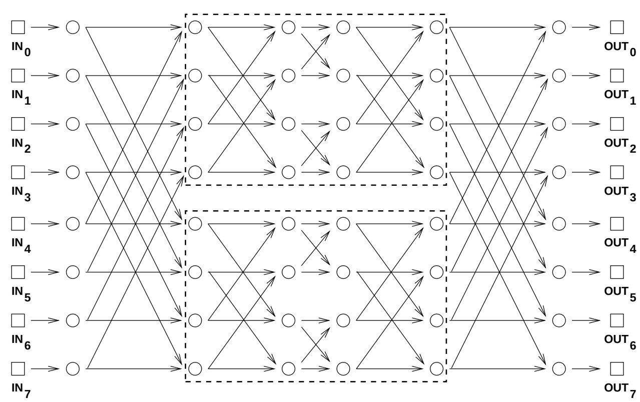
Notes for Recitation 8

Routing in a Beneš Network

In lecture, we saw that the Beneš network has a max congestion of 1; that is, every permutation can be routed in such a way that a single packet passes through each switch. Let's work through an example. A Beneš network of size $N = 8$ is attached.

1. Within the Beneš network of size $N = 8$, there are two subnetworks of size $N = 4$. Put boxes around these. Hereafter, we'll refer to these as the *upper* and *lower* subnetworks.

   Solution.

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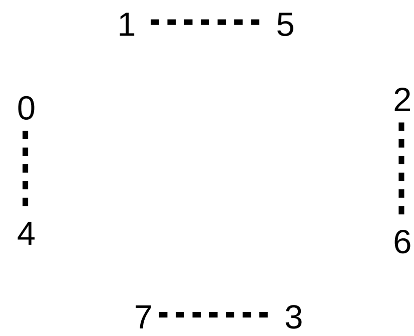
2. Now consider the following permutation routing problem:

$$\pi(0) = 3 \qquad \pi(4) = 2$$
$$\pi(1) = 1 \qquad \pi(5) = 0$$
$$\pi(2) = 6 \qquad \pi(6) = 7$$
$$\pi(3) = 5 \qquad \pi(7) = 4$$

   Each packet must be routed through either the upper subnetwork or the lower subnetwork. Construct a graph with vertices 0, 1, …, 7 and draw a *dashed* edge between each pair of packets that can not go through the same subnetwork because a collision would occur in the second column of switches.

Solution.



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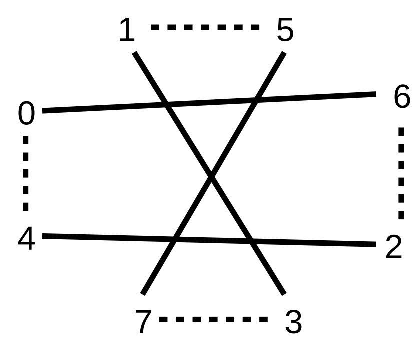
3. Add a *solid* edge in your graph between each pair of packets that can not go through the same subnetwork because a collision would occur in the next-to-last column of switches.

Solution.



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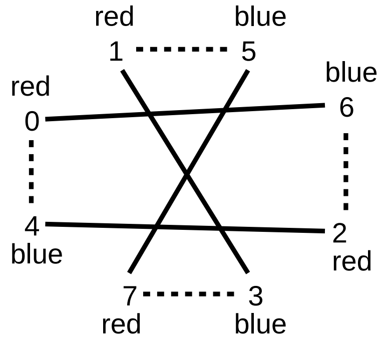
4. Color (i.e., label) the vertices of your graph red and blue so that adjacent vertices get different colors. Why must this be possible, regardless of the permutation π?

Solution. This must be possible, because the dashed edges form a matching and the solid edges form another matching. Because of the result you proved in homework, when you combine the edges, the result is a bipartite graph, which must be 2-colorable.



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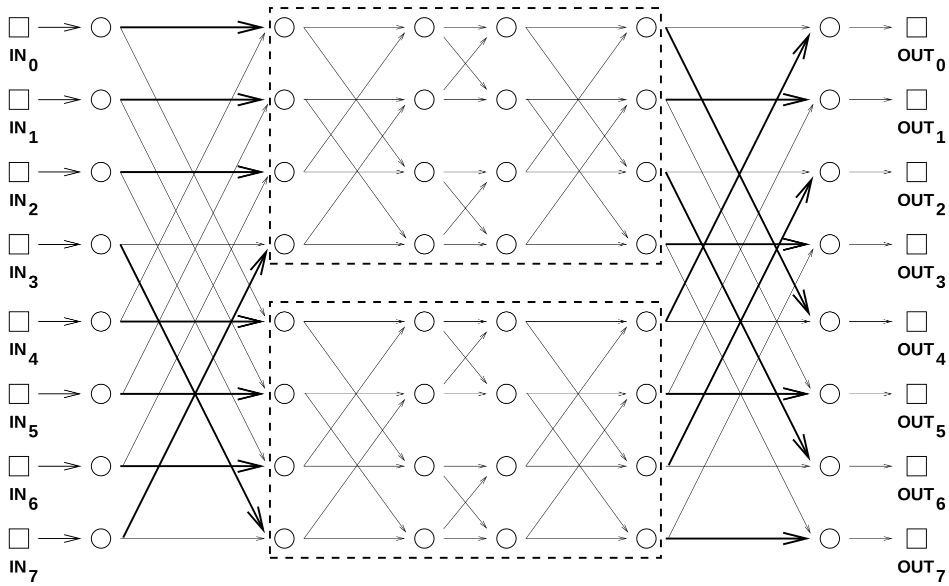
5. Suppose that red vertices correspond to packets routed through the upper subnetwork and blue vertices correspond to packets routed through the lower subnetwork. On the attached copy of the Beneš network, highlight the first and last edge traversed by each packet.

Solution.



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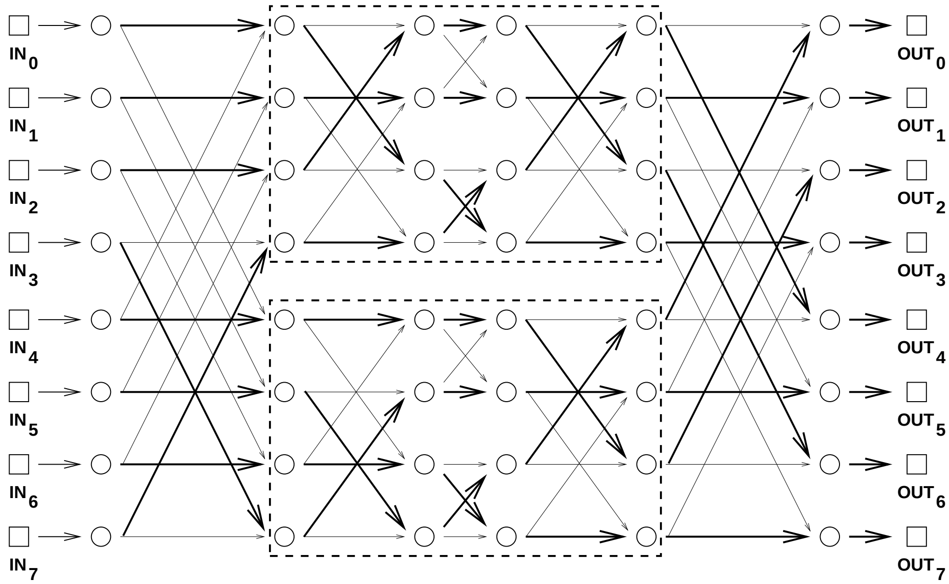
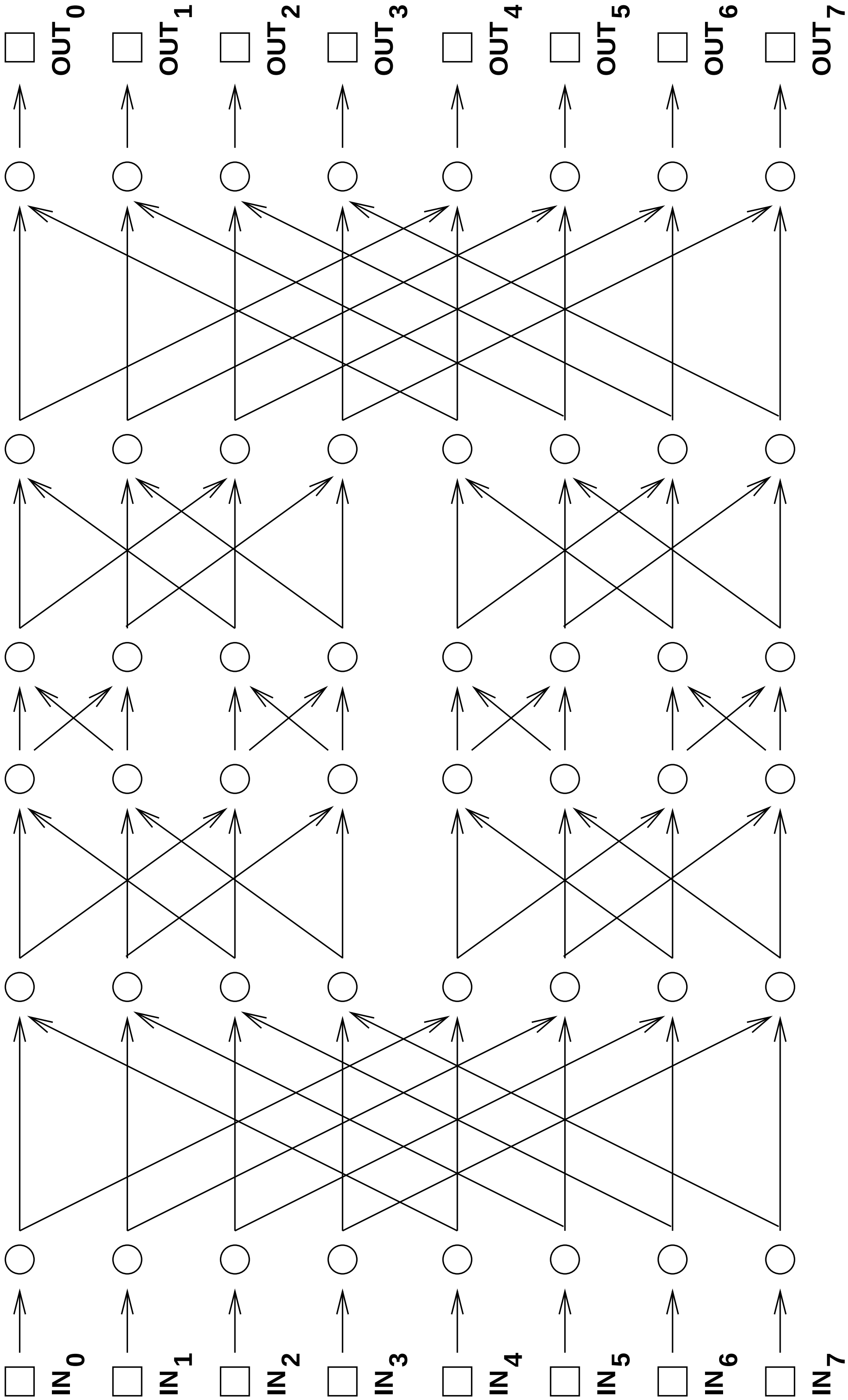
6. All that remains is to route packets through the upper and lower subnetworks. One way to do this is by applying the procedure described above recursively on each subnetwork. However, since the remaining problems are small, see if you can complete all the paths on your own.

Solution.



IN0
IN1
IN2
IN3
IN4
IN5
IN6
IN7
OUT0
OUT1
OUT2
OUT3
OUT4
OUT5
OUT6
OUT7

■


OUT0
OUT1
OUT2
OUT3
OUT4
OUT5
OUT6
OUT7
IN0
IN1
IN2
IN3
IN4
IN5
IN6
IN7

1 Euler tours

(a) Prove that a graph G has an Euler tour if and only if: i) every vertex of G has even degree, and ii) the subgraph obtained after removing all isolated vertices is connected. (An *isolated vertex* is a vertex of degree 0.)

Note that there are two directions to prove!

Solution. Let G' be the subgraph induced by the vertices that are not isolated vertices. Note that G' has an Euler tour if and only if G has. Any Euler tour must visit every vertex in G', since all edges must be visited. Thus ii) is certainly a necessary condition for the existence of an Euler tour.

The rest of the proof is as in the proof of Theorem 5.6.3 in the book (pp. 159–160). ■

(b) Come up with a necessary and sufficient condition for the existence of an Euler tour in a *directed* graph. Adapt your proof above to prove that your condition is the right one.

Solution. The condition is: an Euler tour exists if and only if i) for every vertex, the indegree equals the outdegree, and ii) the subgraph obtained after removing all isolated vertices is strongly connected.

The proof is basically the same. Again, let G' be the subgraph of G induced on the non-isolated vertices of G; G' has an Euler tour if and only if G does. Any Euler tour provides a directed path between any two vertices of G' (since we need to visit every arc), and must enter and exit a vertex the same number of times; so the condition is certainly necessary.

Now suppose the condition holds, and let $W = w_0, w_1, \ldots, w_k$ be a longest walk in G' using every directed edge at most once. Then W must be a closed walk; for suppose that $w_k \neq w_0$. Then we must have entered w_k one more time than we left it, which means that there is some outgoing directed edge that we have not used. This would allow us to extend the walk, contradicting that W was as long as possible.

Suppose that W is not an Euler tour. There must be an unused edge directed away from some vertex in the walk W; for if not, there would be no path from any vertex on W to a vertex not in W, contradicting the assumption that G' is strongly connected. Let $w_i \to u$ be this edge. Construct a walk W' beginning with this edge and traversing only unused edges, stopping when we cannot make a move. Again by the condition that indegree equals outdegree, this walk will end at w_i. We thus obtain a longer walk

$$W' = w_0, w_1, \ldots, w_i, u, \ldots, w_i, w_{i+1}, \ldots, w_k.$$

This is again a contradiction.

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