

In-Class Problems Week 7, Mon.

Problem 1.

A *Latin square* is $n \times n$ array whose entries are the number $1, \dots, n$. These entries satisfy two constraints: every row contains all n integers in some order, and also every column contains all n integers in some order. Latin squares come up frequently in the design of scientific experiments for reasons illustrated by a little story in a footnote¹

For example, here is a 4×4 Latin square:


1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

(a) Here are three rows of what could be part of a 5×5 Latin square:

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

Fill in the last two rows to extend this “Latin rectangle” to a complete Latin square.
 theory.

(b) Show that filling in the next row of an $n \times n$ Latin rectangle is equivalent to finding a matching in some $2n$ -vertex bipartite graph.

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¹At Guinness brewery in the early 1900's, W. S. Gosset (a chemist) and E. S. Beavan (a “maltster”) were trying to improve the barley used to make the brew. The brewery used different varieties of barley according to price and availability, and their agricultural consultants suggested a different fertilizer mix and best planting month for each variety.

Somewhat sceptical about paying high prices for customized fertilizer, Gosset and Beavan planned a season long test of the influence of fertilizer and planting month on barley yields. For as many months as there were varieties of barley, they would plant one sample of each variety using a different one of the fertilizers. So every month, they would have all the barley varieties planted and all the fertilizers used, which would give them a way to judge the overall quality of that planting month. But they also wanted to judge the fertilizers, so they wanted each fertilizer to be used on each variety during the course of the season. Now they had a little mathematical problem, which we can abstract as follows.

Suppose there are n barley varieties and an equal number of recommended fertilizers. Form an $n \times n$ array with a column for each fertilizer and a row for each planting month. We want to fill in the entries of this array with the integers $1, \dots, n$ numbering the barley varieties, so that every row contains all n integers in some order (so every month each variety is planted and each fertilizer is used), and also every column contains all n integers (so each fertilizer is used on all the varieties over the course of the growing season).

(c) Prove that a matching must exist in this bipartite graph and, consequently, a Latin rectangle can always be extended to a Latin square.

Problem 2.

A portion of a computer program consists of a sequence of calculations where the results are stored in variables, like this:

	Inputs:	a, b
Step 1.	c	$= a + b$
2.	d	$= a * c$
3.	e	$= c + 3$
4.	f	$= c - e$
5.	g	$= a + f$
6.	h	$= f + 1$
	Outputs:	d, g, h

A computer can perform such calculations most quickly if the value of each variable is stored in a *register*, a chunk of very fast memory inside the microprocessor. Programming language compilers face the problem of assigning each variable in a program to a register. Computers usually have few registers, however, so they must be used wisely and reused often. This is called the *register allocation* problem.

In the example above, variables a and b must be assigned different registers, because they hold distinct input values. Furthermore, c and d must be assigned different registers; if they used the same one, then the value of c would be overwritten in the second step and we'd get the wrong answer in the third step. On the other hand, variables b and d may use the same register; after the first step, we no longer need b and can overwrite the register that holds its value. Also, f and h may use the same register; once $f + 1$ is evaluated in the last step, the register holding the value of f can be overwritten.

(a) Recast the register allocation problem as a question about graph coloring. What do the vertices correspond to? Under what conditions should there be an edge between two vertices? Construct the graph corresponding to the example above.

(b) Color your graph using as few colors as you can. Call the computer's registers $R1$, $R2$, etc. Describe the assignment of variables to registers implied by your coloring. How many registers do you need?

(c) Suppose that a variable is assigned a value more than once, as in the code snippet below:

```

...
t = r + s
u = t * 3
t = m - k
v = t + u
...

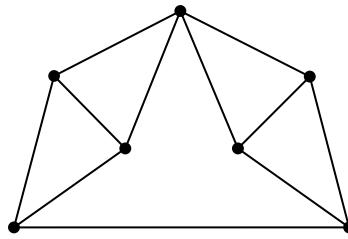
```

How might you cope with this complication?

Problem 3.

Let G be the graph below². Carefully explain why $\chi(G) = 4$.

²From *Discrete Mathematics: Elementary and Beyond*, Lovász et al., Springer, 2003, Exercise 13.3.1

**Problem 4.**

Suppose an n -vertex bipartite graph has exactly k connected components, each of which has two or more vertices. How many ways are there color it using a given set of two colors?