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Problem Set 9

Due: Thursday, November 6

Problem 1. **[10 points]**

(a) [5 pts] Show that of any $n + 1$ distinct numbers chosen from the set $\{1, 2, \ldots, 2n\}$, at least 2 must be relatively prime. (*Hint:* $\gcd(k, k + 1) = 1$.)

(b) [5 pts] Show that any finite connected undirected graph with $n \geq 2$ vertices must have 2 vertices with the same degree.

Problem 2. **[15 points]** **Under Siege!**

Fearing retribution for the many long hours his students spent completing problem sets, Prof. Leighton decides to convert his office into a reinforced bunker. His only remaining task is to set the 10-digit numeric password on his door. Knowing the students are a clever bunch, he is not going to pick any passwords containing the forbidden consecutive sequences "18062", "6042" or "35876" (his MIT extension).

How many 10-digit passwords can he pick that don't contain forbidden sequences if each number $0, 1, \ldots, 9$ can only be chosen once (i.e. without replacement)?

Problem 3. **[50 points]** Be sure to show your work to receive full credit. In this problem we assume a standard card deck of 52 cards.

(a) [5 pts] How many 5-card hands have a single pair and no 3-of-a-kind or 4-of-a-kind?

(b) [5 pts] How many 5-card hands have two or more kings?

(c) [5 pts] How many 5-card hands contain the ace of spades, the ace of clubs, or both?

(d) [5 pts] For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i = n$$

(e) [5 pts] For fixed positive integers n and k, how many nonnegative integer solutions $x_0, x_1, \ldots, x_k$ are there to the following equation?

$$\sum_{i=0}^{k} x_i \leq n$$

(f) [5 pts] In how many ways can $3n$ students be broken up into n groups of 3?

(g) [5 pts] How many simple undirected graphs are there with n vertices?

(h) [5 pts] How many directed graphs are there with n vertices (self loops allowed)?

(i) [5 pts] How many tournament graphs are there with n vertices?

(j) [5 pts] How many acyclic tournament graphs are there with n vertices?

Problem 4. [10 points] Suppose we have a deck of cards that has 4 suits, each suit having 13 cards. The magician asks the audience to select an arbitrary set of 7 cards. His assistant selects v cards out of the 7 cards and puts these v cards on a table. Is it possible for the magician to figure out the identities of the $7 - v$ remaining cards that are hidden from him (by only considering the v cards that his assistant put on the table)?

(a) [5 pts] Use a counting argument to show that for $v = 5$ the magician and assistant can work together such that the magician is able find the identities of the hidden cards.

(b) [5 pts] Is it possible to make the card trick work for $v = 4$? Explain your answer.

Problem 5. [15 points] Give a combinatorial proof of the following theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$

(*Hint: Consider the set of all length-n sequences of 0's, 1's and a single *.*)