6.046	Lecture 9	March 7, 2013

(supplementary material to lecture slides)

Proof of (1)

- Suppose $A = (A_{ij})_{i,j \in \{1,\dots,n\}}$ where

$$A_{ij} = \mathbb{P}r[v_{t+1} = j \mid v_t = i], \quad \forall i,j, \forall t.$$

location at time t+1

location at time t

- Claim: $x_{t+1} = x_t \cdot A$

distn' of v_{t+1}

distn' of v_t

- Proof:

$$\forall j: \ x_{t+1}(j) = \mathbb{P}r[v_{t+1} = j] = \sum_{i=1}^{n} \underbrace{\mathbb{P}r[v_t = i]}_{\substack{\shortparallel \\ x_t(i)}} \cdot \underbrace{\mathbb{P}r[v_{t+1} = j \mid v_t = i]}_{\substack{\shortparallel \\ A_{ij}}}$$

$$= x_t \cdot A \quad \boxtimes$$

Review: Eigenvalues/Eigenvectors:

- Let A be a square matrix
- Def: ∘ λ is an eigenvalue of A if $\underset{\sim}{x} \cdot A = \lambda \cdot \underset{\sim}{x}$ for some vector $\underset{\sim}{0} \neq \underset{\sim}{x} \in \mathbb{R}^n$.
  ∘ x is called a left-eigenvector of A corresponding to eigenvalue λ

Lazy Random Walk on a 4-Cycle:

$$A = \begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix}$$

eigenvalues $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 1/3$, $\lambda_4 = -1/3$

Claim: No matter what x_0 is, $x_t \to (1/4, 1/4, 1/4, 1/4)$ as $t \to \infty$.

Proof: ∘ $e_1 = (1/4, 1/4, 1/4, 1/4)$ is a left-eigenvector corr. to λ_1
∘ By the spectral theorem, because A is symmetric there exist eigenvectors e_2, e_3, e_4 corresponding to $\lambda_2, \lambda_3, \lambda_4$ respectively, so that $\{e_1, e_2, e_3, e_4\}$

forms a basis for $\mathbb{R}^4$.

- Hence x_0 can be expressed as a linear combination of e_1, e_2, e_3, e_4. I.e.

$$x_0 = a_1 \cdot e_1 + a_2 \cdot e_2 + a_3 \cdot e_3 + a_4 \cdot e_4 \quad \text{for some scalars } a_1, a_2, a_3, a_4$$

- $$\begin{aligned} x_t &= x_0 \cdot A^t = \\ &= (a_1 \cdot e_1 + a_2 e_2 + a_3 e_3 + a_4 \cdot e_4) \cdot A^t = \\ &= a_1 \cdot e_1 A^t + a_2 \cdot e_2 A^t + a_3 \cdot e_3 A^t + a_4 \cdot e_4 A^t \\ &= a_1 \cdot e_1 A A^{t-1} + a_2 \cdot e_2 A \cdot A^{t-1} + \dots \\ &= a_1 \cdot \lambda_1 \cdot e_1 A^{t-1} + a_2 \cdot \lambda_2 \cdot e_2 A^{t-1} + \dots \\ &= \dots \\ &= a_1 \cdot \lambda_1^t \cdot e_1 + a_2 \lambda_2^t \cdot e_2 + a_3 \lambda_3^t \cdot e_3 + a_4 \lambda_4^t \cdot e_4 \\ &\to a_1 \cdot e_1, \quad \text{as } t \to \infty \end{aligned}$$

(because $|\lambda_2|, |\lambda_3|, |\lambda_4| < 1$)

- So $x_t \to \left(\frac{a_1}{4}, \frac{a_1}{4}, \frac{a_1}{4}, \frac{a_1}{4}\right)$ as $t \to \infty$

- <u>Q</u>: what may a_1 be?

  <u>A</u>: The limit must be a distribution. So $a_1 = 1$.

- hence $x_t \to \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ as $t \to \infty$

$\boxtimes$

Note: Above proof works for all lazy random walks on undirected, connected graphs.

Namely: i. transition matrix A is symmetrix

ii. it has n real eigenvalues of which $\lambda_1 = 1$

rest satisfy $|\lambda_2|, |\lambda_3|, \dots, |\lambda_n| < 1$.

iii. because A is symmetric, eigenvectors $e_1, e_2, \dots, e_n$ corresponding to these eigenvalues form a basis

iv. hence starting distn' x_0 can be written as:

$$x_0 = \sum_{i=1}^{n} a_i \cdot e_i$$

v. So

$$x_t = x_0 A^t = \sum_{i=1}^{n} a_i (e_i A^t)$$

$$= \sum_i a_i \lambda_i^t e_i$$

$\Rightarrow$ as $t \to \infty$, $x_t \to a_1 \cdot e_1$ (because $\lambda_1 = 1$ and $|\lambda_2|, |\lambda_3|, |\lambda_n|$ are < 1)

vi. So $x_\infty = a_1 \cdot e_1$ for whatever a_1 makes this a distn'.

Graph Coloring

Input: $G=(V,E)$ graph; $Q=\{1,2,\dots,q\}$ set of colors
Goal: Sample a u.a.r. legal coloring of G

↳ assign a color to each vertex so that no two adjacent vertices get the same color

Denote by Δ: the maximum degree of G.

Remarks:

- if $q \geq \Delta+1$: a legal coloring guaranteed to exist

- if $q=\Delta$: Brook's theorem:
  - if $\Delta > 2$ the graph has a Δ-coloring iff it does not contain a $(\Delta+1)$-clique
  - if $\Delta=2$:

    2-coloring exists $\Longleftrightarrow$ no odd cycle

- if $q<\Delta$: NP-hard to tell if Δ-coloring exists

Working Regime for us: $q \geq \Delta + 1$

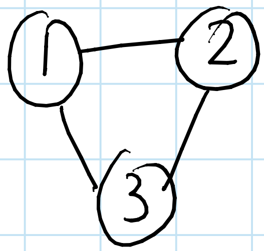
Markov Chain:

- start at arbitrary legal coloring (can construct it greedily)
- pick vertex $v \in V$ u.a.r. and color $c \in Q$ u.a.r.
- recolor v with c if this gives legal coloring otherwise do nothing
- repeat

Claim: chain is lazy

Claim 2: not necessarily strongly connected

e.g.



$\Delta = 2$, $q = 3$

NO MOVES AVAILABLE!

Claim 3: If $q \geq \Delta + 2$ it is strongly connected.

left as exercise

Claim 4: If $q \geq \Delta + 2$, the Markov chain converges to the uniform distn' over colorings.

Proof:

- lazyness + strong connectivity $\Rightarrow$ convergence to stationary distn'
- stationary distn' is left-eigenvector of transition matrix corresponding to eigenvalue 1.
- let's inspect A:

$$A = \begin{array}{cc} & \text{legal colorings} \\ & j \\ \text{legal colorings} \; i & \begin{pmatrix} \cdots & \cdots & \circ & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

entry i,j: probability of transitioning from coloring i to coloring j

lemma: $A_{ij} = A_{ji}$

Proof: • If $A_{ij} \neq 0$:

- Suppose $i \to j$ by changing color of node v from c to c'

happens w/ prob $\frac{1}{n} \times \frac{1}{q}$

choose node v

choose color c'

- then we can also move from j to i by changing v's color from c' to c

happens w/prob $\frac{1}{n} \times \frac{1}{q}$

hence $A_{ij} = A_{ji}$.

• If $A_{ij} = 0$, then can't go from col. i to col. j. So we can't go from col. j to col. i either. Hence $A_{ji} = 0$ $\square$

Lemma $\Rightarrow$ all columns of A sum to 1

follows from the fact that all rows sum to 1

$\Rightarrow$ If x is uniform distn' over legal colorings, then

$$x \cdot A = x$$

$\boxtimes$

<u>Thm:</u> If $q \geq 2\Delta + 1$, $\tau_{mix} = O(n \cdot \log n)$

<u>Conjecture:</u> same is true for $q \geq \Delta + 2$.