Solutions to Mini-Quiz 4, afternoon

Problem 1 (6 points).
The nth Fibonacci number, F_n, is defined recursively as follows:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

These numbers satisfy many unexpected identities, such as

$$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1} \tag{1}$$

Equation (1) can be proved to hold for all $n \in \mathbb{N}$ by induction, using the equation itself as the induction hypothesis, $P(n)$.

(a) Prove the **base case** ($n = 0$).

Solution.

$$\sum_{i=0}^{0} F_i^2 = (F_0)^2 = 0 = (0)(1) = F_0 F_1$$

Therefore, $P(0)$ is true. ∎

(b) Now prove the **inductive step**.

Solution. We need to prove that $P(n)$:

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

implies $P(n+1)$:

$$\sum_{i=0}^{n+1} F_i^2 = F_{n+1} F_{n+2}$$

Proof.

$$\begin{aligned} \sum_{i=0}^{n+1} F_i^2 &= \sum_{i=0}^{n} F_i^2 + F_{n+1}^2 \\ &= F_n F_{n+1} + F_{n+1}^2 && \text{By } P(n). \\ &= F_{n+1}(F_n + F_{n+1}) \\ &= F_{n+1} F_{n+2} && \text{By the definition of the Fibonacci sequence.} \end{aligned}$$

∎

Problem 2 (4 points).
The following state machine describes a procedure that terminates with the product of two nonnegative integers x and y in register a. Its states are triples of nonnegative integers (r, s, a). The initial state is $(x, y, 0)$. The transitions are given by the rule that for $s > 0$:

$$(r, s, a) \to \begin{cases} (2r, s/2, a) & \text{if } s \text{ is even,} \\ (2r, (s-1)/2, a+r) & \text{otherwise.} \end{cases}$$

Circle the predicates below that are invariant for this state machine:

- $P((r, s, a)) ::= \quad [rs + a = xy]$
- $P((r, s, a)) ::= \quad [r = 2r + 1]$
- $P((r, s, a)) ::= \quad [s + a = xy]$
- $P((r, s, a)) ::= \quad [r + a = s]$

Solution. $[rs + a = xy]$ and $[r = 2r + 1]$ are invariants. The second of these is vacuously invariant because it is always false. ∎