

Notes for Recitation 18

The Total Probability Law is a handy tool for breaking down the computation of a probability into distinct cases.

Theorem 1 (Total Probability Law). *Let E and X be events. Then*

$$\Pr\{E\} = \Pr\{E \mid X\} \cdot \Pr\{X\} + \Pr\{E \mid \overline{X}\} \cdot \Pr\{\overline{X}\}$$

provided $0 < \Pr\{X\} < 1$.

Proof. Let's simplify the right side.

$$\begin{aligned} & \Pr\{E \mid X\} \cdot \Pr\{X\} + \Pr\{E \mid \overline{X}\} \cdot \Pr\{\overline{X}\} \\ &= \frac{\Pr\{E \cap X\}}{\Pr\{X\}} \cdot \Pr\{X\} + \frac{\Pr\{E \cap \overline{X}\}}{\Pr\{\overline{X}\}} \cdot \Pr\{\overline{X}\} \\ &= \Pr\{E \cap X\} + \Pr\{E \cap \overline{X}\} \\ &= \Pr\{E\} \end{aligned}$$

The first step uses the definition of conditional probability. On the next-to-last line, we're adding the probabilities of all outcomes in E and X to the probabilities of all outcomes in E and *not* in X . Since every outcome in E is either in X or not in X , this is the sum of the probabilities of all outcomes in E , which equals $\Pr\{E\}$ by the definition of the probability of an event. \square

The theorem generalizes as follows:

Theorem 2. *Let E be an event and let X_1, \dots, X_n be disjoint events whose union is the entire sample space. Then*

$$\Pr\{E\} = \sum_{i=1}^n \Pr\{E \mid X_i\} \cdot \Pr\{X_i\}$$

provided $0 < \Pr\{X_i\} < 1$.

1 Nerditosis

There is a rare and deadly disease called *Nerditosis* which afflicts about 1 person in 1000. One symptom is a compulsion to refer to everything— fields of study, classes, buildings, etc.— using numbers. It’s horrible. As victims enter their final, downward spiral, they’re awarded a degree from MIT. Two doctors claim that they can diagnose Nerditosis.

1. Doctor X received his degree from Harvard Medical School. He practices at Massachusetts General Hospital and has access to the latest scanners, lab tests, and research. Suppose you ask Doctor X whether you have the disease.

- If you have Nerditosis, he says “yes” with probability 0.99.
- If you don’t have it, he says “no” with probability 0.97.

Let D be the event that you have the disease, and let E be the event that the diagnosis is erroneous. Use the Total Probability Law to compute $\Pr\{E\}$, the probability that Doctor X makes a mistake.

Solution. By the Total Probability Law:

$$\begin{aligned}\Pr\{E\} &= \Pr\{E \mid D\} \cdot \Pr\{D\} + \Pr\{E \mid \overline{D}\} \cdot \Pr\{\overline{D}\} \\ &= 0.01 \cdot 0.001 + 0.03 \cdot 0.999 \\ &= 0.02998\end{aligned}$$

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2. “Doctor” Y received his genuine degree from a fully-accredited university for \$49.95 via a special internet offer. He knows that Nerditosis strikes 1 person in 1000, but is a little shaky on how to interpret this. So if you ask him whether you have the disease, he’ll helpfully say “yes” with probability 1 in 1000 regardless of whether you actually do or not.

Let D be the event that you have the disease, and let F be the event that the diagnosis is faulty. Use the Total Probability Law to compute $\Pr\{F\}$, the probability that Doctor Y made a mistake.

Solution. By the Total Probability Law:

$$\begin{aligned}\Pr\{F\} &= \Pr\{F \mid D\} \cdot \Pr\{D\} + \Pr\{F \mid \overline{D}\} \cdot \Pr\{\overline{D}\} \\ &= 0.999 \cdot 0.001 + 0.001 \cdot 0.999 \\ &= 0.001998\end{aligned}$$

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3. Which doctor is more reliable?

Solution. Doctor X makes more than 15 times as many errors as Doctor Y .

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2 Barglesnort

A Barglesnort makes its lair in one of three caves:



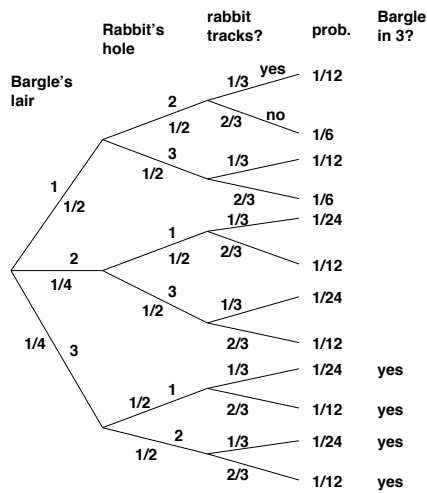
The Barglesnort inhabits cave 1 with probability $\frac{1}{2}$, cave 2 with probability $\frac{1}{4}$, and cave 3 with probability $\frac{1}{4}$. A rabbit subsequently moves into one of the two unoccupied caves, selected with equal probability. With probability $\frac{1}{3}$, the rabbit leaves tracks at the entrance to its cave. (Barglesnorts are much too clever to leave tracks.) What is the probability that the Barglesnort lives in cave 3, given that there are no tracks in front of cave 2?

Use a tree diagram and the four-step method.

Solution. A tree diagram is given below. Let B_3 be the event that the Barglesnort inhabits cave 3, and let T_2 be the event that there are tracks in front of cave 2. Taking data from the tree diagram, we can compute the desired probability as follows:

$$\begin{aligned} \Pr \{B_3 \mid \overline{T_2}\} &= \frac{\Pr \{B_3 \cap \overline{T_2}\}}{\Pr \{\overline{T_2}\}} \\ &= \frac{\frac{1}{24} + \frac{1}{12} + \frac{1}{12}}{1 - \frac{1}{12} - \frac{1}{24}} \\ &= \frac{5}{21} \end{aligned}$$

In the denominator, we apply the formula $\Pr \{\overline{T_2}\} = 1 - \Pr \{T_2\}$ for convenience.



3 Prisoners

There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $\frac{2}{3}$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). However, Sauron declines this offer. He reasons that if the guard says, for example, "Little Bunny Foo-Foo will be released", then his own probability of release will drop to $\frac{1}{2}$. This is because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

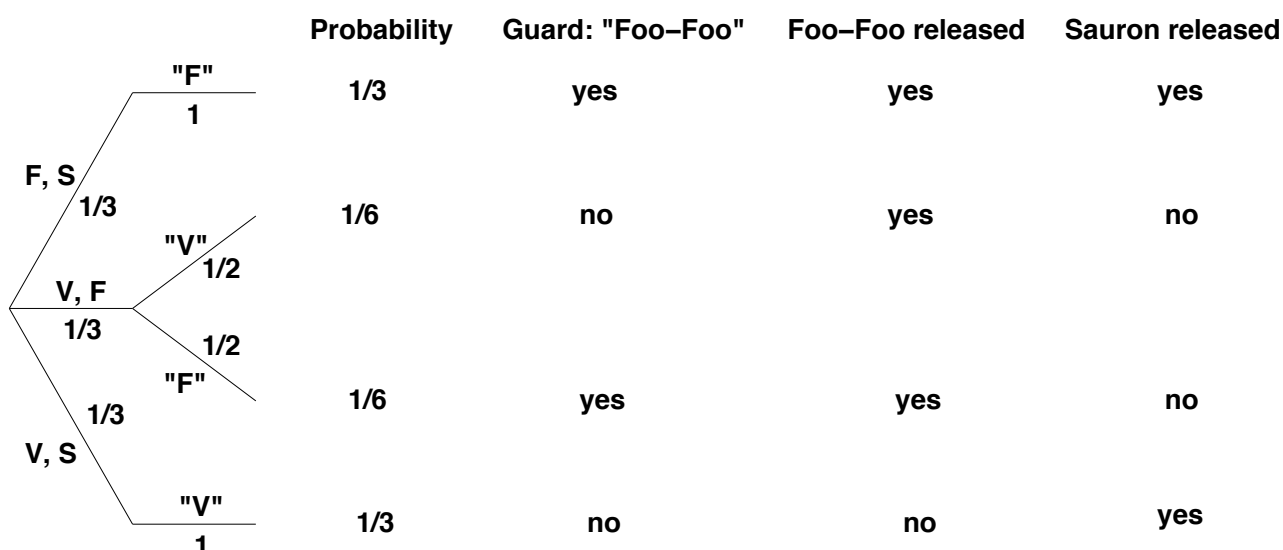
Using a tree diagram and the four-step method, either prove that the Dark Lord Sauron has reasoned correctly or prove that he is wrong. Assume that if the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), then he names one of the two uniformly at random.

Solution. Sauron has reasoned incorrectly.

We can explain the issue in terms of conditional probability. Let S , F , and " F " be the events:

- S = Sauron is released,
- F = Foo-Foo is released,
- " F " = Guard says Foo-Foo is released.

We can use the tree diagram below to analyze the outcomes and probabilities we should consider. The first split in the tree is based on which two of the three prisoners are to be released, and the second is based on what the guard tells Sauron.



From the diagram we can see that Sauron has correctly observed that

$$\Pr \{S\} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

And Sauron is correct in reasoning that if event "F" happens, then event F has also happened. And he's correct that the probability of his release given F shrinks to $1/2$. Namely, from the diagram we have:

$$\begin{aligned}\Pr \{F\} &= \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}, \\ \Pr \{S \cap F\} &= \frac{1}{3} \\ \Pr \{S \mid F\} &= \frac{\Pr \{S \cap F\}}{\Pr \{F\}} = \frac{1}{2}.\end{aligned}$$

So he worries that if he lets "F" happen, then his probability of release will shrink from $2/3$ to $\Pr \{S \mid F\} = 1/2$.

Sauron's confusion is not realizing that the events F and "F" are different, as the tree makes clear. These events even have different probabilities:

$$\Pr \{"F"\} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \neq \frac{2}{3} = \Pr \{F\}.$$

Now in determining his probability of release, Sauron should use *all* the information available. He should be concerned about his chances given that *the Guard says* Foo-Foo is released, not merely given the weaker fact that Foo-Foo is released. That is, Sauron should care about $\Pr \{S \mid \text{"F"}\}$, not $\Pr \{S \mid F\}$.

We can see from the diagram that

$$\Pr \{S \mid \text{"F"}\} = \frac{\Pr \{S \cap \text{"F"}\}}{\Pr \{\text{"F"}\}} = \frac{1/3}{1/2} = \frac{2}{3} = \Pr \{S\}.$$

So Sauron's probability of release is not changed by hearing from the guard. ■