Problem Set 7

Due: Thursday, October 23

Problem 1. [15 points] Express

$$\sum_{i=0}^{n} i^2 x^i$$

as a closed-form function of n.

Problem 2. [20 points]

(a) [5 pts] What is the product of the first n odd powers of two: $\prod_{k=1}^{n} 2^{2k-1}$?

(b) [5 pts] Find a closed expression for

$$\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j}$$

(c) [5 pts] Find a closed expression for

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (i+j)$$

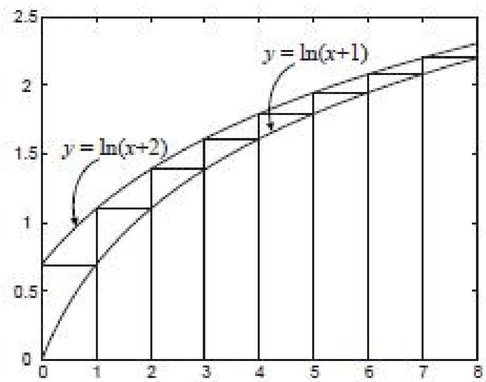
(d) [5 pts] Find a closed expression for

$$\prod_{i=1}^{n} \prod_{j=1}^{n} 2^i \cdot 3^j$$

Problem 3. [10 points]

(a) [6 pts] Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$



(b) [4 pts] Assume n is an integer larger than 1. Which of the following inequalities, if any, hold. You may find the graph helpful.

1. $\sum_{i=1}^{n} \ln(i+1) \leq \int_0^n \ln(x+2)dx$
2. $\sum_{i=1}^{n} \ln(i+1) \leq \ln 2 + \int_1^n \ln(x+1)dx$

Problem 4. [15 points] We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of n times.

(a) [10 pts] Describe a closed form formula for the amount of wine in the first glass after n back-and-forth pourings.

(b) [5 pts] What is the limit of the amount of wine in each glass as n approaches infinity?

Problem 5. [20 points] For each of the following six pairs of functions f and g (parts (a) through (f)), state which of these order-of-growth relations hold (more than one may hold, or none may hold):

$$f = o(g) \qquad f = O(g) \qquad f = \omega(g) \qquad f = \Omega(g) \qquad f = \Theta(g) \qquad f \sim g$$

(a)	$f(n) = \log_2 n$	$g(n) = \log_{10} n$
(b)	$f(n) = 2^n$	$g(n) = 10^n$
(c)	$f(n) = 0$	$g(n) = 17$
(d)	$f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$	$g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$
(e)	$f(n) = 1.0000000001^n$	$g(n) = n^{10000000000}$

Problem 6. [20 points] This problem continues the study of the asymptotics of factorials.

(a) [5 pts]

Either prove or disprove each of the following statements.

- $n! = O((n+1)!)$
- $n! = \Omega((n+1)!)$
- $n! = \Theta((n+1)!)$
- $n! = \omega((n+1)!)$
- $n! = o((n+1)!)$

(b) [5 pts] Show that $n! = \omega\left(\left(\frac{n}{3}\right)^{n+e}\right)$.

(c) [5 pts] Show that $n! = \Omega(2^n)$

(d) [5 pts] Show that

$$\sum_{k=1}^{n} k^6 = \Theta(n^7).$$