Problem 1. [10 points]

(a) [2 pts] Does the above graph have a Hamiltonian path?

(b) [2 pts] Does the above graph have a Eulerian path?

(c) [6 pts] Consider the complete graph on n vertices K_n for odd $n \geq 3$. Prove that we cannot find a series of Hamiltonian paths in K_n that together cover all the edges of K_n.

Problem 2. [10 points]

(a) [3 pts] Find 7^{100} mod 13.

(b) [3 pts] Find the inverse of 33 mod 121 or prove that no such inverse exists.

(c) [4 pts] Prove that for any non-zero integers a, b, c, d, if $a - c \mid ab + cd$, then $a - c \mid ad + bc$.

Problem 3. [10 points] Suppose we are planning a trip to California for Thanksgiving. Unfortunately, we are booking our tickets late and so the prices are all really high. Suppose we are given the following list of ticket prices and travel times:

A 600 dollars, 9 hours 20 minutes

B 650 dollars, 8 hours 40 minutes

C 550 dollars, 9 hours 10 minutes

D 575 dollars, 8 hours 20 minutes

E 660 dollars, 9 hours 5 minutes

Our goal is to find the tickets that are the cheapest while minimizing travel time.

(a) [4 pts] Show that the relation of being more expensive and having more travel time creates a partial order for tickets.

(b) [3 pts] Draw the Hasse diagram for the above tickets with the ticket i is $<$ ticket j if ticket i is both more expensive and has more travel time than ticket j.

(c) [3 pts] Find the maximal elements of the poset. Is there a maximum element?

Problem 4. [10 points] A portion of a computer program consists of a sequence of calculations where the results are stored in variables, like this:

$$
\begin{array}{rrcl}
 & \text{Inputs:} & & x, y \\
\text{Step 1.} & a & = & x - 24 \\
2. & b & = & x * a \\
3. & c & = & 3 \\
4. & d & = & y - c \\
5. & e & = & y * *c \\
6. & f & = & e + 1 \\
 & \text{Outputs:} & & b, d, e
\end{array}
$$

A computer can perform such calculations most quickly if the value of each variable is stored in a *register*, a chunk of very fast memory inside the microprocessor. Programming language compilers face the problem of assigning each variable in a program to a register. Computers usually have few registers, however, so they must be used wisely and reused often. This is called the *register allocation* problem.

In the example above, variables x and y must be assigned different registers, because they hold distinct input values. Furthermore, c and d must be assigned different registers; if they used the same one, then the value of c would be overwritten in the fourth step and we'd get the wrong answer in the fifth step. On the other hand, variables b and d may use the same register; we no longer need b and can overwrite the register that holds its value. Assume that the computer carries out each step in the order listed and that each step is completed before the next is begun.

(a) [7 pts] Recast the register allocation problem as a question about graph coloring. What do the vertices correspond to? Construct the graph corresponding to the example above.

(b) [3 pts] How many registers do you need?

Problem 5. [10 points] Use the web graph below to answer parts (a), (b), and (c) of this question. The pagerank of vertex i can be written as p_i.

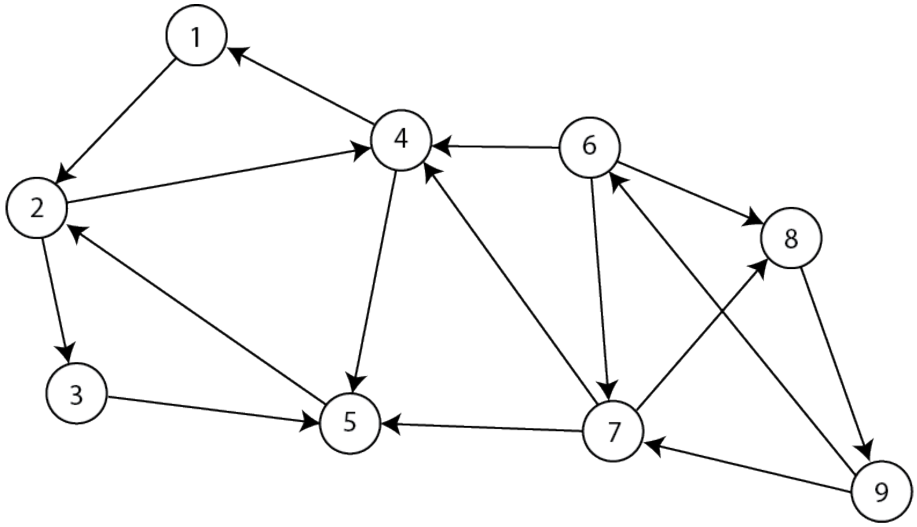

Figure 1: Web Graph

(a) [2 pts] For $\vec{p}\,'$ in terms of $\vec{p}$ can be written as a matrix product: $\vec{p}' = W\vec{p}$, for some matrix W, which is the *update matrix*. Find the update matrix.

(b) [2 pts] Which (if any) of the vertices of the web graph above will have PageRank value tending to zero if we run the PageRunk algorithm for many (let's say 200,000) iterations?

(c) [6 pts] Suppose we remove all the vertices that you found in part b from our Web Graph along with all corresponding edges from those vertices and consider the remaining graph G. Find the stationary distribution across the nodes assuming that each node in G starts with a value of $\frac{1}{5}$.

Note: If you are not confident about your answer in part b, feel free to instead find the stationary distribution for the directed triangle formed by vertices 1, 2, 4 of the Web Graph. Again assume each of the nodes 1, 2, 4 starts with inital PageRank value $\frac{1}{3}$.