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Quiz 2

Problem 1. [15 points] Circle every symbol on the left that could correctly appear in the box to its right. For each of the six parts you may need to circle any number of symbols.

(**a**) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$6n^2 + 7n - 10 = \boxed{O, \Omega, \Theta}\left(n^2\right)$$

(**b**) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$6^n = \boxed{\Omega, \omega}\left(n^6\right)$$

(**c**) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$n! = \boxed{O, o}\left(n^n\right)$$

(**d**) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$\sum_{j=1}^{n} \frac{1}{j} = \boxed{O, \Omega, \Theta, \sim}(\ln n)$$

(**e**) $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$

$$\ln(n^3) = \boxed{O, \Omega, \Theta}(\ln n)$$

Problem 2. [10 points] Give upper and lower bounds for the following expression which differ by at most 1.

$$\sum_{i=1}^{n} \frac{1}{i^3}$$

Solution. To find upper and lower bounds, we use the integral method:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{i^3} &\leq 1 + \int_1^n \frac{1}{x^3} dx \\
&= 1 - \frac{1}{2} x^{-2} \Big|_1^n \\
&= 1 - \frac{1}{2}\left(\frac{1}{n^2} - 1\right) = \frac{3}{2} - \frac{1}{2n^2} \\
\sum_{i=1}^{n} \frac{1}{i^3} &\geq \frac{1}{n^3} + \int_1^n \frac{1}{x^3} dx \\
&= \frac{1}{n^3} - \frac{1}{2} x^{-2} \Big|_1^n \\
&= \frac{1}{n^3} - \frac{1}{2}\left(\frac{1}{n^2} - 1\right) = \frac{1}{2} + \frac{1}{n^3} - \frac{1}{2n^2}
\end{aligned}$$

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Problem 3. [10 points] Let $T(n)$ be a recurrence such that for all integers $n > 8$,

$$T(n) = 16T(\lfloor n/2 + \log n \rfloor) + n^4$$

Assume that $T(n) = 0$ for $n \leq 8$. Find a Θ bound for $T(n)$. Show your work.

Solution. Use Akra-Bazzi: $a_1 = 16$, $b_1 = 1/2$, $h_1(n) = \lfloor n/2 + \log n \rfloor - n/2$, $g(n) = n^4$, $p = 4$,

$$T(n) = \Theta\left(n^4\left(1 + \int_1^n \frac{u^4}{u^5} du\right)\right) = \Theta\left(n^4\left(1 + \int_1^n \frac{1}{u} du\right)\right) = \Theta(n^4 \log n)$$

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Problem 4. [10 points] At the end of year 0, Karen and Joe both have no money. In each subsequent year, the following happens:

1. On November 15th, Joe, an extremely good investor, has three times the amount of money he had at the beginning of the year.
2. On December 1st, Joe gives Karen the amount of money he had at the beginning of the year.
3. On December 15th, Karen makes $10, which she promptly gives to Joe.

Find a linear recurrence for the *total* amount of money T_n that the two have between them at the end of year n, including base cases. You do not have to solve the recurrence.

You may choose to define recurrences, K_n and J_n, for the amount of money that each of Karen and Joe have, respectively, but your final answer must be solely in terms of T_n.

(*Hint: First find an expression for the amount of money Joe has at the end of year* $n-1$, J_{n-1}, *in terms of* T_{n-1} *and* T_{n-2}.)

Solution. At the end of year n, Karen has the amount she had at the end of year $n-1$, plus the amount that Joe had at the beginning of year n (which is the same as the amount he had at the end of year $n-1$). Hence, $K_n = K_{n-1} + J_{n-1}$.

At the end of year n, Joe has three times the amount he had at the beginning of the year, minus the amount he had at the beginning of the year, plus the $10 Karen gave him. Hence, $J_n = 3J_{n-1} - J_{n-1} + 10 = 2J_{n-1} + 10$.

Now to determine the total that the two have, $T_n = K_n + J_n$. According to the hint, we rearrange the equation as $J_n = T_n - K_n$. Notice that $K_n = T_{n-1} = K_{n-1} + J_{n-1}$. Hence, we can substitute for K_n to get the equation $J_n = T_n - T_{n-1}$.

Finally, we substitute back into the equation for Joe to get:

$$\begin{aligned} J_n &= 2J_{n-1} + 10 \\ T_n - T_{n-1} &= 2(T_{n-1} - T_{n-2}) + 10 \\ T_n &= 3T_{n-1} - 2T_{n-2} + 10 \end{aligned}$$

The base cases are $T_0 = 0$ and $T_1 = 10$.

Note that this is one possible recurrence for T_n; other equivalent recurrences are possible.

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Problem 5. [10 points]

Let $T(n)$ be defined by the recurrence

$$T(n) = 2\sqrt{T(n-1)T(n-2)}$$

for $n \geq 2$ with $T(0) = T(1) = 1$. Prove by induction that $T(n) = O(2^{2n/3})$.

Solution. *Proof.* Proof by strong induction.

Let $P(n)$ be the proposition that $T(n) \leq c2^{2n/3}$, where c is a very large constant, say 100.

Base cases: $T(0) = 1 \leq c$, $T(1) = 1 \leq c2^{2/3}$.

Inductive step: Assume $P(k)$ for $0 \leq k \leq n$ in order to prove $P(n+1)$. We derive

$$\begin{aligned} T(n+1) &= 2\sqrt{T(n)T(n-1)} \\ &\leq 2\sqrt{c2^{2n/3} \cdot c2^{2(n-1)/3}} \quad \text{(By } P(n) \text{ and } P(n-1).\text{)} \\ &= c2^{1+(2n/3+2n/3-2/3)/2} \\ &= c2^{2(n+1)/3}. \end{aligned}$$

This proves $P(n+1)$. □

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Problem 6. [10 points] Solve the recurrence $T(n) = T(n-1) + 12T(n-2)$ for $n \geq 2$ with $T(0) = 2$ and $T(1) = 1$.

Solution. The characteristic polynomial is $x^2 = x + 12$ with solutions $x = 4$ and $x = -3$. This gives $T(n) = c_1 4^n + c_2(-3)^n$ for $n \geq 0$.

Plugging in the base cases, we get the system of equations

$$\begin{aligned} 2 &= c_1 + c_2 \\ 1 &= 4c_1 - 3c_2 \end{aligned}$$

Hence, $c_1 = c_2 = 1$, so $T(n) = 4^n + (-3)^n$. ■

Problem 7. [10 points]

How many length $2n$ sequences of red and green balls with exactly m red balls satisfy the constraint that every red ball has a green ball adjacent to it on its left? You may assume that $m \leq n$. Express your answer using a single binomial coefficient; that is, using a single term of the form $\binom{x}{y}$.

Solution. Let B be the set of length $2n$ sequences with m red and $2n - m$ green balls such that every red ball has a green ball on its left. Let A be the set of length $2n - m$ bit sequences with exactly m ones and $2n - 2m$ zeroes. Map a sequence from A to a sequence in B by mapping each 0 to a green ball and each 1 to a green ball followed by a red ball. This mapping is a bijection. By the bijection rule, $|B| = |A| = \binom{2n-m}{m}$. ■

Problem 8. [10 points]

Find a combinatorial proof of the following identity by counting the number of pairs of sets (X, Y) such that $X \subseteq Y \subseteq \{1, 2, \ldots, n\}$ and $|Y| = m$:

$$\binom{n}{m} 2^m = \sum_{i=0}^{m} \binom{n}{i} \binom{n-i}{m-i}$$

Solution. Let's consider each side of the equation.

Looking at the left side:

One way to count the number of pairs of sets (X, Y) is to first count the number of possible sets Y, and then count the number of possible sets X for each Y.

The number of sets $Y \subseteq \{1, 2, \ldots, n\}$ with $|Y| = m$ is equal to $\binom{n}{m}$. For each Y, every value in Y can either be in X or not be in X. Since $|Y| = m$, the number of sets X such that $X \subseteq Y$ is equal to the number of binary bit strings of length m, which equals 2^m.

By the generalized product rule, the number of pairs of sets (X, Y) such that $X \subseteq Y \subseteq \{1, 2, \ldots, n\}$ and $|Y| = m$ is equal to $\binom{n}{m} 2^m$.

Looking at the right side:

Another way to count the number of pairs of sets (X, Y) is to first count the number of possible sets X, and then count the number of possible sets Y for each X.

Let $|X| = i$, where i must range from 0 to m, since $X \subseteq Y$. For each value i, the number of sets $X \subseteq \{1, 2, \ldots, n\}$ with $|X| = i$ is equal to $\binom{n}{i}$. For each X, we must pick the remaining $m - i$ elements from the remaining $n - i$ possible values to complete the set Y. Therefore, the number of sets Y such that $X \subseteq Y$ is equal to $\binom{n-i}{m-i}$.

By the generalized product rule, the number of pairs of sets (X, Y) such that $X \subseteq Y \subseteq \{1, 2, \ldots, n\}$ and $|Y| = m$ is also equal to $\sum_{i=0}^{m} \binom{n}{i} \binom{n-i}{m-i}$.

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Problem 9. [15 points]

Your answers for the following questions may contain binomial coefficients of the form $\binom{x}{y}$, factorials, additions, multiplications, and divisions. Please explain your terms for partial credit.

(a) [5 pts] How many 8-digit decimal sequences satisfy the following constraints?

1. Exactly four of the possible digits (the numbers from 0 to 9) appear.
2. One of the digits appears *exactly* five times.

For example, 01921111 is such a sequence.

Solution. If one of the four digits appears *exactly* five times, then each of the other three digits must appear only once each (such as in 11111234). There are 10 choices for the repeated digit and $\binom{9}{3}$ choices for the remaining digits. By the bookkeeper rule, these digits may be arranged in $\frac{8!}{5!1!1!1!}$ ways. Thus, the number of such sequences is $10\binom{9}{3}\frac{8!}{5!1!1!1!} = 282,240$. ■

(b) [10 pts] How many 8-digit decimal sequences satisfy the following constraints?

1. Exactly four of the possible digits appear.
2. One of the digits appears *at least* four times.

For example, 01921111 and 01921112 are both such sequences.

Solution. If one of the four digits apears *at least* four times, then it must be the case that either one digit appears five times as in the previous part, *or* one digit appears exactly four times, a second digit appears exactly twice, and each of the other two digits appears only once each (such as in 11112234).

In the new case, there are 10 choices for the digit that appears four times, 9 remaining choices for the digit that appears twice, and $\binom{8}{2}$ remaining choices for the other digits. By the bookkeeper rule, these digits may be arranged in $\frac{8!}{4!2!1!1!}$ ways. Thus, there are $10 \cdot 9 \cdot \binom{8}{2}\frac{8!}{4!2!1!1!}$ such sequences.

Therefore, the total number of sequences is the sum of the two disjoint sets, which equals $10 \cdot 9 \cdot \binom{8}{2}\frac{8!}{4!2!1!1!} + 10\binom{9}{3}\frac{8!}{5!1!1!1!} = 2,399,040$. ■