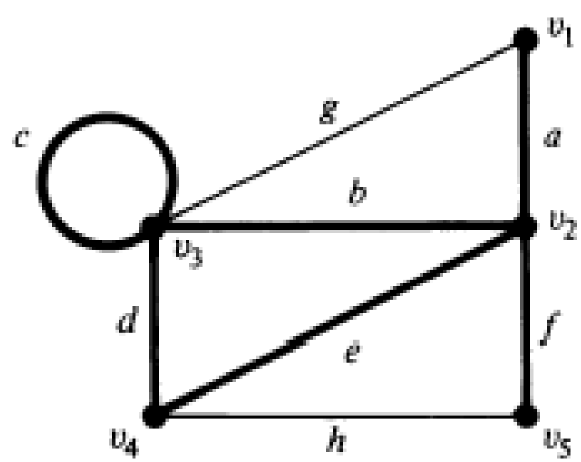
DAY 4

Walk

A walk is a finite, alternating sequence of vertices and edges, beginning and ending with vertices such that each edge is incident with the vertices preceding and following it. Whenever we traverse a graph, we get a walk. In a walk, Vertex can be repeated and Edges can be repeated.



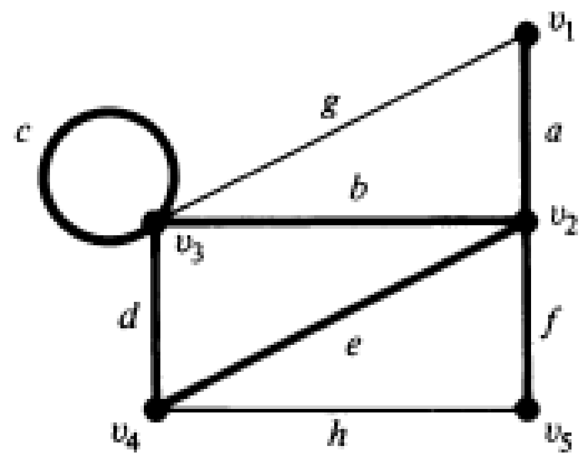
v2-f-v5-h-v4-e-v2-f-v5 is a walk

Contd..

Open walk-A walk is said to be an open walk if the starting and ending vertices are different i.e. the origin vertex and terminal vertex are different.

Closed walk-A walk is said to be a closed walk if the starting and ending vertices are identical i.e. if a walk starts and ends at the same vertex, then it is said to be a closed walk.

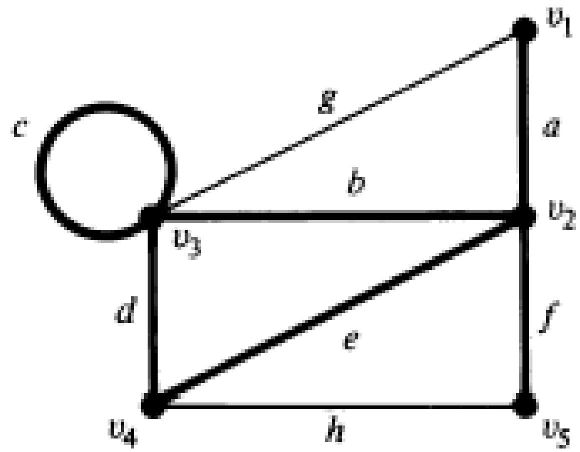
v1-a-v2-f-v5 is an open-walk



v2-f-v5-h-v4-e-v2 is a closed-walk

Trail

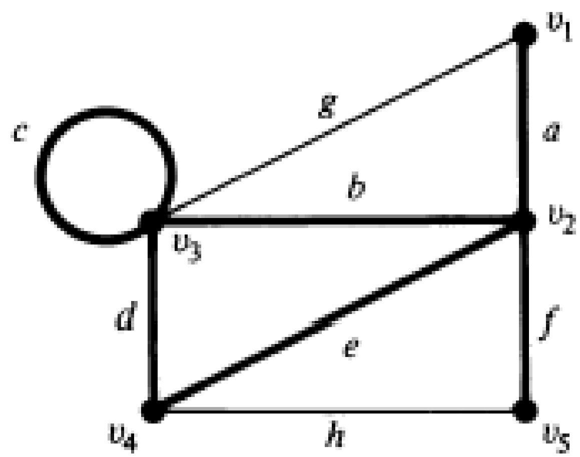
Trail is a walk in which no edge is repeated but vertex can be repeated. Trail can be open or closed.



$v_1\ a\ v_2\ b\ v_3\ c\ v_3\ d\ v_4\ e\ v_2\ f\ v_5$ is a trail

Path

It is a trail in which neither vertices nor edges are repeated. No. of edges present in a path is known as the path-length.

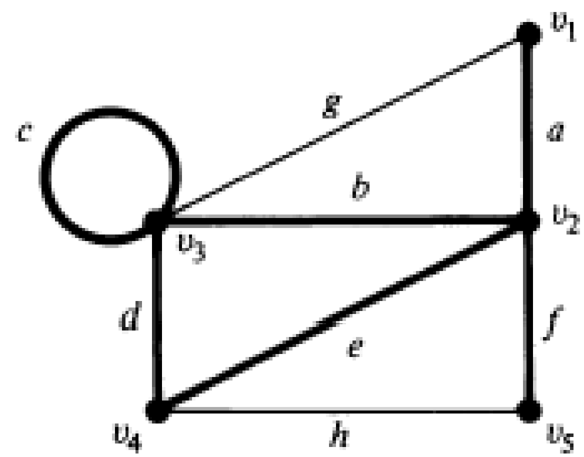


v1-a-v2-b-v3-d-v4 is a path of path-length 3

v1-a-v2-b-v3-c-v3-d-v4 is *not* a path

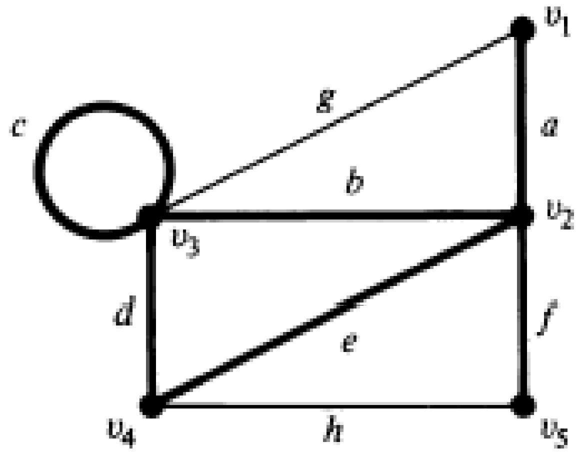
Circuit

It is a closed trail where no vertex (except initial and the final vertex) appears more than once.



v2-f-v5-h-v4-e-v2 is a circuit

Question for Self Study

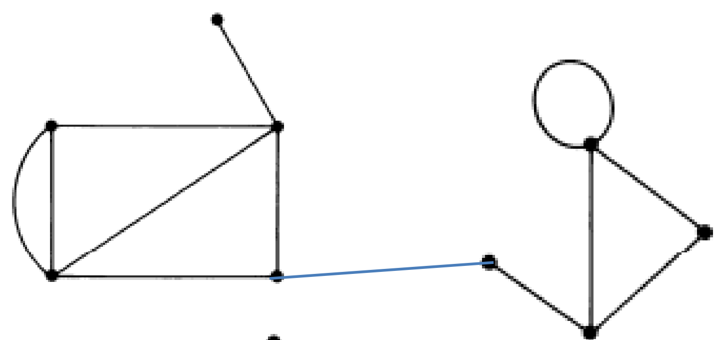


From the given graph find:

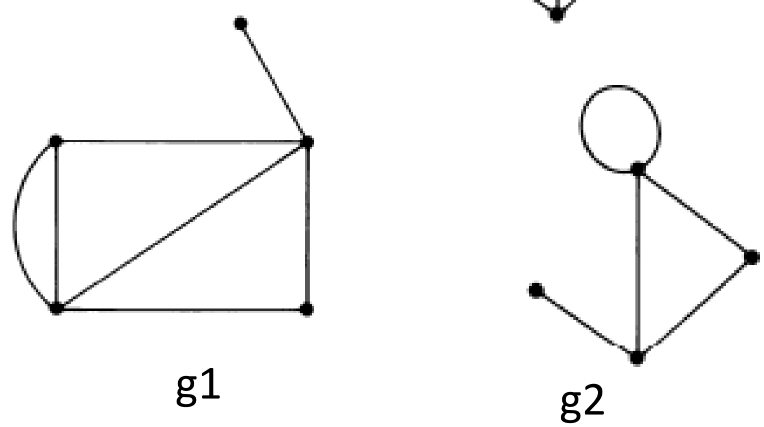
a) **Walks – open and closed**
b) **Trails**
c) **Paths of various length**
d) **Circuit**

Disconnected Graphs

A graph G is said to be connected if there is at least one path between every pair of vertices in G, otherwise G is disconnected. A null graph of more than one vertex is disconnected. Each connected sub-graph in a disconnected graph is called component.



A connected graph



A disconnected graph with two components g1 and g2

Theorem

A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

Proof: Let the number of vertices in each of the k components of a graph G be $n_1, n_2, \ldots, n_k$. Thus we have

$$n_1 + n_2 + \cdots + n_k = n,$$
$$n_i \geq 1.$$

Therefore, $\sum_{i=1}^{k} (n_i - 1) = n - k$

Squaring both sides we have, $\left(\sum_{i=1}^{k} (n_i - 1)\right)^2 = n^2 + k^2 - 2nk$

or $\sum_{i=1}^{k} (n_i^2 - 2n_i) + k = n^2 + k^2 - 2nk$ because $(n_i - 1) \geq 0$,

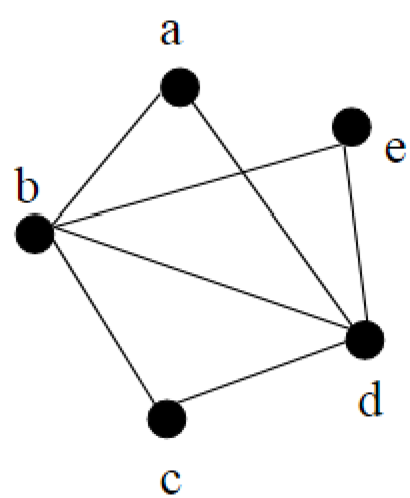
Therefore, $\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk - k + 2n = n^2 - (k-1)(2n-k)$.

Now the maximum number of edges in the ith component of G (which is a simple connected graph) is $\frac{1}{2}n_i(n_i - 1)$. (See Problem 1-12.) Therefore, the maximum number of edges in G is

$$\frac{1}{2}\sum_{i=1}^{k}(n_i - 1)n_i = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2\right) - \frac{n}{2}$$
$$\leq \frac{1}{2}[n^2 - (k-1)(2n-k)] - \frac{n}{2},$$
$$= \frac{1}{2}\cdot(n-k)(n-k+1).$$

Contd..

A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.



Here, n= 5, k= 1

So, max edges = (5-1)(5-1+1)/2 = 10

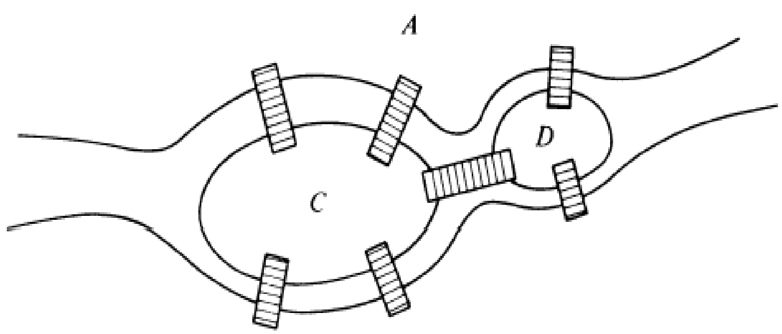
Euler Graph

As mentioned in Chapter 1, graph theory was born in 1736 with Euler's famous paper in which he solved the Königsberg bridge problem. In the same paper, Euler posed (and then solved) a more general problem: In what type of graph G is it possible to find a closed walk running through every edge of G exactly once? Such a walk is now called an *Euler line*, and a graph that consists of an Euler line is called an *Euler graph*. More formally:

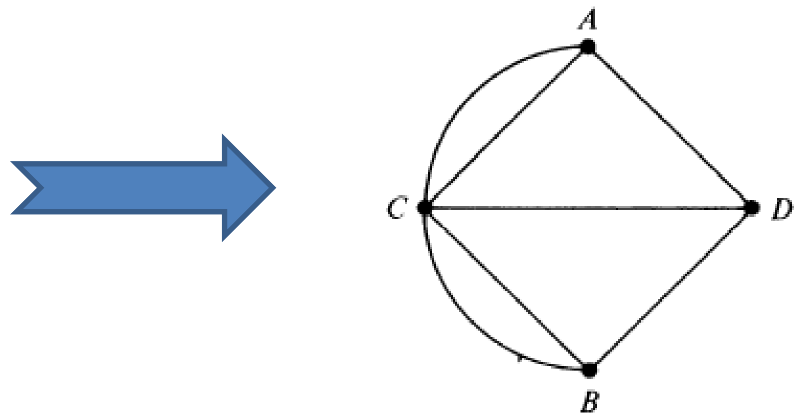
If some closed walk in a graph contains all the edges of the graph, then the walk is called an *Euler line* and the graph an *Euler graph*.

Contd..

Königsberg Bridge Problem.



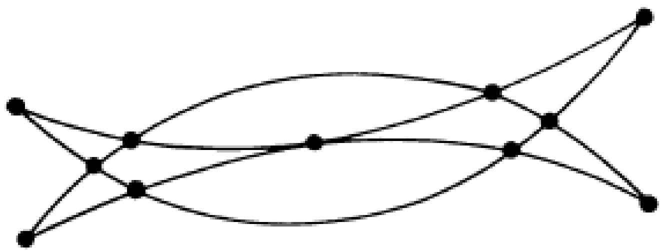
Königsberg bridge problem



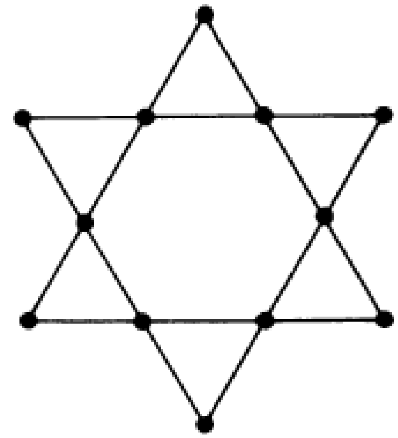
A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.

Königsberg Bridge Problem: Now looking at the graph of the Königsberg bridges (Fig. 1-5), we find that not all its vertices are of even degree. Hence, it is not an Euler graph. Thus it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

Contd..

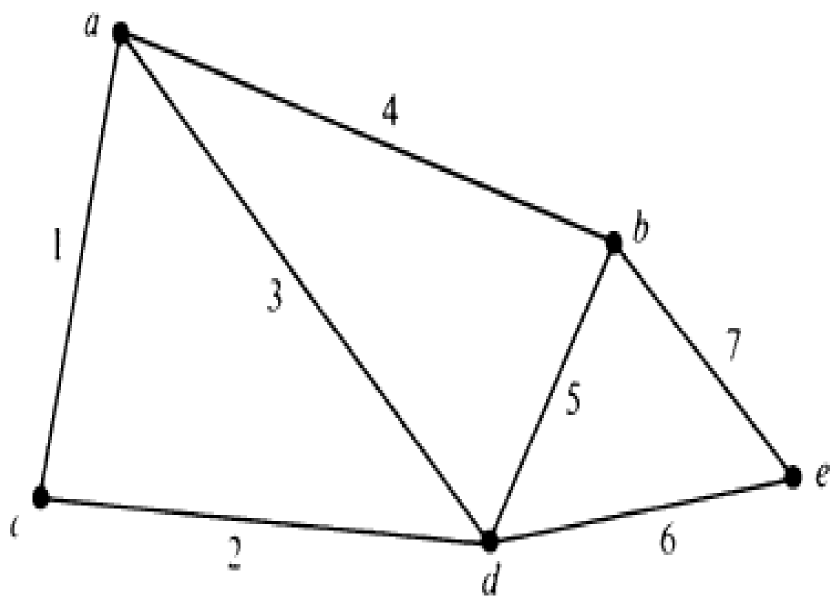


(a)



(b)

Contd..



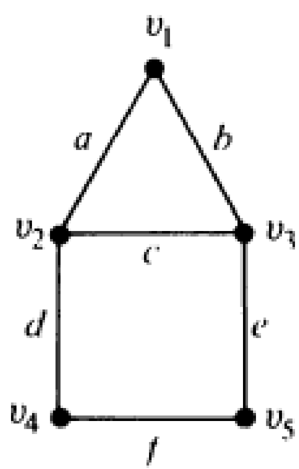
Unicursal graph.

In a Euler graph, if the starting and ending vertices are different, we call it, Unicursal Graph or Open Euler Graph. Here degree of starting and ending vertices are odd.

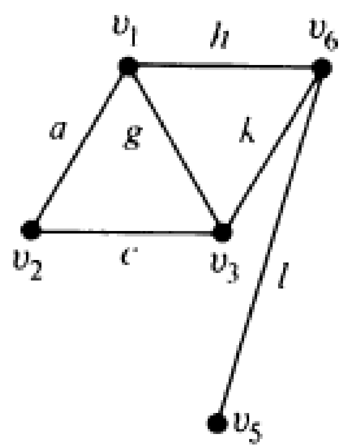
Operations on Graph - Union

The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph G_3 (written as $G_3 = G_1 \cup G_2$) whose vertex set $V_3 = V_1 \cup V_2$ and the edge set $E_3 = E_1 \cup E_2$.

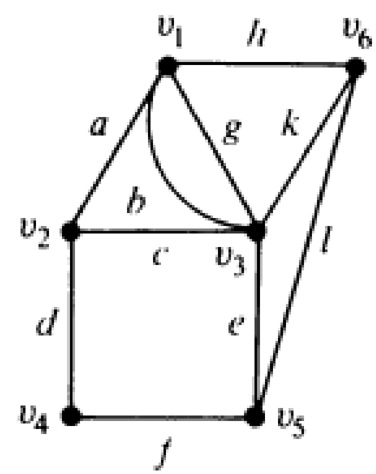
$$G_1 \cup G_2 = G_2 \cup G_1$$



G_1



G_2

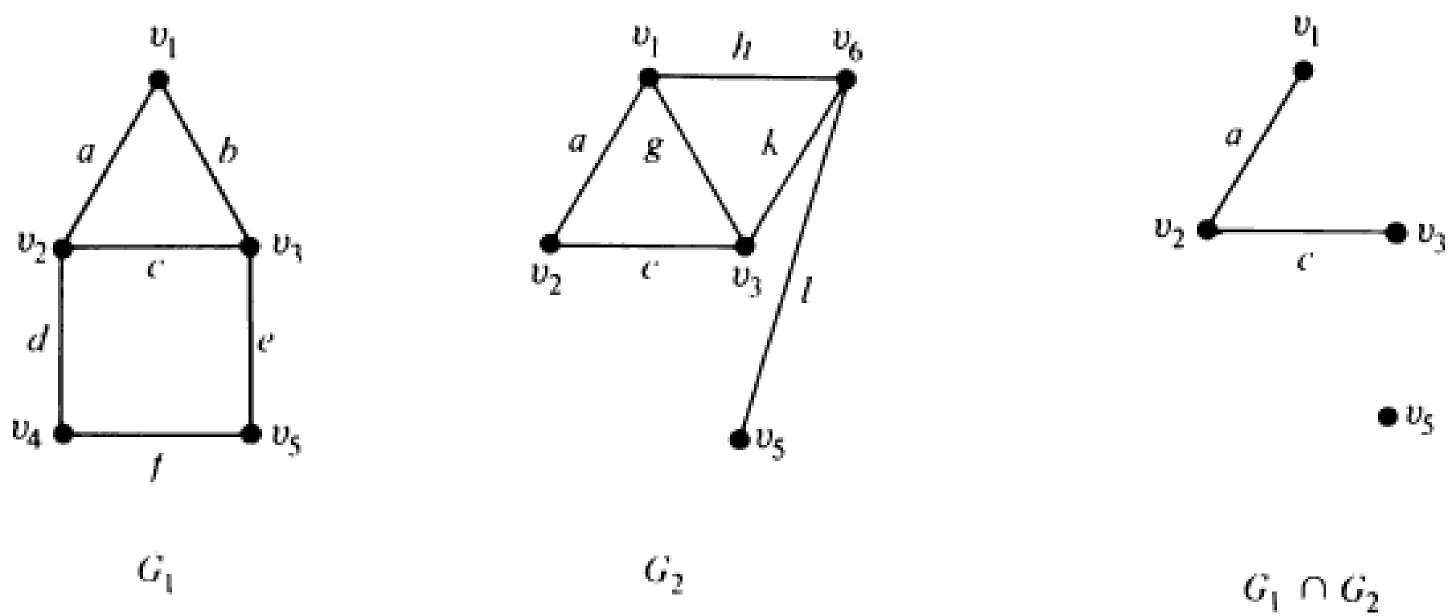


$G_1 \cup G_2$

Intersection

Likewise, the *intersection* $G_1 \cap G_2$ of graphs G_1 and G_2 is a graph G_4 consisting only of those vertices and edges that are in both G_1 and G_2.

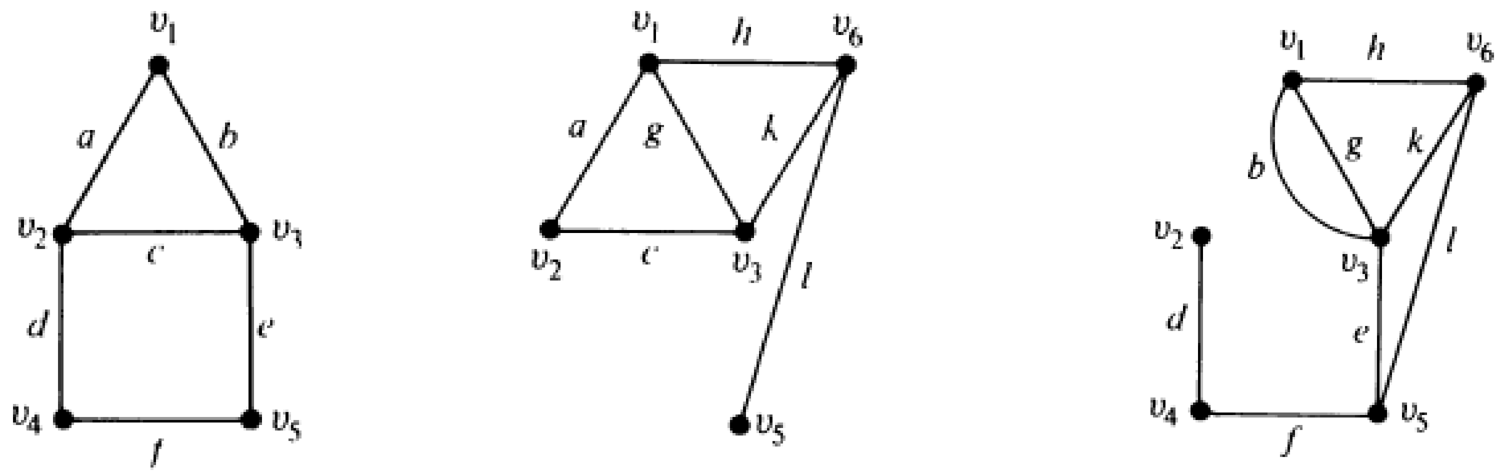
$$G_1 \cap G_2 = G_2 \cap G_1$$



Ring Sum

The *ring sum* of two graphs G_1 and G_2 (written as $G_1 \oplus G_2$) is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but *not* in both.

$$G_1 \oplus G_2 = G_2 \oplus G_1$$



G_1 G_2 $G_1 \oplus G_2$

Operations on Graph

If G_1 and G_2 are edge disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$. If G_1 and G_2 are vertex disjoint, then $G_1 \cap G_2$ is empty.

For any graph G,

$$G \cup G = G \cap G = G,$$

$$G \oplus G = \text{a null graph.}$$

Please draw appropriate graphs and explain the above

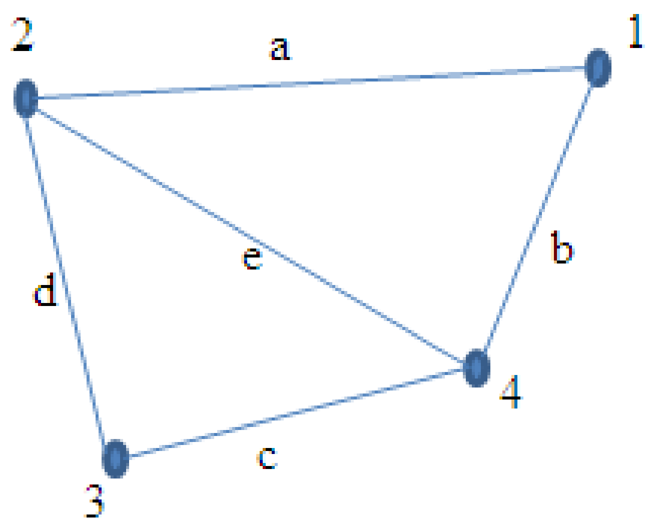
Decomposition

Decomposition: A graph G is said to have been *decomposed* into two subgraphs g_1 and g_2 if

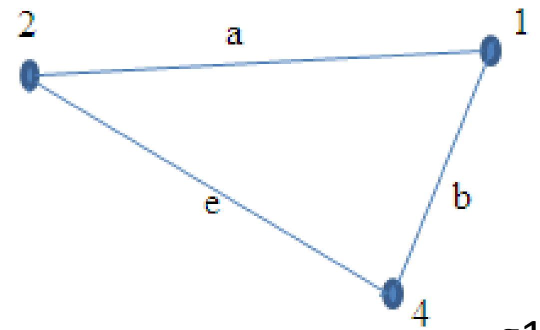
$$g_1 \cup g_2 = G,$$

and

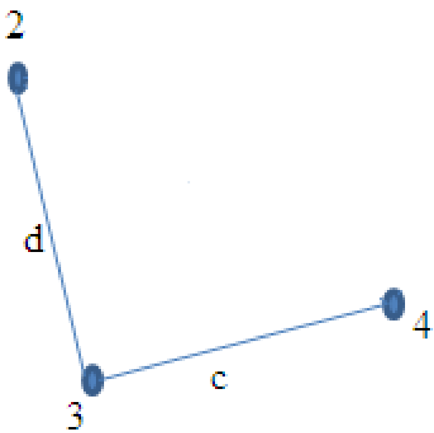
$$g_1 \cap g_2 = \text{a null graph.}$$



G



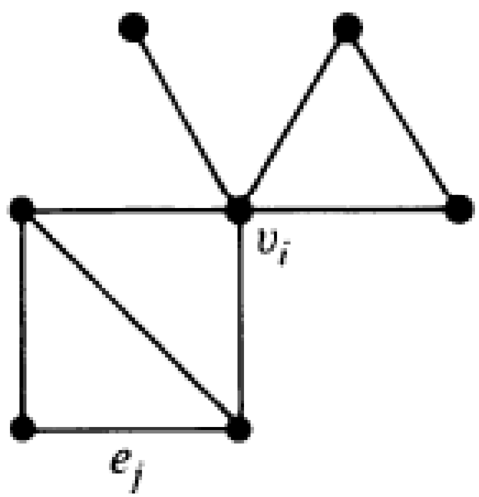
g1



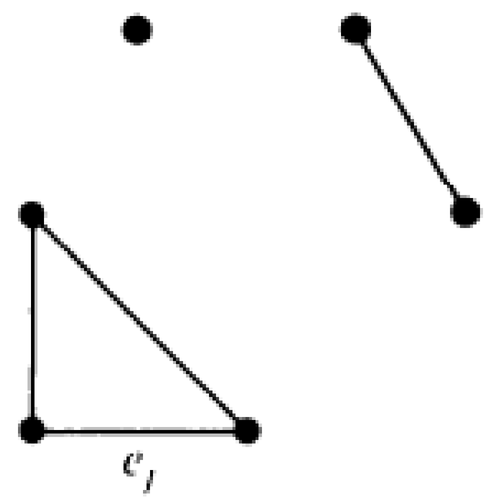
g2

Deletion

If v_i is a vertex in graph G, then $G - v_i$ denotes a subgraph of G obtained by deleting (i.e., removing) v_i from G. Deletion of a vertex always implies the deletion of all edges incident on that vertex.



G



$(G - v_i)$

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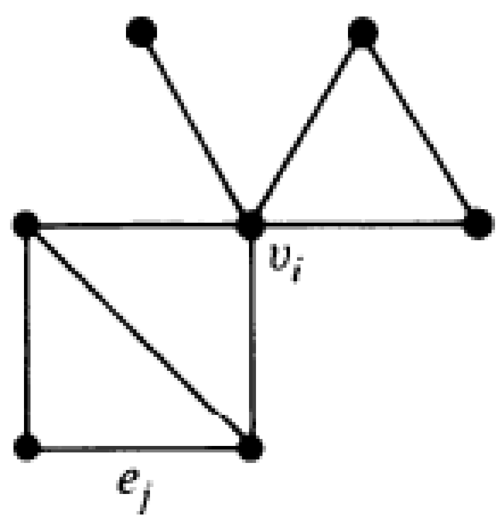
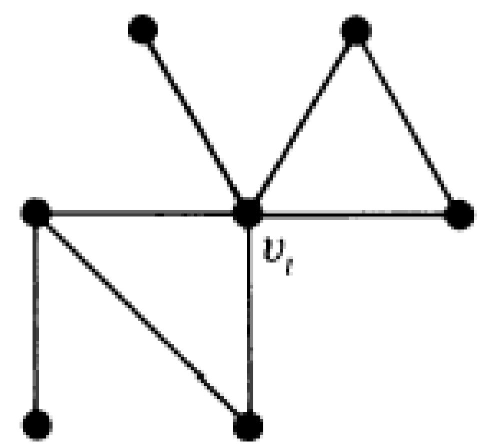
$G - e_j$ is a subgraph of G obtained by deleting e_j from G. Deletion of an edge does not imply deletion of its end vertices.



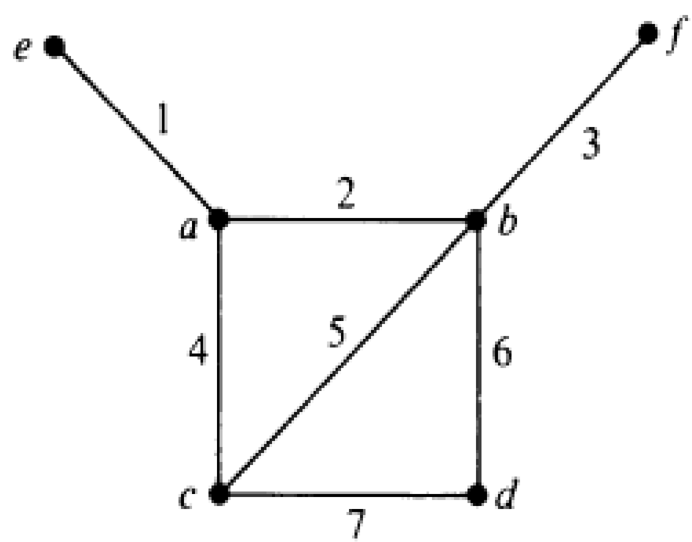
G

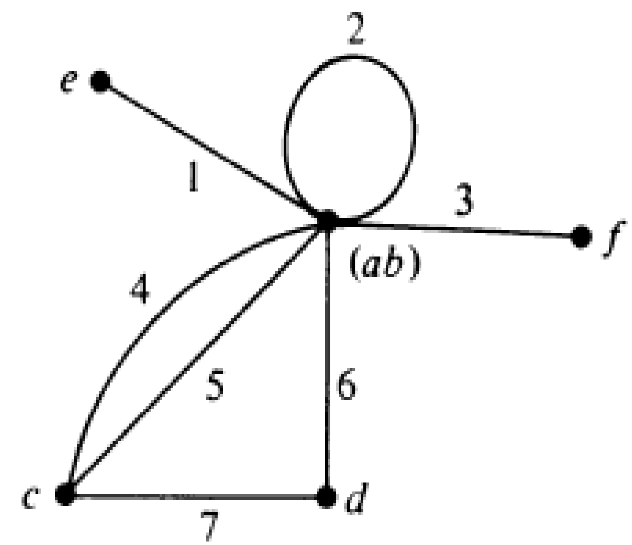


$(G - e_j)$

Fusion

A pair of vertices a, b in a graph are said to be *fused* (*merged* or *identified*) if the two vertices are replaced by a single new vertex such that every edge that was incident on either a or b or on both is incident on the new vertex. Thus fusion of two vertices does not alter the number of edges, but it reduces the number of vertices by one.





Fusion of vertices a and b.