Problem Set 9 Solutions

Due: Thursday, November 6

Problem 1. [10 points]

(a) [5 pts] Show that of any $n+1$ distinct numbers chosen from the set $\{1, 2, \ldots, 2n\}$, at least 2 must be relatively prime. (*Hint:* $\gcd(k, k+1) = 1$.)

Solution. Treat the $n+1$ numbers as the pigeons and the n disjoint subsets of the form $\{2j-1, 2j\}$ as the pigeonholes. The pigeonhole principle implies that there must exist a pair of consecutive integers among the $n+1$ chosen which, as suggested in the hint, must be relatively prime. ■

(b) [5 pts] Show that any finite connected undirected graph with $n \geq 2$ vertices must have 2 vertices with the same degree.

Solution. In a finite connected graph with $n \geq 2$ vertices, the domain for the vertex degrees is the set $\{1, 2, \ldots, n-1\}$ since each vertex can be adjacent to at most all of the remaining $n-1$ vertices and the existence of a degree 0 vertex would violate the assumption that the graph be connected. Therefore, treating the n vertices as the pigeons and the $n-1$ possible degrees as the pigeonholes, the pigeonhole principle implies that there must exist a pair of vertices with the same degree. ■

Problem 2. [15 points] Under Siege!

Fearing retribution for the many long hours his students spent completing problem sets, Prof. Leighton decides to convert his office into a reinforced bunker. His only remaining task is to set the 10-digit numeric password on his door. Knowing the students are a clever bunch, he is not going to pick any passwords containing the forbidden consecutive sequences "18062", "6042" or "35876" (his MIT extension).

How many 10-digit passwords can he pick that don't contain forbidden sequences if each number $0, 1, \ldots, 9$ can only be chosen once (i.e. without replacement)?

Solution. The number of passwords he can choose is the number of permutations of the 10 digits minus the number of passwords containing one or more of the forbidden words, which we will find using inclusion-exclusion.

There are 6 positions 18062 could appear and the remaining digits could be any permutation of the remaining 5 digits. Therefore, there are $6 \cdot 5!$ passwords containing 18062. Similarly, there are $7 \cdot 6!$ passwords containing 6042 and $6 \cdot 5!$ passwords containing 35876.

Each of the forbidden words contain the digit 6 and since he must choose each number exactly once, the only way two forbidden words can appear in the same password is if they overlap at 6. The only case where this can happen is if the password contains 35876042 and there are $3 \cdot 2!$ such passwords.

By inclusion-exclusion the total number of passwords not containing any of the forbidden words is

$$10! - (6 \cdot 5! + 7 \cdot 6! + 6 \cdot 5!) + 3 \cdot 2! = 3622326$$

■

Problem 3. [50 points] Be sure to show your work to receive full credit. In this problem we assume a standard card deck of 52 cards.

(a) [5 pts] How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

Solution. There is a bijection with sequence of the form:

$$(\text{value of pair}, \text{suits of pair}, \text{value of other three cards}, \text{suits of other three cards})$$

Thus, the number of hands with a single pair is:

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1,098,240$$

Alternatively, there is also a 3!-to-1 mapping to the tuple:

$$(\text{value of pair}, \text{suits of pair},$$
$$\text{value 3rd card}, \text{suit 3rd card}, \text{value 4th card}, \text{suit 4th card}, \text{value 5th card}, \text{suit 5th card})$$

Thus, the number of hands with a single pair is:

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot 4 \cdot 11 \cdot 4 \cdot 10 \cdot 4}{3!} = 1,098,240$$

■

(b) [5 pts] How many 5-card hands have two or more kings?

Solution. This is the set of all hands minus the hands with either no kings or one king:

$$\binom{52}{5} - \binom{48}{5} - 4 \cdot \binom{48}{4} = 108,336$$

Alternatively, this is also the set of all hands of two, three, or four kings:

$$\binom{48}{3}\binom{4}{2} + \binom{48}{2}\binom{4}{3} + \binom{48}{1}\binom{4}{4} = 108,336$$

■

(c) [5 pts] How many 5-card hands contain the ace of spades, the ace of clubs, or both?

Solution. There are $\binom{51}{4}$ hands containing the ace of spades, an equal number containing the ace of clubs and $\binom{50}{3}$ containing both. By the inclusion-exclusion formula:

$$\binom{51}{4} + \binom{51}{4} - \binom{50}{3}$$

hands contain one or the other or both. ■

(d) [5 pts] For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i = n$$

Solution. There is a bijection from the solutions of the equation to the binary strings containing n zeros and k ones where x_0 is the number of 0s preceding the first 1, x_k is the number of 0s following the last 1 and x_i is the number of 0s between the i^{th} and $(i+1)^{th}$ 1 for $0 < i < k$.

$$\binom{n+k}{k}$$

■

(e) [5 pts] For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i \leq n$$

Solution. There is a bijection from the solutions of

$$\begin{aligned} \sum_{i=0}^{k} x_i &\leq n \\ &= n - x_{k+1} && \text{(for some } x_{k+1} \geq 0\text{)} \end{aligned}$$

and the solutions of

$$\sum_{i=0}^{k+1} x_i = n.$$

$$\binom{n+k+1}{k+1}$$

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(f) [5 pts] In how many ways can $3n$ students be broken up into n groups of 3?

Solution.

$$\frac{(3n)!}{(3!)^n n!}.$$

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(g) [5 pts] How many simple undirected graphs are there with n vertices?

Solution. There are $\binom{n}{2}$ potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{\binom{n}{2}}$$

■

(h) [5 pts] How many directed graphs are there with n vertices (self loops allowed)?

Solution. There are n^2 potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{n^2}$$

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(i) [5 pts] How many tournament graphs are there with n vertices?

Solution. There are no self-loops in a tournament graph and for each of the $\binom{n}{2}$ pairs of distinct vertices a and b, either $a \rightarrow b$ or $b \rightarrow a$ but not both. Therefore, the number of tournament graphs is:

$$2^{\binom{n}{2}}$$

■

(j) [5 pts] How many acyclic tournament graphs are there with n vertices?

Solution. For any path from x to y in a tournament graph, an edge $y \rightarrow x$ would create a cycle. Therefore in any acyclic tournament graph, the existence of a path between distinct vertices x and y would require the edge $x \rightarrow y$ also be in the graph. That is, the "beats" relation for such a graph would be transitive. Since each pair of distinct players are comparable (either $x \rightarrow y$ or $y \rightarrow x$) we can uniquely rank the players $x_1 < x_2 < \cdots < x_n$. There are $n!$ such rankings. ■

Problem 4. [10 points] Suppose we have a deck of cards that has 4 suits, each suit having 13 cards. The magician asks the audience to select an arbitrary set of 7 cards. His assistant selects v cards out of the 7 cards and puts these v cards on a table. Is it possible for the magician to figure out the identities of the $7 - v$ remaining cards that are hidden from him (by only considering the v cards that his assistant put on the table)?

(a) [5 pts] Use a counting argument to show that for $v = 5$ the magician and assistant can work together such that the magician is able find the identities of the hidden cards.

Solution. The assistant has $\binom{7}{5}$ choices to select $v = 5$ cards and 5! choices to order these cards. This gives $\binom{7}{5}5! = 7!/2 = 2520$ ways to map a set of 7 cards to a sequence of 5 cards.

There are $7 - v = 2$ hidden cards. The complete deck has 52 cards and 5 cards are visible. So, the two hidden cards are from a set of 47 cards. There are $\binom{47}{2} = 47 \cdot 46/2 = 1081$ ways to select 2 out of 52 cards.

Since the number of possible mappings is at least the number of possible pairs of hidden cards, it is possible for the assistant and magician to agree on a mapping such that the magician is able to find the identities of the hidden cards. ■

(b) [5 pts] Is it possible to make the card trick work for $v = 4$? Explain your answer.

Solution. No. The assistant has $\binom{7}{4}$ choices to select $v = 4$ cards and 4! choices to order these cards. This gives $\binom{7}{4}4! = 7!/3! = 840$ ways to map a set of 7 cards to a sequence of 4 cards.

There are $7 - v = 3$ hidden cards. The complete deck has 52 cards and 4 cards are visible. So, the two hidden cards are from a set of 48 cards. There are $\binom{48}{3} = 48 \cdot 47 \cdot 46/3! = 17296$ ways to select 3 out of 52 cards.

Since the number of possible mappings is more than the number of possible triples of hidden cards, it is not possible for the assistant and magician to agree on a mapping such that the magician is able to find the identities of the hidden cards. ■

Problem 5. [15 points] Give a combinatorial proof of the following theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

(*Hint: Consider the set of all length-n sequences of 0's, 1's and a single *.*)

Solution. Let $P = \{0, \ldots, n-1\} \times \{0,1\}^{n-1}$. On the one hand, there is a bijection from P to S by mapping (k, x) to the word obtained by inserting a * just after the kth bit in the length-$n-1$ binary word, x. So

$$|S| = |P| = n2^{n-1} \tag{1}$$

by the Product Rule.

On the other hand, every sequence in S contains between 1 and n nonzero entries since the $*$, at least, is nonzero. The mapping from a sequence in S with exactly k nonzero entries to a pair consisting of the set of positions of the nonzero entries and the position of the * among these entries is a bijection, and the number of such pairs is $\binom{n}{k}k$ by the Generalized Product Rule. Thus, by the Sum Rule:

$$|S| = \sum_{k=1}^{n} k\binom{n}{k}$$

Equating this expression and the expression (1) for $|S|$ proves the theorem. ■