Final

- The exam is **closed book**, but you may have four 8.5″ × 11″ sheet with notes (either printed or in your own handwriting) on both sides.
- Calculators and electronic devices (including cell phones) are not allowed.
- You may assume all of the results presented in class. This does **not** include results demonstrated in practice quiz material.
- Write your name on each page of the exam
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

NAME: ____________________

TA: ____________________

Problem	Value	Score	Grader
1	10		
2	10		
3	10		
4	15		
5	20		
6	10		
7	15		
8	10		
9	10		
10	10		
11	15		
Total	135		

Name: ____________________________

Problem 1. [10 points] Let $a_0 = a_1 = 1$, and let $a_{n+2} = a_{n+1} + 5a_n$ for $n \geq 0$. Prove by strong induction that $a_n \leq 3^n$ for all $n \geq 0$. You do not need to solve the recursion to do this.

Name: ______________________

Problem 2. [10 points] The MIT Social Statistics Society (MIT S^3) is doing a study to see how social engineers are. So they host a party with 15 students. After the party, 8 students report not having met anyone at the party, 4 students report having met one other student each, 2 students report have met two other students each, and one student reports having met three other students. Is this possible? You must prove your answer. Assume that the "met" relation is symmetric and anti-reflexive.

Name: ______________________________

Problem 3. [10 points] Find a closed form for $\prod_{i=1}^{n} \prod_{j=i}^{n} 2^{(i-j)}$.

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Problem 4. [10 points]

(a) [7 pts] Find a solution to $f_n = 4f_{n-1} + 5f_{n-2}$, with $f_0 = 1, f_1 = 1$.

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(b) [8 pts] Give an asymptotic expression for the following recurrence, in Θ notation:

$$T(n) = 8T(\frac{n}{4}) + 18T(\frac{n}{6}) + n^3, \qquad T(1) = 0$$

Name: ________________________

Problem 5. [20 points]

(a) [6 pts] A cashier wants to work 5 days a week, but he wants to have at least one of Saturday or Sunday off. How many ways can he choose the days he will work? Your answer should be an integer.

Name: ________________________

(b) [6 pts] How many permutations of $1, 2, 3, \ldots n$ are there if 1 must precede 2 and 3 must precede 4 (for positive integers $n \geq 4$). Your answer should be in terms of n.

Name: __________________________

(c) [8 pts] Let $a_1, a_2, \ldots a_k$ be positive integers with sum at most n (with $k > 1$). Use a combinatorial argument to show that $a_1!a_2!\ldots a_k! < n!$.

Name: ______________________

Problem 6. [10 points] Find the generating function of the number of solutions to

$$x_1 + 2x_2 + 3x_3 + 4x_4 = n$$

where x_1, x_2, x_3, x_4 are positive integers. Express your answer as the inverse of a product of polynomials.

Name: ______________________

Problem 7. [15 points] Vlad's tiger has wandered into one of two forests overnight. It is in forest A with probability .4 and in forest B with probability .6.

If his tiger is in forest A and Vlad spends a day searching for it in forest A, the conditional probability that he will find his tiger that day is .25. Similarly, if his tiger is in forest B and he spends a day searching for it in forest B, then he will find his tiger that day with conditional probability .15.

You don't have to reduce your answer for the following problems.

(a) [5 pts] In which forest should Vlad look on the first day in order to maximize the probability of finding his tiger that day?

Name: ______________________

(b) [5 pts] Vlad looked in forest A on the first day but didn't find his tiger. What is the probability that the tiger is in forest A?

Name: ______________________

(c) [5 pts] Vlad flips a fair coin to determine where to look on the first day and finds his tiger on the first day. What is the probability that he looked in forest A?

Name: ______________________

Problem 8. [10 points] In a permutation of n elements, a pair (i, j) is called an inversion if and only if $i < j$ and i comes after j. For example, the permutation 31542 in the case of $n = 5$ has five inversions: $(3, 1), (3, 2), (5, 4), (5, 2)$ and $(4, 2)$. What is the expected number of inversions in a uniform random permutation of the number $1, 2, \ldots n$?

Hint: Use appropriate indicator variables and linearity of expectation.

Name: ______________________

Problem 9. [10 points] Consider tossing a non-fair coin C until one throws a heads. Tossing C results in heads with probability $\frac{1}{3}$. Let X be a random variable corresponding to the number of tosses needed until one throws a heads (so $X \geq 1$).

(a) [5 pts] Calculate $\mathbb{E}(X)$.

Name: ______________________________

(b) [5 pts] Calculate the variance of X.

Name: ______________________

Problem 10. [10 points] One raffle ticket is drawn randomly from a bowl containing four tickets numbered 1, 2, 3, and 4. Consider the following three random variables:

1. Let A be a binary random variable that is 1 if a 1 or 2 is drawn and 0 otherwise.
2. Let B be a binary random variable that is 1 if a 1 or 3 is drawn and 0 otherwise.
3. Let C be a binary random variable that is 1 if a 1 or 4 is drawn and 0 otherwise.

Assume that all raffle tickets have equal probability of being drawn

(a) [5 pts] Are A, B, C mutually independent? Briefly justify your answer.

Name: ______________________

(b) [5 pts] Are A, B, C pairwise independent? Briefly justify your answer.

Name: ______________________

Problem 11. [15 points] Let X be a random variable indicating the runtime of an algorithm on an input of size n. You know that $X \geq 0$ and that $\mathbb{E}(X) \leq 10n$.

(a) [5 pts] Give as good a bound as you can on the probability that $X \geq 20n$.

Name: ____________________________

(b) [5 pts] Now suppose you are told that $\mathrm{Var}(X) \leq 10n$. Use Chebyshev's inequality to bound the probability that $X \geq 20n$.

Name: ____________________________

(c) [5 pts] Now suppose there is another random variable T indicating the runtime of a different algorithm on the same input of size n. You are given that $\mathbb{E}(e^T) \leq e^{10n}$. Write down as strong a bound as you can on the probability that $T \geq 20n$.

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