# Notes for Recitation 1

## 1 Team Problem: Contrapositive

Prove by truth table that an implication is equivalent to its contrapositive.

**Solution.**

| $x$ | $y$ | $x \rightarrow$ y | ¬y | ¬x | ¬y → ¬x | ($x \rightarrow$ y) $\longleftrightarrow$ (¬y → ¬x) |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |

In every row, x → y is T precisely when ¬y → ¬x is T. Thus, we conclude that an implication is equivalent to its contrapositive. ■

# 2 Team Problem: A Mystery

A certain cabal within the 6.042 course staff is plotting to make the final exam *ridiculously hard*. ("Problem 1. Prove that the axioms of mathematics are complete and consistent. Express your answer in Mayan hieroglyphics.") The only way to stop their evil plan is to determine exactly who is in the cabal. The course staff consists of nine people:

$$\{\text{Devin}, \text{Elizabeth}, \text{Emanuele}, \text{Hao}, \text{Henry}, \text{Hyungie}, \text{Michael}, \text{Patrick}, \text{Rachel}\}$$

The cabal is a subset of these nine. A membership roster has been found and appears below, but it is deviously encrypted in logic notation. The predicate *incabal* indicates who is in the cabal; that is, $\textit{incabal}(x)$ is true if and only if $x$ is a member. Translate each statement below into English and deduce who is in the cabal.

(i) $\exists x\ \exists y\ \exists z\ (x \neq y \wedge x \neq z \wedge y \neq z \wedge \textit{incabal}(x) \wedge \textit{incabal}(y) \wedge \textit{incabal}(z))$

**Solution.** A direct English paraphrase would be "There exist people we'll call $x, y$, and $z$, who are all different, such that $x, y$ and $z$ are each in the cabal." A better version would use the fact that there's no need in this case to give names to the people. Namely, a better paraphrase is, "There are 3 different people in the cabal." Perhaps a simpler way to say this is, "The cabal is of size at least 3." ■

(ii) $\neg(\textit{incabal}(\text{Michael}) \wedge \textit{incabal}(\text{Henry}))$

**Solution.** Michael and Henry are not both in the cabal. Equivalently: at least one of Michael and Henry is not in the cabal. ■

(iii) $(\textit{incabal}(\text{Hyungie}) \vee \textit{incabal}(\text{Emanuele})) \rightarrow \forall x\ \textit{incabal}(x)$

**Solution.** If either Hyungie or Emanuele is in the cabal, then everyone is. ■

(iv) $\textit{incabal}(\text{Michael}) \rightarrow \textit{incabal}(\text{Henry})$

**Solution.** If Michael is in the cabal, then Henry is also. ■

(v) $\textit{incabal}(\text{Elizabeth}) \rightarrow \textit{incabal}(\text{Hyungie})$

**Solution.** If Elizabeth is in the cabal, then Hyungie is also. ■

(vi) $(\textit{incabal}(\text{Devin}) \vee \textit{incabal}(\text{Hao})) \rightarrow \neg\textit{incabal}(\text{Rachel})$

**Solution.** If either of Devin or Hao is in the cabal, then Rachel is not. Equivalently, if Rachel *is* in the cabal, then neither Devin nor Hao is. ■

(vii) $(\textit{incabal}(\text{Devin}) \vee \textit{incabal}(\text{Henry})) \rightarrow \neg\textit{incabal}(\text{Patrick})$

**Solution.** If either of Devin or Henry is in the cabal, then Patrick is not. Equivalently, if Patrick *is* in the cabal, then neither Devin nor Henry is. ■

So much for the translations. We now argue that the only cabal satisfying all seven propositions above is one whose members are exactly Devin, Henry, and Hao.

We first observe that by (ii), there must be someone — either Michael or Henry — who is not in the cabal. But if either Hyungie or Emanuele were in the cabal, then by (iii), everyone would be. So we conclude by contradiction that

$$\text{Hyungie and Emanuele are not in the cabal.} \tag{1}$$

Now consider that (v) implies its contrapositive: if Hyungie is not in the cabal, then neither is Elizabeth. Therefore, since Hyungie is not in the cabal,

$$\text{Elizabeth is not in the cabal.} \tag{2}$$

Next observe that if Michael were in the cabal, then by (iv), Henry would be too, contradicting (ii). So by again contradiction, we conclude that

$$\text{Michael is not in the cabal.} \tag{3}$$

Now suppose Rachel is in the cabal. Then by (vi), Devin and Hao are not. We already know Hyungie, Emanuele, Elizabeth, and Michael are not in the cabal, leaving only three who could be — Rachel, Patrick, and Henry. But by (i) the cabal must have at least three members, so it follows that the cabal must consist of exactly these three. This proves:

**Lemma 1.** *If Rachel is in the cabal, then Patrick and Henry are in the cabal.*

But by (vii), if Henry is the cabal, then Patrick is not. That is,

**Lemma 2.** *Henry and Patrick cannot both be in the cabal.*

Now from Lemma 2 we conclude that the conclusion of Lemma 1 is false. So by contrapositive, the hypothesis of Lemma 1 must also be false, namely,

$$\text{Rachel is not in the cabal.} \tag{4}$$

Finally, suppose Patrick is in the cabal. Then by (vii), Devin and Henry are not, and we already know Hyungie, Emanuele, Elizabeth, Michael, and Rachel are not. So the cabal must consist of at most two people (Patrick and Hao). This contradicts (i), and we conclude by contradiction that

$$\text{Patrick is not in the cabal.} \tag{5}$$

So the only remaining people who could be in the cabal are Devin, Henry, and Hao. Since the cabal must have at least three members, we conclude that

**Lemma 3.** *The only possible cabal consists of Devin, Henry, and Hao.*

But we're not done yet: we haven't shown that a cabal consisting of Devin, Henry, and Hao actually does satisfy all seven conditions. So let $\mathcal{A} = \{\text{Devin}, \text{Henry}, \text{Hao}\}$, and let's quickly check that $\mathcal{A}$ satisfies (i)–(vii):

- $|A| = 3$, so $A$ satisfies (i).
- Michael is not in $A$, so $A$ satisfies (ii) and (iv).
- Neither Hyungie nor Emanuele is in $A$, so the hypothesis of (iii) is false, which means that $A$ satisfies (iii).
- Elizabeth is not in $A$, so $A$ satisfies (v).
- Finally, Rachel and Patrick are not in $A$, so the conclusions of both (vi) and (vii) are true, and so $A$ satisfies (vi) and (vii).

So now we have proved

**Proposition.** $\{\textit{Devin}, \textit{Henry}, \textit{Hao}\}$ *is the* unique *cabal satisfying conditions* (i)–(vii).