Second Order Differential Equations

General second order Linear Diff eqn is given by

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = b(x) \quad (1)$$

Here $a_0(x) \neq 0$. a_1, a_2 and b are given functions.

Soln of (1) will exist if a_0, a_1, a_2 and b are given continuous functions.

Standard Second LDEqu^ns.

① — $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y(x) = R(x)$ is the standard second order LDEs.

If $R(x) \neq 0$, then equ^n ① is called non-homogeneous second order LDEs.

If $R(x) = 0$, then equ^n ① becomes

$$y'' + P(x)\,y' + Q(x)\,y = 0 \quad \text{—— ②.}$$

Equ^n ② is homogeneous 2nd Order LDEs.

Eq^n ② also known as reduced form of eq^n ①

Sol^{n}s of Eq^n ① & ②

Theorem A : Let $P(x), Q(x)$ and $R(x)$ are continuous functions on $[a,b]$. If $\exists$ $x_0 \in [a,b]$, then eq^n

$$\begin{cases} y'' + P(x)y' + Q(x)y = 0 \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

has unique sol^n satisfying the Initial conditions.

Note: If $y_1(x)$ and $y_2(x)$ be the two solu^{n}s of equ^n ①, then $y_1(x)+y_2(x)$ will not be a solu^n of ①(NH).

Verify it:
$$\frac{d^2}{dx^2}(y_1+y_2)+P(x)[y_1+y_2]'+Q(x)[y_1+y_2]$$

$$=\underbrace{[y_1''+P(x)y_1'+Q(x)y_1]}_{R(x)}+\underbrace{[y_2''+P(x)y_2'(x)+Q(x)y_2]}_{R(x)}$$

$$=2R(x)$$

$$\neq R(x).$$ Hence y_1+y_2 is not a solu^n of NH (1).

If $y_1(x)$ & $y_2(x)$ be the soln's of $\underline{y''+P(x)y'+Q(x)y=0}$

then $c_1y_1+c_2y_2$ be the soln of eqn ②.

$y''+P(x)y'+Q(x)y=0$, Trivial soln $y=0$ is always a soln of Homogeneous Eqn's. NH may not have a soln always.

We have $y''+P(x)y'+Q(x)y=0$ —— ②

Let $y=y_g(x,c_1,c_2)$ be the general soln of eqn ②.

and y_p be the particular soln of eqn ①(NH).

Let y be any soln of eqn ① (NH). Then,

we can see that $y - y_p$ be the soln of eqn ②.

Here $\underline{(y-y_p)'' + P(x)(y-y_p)' + Q(x)[y-y_p]}$

$$= \underbrace{[y'' + P(x)y' + Q(x)y]}_{= R(x)} - \underbrace{[y_p'' + P(x)y_p' + Q(x)y_p]}_{= R(x)}$$

$= 0$ $\Rightarrow$ $(y-y_p)$ is the soln of $y'' + P(x)y' + Q(x)y = 0$

Here $y - y_p$ is the general soln of eqn (2).

$$\therefore \quad y - y_p = y_g(x, c_1, c_2)$$

$$\Rightarrow \boxed{y = y_g(x, c_1, c_2) + y_p} \quad \text{--- (3)}$$

where y_g is the general soln of eqn (2) & y_p be the particular soln of NH eqn (1).

y in eqn (3) is the general soln of NH eqn (1).

Example. Show that $x^2y'' - 4xy' + (x^2+6)y = 0$

has two sol^{n}s $y = x^2 \sin x$ and $y = 0$

satisfying the ICs $y(0) = 0$ & $y'(0) = 0$.

Does this example contradict the theorem A.

Def^n: Let y_1 and y_2 be two functions. Then,

the wronskian of y_1 and y_2 is defined

as
$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1y_2' - y_2y_1'.$$

Def^n: Let y_1 and y_2 be two functions.

If $c_1 y_1(x) + c_2 y_2(x) = 0 \Rightarrow c_1 = c_2 = 0$, then $y_1(x)$ and $y_2(x)$ are <u>Linearly Independent</u> (LI) on $[a,b]$ when $x \in [a,b]$.

Note: If y_1 & y_2 are not LI, then they are LD (linearly dependent).