Problem Set 1

Due: February 15

Reading: UNDER CONSTRUCTION

Part I. *Proofs: Introduction*, Chapter 1, *What is a Proof?*; Chapter 2, *The Well Ordering Principle*; and Chapter 3 through 3.5, covering *Propositional Logic*. These assigned readings **do not include the Problem sections**. (Many of the problems in the text will appear as class or homework problems.)

Reminders:

- Comments on the different reading assignments are due by 10PM the night before class on days indicated in the class calendar (also in online tutor problem set TP.2). Make comments using the class NB annotation system. Reading Comments count for 5% of the final grade.
- Problems should be submitted separately following the pset submission instructions, and each problem should have a *collaboration statement* at the beginning, with the requisite information written in or attached using the collaboration statement template.

Problem 1.
The fact that that there are irrational numbers a, b such that a^b is rational was proved in Problem 1.5 of the course text. Unfortunately, that proof was *nonconstructive*: it didn't reveal a specific pair, a, b, with this property. But in fact, it's easy to do this: let $a ::= \sqrt{2}$ and $b ::= 2\log_2 3$.

We know $\sqrt{2}$ is irrational, and obviously $a^b = 3$. Finish the proof that these values for a, b work, by showing that $2\log_2 3$ is irrational.

Problem 2.
Prove that the propositional formulas

$$P \text{ OR } Q \text{ OR } R$$

and

$$(P \text{ AND NOT}(Q)) \text{ OR } (Q \text{ AND NOT}(R)) \text{ OR } (R \text{ AND NOT}(P)) \text{ OR } (P \text{ AND } Q \text{ AND } R).$$

are equivalent.

Problem 3. **(a)** Verify by truth table that

$$(P \text{ IMPLIES } Q) \text{ OR } (Q \text{ IMPLIES } P)$$

is valid.

(b) Let P and Q be propositional formulas. Describe a single formula, R, using AND's, OR's, and NOT's such that R is valid iff P and Q are equivalent.

(c) A propositional formula is *satisfiable* iff there is an assignment of truth values to its variables —an *environment* —which makes it true. Explain why

$$P \text{ is valid} \quad \text{iff} \quad \text{NOT}(P) \text{ is } \textit{not} \text{ satisfiable.}$$

(d) A set of propositional formulas $P_1, \ldots, P_k$ is *consistent* iff there is an environment in which they are all true. Write a formula, S, so that the set $P_1, \ldots, P_k$ is *not* consistent iff S is valid.

Problem 4.
For $n = 40$, the value of polynomial $p(n) ::= n^2 + n + 41$ is not prime, as noted in Section 1.1 of the course text. But we could have predicted based on general principles that no nonconstant polynomial can generate only prime numbers.

In particular, let $q(n)$ be a polynomial with integer coefficients, and let $c ::= q(0)$ be the constant term of q.

(a) Verify that $q(cm)$ is a multiple of c for all $m \in \mathbb{Z}$.

(b) Show that if q is nonconstant and $c > 1$, then as n ranges over the nonnegative integers, $\mathbb{N}$, there are infinitely many $q(n) \in \mathbb{Z}$ that are not primes.

Hint: You may assume the familiar fact that the magnitude of any nonconstant polynomial, $q(n)$, grows unboundedly as n grows.

(c) Conclude immediately that for every nonconstant polynomial, q, there must be an $n \in \mathbb{N}$ such that $q(n)$ is not prime.