CHAPTER 17

Partial Differential Equations

17.1 INTRODUCTION

The reader has, already been introduced to the notion of partial differential equations. Here, we shall begin by studying the ways in which partial differential equations are formed. Then we shall investigate the solutions of special types of partial differential equations of the first and higher orders.

In what follows x and y will, usually be taken as the independent variables and z, the dependent variable so that $z = f(x, y)$ and we shall employ the following notation :

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

17.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Unlike the case of ordinary differential equations which arise from the elimination of arbitrary constants; *the partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.* The method is best illustrated by the following examples :

Example 17.1. *Derive a partial differential equation (by eliminating the constants) from the equation*

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \qquad ...(i)$$

Solution. Differentiating (*i*) partially with respect to x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

and

$$\frac{2\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (*i*), we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

Example 17.2. *Form the partial differential equations (by eliminating the arbitrary functions) from*
(*a*) $z = (x + y)\,\phi\,(x^2 - y^2)$ (*P.T.U., 2009*)
(*b*) $z = f(x + at) + g(x - at)$ (*V.T.U., 2009*) (*c*) $f(x^2 + y^2, z - xy) = 0$ (*S.V.T.U., 2007*)

Solution. (*a*) We have $z = (x + y)\,\phi\,(x^2 - y^2)$

Differentiating z partially with respect to x and y,

$$p = \frac{\partial z}{\partial x} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,2x + \phi\,(x^2 - y^2), \qquad ...(i)$$

$$q = \frac{\partial z}{\partial y} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,(-\,2y) + \phi\,(x^2 - y^2) \qquad ...(ii)$$

From (*i*), $\quad p - \dfrac{z}{x + y} = 2x\,(x + y)\,\phi'\,(x^2 - y^2)$

From (*ii*), $\quad q - \dfrac{z}{x + y} = -\,2y\,(x + y)\,\phi'\,(x^2 - y^2)$

Division gives $\dfrac{p - z/(x + y)}{q - z/(x + y)} = -\dfrac{x}{y}$

i.e., $\quad [p(x + y) - z]y + [q\,(x + y) - z]x$

i.e., $\quad (x + y)\,(py + qx) - z(x + y) = 0$

Hence $\quad py + qz = z$ is required equation.

(*b*) We have $\quad z = f\,(x + at) + g(x - at) \qquad ...(i)$

Differentiating z partially with respect to x and t,

$$\frac{\partial z}{\partial x} = f'\,(x + at) + g'(x - at),\ \frac{\partial^2 z}{\partial x^2} = f''\,(x + at) + g''(x - at) \qquad ...(ii)$$

$$\frac{\partial z}{\partial t} = af'\,(x + at) - ag'(x - at),\ \frac{\partial^2 z}{\partial t^2} = a^2 f''\,(x + at) + a^2 g''(x - at) = a^2\frac{\partial^2 z}{\partial x^2} \qquad [\text{By } (ii)]$$

Thus the desired partial differential equation is $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$

which is an equation of the second order and (*i*) is its solution.

(*c*) Let $x^2 + y^2 = u$ and $z - xy = v$ so that $f\,(u, v) = 0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

or
$$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-\,y + p) = 0 \qquad ...(i)$$

and
$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(-\,x + q) = 0 \qquad ...(ii)$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (*i*) and (*ii*), we get

$$\begin{vmatrix} 2x & -\,y + p \\ 2y & -\,x + q \end{vmatrix} = 0 \qquad \text{or} \qquad xq - yp = x^2 - y^2.$$

Example 17.3. *Find the differential equation of all planes which are at a constant distance a from the origin.* (*V.T.U., 2009 S ; Kurukshetra, 2006*)

Solution. The equation of the plane in 'normal form' is

$$lx + my + nz = a \qquad ...(i)$$

where l, m, n are the d.c.s of the normal from the origin to the plane.

Then $\quad l^2 + m^2 + n^2 = 1$ or $n = \sqrt{(1 - l^2 - m^2)}$

$\therefore$ (*i*) becomes $\quad lx + my + \sqrt{(1 - l^2 - m^2)}\ z = a \qquad ...(ii)$

Differentiating partially w.r.t. x, we get

$$l + \sqrt{(1 - l^2 - m^2)}\ .\ p = 0 \qquad ...(iii)$$

Differentiating partially w.r.t. y, we get

$$m + \sqrt{(1 - l^2 - m^2)}\ .\ q = 0 \qquad ...(iv)$$

Now we have to eliminate l, m from (*ii*), (*iii*) and (*iv*).

From (*iii*), $l = -\sqrt{(1 - l^2 - m^2)}\ .\ p$ and $m = -\sqrt{(1 - l^2 - m^2)}\ .\ q$

Squaring and adding, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$

or $\quad (l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2$ or $1 - l^2 - m^2 = 1 - \dfrac{p^2 + q^2}{1 + p^2 + q^2} = \dfrac{1}{1 + p^2 + q^2}$

Also $\quad l = -\dfrac{p}{\sqrt{(1 + p^2 + q^2)}}$ and $m = -\dfrac{q}{\sqrt{(1 + p^2 + q^2)}}$

Substituting the values of l, m and $1 - l^2 - m^2$ in (*ii*), we obtain

$$\frac{-px}{\sqrt{(1 + p^2 + q^2)}} - \frac{qy}{\sqrt{(1 + p^2 + q^2)}} + \frac{1}{\sqrt{(1 + p^2 + q^2)}}\ z = a$$

or $\quad z = px + qy + a\sqrt{(1 + p^2 + q^2)}$ which is the required partial differential equation.

PROBLEMS 17.1

From the partial differential equation (by eliminating the arbitrary constants from :

1. $z = ax + by + a^2 + b^2$.
2. $(x - a)^2 + (y - b)^2 + z^2 = c^2$. (*Kottayam, 2005*)
3. $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$ (*Anna, 2009*)
4. $z = a \log \left\{\dfrac{b(y-1)}{1-x}\right\}$ (*J.N.T.U., 2002 S*)
5. Find the differential equation of all spheres of fixed radius having their centres in the xy-plane. (*Madras 2000 S*)
6. Find the differential equation of all spheres whose centres lie on the z-axis. (*Kerala, 2005*)

Form the partial differential equations (by eliminating the arbitrary functions) from :

7. $z = f(x^2 - y^2)$ (*S.V.T.U., 2008*)
8. $z = f(x^2 + y^2) + x + y$ (*Anna, 2009*)
9. $z = yf(x) + xg(y)$. (*V.T.U., 2004*)
10. $z = x^2 f(y) + y^2 g(x)$. (*Anna, 2003*)
11. $z = f(x) + e^y g(x)$.
12. $xyz = \phi(x + y + z)$.
13. $z = f_1(x) f_2(y)$.
14. $z = e^{my} \phi(x - y)$. (*P.T.U., 2002*)
15. $z = y^2 + 2f\left(\dfrac{1}{x} + \log y\right)$. (*V.T.U., 2010 ; J.N.T.U., 2010 ; Madras, 2000*)
16. $z = f_1(y + 2x) + f_2(y - 3x)$. (*Kurukshetra, 2005*)
17. $v = \dfrac{1}{r}[f(r - at) + F(r + at)]$.
18. $z = xf_1(x + t) + f_2(x + t)$.
19. $F(xy + z^2, x + y + z) = 0$. (*V.T.U., 2006*)
20. $F(x + y + z, x^2 + y^2 + z^2) = 0$. (*S.V.T.U., 2007*)
21. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\dfrac{\partial u}{\partial x} + (x^2 - yz)\dfrac{\partial u}{\partial y} + (z^2 - xy)\dfrac{\partial u}{\partial z} = 0$.

17.3 SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

It is clear from the above examples that a partial differential equation can result both from elimination of arbitrary constants and from the elimination of arbitrary functions.

The solution $\quad f(x, y, z, a, b) = 0 \qquad ...(1)$

of a first order partial differential equation which contains two arbitrary constants is called a *complete integral*.

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a particular integral.

If we put $b = \phi(a)$ in (1) and find the envelope of the family of surfaces $f[x, y, z, \phi(a)] = 0$, then we get a solution containing an arbitrary function ϕ, which is called the *general integral*.

The envelope of the family of surfaces (1), with parameters a and b, if it exists, is called a *singular integral*. The singular integral differs from the particular integral in that it is not obtained from the complete integral by giving particular values to the constants.

17.4 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

We now consider such partial differential equations which can be solved by direct integration. In place of the usual constants of integration, we must, however, use arbitrary functions of the variable held fixed.

Example 17.4. *Solve* $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0.$ *(V.T.U., 2010)*

Solution. Integrating twice with respect to x (keeping y fixed),

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x - y) = xf(y) + g(y).$$

Now integrating with respect to y (keeping x fixed)

$$z + x^3y^3 - \frac{1}{4}\cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x)$$

The result may be simplified by writing

$$\int f(y)dy = u(y) \text{ and } \int g(y)dy = v(y).$$

Thus $z = \dfrac{1}{4}\cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$ where u, v, w are arbitrary functions.

Example 17.5. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0$, $z = e^y$ *and* $\dfrac{\partial z}{\partial x} = 1.$

Solution. If z were function of x alone, the solution would have been $z = A\sin x + B\cos x$, where A and B are constants. Since z is a function of x and y, A and B can be arbitrary functions of y. Hence the solution of the given equation is $z = f(y)\sin x + \phi(y)\cos x$

$$\therefore \qquad \frac{\partial z}{\partial x} = f(y)\cos x - \phi(y)\sin x$$

When $x = 0$; $z = e^y$, $\quad \therefore e^y = \phi(y)$. $\quad$ When $x = 0$, $\dfrac{\partial z}{\partial x} = 1$, $\quad \therefore 1 = f(y)$.

Hence the desired solution is $z = \sin x + e^y \cos x$.

Example 17.6. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, *for which* $\dfrac{\partial z}{\partial y} = -2\sin y$ *when* $x = 0$ *and* $z = 0$ *when* y *is an odd multiple of* $\pi/2$. *(V.T.U., 2010 S)*

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t. x, keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \qquad ...(i)$$

When $x = 0$, $\dfrac{\partial z}{\partial y} = -2 \sin y$, $\quad \therefore \quad -2 \sin y = -\sin y + f(y)$ or $f(y) = -\sin y$

$\therefore$ (*i*) becomes $\dfrac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Now integrating w.r.t. y, keeping x constant, we get

$$z = \cos x \cos y + \cos y + g(x) \qquad \text{...(ii)}$$

When y is an odd multiple of $\pi/2$, $z = 0$.

$\therefore \quad 0 = 0 + 0 + g(x)$ or $g(x) = 0$ $\quad [\because \cos(2n+1)\pi/2 = 0]$

Hence from (*ii*), the complete solution is $z = (1 + \cos x) \cos y$.

PROBLEMS 17.2

Solve the following equations :

1. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a.$
2. $\dfrac{\partial^2 z}{\partial x^2} = xy.$
3. $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x.$
4. $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y).$
5. $\dfrac{\partial^2 z}{\partial y^2} = z$, gives that when $y = 0$, $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$
6. $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$, $\dfrac{\partial z}{\partial x} = a \sin y$ and $\dfrac{\partial z}{\partial y} = 0.$

17.5 LINEAR EQUATIONS OF THE FIRST ORDER

A linear partial differential equation of the first order, commonly known as *Lagrange's Linear equation**, is of the form

$$Pp + Qq = R \qquad \text{...(1)}$$

where P, Q and R are functions of x, y, z. This equation is called a *quasi-linear equation*. When P, Q and R are independent of z it is known as *linear equation*.

Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u, v) = 0$ ...(2)

where u, v are some functions of x, y, z.

Differentiating (2) partially with respect to x and y.

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0 \text{ and } \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0.$$

Eliminating $\dfrac{\partial \phi}{\partial u}$ and $\dfrac{\partial \phi}{\partial v}$, we get $\begin{vmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z} p & \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z} p \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} q & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial z} q \end{vmatrix} = 0$

which simplifies to $\left(\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}\right)p + \left(\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}\right)q = \left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right)$ ...(3)

This is of the same form as (1).

Now suppose $u = a$ and $v = b$, where a, b are constants, so that

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = dv = 0.$$

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*See footnote p. 142.

By cross-multiplication, we have

$$\frac{dx}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z}-\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}}=\frac{dy}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}}=\frac{dz}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}-\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}.$$

or $$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \text{...(4) [By virtue of (1) and (3)]}$$

The solutions of these equations are $u = a$ and $v = b$.

$\therefore$ $\phi(u, v) = 0$ is the required solution of (1).

Thus to solve the equation $Pp + Qq = R$.

(*i*) *form the subsidiary equations* $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$.

(*ii*) *solve these simultaneous equations by the method of § 16.10 giving* $u = a$ *and* $v = b$ *as its solutions.*

(*iii*) *write the complete solution as* $\phi(u, v) = 0$ or $u = f(v)$.

Example 17.7. *Solve* $\dfrac{y^2 z}{x}p + xzq = y^2$. *(Kottayam, 2005)*

Solution. Rewriting the given equation as

$$y^2zp + x^2zq = y^2x,$$

The subsidiary equations are $\dfrac{dx}{y^2z}=\dfrac{dy}{x^2z}=\dfrac{dz}{y^2x}$

The first two fractions give $x^2dx = y^2dy$.

Integrating, we get $\quad x^3 - y^3 = a \qquad$ ...(*i*)

Again the first and third fractions give $xdx = zdz$

Integrating, we get $\quad x^2 - z^2 = b \qquad$ ...(*ii*)

Hence from (*i*) and (*ii*), the complete solution is

$$x^3 - y^3 = f(x^2 - z^2).$$

Example 17.8. *Solve* $(mz - ny)\dfrac{\partial z}{\partial x} + (nx - lz)\dfrac{\partial z}{\partial y} = ly - mx$. *(V.T.U., 2010 ; S.V.T.U., 2009)*

Solution. Here the subsidiary equations are $\dfrac{dx}{mz-ny}=\dfrac{dy}{mx-lz}=\dfrac{dz}{ly-mx}$

Using multipliers x, y, and z, we get each fraction $= \dfrac{xdx+ydy+zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$ which on integration gives $x^2 + y^2 + z^2 = a \qquad$ ...(*i*)

Again using multipliers l, m and n, we get each fraction $= \dfrac{ldx+mdy+ndz}{0}$

$\therefore$ $ldx + mdy + ndx = 0$ which on integration gives $lx + my + nz = b \qquad$ ...(*ii*)

Hence from (*i*) and (*ii*), the required solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$.

Example 17.9. *Solve* $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. *(V.T.U., 2010 ; Anna, 2009 ; S.V.T.U., 2008)*

Solution. Here the subsidiary equations are $\dfrac{dx}{x^2-y^2-z^2}=\dfrac{dy}{2xy}=\dfrac{dz}{2xz}$

From the last two fractions, we have $\dfrac{dy}{y}=\dfrac{dz}{z}$

which on integration gives $\log y = \log z + \log a$ or $y/z = a \qquad$ ...(*i*)

Using multipliers x, y and z, we have

$$\text{each fraction} = \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)} \quad \therefore \quad \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}=\frac{dz}{z}$$

which on integration gives $\log(x^2+y^2+z^2) = \log z + \log b$

or $$\frac{x^2+y^2+z^2}{z} = b \qquad \text{...(ii)}$$

Hence from (*i*) and (*ii*), the required solution is $x^2+y^2+z^2 = zf(y/z)$.

Example 17.10. *Solve* $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$. (*P.T.U., 2009 ; Bhopal, 2008 ; S.V.T.U. 2007*)

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using the multipliers $1/x$, $1/y$ and $1/z$, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \text{ which on integration gives}$$

$$\log x + \log y + \log z = \log a \quad \text{or} \quad xyz = a \qquad \text{...(i)}$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\therefore \quad \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0, \text{ which on integrating gives}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \qquad \text{...(ii)}$$

Hence from (*i*) and (*ii*), the complete solution is

$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Example 17.11. *Solve* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. (*Bhopal, 2008 ; V.T.U., 2006 ; Madras, 2000*)

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \qquad \text{...(i)}$$

$$\text{Each of these equations} = \frac{dx-dy}{x^2-y^2-(y-x)z} = \frac{dy-dz}{y^2-z^2-x(z-y)}$$

i.e., $$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} \quad \text{or} \quad \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, $$\log(x-y) = \log(y-z) + \log c \quad \text{or} \quad \frac{x-y}{y-z} = c \qquad \text{...(ii)}$$

$$\text{Each of the subsidiary equations (i)} = \frac{xdx + ydy + zdz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{xdx + ydy + zdz}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)} \qquad \text{...(iii)}$$

$$\text{Also each of the subsidiary equations} = \frac{dx+dy+dz}{x^2+y^2+z^2-yz-zx-xy} \qquad \text{...(iv)}$$

Equating (*iii*) and (*iv*) and cancelling the common factor, we get

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or $$\int (xdx + ydy + zdz) = \int (x + y + z) d\,(x + y + z) + c'$$

or $$x^2 + y^2 + z^2 = (x + y + z)^2 + 2c' \quad \text{or} \quad xy + yz + zx + c' = 0 \qquad ...(v)$$

Combining (*ii*) and (*v*), the general solution is

$$\frac{x - y}{y - z} = f\,(xy + yz + zx).$$

PROBLEMS 17.3

Solve the following equations :

1. $xp + yq = 3z$.
2. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.
3. $(z - y)p + (x - z)q = y - x$.
4. $p \cos(x + y) + q \sin(x + y) = z$.
5. $pyz + qzx = xy$.
6. $p \tan x + q \tan y = \tan z$.
7. $p - q = \log(x + y)$.
8. $xp - yq = y^2 - x^2$ (*J.N.T.U., 2002 S*)
9. $(y + z)p - (z + x)q = x - y$.
10. $x(y - z)p + y(z - x)q = z(x - y)$. (*Bhopal, 2007*)
11. $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$. (*V.T.U., 2010 ; Anna, 2008*)
12. $y^2p - xyq = x(z - 2y)$. (*S.V.T.U., 2008*)
13. $(y^2 + z^2)p - xyq + zx = 0$. (*P.T.U., 2009 ; V.T.U., 2009*)
14. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. (*Kerala, 2005*)
15. $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$.

17.6 NON-LINEAR EQUATIONS OF THE FIRST ORDER

Those equations in which p and q occur other than in the first degree are called *non-linear partial differential equations of the first order*. The *complete solution* of such an equation contains only two arbitrary constants (*i.e.*, equal to the number of independent variables involved) and the particular integral is obtained by giving particular values to the constants.]

Here we shall discuss four standard forms of these equations.

Form I. f(p, q) = 0, *i.e., equations containing p and q only.*

Its complete solution is $z = ax + by + c$ ...(1)

where a and b are connected by the relation $f(a, b) = 0$ ...(2)

[Since from (1), $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$, which when substituted in (2) give $f(p, q) = 0$].

Expressing (2) as $b = \phi(a)$ and substituting this value of b in (1), we get the required solution as $z = ax + \phi(a)y + c$ in which a and c are arbitrary constants.

Example 17.12. *Solve* $p - q = 1$. (*Anna, 2009*)

Solution. The complete solution is $z = ax + by + c$ where $a - b = 1$

Hence $z = ax + a - 1y + c$ is the desired solution.

Example 17.13. *Solve* $x^2p^2 + y^2q^2 = z^2$. (*Anna, 2008 ; Bhopal, 2008 ; Kerala, 2005 ; Kurukshetra, 2005*)

Solution. Given equation can be reduced to the above form by writing it as

$$\left(\frac{x}{z} \cdot \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1 \qquad ...(i)$$

and setting $$\frac{dx}{x} = du, \frac{dy}{y} = dv, \frac{dz}{z} = dw \text{ so that } u = \log x, v = \log y, w = \log z.$$

Then (*i*) becomes $$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 1$$

i.e., $P^2 + Q^2 = 1$ where $P = \dfrac{\partial w}{\partial u}$ and $Q = \dfrac{\partial w}{\partial v}$.

Its complete solution is $w = au + bv + c$ ...(*ii*)

where $a^2 + b^2 = 1$ or $b = \sqrt{(1-a^2)}$.

$\therefore$ (*ii*) becomes $w = au + \sqrt{(1-a^2)}\,v + c$

or $\log z = a \log x + \sqrt{(1-a^2)} \log y + c$ which is the required solution.

Form II. f(z, p, q) = 0, *i.e., equations not containing x and y.*

As a trial solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = \frac{dz}{du} \qquad q = \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the values of p and q in $f(z, p, q) = 0$, we get

$$f\left(z, \frac{\partial z}{\partial u}, a\frac{dz}{du}\right) = 0$$ which is an ordinary differential equation of the first order.

Rewiting it as $\dfrac{dz}{du} = \phi(z, a)$ it can be easily integrated giving

$F(z, a) = u + b$, or $x + ay + b = F(z, a)$ which is the desired complete solution.

Thus to solve $f(z, p, q) = 0$,

(*i*) *assume* $u = x + ay$ *and substitute* $p = dz/du$, $q = a\,dz/du$ *in the given equation;*

(*ii*) *solve the resulting ordinary differential equation in z and u;*

(*iii*) *replace u by* $x + ay$.

Example 17.14. *Solve* $p(1 + q) = qz$. (*Madras, 2000 S*)

Solution. Let $u = x + ay$, so that $p = dz/du$ and $q = a\,dz/du$.

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du} \quad \text{or} \quad a\frac{dz}{du} = az - 1 \quad \text{or} \quad \int\frac{a\,dz}{az-1} = \int du + b$$

or $\log(az - 1) = u + b$ or $\log(az - 1) = x + ay + b$

which is the required complete solution.

Example 17.15. *Solve* $q^2 = z^2p^2(1 - p^2)$. (*J.N.T.U., 2005 ; Kerala, 2005*)

Solution. Setting $u = y + ax$ and $z = f(u)$, we get

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = a\frac{dz}{du} \text{ and } q = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = \frac{dz}{du}$$

$\therefore$ The given equation becomes $\left(\dfrac{dz}{du}\right)^2 = a^2z^2\left(\dfrac{dz}{du}\right)^2\left\{1 - a^2\left(\dfrac{dz}{du}\right)^2\right\}$ ...(*i*)

or $$a^4z^2\left(\frac{dz}{du}\right)^2 = a^2z^2 - 1 \quad \text{or} \quad \frac{dz}{du} = \frac{\sqrt{(a^2z^2-1)}}{a^2z}$$

Integrating, $$\int\frac{a^2z}{\sqrt{(a^2z^2-1)}}\,dz = \int du + c \quad \text{or} \quad (a^2z^2 - 1)^{1/2} = u + c$$

i.e., $$a^2z^2 = (y + ax + c)^2 + 1 \qquad [\because u = y + ax]$$

The second factor in (*i*) is $dz/du = 0$. Its solution is $z = c'$.

Example 17.16. *Solve* $z^2(p^2x^2 + q^2) = 1$. (*Bhopal, 2008 S*)

Solution. Given equation can be reduced to the above form by writing it as

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \quad ...(i)$$

Putting $X = \log x$, so that $x\,\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}$, (*i*) takes the standard form

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \quad ...(ii)$$

Let $u = X + ay$ and put $\frac{\partial z}{\partial X}=\frac{dz}{du}$ and $\frac{\partial z}{\partial y}=a\,\frac{dz}{du}$ in (*ii*), so that

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2\right]=1 \quad \text{or} \quad \sqrt{(1+a^2)}\,zdz=\pm\,du$$

Integrating, $\quad \sqrt{(1+a^2)}\,z^2=\pm\,2u+b=\pm\,2(X+ay)+b$

or $\quad z^2\sqrt{(1+a^2)}=\pm\,2\,(\log x+ay)+b$

which is the complete solution required.

Form III. f(x, p) = F(y, q), *i.e., equations in which z is absent and the terms containing x and p can be separated from those containing y and q.*

As a trial solution assume that $f(x, p) = F(y, q) = a$, say

Then solving for p, we get $\quad p = \phi(x)$

and solving for q, we get $\quad q = \psi(y)$

Since $\quad dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$\therefore \quad dz = \phi(x)dx + \psi(y)dy$

Integrating, $\quad z=\int\phi(x)dx+\int\psi(y)dy+b$

which is the desired complete solution containing two constants a and b.

Example 17.17. *Solve $p^2 + q^2 = x + y$.* *(Bhopal, 2006 ; Madras, 2003)*

Solution. Given equation is $p^2 - x = y - q^2 = a$, say

$\therefore \quad p^2 - x = a$ gives $p=\sqrt{(a+x)}$

and $\quad y - q^2 = a$ gives $q=\sqrt{(y-a)}$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz=\sqrt{(a+x)}\,dx+\sqrt{(y-a)}\,dy$$

$\therefore$ integrating gives, $\quad z=\frac{2}{3}(a+x)^{3/2}+\frac{2}{3}(y-a)^{3/2}+b$

which is the required complete solution.

Example 17.18. *Solve $z^2(p^2 + q^2) = x^2 + y^2$.* *(Bhopal, 2008)*

Solution. The equation can be reduced to the above form by writing it as

$$\left(z\frac{\partial z}{\partial x}\right)^2+\left(z\frac{\partial z}{\partial y}\right)^2=x^2+y^2 \quad ...(i)$$

and putting $\quad zdz = dZ$, *i.e.*, $Z=\frac{1}{2}z^2$

$\therefore \quad \frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}=z\frac{\partial z}{\partial x}=P$

and $\quad \frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial y}=z\frac{\partial z}{\partial y}=Q$

$\therefore$ (i) becomes $$P^2 + Q^2 = x^2 + y^2$$

or $$P^2 - x^2 = y^2 - Q^2 = a, \text{ say.}$$

$\therefore$ $$P = \sqrt{(x^2 + a)} \text{ and } Q = \sqrt{(y^2 - a)}.$$

$\therefore$ $$dZ = Pdx + Qdy \text{ gives}$$

$$dZ = \sqrt{(x^2 + a)}\, dx + \sqrt{(y^2 - a)}\, dy$$

Integrating, we have $$Z = \frac{1}{2} x \sqrt{(x^2 + a)} + \frac{1}{2} a \log [x + \sqrt{(x^2 + a)}]$$
$$+ \frac{1}{2} y \sqrt{(y^2 - a)} - \frac{1}{2} a \log [y + \sqrt{(y^2 - a)}] + b$$

or $$z^2 = x\sqrt{(x^2 + a)} + y\sqrt{(y^2 - a} + a \log \frac{x + \sqrt{(x^2 + a)}}{y + \sqrt{(y^2 - a)}} + 2b$$

which is the required complete solution.

Example 17.19. *Solve* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.

(Bhopal, 2006 ; Rajasthan, 2006 ; V.T.U., 2003)

Solution. This equation can be reduced to the form $f(x, q) = F(y, q)$ by putting $u = x + y$, $v = x - y$ and taking $z = z(u, v)$.

Then $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = P + Q$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = P - Q, \text{ where } P = \frac{\partial z}{\partial u}, Q = \frac{\partial z}{\partial v}$$

Substituting these, the given equation reduces to

$$u(2P)^2 + v(2Q)^2 = 1 \quad \text{or} \quad 4P^2 u = 1 - 4Q^2 v = a \text{ (say)}$$

$$P = \pm \frac{1}{2}\sqrt{\frac{a}{u}}, Q = \pm \frac{1}{2}\sqrt{\frac{1 - a}{v}}$$

$\therefore$ $$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = Pdu + Qdv$$

$$= \pm \frac{\sqrt{a}}{2} \frac{du}{\sqrt{u}} \pm \frac{\sqrt{1 - a}}{2} \frac{dv}{\sqrt{v}}$$

Integrating, we have $$z = \pm \sqrt{a}\sqrt{u} \pm \sqrt{1 - a}\sqrt{v} + b$$

or $$z = \pm \sqrt{a(x + y)} \pm \sqrt{(1 - a)(x - y)} + b$$

which is the required complete solution.

Form IV. z = px + qy + f(p, q): *an equation analogous to the Clairaut's equation (§ 11.14).*

Its complete solution is $z = ax + by + f(a, b)$ which is obtained by writing a for p and b for q in the given equation.

Example 17.20. *Solve* $z = px + qy + \sqrt{(1 + p^2 + q^2)}$. *(Anna, 2009)*

Solution. Given equation is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{(1 + p^2 + q^2)}$

$\therefore$ Its complete solution is $z = ax + by + \sqrt{(1 + a^2 + b^2)}$.

PROBLEMS 17.4

Obtain the complete solution of the following equations :

1. $pq + p + q = 0$.
2. $p^2 + q^2 = 1$. *(Osmania, 2000)*
3. $z = p^2 + q^2$. *(Anna, 2005 S ; J.N.T.U., 2002 S)*
4. $p(1 - q^2) = q(1 - z)$ *(Anna, 2006)*
5. $yp + xq + pq = 0$.
6. $p + q = \sin x + \sin y$.

7. $p^2 - q^2 = x - y$.
8. $\sqrt{p} + \sqrt{q} = x + y$.
9. $p^2 + q^2 = x^2 + y^2$. (*Osmania, 2003*)
10. $z = px + qy + \sin(x + y)$.
11. $\sqrt{p} + \sqrt{q} = 2x$. (*J.N.T.U., 2006*)
12. $z = px + qy - 2\sqrt{(pq)}$.
13. $(x - y)(px - qy) = (p - q)^2$. [**Hint.** Use $x + y = u, xy = v$]

17.7 CHARPIT'S METHOD*

We now explain a general method for finding the complete integral of a non-linear partial differential equation which is due to Charpit.

Consider the equation

$$f(x, y\, z, p, q) = 0 \qquad \text{...(1)}$$

Since z depends on x and y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy \qquad \text{...(2)}$$

Now if we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ ...(3)

then we can solve (1) and (3) for p and q and substitute in (2). This will give the solution provided (2) is integrable.

To determine ϕ, we differentiate (1) and (3) with respect to x and y giving

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \text{...(4)}$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \text{...(5)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(6)}$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}q + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(7)}$$

Eliminating $\frac{\partial p}{\partial x}$ between the equations (4) and (5), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \qquad \text{...(8)}$$

Also eliminating $\frac{\partial q}{\partial y}$ between the equations (6) and (7), we obtain

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \qquad \text{...(9)}$$

Adding (8) and (9) and using $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$,

we find that the last terms in both cancel and the other terms, on rearrangement, give

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} = 0 \qquad \text{...(10)}$$

i.e.,

$$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} = 0 \qquad \text{...(11)}$$

This is Lagrange's linear equation (§ 17.5) with x, y, z, p, q as independent variables and ϕ as the dependent variable. Its solution will depend on the solution of the subsidiary equations

*Charpit's memoir containing this method was presented to the Paris Academy of Sciences in 1784.

$$\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{\partial \phi}{0}$$

An integral of these equations involving p or q or both, can be taken as the required relation (3), which alongwith (1) will give the values of p and q to make (2) integrable. Of course, we should take the simplest of the integrals so that it may be easier to solve for p and q.

Example 17.21. *Solve* $(p^2 + q^2)y = qz$. *(V.T.U., 2007 ; Hissar, 2005)*

Solution. Let $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{-2py}=\frac{dy}{z-2qy}=\frac{dz}{-qz}=\frac{dp}{-pq}=\frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$

Integrating, $p^2 + q^2 = c^2$ ...(*ii*)

Now to solve (*i*) and (*ii*), put $p^2 + q^2 = c^2$ in (*i*), so that $q = c^2y/z$

Substituting this value of q in (*ii*), we get $p = c\sqrt{(z^2 - c^2y^2)}/z$

Hence $$dz = pdx + qdy = \frac{c}{z}\sqrt{(z^2-c^2y^2)}\,dx + \frac{c^2y}{z}dy$$

or $$zdz - c^2y\,dy = c\sqrt{(z^2-c^2y^2)}\,dx \quad \text{or} \quad \frac{\frac{1}{2}d(z^2-c^2y^2)}{\sqrt{(z^2-c^2y^2)}} = c\,dx$$

Integrating, we get $\sqrt{(z^2-c^2y^2)} = cx + a$ or $z^2 = (a + cx)^2 + c^2y^2$ which is the required complete integral.

Example 17.22. *Solve* $2xz - px^2 - 2qxy + pq = 0$. *(Rajasthan, 2006)*

Solution. Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dz}{px^2-2pq+2qxy}=\frac{dp}{2z-2qy}=\frac{dq}{0}$$

$\therefore$ $dq = 0$ or $q = a$.

Putting $q = a$ in (*i*), we get $p = \dfrac{2x(z-ay)}{x^2-a}$

$\therefore$ $$dz = pdx + qdy = \frac{2x(z-ay)}{x^2-a}dx + ady \quad \text{or} \quad \frac{dz-ady}{z-ay} = \frac{2x}{x^2-a}dx$$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

or $z - ay = b(x^2 - a)$ or $z = ay + b(x^2 - a)$

which is the required complete solution.

Example 17.23. *Solve* $2z + p^2 + qy + 2y^2 = 0$. *(J.N.T.U., 2005 ; Kurukshetra, 2005)*

Solution. Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$

Charpit's subsidiary equations are

$$\frac{dx}{-2p}=\frac{dy}{-y}=\frac{dz}{-(2p^2+qy)}=\frac{dp}{2p}=\frac{dq}{4y+3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in the given equation, we get

$$q = \frac{1}{y}[-2z - 2y^2 - (a-x)^2]$$

$\therefore \qquad dz = pdx + qdy = (a - x)dx - \dfrac{1}{y}[2z + 2y^2 + (a - x)^2]dy$

Multiplying both sides by $2y^2$,

$$2y^2dz + 4yz\,dy = 2y^2(a - x)dx - 4y^3dy - 2y(a - x)^2dy$$

Integrating $\qquad 2zy^2 = -[y^2(a - x)^2 + y^4] + b$

or $\qquad y^2[(x - a)^2 + 2z + y^2] = b$, which is the desired solution.

PROBLEMS 17.5

Solve the following equations :

1. $z = p^2x + q^2x$.
2. $z^2 = pq\,xy$. *(Anna, 2009 ; V.T.U., 2004)*
3. $1 + p^2 = qz$.
4. $pxy + pq + qy = yz$. *(J.N.T.U., 2006 ; Kurukshetra, 2006)*
5. $p(p^2 + 1) + (b - z)q = 0$.
6. $q + xp = p^2$. *(Osmania, 2003)*

17.8 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad ...(1)$$

in which k's are constants, is called a *homogeneous linear partial differential equation of the nth order with constant coefficients.* It is called homogeneous because all terms contain derivatives of the same order.

On writing, $\dfrac{\partial^r}{\partial x^r} = D^r$ and $\dfrac{\partial^r}{\partial y^r} = D'^r$. (1) becomes $(D^n + k_1D^{n-1}D'^r + D' + \dots + k_nD'^n)z = F(x, y)$

or briefly $\qquad f(D, D')z = F(x, y) \qquad ...(2)$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely : the *complementary function* and *the particular integral.*

The complementary function is the complete solution of the equation $f(D, D')z = 0$, which must contain n arbitrary functions. The particular integral is the particular solution of equation (2).

17.9 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $\dfrac{\partial^2 z}{\partial x^2} + k_1 \dfrac{\partial^2 z}{\partial x \partial y} + k_2 \dfrac{\partial^2 z}{\partial y^2} = 0 \qquad ...(1)$

which in *symbolic form is* $(D^2 + k_1DD' + k_2D'^2)z = 0 \qquad ...(2)$

Its symbolic operator equated to zero, *i.e.*, $D^2 + k_1DD' + k_2D'^2 = 0$ is called the *auxiliary equation* (A.E.)

Let its root be $\qquad D/D' = m_1, m_2$.

Case I. *If the roots be real and distinct* then (2) is equivalent to

$$(D - m_1D')(D - m_2D')z = 0 \qquad ...(3)$$

It will be satisfied by the solution of

$$(D - m_2D')z = 0, \text{ i.e., } p - m_2q = 0.$$

This is a Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}, \text{ whence } y + m_2x = a \text{ and } z = b.$$

$\therefore$ its solution is $z = \phi(y + m_2x)$.

Similarly (3) will also be satisfied by the solution of

$$(D - m_1D')z = 0, \quad i.e., \quad \text{by } z = f(y + m_1x)$$

Hence the complete solution of (1) is $z = f(y + m_1x) + \phi(y + m_2x)$.

Case II. *If the roots be equal* (*i.e.*, $m_1 = m_2$), then (2) is equivalent to

$$(D - m_1 D')^2 z = 0 \quad ...(4)$$

Putting $(D - m_1 D')z = u$, it becomes $(D - m_1 D')u = 0$ which gives

$$u = \phi(y + m_1 x)$$

$\therefore$ (4) takes the form $(D - m_1' D)z = \phi(y + m_1 x)$ or $p - m_1 q) = \phi(y + m_1 x)$

This is again Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi(y + m_1 x)}$$

giving $y + m_1 x = a$ and $dz = \phi(a)\,dx$, *i.e.*, $z = \phi(a)x + b$

Thus the complete solution of (1) is

$$z - x\phi(y + m_1 x) = f(y + m_1 x). \quad i.e., \quad z = f(y + m_1 x) + x\phi(y + m_1 x).$$

Example 17.24. *Solve* $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$

Solution. Given equation in *symbolic form* is $(2D^2 + 5DD' + 2D'^2)z = 0$.

Its *auxiliary equation* is $2m^2 + 5m + 2 = 0$, where $m = D/D'$.

which gives $m = -2, -1/2.$

Here the complete solution is $z = f_1(y - 2x) + f_2(y - \frac{1}{2}x)$

which may be written as $z = f_1(y - 2x) + f_2(2y - x)$.

Example 17.25. *Solve* $4r + 12s + 9t = 0.$ *(P.T.U., 2010)*

Solution. Given equation in *symbolic form* is $(4D^2 + 12DD' + 9D'^2)z = 0$

for $r = \frac{\partial^2 z}{\partial x^2} = D^2 z,\ s = \frac{\partial^2 z}{\partial x \partial y} = DD'z$ and $t = \frac{\partial^2 z}{\partial y^2} = D'^2 z.$

$\therefore$ Its *auxiliary equation* is $4m^2 + 12m + 9 = 0$, whence $m = -3/2, -3/2$

Hence the complete solution is $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$.

17.10 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $(D^2 + k_1 DD' + k_2 D'^2)z = F(x, y)$ *i.e.*, $f(D, D')z = F(x, y)$.

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

Case I. *When* $F(x, y) = e^{ax + by}$

Since $De^{ax+by} = ae^{ax+by}$; $D'e^{ax+by} = be^{ax+by}$

$\therefore$ $D^2 e^{ax+by} = a^2 e^{ax+by}$; $DD'e^{ax+by} = abe^{ax+by}$

and $D'^2 e^{ax+by} = b^2 e^{ax+by}$

$\therefore$ $(D^2 + k_1 DD' + k_2 D'^2)e^{ax+by} = (a^2 + k_1 ab + k_2 b^2)\, e^{ax+by}$

i.e., $f(D, D')e^{ax+by} = f(a, b)\, e^{ax+by}$

Operating both sides by $1/f(D, D')$, we get

$$\text{P.I.} = \frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}$$

Case II. *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

Since $D^2 \sin(mx + ny) = -m^2 \sin(mx + ny)$

$DD' \sin(mx + ny) = -mn \sin(mx + ny)$

and $D'^2 \sin(mx + ny) = -n^2 \sin(mx + ny).$

$\therefore$ $f(D^2, DD', D'^2) \sin(mx + ny) = f(-m^2, -mn, -n^2) \sin(mx + ny)$

Operating both sides by $1/f(D^2, DD', D'^2)$, we get

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin(mx + ny) = \frac{1}{f(-m^2 - mn, -n^2)} \sin(mx + ny)$$

Similarly about the P.I. for $\cos(mx + ny)$.

Case III. *When* $F(x, y) = x^m y^n$, m and n being constants.

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n.$$

To evaluate it, we expand $[f(D, D')]^{-1}$ in ascending powers of D or D' by Binomial theorem and then operate on $x^m y^n$ term by term.

Case IV. *When $F(x, y)$ is any function of x and y.*

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

To evaluate it, we resolve $1/f(D, D')$ into partial fractions treating $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$$

where c is replaced by $y + mx$ after integration.

17.11 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots + k^n \frac{\partial^n z}{\partial y^n} = F(x, y).$$

Its *symbolic form* is $\quad (D^n + k_1 D^{n-1} D' + \ldots + k_n D'^n) z = F(x, y)$

or briefly $\quad f(D, D') z = F(x, y)$

Step I. To find the C.F.

(*i*) *Write the A.E.*

i.e., $\quad m^n + k_1 m^{n-1} + \ldots + k_n = 0$ *and solve it for m.*

(*ii*) *Write the C.F. as follows*

Roots of A.E.	*C.F.*
1. $m_1, m_2, m_3 \ldots$ (distinct roots)	$f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \ldots$
2. $m_1, m_1, m_3 \ldots$ (two equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \ldots$
3. $m_1, m_1, m_1 \ldots$ (three equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \ldots$

Step II. To find the P.I.

From the *symbolic form*, P.I. $= \dfrac{1}{f(D, D')} F(x, y)$.

(*i*) *When* $F(x, y) = e^{ax + by}$ P.I. $= \dfrac{1}{f(D, D')} e^{ax + by}$ [Put $D = a$ and $D' = b$]

(*ii*) *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos(mx + ny) \qquad [\text{Put } D^2 = -m^2, DD' = -mn, D'^2 = -n^2]$$

(*iii*) *When* $F(x, y) = x^m y^n$, P.I. $= \dfrac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$.

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D′ and operate on $x^m y^n$ term by term.

(*iv*) *When $F(x, y)$ is any function of x and y* P.I. $= \dfrac{1}{f(D, D')} F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\,dx \text{ where } c \text{ is replaced by } y + mx \text{ after integration.}$$

Example 17.26. *Solve $(D^2 + 4DD' - 5D'^2)z = \sin(2x + 3y)$.* *(Madras, 2006)*

Solution. A.E. of the given equation is $m^2 + 4m - 5 = 0$ *i.e.*, $m = 1, -5$

$\therefore$ C.F. $= f_1(y + x) + f_2(y - 5x)$

$$\text{P.I.} = \frac{1}{D^2 + 4DD' - 5D'^2}\sin(2x + 3y) \quad [\text{Put } D^2 = -2^2, DD' = -2 \times 3, D'^2 = -3^2$$

$$= \frac{1}{-4 + 4(-6) - 5(-9)}\sin(2x + 3y) = \frac{1}{17}\sin(2x + 3y).$$

Hence the C.S. is $z = f_1(y + x) + f_2(y - 5x) + \frac{1}{17}\sin(2x + 3y)$.

Example 17.27. *Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$.* *(Bhopal, 2008 S)*

Solution. Given equation in *symbolic form* is $(D^2 - DD')z = \cos x \cos 2y$.

Its A.E. is $m^2 - m = 0$, whence $m = 0, 1$.

$\therefore$ C.F. $= f_1(y) + f_2(y + x)$

$$\text{P.I.} = \frac{1}{D^2 - DD'}\cos x \cos 2y = \frac{1}{2}\frac{1}{D^2 - DD'}[\cos(x + 2y) + \cos(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x + 2y) \right. \qquad [\text{Put } D^2 = -1, DD' = -2]$$

$$\left. + \frac{1}{D^2 - DD'}\cos(x - 2y)\right] \qquad [\text{Put } D^2 = -1, DD' = 2]$$

$$= \frac{1}{2}\left[\frac{1}{-1 + 2}\cos(x + 2y) + \frac{1}{-1 - 2}\cos(x - 2y)\right] = \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y)$$

Hence the C.S. is $z = f_1(y) + f_2(y + x) + \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y)$.

Example 17.28. *Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$.* *(S.V.T.U., 2007)*

Solution. Given equation in *symbolic form* is

$$(D^3 - 2D^2 D')z = 2e^{2x} + 3x^2 y$$

Its A.E. is $m^3 - 2m^2 = 0$, whence $m = 0, 0, 2$.

$\therefore$ C.F. $= f_1(y) + x f_2(y) + f_3(y + 2x)$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2 D'}(2e^{2x} + 3x^2 y) = 2\frac{1}{D^3 - 2D^2 D'}e^{2x} + 3\frac{1}{D^3(1 - 2D'/D)}x^2 y$$

$$= 2\frac{1}{2^3 - 2\,.\,2^2(0)}e^{2x} + \frac{3}{D^3}(1 - 2D'/D)^{-1}x^2 y = \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \ldots\right)x^2 y$$

$$= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2 y + \frac{2}{D}x^2\,.\,1\right) = \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2 y + \frac{2}{3}x^3\right) \qquad \left[\because \frac{1}{D}f(x) = \int f(x)\,dx\right]$$

$$= \frac{1}{4}e^{2x} + 3y\frac{x^5}{3\,.\,4\,.\,5} + 2\,.\,\frac{x^6}{4\,.\,5\,.\,6} \qquad \left[\because \frac{1}{D^3}f(x) = \int\left[\int\left(\int f(x)\,dx\right)dx\right]dx\right]$$

$$= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Hence the C.S. is $z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5 y + x^6)$.

Example 17.29. *Solve* $r - 4s + 4t = e^{2x+y}$.

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = e^{2x+y}$.

i.e., in *symbolic form* $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$.

Its A.E. is $(m - 2)^2 = 0$, whence $m = 2, 2$.

$\therefore$ C.F. $= f_1(y + 2x) + xf_2(y + 2x)$

$$\text{P.I.} = \frac{1}{(D - 2D')^2} e^{2x+y}$$

The usual rule fails because $(D - 2D')^2 = 0$ for $D = 2$ and $D' = 1$.

$\therefore$ to obtain the P.I., we find from $(D - 2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c - mx)\, dx = \int e^{2x+(c-2x)}\, dx = xe^c = xe^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

and from $(D - 2D')z = u = xe^{2x+y}$, the solution

$$z = \int xe^{2x+(c-2x)} dy = \frac{1}{2}x^2 e^c = \frac{1}{2}x^2 e^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

Hence the C.S. is $z = f_1(y + 2x) + xf_2(y + 2x) + \dfrac{1}{2}x^2 e^{2x+y}$.

Example 17.30. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(2x + y)$. *(P.T.U., 2010 ; S.V.T.U., 2009)*

Solution. Given equation in *symbolic form* is $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$

Its A.E. is $m^2 + m - 6 = 0$ whence $m = -3, 2$.

$\therefore$ C.F. $= f_1(y - 3x) + f_2(y + 2x)$.

Since $D^2 + DD' - 6D'^2 = -2^2 - (2)(1) - 6(-1)^2 = 0$

$\therefore$ It is a case of failure and we have to apply the general method.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x + y) = \frac{1}{(D + 3D')(D - 2D')}\cos(2x + y)$$

$$= \frac{1}{D + 3D'}\left[\int \cos(2x + \overline{c - 2x})dx\right]_{c \to y+2x} = \frac{1}{D + 3D'}\left[\int \cos c\, dx\right]_{c \to y+2x}$$

$[\because\ y = c - mx = c - 2x$

$$= \frac{1}{D + 3D'} x\cos(y + 2x) = \left[\int x\cos(\overline{c + 3x} + 2x)dx\right]_{c \to y-3x} = \left[\int x\cos(5x + c)dx\right]_{c \to y-3x}$$

$$= \left[\frac{x\sin(5x + c)}{5} + \frac{\cos(5x + c)}{25}\right]_{c \to y-3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{x}{5}\sin(5x + \overline{y - 3x}) + \frac{1}{25}\cos(5x + \overline{y - 3x}) = \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

Hence the C.S. is

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y).$$

Example 17.31. *Solve* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x.$ *(Anna, 2005 S ; U.P.T.U., 2003)*

or $r + s - 6t = y \cos x.$ *(Bhopal, 2008 ; S.V.T.U., 2008)*

Solution. Its *symbolic form* is $(D^2 + DD' - 6D'^2)z = y \cos x$
and the A.E. is $m^2 + m - 6 = 0$, whence $m = -3, 2$.

$$\therefore \quad \text{C.F.} = f_1(y - 3x) + f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D - 2D')(D + 3D')} y \cos x = \frac{1}{D - 2D'}\left[\int (c + 3x) \cos x \, dx\right]_{c \to y - 3x}$$

$[\because \; y = c - mx = c + 3x]$

$$= \frac{1}{D - 2D'}[(c + 3x) \sin x + 3 \cos x]_{c \to y - 3x} \quad \text{[Integrating by parts]}$$

$$= \frac{1}{D - 2D'}(y \sin x + 3 \cos x) = \left[\int \{(c - 2x) \sin x \; 3 \cos x\} dx\right]_{c \to y - 2x}$$

$$= [(c - 2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x]_{c \to y + 2x}$$

$$= -y \cos x + \sin x$$

Hence the C.S. is $z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y \cos x$.

Example 17.32. *Solve* $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16 \log (x + 2y).$

Solution. Its *symbolic form* is $4D^2 - 4DD' + D'^2 = 16 \log (x + 2y)$
and the A.E. is $4m^2 - 4m + 1 = 0$, $m = 1/2, 1/2$.

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right)$$

$$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log (x + 2y) = 4\frac{1}{\left(D - \frac{1}{2}D'\right)}\left\{\frac{1}{D - \frac{1}{2}D'} \log (x + 2y)\right\}$$

$$= 4\frac{1}{D - \frac{1}{2}D'}\left[\int \log\left\{x + 2\left(c - \frac{x}{2}\right)\right\} dx\right]_{c \to y + x/2} \quad [\because \; y = c - mx = c - x/2]$$

$$= 4\frac{1}{D - \frac{1}{2}D'}\left[\int \log (2c) dx\right]_{c \to y + x/2} = 4\frac{1}{D - \frac{1}{2}D'}[x \log (x + 2y)]$$

$$= 4\left[\int \left\{x \log\left[x + 2\left(c - \frac{x}{2}\right)\right]\right\} dx\right]_{c \to y + x/2} = 4\left[\log 2c \int x \, dx\right]_{c \to y + x/2} = 2x^2 \log (x + 2y)$$

Hence the C.S. is $z = f_1\left(y + \frac{x}{2}\right) + x f_2\left(y + \frac{x}{2}\right) + 2x^2 \log (x + 2y)$.

PROBLEMS 17.6

Solve the following equations :

1. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 x}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 0.$
2. $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x + 2y}.$ *(Burdwan, 2003)*
3. $(D^2 - 2DD' + D'^2)z = e^{x + y}.$ *(Bhopal, 2007)*
4. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 5\frac{\partial^3 z}{\partial x \partial y^2} - 2\frac{\partial^3 z}{\partial y^3} = e^{2x + y}.$ *(Bhopal, 2008)*

5. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x.$ (*P.T.U., 2009 S*)

6. $\frac{\partial^2 y}{\partial t^2} - a^2 \frac{\partial^2 y}{\partial x^2} = E \sin pt.$

7. $\frac{\partial^3 z}{\partial x^3} - \frac{4\partial^3 z}{\partial z^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 2 \sin (3x + 2y).$ (*S.V.T.U., 2007*)

8. $(D^3 - 7DD'^2 - 6D'^3)z = \cos (x + 2y) + 4.$ (*Anna, 2008*)

9. $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos (x + 2y).$ (*U.P.T.U., 2006*)

10. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y.$ (*U.P.T.U., 2003*)

11. $(D^2 - DD')z = \cos 2y (\sin x + \cos x).$

12. $(D^2 - D'^2)z = e^{x-y} \sin (x + 2y).$ (*Anna, 2009*)

13. $(D^2 + 3DD' + 2D'^2)z = 24xy.$

14. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2.$

15. $(D^2 - DD' - 2D'^2)z = (y - 1) e^x.$ (*Bhopal, 2006*)

16. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y.$

17. $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y.$ (*P.T.U., 2005*)

17.12 NON-HOMOGENEOUS LINEAR EQUATIONS

If in the equation $\qquad f(D, D')z = F(x, y) \qquad$ ...(1)

the polynomial expression $f(D, D')$ is not homogeneous, then (1) is a non-homogeneous linear partial differential equation. As in the case of homogeneous linear partial differential equations, its complete solution = C.F. + P.I.

The methods to find P.I. are the same as those for homogeneous linear equations.

To find the C.F., we factorize $f(D, D')$ into factors of the form $D - mD' - c$. To find the solution of $(D - mD' - c)z = 0$, we write it as $p - mq = cz$ ...(2)

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$$

Its integrals are $y + mx = a$ and $z = be^{cx}$.

Taking $b = \phi(a)$, we get $z = e^{cx} \phi(y + mx)$

as the solution of (2). The solution corresponding to various factors added up, give the C.F. of (1).

Example 17.32. *Solve* $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin (x + 2y).$ (*U.P.T.U., 2004*)

Solution. Here $\qquad f(D, D') = (D + D')(D + D' - 2)$

Since the solution corresponding to the factor $D - mD' - c$ is known to be

$$z = e^{cx} \phi(y + mx)$$

$$\therefore \qquad \text{C.F.} = \phi_1(y - x) + e^{2x} f_2(y - x)$$

$$\therefore \qquad \text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin (x + 2y)$$

$$= \frac{1}{-1 + 2(-2) + (-4) - 2D - 2D'} \sin (x + 2y)$$

$$= -\frac{1}{2(D + D') + 9} \sin (x + 2y) = -\frac{2(D + D') - 9}{4(D^2 + 2DD' + D'^2) - 81} \sin (x + 2y)$$

$$= \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)]$$

Hence the complete solution is

$$z = \phi_1(y - x) + e^{2x} \phi_2(y - x) + \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)].$$

PROBLEMS 17.7

Solve the following equations :

1. $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial z}{\partial y}-z=e^{-x}$.
2. $(D-D'-1)(D-D'-2)z=e^{2x-y}$.
3. $(D+D'-1)(D+2D'-3)z=4+3x+6y$.
4. $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial z}{\partial y}=x^2+y^2$. *(Madras, 2000 S)*
5. $(D^2+DD'+D'-1)z=\sin(x+2y)$. *(S.V.T.U., 2009)*
6. $(2DD'+D'^2-3D')z=3\cos(3x-2y)$.

17.13 NON-LINEAR EQUATIONS OF THE SECOND ORDER

We now give a method due to *Monge**, for integrating the equation $Rr+Ss+Tt=V$ ...(1)
in which R, S, T, V are functions of x, y, z, p and q.

Since $$dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy=rdx+tdy, \text{ and } dq=sdx+tdy,$$
we have $$r=(dp-tdy)/dx \text{ and } t=(dq-sdx)/dy.$$

Substituting these values of r and t in (1), and rearranging the terms, we get
$$(Rdpdy+Tdqdx-Vdxdy)-s(Rdy^2-Sdydx+Tdx^2)=0 \quad \text{...(2)}$$

Let us consider the equations
$$Rdy^2-Sdydx+Tdx^2=0 \quad \text{...(3)}$$
$$Rdpdy+Tdqdx-Vdxdy=0 \quad \text{...(4)}$$
which are known as *Monge's equations.*

Since (3) can be factorised, we obtain its integral first. In case the factors are different, we may get two distinct integrals of (3). Either of these together with (4) will give an integral of (4). If need be, we may also use the relation $dz=pdx+qdy$ while solving (3) and (4).

Let $u(x, y, z, p, q)=a$ and $v(x, y, z, p, q)=b$ be the integrals of (3) and (4) respectively. Then $u=a$, $v=b$ evidently constitute a solution of (2) and therefore, of (1) also. Taking $b=\phi(a)$, we find a general solution of (1) to be $v=\phi(u)$, which should be further integrated by methods of first order equations.

Example 17.34. *Solve $(x-y)(xr-xs-ys+yt)=(x+y)(p-q)$.* *(S.V.T.U., 2007)*

Solution. Monge's equations are
$$xdy^2+(x+y)dy\,dx+ydx^2=0 \quad \text{...}(i)$$
$$xdpdy+ydqdx-\frac{x+y}{x-y}(p-q)\,dydx=0 \quad \text{...}(ii)$$

(i) may be factorised as $(xdy+ydx)(dx+dy)=0$ whose integrals are $xy=c$ and $x+y=c$.

Taking $xy=c$ and dividing each term of (ii) by xdy or its equivalent $-ydx$, we get
$$dp-dq-\frac{dx-dy}{x-y}(p-q)=0 \quad \text{or} \quad \frac{d(p-q)}{p-q}-\frac{d(x-y)}{x-y}=0$$
This gives on integration $(p-q)/(x-y)=c$.

Hence a first integral of the given equation is $p-q=(x-y)\,\phi(xy)$ which is a Lagrange's linear equation. Its subsidiary equations are
$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{(x-y)\phi(xy)}$$
From the first two equations, we have $x+y=a$

Using this, we have
$$dz=-\phi(ax-x^2)\,.\,(a-2x)\,dx \quad \text{which gives } z=\phi_1(ax-x^2)+b$$
Writing $b=\phi_2(a)$ and $a=x+y$, we get
$$z=\phi_1(xy)+\phi_2(x+y).$$

* Named after *Gaspard Monge* (1746–1818), Professor at Paris.

Obs. Had we started with the integral $x + y = c$ and divided each term of (*ii*) by dx or $-dy$, we would have arrived at the same solution.

Example 17.35. *Solve* $y^2r - 2ys + t = p + 6y$. (*Osmania, 2002*)

Solution. Monge's equations are $y^2dy^2 + 2ydydx + dx^2 = 0$ ...(*i*)

and $$y^2dpdy + dqdx - (p + 6y)dydx = 0 \quad ...(ii)$$

(*i*) gives $$(ydy + dx)^2 = 0 \text{ i.e. } y^2 + 2x = c \quad ...(iii)$$

Putting $ydy = -dx$ in (*ii*), we get

$$ydp - dq + (p + 6y)dy = 0 \quad \text{or} \quad (ydp + pdy) - dq + 6ydy = 0$$

whose integral is $$py - q + 3y^2 = a$$

Combining this with (*iii*), we get the integral $py - q + 3y^2 = \phi(y^2 + 2x)$

The subsidiary equations for this Lagrange's linear equation are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{\phi(y^2 + 2x) - 3y^2}$$

From the first two equations, we have $y^2 + 2x = c$

Using this, we have $dz + [\phi(c) - 3y^2]\, dy = 0$

whose solution is $$z + y\phi(c) - y^3 = b.$$

Hence the required solution is $z = y^3 - y\phi(y^2 + 2x) + \psi(y^2 + 2x)$.

PROBLEMS 17.8

Solve :

1. $(q + 1)s = (p + 1)t$.
2. $r - t\cos^2 x + p\tan x = 0$.
3. $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$. (*J.N.T.U., 2006*)
4. $xy(t - r) + (x^2 - y^2)(s - 2) = py - qx$.
5. $q^2r - 2pqs + p^2t = pq^2$.
6. $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$.

17.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 17.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. The equation $\dfrac{\partial^2 z}{\partial x^2} + 2xy\left(\dfrac{\partial z}{\partial x}\right)^2 + \dfrac{\partial z}{\partial y} = 5$ is of order .... and degree ....
2. The complementary function of $(D^2 - 4DD' + 4D'^2)z = x + y$ is ......
3. The solution of $\dfrac{\partial^2 z}{\partial y^2} = \sin(xy)$ is ......
4. A solution of $(y - z)p + (z - x)q = x - y$ is ....
5. The particular integral of $(D^2 + DD')z = \sin(x + y)$ is ....
6. The partial differential equation obtained from $z = ax + by + ab$ by eliminating a and b is ......
7. Solution of $\sqrt{p} + \sqrt{q} = 1$ is ......
8. Solution of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ is ......
9. Solution of $p - q = \log(x + y)$.
10. The order of the partial differential equation obtained by eliminating f from $z = f(x^2 + y^2)$, is ......
11. The solution of $x\,\partial z/\partial x = 2x + y$ is
12. By eliminating a and b from $z = a(x + y) + b$, the p.d.e. formed is ......
13. The solution of $[D^3 - 3D^2D' + 2DD'^2]z = 0$ is ......
14. By eliminating the arbitrary constants from $z = a^2x + ay^2 + b$, the partial differential equation formed is ...... (*Anna, 2008*)
15. A solution of $u_{xy} = 0$ is of the form ....
16. If $u = x^2 + t^2$ is a solution of $c^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, then $c =$ ......

17. The general solution of $u_{xx} = xy$ is ....

18. The complementary function of $r - 7s + 6t = e^{x+y}$ is ....

19. The solution of $xp + yq = z$ is

(*i*) $f(x^2, y^2) = 0$ (*ii*) $f(xy, yz)$ (*iii*) $f(x, y) = 0$ (*iv*) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

20. The solution of $(y - z)p + (z - x)q = x - y$, is

(*i*) $f(x^2 + y^2 + z^2) = xyz$ (*ii*) $f(x + y + z) = xyz$

(*iii*) $f(x + y + z) = x^2 + y^2 + z^2$ (*iv*) $f(x^2 + y^2 + z^2, xyz) = 0$.

21. The partial differential equation from $z = (c + x)^2 + y$ is

(*i*) $z = \left(\frac{\partial z}{\partial x}\right)^2 + y$ (*ii*) $z = \left(\frac{\partial z}{\partial y}\right)^2 + y$ (*iii*) $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$ (*iv*) $z = \frac{1}{4}\left(\frac{\partial z}{\partial y}\right)^2 + y$.

22. The solution of $p + q = z$ is

(*i*) $f(xy, y \log z) = 0$ (*ii*) $f(x + y, y + \log z) = 0$

(*iii*) $f(x - y, y - \log z) = 0$ (*iv*) None of these.

23. Particular integral of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$ is

(*i*) $\frac{1}{2}e^{x+2y}$ (*ii*) $-\frac{x}{2}e^{x+2y}$ (*iii*) xe^{x+2y} (*iv*) x^2e^{x+2y}.

24. The solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is

(*i*) $z = (1 + x + x^2) f(y)$ (*ii*) $z = (1 + y + y^2) f(x)$

(*iii*) $z = f_1(x) + y f_2(x) + y^2 f_3(x)$ (*iv*) $z = f_1(y) + x f_2(y) + x^2 f_3(y)$.

25. Particular integral of $(D^2 - D'^2)z = \cos(x + y)$ is

(*i*) $x \cos(x + y)$ (*ii*) $\frac{x}{2}\cos(x + y)$ (*iii*) $x \sin(x + y)$ (*iv*) $\frac{x}{2}\sin(x + y)$

26. The solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ is

(*i*) $z = f_1(y + x) + f_2(y - x)$ (*ii*) $z = f_1(y + x) + f_1(y - x)$

(*iii*) $z = f(x^2 - y^2)$ (*iv*) $z = f(x^2 + y^2)$.

27. $xu_x + yu_y = u^2$ is a non-linear partial differential equation. (True or False)

28. $xu_x + u_{xx} = 0$ is a non-linear partial differential equation. (True or False)

29. $u = x^2 - y^2$ is a solution of $u_{xx} + u_{yy} = 0$. (True or False)

30. $u = e^{-t} \sin x$ is a solution of $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$. (True or False)

31. $x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = 2u$ is an ordinary differential equation. (True or False)