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Problems for Recitation 24

1 Properties of Variance

In this problem we will study some properties of the variance and the standard deviation of random variables.

(a) Show that for any random variable R, $\mathrm{Var}[R] = \mathrm{E}[R^2] - \mathrm{E}^2[R]$.

(b) Show that for any random variable R and constants a and b, $\mathrm{Var}[aR + b] = a^2 \mathrm{Var}[R]$. Conclude that the standard deviation of $aR + b$ is a times the standard deviation of R.

(c) Show that if R_1 and R_2 are independent random variables, then

$$\mathrm{Var}[R_1 + R_2] = \mathrm{Var}[R_1] + \mathrm{Var}[R_2].$$

(d) Give an example of random variables R_1 and R_2 for which

$$\mathrm{Var}\left[R_1 + R_2\right] \neq \mathrm{Var}\left[R_1\right] + \mathrm{Var}\left[R_2\right].$$

(e) Compute the variance and standard deviation of the Binomial distribution $H_{n,p}$ with parameters n and p.

(f) Let's say we have a random variable T such that $T = \sum_{j=1}^{n} T_j$, where all of the T_j's are mutually independent and take values in the range $[0, 1]$. Prove that $\mathrm{Var}[T] \leq \mathrm{E}[T]$. *Hint: Upper bound* $\mathrm{Var}[T_j]$ *with* $\mathrm{E}[T_j]$ *using the definition of variance in part (a) and the rule for computing the expectation of a function of a random variable.*

2 Law of Total Expectation

Suppose that an unpredictable friend has gone to buy you candy, and in eager wait, you're trying to imagine how many pieces of candy you might be about to receive. Your friend might buy Swedish Fish, say with 0.3 probability, and on average 25 individual fish in the package; or your friend might buy Sour Patch Kids, with 0.7 probability, and on average 43 individual pieces in the package. You might reasonably compute that the expected number of pieces of candy is

$$0.3 \cdot 25 + 0.7 \cdot 43.$$

The Law of Total Expectation justifies this:

Rule (Law of Total Expectation)**.** *Let X and Y be two random variables, not necessarily independent. Then*

$$\mathrm{E}\left[X\right] = \mathrm{E}_Y\left[\mathrm{E}_{X|Y}\left[X|Y\right]\right].$$

In our candy scenario, Y represents the type of candy bought – Swedish Fish or Sour Patch Kids – and X represents the number of pieces of candy.

For Problem 2, prove the Law of Total Expectation. (You may assume that each random variable has only finitely many outcomes.)