Problem 1. [10 points]

(a) [5 pts] Prove or disprove $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$. *Hint*: Use a truth table.

(b) [5 pts] Translate the following statements from English into propositional logic or vice versa.

1. If $n > 1$, then there is always at least one prime p such that $n < p < 2n$. *Hint*: Let $\text{Prime}(p) :=$ p is a prime
2. The domain is $\mathbb{N}$. $\forall m\ \exists\, p > m.\ \text{Prime}(p) \wedge \text{Prime}(p+2)$
3. Let T be the set of TA's, S be the set of students, and $G(x, y) := x$ grades y's exam

$$\exists\, t \in T\ \forall s \in S.\ G(t, s)$$

Problem 2. [10 points] Consider the following matrix A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Prove by induction that:

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. F_i for $i \in \mathbb{N}$ is actually the sequence of Fibonacci numbers, but this knowledge is not needed for the proof.

Problem 3. [10 points] Consider the matrix below..

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Suppose we are allowed to flip all of the signs of entries in any row or column. For example, flipping the signs of elements in column two will give the matrix:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

If we are allowed to perform any number of these operations, prove that at least one -1 will remain in the matrix. Do this by identifying an invariant and explain why you can use this invariant to show that at least one -1 will remain in the matrix.

Problem 4. [20 points]

(a) [5 pts] Evaluate 2^{6042} mod 63.

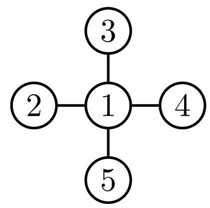
(b) [3 pts] What is 63^{6042} mod 6043? (Hint: 6043 is prime; don't do a messy calculation–it will just waste your time.)

(c) [6 pts] Give a proof by contradiction that 33 does not have an inverse mod 121.

(d) [6 pts] Find the inverse of 32 mod 121 in the range $\{1, 2, \ldots 120\}$. (Hint: use the Pulverizer)

Problem 5. **[10 points]** A simple graph is said to have width k if you can order the nodes on a straight line so that each node is adjacent to at most k nodes to the left. Each node can be adjacent to any number of nodes to the right.

For example, the star graph shown below with corresponding ordering $1, 2, 3, 4, 5$ has width 1.



Prove by induction that any simple graph with width k can be colored in at most k+1 colors. (Hint: do not induct on k)

Problem 6. [10 points] The following questions concern the following preferences. Since there are unequal numbers of boys and girls, assume that being matched is preferred to being unmatched.

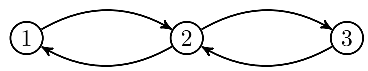
girls	*boys*
Wendy : Stan, Kenny, Butters, Eric	Stan: Wendy, Bebe, Heidi
Bebe : Kenny, Eric, Butters, Stan	Kenny : Heidi, Wendy, Bebe
Heidi : Eric, Butters, Stan, Kenny	Butters : Bebe, Wendy, Heidi
	Eric : Bebe, Heidi, Wendy

(a) [3 pts] Is the following a stable marriage? If not, list a rogue couple.

Wendy - Stan
Bebe - Eric
Heidi - Butters
None - Kenny

(b) [7 pts] Find the matching produced by the stable matching algorithm.

Problem 7. [10 points] Consider the following graph:



Suppose that each node starts with a PageRank value of $\frac{1}{3}$.

(a) [3 pts] What weights will the nodes have after one iteration of the Pagerank algorithm?

(b) [7 pts] What will be the PageRank values of each node for the stationary distribution? Show your work.

Problem 8. [10 points] Consider the following relation:

$$R = \{(x, y) : x + y = 0, x, y \in \mathbb{R}\}.$$

Which of the following properties holds for R? If it has the property, prove it. If not, provide a counterexample.

(a) [2 pts] Symmetry.

(b) [2 pts] Antisymmetry.

(c) [2 pts] Irreflexivity.

(d) [2 pts] Transitivity.

(e) [2 pts] The property of being an equivalence relation.

Problem 9. [10 points] Suppose we are planning a trip to California for Thanksgiving. Unfortunately, we are booking our tickets late and so the prices are all really high. Suppose we are given the following list of ticket prices and travel times:

A 600 dollars, 9 hours 20 minutes

B 650 dollars, 8 hours 40 minutes

C 550 dollars, 9 hours 10 minutes

D 575 dollars, 8 hours 20 minutes

E 660 dollars, 9 hours 5 minutes

Our goal is to find the tickets that are the cheapest while minimizing travel time.

(a) [4 pts] Let's define the following ordering $\leq$. For tickets i and j, we say that $i \leq j$ if i is at least as expensive as j is and i's travel time is at least as long as j's travel time. Prove that $\leq$ is a partial order.

(b) [3 pts] Draw the Hasse diagram for the partial order defined above, whose elements are the tickets. You may omit arrows that are self-loops and those that are implied by transitivity.

(c) [3 pts] Find the maximal elements of the poset. Is there a maximum element?