Problem Set 4 Solutions

Due: Monday, September 29

Reading Assignment: Sections 5.0-5.3

Problem 1. [15 points] Recall that a ***coloring*** of a simple graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color. A k***-coloring*** is a coloring that uses at most k colors.

False Claim. *Let G be a (simple) graph with maximum degree at most k. If G also has a vertex of degree less than k, then G is k-colorable.*

(a) [5 pts] Give a counterexample to the False Claim when $k = 2$.

Solution. One node by itself, and a separate triangle (K_3). The graph has max degree 2, and a node of degree zero, but is not 2-colorable. ■

(b) [10 pts] Consider the following proof of the False Claim:

Proof. Proof by induction on the number n of vertices:

Induction hypothesis: $P(n)$ is defined to be: Let G be a graph with n vertices and maximum degree a most k. If G also has a vertex of degree less than k, then G is k-colorable.

Base case: (n=1) G has only one vertex and so is 1-colorable. So $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices and maximum degree at most k. Also, suppose G_{n+1} has a vertex, v, of degree less than k. We need only prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Removing v reduces the degree of all vertices adjacent to v by 1. So in G_n, each of these vertices has degree less than k. Also the maximum degree of G_n remains at most k. So G_n satisfies the conditions of the induction hypothesis $P(n)$. We conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. □

Identify the exact sentence where the proof goes wrong.

Solution. "So G_n satisfies the conditions of the induction hypothesis $P(n)$." The flaw is that if v has degree 0, then removing v will not reduce the degree of any vertex, and so there may not be any vertex of degree less than k in G_n, as in the counterexample of part (a). ■

Problem 2. [15 points] Prove or disprove the following claim: for some $n \geq 3$ (n boys and n girls, for a total of $2n$ people), there exists a set of boys' and girls' preferences such that every dating arrangement is stable.

Solution. The claim is false.

Proof. We will use letters to denote girls and numbers to denoted boys.

There must be some girl A rated worst by at most $n - 2$ boys. The reason is as follows. Each boy can rate exactly one girl worst. If each of the n girls was rated worst by at least $n - 1$ boys, then there would have to be at least $n(n - 1)$ boys in all. But this is false when $n \geq 3$, because then $n(n - 1)$ exceeds n, the actual number of boys.

Suppose that this girl A is paired with the boy, 1, that she rates worst. Since at most $n - 2$ boys rate girl A worst, there is some other boy, 2, that rates a different girl, B, worst. Suppose that boy 2 is paired with girl B.

Now girl A and boy 2 form a rogue couple. Girl A prefers every other boy to her date, 1. Similarly, boy 2 prefers every other girl to his date B. Therefore, A and 2 prefer one another to their current dates. □

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Problem 3. [20 points] 6.042 is often taught using recitations. Suppose it happened that 8 recitations were needed, with two or three staff members running each recitation. The assignment of staff to recitation sections is as follows:

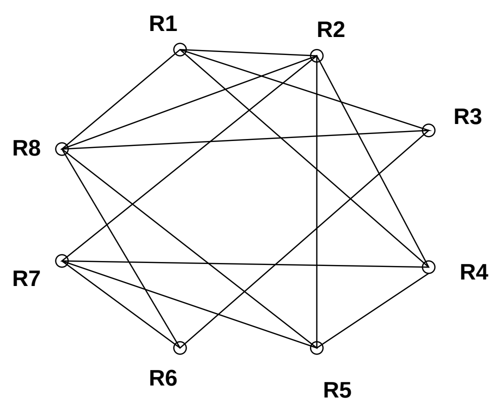
- R1: Maverick, Goose, Iceman
- R2: Maverick, Stinger, Viper
- R3: Goose, Merlin
- R4: Slider, Stinger, Cougar
- R5: Slider, Jester, Viper
- R6: Jester, Merlin
- R7: Jester, Stinger

- R8: Goose, Merlin, Viper

Two recitations can not be held in the same 90-minute time slot if some staff member is assigned to both recitations. The problem is to determine the minimum number of time slots required to complete all the recitations.

(a) [10 pts] Recast this problem as a question about coloring the vertices of a particular graph. Draw the graph and explain what the vertices, edges, and colors represent.

Solution. Each vertex in the graph below represents a recitation section. An edge connects two vertices if the corresponding recitation sections share a staff member and thus can not be scheduled at the same time. The color of a vertex indicates the time slot of the corresponding recitation.



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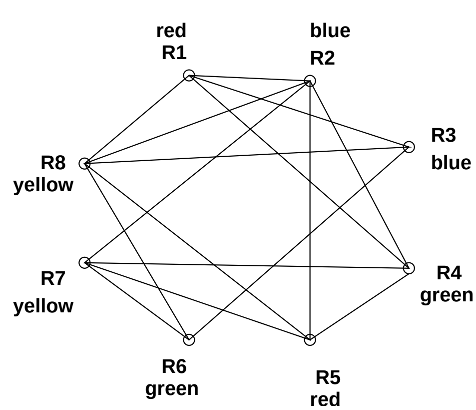
(b) [10 pts] Show a coloring of this graph using the fewest possible colors. What schedule of recitations does this imply?

Solution. Four colors are necessary and sufficient. To see why they are *sufficient*, consider the coloring:



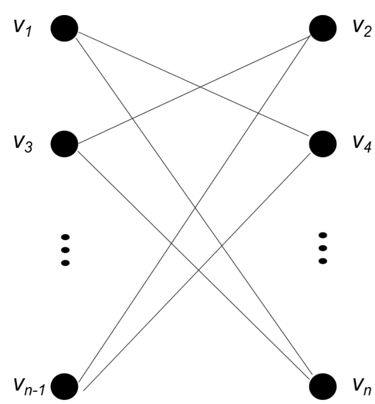
This corresponds to the following assignment of recitations to four time slots:

1. R1, R5
2. R2, R3
3. R4, R6
4. R7, R8

Other schedules are also possible.

To see why 4 colors are *necessary*, look at the subgraph defined by the vertices for R2, R4, R5, and R7. This is the complete graph on 4 vertices, and it obviously needs 4 colors. ■

Problem 4. [20 points] Suppose you have a graph as shown below. Every node on the left is adjacent to every node on the right except the node directly across from it.



(a) [5 pts] Find the chromatic number of the graph.

Solution. We can color each node on the left side with red and each node on the right with blue since no edges form between nodes on the same side. Thus, the chromatic number is 2. ■

(b) [5 pts] The graph pictured above is often referred to as *bipartite.*

Definition. *A graph $G = (V, E)$ is bipartite if the set of vertices, V, can be split into two subsets V_l and V_r such that all edges in G connect nodes in V_l to nodes in V_r.*

Now recall from lecture the Greedy Coloring Algorithm:
Greedy Coloring Algorithm: For a graph $G = (V, E)$ and an ordering of vertices $v_1, v_2, \cdots, v_n$

1. Color v_1 with a new color c_1.
2. For each vertex v_i, if v_i shares an edge with with any earlier vertex, v_j, colored c_k, then it cannot be colored c_k. Choose the lowest available color for v_i.

Find an ordering of the vertices $\{v_1, v_2, \cdots, v_n\}$ such that the Greedy Coloring Algorithm uses exactly 2 colors.

Solution. Since none of the vertices on the left share an edge, we can color all of them with the same color, so we iterate through all the left nodes, then all the right nodes with the ordering, $v_1, v_3, v_5, \cdots v_{n-1}, \cdots, v_2, v_4, \cdots, v_n$. The left side uses only one color and the right side uses only one color similar to the solution in part (a). ∎

(c) [5 pts] Find an ordering such that the Greedy Coloring Algorithm uses exactly $n/2$ colors.

Solution. Notice that alternating across left and right with the greedy algorithm forces us to pick a new color every time we are on the left side. Thus, we can choose the ordering $v_1, v_2, \cdots v_{n/2}, \cdots v_n$ to get $n/2$ colors. ∎

(d) [5 pts] Prove your answer in part (c) by induction for all even integers n.

Solution. *Proof.* $P(n)$ is defined to be: For a bipartite graph with n vertices (labeled in the pattern as shown in the graph), with edges as shown in the graph, applying the Greedy Coloring Algorithm to the vertex ordering $v_1, v_2 \cdots v_n$ uses exactly $n/2$ colors for all even integers n. In addition, the coloring consists of the first $n/2$ colors on each side across from one another.

Base Case (n = 2) There are no edges in this case, so the Greedy Coloring Algorithm uses only one color.

Inductive Step Assume that $P(n)$ is true. We will show that this implies $P(n+2)$. Denote our two new nodes v_{n+1} and v_{n+2}. v_{n+1} is connected to $\{v_2, v_4, \cdots v_n\}$ and v_{n+2} is connected to $\{v_1, v_3, \cdots v_{n-1}\}$. Since by $P(n)$, v_{n+1} is connected to $n/2$ different colors, the Greedy Coloring Algorithm will choose a different color $c_{n/2+1}$. For v_{n+2}, the Greedy Algorithm will color it $c_{n/2+1}$ since its connected to $v_1, v_3, \cdots v_{n-1}$ which are the first $n/2$ colors. Thus, the new nodes are a both a new color.

Thus, by induction, $P(n+2)$ is true.

□

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Problem 5. [15 points] For each of the following pairs of graphs, either define an isomorphism between them, or prove that there is none. (We write ab as shorthand for the edge from a to b).

(a) [5 pts]

$$G_1 \text{ with } V_1 = \{1, 2, 3, 4, 5, 6\},\ E_1 = \{12, 23, 34, 14, 15, 35, 45\}$$
$$G_2 \text{ with } V_2 = \{1, 2, 3, 4, 5, 6\},\ E_2 = \{12, 23, 34, 45, 51, 24, 25\}$$

Solution. Not isomorphic: G_2 has a node, 2, of degree 4, but the maximum degree in G_1 is 3. ■

(b) [5 pts]

$$G_3 \text{ with } V_3 = \{1,2,3,4,5,6\}\,,\ E_3 = \{12,23,34,14,45,56,26\}$$
$$G_4 \text{ with } V_4 = \{a,b,c,d,e,f\}\,,\ E_4 = \{ab,bc,cd,de,ae,ef,cf\}$$

Solution. Isomorphic (two isomorphisms) with the vertex correspondences:
$1f, 2c, 3d, 4e, 5a, 6b$
or $1f, 2e, 3d, 4c, 5b, 6a$ ■

(c) [5 pts]

$$G_5 \text{ with } V_5 = \{a,b,c,d,e,f,g,h\}\,,\ E_5 = \{ab,bc,cd,ad,ef,fg,gh,he,dh,bf\}$$
$$G_6 \text{ with } V_6 = \{s,t,u,v,w,x,y,z\}\,,\ E_6 = \{st,tu,uv,sv,wx,xy,yz,wz,sw,vz\}$$

Solution. Not isomorphic: they have the same number of vertices, edges, and set of vertex degrees. But the degree 2 vertices of G_1 are all adjacent to two degree 3 vertices, while the degree 2 vertices of G_2 are all adjacent to one degree 2 vertex and one degree 3 vertex. ■

Problem 6. [15 points] Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subset E$ such that no two edges in M are incident on a common vertex.

Let M_1, M_2 be two matchings of G. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Show that G' is bipartite.

Helpful definition: A *connected component* is a subgraph of a graph consisting of some vertex and every node and edge that is connected to that vertex.

Solution. *Proof.* Proof by induction on the number of vertices n:

Induction hypothesis: $P(n)$ is defined to be: Let G be a graph with n vertices and matchings M_1 and M_2. Let $G' = (V, M_1 \cup M_2)$. Then G' is bipartite.

Base case: G has only one vertex and so is bipartite. $P(1)$ holds.

Inductive step: We will assume $P(n)$ in order to prove $P(n+1)$.

Let G be a graph with $n+1$ vertices. We will remove a vertex v from G to obtain an n vertex graph, G_1, with vertex set V_1. If we remove v we will be in one of the following cases:

Case 1: v is in none of the edges in M_1 nor M_2.

By our inductive hypothesis we know that since G_1 has n vertices and M_1 and M_2 are matchings of G_1, then $G_1' = (V_1, M_1 \cup M_2)$ is bipartite. Since G_1' is bipartite, there exists a partition of the vertices into two sets, L and R such that every edge is incident to a vertex in L and to a vertex in R. We can now add v to either set and obtain a bipartite representation of G'.

Case 2: v is in an edge in either M_1 or M_2, we will assume, without loss of generality, that v is in an edge in M_1.

Suppose the edge $v - x$ is in M_1, now remove $v - x$ from M_1 to obtain M_1'.

Now by our inductive hypothesis we know that since G_1 has n vertices and M_1' and M_2 are matchings of G_1, then $G_1' = (V_1, M_1' \cup M_2)$ is bipartite. Since G_1' is bipartite, there exists a partition of the vertices into two sets, L and R such that every edge is incident to a vertex in L and to a vertex in R.

We know that the vertex x is in either L or R. We can just add vertex v to the other set, along with edge $v - x$, and we obtain a valid partitioning of L and R for our graph G'.

Case 3: v is in both M_1 and M_2

Suppose the edge $v - x$ is in M_1 and $v - y$ is in M_2, now remove those edges from M_1 and M_2 to obtain M_1' and M_2'.

Now by our inductive hypothesis we know that since G_1 has n vertices and M_1' and M_2' are matchings of G_1, then $G_1' = (V_1, M_1' \cup M_2')$ is bipartite. Since G_1' is bipartite, there exists a partition of the vertices into two sets, L and R such that every edge is incident to a vertex in L and to a vertex in R.

If x and y in the same set, either L or R, then we can just add v to the other set, and add edges $v - x$ and $v - y$ to obtain G'. So our graph remains bipartite.

If x and y are on different sides of L and R, then either x and y are in the same connected component or they are in different connected components. If x and y are in different connected components, then each connected component has a corresponding set of L and R vertices, such that there are edges only within that component. Let's say that the first component has left and right vertices in the set L_1 and R_1, and the second component has sets L_2 and R_2, where $L = L_1 \cup L_2$ and $R = R_1 \cup R_2$. Now if we swap L_1 and R_1 – that is we define $L = R_1 \cup L_2$ and $R = L_1 \cup R_2$ – then our graph will remain bipartite, as there were edges only within the connected components. But after the swapping x and y will be in the same set, L or R, and as before we can just add v to the other set to get a bipartite graph for G' as desired.

Now we will show that it is impossible for x and y to be on opposite sides and in the same component.

Suppose for a contradiction that x and y are in the same connected component and x and y are both in L. Then since they are in the same connected component there is a path from x to y say $x - v_1, v_1 - v_2, \ldots, v_k - y$, where k is even. Then the edges $x - v_1$ and $v_2 - v_3$, $v_4 - v_5, \ldots, v_k - y$ must all be in the same matching (otherwise we will have two edges incident on the same vertex in the same matching). This is a contradiction since our original M_1 cannot have any edge with x and M_2 cannot have any edge with y (since a matching has only one edge incident to a vertex). So this cannot be the case and x and y must be on the same side.

Hence we conclude that in all cases G' is bipartite.

Therefore by induction our claim holds. $\square$

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