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Midterm

Problem 1. [10 points]

(a) [5 pts] Show that $(p \wedge q) \rightarrow r$ is equivalent to $p \rightarrow (q \rightarrow r)$ by using a truth table.

Solution. By truth table. ■

(b) [5 pts] Suppose $P(x, y)$ is the predicate "$xy = 1$", where the universe of discourse for x is the set of positive integers, and the universe of discourse for y is the set of real numbers. Transform the following propositions into English and establish if they are true or false:

1. $\forall x \exists y P(x, y)$
2. $\exists y \forall x P(x, y)$

Solution. The first is translated as: For every positive integer x there is a real number y such that $xy = 1$. This is true.

The second one reads: There exists a real number y such that, for every positive integer x, $xy = 1$. This proposition is false. ■

Problem 2. **[10 points]** Let F_n be the n'th Fibonacci number. Recall that the Fibonacci sequence satisfies $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}, \forall n \geq 3$. Prove by induction that for all $n \geq 1$ we have

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

Solution. We use induction.

Base case: When $n = 1$, the left side of the equation is $F_1^2 = 1$, and the right side is $F_1 F_2 = 1 \times 1 = 1$, so both sides are equal and the statement is true for $n = 1$.

Inductive step: Suppose the statement is true for $n = k$ and we want to prove it for $n = k + 1$. We have

$$\begin{aligned}
\sum_{i=1}^{k+1} F_i^2 &= \sum_{i=1}^{k} F_i^2 + F_{k+1}^2 \\
&= F_k F_{k+1} + F_{k+1}^2 \text{ (by induction hypothesis)} \\
&= F_{k+1}(F_k + F_{k+1}) \\
&= F_{k+1} F_{k+2} \text{ (by recurrence for } F_n)
\end{aligned}$$

Thus the equality holds for $n = k + 1$, and the proof of the inductive step is complete. ■

Problem 3. [18 points] Let G be a graph with n vertices, where n is even, and where G contains no cycle of length 3.

(a) [4 pts] If $e = (u, v)$ is an edge of G, what is the maximum number of edges other than e that can be incident to one of u or v?

Solution. The answer is $n-2$. Since u, v are connected, if any other vertex were connected by an edge with both u and v then they would form a cycle of length 3. Hence each of the other $n-2$ vertices is connected to at most one of u or v, so we have at most $n-2$ edges with exactly one endpoint at u or v. ■

(b) [10 pts] Prove by induction that G has at most $\frac{n^2}{4}$ edges.

Solution. We use induction on the number of vertices. For the base case, G has $n = 2$ vertices. G can have at most 1 edge, and since $1 = \frac{2^2}{4}$, this case is proven.

For the induction step, we assume that any graph with $2n$ vertices has at most $\frac{n^2}{4}$ edges, and we want to prove the statement when G has $2n+2$ edges. Let u, v be two vertices of G connected by an edge. Take them out to obtain G'. G' still satisfies the condition that it has no triangles and has $2n$ vertices, hence by the induction hypothesis has at most $\frac{n^2}{4}$ edges. Adding back vertices u and v, part (a) guarantees that we add at most $2n+1$ edges, hence overall G has at most $\frac{n^2}{4} + n + 1 = \frac{(n+2)^2}{4}$ edges. ■

(c) [4 pts] Give an example of a graph G with n vertices and $\frac{n^2}{4}$ edges that has no cycle of length 3.

Solution. A complete bipartite graph with two sets of $\frac{n}{2}$ vertices has $\frac{n^2}{4}$ edges and no 3 cycles. ■

Problem 4. [12 points]

For the following parts, a correct numerical answer will only earn credit if accompanied by its derivation. Show your work.

(a) [6 pts] Use the Pulverizer to find integers s and t such that $141s + 61t = \gcd(141, 61)$.

Solution.

$$
\begin{array}{ccrcl}
x & y & \operatorname{rem}(x,y) & = & x - q \cdot y \\
\hline
141 & 61 & 19 & = & 141 - 2 \cdot 61 \\
61 & 19 & 4 & = & 61 - 3 \cdot 19 \\
 & & & = & 61 - 3 \cdot (141 - 2 \cdot 61) \\
 & & & = & -3 \cdot 141 + 7 \cdot 61 \\
19 & 4 & 3 & = & 19 - 4 \cdot 3 \\
 & & & = & (141 - 2 \cdot 61) - 4(-3 \cdot 141 + 7 \cdot 61) \\
 & & & = & 13 \cdot 141 - 30 \cdot 61 \\
4 & 3 & 1 & = & 4 - 3 \\
 & & & = & (-3 \cdot 141 + 7 \cdot 61) - (13 \cdot 141 - 30 \cdot 61) \\
 & & & = & -16 \cdot 141 + 37 \cdot 61
\end{array}
$$

■

(b) [6 pts] Find the remainder of 10^{1001} when divided by 101.

Solution. Euler's theorem states tells us that $10^{100} \equiv 1 \mod 101$:

$$\begin{aligned} 10^{1001} &\equiv (10^{100})^{10} * 10 \mod 101 \\ &\equiv 1^{10} * 10 \mod 101 \\ &\equiv \boxed{10} \mod 101 \end{aligned}$$

■

Problem 5. [12 points] Alice plays a game with piles of coins. Initially, there are 3 piles of coins, containing 5, 7, and 93 coins, respectively. The game ends when there are 105 piles, each with one coin. There are two moves she is allowed to make.

- She can merge two piles together.
- She can divide a pile with an even number of coins into two piles of equal size.

Using an invariant, show that the game will never end.

(*Hint: Consider each of the three cases for the first move separately. Notice that in each case, the sizes of the two resulting piles share a common factor.*)

Solution. The initial piles are of odd sizes $(5, 7, 93)$, so the only possible move is to merge two piles. There are three choices for the pair of piles to merge, which gives $(12, 93)$, $(7, 98)$, or $(5, 100)$.

Suppose there are piles whose sizes are all divisible by some odd prime p. Then merging two piles yields a pile whose size is still divisible by p. Splitting a pile leaves two piles of sizes divisible by p. So, having all piles of sizes divisible by p is invariant

$(12, 93)$ are both divisible by 3. $(7, 98)$ are both divisible by 7. And $(5, 100)$ are both divisible by 5. So in each case, the number of coins in each pile will always be divisible by an odd prime, so the game will never finish. ∎

Problem 6. [10 points]

Consider a stable marriage problem with the following preferences.

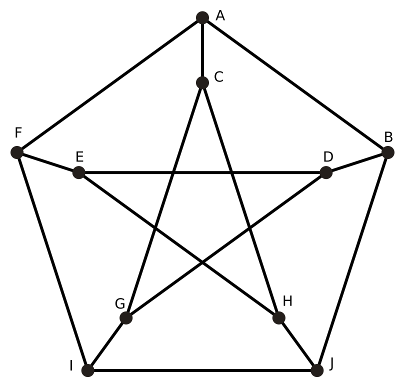
Alfred:	Fiona, Helen, Emily, Grace
Billy:	Helen, Emily, Fiona, Grace
Calvin:	Fiona, Helen, Grace, Emily
David:	Fiona, Helen, Emily, Grace

Emily:	Billy, Alfred, Calvin, David
Fiona:	Alfred, Calvin, Billy, David
Grace:	Billy, Alfred, David, Calvin
Helen:	Alfred, Billy, David, Calvin

What is the stable matching generated by the mating algorithm on these preferences?

Solution. (Alfred, Fiona), (Billy, Helen), (Calvin, Grace), (David, Emily) ■

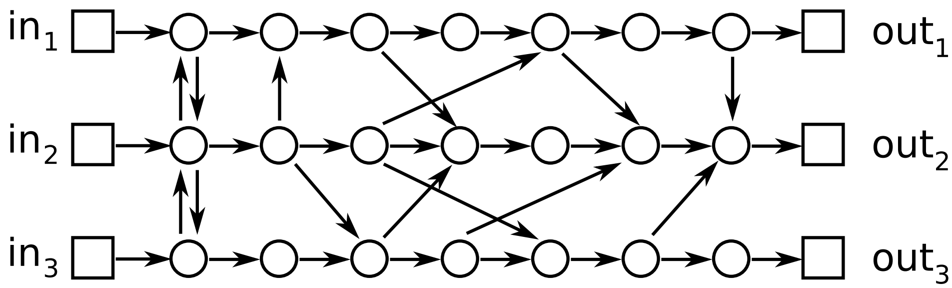
Problem 7. [10 points] Show that the chromatic number of this graph is 3. (Remember to show that you can't use fewer than 3 colors.)



Solution. Since it has an odd length cycle, the chromatic number cannot be 2. One coloring with three colors is: Red: {A, G, E, J}, Blue: {B, C, H, F}, Green: {D, I} ■

Problem 8. [10 points]

Consider the routing network below:



A routing network.

(a) [4 pts] What is the diameter?

Solution. The diameter is 9; several pairs of nodes achieve this, such as in_1 and out_3. ■

(b) [6 pts] Show that there is a 1–1 routing problem on this network which has a congestion of 3. (Be sure to show that no matter what set of paths are chosen for your routing problem, that they must all go through a single node.)

Solution. Consider the following routing problem:

$$\mathsf{in}_1 \to \mathsf{out}_3, \qquad \mathsf{in}_2 \to \mathsf{out}_2, \qquad \mathsf{in}_3 \to \mathsf{out}_1.$$

All paths from in_1 to out_3 pass through the switch adjacent to in_2, as do all paths from in_3 to out_1. So all three routes must pass through that vertex, which forces the congestion to be at least 3. The congestion can not be higher as there are only three routes. ■

Problem 9. [16 points]

In this problem, PageRank refers to *unscaled* PageRank.

(a) [6 pts] Consider the directed graph A_n consisting of a single doubly-linked path on n elements, as pictured below, and let v_i denote the ith vertex in order. Show that the PageRank values

$$\text{pagerank}(v_i) = a_i = \begin{cases} \frac{1}{2(n-1)} & \text{if } i = 1 \text{ or } n \\ \frac{1}{n-1} & \text{if } 1 < i < n \end{cases}$$

are in equilibrium.

$$v_1 \rightleftarrows v_2 \rightleftarrows v_3 \rightleftarrows \cdots \rightleftarrows v_{n-1} \rightleftarrows v_n$$

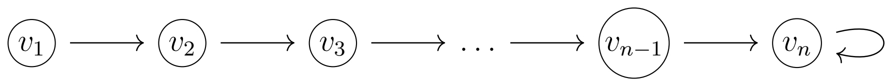
Solution. Equilibrium values must be fixed under the PageRank update step, yielding the equations:

$$\begin{aligned}
a_1 &= \frac{1}{2}a_2, \\
a_2 &= a_1 + \frac{1}{2}a_3, \\
a_i &= \frac{1}{2}a_{i-1} + \frac{1}{2}a_{i+1} \qquad \text{for } 3 \leq i \leq n-2, \\
a_{n-1} &= \frac{1}{2}a_{n-2} + a_n, \\
a_n &= \frac{1}{2}a_{n-1}.
\end{aligned}$$

Each of these equations is easily verified for the given values of a_i. ■

(b) [10 pts] Now consider the directed graph B_n consisting of a single directed path on n elements, as pictured below, with v_i again denoting the ith vertex in order. We consider v_n to have a self-loop, so that every vertex has an outgoing edge. Starting from arbitrary PageRank values, prove that after $n-1$ steps, the PageRank of the graph is concentrated at the last vertex v_n, i.e. the first $n-1$ vertices all have PageRank value zero.

(*Hint: What can you say about the PageRank values after t update steps? Which vertices can still have nonzero PageRank?*)



Solution. We prove by induction that, after t steps, the first t vertices have zero PageRank, for all $t < n$. The base case of $t = 0$ is vacuous. For the inductive step, we assume that after $t-1$ steps, the first $t-1$ vertices have zero PageRank. All edges leading into the first t vertices come from the first $t-1$ vertices; so in the tth PageRank update step, the first t vertices each receive no contributions from any other vertex. Moreover, as we have assumed that $t < n$, each of the first t vertices has no self-loop, and therefore makes no PageRank contribution to itself. Thus the PageRank value after t steps must be zero at each of the first t vertices, completing the induction.

Setting $t = n - 1$ gives the desired result. ■

Problem 10. [12 points]

Define a relation R on the positive integers as follows: we say $a \; R \; b$ if either a divides b or b divides a. State and briefly justify whether or not the relation R has the following properties:

(a) [2 pts] Reflexivity.

Solution. Yes. Any a divides itself, so $a \; R \; a$. ■

(b) [2 pts] Symmetry.

Solution. Yes. The definition of R is clearly symmetric. We could argue more explicitly as follows: if $a \; R \; b$, then there are two cases – either a divides b, in which case $b \; R \; a$ by definition, or else b divides a, in which case we still have $b \; R \; a$ by definition. ■

(c) [2 pts] Antisymmetry.

Solution. No – for example $2 \; R \; 6$ and $6 \; R \; 2$, but $2 \neq 6$, violating antisymmetry. ■

(d) [3 pts] Transitivity.

Solution. No – for example 2 divides both 4 and 6, so that $4\ R\ 2$ and $2\ R\ 6$, but it is false that $4\ R\ 6$, as neither of 4 and 6 divides the other. ■

(e) [3 pts] The property of being an equivalence relation.

Solution. No. An equivalence relation must be transitive, which relation R is not. ■