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Problems for Recitation 16

The ***(ordinary) generating function*** for a sequence $\langle a_0, a_1, a_2, a_3, \ldots \rangle$ is the power series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

1 Problem: Sequences to Generating Functions

Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) $\langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle$

(b) $\langle 1, 1, 1, 1, 1, 1, 1, \ldots \rangle$

(c) $\langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle$

(d) $\langle 1, 0, 1, 0, 1, 0, 1, 0, \ldots \rangle$

(e) $\langle 0, 0, 0, 1, 1, 1, 1, 1, \ldots \rangle$

(f) $\langle 1, 3, 5, 7, 9, 11, \ldots \rangle$

2 Problem: Generating Functions to Sequences

Suppose that:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$
$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + \cdots$$

What sequences do the following functions generate?

(a) $f(x) + g(x)$

(b) $f(x) \cdot g(x)$

(c) $f(x)/(1 - x)$

3 Problem: Candy Jar

There is a jar containing n different flavors of candy (and lots of each kind). I'd like to pick out a set of k candies.

(a) In how many different ways can this be done?

(b) Now let's approach the same problem using generating functions. Give a closed-form generating function for the sequence $\langle s_0, s_1, s_2, s_3, \ldots \rangle$ where s_k is the number of ways to select k candies when there is only $n = 1$ flavor available.

(c) Give a closed-form generating function for the sequence $\langle t_0, t_1, t_2, t_3, \ldots \rangle$ where t_k is the number of ways to select k candies when there are $n = 2$ flavors.

(d) Give a closed-form generating function for the sequence $\langle u_0, u_1, u_2, u_3, \ldots \rangle$ where u_k is the number of ways to select k candies when there are n flavors.

4 Problem: Recurrence

Consider the following recurrence equation:

$$T_n = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 2T_{n-1} + 3T_{n-2} & (n \geq 2) \end{cases}$$

Let $f(x)$ be a generating function for the sequence $\langle T_0, T_1, T_2, T_3, \ldots \rangle$.

(a) Give a generating function in terms of $f(x)$ for the sequence:

$$\langle 1, \quad 2, \quad 2T_1 + 3T_0, \quad 2T_2 + 3T_1, \quad 2T_3 + 3T_2, \ldots \rangle$$

(b) Form an equation in $f(x)$ and solve to obtain a closed-form generating function for $f(x)$.

(c) Expand the closed form for $f(x)$ using partial fractions.

(d) Find a closed-form expression for T_n from the partial fractions expansion.

5 Problem: Bouquet

You would like to buy a bouquet of flowers. You find an online service that will make bouquets of **lilies**, **roses** and **tulips**, subject to the following constraints:

- there must be at most 1 lily,
- there must be an odd number of tulips,
- there must be at least two roses.

Example: A bouquet of no lilies, 3 tulips, and 5 roses satisfies the constraints.

Express $B(x)$, the generating function for the number of ways to select a bouquet of n flowers, as a quotient of polynomials (or products of polynomials). You do not need to simplify this expression.