Problem Set 8

Due: November 22

Reading:

- Chapter **??**. *Conditional Probability*
- Chapter **??**. *Random Variables & Expectation*

Problem 1.
There were n Immortal Warriors born into our world, but in the end there can be *only one*. The Immortals' original plan was to stalk the world for centuries, dueling one another with ancient swords in dramatic landscapes until only one survivor remained. However, after a thought-provoking discussion probability, they opt to give the following protocol a try:

(i) The Immortals forge a coin that comes up heads with probability p.

(ii) Each Immortal flips the coin once.

(iii) If *exactly one* Immortal flips heads, then they are declared The One. Otherwise, the protocol is declared a failure, and they all go back to hacking each other up with swords.

One of the Immortals (Kurgan from the Russian steppe) argues that as n grows large, the probability that this protocol succeeds must tend to zero. Another (McLeod from the Scottish highlands) argues that this need not be the case, provided p is chosen carefully.

(a) A natural sample space to use to model this problem is $\{H, T\}^n$ of length-n sequences of H and T's, where the successive H's and T's in an outcome correspond to the Head or Tail flipped on each one of the n successive flips. Explain how a tree diagram approach leads to assigning a probability to each outcome that depends only on p, n and the number h of H's in the outcome.

(b) What is the probability that the experiment succeeds as a function of p and n?

(c) How should p, the bias of the coin, be chosen in order to maximize the probability that the experiment succeeds?

(d) What is the probability of success if p is chosen in this way? What quantity does this approach when n, the number of Immortal Warriors, grows large?

Problem 2.
We're interested in the probability that a randomly chosen poker hand (5 cards from a standard 52-card deck) contains cards from at most two suits.

(a) What is an appropriate sample space to use for this problem? What are the outcomes in the event $\mathcal{E}$ we are interested in? What are the probabilities of the individual outcomes in this sample space?

(b) What is $\Pr[\mathcal{E}]$?

Problem 3.
Suppose you have three cards: A$\heartsuit$, A$\spadesuit$ and a jack. From these, you choose a random hand (that is, each card is equally likely to be chosen) of two cards, and let n be the number of aces in your hand. You then randomly pick one of the cards in the hand and reveal it.

(a) Describe a simple probability space (that is, outcomes and their probabilities) for this scenario, and list the outcomes in each of the following events:

1. $[n \geq 1]$, (that is, your hand has an ace in it),
2. A$\heartsuit$ is in your hand,
3. the revealed card is an A$\heartsuit$,
4. the revealed card is an ace.

(b) Then calculate $\Pr\left[n = 2 \mid E\right]$ for E equal to each of the four events in part (a). Notice that most, but *not all*, of these probabilities are equal.

Now suppose you have a deck with d distinct cards, a different kinds of aces (including an A$\heartsuit$), you draw a random hand with h cards, and then reveal a random card from your hand.

(c) Prove that $\Pr[\text{A}\heartsuit \text{ is in your hand}] = h/d$.

(d) Prove that

$$\Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right] = \Pr[n = 2] \cdot \frac{2d}{ah}. \tag{1}$$

(e) Conclude that

$$\Pr\left[n = 2 \mid \text{the revealed card is an ace}\right] = \Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right].$$

Problem 4.
There is a subject—naturally not *Math for Computer Science*—in which 10% of the assigned problems contain errors. If you ask a Teaching Assistant (TA) whether a problem has an error, then they will answer correctly 80% of the time, regardless of whether or not a problem has an error. If you ask a lecturer, he will identify whether or not there is an error with only 75% accuracy.

We formulate this as an experiment of choosing one problem randomly and asking a particular TA and Lecturer about it. Define the following events:

$$\begin{aligned} E &::= [\text{the problem has an error}], \\ T &::= [\text{the TA says the problem has an error}], \\ L &::= [\text{the lecturer says the problem has an error}]. \end{aligned}$$

(a) Translate the description above into a precise set of equations involving conditional probabilities among the events E, T and L.

(b) Suppose you have doubts about a problem and ask a TA about it, and they tell you that the problem is correct. To double-check, you ask a lecturer, who says that the problem has an error. Assuming that the correctness of the lecturer's answer and the TA's answer are independent of each other, regardless of whether there is an error, what is the probability that there is an error in the problem?

(c) Is event T independent of event L (that is, $\Pr\left[T \mid L\right] = \Pr[T]$)? First, give an argument based on intuition, and then calculate both probabilities to verify your intuition.

probability$_{product_rule}$

Problem 5.
We want to count step-by-step paths between points with integer coordinates in three dimensions. A step may move a unit distance in the positive x, y or z direction. For example, a step from point $(2,3,7)$ in the y direction leads to $(2,4,7)$.

For points $\mathbf{p}$ and $\mathbf{q}$ we write $\mathbf{p} \leq \mathbf{q}$ to mean that $\mathbf{p}$ is coordinatewise less than or equal to $\mathbf{q}$. That is, if $\mathbf{p} = (p_x, p_y, p_z)$ and $\mathbf{q} = (q_x, q_y, q_z)$, then

$$\mathbf{p} \leq \mathbf{q} \quad ::= \quad [p_x \leq q_x \text{ AND } p_y \leq q_y \text{ AND } p_z \leq q_z].$$

So there is a path from $\mathbf{p}$ to $\mathbf{q}$ iff $\mathbf{p} \leq \mathbf{q}$.

(a) Let $P_{\{\mathbf{p},\mathbf{q}\}}$ be the set of paths from $\mathbf{p}$ to $\mathbf{q}$. Suppose that $\mathbf{p} \leq \mathbf{q}$, and let $d_x ::= q_x - p_x$, and likewise for d_y and d_z. Express the number of paths $|P_{\{\mathbf{p},\mathbf{q}\}}|$ as a multinomial coefficient involving the preceding quantities.

More generally, for any set S of points, let

$$P_S ::= \text{the paths that go through all the points in } S.$$

(b) Suppose $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c} \leq \mathbf{d}$. Express $|P_{\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}}|$ in terms of $|P_{\{\mathbf{p},\mathbf{q}\}}|$ for various $\mathbf{p}, \mathbf{q} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$.

(c) Let

$$\begin{aligned}
&\mathbf{o} ::= (0,0,0), \\
&\mathbf{a} ::= (3,7,11), \quad \mathbf{b} ::= (11,6,3), \quad \mathbf{c} ::= (10,5,40), \\
&\mathbf{d} ::= (12,13,14), \quad \mathbf{e} ::= (12,6,45), \\
&\mathbf{f} ::= (50,50,50).
\end{aligned}$$

Let N be the paths in $P_{\mathbf{o},\mathbf{f}}$ that do *not* go through any of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$. Express $|N|$ as an arithmetic combination of $|P_S|$ for various $S \subseteq \{\mathbf{o}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$. Do not include any terms $|P_S|$ that equal zero.