Notes for Recitation 16

The ***(ordinary) generating function*** for a sequence $\langle a_0, a_1, a_2, a_3, \ldots \rangle$ is the power series:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

1 Problem: Sequences to Generating Functions

Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) $\langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle$

Solution.

$$2 + 3x + 5x^2$$

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(b) $\langle 1, 1, 1, 1, 1, 1, 1, \ldots \rangle$

Solution.

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

■

(c) $\langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle$

Solution.

$$\begin{aligned} 1 + 2x + 4x^2 + 8x^3 + \ldots &= (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots \\ &= \frac{1}{1-2x} \end{aligned}$$

■

(d) $\langle 1, 0, 1, 0, 1, 0, 1, 0, \ldots \rangle$

Solution.

$$1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1-x^2}$$

■

(e) $\langle 0, 0, 0, 1, 1, 1, 1, 1, \ldots \rangle$

Solution.

$$x^3 + x^4 + x^5 + x^6 + \ldots = x^3(1 + x + x^2 + x^3 + \ldots) \qquad = \frac{x^3}{1-x}$$

■

(f) $\langle 1, 3, 5, 7, 9, 11, \ldots \rangle$

Solution.

$$\begin{aligned}
1 + x + x^2 + x^3 + \ldots &= \frac{1}{1-x} \\
\frac{d}{dx}\ 1 + x + x^2 + x^3 + \ldots &= \frac{d}{dx}\ \frac{1}{1-x} \\
1 + 2x + 3x^2 + 4x^2 + \ldots &= \frac{1}{(1-x)^2} \\
2 + 4x + 6x^2 + 8x^2 + \ldots &= \frac{2}{(1-x)^2} \\
1 + 3x + 5x^2 + 7x^3 + \ldots &= \frac{2}{(1-x)^2} - \frac{1}{1-x} \\
&= \frac{1+x}{(1-x)^2}
\end{aligned}$$

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2 Problem: Generating Functions to Sequences

Suppose that:

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ g(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots \end{aligned}$$

What sequences do the following functions generate?

(a) $f(x) + g(x)$

Solution.

$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + \ldots$$

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(b) $f(x) \cdot g(x)$

Solution.

$$a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \ldots + \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n + \ldots$$

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(c) $f(x)/(1 - x)$

Solution. This is a special case of the preceding problem part where:

$$\begin{aligned} g(x) &= \frac{1}{1-x} \\ &= 1 + x + x^2 + x^3 + x^4 + \cdots \end{aligned}$$

and so $b_0 = b_1 = b_2 = \ldots = 1$. In this case, we have:

$$f(x) \cdot g(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \ldots + \left(\sum_{k=0}^{n} a_k \right) x^n + \ldots$$

Thus, $f(x)/(1 - x)$ is the generating function for sums of prefixes of the sequence generated by f. ■

3 Problem: Candy Jar

There is a jar containing n different flavors of candy (and lots of each kind). I'd like to pick out a set of k candies.

(a) In how many different ways can this be done?

Solution. There is a bijection with sequences containing k zeroes (representing candies) and $n-1$ ones (separating the different varieties). The number of such sequences is:

$$\binom{n+k-1}{k}$$

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(b) Now let's approach the same problem using generating functions. Give a closed-form generating function for the sequence $\langle s_0, s_1, s_2, s_3, \ldots \rangle$ where s_k is the number of ways to select k candies when there is only $n = 1$ flavor available.

Solution.

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

■

(c) Give a closed-form generating function for the sequence $\langle t_0, t_1, t_2, t_3, \ldots \rangle$ where t_k is the number of ways to select k candies when there are $n = 2$ flavors.

Solution.

$$(1 + x + x^2 + x^3 + \ldots)^2 = \frac{1}{(1-x)^2}$$

■

(d) Give a closed-form generating function for the sequence $\langle u_0, u_1, u_2, u_3, \ldots \rangle$ where u_k is the number of ways to select k candies when there are n flavors.

Solution.

$$\frac{1}{(1-x)^n}$$

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4 Problem: Recurrence

Consider the following recurrence equation:

$$T_n = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 2T_{n-1} + 3T_{n-2} & (n \geq 2) \end{cases}$$

Let $f(x)$ be a generating function for the sequence $\langle T_0, T_1, T_2, T_3, \ldots \rangle$.

(a) Give a generating function in terms of $f(x)$ for the sequence:

$$\langle 1, \quad 2, \quad 2T_1 + 3T_0, \quad 2T_2 + 3T_1, \quad 2T_3 + 3T_2, \ldots \rangle$$

Solution. We can break this down into a linear combination of three sequences:

$$\begin{array}{ccccccccl} \langle & 1, & 2, & 0, & 0, & 0, & \ldots & \rangle & = 1 + 2x \\ \langle & 0, & T_0, & T_1, & T_2, & T_3, & \ldots & \rangle & = xf(x) \\ \langle & 0, & 0, & T_0, & T_1, & T_2, & \ldots & \rangle & = x^2 f(x) \end{array}$$

In particular, the sequence we want is very nearly generated by $1+2x+2xf(x)+3x^2f(x)$. However, the second term is not quite correct; we're generating $2 + 2T_0 = 4$ instead of the correct value, which is 2. We correct this by subtracting $2x$ from the generating function, which leaves:

$$1 + 2xf(x) + 3x^2 f(x)$$

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(b) Form an equation in $f(x)$ and solve to obtain a closed-form generating function for $f(x)$.

Solution. The equation

$$f(x) = 1 + 2xf(x) + 3x^2 f(x)$$

equates the left sides of all the equations defining the sequence $T_0, T_1, T_2, \ldots$ with all the right sides. Solving for $f(x)$ gives the closed-form generating function:

$$f(x) = \frac{1}{1 - 2x - 3x^2}$$

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(c) Expand the closed form for $f(x)$ using partial fractions.

Solution. We can write:

$$1 - 2x - 3x^2 = (1 + x)(1 - 3x)$$

Thus, there exist constants A and B such that:

$$f(x) = \frac{1}{1 - 2x - 3x^2} = \frac{A}{1 + x} + \frac{B}{1 - 3x}$$

Now substituting $x = 0$ and $x = 1$ gives the system of equations:

$$\begin{aligned} 1 &= A + B \\ -\frac{1}{4} &= \frac{A}{2} - \frac{B}{2} \end{aligned}$$

Solving the system, we find that $A = 1/4$ and $B = 3/4$. Therefore, we have:

$$f(x) = \frac{1/4}{1 + x} + \frac{3/4}{1 - 3x}$$

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(d) Find a closed-form expression for T_n from the partial fractions expansion.

Solution. Using the formula for the sum of an infinite geometric series gives:

$$f(x) = \frac{1}{4}\left(1 - x + x^2 - x^3 + x^4 - \ldots\right) + \frac{3}{4}\left(1 + 3x + 3^2x^2 + 3^3x^3 + 3^4x^4 + \ldots\right)$$

Thus, the coefficient of x^n is:

$$T_n = \frac{1}{4} \cdot (-1)^n + \frac{3}{4} \cdot 3^n$$

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5 Problem: Bouquet

You would like to buy a bouquet of flowers. You find an online service that will make bouquets of **lilies**, **roses** and **tulips**, subject to the following constraints:

- there must be at most 1 lily,
- there must be an odd number of tulips,
- there must be at least two roses.

Example: A bouquet of no lilies, 3 tulips, and 5 roses satisfies the constraints.

Express $B(x)$, the generating function for the number of ways to select a bouquet of n flowers, as a quotient of polynomials (or products of polynomials). You do not need to simplify this expression.

Solution. Generating function for the number of ways to choose lilies:

$$L(x) = 1 + x$$

Generating function for the number of ways to choose tulips:

$$T(x) = x + x^3 + x^5 + \cdots = \frac{x}{1 - x^2}$$

Generating function for the number of ways to choose roses:

$$R(x) = x^2 + x^3 + x^4 + \cdots = \frac{x^2}{1 - x}$$

By the Convolution Property, the generating function $B(x)$ is the product of these functions, namely,

$$\begin{aligned} B(x) &= L(x)R(x)T(x) \\ &= (1 + x)\frac{x}{1 - x^2}\frac{x^2}{1 - x} \\ &= \frac{x^3(1 + x)}{(1 + x)(1 - x)^2} \\ &= \frac{x^3}{(1 - x)^2} \end{aligned}$$

■