Notes for Recitation 6

Hall's theorem

Let $G = (V, E)$ be a bipartite graph, with left vertex set L and right vertex set R. Recall that for a subset S of the vertices, $N(S)$ is the set of vertices which are adjacent to some vertex in S:

$$N(S) = \{r \in V \mid \{r, s\} \in E \text{ for some } s \in S\}.$$

Halls' theorem says that if for every subset S of L we have $|N(S)| \geq |S|$, then there is a matching in G that covers L.

Problem 1

Recall that a graph is called *d-regular* if every vertex in the graph has degree exactly d. Let $G = (V, E)$ be a d-regular bipartite graph, with the same number of vertices in the left part L as in the right part R.

Prove, using Hall's theorem and induction, that G can be partitioned into d perfect matchings. In other words, we can find $E_1, E_2, \ldots, E_d \subseteq E$, all disjoint ($E_i \cap E_j = \emptyset$) and which together form E, so that E_i is a perfect matching of G for each $1 \leq i \leq d$.

Solution. The proof is by induction on d. So let

$P(d) =$ "Any d-regular bipartite graph with $n/2$ left nodes and $n/2$ right nodes can be partitioned into d perfect matchings".

Let us take $d = 0$ for the base case (the statement still makes sense). In a 0-regular graph, there are no edges. We can indeed partition the empty set into 0 perfect matchings! So $P(0)$ is true.

So assume $P(d)$ holds; we want to prove $P(d + 1)$, so let G be a $(d + 1)$-regular bipartite graph, with left subset L and right subset R, where $|L| = |R| = n/2$.

Let S be any subset of L; we want to show that $|N(S)| \geq |S|$. Consider the subgraph $G' = (S \cup N(S), E')$ induced by $S \cup N(S)$. Every node of S has degree d in G' (since we kept all the neighbours of S). Every node in $N(S)$ has degree at most $d + 1$ in G' (since it had degree $d+1$ in G, and G' is a subgraph). Now count the number of edges of G' in two ways, just like in class: we get

$$(d+1)|S| = |E'| = \sum_{v \in N(S)} \deg_{G'}(v) \leq (d+1)|N(S)|.$$

Simplifying, $|N(S)| \geq |S|$.

So Hall's condition is satisfied, and there exists some perfect matching M on G. Now look at the subgraph $H = (V, E_H)$ obtained by removing M from G. This is d-regular, since for each vertex we remove a single edge adjacent to it. So by induction, H can be partitioned into d perfect matchings $E_1, E_2, \ldots, E_d$. Writing $E_{d+1} = M$, we obtain that $E = E_H \cup M = E_1 \cup \cdots \cup E_{d+1}$ (and all the E_i's are disjoint).

This proves $P(d+1)$, and so by induction, $P(d)$ holds for all $d \geq 0$. ■

Problem 2

Given the preference lists of each boy and girl, there can be in general many different stable matchings.

Consider a particular boy i, and let P_i be the set of girls for which there is *some* stable matching where this girl is matched to i. We say that boy i's favorite girl in P_i is his *optimal mate*; this represents the best outcome for boy i, given that only stable matchings are allowed.

Prove that The Mating Algorithm returns a matching where every boy is matched with his optimal mate.

Solution. See the book, pp 140–142. ■

Problem 3

Similarly to the previous problem, we say that the *pessimal mate* of girl j is her least favorite boy from the set P_j of boys she can be matched to in some stable matching.

Prove that The Mating Algorithm returns a matching where every girl is matched with her pessimal mate.

Solution. See the book, pp 140–142. ■