Final Solutions

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Problem 1. [10 points] Find a closed form for $\sum_{i=1}^{n} \sum_{j=i}^{m} \frac{i}{j}$. Leave your answer in terms of n, m.

Hint: Use H_k to represent the kth harmonic number

Solution.

$$\frac{n(n+1)}{2} H_m$$

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Problem 2. [10 points] We define the sequence of numbers

$$a_n = \begin{cases} 1, & \text{for } n \leq 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}, & \text{for } n > 3. \end{cases}$$

Use *strong induction* to prove that remainder$(a_n, 3) = 1$ for all $n \geq 0$.

Solution. We use the fact that $\text{rem}\ m3 = 1$ iff $m \equiv 1 \pmod 3$. Letting

$$P(n) ::= \quad a_n \equiv 1 \pmod 3,$$

we need only show by strong induction that $P(n)$ is true for all n.

Base case ($n \leq 3$): $a_n ::= 1$, so $a_n \equiv 1 \pmod 3$.

Inductive step: For $n > 3$, assume $P(k)$ for $0 \leq k < n$ in order to prove $P(n)$.

In particular, we may assume that $a_k \equiv 1 \pmod 3$ for $k = n-4, n-3, n-2, n-1$. But $P(n)$ is equivalent to $a_{n-4} + a_{n-3} + a_{n-2} + a_{n-1} = 1 \pmod 3$, so $P(n)$ is true because

$$a_n = a_{n-4} + a_{n-3} + a_{n-2} + a_{n-1} \equiv (1 + 1 + 1 + 1) = 1 \pmod 3,$$

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Problem 3. [15 points]

(a) [10 pts] Find a solution to $x_n = 4x_{n-1} + n + 1$ with $x_0 = 2$.

Solution. We begin by first solving the homogeneous linear recurrence $x_n = 4x_{n-1}$. We have that the characteristic polynomial is $r - 4 = 0$ and so $x_n = C_1 \cdot 4^n$ for some constant C_1.

Now we solve for the particular solution. As the inhomogeneous term is $n+1$, we make a guess that the solution is of the form $An + B$ for constants A, B. Substituting we find:

$$\begin{aligned} An + B &= 4(An - A + B) + n + 1 \\ An + B &= 4An - 4A + 4B + n + 1 \\ 3An - 4A + 3B + n + 1 &= 0 \end{aligned}$$

Hence we must have that $3A + 1 = 0$ and $-4A + 3B + 1 = 0$, implying that $A = \frac{-1}{3}$ and $B = \frac{-7}{9}$.

Substituting back, we find that our solution is $x_n = C_1 \cdot 4^n - \frac{1}{3}n - \frac{7}{9}$. Now we are given that $x_0 = 2$ and so $C_1 = \frac{25}{9}$.

Hence the solution is $x_n = \frac{25*4^n - 3n - 7}{9}$.

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(b) [5 pts] Give an asymptotic expression for the following recurrence, in Θ notation:

$$T(n) = 2T(\frac{n}{4}) + 3T(\frac{n}{6}) + n^3, \qquad T(1) = 0$$

Solution. Using Akra-Bazzi, we see that $a_1 = 2, b_1 = \frac{1}{4}, a_2 = 3, b_2 = \frac{1}{6}$ and so $p = 1$. Hence, $T(n) = \Theta\left(n^2\left(1 + \int_1^n \frac{n^3}{n^3} dn\right)\right) = \Theta(n^3)$. ■

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Problem 4. [20 points]

(a) [10 pts] Suppose that we are flipping a fair coin n times. What is the probability that there are exactly k heads, where the heads must be separated by at least 2 tails?

Solution. There are $\binom{n-2k+2}{k}$ number of ways to choose where to place the heads (consider a bijection with having k 1s in an $n-2k+2$ bit binary string). Note that it is $n-2(k-1)$ because the last head does not need to be followed by 2 tails.

The total number of results is 2^n, so the probability is:

$$\frac{\binom{n-2k+2}{k}}{2^n}$$

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(b) [10 pts] Give a combinatorial proof for this identity:

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

Hint: Let S_i be the set of binary sequences with exactly n zeroes, $k+1$ ones, and a total of exactly i occurrences of zeroes appearing before the rightmost occurrence of a one.

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Problem 5. [**10 points**] Is there a bipartite graph with ordered degree sequence $3, 3, 3, 3, 3, 3, 4, 4, 4$?
Hint: The vertices of a bipartite graph can be divided into two subsets. Consider the sum of degrees of the vertices in each subset.

Solution. This is not possible. We know that the vertices of a bipartite graph can be divided into two subsets, A, B. Now as there are only edges between A and B in the graph, the sum of degrees of vertices in subset A must be the same as the sum of degrees of vertices in subset B. ■

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Problem 6. [20 points] The hat-check staff has had a long day serving at a party, and at the end of the party they simply return the n checked hats uniformly at random, such that the probability that any particular person gets their own hat back is $1/n$.

Let X_i be the *indicator variable* for the ith person getting their own hat back. Let S_n be the total number of people who get their own hat back.

(a) [2 pts] What is the expected number of people who get their own hat back?

Solution. $S_n = \sum_{i=1}^{n} X_i$, so by *linearity of expectation*,

$$\mathrm{E}[S_n] = \sum_{i=1}^{n} \mathrm{E}[X_i].$$

Since the probability a person gets their own hat back is $1/n$, therefore $\Pr[X_i = 1] = 1/n$. Now, since X_i is an indicator, we have $\mathrm{E}[X_i] = 1/n$. By linearity of expectation,

$$\mathrm{E}[S_n] = \sum_{i=1}^{n} \mathrm{E}[X_i] = n \cdot \frac{1}{n} = 1.$$

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(b) [3 pts] Write a simple formula for $\mathrm{E}[X_i \cdot X_j]$ for $i \neq j$.

Hint: What is $\Pr\{X_j = 1 | X_i = 1\}$?

Solution. We observed above that $\Pr[X_i = 1] = 1/n$. Also, given that the ith person got their own hat, each other person has an equal chance of getting their own hat among the remaining $n-1$ hats. So

$$\Pr[X_j = 1 \mid X_i = 1] = \frac{1}{n-1},$$

for $j \neq i$. Therefore,

$$\Pr[X_i = 1 \cap X_j = 1] = \Pr[X_j = 1 \mid X_i = 1] \cdot \Pr[X_i = 1] = \frac{1}{n(n-1)}.$$

But $X_i = 1 \cap X_j = 1$ iff $X_i X_j = 1$, so

$$\mathrm{E}[X_i X_j] = \Pr[X_i X_j = 1] = \Pr[X_i = 1 \cap X_j = 1],$$

and hence

$$\mathrm{E}[X_i X_j] = \frac{1}{n(n-1)}.$$

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(c) [5 pts] Show that $\mathrm{E}[(S_n)^2] = 2$.

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Solution.

$$\begin{aligned}
\mathrm{E}\left[S_n^2\right] &= \mathbb{E}[\sum_{i=1}^{n}(X_i)^2 + 2\sum_{1\le i<j\le n} X_iX_j] && \text{(expanding the sum for } S_n\text{)}\\
&= \sum_{i=1}^{n}\mathrm{E}\left[(X_i)^2\right] + 2\sum_{1\le i<j\le n}\mathrm{E}\left[X_iX_j\right] && \text{(linearity of } \mathrm{E}\left[\cdot\right]\text{)}\\
&= \sum_{i=1}^{n}\mathrm{E}\left[X_i\right] + 2\sum_{1\le i<j\le n}\frac{1}{n(n-1)} && \text{(since } (X_i)^2 = X_i\text{)}\\
&= n\cdot\frac{1}{n} + 2\binom{n}{2}\frac{1}{n(n-1)}\\
&= 1+1 = 2.
\end{aligned}$$

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(d) [5 pts] What is the variance of S_n?

Solution.

$$\text{Var}[S_n] = \text{E}\left[(S_n)^2\right] - \text{E}^2[S_n] = 2 - 1^2 = 1. \tag{1}$$

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(e) [5 pts] Use the *Chebyshev bound* to show that there is at most a 1% chance that more than 10 people get their own hat back.

Solution.

$$\begin{aligned}
\Pr[S_n \geq 11] &= \Pr[S_n - \text{E}[S_n] \geq 11 - \text{E}[S_n]] \\
&= \Pr[S_n - \text{E}[S_n] \geq 10] && \text{(by (1))} \\
&\leq \Pr[|S_n - \text{E}[S_n]| \geq 10] \\
&\leq \frac{\text{Var}[S_n]}{10^2} = .01 && \text{(by Chebyshev)}
\end{aligned}$$

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Problem 7. [10 points] Find the generating function for the number of ways to pay any amount of money using only pennies, nickels, dimes, and quarters.

Solution. A generating function for x_1 is just $(x + x^2 + x^3 + \ldots) = \frac{x}{1-x}$. A generating function for $2x_2$ is just $(x^2 + x^4 + x^6 + \ldots) = \frac{x^2}{1-x^2}$. A generating function for $3x_3$ is just $(x^3 + x^6 + x^9 + \ldots) = \frac{x^3}{1-x^3}$. A generating function for $4x_4$ is just $(x^4 + x^8 + x^{12} + \ldots) = \frac{x^4}{1-x^4}$. Hence a generating function for the number of solutions to the above equation is just $\frac{x \cdot x^2 \cdot x^3 \cdot x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$, and the number of solutions to the particular solution above is the coefficient of the x^n term in the expansion of the generating function. ■

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Problem 8. [15 points] Vlad has been sick for the past few days and is curious to know which disease he is suffering from. He knows that he has the flu with probability .3 and the common cold with probability .7.

If he has the flu, the conditional probability that the flu medicine will work and cure all symptoms is .4. Similarly, if he has the common cold, then he will be free of symptoms by taking the common cold medicine with conditional probability .15. Vlad can only take medicine for one disease each day.

You don't have to reduce your answer for the following problems.

(a) [5 pts] Which disease should Vlad take medicine for on the first day in order to maximize the probability of curing his disease?

Solution. We just compare the probabilities that he is cured taking the flu medicine and the cold medicine. As $.4 \cdot .3 \geq .15 \cdot .7$, he should take the flu medicine first. ■

(b) [5 pts] Vlad took medicine for the flu on the first day but is still sick. What is the probability that he has the flu?

Solution. This is just $\frac{.6 \cdot .3}{.6 \cdot .3 + .7 \cdot 1}$. ■

(c) [5 pts] Vlad flips a fair coin to determine which medicine to take the first day and gets better on the first day. What is the probability that he took the flu medicine?

Solution. This is just the conditional probability that Vlad looked took the flu medicine and was cured. This is given by $\frac{.5 \cdot .3 \cdot .4}{.5 \cdot .3 \cdot .4 + .5 \cdot .6 \cdot .15}$. ■

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Problem 9. [10 points] In a permutation of n elements, a pair (i, j) is called an inversion if and only if $i < j$ and i comes after j. For example, the permutation 31542 in the case of $n = 5$ has five inversions: $(3,1), (3,2), (5,4), (5,2)$ and $(4,2)$. What is the expected number of inversions in a uniform random permutation of the number $1, 2, \ldots n$?

Hint: Use appropriate indicator variables and linearity of expectation.

Solution. We use $\mathbb{I}(i,j)$ to be an indicator variable that the pair (i,j) is an inversion. Then if T is the number of inversions, by linearity of expectation, the number of inversions $\mathbb{E}(T) = \sum_{i=1}^{n-1} \sum_{j=i}^{n} \mathbb{E}(\mathbb{I}(i,j))$. Now the probability that (i,j) is an inverted pair is just $\frac{1}{2}$ as we draw a uniform random permutation. Hence $\mathbb{E}(\mathbb{I}(i,j)) = \frac{1}{2}$ and so the $\mathbb{E}(T) = \frac{1}{2}\binom{n}{2}$. ■

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Problem 10. [10 points] Consider the following three random variables:

1. Let A be a binary random variable that is 1 if a coin C_1 comes up heads and 0 otherwise.
2. Let B be a binary random variable that is 1 if a coin C_2 comes up heads and 0 otherwise.
3. Let C be a binary random variable that is 1 if both A and B are different values and 0 otherwise.

Assume that C_1 and C_2 are independent coins.

(a) [5 pts] Are A, B, C mutually independent?

Solution. No A, B, C are not mutually independent as fixing values for A, B, uniquely determine a value for variable C. ■

(b) [5 pts] Are A, B, C pairwise independent?

Solution. A, B, and C are pairwise independent. We first see that A, B are independent. Now we show that A, C are independent and by symmetry this will imply B, C are independent. We have that $P(C = 1|A = 1) = \frac{1}{2} = P(A = 1)$ as now C is uniquely determined by the value of B and so is 1 with probability $\frac{1}{2}$ and 0 with probability $\frac{1}{2}$. Similarly, $P(C = 1|A = 0) = \frac{1}{2} = P(A = 0)$. Hence we have that C and A are independent events. ■

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Problem 11. [10 points] Consider tossing a fair coin C until one throws a heads. Tossing C results in heads with probability $\frac{1}{2}$. Let X be a random variable corresponding to the number of tosses needed until one throws a heads (so $X \geq 1$).

(a) [5 pts] Calculate $\mathbb{E}(X)$.

Solution. We use a recursion to calculate $\mathbb{E}(X)$. Namely, $\mathbb{E}(X) = \frac{1}{2} + \frac{1}{2}(\mathbb{E}(X) + 1)$. This implies that $\mathbb{E}(X) = 2$. ■

(b) [10 pts] Calculate $\mathbb{E}(X^3)$.

Solution. We use the hint. We already know $\mathbb{E}(X)$. Now we just calculate $\mathbb{E}(X^2)$ again by using a recursion. Namely, $\mathbb{E}(X^2) = \frac{1}{3} + \frac{2}{3}(\mathbb{E}((X+1)^2))$.

Hence we have, $\mathbb{E}(X^2) = \frac{1}{3} + \frac{2}{3}(\mathbb{E}(X^2) + 2\mathbb{E}(X) + 1)$, and so $\mathbb{E}(X^2) = 15$.

Thus $\mathbb{E}(X^2) - \mathbb{E}(X)^2 = 15 - 9 = 6$ is the variance. ■

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