Midterm

Problem 1. [10 points]

Consider these two propositions:

$$\texttt{P:}\quad (A \land B) \Rightarrow C$$
$$\texttt{Q:}\quad (\neg C \Rightarrow \neg A) \lor (\neg C \Rightarrow \neg B)$$

Use the truth table below to show whether P and Q are equivalent, $P \Rightarrow Q$, $Q \Rightarrow P$, or none of the above.

A	B	C	$(A \land B) \Rightarrow C$	$(\neg C \Rightarrow \neg A) \lor (\neg C \Rightarrow \neg B)$

Problem 2. [15 points]

(a) [8 pts] Find a number $x \in \{0, 1, \ldots, 112\}$ such that $18x \equiv 1 \pmod{113}$.

Solution. We can do this using the pulverizer. Specifically, if we find a pair (s, t) such that $18s + 113t = 1$, then we know that $18s \equiv 1 \pmod{113}$.

$$\begin{array}{cccll}
x & y & \text{rem}\,(x,y) & = & x - q \cdot y \\
\hline
113 & 18 & 5 & = & 113 - 6 \cdot 18 \\
18 & 5 & 3 & = & 18 - 3 \cdot 5 \\
 & & & = & 18 - 3 \cdot (113 - 6 \cdot 18) \\
 & & & = & -1 \cdot 113 + 19 \cdot 18 \\
5 & 3 & 2 & = & 5 - 3 \\
 & & & = & (113 - 6 \cdot 18) - (-1 \cdot 113 + 19 \cdot 18) \\
 & & & = & (2 \cdot 113 - 25 \cdot 18) \\
3 & 2 & 1 & = & 3 - 2 \\
 & & & = & (-1 \cdot 113 + 19 \cdot 18) - (2 \cdot 113 - 25 \cdot 18) \\
 & & & = & \boxed{-3 \cdot 113 + 44 \cdot 18}
\end{array}$$

So the multiplicative inverse of 18 modulo 113 is 44. ■

(b) [7 pts] Find a number $y \in \{0, 1, \ldots, 112\}$ such that $18^{112111} \equiv y \pmod{113}$ (*Hint: What power of* 18 *is x equivalent to modulo* 113*?*)

Solution. By Fermat's Theorem, since 113 is prime and 113 and 18 are relatively prime, it must be that

$$18 \cdot 18^{111} \equiv 18^{113-1} \equiv 1 \pmod{113},$$

so $x \equiv 111 \pmod{113}$. As a result,

$$18^{112111} \equiv 18^{112\cdot 1000+111} \equiv 18^{112^1}000 \cdot 18^{111} \equiv 1^{1000} \cdot x \equiv x \equiv 44 \pmod{113},$$

so the answer is 44. ■

Problem 3. [10 points] Define a number $S_p = 1^p + 2^p + 3^p + \ldots (p-1)^p$. You will show in this problem that if p is an odd prime, then $p | S_p$.

(a) [5 pts] Use Fermat's Theorem to show that $S_p \equiv 1 + 2 + \ldots + (p-1) \pmod{p}$.

Solution.

$$\begin{aligned} S_p &\equiv 1^p + 2^p + \ldots (p-1)^p \\ &\equiv 1 \cdot 1^{p-1} + 2 \cdot 2^{p-1} + \ldots + (p-1) \cdot (p-1)^{p-1} \\ &\equiv 1 \cdot 1 + 2 \cdot 1 + \ldots + (p-1) \cdot 1 \\ &\equiv 1 + 2 + \ldots + (p-1) \pmod{p} \end{aligned}$$

■

(b) [5 pts] Show that $p | (1 + 2 + \ldots + (p-1))$ and explain why this implies that p divides S_p.

Solution. $1 + 2 + \ldots + (p-1) = \frac{(p-1)p}{2}$.
p is odd, so $p-1$ is even, which means $\frac{p-1}{2}$ is some integer k. Therefore

$$1 + 2 + \ldots + (p-1) = kp,$$

so $p | (1 + 2 + \ldots + (p-1))$. As a consequence, $1 + 2 + \ldots + (p-1) \equiv 0 \pmod{p}$. Therefore

$$S_p \equiv 1 + 2 + \ldots + (p-1) \equiv 0 \pmod{p},$$

so $p | S_p$. ■

Problem 4. [20 points]

Let $G_0 = 1$, $G_1 = 2$, $G_2 = 4$, and define

$$G_n = G_{n-1} + 2G_{n-2} + G_{n-3} \tag{1}$$

for $n \geq 3$. Show by induction that $G_n \leq 3^n$ for all $n \geq 0$.

Solution. The proof is by strong induction with hypothesis $P(n) := G_n \leq 3^n$.

Proof. **Base Cases**

$n = 0$**:** $G_0 = 1 = 3^0$.

$n = 1$**:** $G_1 = 2 < 3 = 3^1$.

$n = 2$**:** $G_2 = f4 < 9 < 3^2$.

Inductive Step: Assume $n \geq 2$ and $P(k)$ for all k such that $0 \leq k \leq n$.

$$\begin{aligned}
G_{n+1} &= G_n + 2G_{n-1} + G_{n-2} && \text{by (1)} \\
&\leq 3^n + (2)3^{n-1} + 3^{n-2} && \text{by induction hypothesis} \\
&= 3^{n-2}[3^2 + (2)3 + 1] \\
&= 3^{n-2}[(3+1)^2] \\
&= 3^{n-2}4^2 \\
&= 3^{n-2}16 \\
&< 3^{n-2}27 \\
&= 3^{n-2}3^3 \\
&= 3^{n+1}
\end{aligned}$$

□

■

Problem 5. [0 points] Induction: Prove that a sum of consecutive odd numbers (beginning with 1); i.e.

$$\sum_{i=0}^{n} 2i + 1$$

with $n \geq 1$; is a perfect square.

Hint: prove something stronger

Problem 6. [0 points]

In the game of Squares and Circles, the players (you and your computer) start with a sequence of shapes: some circles and some squares. On each move a player selects two shapes. These two are replaced with a single one according to the following rule:

Identical shapes are replaced with a square. Different shapes are replaced with a circle.

At the end of the game, when only one shape remains, you are a winner if the remaining shape is a circle. Otherwise, your computer wins.

(a) [0 pts] Prove that the game will end.

Solution. Todo. ■

(b) [0 pts] Prove that you will win if the number of circles initially is odd. Hint: Use an invariant about the number of circles.

Solution. Todo. ■

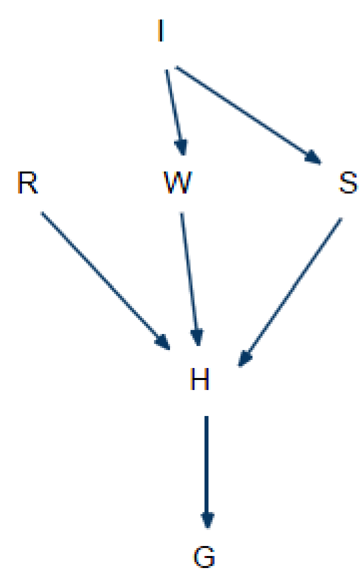
Problem 7. [10 points] Use integration to find upper and lower bounds that differ by at most 0.5 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{i^3}$$

Problem 8. [20 points] The 6.042 professors are planning to have a midterm exam and want the midterm grades to be recognized by the EECS department. This requires the completion of a number tasks, each of which takes one hour to complete. The prerequisites associated with these tasks are listed below.

ABBRV.	TASK	PREREQUISITES
S	Hold an ice cream study session	I
W	Write midterm questions	I (for the TAs)
G	Grade the midterms	H, W
H	Hold the midterm	W,R,S
I	Buy ice cream	
R	Release a sample midterm	

(a) [4 pts] Draw the Hasse diagram for the tasks and their prerequisites.



Solution.

■

(b) [2 pts] Give one ordering of the tasks that will fulfill the department's prerequisites.

Solution. I R W S H G ■

(c) [2 pts] The professors have decided that since their TAs are quite smart and they have so many of them, they can get as many tasks done at a time as they wish. What is the minimum amount of time required for them to finish all the tasks? Give a sample scheduling, listing the tasks performed in each time slot.

Solution. The minimum required time is 4 (length of the critical path I-W-H-G) A possible scheduling is:

1. R, I
2. W, S
3. H
4. G

■

Assume now that you are given a Hasse diagram with n vertices in which the longest antichain has length t. Without knowing anything else about the graph...

(d) [4 pts] ...write a simple formula in n and t for the maximum possible length of the longest chain in such a graph.

Solution. In the worst case there could be a chain of size $n - t + 1$ but no larger. ■

(e) [3 pts] ...write a simple formula in n and t for the minimum possible length of the longest chain in such a graph.

Solution. $\lceil n/t \rceil$. Since there is no antichain of size $t + 1$, we know that at least one chain must have $\geq n/t$ elements. ■

Problem 9. [G points] ive a proof of the following propositions.

1. x is $O(x \ln x)$
2. $x/\ln x$ is $o(x)$
3. x^{n+1} is $\Omega(x^n)$
4. $n!$ is $\omega(n^n)$

Problem 10. [20 points] Consider a strongly connected directed graph with indegree$(v) =$ outdegree(v) for all $v \in V$. We will prove such a graph has a (directed) Eulerean tour, by considering its longest path.

[10 pts] Show the longest sequence of adjacent edges (walk or tour) where no edges are repeated is a tour.

Solution. Consider the longest walk or tour in the graph (whichever is longer). We want to show it is indeed a tour. By way of contradiction, suppose not. Then we have a walk where each edge is different and the starting node v_s is different from the end node. The starting node may have appeared in the walk on several occasions but not at the end. We can count the number of edges used on each occasion. There is the initial start edge. From then on the rest of the edges are paired: one in for each one out of v_s. Hence we used up one more outgoing edge than we used incoming edges. Hence we can extend the path backwards by prefixing that missing incoming edge to v_s, which contradicts our walk was as long as possible. ∎

[10 pts] Show no directed edge is left out of the longest possible walk or tour.

Solution. Suppose some edge (u, w) is left out from the longest path. Since the graph is strongly connected, there is a shortest path from u to some v in the longest tour. We can then construct an even longer path: $u, w, \cdots v, \cdots, v$ Where we prefix the path onto the existing cycle. ∎