Day 15 Class

MATRIX REPRESENTATION OF GRAPHS

Situation comes when we need a better way out to convert the graphical representation into digital or algorithmic format. A matrix is a convenient and useful way to represent a graph to a computer.

Since, matrix is widely used in mechanical manipulations and also it is efficient enough to define various applications so this chapter is all about

- Formation of matrix
- Different kind of observations on that matrix

Various types of matrix

- Adjacency Matrix
- Incidence Matrix
- Reduced Incidence Matrix
- Sub Matrix
- Circuit Matrix
- Fundamental Circuit Matrix
- Cut Set Matrix
- Path Matrix

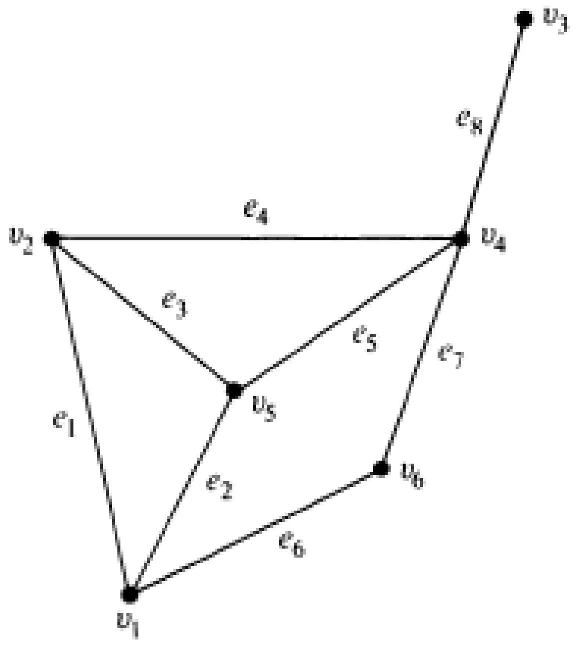
ADJACENCY MATRIX

- With a given graph G , Adjacency or connection matrix is a n by n symmetric binary matrix .
  - Consider graph has no parallel edge and self loop , because those do not carry redundant information .
- Then each element of the graph is defined as:

  - $X_{i,j}$ = 1, if there is an edge between i^{th} and j^{th} vertices and
    = 0, if there is no edge between them.

Contd..

$$
X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$



Observations

1. The entries along the principal diagonal of X are all 0's if and only if the graph has no self-loops. A self-loop at the ith vertex corresponds to $x_{ii} = 1$.
2. The definition of adjacency matrix makes no provision for parallel edges. This is why the adjacency matrix X was defined for graphs
3. If the graph has no self-loops (and no parallel edges, of course), the degree of a vertex equals the number of 1's in the corresponding row or column of X.

Observations

4. Permutations of rows and of the corresponding columns imply reordering the vertices. It must be noted, however, that the rows and columns must be arranged in the same order. Thus, if two rows are interchanged in X, the corresponding columns must also be interchanged. Hence two graphs G_1 and G_2 with no parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related:

$$X(G_2) = R^{-1} \cdot X(G_1) \cdot R,$$

where R is a permutation matrix.

5. A graph G is disconnected and is in two components g_1 and g_2 if and only if its adjacency matrix $X(G)$ can be partitioned as

$$X(G) = \left[\begin{array}{c|c} X(g_1) & 0 \\ \hline 0 & X(g_2) \end{array}\right],$$

where $X(g_1)$ is the adjacency matrix of the component g_1 and $X(g_2)$ is that of the component g_2.

This partitioning clearly implies that there exists no edge joining any vertex in subgraph g_1 to any vertex in subgraph g_2.

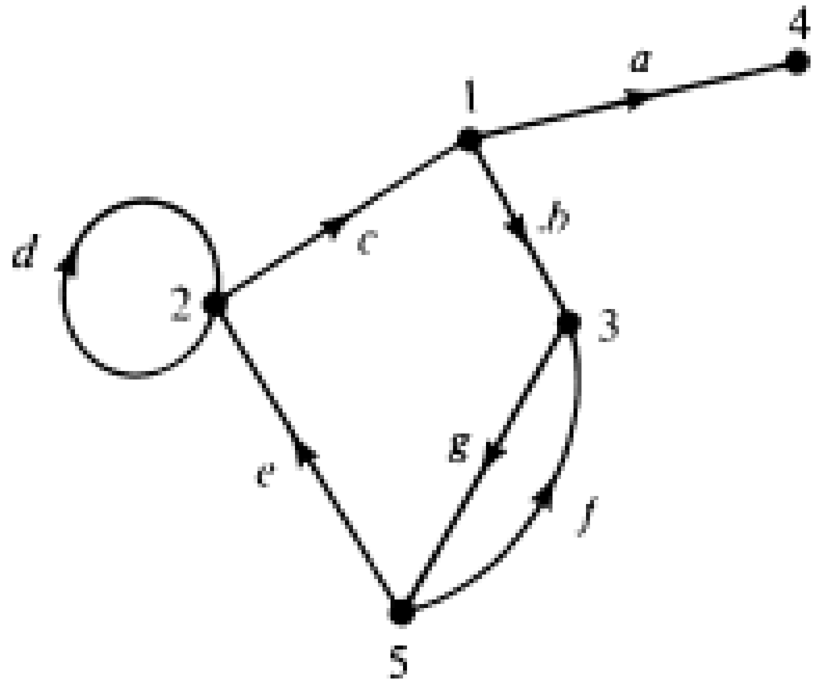
6. Given any square, symmetric, binary matrix Q of order n, one can always construct a graph G of n vertices (and no parallel edges) such that Q is the adjacency matrix of G.

ADJACENCY MATRIX IN DIGRAPH

Another important matrix used in the representation and study of digraphs is the *adjacency matrix* defined as follows: Let G be a digraph with n vertices, containing no parallel edges. Then the adjacency matrix $\mathrm{X} = [x_{ij}]$ of the digraph G is an n by n (0, 1)–matrix whose element

$$
\begin{aligned}
x_{ij} &= 1, &&\text{if there is an edge directed from } i\text{th vertex to } j\text{th vertex,}\\
&= 0, &&\text{otherwise.}
\end{aligned}
$$

Contd..



$$X = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

INCIDENCE MATRIX

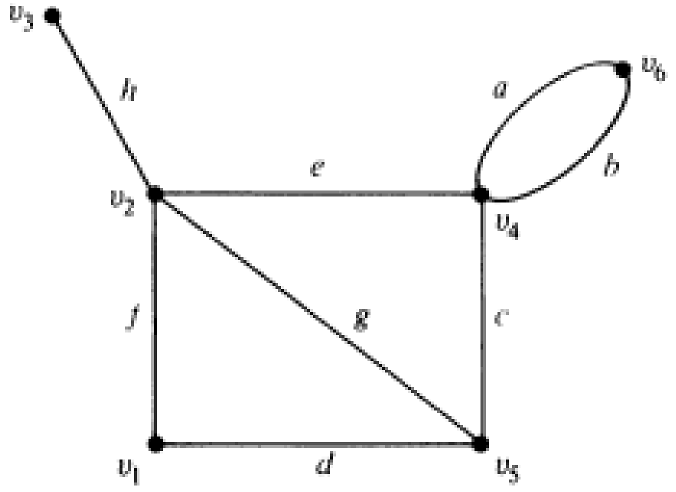
Let G be a graph with n vertices, e edges, and no self-loops. Define an n by e matrix $A = [a_{ij}]$, whose n rows correspond to the n vertices and the e columns correspond to the e edges, as follows:

The matrix element

$$
\begin{aligned}
a_{ij} &= 1, \quad \text{if } j\text{th edge } e_j \text{ is incident on } i\text{th vertex } v_i \text{, and} \\
&= 0, \quad \text{otherwise.}
\end{aligned}
$$

INCIDENCE MATRIX

$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$



Observations:

1. Since every edge is incident on exactly two vertices, each column of A has exactly two 1's.
2. The number of 1's in each row equals the degree of the corresponding vertex.
3. A row with all 0's, therefore, represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix, for example, columns 1 and 2 in Fig. 7-1.
5. If a graph G is disconnected and consists of two components g_1 and g_2, the incidence matrix $\mathsf{A}(G)$ of graph G can be written in a block-diagonal form as

$$\mathsf{A}(G) = \left[\begin{array}{c|c} \mathsf{A}(g_1) & 0 \\ \hline 0 & \mathsf{A}(g_2) \end{array}\right],$$

   where $\mathsf{A}(g_1)$ and $\mathsf{A}(g_2)$ are the incidence matrices of components g_1 and g_2. This observation results from the fact that no edge in g_1 is incident on vertices of g_2, and vice versa. Obviously, this remark is also true for a disconnected graph with any number of components.
6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

Reduced Incidence Matrix

an $(n-1)$ by e submatrix A_f of A is called a *reduced incidence matrix*. The vertex corresponding to the deleted row in A_f is called the *reference vertex*. Clearly, any vertex of a connected graph can be made the reference vertex.

Let g be a subgraph of a graph G, and let $\mathsf{A}(g)$ and $\mathsf{A}(G)$ be the incidence matrices of g and G, respectively. Clearly, $\mathsf{A}(g)$ is a submatrix of $\mathsf{A}(G)$ (possibly with rows or columns permuted). In fact, there is a one-to-one correspondence between each n by k submatrix of $\mathsf{A}(G)$ and a subgraph of G with k edges, k being any positive integer less than e and n being the number of vertices in G.

Question

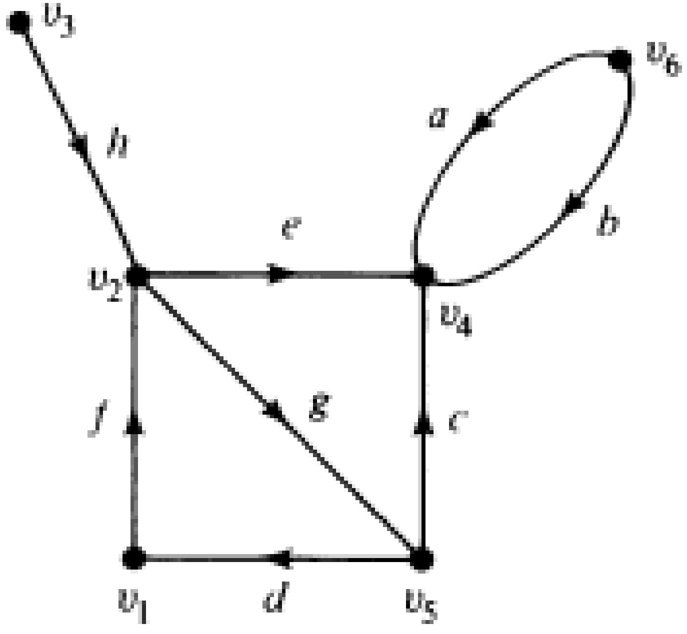
What is sub-matrix?

INCIDENT MATRIX IN DIGRAPH

Incidence Matrix: The *incidence matrix of a digraph* with n vertices, e edges, and no self-loops is an n by n matrix $A = [a_{ij}]$, whose rows correspond to vertices and columns correspond to edges, such that

$$
\begin{aligned}
a_{ij} &= 1, && \text{if } j\text{th edge is incident out of } i\text{th vertex,}\\
&= -1, && \text{if } j\text{th edge is incident into } i\text{th vertex,}\\
&= 0, && \text{if } j\text{th edge is not incident on } i\text{th vertex.}
\end{aligned}
$$

Contd..



(a)

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\left[\begin{array}{cccccccc}
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

CIRCUIT MATRIX

Let the number of different circuits in a graph G be q and the number of edges in G be e. Then a *circuit matrix* $\mathsf{B} = [b_{ij}]$ of G is a q by e, (0, 1)-matrix defined as follows:

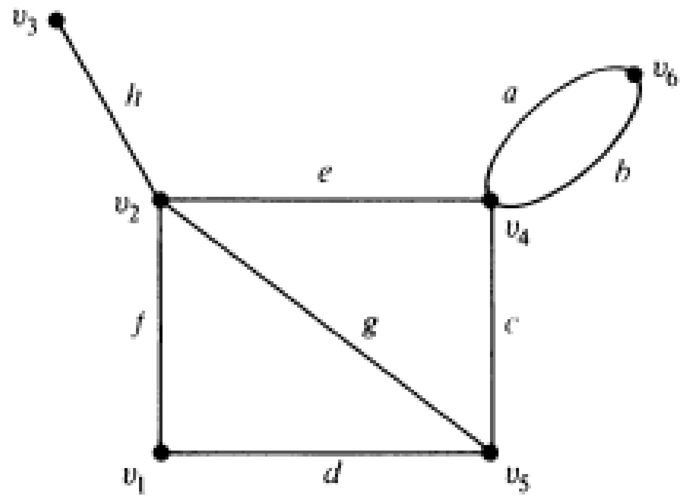
$$\begin{aligned} b_{ij} &= 1, \quad \text{if } i\text{th circuit includes } j\text{th edge, and} \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

To emphasize the fact that B is a circuit matrix of graph G, the circuit matrix may also be written as $\mathsf{B}(G)$.

The graph in Fig. 7-1(a) has four different circuits, $\{a, b\}$, $\{c, e, g\}$, $\{d, f, g\}$, and $\{c, d, f, e\}$. Therefore, its circuit matrix is a 4 by 8, (0, 1)-matrix as shown:

CIRCUIT MATRIX

$$B(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array},$$



Observations:

1. A column of all zeros corresponds to a noncircuit edge (i.e., an edge that does not belong to any circuit).
2. Each row of $B(G)$ is a circuit vector.
3. Unlike the incidence matrix, a circuit matrix is capable of representing a self-loop—the corresponding row will have a single 1.
4. The number of 1's in a row is equal to the number of edges in the corresponding circuit.
5. If graph G is separable (or disconnected) and consists of two blocks (or components) g_1 and g_2, the circuit matrix $B(G)$ can be written in a block-diagonal form as

$$B(G) = \left[\begin{array}{c|c} B(g_1) & 0 \\ \hline 0 & B(g_2) \end{array}\right],$$

   where $B(g_1)$ and $B(g_2)$ are the circuit matrices of g_1 and g_2. This observation results from the fact that circuits in g_1 have no edges belonging to g_2, and vice versa (Problem 4-14).
6. Permutation of any two rows or columns in a circuit matrix simply corresponds to relabeling the circuits and edges.

Contd..

THEOREM 7-4

Let B and A be, respectively, the circuit matrix and the incidence matrix (of a self-loop-free graph) whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row A; that is,

$$\mathsf{A} \cdot \mathsf{B}^T = \mathsf{B} \cdot \mathsf{A}^T = 0 \qquad (\text{mod } 2), \tag{7-4}$$

where superscript T denotes the transposed matrix.

Contd..

$$A \cdot B^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \pmod 2.$$

FUNDAMENTAL CIRCUIT MATRIX

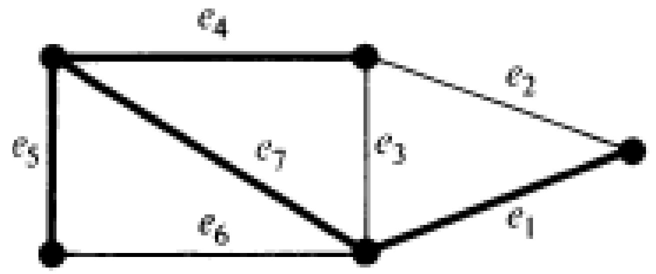
A submatrix (of a circuit matrix) in which all rows correspond to a set of fundamental circuits is called a *fundamental circuit matrix* B_f. A graph and its fundamental circuit matrix with respect to a spanning tree (indicated by heavy lines) are shown in Fig. 7-2.

As in matrices A and B, permutations of rows (and/or of columns) do not affect B_f. If n is the number of vertices and e the number of edges in a connected graph, then B_f is an $(e - n + 1)$ by e matrix, because the number of fundamental circuits is $e - n + 1$, each fundamental circuit being produced by one chord.

Contd..

$$\begin{array}{c} \begin{array}{ccccccc} e_2 & e_3 & e_6 & e_1 & e_4 & e_5 & e_7 \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

(b)



(a)

Fig. 7-2 Graph and its fundamental circuit matrix (with respect to the spanning tree shown in heavy lines).

Contd..

A matrix B_f thus arranged can be written as

$$B_f = [I_\mu \mid B_t], \tag{7-5}$$

where I_μ is an identity matrix of order $\mu = e - n + 1$, and B_t is the remaining μ by $(n - 1)$ submatrix, corresponding to the branches of the spanning tree.

From Eq. (7-5) it is clear that the

$$\text{rank of } B_f = \mu = e - n + 1.$$

Since B_f is a submatrix of the circuit matrix B, the

$$\text{rank of } B \geq e - n + 1.$$

CIRCUIT MATRIX IN DIGRAPH

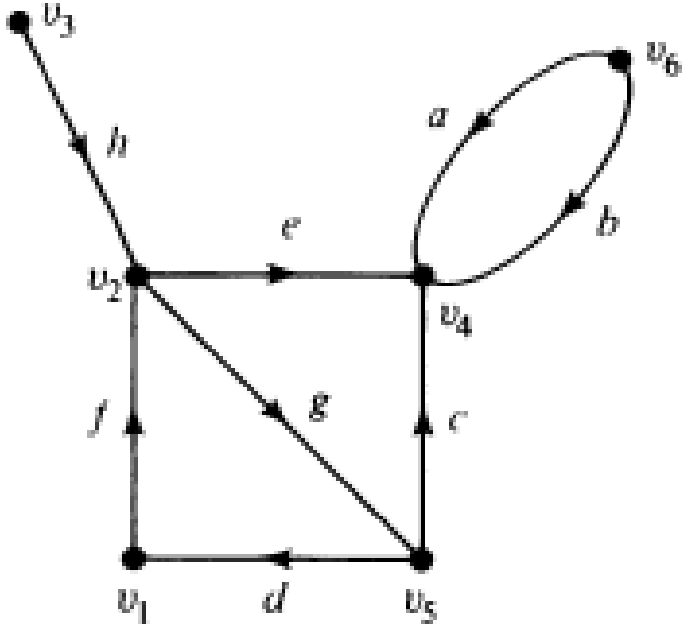
Circuit Matrix of a Digraph: Let G be a digraph with e edges and q circuits (directed circuits or semicircuits). An arbitrary orientation (clockwise or counterclockwise) is assigned to each of the q circuits. Then a circuit matrix $\mathsf{B} = [b_{ij}]$ of the digraph G is a q by e matrix defined as

$b_{ij} = 1$, if ith circuit includes jth edge, and the orientations of the edge and circuit coincide,

$= -1$, if ith circuit includes jth edge, but the orientations of the two are opposite,

$= 0$, if ith circuit does not include the jth edge.

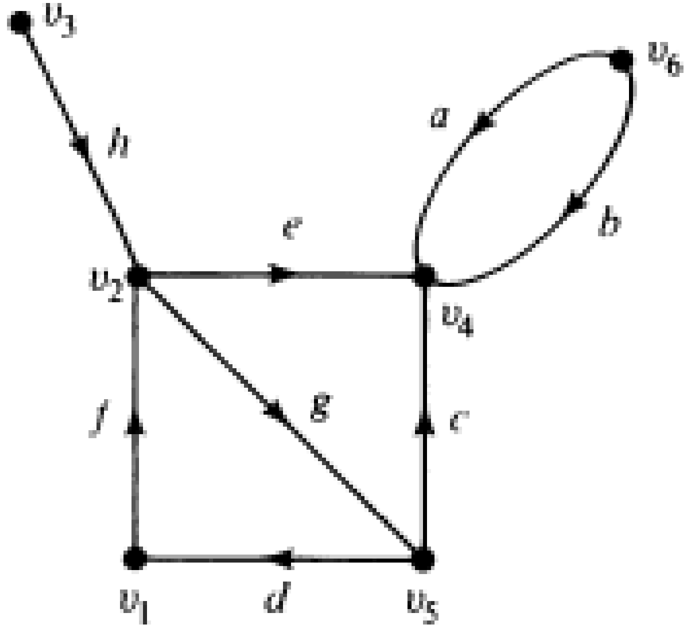
Contd..



(a)

$$
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\end{array}
$$

FUNDAMENTAL CIRCUIT MATRIX IN DIGRAPH



(a)

Fundamental Circuit Matrix: The μ fundamental circuits each made by a chord (with respect to some specified spanning tree) define a fundamental circuit matrix B_f for a digraph. The orientation assigned to each of the fundamental circuits is chosen to coincide with that of the chord. Therefore, B_f, a μ by e matrix, can be expressed exactly in the same form as in the case of an undirected graph in Section 7-4:

$$B_f = [I_\mu \mid B_t],$$

where I_μ is the identity matrix of order μ, and the columns of B_t correspond to the edges in a spanning tree. This is illustrated in Fig. 9-18.

$$B_f = \begin{array}{c} \begin{array}{cccccccc} b & d & g & a & c & e & f & h \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

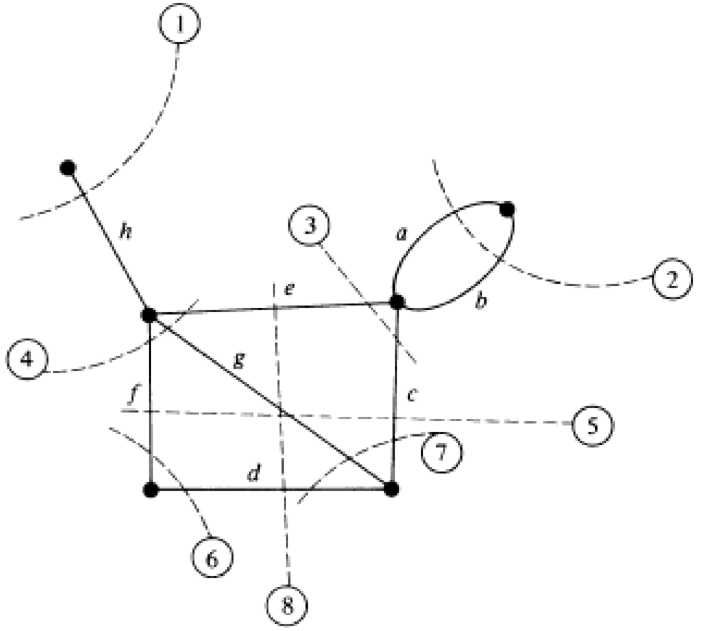
CUT SET MATRIX

Analogous to a circuit matrix, we can define a *cut-set matrix* $\mathbf{C} = [c_{ij}]$ in which the rows correspond to the cut-sets and the columns to the edges of the graph, as follows:

$$
\begin{aligned}
c_{ij} &= 1, \quad \text{if } i\text{th cut-set contains } j\text{th edge, and} \\
&= 0, \quad \text{otherwise.}
\end{aligned}
$$

CUT SET MATRIX

$$
C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$



Question

What is a fundamental cut-set matrix?

PATH MATRIX

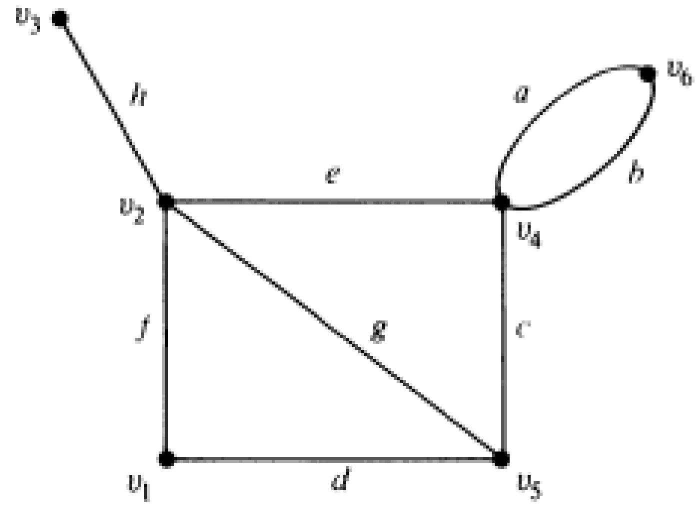
Another (0, 1)-matrix often convenient to use in communication and transportation networks is the *path matrix*. A path matrix is defined for a specific pair of vertices in a graph, say (x, y), and is written as $\mathsf{P}(x, y)$. The rows in $\mathsf{P}(x, y)$ correspond to different paths between vertices x and y, and the columns correspond to the edges in G. That is, the path matrix for (x, y) vertices is $\mathsf{P}(x, y) = [p_{ij}]$, where

$$\begin{aligned} p_{ij} &= 1, \quad \text{if } j\text{th edge lies in } i\text{th path, and} \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

As an illustration, consider all paths between vertices v_3 and v_4 in Fig. 7-1(a). There are three different paths; $\{h, e\}$, $\{h, g, c\}$, and $\{h, f, d, c\}$. Let us number them 1, 2, and 3, respectively. Then we get the 3 by 8 path matrix $\mathsf{P}(v_3, v_4)$:

PATH MATRIX

$$P(v_3, v_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$



Observations

1. A column of all 0's corresponds to an edge that does not lie in any path between x and y.
2. A column of all 1's corresponds to an edge that lies in every path between x and y.
3. There is no row with all 0's.
4. The ring sum of any two rows in $P(x, y)$ corresponds to a circuit or an edge-disjoint union of circuits.

Theorem 7-7

If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix $P(x, y)$, then the product (mod 2)

$$A \cdot P^T(x, y) = M,$$

where the matrix M has 1's in two rows x and y, and the rest of the $n - 2$ rows are all 0's.

Proof: The proof is left as an exercise for the reader (Problem 7-14).

As an example, multiply the incidence matrix in Fig. 7-1 to the transposed $P(v_3, v_4)$, just discussed.

$$A \cdot P^T(v_3, v_4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{(mod 2).}$$