Solutions to Problem Set 2

Reading: Chapters **??**, *Mathematical Data Types*; **??**, *First-Order Logic*.

These assigned readings do not include the Problem sections. (Many of the problems in the text will appear as class or homework problems.)

Reminder: Comments on the reading using the *NB online annotation system* are due at times indicated in the online tutor problem set TP.2. Reading Comments count for 5% of the final grade.

Problem 1.
Recall that the composition of relations $R : A \to B$ and $S : B \to C$ is the relation $S \circ R : A \to C$ defined by the rule

$$a\ (S \circ R)\ c \quad \text{IFF} \quad \exists b.\ (a\ R\ b) \text{ AND } (b\ S\ c).$$

We can represent a relation S between two sets A and B of size n as an $n \times n$ square matrix M_S, where the elements of M_S are defined by the rule

$$i\ S\ j \quad \text{IFF} \quad M_S(i,j) = 1.$$

If we represent relations as matrices in this fashion, then we can compute the composition of two relations by a "boolean" matrix multiplication of their matrices. Boolean matrix multiplication is the same as matrix multiplication except that "$+$" is replaced by OR and "$\times$" is replaced by AND.

Prove that the matrix representation of $S \circ R$ is equal to the boolean product of M_R and M_S (note the reversal of R and S), where M_R is the matrix representing R and M_S is the matrix representing S.

Solution. *Proof.* Let M_P be the boolean product of M_R and M_S (notice that M_P, M_R and M_S are all $n \times n$ square matrices). What we want to prove is that

$$i\ (S \circ R)\ j \quad \text{IFF} \quad M_P(i,j) = 1.$$

Recall that by the definition of composition, $i\ (S \circ R)\ j$ iff there exists a k such that $i\ R\ k$ and $k\ S\ j$. Also, by the definition of boolean matrix multiplication,

$$M_P(i,j) = \underbrace{[M_R(i,k_1) \text{ AND } M_S(k_1,j)]}_{k_1 \text{ is the "link"}} \text{ OR } \underbrace{[M_R(i,k_2) \text{ AND } M_S(k_2,j)]}_{k_2 \text{ is the "link"}} \text{ OR} \ldots \text{OR} \underbrace{[M_R(i,k_n) \text{ AND } M_S(k_n,j)]}_{k_n \text{ is the "link"}}$$

Case 1:(IMPLIES) If $i\ (S \circ R)\ j$, then for at least one k, say k', $i\ R\ k'$ and $k'\ S\ j$. Consequently, $M_R(i,k') = 1$ and $M_S(k',j) = 1$. This turns $[M_R(i,k') \text{ AND } M_S(k',j)]$ true, and hence $M_P(i,j) = 1$.

Case 2: ($\Longleftarrow$) If $M_P(i,j) = 1$, then there is at least one k, say k', for which $[M_R(i,k')$ AND $M_S(k',j)] = 1$. This means that both $M_R(i,k') = 1$ and $M_S(k',j) = 1$. Since M_R and M_S are the matrix representations of R and S, we can conclude that $i\ R\ k'$ and $k'\ S\ j$, and so, by the definition of composition, $i\ (S \circ R)\ j$.

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Problem 2.
Prove that for any sets A, B, C, and D, if $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

Solution. *Proof.* We will prove the contrapositive. In other words, we will assume

$$[(A \cap C) \neq \emptyset \text{ AND } (B \cap D) \neq \emptyset] \tag{1}$$

and prove that

$$(A \times B) \cap (C \times D) \neq \emptyset. \tag{2}$$

Now by 1, there must be an element $e \in A$ AND $e \in C$, as well as an element $f \in B$ AND $f \in D$. So, $(e,f) \in A \times B$ by definition of Cartesian product, and likewise $(e,f) \in C \times D$. This means that

$$(e,f) \in (A \times B) \cap (C \times D),$$

so $(A \times B) \cap (C \times D) \neq \emptyset$

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Problem 3.
Find the flaw in the following false proof, and give a counterexample to the claim.

Claim. *Suppose R is a relation on a set, A. If R is symmetric and transitive, then R is reflexive.*

False proof. Let a be an arbitrary element of A. Let b be any element of A such that $a\ R\ b$. Since R is symmetric, it follows that $b\ R\ a$. Then since $a\ R\ b$ and $b\ R\ a$, we conclude by transitivity that $a\ R\ a$. Since a was arbitrary, we have shown that $\forall a \in A.\ a\ R\ a$, which means that R is reflexive. ■

Solution. The flaw is assuming that b exists. It is possible that there is an $a \in A$ that is not related by R to anything. No such R will be reflexive. The simplest such R that is also symmetric and transitive is the empty relation on any nonempty set A. We can easily construct other examples, such as letting $A ::= \{a,b,c\}$ and

$$\text{graph}(R_0) ::= \{(c,c),(c,b),(b,c),(b,b)\}.$$

Now R_0 is not reflexive because NOT$(a\ R_0\ a)$. So R_0 is a counterexamples to the claim.

Note that the theorem can be fixed: R restricted to its domain of definition *is* reflexive.

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Problem 4.
Suppose that f is a function of the form $f : A \mapsto B$, and g is a function of the form $g : B \mapsto C$. The composed function $g \circ f$ has domain A, range C, and is defined by $(g \circ f)(a) = g(f(a))$.

(a) Prove that if the composition $g \circ f$ is a bijection, then f is an injection and g is a surjection.

Solution. *Proof.* Suppose that $g \circ f$ is a bijection.

Assume for the purpose of contradiction that f is not an injection. Then there exist elements $a_1, a_2 \in A$, such that $f(a_1) = f(a_2)$. This implies that $g(f(a_1)) = g(f(a_2))$. Therefore, $g \circ f$ is not an injection and thus not a bijection. This is a contradiction; therefore, f must be an injection.

Now assume for the purpose of contradiction that g is not a surjection. Then there exists an element $c \in C$ such that for all $b \in B$, $g(b) \neq c$. Therefore, for all $a \in A$, $g(f(a)) \neq c$. This implies that $g \circ f$ is not a surjection and thus not a bijection. This is again a contradiction; therefore, g must be a surjection. ■

If f is an injection and g is a surjection, then is $g \circ f$ necessarily a bijection?

Solution. No. For example, consider the following setup.

$$\begin{aligned} A &= \{1\} \\ B &= \{1, 2\} \\ C &= \{1, 2\} \end{aligned}$$

$$\begin{aligned} f(x) &= x \\ g(x) &= x \end{aligned}$$

In this case, f is injective, g is surjective, but $g \circ f$ is not bijective.

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