Notes for Recitation 8

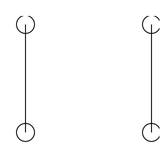
1 Build-up error

Recall a graph is ***connected*** iff there is a path between every pair of its vertices.

False Claim. *If every vertex in a graph has positive degree, then the graph is connected.*

1. Prove that this Claim is indeed false by providing a counterexample.

   Solution. There are many counterexamples; here is one:

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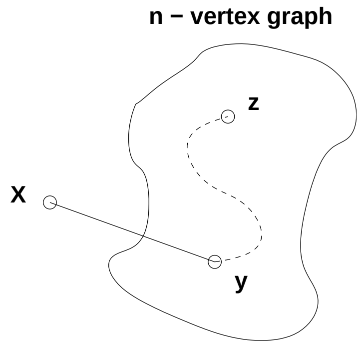
2. Since the Claim is false, there must be a logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

   Proof. We prove the Claim above by induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has positive degree, then the graph is connected.

   Base cases: $(n \leq 2)$. In a graph with 1 vertex, that vertex cannot have positive degree, so $P(1)$ holds vacuously.

   $P(2)$ holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

   Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 2$. Consider an n-vertex graph in which every vertex has positive degree. By the assumption $P(n)$, this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to obtain an $(n+1)$-vertex graph:



All that remains is to check that there is a path from x to every other vertex z. Since x has positive degree, there is an edge from x to some other vertex, y. Thus, we can obtain a path from x to z by going from x to y and then following the path from y to z. This proves $P(n+1)$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, which proves the Claim.

□

Solution. This one is tricky: the proof is actually a good proof of something else. The first error in the proof is only in the final statement of the inductive step: "This proves $P(n+1)$".

The issue is that to prove $P(n+1)$, *every* $(n+1)$-vertex positive-degree graph must be shown to be connected. But the proof doesn't show this. Instead, it shows that every $(n+1)$-vertex positive-degree graph *that can be built up by adding a vertex of positive degree to an n-vertex connected graph*, is connected.

The problem is that *not every* $(n+1)$-vertex positive-degree graph can be built up in this way. The counterexample above illustrates this: there is no way to build that 4-vertex positive-degree graph from a 3-vertex positive-degree graph.

More generally, this is an example of "buildup error". This error arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" in some particular way from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others— such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a "shrink down, grow back" process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds. Let's see what would have happened if we'd tried to prove the claim above by this method:

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$-vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh!

The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck— and properly so, since the claim is false! ■

2 The Grow Algorithm

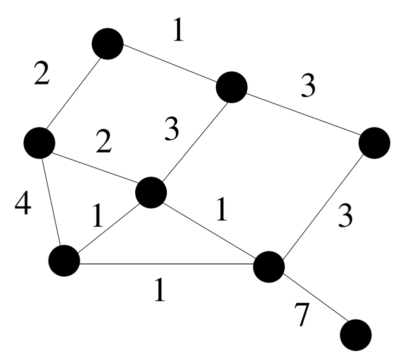
Yesterday in lecture, we saw the following algorithm for constructing a minimum-weight spanning tree (MST) from an edge-weighted N-vertex graph G.

ALG-GROW:

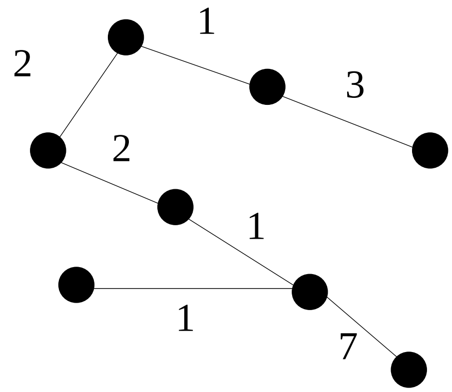
1. Label the edges of the graph $e_1, e_2, \ldots, e_t$ so that $wt(e_1) \leq wt(e_2) \ldots \leq wt(e_t)$.
2. Let S be the empty set.
3. For $i = 1 \ldots t$, if $S \cup \{e_i\}$ does not contain a cycle, then extend S with the edge e_i.
4. Output S.

In summary, ALG-GROW selects edges one at a time, always choosing the minimum weight edge that does not create a cycle with previously selected edges. Notice that as edges are added S may not be connected. When the algorithm terminates, S contains $N-1$ edges. If it is connected, then it is a spanning tree.

Consider, for example, the following edge-weighted graph.



Now suppose we run ALG-GROW on our graph. We may choose the weight 1 edge on the bottom of the triangle of weight 1 edges in our graph. In the next step, we may choose the weight 1 edge on the top of the graph. Note that this edge still has minimum weight, and does not cause us to form a cycle, so ALG-GROW can choose it. We will then choose one of the remaining weight 1 edges. Note that neither causes us to form a cycle. Continuing the algorithm, we may end up with the same spanning tree shown below.



In this recitation, we will analyze ALG-GROW.

3 Analysis of ALG-GROW

In this problem you may assume the following lemma from the problem set:

Lemma 1. *Suppose that $T = (V, E)$ is a simple, connected graph. Then T is a tree iff $|E| = |V| - 1$.*

In this exercise you will prove the following theorem.

Theorem. *For any connected, weighted graph G, ALG-GROW produces an MST of G.*

(a) Prove the following lemma.

Lemma 2. *Let $T = (V, E)$ be a tree and let e be an edge not in E. Then, $G = (V, E \cup \{e\})$ contains a cycle.*

(Hint: Suppose G does *not* contain a cycle. Is G a tree?)

Solution. *Proof.* (by contradiction) Suppose G does not contain a cycle. By the definition of a tree, T is connected. Notice that T is a subgraph of G. Because any two nodes in G are connected by a path in T, G is a connected graph. So G is connected and acyclic and therefore a tree by definition. Both G and T are trees and have the same number of nodes. Therefore, they have the same number of edges (by Lemma 1). This is a contradiction because G has one more edge than T. $\square$

$\blacksquare$

(b) Prove the following lemma.

Lemma 3. *Let $T = (V, E)$ be a spanning tree of G and let e be an edge not in E. Then there exists an edge $e' \neq e$ in E such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of G.*

(Hint: Adding e to E introduces a cycle in $(V, E \cup \{e\})$.)

Solution. *Proof.* By Lemma 2, we know that the set of edges $E \cup \{e\}$ contains a cycle. If this cycle does not contain the edge e, then this cycle is a subset of E. Since E is the set of edges of a tree, this cannot occur. So, this cycle contains e. If e' is another edge distinct from e in this cycle, then the graph T^* that results after removing e' from $E \cup \{e\}$ is still connected. The number of edges in T^* is equal to the number of edges in T, which is equal to $|V| - 1$ by Lemma 1. Since T^* is connected, T^* is a tree by Lemma 1. Since T^* is a subgraph of G with vertices V, it spans G. □

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(c) Prove the following lemma.

Lemma 4. *Let $T = (V, E)$ be a spanning tree of G, let e be an edge not in E and let $S \subseteq E$ such that $S \cup \{e\}$ does not contain a cycle. Then there exists an edge $e' \neq e$ in $E - S$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of G.*

(Hint: Modify your proof to part (b). Of all possible edges $e' \neq e$ that can be removed to construct T^*, at least one is not in S.)

Solution. *Proof.* We need to change the proof in part (b) slightly. The proof of part (b) holds for any edge $e' \neq e$ in the cycle. We need to show that we can select an edge $e' \neq e$ that is in the cycle but not in S. We will prove this by contradiction. Suppose that all the edges not equal to e that are in the cycle are in S. Then, $S \cup e$ is a cycle. This contradicts the assumption of the lemma. □

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(d) Prove the following lemma.

Lemma 5. *Define S_m to be the set consisting of the first m edges selected by ALG-GROW from a connected graph G. Let $P(m)$ be the predicate that if $m \leq |V|$ then $S_m \subseteq E$ for some MST $T = (V, E)$ of G. Then $\forall m \,.\, P(m)$.*

(Hint: Use induction. There are two cases: $m + 1 > |V|$ and $m + 1 \leq |V|$. In the second case, there are two subcases.)

Solution. *Proof.* (By induction.) Let $P(m)$ be the predicate as defined above.

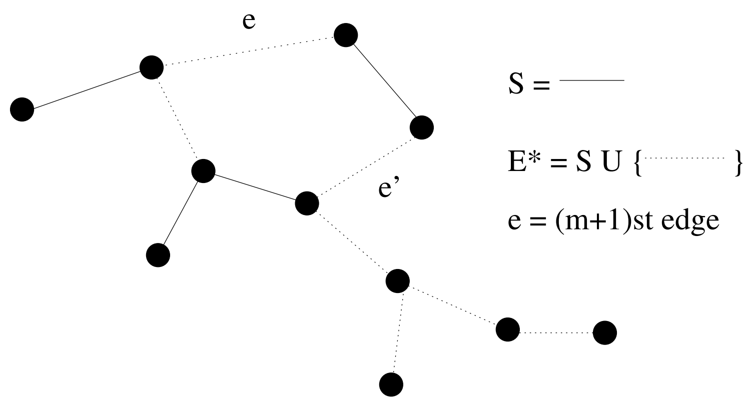
Base Case: S_0 contains 0 edges and is equal to the empty set, which is a subset of any set of edges E.

Inductive Step: Assume $P(m)$ in order to prove $P(m + 1)$.

If $m \geq |V|$ then $m+1 > |V|$ and $P(m+1)$ holds vacuously. Otherwise, if $m < |V|$ then let e denote the $(m+1)$th edge selected by ALG-GROW. By the inductive hypothesis, there exists an MST $T = (V, E)$ such that $S_m \subseteq E$. There are now two cases.

In the first case, $e \in E$ which case $S_m \cup \{e\} \subseteq E$, and thus $P(m + 1)$ holds.

In the second case, $e \notin E$, as illustrated by the following diagram. Now we need to find a different MST that contains $S_m \cup \{e\}$.



What happens when we add e to T? By the description of ALG-GROW, $S_m \cup \{e\}$ does not contain a cycle. Therefore, by Lemma 4, there exists an edge $e' \neq e$ in $E - S_m$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree for G.

In order to prove that T^* is a MST, we need to show that $wt(e) \leq wt(e')$. We will prove this by contradiction. Suppose that $wt(e') < wt(e)$. Since $e' \in E$, which is the set of edges of the MST T, and $S_m \subseteq E$, the set of edges $S_m \cup \{e'\}$, does not contain a cycle. Therefore e' would have already been added to S_m in a previous iteration of ALG-GROW as one of the first m edges. However, e' is in $E - S_m$. This is a contradiction. □

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(e) Prove the theorem. (Hint: Lemma 5 says there exists an MST $T = (V, E)$ for G such that $S \subseteq E$. Use contradiction to rule out the case in which S is a proper subset of E.)

Solution. *Proof.* (by contradiction) Let S be the set of edges produced by ALG-GROW. By Lemma 5, there exists an MST $T = (V, E)$ for G such that $S \subseteq E$. If $S = E$, then ALG-GROW outputs the edges of the MST T.

We will show that the other case, $S \neq E$, leads to a contradiction. Suppose $S \neq E$. Then there exists an edge $e \in E - S$. This implies that $S \cup \{e\} \subseteq E$. Since E is the set of edges of a tree, $S \cup \{e\}$ does not contain a cycle. Therefore, e would be added to S by ALG-GROW. So $e \in S$, and this contradicts $e \in E - S$.

□

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