Problem Set 8 Solutions

Due: Tuesday, November 8

Problem 1. [15 points] This problem continues the study of the asymptotics of factorials.

(a) [5 pts]

Either prove or disprove each of the following statements.

- $n! = O((n+1)!)$
- $n! = \Omega((n+1)!)$
- $n! = \Theta((n+1)!)$
- $n! = \omega((n+1)!)$
- $n! = o((n+1)!)$

Solution. Observe that $n! = (n+1)!/(n+1)$, and thus $n! = o((n+1)!)$. Thus, $n! = O((n+1)!)$ as well, but the remaining statements are false. ■

(b) [5 pts] Is $n! = o\left(\left(\frac{n}{2}\right)^{n+e}\right)$ or is $n! = \omega\left(\left(\frac{n}{2}\right)^{n+e}\right)$? (Hint: Use Stirling's formula)

Solution. We show that $n! = o\left(\left(\frac{n}{2}\right)^{n+e}\right)$.

By Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{2}\right)^{n+e}} &= \lim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{2}{n}\right)^{n+e} \\
&= \lim_{n\to\infty} \sqrt{2\pi n}\left(\frac{2}{e}\right)^n \left(\frac{2}{n}\right)^{e} \qquad (1)\\
&= 0 \quad \text{Since } \frac{2}{e} < 1
\end{aligned}$$

■

(c) [5 pts] Show that $n! = \Omega(3^n)$

Solution. We follow the definition of Ω:

$$\begin{aligned}\lim_{n\to\infty}\frac{3^n}{n!} &= \lim_{n\to\infty}\frac{3^n}{1\cdot 2\cdot 3\ldots\cdot n}\\ &< \lim_{n\to\infty}\frac{3^n}{1\cdot 2\cdot 3\cdot 3\ldots\cdot 3}\\ &= \lim_{n\to\infty}\frac{3^2}{1\cdot 2}\\ &= \frac{9}{2} < \infty\end{aligned} \tag{2}$$

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Problem 2. [25 points] Find Θ bounds for the following divide-and-conquer recurrences. Assume $T(1) = 1$ in all cases. Show your work.

(a) [5 pts] $T(n) = 8T(\lfloor n/2\rfloor) + n$

(b) [5 pts] $T(n) = 2T(\lfloor n/8\rfloor + 1/n) + n$

(c) [5 pts] $T(n) = 7T(\lfloor n/20\rfloor) + 2T(\lfloor n/8\rfloor) + n$

(d) [5 pts] $T(n) = 2T(\lfloor n/4\rfloor + 1) + n^{1/2}$

(e) [5 pts] $T(n) = 3T(\lfloor n/9 + n^{1/9}\rfloor) + 1$

Solution. We use the method of Akra-Bazzi for these problems.

1. We see that $a = 8$, $b = 1/2$, $h = \lfloor n/2\rfloor - n/2$ so $p = 3$ gives $ab^p = 1$.

$$T(n) = \Theta(n^3(1 + \int_1^n \frac{u}{u^4}du)) = \Theta(n^3(1 + \int_1^n u^{-3}du)) = \Theta(n^3).$$

2. $a_1 = 2$, $b_1 = 1/8$, $h_1(n) = \lfloor n/8\rfloor - n/8 + 1/n$, $g(n) = n$, $p = 1/3$,

$$\begin{aligned}T(n) &= \Theta\left(n^p\left(1 + \int_1^n \frac{g(u)}{u^{p+1}}du\right)\right)\\ &= \Theta\left(n^{1/3}\left(1 + \int_1^n \frac{u}{u^{4/3}}\,du\right)\right)\\ &= \Theta\left(n^{1/3} + n^{1/3}\int_1^n u^{-1/3}\,du\right)\\ &= \Theta(n^{1/3} + n^{1/3}\frac{3}{2}(n^{2/3} - 1))\\ &= \Theta(n).\end{aligned}$$

3. $a_1 = 7$, $b_1 = 1/20$,$a_2 = 2$, $b_2 = 1/8$, $h_1(n) = \lfloor n/20 \rfloor - n/20$, $h_2(n) = \lfloor n/8 \rfloor - n/8$, and $g(n) = n$. Finally, note that although we do not know what p is, we are guaranteed that $p < 1$.

$$\begin{aligned} T(n) = \Theta(n^p(1 + \int_1^n \frac{u}{u^{p+1}} du)) &= \Theta(n^p(1 + \int_1^n u^{-p} du)) \\ &= \Theta(n^p + n^p \frac{1}{1-p}(n^{1-p} - 1)) \\ &= \Theta(n). \end{aligned}$$

4. $a_1 = 2$, $b_1 = 1/4$, $h_1(n) = \lfloor n/4 \rfloor - n/4 + 1$, $g(n) = n^{1/2}$, $p = 1/2$,

$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{u^{1/2}}{u^{3/2}} du)) = \Theta(n^{1/2} \log n).$$

5. $a_1 = 3$, $b_1 = 1/9$, $h_1(n) = \left\lfloor n/9 + n^{1/9} \right\rfloor - n/9$, $g(n) = 1$, $p = 1/2$,

$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{1}{u^{3/2}} du)) = \Theta(n^{1/2}).$$

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Problem 3. [20 points] It is easy to misuse induction when working with asymptotic notation.

False Claim If

$$T(1) = 1 \text{ and}$$
$$T(n) = 4T(n/2) + n$$

Then T(n) = O(n).

False Proof We show this by induction. Let $P(n)$ be the proposition that $T(n) = O(n)$.

Base Case: $P(1)$ is true because $T(1) = 1 = O(1)$.

Inductive Case: For $n \geq 1$, assume that $P(n-1), \ldots, P(1)$ are true. We then have that

$$T(n) = 4T(n/2) + n = 4O(n/2) + n = O(n)$$

And we are done.

(a) [5 pts] Identify the flaw in the above proof.

(b) [5 pts] Using Akra-Bazzi theorem, find the correct asymptotic behavior of this recurrence.

(c) [10 pts] We have now seen several recurrences of the form $T(n) = aT(\lfloor n/b \rfloor) + n$. Some of them give a runtime that is $O(n)$, and some don't. Find the relationship between a and b that yields $T(n) = O(n)$, and prove that this is sufficient.

Solution.

1. The flaw is that $P(n)$ is a predicate on n, whereas $O(n)$ is a statement not on n, but on the limit of n as n approaches infinity. $T(n) = O(n)$ does not depend on the value of n - it is either true or false.
2. We have that $p = 2$, so $T = \Theta(n^2(1 + \int_1^n (u/u^3)du)) = \Theta(n^2)$.
3. From analyzing the integral we can see that any case where $p < 1$ will give a linear solution, so having the condition $a < b$ is sufficient.

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Problem 4. [15 points] Define the sequence of numbers A_i by

$A_0 = 2$

$A_{n+1} = A_n/2 + 1/A_n$ (for $n \geq 1$)

Prove that $A_n \leq \sqrt{2} + 1/2^n$ for all $n \geq 0$.

Solution. *Proof.* The proof is by induction on n. Let $P(n)$ be the proposition that $A_n \leq \sqrt{2} + 1/2^n$.

Base case: $A_0 = 2 \leq \sqrt{2} + 1/2^0$ is true.

Inductive step: Let $n \geq 0$ and assume the inductive hypothesis $A_n \leq \sqrt{2} + 1/2^n$. We need the following lemma.

Lemma. *For real numbers $x > 0$, $x/2 + 1/x \geq \sqrt{2}$.*

Proof. For real numbers $x > 0$,

$$\begin{aligned} & x/2 + 1/x \geq \sqrt{2} \\ \Leftrightarrow\ & x^2 + 2 \geq 2\sqrt{2} \cdot x \\ \Leftrightarrow\ & x^2 - 2\sqrt{2} \cdot x + 2 \geq 0 \\ \Leftrightarrow\ & (x - \sqrt{2})^2 \geq 0, \end{aligned}$$

which is true. □

By using induction it is straightforward to prove that $A_n > 0$ for $n \geq 0$ (base case: $A_0 = 2 > 0$; inductive step: if $A_n > 0$, then $A_{n+1} = A_n/2 + 1/A_n > 0$). By the lemma, $A_n \geq \sqrt{2}$ for $n \geq 0$. Together with the inductive hypothesis this can be used in the following derivation:

$$\begin{aligned} A_{n+1} &= A_n/2 + 1/A_n \\ &\leq (\sqrt{2} + 1/2^n)/2 + 1/\sqrt{2} \\ &= \sqrt{2} + 1/2^{n+1}. \end{aligned}$$

This completes the proof. □

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Problem 5. [25 points] Find closed-form solutions to the following linear recurrences.

(a) [5 pts] $x_n = 5x_{n-1} - 6x_{n-2} \quad (x_0 = 0, x_1 = 1)$

Solution. The characteristic equation is just $r^2 - 5r + 6 = 0$, which has solutions $r = 2, 3$.

Hence $x_n = A2^n + B3^n$ for $n \in \mathbb{N}$. Now $x_0 = 0$ implies $A + B = 0$, and $x_1 = 1$ implies, $2A + 3B = 1$. Hence $B = 1, A = -1$.

Thus, the complete solution is $x_n = -2^n + 3^n$ for $n \in \mathbb{N}$.

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(b) [10 pts] $x_n = 4x_{n-1} - 4x_{n-2} \quad (x_0 = 0, x_1 = 2)$

Solution. The characteristic equation is $r^2 - 4r + 4 = 0$, which has one solution $r = 2$ of multiplicity 2.

Hence $x_n = A2^n + Bn2^n$ for $n \in \mathbb{N}$. Now $x_0 = 0$ implies that $A = 0$, and $x_1 = 2$ implies that $B = 1$.

Thus, the complete solution is $x_n = n2^n$ for $n \in \mathbb{N}$. ■

(c) [10 pts] $x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3} \quad (x_0 = 3, x_1 = 4, x_2 = 14)$

Solution. The characteristic equation is $r^3 - 4r^2 + r + 6 = 0$.

Generally, solving a cubic equation is a difficult problem. However, we can find from inspection that the roots are:

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 2 \\ r_3 &= 3 \end{aligned}$$

Therefore a general form for a solution is

$$x_n = A(-1)^n + B(2)^n + C(3)^n.$$

Substituting the initial conditions into this general form gives a system of linear equations.

$$\begin{array}{rcl} 3 &=& A + B + C \\ 4 &=& -A + 2B + 3C \\ 14 &=& A + 4B + 9C \end{array}$$

The solution to this linear system is $A = 1$, $B = 1$, and $C = 1$. The complete solution to the recurrence is therefore

$$x_n = (-1)^n + 2^n + 3^n.$$

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Problem 6. [25 points] In this problem, we will solve inhomogeneous linear recurrences. For the following problems, use the technique learned in class. That is, first solve the homogeneous linear recurrence, and guess the form of the particular solution. Then add the particular solution and the homogeneous solution to get the general solution, and then use the boundary case to determine remaining constants.

(a) [10 pts] Find the solution to $x_n = 3x_{n-1} + n \quad (x_0 = 2)$.

Solution. First we find the general solution to the homogeneous recurrence, which is $x_n = 3x_{n-1}$. The characteristic equation is $r - 3 = 0$ so $r = 3$, and $x_n = A3^n$.

Now, we find the particular solution. As the inhomogeneous term is n, we make a guess that the particular solution is of the form $an + b$.

If this is true, then we must have:

$$\begin{aligned} an + b &= 3a(n-1) + 3b + n \\ 3an - an + n - 3a + 3b - b &= 0 \\ (2a+1)n + 2b - 3a &= 0 \end{aligned} \tag{3}$$

Thus $2a + 1 = 0$ and $2b - 3a = 0$, and so $a = \frac{-1}{2}, b = \frac{-3}{4}$. Hence the particular solution is of the form $\frac{-1}{2}n + \frac{-3}{4}$.

This means the general solution is of the form $A3^n + \frac{-1}{2}n + \frac{-3}{4}$.

Now substituting in $x_0 = 2$, we have that $A = \frac{11}{4}$.

Hence, the complete solution is:

$x_n = \frac{11 \cdot 3^n - 2n - 3}{4}$ for $n \in \mathbb{N}$. ■

(b) [15 pts] Find the solution to $x_n = -x_{n-1} + 2x_{n-2} + n \quad (x_0 = 5, x_1 = -4/9)$.

Solution. First, we find the general solution to the homogeneous recurrence. The characteristic equation is $r^2 + r - 2 = 0$. The roots of this equation are $r_1 = 1$ and $r_2 = -2$. Therefore, the general solution to the homogenous recurrence is

$$x_n \quad = \quad A(-1)^n + B2^n.$$

Now we must find a particular solution to the recurrence, ignoring the boundary conditions. Since the inhomogenous term is linear, we guess there is a linear solution, that is, a solution of the form $an + b$. If the solution is of this form, we must have

$$an + b = -a(n-1) - b + 2a(n-2) + 2b + n$$

Gathering up like terms, this simplifies to

$$n(a + a - 2a - 1) + (b + a + b + 4a - 2b) = 0$$

which implies that

$$n = -5a$$

But a is a constant, so this cannot be so. So we make another guess, this time that there is a quadratic solution of the form $an^2 + bn + c$. If the solution is of this form, we must have

$$an^2 + bn + c = -[a(n-1)^2 + b(n-1) + c] + 2[a(n-2)^2 + b(n-2) + c] + n$$

which simplifies to

$$n^2(a + a - 2a) + n(b + b - 2a + 8a - 2b - 1) + (c + a - b + c - 8a + 4b - 2c) = 0$$

This simplifies to

$$n(6a - 1) + (-7a + 3b) = 0$$

This last equation is satisfied only if the coefficient of n and the constant term are both zero, which implies $a = 1/6$ and $b = 7/18$. Apparently, any value of c gives a valid particular solution. For simplicity, we choose $c = 0$ and obtain the particular solution:

$$x_n = \frac{1}{6}n^2 - \frac{7}{18}n.$$

The complete solution to the recurrence is the homogenous solution plus the particular solution:

$$x_n = A(-1)^n + B2^n + \frac{1}{6}n^2 - \frac{7}{18}n$$

Substituting the initial conditions gives a system of linear equations:

$$\begin{aligned} 5 &= A + B \\ -4/9 &= -A + 2B - +1/6 + 7/18 \end{aligned}$$

The solution to this linear system is $A = 3$ and $B = 2$. Therefore, the complete solution to the recurrence is

$$x_n = 3 + 2(-2)^n + \frac{1}{6}n^2 + \frac{7}{18}n$$

■