Problems for Recitation 12

1 A False Proof

False Claim
$\sum_{i=1}^{n}$ i is O(n)

Induction Hypothesis
P(n) ::= $\sum_{i=1}^{n}$ i is O(n)

Base Case
P(1): 1 is O(1)

Inductive Step
$P(n+1) : \sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = O(n) + (n+1) = O(n)$

Therefore $\sum_{i=1}^{n} i =$ O(n)

However, this is not true! It breaks down at the base case, since the induction hypothesis is not true! We are abusing notation. Really should be "P(n): $\sum_{i=1}^{x}$ is $O(x)$" which doesn't tell us anything in the induction step. The lesson here is to never use asymptotic notation with n (e.g. O(n)) in a predicate P(n)!

2 The L-tower problem

Observe the structures shown in Figure 1. One has 2 L-shapes, the other 5 L-shapes. Consider a tower with k L-shapes. Assume that the blocks are all of size $x \times 1$ where $x > 1$. As the picture indicates, if k is too small then the tower falls to the left. On the other hand, if k is too large the tower would fall to the right. Will the tower be stable for some k? Prove there is at least one value of k for which the L-tower is stable. Assume that a structure is stable if and only if its center of gravity is not hanging in the air horizontally. The L-tower is stable if and only if each of its subparts is stable.

Hint: Show the tower is stable if and only if $\frac{3x-3}{2} \leq k \leq \frac{3x-1}{2}$.

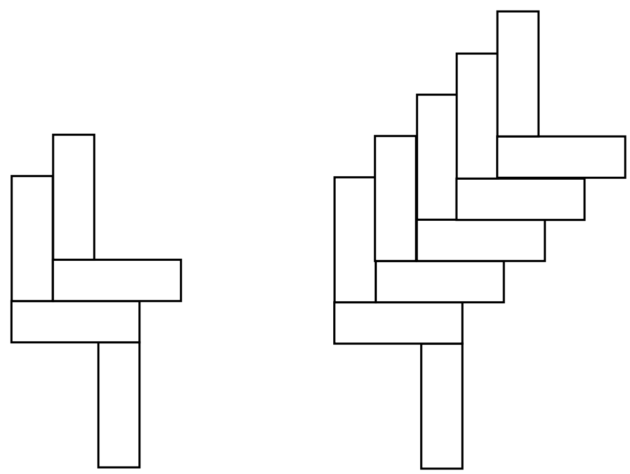

Figure 1: Too few or too many L shapes make the tower unstable

3 Double Sums

Sometimes we have to evaluate sums of sums, otherwise known as *double summations.* It's good to know how to tame these beasts! Here's an example of a double summation:

$$\sum_{i=1}^{n}\sum_{j=1}^{i} j$$

It looks ferocious...all those sharp teeth! But actually, this double summation is just a sheep in wolf's clothing: to evaluate it, we can just evaluate the inner sum, replace it with a closed form we already know, and then evaluate the outer sum which no longer has a summation inside it.

(a) Evaluate the summation. (*Hint:* $\sum(a+b) = \sum a + \sum b$.)

Unfortunately, not all summations are so docile. Fortunately, we have ways to deal with this. There's a special trick that is often extremely useful for sums, and that is to *exchange the order of summation.*

Our goal will be to compute the following sum, for n a positive integer:

$$S_n = \sum_{k=1}^{n} k2^k.$$

There are various ways to handle it, but here we'll see a way that (surprisingly!) involves double sums. You may object that there is only a single sum. But here's a trick: we can rewrite S_n as

$$S_n = \sum_{k=1}^{n} \sum_{j=1}^{k} 2^k.$$

(b) If we think about the pairs (k, j) over which we are summing, they form a triangle in the table below. The values in the cells of the table correspond to the terms in the double summation. Complete this table to see the pattern.

$k \setminus j$	1	2	3	4	...	n
1						
2						
3						
4						
	...					
n					...	

(c) The summation above is summing each row and then adding the row sums. But we can tame this beast if, instead, we first sum the columns and then add the column sums. Use the table to rewrite the double summation. The inner summation should sum over k, and the outer summation should sum over j.

(d) Now simplify the summation to derive a closed formula for S_n.

Now try your hand at another double sum.

(e) Find a formula for F_n, for n a positive integer:

$$F_n = \sum_{k=1}^{n} k \sum_{j=k}^{n} 2^j/(j+1).$$

4 Asymptotic Notation

Which of these symbols

$$\Theta \qquad O \qquad \Omega \qquad o \qquad \omega$$

can go in these boxes? (List all that apply.)

$$2n + \log n \quad = \quad \boxed{}\,(n)$$

$$\log n \quad = \quad \boxed{}\,(n)$$

$$\sqrt{n} \quad = \quad \boxed{}\,(\log^{300} n)$$

$$n2^n \quad = \quad \boxed{}\,(n)$$

$$n^7 \quad = \quad \boxed{}\,(1.01^n)$$

5 More Asymptotic Notation

1. Show that
$$(an)^{b/n} \sim 1.$$
where a, b are positive constants and $\sim$ denotes asymptotic equality. Hint $an = a2^{\log_2 n}$.

2. You may assume that if $f(n) \geq 1$ and $g(n) \geq 1$ for all n, then $f \sim g \Rightarrow f^{\frac{1}{n}} \sim g^{\frac{1}{n}}$. Show that
$$\sqrt[n]{n!} = \Theta(n).$$