Notes for Recitation 16

1 Combinatorial Proof

A ***combinatorial proof*** is an argument that establishes an algebraic fact by relying on counting principles. Many such proofs follow the same basic outline:

1. Define a set S.
2. Show that $|S| = n$ by counting one way.
3. Show that $|S| = m$ by counting another way.
4. Conclude that $n = m$.

Consider the following theorem:

Theorem.

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

We can prove it with a combinatorial approach:

Proof. We give a combinatorial proof. Let S be the set of all binary sequences with exactly n zeroes and $k+1$ ones.

On the one hand, we know from a previous recitation that the number of such sequences is equal to $\binom{k+n+1}{k+1}$.

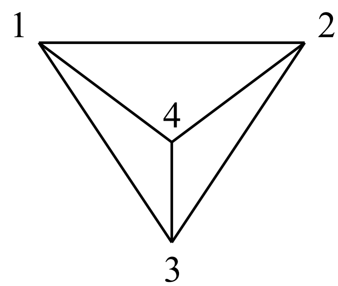
On the other hand, the number of zeroes i to the left of the rightmost one ranges from 0 to n. For a fixed value of i, there are $\binom{k+i}{k}$ possible choices for the sequence of bits before the rightmost one. If we sum over all possible i, we find that $|S| = \sum_{i=0}^{n} \binom{k+i}{k}$.

Equating these two expressions for $|S|$ proves the theorem. $\square$

2 Triangles

Let $T = \{X_1, \ldots, X_t\}$ be a set whose elements X_i are themselves sets such that each X_i has size 3 and is $\subseteq \{1, 2, \ldots, n\}$. We call the elements of T "triangles". Suppose that for all "edges" $E \subseteq \{1, 2, \ldots, n\}$ with $|E| = 2$ there are exactly λ triangles $X \in T$ with $E \subseteq X$.

For example, if we might have the triangles depicted in the following diagram, which has $\lambda = 2$, $n = 4$, and $t = 4$:



In this example, each edge appears in exactly two of the following triangles:

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$$

Prove

$$\lambda \cdot \frac{n(n-1)}{2} = 3t$$

by counting the set

$$C = \{(E, X) : X \in T, E \subseteq X, |E| = 2\}$$

in two different ways.

Solution. We give a combinatorial proof. Let C be $\{(E, X) : X \in T, E \subseteq X, |E| = 2\}$.

On the one hand, there are $\binom{n}{2}$ sets $E \subseteq \{1, \ldots, n\}$ of size $|E| = 2$. For each such E there are exactly λ triangles $X \in T$ with $E \subseteq X$. So, $|C| = \lambda\binom{n}{2} = \lambda \cdot \frac{n(n-1)}{2}$.

On the other hand, there are t triangles. Each triangle has exactly $\binom{3}{2} = 3$ subsets E of size 2. So, $|C| = 3t$.

Equating these two expressions for $|C|$ proves the theorem. ■

3 Counting, counting, counting

Learning to count takes practice! Briefly justify your answers to the following questions. Not every problem can be solved with a cute formula; you may have to fall back on case analysis, explicit enumeration, or ad hoc methods. Do as many problems as you can and save the rest to study for Quiz II. You may leave factorials and binomial coefficients in your answers.

1. How many different arrangements are there of the letters in $BANANA$?

   Solution. By the Bookkeeper Rule, there are:

   $$\frac{6!}{1!\ 3!\ 2!} = 60$$

   ■

2. How many different paths are there from point $(0,0,0)$ to point $(10,20,30)$ if every step increments one coordinate and leaves the other two unchanged?

   Solution. There is a bijection between the set of all such paths and the set of strings containing 10 X's, 20 Y's, and 30 Z's. In particular, we obtain a path by working through a string from left to right. An X corresponds to a step that increments the first coordinate, a Y increments the second coordinate, and a Z increments the third. The number of such strings is:

   $$\frac{60!}{10!\ 20!\ 30!}$$

   Therefore, this is also the number of paths. ■

3. Find the number of 5-card hands with exactly three aces.

   Solution. We can choose the three aces in $\binom{4}{3}$ ways, and we can choose the remaining two cards in $\binom{48}{2}$ ways. Thus, there are $\binom{4}{3}\binom{48}{2}$ such hands. ■

4. Find the number of 5-card hands in which every suit appears at most twice.

   Solution. There are two cases. Either one suit appears twice or else two suits appear twice. The number of hands in which one suit appears twice is $\binom{13}{2} \cdot 13^3 \cdot 4$, since there are 4 ways to choose the doubly represented suit, $\binom{13}{2}$ ways to choose two values from this suit, and 13^3 ways to choose values for cards in the three remaining suits. Similarly, the number of hands in which two suits appear twice is $\binom{13}{2}^2 \cdot 13 \cdot \binom{4}{2} \cdot 2$. Therefore, there are a total of

   $$\binom{13}{2} \cdot 13^3 \cdot 4 + \binom{13}{2}^2 \cdot 13 \cdot \binom{4}{2} \cdot 2$$

   such hands. ■

5. There are 15 sidewalk squares in a row. Suppose that a ball is thrown down the row so that it bounces on 0, 1, 2, or 3 distinct sidewalk squares. How many different throws are possible? Two throws are considered to be equivalent if they bounce on the same squares in a different order.

Solution.

$$\binom{15}{0} + \binom{15}{1} + \binom{15}{2} + \binom{15}{3}$$

■

6. In how many different ways can the numbers shown on a red die, a green die, and a blue die total up to 15? Assume that these are ordinary, 6-sided dice.

   Solution. We fall back on explicit enumeration. Let R, G, and B be the values shown on the three dice.

$$\begin{aligned}
R &= 1, & B+G &= 14 & &\to 0 \text{ ways} \\
R &= 2, & B+G &= 13 & &\to 0 \text{ ways} \\
R &= 3, & B+G &= 12 & &\to 1 \text{ ways} \\
R &= 4, & B+G &= 11 & &\to 2 \text{ ways} \\
R &= 5, & B+G &= 10 & &\to 3 \text{ ways} \\
R &= 6, & B+G &= 9 & &\to 4 \text{ ways}
\end{aligned}$$

   A total of 10 ways.

   Another approach (suggested by a student in recitation) begins by observing that the number of ways the dice can sum to 15 is the same as the number of positive integer solutions to the equation

$$x_1 + x_2 + x_3 = 15$$

   such that $x_i \leq 6$. In general, counting solutions with inequality constraints on the variables involves a tedious case analysis, but in this case there's a trick to remove the constraints: let $y_i = 6 - x_i$. Now the number of desired x_i solutions is the same as the number of nonnegative integer solutions to

$$y_1 + y_2 + y_3 = 3 \tag{1}$$

   such that $y_i \leq 5$. But since the sum of the y_i's must be three, the constraint that $y_i \leq 5$ will be met by every nonnegative integer solution to (1). So we need only count the number of nonnegative integer solutions to (1), which we already know is the same as the number of binary sequences of two zeros and three ones, namely

$$\binom{2+3}{3} = 10.$$

   ■

7. In how many ways can 20 indistinguishable pre-frosh be stored in four different crates if each crate must contain an *even* number of pre-frosh?

   Solution. There is a bijection from 13-bit strings with exactly 3 ones. In particular, the string $0^a10^b10^c10^d$ corresponds to to storing $2a$ prefrosh in the first crate, $2b$ in the second, $2c$ in the third, and $2d$ in the fourth. Therefore, the number of ways to store the pre-frosh is equal to the number of 13-bit strings with exactly 3 ones, which is $\binom{13}{3}$. ■

8. How many paths are there from point $(0, 0)$ to $(50, 50)$ if every step increments one coordinate and leaves the other unchanged and there are impassable boulders sitting at points $(10, 10)$ and $(20, 20)$?

   Solution. We use inclusion-exclusion. The total number of paths is $\binom{100}{50}$, but we must subtract off the obstructed paths. There are $\binom{20}{10} \cdot \binom{80}{40}$ paths through the first boulder, since there are $\binom{20}{10}$ paths from the start to the boulder and $\binom{80}{40}$ paths from the boulder to the finish. Similarly, there are $\binom{40}{20} \cdot \binom{60}{30}$ paths through the second boulder. However, we must add back in paths going through both boulders, and there are $\binom{20}{10} \cdot \binom{20}{10} \cdot \binom{60}{30}$ of those. Therefore, the total number of paths is:

$$\binom{100}{50} - \binom{20}{10} \cdot \binom{80}{40} - \binom{40}{20} \cdot \binom{60}{30} + \binom{20}{10} \cdot \binom{20}{10} \cdot \binom{60}{30}$$

■

9. In how many ways can the 180 students in 6.042 be divided into 36 groups of 5?

   Solution. We can group the students using the following procedure: line up the students in some order. Group the first five students, the sixth through tenth students, the eleventh through fifteenth students, and so on. The students can be lined up in 180! ways. However, this overcounts by a factor of $(5!)^{36}$, since the students within each of the 36 groups can be ordered in 5! ways. We are also overcounting by an additional factor of 36!, since the 36 groups can be ordered in 36! ways. Thus, the number of groupings is

$$\frac{180!}{(5!)^{36} \cdot 36!}$$

■

10. In how many different ways can 10 indistinguishable balls be placed in four distinguishable boxes, such that every box gets 1, 2, 3, or 4 balls?

    Solution. First, we might as well put 1 ball in every box. Now the problem is to put the remaining 6 balls into 4 boxes so that no box gets more than 3 balls. Now we turn to case analysis. For example, we could put 3 balls in two boxes and 0 balls in the other two boxes. There are $4!/(2!\ 2!) = 6$ ways to do this. All cases are listed below:

distribution of balls	# of ways	
$3, 3, 0, 0$	$\frac{4!}{2!\ 2!}$	$= 6$
$3, 2, 1, 0$	$\frac{4!}{1!\ 1!\ 1!\ 1!}$	$= 24$
$3, 1, 1, 1$	$\frac{4!}{3!\ 1!}$	$= 4$
$2, 2, 2, 0$	$\frac{4!}{3!\ 1!}$	$= 4$
$2, 2, 1, 1$	$\frac{4!}{2!\ 2!}$	$= 6$

■

11. In how many different ways can Blockbuster arrange 64 copies of *Cat in the Hat*, 96 copies of *Matrix Revolutions*, and 1 copy of *Amelie* on 5 shelves?

 Solution. This is the number of ways to arrange 64 C's (Cat in the Hat), 96 M's (Matrix), 1 A's (Amelie), and 4 X's (dividers between shelves). This is equal to:

$$\frac{(64+96+1+4)!}{64!\ 96!\ 1!\ 4!}$$

■

4 There's more than one way...

In the beginning of today's recitation, we gave a combinatorial proof of the following theorem:

Theorem.

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

We can also prove this theorem using induction. Give such a proof.

Solution. *Proof.* The proof is by induction on n. Let $P(n)$ be the proposition that $\forall k > 0$, $\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}, n \geq 0$.

Base case: $P(0)$ is true because

$$\forall k > 0, \sum_{i=0}^{0} \binom{k+i}{k} = \binom{k}{k} = \binom{k+1}{k+1} = 1$$

Inductive step: Assume $P(n)$ is true. Show $P(n+1)$ must be true $\forall n > 0$.

$$\begin{align}
\forall k > 0, \sum_{i=0}^{n+1} \binom{k+i}{k} &= \binom{k+n+1}{k} + \sum_{i=0}^{n} \binom{k+i}{k} \tag{2}\\
&= \binom{k+n+1}{k} + \binom{k+n+1}{k+1} \tag{3}\\
&= \frac{(k+n+1)!}{k!(n+1)!} + \frac{(k+n+1)!}{(k+1)!n!} \tag{4}\\
&= \frac{(k+1)(k+n+1)! + (k+n+1)!(n+1)}{(k+1)!(n+1)!} \tag{5}\\
&= \frac{(k+n+1)!(k+n+2)}{(k+1)!(n+1)!} \tag{6}\\
&= \frac{(k+n+2)!}{(k+1)!(n+1)!} \tag{7}\\
&= \binom{k+n+2}{k+1} \tag{8}
\end{align}$$

The inductive hypothesis is applied in Step 3. Step 4 follows by definition of choose, and the remaining steps are algebraic simplifications. □

It is always good to have more than one way to solve a problem! ■