ON CHEBYSHEV-TYPE INEQUALITIES FOR PRIMES

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For $N \in \mathbb{N}$, let $\pi(N)$ denote the number of prime numbers less than or equal to N. In 1851, Chebyshev [**1**] proved that, given $\varepsilon > 0$,

$$(c_1 - \varepsilon)\frac{N}{\log N} \leqslant \pi(N) \leqslant (c_2 + \varepsilon)\frac{N}{\log N} \tag{1}$$

where $c_1 = \log(2^{1/2}3^{1/3}5^{1/5}/30^{1/30})$, $c_2 = 6c_1/5$, and N is sufficiently large. This result was extended and improved by a number of authors culminating, of course, in the prime-number theorem of Hadamard-de la Vallée Poussin, namely, that

$$\pi(N) \sim \frac{N}{\log N} \qquad (N \to \infty).$$

The standard version of (1), which is nowadays found in most textbooks, e.g., [**2**], stems from Erdős [**3**] as modified by Erdős-Kalmár (unpublished). They showed that, given $\varepsilon > 0$, $\exists N_0 = N_0(\varepsilon)$ such that

$$(\log 2 - \varepsilon)\frac{N}{\log N} \leqslant \pi(N) \leqslant (\log 4 + \varepsilon)\frac{N}{\log N}, \qquad (N \geqslant N_0). \tag{2}$$

This is indeed weaker than Chebyshev's original result, but the proof is easier and certainly more elegant.

In this note, we shall prove (2) in a novel way that does not seem to have been noticed till now. The lower bound in (2), in particular, is obtained with rather surprising ease. A feature of this new approach is in the use of certain binomial identities to obtain estimates for d_n defined by

$$d_n = \operatorname*{lcm}_{1 \leqslant m \leqslant n} \{m\} \tag{3}$$

where lcm, as usual, stands for the least common multiple. The inequalities for $\pi(N)$ are then deduced by standard methods from those for d_n. The quickest way to get a lower bound for $\pi(N)$ of the correct order is the following.

THEOREM 1. *For $N \geqslant 2$,*

$$\pi(N) \geqslant c\frac{N}{\log N}$$

for some $c > 0$.

Proof. Consider, for $n \geqslant 1$, the integral

$$I = \int_0^1 x^n(1-x)^n\,dx = \int_0^1 \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^{n+r}\,dx = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{n+r+1}. \tag{4}$$

Since every denominator on the R.H.S. of (4) is at most $2n+1$ we have that $Id_{2n+1} \in \mathbb{N}$. On the other hand, for $x \in [0,1]$, we have $x(1-x) \leqslant 1/4$ so that $I \leqslant 1/4^n$. Hence, we deduce that

$$d_{2n+1} \geqslant 4^n. \tag{5}$$

Now, if $p^a \| d_{2n+1}$, $p^a \mid m$ for some m, $1 \leqslant m \leqslant 2n+1$, so that $p^a \leqslant 2n+1$. Hence

$$d_{2n+1} \leqslant \prod_{p \leqslant 2n+1} p^{\log(2n+1)/\log p}. \tag{6}$$

Comparing (5) and (6), we deduce that

$$n\log 4 \leqslant \log d_{2n+1} \leqslant \sum_{p\leqslant 2n+1} \frac{\log(2n+1)}{\log p}\log p = \log(2n+1)\pi(2n+1)$$

or

$$\pi(2n+1) \geqslant \frac{2n\log 2}{\log(2n+1)}.$$

This immediately implies that

$$\pi(N) \geqslant \frac{(N-2)\log 2}{\log N} \qquad (N \geqslant 2),$$

as required.

Note that the usual lemma on the exact power of a prime that divides $n!$, which is present in the traditional proofs, is absent here. With a bit more effort, we now obtain a clear, explicit constant in the lower bound for $\pi(N)$.

THEOREM 2. *For $N \geqslant 4$,*

$$\pi(N) \geqslant \log 2 \frac{N}{\log N}.$$

Proof. For $1 \leqslant m \leqslant n$, consider the integral

$$I = I(m,n) = \int_0^1 x^{m-1}(1-x)^{n-m}\,dx = \sum_{r=0}^{n-m}(-1)^r\binom{n-m}{r}\frac{1}{m+r}. \tag{7}$$

Clearly, $Id_n \in \mathbb{N}$. On the other hand, repeated integration by parts yields

$$I = 1/m\binom{n}{m}. \tag{8}$$

Hence, $m\binom{n}{m}\Big| d_n \forall m,\ 1 \leqslant m \leqslant n$. In particular, since $n\binom{2n}{n}\Big| d_{2n}$ and $(2n+1)\binom{2n}{n} = (n+1)\binom{2n+1}{n+1}$, both $n\binom{2n}{n}$ and $(2n+1)\binom{2n}{n}$ divide d_{2n+1}. Since $(n, 2n+1) = 1$, we deduce that $n(2n+1)\binom{2n}{n}\Big| d_{2n+1}$ so that

$$d_{2n+1} \geqslant n(2n+1)\binom{2n}{n} \geqslant n4^n. \tag{9}$$

The last inequality follows from the fact that $\binom{2n}{n}$ is the largest of the $2n+1$ terms in the binomial expansion of $(1+1)^{2n}$. From (9), we infer that, if $n \geqslant 2$, $d_{2n+1} \geqslant 2^{2n+1}$, and that, if $n \geqslant 4$, $d_{2n+2} \geqslant d_{2n+1} \geqslant 2^{2n+2}$. Hence

$$d_N \geqslant 2^N$$

for $N \geqslant 9$. Taking logarithms, as in the proof of Theorem 1, we get

$$\pi(N) \geqslant \frac{N\log 2}{\log N} \qquad (N \geqslant 9).$$

This inequality is also easily checked for $4 \leqslant N \leqslant 8$. This completes the proof of Theorem 2.

We now give two different characterizations of d_n.

THEOREM 3. *For $n \in \mathbb{N}$, we have*

(i) $d_n = \operatorname{lcm}_{1\leqslant m\leqslant n}\left\{m\binom{n}{m}\right\}$ *and*

(ii) $d_n = \operatorname{hcf}_{n/2\leqslant m\leqslant n}\left\{d_m\binom{n}{m}\right\}$, *where* hcf *denotes, as usual, the highest common factor.*

Proof. In the proof of Theorem 2, we showed that $m\binom{n}{m}\Big| d_n$ for all m with $1 \leqslant m \leqslant n$. Hence

$$\operatorname*{lcm}_{1 \leqslant m \leqslant n}\left\{m\binom{n}{m}\right\}\Big| d_n.$$

But each m divides $m\binom{n}{m}$ so that $d_n = \operatorname{lcm}\{m\}$ divides $\operatorname{lcm}\left\{m\binom{n}{m}\right\}$. Hence we have that

$$d_n = \operatorname*{lcm}_{1 \leqslant m \leqslant n}\left\{m\binom{n}{m}\right\},$$

as required.

The proof of (ii) rather surprisingly uses (i). For $1 \leqslant k \leqslant m \leqslant n$, consider the identity

$$k\binom{n}{k}\binom{n-k}{m-k} = k\binom{m}{k}\binom{n}{m}. \tag{10}$$

From (i), the R.H.S. of (10) is a factor of $\binom{n}{m}d_m$. Hence we have that

$$k\binom{n}{k}\Big|\binom{n}{m}d_m, \qquad \forall 1 \leqslant k \leqslant m. \tag{11}$$

But

$$k\binom{n}{k} = (n-k+1)\binom{n}{n-k+1}.$$

Hence, if $n/2 \leqslant m \leqslant n$ (11) implies that

$$k\binom{n}{k}\Big|\binom{n}{m}d_m, \qquad \forall 1 \leqslant k \leqslant n$$

so that, from (i), we have that

$$d_n\Big|\binom{n}{m}d_m,$$

i.e., that

$$d_n\Big| \operatorname*{hcf}_{n/2 \leqslant m \leqslant n}\left\{d_m\binom{n}{m}\right\}. \tag{12}$$

But one of the terms on the R.H.S. of (12) is precisely d_n. Hence

$$d_n = \operatorname*{hcf}_{n/2 \leqslant m \leqslant n}\left\{d_m\binom{n}{m}\right\},$$

as required.

COROLLARY. *We have, for $n \in \mathbb{N}$,*

$$d_n \leqslant 4^n. \tag{13}$$

Proof. Note that $d_1 \leqslant 4$ and that $d_2 \leqslant 4^2$. This forms the basis for an induction argument from (ii) of Theorem 3. From (ii), we have, assuming the truth of (13) for $n < 2m$,

$$d_{2m} \leqslant d_m\binom{2m}{m} \leqslant 4^m(1+1)^{2m} = 4^{2m}$$

and that

$$d_{2m+1} \leqslant d_{m+1}\binom{2m+1}{m+1} \leqslant 4^{m+1}\frac{(1+1)^{2m+1}}{2} = 4^{2m+1}.$$

The last inequality follows from the fact that

$$\binom{2m+1}{m+1} = \binom{2m+1}{m}$$

and both occur in the expansion of $(1+1)^{2m+1}$. The two inequalities above imply that

$$d_n \leqslant 4^n, \qquad \forall n \in \mathbb{N},$$

as required.

Note that we may now, by familiar methods, deduce the upper bound for $\pi(N)$. From (13) we have, for any t with $1 < t \leqslant n$,

$$\prod_{t<p\leqslant n} p \leqslant d_n \leqslant 4^n$$

so that

$$(\log t)(\pi(n) - \pi(t)) \leqslant \sum_{t<p\leqslant n} \log p \leqslant n \log 4.$$

Hence

$$\pi(n) \leqslant \frac{n \log 4}{\log t} + t.$$

Put $t = n/\log^2 n$ to deduce that

$$\pi(n) \leqslant \frac{n \log 4}{\log n} + O\left(\frac{n \log \log n}{\log^2 n}\right).$$

This implies the upper bound in (2).

In conclusion, it is perhaps appropriate to point out that Theorem 3 can also be proved by the standard methods of proof. The interest here lies essentially in the rather curious nature of this proof. It is unexpected to use (i) to prove (ii), and it certainly is strange that there is no mention of primes in the proof of Theorem 3. It also seems worthwhile to point out that there are different ways to prove the identity implied by equations (7) and (8), for example, by expressing $1/x(x+1)\cdots(x+m)$ in partial fractions or by using the difference operator.

References

1. P. L. Chebyshev, Mémoire sur les nombres premiers, Oeuvres I, 1852, pp. 51–70
2. W. J. LeVeque, Fundamentals of Number Theory, Addison-Wesley, Reading, Mass., 1977.
3. P. Erdős, Acta Universitatis Szegediensis (Szeged, Hungary), 5 (1932) 194–198.

THE WISE GURU AND HIS DISCIPLES (PAGE 125)

The central figure is André Weil. On his right is Serge Lang, and on his left Peter Swinnerton-Dyer (on the floor), and E. C. Zeeman. The picture was taken in approximately 1955.