Quiz 1

Problem 1. [20 points] For $n \geq 1$, let a_n be the largest odd divisor of n, and let $b_n = a_1 + a_2 + \ldots + a_n$.

(a) [2 pts] Prove the proposition that $a_{2n+1} = 2n + 1$.

Solution. *Proof.* The largest odd divisor of an odd number is the odd number itself. □

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(b) [5 pts] Prove the proposition that $a_{2n} = a_n$.

Solution. *Proof.* Let d be the largest odd divisor of $2n$. Since d is odd, d must divide n. So, $a_{2n} \leq a_n$. Since the largest odd divisor of n also divides $2n$, $a_n \leq a_{2n}$. Combining both inequalities proves the proposition. □

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(c) [10 pts] Prove the proposition that $b_n \geq (n^2 + 2)/3$.

Hint: use strong induction and when you prove the inductive hypothesis for $n+1$, distinguish the two cases $n + 1$ is even (that is, $n + 1 = 2k$ with $k \geq 1$) and $n + 1$ is odd (that is, $n + 1 = 2k + 1$ with $k \geq 1$).

Solution. *Proof.* We use strong induction. Let $P(n)$, for $n \geq 1$, be the predicate $b_n \geq (n^2 + 2)/3$.

Base case: $b_1 = a_1 = 1 \geq (1^2 + 2)/3$.

Inductive step: For the purposes of proving $P(n + 1)$, assume $P(k)$ for $1 \leq k \leq n$. So, we assume that $b_k \geq (k^2 + 2)/3$ is true for $1 \leq k \leq n$.

We distinguish the two cases $n + 1$ is even and $n + 1$ is odd.

If $n + 1 = 2k$ with $k \geq 1$, then

$$\begin{aligned}
b_{n+1} &= (a_1 + a_3 + \ldots + a_{2k-1}) + (a_2 + a_4 + \ldots + a_{2k}) \\
&= 1 + 3 + \ldots + (2k - 1) + (a_1 + a_2 + \ldots + a_k) \\
&= k^2 + b_k \\
&\geq k^2 + (k^2 + 2)/3 \text{ (by the inductive hypothesis)} \\
&= ((2k)^2 + 2)/3 \\
&= ((n + 1)^2 + 2)/3.
\end{aligned}$$

If instead $n+1 = 2k+1$ with $k \geq 1$, then

$$\begin{aligned} b_{n+1} &= (a_1 + a_3 + \ldots + a_{2k+1}) + (a_2 + a_4 + \ldots + a_{2k}) \\ &= 1 + 3 + \ldots + (2k+1) + (a_1 + a_2 + \ldots + a_k) \\ &= (k+1)^2 + b_k \\ &\geq (k+1)^2 + (k^2+2)/3 \text{ (by the inductive hypothesis)} \\ &= ((2k+1)^2+2)/3 + (2k+2)/3 \\ &> ((n+1)^2+2)/3. \end{aligned}$$

□

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(d) [0 pts] DISREGARD THIS PART? Determine for which n the equality $b_n = (n^2+2)/3$ holds. You do not need to prove your answer.

Solution. The previous derivations show that, for odd $n+1$, $b_{n+1} > ((n+1)^2+2)/3$, and, for even $n+1 = 2k$, $b_{n+1} = ((n+1)^2+2)/3$ if and only if $b_k = (k^2+2)/3$. This leads us to believe

$$P(n) = \text{“}b_n = (n^2+2)/3 \leftrightarrow n \text{ is a power of } 2\text{''}$$

for $n \geq 1$.

Proof. We use strong induction.

Base case: $b_1 = 1 = (1^2+2)/3$ and $1 = 2^0$, so $P(1)$ is true.

Inductive step: For the purpose of proving $P(n+1)$, suppose that $P(k)$ is true for $1 \leq k \leq n$. If $n+1$ is odd $b_{n+1} > ((n+1)^2+2)/3$ and $P(n+1)$ holds. If $n+1 = 2k$ is even with $k \geq 1$, then $b_{n+1} = ((n+1)^2+2)/3$ if and only if $b_k = (k^2+2)/3$, that is, if and only if k is a power of 2 by the inductive hypothesis. This proves $P(n+1)$, that is, $b_{n+1} = ((n+1)^2+2)/3$ if and only if $n+1$ is a power of 2. □

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Problem 2. [15 points] Define the sequence of numbers A_i, by

$A_0 = 2$ and

$A_{n+1} = A_n/2 + 1/A_n$, for $n \geq 1$.

Prove that $A_n \leq \sqrt{2} + 1/2^n$ for all $n \geq 0$. You may use the following result:

Lemma. *For real numbers $x > 0$, $x/2 + 1/x \geq \sqrt{2}$.*

Solution. *Proof.* We will use induction. For $n \geq 0$, let $P(n)$ be the predicate $A_n \leq \sqrt{2} + 1/2^n$.

Base case: $A_0 = 2 \leq \sqrt{2} + 1/2^0$ is true.

Inductive step: Let $n \geq 0$ and suppose the inductive hypothesis $P(n)$, that is, $A_n \leq \sqrt{2} + 1/2^n$. We need the following lemma.

Lemma. *For real numbers $x > 0$, $x/2 + 1/x \geq \sqrt{2}$.*

Proof. For real numbers $x > 0$,

$$\begin{array}{ll} & x/2 + 1/x \geq \sqrt{2} \\ \leftrightarrow & x^2 + 2 \geq 2\sqrt{2} \cdot x \\ \leftrightarrow & x^2 - 2\sqrt{2} \cdot x + 2 \geq 0 \\ \leftrightarrow & (x - \sqrt{2})^2 \geq 0, \end{array}$$

which is true. □

By using induction it is straightforward to prove that $A_n > 0$ for $n \geq 0$ (base case: $A_0 = 2 > 0$; inductive step: if $A_n > 0$, then $A_{n+1} = A_n/2 + 1/A_n > 0$). By the lemma, $A_n \geq \sqrt{2}$ for $n \geq 0$. Together with the induction hypothesis this can be used in the following derivation:

$$\begin{array}{rcl} A_{n+1} & = & A_n/2 + 1/A_n \\ & \leq & (\sqrt{2} + 1/2^n)/2 + 1/\sqrt{2} \\ & = & \sqrt{2} + 1/2^{n+1}. \end{array}$$

This completes the proof. □

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Problem 3. [15 points]

An n-player *tournament* consists of some set of $n \geq 2$ *players*, and has the property that for every two players, $p \neq q$, either p beats q or q beats p, but not both.

(a) [10 pts] Prove the proposition that if there exists a cycle of at least two nodes in the tournament graph, then there exists a cycle of three nodes.

Solution. Let $v_1 \rightarrow v_2 \ldots \rightarrow v_h \rightarrow v_1$ be a smallest length cycle. Since the graph is a tournament, there do not exist cycles of length 2. So, $h \geq 3$. If $v_1 \rightarrow v_3$, then $v_1 \rightarrow v_3 \rightarrow v_4 \ldots \rightarrow v_h \rightarrow v_1$ is a shorter cycle of length $h - 1$. This contradicts h being the length of the shortest cycle. So, there is no edge $v_1 \rightarrow v_3$. Since the graph represents a tournament $v_3 \rightarrow v_1$. So, $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_1$ is a cycle of length 3, therefore $h \leq 3$. So, $h = 3$. ■

(b) [5 pts] A *consistent ranking* is a sequence $p_1, p_2, \ldots, p_n$ of all n players in the tournament such that each player beats all the later players in the sequence (that is, p_i beats p_j iff $i < j$, for $1 \leq i, j \leq n$). Prove by using the previous problem parts, that a tournament has no consistent ranking *iff* some subset of three of its players has no consistent ranking.

Solution. A tournament has no consistent ranking iff there exists a cycle. There exists a cycle iff there exists a 3-cycle by part a. There exists a 3-cycle iff some subset of three players has no consistent ranking. ■

Problem 4. [15 points] Prove the proposition that if a finite digraph has no cycle at all, then it has a node with no incoming edges.

Solution. By contradiction. Suppose that there are no cycles and suppose that each node has at least one incoming edge. We use induction and prove that there exists a walk of any length. For $h \geq 0$, let $P(h)$ be the predicate that there exists a walk of length h.

Base case $h = 0$: there exists a walk of length 0.

Inductive step: Assume that $P(h)$ is true in order to show that $P(h+1)$ is true. Let $v_1 \to v_2 \to \ldots \to v_h$ be a walk of length h. Since v_1 has an incoming edge $v_0 \to v_1$, $v_0 \to \ldots \to v_h$ is a walk of length $h+1$. This proves $P(h+1)$.

If n is the number of nodes in the digraph, then there exists a node in a walk of length $n+1$ that repeats itself. This shows that there exists a cycle. ■

Problem 5. [20 points] Suppose m, n are relatively prime and let s and t be integers such that $sm + tn = 1$.

(a) [10 pts] Prove that $(sm)^k \equiv sm \pmod{mn}$ for integers $k \geq 1$.

Solution. By strong induction. For $k \geq 1$, let $P(k)$ be the predicate $(sm)^k \equiv sm \pmod{mn}$.

Base case For $k = 1$, $(sm)^1 \equiv sm \pmod{mn}$. For $k = 2$, we derive

$$\begin{aligned} (sm)^2 &= sm(1 - tn) \\ &= sm - stmn \\ &\equiv sm \pmod{mn}. \end{aligned}$$

Inductive step: Let $k \geq 2$. Assume that $P(i)$ is true for $1 \leq i \leq k$ in order to show that $P(k+1)$ is true. We derive

$$\begin{aligned} (sm)^{k+1} &= (sm)(sm)^k \\ &\equiv (sm)(sm) \pmod{mn} \text{ (by } P(k)) \\ &\equiv sm \pmod{mn} \text{ (by } P(2)) \end{aligned}$$

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(b) [10 pts] For integers a and b, prove that $(sma + tnb)^k \equiv sma^k + tnb^k \pmod{mn}$ for $k \geq 1$. You may use part a in your solution.

Solution. By induction. For $k \geq 1$, let $P(k)$ be the predicate $(sma + tnb)^k \equiv sma^k + tnb^k \pmod{mn}$.

Base case For $k = 1$, $P(k)$ holds true.

Inductive step: Let $k \geq 1$ and assume $P(k)$ in order to prove $P(k+1)$. We derive

$$\begin{aligned}
(sma + tnb)^{k+1} &= (sma + tnb)(sma + tnb)^k \\
&\equiv (sma + tnb)(sma^k + tnb^k) \pmod{mn} \text{ (by } P(k)\text{)} \\
&= (sm)^2 a^{k+1} + (tn)^2 b^{k+1} + (tbsa^k + satb^k)mn \\
&\equiv (sm)^2 a^{k+1} + (tn)^2 b^{k+1} \pmod{mn} \text{ (by } P(k)\text{)} \\
&\equiv sma^{k+1} + tnb^{k+1} \pmod{mn} \text{ (by part a)}
\end{aligned}$$

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