is justified by noting that, in any hand with 2 aces, that is, when $n = 2$, the revealed card must come from one of exactly 2 positions (with aces) out of h total positions.

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