Notes for Recitation 2

1   Induction

Recall the principle of induction:

Principle of Induction. Let $P(n)$ be a predicate. If

- $P(0)$ is true, and

- for all $n \in \mathbb{N}$, $P(n)$ implies $P(n+1)$,

then $P(n)$ is true for all $n \in \mathbb{N}$.

We'll use induction to prove the following conjecture:

Conjecture. *For all positive integers, n:*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Remember that an induction proof has five parts, though the last one is often omitted:

1. Say that the proof is by induction.

2. Define the induction hypothesis, a predicate P defined on the natural numbers.

3. Handle the base case: prove that $P(0)$ is true.

4. Handle the inductive step: prove that $P(n)$ implies $P(n+1)$ for all integers $n \geq 0$.

5. Conclude that $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of induction.

We noted in Lecture that while the base case is usually $n = 0$, it could be any nonnegative integer, k, in which case the conclusion would simply be that $P(n)$ holds for all $n \geq k$.

Proof. We use induction. Let $P(n)$ be the proposition that:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3} \tag{1}$$

Base case $n = 1$: $P(1)$ is true, because the left-hand side of (??) is $1 \cdot 2 = 2$, and the right-hand side is $(1 \cdot 2 \cdot 3)/3 = 2$.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. So assume that $P(n)$ is true, where n denotes a positive integer. Then we can reason as follows:

$$
\begin{aligned}
& 1 \cdot 2 + 2 \cdot 3 + \cdots + (n+1)(n+2) \\
&= [1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1)] + (n+1)(n+2) \\
&= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \qquad\qquad\qquad \text{by ind. hypothesis (??)} \\
&= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} \\
&= \frac{(n+1)(n+2)(n+3)}{3}
\end{aligned}
$$

This shows that $P(n+1)$ is true, and so $P(n)$ implies $P(n+1)$ for all $n \geq 1$.

By the induction principle, $P(n)$ is true for all $n \geq 1$, which proves the claim. $\qquad\square$

2 Problem: A Geometric Sum

Perhaps you encountered this classic formula in school:

$$1 + r + r^2 + r^3 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

First use the well ordering principle, and then induction, to prove that this formula is correct for all real values $r \neq 1$.

Prepare a complete, careful solution. You'll be passing your proof to another group for "constructive criticism"!

Solution. Proof by Well Ordering Principle

Proof. The proof is by contradiction and use of the Well Ordering Principle. Assume that the theorem is false. Then, some nonnegative integers serve as counterexamples to it. Lets collect these counterexamples in a set: $C ::= \{n \in \mathbb{N} \mid 1 + r + r^2 + r^3 + \ldots + r^n \neq \frac{1-r^{n+1}}{1-r}\}$

By our assumption that the theorem admits counterexamples, C is a nonempty set of nonnegative integers. So, by the Well Ordering Principle, C has a minimum element, call it c. That is, c is the smallest counterexample to the theorem.

Since c is the smallest counterexample, we know that formula is false for $n = c$ but true for all nonnegative integers $n < c$. But the formula is true for $n = 0$ since $1 = \frac{1-r^{0+1}}{1-r}$. Hence $c > 0$. This means $c - 1$ is a nonnegative integer, and since it is less than c, the formula is true for $c - 1$. That is,

$$1 + r + r^2 + r^3 + \ldots + r^{c-1} = \frac{1 - r^c}{1 - r} \tag{2}$$

But then, adding r^c to both sides of equation (**??**) gives us

$$
\begin{aligned}
1 + r + r^2 + r^3 + \ldots + r^{c-1} + r^c &= \frac{1 - r^c}{1 - r} + r^c \\
&= \frac{1 - r^c + (1 - r)r^c}{1 - r} \\
&= \frac{1 - r^c + r^c - r^{c+1}}{1 - r} \\
&= \frac{1 - r^{c+1}}{1 - r}
\end{aligned}
$$

which means that the formula does hold for c, after all! This is a contradiction, and we are done.

$\square$

Proof by Induction

Proof. We use induction. Let $P(n)$ be the proposition that the following equation holds for all $r \neq 1$:

$$1 + r + r^2 + r^3 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Base case: $P(0)$ is true, because both sides of the equation are equal to 1.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$. So assume that $P(n)$ is true, where n denotes an arbitrary natural number. We can reason as follows:

$$
\begin{aligned}
1 + r + r^2 + r^3 + \ldots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\
&= \frac{1 - r^{n+1} + (1 - r) \cdot r^{n+1}}{1 - r} \\
&= \frac{1 - r^{n+2}}{1 - r}
\end{aligned}
$$

The first equation follows from the assumption $P(n)$, and the remaining steps are simplifications. This proves that $P(n+1)$ is also true. Therefore, $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$. By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Note: You may have encountered a different proof of this formula. We'll write down a sequence of equations and then explain the reasoning.

$$
\begin{aligned}
S &= 1 + r + r^2 + r^3 + \ldots + r^n \\
rS &= r + r^2 + r^3 + \ldots + r^{n+1} \\
S - rS &= 1 - r^{n+1} \\
S &= \frac{1 - r^{n+1}}{1 - r}
\end{aligned}
$$

We define S on the first line, multiply by r to get the second equation, subtract the second equation from the first to get the third, and then solve for S. This gives the formula above!

This argument is great! It is a derivation of the formula rather than just a verification. But, at some level, we've only hidden the use of induction, since the operations we're doing on n-term sums are justified using — you guessed it — induction. ∎

3 Problem: Surveyevor

In a new reality TV series called *Surveyevor*, a group of contestants is placed on a small island. Before the series begins, each contestant agrees to have a small purple or red tattoo, in the shape of an eye, applied to the middle of his or her forehead. In all, there are $p \geq 1$ purple eyes and $r \geq 0$ red eyes. However, none of the contestants knows the color of his or her third eye, nor how many total purple and red eyes there are. Furthermore, there are no mirrors and no one is allowed to discuss the tattoos ever. Therefore, everyone knows the colors of everyone else's third eye, but not their own. Good thing, because a contestant who learns that he or she has a purple eye must leave the island at the end of the show that day, and is therefore no longer eligible to win the $1 million cash prize at the end of the show!

The contestants live in uneasy ignorance for several weeks. As time goes on, however, most of them lose their fear of being exiled, adapt to island living, and even make friends with one another. Things are going quite well for the islanders, but as you might suppose, the television audience grows bored, and the show's ratings plummet. When the network threatens to cancel the series, the producer decides she needs to do something, fast: on the next show, to the surprise of the happy islanders, the producer herself appears and convenes a meeting. Very loudly, she proclaims, "I see that at least one person here has a purple eye." Assuming that all the contestants are master logicians, what happens?

Solution. All the purple-tattooed contestants leave the island at the end of the pth day.∎

Use induction to prove that your conclusion is correct. We suggest a hypothesis $P(n)$ that asserts all of the following are true on day n:

1. If $p > n$, then _____.

2. If $p = n$, then _____.

3. If $p < n$, then _____.

(We leave the task of filling in the blanks to you.)

Solution. Note that a red-eyed islander shouldn't ever conclude that she has a purple eye, since she doesn't, and we're assuming the contestants always reason correctly from what they know (and that what they know from the producer is also true). So no red-eyed contestant should ever leave the island.

Theorem 1. *All the purple-eyed people leave the island on day p.*

Proof. We use induction. Let $P(n)$ be the proposition that all of the following are true on day n:

1. If $p > n$, then all purple-eyed people survive the day.

2. If $p = n$, then all purple-eyed people leave the island.

3. If $p < n$, then all purple-eyed people are already gone.

Base case: We must verify that the three parts of $P(n)$ hold on day $n = 1$.

1. Suppose $p > 1$. Consider events on day 1 from the perspective of a purple-eyed islander. The producer says that someone has a purple eye, and she can indeed see at least one other person with a purple eye. Therefore, the facts available to her are consistent with her having either a purple or red eye. So she survives the day.

2. Suppose $p = 1$. The single purple-eyed islander sees no one else with a purple eye, concludes that he must have a purple eye, and leaves the island at the end of the show. No one else leaves because everyone else does see the purple-eyed islander, and they have no reason at this point to think they too are purple-eyed.

3. This statement is vacuously true, because the if-part ($p < 1$) is false; the problem statement says that $p \geq 1$.

Therefore, $P(1)$ is true.

Inductive step: Now suppose that $P(n)$ is true where $n \geq 0$. We must verify the three parts of $P(n+1)$.

1. Suppose $p > n + 1$. Then $p > n$ so all the purple-eyed contestants survived the preceding day by part 1 of $P(n)$. Furthermore, each purple-eyed islander can see at least $n + 1 > n$ other purple-eyed people, so the observation that everyone survives is consistent with she herself having either a purple or a red eye by $P(n)$ as well. Thus, each purple-eyed islander survives the day.

2. Suppose $p = n + 1$. Then $p > n$, so all the purple-eyed contestants survived the preceding day by part 1 of $P(n)$. Thus, on day $n + 1$ each purple-eyed islander knows $p > n$, but sees only n other people with a purple eye. Thus, each purple-eyed islander realizes that she has a purple eye and leaves the island at the end of the show.

3. Suppose $p < n + 1$. Then either $p = n$ (in which case all the purple-eyed contestants left the island on day n by part 2 of $P(n)$) or else $p < n$ (in which case all the purple-eyed contestants were already gone on day n by part 3 of $P(n)$). In either case, all the purple-eyed people are already out of luck.

Therefore $P(n)$ implies $P(n+1)$ for all $n \geq 0$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, and the theorem follows. $\square$

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