Quiz 1

- The quiz is **closed book**, but you may have one $8.5'' \times 11''$ sheet with notes (either printed or in your own handwriting) on both sides.
- Calculators and electronic devices (including cell phones) are not allowed.
- You may assume all of the results presented in class. This does **not** include results demonstrated in practice quiz material.
- Write your name on each page of the exam
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

NAME: ____________________

TA: ____________________

Problem	**Value**	**Score**	**Grader**
1	10		
2	10		
3	10		
4	20		
5	10		
6	10		
7	10		
8	10		
9	10		
Total	100		

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Problem 1. [10 points]

(a) [4 pts] Use a truth table to prove or disprove $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$.

(b) Translate the following statements from English into propositional logic or vice versa. You may use Prime(p) to denote that p is prime (i.e., Prime(p) is True if and only if p is a prime number) in the logical statements.

1. [2pts] If $n > 1$, then there is always at least one prime p such that $n < p < 2n$. n is an integer.

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2. [2pts] The domain is $\mathbb{N}$. $\forall m \; \exists \; p > m.$ Prime$(p) \wedge$ Prime$(p+2)$

3. [2pts] Let T be the set of TA's, S be the set of students, and $G(x, y) := x$ grades y's exam

$$\exists \; t \in T \; \forall s \in S. \; G(t, s)$$

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Problem 2. [10 points] Prove by induction that $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$ for all integers $n \geq 1$

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Problem 3. [10 points] Consider the matrix below.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Suppose we are allowed to flip all of the signs of entries in any row or column. For example, flipping the signs of elements in column two will give the matrix:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Prove that no matter how many operations you perform, there will always be at least one negative entry in the matrix. Do this by identifying an invariant and explain why you can use this invariant to show that at least one -1 will remain in the matrix.

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Problem 4. [20 points]

(a) [5 pts] Evaluate 2^{6042} mod 63.

(b) [3 pts] What is 63^{6042} mod 6043? (Hint: 6043 is prime; don't do a messy calculation–it will just waste your time.)

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(c) [6 pts] Give a proof by contradiction that 33 does not have an inverse mod 121.

(d) [6 pts] Find the inverse of 32 mod 121 in the range $\{1, 2, \ldots 120\}$. (Hint: use the Pulverizer)

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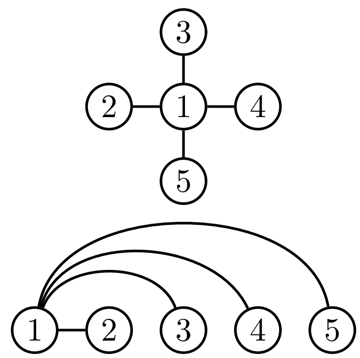
Problem 5. [10 points] A simple graph is said to have width k if you can order the nodes on a straight line so that each node is adjacent to at most k nodes to the left. Each node can be adjacent to any number of nodes to the right.

For example, the star graph with 5 nodes shown below has width 1 as can be seen from the ordering shown below the graph.



Prove by induction that any simple graph with width k can be colored in at most k+1 colors. (Hint: do not induct on k)

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Problem 6. [10 points] The following questions concern the following preferences.

girls	*boys*
Wendy : Stan, Kenny, Butters, Eric	Stan: Wendy, Bebe, Heidi, Annie
Bebe : Kenny, Eric, Butters, Stan	Kenny : Heidi, Wendy, Bebe, Annie
Heidi : Eric, Butters, Stan, Kenny	Butters : Bebe, Wendy, Heidi, Annie
Annie: Butters, Stan, Eric, Kenny	Eric : Bebe, Heidi, Wendy, Annie

(a) [3 pts] Is the following a stable marriage? If not, list a rogue couple.

Wendy – Stan, Bebe – Eric, Heidi – Butters, Annie – Kenny

(b) [7 pts] Find the matching produced by the stable matching algorithm.

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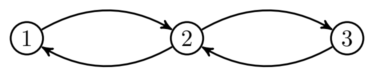
Problem 7. [10 points] Consider the following graph:



Suppose that each node starts with a PageRank value of $\frac{1}{3}$.

(a) [3 pts] What weights will the nodes have after one iteration of the Pagerank algorithm?

(b) [7 pts] What will be the PageRank values of each node for the stationary distribution?

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Problem 8. [10 points] Consider the following relation:

$$R = \{(x, y) : x + y = 0, x, y \in \mathbb{R}\}.$$

Which of the following properties holds for R? If it has the property, prove it. If not, provide a counterexample.

(a) [2 pts] Symmetry.

(b) [2 pts] Antisymmetry.

(c) [2 pts] Reflexivity.

(d) [2 pts] Transitivity.

(e) [2 pts] The property of being an equivalence relation.

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Problem 9. [10 points] Prove that a simple graph (i.e. undirected edges) cannot have an odd number of vertices with an odd degree. (Recall that the degree of a vertex is the number of edges incident to that vertex).