Problem Set 4

Due: October 4

Reading: Chapter 6. *Recursive Data Types*; Chapter 7. *Infinite Sets*; Chapter 8. *Number Theory* through 8.4. *The Fundamental Theorem of Arithmetic*

Problem 1.
The set Supersymm of "super-symmetric strings" is defined recursively as follows:

Base Case: The 26 lower case letters of the Roman alphabet, a, b,..., z, are in Supersymm.

Constructor Case: If α and β are strings in Supersymm, then the string $\alpha\beta\alpha$ is in Supersymm.

(a) Which of the following are super-symmetric strings? Briefly explain your answers.

(i) a

Solution. Yes, by the Base Case. ■

(ii) aaaba

Solution. No. This string is not of the form $\alpha\beta\alpha$. ■

(iii) abcbacabcba

Solution. Yes. Let $\beta =$ aca, $\alpha =$ bcb. Then we have a string of the form a$\alpha\beta\alpha$a. ■

(iv) λ, the empty string

Solution. No. A trivial structural induction implies that all super-symmetric strings have positive length. ■

(v) abaabcbaaba

Solution. Yes. Similar reasoning to case (iii) shows that the string bcb is in the middle of the super-symmetric string, with the string a wrapped around it, and with the string aba wrapped around that. ■

(b) Prove by structural induction that in any super-symmetric string, exactly one letter appears an odd number of times.

Solution. Proof: Define $P(e)$: String e has exactly one letter which appears an odd number of times. Prove: $\forall e \in SSS\ P(e)$.
(Base) $P(x)$: The string x, where x is a single letter of the alphabet, has exactly one letter which appears an odd number of times - namely, x.

(Inductive Step) $\forall e, e' \in SSS\ P(e) \wedge P(e') \longrightarrow P(ee'e)$
Fix $e, e' \in SSS$. Assume $P(e) \wedge P(e')$. That is, each of e and e' has exactly one letter which appears and

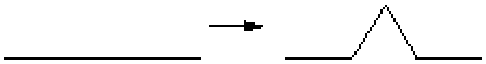

Figure 1 Constructing the Koch Snowflake.

odd number of times. By the inductive hypothesis, both e and e' have exactly one letter which appears an odd number of times. When we form the string $ee'e$, the letters in e all get repeated, and thus they all appear an even number of times. By the IH, there is one letter in e' which appears an odd number of times in e'. Now, even if this letter is also present in e, it will still appear an odd number of times in $ee'e$ (since the sum of an odd number and an even number is odd). Therefore, we have shown $P(ee'e)$. QED.

■

Problem 2.
Fractals are an example of mathematical objects that can be defined recursively. In this problem, we consider the Koch snowflake. Any Koch snowflake can be constructed by the following recursive definition.

- **base case**: An equilateral triangle with a positive integer side length is a Koch snowflake.
- **constructor case**: Let K be a Koch snowflake, and let l be a line segment on the snowflake. Remove the middle third of l, and replace it with two line segments of the same length as is done in Figure 1

  The resulting figure is also a Koch snowflake.

Prove by structural induction that the area inside any Koch snowflake is of the form $q\sqrt{3}$, where q is a rational number.

Solution. We first show that the side length of any Koch snowflake is rational, and prove it in the lemma below.

Lemma. *Every side length of a Koch snowflake is rational.*

Proof. For the base case, every side length is the same positive integer. For the inductive case, let K be a Koch snowflake. Then K was constructed by modifying a Koch snowflake K' as in the recursive case. By the induction hypothesis, each side length of K' is rational. Let l be the line segment of K' modified via the recursive case. The two new line segments are of length $\frac{l}{3}$, which is rational since l is rational. ■

Now, we prove the main theorem.

Proof. We prove the claim by structural induction.

For the base case, the area of an equilateral triangle with side length l is $q\sqrt{3}$, where $q = \frac{l}{2}$.

For the inductive case, let K be a Koch snowflake. K was constructed by modifying a Koch snowflake K' as in the recursive case above. By the induction hypothesis, the area of K' is $q\sqrt{3}$ for some rational q. Let l' be the length of the line segment of K' that was modified according to the recursive case. The area added by the modification is $q'\sqrt{3}$, where $q' = \frac{l'}{6}$. By the above lemma, l' is rational, so q' is rational. Thus, the area of K is $(q' + q) \cdot \sqrt{3}$, and $q' + q$ is rational, so we have proved our claim. ■

Problem 3.
In this problem you will prove a fact that may surprise you—or make you even more convinced that set

theory is nonsense: the half-open unit interval is actually the "*same size*" as the nonnegative quadrant of the real plane![1] Namely, there is a bijection from $(0, 1]$ to $[0, \infty) \times [0, \infty)$.

(a) Describe a bijection from $(0, 1]$ to $[0, \infty)$.

Hint: $1/x$ almost works.

Solution. $f(x) ::= 1/x$ defines a bijection from $(0, 1]$ to $[1, \infty)$, so $g(x) ::= f(x) - 1$ does the job. ■

(b) An infinite sequence of the decimal digits $\{\texttt{0}, \texttt{1}, \ldots, \texttt{9}\}$ will be called *long* if it does not end with all 0's. An equivalent way to say this is that a long sequence is one that has infinitely many occurrences of nonzero digits. Let L be the set of all such long sequences. Describe a bijection from L to the half-open real interval $(0, 1]$.

Hint: Put a decimal point at the beginning of the sequence.

Solution. Putting a decimal point in front of a long sequence defines a bijection from L to $(0, 1]$. This follows because every real number in $(0, 1]$ has a unique long decimal expansion. Note that if we didn't exclude the non-long sequences, namely, those sequences ending with all zeroes, this wouldn't be a bijection. For example, putting a decimal point in front of the sequences `1000`... and `099999`... maps both sequences to the same real number, namely, $1/10$. ■

(c) Describe a surjective function from L to L^2 that involves alternating digits from two long sequences. *Hint:* The surjection need not be total.

Solution. Given any long sequence $s = x_0, x_1, x_2, \ldots$, let

$$h_0(s) ::= x_0, x_2, x_4, \ldots$$

be the sequence of digits in even positions. Similarly, let

$$h_1(s) ::= x_1, x_3, x_5, \ldots$$

be the sequence of digits in odd positions. Then h is a surjective function from L to L^2, where

$$h(s) ::= \begin{cases} (h_1(s), h_2(s)), & \text{if } h_1(s) \in L \text{ and } h_2(s) \in L, \\ \text{undefined}, & \text{otherwise.} \end{cases} \tag{1}$$

■

(d) Prove the following lemma and use it to conclude that there is a bijection from L^2 to $(0, 1]^2$.

Lemma 3.1. *Let A and B be nonempty sets. If there is a bijection from A to B, then there is also a bijection from $A \times A$ to $B \times B$.*

Solution. *Proof.* Suppose $f : A \to B$ is a bijection. Let $g : A^2 \to B^2$ be the function defined by the rule $g(x, y) = (f(x), f(y))$. It is easy to show that g is a bijection:

- g **is total**: Since f is total, $f(a_1)$ and $f(a_2)$ exist $\forall a_1, a_2 \in A$ and so $g(a_1, a_2) = (f(a_1), f(a_2))$ also exists.
- g **is surjective**: Since f is surjective, for any $b_i \in B$ there exists $a_i \in A$ such that $b_i = f(a_i)$. So for any (b_1, b_2) in B^2, there is a pair $(a_1, a_2) \in A^2$ such that $g(a_1, a_2) ::= (f(a_1), f(a_2)) = (b_1, b_2)$. This shows that g is a surjection.

[1] The half open unit interval, $(0, 1]$, is $\{r \in \mathbb{R} \mid 0 < r \leq 1\}$. Similarly, $[0, \infty) ::= \{r \in \mathbb{R} \mid r \geq 0\}$.

- g **is injective**:

$$\begin{aligned}
g(a_1, a_2) = g(a_3, a_4) \quad &\text{iff} \quad (f(a_1), f(a_2)) = (f(a_3), f(a_4)) && \text{(by def of } g\text{)} \\
&\text{iff} \quad f(a_1) = f(a_3) \text{ AND } f(a_2) = f(a_4) \\
&\text{iff} \quad a_1 = a_3 \text{ AND } a_2 = a_4 (\text{since } f \text{ is injective}) \\
(a_1, a_2) = (a_3, a_4),
\end{aligned}$$

which confirms that g is injective.

■

Since it was shown in part (b) that there is a bijection from L to $(0, 1]$, an immediate corollary of the Lemma is that there is a bijection from L^2 to $(0, 1]^2$. ■

(e) Conclude from the previous parts that there is a surjection from $(0, 1]$ and $(0, 1]^2$. Then appeal to the Schröder-Bernstein Theorem to show that there is actually a bijection from $(0, 1]$ and $(0, 1]^2$.

Solution. There is a bijection between $(0, 1]$ and L by part (b), a surjective function from L to L^2 by part (c), and a bijection from from L^2 to $(0, 1]^2$ by part (d). These surjections compose to yield a surjection from $(0, 1]$ to $(0, 1]^2$.

Conversely, there is obviously a surjective function $f : (0, 1]^2 \to (0, 1]$, namely

$$f(\langle x, y \rangle) ::= x.$$

The Schröder-Bernstein Theorem now implies that there is a bijection from $(0, 1]$ to $(0, 1]^2$. ■

(f) Complete the proof that there is a bijection from $(0, 1]$ to $[0, \infty)^2$.

Solution. There is a bijection from $(0, 1]$ to $(0, 1]^2$ by part (e), and there is a bijection from $(0, 1]^2$ to $[0, \infty)^2$ by part (a) and the Lemma. These bijections compose to yield a bijection from $(0, 1]$ to $[0, \infty)^2$. ■

Problem 4.

Here is a game you can analyze with number theory and always beat me. We start with two distinct, positive integers written on a blackboard. Call them a and b. Now we take turns. (I'll let you decide who goes first.) On each turn, the player must write a new positive integer on the board that is the difference of two numbers that are already there. If a player cannot play, then they lose.

For example, suppose that 12 and 15 are on the board initially. Your first play must be 3, which is $15 - 12$. Then I might play 9, which is $12 - 3$. Then you might play 6, which is $15 - 9$. Then I can't play, so I lose.

(a) Show that every number on the board at the end of the game is a multiple of $\gcd(a, b)$.

Solution. Thinking of the game as a state machine, we observe that the property that $\gcd(a, b)$ divides all the numbers on the board is an invariant. This follows because the next state (board) is the same as the previous state, except for an additional number which is the difference of two numbers already there. Assuming these two numbers are divisible by $\gcd(a, b)$, we know that their difference will be as well, which proves that the next state satisfies the invariant.

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(b) Show that every positive multiple of $\gcd(a, b)$ up to $\max(a, b)$ is on the board at the end of the game.

Solution. Assume without loss of generality that $a > b$. Let s be the smallest number on the board at the end of the game. So $a = qs + r$ where $0 \leq r < s$ by the division algorithm. Then $a - s$ must be on the board and thus so must $a-2s, a-3s, \ldots, a-(q-1)s$. However, $r = a - qs$ cannot be on the board, since $r < s$ and s is defined to be the smallest number there. The only explanation is that $r = 0$, which implies that $s \mid a$. By the same argument, $s \mid b$. Therefore, s is a common divisor of a and b. Since s is a multiple of the greatest common divisor of a and b by the preceding problem part, s must actually be the greatest common divisor. We already argued that $a, a-s, a-2s, \ldots, a-(q-1)s$ must be on the board, and these are all the positive multiples of $\gcd(a, b)$ up to $\max(a, b)$.

■

(c) Describe a strategy that lets you win this game every time.

Solution. Assume without loss of generality that $a \geq b$. By the previous parts, the numbers that appear on the final board are precisely all the multiples $\leq a$ of $\gcd(a, b)$. Thus, for each game, we know *exactly* how many values will be placed on the board before the game ends. So if an odd number of values will appear on the final board (which happens precisely when a is an even multiple of $\gcd(a, b)$), then choose to go first. ■