Midterm

Name: ______________________________

Circle the name of your recitation instructor:

David Darren Martyna Nick Oscar Stav

- This quiz is **closed book**, but you may have one $8.5 \times 11''$ sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- The exam ends at 9:30 PM.

Problem	Points	Grade	Grader
1	10		
2	10		
3	20		
4	15		
5	20		
6	25		
7	10		
8	10		
Total	120		

Problem 1. [10 points]

Consider these two propositions:

$$\texttt{P:}\quad (A \vee B) \Rightarrow C$$

$$\texttt{Q:}\quad (\neg C \Rightarrow \neg A) \vee (\neg C \Rightarrow \neg B)$$

Which of the following best describes the relationship between P and Q? Please circle exactly one answer.

1. P and Q are equivalent
2. $P \Rightarrow Q$
3. $Q \Rightarrow P$
4. All of the above
5. None of the above

Draw a truth table to illustrate your reasoning. You can use as many columns as you need.

Problem 2. [10 points]

Let $G_0 = 1$, $G_1 = 3$, $G_2 = 9$, and define

$$G_n = G_{n-1} + 3G_{n-2} + 3G_{n-3} \tag{1}$$

for $n \geq 3$. Show by induction that $G_n \leq 3^n$ for all $n \geq 0$.

Problem 3. [20 points]

In the game of Squares and Circles, the players (you and your computer) start with a shared finite collection of shapes: some circles and some squares. Players take turns making moves. On each move, a player chooses any two shapes from the collection. These two are replaced with a single one according to the following rule:

A pair of identical shapes is replaced with a square. A pair of different shapes is replaced with a circle.

At the end of the game, when only one shape remains, you are a winner if the remaining shape is a circle. Otherwise, your computer wins.

(a) [5 pts] Prove that the game will end.

(b) [15 pts] Prove that you will win if and only if the number of circles initially is odd.

Problem 4. [15 points]

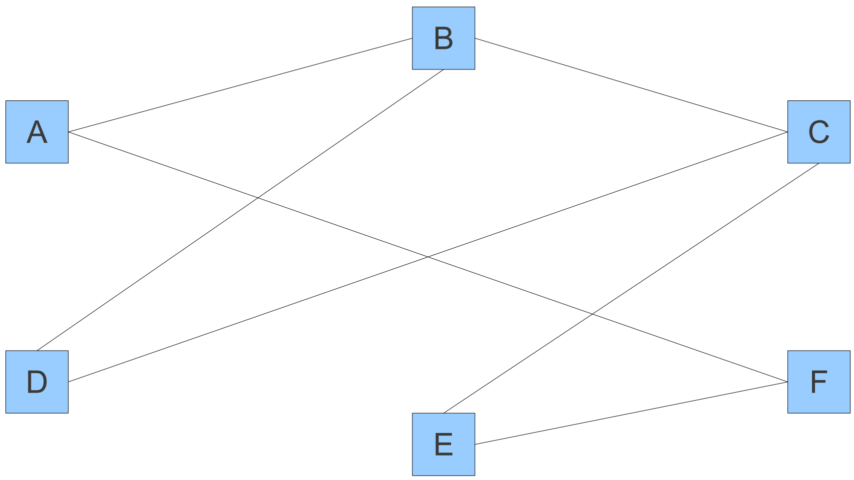
(a) [8 pts] Find a number $x \in \{0, 1, \ldots, 112\}$ such that $11x \equiv 1 \pmod{113}$.

(b) [7 pts] Find a number $y \in \{0, 1, \ldots, 112\}$ such that $11^{112111} \equiv y \pmod{113}$ (*Hint: Note that* 113 *is a prime.*)

Problem 5. [20 points]

Consider the simple graph G given in figure 1.

Figure 1: Simple graph G



(a) [4 pts] Give the diameter of G.

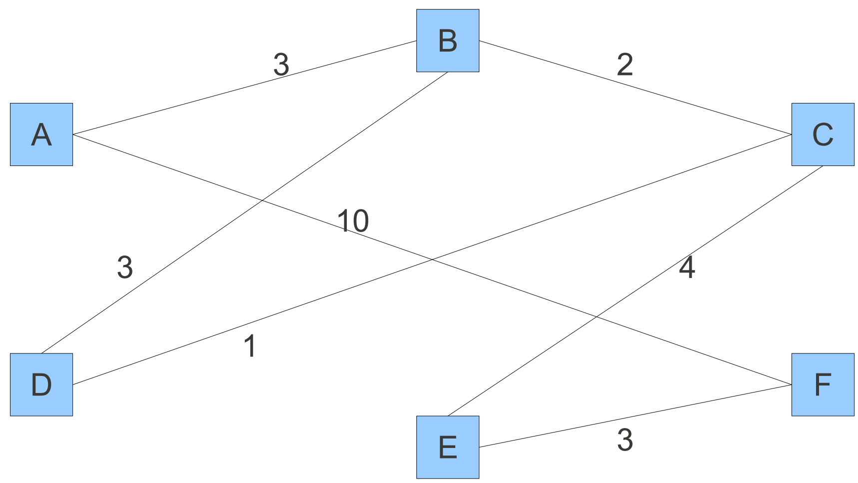
(b) [4 pts] Give a Hamiltonian Cycle on G.

(c) [4 pts] Give a coloring on G and show that it uses the smallest possible number of colors.

(d) [4 pts] Does G have an Eulerian cycle? Justify your answer.

Now consider graph H, which is like G but with weighted edges, in figure 2:

Figure 2: Weighted graph H



(e) [4 pts] Give a list of edges reflecting the order in which one of the greedy algorithms presented in class (i.e. in lecture, recitation, or the course text) would choose edges when finding an MST on H.

Problem 6. [25 points] Let G be a graph with m edges, n vertices, and k components. Prove that G contains at least $m - n + k$ cycles. (Hint: Prove this by induction on the number of edges, m)

Problem 7. [10 points] For the following sum, find an upper and a lower bound that differ by at most 1.

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{i^3}}$$

Problem 8. [10 points] State whether each of the following claims is True or False and prove your answer.

(a) [2 pts] $x \ln x$ is $O(x)$

(b) [2 pts] $x/100$ is $o(x)$

(c) [2 pts] x^{n+1} is $\Omega\left(x^n\right)$

(d) [4 pts] $n!$ is $\Theta\left(n^n\right)$.

(Notes)