Notes for Recitation 11

Chains and antichains

Recall some definitions and theorems from lecture.

Definition 1. *A **chain** in a poset $(S, \preceq)$ is a subset $C \subseteq S$ such that for all $x, y \in C$, either $x \preceq y$ or $y \preceq x$. In other words, it is a sequence $a_1 \preceq a_2 \preceq a_3 \cdots \preceq a_t$, where $a_i \neq a_j$ for all $i \neq j$, such that each item is comparable to the next in the chain, and it is smaller with respect to $\preceq$.*

Theorem 1. *Given any finite poset $(A, \preceq)$ for which the longest chain has length t, it is possible to partition A into t subsets $A_1, A_2, \ldots, A_t$ such that for all $i \in \{1, 2, \ldots, t\}$ and for all $a \in A_i$, we have that all $b \preceq a$ appear in the set $A_1 \cup \cdots \cup A_{i-1}$.*

Corollary 2. *The total amount of parallel time needed to complete the tasks is the same as the length of the longest chain.*

Definition 2. *An **antichain** in a poset $(S, \preceq)$ is a subset $A \subseteq S$ such that for all $x, y \in A$ with $x \neq y$, neither $x \preceq y$ nor $y \preceq x$.). In other words, it is a set of incomparable items, e.g., things that can be scheduled at the same time.*

1 Problem: Antichains

The above theorem can be recast in the language of antichains. Prove the following corollary.

Corollary 3. *If i is the length of the longest chain in a poset $(A, \preceq)$, then A can be partitioned into t antichains.*

Solution. *Proof.* By the theorem, we can partition A into t sets, where each set consists of tasks that could be scheduled at the same time, i.e., incomparable items. Indeed, if some $x \neq y$ were comparable, then either $x \preceq y$ or $y \preceq x$, and x and y could not be scheduled at the same time. □

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Note: It turns out that the dual of this corollary, which states that if the longest antichain has size t, then A can be partitioned into t chains, is also true! However, this is much harder to prove. It is known as ***Dilworth's theorem***.

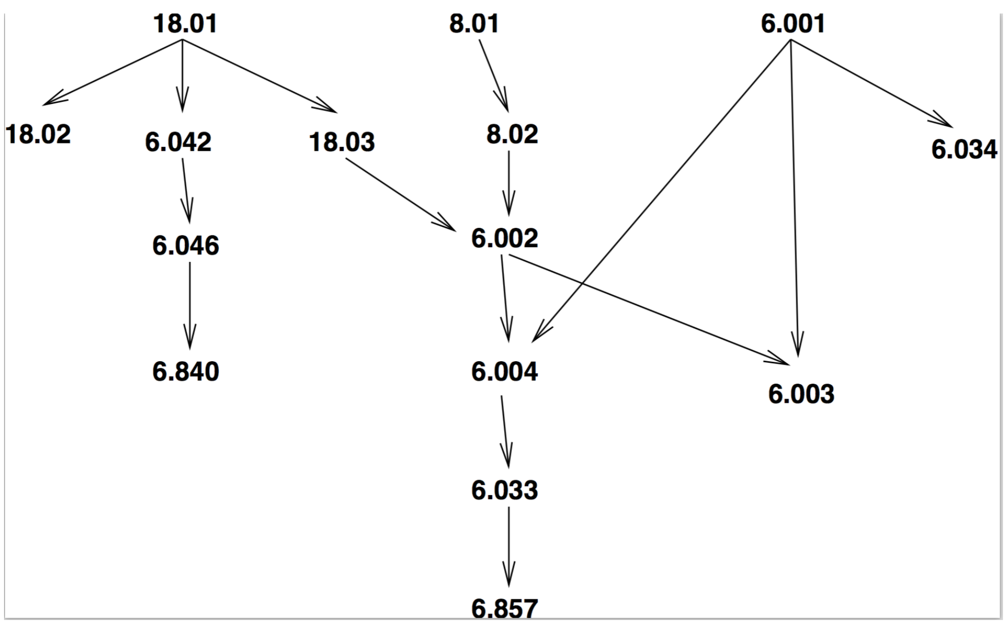
2 Problem: Taking classes

Here is prerequistite information for some MIT courses:

$$
\begin{array}{rr}
18.01 \to 6.042 & 18.01 \to 18.02 \\
18.01 \to 18.03 & 6.046 \to 6.840 \\
8.01 \to 8.02 & 6.01 \to 6.034 \\
6.042 \to 6.046 & 18.03, 8.02 \to 6.02 \\
6.01, 6.02 \to 6.003 & 6.01, 6.02 \to 6.004 \\
6.004 \to 6.033 & 6.033 \to 6.857
\end{array}
$$

1. Draw a Hasse diagram for the corresponding partially-ordered set. (Recall: A ***Hasse diagram*** is a way of representing a poset $(A, \preceq)$ as a directed acyclic graph. The vertices are the element of A, and there is generally an edge $u \to v$ if $u \preceq v$. However, self-loops and edges implied by transitivity are omitted.) You'll need this diagram for all the subsequent problem parts, so be neat!

   Solution. ■



2. Identify a largest chain.

   Solution. There are two largest chains:

$$8.01 \preceq 8.02 \preceq 6.02 \preceq 6.004 \preceq 6.033 \preceq 6.857$$

and

$$18.01 \preceq 18.03 \preceq 6.02 \preceq 6.004 \preceq 6.033 \preceq 6.857$$

■

3. Suppose that you want to take all the courses. What is the minimum number of terms required to graduate, if you can take as many courses as you want per term?

   Solution. Six terms are necessary, because at most one course in the longest chain can be taken each term. Six terms are sufficient by a theorem proved in lecture. ■

4. Identify a largest antichain.

   Solution. There are several five-element antichains. One is:

   $$\{18.02, 6.042, 18.03, 8.01, 6.01\}$$

   ■

5. What is the maximum number of classes that you could possibly take at once?

   Solution. Classes you are taking simultaneously must form an antichain, so you can take at most five at once. ■

6. Identify a topological sort of the classes. (A ***topological sort*** of a poset $(A, \preceq)$ is a total order of all the elements such that if $a_i \preceq a_j$ in the partial order, then a_i precedes a_j in the total order.)

   Solution. Many answers are possible. One is 18.01, 8.01, 6.01, 18.02, 6.042, 18.03, 8.02, 6.034, 6.046, 6.02, 6.840, 6.004, 6.003, 6.033, 6.857. ■

7. Suppose that you want to take all of the courses, but can handle only two per term. How many terms are required to graduate?

   Solution. There are 15 courses, so at least 8 terms are necessary. The schedule below shows that 8 terms are sufficient as well:

1:	18.01	8.01
2:	6.01	18.02
3:	6.042	18.03
4:	8.02	6.034
5:	6.046	6.02
6:	6.840	6.004
7:	6.003	6.033
8:	6.857	

■

8. What if you could take three courses per term?

   Solution. In part (c) we argued that six terms are required even if there is no limit on the number of courses per term. Six terms are also sufficient, as the following schedule shows:

1:	18.01	8.01	6.01
2:	6.042	18.03	8.02
3:	18.02	6.046	6.02
4:	6.004	6.003	6.034
5:	6.840	6.033	
6:	6.857		

   ■

9. Stanford's computer science department offers n courses, limits students to at most k classes per term, and has its own complicated prerequisite structure. Describe two different lower bounds on the number of terms required to complete all the courses. One should be based on your answers to parts (b) and (c) and a second should be based on your answer to part (g).

   Solution. One lower bound is the length of the longest chain and another is $\lceil n/k \rceil$. ■

10. We now complement these lower bounds with an upper bound. This upper bound is known as ***Brent's Theorem***, and it implies that the *sum* of these two lower bounds is an *upper bound* on the number of terms required to complete all courses.

    Suppose the length of the longest prerequisite chain is c. Show that the maximum number of terms required to complete all the courses is

    $$M(n, c, k) := (c - 1) + \left\lceil \frac{n - (c - 1)}{k} \right\rceil .$$

    Hint: Try induction on c. You may find it helpful to use the fact that if $a \geq b \geq 0$, then

    $$\lceil a - b \rceil \leq 1 + \lceil a \rceil - \lceil b \rceil \tag{1}$$

    for all real numbers a, b.

    Solution. Proof by induction. Induction hypothesis:

    > $P(c) :=$ $\forall$ Prerequisite structures , $\forall n, k \in \mathbb{N}^+$, if there are n classes with longest prerequisite chain c, and if a student can take at most k classes per term, then the student can take all the classes in $M(n, c, k)$ terms.

Base case $c = 1$: In this case there are n classes and no prerequisites. So any partition of the classes into $\lceil n/k \rceil$ blocks of size at most k will take time $\lceil n/k \rceil = 0 + \lceil (n-0)/k \rceil = M(n, 1, k)$.

Inductive step: Assume $P(c)$ and conclude $P(c+1)$ where $c \geq 1$.

Suppose now that Stanford offers n classes with a longest prerequisite chain size $c+1$. Suppose r of these classes are endpoints of maximum-size chains. Note that none of these r classes can be a prerequisite for another, as otherwise there would be a chain of length one more than the maximum chain size. Consider the prerequisite schedule obtained by removing these r classes.

Now H is a prerequisite schedule with $n - r$ classes and maximum chain size c, so by the Induction Hypothesis, a student taking at most k classes per term can finish within $M(n - r, c, k)$ terms.

This class schedule can be extended to one for the original prerequisite structure by arbitrarily grouping $\lceil r/k \rceil$ of the r classes at endpoints of maximum-size chains, each group of size at most k, and taking all classes in a single group in one term. So the total number of terms required is

$$\begin{aligned}
&M(n-r, c, k) + \left\lceil \frac{r}{k} \right\rceil \\
&= (c-1) + \left\lceil \frac{n - r - (c-1)}{k} \right\rceil + \left\lceil \frac{r}{k} \right\rceil && \text{(def of } M\text{)} \\
&= (c-1) + \left\lceil \frac{n-c}{k} - \frac{r-1}{k} \right\rceil + \left\lceil \frac{r}{k} \right\rceil && (2)
\end{aligned}$$

We complete the proof by showing that the expression (2) is $\leq M(n, c+1, k)$. To do this, we consider two cases:

- **Case 1** ($r - 1$ is not a multiple of k): We have

$$\left\lceil \frac{r-1}{k} \right\rceil = \left\lceil \frac{r}{k} \right\rceil, \tag{3}$$

  so

$$\begin{aligned}
(2) &\leq (c-1) + \left(1 + \left\lceil \frac{n-c}{k} \right\rceil - \left\lceil \frac{r-1}{k} \right\rceil \right) + \left\lceil \frac{r}{k} \right\rceil && \text{(by (1))} \\
&= (c-1) + \left(1 + \left\lceil \frac{n-c}{k} \right\rceil - \left\lceil \frac{r}{k} \right\rceil \right) + \left\lceil \frac{r}{k} \right\rceil && \text{(by (3))} \\
&= c + \left\lceil \frac{n-c}{k} \right\rceil \\
&= M(n, c+1, k). && \text{(def of } M\text{)}
\end{aligned}$$

- **Case 2** ($r - 1$ is a multiple of k): Now we have

$$\left\lceil \frac{r}{k} \right\rceil = 1 + \frac{r-1}{k}, \tag{4}$$

so

$$\begin{aligned}
(2) &= (c-1) + \left(\left\lceil \frac{n-c}{k} \right\rceil - \frac{r-1}{k}\right) + \left\lceil \frac{r}{k} \right\rceil && \text{(since } (r-1)/k \in \mathbb{Z}\text{)} \\
&= (c-1) + \left\lceil \frac{n-c}{k} \right\rceil - \frac{r-1}{k} + \left(1 + \frac{r-1}{k}\right) && \text{(by (4))} \\
&= c + \left\lceil \frac{n-c}{k} \right\rceil \\
&= M(n, c+1, k). && \text{(def of } M\text{)}
\end{aligned}$$

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3 Problem: Relations

1. Give a description of the equivalence classes associated with each of the following equivalence relations.

   (a) Integers x and y are equivalent if $x \equiv y \pmod 3$.

   Solution.

   $$\begin{gathered}\{\dots, -6, -3, 0, 3, 6, \dots\} \\ \{\dots, -5, -2, 1, 4, 7, \dots\} \\ \{\dots, -4, -1, 2, 5, 8, \dots\}\end{gathered}$$

   ■

   (b) Real numbers x and y are equivalent if $\lceil x \rceil = \lceil y \rceil$, where $\lceil z \rceil$ denotes the smallest integer greater than or equal to z.

   Solution. For each integer n, all the real numbers r such that $n - 1 < r \leq n$ form an equivalence class. ■

   (c) Vertices x and y in the graph G are equivalent if there is a path from x to y.

   Solution. The vertices in each connected component of G form an equivalence class. ■

2. Show that neither of the following relations is an equivalence relation by identifying a missing property (reflexivity, symmetry, or transitivity).

   (a) The "divides" relation on the positive integers.

   Solution. This relation is reflexive (since $a \mid a$) and transitive (since $a \mid b$ and $b \mid c$ implies $a \mid c$), but not symmetric (since $3 \mid 6$, but not $6 \mid 3$). ■

   (b) The "implies" relation on propositional formulas.

Solution. This relation is reflexive since $p \Rightarrow p$. It is also transitive, since if $p \Rightarrow q$ and $q \Rightarrow r$, then $p \Rightarrow r$. However, it isn't symmetric since, for example, `false` $\Rightarrow$ `true`, but not `true` $\Rightarrow$ `false`. ■

(c) The "intersects" relation on nonempty subsets of $\mathbb{N}$.

Solution. The relation is reflexive, since every nonempty subset A of $\mathbb{N}$ intersects itself. The relation is symmetric, because if A intersects B, then B intersects A. However, the relation is not transitive; for example, $\{1, 2\}$ intersects $\{2, 3\}$ and $\{2, 3\}$ intersects $\{3, 4\}$, but $\{1, 2\}$ does not intersect $\{3, 4\}$. ■