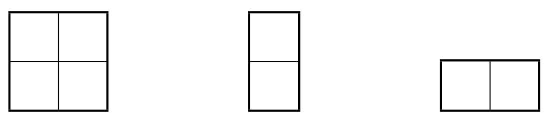
6.042/18.062J Mathematics for Computer Science November 3, 2016
Tom Leighton and Ankur Moitra

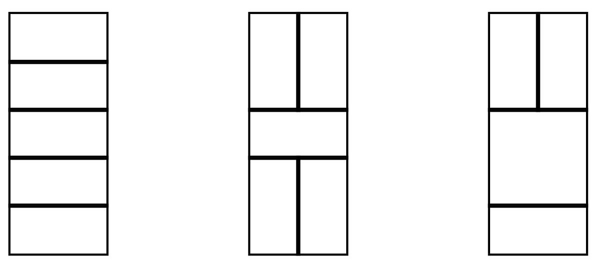
Problems for Recitation 14

Mini-Tetris

A *winning configuration* in the game of Mini-Tetris is a complete tiling of a $2 \times n$ board using only the three shapes shown below:



For example, here are several possible winning configurations on a 2×5 board:



1. Let T_n denote the number of different winning configurations on a $2 \times n$ board. Determine the values of T_1, T_2, and T_3.

2. Find a recurrence equation that expresses T_n in terms of T_{n-1} and T_{n-2}.

3. Find a closed-form expression for the number of winning configurations on a $2 \times n$ Mini-Tetris board.

Linear Recurrences

Find closed-form solutions to the following linear recurrences.

1.
$$T_0 = 0$$
$$T_1 = 1$$
$$T_n = T_{n-1} + T_{n-2} + 1$$

2.
$$S_0 = 0$$
$$S_1 = 1$$
$$S_n = 6S_{n-1} - 9S_{n-2}$$

Short Guide to Solving Linear Recurrences

A *linear recurrence* is an equation

$$\underbrace{f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)}_{\text{homogeneous part}} \qquad \underbrace{+g(n)}_{\text{inhomogeneous part}}$$

together with boundary conditions such as $f(0) = b_0$, $f(1) = b_1$, etc.

1. Find the roots of the *characteristic equation*:

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_k$$

2. Write down the *homogeneous solution*. Each root generates one term and the homogeneous solution is the sum of these terms. A nonrepeated root r generates the term $c_r r^n$, where c_r is a constant to be determined later. A root r with multiplicity k generates the terms:

$$c_{r_1} r^n, \quad c_{r_2} n r^n, \quad c_{r_3} n^2 r^n, \quad \ldots, \quad c_{r_k} n^{k-1} r^n$$

   where $c_{r_1}, \ldots, c_{r_k}$ are constants to be determined later.

3. Find a *particular solution*. This is a solution to the full recurrence that need not be consistent with the boundary conditions. Use guess and verify. If $g(n)$ is a polynomial, try a polynomial of the same degree, then a polynomial of degree one higher, then two higher, etc. For example, if $g(n) = n$, then try $f(n) = bn+c$ and then $f(n) = an^2+bn+c$. If $g(n)$ is an exponential, such as 3^n, then first guess that $f(n) = c3^n$. Failing that, try $f(n) = bn3^n + c3^n$ and then $an^2 3^n + bn3^n + c3^n$, etc.

4. Form the *general solution*, which is the sum of the homogeneous solution and the particular solution. Here is a typical general solution:

$$f(n) = \underbrace{c2^n + d(-1)^n}_{\text{homogeneous solution}} + \underbrace{3n+1}_{\text{particular solution}}$$

5. Substitute the boundary conditions into the general solution. Each boundary condition gives a linear equation in the unknown constants. For example, substituting $f(1) = 2$ into the general solution above gives:

$$\begin{aligned} 2 &= c \cdot 2^1 + d \cdot (-1)^1 + 3 \cdot 1 + 1 \\ \Rightarrow \quad -2 &= 2c - d \end{aligned}$$

   Determine the values of these constants by solving the resulting system of linear equations.