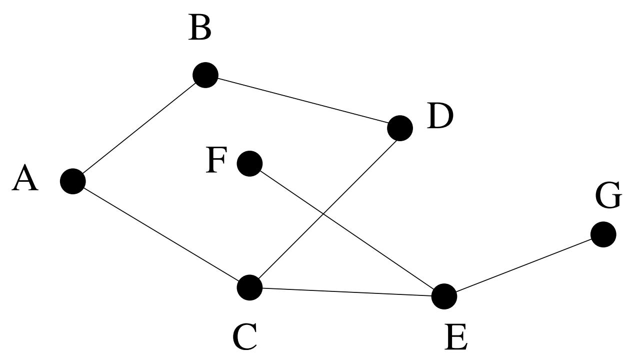
Problems for Recitation 6

Graph Basics

Let $G = (V, E)$ be a graph. Here is a picture of a graph.



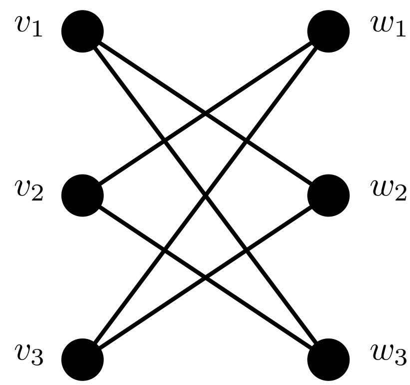
Recall that the elements of V are called vertices, and those of E are called edges. In this example the vertices are $\{A, B, C, D, E, F, G\}$ and the edges are

$$\{A—B, B—D, C—D, A—C, E—F, C—E, E—G\}.$$

Deleting some vertices or edges from a graph leaves a *subgraph*. Formally, a subgraph of $G = (V, E)$ is a graph $G' = (V', E')$ where V' is a nonempty subset of V and E' is a subset of E. Since a subgraph is itself a graph, the endpoints of every edge in E' must be vertices in V'. For example, $V' = \{A, B, C, D\}$ and $E' = \{A—B, B—D, C—D, A—C\}$ forms a subgraph of G.

In the special case where we only remove edges incident to removed nodes, we say that G' is the *subgraph induced on* V' if $E' = \{(x—y | x, y \in V' \text{ and } x—y \in E\}$. In other words, we keep all edges unless they are incident to a node not in V'. For instance, for a new set of vertices $V' = \{A, B, C, D\}$, the induced subgraph G' has the set of edges $E' = \{A—B, B—D, C—D, A—C\}$.

Remember in lecture that we covered a graph of actors and actresses. In the graph, there were two kinds of vertices, male and female, and the edges went only went between both. This type of graph is known as a *bipartite graph*. A graph $G = (V, E)$ is called bipartite if we can divide the vertex set into two parts, the "left" part and the "right" part, so that every edge has one endpoint in the left part, and one endpoint in the right part. The figure below shows an example of a bipartite graph. A *matching* means that there is a way of assigning every vertex in the "left" part to a vertex in the "right" part so that different vertices in the "left" part are assigned to different vertices in the "right" part and there is an edge between assigned vertices. In the graph from lecture, a matching will mean a way of assigning every man to a woman so that different men are assigned to different women, and a man is always assigned to a woman that he likes.



Problem 1

A **planar graph** is a graph that can be drawn without any edges crossing.

1. First, show that any subgraph of a planar graph is planar.

2. Also, any planar graph has a node of degree at most 5. Now, prove by induction that any graph can be colored in at most 6 colors.

Problem 2

An undirected graph G has ***width*** w if the vertices can be arranged in a sequence

$$v_1,\ v_2,\ v_3,\ \ldots,\ v_n$$

such that each vertex v_i is joined by an edge to at most w preceding vertices. (Vertex v_j *precedes* v_i if $j < i$.) Use induction to prove that every graph with width at most w is $(w+1)$-colorable.

(Recall that a graph is *k-colorable* iff every vertex can be assigned one of k colors so that adjacent vertices get different colors.)

Problem 3

A certain Institute of Technology has a lot of student clubs; these are loosely overseen by the Student Association. Each eligible club would like to delegate one of its members to appeal to the Dean for funding, but the Dean will not allow a student to be the delegate of more than one club. Fortunately, the Association VP took Math for Computer Science and recognizes a matching problem when she sees one.

1. Explain how to model the delegate selection problem as a bipartite matching problem. (This is a *modeling problem*; we aren't looking for a description of an algorithm to solve the problem.)

Problem 4

A ***Latin square*** is $n \times n$ array whose entries are the number $1, \ldots, n$. These entries satisfy two constraints: every row contains all n integers in some order, and also every column contains all n integers in some order. Latin squares come up frequently in the design of scientific experiments for reasons illustrated by a little story in a footnote[1]

For example, here is a 4×4 Latin square:

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

[1] At Guinness brewery in the eary 1900's, W. S. Gosset (a chemist) and E. S. Beavan (a "maltster") were trying to improve the barley used to make the brew. The brewery used different varieties of barley according to price and availability, and their agricultural consultants suggested a different fertilizer mix and best planting month for each variety.

Somewhat skeptical about paying high prices for customized fertilizer, Gosset and Beavan planned a season long test of the influence of fertilizer and planting month on barley yields. For as many months as there were varieties of barley, they would plant one sample of each variety using a different one of the fertilizers. So every month, they would have all the barley varieties planted and all the fertilizers used, which would give them a way to judge the overall quality of that planting month. But they also wanted to judge the fertilizers, so they wanted each fertilizer to be used on each variety during the course of the season. Now they had a little mathematical problem, which we can abstract as follows.

Suppose there are n barley varieties and an equal number of recommended fertilizers. Form an $n \times n$ array with a column for each fertilizer and a row for each planting month. We want to fill in the entries of this array with the integers 1,...,n numbering the barley varieties, so that every row contains all n integers in some order (so every month each variety is planted and each fertilizer is used), and also every column contains all n integers (so each fertilizer is used on all the varieties over the course of the growing season).

1. Here are three rows of what could be part of a 5×5 Latin square:

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

   Fill in the last two rows to extend this "Latin rectangle" to a complete Latin square.

2. Show that filling in the next row of an $n \times n$ Latin rectangle is equivalent to finding a matching in some $2n$-vertex bipartite graph.