The Gunfight at the OK Corral

JAMES T. SANDEFUR
Georgetown University
Washington, DC 20057

The Movie. The final scene from the movie "The Good, the Bad, and the Ugly," is a three-way gunfight. Each gunfighter has exactly one bullet. They draw and fire, simultaneously, with each gunfighter firing at the better shot of his opponents.

In this article, we wish to consider a variation of this movie, in which the survivors of one round of the gunfight repeat the entire process (assuming there are at least two survivors). Thus, the gunfight continues, one round at a time, until there are less than two survivors. The questions we wish to answer are: for each gunfighter, what is the probability of surviving the entire gunfight? and what is the probability that no one survives?

This gunfight is an example of an absorbing Markov chain, and is discussed (as a three-way tank battle) in some finite mathematics books [1]. In this paper, we develop well-known results of absorbing Markov chains using the theory of difference equations, and then use these results to answer our questions. This is a natural approach in the sense that dynamical methods are used to analyze a dynamical model.

The Model. In this section, we will construct a mathematical model of the above gunfight. We need to have some information concerning each of the gunfighters. Let's designate our gunfighters as G, B and U (for Good, Bad, and Ugly). Suppose from previous observations we have determined (approximately) that: gunfighter G is a good shot and hits his target 60% of the time; gunfighter U is not as good a shot and hits his target 50% of the time; and that gunfighter B is a bad shot and hits his target only 30% of the time.

Thus, on the first round of the gunfight, G shoots at U, while U and B shoot at G. (Those who have seen the movie know that this is not the way it happened.)

We call each possible outcome of a round of the gunfight a **state**. There are 7 possible outcomes or states. The first is that no one survives the round, designated by N. We list each of the other states with the letters of the survivors; that is, G, U, B, GB, UB, and GUB.

Notice that GU is not a possible state. For GU to be a state, B must be shot first. But if all three gunfighters are alive, no one is shooting at B. Thus, B cannot be shot first. (We are excluding the possibility that B is hit by a ricocheting bullet.)

A state is called an **absorbing state** if, once that state is the result of one round, it will be the result of all following rounds. In other words, once that state is reached the game is over. The states, N, G, U, and B are absorbing states.

A state that is not an absorbing state is called a **nonabsorbing state**. These states are GB, UB, and GUB.

Let $p_1(n)$ be the probability that state N (nobody survives) occurs after n rounds of shots have been fired. Likewise, let $p_2(n)$, $p_3(n)$, and $p_4(n)$ be the probabilities that states G, U, and B have been reached after n rounds of shots have been fired, respectively. Let $q_1(n)$, $q_2(n)$, and $q_3(n)$ be the probabilities that the nonabsorbing states GB, UB, and GUB, respectively, are the result of the nth round of the gunfight.

Let's compute $p_1(n+1)$. There are three cases (or ways) in which nobody survives round $n+1$. The first case is that nobody survives round n (with probability $p_1(n)$). The second case is that only G and B survive round n (with probability $q_1(n)$) and then G and B shoot each other. The third case is that only U and B survive round n (with probability $q_2(n)$) and then U and B shoot each other.

The second case is a three-stage process. The first stage is that only G and B survive round n ($q_1(n)$). The second stage is that G shoots B (with probability .6). The third stage is that B shoots G (with probability .3). Observe that stages 2 and 3 occur simultaneously. Thus, the probability of the second case occurring is (using the multiplication principle for independent events) $.18q_1(n)$. Likewise, using three stages, we get that the probability of the third case is $(.5)(.3)q_2(n) = .15q_2(n)$. Adding the probabilities of the three cases, we get that

$$p_1(n+1) = p_1(n) + .18q_1(n) + .15q_2(n).$$

In a similar manner,

$$p_2(n+1) = p_2(n) + .42q_1(n),$$

and

$$p_3(n+1) = p_3(n) + .35q_2(n).$$

To get $p_4(n+1)$, the probability that only B survives the nth round, we must consider four cases. Three of them are easy. It is more difficult to compute the fourth case: that G, U, and B survive round n; that G shoots U; and that U and/or B shoots G.

Case 4 is a three-stage process. The first stage is that all three survive round n, which occurs with probability $q_3(n)$. The second stage is that G shoots U, which occurs with probability .6. The third stage is that U and/or B shoots G. The easiest way to compute this is to use the complement rule; that is, the probability we seek is 1 minus the probability that neither U nor B shoots G. The probability that U misses and B also misses is .5(.7). Thus the probability of the third stage is $1 - .35 = .65$. The product of the three stages, $.6(.65)q_3(n) = .39q_3(n)$, is the probability of the fourth case. Thus,

$$p_4(n+1) = p_4(n) + .12q_1(n) + .15q_2(n) + .39q_3(n).$$

In a similar manner,

$$q_1(n+1) = .28q_1(n) + .21q_3(n),$$

$$q_2(n+1) = .35q_2(n) + .26q_3(n),$$

and

$$q_3(n+1) = .14q_3(n).$$

We now notice that we can write the last three equations in the matrix form

$$Q(n+1) = RQ(n), \tag{1}$$

where

$$Q(n)=\begin{pmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{pmatrix}, \quad \text{and } R=\begin{pmatrix} .28 & 0 & .21 \\ 0 & .35 & .26 \\ 0 & 0 & .14 \end{pmatrix}.$$

Note that $q_1(0)=0$, $q_2(0)=0$, and $q_3(0)=1$, since the probability is 1 that *GUB* are alive before the first round of the gunfight. Thus, we are given $Q(0)$. Equation (1) is a **first-order linear system of difference equations,** and $Q(0)$ is the **initial value** of the system of difference equations. Observe that we could now compute $Q(1)$, $Q(2)$, and so forth.

In a similar fashion, we can rewrite the first four equations as

$$P(n+1)=P(n)+SQ(n), \tag{2}$$

where

$$P(n)=\begin{pmatrix} p_1(n) \\ p_2(n) \\ p_3(n) \\ p_4(n) \end{pmatrix}, \quad \text{and } S=\begin{pmatrix} .18 & .15 & 0 \\ .42 & 0 & 0 \\ 0 & .35 & 0 \\ .12 & .15 & .39 \end{pmatrix}.$$

Note that $p_j(0)=0$ for $j=1$, 2, 3, and 4, so we are given $P(0)$. Since we (theoretically) know $Q(0)$, $Q(1)$, and so forth, we can compute $P(1)$, $P(2)$, and so forth.

Equation (2) is an example of an **absorbing Markov chain,** although in most undergraduate texts it is written in a different form. We will give a precise definition later.

We observe that the vector, P, given by

$$P=\lim_{n\to\infty} P(n),$$

gives the probabilities of each of the 4 absorbing states being the (eventual) outcome of the gunfight (if this limit exists). We could approximate P by computing $P(1)$, $P(2)$, and so forth.

The Method. We will find a closed form solution to equation (2), and use this to compute the vector, P. We will find the solution using the **method of undetermined coefficients.**

Suppose we are given a first-order linear difference equation

$$q(n+1)=rq(n),$$

as well as the initial value, $q(0)$. Induction shows that $q(1)=rq(0)$, $q(2)=rq(1)=r^2q(0)$, and

$$q(k)=r^kq(0).$$

Consider the first-order nonhomogeneous difference equation,

$$p(n+1)=sp(n)+ar^n,$$

where $p(0)$ is given and $s\neq r$. Using the method of undetermined coefficients, we try to find a **particular solution** to the equation of the form $p(n)=br^n$. We substitute this "guess" into the difference equation and solve for the constant b. In this case, $b = a/(r-s)$. The **general solution** is then found by adding the solution to the linear equation, $p(n+1) = sp(n)$, to this particular solution, giving

$$p(n) = cs^n + br^n.$$

We then use $p(0)$ to compute $c = p(0) - b$, to get the particular solution to the nonhomogeneous difference equation.

Suppose we have a Markov chain in which the nonabsorbing states satisfy the system of equations

$$Q(n+1) = RQ(n). \tag{1}$$

We note that the solution to this system of difference equations is

$$Q(n) = R^n Q(0).$$

Suppose that the absorbing states satisfy the system of equations

$$P(n+1) = P(n) + SQ(n). \tag{2}$$

Substituting $R^n Q(0)$ for $Q(n)$ gives the nonhomogeneous system of difference equations

$$P(n+1) = P(n) + SR^n Q(0). \tag{3}$$

Assume that the matrix, R, is such that R^n goes to zero; that is, each component of R^n goes to zero. We then call (3) an **absorbing Markov chain**. (R^n going to zero implies that $Q(n)$, the probabilities of being in each of the nonabsorbing states after n rounds, goes to zero.)

Alternatively, an absorbing Markov chain could be defined as a Markov chain in which there is at least one absorbing state, and that you can reach some absorbing state from each of the nonabsorbing states. These definitions are equivalent, although the proof of this fact is beyond the scope of this paper. The advantage of our definition is that if R^n goes to zero, then it is clear that $(I - R)$ is invertible.

The general solution of the linear part of (3) is $P(n) = A$, where A is a constant vector. By the method of undetermined coefficients, we look for a solution to the nonhomogeneous difference equation of the form

$$P(n) = S(R^n)C.$$

Since

$$P(n+1) = S(R^{n+1})C = S(R^n)RC,$$

then substitution into (3) gives

$$S(R^n)RC = S(R^n)C + S(R^n)Q(0).$$

Bringing all the terms to the left gives

$$S(R^n)RC - S(R^n)C - S(R^n)Q(0) = 0,$$

(where 0 is the zero vector). Factoring out $S(R^n)$ gives

$$S(R^n)(RC - C - Q(0)) = 0.$$

Thus, the equation is satisfied if

$$RC - C - Q(0) = 0; \quad \text{that is,} \quad \text{if } (R - I)C = Q(0),$$

or, after multiplying both sides by $(R - I)^{-1}$ on the left,

$$C = (R - I)^{-1}Q(0).$$

Thus, adding this particular solution to the nonhomogeneous difference equation to the general solution to the linear difference equation, implies that the general solution to (3) is

$$P(n) = A + S(R^n)(R - I)^{-1}Q(0).$$

Since $P(0)$ is the zero vector and $R^0 = I$,

$$P(0) = A + S(R^0)(R - I)^{-1}Q(0),$$

or

$$A = S(I - R)^{-1}Q(0).$$

Hence, the particular solution to the system (3) is

$$P(n) = S(I - R^n)(I - R)^{-1}Q(0).$$

Since R^n goes to 0 as n gets large,

$$\lim_{n \to \infty} P(n) = S(I - R)^{-1}Q(0).$$

In summary, we have derived the following well-known result. Suppose we have an absorbing Markov chain, given by the systems of difference equations (1) and (2). Suppose we *start* in one of the nonabsorbing states, given by the vector $Q(0)$, which will have a 1 in the position corresponding to the starting nonabsorbing state and 0's in all other positions. Then the probabilities of *eventually ending* in each of the absorbing states are given by the vector

$$P = S(I - R)^{-1}Q(0).$$

Remark. Suppose, instead of starting in one nonabsorbing state, you randomly choose your nonabsorbing state by flipping a coin, rolling a die, drawing a marble, or by some other random method. Then in the above computation, you only need to compute the correct $Q(0)$, where each component, $q_j(0)$, is the probability of picking the jth nonabsorbing state as the initial state.

To solve our gunfighter problem we compute

$$(I - R)^{-1} = \begin{pmatrix} 25/18 & 0 & 175/516 \\ 0 & 20/13 & 20/43 \\ 0 & 0 & 50/43 \end{pmatrix}$$

and

$$S(I - R)^{-1} = \begin{pmatrix} 1/4 & 3/13 & 45/344 \\ 7/12 & 0 & 49/344 \\ 0 & 7/13 & 7/43 \\ 1/6 & 3/13 & 97/172 \end{pmatrix}$$

$S(I-R)^{-1}Q(0)$ is the last column of the matrix $S(I-R)$.$^{-1}$ Therefore, the probability that everyone dies is $45/344 = 0.13$; that G wins is 0.14; that U wins is 0.16; and that the worst gunfighter, B, wins is 0.56. (There is .01 roundoff error.)

Remark. Let the vector, $A(n)$, be the expected number of times that we are in each nonabsorbing state after n rounds. The system of difference equations that gives $A(n)$ is

$$A(n+1) = A(n) + RQ(n).$$

Using the techniques given above, we get that

$$A(n) = (I-R)^{-1}(I-R^{n+1})Q(0).$$

Since R^n goes to zero, we have derived the well-known result that

$$A = (I-R)^{-1}Q(0)$$

gives the expected number of times the process will be in each of the nonabsorbing states. The sum of the components of A, 1.96 in this example, gives the expected number of rounds the gunfight will last.

The Motive. The standard method for finding P, the probabilities of absorption into each of the absorbing states, is by constructing and solving a system of linear equations [1]. This is a static method, in which the probabilities are assumed to exist and then computed. Students can understand the mathematics involved in this approach, while still not being intuitively sure why the answer is right.

The method used in this paper is dynamic. Loosely speaking, in a dynamic process the result after a finite time evolves toward the answer. For example, $P(n)$ converges to P in the discussion above.

Discrete dynamics, in the form of difference equations or recurrence relations, is a natural (but underused) approach to many mathematical models. This approach is easily accessible to those with only a good algebra background. For a more complete survey of elementary discrete dynamical methods, see [2].

Personally, I would like to see introductory mathematics courses (both for science and nonscience majors) use dynamical methods more extensively. This would help students to think in terms of "cause and effect", as well as to see the value of mathematics. We might even find that our students enjoy mathematics.

REFERENCES

1. A. W. Goodman and J. S. Ratti, *Finite Mathematics with Applications*, Macmillan, New York, 1975.
2. J. T. Sandefur, *Discrete Mathematics with Finite Difference Equations*, Lecture Notes, Mathematics Department, Georgetown University, 1983.