NOTES

Stirred, Not Shaken, by Stirling's Formula

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In this note we present a smooth and easy derivation of Stirling's asymptotic formula for $n!$,

$$n! \sim \frac{n^n \sqrt{2\pi n}}{e^n}, \quad n \to \infty. \tag{1}$$

We use the notation for asymptotic equivalence, so that

$$f(n) \sim g(n), \quad n \to \infty \qquad \text{means} \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Thus (1) is equivalent to

$$\lim_{n\to\infty} \frac{n^n \sqrt{2\pi n}}{n!\, e^n} = 1.$$

Over the years many proofs of this result have been published. Quite a number of them use tools such as the gamma function (see [**5**] for a recent one), or the Euler-Maclaurin summation formula (see the historic note in [**2**]). For an extensive bibliography, see [**3**], [**6**].

Our method is based on Wallis's product formula for π (most published proofs use this formula, see for instance [**1**]) and the trapezoidal rule for approximating an integral.

The trapezoidal rule

The trapezoidal rule for approximating a definite integral can be found in most calculus textbooks (see [**1**]). We will need an error estimate, and the one given in the following theorem will suffice:

THEOREM. *If for the function f the second derivative exists for all $x \in [a, b]$, then*

$$\int_a^b f(t)\,dt - \frac{f(a) + f(b)}{2} \cdot (b-a) = -\frac{f''(c)}{12} \cdot (b-a)^3$$

for some $c \in [a, b]$.

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Proof. The proof uses Rolle's mean value theorem. Define the constant K by:

$$\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\cdot(b-a) = K\cdot(b-a)^3.$$

As a consequence we have that the function

$$F(x) = \int_a^x f(t)\,dt - \frac{f(a)+f(x)}{2}\cdot(x-a) - K\cdot(x-a)^3$$

satisfies $F(a) = F(b) = 0$. Hence we may apply Rolle's theorem to F and so we know that for some $t \in (a,b)$ the derivative

$$F'(x) = f(x) - \frac{f'(x)}{2}\cdot(x-a) - \frac{f(a)+f(x)}{2} - 3K\cdot(x-a)^2 \tag{2}$$

is equal to zero: $F'(t) = 0$. Furthermore by evaluating F' for $x = a$ we find that $F'(a) = 0$. Applying Rolle's theorem a second time, but now to F', we find a value $c \in (a,t)$ with $F''(c) = 0$. From (2) we get

$$F''(x) = -\frac{f''(x)}{2}\cdot(x-a) - 6K\cdot(x-a).$$

So $F''(c) = 0$ implies that

$$K = -\frac{f''(c)}{12}.$$

■

Wallis's formula for π

The well known Wallis product formula for π states:

$$\lim_{n\to\infty} \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdots\frac{2n\cdot 2n}{(2n-1)\cdot(2n+1)} = \frac{\pi}{2}. \tag{3}$$

It follows immediately from Euler's product formula for the sine function [**1**]

$$\frac{\sin x}{x} = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right)\cdots$$

by taking $x = \frac{\pi}{2}$. (See also [**7**] for a nice proof without calculus or [**4**] for a proof without Euler's product formula.)

We need a concise form of Wallis's formula. To find it, we take the square root in (3) and rearrange the result:

$$\sqrt{\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2n\cdot 2n}{(2n-1)\cdot(2n+1)}} = \frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\cdot\frac{1}{\sqrt{2n+1}}.$$

We insert some matching factors into the denominator and the numerator:

$$\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}\cdot\frac{1}{\sqrt{2n+1}} = \frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)\cdot 2n}\cdot\frac{1}{\sqrt{2n+1}}.$$

Note that the numerator is equal to $2^{2n}\cdot(n!)^2$. Hence Wallis's formula reduces to:

$$\lim_{n\to\infty}\frac{(n!)^2}{(2n)!}\cdot\frac{2^{2n}}{\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}.$$

From this it follows that

$$\frac{(n!)^2}{(2n)!} \sim \frac{\sqrt{\pi n}}{2^{2n}}, \quad n \to \infty. \tag{4}$$

Stirling's formula

If we multiply the left hand side of (4) with $\frac{(2n)!}{n!}$, we get $n!$. We now use the trapezoidal rule to find an asymptotic estimate for $\frac{(2n)!}{n!}$. We start by rewriting this expression in the following way:

$$\frac{(2n)!}{n!} = (2n)(2n-1)\cdots(n+1) = n^n\left(1+\frac{n}{n}\right)\left(1+\frac{n-1}{n}\right)\cdots\left(1+\frac{1}{n}\right) \tag{5}$$

and we take a closer look at the factors in parentheses. If we take the logarithm of this part of the previous equation, we get a sum that reminds us of a Riemann sum:

$$\ln\left(1+\frac{n}{n}\right)+\ln\left(1+\frac{n-1}{n}\right)+\cdots+\ln\left(1+\frac{1}{n}\right).$$

Indeed, if we write it like this:

$$n\cdot\left[\frac{1}{n}\ln\left(1+\frac{1}{n}\right)+\frac{1}{n}\ln\left(1+\frac{2}{n}\right)+\cdots+\frac{1}{n}\ln\left(1+\frac{n}{n}\right)\right] \tag{6}$$

we have a right Riemann sum for $\ln x$ in the interval $[1,2]$. Since

$$\int_1^2 \ln x\,dx = 2\ln 2 - 1$$

we find that

$$\ln\left[\left(1+\frac{n}{n}\right)\left(1+\frac{n-1}{n}\right)\cdots\left(1+\frac{1}{n}\right)\right] \sim n\cdot(2\ln 2-1), \quad n\to\infty.$$

This asymptotic estimate isn't quite good enough; we need to do better. Using the trapezoidal rule instead of the right Riemann sum makes the difference in this case.

The trapezoidal sum is the average of the left and right Riemann sum, and hence equals

$$\frac{1}{2n}\ln 1+\frac{1}{n}\ln\left(1+\frac{1}{n}\right)+\cdots+\frac{1}{n}\ln\left(1+\frac{n-1}{n}\right)+\frac{1}{2n}\ln 2.$$

Since the first term is zero, this expression differs only in one term from the factor between brackets in (6). Its limit is still $2\ln 2-1$, and the theorem above provides us with an error estimate. We use the theorem with $f(x)=\ln x$ for the intervals in our Riemann sum, and sum the results. Since $-f''(x)=\frac{1}{x^2}$ is bounded above by 1 in the interval $[1,2]$, we get the inequality:

$$0 \le 2\ln 2 - 1 - \left[\frac{1}{n}\ln\left(1+\frac{1}{n}\right)+\cdots+\frac{1}{n}\ln\left(1+\frac{n}{n}\right)-\frac{1}{2n}\ln 2\right] \le \frac{1}{12n^2}.$$

We now multiply by n and rearrange the result:

$$0 \le n(2\ln 2-1)+\frac{1}{2}\ln 2-\ln\left[\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right] \le \frac{1}{12n}.$$

Applying the exponential function and taking the limit for $n \to \infty$ leads to

$$\left(1+\frac{n}{n}\right)\left(1+\frac{n-1}{n}\right)\cdots\left(1+\frac{1}{n}\right) \sim e^{n\cdot(2\ln 2-1)+\frac{\ln 2}{2}}, \quad n \to \infty.$$

If we multiply this result by n^n and rewrite the right hand side, we get the following asymptotic estimate for (5):

$$\frac{(2n)!}{n!} \sim \frac{2^{2n}n^n\sqrt{2}}{e^n}, \quad n \to \infty. \tag{7}$$

By multiplying the estimates in (4) and (7), we get Stirling's formula (1).

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Summary In this note an elementary proof of Stirling's asymptotic formula for $n!$ is given. The proof uses the Wallis formula for π and the trapezoidal rule for the calculation of a definite integral, with error estimate.

A Note on Disjoint Covering Systems— Variations on a 2002 AIME Problem

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We will consider two questions about covering systems, both suggested by Problem 9 from the 2002 American Invitational Mathematics Examination (AIME) [**1**]. The AIME problem, in effect, is the following:

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