# Finding Your Seat Versus Tossing a Coin

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**Abstract.** In a classroom of $n$ seats and $n$ students, the first student sits at random, whereas every other student must sit at her/his seat, but may sit randomly if her/his seat is already taken. The probability that a student finds her/his seat is given by a simple formula. Two entertaining proofs are given.

**1. INTRODUCTION.** Let us suppose that we have $n$ students $\{A_1, A_2, \ldots, A_n\}$, to be seated in a classroom that has $n$ seats $\{\text{Seat}_1, \text{Seat}_2, \ldots, \text{Seat}_n\}$. Suppose that $A_1$ is given the privilege of sitting at random. All others $A_2, A_3, \ldots, A_n$ in numerical order are instructed to be seated at their own seat, but can also sit at random, if some other student has already taken their seat. For $j \geq 2$, let $P(n, j)$ denote the probability that $A_j$ sits on $\text{Seat}_j$. Then, we have the following proposition.

**Proposition 1.** *If $n \geq 2$ and $2 \leq j \leq n$, then $P(n, j) = \frac{n-j+1}{n-j+2}$.*

In the next section, two short proofs of the proposition are given. We conclude with a brief discussion.

**2. TWO SHORT PROOFS OF PROPOSITION 1.**

*First proof.* The result is obvious for $n = 2$. If $A_1$ sits on $\text{Seat}_k$, for $1 \leq k \leq n$, then

$$P(n, j) = \begin{cases} 1 & \text{if} \quad k \in \{1\} \cup \{j+1, j+2, \ldots, n\}, \\ 0 & \text{if} \quad k = j, \\ P(n-k+1, j-k+1) & \text{if} \quad k \in \{2, 3, \ldots, j-1\}. \end{cases}$$

Note that when $2 \leq k \leq j-1$, we remove $\{\text{Seat}_2, \text{Seat}_3, \ldots, \text{Seat}_k\}$ and $\{A_1, A_2, \ldots, A_{k-1}\}$, and then for each $p \geq k$ give $A_p$ the role of $A_{p-(k-1)}$, in a classroom of $n-(k-1)$ seats and $n-(k-1)$ students. Clearly, for $k = 1, 2, \ldots, n$, the probability $A_1$ sits at $\text{Seat}_k$ is $\frac{1}{n}$. So,

$$P(n, j) = \frac{1}{n}\left(1 + \sum_{k=2}^{j-1} P(n-k+1, j-k+1) + 0 + \sum_{k=j+1}^{n} 1\right).$$

Since

$$\frac{n-k+1-(j-k+1)+1}{n-k+1-(j-k+1)+2} = \frac{n-j+1}{n-j+2},$$

by induction on $n$, we have

$$P(n, j) = \frac{1}{n}\left[(j-2)\frac{n-j+1}{n-j+2} + n - j + 1\right].$$

http://dx.doi.org/10.4169/amer.math.monthly.121.06.545
MSC: Primary 60A05, Secondary 60B99; 60C05

Then, by elementary algebra (factoring $(n - j + 1)$), the formula follows. ∎

The following constitutes a combinatorial proof with less calculation.

*Second proof.* It is trivial to see by induction on $j$, for $j > 2$, that when it is $A_j$'s turn to sit, $\text{Seat}_2, \text{Seat}_3, \ldots, \text{Seat}_{j-1}$ are all taken, in all cases. But so is one more seat taken among the remaining $n - j + 2$ seats, by exactly one of $\{A_1, A_2, \ldots, A_{j-1}\}$, whom we know has *equal* chance of sitting at any one of those $n - j + 2$ seats. So, $P(j, n) = 1 - \frac{1}{n-j+2}$. ∎

**3. CONCLUSION.** For the last student, $P(n, n) = \frac{1}{2}$, regardless the value $n \geq 2$, (calls head or tail on the first or last seat); $P(n, n-1) = \frac{2}{3}$ (on the first or last two seats), and so on. So, it is remarkable that the reverse order chances are invariant to $n$.

**ACKNOWLEDGMENT.** I thank a former mathematician and friend Jin Qian, for posing a similar version of this fun problem and later for suggesting the first nice proof presented in this article.

REFERENCE

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**100 Years ago this Month in *The American Mathematical Monthly***
**Edited by Vadim Ponomarenko**

Special efforts are being made to secure as large a collection as possible of letters by and to LEONHARD EULER for publication in his complete works. In view of the fact that this publication is much more expensive than was expected, the committee in charge is anxious to secure more subscribers for these works, as well as more members of the Euler Society. The members of this society promise to contribute at least ten francs per year towards the expense of publishing Euler's complete works.

Complete sets of the MONTHLY for the year 1913, being the first volume under the reorganization, are in constant demand, but the supply of certain numbers in that volume has been exhausted. These numbers are January and June, 1913. Since the last issue several extra copies of the January number have been sent in and now a few complete sets of volume xx are again available, and several other sets are complete except for the June number. Any one who can contribute an extra copy of either the January or the June number will confer a great favor upon the editors.

—Excerpted from "Notes and News" **21** (1914) 205–208.