# Problem Set 7 Solutions

**Due:** Monday, October 20

**Problem 1. [10 points]**

**(a)** [10 pts] What is the product of the first $n$ odd powers of two: $\prod_{k=1}^{n} 2^{2k-1}$?

**Solution.**

$$\Pi_{k=1}^{n} 2^{2k-1} = 2^{\sum_{k=1}^{n} 2k-1} = 2^{2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1} = 2^{n(n+1)-n} = 2^{n^2}$$

■

**(b)** [10 pts] What is $\sum_{k=a}^{b} k$?

**Solution.**

$$\sum_{k=a}^{b} k = \sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k = \frac{b(b+1)}{2} - \frac{a(a-1)}{2}$$

Also, some students computed the same answer directly in the form

$$\frac{(a+b)(b-a+1)}{2}$$

by noticing that the first and last terms sum to $(a+b)$, the second and next to last terms also sum to $(a+b)$, etc. This means that we are adding the term $(a+b)$ a number of times equal to $\frac{(b-a+1)}{2}$ sinc there are $(b-a+1)$ terms and we take two for each $(a+b)$.

The most common mistake on this part was to sum up to $a$ instead of $a-1$ using the first approach. ■

**(c)** [10 pts]

Find upper and lower bounds for

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

which differ by at most 1.

**Solution.**

$$\begin{aligned}
\int_0^n \frac{1}{\sqrt{x+1}}\,dx \le &\ \textstyle\sum_{k=1}^n \frac{1}{\sqrt{k}} &\le \int_0^n \frac{1}{\sqrt{x}}\,dx \\
\int_0^n \frac{1}{\sqrt{x+1}}\,dx \le &\ \textstyle\sum_{k=1}^n \frac{1}{\sqrt{k}} &\le 1 + \int_1^n \frac{1}{\sqrt{x}}\,dx \\
2\sqrt{n+1} - 2 \le &\ \textstyle\sum_{k=1}^n \frac{1}{\sqrt{k}} &\le 1 + 2\sqrt{n} - 2 \\
2\sqrt{n+1} - 2 \le &\ \textstyle\sum_{k=1}^n \frac{1}{\sqrt{k}} &\le 2\sqrt{n} - 1
\end{aligned}$$

The most common mistake on this part of the problem was having the wrong limits of integration (going from 1 to $n$ instead of from 0 to $n$). The more severe (and less common) mistake was to switch the two functions for the upper and lower bound. ■

**Problem 2. [10 points]** Suppose you deposit \$100 into your MIT Credit Union account today, \$99 in one month from now, \$98 in two months from now, and so on. Given that the interest rate is constantly 0.3% per month, how long will it take to save \$5,000?

**Solution.** First note that you will certainly manage to have saved \$5,000 *some* day, since, even without your earnings from the interest, you will have $100+99+\cdots+1 = 100\times 101/2 = 5,050$ dollars after 99 months.

But fewer months will be needed: After the first deposit you will have \$100. After the second deposit, you will have $\$(100 \times 1.003 + 99)$. After your third deposit, your saved money will be $\$\big((100 \times 1.003 + 99) \times 1.003 + 98\big) = \$\big(100 \times (1.003)^2 + 99 \times 1.003 + 98\big)$, and so on. So, after the $n$th deposit,

$$S_n = \sum_{i=0}^{n-1} (100 - i)(1.003)^{n-i-1}$$

dollars will be in your account. Substituting $j = n - i - 1$, we can rewrite this as

$$\sum_{j=0}^{n-1} \big(100 - (n - j - 1)\big)(1.003)^j;$$

and then as

$$(101 - n)\Big(\sum_{j=0}^{n-1} (1.003)^j\Big) + \Big(\sum_{j=0}^{n-1} j(1.003)^j\Big).$$

Using the closed forms from Course Notes 6, we can finally write $S_n$ as

$$(101 - n)\Big(\frac{1 - 1.003^n}{1 - 1.003}\Big) + \Big(\frac{1.003 - n1.003^n + (n-1)1.003^{n+1}}{(1 - 1.003)^2}\Big).$$

Solving

$$S_n \ge 5,000$$

for $n$, we get $n \ge 67$. That is, you'll need more than 5.5 years to save \$5,000. ■

**Problem 3. [F points]** alse Claim:

$$2^n = O(1). \tag{1}$$

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

*Proof.* The proof by induction on $n$ where the induction hypothesis, $P(n)$, is the assertion (1).

**base case:** $P(0)$ holds trivially.

**inductive step:** We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

$$2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,$$

which implies that $2^{n+1} = O(1)$. That is, $P(n + 1)$ holds, which completes the proof of the inductive step.

We conclude by induction that $2^n = O(1)$ for all $n$. That is, the exponential function is bounded by a constant.

**Solution.** A function is $O(1)$ iff it is bounded by a constant, and since the function $2^n$ grows unboundedly with $n$, it is not $O(1)$.

The mistake in the bogus proof is in its misinterpretation of the expression $2^n$ in assertion (1). The intended interpration of (1) is

$$\text{Let } f \text{ be the function defined by the rule } f(n) = 2^n. \text{ Then } f = O(1). \tag{2}$$

But the bogus proof treats (1) as an assertion, $P(n)$, about $n$. Namely, it misinterprets (1) as meaning:

> Let $f_n$ be the constant function equal to $2^n$. That is, $f_n(k) = 2^n$ for all $k \in \mathbb{N}$. Then
>
> $$f_n = O(1). \tag{3}$$

Now (3) is true since every constant function is $O(1)$, and the bogus proof is an unnecessarily complicated, but *correct*, proof that that for each $n$, the constant function $f_n$ is $O(1)$. But in the last line, the bogus proof switches from the misinterpretation (3) and claims to have proved (2).

So you could say that the exact place where the proof goes wrong is in its first line, where it defines $P(n)$ based on misinterpretation (3). Alternatively, you could say that the proof was a correct proof (of the misinterpretation), and its first mistake was in its last line, when it switches from the misinterpretation to the proper interpretation (2). ■

**Problem 4. [30 points]** An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine $d$ days walk from the nearest oasis.[1] In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to create water caches out in the desert.

For example, if the shrine were 2/3 of a day's walk into the desert, then she could recover the Holy Grail with the following strategy. She leaves the oasis with 1 gallon of water, travels 1/3 day into the desert, caches 1/3 gallon, and then walks back to the oasis— arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks 1/3 day into the desert, tops off her water supply by taking the 1/3 gallon in her cache, walks the remaining 1/3 day to the shine, grabs the Holy Grail, and then walks for 2/3 of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

**(a)** [5 pts] What is the most distant point that the explorer can reach and return from if she takes only 1 gallon from the oasis.?

**Solution.** At best she can walk 1/2 day into the desert and then walk back. ■

**(b)** [5 pts] What is the most distant point the explorer can reach and return form if she takes only 2 gallons from the oasis? No proof is required; just do the best you can.

**Solution.** The explorer walks 1/4 day into the desert, drops 1/2 gallon, then walks home. Next, she walks 1/4 day into the desert, picks up 1/4 gallon from her cache, walks an additional 1/2 day out and back, then picks up another 1/4 gallon from her cache and walks home. Thus, her maximum distance from the oasis is 3/4 of a day's walk. ■

**(c)** [5 pts] What about 3 gallons? (Hint: First, try to establish a cache of 2 gallons *plus* enough water for the walk home as far into the desert as possible. Then use this cache as a springboard for your solution to the previous part.)

**Solution.** Suppose the explorer makes three trips 1/6 day into the desert, dropping 2/3 gallon off units each time. On the third trip, the cache has 2 gallons of water, and the explorer still has 1/6 gllon for the trip back home. So, instead of returning immediately, she uses the solution described above to advance another 3/4 day into the desert and then returns home. Thus, she reaches

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{11}{12}$$

days' walk into the desert. ■

**(d)** [5 pts] How can the explorer go as far as possible is she withdraws $n$ gallons of water? Express your answer in terms of the Harmonic number $H_n$, defined by:

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}$$

[1] She's right about the location, but doesn't realize that the Holy Grail is actually just the Beneš network.

**Solution.** With $n$ gallons of water, the explorer can reach a point $H_n/2$ days into the desert.

Suppose she makes $n$ trips $1/(2n)$ days into the desert, dropping of $(n-1)/n$ gallons each time. Before she leaves the cache for the last time, she has $n-1$ gallons plus enough for the walk home. So she applies her $(n-1)$-day strategy to go an additional $H_{n-1}/2$ days into the desert and then returns home. Her maximum distance from the oasis is then:

$$\frac{1}{2n} + \frac{H_{n-1}}{2} = \frac{H_n}{2}$$

■

**(e)** [5 pts] Use the fact that

$$H_n \sim \ln n$$

to approximate your previous answer in terms of logarithms.

**Solution.** An approximate answer is $\ln n/2$. ■

**(f)** [5 pts] Suppose that the shrine is $d = 10$ days walk into the desert. Relying on your approximate answer, how many days must the explorer travel to recover the Holy Grail?

**Solution.** She can obtains the Grail when:

$$\frac{H_n}{2} \approx \frac{\ln n}{2} \geq 10$$

This requires about $n \geq e^{20} = 4.8 \cdot 10^8$ days. ■

**Problem 5. [T points]** his problem continues the study of the asymptotics of factorials.

**(a)** [ pts]Either prove or disprove each of the following statements.

- $n! = O((n+1)!)$
- $n! = \Omega((n+1)!)$
- $n! = \Theta((n+1)!)$
- $n! = \omega((n+1)!)$
- $n! = o((n+1)!)$

**Solution.** Observe that $n! = (n+1)!/(n+1)$, and thus $n! = o((n+1)!)$. Thus, $n! = O((n+1)!)$ as well, but the remaining statements are false. ■

**(b)** [S pts]how that $n! = \omega\left(\frac{n}{3}\right)^{n+e}$.

**Solution.** By Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

On the other hand, note that $\left(\frac{n}{3}\right)^{n+e} = \left(\frac{n}{3}\right)^e \left(\frac{n}{3}\right)^n$. Dividing $n!$ by this quantity,

$$\frac{3^e\sqrt{2\pi}}{n^{e-1/2}} \cdot \left(\frac{3}{e}\right)^n,$$

we see that since $3 > e$, this expression goes to $\infty$. Thus, $n! = \omega\left(\frac{n}{3}\right)^{n+e}$. ■

**(c)** [100 pts] $n!$ is $\Omega(2^n)$

**Problem 6.** **[T points]** here is a bug on the edge of a 1-meter rug. The bug wants to cross to the other side of the rug. It crawls at 1 cm per second. However, at the end of each second, a malicious first-grader named Mildred Anderson *stretches* the rug by 1 meter. Assume that her action is instantaneous and the rug stretches uniformly. Thus, here's what happens in the first few seconds:

- The bug walks 1 cm in the first second, so 99 cm remain ahead.
- Mildred stretches the rug by 1 meter, which doubles its length. So now there are 2 cm behind the bug and 198 cm ahead.
- The bug walks another 1 cm in the next second, leaving 3 cm behind and 197 cm ahead.
- Then Mildred strikes, stretching the rug from 2 meters to 3 meters. So there are now $3 \cdot (3/2) = 4.5$ cm behind the bug and $197 \cdot (3/2) = 295.5$ cm ahead.
- The bug walks another 1 cm in the third second, and so on.

Your job is to determine this poor bug's fate.

**(a)** During second $i$, what *fraction* of the rug does the bug cross?

**Solution.** During second $i$, the length of the rug is $100i$ cm and the bug crosses 1 cm. Therefore, the fraction that the bug crosses is $1/100i$. ■

**(b)** Over the first $n$ seconds, what fraction of the rug does the bug cross altogether? Express your answer in terms of the Harmonic

number $H_n$.

**Solution.** The bug crosses 1/100 of the rug in the first second, 1/200 in the second, 1/300 in the third, and so forth. Thus, over the first $n$ seconds, the fraction crossed by the bug is:

$$\sum_{k=1}^{n} \frac{1}{100k} = H_n/100$$

(This formula is valid only until the bug reaches the far side of the rug.) ■

**(c)** Approximately how many seconds does the bug need to cross the entire rug?

**Solution.** The bug arrives at the far side when the fraction it has crossed reaches 1. This occurs when $n$, the number of seconds elapsed, is sufficiently large that $H_n/100 \geq 1$. Now $H_n$ is approximately $\ln n$, so the bug arrives about when:

$$\begin{aligned} \frac{\ln n}{100} &\geq 1 \\ \ln n &\geq 100 \\ n &\geq e^{100} \approx 10^{43} \text{ seconds} \end{aligned}$$

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**Problem 7. [F points]** ind closed-form expressions for the following. Show your work.

**(a)** [10 pts]

$$\sum_{i=0}^{n} \frac{9^i - 7^i}{11^i}$$

**Solution.** Split the expression into two geometric series and then apply the formula for the sum of a geometric series.

$$\begin{aligned} \sum_{i=0}^{n} \frac{9^i - 7^i}{11^i} &= \sum_{i=0}^{n} \left(\frac{9}{11}\right)^i - \sum_{i=0}^{n} \left(\frac{7}{11}\right)^i \\ &= \frac{1 - \left(\frac{9}{11}\right)^{n+1}}{1 - \frac{9}{11}} - \frac{1 - \left(\frac{7}{11}\right)^{n+1}}{1 - \frac{7}{11}} \\ &= -\frac{11}{2} \cdot \left(\frac{9}{11}\right)^{n+1} + \frac{11}{4} \cdot \left(\frac{7}{11}\right)^{n+1} + \frac{11}{4} \end{aligned}$$

■

**(b)** [10 pts]

$$\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j}$$

**Solution.**

$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=0}^{m} 3^{i+j} &= \sum_{i=0}^{n}\left(3^i \cdot \sum_{j=0}^{m} 3^j\right)\\ &= \left(\sum_{j=0}^{m} 3^j\right)\cdot\left(\sum_{i=0}^{n} 3^i\right)\\ &= \left(\frac{3^{m+1}-1}{2}\right)\cdot\left(\frac{3^{n+1}-1}{2}\right)\end{aligned}$$

■

**(c)** [10 pts]

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)$$

**Solution.**

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j) &= \left(\sum_{i=1}^{n}\sum_{j=1}^{n} i\right) + \left(\sum_{i=1}^{n}\sum_{j=1}^{n} j\right)\\ &= \left(\sum_{i=1}^{n} ni\right) + \left(\sum_{i=1}^{n}\frac{n(n+1)}{2}\right)\\ &= \frac{2n^2(n+1)}{2}\\ &= n^2(n+1)\end{aligned}$$

■

**(d)** [10 pts]

$$\prod_{i=1}^{n}\prod_{j=1}^{n} 2^i \cdot 3^j$$

**Solution.**

$$\begin{aligned}\prod_{i=1}^{n}\prod_{j=1}^{n} 2^i \cdot 3^j &= \left(\prod_{i=1}^{n} 2^{ni}\right)\left(\prod_{j=1}^{n} 3^{nj}\right)\\ &= 2^{n\sum_{i=1}^{n} i}3^{n\sum_{j=1}^{n} j}\\ &= 2^{n^2(n+1)/2}3^{n^2(n+1)/2}\end{aligned}$$

■

**(e)** [10 pts] In addition to expressing the following in closed form, compute the $\sim$ value for it.

$$\prod_{i=1}^{n}(2i-1)$$

**Solution.**

$$\begin{aligned}\prod_{i=1}^{n}(2i-1) &= \frac{(2n)!}{\prod_{i=1}^{n}(2i)} \\ &= \frac{(2n)!}{2^n \prod_{i=1}^{n} i} \\ &= \frac{(2n)!}{2^n n!}\end{aligned}$$

Using Stirling's formula, $(2n)! \sim \sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}$ and $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}$. Thus the $\sim$ value of this expression is $\sqrt{2}\left(\frac{2}{e}\right)^n n^n$. ■

**Problem 8. [20 points]** For each of the following six pairs of functions $f$ and $g$ (parts (a) through (f)), state which of these order-of-growth relations hold (more than one may hold, or none may hold):

$$f = o(g) \qquad f = O(g) \qquad f = \omega(g) \qquad f = \Omega(g) \qquad f = \Theta(g) \qquad f \sim g$$

| | | |
|---|---|---|
| **(a)** | $f(n) = n!$ | $g(n) = (n+1)!$ |
| **(b)** | $f(n) = \log_2 n$ | $g(n) = \log_{10} n$ |
| **(c)** | $f(n) = 2^n$ | $g(n) = 10^n$ |
| **(d)** | $f(n) = 0$ | $g(n) = 17$ |
| **(e)** | $f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$ | $g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$ |
| **(f)** | $f(n) = 1.0000000001^n$ | $g(n) = n^{10000000000}$ |

**Solution.**

- $f(n) = n!$ and $g(n) = (n+1)!$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{1}{n+1} \\ &= 0\end{aligned}$$

So $f(n) = o(g(n))$ and $f(n) = O(g(n))$.

- $f(n) = \log_2 n$ and $g(n) = \log_{10} n$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{\ln n/\ln 2}{\ln n/\ln 10}\\ &= \frac{\ln 10}{\ln 2}\end{aligned}$$

  So $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$ and $f(n) = \Theta(g(n))$.

- $f(n) = 2^n$ and $g(n) = 10^n$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{2^n}{10^n}\\ &= \lim_{n\to\infty}(1/5)^n\\ &= 0\end{aligned}$$

  So $f(n) = o(g(n))$ and $f(n) = O(g(n))$.

- $f(n) = 0$ and $g(n) = 17$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \frac{0}{17}\\ &= 0\end{aligned}$$

  So $f(n) = o(g(n))$ and $f(n) = O(g(n))$.

- $f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$ and $g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$:
  For all $n \equiv 1 \pmod 4$, $f(n)/g(n) = 0$, so $f(n) \neq \Omega(g(n))$. Likewise, for all $n \equiv 0 \pmod 4$, $g(n)/f(n) = 0$, so $f(n) \neq O(g(n))$. Therefore, none of the relations hold.

- $f(n) = 1.0000000001^n$ and $g(n) = n^{10000000000}$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{1.0000000001^n}{n^{10000000000}}\\ &= \lim_{n\to\infty}\frac{1.0000000001^n\ln 1.0000000001}{10000000000n^{9999999999}}\\ &= \lim_{n\to\infty}\frac{1.0000000001^n(\ln 1.0000000001)^{10000000000}}{10000000000!}\\ &= \infty\end{aligned}$$

  So $f(n) = \omega(g(n))$ and $f(n) = \Omega(g(n))$.

■