Problem Set 11

Due: Tuesday, November 29, 7:30pm

Problem 1. [20 points] You are organizing a neighborhood census and instruct your census takers to knock on doors and note the sex of any child that answers the knock. Assume that there are two children in a household, that children are equally likely to be girls and boys, and that girls and boys are equally likely to open the door.

A sample space for this experiment has outcomes that are triples whose first element is either B or G for the sex of the elder child, likewise for the second element and the sex of the younger child, and whose third coordinate is E or Y indicating whether the *e*lder child or *y*ounger child opened the door. For example, $(\mathtt{B}, \mathtt{G}, \mathtt{Y})$ is the outcome that the elder child is a boy, the younger child is a girl, and the girl opened the door.

(a) [5 pts] Let T be the event that the household has two girls, and O be the event that a girl opened the door. List the outcomes in T and O.

(b) [5 pts] What is the probability $\Pr(T \mid O)$, that both children are girls, given that a girl opened the door?

(c) [10 pts] Where is the mistake in the following argument for computing $\Pr(T \mid O)$?

> If a girl opens the door, then we know that there is at least one girl in the household. The probability that there is at least one girl is
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> $$1 - \Pr(\text{both children are boys}) = 1 - (1/2 \times 1/2) = 3/4.$$
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> So,
>
> $$\begin{aligned} &\Pr(T \mid \text{there is at least one girl in the household}) \\ &\quad = \frac{\Pr(T \cap \text{there is at least one girl in the household})}{\Pr\{\text{there is at least one girl in the household}\}} \\ &\quad = \frac{\Pr(T)}{\Pr\{\text{there is at least one girl in the household}\}} \\ &\quad = (1/4)/(3/4) = 1/3. \end{aligned}$$
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> Therefore, given that a girl opened the door, the probability that there are two girls in the household is 1/3.

Problem 2. [20 points]

(a) [7 pts] Suppose you repeatedly flip a fair coin until you see the sequence `HHT` or the sequence `TTH`. What is the probability you will see `HHT` first?

Hint: Use a bijection argument.

(b) [7 pts] What is the probability you see the sequence `HTT` before you see the sequence `HHT`?

Hint: Try to find the probability that `HHT` comes before `HTT` conditioning on whether you first toss an `H` or a `T`. Somewhat surprisingly, the answer is not $1/2$.

(c) [6 pts] Suppose you flip three fair coins. Define the following events:

- Let A be the event that *the first* coin is heads.
- Let B be the event that *the second* coin is heads.
- Let C be the event that *the third* coin is heads.
- Let D be the event that *an even number of* coins are heads.

Use the four step method to determine the probability of each of A, B, C, D.

Problem 3. [20 points] Suppose you have seven standard dice with faces numbered 1 to 6. Each die has a label corresponding to a letter of the alphabet (A through G). A *roll* is a sequence specifying a value for each die in alphabet order. For example, one roll is $(6, 1, 4, 1, 3, 5, 2)$ indicating that die A showed a 6, die B showed 1, die C showed 4, $\ldots$.

(a) [5 pts] What is the probability of a roll where *exactly* two dice have the value 3 and the remaining five dice all have different values?

Example: $(3, 2, 3, 1, 6, 4, 5)$ is a roll of this type, but $(1, 1, 2, 6, 3, 4, 5)$ and $(3, 3, 1, 2, 4, 6, 4)$ are not.

(b) [5 pts] What is the probability of a roll where two dice have an even value and the remaining five dice all have different values?

Example: $(4, 2, 4, 1, 3, 6, 5)$ is a roll of this type, but $(1, 1, 2, 6, 1, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

(c) [10 pts] What is the probability of a roll where two dice have one value, two different dice have a second value, and the remaining three dice a third value?

Example: $(6, 1, 2, 1, 2, 6, 6)$ is a roll of this type, but $(4, 4, 4, 4, 1, 3, 5)$ and $(5, 5, 5, 6, 6, 1, 2)$ are not.