Problem Set 7 Solutions

Due: Tuesday, November 1

Problem 1. [15 points] Express

$$\sum_{i=0}^{n} i^2 x^i$$

as a closed-form function of n.

Solution. We use the derivative method. Let us start with the following formula, derived in lecture (for $x \neq 1$):

$$\sum_{i=0}^{n} i x^i = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

Differentiating both sides:

$$\begin{aligned}
x^{-1} \sum_{i=0}^{n} i^2 x^i &= \frac{(1 - (n+1)^2 x^n + n(n+2)x^{n+1})(1-x)^2 - (x - (n+1)x^{n+1} + nx^{n+2})(2(1-x)(-1))}{(1-x)^4} \\
&= \frac{(1 - (n+1)^2 x^n + n(n+2)x^{n+1})(1-x) + 2(x - (n+1)x^{n+1} + nx^{n+2})}{(1-x)^3} \\
&= \frac{1 - (n+1)^2 x^n + n(n+2)x^{n+1} - x + (n+1)^2 x^{n+1} - n(n+2)x^{n+2}}{(1-x)^3} \\
&\quad + \frac{2x - 2(n+1)x^{n+1} + 2nx^{n+2}}{(1-x)^3} \\
&= \frac{1 + x - (n+1)^2 x^n + (n(n+2) + (n+1)^2 - 2(n+1))x^{n+1} + (2n - n(n+2))x^{n+2}}{(1-x)^3} \\
&= \frac{1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2}}{(1-x)^3}.
\end{aligned}$$

Multiplying both sides by x, we get

$$\sum_{i=0}^{n} i^2 x^i = \frac{x(1 + x - (n+1)^2 x^n + (2n^2 + 2n - 1)x^{n+1} - n^2 x^{n+2})}{(1-x)^3}.$$

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Problem 2. [20 points]

(a) [5 pts] What is the product of the first n odd powers of two: $\prod_{k=1}^{n} 2^{2k-1}$?

Solution.

$$\Pi_{k=1}^{n} 2^{2k-1} = 2^{\sum_{k=1}^{n} 2k-1} = 2^{2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1} = 2^{n(n+1)-n} = 2^{n^2}$$

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(b) [5 pts] Find a closed expression for

$$\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j}$$

Solution.

$$\begin{aligned}
\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j} &= \sum_{i=0}^{n} \left(3^i \cdot \sum_{j=0}^{m} 3^j \right) \\
&= \left(\sum_{j=0}^{m} 3^j \right) \cdot \left(\sum_{i=0}^{n} 3^i \right) \\
&= \left(\frac{3^{m+1} - 1}{2} \right) \cdot \left(\frac{3^{n+1} - 1}{2} \right)
\end{aligned}$$

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(c) [5 pts] Find a closed expression for

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (i + j)$$

Solution.

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{n} (i + j) &= \left(\sum_{i=1}^{n} \sum_{j=1}^{n} i \right) + \left(\sum_{i=1}^{n} \sum_{j=1}^{n} j \right) \\
&= \left(\sum_{i=1}^{n} ni \right) + \left(\sum_{i=1}^{n} \frac{n(n+1)}{2} \right) \\
&= \frac{2n^2(n+1)}{2} \\
&= n^2(n+1)
\end{aligned}$$

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(d) [5 pts] Find a closed expression for

$$\prod_{i=1}^{n}\prod_{j=1}^{n} 2^i \cdot 3^j$$

Solution.

$$\begin{aligned}
\prod_{i=1}^{n}\prod_{j=1}^{n} 2^i \cdot 3^j &= \left(\prod_{i=1}^{n} 2^{ni}\right)\left(\prod_{j=1}^{n} 3^{nj}\right) \\
&= 2^{n\sum_{i=1}^{n} i} 3^{n\sum_{j=1}^{n} j} \\
&= 2^{n^2(n+1)/2} 3^{n^2(n+1)/2}
\end{aligned}$$

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Problem 3. [10 points]

(a) [6 pts] Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

Solution. Let's first try standard bounds:

$$\int_1^{\infty} \frac{1}{(2x+1)^2}\,dx \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad f(1) + \int_1^{\infty} \frac{1}{(2x+1)^2}\,dx$$

Evaluating the integrals gives:

$$-\frac{1}{2(2x+1)}\bigg|_1^{\infty} \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + -\frac{1}{2(2x+1)}\bigg|_1^{\infty}$$

$$\frac{1}{6} \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{9} + \frac{1}{6}$$

These bounds are too far apart, so let's sum the first couple terms explicitly and bound the rest with integrals.

$$\frac{1}{3^2} + \int_2^{\infty} \frac{1}{(2x+1)^2}\,dx \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + f(2) + \int_2^{\infty} \frac{1}{(2x+1)^2}\,dx$$

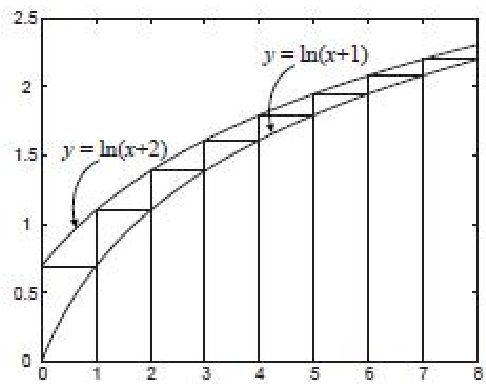
Integration now gives:

$$\frac{1}{3^2} + \left(-\frac{1}{2(2x+1)}\bigg|_2^{\infty}\right) \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + \frac{1}{5^2} + \left(-\frac{1}{2(2x+1)}\bigg|_2^{\infty}\right)$$

$$\frac{1}{3^2} + \frac{1}{10} \quad \leq \quad \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \quad \leq \quad \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{10}$$

Now we have bounds that differ by $1/5^2 = 0.04$. ■

(b) [4 pts] Assume n is an integer larger than 1. Which of the following inequalities, if any, hold. You may find the graph helpful.



1. $\displaystyle\sum_{i=1}^{n} \ln(i+1) \leq \int_{0}^{n} \ln(x+2)dx$
2. $\displaystyle\sum_{i=1}^{n} \ln(i+1) \leq \ln 2 + \int_{1}^{n} \ln(x+1)dx$

Solution. The 1st inequality holds. ■

Problem 4. [20 points] For each of the following six pairs of functions f and g (parts (a) through (f)), state which of these order-of-growth relations hold (more than one may hold, or none may hold):

$$f = o(g) \qquad f = O(g) \qquad f = \omega(g) \qquad f = \Omega(g) \qquad f = \Theta(g) \qquad f \sim g$$

(a)	$f(n) = \log_2 n$	$g(n) = \log_{10} n$
(b)	$f(n) = 2^n$	$g(n) = 10^n$
(c)	$f(n) = 0$	$g(n) = 17$
(d)	$f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$	$g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$
(e)	$f(n) = 1.0000000001^n$	$g(n) = n^{10000000000}$

Solution.

- $f(n) = \log_2 n$ and $g(n) = \log_{10} n$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{\ln n/\ln 2}{\ln n/\ln 10}\\ &= \frac{\ln 10}{\ln 2}\end{aligned}$$

So $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$ and $f(n) = \Theta(g(n))$.

- $f(n) = 2^n$ and $g(n) = 10^n$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{2^n}{10^n}\\ &= \lim_{n\to\infty}(1/5)^n\\ &= 0\end{aligned}$$

So $f(n) = o(g(n))$ and $f(n) = O(g(n))$.

- $f(n) = 0$ and $g(n) = 17$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \frac{0}{17}\\ &= 0\end{aligned}$$

So $f(n) = o(g(n))$ and $f(n) = O(g(n))$.

- $f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$ and $g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$:

For all $n \equiv 1 \pmod 4$, $f(n)/g(n) = 0$, so $f(n) \neq \Omega(g(n))$. Likewise, for all $n \equiv 0 \pmod 4$, $g(n)/f(n) = 0$, so $f(n) \neq O(g(n))$. The quotient never converges to some particular limit, so no relation holds.

- $f(n) = 1.0000000001^n$ and $g(n) = n^{10000000000}$:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| &= \lim_{n\to\infty}\frac{1.0000000001^n}{n^{10000000000}}\\ &= \lim_{n\to\infty}\frac{1.0000000001^n \ln 1.0000000001}{10000000000 n^{9999999999}}\\ &= \lim_{n\to\infty}\frac{1.0000000001^n (\ln 1.0000000001)^{10000000000}}{10000000000!}\\ &= \infty\end{aligned}$$

So $f(n) = \omega(g(n))$ and $f(n) = \Omega(g(n))$.

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