Practice Final Exam Solutions

- The exam is **closed book**, but you may have three $8.5'' \times 11''$ sheet with notes (either printed or in your own handwriting) on both sides.
- Calculators and electronic devices (including cell phones) are not allowed.
- You may assume all of the results presented in class. This does **not** include results demonstrated in practice quiz material.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

NAME: ______________________________

TA: ______________________________

Problem	**Value**	**Score**	**Grader**
1	10		
2	12		
3	12		
4	10		
5	15		
6	15		
7	11		
8	14		
9	16		
10	15		
11	15		
12	15		
Total	160		

Problem 1. [**10 points**] It is well-known that *elitotis* is a common disease amongst Harvard students: in fact, 1 in 10 of their students have it. Fortunately, MIT has developed a reliable test for the presence of elevated *arrogentes* levels, which is helpful for testing for elitotis because:

- a student with elitotis has elevated arrogentes levels with probability 4/5, and
- a student with no elitosis has elevated arrogentes levels with probability 1/3.

What is the probability that a student selected uniformly at random from Harvard has elitosis, given that he or she has elevated arrogentes levels?

Solution. Let C be the event that the randomly-selected student has elitosis, and let A be the event that he/she has elevated arrogentes levels. The probability that a student has elitosos, given that he/she has elevated arrogentes levels is:

$$\begin{aligned}
\Pr[C \mid A] &= \frac{\Pr[C \cap A]}{\Pr[A]} \\
&= \frac{\Pr[A \mid C]\Pr[C]}{\Pr[A \cap C] + \Pr[A \cap \bar{C}]} \\
&= \frac{\Pr[A \mid C]\Pr[C]}{\Pr[A \mid C]\Pr[C] + \Pr[A \mid \bar{C}]\Pr[\bar{C}]} \\
&= \frac{(4/5) \times (1/10)}{(4/5) \times (1/10) + (1/3) \times (9/10)} \\
&= \frac{4}{4+15} \\
&= \frac{4}{19}.
\end{aligned}$$

■

Problem 2. [**12 points**] Amy, Bill, and Poor Pete play a game:

1. Each player puts \$2 on the table.
2. Each player secretly writes a number between 1 and 4.
3. They roll a fair, four-sided die with faces numbered 1, 2, 3, and 4.
4. The money on the table is divided among the players that guessed correctly. If no one guessed correctly, then everyone gets their money back *and Poor Pete is paid \$0.25 in "service fees"*.

Suppose that, Amy and Bill cheat by picking a pair of *distinct* numbers uniformly at random.

(a) [8 pts] For each event listed below, indicate the probability of the event (in the box on the left) and Poor Pete's profit (in the box on the right) if that event occurs.

Pete guesses right AND either Amy or Bill guesses right		
Pete guesses right AND both Amy and Bill guess wrong		
Pete guesses wrong AND either Amy or Bill guesses right		
Pete guesses wrong AND both Amy and Bill guess wrong		

Solution.

Pete guesses right AND either Amy or Bill guesses right	1/8	1
Pete guesses right AND both Amy and Bill guess wrong	1/8	4
Pete guesses wrong AND either Amy or Bill guesses right	3/8	-2
Pete guesses wrong AND both Amy and Bill guess wrong	3/8	0.25

■

(b) [4 pts] What is Poor Pete's expected profit?

Solution.

$$\frac{1}{8}\cdot 1+\frac{1}{8}\cdot 4+\frac{3}{8}\cdot(-2)+\frac{3}{8}\cdot(0.25)=-\frac{1}{32}.$$

■

Problem 3. [12 points] Let T be a positive integer. Consider the following recurrence equation:

$$t(n)=\frac{t(n+1)}{3}+\frac{2t(n-1)}{3}+1 \text{ for } 1\le n\le T-1; \qquad t(0)=t(T)=0.$$

Find a closed form solution for $t(n)$ for $0\le n\le T$ as a function of T.

Solution. The characteristic equation is $x = \frac{x^2}{3} + \frac{2}{3}$. Thus we have that:

$$x^2 - 3x + 2 = 0,$$

and so

$$x = 1 \text{ or } x = 2.$$

Thus the general homogenous solution is

$$t_h(n) = a + b2^n.$$

We need to find some specific solution. A constant won't work, since it's already a soution to the homogenous problem. Thus, let's try $f(n) = pn + q$:

$$pn + q = \frac{pn + p + q}{3} + \frac{2pn - 2p + 2q}{3} + 1$$

This has a solution $p = 3$ (and for any value of q), so $t_p(n) = 3n$ is a particular solution. Adding the homogenous solution, we obtain the general solution:

$$t(n) = a + b2^n + 3n.$$

Using $t(0) = 0$ and $t(T) = 0$, then:

$$a + b = 0 \text{ and } a + b2^T + 3T = 0.$$

Solving this, we obtain

$$a = -b \text{ and } b(2^T - 1) = -3T,$$

and hence

$$a = -\frac{3T}{1 - 2^T} \text{ and } b = \frac{3T}{1 - 2^T}.$$

Thus

$$t(n) = -\frac{3T}{1 - 2^T} + \frac{3T}{1 - 2^T} 2^n + 3n.$$

■

Problem 4. [10 points] Determine a closed form formula for the following sum (here, n is a positive integer):

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \frac{1}{j}.$$

Solution. Swapping the order of summation, we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{1}{j} &= \sum_{j=1}^{n}\sum_{i=1}^{j}\frac{1}{j} \\
&= \sum_{j=1}^{n}\frac{1}{j}\sum_{i=1}^{j}1 \\
&= \sum_{j=1}^{n}\frac{1}{j}\cdot j \\
&= \sum_{j=1}^{n}1 \\
&= n.
\end{aligned}$$

■

Problem 5. [15 points] The *Lucas numbers* L_n are defined by the following recurrence:

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for all } n \geq 2.$$

Prove by induction that for all $n \geq 1$,

$$\sum_{j=0}^{n} L_j = L_{n+2} - 1.$$

Solution. Let $P(n)$ be the statement that $\sum_{j=0}^{n} L_j = L_{n+2} - 1$, for some fixed n.

Base case: $\sum_{j=0}^{1} L_1 = 2 + 1 = 3$. $L_3 - 1 = L_1 + L_2 - 1 = 2L_1 + L_0 - 1 = 3$. So $P(1)$ holds. (Alternatively: you could show that in fact $P(0)$ holds and use that as the base case.)

Now assume for the purposes of induction that $P(n)$ holds. Thus

$$\sum_{j=0}^{n} L_j = L_{n+2} - 1.$$

Then

$$\begin{aligned}
\sum_{j=0}^{n+1} L_j &= \sum_{j=0}^{n} L_j \ + \ L_{n+1} \\
&= (L_{n+2} - 1) \ + \ L_{n+1} \qquad \text{by inductive assumption} \\
&= L_{n+3} - 1 \qquad \text{by recurrence relation.}
\end{aligned}$$

Hence $P(n+1)$ holds. Thus $P(n)$ holds for all $n \geq 1$ by induction. ■

Problem 6. [15 points]

Let T be a tree with n nodes, where n is a positive integer. Suppose we color each node of T in a random way: each node is red with probability $1/3$, green with probability $1/3$ and blue with probability $1/3$, and the colors of distinct nodes are mutually independent.

Find a formula for the probability that the resulting coloring is a proper coloring. You do not need to give a full proof of your answer, but do include your reasoning.

(Hint: you may use the fact that any tree on at least 2 *nodes has at least one leaf.)*

Solution. The answer is $(2/3)^{n-1}$, as we will prove by induction.

Let $P(n)$ be the statement that a random coloring of any tree of n nodes is a proper coloring with probability $(2/3)^{n-1}$. Note that $P(1)$ holds; the coloring is always proper, and so the probability is 1.

Now suppose $P(n)$ holds, and let T be any tree with $n+1$ nodes. Let v be any leaf of T, let u be the node adjacent to v, and let T' be the tree obtained by removing v. Color T randomly as prescribed. Since T' has n nodes, we know that the probability that the coloring restricted to T' is proper is exactly $(2/3)^{n-1}$. Let us condition on this event; given that T' is properly colored, what is the probability that T is properly colored? This is just $2/3$; no matter the coloring of T', we get a proper coloring of T precisely if v is colored a different color to u, which happens with probability $2/3$.

So the probability that T is colored properly is

$$\begin{aligned}
\Pr[T \text{ properly colored}] &= \Pr[T \text{ properly colored} \mid T' \text{ properly colored}] \Pr[T' \text{ properly colored}] \\
&= \frac{2}{3} \cdot (2/3)^{n-1} \\
&= (2/3)^n.
\end{aligned}$$

Hence $P(n+1)$ holds. Thus $P(n)$ holds for all $n \geq 1$ by induction. ■

Problem 7. [11 points] You have twelve cards:

1	1	2	2	3	3	4	4	5	5	6	6

You shuffle them well, and deal them in a row (so the ordering will be a uniformly random permutation). For example, you might get:

1	2	3	3	4	6	1	4	5	5	2	6

What is the expected number of adjacent pairs with the same value? In the example, there are two adjacent pairs with the same value, the 3's and the 5's.

Solution. Let I_j be the indicator of the event A_j that the j'th card from the left is the same as the $(j+1)$'st card, for $1 \leq j \leq 11$. Let R be the r.v. for the number of adjacent pairs. Then

$$R = \sum_{j=1}^{11} I_j;$$

thus by linearity of expectation,

$$\text{Ex}[R] = \sum_{j=1}^{11} \text{Ex}[I_j] = \sum_{j=1}^{11} \Pr[A_j].$$

Fix some $j \in \{1, 2, \ldots, 11\}$. Now we count the number of possible layouts for which cards j and $j+1$ are the same. This is just

$$6 \cdot \frac{10!}{(2!)^5},$$

by the BOOKKEEPER lemma. On the other hand, the total number of possible layouts is

$$\frac{12!}{(2!)^6},$$

by similar reasoning. Thus

$$\Pr[A_j] = \frac{610!/(2!)^5}{12!/(2!)^6} = \frac{2 \cdot 6}{12 \cdot 11} = \frac{1}{11}.$$

Thus

$$\text{Ex}[R] = 11 \cdot \frac{1}{11} = 1.$$

■

Problem 8. [14 points]

T-Pain is planning an epic boat trip and he needs to decide what to bring with him.

- He *definitely* wants to bring burgers, but they only come in packs of 6.
- He and his two friends can't decide whether they want to dress formally or casually. He'll either bring 0 pairs of flip flops or 3 pairs.
- He doesn't have very much room in his suitcase for towels, so he can bring at most 2 (and might not bring any!)
- In order for the boat trip to be truly epic, he has to bring at least 1 nautical-themed pashmina afghan.

(a) [7 pts] Let g_n be the the number of different ways for T-Pain to bring n items (burgers, pairs of flip flops, towels, and/or afghans) on his boat trip, satisfying the restrictions above. Express the generating function $G(x) := \sum_{n=0}^{\infty} g_n x^n$ as a quotient of polynomials.

Solution.

$$\begin{aligned}
G(x) &= \frac{x^6}{1-x^6}(1+x^3)(1+x+x^2)\frac{x}{1-x} \\
&= \frac{(1+x^3)(1+x+x^2)x^7}{(1-x^3)(1+x^3)(1-x)} \\
&= \frac{(1+x+x^2)x^7}{(1-x)(1+x+x^2)(1-x)} \\
&= \frac{x^7}{(1-x)^2}
\end{aligned}$$

■

(b) [7 pts] Let $H(x) := \sum_{n=0}^{\infty} h_n x^n$ be the generating function for the sequence $h_0, h_1, h_2, \ldots$ representing the number of ways T-Pain could write an epic book of n chapters about his boat trip. It turns out that

$$H(x) = \frac{3-2x}{(1-x)^2} - 2.$$

Using this information, determine a closed formula for h_n.

Solution. We have

$$H(x) = \frac{1}{(1-x)^2} + \frac{2}{1-x} - 2.$$

Recall that

$$\frac{1}{1-x} = 1 + x + x^2 \cdots = \sum_{n=0}^{\infty} x^n.$$

Thus

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= \sum_{n=0}^{\infty} \frac{d}{dx} x^n \\
&= \sum_{n=1}^{\infty} n x^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1) x^n.
\end{aligned}$$

Hence

$$H(x) = \sum_{n=0}^{\infty}(n+1)x^n + 2\sum_{n=1}^{\infty} x^n \; - \; 2$$
$$= 1 + \sum_{n=1}^{\infty}(n+3)x^n.$$

So $h_n = n+3$ for $n \geq 1$, and $h_0 = 1$. ■

Problem 9. [16 points] Clumsy Clarke is rather injury prone:

- Every time he enters his car, which he does 72 times a month, he bumps his head with probability 1/6.
- Each time he enters his house, which he does 32 times a month, he nicks his finger with probability 1/2.
- Every time he does some shopping, which he does 25 times a month, he drops a bag on his foot with probability 1/5.

All of these events are mutually independent.

(a) [4 pts] What is the expected number of injuries Clumsy Clarke experiences in a month?

Solution. Let X be the number of injuries Clarke experiences.

$$\text{Ex}[X] = 72/6 + 32/2 + 25/5 = 33.$$

■

(b) [4 pts] What is the variance in the number of injuries Clarke has in a month?

Solution.

$$\text{Var}[X] = 72 \cdot \tfrac{1}{6} \cdot \tfrac{5}{6} + 32 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + 25 \cdot \tfrac{1}{5} \cdot \tfrac{4}{5} = 22.$$

■

(c) [4 pts] What would the Markov bound be on the probability that Clarke has 100 or more injuries in a month?

Solution.

$$\Pr[X \geq 100] \leq \frac{33}{100}.$$

■

(d) [4 pts] What would the Chebyshev bound be on the probability that Clarke has 100 or more injuries in a month?

Solution.

$$\Pr[X - 33 \geq 67] \leq \Pr[|X - \text{Ex}[X]| \geq 67] \leq \frac{\text{Var}[X]}{67^2} = \frac{22}{67^2}.$$

■

Problem 10. [15 points] Alyssa, an industrious software developer, writes 100 lines of code each hour, with probability p of making a mistake each hour (she never makes more than one mistake in an hour, though). Assuming Alyssa works n hours in a day and the mistakes she makes in each hour are independent, give formulas for the following:

(a) [4 pts] The probability of exactly k mistakes in a day.

Solution. Let M be the random variable equal to the number of mistakes in n hours. Then M has binomial distribution with parameters n, p, so

$$\Pr[M = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

for $0 \leq k \leq n$. ■

(b) [5 pts] The probability of at least one mistake in a day.

Solution.

$$\Pr[M > 0] = 1 - \Pr[\text{No Mistake}] = 1 - (1-p)^n.$$

■

(c) [6 pts] The expected number of hours until either the first mistake, or the end of the work day, whichever comes first. (Assume that if Alyssa makes a mistake in some hour, she makes it at the beginning of that hour).

Solution. Let H be a random variable representing the number of hours before the first mistake. Calculating $\text{E}[H]$ is similar to finding mean time to failure, except that we stop after n hours.

$$\begin{aligned}
\text{E}[T] &= \sum_{i=0}^{\infty} \Pr[H > i] \\
&= \sum_{i=0}^{n-1} \Pr[H > i] \\
&= \sum_{i=0}^{n-1} (1-p)^i \\
&= \frac{1 - (1-p)^n}{p}.
\end{aligned}$$

■

Problem 11. [15 points] Consider the following game. You have the following grid:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

There are 25 balls in a bucket, numbered from 1 to 25. 7 of these balls are randomly chosen from the bucket. You cross out the 7 numbers on your grid corresponding to the selected balls. If you cross out an entire row, an entire column, or either of the diagonals, then you win! E.g., if the draw yields balls 2, 6, 7, 12, 15, 17 and 22, then you win.

What is the probability of winning?

Solution. Call a row, column, or diagonal of boxes a *line*. Fix any line. The probability that we cross out all the squares in this line is

$$\frac{\binom{20}{2}}{\binom{25}{7}}.$$

We can never cross out more than one line at the same time, so we just need to multiply by the number of possible lines. Thus the total probability is

$$12 \cdot \frac{\binom{20}{2}}{\binom{25}{7}}.$$

■

Problem 12. [15 points] Consider the following tennis tournament. There are n players, and every pair of players play against each other once. Moreover, all the players are equally matched, and so the winner of each matchup is uniformly random; the game outcomes are also mutually independent.

Call a player *awesome* if they win all their games, and *terrible* if they lose all of them.

(a) [5 pts] What is the probability that there will be an awesome player?

Solution. Consider an arbitrary fixed player i, and let E_i be the event that player i is awesome. Then $\Pr[E_i] = 2^{-(n-1)}$, since to be awesome player i must win all their $n-1$ games. Now observe that there cannot be more than one awesome player (since between two players, whoever lost their match cannot be awesome). So all the events E_i are disjoint; thus

$$\Pr[E_1 \cup E_2 \cdots \cup E_n] = \sum_{i=1}^{n} \Pr[E_i] = n2^{-(n-1)}.$$

■

(b) [5 pts] What is the probability that there will be both an awesome player and a terrible player?

Solution. Let $E_{i,j}$ be the probability that player i is awesome and player j is terrible, for $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. Then

$$\Pr[E_{i,j}] = 2^{-(2n-3)},$$

since player i must win all her matches, and player j must lose all their matches, thus fixing the outcome of precisely $2(n-1) - 1 = 2n - 3$ matches (the match between player i and player j must only be counted once). Again, the events $E_{i,j}$ are all disjoint, and so

$$\Pr[\text{both an awesome and a terrible player}] = \sum_{i=1}^{n} \sum_{j \in \{1,2,\ldots n\}: j \neq i} E_{i,j} = n(n-1)2^{-(2n-3)}.$$

■

(c) [5 pts] What is the probability that there will be neither an awesome player nor a terrible player?

Solution. The probability that there is a terrible player is the same as the probability that there is an awesome player. Thus, using the inclusion-exclusion principle,

$$\begin{aligned}\Pr[\text{no awesome or terrible player}] &= 1 - 2n2^{-(n-1)} + n(n-1)2^{-(2n-3)} \\ &= 1 - n2^{-(n-2)} + n(n-1)2^{-(2n-3)}.\end{aligned}$$

■