Graph Theory and Stable Matching

October 23, 2016

1 Graph Theory

Relevant material: pset 4 – problems 4, 5 and 6
pset 5 – problems 1 and 2
pset 6 – problems 1 and 2
recitation 6
Midterm practice – problems 5 and 7
2014 – problem 3 and 7

- A **graph** $G = (V, E)$ is a set of vertices and edges between them.
- The **degree** of a vertex is the number of edges incident to it.
- Two graphs G_1 and G_2 are **isomorphic** if there exists a bijection $f : V_1 \to V_2$ such that every edge in E_1 corresponds to exactly one edge in E_2. That is, G_1 is isomorphic to G_2 if we can relabel G_2 with the vertex names of G_1, and find that the edges(relationships) between each pair of vertices are the same.
- To disprove isomorphism, look for properties that are not maintained. For example, if one graph has a vertex of degree 6 and the other does not, they cannot be isomorphic.
- $G' = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.
- A graph G is **bipartite** if its vertices can be split into sets V_1, V_2 such that vertices in V_1 only have edges going to V_2, and vice versa.
- A graph is **k-colorable** if there is a way to assign k colors to the vertices of the graph such that if a vertex is colored c, it is not adjacent to any other vertices of color c.
- Bipartite $\Leftrightarrow$ 2-colorable $\Leftrightarrow$ the graph has no odd-length cycles.
- Vertices v_1 and v_2 are **connected** if there is a path in the graph from v_1 to v_2.
- A graph G is connected if every vertex has a path to every other vertex.
- A **connected component** is a connected subgraph of a graph.
- A **closed walk** is a set of vertices $v_0 v_1 \dots v_k$ where each v_i is adjacent to v_{i+1} and where $v_0 = v_k$. If each of the v_i are distinct, a closed walk is called a **cycle**.

- An **Euler walk** is a walk on a graph that traverses every edge exactly once. An **Euler tour** is an Euler walk where the start and end nodes are the same.
- A connected graph has an Euler tour $\Leftrightarrow$ every vertex has even degree.
- A **Hamiltonian cycle** is a cycle that visits every node exactly once.
- An undirected graph G has **width** w if the vertices can be arranged in a sequence

  $$v_1,\ v_2,\ v_3,\ \ldots,\ v_n$$

  such that each vertex v_i is joined by an edge to at most w preceding vertices. (Vertex v_j *precedes* v_i if $j < i$.)

2 Stable Matching

Relevant material: pset 5 – problems 3
recitation 7
Midterm practice – problem 6
2014 – problem 6

- Let $G = (V, E)$ be a bipartite graph, with left vertex set L and right vertex set R. Recall that for a subset S of the vertices, $N(S)$ is the set of vertices which are adjacent to some vertex in S:

  $$N(S) = \{r \in V \mid \{r, s\} \in E \text{ for some } s \in S\}.$$

  Halls' theorem says that if for every subset S of L we have $|N(S)| \geq |S|$, then there is a matching in G that covers L.
- The Mating Algorithm returns a matching where every boy is matched with his optimal mate.
- The Mating Algorithm returns a matching where every girl is matched with her pessimal mate.
- If the boy optimal matching is the same as the boy pessimal matching, there is only one stable matching