Notes for Recitation 9

Getting around a graph

Definition 1. *A* ***walk***[1] *in a graph, G, is a sequence of vertices*

$$v_0, v_1, \ldots, v_k$$

and edges

$$\{v_0,v_1\}, \{v_1,v_2\}, \ldots, \{v_{k-1},v_k\}$$

such that $\{v_i,v_{i+1}\}$ is an edge of G for all i where $0 \le i < k$. The walk is said to start *at v_0 and to* end *at v_k, and the* length *of the walk is defined to be k. An edge, $\{u,v\}$, is* ***traversed*** *n times by the walk if there are n different values of i such that $\{v_i,v_{i+1}\} = \{u,v\}$.*

Definition 2. *A* ***path*** *is a walk where all the v_i's are different, that is, $i \ne j$ implies $v_i \ne v_j$. For simplicity, we will refer to paths and walks by the sequence of vertices.*[2]

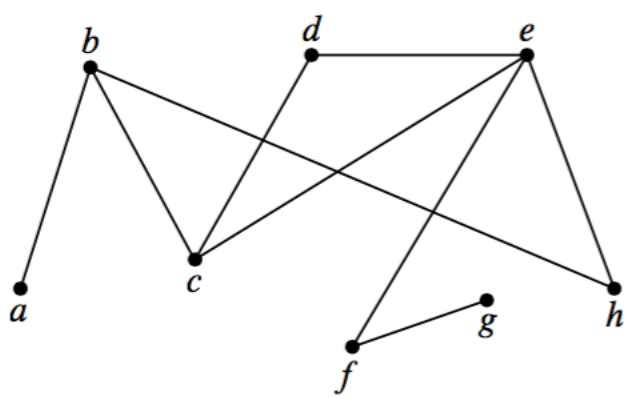

Figure 1: A graph containing a path a, b, c, d, e, f, g of length 6.

[1] Some texts use the word *path* for our definition of walk and the term *simple path* for our definition of path.

[2] This works fine for simple graphs since the edges in a walk are completely determined by the sequence of vertices and there is no ambiguity. For graphs with multiple edges, we would need to specify the edges as well as the nodes.

For example, the graph in Figure 1 has a length 6 path a, b, c, d, e, f, g. This is the longest path in the graph. Of course, the graph has walks with arbitrarily large lengths; for example, a, b, a, b, a, $b, \ldots$.

The length of a walk or path is the total number of times it traverses edges, which is *one less* than its length as a sequence of vertices. For example, the length 6 path a, b, c, d, e, f, g contains a sequence of 7 vertices.

1 Problem: Finding a Path

Use the Well Ordering Principle to prove the following lemma.

Lemma 1. *If there is a walk from a vertex u to a vertex v in a graph, then there is a path from u to v.*

Solution. *Proof.* Since there is a walk from u to v, there must, by the Well-ordering Principle, be a minimum length walk from u to v. If the minimum length is zero or one, this minimum length walk is itself a path from u to v. Otherwise, there is a minimum length walk

$$v_0, v_1, \ldots, v_k$$

from $u = v_0$ to $v = v_k$ where $k \geq 2$. We claim this walk must be a path. To prove the claim, suppose to the contrary that the walk is not a path; that is, some vertex on the walk occurs twice. This means that there are integers i, j such that $0 \leq i < j \leq k$ with $v_i = v_j$. Then deleting the subsequence

$$v_{i+1}, \ldots, v_j$$

yields a strictly shorter walk

$$v_0, v_1, \ldots, v_i, v_{j+1}, v_{j+2}, \ldots, v_k$$

from u to v, contradicting the minimality of the given walk. □

Actually, we proved something stronger:

Corollary 2. *For any walk of length k in a graph, there is a path of length* at most *k with the same endpoints. Moreover, the shortest walk between a pair of vertices is, in fact, a path.*

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Matrix Multiplication

Although this may be a review for some, it will be an important part of tomorrow's lecture when we describe graphs using an *adjacency matrix*.

Recall the definition of matrix multiplication. Given two matrices A and B, where A is $m \times n$ and B is $n \times k$ their product is given by:

$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{bmatrix} = C$$

where

$$c_{ij} = \sum_{p=1}^{n} a_{ip} b_{pj}$$

Note that C is a $m \times k$ matrix.

It is also important to note that in general $A \cdot B \neq B \cdot A$. In fact, this operation may not even be defined, since the row dimension of the first matrix must equal the column dimension of the second (above, A has n rows and B has n columns).

Multiplying a matrix and a vector is identical, noting that a column vector is simply a $n \times 1$ matrix.

Let's do some examples.

2 Problem: Matrix Multiplication Practice

1. Evaluate the following expressions.

(a)

$$\begin{bmatrix} \alpha & \rho \\ \beta & \sigma \\ \gamma & \tau \end{bmatrix} \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$$

Solution.

$$\begin{bmatrix} \alpha a + \rho x & \alpha b + \rho y & \alpha c + \rho z \\ \beta a + \sigma x & \beta b + \sigma y & \beta c + \sigma z \\ \gamma a + \tau x & \gamma b + \tau y & \gamma c + \tau z \end{bmatrix}$$

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(b)

$$\begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} \begin{bmatrix} \alpha & \rho \\ \beta & \sigma \\ \gamma & \tau \end{bmatrix}$$

Solution.

$$\begin{bmatrix} a\alpha + b\beta + c\gamma & a\rho + b\sigma + c\tau \\ x\alpha + y\beta + z\gamma & x\rho + y\sigma + z\tau \end{bmatrix}$$

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(c)

$$\begin{bmatrix} a & b & c & d \\ w & x & y & z \end{bmatrix} \begin{bmatrix} \alpha & \rho \\ \beta & \sigma \\ \gamma & \tau \end{bmatrix}$$

Solution. This is not defined, since the first matrix is 2×4 and the second is 3×2 ■

2. Prove the following lemma:

Lemma 3. *Let b be a $m \times 1$ vector whose entries are nonnegative and sum to 1. Let A be a $n \times m$ matrix whose entries are nonnegative and each column sums to one. Then, the product $c = Ab$ is a $n \times 1$ vector whose entries are nonnegative and sum to one.*

Solution. *Proof.* The i^{th} entry $c_i = \sum_{j=1}^{m} A_{ij} b_j$ is nonnegative since each A_{ij} and b_j is nonnegative.

The sum of the entries in c is

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} b_j = \sum_{j=1}^{m} \sum_{i=1}^{n} A_{ij} b_j = \sum_{j=1}^{m} b_j \sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{m} b_j = 1$$

Where we used in the second to last equality the fact that the sum of every column in A is 1, and in the final equality the fact that the sum of the entries in b is 1.

□

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3 Problem: Connectivity

Prove that any simple graph with n nodes and strictly more than $\frac{1}{2}(n-1)(n-2)$ edges is connected.

Solution. We'll show the equivalent statement that any disconnected graph on n nodes has at most $(n-1)(n-2)/2$ edges.

Let $G = (V, E)$ be any graph on n nodes that is not connected. Then there must be more than one connected component; let $G_1 = (V_1, E_1)$ be any connected component, and let $G_2 = (V_2, E_2)$ be the graph induced on $V_2 := V - V_1$. Note that there are no edges going between G_1 and G_2, and so $E_1 \cup E_2 = E$.

How many edges can G_1 have? At most $\binom{|V_1|}{2}$ edges (one for each pair of nodes). Similarly, G_2 can have at most $\binom{|V_2|}{2}$ edges.

Write $t := |V_1|$; then $|V_2| = |V| - |V_1| = n - t$. So the total number of edges in G is at most

$$\frac{t(t-1)}{2} + \frac{(n-t)(n-t-1)}{2}.$$

If we simplify this, we get

$$|E| \leq \frac{n(n-1)}{2} - t(n-t).$$

But since $1 \leq t \leq n-1$, $t(n-t) \geq n-1$. (You can confirm this with some calculus; it might help to draw $t(n-t)$ as a function of t; it's just a parabola.) So

$$|E| \leq \frac{n(n-1)}{2} - (n-1) = \frac{(n-2)(n-1)}{2}.$$

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