Problem Set 7

Due: Monday, October 20

Problem 1. **[10 points]** Express

$$\sum_{i=0}^{n} i^2 x^i$$

as a closed-form function of n.

Problem 2. **[10 points]** Express

$$\sum_{j=0}^{n} \sum_{i=j}^{n} \frac{j}{n+1-(i-j)}.$$

as a closed-form function of n.

Problem 3. **[10 points]** Find asymptotically tight bounds for

$$f(n) = \prod_{i=1}^{n} e^{1/i}.$$

That is, find a lower bound $l(n) \leq f(n)$ and an upper bound $u(n) \geq f(n)$ such that $l(n) = \Theta(u(n))$.

Problem 4. **[10 points]** Use the integral method to find upper and lower bounds that differ by at most 0.1 for the following sum. (Note that you may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{i^2}.$$

The actual value of the summation turns out to be $\pi^2/6 = 1.644\ldots$.

Problem 5. **[10 points]**

(a) [5 pts] Prove that the statement

$$n + n\cos\left(\frac{\pi n}{2}\right) = o(n)$$

is false.

(b) [5 pts] Prove that the statement

$$n + n\cos\left(\frac{\pi n}{2}\right) = \Omega(1)$$

is also false.

Problem 6. [20 points] For each of the following six pairs of functions f and g (parts (a) through (f)), state which of these order-of-growth relations hold (more than one may hold, or none may hold):

$$f = o(g) \qquad f = O(g) \qquad f = \omega(g) \qquad f = \Omega(g) \qquad f = \Theta(g) \qquad f \sim g$$

(a)	$f(n) = n!$	$g(n) = (n+1)!$
(b)	$f(n) = \log_2 n$	$g(n) = \log_{10} n$
(c)	$f(n) = 2^n$	$g(n) = 10^n$
(d)	$f(n) = 0$	$g(n) = 17$
(e)	$f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$	$g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$
(f)	$f(n) = 1.0000000001^n$	$g(n) = n^{10000000000}$

Problem 7. [30 points] An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis.[1] In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to create water caches out in the desert.

For example, if the shrine were 2/3 of a day's walk into the desert, then she could recover the Holy Grail with the following strategy. She leaves the oasis with 1 gallon of water, travels 1/3 day into the desert, caches 1/3 gallon, and then walks back to the oasis— arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks 1/3 day into the desert, tops off her water supply by taking the 1/3 gallon in her cache, walks the remaining 1/3 day to the shine, grabs the Holy Grail, and then walks for 2/3 of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

(a) [5 pts] What is the most distant point that the explorer can reach and return from if she takes only 1 gallon from the oasis.?

(b) [5 pts] What is the most distant point the explorer can reach and return form if she takes only 2 gallons from the oasis? No proof is required; just do the best you can.

[1] She's right about the location, but doesn't realize that the Holy Grail is actually just the Beneš network.

(c) [5 pts] What about 3 gallons? (Hint: First, try to establish a cache of 2 gallons *plus* enough water for the walk home as far into the desert as possible. Then use this cache as a springboard for your solution to the previous part.)

(d) [5 pts] How can the explorer go as far as possible is she withdraws n gallons of water? Express your answer in terms of the Harmonic number H_n, defined by:

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}$$

(e) [5 pts] Use the fact that

$$H_n \sim \ln n$$

to approximate your previous answer in terms of logarithms.

(f) [5 pts] Suppose that the shrine is $d = 10$ days walk into the desert. Relying on your approximate answer, how many days must the explorer travel to recover the Holy Grail?