Notes for Recitation 7

Hall's theorem

Let $G = (V, E)$ be a bipartite graph, with left vertex set L and right vertex set R. Recall that for a subset S of the vertices, $N(S)$ is the set of vertices which are adjacent to some vertex in S:

$$N(S) = \{r \in V \mid \{r, s\} \in E \text{ for some } s \in S\}.$$

Halls' theorem says that if for every subset S of L we have $|N(S)| \geq |S|$, then there is a matching in G that covers L.

Problem 1

Recall that a graph is called *d-regular* if every vertex in the graph has degree exactly d. Let $G = (V, E)$ be a d-regular bipartite graph, with the same number of vertices in the left part L as in the right part R.

Prove, using Hall's theorem and induction, that G can be partitioned into d perfect matchings. In other words, we can find $E_1, E_2, \ldots, E_d \subseteq E$, all disjoint ($E_i \cap E_j = \emptyset$) and which together form E, so that E_i is a perfect matching of G for each $1 \leq i \leq d$.

Solution. The proof is by induction on d. So let

$$P(d) = \text{“Any } d\text{-regular bipartite graph with } n/2 \text{ left nodes and } n/2 \text{ right nodes can be partitioned into } d \text{ perfect matchings”.}$$

Let us take $d = 0$ for the base case (the statement still makes sense). In a 0-regular graph, there are no edges. We can indeed partition the empty set into 0 perfect matchings! So $P(0)$ is true.

So assume $P(d)$ holds; we want to prove $P(d+1)$, so let G be a $(d+1)$-regular bipartite graph, with left subset L and right subset R, where $|L| = |R| = n/2$.

Let S be any subset of L; we want to show that $|N(S)| \geq |S|$. Consider the subgraph $G' = (S \cup N(S), E')$ induced by $S \cup N(S)$. Every node of S has degree $d+1$ in G' (since we kept all the neighbours of S). Every node in $N(S)$ has degree at most $d+1$ in G' (since it had degree $d+1$ in G, and G' is a subgraph). Now count the number of edges of G' in two ways, just like in class: we get

$$(d+1)|S| = |E'| = \sum_{v \in N(S)} \deg_{G'}(v) \leq (d+1)|N(S)|.$$

Simplifying, $|N(S)| \geq |S|$.

So Hall's condition is satisfied, and there exists some perfect matching M on G. Now look at the subgraph $H = (V, E_H)$ obtained by removing M from G. This is d-regular, since for each vertex we remove a single edge adjacent to it. So by induction, H can be partitioned into d perfect matchings $E_1, E_2, \ldots, E_d$. Writing $E_{d+1} = M$, we obtain that $E = E_H \cup M = E_1 \cup \cdots \cup E_{d+1}$ (and all the E_i's are disjoint).

This proves $P(d+1)$, and so by induction, $P(d)$ holds for all $d \geq 0$. ■

Problem 2

Given the preference lists of each boy and girl, there can be in general many different stable matchings.

Consider a particular boy i, and let P_i be the set of girls for which there is *some* stable matching where this girl is matched to i. We say that boy i's favorite girl in P_i is his *optimal mate*; this represents the best outcome for boy i, given that only stable matchings are allowed.

Prove that The Mating Algorithm returns a matching where every boy is matched with his optimal mate.

Solution.

Theorem 1. *The Mating Ritual marries every man to his optimal spouse.*

Proof. By contradiction. Assume for the purpose of contradiction that some man does not get his optimal spouse. Then there must have been a day when he crossed off his optimal spouse—otherwise he would still be serenading (and would ultimately marry) her or some even more desirable woman.

By the Well Ordering Principle, there must be a *first* day when a man (call him "Keith") crosses off his optimal spouse (call her Nicole). According to the rules of the Ritual, Keith crosses off Nicole because Nicole has a preferred suitor (call him Tom), so

$$\text{Nicole prefers Tom to Keith.} \tag{$*$}$$

Since this is the first day an optimal woman gets crossed off, we know that Tom had not previously crossed off his optimal spouse, and so

$$\text{Tom ranks Nicole at least as high as his optimal spouse.} \tag{$**$}$$

By the definition of an optimal spouse, there must be some stable set of marriages in which Keith gets his optimal spouse, Nicole. But then the preferences given in (∗) and (∗∗) imply that Nicole and Tom are a rogue couple within this supposedly stable set of marriages (think about it). This is a contradiction. □

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Problem 3

Similarly to the previous problem, we say that the *pessimal mate* of girl j is her least favorite boy from the set P_j of boys she can be matched to in some stable matching.

Prove that The Mating Algorithm returns a matching where every girl is matched with her pessimal mate.

Solution.

Theorem 2. *The Mating Ritual marries every woman to her pessimal spouse.*

Proof. By contradiction. Assume that the theorem is not true. Hence there must be a stable set of marriages $\mathcal{M}$ where some woman (call her Nicole) is married to a man (call him Tom) that she likes less than her spouse in The Mating Ritual (call him Keith). This means that

$$\text{Nicole prefers Keith to Tom.} \tag{+}$$

By Theorem 1 and the fact that Nicole and Keith are married in the Mating Ritual, we know that

$$\text{Keith prefers Nicole to his spouse in } \mathcal{M}. \tag{++}$$

This means that Keith and Nicole form a rogue couple in $\mathcal{M}$, which contradicts the stability of $\mathcal{M}$. □

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