Problems for Recitation 6

Hall's theorem

Let $G = (V, E)$ be a bipartite graph, with left vertex set L and right vertex set R. Recall that for a subset S of the vertices, $N(S)$ is the set of vertices which are adjacent to some vertex in S:

$$N(S) = \{r \in V \mid \{r, s\} \in E \text{ for some } s \in S\}.$$

Halls' theorem says that if for every subset S of L we have $|N(S)| \geq |S|$, then there is a matching in G that covers L.

Problem 1

Recall that a graph is called *d-regular* if every vertex in the graph has degree exactly d. Let $G = (V, E)$ be a d-regular bipartite graph, with the same number of vertices in the left part L as in the right part R.

Prove, using Hall's theorem and induction, that G can be partitioned into d perfect matchings. In other words, we can find $E_1, E_2, \ldots, E_d \subseteq E$, all disjoint ($E_i \cap E_j = \emptyset$) and which together form E, so that E_i is a perfect matching of G for each $1 \leq i \leq d$.

Problem 2

Given the preference lists of each boy and girl, there can be in general many different stable matchings.

Consider a particular boy i, and let P_i be the set of girls for which there is *some* stable matching where this girl is matched to i. We say that boy i's favorite girl in P_i is his *optimal mate*; this represents the best outcome for boy i, given that only stable matchings are allowed.

Prove that The Mating Algorithm returns a matching where every boy is matched with his optimal mate.

Problem 3

Similarly to the previous problem, we say that the *pessimal mate* of girl j is her least favorite boy from the set P_j of boys she can be matched to in some stable matching.

Prove that The Mating Algorithm returns a matching where every girl is matched with her pessimal mate.