Problem Set 11

Due: December 2

Reading: *For this pset*: also add combin proof: Chapter **??–??** Operations & Fibbonaci, Chapter **?? –??** on Counting;

Problem 1.
Give combinatorial proofs of the identities below. Use the following structure for each proof. First, define an appropriate set S. Next, show that the left side of the equation counts the number of elements in S. Then show that, from another perspective, the right side of the equation also counts the number of elements in set S. Conclude that the left side must be equal to the right, since both are equal to $|S|$.

(a)

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}$$

(b)

$$\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}$$

Hint: consider a set of binary strings that could be counted using the right side of the equation, then try partitioning them into subsets countable by the elements of the sum on the left.

Problem 2.
Give a combinatorial proof for this identity:

$$\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i,j,k} = 3^n$$

Problem 3.
Give a combinatorial proof for this identity:

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

Problem 4. **(a)** Find a combinatorial (*not* algebraic) proof that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

(b) Below is a combinatorial proof of an equation. What is the equation?

Proof. Stinky Peterson owns n newts, t toads, and s slugs. Conveniently, he lives in a dorm with $n + t + s$ other students. (The students are distinguishable, but creatures of the same variety are not distinguishable.) Stinky wants to put one creature in each neighbor's bed. Let W be the set of all ways in which this can be done.

On one hand, he could first determine who gets the slugs. Then, he could decide who among his remaining neighbors has earned a toad. Therefore, $|W|$ is equal to the expression on the left.

On the other hand, Stinky could first decide which people deserve newts and slugs and then, from among those, determine who truly merits a newt. This shows that $|W|$ is equal to the expression on the right.

Since both expressions are equal to $|W|$, they must be equal to each other. ■

(Combinatorial proofs are real proofs. They are not only rigorous, but also convey an intuitive understanding that a purely algebraic argument might not reveal. However, combinatorial proofs are usually less colorful than this one.)

Problem 5.
According to the Multinomial Theorem **??**, $(x_1 + x_2 + \cdots + x_k)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}.$$

(a) How many terms are there in the sum?

(b) The sum of these multinomial coefficients has an easily expressed value:

$$\sum_{\substack{r_1+r_2+\cdots+r_k=n, \\ r_i \in \mathbb{N}}} \binom{n}{r_1,\ r_2,\ \ldots,\ r_k} = k^n \tag{1}$$

Give a combinatorial proof of this identity.

Hint: How many terms are there when $(x_1 + x_2 + \cdots + x_k)^n$ is expressed as a sum of monomials in x_i *before* terms with like powers of these variables are collected together under a single coefficient?

Problem 6.
Taking derivatives of generating functions is another useful operation. This is done termwise, that is, if

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots,$$

then

$$F'(x) ::= f_1 + 2 f_2 x + 3 f_3 x^2 + \cdots.$$

For example,

$$\frac{1}{(1-x)^2} = \left(\frac{1}{(1-x)}\right)' = 1 + 2x + 3x^2 + \cdots$$

so

$$H(x) ::= \frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + \cdots$$

is the generating function for the sequence of nonnegative integers. Therefore

$$\frac{1+x}{(1-x)^3} = H'(x) = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots,$$

so

$$\frac{x^2+x}{(1-x)^3} = xH'(x) = 0 + 1x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n + \cdots$$

is the generating function for the nonnegative integer squares.

(a) Prove that for all $k \in \mathbb{N}$, the generating function for the nonnegative integer kth powers is a quotient of polynomials in x. That is, for all $k \in \mathbb{N}$ there are polynomials $R_k(x)$ and $S_k(x)$ such that

$$[x^n]\left(\frac{R_k(x)}{S_k(x)}\right) = n^k. \tag{2}$$

Hint: Observe that the derivative of a quotient of polynomials is also a quotient of polynomials. It is not necessary work out explicit formulas for R_k and S_k to prove this part.

(b) Conclude that if $f(n)$ is a function on the nonnegative integers defined recursively in the form

$$f(n) = af(n-1) + bf(n-2) + cf(n-3) + p(n)\alpha^n$$

where the $a, b, c, \alpha \in \mathbb{C}$ and p is a polynomial with complex coefficients, then the generating function for the sequence $f(0), f(1), f(2), \ldots$ will be a quotient of polynomials in x, and hence there is a closed form expression for $f(n)$.

Hint: Consider

$$\frac{R_k(\alpha x)}{S_k(\alpha x)}$$

Problem 7.

Generating functions provide an interesting way to count the number of strings of matched brackets. To do this, we'll use a description of these strings as the set, GoodCount, of strings of brackets with a good count.

Namely, one precise way to determine if a string is matched is to start with 0 and read the string from left to right, adding 1 to the count for each left bracket and subtracting 1 from the count for each right bracket. For example, here are the counts for the two strings above

$$\begin{array}{ccccccccccccc} & [&] &] & [& [& [& [& [&] &] &] &] \\ 0 & 1 & 0 & -1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \end{array}$$

$$\begin{array}{ccccccccccc} & [& [& [&] &] & [&] &] & [&] \\ 0 & 1 & 2 & 3 & 2 & 1 & 2 & 1 & 0 & 1 & 0 \end{array}$$

A string has a *good count* if its running count never goes negative and ends with 0. So the second string above has a good count, but the first one does not because its count went negative at the third step.

Definition 7.1. Let

$$\text{GoodCount} ::= \{s \in \{\,],[\,\}^* \mid s \text{ has a good count}\}.$$

The matched strings can now be characterized precisely as this set of strings with good counts.

Let c_n be the number of strings in GoodCount with exactly n left brackets, and let $C(x)$ be the generating function for these numbers:

$$C(x) ::= c_0 + c_1x + c_2x^2 + \cdots.$$

(a) The *wrap* of a string, s, is the string, **[** s **]**, that starts with a left bracket followed by the characters of s, and then ends with a right bracket. Explain why the generating function for the wraps of strings with a good count is $xC(x)$.

Hint: The wrap of a string with good count also has a good count that starts and ends with 0 and remains *positive* everywhere else.

(b) Explain why, for every string, s, with a good count, there is a unique sequence of strings $s_1, \ldots, s_k$ that are wraps of strings with good counts and $s = s_1 \cdots s_k$. For example, the string r ::= **[[]][][[][]]** $\in$ GoodCount equals $s_1 s_2 s_3$ where s_1 ::= **[[]]**, s_2 ::= **[]**, s_3 ::= **[[][]]**, and this is the only way to express r as a sequence of wraps of strings with good counts.

(c) Conclude that

$$C = 1 + xC + (xC)^2 + \cdots + (xC)^n + \cdots, \tag{3}$$

so

$$C = \frac{1}{1 - xC}, \tag{4}$$

and hence

$$C = \frac{1 \pm \sqrt{1 - 4x}}{2x}. \tag{5}$$

Let $D(x) ::= 2xC(x)$. Expressing D as a power series

$$D(x) = d_0 + d_1 x + d_2 x^2 + \cdots,$$

we have

$$c_n = \frac{d_{n+1}}{2}. \tag{6}$$

(d) Use (5), (6), and the value of c_0 to conclude that

$$D(x) = 1 - \sqrt{1 - 4x}.$$

(e) Prove that

$$d_n = \frac{(2n-3) \cdot (2n-5) \cdots 5 \cdot 3 \cdot 1 \cdot 2^n}{n!}.$$

Hint: $d_n = D^{(n)}(0)/n!$

(f) Conclude that

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Problem 8.

We will use generating functions to determine how many ways there are to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

(a) Write the sequence P_n for the number of ways to use only pennies to change n cents. Write the generating function for that sequence.

(b) Write the sequence N_n for the number of ways to use only nickels to change n cents. Write the generating function for that sequence.

(c) Write the generating function for the number of ways to use only nickels and pennies to change n cents.

(d) Write the generating function for the number of ways to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

(e) Explain how to use this function to find out how many ways are there to change 50 cents; you do *not* have to provide the answer or actually carry out the process.

Problem 9.
Let $x_0 ::= 0, x_1 ::= 1$ and for $n \geq 2$, let x_n be defined by the linear recurrence:

$$x_n = 3x_{n-1} - 2x_{n-2} + n.$$

Find a closed form expression for x_n.