Tom Leighton and Marten van Dijk

# Midterm

**Name:** ______________________________

**Circle the name of your recitation instructor**:

David Darren Martyna Nick Oscar Stav

- This quiz is **closed book**, but you may have one $8.5 \times 11''$ sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- The exam ends at 9:30 PM.

| Problem | Points | Grade | Grader |
|---|---|---|---|
| 1 | 10 | | |
| 2 | 15 | | |
| 3 | 20 | | |
| 4 | 10 | | |
| 5 | 15 | | |
| 6 | 10 | | |
| 7 | 20 | | |
| Total | 100 | | |

**Problem 1. [10 points]**

Consider these two propositions:

$$\texttt{P:}\quad (A \wedge B) \Rightarrow C$$

$$\texttt{Q:}\quad (\neg C \Rightarrow \neg A) \vee (\neg C \Rightarrow \neg B)$$

Use the truth table below to show whether $P$ and $Q$ are equivalent, $P \Rightarrow Q$, $Q \Rightarrow P$, or none of the above.

| $A$ | $B$ | $C$ | $(A \wedge B) \Rightarrow C$ | $(\neg C \Rightarrow \neg A) \vee (\neg C \Rightarrow \neg B)$ |
|---|---|---|---|---|
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**Problem 2. [20 points]**

Let $G_0 = 1$, $G_1 = 2$, $G_2 = 4$, and define

$$G_n = G_{n-1} + 2G_{n-2} + G_{n-3} \tag{1}$$

for $n \geq 3$. Show by induction that $G_n \leq 3^n$ for all $n \geq 0$.

**Problem 3. [0 points]**

In the game of Squares and Circles, the players (you and your computer) start with a sequence of shapes: some circles and some squares. On each move a player selects two shapes. These two are replaced with a single one according to the following rule:

Identical shapes are replaced with a square. Different shapes are replaced with a circle.

At the end of the game, when only one shape remains, you are a winner if the remaining shape is a circle. Otherwise, your computer wins.

**(a)** [0 pts] Prove that the game will end.

**(b)** [0 pts] Prove that you will win if the number of circles initially is odd. Hint: Use an invariant about the number of circles.

**Problem 4. [15 points]**

**(a)** [8 pts] Find a number $x \in \{0, 1, \ldots, 112\}$ such that $18x \equiv 1 \pmod{113}$.

**(b)** [7 pts] Find a number $y \in \{0, 1, \ldots, 112\}$ such that $18^{112111} \equiv y \pmod{113}$ (*Hint: What power of* 18 *is x equivalent to modulo* 113*?*)

**Problem 5. [10 points]** Define a number $S_p = 1^p + 2^p + 3^p + \dots (p-1)^p$. You will show in this problem that if $p$ is an odd prime, then $p|S_p$.

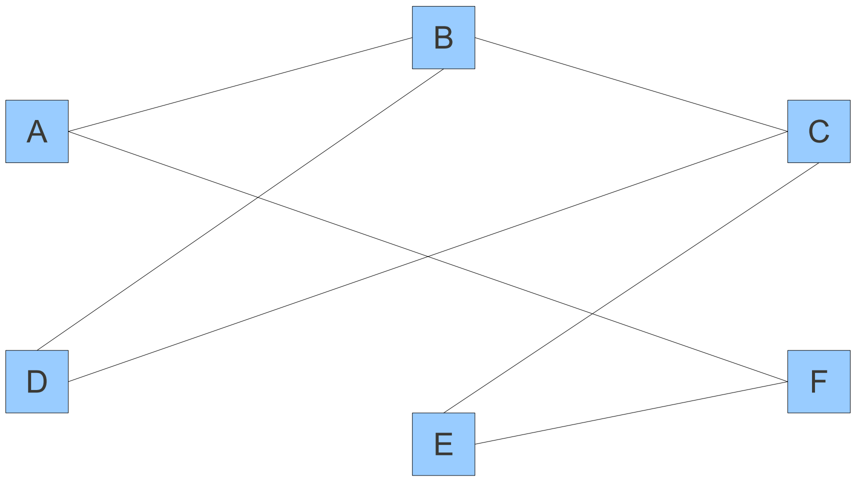
**(a)** [5 pts] Use Fermat's Theorem to show that $S_p \equiv 1 + 2 + \dots + (p-1) \pmod{p}$.

**(b)** [5 pts] Show that $p|(1 + 2 + \dots + (p-1))$ and explain why this implies that $p$ divides $S_p$.

**Problem 6. [[ points]** 20]

Consider the simple graph $G$ given in figure $X$.

Figure 1: Simple graph G




**(a)** [0 pts] Give the diameter of $G$.

**(b)** [0 pts] Give a longest path on $G$.

**(c)** [0 pts] Give a coloring on $G$ and show that it uses the smallest possible number of colors.

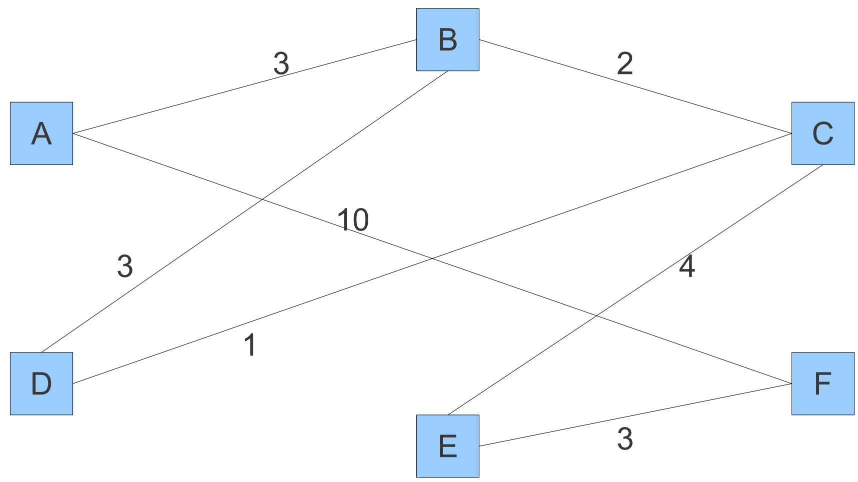
**(d)** [0 pts] Does $G$ have an Eulerian cycle? Justify your answer.

Now consider graph $H$, which is like $G$ but with weighted edges, in figure $Y$:

**(e)** [0 pts] Draw a minimum spanning tree on $H$.

**(f)** [0 pts] Give a list of edges reflecting the order in which a greedy algorithm would choose edges when finding an MST on $H$.

Figure 2: Weighted graph H




**Problem 7. [20 points]** Consider a strongly connected directed graph with indegree$(v) =$ outdegree$(v)$ for all $v \in V$. We will prove such a graph has a (directed) Eulerean tour, by considering its longest path.

[10 pts] Show the longest sequence of adjacent edges (walk or tour) where no edges are repeated is a tour.

[10 pts] Show no directed edge is left out of the longest possible walk or tour.

**Problem 8. [10 points]** Let G be a graph with $n$ vertices, $m$ edges and $k$ components. Prove that $G$ contains at least $n + m - k = c$ cycles. (Hint: Prove this by induction on the number of edges, $m$)

**Problem 9. [20 points]** The 6.042 professors are planning to have a midterm exam and want the midterm grades to be recognized by the EECS department. This requires the completion of a number tasks, each of which takes one hour to complete. The prerequisites associated with these tasks are listed below.

| ABBRV. | TASK | PREREQUISITES |
|---|---|---|
| S | Hold an ice cream study session | I |
| W | Write midterm questions | I (for the TAs) |
| G | Grade the midterms | H, W |
| H | Hold the midterm | W,R,S |
| I | Buy ice cream | |
| R | Release a sample midterm | |

**(a)** [4 pts] Draw the Hasse diagram for the tasks and their prerequisites.

**(b)** [2 pts] Give one ordering of the tasks that will fulfill the department's prerequisites.

**(c)** [2 pts] The professors have decided that since their TAs are quite smart and they have so many of them, they can get as many tasks done at a time as they wish. What is the minimum amount of time required for them to finish all the tasks? Give a sample scheduling, listing the tasks performed in each time slot.

Assume now that you are given a Hasse diagram with $n$ vertices in which the longest antichain has length $t$. Without knowing anything else about the graph...

**(d)** [4 pts] ...write a simple formula in $n$ and $t$ for the maximum possible length of the longest chain in such a graph.

**(e)** [3 pts] ...write a simple formula in $n$ and $t$ for the minimum possible length of the longest chain in such a graph.

**Problem 10. [0 points]** Induction: Prove that a sum of consecutive odd numbers (beginning with 1); i.e.

$$\sum_{i=0}^{n} 2i + 1$$

with $n \geq 1$; is a perfect square.

*Hint: prove something stronger*

**Problem 11. [10 points]** Use integration to find upper and lower bounds that differ by at 1 for the following sum.

$$\sum_{i=1}^{\infty} \frac{1}{i^4}$$

**Problem 12. [G points]** ive a proof of the following propositions.

1. $x$ is $O(x \ln x)$
2. $x / \ln x$ is $o(x)$
3. $x^{n+1}$ is $\Omega(x^n)$
4. $n!$ is $\omega(n^n)$