

Problems for Recitation 3

1 Problem: Breaking a chocolate bar

We are given a chocolate bar with $m \times n$ squares of chocolate, and our task is to divide it into mn individual squares. We are only allowed to split one piece of chocolate at a time using a vertical or a horizontal break.

For example, suppose that the chocolate bar is 2×2 . The first split makes two pieces, both 2×1 . Each of these pieces requires one more split to form single squares. This gives a total of three splits.

Prove that the number of times you split the bar does not depend on the sequence of splits you make.

2 Problem: The Temple of Forever

Each monk entering the Temple of Forever is given a bowl with 15 red beads and 12 green beads. Each time the Gong of Time rings, a monk must do one of two things:

1. *Exchange*: If he has at least 3 red beads in his bowl, then he may exchange 3 red beads for 2 green beads.
2. *Swap*: He may replace each green bead in his bowl with a red bead and replace each red bead in his bowl with a green bead. That is, if he starts with i red beads and j green beads, then after he performs this operation, he will have j red beads and i green beads.

A monk may leave the Temple of Forever only when he has exactly 5 red beads and 5 green beads in his bowl.

Let's look at how we can represent this problem as a state machine.

- What do the states of the machine look like?
- Use the notation you developed above to represent the allowable transitions in the state machine.
- Expand the state machine diagram to the first three or four levels. Label the transitions according to the operation type (E for *exchange* or S for *swap*).

Now we'll show that no monk can ever escape the Temple of Forever because the state $(5, 5)$ violates an invariant of the Temple of Forever machine.

Theorem 1. *No one ever leaves the Temple of Forever.*

Prove this theorem by induction. Begin by searching for an invariant that holds initially and is maintained by each operation, but would be violated by the condition required for departure.

Now let's take a look at a different property of the Temple of Forever machine.

Theorem 2. *There is a finite number of reachable states in the Temple of Forever machine.*

Prove this theorem. (Hint: First find an invariant that suggests an upper bound on the number of reachable states. Be sure to prove the invariant.)

Inside the Temple of Forever, the Gong of Time rings on. As you may well imagine, the monks begin to recognize that no matter how many ways they try to exchange or swap their beads, they always seem to end up in some state they've already been in before! For one or two monks, this realization is all they need to propel them instantly into a state of

enlightenment. For the overwhelming majority, however, this knowledge does nothing but weaken their resolve. They just get depressed. Taking note of the mental state of this second group, the Keeper of the Temple makes an unannounced appearance and proclaims to the group, “From now on, any monk who is able to visit 108 (108 being the mystical number that encompasses all of existence¹) unique states will be allowed to leave the Temple of Forever.”

Do the monks have any chance of leaving the Temple of Forever?

Theorem 3. *It is not possible to visit 108 unique states in the Temple of Forever machine.*

Prove this theorem. (Hint: Consider a proof by contradiction.)

What is the true maximal number of unique states a monk can visit in any execution of the Temple of Forever machine? How can this number be achieved?

¹See <http://astrologyforthesoul.com/vp/mysticalnumber108.html>. Also consider: $42 + 24 + 42 = 108$.