Problem Set 9

Due: Monday, November 14

Reading Assignment: Sections 7.2, 11.1-11.10

Problem 1. [**10 points**]

(a) [5 pts] Show that of any $n+1$ distinct numbers chosen from the set $\{1, 2, \ldots, 2n\}$, at least 2 must be relatively prime. (*Hint:* $\gcd(k, k+1) = 1$.)

(b) [5 pts] Show that any finite connected undirected graph with $n \geq 2$ vertices must have 2 vertices with the same degree.

Problem 2. [**15 points**]

Consider the 40 most popular cities on Earth. Use the pigeonhole principle to show that there are two subsets of these cities that have exactly the same number of people. Assume that there are 10^{10} people on the Earth.

Problem 3. [**15 points**] In this problem, we will use the principle of inclusion-exclusion (PIE) to solve the problem of derangements.

Suppose you attend a Halloween party with n guests, where each guest dressed up as Harley-Quinn. Each Harley-Quinn brought a bat as an accessory to the party, but didn't want to carry it through the night, and so all the bats were left stacked together in a single room. Unfortunately, later on in the party, one of the party-goers bumped into the stack of bats causing them all to fall in a pile. Thus, at the end of the party, each Harley-Quinn grabbed a random bat and left. It turns out none of them got his/her own bat back. We will count the number of ways this can happen.

(a) [2 pts] How many ways could the guests have picked up the bats at random?

(b) [3 pts] How many ways there are to choose 1 person to get their bat back and randomly assign everyone else?

(c) [10 pts] Use the principle of inclusion-exclusion to determine the number of ways in which no guest got their hat back.

Problem 4. [**45 points**] Be sure to show your work to receive full credit. In this problem we assume a standard card deck of 52 cards.

(a) [4 pts] How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

(b) [4 pts] For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i = n$$

(c) [4 pts] For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i \leq n$$

(d) [4 pts] How many simple undirected graphs are there with n vertices?

(e) [4 pts] How many directed graphs are there with n vertices (self loops allowed)?

(f) [4 pts] How many tournament graphs are there with n vertices?

(g) [4 pts] How many acyclic tournament graphs are there with n vertices?

(h) [4 pts] How many numbers are there that are in the range [1..700] which are divisible by $2, 5$ or 7?

(i) [4 pts] How many ways are there to list the digits $\{1, 1, 2, 2, 3, 4, 5\}$ so that the 1's are always consecutive?

(j) [4 pts] What is the coefficient of x^3y^2z in the expansion of $(x + y + z)^6$?

(k) [5 pts] How many unique terms are there in the expansion of $(x + y + z)^6$?

Problem 5. [15 points] Give a combinatorial proof of the following theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$