Staff Solutions to Problem Set 8

Reading:

- Chapter **??**. *Conditional Probability*
- Chapter **??**. *Random Variables & Expectation*

STAFF NOTE: `Conditional Probabillity                  Ch.18.1-18.7`

Problem 1.
There were n Immortal Warriors born into our world, but in the end there can be *only one*. The Immortals' original plan was to stalk the world for centuries, dueling one another with ancient swords in dramatic landscapes until only one survivor remained. However, after a thought-provoking discussion probability, they opt to give the following protocol a try:

(i) The Immortals forge a coin that comes up heads with probability p.

(ii) Each Immortal flips the coin once.

(iii) If *exactly one* Immortal flips heads, then they are declared The One. Otherwise, the protocol is declared a failure, and they all go back to hacking each other up with swords.

One of the Immortals (Kurgan from the Russian steppe) argues that as n grows large, the probability that this protocol succeeds must tend to zero. Another (McLeod from the Scottish highlands) argues that this need not be the case, provided p is chosen carefully.

(a) A natural sample space to use to model this problem is $\{H, T\}^n$ of length-n sequences of H and T's, where the successive H's and T's in an outcome correspond to the Head or Tail flipped on each one of the n successive flips. Explain how a tree diagram approach leads to assigning a probability to each outcome that depends only on p, n and the number h of H's in the outcome.

COMMENTS:

- PS_immortal_probability
- overlaps of PS_ethernet
- F12.rec22

keywords = [*probability sample_space tie_breaking*]

Solution. The tree would have depth n, with each vertex having a branch assigned probability p to a child labelled H and a branch assigned probability $1 - p$ to a child labelled T. An outcome with h H's must have $n - h$ T's and would therefore be assigned a probability of

$$p^h (1 - p)^{n-h}.$$

■

(b) What is the probability that the experiment succeeds as a function of p and n?

Solution. Let E be the event that the experiment successfully selects The One. Then E consists of the n outcomes which contain a single head. Each of these has probability $p(1-p)^{n-1}$, so the probability that the procedure succeeds is

$$\Pr[E] = n\left(p(1-p)^{n-1}\right). \tag{1}$$

■

(c) How should p, the bias of the coin, be chosen in order to maximize the probability that the experiment succeeds?

STAFF NOTE: You're going to have to compute a derivative! ■

Solution. We compute the derivative of the success probability:

$$\frac{d}{dp}\, np(1-p)^{n-1} \quad = \quad n(1-p)^{n-1} - np(n-1)(1-p)^{n-2}$$

Now we set the right side equal to zero to find the best probability p:

$$\begin{aligned} n(1-p)^{n-1} &= np(n-1)(1-p)^{n-2} \\ (1-p) &= p(n-1) \\ p &= \frac{1}{n}. \end{aligned}$$

This answer makes some intuitive sense, since we want the coin to come up heads exactly 1 time in n. ■

(d) What is the probability of success if p is chosen in this way? What quantity does this approach when n, the number of Immortal Warriors, grows large?

Solution. Setting $p = 1/n$ in the formula (1) for the probability that the experiment succeeds gives:

$$\Pr[E] = \left(1 - \frac{1}{n}\right)^{n-1}$$

In the limit, this tends to $1/e$. McLeod is right. ■

Problem 2.
We're interested in the probability that a randomly chosen poker hand (5 cards from a standard 52-card deck) contains cards from at most two suits.

(a) What is an appropriate sample space to use for this problem? What are the outcomes in the event $\mathcal{E}$ we are interested in? What are the probabilities of the individual outcomes in this sample space?

COMMENTS:

- PS_random_poker_hand
- from: F06.ps11 (ported by Rich)

keywords = [*probability playing cards*]

Solution. The natural sample space to use consists of the $\binom{52}{5}$ possible poker hands. The sample space is *uniform*: Each hand is equally likely and comes up with probability $1/\binom{52}{5}$.

We define $\mathcal{E}$ to be the subset of outcomes in which the 5 cards on the outcome come from at most two suits. ■

(b) What is $\Pr[\mathcal{E}]$?

Solution. Since the sample space is uniform,

$$\Pr[\mathcal{E}] = \frac{|\mathcal{E}|}{\binom{52}{5}}.$$

So we just need to determine the number of outcomes in $\mathcal{E}$. For this, we resort to our usual counting techniques. Doing it by cases works well. There are three cases: 5 cards of one suit, 4 cards of one suit and 1 of another suit, 3 cards of one suit and 2 of another suit.

For 5 of one suit, there are 4 ways to choose the suit and then $\binom{13}{5}$ ways to choose 5 cards of that suit.

For 4 of one suit and 1 of another, there are 4 ways to choose the suit of the 4 and $\binom{13}{4}$ ways to choose 4 cards of that suit, and there are 3 remaining suits to choose for the 1, and 13 choices for the 1 card of that suit.

Finally, for 3 of one suit and 2 of another, there are 4 ways to choose the suit of the 3 and $\binom{13}{3}$ ways to choose 3 cards of that suit, and there are 3 remaining suits to choose for the 2 cards, and $\binom{13}{2}$ choices for the 2 cards of that suit. So the total is

$$4 \cdot \binom{13}{5} + 4 \cdot \binom{13}{4} \cdot 3 \cdot 13 + 4 \cdot \binom{13}{3} \cdot 3 \cdot \binom{13}{2},$$

and the probability of at most two suits is

$$\frac{4 \cdot \binom{13}{5} + 4 \cdot \binom{13}{4} \cdot 3 \cdot 13 + 4 \cdot \binom{13}{3} \cdot 3 \cdot \binom{13}{2}}{\binom{52}{5}} = 88/595 \approx 0.15.$$

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Problem 3.

Suppose you have three cards: A♡, A♠ and a jack. From these, you choose a random hand (that is, each card is equally likely to be chosen) of two cards, and let n be the number of aces in your hand. You then randomly pick one of the cards in the hand and reveal it.

(a) Describe a simple probability space (that is, outcomes and their probabilities) for this scenario, and list the outcomes in each of the following events:

1. $[n \geq 1]$, (that is, your hand has an ace in it),
2. A♡ is in your hand,
3. the revealed card is an A♡,
4. the revealed card is an ace.

COMMENTS:

- PS_conditional_aces
- from S07.ps11
- soln edited ARM 10/23/15

keywords = [*probability probability_space conditional_probability outcomes*]

Solution. Consider each outcome as a pair of cards, the first of which is the revealed card. Each outcome is equally likely (probability $1/6$).

The sets of outcomes are then as follows:

1. $[n \geq 1]$: all pairs:

$$\{(\text{A}\heartsuit, \text{A}\spadesuit), (\text{A}\heartsuit, \text{jack}), (\text{A}\spadesuit, \text{A}\heartsuit), \\ (\text{A}\spadesuit, \text{jack}), (\text{jack}, \text{A}\heartsuit), (\text{jack}, \text{A}\spadesuit)\},$$

2. A$\heartsuit$ is in your hand: $\{(\text{A}\heartsuit, \text{A}\spadesuit), (\text{A}\heartsuit, \text{jack}), (\text{A}\spadesuit, \text{A}\heartsuit), (\text{jack}, \text{A}\heartsuit)\}$,
3. the revealed card is an A$\heartsuit$: $\{(\text{A}\heartsuit, \text{A}\spadesuit), (\text{A}\heartsuit, \text{jack})\}$,
4. the revealed card is an ace: $\{(\text{A}\heartsuit, \text{A}\spadesuit), (\text{A}\heartsuit, \text{jack}), (\text{A}\spadesuit, \text{A}\heartsuit), (\text{A}\spadesuit, \text{jack})\}$.

■

(b) Then calculate $\Pr\left[n = 2 \mid E\right]$ for E equal to each of the four events in part (a). Notice that most, but *not all*, of these probabilities are equal.

Solution. First, note that $\Pr[n = 2] = 1/3$.

1. $\Pr\left[n = 2 \mid n \geq 1\right] = \Pr[n = 2]/1 = 1/3$,
2. $\Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right] = \Pr[n = 2]/(2/3) = 1/2$,
3. $\Pr\left[n = 2 \mid \text{the revealed card is an A}\heartsuit\right] = \Pr[(\text{A}\heartsuit, \text{A}\spadesuit)]/(1/3) = 1/2$,
4. $\Pr\left[n = 2 \mid \text{the revealed card is an ace}\right] = \Pr[n = 2]/(2/3) = 1/2$.

■

Now suppose you have a deck with d distinct cards, a different kinds of aces (including an A$\heartsuit$), you draw a random hand with h cards, and then reveal a random card from your hand.

(c) Prove that $\Pr[\text{A}\heartsuit \text{ is in your hand}] = h/d$.

Solution. The number N of hands is

$$N = \binom{d}{h}.$$

$$
\begin{aligned}
&\Pr[\text{A}\heartsuit \text{ is in your hand}] \\
&= \frac{\#\text{ hands with A}\heartsuit}{N} \\
&= \frac{\#\ (h-1)\text{-card hands from deck w/o A}\heartsuit}{N} \\
&= \frac{\binom{d-1}{h-1}}{N} \\
&= \frac{(d-1)!h!(d-h)!}{(h-1)!(d-h)!d!} && \text{(def. of } \tbinom{m}{n}\text{)} \\
&= h/d. && \text{(simplification)}
\end{aligned}
$$

■

(d) Prove that

$$\Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right] = \Pr[n = 2] \cdot \frac{2d}{ah}. \tag{2}$$

Solution.

$$
\begin{aligned}
&\Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right] \\
&= \frac{\Pr[n = 2 \text{ and A}\heartsuit \text{ is in your hand}]}{\Pr[\text{A}\heartsuit \text{ is in your hand}]} \\
&= \frac{\Pr[n = 2 \text{ and A}\heartsuit \text{ is in your hand}]}{h/d} && \text{(part (c))} \\
&= \frac{\Pr[n = 2] \cdot \Pr\left[\text{A}\heartsuit \text{ is in your hand} \mid n = 2\right]}{h/d} \\
&= \frac{\Pr[n = 2] \cdot 2/a}{h/d} && \text{(see below)} \\
&= \Pr[n = 2] \cdot \frac{2d}{ah}
\end{aligned}
$$

The $\frac{2}{a}$ substitution above is justified by observing that in a hand with two aces, each of the $\binom{a}{2}$ pairs of aces is equally likely to be the one in the hand. Of these pairs, $a - 1$ include the A♡. So

$$\Pr\left[\text{A}\heartsuit \text{ is in your hand} \mid n = 2\right] = \frac{a-1}{\binom{a}{2}} = \frac{2}{a}.$$

■

(e) Conclude that

$$\Pr\left[n = 2 \mid \text{the revealed card is an ace}\right] = \Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right].$$

Solution. Note that

$$\Pr[\text{the revealed card is an ace}] = \frac{a}{d}, \tag{3}$$

since the probability of revealing an ace from the random hand is simply the probability that a random card is an ace. Now,

$$
\begin{aligned}
&\Pr\left[n = 2 \mid \text{the revealed card is an ace}\right] \\
&= \frac{\Pr[n = 2 \text{ and the revealed card is an ace}]}{\Pr[\text{the revealed card is an ace}]} \\
&= \frac{\Pr[n = 2 \text{ and the revealed card is an ace}]}{a/d} && \text{(by (3))} \\
&= \frac{\Pr[n = 2]\Pr\left[\text{the revealed card is an ace} \mid n = 2\right]}{a/d} \\
&= \frac{\Pr[n = 2](2/h)}{a/d} && \text{(see below)} \\
&= \frac{2d}{ah} \cdot \Pr[n = 2] \\
&= \Pr\left[n = 2 \mid \text{A}\heartsuit \text{ is in your hand}\right]. && \text{(by (2))}
\end{aligned}
$$

The $\frac{2}{h}$ substitution for $\Pr\left[\text{the revealed card is an ace} \mid n = 2\right]$ is justified by noting that, in any hand with 2 aces, that is, when $n = 2$, the revealed card must come from one of exactly 2 positions (with aces) out of h total positions.

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Problem 4.
There is a subject—naturally not *Math for Computer Science*—in which 10% of the assigned problems contain errors. If you ask a Teaching Assistant (TA) whether a problem has an error, then they will answer correctly 80% of the time, regardless of whether or not a problem has an error. If you ask a lecturer, he will identify whether or not there is an error with only 75% accuracy.

We formulate this as an experiment of choosing one problem randomly and asking a particular TA and Lecturer about it. Define the following events:

$$
\begin{aligned}
E &::= [\text{the problem has an error}], \\
T &::= [\text{the TA says the problem has an error}], \\
L &::= [\text{the lecturer says the problem has an error}].
\end{aligned}
$$

(a) Translate the description above into a precise set of equations involving conditional probabilities among the events E, T and L.

COMMENTS:

- PS_conditional_probability_problem_errors
- from: S09.ps11

keywords = [*conditional_probability independence Bayes*]

Solution. The assumptions above tell us:

$$
\begin{aligned}
\Pr[E] &= \frac{10}{100} = \frac{1}{10}, \\
\Pr\left[T \mid E\right] = \Pr\left[\overline{T} \mid \overline{E}\right] &= \frac{80}{100} = \frac{4}{5}, \\
\Pr\left[L \mid E\right] = \Pr\left[\overline{L} \mid \overline{E}\right] &= \frac{75}{100} = \frac{3}{4},
\end{aligned}
$$

Also, T and L are independent given E and given $\overline{E}$:

$$\Pr\left[T \cap L \mid E\right] = \Pr\left[T \mid E\right]\Pr\left[L \mid E\right]$$
$$\Pr\left[T \cap L \mid \overline{E}\right] = \Pr\left[T \mid \overline{E}\right]\Pr\left[L \mid \overline{E}\right]$$

Note that while we know that T and L are independent *given* E or *given* $\overline{E}$, they are not independent by themselves, see part (c). ■

(b) Suppose you have doubts about a problem and ask a TA about it, and they tell you that the problem is correct. To double-check, you ask a lecturer, who says that the problem has an error. Assuming that the correctness of the lecturer's answer and the TA's answer are independent of each other, regardless of whether there is an error, what is the probability that there is an error in the problem?

Solution. We want to calculate

$$\Pr\left[E \mid \overline{T} \cap L\right].$$

From the definition of conditional probability (this is known as *Bayes' rule*):

$$\Pr\left[E \mid \overline{T} \cap L\right] = \Pr[E]\frac{\Pr\left[\overline{T} \cap L \mid E\right]}{\Pr[\overline{T} \cap L]}. \tag{4}$$

By the independence assumptions, we have:

$$\Pr\left[\overline{T} \cap L \mid E\right] = \Pr\left[\overline{T} \mid E\right]\Pr\left[L \mid E\right] = \frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20},$$
$$\Pr\left[\overline{T} \cap L \mid \overline{E}\right] = \Pr\left[\overline{T} \mid \overline{E}\right]\Pr\left[L \mid \overline{E}\right] = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5},$$
$$\Pr[\overline{T} \cap L] = \Pr\left[\overline{T} \cap L \mid E\right]\Pr[E] + \Pr\left[\overline{T} \cap L \mid \overline{E}\right]\Pr[\overline{E}]$$
$$= \frac{3}{20} \cdot \frac{1}{10} + \frac{1}{5} \cdot \frac{9}{10} = \frac{39}{200}.$$

Substituting these values in equation (4), we get

$$\Pr\left[E \mid \overline{T} \cap L\right] = \frac{1}{10} \cdot \frac{3/20}{39/200} = \frac{1}{13} \approx 0.077.$$

So this contradictory information has decreased the probability of an error from 10% to about 7.7%.

The calculations here support the common-sense rule that when two people make contradictory statements, you should be influenced more by the most "authoritative" person—the one who is right more often. But note that this does not mean that you should *believe* in what the most authoritative person says, since the probability of an error remains uncomfortably large. ■

(c) Is event T independent of event L (that is, $\Pr\left[T \mid L\right] = \Pr[T]$)? First, give an argument based on intuition, and then calculate both probabilities to verify your intuition.

Solution. The answer is no. Because the TA is usually right, when the TA says that the problem has an error, the likelihood that there really is an error is increased. But the lecturer is also usually right, so increasing the likelihood of there *being* an error also increases the likelihood that the lecturer will *report* an error.

We verify this informal argument by actually calculating the probability of each of these events and their conjunction, and observing that the probability that the two events occur is different from the product of the probabilities. Let events E, T, L be as above.

$$\begin{aligned}
\Pr[T] &= \Pr[T \cap E] + \Pr[T \cap \overline{E}] \\
&= \Pr\left[T \mid E\right] \Pr[E] + \Pr\left[T \mid \overline{E}\right] \Pr[\overline{E}] \\
&= \frac{4}{5}\frac{1}{10} + (1 - \frac{4}{5})(1 - \frac{1}{10}) = \frac{13}{50}, \\
\Pr[L] &= \Pr[L \cap E] + \Pr[L \cap \overline{E}] \\
&= \frac{3}{4}\frac{1}{10} + (1 - \frac{3}{4})(1 - \frac{1}{10}) = \frac{3}{10}, \\
\Pr[L \cap T] &= \Pr[L \cap T \cap E] + \Pr[L \cap T \cap \overline{E}] \\
&= \Pr\left[L \cap T \mid E\right] \Pr[E] + \Pr\left[L \cap T \mid \overline{E}\right] \Pr[\overline{E}] \\
&= \Pr\left[L \mid E\right] \Pr\left[T \mid E\right] \Pr[E] + \Pr\left[L \mid \overline{E}\right] \Pr\left[T \mid \overline{E}\right] \Pr[\overline{E}] \\
&= \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{1}{10} + (1 - \frac{3}{4})(1 - \frac{4}{5}) \cdot (1 - \frac{1}{10}) = \frac{105}{1000} = 0.105,
\end{aligned}$$

which is higher than

$$\Pr[L] \Pr[T] = \frac{3}{10} \cdot \frac{13}{50} = .078.$$

■

probability$_{p}roduct_{r}ule$

Problem 5.

We want to count step-by-step paths between points with integer coordinates in three dimensions. A step may move a unit distance in the positive x, y or z direction. For example, a step from point $(2, 3, 7)$ in the y direction leads to $(2, 4, 7)$.

For points $\mathbf{p}$ and $\mathbf{q}$ we write $\mathbf{p} \leq \mathbf{q}$ to mean that $\mathbf{p}$ is coordinatewise less than or equal to $\mathbf{q}$. That is, if $\mathbf{p} = (p_x, p_y, p_z)$ and $\mathbf{q} = (q_x, q_y, q_z)$, then

$$\mathbf{p} \leq \mathbf{q} \quad ::= \quad [p_x \leq q_x \text{ AND } p_y \leq q_y \text{ AND } p_z \leq q_z].$$

So there is a path from $\mathbf{p}$ to $\mathbf{q}$ iff $\mathbf{p} \leq \mathbf{q}$.

(a) Let $P_{\{\mathbf{p},\mathbf{q}\}}$ be the set of paths from $\mathbf{p}$ to $\mathbf{q}$. Suppose that $\mathbf{p} \leq \mathbf{q}$, and let $d_x ::= q_x - p_x$, and likewise for d_y and d_z. Express the number of paths $|P_{\{\mathbf{p},\mathbf{q}\}}|$ as a multinomial coefficient involving the preceding quantities.

COMMENTS:

- PS_inclusion-exclusion_paths
- F13.ps10
- perturbed from PS_inclusion-exclusion_paths_afternoon, PS_path_counting
- ARM 11/7/13

keywords = [*counting inclusion_exclsuion paths multinomial*]

Solution. There is an obvious bijection between the set $P_{\{\mathbf{p},\mathbf{q}\}}$ and the strings consisting of d_x x's, d_y y's, and d_z z's. So

$$|P_{\{\mathbf{p},\mathbf{q}\}}| = \binom{d_x + d_y + d_y}{d_x, d_y, d_z}.$$

■

More generally, for any set S of points, let

$$P_S ::= \text{the paths that go through all the points in } S.$$

(b) Suppose $\mathbf{a} \le \mathbf{b} \le \mathbf{c} \le \mathbf{d}$. Express $|P_{\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}}|$ in terms of $|P_{\{\mathbf{p},\mathbf{q}\}}|$ for various $\mathbf{p}, \mathbf{q} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$.

Solution.

$$|P_{\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}}| = |P_{\{\mathbf{a},\mathbf{b}\}}| \cdot |P_{\{\mathbf{b},\mathbf{c}\}}| \cdot |P_{\{\mathbf{c},\mathbf{d}\}}|$$

■

(c) Let

$$\begin{aligned}
&\mathbf{o} ::= (0,0,0),\\
&\mathbf{a} ::= (3,7,11), \quad \mathbf{b} ::= (11,6,3), \quad \mathbf{c} ::= (10,5,40),\\
&\mathbf{d} ::= (12,13,14), \quad \mathbf{e} ::= (12,6,45),\\
&\mathbf{f} ::= (50,50,50).
\end{aligned}$$

Let N be the paths in $P_{\mathbf{o},\mathbf{f}}$ that do *not* go through any of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$. Express $|N|$ as an arithmetic combination of $|P_S|$ for various $S \subseteq \{\mathbf{o}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$. Do not include any terms $|P_S|$ that equal zero.

Solution. N is the union of the points that do *not* go through at least one of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$. So it equals $|P_{\mathbf{o},\mathbf{f}}|$ minus the number k of paths that go through at least one of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$.

By Inclusion-Exclusion

$$\begin{aligned}
k = {} & |P_{\mathbf{o},\mathbf{a},\mathbf{f}}| + |P_{\mathbf{o},\mathbf{b},\mathbf{f}}| + |P_{\mathbf{o},\mathbf{c},\mathbf{f}}|\\
& + |P_{\mathbf{o},\mathbf{d},\mathbf{f}}| + |P_{\mathbf{o},\mathbf{e},\mathbf{f}}|\\
-|P_{\mathbf{o},\mathbf{a},\mathbf{d},\mathbf{f}}| - |P_{\mathbf{o},\mathbf{b},\mathbf{d},\mathbf{f}}| - |P_{\mathbf{o},\mathbf{b},\mathbf{e},\mathbf{f}}| - |P_{\mathbf{o},\mathbf{c},\mathbf{e},\mathbf{f}}|
\end{aligned}$$

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