Problem Set 9 Solutions

Due: *Monday November 6 at 8 PM*

Problem 1. [15 points]

(a) Describe a bijection between the sequences $x_1, x_2, \ldots, x_k$ of positive integers such that

$$x_1 < x_2 < \cdots < x_k \leq n \tag{1}$$

and the length n bit-strings (i.e., strings of 0's and 1's) containing exactly k 1's.

Solution. Map $(x_1, x_2, \ldots, x_k)$ to

$$\texttt{0}^{x_1-1}\texttt{1}\texttt{0}^{x_2-x_1-1}\texttt{1}\texttt{0}^{x_3-x_2-1}\texttt{1}\texttt{0}\ldots\texttt{1}\texttt{0}^{x_k-x_{k-1}-1}\texttt{1}\texttt{0}^{n-x_k}.$$

Notice that there are exactly k 1's in the final string. Also, the prefix of the string ending with ith 1 is exactly x_i letters long. So the length of the prefix ending with the last 1 is x_k, and the final string of 0's ensures the length of the whole string is n.

It's not hard to see how to recover the sequence from the bit-string it maps to. Also, we can recover a sequence from *any* length n bit-string with k 1's. This means that the mapping from sequences to bit-strings has an inverse mapping bit-strings back to sequences, which implies that the mapping is indeed a bijection.

Another, indirect, way to obtain a bijection is to observe that there is bijection between the sequences satisfying (1) and the k-element subsets of $\{1, \ldots, n\}$. Namely, map the sequence $(x_1, x_2, \ldots, x_k)$ to $\{x_1, x_2, \ldots, x_k\}$. We know there is a bijection between these k-element subsets and bit-strings of length n with k occurrences of 1's. Composing these two bijections yields a bijection from sequences to bit-strings.

(b) Use the bijection to write a closed form (which may involve factorials) for the number of sequences satisfying (1).

Solution.

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

Problem 2. [20 points] Every Halloween, 3 children visit your neighborhood trick-or-treating. This year, you have 4 pieces of candy to give them, all of which you will place into 3 bags. You can choose any placement of candy in the bags, including leaving a bag empty (poor kid!). How many placements are there if:

(a) You have 4 different types of candy, and the bags are labeled for the children, Ann, Bob, and Cuauhtemoc?

Note: once the pieces of candy are in a bag, there is no way to tell the order in which they were placed.

Solution. The answer is 81. We have 3 choices for the first piece of candy, 3 choices for the second piece, etc. The product rule shows that there are $3^4 = 81$ ways in total.

(b) You only have 1 type of candy, and the bags are unlabeled because you couldn't figure out how to spell "Ann" properly (i.e. the bags are indistinguishable except for the number of pieces of candy in them)?

Solution. The answer is 4. there are four possibilities for the number of pieces of candy in the bags: all 4 pieces in one bag; 3 in one bag and 1 in another; 2 in one bag and 2 in another; or 2 in one and 1 in each of the other two. We do not care which pieces of candy end up in which bag, so the number of ways is just 4.

(c) You have 4 types of candy, but the bags are unlabeled?

Solution. The answer is 14. As observed in part (b), there are four possibilities for the number of piece of candy in the bags: all 4 pieces in one bag; 3 in one bag and 1 in another; 2 in one bag and 2 in another; or 2 in one and 1 in each of the other two. Since the bags are not distinguishable, there is only one way to put all 4 pieces into the same bag. Since the pieces *are* distinguishable, there are $\binom{4}{3}$ ways of putting 3 pieces into one bag, and this determines which ball is by itself in the other non-empty bag. There are $\binom{4}{2}/2$ of putting 2 pieces in one bag and 2 in another. There are $\binom{4}{2}$ ways of putting 2 in one and 1 in each of the other two. So the total number of ways is

$$1 + \binom{4}{3} + \frac{1}{2}\binom{4}{2} + \binom{4}{2} = 14.$$

(d) The bags are labeled, but you only have 1 type of candy?

Solution. The answer is 15. This is like the problem of choosing a dozen donuts seen in lecture. Here the bags are the "types of donuts" and the pieces correspond to the donuts whose type we can choose. (This sort of problem is also known as a "stars and bars" problem: the donuts are the stars, and the bars are the separators between the types of donuts.) The answer is

$$\binom{4+3-1}{4} = \binom{6}{4} = 15.$$

Problem 3. [15 points] An urn contains balls numbered from 1 to 9. You draw 6 balls from the urn. How many different *ordered* draws can you get if:

(a) Ball 1 is included in the draw?

Solution. There are 6 ways to choose the position of ball 1 in the ordering, and $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ ways to choose and arrange the others. So the total is:

$$6 \cdot \frac{8!}{3!} = 40320.$$

(b) Ball 1 and ball 2 are both included in the draw?

Solution. There are 6 ways to place ball 1 and 5 ways to place ball 2. There are $7 \cdot 6 \cdot 5 \cdot 4$ ways to choose and arrange the others. So the total is:

$$6 \cdot 5 \cdot \frac{7!}{3!} = 25200.$$

(c) Exactly one of ball 1 and ball 2 is included in the draw?

Solution. There are 2 ways to choose between ball 1 and ball 2, 6 ways to place whichever one is chosen, and $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$ to choose and arrange the remaining people. Total:

$$2 \cdot 6 \cdot \frac{7!}{2!} = 6 \cdot 7! = 30240.$$

Problem 4. [10 points] (The two parts of this problem are independent.)

(a) Describe a bijection between the length n bit-strings with an even number of `1`'s and the length n bit-strings with an odd number of `1`'s. Explain why any such bijection leads to a simple formula for the number of even-size subsets of any set of size n.

Solution. Map a string to the same string with the first bit complemented. This map is its own inverse, and hence is a bijection from length n bit-strings to length n bit-strings. Restricting it to the length n bit-strings with an even number of `1`'s yields a bijection between the length n bit-strings with an even number of `1`'s and the length n bit-strings with an odd number of `1`'s. So there are the same number of strings of each kind, and therefore exactly half of the length n bit-strings have an even number of `1`'s, namely, $2^n/2 = 2^{n-1}$ strings with an an even number of `1`'s.

But with the standard bijection between bit-strings of length n and subsets of a set of size n, the strings with an even number of `1`'s correspond to even-size subsets. So the number of even-size subsets is also 2^{n-1}.

(b) A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?

Solution. There are 5 bars (chosen books), and therefore there are 6 places where the 7 stars (non-chosen books) can fit (before the first bar, between the first and second bars,..., after the fifth bar). Each of the second through fifth of these slots must have at least one star in it, so that adjacent books are not chosen. Once we have placed these 4 stars, there are 3 stars left to be placed in 6 slots. The number of ways to do this is therefore

$$\binom{6+3-1}{3} = \binom{8}{3} = 56.$$

Problem 5. [15 points] Suppose we have n distinct points on the sphere, where the sphere is taken to mean all points of the form $(x, y, z) \in \mathcal{R}^3$ with $x^2 + y^2 + z^2 = 1$. Show that there is a closed hemisphere containing $2 + \lceil \frac{n-2}{2} \rceil$ of the n points.

Here, by closed hemisphere we mean exactly half of the sphere, including its boundary points. For instance, if we take two distinct points p, q on the sphere, and draw the unique circle C containing p and q centered at the origin, then the sphere decomposes into two closed hemispheres with common intersection C.

Solution. Consider any 2 of the n points, say p and q, and draw the circle C containing p and q centered at the origin. This divides the sphere into two closed hemispheres with common intersection C. By the pigeonhole principle, $\lceil \frac{n-2}{2} \rceil$ of the remaining points must lie in one of the hemispheres, and since the hemisphere is closed, there are $2 + \lceil \frac{n-2}{2} \rceil$ points in that hemisphere.

Problem 6. [15 points] We want to split the class into teams of four. Suppose there are $4n$ students in the class, so there should be n teams. Write a simple closed-form formula (which may involve factorials) for the number of ways the class could be split. Explain your answer.

Solution.

$$\frac{(4n)!}{n! \cdot 24^n} \tag{2}$$

There are $(4n)!$ sequences consisting of the distinct students. From such a sequence, we can split into teams by teaming up the first four students, the next four students, ..., the last four students.

Now we observe that for a given split, there are $n!$ sequences of *teams* that describe that split. Each team can be ordered in $4!$ ways, so there are $(4!)^n$ ways to order every one of the teams. That is, each split arises from exactly $n! \cdot 24^n$ sequences. So (2) follows by the Division Rule.

Problem 7. [10 points] In preparation for a 6.042 study session, you want to calculate the number of different ways to make sundaes for you and your friends. You have 10 different toppings, and you want to make four sundaes such that each sundae has between one and four (inclusive) toppings, and you don't reuse any toppings. The sundaes are going to 4 different people, so their order matters! How many ways can this be done?

Solution. We first enumerate the different ways to divide up 10 toppings to the 4 sundaes, where we don't distinguish between toppings, and the order of the sundaes doesn't matter. One such way would be to put 1 topping on one sundae, and 3 on each of the other 3, denoted by $[1, 3, 3, 3]$. Similarly, we have $[1, 1, 4, 4]$, $[1, 2, 3, 4]$, $[2, 2, 3, 3]$, and $[2, 2, 2, 4]$.

Now, let's consider the order of the sundaes. For $[1, 3, 3, 3]$, we also have [3, 1, 3, 3], [3, 3, 1, 3], and [3, 3, 3, 1], so there are 4 ways to get 1 topping on 1 sundae and 3 on each of the other 3 (also, $4!/3!$, as we have 4 elements to order, 3 of which are the same). Similarly, we have $4!/(2!2!) = 6$ ways of getting $[1, 1, 4, 4]$, $4! = 24$ ways for $[1, 2, 3, 4]$, $4!/(2!2!) = 6$ ways for $[2, 2, 3, 3]$, and $4!/3! = 4$ ways for $[2, 2, 2, 4]$.

Finally, let's distinguish between the toppings. We assume that the order of toppings on a given sundae doesn't matter (i.e. a sundae with fudge and whipped cream is the same as a sundae with whipped cream and fudge). We then have $10!/(1!3!3!3!)$ ways to distinguish between toppings when we have the division $[1, 3, 3, 3]$, and so on. Therefore, the final answer is:

$$4\frac{10!}{1!3!3!3!} + 6\frac{10!}{1!1!4!4!} + 24\frac{10!}{1!2!3!4!} + 6\frac{10!}{2!2!3!3!} + 4\frac{10!}{2!2!2!4!}$$
$$= 10!(\frac{4}{216} + \frac{6}{576} + \frac{24}{288} + \frac{6}{144} + \frac{4}{192}) = 634200.$$

(Note that this is a large proportion of the total $4^{10} = 1048576$ ways to distribute 10 toppings to 4 sundaes, so our restrictions weren't that restrictive!)