Midterm Practice Problems

Problem 1. [12 points] Let X be the set of students in 6.042. Let Y be the set of problems on this quiz. For $x \in X$ and $y \in Y$, let $P(x, y)$ be the statement "Student x got full points on problem y". Let $Q(x)$ be the statement "Student x drops 6.042".

(a) [6 pts] Convert the following statements into English.

1. $(\exists x \in X, Q(x)) \Rightarrow (\forall x \in X, Q(x))$.

   Solution. If a student in 6.042 drops the class, then all the students in 6.042 will drop the class. ∎

2. $\forall x \in X((\exists y \in Y \neg P(x, y)) \Rightarrow Q(x))$

   Solution. For every student in 6.042, if there is a problem on the quiz that he / she doesn't get full points on, he / she will drop the class. ∎

3. $\exists x \in X(\neg Q(x))$.

   Solution. There is a student in 6.042 who won't drop the class. ∎

(b) [6 pts] Assuming 1,2 and 3 are true, what can you say about your score on this quiz?

Solution. I will get full points on the quiz. ∎

Problem 2. [20 points] 6 people are sitting in a circle. Each person has a number. They do a ritual during each round that makes their numbers update. Each person, to get his number for round $n + 1$, takes his number from round n, then adds his right neighbor's number from round n to it, and then subtracts his left neighbor's number from round n from it.

For example if on the current round there is a person A whose number is 5, and A's right neighbor's number is 4, and A's left neighbor's number is 6, then in the following round, A's number will be $5 + 4 - 6 = 3$.

Initially the people are given the numbers $1, 2, 3, 4, 5, 6$ (in clockwise order, and the person with 6 is sitting next to the person with 1).

Thus, after 1 round, their numbers will be $5, 0, 1, 2, 3, 10$.

Prove that there will never be a round when all their numbers are equal.

BIG HINT: Consider what happens to the sum of the numbers.

Solution.

Lemma 1. *After every round, each number is an integer, and the sum of the numbers is 21.*

Proof. The inductive hypothesis $P(n)$ states that after round n, the sum of the numbers is 21 and all numbers are integers.

The base case $P(0)$ follows from summing up the values of the initial settings of the numbers and noting that all initial values are integers.

Assuming that $P(n)$ holds, we show that $P(n + 1)$ holds. Let the values of the numbers at round n be $a_1, \ldots, a_6$. For each i, the new value of the ith person is $a_i + a_{i+1 \text{ rem } 6} - a_{i-1 \text{ rem } 6}$. Since the inductive hypothesis tells us that $a_i, a_{i+1 \text{ rem } 6}$ and $a_{i-1 \text{ rem } 6}$ are all integers, the new value of the ith person is also an integer. The sum at round $n + 1$ is

$$\sum_{1 \leq i \leq 6} a_i + a_{i+1 \text{ rem } 6} - a_{i-1 \text{ rem } 6} = \sum_{1 \leq i \leq 6} a_i + a_i - a_i = \sum_{1 \leq i \leq 6} a_i = 21.$$

The second equality is by reordering the summation, and the third equality is from the inductive hypothesis. And so $P(n + 1)$ holds by the principle of induction. $\square$

Assume for the purposes of contradiction that there is a round in which all numbers are equal to the value v. By the lemma, $v = 21/6$ and v is an integer, which is impossible since 6 does not divide 21. Thus, there cannot be a round in which all numbers are equal. ∎

Problem 3. [15 points] We define the sequence of numbers

$$a_n = \begin{cases} 1 & \text{if } 0 \le n \le 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \ge 4. \end{cases}$$

Prove that $a_n \equiv 1 \pmod 3$ for all $n \ge 0$.

Solution. Proof by strong induction. Let $P(n)$ be the predicate that $a_n \equiv 1 \pmod 3$.

Base case: For $0 \le n \le 3$, $a_n = 1$ and is therefore $\equiv 1 \pmod 3$.

Inductive step: For $n \ge 4$, assume $P(k)$ for $0 \le k \le n$ in order to prove $P(n+1)$.

In particular, since each of a_{n-4}, a_{n-3}, a_{n-2} and a_{n-1} is $\equiv 1 \pmod 3$, their sum must be $\equiv 4 \equiv 1 \pmod 3$. Therefore, $a_n \equiv 1 \pmod 3$ and $P(n+1)$ holds. ∎

Problem 4. [10 points] Find the multiplicative inverse of 17 modulo 72 in the range $\{0, 1, \ldots, 71\}$.

Solution. Since 17 and $72 = 2^3 3^2$ are relatively prime, an inverse exists and can be found by either Euler's theorem or the Pulverizer.

Solution 1: Euler's Theorem

$$
\begin{aligned}
\phi(72) &= \phi(2^3 \cdot 3^2) \\
&= \phi(2^3) \cdot \phi(3^2) && \text{(since } 2^3 \text{ and } 3^2 \text{ are rel. prime)} \\
&= (2^3 - 2^2)(3^2 - 3^1) && \text{(since 2 and 3 are prime)} \\
&= 4 \cdot 6 = 24
\end{aligned}
$$

Therefore, $17^{\phi(72)-1} = 17^{23}$ is *an* inverse of 17. To find *the* inverse in the range $\{0, 1, \ldots, 71\}$ we take the remainder using the method of repeated squaring:

$$
\begin{aligned}
17 &= 17 \\
17^2 &= 289 \\
&\equiv 1 && \text{(since } 289 = 4 \cdot 72 + 1) \\
17^4 &\equiv 1^2 = 1 \\
17^8 &\equiv 1 \\
&\ldots etc.
\end{aligned}
$$

Therefore the inverse of 17 in the range $\{0, 1, \ldots, 71\}$ is given by,

$$
\begin{aligned}
17^{23} &= 17^{16} 17^4 17^2 17^1 \\
&\equiv 1 \cdot 1 \cdot 1 \cdot 17 \\
&= 17
\end{aligned}
$$

Solution 2: The Pulverizer

x	y	rem	xy	$=$	$x - q \cdot y$
72	17	4		$=$	$72 - 4 \cdot 17$
17	4	1		$=$	$17 - 4 \cdot 4$
				$=$	$17 - 4 \cdot (72 - 4 \cdot 17)$
				$=$	$17 \cdot 17 - 4 \cdot 72$
4	1	0			

Since $17^2 - 4 \cdot 72 = 1$, $17^2 \equiv 1 \pmod{72}$ and so 17 is self inverse. ∎

Problem 5. [20 points] An outerplanar graph is an undirected graph for which the vertices *can be* placed on a circle in such a way that no edges (drawn as straight lines) cross each other. For example, the complete graph on 4 vertices, K_4, is not outerplanar but any proper subgraph of K_4 with strictly fewer edges is outerplanar. Some examples are provided below:
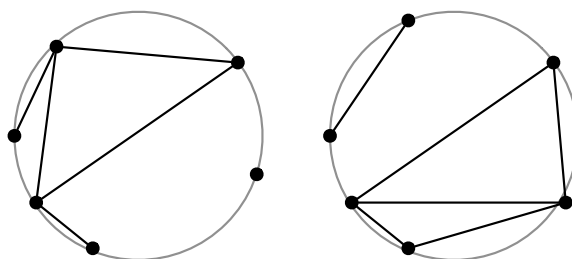
Prove that any outerplanar graph is 3-colorable. A fact you may use without proof is that any outerplanar graph has a vertex of degree at most 2.

Solution. Proof by induction on the number of nodes n with the induction hypothesis $P(n) =$ "every outerplanar graph with n vertices is 3-colorable."

Base case: For $n = 1$ the single node graph with no edges is trivially outerplanar and 3-colorable.

Inductive step: Assume $P(n)$ holds and let G_{n+1} be an outerplanar graph with $n + 1$ vertices. There must exist a vertex v in G_{n+1} with degree at most 2. Removing v and all its incident edges leaves a subgraph G_n with n vertices.

Since G_{n+1} could be drawn with its vertices on a circle and its edges drawn as straight lines without intersections, any subgraph can also be drawn in such a way and so G_n is also an outerplanar graph. $P(n)$ implies G_n is 3-colorable. Therefore we can color all the vertices in G_{n+1} other than v using only 3 colors and since $\deg(v) \leq 2$ we may color it a color different than the vertices adjacent to it using only 3 colors. Therefore, G_{n+1} is 3-colorable and $P(n + 1)$ holds.

■

Problem 6. [20 points]
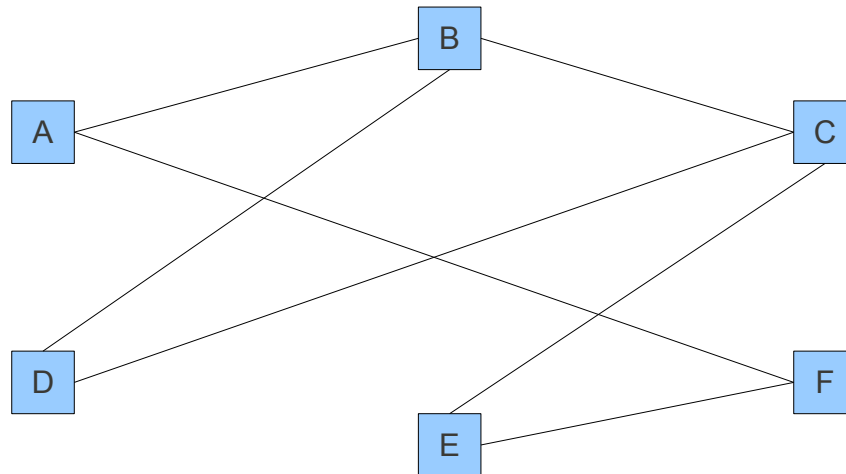
Consider the simple graph G given in Figure 1.

Figure 1: Simple graph G

(a) [4 pts] Give the diameter of G.

Solution. Recall that the diameter is the maximum of all shortest path lengths between pairs of vertices. Note that the shortest path length between D and F is 3, and all other pairs of non-adjacent vertices share a neighbor. ∎

(b) [4 pts] Give a Hamiltonian Cycle on G.

Solution. One possible solution is (A, F, E, C, D, B, A). This cycle and its reverse should constitute all possible solutions. ∎

(c) [4 pts] Give a coloring on G and show that it uses the smallest possible number of colors.

Solution. One possible 3-coloring is: $\{A, D, E\}$ red; $\{B, F\}$ green; C blue. Because there exists an odd-length cycle (e.g. (B, D, C)), no 2-coloring exists. Therefore the given coloring uses the least possible number of colors. ■

(d) [4 pts] Does G have an Eulerian cycle? Justify your answer.

Solution. No. This follows from the fact that there exist vertices with odd degree; e.g. B. ■

Now consider graph H, which is like G but with weighted edges, in Figure 2:
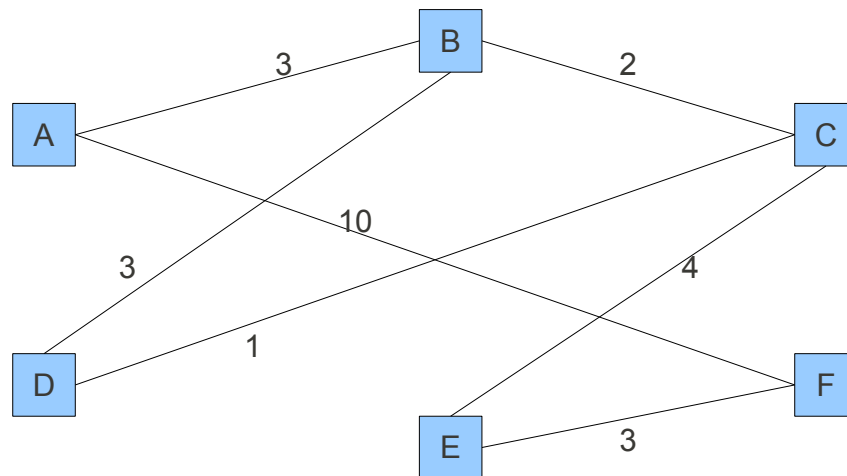
Figure 2: Weighted graph H

(e) [4 pts] Give a list of edges reflecting the order in which one of the greedy algorithms presented in class (i.e. in lecture, recitation, or the course text) would choose edges when finding an MST on H.

Solution. Kruskal's alg (building up an acyclic subgraph) gives two possible orders: $((C, D), (B, C), (A, B), (E, F), (C, E))$ and $((C, D), (B, C), (E, F), (A, B), (C, E))$. Prim's algorithm (building up a connected, acyclic subgraph) gives one possible order: $((C, D), (B, C), (A, B), (C, E), (E, F))$. Figure 3 below gives the MST generated in any case.
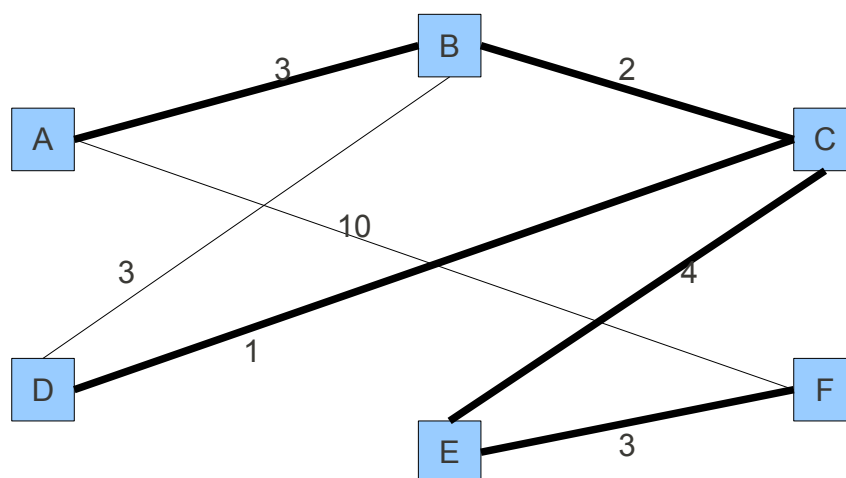
■

Figure 3: MST on graph H

Problem 7. [18 points]

In this problem, we say that an $M \times N$ *grid network* is a routing network consisting of an undirected grid of M rows and N columns, with M inputs on the left and M outputs on the right, as depicted in Figure 4. (Note: this is not the same as the "2-D array" in the notes, which has outputs on the bottom.)

(a) [6 pts] What is the congestion of the 3×3 grid in Figure 4? (Feel free to use the grids drawn on the next page in expressing your answer.)
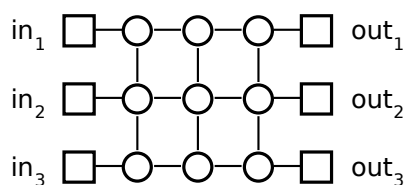
Figure 4: A 3×3 grid network.

Solution. The congestion is 2. We can draw congestion ≤ 2 routings for all six permutations (routing problems), as seen in Figure 5. None of these have a congestion 1 solution except for the identity permutation; for a rigorous argument see part (b).
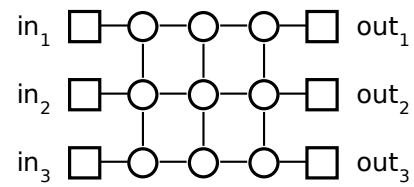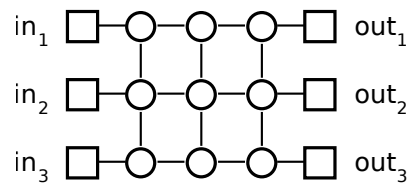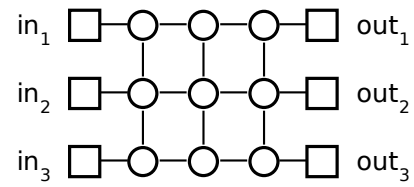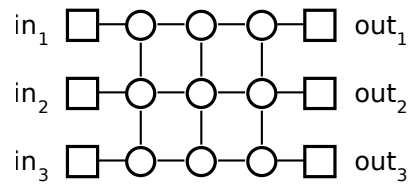
∎
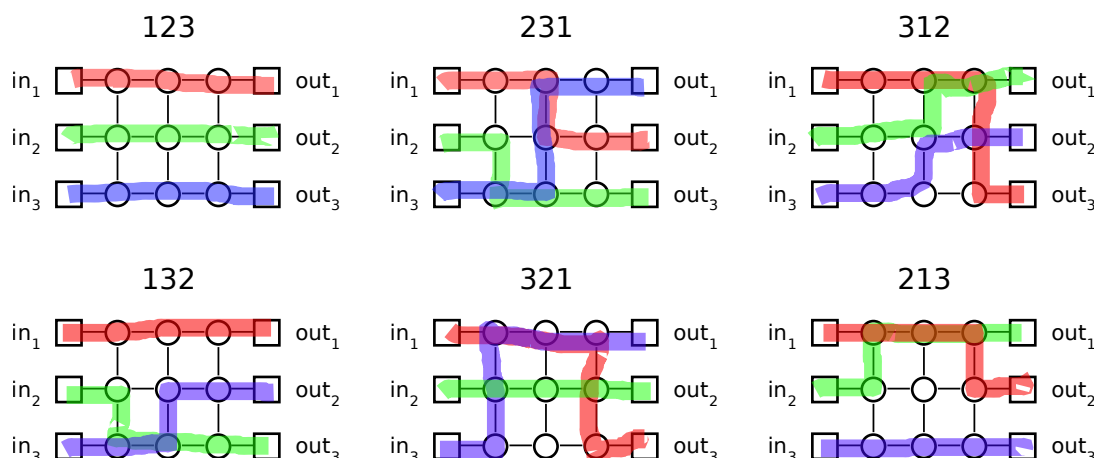
Figure 5: Congestion ≤ 2 routings for part (a).

(b) [6 pts] For the $M \times N$ grid network with M inputs on the left and M outputs on the right, prove carefully that the congestion is always strictly greater than 1, so long as $M > 1$.

Solution. Let (x, y) denote coordinates on the grid, for which $(1, 1)$ is the upper-left corner and (M, N) is the lower-right corner.

Consider a congestion 1 routing on this grid. We show, by strong induction on x, that for all $1 \leq x \leq N$ and $1 \leq y \leq M$, the xth step of route y is to move to vertex (x, y). The base case of $x = 1$ is clear: the switch $(1, y)$ is the only switch connected to $\mat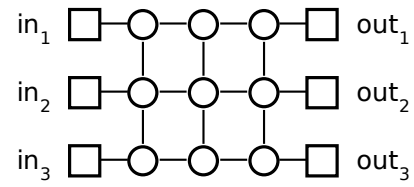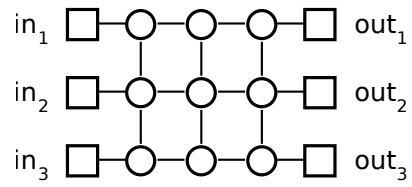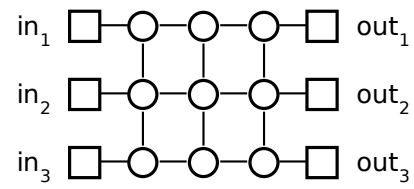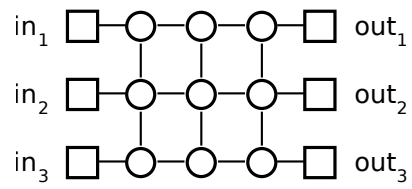hsf{in}_y$, and so the first move is forced. Now in the inductive step, suppose that for all $1 \leq y \leq M$ and all $i < x$, the ith step of route y was to visit vertex (i, y). Then at step x, route y can not move up, down, or left on the grid, as the vertex there (if any) has already been visited, and congestion is constrained to be 1. So the route must move right to (x, y). This completes the induction.

So any congestion 1 routing must have each route move right along the grid rows until it reaches an output. Then the route starting at in_i can only reach out_i, and for $M > 1$, not every routing permutation is possible with congestion 1. ∎

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3

in_1 □—○—○—○—□ out_1
in_2 □—○—○—○—□ out_2
in_3 □—○—○—○—□ out_3
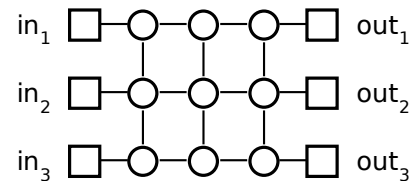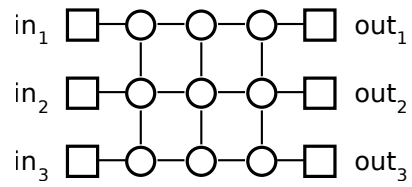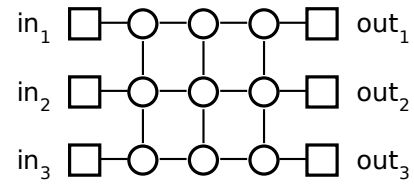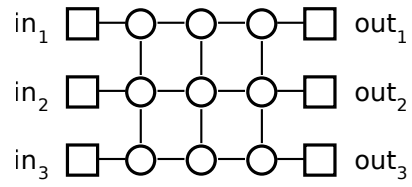
(c) [6 pts] For the $M \times N$ grid, as in part (b), argue that for any $M > 1$, it is possible to choose N large enough so that the grid has congestion 2. (A fully-fledged proof is not required, but give justification.)

Hint: imagine routes traveling along rows and swapping in various places.

Solution. As in the hint, we imagine routes that travel along rows except when participating in a swap move as depicted in Figure 6.
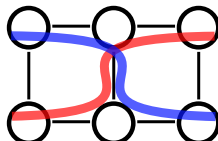

Figure 6: A swap move with congestion 2.

Any routing of this form has congestion 2, so it suffices to show that we can find such a routing for any given routing problem. There are many ways to do this, and some might recognize this as the general mathematical fact that sequences of swaps of adjacent elements can achieve any permutation. We give a concrete approach below.

We construct the routing from left to right along the grid, adding swap moves to move routes between grid rows. As we may assume N to be as large as necessary, we may apply as many swap moves as necessary. Suppose that input in_i corresponds to the bottom output out_M; then we first swap row i with row $i + 1$, then $i + 1$ with $i + 2$, and so on until the route from in_i leads to row M, where it will cease to participate in swaps and ultimately lead to output out_M. We then do the same for the route destined for out_{M-1}; these swaps will not interfere with the bottom-most route. We repeat, moving up the line of outputs, until all routes are on the correct output row.

An example is given in Figure 7, for the following permutation of five elements:

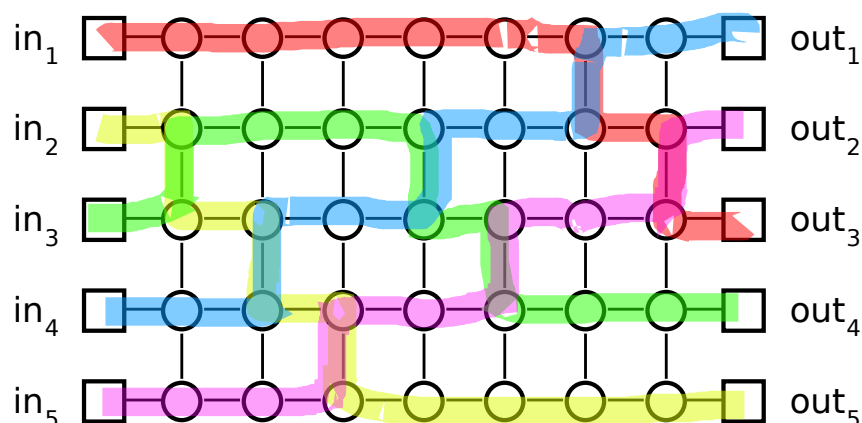$$1 \mapsto 3, \ 2 \mapsto 5, \ 3 \mapsto 4, \ 4 \mapsto 1, \ 5 \mapsto 2.$$

■

Figure 7: An example routing produced by the solution procedure.

Problem 8. [15 points]

Throughout this problem, PageRank is understood to mean *unscaled* PageRank.

(a) [5 pts] Given any strongly connected web graph G on at least 2 vertices, argue that no single vertex can have equilibrium PageRank value greater than $\frac{1}{2}$.

Solution. Let v be a vertex with equilibrium value λ. If we apply the PageRank update step, the PageRank value λ at v is distributed to the rest of the graph in some way, and v takes on contributions from the rest of the graph, of total value $\mu \leq 1 - \lambda$. As the PageRank values are in equilibrium, these two are equal; thus:

$$\lambda = \mu \leq 1 - \lambda,$$

$$\lambda \leq \frac{1}{2}.$$

∎

(b) [5 pts] Let $N \geq 2$ be given. Exhibit a web graph on N vertices in which one vertex has equilibrium PageRank value equal to $\frac{1}{2}$.

Solution. Consider the graph consisting of a single hub H and $N - 1$ peripheral pages P_i each with a link to and from H. Then if we assign the PageRank value $\frac{1}{2}$ to the hub H, and value $\frac{1}{2(N-1)}$ to each page P_i, the PageRank values are in equilibrium. ∎

(c) [5 pts] Argue that for each N, there is only one solution to part (b) that is strongly connected.

Solution. In order for a vertex v to have equilibrium PageRank $\frac{1}{2}$, the inequalities in part (a) must be tight: during the update step, v must take on the entirety of the PageRank from the rest of the graph. This is only possible if every other vertex has an edge to v and has no other outbound edges. This almost entirely determines the graph; the only remaining question concerns the edges out of v; but v must have an edge to every other vertex w in order for w to be reachable from v, i.e. in order for the graph to be strongly connected. ■

Problem 9. [10 points] Consider the following relation on the set of natural numbers:

$$R = \{(x, y) : x \leq y^2 \text{ for } x, y \in \mathbb{N}\}.$$

Which of the following properties holds for R? If it has the property, prove it. If not, provide a counterexample.

(a) [2 pts] Reflexivity.

Solution. Yes. R is reflexive if $\forall x \; xRx$, that is, if $x \leq x^2$. If $x = 0, 0 \leq 0$, so reflexivity holds. If $x \geq 1$, then it follows that $x^2 \geq x$, from multiplying both sides of the inequality by x, which is positive. This covers all $x \in \mathbb{N}$. ∎

(b) [2 pts] Symmetry.

Solution. No. Counterexample: x = 0, y = 1. $0 \leq 1^2$, but $1 > 0^2$. ∎

(c) [2 pts] Transitivity.

Solution. No. Counterexample: $x = 10, y = 5, z = 3$. $10 \leq 5^2$, and $5 \leq 3^2$, but $10 \geq 3^2$. ∎

(d) [2 pts] Antisymmetry.

Solution. No. Counterexample: $x = 2, y = 3$. $2 \leq 3^2$, and $3 \leq 2^2$, but $3 \neq 2$.   ∎

(e) [2 pts] The property of being an equivalence relation.

Solution. No. An equivalence relation must be reflexive, symmetric, and transitive. Of those three, the relation is only reflexive.   ∎