Notes for Recitation 14

Guessing a Particular Solution

A general linear recurrence has the form:

$$f(n) = b_1 f(n-1) + b_2 f(n-2) + \ldots + b_d f(n-d) + g(n)$$

One step in solving this recurrence is finding a *particular solution*. This is a function $f(n)$ that satisfies the recurrence equation, but may not be consistent with the boundary conditions. Here's a recipe to help you guess a particular solution:

- If $g(n)$ is a constant, guess that $f(n)$ is some constant c. Plug this into the recurrence equation and see if any constant actually works. If not, try $f(n) = bn + c$, then $f(n) = an^2 + bn + c$, etc.

- More generally, if $g(n)$ is a polynomial, try a polynomial of the same degree. If that fails, try a polynomial of degree one higher, then two higher, etc. For example, if $g(n) = n$, then try $f(n) = bn + c$ and then $f(n) = an^2 + bn + c$.

- If $g(n)$ is an exponential, such as 3^n, then first guess that $f(n) = c3^n$. Failing that, try $f(n) = bn3^n + c3^n$ and then $an^2 3^n + bn3^n + c3^n$, etc.
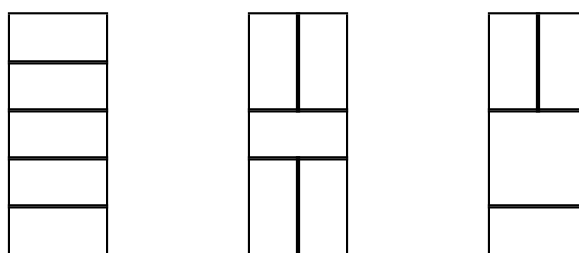
In practice, your first or second guess will almost always work.

Mini-Tetris

A *winning configuration* in the game of Mini-Tetris is a complete tiling of a $2 \times n$ board using only the three shapes shown below:
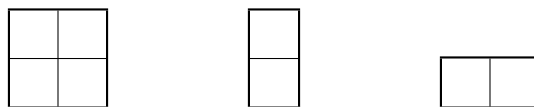
For example, here are several possible winning configurations on a 2×5 board:
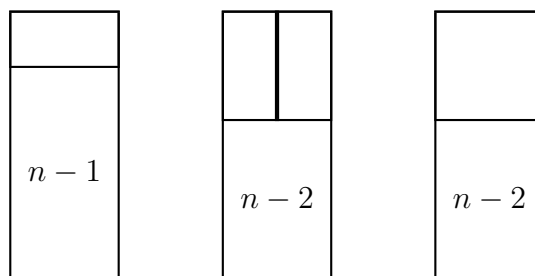
1. Let T_n denote the number of different winning configurations on a $2 \times n$ board. Determine the values of T_1, T_2, and T_3.

   Solution. $T_1 = 1$, $T_2 = 3$, and $T_3 = 5$. ■

2. Find a recurrence equation that expresses T_n in terms of T_{n-1} and T_{n-2}.

   Solution. Every winning configuration on a $2 \times n$ board is of one three types, distinguished by the arrangment of pieces at the top of the board.

   There are T_{n-1} winning configurations of the first type, and there are T_{n-2} winning configurations of each of the second and third types. Overall, the number of winning configurations on a $2 \times n$ board is:

   $$T_n = T_{n-1} + 2T_{n-2}$$

   ■

3. Find a closed-form expression for the number of winning configurations on a $2 \times n$ Mini-Tetris board.

   Solution. The characteristic polynomial is $r^2 - r - 2 = (r-2)(r+1)$, so the solution is of the form $A2^n + B(-1)^n$. Setting $n = 1$, we have $1 = T_1 = 2A - B$. Setting $n = 2$, we have $3 = T_2 = A2^2 + B(-1)^2 = 4A + B$. Solving these two equations, we conclude $A = 2/3$ and $B = 1/3$. That is,

   $$T_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n = \frac{2^{n+1} + (-1)^n}{3}.$$

   ∎

Linear Recurrences

Find closed-form solutions to the following linear recurrences.

1. $T_0 = 0$
      $T_1 = 1$
      $T_n = T_{n-1} + T_{n-2} + 1$

   Solution. Following the guide to solving linear recurrences:

   (a) First, we find the general solution to the homogeneous recurrence. The characteristic equation is $r^2 - r - 1 = 0$. The roots of this equation are:

   $$r_1 = \frac{1 + \sqrt{5}}{2}$$
   $$r_2 = \frac{1 - \sqrt{5}}{2}$$

   (b) Using the roots, we write down the homogeneous solution in the form

   $$T_n = A\left(\frac{1 + \sqrt{5}}{2}\right)^n + B\left(\frac{1 - \sqrt{5}}{2}\right)^n.$$

   (c) Next, we need a particular solution to the inhomogenous recurrence. Since the inhomogenous term is constant, we guess a constant solution, c. So replacing the $T(n)$ terms in $T_n = T_{n-1} + T_{n-2} + 1$ by c, we require

   $$c = c + c + 1,$$

   namely, $c = -1$. That is, $T_n = -1$ is a particular solution to the equation.

(d) Putting it together, the complete solution to the recurrence is the homogenous solution plus the particular solution:

$$T_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n - 1$$

(e) All that remains is to find the constants A and B. Substituting the initial conditions gives a system of linear equations.

$$0 = A + B - 1$$
$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) - 1$$

The solution to this linear system is:

$$A = \frac{5+3\sqrt{5}}{10}$$
$$B = \frac{5-3\sqrt{5}}{10}$$

Therefore, the complete solution to the recurrence is

$$T_n = \left(\frac{5+3\sqrt{5}}{10}\right) \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right) \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n - 1.$$

■

2. $S_0 = 0$

           $S_1 = 1$

           $S_n = 6S_{n-1} - 9S_{n-2}$

Solution. The characteristic polynomial is $r^2 - 6r + 9 = (r-3)^2$, so the solution is of the form $A3^n + Bn3^n$ for some constants A and B. Setting $n = 0$, we have $0 = S_0 = A3^0 + B \cdot 0 \cdot 3^0 = A$. Setting $n = 1$, we have $1 = S_1 = A3^1 + B \cdot 1 \cdot 3^1 = 3B$, so $B = 1/3$. That is,

$$S_n = 0 \cdot 3^n + \frac{1}{3} \cdot n3^n = n3^{n-1}.$$

■

Short Guide to Solving Linear Recurrences

A *linear recurrence* is an equation

$$\underbrace{f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)}_{\text{homogeneous part}} \quad \underbrace{+g(n)}_{\text{inhomogeneous part}}$$

together with boundary conditions such as $f(0) = b_0$, $f(1) = b_1$, etc.

1. Find the roots of the *characteristic equation*:

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_k$$

2. Write down the *homogeneous solution*. Each root generates one term and the homogeneous solution is the sum of these terms. A nonrepeated root r generates the term $c_r r^n$, where c_r is a constant to be determined later. A root r with multiplicity k generates the terms:

$$c_{r_1} r^n, \quad c_{r_2} n r^n, \quad c_{r_3} n^2 r^n, \quad \ldots, \quad c_{r_k} n^{k-1} r^n$$

where $c_{r_1}, \ldots, c_{r_k}$ are constants to be determined later.

3. Find a *particular solution*. This is a solution to the full recurrence that need not be consistent with the boundary conditions. Use guess and verify. If $g(n)$ is a polynomial, try a polynomial of the same degree, then a polynomial of degree one higher, then two higher, etc. For example, if $g(n) = n$, then try $f(n) = bn+c$ and then $f(n) = an^2+bn+c$. If $g(n)$ is an exponential, such as 3^n, then first guess that $f(n) = c3^n$. Failing that, try $f(n) = bn3^n + c3^n$ and then $an^2 3^n + bn3^n + c3^n$, etc.

4. Form the *general solution*, which is the sum of the homogeneous solution and the particular solution. Here is a typical general solution:

$$f(n) = \underbrace{c2^n + d(-1)^n}_{\text{homogeneous solution}} + \underbrace{3n+1}_{\text{particular solution}}$$

5. Substitute the boundary conditions into the general solution. Each boundary condition gives a linear equation in the unknown constants. For example, substituting $f(1) = 2$ into the general solution above gives:

$$2 = c \cdot 2^1 + d \cdot (-1)^1 + 3 \cdot 1 + 1$$
$$\Rightarrow \quad -2 = 2c - d$$

Determine the values of these constants by solving the resulting system of linear equations.