6.042/18.062J Mathematics for Computer Science — October 21, 2014
Tom Leighton and Ankur Moitra

Midterm

Name: ______________________________

Recitation instructor: ______________________ **Section:** ______

- This quiz is **closed book**, but you may have one 8.5 × 11” sheet with notes in your own handwriting on both sides.
- Calculators and electronic devices are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn’t shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem’s page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

Problem	Points	Grade	Grader
1	10		
2	10		
3	18		
4	12		
5	12		
6	10		
7	10		
8	10		
9	16		
10	12		
Total	120		

Problem 1. [10 points]

(a) [5 pts] Show that $(p \wedge q) \rightarrow r$ is equivalent to $p \rightarrow (q \rightarrow r)$ by using a truth table.

(b) [5 pts] Suppose $P(x, y)$ is the predicate "$xy = 1$", where the universe of discourse for x is the set of positive integers, and the universe of discourse for y is the set of real numbers. Transform the following propositions into English and establish if they are true or false:

1. $\forall x \exists y P(x, y)$
2. $\exists y \forall x P(x, y)$

Problem 2. [10 points] Let F_n be the n'th Fibonacci number. Recall that the Fibonacci sequence satisfies $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}, \forall n \geq 3$. Prove by induction that for all $n \geq 1$ we have

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

Problem 3. [18 points] Let G be a graph with n vertices, where n is even, and where G contains no cycle of length 3.

(a) [4 pts] If $e = (u, v)$ is an edge of G, what is the maximum number of edges other than e that can be incident to one of u or v?

(b) [10 pts] Prove by induction that G has at most $\frac{n^2}{4}$ edges.

(c) [4 pts] Give an example of a graph G with n vertices and $\frac{n^2}{4}$ edges that has no cycle of length 3.

Problem 4. [12 points]

For the following parts, a correct numerical answer will only earn credit if accompanied by its derivation. Show your work.

(a) [6 pts] Use the Pulverizer to find integers s and t such that $141s + 61t = \gcd(141, 61)$.

(b) [6 pts] Find the remainder of 10^{1001} when divided by 101.

Problem 5. [12 points] Alice plays a game with piles of coins. Initially, there are 3 piles of coins, containing 5, 7, and 93 coins, respectively. The game ends when there are 105 piles, each with one coin. There are two moves she is allowed to make.

- She can merge two piles together.
- She can divide a pile with an even number of coins into two piles of equal size.

Using an invariant, show that the game will never end.

(*Hint: Consider each of the three cases for the first move separately. Notice that in each case, the sizes of the two resulting piles share a common factor.*)

Problem 6. [10 points]

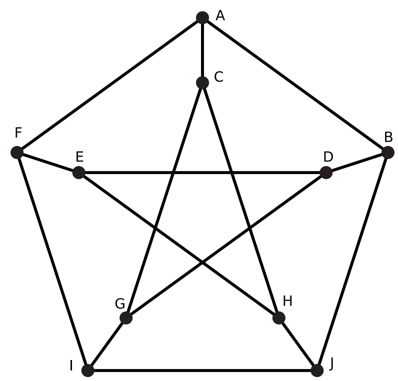
Consider a stable marriage problem with the following preferences.

Alfred:	Fiona, Helen, Emily, Grace
Billy:	Helen, Emily, Fiona, Grace
Calvin:	Fiona, Helen, Grace, Emily
David:	Fiona, Helen, Emily, Grace

Emily:	Billy, Alfred, Calvin, David
Fiona:	Alfred, Calvin, Billy, David
Grace:	Billy, Alfred, David, Calvin
Helen:	Alfred, Billy, David, Calvin

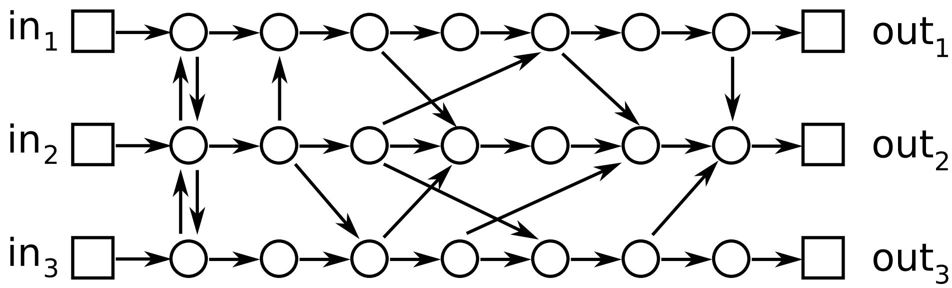
What is the stable matching generated by the mating algorithm on these preferences?

Problem 7. [10 points] Show that the chromatic number of this graph is 3. (Remember to show that you can't use fewer than 3 colors.)



Problem 8. [10 points]

Consider the routing network below:



A routing network.

(a) [4 pts] What is the diameter?

(b) [6 pts] Show that there is a 1–1 routing problem on this network which has a congestion of 3. (Be sure to show that no matter what set of paths are chosen for your routing problem, that they must all go through a single node.)

Problem 9. [16 points]

In this problem, PageRank refers to *unscaled* PageRank.

(a) [6 pts] Consider the directed graph A_n consisting of a single doubly-linked path on n elements, as pictured below, and let v_i denote the ith vertex in order. Show that the PageRank values

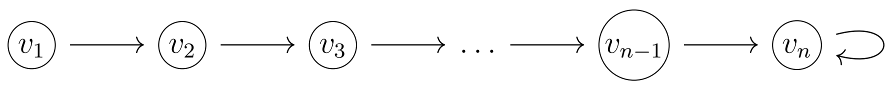
$$\text{pagerank}(v_i) = a_i = \begin{cases} \frac{1}{2(n-1)} & \text{if } i = 1 \text{ or } n \\ \frac{1}{n-1} & \text{if } 1 < i < n \end{cases}$$

are in equilibrium.

$$v_1 \rightleftarrows v_2 \rightleftarrows v_3 \rightleftarrows \cdots \rightleftarrows v_{n-1} \rightleftarrows v_n$$

(b) [10 pts] Now consider the directed graph B_n consisting of a single directed path on n elements, as pictured below, with v_i again denoting the ith vertex in order. We consider v_n to have a self-loop, so that every vertex has an outgoing edge. Starting from arbitrary PageRank values, prove that after $n-1$ steps, the PageRank of the graph is concentrated at the last vertex v_n, i.e. the first $n-1$ vertices all have PageRank value zero.

(*Hint: What can you say about the PageRank values after t update steps? Which vertices can still have nonzero PageRank?*)



Problem 10. [12 points]

Define a relation R on the positive integers as follows: we say $a\ R\ b$ if either a divides b or b divides a. State and briefly justify whether or not the relation R has the following properties:

(a) [2 pts] Reflexivity.

(b) [2 pts] Symmetry.

(c) [2 pts] Antisymmetry.

(d) [3 pts] Transitivity.

(e) [3 pts] The property of being an equivalence relation.