DAY 12

Abstract or Combinatorial Graph

As mentioned in Chapter 1, a graph exists as an abstract object, devoid of any geometric connotation of its ability of being drawn in a three-dimensional Euclidean space. For example, an abstract graph G_1 can be defined as

$$G_1 = (V, E, \Psi),$$

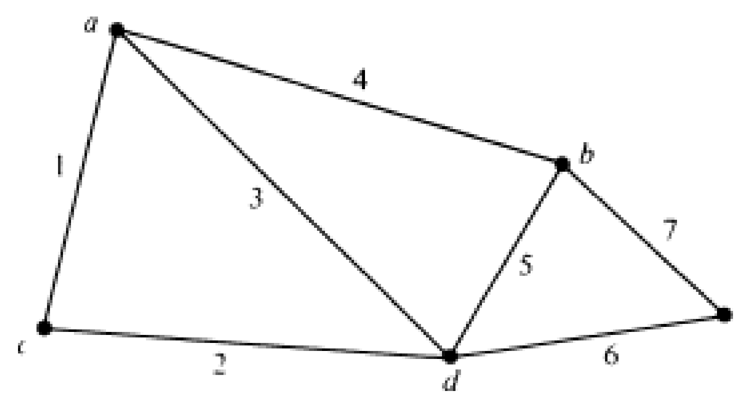
where the set V consists of the five objects named a, b, c, d, and e, that is,

and the set E consists of seven objects (none of which is in set V) named 1, 2, 3, 4, 5, 6, and 7, that is,

$$E = \{1, 2, 3, 4, 5, 6, 7\},$$

and the relationship between the two sets is defined by the mapping Ψ, which consists of

$$\Psi = \begin{bmatrix} 1 \longrightarrow (a, c) \\ 2 \longrightarrow (c, d) \\ 3 \longrightarrow (a, d) \\ 4 \longrightarrow (a, b). \\ 5 \longrightarrow (b, d) \\ 6 \longrightarrow (d, e) \\ 7 \longrightarrow (b, e) \end{bmatrix}$$



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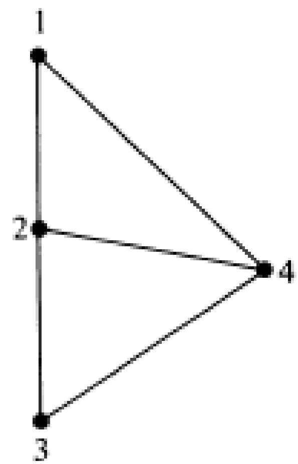
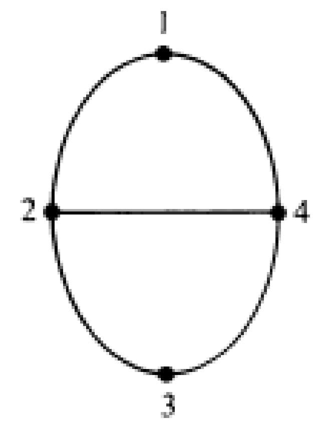


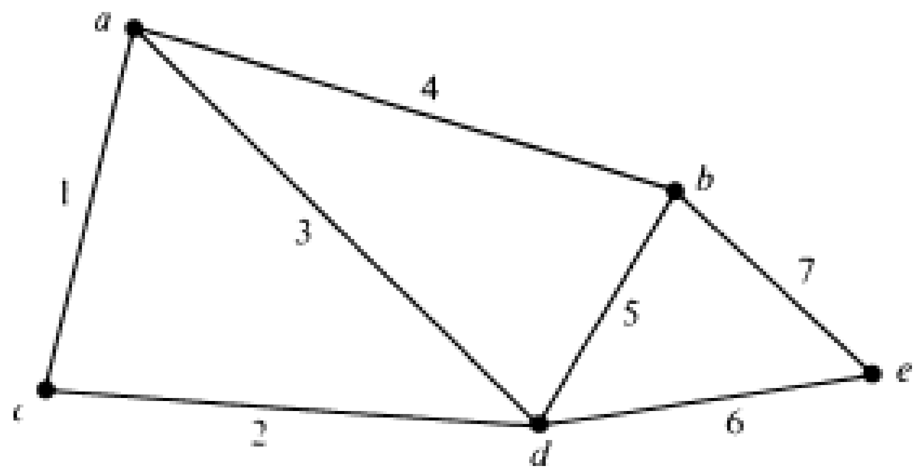
(a)



(b)

Planar Graph

A graph G is said to be *planar* if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect. A graph that cannot be drawn on a plane without a crossover between its edges is called *nonplanar*.



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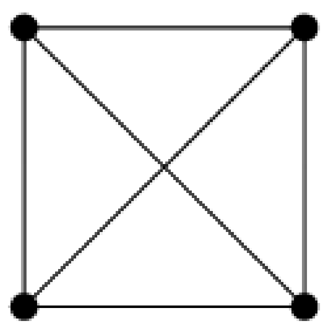
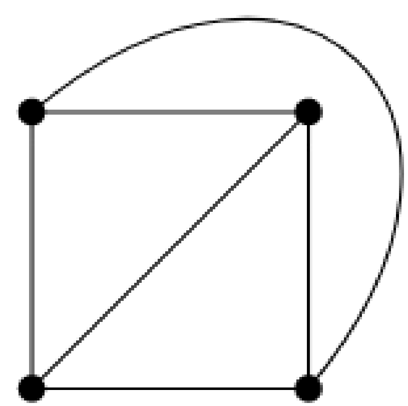
A drawing of a geometric representation of a graph on any surface such that no edges intersect is called *embedding*.





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How can one say whether a given graph is planar or not?

The answer is given by Kasimir Kuratowski.

KURATOWSKI'S TWO GRAPHS

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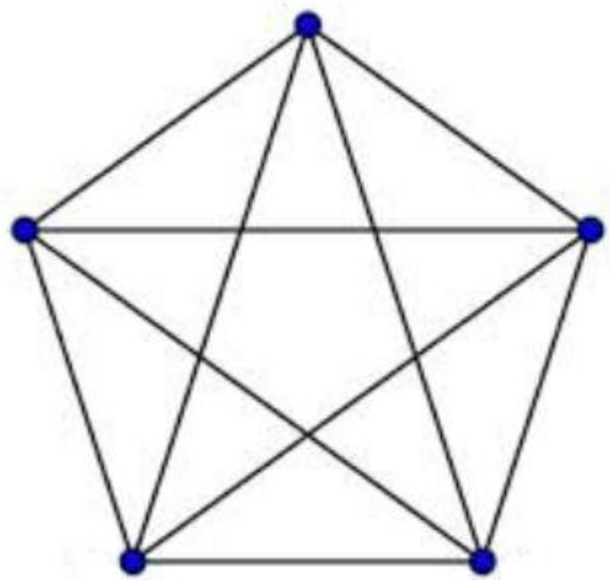
THEOREM 5-1

The complete graph of five vertices is nonplanar.



K_5

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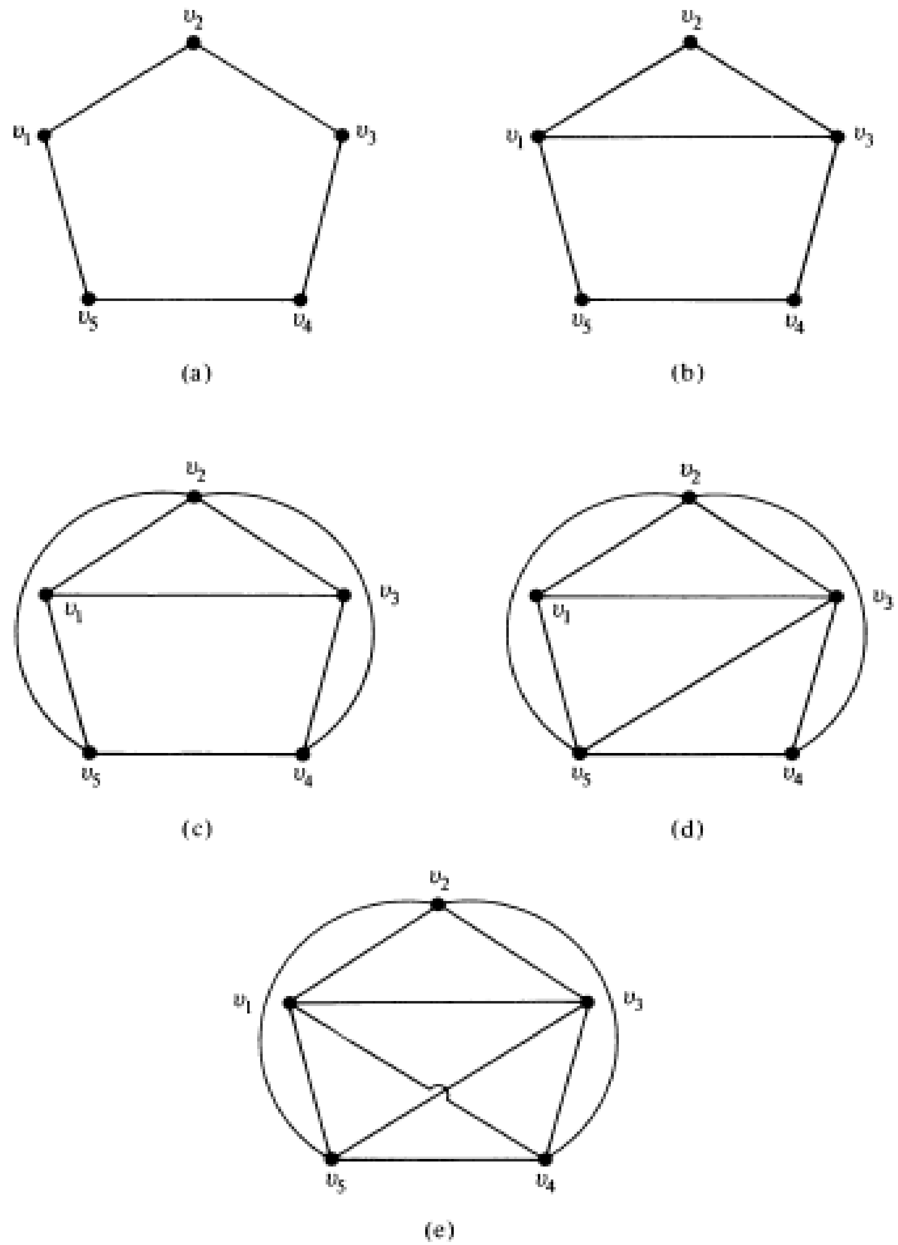

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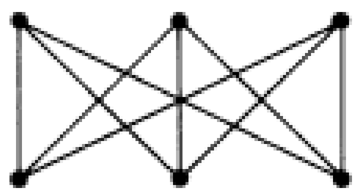
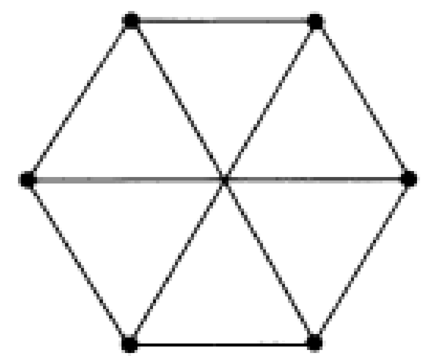
THEOREM 5-2

Kuratowski's second graph is also nonplanar.



(a)



(b)

$K_{3,3}$

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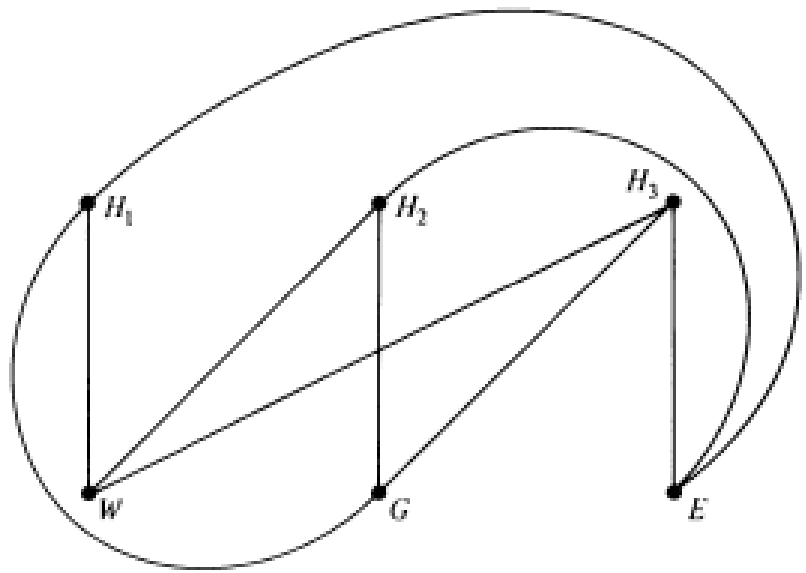
THEOREM 5-2

Kuratowski's second graph is also nonplanar.



$K_{3,3}$

Properties of two graphs

1. Both are regular graphs.
2. Both are nonplanar.
3. Removal of one edge or a vertex makes each a planar graph.
4. Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski's second graph is the nonplanar graph with the smallest number of edges. Thus both are the simplest nonplanar graphs.

Region of a graph

Region: A plane representation of a graph divides the plane into *regions* (also called *windows*, *faces*, or *meshes*), as shown in Fig. 5-4.

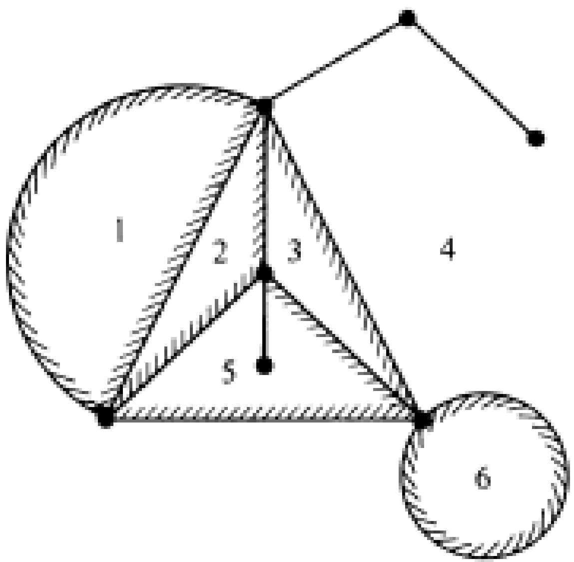


Fig. 5-4 Plane representation (the numbers stand for regions).

Graph Embedding on a Sphere

Embedding on a Sphere: To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere. It is accomplished by stereographic projection of a sphere on a plane. Put the sphere on the plane and call the point of contact SP (south pole). At point SP, draw a straight line perpendicular to the plane, and let the point where this line intersects the surface of the sphere be called NP (north pole). See Fig. 5-5.

Now, corresponding to any point p on the plane, there exists a unique point p' on the sphere and vice versa, where p' is the point at which the straight line from point p to point NP intersects the surface of the sphere. Thus there is a one-to-one correspondence between the points of the sphere and the finite points on the plane, and points at infinity in the plane correspond to the point NP on the sphere.

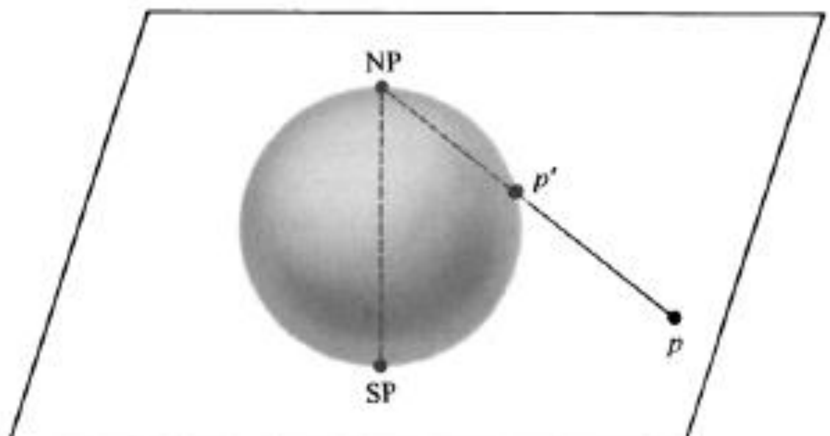


Fig. 5-5 Stereographic projection.

Infinite Region

Infinite Region: The portion of the plane lying outside a graph embedded in a plane, such as region 4 in Fig. 5-4, is infinite in its extent. Such a region is called the *infinite*, *unbounded*, *outer*, or *exterior* region for that particular plane representation.

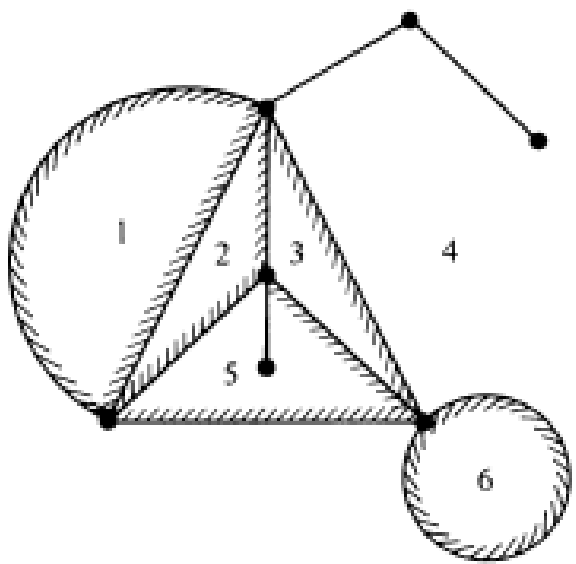
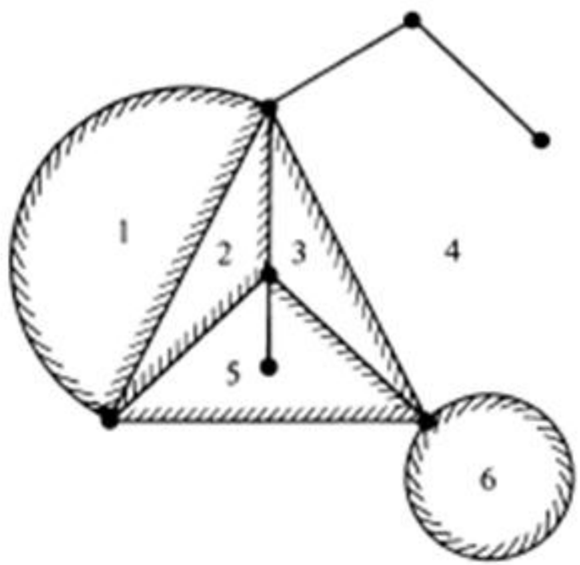


Fig. 5-4 Plane representation (the numbers stand for regions).

Number of Region

THEOREM 5-6

A connected planar graph with n vertices and e edges has $e - n + 2$ regions.



$f = e - n + (k+1)$

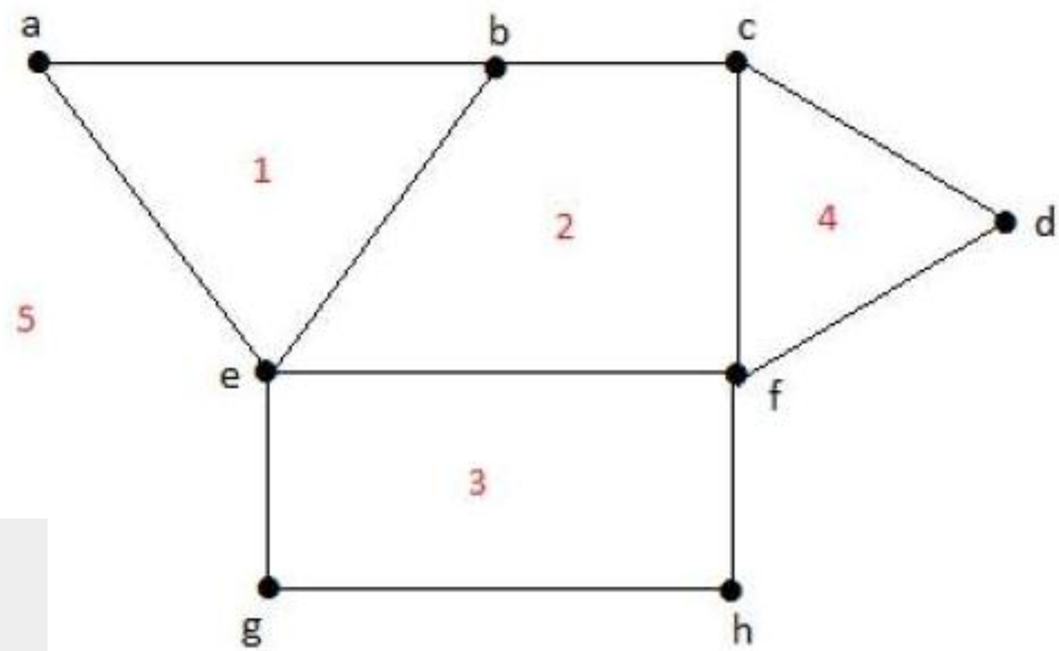
$f = e - n + 2$

$= 11 - 7 + 2$

$= 6$

Degree of a Region

Degree of a bounded region **r = deg(r)** = Number of edges enclosing that regions **r**.

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deg(1) = 3
deg(2) = 4
deg(3) = 4
deg(4) = 3
deg(5) = 8
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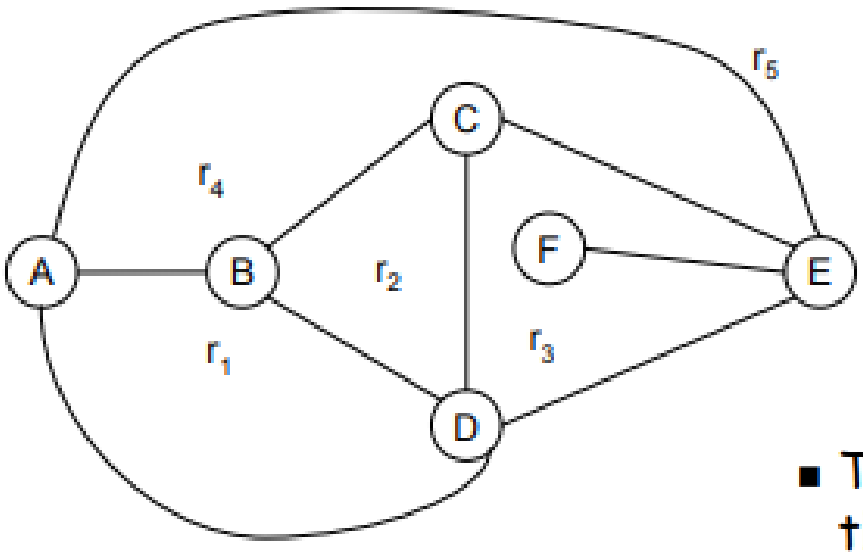

- The length of the cycle which borders the region
  - r_1: 3 (ABDA)
  - r_3: 5 (CDEFEC)
- The sum of the degrees of the regions of a map is equal to twice the number of edges.
  - Each edge borders two regions, and counted twice in the sum.

Solve

1. Is it possible to have a planer graph with 6 vertices, 10 edges and 5 faces?

2. For the given graph verify that sum of degrees of regions equals to twice edges.