

## Midterm Practice Problems

Name: \_\_\_\_\_

- This quiz is **closed book**, but you may have one  $8.5 \times 11$ " sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.

**Problem 1. [10 points]** In problem set 1 you showed that the **nand** operator by itself can be used to write equivalent expressions for all other Boolean logical operators. We call such an operator *universal*. Another universal operator is **nor**, defined such that  $P \text{ nor } Q \Leftrightarrow \neg(P \vee Q)$ . Show how to express  $P \wedge Q$  in terms of: **nor**,  $P$ ,  $Q$ , and grouping parentheses.

**Problem 2. [15 points]** We define the sequence of numbers

$$a_n = \begin{cases} 1 & \text{if } 0 \leq n \leq 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \geq 4. \end{cases}$$

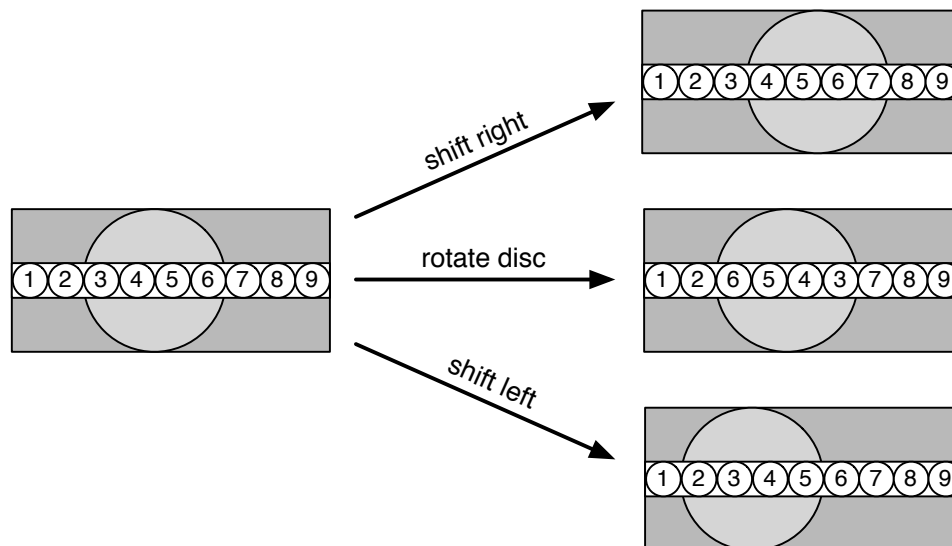
Prove that  $a_n \equiv 1 \pmod{3}$  for all  $n \geq 0$ .

**Problem 3. [20 points]** The Slipped Disc Puzzle™ consists of a track holding 9 circular tiles. In the middle is a disc that can slide left and right and rotate  $180^\circ$  to change the positions of *exactly* four tiles. As shown below, there are three ways to manipulate the puzzle:

**Shift Right:** The center disc is moved one unit to the right (if there is space)

**Rotate Disc:** The four tiles in the center disc are reversed

**Shift Left:** The center disc is moved one unit to the left (if there is space)



Prove that if the puzzle starts in an initial state with all but tiles 1 and 2 in their natural order, then it is impossible to reach a goal state where all the tiles are in their natural order. The initial and goal states are shown below:

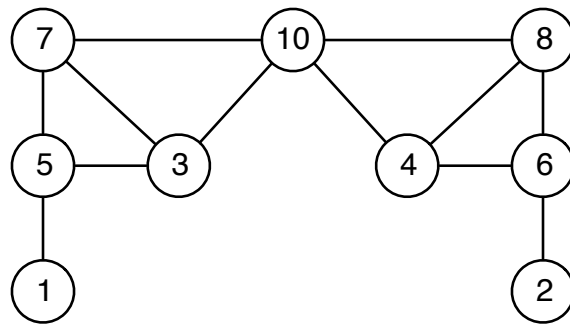


Write your proof on the next page...

room for problem 3...

**Problem 4.** [10 points] Find the multiplicative inverse of 17 modulo 72 in the range  $\{0, 1, \dots, 71\}$ .

**Problem 5. [15 points]** Consider a graph representing the main campus buildings at MIT.

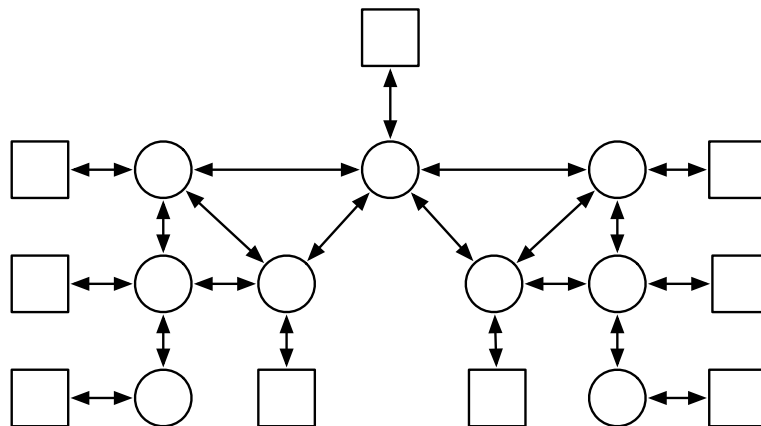


(a) [3 pts] Is this graph bipartite? Provide a brief argument for your answer.

(b) [4 pts] Does this graph have an Euler circuit? Provide a brief argument for your answer.

**Problem 5 continued...**

Now suppose each building has separate mail collection and drop-off boxes and each collection box has a single package destined for a unique drop-off box (i.e. a permutation). We can model this as a permutation routing problem by treating the buildings as switches, attaching an input and output terminal to each of the nine buildings, and treating the existing edges as bidirectional as in the graph below:



(c) [4 pts] Give the diameter of this graph:

(d) [4 pts] What is the max congestion of this graph? That is, in the worst case permutation, how many packages would need to pass through a single building? Provide a brief argument for your answer.

**Problem 6. [10 points]**

A tournament graph  $G = (V, E)$  is a directed graph such that there is either an edge from  $u$  to  $v$  or an edge from  $v$  to  $u$  for *every* distinct pair of nodes  $u$  and  $v$ . (The nodes represent players and an edge  $u \rightarrow v$  indicates that player  $u$  beats player  $v$ .)

Consider the “beats” relation implied by a tournament graph. Indicate whether or not each of the following relational properties hold *for all* tournament graphs and briefly explain your reasoning. You may assume that a player never plays herself.

1. **transitive**

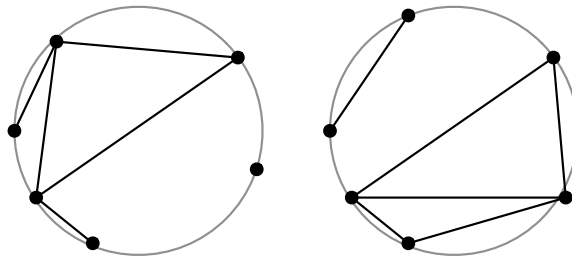
2. **symmetric**

3. **antisymmetric**

4. **reflexive**



**Problem 7. [20 points]** An outerplanar graph is an undirected graph for which the vertices *can be* placed on a circle in such a way that no edges (drawn as straight lines) cross each other. For example, the complete graph on 4 vertices,  $K_4$ , is not outerplanar but any proper subgraph of  $K_4$  with strictly fewer edges is outerplanar. Some examples are provided below:



Prove that any outerplanar graph is 3-colorable. A fact you may use without proof is that any outerplanar graph has a vertex of degree at most 2.

**Problem 8. [10 points]** Give upper and lower bounds for the following expression which differ by at most 1.

$$\sum_{i=1}^n \frac{1}{i^3}$$

**Problem 9. [15 points]** Circle every symbol on the left that could correctly appear in the box to its right. For each of the six parts you may need to circle any number of symbols.

(a)  $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$   $6n^2 + 7n - 10 = \boxed{\phantom{0000}} (n^2)$

(b)  $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$   $6^n = \boxed{\phantom{0000}} (n^6)$

(c)  $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$   $n! = \boxed{\phantom{0000}} (n^n)$

(d)  $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$   $\sum_{j=1}^n \frac{1}{j} = \boxed{\phantom{0000}} (\ln n)$

(e)  $O \quad \Omega \quad \Theta \quad o \quad \omega \quad \sim$   $\ln(n^3) = \boxed{\phantom{0000}} (\ln n)$