Problem Set 4

Due: Monday, September 29

Reading Assignment: Sections 5.0-5.3

Problem 1. [15 points] Recall that a ***coloring*** of a simple graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color. A ***k-coloring*** is a coloring that uses at most k colors.

False Claim. *Let G be a (simple) graph with maximum degree at most k. If G also has a vertex of degree less than k, then G is k-colorable.*

(a) [5 pts] Give a counterexample to the False Claim when $k = 2$.

(b) [10 pts] Consider the following proof of the False Claim:

Proof. Proof by induction on the number n of vertices:

Induction hypothesis: $P(n)$ is defined to be: Let G be a graph with n vertices and maximum degree a most k. If G also has a vertex of degree less than k, then G is k-colorable.

Base case: (n=1) G has only one vertex and so is 1-colorable. So $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices and maximum degree at most k. Also, suppose G_{n+1} has a vertex, v, of degree less than k. We need only prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Removing v reduces the degree of all vertices adjacent to v by 1. So in G_n, each of these vertices has degree less than k. Also the maximum degree of G_n remains at most k. So G_n satisfies the conditions of the induction hypothesis $P(n)$. We conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. □

Identify the exact sentence where the proof goes wrong.

Problem 2. [15 points] Prove or disprove the following claim: for some $n \geq 3$ (n boys and n girls, for a total of $2n$ people), there exists a set of boys' and girls' preferences such that every dating arrangement is stable.

Problem 3. [20 points] 6.042 is often taught using recitations. Suppose it happened that 8 recitations were needed, with two or three staff members running each recitation. The assignment of staff to recitation sections is as follows:

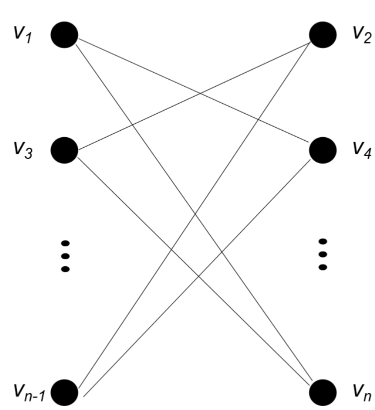
- R1: Maverick, Goose, Iceman
- R2: Maverick, Stinger, Viper
- R3: Goose, Merlin
- R4: Slider, Stinger, Cougar
- R5: Slider, Jester, Viper
- R6: Jester, Merlin
- R7: Jester, Stinger
- R8: Goose, Merlin, Viper

Two recitations can not be held in the same 90-minute time slot if some staff member is assigned to both recitations. The problem is to determine the minimum number of time slots required to complete all the recitations.

(a) [10 pts] Recast this problem as a question about coloring the vertices of a particular graph. Draw the graph and explain what the vertices, edges, and colors represent.

(b) [10 pts] Show a coloring of this graph using the fewest possible colors. What schedule of recitations does this imply?

Problem 4. [20 points] Suppose you have a graph as shown below. Every node on the left is adjacent to every node on the right except the node directly across from it.



(a) [5 pts] Find the chromatic number of the graph.

(b) [5 pts] The graph pictured above is often referred to as *bipartite.*

Definition. *A graph $G = (V, E)$ is bipartite if the set of vertices, V, can be split into two subsets V_l and V_r such that all edges in G connect nodes in V_l to nodes in V_r.*

Now recall from lecture the Greedy Coloring Algorithm:
Greedy Coloring Algorithm: For a graph $G = (V, E)$ and an ordering of vertices $v_1, v_2, \cdots, v_n$

1. Color v_1 with a new color c_1.
2. For each vertex v_i, if v_i shares an edge with with any earlier vertex, v_j, colored c_k, then it cannot be colored c_k. Choose the lowest available color for v_i.

Find an ordering of the vertices $\{v_1, v_2, \cdots, v_n\}$ such that the Greedy Coloring Algorithm uses exactly 2 colors.

(c) [5 pts] Find an ordering such that the Greedy Coloring Algorithm uses exactly $n/2$ colors.

(d) [5 pts] Prove your answer in part (c) by induction for all even integers n.

Problem 5. [15 points] For each of the following pairs of graphs, either define an isomorphism between them, or prove that there is none. (We write ab as shorthand for the edge from a to b).

(a) [5 pts]

$$G_1 \text{ with } V_1 = \{1, 2, 3, 4, 5, 6\}, \ E_1 = \{12, 23, 34, 14, 15, 35, 45\}$$
$$G_2 \text{ with } V_2 = \{1, 2, 3, 4, 5, 6\}, \ E_2 = \{12, 23, 34, 45, 51, 24, 25\}$$

(b) [5 pts]

$$G_3 \text{ with } V_3 = \{1, 2, 3, 4, 5, 6\}, \ E_3 = \{12, 23, 34, 14, 45, 56, 26\}$$
$$G_4 \text{ with } V_4 = \{a, b, c, d, e, f\}, \ E_4 = \{ab, bc, cd, de, ae, ef, cf\}$$

(c) [5 pts]

$$G_5 \text{ with } V_5 = \{a, b, c, d, e, f, g, h\}, \ E_5 = \{ab, bc, cd, ad, ef, fg, gh, he, dh, bf\}$$
$$G_6 \text{ with } V_6 = \{s, t, u, v, w, x, y, z\}, \ E_6 = \{st, tu, uv, sv, wx, xy, yz, wz, sw, vz\}$$

Problem 6. [15 points] Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subset E$ such that no two edges in M are incident on a common vertex.

Let M_1, M_2 be two matchings of G. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Show that G' is bipartite.

Helpful definition: A *connected component* is a subgraph of a graph consisting of some vertex and every node and edge that is connected to that vertex.