Graph Theory III

1 Build-Up Error

Here is a false proof about connectivity. It exposes a very common flaw made on proofs by induction on graphs – it even has a name – it is known as "build-up error".

False Claim. *If every vertex in a graph has degee at least 1, then the graph is connected.*

There are many counterexamples; here is one:

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Since the claim is false, there must be at least one error in the following "proof".

Proof. We use induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has degree at least 1, then the graph is connected.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, $P(1)$ is vacuously true, since the if-part is false.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an n-vertex graph in which every vertex has degree at least 1. By the assumption $P(n)$, this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to obtain an $(n+1)$-vertex graph:

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All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least one, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge xy to the path from y to z. This proves $P(n+1)$.

By the principle of induction $P(n)$ is true for all $n \geq 1$, which proves the theorem. □

That looks fine! Where is the bug? It turns out that faulty assumption underlying this argument is that *every* $(n+1)$*-vertex graph with minimum degree 1 can be obtained from an* n*-vertex graph with minimum degree 1 by adding 1 more vertex.* Instead of starting by considering an arbitrary $(n+1)$- node graph, this proof only considered an $(n+1)$-node graph that you can make by starting with an n-node graph with minimum degree 1.

The counterexample above shows that this assumption is false; there is no way to build that 4-vertex graph from a 3-vertex graph with minimum degree 1. Thus the first error in the proof is the statement "This proves $P(n+1)$".

More generally, this is an example of "build-up error". Generally, this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others— such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a "shrink down, grow back" process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds. Let's see what would have happened if we'd tried to prove the claim above by this method:

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$-vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh!

The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck— and properly so, since the claim is false!