Fourier Series

Chapter 13

13.1 INTRODUCTION

Fourier series is used in the analysis of periodic functions. Many of the phenomena studied in engineering and sciences are periodic in nature e.g., current and voltage in an ac circuit. These periodic functions can be analysed into their constituent components by a Fourier analysis. Fourier series makes use of orthogonality relationships of the sine and cosine functions and exponential functions. It decomposes a periodic function into a sum of sine-cosine functions or exponential functions. The computation and study of Fourier series is known as harmonic analysis. It has many applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, etc.

13.2 ORTHOGONALITY OF FUNCTIONS

Consider two functions $f_1(x)$ and $f_2(x)$. Suppose we want to approximate $f_1(x)$ in terms of $f_2(x)$ in a certain interval (a, b)

$$f_1(x) \cong c_{12} f_2(x)$$

The error in this approximation will be

$$f_e(x) = f_1(x) - c_{12} f_2(x)$$

c_{12} is selected such that error between $f_1(x)$ and $c_{12} f_2(x)$ is minimum. For minimising the error $f_e(x)$ in the interval (a, b), we have to minimise the average (or mean) of the square of the error function $f_e(x)$.

The mean square error $\in$ is given by,

$$\begin{aligned}
\in &= \frac{1}{b-a} \int_a^b f_e^2(x)\,\mathrm{d}x \\
&= \frac{1}{b-a} \int_a^b [f_1(x) - c_{12} f_2(x)]^2\,\mathrm{d}x \\
&= \frac{1}{b-a} \int_a^b [\{f_1(x)\}^2 - 2c_{12} f_1(x) f_2(x) + c_{12}^2 \{f_2(x)\}^2]\,\mathrm{d}x \\
&= \frac{1}{b-a} \left[\int_a^b \{f_1(x)\}^2\,\mathrm{d}x - 2c_{12} \int_a^b f_1(x) f_2(x)\,\mathrm{d}x + c_{12}^2 \int_a^b \{f_2(x)\}^2\,\mathrm{d}x \right]
\end{aligned}$$

To find the value of c_{12} which will minimise $\in$, we must have $\dfrac{\partial \in}{\partial c_{12}} = 0$

$$\frac{\partial \in}{\partial c_{12}} = \frac{\partial}{\partial c_{12}} \frac{1}{b-a}\left[\int_a^b \{f_1(x)\}^2 \mathrm{d}x - 2c_{12}\int_a^b f_1(x) f_2(x) \mathrm{d}x + c_{12}^2 \int_a^b \{f_2(x)\}^2 \mathrm{d}x\right]$$

$$= \frac{1}{b-a}\left[\int_a^b \frac{\partial}{\partial c_{12}} \{f_1(x)\}^2 \mathrm{d}x - 2\int_a^b f_1(x) f_2(x) \mathrm{d}x + 2c_{12}\int_a^b \{f_2(x)\}^2 \mathrm{d}x\right]$$

$$= \frac{1}{b-a}\left[0 - 2\int_a^b f_1(x) f_2(x) \mathrm{d}x + 2c_{12}\int_a^b \{f_2(x)\}^2 \mathrm{d}x\right]$$

When $\dfrac{\partial \in}{\partial c_{12}} = 0$, we get

$$\int_a^b f_1(x) f_2(x) \mathrm{d}x = c_{12}\int_a^b [f_2(x)]^2 \mathrm{d}x$$

$$c_{12} = \frac{\int_a^b f_1(x) f_2(x) \mathrm{d}x}{\int_a^b [f_2(x)]^2 \mathrm{d}x}$$

When c_{12} is zero, the function $f_1(x)$ contains no component of function $f_2(x)$ and two functions are said to be orthogonal in the interval (a, b). Thus two functions are orthogonal in the interval (a, b) if

$$\int_a^b f_1(x) f_2(x) \mathrm{d}x = 0$$

If, in addition, $\int_a^b [f_1(x)]^2 \mathrm{d}x = 1$ and $\int_a^b [f_2(x)]^2 \mathrm{d}x = 1$, the functions are said to be normalised and hence are called, orthonormal.

Note:

1. A set of functions $f_1(x), f_2(x) \ldots\ldots. f_n(x), \ldots$ is said to be orthogonal in the interval (a, b) if these functions are mutually orthogonal, i.e.,

$$\int_a^b f_m(x) f_n(x) \mathrm{d}x = 0, \quad m \neq n$$

2. The orthonormal set of functions is constructed by dividing the orthogonal set of functions by its norm, $\|f(x)\|$, i.e., $\sqrt{\int_a^b [f(x)]^2 \mathrm{d}x}$. Hence, the orthonormal set of functions is,

$$\frac{f_1(x)}{\|f_1(x)\|}, \frac{f_2(x)}{\|f_2(x)\|}, \cdots \frac{f_n(x)}{\|f_n(x)\|}, \cdots$$

3. Any function $f(x)$ can be represented by a complete set of orthogonal functions, in a certain interval which is the basis of a Fourier series representation.

Example 1: Show that the following functions are orthogonal in the given interval.

(i) $f_1(x) = x, \quad f_2(x) = \cos 2x$ **in** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $f_1(x) = e^x, \quad f_2(x) = \sin x$ **in** $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

(iii) $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \frac{3x^2 - 1}{2}$ **in** $[-1, 1]$

Solution:

(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_1(x) f_2(x)\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos 2x\, dx = \left| x\left(\frac{\sin 2x}{2}\right) - 1\left(-\frac{\cos 2x}{4}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$

Hence, x and $\cos 2x$ are orthogonal in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} f_1(x) f_2(x) dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} e^x \sin x\, dx = \left|\frac{e^x}{2}(\sin x - \cos x)\right|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

$$= \frac{1}{2}\left[e^{\frac{5\pi}{4}}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) - e^{\frac{\pi}{4}}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right] = 0$$

Hence, e^x and $\sin x$ are orthogonal in the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

(iii) $\int_{-1}^{1} f_1(x) f_2(x) dx = \int_{-1}^{1} 1 \cdot x\, dx = \left|\frac{x^2}{2}\right|_{-1}^{1} = 0$

$$\int_{-1}^{1} f_1(x) f_3(x) dx = \int_{-1}^{1} 1\left(\frac{3x^2 - 1}{2}\right) dx = \frac{1}{2}\left|x^3 - x\right|_{-1}^{1} = 0$$

$$\int_{-1}^{1} f_2(x) f_3(x) dx = \int_{-1}^{1} x\left(\frac{3x^2 - 1}{2}\right) dx = \frac{1}{2}\int_{-1}^{1} (3x^3 - x) dx = \frac{1}{2}\left|\frac{3x^4}{4} - \frac{x^2}{2}\right|_{-1}^{1} = 0$$

Hence, $1, x$ and $\frac{3x^2 - 1}{2}$ are orthogonal in the interval $[-1, 1]$.

Example 2: Show that the functions $f_1(x) = 1$ and $f_2(x) = x$ are orthogonal in the interval $(-1, 1)$ and determine the constant a and b so that the function $f_3(x) = 1 + ax + bx^2$ is orthogonal to both $f_1(x)$ and $f_2(x)$ in that interval.

Solution: $\int_{-1}^{1} f_1(x) f_2(x)\, dx = \int_{-1}^{1} 1 \cdot x\, dx = \left|\frac{x^2}{2}\right|_{-1}^{1} = 0$

Hence, the functions $f_1(x)$ and $f_2(x)$ are orthogonal in the interval $(-1, 1)$.

The function $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$.

$$\int_{-1}^{1} f_1(x) f_3(x) \mathrm{d}x = 0$$

$$\int_{-1}^{1} 1(1 + ax + bx^2) \mathrm{d}x = 0$$

$$\left| x + \frac{ax^2}{2} + \frac{bx^3}{3} \right|_{-1}^{1} = 0$$

$$2 + \frac{2b}{3} = 0$$

$$b = -3$$

and

$$\int_{-1}^{1} f_2(x) f_3(x) \mathrm{d}x = 0$$

$$\int_{-1}^{1} x(1 + ax + bx^2) \mathrm{d}x = 0$$

$$\left| \frac{x^2}{2} + \frac{ax^3}{3} + \frac{bx^4}{4} \right|_{-1}^{1} = 0$$

$$\frac{2a}{3} = 0$$

$$a = 0$$

Example 3: Show that the set of functions $\{\sin(2n + 1)x\}$, $n = 0, 1, 2, \ldots$ is orthogonal in the interval $\left[0, \frac{\pi}{2}\right]$. Hence, construct corresponding orthonormal set of functions.

Solution: Let $f_m(x) = \sin(2m+1)x, \quad m = 0, 1, 2, \ldots$

$f_n(x) = \sin(2n+1)x, \quad n = 0, 1, 2, \ldots$

Case I: If $m \neq n$

$$\int_0^{\frac{\pi}{2}} f_m(x) f_n(x) \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin(2m+1)x \cdot \sin(2n+1)x \, \mathrm{d}x$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} [\cos(2m-2n)x - \cos(2m+2n+2)x] \mathrm{d}x$$

$$= \frac{1}{2} \left| \frac{\sin(2m-2n)x}{2m-2n} - \frac{\sin(2m+2n+2)x}{2m+2n+2} \right|_0^{\frac{\pi}{2}}$$

$$= 0 \qquad [\because \sin p\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_0^{\frac{\pi}{2}} [f_n(x)]^2 \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^2(2n+1)x \,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [1-\cos 2(2n+1)x]\mathrm{d}x = \frac{1}{2}\left| x - \frac{\sin 2(2n+1)x}{2(2n+1)} \right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} \neq 0$$

Hence, the set of functions $\{\sin(2n+1)x\}$, $n = 0, 1, 2, \ldots$ is orthogonal in the interval $\left[0, \frac{\pi}{2}\right]$. The orthonormal set of functions is constructed by dividing the orthogonal set of functions by its norm.

$$\|f(x)\| = \sqrt{\int_0^{\frac{\pi}{2}} [f(x)]^2 \mathrm{d}x} = \frac{\sqrt{\pi}}{2}$$

Hence, the required orthonormal set of functions is,

$$\left\{ \frac{2}{\sqrt{\pi}} \sin(2n+1)x \right\}, \quad n = 0, 1, 2, \ldots$$

Example 4: Show that $\{\cos x, \cos 2x, \cos 3x, \ldots\}$ is a set of orthogonal function in the interval $(-\pi, \pi)$. Hence, construct an orthonormal set.

Solution: Let $f_m(x) = \cos mx, \quad m = 1, 2, 3, \ldots$

$$f_n(x) = \cos nx, \quad n = 1, 2, 3, \ldots$$

Case I: If $m \neq n$

$$\int_{-\pi}^{\pi} f_m(x) f_n(x)\mathrm{d}x = \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \,\mathrm{d}x$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x]\mathrm{d}x$$

$$= \frac{1}{2}\left| \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right|_{-\pi}^{\pi}$$

$$= 0 \qquad [\because \sin p\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_{-\pi}^{\pi} [f_n(x)]^2 \mathrm{d}x = \int_{-\pi}^{\pi} \cos^2 nx \,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos 2nx)\mathrm{d}x$$

$$= \frac{1}{2}\left| x + \frac{\sin 2nx}{2n} \right|_{-\pi}^{\pi} = \pi \neq 0$$

Hence, the set of functions is orthogonal in the interval $(-\pi, \pi)$. The orthonormal set of functions is constructed by dividing the orthogonal set of functions by its norm.

$$\|f(x)\| = \sqrt{\int_{-\pi}^{\pi} \cos^2 nx \,\mathrm{d}x} = \sqrt{\pi}$$

Hence, the required set of orthonormal functions is

$$\left\{\frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\cos 3x, \ldots\right\}$$

Example 5: Prove that the set of functions $\left\{\sin\frac{\pi x}{l}, \sin\frac{3\pi x}{l}, \sin\frac{5\pi x}{l}, \ldots\right\}$ is orthogonal in the interval $[0, l]$ and construct the corresponding orthonormal set.

Solution: Let $f_m(x) = \sin\frac{(2m+1)\pi x}{l}, \quad m = 0, 1, 2, \ldots$

$$f_n(x) = \sin\frac{(2n+1)\pi x}{l}, \quad n = 0, 1, 2, \ldots$$

Case I: If $m \neq n$

$$\int_0^l f_m(x) f_n(x)\mathrm{d}x = \int_0^l \sin\frac{(2m+1)\pi x}{l}\sin\frac{(2n+1)\pi x}{l}\mathrm{d}x$$

$$= \frac{1}{2}\int_0^l\left[\cos\frac{(2m-2n)\pi x}{l} - \cos\frac{(2m+2n+2)\pi x}{l}\right]\mathrm{d}x$$

$$= \frac{1}{2}\left|\frac{\sin\frac{(2m-2n)\pi x}{l}}{\frac{(2m-2n)\pi}{l}} - \frac{\sin\frac{(2m+2n+2)\pi x}{l}}{\frac{(2m+2n+2)\pi}{l}}\right|_0^l$$

$$= 0 \qquad [\because \sin p\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_0^l [f_n(x)]^2\,\mathrm{d}x = \int_0^l \sin^2\frac{(2n+1)\pi x}{l}\,\mathrm{d}x = \frac{1}{2}\int_0^l\left[1 - \cos\frac{2(2n+1)}{l}\pi x\right]\mathrm{d}x$$

$$= \frac{1}{2}\left|x - \frac{\sin\frac{2(2n+1)}{l}\pi x}{\frac{2(2n+1)\pi}{l}}\right|_0^l = \frac{l}{2} \neq 0$$

Hence, the set of functions is orthogonal in the interval $[0, l]$. The othonormal set of functions is constructed by dividing the orthogonal set of functions by its norm.

$$\|f(x)\| = \sqrt{\int_0^l [f(x)]^2\,\mathrm{d}x} = \sqrt{\frac{l}{2}}$$

Hence, the required orthonormal set of functions is,

$$\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi x}{l}, \sqrt{\frac{2}{l}}\sin\frac{3\pi x}{l}, \sqrt{\frac{2}{l}}\sin\frac{5\pi x}{l}, \ldots\right\}$$

Example 6: Prove that the set of functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\}$ is orthogonal in the interval $(0, 2\pi)$ and construct the corresponding orthonormal set.

Solution: Let $f_n(x) = \sin nx, \quad n = 1, 2, 3, \ldots$

$g_n(x) = \cos nx, \quad n = 1, 2, 3, \ldots$

$h(x) = 1$

(a) Case I: If $m \neq n$

$$\int_0^{2\pi} f_m(x) f_n(x)\,\mathrm{d}x = \int_0^{2\pi} \sin mx \sin nx\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(m-n)x - \cos(m+n)x]\mathrm{d}x$$

$$= \frac{1}{2}\left| \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right|_0^{2\pi}$$

$$= 0 \qquad [\because \sin \mathrm{p}\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_0^{2\pi} [f_n(x)]^2\,\mathrm{d}x = \int_0^{2\pi} \sin^2 nx\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{2\pi} (1 - \cos 2nx)\mathrm{d}x = \frac{1}{2}\left| x - \frac{\sin 2nx}{2} \right|_0^{2\pi}$$

$$= \pi \neq 0$$

(b) Case I: If $m \neq n$

$$\int_0^{2\pi} g_m(x)\, g_n(x)\,\mathrm{d}x = \int_0^{2\pi} \cos mx \cos nx\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]\mathrm{d}x$$

$$= \frac{1}{2}\left| \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right|_0^{2\pi}$$

$$= 0 \qquad [\because \sin p\pi = 0 \text{ where } p \text{ is an integer}]$$

Case II: If $m = n$

$$\int_0^{2\pi} [g_n(x)]^2\,\mathrm{d}x = \int_0^{2\pi} \cos^2 nx\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2nx)\mathrm{d}x = \frac{1}{2}\left| x + \frac{\sin 2nx}{2n} \right|_0^{2\pi}$$

$$= \pi \neq 0$$

(c) $\int_0^{2\pi} f_m(x)\, g_n(x)\, dx = \int_0^{2\pi} \sin mx \cos nx \, dx$

$$= \frac{1}{2}\int_0^{2\pi} [\sin(m+n)x + \sin(m-n)x] dx = \frac{1}{2}\left| -\frac{\cos(m+n)x}{(m+n)} - \frac{\cos(m-n)x}{m-n} \right|_0^{2\pi}$$

$$= \frac{1}{2}\left[\frac{-\cos(m+n)2\pi + \cos 0}{m+n} - \frac{\cos(m-n)2\pi - \cos 0}{m-n} \right]$$

$$= 0 \qquad [\because \cos 2p\pi = 1 \quad \text{where } p \text{ is an integer}]$$

(d) $\int_0^{2\pi} h(x) f_n(x) dx = \int_0^{2\pi} 1 \cdot \sin nx \, dx = \left| -\frac{\cos nx}{n} \right|_0^{2\pi} = \frac{1}{n}(-1+1) = 0$

(e) $\int_0^{2\pi} h(x)\, g_n(x)\, dx = \int_0^{2\pi} 1 \cdot \cos nx \, dx = \left| \frac{\sin nx}{n} \right|_0^{2\pi} = 0$

(f) $\int_0^{2\pi} [h(x)]^2 dx = \int_0^{2\pi} 1 \cdot dx = |x|_0^{2\pi} = 2\pi \neq 0$

Hence, the set of functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\}$ is orthogonal in the interval $(0, 2\pi)$.

The orthonormal set of functions is constructed by dividing the each term of orthogonal set of functions by its norm.

Hence, the required orthonormal set of functions is,

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots \right\}$$

Example 7: Prove that the set of functions $\left\{ e^{-\frac{x}{2}}, e^{-\frac{x}{2}}(-x+1), \frac{1}{2}e^{-\frac{x}{2}}(x^2-4x+2) \right\}$ is orthonormal in the interval $(0,\infty)$.

Solution: $\int_0^\infty f_1(x) f_2(x) dx = \int_0^\infty e^{-\frac{x}{2}} \cdot e^{-\frac{x}{2}}(-x+1) dx$

$$= \int_0^\infty e^{-x}(-x+1) dx = \left| -e^{-x}(-x+1) - e^{-x}(-1) \right|_0^\infty$$

$$= \left| xe^{-x} \right|_0^\infty$$

$$= 0$$

$$\int_0^\infty f_2(x) f_3(x) dx = \int_0^\infty e^{-\frac{x}{2}}(-x+1) \cdot \frac{1}{2} e^{-\frac{x}{2}}(x^2-4x+2) dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x}(-x+1)(x^2-4x+2) dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x}(-x^3+5x^2-6x+2) dx$$

$$= \frac{1}{2}\Big| -e^{-x}(-x^3+5x^2-6x+2) - e^{-x}(-3x^2+10x-6) + (-e^{-x})(-6x+10) - e^{-x}(-6) \Big|_0^{\infty}$$

$$= 0$$

$$\int_0^{\infty} f_1(x) f_3(x)\mathrm{d}x = \int_0^{\infty} e^{-\frac{x}{2}} \cdot \frac{1}{2} e^{-\frac{x}{2}}(x^2-4x+2)\mathrm{d}x = \frac{1}{2}\int_0^{\infty} e^{-x}(x^2-4x+2)\mathrm{d}x$$

$$= \frac{1}{2}\Big| -e^{-x}(x^2-4x+2) - e^{-x}(2x-4) + (-e^{-x})(2) \Big|_0^{\infty} = 0$$

$$\int_0^{\infty} [f_1(x)]^2 \mathrm{d}x = \int_0^{\infty} \left(e^{-\frac{x}{2}} \right)^2 \mathrm{d}x = \int_0^{\infty} e^{-x}\mathrm{d}x = \Big| -e^{-x} \Big|_0^{\infty} = 1 \neq 0$$

$$\int_0^{\infty} [f_2(x)]^2 \mathrm{d}x = \int_0^{\infty} \left[e^{-\frac{x}{2}}(-x+1) \right]^2 \mathrm{d}x = \int_0^{\infty} e^{-x}(-x+1)^2 \mathrm{d}x$$

$$= \Big| -e^{-x}(-x+1)^2 - e^{-x} \cdot 2(-x+1)(-1) + (-e^{-x})\, 2(-1)(-1) \Big|_0^{\infty}$$

$$= \; 1 - e^{-x}(-x+1)^2 + 2e^{-x}(-x+1) - 2e^{-x} \; \Big|_0^{\infty}$$

$$= 1 \neq 0$$

$$\int_0^{\infty} [f_3(x)]^2 \mathrm{d}x = \int_0^{\infty} \left[\frac{1}{2} e^{-\frac{x}{2}}(x^2-4x+2) \right]^2 \mathrm{d}x = \frac{1}{4}\int_0^{\infty} e^{-x}(x^2-4x+2)^2 \mathrm{d}x$$

$$= \frac{1}{4}\int_0^{\infty} e^{-x}(x^4+16x^2+4-8x^3-16x+4x^2)\mathrm{d}x$$

$$= \frac{1}{4}\int_0^{\infty} e^{-x}(x^4-8x^3+20x^2-16x+4)\mathrm{d}x$$

$$= \frac{1}{4}\left| \begin{array}{l} -e^{-x}(x^4-8x^3+20x^2-16x+4) - e^{-x}(4x^3-24x^2+40x-16) \\ \qquad +(-e^{-x})(12x^2-48x+40) - e^{-x}(24x-48) + e^{-x}(24) \end{array} \right|_0^{\infty}$$

$$= \frac{1}{4}(4) = 1 \neq 0$$

Hence, the set of functions is orthonormal in the interval $(0, \infty)$.

Example 8: If $f(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$ where c_1, c_2 and c_3 are constants and $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ are orthonormal functions in the internal (a, b), show that $\int_a^b [f(x)]^2 \mathrm{d}x = c_1^2 + c_2^2 + c_3^2$.

Solution: Since $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are orthonormal in the interval (a, b),

$$\int_a^b [\phi_1(x)]^2 \mathrm{d}x = \int_a^b [\phi_2(x)]^2 \mathrm{d}x = \int_a^b [\phi_3(x)]^2 \mathrm{d}x = 1$$

and $\int_a^b \phi_1(x)\phi_2(x)\mathrm{d}x = \int_a^b \phi_2(x)\phi_3(x)\mathrm{d}x = \int_a^b \phi_3(x)\phi_1(x)\mathrm{d}x = 0$

$$\int_a^b [f(x)]^2 \mathrm{d}x = \int_a^b [c_1\phi_2(x) + c_2\phi_2(x) + c_3\phi_3(x)]^2 \mathrm{d}x$$

$$= \int_a^b [c_1^2 \{\phi_1(x)\}^2 + c_2^2 \{\phi_2(x)\}^2 + c_3^2 \{\phi_3(x)\}^2 + 2c_1c_2\phi_1(x)\phi_2(x)$$
$$+ 2c_1c_3\phi_1(x)\phi_3(x) + 2c_2c_3\phi_2(x)\phi_3(x)]\mathrm{d}x$$
$$= c_1^2 \int_a^b [\phi_1(x)]^2 \,\mathrm{d}x + c_2^2 \int_a^b [\phi_2(x)]^2 \,\mathrm{d}x + c_3 \int_a^b [\phi_3(x)]^2 \,\mathrm{d}x$$
$$+ 2c_1c_2 \int_a^b \phi_1(x)\phi_2(x)\mathrm{d}x + 2c_1c_3 \int_a^b \phi_1(x)\phi_3(x)\mathrm{d}x + 2c_2c_3 \int_a^b \phi_2(x)\phi_3(x)\mathrm{d}x$$
$$= c_1^2(1) + c_2^2(1) + c_3^2(1) + 2c_1c_2(0) + 2c_1c_3(0) + 2c_2c_3(0)$$
$$= c_1^2 + c_2^2 + c_3^2$$

Exercise 13.1

1. Show that the following functions are orthogonal in the given interval.

(i) $f_1(x) = x^3$, $f_2(x) = x^2 + 1$ in $[-1, 1]$

(ii) $f_1(x) = \sin^2 x$, $f_2(x) = \cos x$ in $[0, 1]$

2. Show that the set of functions $\{\sin(2n-1)x\}$, $n = 0, 1, 2, \ldots$ is orthogonal over the interval $\left[0, \frac{\pi}{2}\right]$.

Hence, construct the corresponding orthonormal set of functions.

3. Show that $\{\sin nx\}$, $n = 1, 2, \ldots$ is orthogonal in the interval $(0, 2\pi)$

4. Show that the set of functions

$$\left\{1, \frac{\sin \pi x}{l}, \frac{\cos \pi x}{l}, \frac{\sin 2\pi x}{l}, \frac{\cos 2\pi x}{l}, \ldots\right\}$$

form an orthogonal set in the interval $(-l, l)$ and construct an orthonormal set.

5. Show that if $\phi_1(x)$, $\phi_2(x)$ form an orthogonal set in the interval $[a, b]$, then the functions $\phi_1(\alpha x + \beta)$, $\phi_2(\alpha x + \beta)$ form an orthogonal set for $\beta > 0$ in the interval $\left[\frac{a-\beta}{\alpha}, \frac{b-\beta}{\alpha}\right]$.

6. Prove that the functions $f_1(x) = b$ and $f_2(x) = x^3$ are orthogonal in the interval $(-a, a)$ where a and b are real constants. Determine constants A and B such that the function $f_3(x) = 1 + Ax + Bx^2$ is orthogonal to both $f_1(x)$ and $f_2(x)$ in the interval $(-a, a)$.

$$\left[\textbf{Ans.: } A = 0, B = -\frac{3}{a^2}\right]$$

7. Determine the constants a, b, c, l, m, n so that $\phi_1(x) = a$, $\phi_2(x) = b + cx$, $\phi_3(x) = l + mx + nx^2$ form an orthonormal set in the interval $[-1, 1]$.

$$\left[\begin{aligned} &\textbf{Ans.: } a = \frac{1}{\sqrt{2}}, \quad b = 0, c = \sqrt{\frac{3}{2}}, \\ &l = \frac{\sqrt{5}}{2\sqrt{2}}, m = 0, n = \frac{-3\sqrt{5}}{2\sqrt{2}} \end{aligned}\right]$$

13.3 FOURIER SERIES

Representation of a function over a certain interval by a linear combination of mutually orthogonal functions is called Fourier series representation.

13.3.1 Convergence of the Fourier Series (Dirichlet's Conditions)

A function $f(x)$ can be represented by a complete set of orthogonal functions within the interval $(c, c + 2l)$. The Fourier series of the function $f(x)$ exists only if the following conditions are satisfied:

(i) $f(x)$ is periodic, i.e., $f(x) = f(x + 2l)$, where $2l$ is the period of function $f(x)$.
(ii) $f(x)$ and its integrals are finite and single valued.
(iii) $f(x)$ has a finite number of discontinuities, i.e., $f(x)$ is piecewise continuous in the interval $(c, c + 2l)$.
(iv) $f(x)$ has a finite number of maxima and minima.

These conditions are known as Dirichlet's conditions.

13.3.2 Trignometric Fourier Series

We know that the set of function $\sin\frac{n\pi x}{l}$ and $\cos\frac{n\pi x}{l}$ are orthogonal in the interval $(c, c + 2l)$ for any value of c where $n = 1, 2, 3, \ldots$.

i.e., $$\int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0, \quad m \neq n$$
$$= l, \quad m = n,$$

$$\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0, \quad m \neq n$$
$$= l, \quad m = n$$

$$\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0 \text{ for all } m, n$$

Hence, any function $f(x)$ can be represented in terms of these orthogonal functions in the interval $(c, c + 2l)$ for any value of c.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This series is known as a trignometric Fourier series or simply a Fourier series. For example, a square function can be constructed by adding orthogonal sine components as shown in Fig. 13.1.

13.3.3 Euler's Formula

Let $f(x)$ be a periodic function with period $2l$ in the interval $(c, c + 2l)$. Then the Fourier series of $f(x)$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \ldots (1)$$

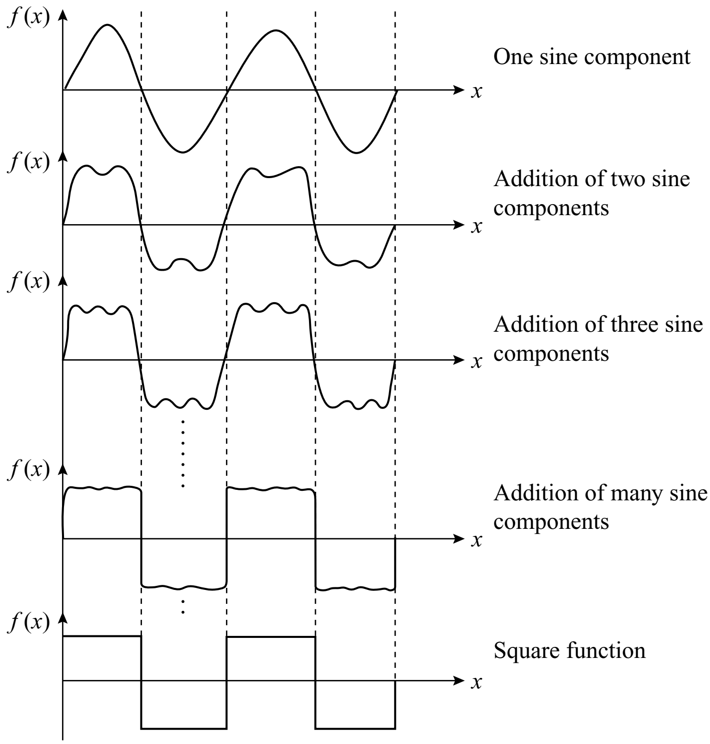


Fig. 13.1

(i) Determination of a_0: Integrating both the sides of Eq. (1) w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} f(x)\mathrm{d}x = a_0\int_c^{c+2l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\right)\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= a_0(c+2l-c)+0+0$$

$$= a_0(2l)$$

Hence, $a_0 = \dfrac{1}{2l}\displaystyle\int_c^{c+2l} f(x)\mathrm{d}x$ … (2)

(ii) Determination of a_n: Multiplying both the sides of Eq. (1) by $\cos\dfrac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = a_0\int_c^{c+2l}\cos\frac{n\pi x}{l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= 0 + la_n + 0$$

Hence, $a_n = \dfrac{1}{l}\displaystyle\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$ ... (3)

(iii) Determination of b_n: Multiplying both the sides of Eq. (1) by $\sin\dfrac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c+2l)$,

$$\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx = a_0\int_c^{c+2l}\sin\frac{n\pi x}{l}\,dx + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\,dx$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\,dx$$

$$= 0 + 0 + lb_n$$

Hence, $b_n = \dfrac{1}{l}\displaystyle\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$ ... (4)

The formulae (2), (3) and (4) are known as Euler's Formulae which give the values of coefficients a_0, a_n and b_n. These coefficients are known as Fourier coefficients.

Cor. 1: When $c = 0$ and $2l = 2\pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

where,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Cor. 2: When $c = -\pi$ and $2l = 2\pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

where,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Cor. 3: When $c = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

where

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Cor. 4: When $c = -l$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\mathrm{d}x$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

13.4 PARSEVAL'S IDENTITY

Let $f(x)$ be a periodic function with period $2l$ and is piecewise continuous in the interval $(c, c + 2l)$. Then

$$\frac{1}{2l}\int_c^{c+2l} [f(x)]^2\,\mathrm{d}x = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2 + b_n{}^2)$$

This is known as Parseval's Identity for the function $f(x)$ in the interval $(c, c + 2l)$.

Proof: We know that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots (1)$$

Multiplying both the sides of Eq. (1) by $f(x)$ and integrating term by term w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} [f(x)]^2\,\mathrm{d}x = a_0\int_c^{c+2l} f(x)\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n f(x)\cos\frac{n\pi x}{l}\right)\mathrm{d}x$$
$$+ \int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n f(x)\sin\frac{n\pi x}{l}\right)\mathrm{d}x$$
$$= a_0(2la_0) + \sum_{n=1}^{\infty} a_n(la_n) + \sum_{n=1}^{\infty} b_n(lb_n) = 2la_0^2 + l\left[\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right]$$

$$\frac{1}{2l}\int_c^{c+2l} [f(x)]^2\,\mathrm{d}x = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Cor. 1: When $c = 0$ and $2l = 2\pi$,

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Cor. 2: When $c = -\pi$ and $2l = 2\pi$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Cor. 3: When $c = 0$,

$$\frac{1}{2l}\int_0^{2l}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Cor. 4: When $c = -l$,

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Fourier Series Expansion with Period 2π.

Example 1: Find the Fourier series of $f(x) = x$ in the interval $(0, 2\pi)$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\left|\frac{x^2}{2}\right|_0^{2\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left(\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left|x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left[-2\pi\left(\frac{\cos 2n\pi}{n}\right)\right] = -\frac{2}{n}$$

Hence, $\quad f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$

Example 2: Find the Fourier series of $f(x) = x^2$ in the interval $(0, 2\pi)$ and hence, deduce that $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x^2 \mathrm{d}x = \frac{1}{2\pi}\left|\frac{x^3}{3}\right|_0^{2\pi} = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right) = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left(-\frac{4\pi^2}{n}\right) = -\frac{4\pi}{n}$$

Hence, $f(x) = \dfrac{4\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$ (1) ... (1)

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = \pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\pi^2 = \frac{4\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2}\cdots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Example 3: Find the Fourier series of $f(x) = \frac{1}{2}(\pi - x)$ in the interval $(0, 2\pi)$.

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\mathrm{d}x = \frac{1}{4\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{2\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} = \frac{1}{2\pi}\left(\frac{\pi}{n} + \frac{\pi}{n}\right) = \frac{1}{n}$$

Hence, $$f(x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin nx \quad \ldots (1)$$

Putting $x = \dfrac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Example 4: Find the Fourier series of $f(x) = \dfrac{3x^2 - 6x\pi + 2\pi^2}{12}$ in the interval $(0, 2\pi)$. Hence, deduce that $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty}a_n\cos nx + \sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\frac{3x^2 - 6x\pi + 2\pi^2}{12}\,dx$$

$$= \frac{1}{24\pi}\left|3\left(\frac{x^3}{3}\right) - 6\pi\left(\frac{x^2}{2}\right) + 2\pi^2 x\right|_0^{2\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2 - 6x\pi + 2\pi^2}{12}\right)\cos nx\,dx$$

$$= \frac{1}{12\pi}\left|(3x^2 - 6x\pi + 2\pi^2)\left(\frac{\sin nx}{n}\right) - (6x - 6\pi)\left(-\frac{\cos nx}{n^2}\right) + 6\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{12\pi}\left(\frac{6\pi}{n^2} + \frac{6\pi}{n^2}\right) = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2 - 6x\pi + 2\pi^2}{12}\right)\sin nx\,dx$$

$$= \frac{1}{12\pi}\left|(3x^2 - 6x\pi + 2\pi^2)\left(-\frac{\cos nx}{n}\right) - (6x - 6\pi)\left(-\frac{\sin nx}{n^2}\right) + 6\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi} = 0$$

Hence, $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ ... (1)

Putting $x = 0$ in Eq. (1),

$$f(a) = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

Example 5: Find the Fourier series of $f(x) = e^{-x}$ in the interval $(0, 2\pi)$. Hence, deduce that $\frac{\pi}{2}\frac{1}{\sinh \pi} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}\mathrm{d}x = \frac{1}{2\pi}\left|-e^{-x}\right|_0^{2\pi} = \frac{1-e^{-2\pi}}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right|_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}(1-e^{-2\pi})$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\sin nx - n\cos nx)\right|_0^{2\pi} = \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-n) - \frac{1}{n^2+1}(-n)\right]$$

$$= \frac{n}{\pi(n^2+1)}(1-e^{-2\pi})$$

Hence, $f(x) = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos nx + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx$ ... (1)

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = e^{-\pi} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

$$= \frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}(1-e^{-2\pi})}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}-e^{-\pi}}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Hence, $$\frac{\pi}{2}\frac{1}{\sinh\pi}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Example 6: Find the Fourier series of $f(x)=x\sin x$ in the interval $(0,2\pi)$ and hence, deduce that $\sum_{n=2}^{\infty}\frac{1}{n^2-1}=\frac{3}{4}$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x=\frac{1}{2\pi}\int_0^{2\pi}x\sin x\,\mathrm{d}x=\frac{1}{2\pi}\left|x(-\cos x)-(-\sin x)\right|_0^{2\pi}=-1$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,\mathrm{d}x$$

$$=\frac{1}{\pi}\int_0^{2\pi}x\sin x\cos nx\,\mathrm{d}x=\frac{1}{2\pi}\int_0^{2\pi}x[\sin(n+1)x-\sin(n-1)x]\mathrm{d}x$$

$$=\frac{1}{2\pi}\left|x\left\{-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\}-(1)\left\{-\frac{\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right\}\right|_0^{2\pi},\quad n\neq 1$$

$$=\frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos(n+1)2\pi}{n+1}+\frac{\cos(n-1)2\pi}{n-1}\right\}\right],\quad n\neq 1$$

$$=-\frac{1}{n+1}+\frac{1}{n-1},\quad n\neq 1$$

$$=\frac{2}{n^2-1},\quad n\neq 1$$

For $n=1$,

$$a_1=\frac{1}{\pi}\int_0^{2\pi}x\sin x\cos x\,\mathrm{d}x=\frac{1}{2\pi}\int_0^{2\pi}x\sin 2x\,\mathrm{d}x$$

$$=\frac{1}{2\pi}\left|x\left(-\frac{\cos 2x}{2}\right)-(1)\left(-\frac{\sin 2x}{4}\right)\right|_0^{2\pi}=-\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]\mathrm{d}x$$

$$= \frac{1}{2\pi}\left| x\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right] - (1)\left[-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right]\right|_0^{2\pi}, \qquad n \neq 1$$

$$= \frac{1}{2\pi}\left[\frac{\cos(n-1)2\pi}{(n-1)^2} - \frac{\cos(n+1)2\pi}{n+1} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right], \qquad n \neq 1$$

$$= 0, \qquad n \neq 1$$

For $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\mathrm{d}x$$

$$= \frac{1}{2\pi}\left| x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right|_0^{2\pi} = \frac{1}{2\pi}(2\pi^2) = \pi$$

Hence, $f(x) = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi\sin x$ ... (1)

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = -1 - \frac{1}{2} + \sum_{n=2}^{\infty}\frac{2}{n^2-1}$$

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \frac{3}{4}$$

Example 7: Find the Fourier series of $f(x) = \sqrt{1-\cos x}$ in the interval $(0, 2\pi)$. Hence, deduce that $\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$f(x) = \sqrt{1-\cos x} = \sqrt{2}\sin\frac{x}{2}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\,\mathrm{d}x = \frac{\sqrt{2}}{2\pi}\left|-2\cos\frac{x}{2}\right|_0^{2\pi} = \frac{2\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cos nx\,dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{2n+1}{2}\right)x - \sin\left(\frac{2n-1}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left|-\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right)x + \frac{2}{2n-1}\cos\left(\frac{2n-1}{2}\right)x\right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{2n+1}\cos(2n\pi+\pi) + \frac{2}{2n+1} + \frac{2}{2n-1}\cos(2n\pi-\pi) - \frac{2}{2n-1}\right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{4}{2n+1} - \frac{4}{2n-1}\right]$$

$$= -\frac{4\sqrt{2}}{\pi}\frac{1}{4n^2-1}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\sin nx\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{2n-1}{2}\right)x - \cos\left(\frac{2n+1}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left|\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right|_0^{2\pi} = 0$$

Hence, $$f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

Example 8: Find the Fourier series of

$$\boldsymbol{f(x) = -1 \qquad 0 < x < \pi}$$
$$\boldsymbol{= 2 \qquad \pi < x < 2\pi.}$$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\left[\int_0^{\pi}(-1)dx + \int_{\pi}^{2\pi}2dx\right]$$

$$= \frac{1}{2\pi}\left[\left|-x\right|_0^{\pi} + \left|2x\right|_{\pi}^{2\pi}\right] = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}(-1)\cos nx\,dx + \int_{\pi}^{2\pi} 2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\left|\frac{\sin nx}{n}\right|_0^{\pi} + 2\left|\frac{\sin nx}{n}\right|_{\pi}^{2\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}(-1)\sin nx\,dx + \int_{\pi}^{2\pi} 2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\cos nx}{n}\right|_0^{\pi} + \left|-\frac{2\cos nx}{n}\right|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n} - \frac{2\cos 2n\pi}{n} + \frac{2\cos n\pi}{n}\right]$$

$$= \frac{3}{n\pi}[(-1)^n - 1]$$

Hence, $f(x) = \dfrac{1}{2} + \dfrac{3}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx$

Example 9: Find the Fourier series of $f(x) = x + x^2$ in the interval $(-\pi, \pi)$ and hence, deduce that

(i) $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

(ii) $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.$$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x + x^2)dx$$

$$= \frac{1}{2\pi}\left|\frac{x^2}{2} + \frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left|(x + x^2)\left(\frac{\sin nx}{n}\right) - (1 + 2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 + 2\pi)\frac{\cos n\pi}{n^2} - (1 - 2\pi)\frac{\cos(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\,\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(-\frac{\cos nx}{n}\right)-(1+2x)\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right]$$

$$= \frac{-2(-1)^n}{n}$$

Hence, $f(x) = \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$ … (1)

(i) Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

(ii) Putting $x = \pi$ in Eq. (1),

$$f(\pi) = \pi + \pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) \quad \ldots (2)$$

Putting $x = -\pi$ in Eq. (1),

$$f(-\pi) = -\pi + \pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) \quad \ldots (3)$$

Adding Eqs. (2) and (3),

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

Example 10: Find the Fourier series of $f(x) = e^{ax}$ in the interval $(-\pi, \pi)$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|\frac{e^{ax}}{a}\right|_{-\pi}^{\pi} = \frac{1}{2\pi a}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$= \frac{\sinh a\pi}{\pi a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right|_{-\pi}^{\pi} = -\frac{a\cos n\pi}{\pi(a^2+n^2)}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\,dx$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right|_{-\pi}^{\pi} = -\frac{n\cos n\pi}{\pi(a^2+n^2)}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$= \frac{-2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$$

Hence, $$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx - \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

$$= \frac{\sinh a\pi}{a\pi} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)$$

Example 11: Find the Fourier series of $f(x) = -\pi \quad -\pi < x < 0$

$\qquad = x \quad 0 < x < \pi$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left|-\pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_0^{\pi}\right] = -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_0^{\pi}\right] = \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,\mathrm{d}x + \int_{0}^{\pi} x\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\pi\left|\frac{\cos nx}{n}\right|_{-\pi}^{0} + \left|x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right|_{0}^{\pi}\right] = \frac{1}{n}[1-2\cos n\pi]$$

$$= \frac{1}{n}[1-2(-1)^n]$$

Hence, $$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos nx + \sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin nx \qquad \text{... (1)}$$

At $x = 0$, $f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right] = \frac{-\pi+0}{2} = -\frac{\pi}{2}$

Putting $x = 0$ in Eq. (1),

$$f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 12: Find the Fourier series of $f(x) = -x-\pi \quad -\pi < x < 0$
$\qquad = x+\pi \quad 0 < x < \pi$ **.**

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-x-\pi)\mathrm{d}x + \int_{0}^{\pi}(x+\pi)\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left[\left|-\frac{x^2}{2} - \pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2} + \pi x\right|_{0}^{\pi}\right]$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\cos nx\,\mathrm{d}x + \int_{0}^{\pi}(x+\pi)\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|(-x-\pi)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\pi}^{0} + \left|(x+\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\sin nx\,\mathrm{d}x + \int_{0}^{\pi}(x+\pi)\sin nx\,\mathrm{d}x\right]$$
$$= \frac{1}{\pi}\left[\left|(-x-\pi)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\pi}^{0} + \left|(x+\pi)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right|_{0}^{\pi}\right]$$
$$= \frac{2}{n}[1-(-1)^n]$$

Hence, $$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos nx + 2\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin nx$$

Example 13: Find the Fourier series of $f(x) = 0 \qquad -\pi < x < 0$

$\qquad\qquad = \sin x \qquad 0 < x < \pi$

Hence, deduce that $\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,\mathrm{d}x + \int_{0}^{\pi}\sin x\,\mathrm{d}x\right] = \frac{1}{2\pi}\left|-\cos x\right|_{0}^{\pi} = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,\mathrm{d}x + \int_{0}^{\pi}\sin x\cos nx\,\mathrm{d}x\right]$$
$$= \frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x = \frac{1}{2\pi}\left|-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right|_{0}^{\pi}, \quad n \neq 1$$
$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right], \qquad n \neq 1$$
$$= -\frac{1}{\pi(n^2-1)}[1+(-1)^n], \qquad n \neq 1$$

For $n = 1$,
$$a_1 = \frac{1}{\pi}\int_{0}^{\pi}\sin x\cos x\,\mathrm{d}x = \frac{1}{2\pi}\int_{0}^{\pi}\sin 2x\,\mathrm{d}x$$
$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin nx\,dx + \int_{0}^{\pi}\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{2\pi}\left|\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right|_0^{\pi}, \qquad n \neq 1$$

$$= 0, \qquad n \neq 1$$

For $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{2\pi}\left|x - \frac{\sin 2x}{2}\right|_0^{\pi}$$

$$= \frac{1}{2}$$

Hence,
$$f(x) = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx + \frac{1}{2}\sin x \qquad \text{... (1)}$$

At $x = 0$, $\frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right] = 0$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1} = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \dots\right)$$

$$\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$$

Example 14: Find the Fourier series of $f(x) = x \qquad \frac{-\pi}{2} < x < \frac{\pi}{2}$

$$= \pi - x \qquad \frac{\pi}{2} < x < \frac{3\pi}{2}.$$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)dx = \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\pi - x)dx\right]$$

$$= \frac{1}{2\pi}\left[\left|\frac{x^2}{2}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left| x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\pi\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{3}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{3n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\pi + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi n^2}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]$$

Hence, $f(x) = \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]\sin nx$

Fourier Series Expansion with Period 2l

Example 15: Find the Fourier series of $f(x) = x^2$ in the interval (0, 4). Hence, deduce that $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$.

Solution: The Fourier series of $f(x)$ with period $2l = 4$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x = \frac{1}{4}\int_0^4 x^2\mathrm{d}x = \frac{1}{4}\left|\frac{x^3}{3}\right|_0^4 = \frac{16}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \frac{1}{2}\int_0^4 x^2\cos\frac{n\pi x}{2}\mathrm{d}x$$

$$= \frac{1}{2}\left| x^2\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (2x)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right) + 2\left(\frac{-8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right)\right|_0^4$$

$$= \frac{1}{2}\left[8\left(\frac{4}{n^2\pi^2}\right)\right] = \frac{16}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = \frac{1}{2}\int_0^4 x^2\sin\frac{n\pi x}{2}\mathrm{d}x$$

$$= \frac{1}{2}\left| x^2\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - 2x\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right) + 2\left(\frac{8}{n^3\pi^3}\cos\frac{n\pi x}{2}\right)\right|_0^4$$

$$= \frac{1}{2}\left(-\frac{32}{n\pi}\right) = -\frac{16}{n\pi}$$

Hence,
$$f(x) = \frac{16}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2} \quad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{16}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$-\frac{1}{3} = \frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \quad \ldots (2)$$

Putting $x = 4$ in Eq. (1),

$$f(4) = 16 = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{2}{3} = \frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \quad \ldots (3)$$

Adding Eqs. (2) and (3),

$$\frac{1}{3} = \frac{2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 16: Find the Fourier series of $f(x) = 4 - x^2$ in the interval (0, 2). Hence, deduce that $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$.

Solution: The Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\int_0^2 (4 - x^2)\,dx = \frac{1}{2}\left|4x - \frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^2 (4 - x^2)\cos n\pi x\,dx$$

$$= \left|(4 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^2 = -\frac{4}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^2 (4 - x^2)\sin n\pi x\,dx$$

$$= \left|(4 - x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^2 = \frac{4}{n\pi}$$

Hence,
$$f(x) = \frac{8}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 4 = \frac{8}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{1}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (2)$$

Putting $x = 2$ in Eq. (1),

$$f(2) = 0 = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$-\frac{2}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (3)$$

Adding Eqs. (2) and (3),

$$-\frac{1}{3} = -\frac{2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 17: Find the Fourier series of $f(x) = 2x - x^2$ in the interval (0, 3). Hence, deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$.

Solution: The Fourier series of $f(x)$ with period $2l = 3$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x = \frac{1}{3}\int_0^3 (2x - x^2)\mathrm{d}x = \frac{1}{3}\left| x^2 - \frac{x^3}{3} \right|_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{3}\left| (2x - x^2)\left(\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\right) - (2 - 2x)\left(-\frac{9}{4n^2\pi^2}\cos\frac{2n\pi x}{3}\right) + (-2)\left(-\frac{27}{8n^3\pi^3}\sin\frac{2n\pi x}{3}\right) \right|_0^3$$

$$= \frac{2}{3}\left[\frac{9}{4n^2\pi^2}(-4 - 2)\right]$$

$$= -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\mathrm{d}x$$

$$= \frac{2}{3}\left| (2x - x^2)\left(-\frac{3}{2n\pi}\cos\frac{2n\pi x}{3}\right) - (2 - 2x)\left(-\frac{9}{4n^2\pi^2}\sin\frac{2n\pi x}{3}\right) + (-2)\left(\frac{27}{8n^3\pi^3}\cos\frac{2n\pi x}{3}\right) \right|_0^3$$

$$= \frac{2}{3}\left(\frac{9}{2n\pi}\right) = \frac{3}{n\pi}$$

Hence, $$f(x) = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3} \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = -\frac{9}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$0 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \qquad \ldots (2)$$

Putting $x = 3$ in Eq. (1),

$$f(3) = -3 = -\frac{9}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)$$

$$\frac{\pi^2}{3} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots \qquad \ldots (3)$$

Adding Eqs. (2) and (3),

$$\frac{\pi^2}{3} = 2\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$$

Example 18: Find the Fourier series of $f(x) = \pi x \quad 0 < x < 1$
$= 0 \quad 1 < x < 2$.

Solution: The Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,\mathrm{d}x = \frac{1}{2}\left(\int_0^1 \pi x\,\mathrm{d}x + \int_1^2 0\cdot\mathrm{d}x\right) = \frac{1}{2}\left|\frac{\pi x^2}{2}\right|_0^1 = \frac{\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x\cos n\pi x\,\mathrm{d}x + \int_1^2 0\cdot\cos n\pi x\,\mathrm{d}x = \left|\pi x\left(\frac{\sin n\pi x}{n\pi}\right) - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x = \int_0^1 \pi x\sin n\pi x\,\mathrm{d}x + \int_1^2 0\cdot\sin n\pi x\,\mathrm{d}x$$

$$= \left|\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 = -\frac{\pi\cos n\pi}{n\pi}$$

$$= -\frac{(-1)^n}{n}$$

Hence, $$f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$$

Example 19: Find the Fourier series of $f(x) = \pi x \quad 0 \le x < 1$
$= 0 \quad x = 1$
$= \pi(x-2) \quad 1 < x \le 2.$

Hence, deduce that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.

Solution: The Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(x-2)\,dx\right] = \frac{1}{2}\left[\pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|\frac{x^2}{2} - 2x\right|_1^2\right] = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(x-2)\cos n\pi x\,dx$$

$$= \pi\left[\left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{1}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2}\right]$$

$$= 0$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(x-2)\sin n\pi x\,dx$$

$$= \pi\left[\left|x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi}\right] = -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

Hence,
$$f(x) = 2\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}\right)\sin n\pi x \qquad \dots (1)$$

Putting $x = \frac{1}{2}$ in Eq. (1),

$$f\left(\frac{1}{2}\right) = \frac{\pi}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}$$

$$\frac{\pi}{2} = 2\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Example 20: Find the Fourier series of $f(x) = 4 - x \quad 3 < x < 4$
$\qquad = x - 4 \quad 4 < x < 5.$

Solution: The Fourier series of $f(x)$ with period $2l = 5 - 3 = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\mathrm{d}x = \frac{1}{2}\int_3^5 f(x)\mathrm{d}x = \frac{1}{2}\left[\int_3^4 (4-x)\mathrm{d}x + \int_4^5 (x-4)\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|4x - \frac{x^2}{2}\right|_3^4 + \left|\frac{x^2}{2} - 4x\right|_4^5\right] = \frac{1}{2}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \int_3^4 (4-x)\cos n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\cos n\pi x\,\mathrm{d}x$$

$$= \left[\left|(4-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_3^4 + \left|(x-4)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_4^5\right]$$

$$= -\frac{1}{n^2\pi^2}(\cos 4n\pi - \cos 3n\pi) + \frac{1}{n^2\pi^2}(\cos 5n\pi - \cos 4n\pi)$$

$$= -\frac{1}{n^2\pi^2}[(-1)^{4n} - (-1)^{3n} - (-1)^{5n} + (-1)^{4n}]$$

$$= \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = \int_3^4 (4-x)\sin n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\sin n\pi x\,\mathrm{d}x$$

$$= \left[\left|(4-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_3^4\right.$$

$$\left. + \left|(x-4)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_4^5\right]$$

$$= -\frac{1}{n\pi}\cos 3n\pi - \frac{1}{n\pi}\cos 5n\pi$$

$$= 0$$

Hence, $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x$

Example 21: Find the Fourier series of $f(x) = 0 \quad -5 < x < 0$
$\qquad = 3 \quad 0 < x < 5.$

Solution: The Fourier series of $f(x)$ with period $2l = 10$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{5} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{5}$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\mathrm{d}x = \frac{1}{10}\left(\int_{-5}^{0} 0\,\mathrm{d}x + \int_{0}^{5} 3\mathrm{d}x\right) = \frac{1}{10}\left|3x\right|_0^5 = \frac{3}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{1}{5}\left(\int_{-5}^{0} 0\cdot\cos\frac{n\pi x}{5}\mathrm{d}x + \int_{0}^{5} 3\cos\frac{n\pi x}{5}\mathrm{d}x\right) = \frac{3}{5}\left|\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right|_0^5$$

$$= 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{1}{5}\left(\int_{-5}^{0} 0\cdot\sin\frac{n\pi x}{5}\mathrm{d}x + \int_{0}^{5} 3\sin\frac{n\pi x}{5}\mathrm{d}x\right) = \frac{3}{5}\left|\frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right)\right|_0^5$$

$$= \frac{3}{n\pi}[1-(-1)^n]$$

Hence, $f(x) = \dfrac{3}{2} + \dfrac{3}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{1-(-1)^n}{n}\right]\sin\dfrac{n\pi x}{5}$

Example 22: Find the Fourier series of $f(x) = x \quad -1 < x < 0$
$\qquad = x + 2 \quad 0 < x < 1.$

Solution: The Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_{-1}^{0} x\mathrm{d}x + \int_{0}^{1}(x+2)\mathrm{d}x\right] = \frac{1}{2}\left[\left|\frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2} + 2x\right|_0^1\right] = 1$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \left[\int_{-1}^{0} x\cos n\pi x\,\mathrm{d}x + \int_{0}^{1}(x+2)\cos n\pi x\,\mathrm{d}x\right]$$

$$= \left[\left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1\right] = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x = \left[\int_{-1}^{0} x\sin n\pi x\,\mathrm{d}x + \int_{0}^{1}(x+2)\sin n\pi x\,\mathrm{d}x\right]$$

$$= \left[\left| x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{0}^{1}\right]$$

$$= \left[\frac{-(-1)^n}{n\pi} - \frac{3(-1)^n}{n\pi} + \frac{2}{n\pi}\right]$$

$$= \frac{2}{n\pi}[1-2(-1)^n]$$

Hence, $f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin n\pi x$

Exercise 13.2

Find the Fourier series of the following functions:

1. $f(x) = \left(\frac{\pi - x}{2}\right)^2 \quad 0 \le x \le 2\pi$

Hence, deduce that

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

$\left[\textbf{Ans.: } \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx\right]$

2. $f(x) = e^x \quad 0 < x < 2\pi$

$\left[\textbf{Ans.: } \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx - n\sin nx}{n^2+1}\right]\right]$

3. $f(x) = 1 \quad 0 < x < \pi$
$\quad\; = 2 \quad \pi < x < 2\pi$

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

$\left[\textbf{Ans.: } \frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx\right]$

4. $f(x) = x \quad 0 < x < \pi$
$\quad\; = 2\pi - x \quad \pi < x < 2\pi$

$\left[\textbf{Ans.: } \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx\right]$

5. $f(x) = x - x^2 \quad -\pi < x < \pi$
Hence, deduce that

$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$

$\left[\textbf{Ans.: } -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx\right]$

6. $f(x) = 1 \quad -\pi < x \le 0$
$\quad\; = -2 \quad 0 < x \le \pi$

$\left[\textbf{Ans.: } -\frac{1}{2} - \frac{6}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}\right]$

7. $f(x) = -x \quad -\pi < x \le 0$
$\quad\; = 0 \quad 0 < x \le \pi$

$$\left[\textbf{Ans.:}\ \frac{\pi}{4}-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx\right]$$

8. $f(x)=\frac{1}{2}\qquad -\pi<x<0$

$\quad =\frac{x}{\pi}\qquad 0<x<\pi$

$$\left[\textbf{Ans.:}\ \frac{1}{2}-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx\right]$$

9. $f(x)=x-\pi\qquad -\pi<x<0$

$\quad =\pi-x\qquad 0<x<\pi$

Hence, deduce that

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

$$\left[\textbf{Ans.:}-\frac{\pi}{2}+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x+4\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x\right]$$

10. $f(x)=\cos x\qquad -\pi<x<0$

$\quad =\sin x\qquad 0<x<\pi$

$$\left[\textbf{Ans.:}\ \frac{1}{\pi}+\frac{1}{2}(\cos x+\sin x)-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx\right]$$

11. $f(x)=2-\frac{x^2}{2}\qquad 0\le x\le 2$

$$\left[\textbf{Ans.:}\ \frac{4}{3}-\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}\cos n\pi x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

12. $f(x)=\frac{1}{2}(\pi-x)\qquad 0<x<2$

$$\left[\textbf{Ans.:}\ (\pi-1)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

13. $f(x)=1\qquad 0<x<1$

$\quad =2\qquad 1<x<2$

$$\left[\textbf{Ans.:}\ 3-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\pi x\right]$$

14. $f(x)=\pi x\qquad 0\le x\le 1$

$\quad =\pi(2-x)\qquad 1\le x\le 2$

$$\left[\textbf{Ans.:}\ \frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x\right]$$

15. $f(x)=x\qquad 0<x<1$

$\quad =0\qquad 1<x<2$

$$\left[\textbf{Ans.:}\ \frac{1}{4}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos n\pi x+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin n\pi x\right]$$

16. $f(x)=2\qquad -2<x<0$

$\quad =x\qquad 0<x<2$

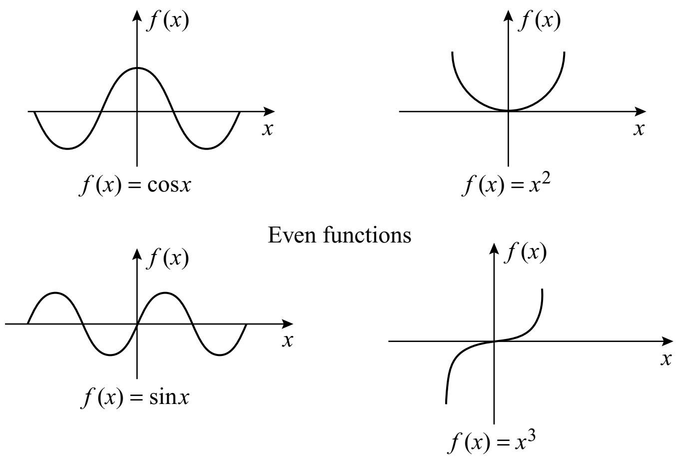
$$\left[\textbf{Ans.:}\ \frac{3}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\frac{\cos n\pi x}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}\right]$$

13.5 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be even if $f(-x)=f(x)$ and odd if $f(-x)=-f(x)$ for all x.

Properties of Even and Odd Functions

(i) The product of two even functions is even.
(ii) The product of two odd functions is even.
(iii) The product of an even function and an odd function is odd.
(iv) The sum or difference of two even functions is even.
(v) The sum or difference of two odd functions is odd.
(vi) If $f(x)$ is even, $\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$
(viii) If $f(x)$ is odd, $\int_{-l}^{l} f(x)\,dx = 0$



Even functions

Odd functions

Fig. 13.2

We know that the Fourier series of a function $f(x)$ in the interval $(-l, l)$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where,
$$a_0 = \frac{1}{2l}\int_{l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Case I: When $f(x)$ is an even function, $\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$

$$a_0 = \frac{1}{l}\int_{0}^{l} f(x)\,dx$$

Since the product of two even functions is even,

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Since the product of an even function and an odd function is odd, $b_n = 0$

Cor: Fourier series of an even function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where,
$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Case II: When $f(x)$ is an odd function,

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Cor: Fourier series of an odd function $f(x)$ in the interval $(-\pi, \pi)$ is given by,

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Thus, the Fourier series of an even function consists entirely of cosine terms while the Fourier series of an odd function consists entirely of sine terms.

Example 1: Find the Fourier series of $f(x) = x^2$ in the interval $(-\pi, \pi)$. Hence, deduce that $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots$

Solution: $f(x) = x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} = \frac{4}{n^2}\cos n\pi$$

$$= \frac{4}{n^2}(-1)^n$$

Hence, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \quad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Example 2: Find the Fourier series of $f(x) = x^3$ in the interval $(-\pi, \pi)$.

Solution: $f(x) = x^3$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx \, dx$$

$$= \frac{2}{\pi}\left| x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\pi^3 \frac{\cos n\pi}{n} + 6\pi \frac{\cos n\pi}{n^3}\right) = 2(-1)^n\left(\frac{-\pi^2}{n} + \frac{6}{n^3}\right)$$

Hence, $$f(x) = 2\sum_{n=1}^{\infty} (-1)^n \left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right) \sin nx$$

Example 3: Find the Fourier series of

$$f(x) = 1 + \frac{2x}{\pi} \qquad -\pi \le x \le 0$$
$$= 1 - \frac{2x}{\pi} \qquad 0 \le x \le \pi$$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dots$.

Solution:

$$f(-x) = 1 - \frac{2x}{\pi} \qquad -\pi \le -x \le 0 \text{ or } 0 \le x \le \pi$$

$$= 1 + \frac{2x}{\pi} \qquad 0 \le -x \le \pi \text{ or } -\pi \le x \le 0$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x-\frac{x^2}{\pi}\right|_0^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

Hence, $$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

$$= \frac{8}{\pi^2}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right) \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 1 = \frac{8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

Example 4: Find the Fourier series of $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in the interval $[-\pi, \pi]$ and deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$.

Solution: $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12}-\frac{x^2}{4}\right)dx = \frac{1}{\pi}\left|\frac{\pi^2 x}{12}-\frac{x^3}{12}\right|_0^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12}-\frac{x^2}{4}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left|\left(\frac{\pi^2}{12}-\frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right)-\left(-\frac{x}{2}\right)\left(-\frac{\cos nx}{n^2}\right)+\left(-\frac{1}{2}\right)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{2n^2}\cos n\pi\right)$$

$$= \frac{-(-1)^n}{n^2}$$

Hence, $f(x) = \sum_{n=1}^{\infty}\frac{-(-1)^n}{n^2}\cos nx$

$$= \frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = \frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots$$

Example 5: Find the Fourier series of $f(x) = |x|$ in the interval $[-\pi, \pi]$.

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$.

Solution: $f(x) = |x| \quad -\pi < x < \pi$

i.e. $\quad f(x) = -x \quad -\pi < x \le 0$

$\quad\quad = x \quad 0 \le x < \pi$

$f(x) = |x|$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left| x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_0^{\pi} = \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right)$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

Hence,
$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right) \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 6: Find the Fourier series of $f(x) = \sin ax$ in the interval $(-\pi, \pi)$.

Solution: $f(-x) = \sin a(-x) = -\sin ax$

$f(-x) = -f(x)$

$f(x) = \sin ax$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\sin ax \sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi}[\cos(n-a)x - \cos(n+a)x]\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right|_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left(\frac{\sin n\pi\cos a\pi - \sin a\pi\cos n\pi}{n-a} - \frac{\sin n\pi\cos a\pi + \sin a\pi\cos n\pi}{n+a}\right)$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^n\sin a\pi}{n-a} - \frac{(-1)^n\sin a\pi}{n+a}\right] = \frac{-(-1)^n\sin a\pi}{\pi}\left(\frac{1}{n-a} + \frac{1}{n+a}\right)$$

$$= \frac{2n(-1)^n\sin a\pi}{\pi(a^2 - n^2)}$$

Hence, $$f(x) = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{a^2 - n^2}\sin nx$$

Example 7: Find the Fourier series of $f(x) = x \sin x$ in the interval $(-\pi, \pi)$.

Hence, deduce that $\frac{\pi - 1}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots$.

Solution: $f(-x) = -x \sin(-x)$
$= x \sin x$
$= f(x)$
$f(x) = x \sin x$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x \sin x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left| x(-\cos x) - (-\sin x) \right|_0^{\pi}$$

$$= 1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left| x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - \left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right|_0^{\pi}, n \neq 1$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{n+1} + \pi\frac{\cos(n-1)\pi}{n-1}\right], \qquad n \neq 1$$

$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1},\ n \neq 1 \qquad [\because\ (-1)^{n+1} = (-1)^{n-1} = -(-1)^n]$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x \sin 2x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left| -x\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right|_0^{\pi}$$

$$= -\frac{1}{2}$$

Hence,
$$f(x) = 1 - \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$= \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx \qquad \ldots (1)$$

Putting $x=\frac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\sin\frac{\pi}{2}=\frac{1}{2}+2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

$$\frac{\pi}{2}=\frac{1}{2}-\frac{2}{3}\cos\pi-\frac{2}{15}\cos 2\pi-\frac{2}{35}\cos 3\pi-\ldots$$

$$\frac{\pi-1}{4}=\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\ldots$$

Example 8: Find the Fourier series of $f(x)=|\cos x|$ in the interval $(-\pi,\pi)$.

Solution: $f(x)=|\cos x|$ is an even function.

Hence, $b_n=0$

$$f(x)=\cos x \qquad 0<x<\frac{\pi}{2}$$

$$=-\cos x \qquad \frac{\pi}{2}<x<\pi$$

The Fourier series of an even function with period 2π is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}f(x)\,\mathrm{d}x=\frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}\cos x\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}(-\cos x)\mathrm{d}x\right]$$

$$=\frac{1}{\pi}\left[\left|\sin x\right|_0^{\frac{\pi}{2}}-\left|\sin x\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$=\frac{2}{\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,\mathrm{d}x=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\cos x\cos nx\,\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}(-\cos x)\cos nx\,\mathrm{d}x\right]$$

$$=\frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}\{\cos(n+1)x+\cos(n-1)x\}\mathrm{d}x-\int_{-\frac{\pi}{2}}^{\pi}\{\cos(n+1)x+\cos(n-1)x\}\mathrm{d}x\right]$$

$$=\frac{1}{\pi}\left[\left|\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right|_0^{\frac{\pi}{2}}-\left|\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right|_{\frac{\pi}{2}}^{\pi}\right],\quad n\neq 1$$

$$=\frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1}-\frac{\cos\frac{n\pi}{2}}{n-1}\right],\quad n\neq 1 \qquad \left[\because\ \sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right)=\cos\frac{n\pi}{2},\ \sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right)=-\cos\frac{n\pi}{2}\right]$$

$$=-\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2},\quad n\neq 1$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos^2 x\,dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos^2 x)dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\frac{\pi}{2}}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left|x + \frac{\sin 2x}{2}\right|_0^{\frac{\pi}{2}} - \left|x + \frac{\sin 2x}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= 0$$

Hence, $$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left(-\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x - \frac{1}{35}\cos 6x + \ldots\right)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x - \ldots\right)$$

Example 9: Find the Fourier series of $f(x) = \cos x \quad -\pi < x < 0$
$= -\cos x \quad 0 < x < \pi$.

Solution:
$$f(-x) = \cos(-x) \qquad -\pi < -x < 0$$
$$= -\cos(-x) \qquad 0 < -x < \pi$$
$$f(-x) = \cos x \qquad 0 < x < \pi$$
$$= -\cos x \qquad -\pi < x < 0$$
$$f(-x) = -f(x)$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin nx\,dx$$

$$= -\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x + \sin(n-1)x]dx$$

$$= -\frac{1}{\pi}\left|-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right|_0^{\pi}, \quad n \neq 1$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right], \quad n \neq 1$$

$$= -\frac{1}{\pi}\left(\frac{1+\cos n\pi}{n+1} + \frac{1+\cos n\pi}{n-1}\right), \quad n \neq 1 \quad [\because \cos(n\pi+\pi) = \cos(n\pi-\pi) = -\cos n\pi]$$

$$= -\frac{2n}{\pi(n^2-1)}(1+\cos n\pi), \qquad n \neq 1$$

$$= -\frac{2n}{\pi(n^2-1)}[1+(-1)^n], \qquad n \neq 1$$

For $n = 1$,

$$b_1 = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin x\,\mathrm{d}x = -\frac{1}{\pi}\int_0^{\pi}\sin 2x\,\mathrm{d}x = -\frac{1}{\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi}$$

$$= 0$$

Hence, $f(x) = -\dfrac{2}{\pi}\displaystyle\sum_{n=2}^{\infty}\frac{n}{n^2-1}[1+(-1)^n]\sin nx$

Example 10: Find the Fourier series of $f(x) = e^{-|x|}$ in the interval $(-\pi, \pi)$.

Solution:

$$f(x) = e^{-|x|}$$

$$f(-x) = e^{-|-x|}$$

$$= e^{-|x|} = f(x)$$

$f(x) = e^{-|x|}$ is an even function

Hence, $b_n = 0$

$$f(x) = e^{x} \qquad -\pi < x < 0$$

$$= e^{-x} \qquad 0 < x < \pi$$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} e^{-x}\mathrm{d}x = \frac{1}{\pi}\left|-e^{-x}\right|_0^{\pi}$$

$$= \frac{1}{\pi}(1-e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right|_0^{\pi} = \frac{2}{\pi(n^2+1)}\left[e^{-\pi}(-\cos n\pi)+1\right]$$

$$= \frac{2}{\pi(n^2+1)}[1-(-1)^n e^{-\pi}]$$

Hence, $f(x) = \dfrac{1}{\pi}(1-e^{-\pi}) + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n e^{-\pi}}{n^2+1}\right]\cos nx$

Example 11: Find the Fourier series of $f(x) = \cosh ax$ in the interval $(-\pi, \pi)$.

Solution: $f(-x) = \cosh a(-x)$
$= \cosh ax$
$= f(x)$
$f(x) = \cosh ax$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \cosh ax\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{e^{ax}+e^{-ax}}{2}\right)dx = \frac{1}{2\pi}\left|\frac{e^{ax}}{a} + \frac{e^{-ax}}{-a}\right|_0^{\pi} = \frac{1}{2\pi a}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$= \frac{\sinh a\pi}{\pi a}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \cosh ax \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{e^{ax}+e^{-ax}}{2}\right)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}(e^{ax}\cos nx + e^{-ax}\cos nx)\,dx$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx) + \frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right|_0^{\pi}$$

$$= \frac{a\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi}) = \frac{a\cos n\pi}{\pi(a^2+n^2)}2\sinh a\pi$$

$$= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)}$$

Hence, $$f(x) = \frac{\sinh a\pi}{\pi a} + \frac{2a}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx$$

Example 12: Find the Fourier series of $f(x) = 1 - x^2$ in the interval $(-1, 1)$.

Solution: $f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x)$
$f(x) = 1 - x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{l}\int_0^{l} f(x)\,dx = \int_0^{1}(1-x^2)\,dx = \left|x - \frac{x^3}{3}\right|_0^1 = \frac{2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = 2\int_0^l (1-x^2)\cos n\pi x\,dx$$

$$= 2\left|(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= 2\left(-2\frac{\cos n\pi}{n^2\pi^2}\right)$$

$$= \frac{-4(-1)^n}{n^2\pi^2}$$

Hence, $$f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

Example 13: Find the Fourier series of $f(x) = x|x|$ in the interval (–1, 1).

Solution: $f(x) = x|x|$

i.e. $f(-x) = -x|-x|$

$= -x|x| = -f(x)$

$f(x) = x|x|$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

$f(x) = -x^2 \qquad -1 < x < 0$

$= x^2 \qquad 0 < x < 1$

The Fourier series of an odd function with period $2l = 2$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= 2\int_0^1 x^2 \sin n\pi x\,dx$$

$$= 2\left|x^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2x\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi} + \frac{2\cos n\pi}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]$$

$$= 2\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]$$

Hence, $$f(x) = 2\sum_{n=1}^{\infty}\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]\sin n\pi x$$

Example 14: Find the Fourier series of $f(x) = \frac{1}{2} + x \qquad -\frac{1}{2} < x < 0$

$= \frac{1}{2} - x \qquad 0 < x < \frac{1}{2}.$

Solution: $f(-x) = \frac{1}{2} - x \qquad -\frac{1}{2} < -x < 0 \text{ or } 0 < x < \frac{1}{2}$

$= \frac{1}{2} + x \qquad 0 < -x < \frac{1}{2} \text{ or } -\frac{1}{2} < x < 0$

$f(-x) = f(x)$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of even function with period $2l = 1$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x = \frac{1}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2} - x\right)\mathrm{d}x = 2\left|\frac{x}{2} - \frac{x^2}{2}\right|_0^{\frac{1}{2}} = \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \frac{2}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2} - x\right)\cos 2n\pi x\ \mathrm{d}x$$

$$= 4\left|\left(\frac{1}{2} - x\right)\left(\frac{\sin 2n\pi x}{2n\pi}\right) - (-1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right|_0^{\frac{1}{2}} = 4\left[\left(-\frac{\cos n\pi}{4n^2\pi^2} + \frac{1}{4n^2\pi^2}\right)\right]$$

$$= \frac{1}{n^2\pi^2}\left[1 - (-1)^n\right]$$

Hence, $$f(x) = \frac{1}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1 - (-1)n}{n^2}\right]\cos\ 2n\pi x$$

Example 15: Find the Fourier series of $f(x) = 0 \qquad -2 < x < -1$

$= 1 + x \qquad -1 < x < 0$

$= 1 - x \qquad 0 < x < 1$

$= 0 \qquad 1 < x < 2.$

Solution: $f(-x) = 0 \qquad -2 < -x < -1 \text{ or } 1 < x < 2$

$= 1 - x \qquad -1 < -x < 0 \text{ or } 0 < x < 1$

$= 1 + x \qquad 0 < -x < 1 \text{ or } -1 < x < 0$

$= 0 \qquad 1 < -x < 2 \text{ or } -2 < x < -1$

$f(-x) = f(x)$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x = \frac{1}{2}\left[\int_0^1 (1 - x)\mathrm{d}x + \int_1^2 0\cdot\mathrm{d}x\right] = \frac{1}{2}\left|x - \frac{x^2}{2}\right|_0^1 = \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2}\int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 0\cdot dx$$

$$= \left|(1-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_0^1 = -\cos\left(\frac{n\pi}{2}\right)\frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]$$

Hence, $$f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\cos\frac{n\pi x}{2}$$

Exercise 13.3

Find the Fourier series of the following functions:

1. $f(x) = x \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\right]$$

2. $f(x) = \dfrac{x(\pi^2 - x^2)}{12} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\right]$$

3. $f(x) = \cos ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{\sin a\pi}{\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}\cos nx\right]$$

4. $f(x) = x\cos x \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{-1}{2}\sin x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\sin nx\right]$$

5. $f(x) = |\sin x| \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx\right]$$

6. $f(x) = \sqrt{1-\cos x} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx\right]$$

7. $f(x) = \sinh ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.: } \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{a^2+n^2}\sin nx\right]$$

8. $f(x) = \dfrac{-(\pi+x)}{2} \qquad -\pi < x < 0$

$\qquad = \dfrac{\pi - x}{2} \qquad 0 < x < \pi$

$$\left[\textbf{Ans.: } \sum_{n=1}^{\infty}\frac{1}{n}\sin nx\right]$$

9. $f(x) = x + \dfrac{\pi}{2} \qquad -\pi < x < 0$

$\qquad = \dfrac{\pi}{2} - x \qquad 0 < x < \pi$

$$\left[\textbf{Ans.: } \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx\right]$$

10. $f(x) = |x| \qquad -2 < x < 2$

$$\left[\textbf{Ans.: } 1 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x\right]$$

11. $f(x) = a^2 - x^2 \qquad -a < x < a$

$$\left[\textbf{Ans.: } \frac{2a^2}{3} - \frac{4a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{a}\right]$$

12. $f(x) = x^2 - 2 \qquad -2 < x < 2$

$$\left[\textbf{Ans.:} \frac{-2}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}\right]$$

13. $f(x) = \sin ax \qquad -l < x < l$

$$\left[\textbf{Ans.:} \sin\, al\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2 - a^2l^2}\sin\frac{n\pi x}{l}\right]$$

14. $f(x) = x - x^3 \qquad -1 < x < 1$

$$\left[\textbf{Ans.:} -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x\right]$$

13.6 HALF-RANGE FOURIER SERIES

Any arbitrary function $f(x)$ with period $2l$ which is defined in half of the interval $(0, l)$ can also be represented in terms of sine and cosine functions. A half-range expansion containing only cosine terms is known as a half-range cosine series. Similarly, a half-range expansion containing only sine terms is known as a half-range sine series.

To represent any function $f(x)$ in half-range cosine series in the interval $(0, l)$, we extend the function by reflecting it in the vertical axis (i.e., y axis) so that $f(-x) = f(x)$. The extended function is an even function in $(-l, l)$ and is periodic with period $2l$. The half-range cosine series of such a function is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where,

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Cor: If any function with period 2π is defined in the interval $(0, \pi)$, then the half-range cosine series of such a function is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Similarly, to represent any function $f(x)$ in the half-range sine series in the interval $(0, l)$, we extend the function by reflecting it in the origin so that $f(-x) = -f(x)$. The extended function is an odd function in $(-l, l)$ and is periodic with period $2l$. The half-range sine series of such a function is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Cor: If any function with period 2π is defined in the interval $(0, \pi)$ then the half-range sine series of such a function is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where,
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Example 1: Find the half-range cosine series of $f(x) = x$ in the interval $(0, \pi)$.

Solution: The half-range cosine series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

Hence,
$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx$$

Example 2: Find half-range sine series of $f(x) = x^2$ in the interval $(0, \pi)$.

Solution: The half-range sine series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$= \frac{2}{\pi}\left|x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$

Hence,
$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]\sin nx$$

Example 3: Find half-range cosine series of $f(x) = x(\pi - x)$ in the interval $(0, \pi)$ and hence, deduce that

(i) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ (ii) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots$.

Solution: The half-range cosine series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x(\pi - x)\mathrm{d}x = \frac{1}{\pi}\left|\pi\frac{x^2}{2} - \frac{x^3}{3}\right|_0^{\pi} = \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi - 2\pi)\frac{\cos n\pi}{n^2} - \frac{\pi}{n^2}\right]$$

$$= -\frac{2}{n^2}[1 + (-1)^n]$$

Hence, $$f(x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{1 + (-1)^n}{n^2}\right]\cos nx \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{1 + (-1)^n}{n^2}\right] = \frac{\pi^2}{6} - 4\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Putting $x = \dfrac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{1 + (-1)^n}{n^2}\right]\cos\frac{n\pi}{2}$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{6} - 4\left(-\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \ldots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 4: Find the half-range sine series of $f(x) = e^{ax}$ in the interval $(0, \pi)$.

Solution: The half-range sine series of $f(x)$ with period 2π is given by,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} e^{ax}\sin nx\,\mathrm{d}x = \frac{2}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi)+\frac{n}{a^2+n^2}\right]$$

$$= \frac{2n}{\pi(a^2+n^2)}[1-(-1)^n e^{a\pi}]$$

Hence, $$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{a^2+n^2}[1-(-1)^n e^{a\pi}]\sin nx$$

Example 5: Find half-range cosine series $f(x)=\sin x$ in the interval $(0,\pi)$ and hence, deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.$

Solution: The half-range cosine series of $f(x)$ with period 2π is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x = \frac{1}{\pi}\left|-\cos x\right|_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x-\sin(n-1)x]\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right|_0^{\pi},\ n\neq 1$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right],\ n\neq 1$$

$$= -\frac{2}{\pi(n^2-1)}[1+(-1)^n],\ n\neq 1 \qquad [\because \cos(n\pi+\pi)=\cos(n\pi-\pi)=\cos n\pi=-(-1)^n]$$

For $n=1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin 2x\,\mathrm{d}x = \frac{1}{\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi} = 0$$

Hence, $$f(x)=\frac{2}{\pi}-\frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx \qquad \ldots (1)$$

Putting $x=\frac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right)=1=\frac{2}{\pi}-\frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos\frac{n\pi}{2}$$

$$1=\frac{2}{\pi}-\frac{2}{\pi}\left(-\frac{2}{3}+\frac{2}{15}-\frac{2}{35}+\ldots\right)$$

$$1=\frac{2}{\pi}+\frac{2}{\pi}\left[\left(1-\frac{1}{3}\right)-\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)-\ldots\right]$$

$$1=\frac{2}{\pi}\left(2-\frac{2}{3}+\frac{2}{5}-\frac{2}{7}+\ldots\right)$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Example 6: Find the half-range cosine series of $f(x)$ where

$$\begin{aligned} f(x) &= x & 0<x<\frac{\pi}{2} \\ &= \pi - x & \frac{\pi}{2}<x<\pi. \end{aligned}$$

Hence, find $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$.

Solution: The half range cosine series of $f(x)$ with period 2π is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}f(x)\,dx=\frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}x\,dx+\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\,dx\right]=\frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\frac{\pi}{2}}+\left|\pi x-\frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]=\frac{\pi}{4}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}x\cos nx\,dx+\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\cos nx\,dx\right]$$

$$=\frac{2}{\pi}\left[\left|x\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\frac{\pi}{2}}+\left|(\pi-x)\left(\frac{\sin nx}{n}\right)-(-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$=\frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2}-1-(-1)^n\right]$$

Hence, $$f(x)=\frac{\pi}{4}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2}-1-(-1)^n\right]\cos nx$$

$$=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{1}{2^2}(-4)\cos 2x+\frac{1}{6^2}(-4)\cos 6x+\frac{1}{10^2}(-4)\cos 10x+\ldots\right]$$

$$=\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x+\frac{1}{3^2}\cos 6x+\frac{1}{5^2}\cos 10x+\ldots\right)$$

By Parseval's identity,

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,\mathrm{d}x = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

$$\frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} x^2 dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)^2 dx\right] = \frac{\pi^2}{16} + \frac{1}{2}\cdot\frac{4}{\pi^2}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right)$$

$$\frac{1}{\pi}\left[\left|\frac{x^3}{3}\right|_0^{\frac{\pi}{2}} + \left|\frac{(\pi - x)^3}{-3}\right|_{\frac{\pi}{2}}^{\pi}\right] = \frac{\pi^2}{16} + \frac{2}{\pi^2}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right)$$

$$\frac{\pi^2}{12} - \frac{\pi^2}{16} = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Example 7: Find the half-range sine series of $f(x)$ where

$$\begin{aligned} f(x) &= \frac{\pi}{3} && 0 \le x < \frac{\pi}{3} \\ &= 0 && \frac{\pi}{3} \le x < \frac{2\pi}{3} \\ &= -\frac{\pi}{3} && \frac{2\pi}{3} \le x \le \pi. \end{aligned}$$

Solution: The half-range sine series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{3}}\frac{\pi}{3}\sin nx\,\mathrm{d}x + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 0\cdot\sin nx\,\mathrm{d}x + \int_{\frac{2\pi}{3}}^{\pi}\left(-\frac{\pi}{3}\right)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{2}{3}\left[\left|-\frac{\cos nx}{n}\right|_0^{\frac{\pi}{3}} - \left|\frac{-\cos nx}{n}\right|_{\frac{2\pi}{3}}^{\pi}\right] = \frac{2}{3n}\left[-\cos\frac{n\pi}{3} + 1 + (-1)^n - \cos\frac{2n\pi}{3}\right]$$

$$= \frac{2}{3n}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right]$$

Hence, $$f(x) = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right]\sin nx.$$

Example 8: Find the half-range sine series of $f(x) = lx - x^2$ in the interval $(0, l)$ and hence, deduce that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots.$$

Solution: The half-range sine series of $f(x)$ with period $2l$ is given by,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left|(lx-x^2)\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]-(l-2x)\frac{l^2}{n^2\pi^2}\left[-\sin\frac{n\pi x}{l}+(-2)\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]\right|_0^l$$

$$=\frac{2}{l}\left[-\frac{2l^3}{n^3\pi^3}(\cos n\pi-1)\right]=\frac{4l^2}{n^3\pi^3}[1-(-1)^n]$$

Hence,
$$f(x)=\frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin\frac{n\pi x}{l} \qquad \dots (1)$$

Putting $x=\frac{l}{2}$ in Eq. (1),

$$f\left(\frac{l}{2}\right)=\frac{l^2}{2}-\frac{l^2}{4}=\frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin\frac{n\pi}{2}$$

$$\frac{l^2}{4}=\frac{8l^2}{\pi^3}\left(\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\dots\right)$$

$$\frac{\pi^3}{32}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\dots$$

Example 9: Find half-range cosine series of $f(x)$ where

$$\begin{aligned} f(x)&=kx & 0\le x\le \frac{l}{2}\\ &=k(l-x) & \frac{l}{2}\le x\le l. \end{aligned}$$

Hence, deduce that $\dfrac{\pi^2}{8}=\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dots.$

Solution: The half-range cosine series of $f(x)$ with period $2l$ is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

$$a_0=\frac{1}{l}\int_0^l f(x)\,dx=\frac{1}{l}\left[\int_0^{\frac{l}{2}} kx\,dx+\int_{\frac{l}{2}}^{l} k(l-x)\,dx\right]=\frac{1}{l}\left[k\left|\frac{x^2}{2}\right|_0^{\frac{l}{2}}+k\left|lx-\frac{x^2}{2}\right|_{\frac{l}{2}}^{l}\right]$$

$$=\frac{k}{l}\left[\frac{2l^2}{8}\right]=\frac{kl}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx\cos\frac{n\pi x}{l}\mathrm{d}x + \int_{\frac{l}{2}}^l k(l-x)\cos\frac{n\pi x}{l}\mathrm{d}x\right]$$

$$= \frac{2k}{l}\left[\left| x\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - \left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - (-1)\left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)\right|_{\frac{l}{2}}^l\right]$$

$$= \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - \{1+(-1)\}^n\right]$$

Hence,
$$f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - \{1+(-1)^n\}\right]\cos\frac{n\pi x}{l} \quad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{kl}{4} + \frac{2kl}{\pi^2}\left(-\frac{4}{2^2} - \frac{4}{6^2} - \frac{4}{10^2} - \ldots\right)$$

$$0 = \frac{kl}{4} - \frac{2kl}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \quad \ldots (2)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 10: Find the half-range sine series of

$$f(x) = \frac{2x}{l} \qquad 0 \le x \le \frac{l}{2}$$
$$= \frac{2(l-x)}{l} \qquad \frac{l}{2} \le x \le l.$$

Solution: The the half-range sine series of $f(x)$ with period $2l$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} \frac{2x}{l}\sin\frac{n\pi x}{l}\mathrm{d}x + \int_{\frac{l}{2}}^l \frac{2(l-x)}{l}\sin\frac{n\pi x}{l}\mathrm{d}x\right]$$

$$= \frac{4}{l^2}\left[\left| x\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - \left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{4}{l^2}\frac{l^2}{n^2\pi^2}\left(2\sin\frac{n\pi}{2}\right) = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $f(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$

Example 11: Find the half-range sine series of $f(x) = x \qquad 0 < x < 1$

$\qquad = 2 - x \qquad 1 < x < 2$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots.$

Solution: The half-range sine series of $f(x)$ with period $2l = 4$ is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx$$

$$= \left[\left| x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_0^1\right.$$

$$\left. + \left|(2-x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_1^2\right]$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $f(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}$ $\qquad \ldots (1)$

At $x = 1$, $f(1) = \dfrac{1}{2}\left[\displaystyle\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right] = \dfrac{1+(2-1)}{2} = 1$

Putting $x = 1$ in Eq. (1),

$$f(1) = 1 = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(\frac{n\pi}{2}\right) = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 12: Find the half-range cosine series of

$$f(x) = 1 \qquad 0 \le x \le 1$$
$$= x \qquad 1 \le x \le 2.$$

Solution: The half-range cosine series of $f(x)$ with period $2l = 4$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x = \frac{1}{2}\left[\int_0^1 1\mathrm{d}x + \int_1^2 x\mathrm{d}x\right] = \frac{1}{2}\left[\left|x\right|_0^1 + \left|\frac{x^2}{2}\right|_1^2\right] = \frac{5}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \int_0^1 1\cdot\cos\frac{n\pi x}{2}\mathrm{d}x + \int_1^2 x\cos\frac{n\pi x}{2}\mathrm{d}x$$

$$= \left|\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right|_0^1 + \left[\left|x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right|_1^2\right]$$

$$= \left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right) + \left(\frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

Hence,
$$f(x) = \frac{5}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\cos\frac{n\pi x}{2}.$$

Exercise 13.4

1. Find the half-range cosine series of $f(x) = x\sin x$ in $0 < x < \pi$.

$$\left[\textbf{Ans.:}\ 1 - \frac{1}{2}\cos x + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos nx\right]$$

2. Find the half-range cosine series of $f(x) = (x-1)^2$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x\right]$$

3. Find the half-range cosine series of $f(x) = x$ in $0 < x < 2$. Hence, deduce that

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots.$$

$$\left[\textbf{Ans.:}\ 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]\cos\frac{n\pi x}{2}\right]$$

4. Find the half-range cosine series of $f(x) = e^x$ in $0 < x < 1$.

$$\left[\begin{array}{l}\textbf{Ans.:}\\ (e-1) + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}[e(-1)^n - 1]\cos n\pi x\end{array}\right]$$

5. Find the half-range sine series of

$$f(x) = x \qquad 0 \le x \le 2$$
$$= 4 - x \qquad 2 \le x \le 4.$$

$$\left[\textbf{Ans.:}\ \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{4}\right]$$

6. Find the half-range sine and cosine series of $f(x) = x - x^2$ in $0 < x < 1$.

$$\left[\begin{array}{l}\textbf{Ans.:}\ \frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin(2n+1)\pi x,\\ \frac{1}{6} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\cos 2n\pi x\end{array}\right]$$

7. Find the half-range sine and cosine series of $f(x)=a\left(1-\frac{x}{l}\right)$ in $0<x<l$.

$$\left[\begin{array}{l}\textbf{Ans.:}\ \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l},\\ \frac{a}{2}+\frac{4a}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{l}\end{array}\right]$$

8. Find the half-range sine series of $f(x)=\sin^2 x$ in $0<x<\pi$.

$$\left[\begin{array}{l}\textbf{Ans.:}\\ -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)(2n+1)(2n+3)}\end{array}\right]$$

9. Find the half-range sine series of

$$f(x)=\begin{cases}\frac{2x}{3} & 0\le x\le\frac{\pi}{3}\\ \frac{\pi-x}{3} & \frac{\pi}{3}\le x\le\pi\end{cases}$$

$$\left[\textbf{Ans.:}\ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin nx\right]$$

10. Find the half-range sine series of

$$f(x)=\begin{cases}x & 0\le x<1\\ 1 & 1\le x<2\\ 3-x & 2\le x\le 3\end{cases}$$

$$\left[\begin{array}{l}\textbf{Ans.:}\\ \frac{6}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left(\sin\frac{n\pi}{3}+\sin\frac{2n\pi}{3}\right)\right]\sin\frac{n\pi x}{3}\end{array}\right]$$

13.7 COMPLEX FORM OF FOURIER SERIES

A set of exponential functions $\left\{e^{\frac{in\pi x}{l}}\right\}, n=0,\pm1,\pm2,\ldots$ is orthogonal in the interval $(c, c+2l)$. It is therefore, possible to represent any arbitrary function $f(x)$ by a linear combination of exponential functions in the interval $(c, c+2l)$.

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi x}{l}}$$

where,
$$c_n=\frac{1}{2l}\int_c^{c+2l}f(x)e^{-\frac{in\pi x}{l}}\,dx,\quad n=0,\pm1,\pm2,\ldots$$

Proof: We know that,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\frac{\cos n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

where,
$$a_0=\frac{1}{2l}\int_c^{c+2l}f(x)\,dx$$

$$a_n=\frac{1}{l}\int_c^{c+2l}f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n=\frac{1}{l}\int_c^{c+2l}f(x)\sin\frac{n\pi x}{l}\,dx$$

Now,
$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\left(\frac{e^{\frac{in\pi x}{l}}+e^{-\frac{in\pi x}{l}}}{2}\right)+\sum_{n=1}^{\infty}b_n\left(\frac{e^{\frac{in\pi x}{l}}-e^{-\frac{in\pi x}{l}}}{2i}\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-\frac{in\pi x}{l}}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{-\frac{in\pi x}{l}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

where, $c_0 = a_0 = \dfrac{1}{2l}\displaystyle\int_c^{c+2l} f(x)\,\mathrm{d}x$

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2l}\left[\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x - i\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)e^{-\frac{in\pi x}{l}}\,\mathrm{d}x$$

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2l}\left[\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x + i\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)e^{\frac{in\pi x}{l}}\,\mathrm{d}x$$

In general, we can write

$$c_n = \frac{1}{2l}\int_c^{c+2l} f(x)e^{-\frac{in\pi x}{l}}\,\mathrm{d}x, \quad n = 0, \pm 1, \pm 2, \ldots$$

Cor. 1: When $c = 0$ and $2l = 2\pi$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where, $$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,\mathrm{d}x$$

Cor. 2: When $c = -\pi$ and $2l = 2\pi$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x$$

Cor. 3: When $c = 0$,

$$f(x) = \sum_{n=\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

where, $$c_n = \frac{1}{2l}\int_0^{2l} f(x)e^{-\frac{in\pi x}{l}}\,dx$$

Cor. 4: When $c = -l$,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{x}}$$

where, $$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{in\pi x}{l}}\,dx$$

Example 1: Find the complex form of Fourier series of $f(x) = 2x$ in the interval $(0, 2\pi)$.

Solution: The complex form of Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} 2x e^{-inx}\,dx$$

$$= \frac{1}{\pi}\left| x\left(\frac{e^{-inx}}{-in}\right) - \frac{e^{-inx}}{(-in)^2}\right|_0^{2\pi} \qquad n \neq 0$$

$$= \frac{1}{\pi}\left(2\pi i\frac{e^{-i2n\pi}}{n} + \frac{e^{-i2n\pi}}{n^2} - \frac{1}{n^2}\right) \qquad n \neq 0$$

$$= \frac{1}{\pi}\left(\frac{2\pi i}{n} + \frac{1}{n^2} - \frac{1}{n^2}\right), \qquad n \neq 0 \quad [\because e^{-i2n\pi} = 1]$$

$$= \frac{2i}{n} \qquad n \neq 0$$

For $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} 2x\,dx = \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{2\pi} = 2\pi$$

Hence, $$f(x) = 2\pi + 2i\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}e^{inx}$$

Example 2: Find the complex form of Fourier series of $f(x) = \sin ax$ in the interval $(-\pi, \pi)$ where, a is not an integer.

Solution: The complex form of Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin ax\, e^{-inx}\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|\frac{e^{-inx}}{a^2+i^2n^2}(-in\sin ax - a\cos ax)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{a^2-n^2}(-in\sin a\pi - a\cos a\pi) - \frac{e^{in\pi}}{a^2-n^2}(in\sin a\pi - a\cos a\pi)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{in\sin a\pi}{a^2-n^2}(e^{-in\pi}+e^{in\pi}) + \frac{a\cos a\pi}{a^2-n^2}(-e^{-in\pi}+e^{in\pi})\right]$$

$$= \frac{1}{2\pi}\left[-\frac{in\sin a\pi}{a^2-n^2}(2\cos n\pi) + \frac{a\cos a\pi}{a^2-n^2}(2i\sin n\pi)\right] = \frac{in\sin a\pi\cos n\pi}{\pi(n^2-a^2)}$$

$$= \frac{(-1)^n in\sin a\pi}{\pi(n^2-a^2)}$$

Hence, $f(x) = \dfrac{i\sin a\pi}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-1)^n n}{n^2-a^2}e^{inx}$

Example 3: Find complex form of Fourier series of $f(x) = e^{ax}$ in the interval $(-\pi,\pi)$ where, a is a real constant. Hence, deduce that

$$\frac{\pi}{a\sinh a\pi} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}.$$

Solution: The complex form of Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|\frac{e^{(a-in)x}}{a-in}\right|_{-\pi}^{\pi} = \frac{1}{2\pi(a-in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{in\pi}\right] = \frac{1}{2\pi(a-in)}\left[e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n\right]$$

$$= \frac{(-1)^n}{2\pi(a-in)}\left[e^{a\pi} - e^{-a\pi}\right] = \frac{(-1)^n}{\pi(a-in)}\sinh a\pi$$

$$= \frac{(-1)^n(a+in)\sinh a\pi}{\pi(a^2+n^2)}$$

Hence, $f(x) = \dfrac{\sinh a\pi}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{a^2+n^2}e^{inx}$ … (1)

Putting $x = 0$ in Eq. (1),

$$f(0) = 1 = \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{a^2+n^2}$$

Comparing real part on both the sides,

$$1 = \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n a}{a^2+n^2}$$

$$\frac{\pi}{a\sinh a\pi} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}$$

Example 4: Find the complex form of Fourier series of $f(x) = \cosh ax$ in the interval $(-\pi,\pi)$ where, a is not an integer.

Solution: The complex form of Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cosh ax\, e^{-inx}\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{ax}+e^{-ax}}{2}\right)e^{-inx}\mathrm{d}x = \frac{1}{4\pi}\int_{-\pi}^{\pi}[e^{(a-in)x}+e^{-(a+in)x}]\mathrm{d}x$$

$$= \frac{1}{4\pi}\left|\frac{e^{(a-in)x}}{a-in}+\frac{e^{-(a+in)x}}{-(a+in)}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[\frac{1}{a-in}\left\{e^{(a-in)\pi}-e^{-(a-in)\pi}\right\}-\frac{1}{a+in}\left\{e^{-(a+in)\pi}-e^{(a+in)\pi}\right\}\right]$$

$$= \frac{1}{4\pi}\left[\frac{1}{a-in}(e^{a\pi}e^{-in\pi}-e^{-a\pi}e^{in\pi})-\frac{1}{a+in}\left(e^{-a\pi}e^{-in\pi}-e^{a\pi}e^{in\pi}\right)\right]$$

$$= \frac{(-1)^n}{4\pi}\left[\frac{1}{a-in}(e^{a\pi}-e^{-a\pi})+\frac{1}{a+in}(-e^{-a\pi}+e^{a\pi})\right] \qquad \left[\because e^{in\pi}=e^{-in\pi}=(-1)^n\right]$$

$$= \frac{(-1)^n}{2\pi}\left(\frac{1}{a-in}\sinh a\pi+\frac{1}{a+in}\sinh a\pi\right) = \frac{(-1)^n\sinh a\pi}{2\pi}\left(\frac{1}{a-in}+\frac{1}{a+in}\right)$$

$$= \frac{(-1)^n a\sinh a\pi}{\pi(a^2+n^2)}$$

Hence, $f(x) = \dfrac{a\sinh a\pi}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}e^{inx}$

Example 5: Find the complex form of Fourier series of $f(x) = e^{-x}$ in the interval [–1, 1].

Solution: The complex form of Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{in\pi x}{l}}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} e^{-x}e^{-in\pi x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\,\mathrm{d}x = \frac{1}{2}\left|\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right|_{-1}^{1} = -\frac{1}{2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]$$

$$= -\frac{1}{2(1+in\pi)}[e^{-1}e^{in\pi} - e^{1}e^{in\pi}] = \frac{(-1)^n}{2(1+in\pi)}[e^{1} - e^{-1}] = \frac{(-1)^n \sinh 1}{1+in\pi}$$

$$= \frac{(-1)^n(1-in\pi)\sinh 1}{1+n^2\pi^2}$$

Hence,
$$f(x) = \sinh 1 \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-in\pi)}{1+n^2\pi^2} e^{in\pi x}$$

Example 6: Find the complex form of Fourier series of $f(x) = \sinh ax$ in the interval $(-l, l)$.

Solution: The complex form of Fourier series of $f(x)$ with period $2l$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{in\pi x}{l}}\,\mathrm{d}x = \frac{1}{2l}\int_{-l}^{l} \sinh ax\cdot e^{-\frac{in\pi x}{l}}\,\mathrm{d}x = \frac{1}{2l}\int_{-l}^{l}\left(\frac{e^{ax}-e^{-ax}}{2}\right)e^{-\frac{in\pi x}{l}}\,\mathrm{d}x$$

$$= \frac{1}{4l}\int_{-l}^{l}\left[e^{\left(a-\frac{in\pi}{l}\right)x} - e^{-\left(a+\frac{in\pi}{l}\right)x}\right]\mathrm{d}x = \frac{1}{4l}\left|\frac{e^{\left(a-\frac{in\pi}{l}\right)x}}{\left(a-\frac{in\pi}{l}\right)} - \frac{e^{-\left(a+\frac{in\pi}{l}\right)x}}{-\left(a+\frac{in\pi}{l}\right)}\right|_{-l}^{l}$$

$$= \frac{1}{4}\left[\frac{1}{(al-in\pi)}\left\{\mathrm{e}^{(al-in\pi)} - e^{-(al-in\pi)}\right\} + \frac{1}{(al-in\pi)}\left\{\mathrm{e}^{-(al+in\pi)} - e^{(al+in\pi)}\right\}\right]$$

$$= \frac{1}{4}\left[\frac{1}{al-in\pi}(e^{al}e^{-in\pi} - e^{-al}e^{in\pi}) + \frac{1}{al+in\pi}(e^{-al}e^{-in\pi} - e^{al}e^{in\pi})\right]$$

$$= \frac{1}{4}\left[\frac{(-1)^n}{al-in\pi}(e^{al} - e^{-al}) - \frac{(-1)^n}{al+in\pi}(-e^{-al} + e^{al})\right]$$

$$= \frac{1}{2}\left[\frac{(-1)^n}{al-in\pi}\sinh al - \frac{(-1)^n}{al+in\pi}\sinh al\right] = \frac{(-1)^n \sinh al}{2}\left[\frac{1}{al-in\pi} - \frac{1}{al+in\pi}\right]$$

$$= \frac{(-1)^n in\pi \sinh al}{a^2l^2 + n^2\pi^2}$$

Hence,
$$f(x) = i\pi \sinh al \sum_{n=-\infty}^{\infty} \frac{(-1)^n n}{a^2l^2+n^2\pi^2} e^{\frac{in\pi x}{l}}$$

Example 7: Find the complex form of Fourier series of $f(x) = \cosh 2x + \sinh 2x$ in the interval (–5, 5).

Solution: The complex form of Fourier series of $f(x)$ with period $2l = 10$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{5}}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx = \frac{1}{10}\int_{-5}^{5} (\cosh 2x + \sinh 2x) e^{-\frac{in\pi x}{5}} dx$$

$$= \frac{1}{10}\int_{-5}^{5}\left(\frac{e^{2x}+e^{-2x}}{2} + \frac{e^{2x}-e^{-2x}}{2}\right) e^{-\frac{in\pi x}{5}} dx = \frac{1}{10}\int_{-5}^{5} e^{2x} e^{-\frac{in\pi x}{5}} dx$$

$$= \frac{1}{10}\int_{-5}^{5} e^{\left(\frac{10-in\pi}{5}\right)x} dx = \frac{1}{10}\left|\frac{e^{\left(\frac{10-in\pi}{5}\right)x}}{\frac{10-in\pi}{5}}\right|_{-5}^{5} = \frac{1}{10}\cdot\frac{5}{10-in\pi}\left[e^{(10-in\pi)} - e^{-(10-in\pi)}\right]$$

$$= \frac{1}{2(10-in\pi)}\left[e^{10}e^{-in\pi} - e^{-10}e^{in\pi}\right] = \frac{(-1)^n}{2(10-in\pi)}\left[e^{10} - e^{-10}\right] = \frac{(-1)^n}{10-in\pi}\cosh 10$$

$$= \frac{(-1)^n(10+in\pi)\cosh 10}{100+n^2\pi^2}$$

Hence,
$$f(x) = \cosh 10 \sum_{n=-\infty}^{\infty} \frac{(-1)^n(10+in\pi)}{100+n^2\pi^2)} e^{\frac{in\pi x}{5}}$$

Example 8: Find the complex form of Fourier series of $f(x) = -1 \quad -1 < x < 0$
$= 1 \quad 0 < x < 1.$

Solution: The complex form of Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx = \frac{1}{2}\int_{-1}^{1} f(x) e^{-in\pi x} dx$$

$$= \frac{1}{2}\left[\int_{-1}^{0}(-1)e^{-in\pi x}dx + \int_{0}^{1} 1\cdot e^{-in\pi x}dx\right] = \frac{1}{2}\left[\left|-\frac{e^{-in\pi x}}{-in\pi}\right|_{-1}^{0} + \left|\frac{e^{-in\pi x}}{-in\pi}\right|_{0}^{1}\right] \qquad n \neq 0$$

$$= \frac{1}{2}\left[-\frac{i}{n\pi}(1 - e^{in\pi} - e^{-in\pi} + 1)\right] \qquad n \neq 0$$

$$= -\frac{i}{n\pi}(1 - \cos n\pi) \qquad n \neq 0$$

$$= -\frac{i}{n\pi}[1 - (-1)^n] \qquad n \neq 0$$

For $n = 0$,

$$c_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left[\int_{-1}^{0}(-1)dx + \int_{0}^{1} 1\cdot dx\right] = \frac{1}{2}\left[|-x|_{-1}^{0} + |x|_{0}^{1}\right] = 0$$

Hence, $f(x) = -\frac{i}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{n} e^{in\pi x}$

Example 9: Find complex form of Fourier series of $f(x) = x^2 \qquad 0 \le x < 1$
$\qquad = 1 \qquad 1 < x < 2$.

Solution: The complex form of Fourier series of $f(x)$ with period $2l = 2$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-\frac{in\pi x}{l}} \mathrm{d}x = \frac{1}{2}\left[\int_0^1 x^2 e^{-in\pi x}\mathrm{d}x + \int_1^2 1\cdot e^{-in\pi x}\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left| x^2\left(\frac{e^{-in\pi x}}{-in\pi}\right) - 2x\left(\frac{e^{-in\pi x}}{i^2n^2\pi^2}\right) + 2\left(\frac{e^{-in\pi x}}{-i^3n^3\pi^3}\right)\right|_0^1 + \left|\frac{e^{-in\pi x}}{-in\pi}\right|_1^2\right] \qquad n \neq 0$$

$$= \frac{1}{2}\left[-\frac{e^{-in\pi}}{in\pi} + 2\left(\frac{e^{-in\pi}}{n^2\pi^2}\right) + 2\left(\frac{e^{-in\pi}}{in^3\pi^3}\right) - \frac{2}{in^3\pi^3} - \frac{e^{-2in\pi}}{in\pi} + \frac{e^{-in\pi}}{in\pi}\right], \quad n \neq 0$$

$$= \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} - \frac{2i(-1)^n}{n^3\pi^3} + \frac{2i}{n^3\pi^3} + \frac{i}{n\pi}\right] \qquad n \neq 0$$

$$= \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} - \frac{2i}{n^3\pi^3}\left\{(-1)^n - 1\right\} + \frac{i}{n\pi}\right] \qquad n \neq 0$$

For $n = 0$,

$$c_0 = \frac{1}{2}\int_0^2 f(x)\mathrm{d}x = \frac{1}{2}\left[\int_0^1 x^2\mathrm{d}x + \int_1^2 1\cdot\mathrm{d}x\right] = \frac{1}{2}\left[\left|\frac{x^3}{3}\right|_0^1 + \left|x\right|_1^2\right] = \frac{2}{3}$$

Hence, $$f(x) = \frac{2}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left[\frac{2(-1)^n}{n^2\pi^2} - \frac{2i}{n^3\pi^3}\left\{(-1)^n - 1\right\} + \frac{i}{n\pi}\right] e^{in\pi x}$$

Example 10: Find complex form of Fourier series of $f(x) = \cos x \qquad 0 < x < \frac{\pi}{2}$
$\qquad = 0 \qquad \frac{\pi}{2} < x < \pi$.

Solution: The complex form of Fourier series of $f(x)$ with period $2l = \pi$ is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi x}$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-\frac{in\pi x}{l}}\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\, e^{-i2nx}\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} 0\cdot e^{-i2nx}\mathrm{d}x\right] = \frac{1}{\pi}\left|\frac{e^{-i2nx}}{1+4i^2n^2}(-2in\cos x + \sin x)\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left(\frac{1}{1-4n^2}\right)\left[e^{-in\pi}\left(-2in\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\right)-(-2in\cos 0+\sin 0)\right]$$

$$= \frac{1}{\pi(1+4n^2)}[(-1)^n + 2in]$$

Hence, $$f(x) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1-4n^2}[(-1)^n + 2in]e^{i2nx}$$

Exercise 13.5

Find complex form of Fourier series of following functions:

1. $f(x) = x \qquad -\pi < x < \pi$

$\left[\textbf{Ans.: } i\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n}e^{inx}\right]$

2. $f(x) = e^x \qquad -\pi < x < \pi$

$\left[\textbf{Ans.: } \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1+in)}{1+n^2}e^{inx}\right]$

3. $f(x) = \cos ax \qquad -\pi < x < \pi$

$\left[\textbf{Ans.: } \frac{a\sin a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2-n^2}e^{inx}\right]$

4. $f(x) = \sinh x \qquad -l < x < l$

$\left[\textbf{Ans.: } \sinh l \cdot i\pi\sum_{n=-\infty}^{\infty}\frac{(-1)^n n}{l^2+n^2\pi^2}e^{\frac{in\pi x}{l}}\right]$

5. $f(x) = 1 \qquad 0 < x < 1$

$\quad = 0 \qquad 1 < x < 2$

$\left[\textbf{Ans.: } \frac{1}{2}+\frac{1}{2\pi i}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1-(-1)^n}{n}e^{in\pi x}\right]$

6. $f(x) = e^{-|x|} \qquad -2 < x < 2$

$\left[\textbf{Ans.: } \sum_{n=-\infty}^{\infty}\frac{2}{4+n^2\pi^2}[1-(-1)^n e^{-2}]e^{\frac{in\pi x}{2}}\right]$

7. $f(x) = 0 \qquad -\frac{1}{2} < x < 0$

$\quad = 1 \qquad 0 < x < \frac{1}{4}$

$\quad = 0 \qquad \frac{1}{4} < x < \frac{1}{2}$

$\left[\textbf{Ans.: } \frac{1}{4}+\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1-e^{\frac{-in\pi}{2}}}{2n\pi i}e^{2in\pi x}\right]$

FORMULAE

Orthogonality of Functions

$$\int_a^b f_1(x)f_2(x)\,\mathrm{d}x = 0$$

Orthonormality of Functions

$$\int_a^b f_1(x)f_2(x)\,\mathrm{d}x = 0 \text{ and}$$

$$\int_a^b [f_1(x)]^2\,\mathrm{d}x = 1,$$

$$\int_a^b [f_2(x)]^2\,\mathrm{d}x = 1,$$

Trigonometric Fourier Series in the Interval $(c, c + 2l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l}f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

where $2l$ is the length of the interval.

Parseval's Identity in the Interval $(c, c + 2l)$

$$\frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Fourier Series of Even Functions in the Interval $(-l, l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = 0$$

Fourier Series of Odd Functions in the Interval $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Half Range Cosine Series in the Interval $(0, l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Half Range Sine Series in the Interval $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Complex form of Fourier Series in the Interval $(c,\ c + 2l)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$c_n = \frac{1}{2l}\int_c^{c+2l} f(x)e^{-\frac{in\pi x}{l}}\,dx,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The Fourier series of a real periodic function has only
   (P) cosine terms if it is even
   (Q) sine terms if it is even
   (R) cosine terms if it is odd
   (S) sine terms if it is odd
   which of the above statements are correct
   (a) P and S (b) P and R
   (c) Q and S (d) Q and R
2. Choose the function $f(x)$, $-\infty < x < \infty$, for which a Fourier series cannot be defined
   (a) $3\sin(25x)$
   (b) $4\cos(20x + 3) + 2\sin(10x)$
   (c) $e^{(-|x|)}\sin(25x)$
   (d) 1
3. The Fourier series expansion of a real periodic signal is given by $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$. It is given that $c_3 = 3 + 5i$.
   Then c_{-3} is
   (a) $5 + 3i$ (b) $-3 - 5i$
   (c) $-5 + 3i$ (d) $3 - 5i$

4. Which of the following cannot be the Fourier series expansion of a periodic function?
   (a) $f(x) = 2\cos x + 3\cos 3x$
   (b) $f(x) = 2\cos \pi x + 7\cos x$
   (c) $f(x) = \cos x + 0.5$
   (d) $f(x) = 2\cos 1.5\pi x + \sin 3.5\pi x$
5. If $f(x) = -f(-x)$ and $f(x)$ satisfy the Dirichlet's conditions, then $f(x)$ can be expanded in a Fourier series containing
   (a) only sine terms
   (b) only cosine terms
   (c) cosine terms and a constant term
   (d) sine terms and a constant term
6. If from the function $f(t)$ one forms the function
   $\psi(t) = f(t) + f(-t)$, then $\psi(t)$ is
   (a) even
   (b) odd
   (c) neither even nor odd
   (d) both even and odd
7. The Fourier series expansion for the function $f(x) = \sin^2 x$
   (a) $\sin x + \sin 2x$
   (b) $1 - \cos 2x$
   (c) $\sin 2x + \cos 2x$
   (d) $0.5 - 0.5\cos 2x$
8. Which of the following functions is not periodic?
   (a) $f(x) = \cos 2x + \cos 3x + \cos 5x$
   (b) $f(x) = e^{i8\pi x}$
   (c) $f(x) = e^{(-7x)}\sin 10\pi x$
   (d) $f(x) = \cos 2x \cos 4x$
9. Fourier series of the periodic function (period 2π) defined by
$$f(x) = 0 \qquad -\pi < x < 0 \text{ is}$$
$$= x \qquad 0 < x < \pi$$
$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}(\cos n\pi - 1)\cos nx - \frac{1}{n}\cos n\pi \sin nx\right]$$
   By putting $x = \pi$ in the above, one can deduce that the sum of the series
   $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$ is
   (a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{6}$
   (c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{12}$
10. The Fourier series of an odd periodic function contains only
    (a) odd harmonics
    (b) even harmonics
    (c) cosine terms
    (d) sine terms
11. The trignometric Fourier series of an even function does not have
    (a) constant
    (b) cosine terms
    (c) sine terms
    (d) odd harmonic terms
12. The Fourier series expansion of even function $f(x)$, where
$$f(x) = 1 + \frac{2x}{\pi} \qquad -\pi < x < 0$$
$$= 1 - \frac{2x}{\pi} \qquad 0 < x < \pi$$
    will be
    (a) $\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}(1 + \cos n\pi)$
    (b) $\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}(1 - \cos n\pi)$
    (c) $\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}(1 - \sin n\pi)$
    (d) $\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}(1 + \sin n\pi)$
13. The function $f_3(x) = -1 + ax + bx^2$ is orthogonal to functions $f_1(x) = 1$ and $f_2(x) = x$ in the interval $(-1, 1)$. The value of b will be
    (a) 3 (b) -3
    (c) 0 (d) None of these

14. For the function

$$f(x) = -k \qquad -\pi < x < 0$$
$$= k \qquad 0 < x < \pi,$$

the value of a_0 in Fourier series expansion will be

(a) k (b) $2k$
(c) 0 (d) $-k$

Answers

1. (a)	2. (c)	3. (d)	4. (b)	5. (a)	6. (a)	7. (d)
8. (c)	9. (c)	10. (d)	11. (c)	12. (b)	13. (a)	14. (c)