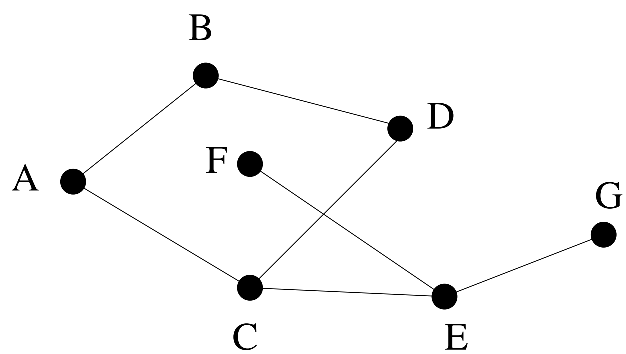
Notes for Recitation 5

Graph Basics

Let $G = (V, E)$ be a graph. Here is a picture of a graph.



Recall that the elements of V are called vertices, and those of E are called edges. In this example the vertices are $\{A, B, C, D, E, F, G\}$ and the edges are

$$\{A—B, B—D, C—D, A—C, E—F, C—E, E—G\}.$$

Deleting some vertices or edges from a graph leaves a *subgraph*. Formally, a subgraph of $G = (V, E)$ is a graph $G' = (V', E')$ where V' is a nonempty subset of V and E' is a subset of E. Since a subgraph is itself a graph, the endpoints of every edge in E' must be vertices in V'. For example, $V' = \{A, B, C, D\}$ and $E' = \{A—B, B—D, C—D, A—C\}$ forms a subgraph of G.

In the special case where we only remove edges incident to removed nodes, we say that G' is the *subgraph induced on* V' if $E' = \{(x—y | x, y \in V' \text{ and } x—y \in E\}$. In other words, we keep all edges unless they are incident to a node not in V'. For instance, for a new set of vertices $V' = \{A, B, C, D\}$, the induced subgraph G' has the set of edges $E' = \{A—B, B—D, C—D, A—C\}$.

Problem 1

A **planar graph** is a graph that can be drawn without any edges crossing.

1. First, show that any subgraph of a planar graph is planar.

   Solution. Take any "planar drawing" of the original graph—a specific drawing of the planar graph such that no edges in the drawing cross. From this, we can obtain a drawing of the subgraph, by keeping only the edges in the subgraph. Still no edges cross, so this is a planar drawing of the subgraph.

   Note: do keep in mind that "planar graph" is not the same thing as "planar drawing".
   ■

2. Also, any planar graph has a node of degree at most 5. Now, prove by induction that any graph can be colored in at most 6 colors.

   Solution. We prove by induction. First, let n be the number of nodes in the graph. Then define

   $$P(n) = \text{Any planar graph with } n \text{ nodes is 6-colorable.}$$

   Base case, $P(1)$: Every graph with $n = 1$ vertex is 6-colorable. Clearly true since it's actually 1-colorable.

   Inductive step, $P(n) \to P(n+1)$: Take a graph G with $n + 1$ nodes. Then take a node v with degree at most 5 (which we know exists because we know any planar graph has a node of degree ≤ 5), and remove it. We know that the induced subgraph G' formed in this way has n nodes, so by our inductive hypothesis, G' is 6-colorable. But v is adjacent to at most 5 other nodes, which can have at most 5 different colors between them. We then choose v to have an unused color (from the 6 colors), and as we have constructed a 6-coloring for G, we are done with the inductive step.

   Because we have shown the base case and the inductive step, we have proved

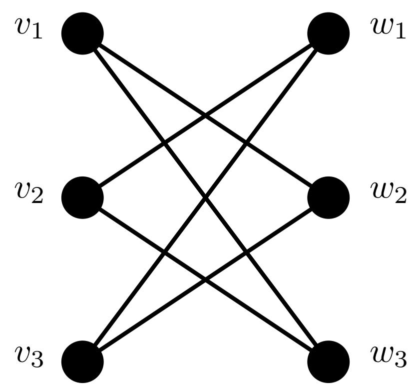
   $$\forall n \in \mathbb{Z}_+ : P(n)$$

   (Note: $\mathbb{Z}_+$ refers to the set of positive integers.) ■

Problem 2

A graph $G = (V, E)$ is called *bipartite* if we can divide the vertex set into two parts, the "left" part and the "right" part, so that every edge has one endpoint in the left part, and one endpoint in the right part. The figure below shows an example of a bipartite graph.

For n even, consider a bipartite graph with $n/2$ vertices on the left, labelled $v_1, v_2, \ldots, v_{n/2}$, and $n/2$ vertices on the right, labelled $w_1, w_2, \ldots, w_{n/2}$. Put an edge between every node on the left and node on the right, *except* between v_i and w_i for each $1 \leq i \leq n/2$ (the figure shows this graph for $n = 6$).



(a) Find an ordering of the vertices where the basic algorithm does well, and uses only 2 colors.

(b) Find an ordering where the basic algorithm does *very* badly, and requires $n/2$ colors.

Solution. (a) One could take the ordering

$$v_1, v_2, \ldots, v_{n/2}, w_1, w_2, \ldots, w_{n/2}.$$

Then each v_i is colored 1, and each w_i is colored 2.

(b) Take the ordering

$$v_1, v_2, \ldots, v_{n/2}, w_1, \ldots, w_{n/2}.$$

Then v_1 and v_2 will both be colored 1. This forces, v_2 and w_2 to both be colored 2. Continuing on, we see that v_i and w_i will both be colored with color i. This could be proved formally by induction.

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Problem 3

An undirected graph G has ***width*** w if the vertices can be arranged in a sequence

$$v_1, \; v_2, \; v_3, \; \ldots, \; v_n$$

such that each vertex v_i is joined by an edge to at most w preceding vertices. (Vertex v_j *precedes* v_i if $j < i$.) Use induction to prove that every graph with width at most w is $(w+1)$-colorable.

(Recall that a graph is *k-colorable* iff every vertex can be assigned one of k colors so that adjacent vertices get different colors.)

Solution. We use induction on n, the number of vertices. Let $P(n)$ be the proposition that every graph with width w is $(w+1)$ colorable.

Base case: Every graph with $n = 1$ vertex has width 0 and is $0+1 = 1$ colorable. Therefore, $P(1)$ is true.

Inductive step: Now we assume $P(n)$ in order to prove $P(n+1)$. Let G be an $(n+1)$-vertex graph with width w. This means that the vertices can be arranged in a sequence

$$v_1, v_2, v_3, \ldots, v_n, v_{n+1}$$

such that each vertex v_i is connected to at most w preceding vertices. Removing vertex v_{n+1} and all incident edges gives a graph G' with n vertices and width at most w. (If original sequence is retained, then removing v_{n+1} does not increase the number of edges from a vertex v_i to a preceding vertex.) Thus, G' is $(w+1)$-colorable by the assumption $P(n)$. Now replace vertex v_{n+1} and its incident edges. Since v_{n+1} is joined by an edge to at most w preceding vertices, we can color v_{n+1} differently from all of these. Therefore, $P(n+1)$ is true.

The theorem follows by the principle of induction. ■