Final Exam

- The exam is **closed book**, but you may have two $8.5'' \times 11''$ sheet with notes (either printed or in your own handwriting) on both sides.
- Calculators and electronic devices (including cell phones) are not allowed.
- You may assume all of the results presented in class. This does **not** include results demonstrated in practice quiz material.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

NAME: ______________________________

TA: ______________________________

Problem	Value	Score	Grader
1	10		
2	13		
3	18		
4	10		
5	24		
6	15		
7	10		
8	15		
9	15		
10	15		
11	15		
12	10		
13	10		
Total	180		

Problem 1. [10 points] *Finalphobia* is a rare disease in which the victim has the delusion that he or she is being subjected to an intense mathematical examination.

- A person selected uniformly at random has finalphobia with probability $\frac{1}{50}$.
- A person with finalphobia has shaky hands with probability $\frac{9}{10}$.
- A person without finalphobia has shaky hands with probability $\frac{1}{20}$.

What is the probability that a person selected uniformly at random has finalphobia, given that he or she has shaky hands?

Your answer should be expressed as a ratio of integers.

Problem 2. [13 points]

(a) [3 pts] Give a closed form for

$$\sum_{i=1}^{n} \sum_{j=1}^{n} i \cdot j$$

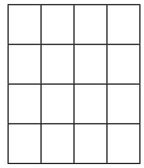
(b) [5 pts] Give a closed form, as a ratio of polynomials in n, for

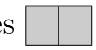
$$\sum_{j=1}^{n} \sum_{i=j}^{n} j.$$

For convenience, we provide the identity: $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

(c) [5 pts] Let $f(n) = \prod_{i=1}^{n} 2i$. Give a function g in closed form such that $f(n) = \Theta(g(n))$. You may not include factorials in your answer.

Problem 3. [18 points] We play a simplified game of battleship. We are given a 4x4 board



on which you have placed two pieces. Your destroyer is 1 square $\times$ 2 squares  and your submarine is 1 square $\times$ 3 squares . The pieces lie entirely on the board, cannot overlap, and are arranged either vertically or horizontally.

Your opponent picks 8 of the 16 squares uniformly at random and then shoots at those 8 squares. A ship is sunk if all the squares it occupies are shot at.

For this problem, you may leave your answer as the sum of expressions that are products or ratios of integers, exponentials, factorials and/or choose expressions.

(a) [8 pts] What is the probability that both of your ships are sunk?

(b) [10 pts] What is the probability of sinking the submarine but not the destroyer?

Problem 4. [10 points] You get 5 cards at random from a standard 52 card deck. What is the probability that you have exactly one pair? (This means that exactly 2 cards share the same rank, so two pairs or three of a kind would not count.)

You may express your answer as the product or ratio of integers, factorials, exponentials, and/or choose expressions.

Problem 5. [24 points] Alice decides to play the lottery. She bought 10,000 tickets, each with a probability of $\frac{1}{1,000,000}$ of winning a payout of $\$1,000,000$, probability $\frac{1}{10}$ of paying out $\$10$, and probability $1 - \frac{1}{10} - \frac{1}{1,000,000}$ of being worth nothing. If multiple tickets win, the payouts remain as above for each ticket. The tickets are mutually independent.

(a) [5 pts] What is Alice's expected return? Express your answer as an integer.

(b) [2 pts] Does your answer to part (a) depend on the tickets being mutually independent?

(c) [5 pts] What is the variance of Alice's return? Express your answer as an integer.

(d) [2 pts] Does your answer to part (c) depend on the tickets being mutually independent?

(e) [4 pts] Give a Markov upper bound for the probability that Alice wins at least $\$100,000,000$. Your answer should be expressed as a ratio of integers.

(f) [6 pts] Give a Chebyshev upper bound for the probability that Alice wins at least $\$100,000,000$. Your answer should be expressed in terms of basic arithmetic operations on integers, but need not be simplified further.

Problem 6. [15 points]

How many of the numbers $1, 2, \ldots, 3000$ are divisible by one or more of 4, 5, or 6?

Problem 7. [10 points]

Prove the following identity for all natural numbers n:

$$\sum_{i=1}^{n} (i \cdot i!) = (n+1)! - 1.$$

Problem 8. [15 points]

(a) [8 pts] Give an exact solution to the following recurrence:

$$T(n) = 5T(n-1) - 6T(n-2), \quad T(0) = 0,\ T(1) = 1.$$

(b) [7 pts] Give an asymptotic expression for the following recurrence, in Θ notation:

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2, \quad T(1) = 0.$$

Problem 9. [15 points] Suppose that we build a graph on N vertices as follows: for each (unordered) pair of distinct vertices, we independently toss a fair coin, and draw an edge between that pair of vertices if the coin lands 'heads'.

Justify your responses for all parts below.

(a) [6 pts] Given a vertex, what is the probability that the vertex has degree exactly 3?

(b) [3 pts] Let D_i denote the degree of vertex i as a random variable. Which standard family of probability distributions does D_i belong to?

(c) [6 pts] What is the expected number of vertices in the graph with degree exactly 3?

Partial credit will be given for answers written in terms of p, where p represents the correct answer to part (a).

Problem 10. [15 points]

(a) [8 pts] Let $d \geq 1$ be a given constant, and consider the sequence $(a_k)_{k \geq 0}$ defined by

$$a_k = \begin{cases} 1 & \text{if } 1 \leq k \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Give a closed form expression for the generating function $f_d(x)$ of this sequence, expressed as a ratio of polynomials in the variable x. The answer will depend on d.

Hint: Be careful of the first term, $a_0 = 0$.

(b) [7 pts] Suppose we have two dice, one with values $1, 2, \ldots, d_1$ on its sides, and the other with values $1, 2, \ldots, d_2$. Define the sequence $(b_k)_{k \geq 0}$ as the number of outcomes for the dice that yield a total sum of k. What is the generating function of this sequence?

Hint: Your answer to part (a) might be helpful.

Partial credit will be given for answers written in terms of $f_d(x)$, the generating function that correctly answers part (a).

Problem 11. [15 points] Suppose we choose a random 9-digit MIT ID number, taken uniformly from the numbers 000000000 to 999999999.

Each answer should be expressed as a ratio of integers or in scientific notation.

(a) [8 pts] What is the expected number of total occurrences of the digit sequence '6042' in the ID number?

(b) [7 pts] What is the expected total number of occurrences of '6042' or '6041'? That is,

$$\mathrm{E}\left[\#\text{ occurrences of '6042'} + \#\text{ occurrences of '6041'}\right].$$

Problem 12. [10 points]

Suppose that we randomly scramble (uniformly) the letters of the word 'ABRACADABRA'; what is the probability that the word remains 'ABRACADABRA' afterwards? Express your answer as a ratio of integers, exponentials, factorials, and/or choose expressions.

Problem 13. [10 points]

Identify each of the following asymptotic statements as true or false, with brief justification.

(a) [2 pts] $4^x = O(3^x)$.

(b) [2 pts] $x^2 = O(x^3)$.

(c) [2 pts] $x^3 - x^2 = o(5x^3)$.

(d) [2 pts] $\frac{1}{x} = \Theta(1)$.

(e) [2 pts] $e^{\ln^2 x} = \Omega(x^{100})$.