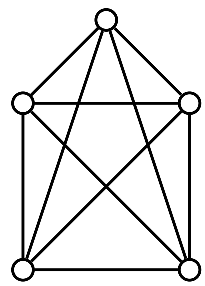
Problem 1. [15 points] Refer to the following graph for parts a, b:



(a) [3 pts] Does the above graph have a Hamiltonian path?

(b) [3 pts] Does the above graph have a Eulerian path?

(c) [9 pts] Consider the complete graph on n vertices K_n for odd $n \geq 3$. Prove that we cannot find a series of Hamiltonian paths in K_n that together cover all the edges of K_n.

Problem 2. [15 points]

(a) [3 pts] Find 7^{100} mod 13.

(b) [3 pts] Find the inverse of 33 mod 121 or prove that no such inverse exists.

(c) [4 pts] Prove that for any non-zero integers a, b, c, d, if $a-c \mid ab+cd$, then $a-c \mid ad+bc$.

(d) [5 pts] Show that any perfect square is of the form $3k$ or $3k+1$ but never of the form $3k+2$ for integer k.

Problem 3. [10 points] Suppose we are planning a trip to California for Thanksgiving. Unfortunately, we are booking our tickets late and so the prices are all really high. Suppose we are given the following list of ticket prices and travel times:

A 600 dollars, 9 hours 20 minutes

B 650 dollars, 8 hours 40 minutes

C 550 dollars, 9 hours 10 minutes

D 575 dollars, 8 hours 20 minutes

E 660 dollars, 9 hours 5 minutes

Our goal is to find the tickets that are the cheapest while minimizing travel time.

(a) [4 pts] Suppose we had the ordering $\leq$ such that for tickets i, j $i \leq j$ if i is at least as expensive as j and i's travel time is at least as long as j's travel time.

(b) [3 pts] Draw the Hasse diagram for the above tickets with the ticket i is $<$ ticket j if ticket i is both more expensive and has more travel time than ticket j.

(c) [3 pts] Find the maximal elements of the poset. Is there a maximum element?

Problem 4. [10 points] A portion of a computer program consists of a sequence of calculations where the results are stored in variables, like this:

$$\begin{array}{rrcl}
 & \text{Inputs:} & & x, y \\
\text{Step 1.} & a & = & x - 24 \\
2. & b & = & x * a \\
3. & c & = & 3 \\
4. & d & = & y - c \\
5. & e & = & y * *c \\
6. & f & = & e + 1 \\
 & \text{Outputs:} & & b, d, e
\end{array}$$

A computer can perform such calculations most quickly if the value of each variable is stored in a *register*, a chunk of very fast memory inside the microprocessor. Programming language compilers face the problem of assigning each variable in a program to a register. Computers usually have few registers, however, so they must be used wisely and reused often. This is called the *register allocation* problem.

In the example above, variables x and y must be assigned different registers, because they hold distinct input values. Furthermore, c and d must be assigned different registers; if they used the same one, then the value of c would be overwritten in the fourth step and we'd get the wrong answer in the fifth step. On the other hand, variables b and d may use the same register; we no longer need b and can overwrite the register that holds its value. Assume that the computer carries out each step in the order listed and that each step is completed before the next is begun.

(a) [7 pts] Recast the register allocation problem as a question about graph coloring. What do the vertices correspond to? Construct the graph corresponding to the example above.

(b) [3 pts] How many registers do you need?

Problem 5. [15 points] Use the web graph below to answer parts (a), (b), and (c) of this question. The pagerank of vertex i can be written as p_i.

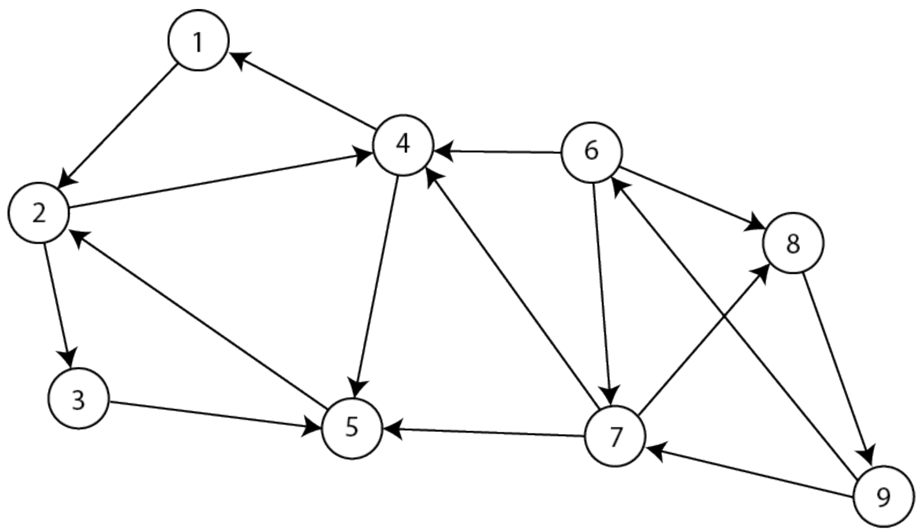


Figure 1: Web Graph

(a) [5 pts] For $\vec{p}\,'$ in terms of $\vec{p}$ can be written as a matrix product: $\vec{p}\,' = W\vec{p}$, for some matrix W, which is the *update matrix*. Find the update matrix.

(b) [3 pts] Which (if any) of the vertices of the web graph above will have PageRank value tending to zero if we run the PageRunk algorithm for many (let's say 200,000) iterations?

(c) [7 pts] Suppose we remove all the vertices that you found in part b from our Web Graph along with all corresponding edges from those vertices and consider the remaining graph G. Find the normalized stationary distribution across the nodes assuming that each node in G starts with uniform values.

Note: If you are not confident about your answer in part b, feel free to instead find the stationary distribution for the directed triangle formed by vertices 1, 2, 4 of the Web Graph. Again assume each of the nodes 1, 2, 4 starts with inital PageRank value $\frac{1}{3}$.

Problem 6. [10 points] A *multiple binary-tree network* has n inputs and n outputs, where n is a power of 2. Each input is connected to the root of a binary tree with $n/2$ leaves and with edges pointing away from the root. Likewise, each output is connected to the root of a binary tree with $n/2$ leaves and with edges pointing toward the root.

Two edges point from each leaf of an input tree, and each of these edges points to a leaf of an output tree. The matching of leaf edges is arranged so that for every input and output tree, there is an edge from a leaf of the input tree to a leaf of the output tree, and every output tree leaf has exactly two edges pointing to it.

(a) [4 pts] Draw such a multiple binary-tree net for $n = 4$.

(b) [6 pts] Fill in the table, and explain your entries.

# switches	switch size	diameter	max congestion

Problem 7. [10 points] There are $n \geq 1$ identical cars on a circular track for some integer n. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around. (Hint: use induction)

Problem 8. [15 points] Use the matrix below and the invariance principle to prove the following statements.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

(a) [5 pts] Suppose we are allowed to flip all of the signs of entries in any row or column. Prove that at least one -1 will remain in the matrix.

(b) [10 pts] Suppose that we are allowed to flip all of the signs of entries in any row, column, or a parallel to one of the diagonals. In particular, you may switch the sign of each corner square. Prove that at least one -1 will remain in the matrix.