Problem Set 1 Solutions

Due: Monday, September 13

Problem 1. [**(15 points) points**] Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subset E$ such that no two edges in M are incident on a common vertex.

Let M_1, M_2 be two matchings of G. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Show that G' is bipartite.

We will need this result in one of the coming lectures.

Solution. We will show that G' has no odd cycle. By the theorem proved in recitation (that a graph is bipartite iff it has no odd cycles) we will be done.

Take a sequence of edges $(e_1, e_2, \ldots, e_k)$ in G' that form a cycle. Let us show that k must be even. We know that $e_1 \in M_1 \cup M_2$. Assume $e_1 \in M_1$ (otherwise $e_1 \in M_2$ and the argument is identical replacing M_1 with M_2). Now since e_1 and e_2 are incident on a common vertex, and M_1 is a matching, e_2 cannot be in M_1. So $e_2 \in M_2$. Similarly $e_3 \in M_1$, $e_4 \in M_2$ etc. By induction we can prove that for all $i \leq k$, $e_i \in M_1$ iff i is odd.

Now if k had been odd, we would have $e_k \in M_1$. But e_k and e_1 are adjacent edges in the cycle, and hence incident on a common vertex, we have a contradiction. Thus k must be even. ■

Problem 2. [**L points**] et $G = (V, E)$ be a graph. Recall that the *degree* of a vertex $v \in V$, denoted d_v, is the number of vertices w such that there is an edge between v and w.

- Prove that
$$2|E| = \sum_{v \in V} d_v.$$

Solution. Let $S = \{(e, v) \in E \times V : e \text{ is incident on } v\}$.

Count the elements in S as follows
$$|S| = \sum_{e \in E} |\{v : (e, v) \in S\}| = 2|E|$$
and also as
$$|S| = \sum_{v \in V} |\{e : (e, v) \in S\}| = \sum_{v \in V} d_v.$$
The result follows.

Once can also prove it by induction on $|E|$. You should try to. ■

- At a 6.042 ice-cream study session (where ice-cream flows by the way, really, you should go ... and yeah, it helps you study too) 111 students showed up. During the session, some students shook hands with each other (everybody being happy and content with the ice-cream and all). Turns out that the University of Chicago did another spectacular study here, and counted that each student shook hands with exactly 17 other students. Can you debunk this too?

  Solution. Assume that the study is accurate. Define a graph $G = (V, E)$ with students as vertices and put an edge between 2 students if they shook hands. By the previous problem, we should have $2|E| = \sum_v d_v = 111 \cdot 17$. But the LHS is even and the RHS is odd, a contradiction. ■

- And on a more dull note, how many edges does K_n, the complete graph on n vertices, have?

  Solution. Apply the first part of the problem. Notice that each vertex is joined to $n-1$ others. $2|E| = \sum_v d_v = n(n-1)$. So $|E| = n(n-1)/2$. ■

Problem 3. [20 points] A planar graph is one which can be drawn in the plane without any edges crossing (i.e. without the lines or arcs representing them intersecting except at common endpoints). Any planar graph with n vertices and m edges satisfies $m \leq 3n - 6$. Show that

(a) [5 pts] any planar graph has a node of degree at most 5.

Solution. Suppose that every vertex has degree at least 6. Then

$$\begin{aligned} 2m &= \sum_{v \in V} \deg(v) \geq \sum_{v \in V} 6 = 6n \\ m &\geq 3n \end{aligned}$$

contradicting our assertion that $m \leq 3n - 6$. ■

(b) [15 pts]Using induction, prove that any planar graph can be colored with six colors.

Solution. The proof is by induction on the number of vertices n.

$P(n) =$ "A planar graph of n vertices can be colored with at most 6 colors".

Base case: $P(1)$ is true because a single vertex can be colored with 1 color.

Inductive step: Assume $P(n)$ is true in order to show that $P(n+1)$ is true.

Let G be a planar graph with $n+1$ vertices and remove a vertex v of degree 5 (or less). The remaining n-vertex graph can be colored with 6 colors by the inductive hypothesis. Re-attach v. v is adjacent to at most 5 vertices, occupying at most 5 out of the six colors. We can use the remaining color to color v.

We have shown that $P(n) \rightarrow P(n+1)$, so the proof is complete.

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Problem 4. [T points] he most famous application of stable matching was in assigning graduating medical students to hospital residencies. Each hospital has a preference ranking of students and each student has a preference order of hospitals, but unlike the setup in the notes where there are an equal number of boys and girls and monogamous marriages, hospitals generally have differing numbers of available residencies, and the total number of residencies may not equal the number of graduating students. Modify the definition of stable matching so it applies in this situation, and explain how to modify the Mating Ritual so it yields stable assignments of students to residencies. No proof is required.

Solution. The Mating Ritual can be applied to this situation by letting the students be the boys and each of the *residencies* (not the hospitals) be the girls.

A matching is an assignment of students to residencies (an injection, A : students $\to$ residencies) such that every student has a residency (A is total), or every residency has an assigned student (A is a surjection). A stable assignment is one with no *rogue couples*, where a rogue couple is a hospital student pair (H, S) such that S is not assigned to one of the residencies at H, which she prefers over her current assignment, and

- H has some students assigned to some of its residencies and prefers S to at least one of its assigned students, or
- H has none of its residencies assigned,

■

Problem 5. [10 points]

A property of a graph is said to be *preserved under isomorphism* if whenever G has that property, every graph isomorphic to G also has that property. For example, the property of having five vertices is preserved under isomorphism: if G has five vertices then every graph isomorphic to G also has five vertices.

Determine which among the four graphs pictured in the Figures are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).

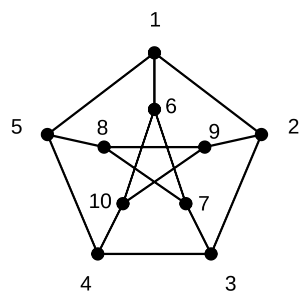
Solution. Here, G_1 and G_3 are isomorphic. In particular, the function $f : V_1 \to V_3$ is an isomomorphism, where

$$\begin{array}{lllll} f(1) = 1 & f(2) = 2 & f(3) = 3 & f(4) = 8 & f(5) = 9 \\ f(6) = 10 & f(7) = 4 & f(8) = 5 & f(9) = 6 & f(10) = 7 \end{array}$$

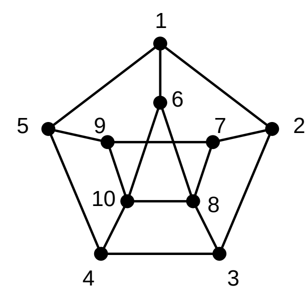
G_1 and G_4 are not isomorphic to G_2: G_2 has a vertex of degree four and neither G_1 nor G_4 has one.

G_1 and G_4 are not isomorphic: G_4 has a simple cycle of length four and G_1 does not.

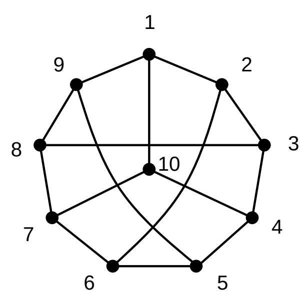
■



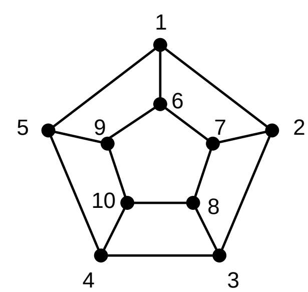
(a) G_1



(b) G_2



(c) G_3



(d) G_4

Figure 1: Which graphs are isomorphic?