Problem Set 9

Due: Wednesday, November 12

Reading assignment: 11.8–11.11, 12.1–12.3, 12.4.1, 12.6

Problem 1. [10 points]

(a) [5 pts] Show that of any $n+1$ distinct numbers chosen from the set $\{1, 2, \ldots, 2n\}$, at least 2 must be relatively prime. (*Hint:* $\gcd(k, k+1) = 1$.)

(b) [5 pts] Show that any finite connected undirected graph with $n \geq 2$ vertices must have 2 vertices with the same degree.

Problem 2. [15 points] Under Siege!

Fearing retribution for the many long hours his students spent completing problem sets, Prof. Leighton decides to convert his office into a reinforced bunker. His only remaining task is to set the 10-digit numeric password on his door. Knowing the students are a clever bunch, he is not going to pick any passwords containing the forbidden consecutive sequences "18062", "6042" or "35876" (his MIT extension).

How many 10-digit passwords can he pick that don't contain forbidden sequences if each number $0, 1, \ldots, 9$ can only be chosen once (i.e. without replacement)?

Problem 3. [15 points] Give a combinatorial proof of the following theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$

(*Hint: Consider the set of all length-n sequences of 0's, 1's and a single *.*)

Problem 4. [20 points] Calculate the following. Make sure to explain your reasoning.

(a) [5 pts] How many n-digit PIN numbers are there where no 2 consecutive digits are the same?

(b) [5 pts] How many numbers are there that are in the range [1..700] which are divisible by $2, 5$ or 7?

(c) [10 pts] How many 10 digit numbers are there in which there are exactly 5 occurances of the digit 9 and the first two digits are not the same?

Problem 5. [10 points] In the card game of bridge, you are dealt a hand of 13 cards from the standard 52-card deck.

(a) [5 pts] A balanced hand is one in which a player has roughly the same number of cards in each suit. How many different hands are there where the player has 4 cards in one suit and 3 cards in each of the other suits?

(b) [5 pts] Not surprisingly, a non-balanced hand is one in which a player has more cards in some suits than others. Hands that are very disired are ones where over half the cards are in one suit. How many different hands are there where there is exactly 7 cards in one suit?

Problem 6. [30 points]

(a) [5 pts] How many ways are there to distibute n dollars among k people assuming each person has to get an integer amount of dollars?

(b) [5 pts] How about if everyone has to get at least one dollar?

(c) [10 pts] In how many ways can you arrange n books on k bookshelves (assuming the books are distinguishable, so that order matters)?

(d) [10 pts] How about if there has to be at least 1 book at each bookshelf?

Problem 7. [25 points] We will use generating functions to determine how many ways there are to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

(a) [4 pts] Write the generating function $P(x)$ for for the number of ways to use only pennies to make n cents.

(b) [4 pts] Write the generating function $N(x)$ for the number of ways to use only nickels to make n cents.

(c) [8 pts] Write the generating function for the number of ways to use only nickels and pennies to change n cents.

(d) [4 pts] Write the generating function for the number of ways to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

(e) [5 pts] Explain how to use this function to find out how many ways are there to change 50 cents; you do *not* have to provide the answer or actually carry out the process.

Problem 8. [20 points] Recall the operation of taking derivatives of generating functions. This is done termwise, that is, if

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots,$$

then

$$F'(x) ::= f_1 + 2f_2 x + 3f_3 x^2 + \cdots.$$

For example,

$$\frac{1}{(1-x)^2} = \left(\frac{1}{(1-x)}\right)' = 1 + 2x + 3x^2 + \cdots$$

so

$$H(x) ::= \frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + \cdots$$

is the generating function for the sequence of nonnegative integers. Therefore

$$\frac{1+x}{(1-x)^3} = H'(x) = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots,$$

so

$$\frac{x^2+x}{(1-x)^3} = xH'(x) = 0 + 1x + 2^2 x^2 + 3^2 x^3 + \cdots + n^2 x^n + \cdots$$

is the generating function for the nonnegative integer squares.

(a) [10 pts] Prove that for all $k \in \mathbb{N}$, the generating function for the nonnegative integer kth powers is a quotient of polynomials in x. That is, for all $k \in \mathbb{N}$ there are polynomials $R_k(x)$ and $S_k(x)$ such that

$$[x^n]\left(\frac{R_k(x)}{S_k(x)}\right) = n^k. \tag{1}$$

(*Hint: Observe that the derivative of a quotient of polynomials is also a quotient of polynomials. It is not necessary work out explicit formulas for R_k and S_k to prove this part.*)

(b) [10 pts] Conclude that if $f(n)$ is a function on the nonnegative integers defined recursively in the form

$$f(n) = af(n-1) + bf(n-2) + cf(n-3) + p(n)\alpha^n$$

where the $a, b, c, \alpha \in \mathbb{C}$ and p is a polynomial with complex coefficients, then the generating function for the sequence $f(0), f(1), f(2), \ldots$ will be a quotient of polynomials in x.

(*Hint: Consider $R_k(\alpha x)/S_k(\alpha x)$.*)

Problem 9. [15 points]

Prove, using generating functions, that

$$\binom{i+j}{k} = \sum_{\ell=0}^{k} \binom{i}{l}\binom{j}{k-\ell}.$$

(*Hint: Think of both expressions as kth terms of some sequences whose generating functions you might want to determine. The right-hand side looks like it could be a convolution.*)