Problem Set 4 Solutions

Due: Monday, October 4 (7pm)

Problem 1. [15 points] Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subset E$ such that no two edges in M are incident on a common vertex.

Let M_1, M_2 be two matchings of G. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Show that G' is bipartite.

Helpful definition: A *connected component* is a subgraph of a graph consisting of some vertex and every node and edge that is connected to that vertex.

Solution. We will show that G' has no odd cycle. By the theorem proved in recitation (that a graph is bipartite iff it has no odd cycles) we will be done.

First, consider any vertices connected by an edge that is in both M_1 and M_2. Since these are matchings, there is no other edges connected to these vertices in either M_1 or M_2 and thus G'. These vertices form connected components of size two which have no odd length cycles. For the rest of the proof, we can assume that every edge in G' came from exactly one of M_1 or M_2.

Take a sequence of edges $(e_1, e_2, \ldots, e_k)$ in G' that form a cycle. Let us show that k must be even. We know that $e_1 \in M_1 \cup M_2$. Assume $e_1 \in M_1$ (otherwise $e_1 \in M_2$ and the argument is identical replacing M_1 with M_2). Now since e_1 and e_2 are incident on a common vertex, and M_1 is a matching, e_2 cannot be in M_1. So $e_2 \in M_2$. Similarly $e_3 \in M_1$, $e_4 \in M_2$ etc. By induction we can prove that for all $i \leq k$, $e_i \in M_1$ iff i is odd.

Now if k had been odd, we would have $e_k \in M_1$. But e_k and e_1 are adjacent edges in the cycle, and hence incident on a common vertex, we have a contradiction. Thus k must be even. ■

Problem 2. [20 points] Let $G = (V, E)$ be a graph. Recall that the *degree* of a vertex $v \in V$, denoted d_v, is the number of vertices w such that there is an edge between v and w.

(a) [10 pts] Prove that

$$2|E| = \sum_{v \in V} d_v.$$

Solution. Let $S = \{(e, v) \in E \times V : e \text{ is incident on } v\}$.

Count the elements in S as follows

$$|S| = \sum_{e \in E} |\{v : (e, v) \in S\}| = 2|E|$$

and also as

$$|S| = \sum_{v \in V} |\{e : (e, v) \in S\}| = \sum_{v \in V} d_v.$$

The result follows.

Once can also prove it by induction on $|E|$. You should try to. ■

(b) [5 pts] At a 6.042 ice cream study session (where the ice cream is plentiful and it helps you study too) 111 students showed up. During the session, some students shook hands with each other (everybody being happy and content with the ice-cream and all). Turns out that the University of Chicago did another spectacular study here, and counted that each student shook hands with exactly 17 other students. Can you debunk this too?

Solution. Assume that the study is accurate. Define a graph $G = (V, E)$ with students as vertices and put an edge between 2 students if they shook hands. By the previous problem, we should have $2|E| = \sum_v d_v = 111 \cdot 17$. But the LHS is even and the RHS is odd, a contradiction. ■

(c) [5 pts] And on a more dull note, how many edges does K_n, the complete graph on n vertices, have?

Solution. Apply the first part of the problem. Notice that each vertex is joined to $n - 1$ others. $2|E| = \sum_v d_v = n(n - 1)$. So $|E| = n(n - 1)/2$. ■

Problem 3. [15 points] Two graphs are isomorphic if they are the same up to a relabeling of their vertices (see Definition 5.1.3 in the book). A property of a graph is said to be *preserved under isomorphism* if whenever G has that property, every graph isomorphic to G also has that property. For example, the property of having five vertices is preserved under isomorphism: if G has five vertices then every graph isomorphic to G also has five vertices.

(a) [5 pts] Some properties of a simple graph, G, are described below. Which of these properties is *preserved under isomorphism*?

1. G has an even number of vertices.
2. None of the vertices of G is an even integer.
3. G has a vertex of degree 3.
4. G has exactly one vertex of degree 3.

Solution. 1. It is invariant under isomorphism. There must be an one-to-one and onto mapping between the vertices of two isomorphic graphs. Therefore, the number of vertices in the two graphs must be the same. If one graph has even number of vertices, then the other must have even number of vertices.

2. It is not invariant under isomorphism. We do not really care what the vertices are. Vertices can be any kind of mathematical objects. All we are interested is that whether there exists a one-to-one and onto function f mapping from vertices of one graph to vertices of another with the property that a and b are adjacent in the first graph if and only if $f(a)$ and $f(b)$ are adjacent in the second graph, for all a and b in the first graph.

   So, for example, let G_1 be a graph with a single vertex, 1, and G_2 be a graph with a single vertex, 2. Obviously the two graphs are isomorphic, but G_2 does not have vertices which are even integers.

3. It is invariant under isomorphism.

   Let G_1, G_2 be simple graphs and $f : V_1 \to V_2$ be an isomorphism between them. Suppose $v \in V_1$ has degree 3; we want to show that there is a vertex of degree 3 in V_2. In fact, we'll show that $f(v)$ has degree 3.

   Since v has degree 3, there are exactly 3 vertices adjacent to v; say these are v_1, v_2, v_3. Since f is a bijection, $f(v_1), f(v_2)$ and $f(v_3)$ are all distinct. Since there is an edge between v and v_i in G_1, the definition of isomrphism implies that there is an edge between $f(v)$ and $f(v_i)$ for $i = 1, 2, 3$, so the degree of $f(v)$ is at least 3.

   We now prove by contradiction that the degree of $f(v)$ is at most 3. Suppose $f(v)$ had degree > 3. This means there is a vertex $w \in V_2$ which is not equal to $f(v_1), f(v_2)$, or $f(v_3)$, but is also adjacent to $f(v)$. Since, f is a bijection, there is a vertex $v_4 \in V_1$ such that $f(v_4) = w$ and $v_4 \neq v_i$ for $i = 1, 2, 3$. Since $w = f(v_4)$ is adjacent to $f(v)$, the definition of isomorphism implies that v_4 is adjacent to v, contradicting the fact the v_1, v_2, v_3 were exactly the vertices adjacent to v.

4. It is invariant under isomorphism.

   Prove by contradiction: Suppose a graph G_1 has exactly one vertex of degree 3 while another graph G_2 does not have exactly one vertex of degree 3. Suppose the two graphs are isomorphic. If G_2 does not have a vertex of degree 3, then from part (c), there is a contradiction. If G_2 has more than one vertices of degree 3, then there must be at least one vertex in G_2 of degree 3 which is mapped to a vertex of degree $\neq 3$ in G_1. Since two vertices of different degrees are mapped from G_1 to G_2, using the same argument from part (c), it reaches a contradiction. Therefore, if G_1 has exactly one vertex of degree 3 , then G_2 must also have exactly one vertex of degree 3.

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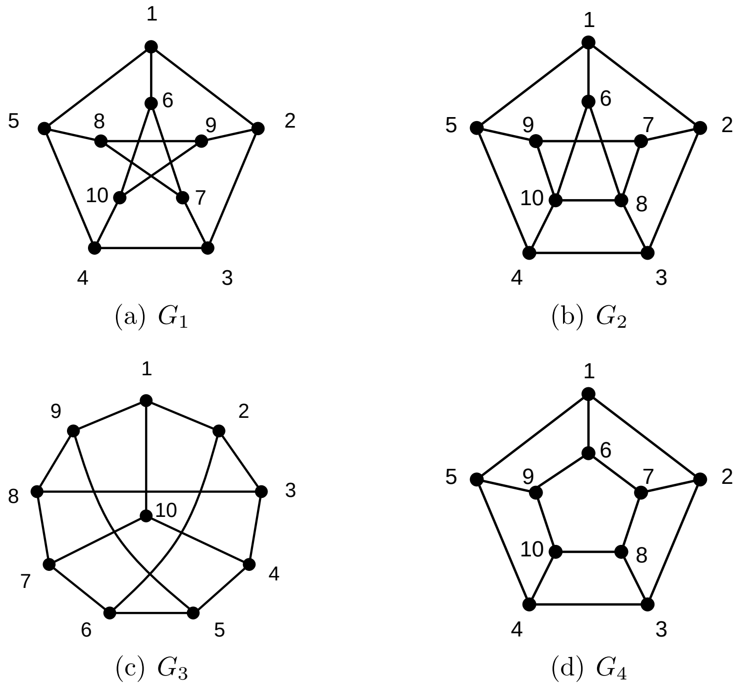
(b) [10 pts] Determine which among the four graphs pictured in the Figures are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).



(a) G_1 (b) G_2 (c) G_3 (d) G_4

Figure 1: Which graphs are isomorphic?

Solution. G_1 and G_3 are isomorphic. In particular, the function $f : V_1 \to V_3$ is an isomomorphism, where

$$\begin{array}{lllll} f(1) = 1 & f(2) = 2 & f(3) = 3 & f(4) = 8 & f(5) = 9 \\ f(6) = 10 & f(7) = 4 & f(8) = 5 & f(9) = 6 & f(10) = 7 \end{array}$$

G_1 and G_4 are not isomorphic to G_2: G_2 has a vertex of degree four and neither G_1 nor G_4 has one.

G_1 and G_4 are not isomorphic: G_4 has a simple cycle of length four and G_1 does not.

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Problem 4. [15 points] Recall that a ***coloring*** of a simple graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color. A k-***coloring*** is a coloring that uses at most k colors.

False Claim. *Let G be a (simple) graph with maximum degree at most k. If G also has a vertex of degree less than k, then G is k-colorable.*

(a) [5 pts] Give a counterexample to the False Claim when $k = 2$.

Solution. One node by itself, and a separate triangle (K_3). The graph has max degree 2, and a node of degree zero, but is not 2-colorable. ■

(b) [10 pts] Consider the following proof of the False Claim:

Proof. Proof by induction on the number n of vertices:

Induction hypothesis: $P(n)$ is defined to be: Let G be a graph with n vertices and maximum degree a most k. If G also has a vertex of degree less than k, then G is k-colorable.

Base case: (n=1) G has only one vertex and so is 1-colorable. So $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices and maximum degree at most k. Also, suppose G_{n+1} has a vertex, v, of degree less than k. We need only prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Removing v reduces the degree of all vertices adjacent to v by 1. So in G_n, each of these vertices has degree less than k. Also the maximum degree of G_n remains at most k. So G_n satisfies the conditions of the induction hypothesis $P(n)$. We conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. □

Identify the exact sentence where the proof goes wrong.

Solution. "So G_n satisfies the conditions of the induction hypothesis $P(n)$." The flaw is that if v has degree 0, then removing v will not reduce the degree of any vertex, and so there may not be any vertex of degree less than k in G_n, as in the counterexample of part (a). ■

Problem 5. [15 points] Prove or disprove the following claim: for some $n \geq 3$ (n boys and n girls, for a total of $2n$ people), there exists a set of boys' and girls' preferences such that every dating arrangement is stable.

Solution. The claim is false.

Proof. We will use letters to denote girls and numbers to denoted boys.

There must be some girl A rated worst by at most $n-2$ boys. The reason is as follows. Each boy can rate exactly one girl worst. If each of the n girls was rated worst by at least $n-1$ boys, then there would have to be at least $n(n-1)$ boys in all. But this is false when $n \geq 3$, because then $n(n-1)$ exceeds n, the actual number of boys.

Suppose that this girl A is paired with the boy, 1, that she rates worst. Since at most $n-2$ boys rate girl A worst, there is some other boy, 2, that rates a different girl, B, worst. Suppose that boy 2 is paired with girl B.

Now girl A and boy 2 form a rogue couple. Girl A prefers every other boy to her date, 1. Similarly, boy 2 prefers every other girl to his date B. Therefore, A and 2 prefer one another to their current dates. □

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Problem 6. [20 points]

Let $(s_1, s_2, ..., s_n)$ be an arbitrarily distributed sequence of the number $1, 2, ..., n-1, n$. For instance, for $n = 5$, one arbitrary sequence could be $(5, 3, 4, 2, 1)$.

Define the graph G=(V,E) as follows:

1. $V = \{v_1, v_2, ..., v_n\}$
2. $e = (v_i, v_j) \in E$ if either:
   (a) $j = i + 1$, for $1 \leq i \leq n - 1$
   (b) $i = s_k$, and $j = s_{k+1}$ for $1 <= k <= n - 1$

(a) [10 pts] Prove that this graph is 4-colorable for any $(s_1, s_2, ..., s_n)$.

Hint: First show that a line graph is 2-colorable. Note that a line graph is defined as follows: The n-node graph containing $n - 1$ edges in sequence is known as the line graph L_n.

Solution. First we argue that any line graph is 2-colorable. Consider the line graph L_n with vertices $v_1, v_2, v_3....v_n$. Suppose we have two colors A and B. Then color all odd numbered vertices with color A and all even numbered vertices with color B. Since each odd vertex is adjacent only to even vertices, and vice versa, this is a valid 2-coloring.

Consider G. G is composed of two - possibly overlapping - line graphs: one line graph contains the vertices of G in order, while the other line graph of the vertices of G in the order of our sequence $(s_1, ..., s_n)$. Pick one of these two graphs to color first using colors A and B. This will be a temporary coloring. Now color the second line graph with colors C and D, noting that these are temporary colors as well. Now each vertex will be assigned two colors, one from the first line graph and one from the second line graph (since both line graphs contain all vertices). Define four new colors, AC, AD, BC, BD. Color the graph with AC if one temporary color is A and the other temporary color is C. Color the graph with AD if one temporary color is A and the other is D. Do the same with colors BC and BD.

We note that our original temporary colors represent adjacencies in the graph. That is we note that each vertex has at most 4 adjacent nodes (two from each line graph). If the first color is A then two of the adjacenct vertices will be colored B, and vice versa. If the second color is C, then the other two adjacenct vertices will be colored D, and vice versa. So if a graph has a specific color, such as AC, then it is adjacent only to vertices of color B and D. Since we color those vertices differently (with one of AB, BC, or BD), then our coloring does not color two adjacent vertices with the same color.

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(b) [10 pts] Suppose $(s_1, s_2, ..., s_n) = (1, a_1, 3, a_2, 5, a_3, ...)$ where $a_1, a2....$ is an arbitrary distributed sequence of the even numbers in $1, ..., n-1$. Prove that the resulting graph is 2-colorable.

Solution. Color all odd vertices with first color. Now color all the even vertices with a second color. Note that by problem definition odd vertices are only adjacent to even vertices and vice versa, hence this is a valid 2-coloring.

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