CHAPTER 6

Partial Differential Equations and Applications

Chapter Outline

6.1 INTRODUCTION

A Partial Differential Equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables. These equations are used to describe phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics.

6.2 PARTIAL DIFFERENTIAL EQUATIONS

A differential equation containing one or more partial derivatives is known as a *partial differential equation*. Partial derivatives $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial x \partial y}, \dfrac{\partial^2 z}{\partial y^2}$ are denoted by p, q, r, s, t respectively. The order of a partial differential equation is the order of the highest-order partial derivative present in the equation. The degree of a partial differential equation is the power of the highest order partial derivative present in the equation.

6.3 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations can be formed using the following methods:

6.3.1 By Elimination of Arbitrary Constants

Let
$$f(x, y, z, a, b) = 0 \qquad ...(6.1)$$
be an equation where a and b are arbitrary constants (Fig. 6.1).

Differentiating Eq. (6.1) partially w.r.t. x,
$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot p = 0 \qquad ...(6.2)$$

Differentiating Eq. (6.1) w.r.t. y,
$$\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$
$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot q = 0 \qquad ...(6.3)$$

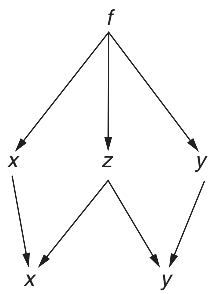

Fig. 6.1 ***Composite function***

By eliminating a, b from Eqs (6.1), (6.2), and (6.3), a partial differential equation of first order is obtained.

Note If the number of arbitrary constants is more than the number of independent variables in Eq. (6.1) then the partial differential equation obtained is of higher order or higher degree (more than one).

Example 1

Form a partial differential equation by eliminating the arbitrary constants from the equation $z = ax^2 + by^2$.

Solution

$$z = ax^2 + by^2 \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = 2ax, \qquad \frac{\partial z}{\partial y} = 2by$$

$$p = 2ax, \qquad q = 2by$$

$$a = \frac{p}{2x}, \qquad b = \frac{q}{2y}$$

Substituting a and b in Eq. (1),

$$z = \frac{p}{2x}x^2 + \frac{q}{2y}y^2$$

$$2z = px + qy$$

which is a partial differential equation of first order.

Example 2

Form a partial differential equation for the equation

$z = (x-2)^2 + (y-3)^2$. **[Summer 2014, 2013]**

Solution

$$z = (x-2)^2 + (y-3)^2 \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = 2(x-2) \qquad \text{...(2)}$$

and
$$\frac{\partial z}{\partial y} = 2(y-3) \qquad \text{...(3)}$$

Squaring and adding Eqs (2) and (3),

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\left[(x-2)^2 + (y-3)^2\right]$$

$$p^2 + q^2 = 4z$$

which is a partial differential equation of order one and degree two.

Example 3

Form a partial differential equation for the equation

$(x-a)(y-b) - z^2 = x^2 + y^2$ **[Winter 2015]**

Solution

$$(x-a)(y-b) - z^2 = x^2 + y^2 \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$(y-b)-2z\frac{\partial z}{\partial x} = 2x$$

$$(y - b) = 2x + 2zp$$

and
$$(x-a)-2z\frac{\partial z}{\partial y} = 2y$$

$$(x - a) = 2y + 2zq$$

Eliminating a and b from Eq. (1),

$$(2y + 2zq)\,(2x + 2zp) - z^2 = x^2 + y^2$$

$$4(x + zp)\,(y + zq) - z^2 = x^2 + y^2$$

which is a partial differential equation.

Example 4

Form a partial differential equation for the equation $z = ax + by + ct$.
[Summer 2017]

Solution

$$z = ax + by + ct \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = a$$

$$p = a$$

and
$$\frac{\partial z}{\partial y} = b$$

$$q = b$$

Differentiating Eq. (1) partially w.r.t. t,

$$\frac{\partial z}{\partial t} = c$$

Substituting a, b and c in Eq. (1),

$$z = px + qy + t\frac{\partial z}{\partial t}$$

which is a partial differential equation of first order.

Example 5

Form a partial differential equation by eliminating the arbitrary constants from the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0, \qquad \frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

$$\frac{x}{a^2}+\frac{z}{c^2}p=0 \qquad ...(2), \qquad \frac{y}{b^2}+\frac{z}{c^2}q=0 \qquad ...(3)$$

Differentiating Eq. (2) partially w.r.t. x,

$$\frac{1}{a^2}+\frac{p}{c^2}\frac{\partial z}{\partial x}+\frac{z}{c^2}\frac{\partial p}{\partial x}=0$$

$$\frac{c^2}{a^2}+p^2+zr=0 \quad \left[\because \frac{\partial p}{\partial x}=\frac{\partial^2 z}{\partial x^2}=r\right]$$

Substituting $\frac{c^2}{a^2}=-\frac{zp}{x}$ from Eq. (2),

$$-\frac{zp}{x}+p^2+zr=0$$

$$-zp+xp^2+xzr=0$$

which is a partial differential equation of second order.
Similarly, differentiating Eq. (3) partially w.r.t. y,

$$\frac{1}{b^2}+\frac{q}{c^2}\frac{\partial z}{\partial y}+\frac{z}{c^2}\frac{\partial q}{\partial y}=0$$

$$\frac{c^2}{b^2}+q^2+zt=0 \quad \left[\because \frac{\partial q}{\partial y}=\frac{\partial^2 z}{\partial y^2}=t\right]$$

Substituting $\frac{c^2}{b^2}=-\frac{zq}{y}$ from Eq. (3),

$$-\frac{zq}{y}+q^2+zt=0$$

$$-zq+yq^2+yzt=0$$

which is also a partial differential equation of order two. Hence, two partial differential equations of order two are obtained.

Example 6

Find the differential equation of all planes which are at a constant distance a from the origin.

Solution

The equation of the plane in normal form is

$$lx + my + nz = a \qquad ...(1)$$

where l, m, and n are the direction cosines of the normal from the origin to the plane.

$$\therefore \quad l^2 + m^2 + n^2 = 1 \qquad ...(2)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$l + n\frac{\partial z}{\partial x} = 0, \qquad m + n\frac{\partial z}{\partial y} = 0$$

$$l + np = 0, \qquad m + nq = 0$$

$$l = -np \quad ...(3) \qquad m = -nq \quad ...(4)$$

Substituting l and m in Eq. (2),

$$n^2 p^2 + n^2 q^2 + n^2 = 1$$

$$n^2 (p^2 + q^2 + 1) = 1$$

$$n = \frac{1}{\sqrt{1 + p^2 + q^2}}$$

Substituting l and m from Eqs (3) and (4) in Eq. (1),

$$-npx - nqy + nz = a$$

$$px + qy - z = -\frac{a}{n}$$

$$px + qy - z = -a\sqrt{1 + p^2 + q^2}$$

$$(px + qy - z)^2 = a^2 (1 + p^2 + q^2)$$

which is a partial differential equation of order one and degree two.

6.3.2 By Elimination of Arbitrary Functions

(a) Let the given equation be $z = f(u)$ ...(6.4)
where u is a function of x, y, and z (Fig. 6.2).
Differentiating Eq. (6.4) w.r.t. x and y,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} \qquad ...(6.5)$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y} \qquad ...(6.6)$$

By eliminating the arbitrary function f from Eqs (6.4), (6.5), and (6.6), a partial differential equation of first order is obtained.

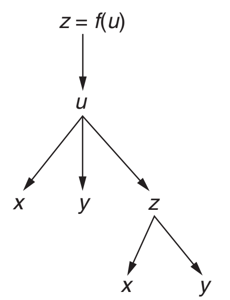

Fig. 6.2 ***Chain rule***

(b) Let the given equation be $F(u, v) = 0$ ...(6.7)

where u and v are functions of x, y, and z.

Differentiating Eq. (6.7) w.r.t. x and y,

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0 \qquad ...(6.8)$$

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0 \qquad ...(6.9)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from Eqs (6.8) and (6.9),

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y} \end{vmatrix}=0$$

Expanding this determinant, a partial differential equation of first order is obtained.

Example 1

Form a partial differential equation by eliminating the arbitrary functions from $z = f(x^2 - y^2)$. **[Winter 2013]**

Solution

$$z = f(x^2 - y^2) \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)\,(2x) \qquad ...(2)$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = f'(x^2 - y^2)\,(-2y) \qquad ...(3)$$

Substituting $f'(x^2 - y^2)$ from Eq. (3) in Eq. (2),

$$\frac{\partial z}{\partial x} = (2x)\left(-\frac{1}{2y}\frac{\partial z}{\partial y}\right)$$

$$p = -\frac{x}{y}q$$

$$py = -xq$$

$$py + xq = 0$$

which is a partial differential equation of first order.

Example 2

Form a partial differential equation by eliminating the arbitrary functions from $xyz = \phi(x + y + z)$. **[Winter 2013]**

Solution

$$xyz = \phi(x + y + z) \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$yz + xy\frac{\partial z}{\partial x} = \phi'(x + y + z)\left(1 + \frac{\partial z}{\partial x}\right)$$

$$yz + xyp = \phi'(x + y + z)(1 + p) \qquad ...(2)$$

Differentiating Eq. (1) partially w.r.t. y,

$$xz + xy\frac{\partial z}{\partial y} = \phi'(x + y + z)\left(1 + \frac{\partial z}{\partial y}\right)$$

$$xz + xyq = \phi'(x + y + z)\,(1 + q) \qquad ...(3)$$

Eliminating $\phi'(x + y + z)$ from Eqs (2) and (3),

$$\frac{yz + xyp}{xz + xyq} = \frac{1 + p}{1 + q}$$

$$(1 + q)(yz + xyp) = (1 + p)(xz + xyq)$$

$$yz + xyp + yzq + xypq = xz + xyq + xzp + xypq$$

$$(xy - xz)p + (yz - xy)q = xz - yz$$

$$x(y - z)p + y(z - x)q = (x - y)z$$

which is a partial differential equation of first order.

Example 3

Form the partial differential equation of $z = f\left(\dfrac{x}{y}\right)$.

[Winter 2012; Summer 2017]

Solution

$$z = f\left(\frac{x}{y}\right) \qquad ...(1)$$

Let $\qquad u = \dfrac{x}{y}$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} = f'(u)\cdot\frac{1}{y}$$

$$p = f'(u)\frac{1}{y} \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y} = f'(u)\left(-\frac{x}{y^2}\right)$$

$$q = f'(u)\left(-\frac{x}{y^2}\right) \qquad \text{...(3)}$$

Eliminating $f'(u)$ from Eqs (2) and (3),

$$q = py\left(-\frac{x}{y^2}\right) = -p\frac{x}{y}$$

$$qy = -px$$

$$px + qy = 0$$

which is a partial differential equation of first order.

Example 4

Form a partial differential equation by eliminating the arbitrary functions from the equation $z = e^{my}\phi(x-y)$.

Solution

$$z = e^{my}\phi(x-y) \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = e^{my}\phi'(x-y)$$

$$p = e^{my}\phi'(x-y) \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = me^{my}\phi(x-y) + \left[e^{my}\phi'(x-y)\right](-1)$$

$$= me^{my}\phi(x-y) - e^{my}\phi'(x-y)$$

$$q = me^{my}\phi(x-y) - e^{my}\phi'(x-y)$$

$$= mz - p \text{ [Using Eqs(1) and (2)]}$$

$$p + q = mz$$

which is a partial differential equation of first order.

Example 5

Eliminate the arbitrary function from the equation $z = xy + f(x^2 + y^2)$.
[Winter 2016]

Solution

Let $\quad u = x^2 + y^2$

$\therefore \quad z = xy + f(u) \qquad$ ...(1)

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = y + \frac{\partial f}{\partial u}(2x)$$

$$p = y + \frac{\partial f}{\partial u}(2x) \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = x + \frac{\partial f}{\partial u}(2y)$$

$$q = x + \frac{\partial f}{\partial u}(2y) \qquad \text{...(3)}$$

Eliminating $\frac{\partial f}{\partial u}$ from Eqs (2) and (3),

$$\frac{p-y}{q-x} = \frac{2x}{2y}$$

$$\frac{p-y}{q-x} = \frac{x}{y}$$

$$qx - x^2 = py - y^2$$

$$qx - py = x^2 - y^2$$

which is a partial differential equation of first order.

Example 6

Form a partial differential equation by eliminating the arbitrary functions from the equation $z = f(x+ay) + \phi(x-ay)$. **[Winter 2016]**

Solution

$$z = f(x+ay) + \phi(x-ay) \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay) \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = f'(x+ay)a + \phi'(x-ay)(-a)$$

$$= af'(x+ay) - a\phi'(x-ay) \qquad \text{...(3)}$$

Differentiating Eq. (2) partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$\frac{\partial^2 z}{\partial y^2} = a\, f''(x+ay)a - a\phi''(x-ay)(-a)$$

$$= a^2\, f''(x+ay) + a^2\phi''(x-ay)$$

$$= a^2\left[f''(x+ay) + \phi''(x-ay)\right]$$

$$= a^2\,\frac{\partial^2 z}{\partial x^2} \qquad \text{[Using Eq. (4)]}$$

$$t = a^2 r$$

which is a partial differential equation of second order.

Example 7

Form the partial differential equations by eliminating the arbitrary functions from $f(x^2 + y^2, z - xy) = 0$. **[Winter 2017]**

Solution

$$f(x^2 + y^2, z - xy) = 0 \qquad \text{...(1)}$$

Let $u = x^2 + y^2, \qquad v = z - xy$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}\left(-y + p\right) = 0 \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}\left(-x + q\right) = 0 \qquad \text{...(3)}$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from Eqs (2) and (3),

$$\frac{2x}{2y} = \frac{-y+p}{-x+q}$$

$$x(q-x) = y(p-y)$$

$$py - qx + x^2 - y^2 = 0$$

which is a partial differential equation of first order.

Example 8

Form a partial differential equation by eliminating the arbitrary functions from $F(x^2 - y^2, xyz) = 0$. **[Winter 2014]**

Solution

$$F(x^2 - y^2, xyz) = 0 \qquad \text{...(1)}$$

Let $u = x^2 - y^2, \qquad v = xyz$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial F}{\partial u}(2x) + \frac{\partial F}{\partial v}\left(yz + xyp\right) = 0 \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial F}{\partial u}(-2y) + \frac{\partial F}{\partial v}\left(xz + xyq\right) = 0 \qquad \text{...(3)}$$

Eliminating $\dfrac{\partial F}{\partial u}$ and $\dfrac{\partial F}{\partial v}$ from Eqs (2) and (3),

$$\frac{2x}{-2y} = \frac{yz + xyp}{xz + xyq}$$

$$x(xz + xyq) = -y(yz + xyp)$$

$$x^2 z + x^2 yq = -y^2 z - xy^2 p$$

$$xy^2 p + x^2 yq = -(x^2 + y^2)z$$

$$xy^2 p + x^2 yq + (x^2 + y^2)z = 0$$

which is a partial differential equation of first order.

Example 9

Form a partial differential equation of $f(x+y+z, x^2+y^2+z^2)=0$, *where f is an arbitrary function.* **[Winter 2012; Summer 2015, 2014]**

Solution

$$f(x + y + z, x^2 + y^2 + z^2) = 0 \qquad \text{...(1)}$$

Let $u = x + y + z, \qquad v = x^2 + y^2 + z^2$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial f}{\partial u}(1+p)+\frac{\partial f}{\partial v}(2x+2zp)=0 \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial f}{\partial u}(1+q)+\frac{\partial f}{\partial v}(2y+2zq)=0 \qquad \text{...(3)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from Eqs (2) and (3),

$$\frac{1+p}{1+q}=\frac{2x+2zp}{2y+2zq}$$

$$(1+p)(2y+2zq)=(1+q)(2x+2zp)$$

$$2y+2zq+2py+2zpq=2x+2zp+2xq+2zpq$$

$$y+zq+py=x+zp+xq$$

$$(y-z)p+(z-x)q=x-y$$

which is a partial differential equation of first order.

Example 10

Eliminate the function f from the relation $f(xy + z^2, x + y + z) = 0$. **[Summer 2013]**

Solution

$$f(xy + z^2, x + y + z) = 0 \qquad \text{...(1)}$$

Let $u = xy + z^2, \qquad v = x + y + z$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0$$

$$\frac{\partial f}{\partial u}(y+2zp)=-\frac{\partial f}{\partial v}(1+p) \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0$$

$$\frac{\partial f}{\partial y}(x+2zq)=-\frac{\partial f}{\partial v}(1+q) \qquad \text{...(3)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from Eqs (2) and (3),

$$\frac{y+2zp}{x+2zq}=\frac{1+p}{1+q}$$

$$(y+2zp)(1+q)=(x+2zq)(1+p)$$

$$y+yq+2zp+2zpq=x+xp+2zq+2zpq$$

$$y+yq+2zp=x+xp+2zq$$

$$p(x-2z)-q(y-2z)=y-x$$

which is a partial differential equation of first order.

EXERCISE 6.1

I. Form partial differential equations by eliminating the arbitrary constants.

1. $z=ax+by+ab$

$\left[\textbf{Ans.:}\ z=px+qy+pq\right]$

2. $z=axe^{y}+\frac{1}{2}a^2e^{2y}+b$

$\left[\textbf{Ans.:}\ q=xp+p^2\right]$

3. $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$

$\left[\textbf{Ans.:}\ p^2+q^2=\tan^2\alpha\right]$

4. $(x-h)^2+(y-k)^2+z^2=c^2$

$\left[\textbf{Ans.:}\ z^2\left(p^2+q^2+1\right)=c^2\right]$

II. Form partial differential equations by eliminating the arbitrary functions.

1. $z = f\left(\frac{y}{x}\right)$

$$\left[\textbf{Ans.:}\ xp + yq = 0\right]$$

2. $z = (x+y)\phi(x^2 - y^2)$

$$\left[\textbf{Ans.:}\ yp'' + xq = z\right]$$

3. $z = y^2 + 2f\left(\frac{1}{x} + \log_e y\right)$

$$\left[\textbf{Ans.:}\ x^2p + yq = 2y^2\right]$$

4. $z = x + y + f(xy)$

$$\left[\textbf{Ans.:}\ xp - yq = x - y\right]$$

5. $z = f(x) + e^y g(x)$

$$[\textbf{Ans.:}\ t = q]$$

6. $z = f(x+y) \cdot g(x-y)$

$$\left[\textbf{Ans.:}\ (r-t)z = (p+q)(p-q)\right]$$

7. $f\left(xy + z^2,\ x + y + z\right) = 0$

$$\left[\textbf{Ans.:}\ (2z - x)p + (y - 2z)q = x - y\right]$$

8. $f\left(x^2 + y^2 + z^2,\ z^2 - 2xy\right) = 0$

$$\left[\textbf{Ans.:}\ (x+y)\left[z(q-p) + (y-x)\right] = 0\right]$$

6.4 SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

The solution of a partial differential equation is a relation between the dependent and independent variables which satisfies the equation.

The solution of a partial differential equation is not always unique. It may have more than one solution or sometimes no solution.

A solution which contains a number of arbitrary constants equal to the independent variables is called a complete integral.

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

Solution of Partial Differential Equations by the Method of Direct Integration

This method is applied to those problems where direct integration is possible. The solutions depend only on the definition of partial differentiation.

Example 1

Solve $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$. **[Winter 2013]**

Solution

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = e^{-t} \cos x$$

Integrating w.r.t. x keeping t constant,

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + \phi(t)$$

Integrating w.r.t. t keeping x constant,

$$u = -e^{-t} \sin x + \int \phi(t)\,\mathrm{d}t + g(x)$$

Example 2

Solve $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, *given that* $\dfrac{\partial z}{\partial y} = -2\sin y$ *when* $x = 0$ *and* $z = 0$, *when y is an odd multiple of* $\dfrac{\pi}{2}$. **[Summer 2013]**

Solution

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \sin y$$

Integrating w.r.t. x keeping y constant,

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \qquad \text{...(1)}$$

When $x = 0$, $\dfrac{\partial z}{\partial y} = -2\sin y$

Putting in Eq. (1),

$$-2\sin y = -1 \cdot \sin y + f(y)$$

$$f(y) = -\sin y$$

Substituting in Eq. (1),

$$\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$$

Integrating w.r.t. y keeping x constant,

$$z = \cos x \cos y + \cos y + g(x) \qquad ...(2)$$

$z = 0$, when y is an odd multiple of $\frac{\pi}{2}$,

i.e.,
$$y = (2n+1)\frac{\pi}{2}, \qquad n = 0, 1, 2, ...$$

$$0 = 0 + 0 + g(x) \qquad \left[\because \cos(2n+1)\frac{\pi}{2} = 0\right]$$

$$g(x) = 0$$

Substituting in Eq. (2),

$$z = \cos x \cos y + \cos y$$
$$= (1 + \cos x)\cos y$$

Example 3

Solve $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0$, $z = e^y$ and $\dfrac{\partial z}{\partial x} = 1$.

Solution

If z is a function of x alone, the solution would have been

$$z = c_1 \cos x + c_2 \sin x$$

Since z is a function of x and y, the solution of the given equation is

$$z = f(y) \cos x + g(y) \sin x \qquad ...(1)$$

When $x = 0$, $\quad z = e^y$

$$f(y) = e^y$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x$$

When $x = 0$, $\quad \dfrac{\partial z}{\partial x} = 1$

$$g(y) = 1$$

Hence, $z = e^y \cos x + \sin x$

Example 4

Find the surface passing through the lines $y = 0$, $z = 0$ and $y = 1$, $z = 1$ and satisfying the equation $\frac{\partial^2 z}{\partial y^2} = 6x^3 y$.

Solution

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 6x^3 y$$

Integrating w.r.t. y keeping x constant,

$$\frac{\partial z}{\partial y} = 3x^3 y^2 + f(x)$$

Integrating w.r.t. y keeping x constant,

$$z = x^3 y^3 + yf(x) + g(x)$$

Since the surface passe through $y = 0$, $z = 0$,

$$g(x) = 0$$

The surface also passes through $y = 1$, $z = 1$.

$$1 = x^3 + f(x)$$

$$f(x) = 1 - x^3$$

Hence, $$z = x^3 y^3 + y(1 - x^3)$$

Example 5

Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$. **[Winter 2014; Summer 2018]**

Solution

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x\,\partial y}\right) = \cos(2x + 3y)$$

Integrating w.r.t. x keeping y constant,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x + 3y)}{2} + f(y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{1}{2}\sin(2x + 3y) + f(y)$$

Integrating w.r.t. x keeping y constant,

$$\frac{\partial z}{\partial y} = -\frac{1}{2}\frac{\cos(2x + 3y)}{2} + x\,f(y) + g(y)$$

Integrating w.r.t. y keeping x constant,

$$z = -\frac{1}{4}\frac{\sin(2x+3y)}{3} + x\int f(y)\,dy + \int g(y)\,dy + \phi(x)$$

EXERCISE 6.2

Solve the following equations:

1. $\dfrac{\partial^2 z}{\partial x \partial y} = \cos x \cos y$

   [Ans.: $z = \sin x \sin y + f(x) + \phi(y)$**]**

2. $\dfrac{\partial^2 z}{\partial x^2} = z$, given that $y = 0$, $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$

   [Ans.: $z = e^y \cosh x + e^{-y} \sinh x$**]**

3. $\dfrac{\partial^2 z}{\partial y^2} = a^2 z$, given that $y = 0$, $\dfrac{\partial z}{\partial y} = a \sin x$ and $\dfrac{\partial z}{\partial x} = 0$

   [Ans.: $z = \sinh ay \sin x + a \cosh ay$**]**

4. $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$, given that $\dfrac{\partial z}{\partial y} = -2\cos y$ when $x = 0$, and $z = 0$ when y is a multiple of π

   [Ans.: $z = -\cos x \sin y - \sin y$**]**

5. $\dfrac{\partial^2 z}{\partial x \partial y} = e^{-y} \cos x$, given that $z = 0$ when $y = 0$ and $\dfrac{\partial z}{\partial y} = 0$ when $x = 0$

   [Ans.: $z = \sin x - e^{-y} \sin x$**]**

6.5 LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A partial differential equation of first order is said to be linear if the dependent variable and its derivatives are of degree one and the products of the dependent variable and its derivatives do not appear in the equation.

The equation is said to be quasi-linear if the degree of the highest-order derivative is one and the products of the highest-order partial derivatives are not present. A quasi-linear partial differential equation is represented as

$$P(x, y, z)\cdot p + Q(x, y, z)\cdot q = R(x, y, z)$$

This equation is known as *Lagrange's linear equation*.

If P and Q are independent of z, and R is linear in z then the equation is known as a *linear equation.*

The general solution of *Lagrange's linear equation* $Pp + Qq = R$ is given by

$$f(u, v) = 0$$

where f is an arbitrary function and u, v are functions of x, y, and z.

Working Rules for Solving Lagrange's Linear Equations

1. Write the given differential equation in the standard form $Pp + Qq = R$
2. Form the Lagrange's auxiliary (subsidiary) equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{...(6.10)}$$

3. Solve the simultaneous equations in Eq. (6.10) to obtain its two independent solutions as $u = c_1$, $v = c_2$.
4. Write the general solution of the given equation as

$$f(u, v) = 0 \quad \text{or} \quad u = \phi(v).$$

Example 1

Solve $xp + yq = 3z$. **[Summer 2018]**

Solution

$$P = x, \quad Q = y, \quad R = 3z$$

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating,

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log \frac{x}{y} = \log c_1$$

$$\frac{x}{y} = c_1 \qquad \text{...(2)}$$

Taking the second and third fractions from Eq. (1),

$$\frac{dy}{y} = \frac{dz}{3z}$$

$$\frac{3}{y}dy = \frac{dz}{z}$$

Integrating,

$$3\log y = \log z + \log c_2$$

$$\log y^3 - \log z = \log c_2$$

$$\log \frac{y^3}{z} = \log c_2$$

$$\frac{y^3}{z} = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left(\frac{x}{y}, \frac{y^3}{z}\right) = 0$$

Example 2

Solve $pz - qz = z^2 + (x + y)^2$. **[Winter 2013]**

Solution

$$P = z, \quad Q = -z, \quad R = z^2 + (x + y)^2$$

The auxiliary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{z} = \frac{dy}{-z}$$

$$dx = -dy$$

Integrating,

$$x = -y + c_1$$

$$x + y = c_1 \qquad \text{...(2)}$$

Taking the first and third fractions from Eq. (1),

$$\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2}$$

$$\frac{dx}{z} = \frac{dz}{z^2 + c_1^2} \qquad \text{[From Eq. (2)]}$$

$$dx = \frac{z\,dz}{z^2 + c_1^2}$$

$$2\,dx = \frac{2z\,dz}{z^2 + c_1^2}$$

Integrating,

$$\log(z^2 + c_1^2) = 2x + c_2$$

$$\log(z^2 + c_1^2) - 2x = c_2$$

$$\log[z^2 + (x + y)^2] - 2x = c_2$$

$$\log(x^2 + y^2 + z^2 + 2xy) - 2x = c_2 \qquad ...(3)$$

From Eqs (2) and (3), the general solution is

$$f\left[(x + y), \log(x^2 + y^2 + z^2 + 2xy) - 2x\right] = 0$$

Example 3

Find the general solution to the partial differential equation $xp + yq = x - y$. **[Winter 2015]**

Solution

$$P = x, \qquad Q = y, \qquad R = x - y$$

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x - y} \qquad ...(1)$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating,

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1 \qquad ...(2)$$

Using the multipliers 1, –1, –1,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x - y} = \frac{dx - dy - dz}{x - y - x + y} = \frac{dx - dy - dz}{0}$$

$$dx - dy - dz = 0$$

Integrating,

$$x - y - z = c_2 \qquad ...(3)$$

From Eqs (2) and (3), the general solution is

$$f\left(\frac{x}{y}, x - y - z\right) = 0$$

Example 4

Solve $yzp - xzq = xy$.

Solution

$$P = yz, \qquad Q = -xz, \qquad R = xy$$

The auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy} \qquad \text{...(1)}$$

Taking the first and second fractions in Eq. (1),

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

$$x\,dx + y\,dy = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$x^2 + y^2 = c_1 \qquad \text{...(2)}$$

where $c_1 = 2c$

Taking the second and third fractions in Eq. (1),

$$\frac{dy}{-xz} = \frac{dz}{xy}$$

$$y\,dy + z\,dz = 0$$

Integrating,

$$\frac{y^2}{2} + \frac{z^2}{2} = c'$$

$$y^2 + z^2 = c_2 \qquad \text{...(3)}$$

where $c_2 = 2c'$

From Eqs (2) and (3), the general solution is

$$f(x^2 + y^2, y^2 + z^2) = 0$$

Example 5

Solve $(z - y)p + (x - z)q = y - x$. **[Summer 2015]**

Solution

$$P = z - y, \qquad Q = x - z, \qquad R = y - x$$

The auxiliary equations are

$$\frac{dx}{z - y} = \frac{dy}{x - z} = \frac{dz}{y - x}$$

$$\text{Each fraction} = \frac{dx+dy+dz}{-y+z+x-z+y-x} = \frac{dx+dy+dz}{0} \quad \text{...(1)}$$

$$dx + dy + dz = 0$$

Integrating,

$$x+y+z=c_1 \quad \text{...(2)}$$

Taking multipliers x, y, z in the pairs,

$$\text{Each fraction} = \frac{x\,dx+ydy+zdz}{-xy+zx+xy-zy+yz-xz} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$$x\,dx + y\,dy + zdz = 0$$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_2'$$

$$x^2 + y^2 + z^2 = 2c_2' = c_2 \quad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f[x + y + z, x^2 + y^2 + z^2] = 0$$

Example 6

Solve $(y + z)p + (z + x)q = x + y.$ **[Winter 2014]**

Solution

$$P = y + z, \qquad Q = z + x, \qquad R = x + y$$

The auxiliary equations are

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}$$

Each of these fractions is equal to

$$\frac{dx-dy}{y-x}=\frac{dy-dz}{z-y}=\frac{dx+dy+dz}{2(x+y+z)} \quad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx-dy}{y-x}=\frac{dy-dz}{z-y}$$

$$\frac{d(x-y)}{x-y}-\frac{d(y-z)}{y-z}=0$$

Integrating,

$$\log(x-y)-\log(y-z)=\log c_1$$

$$\log\left(\frac{x-y}{y-z}\right)=\log c_1$$

$$\frac{x-y}{y-z}=c_1 \quad \text{...(2)}$$

Taking the first and third fractions from Eq. (1),

$$\frac{dx - dy}{y - x} = \frac{dx + dy + dz}{2(x + y + z)}$$

$$\frac{dx + dy + dz}{x + y + z} + \frac{2(dx - dy)}{x - y} = 0$$

$$\frac{d(x + y + z)}{x + y + z} + \frac{2d(x - y)}{x - y} = 0$$

Integrating,

$$\log(x + y + z) + 2\log(x - y) = \log c_2$$

$$(x + y + z)(x - y)^2 = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left[\frac{x - y}{y - z}, (x + y + z)(x - y)^2\right] = 0$$

Example 7

Solve $x^2p + y^2q = z^2$. **[Summer 2016]**

Solution

$$P = x^2, \quad Q = y^2, \quad R = z^2$$

The auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating,

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1 \qquad \text{...(2)}$$

Taking the second and third fractions from Eq. (1),

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

Integrating,

$$-\frac{1}{y} = -\frac{1}{z} + c_2$$

$$\frac{1}{z}-\frac{1}{y}=c_2 \qquad ...(3)$$

From Eqs (2) and (3), the general solution is

$$\int\left(\frac{1}{y}-\frac{1}{x},\ \frac{1}{z}-\frac{1}{y}\right)=0$$

Example 8

Solve $(x^2-y^2-z^2)p+2xyq=2xz.$ **[Winter 2016; Summer 2014]**

Solution

$$P=x^2-y^2-z^2,\quad Q=2xy,\quad R=2xz$$

The auxiliary equations are

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$$

Taking x, y, z as multipliers,

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}=\frac{x\,dx+y\,dy+z\,dz}{x^3-xy^2-xz^2+2xy^2+2xz^2}=\frac{x\,dx+y\,dy+z\,dz}{x^3+xy^2+xz^2} \qquad ...(1)$$

Taking the second and third fractions from Eq. (1),

$$\frac{dy}{2xy}=\frac{dz}{2xz}$$

$$\frac{dy}{y}=\frac{dz}{z}$$

Integrating,

$$\log y=\log z+\log c_1$$

$$\log\left(\frac{y}{z}\right)=\log c_1$$

$$\frac{y}{z}=c_1 \qquad ...(2)$$

Taking the third and fifth fractions from Eq. (1),

$$\frac{dz}{2xz}=\frac{x\,dx+y\,dy+z\,dz}{x^3+xy^2+xz^2}$$

$$\frac{dz}{2xz}=\frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$$

$$\frac{dz}{z}=\frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}$$

Integrating,

$$\log z = \log(x^2 + y^2 + z^2) + \log c_2$$

$$\log\left(\frac{z}{x^2 + y^2 + z^2}\right) = \log c_2$$

$$\frac{z}{x^2 + y^2 + z^2} = c_2 \qquad ...(3)$$

From Eqs (2) and (3), the general solution is

$$f\left(\frac{y}{z}, \frac{z}{x^2 + y^2 + z^2}\right) = 0$$

Example 9

Solve $x(y - z)p + y(z - x)q = z(x - y)$. **[Summer 2013]**

Solution

$$P = x(y - z), \quad Q = y(z - x), \quad R = z(x - y)$$

The auxiliary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{dx + dy + dz}{xy - xz + yz - xy + zx - zy} = \frac{dx + dy + dz}{0}$$

$$dx + dy + dz = 0$$

Integrating,

$$x + y + z = c_1 \qquad ...(1)$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating,

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2$$

$$xyz = c_2 \qquad ...(2)$$

From Eqs (1) and (2), the general solution is

$$f(x + y + z, xyz) = 0$$

Example 10

Solve $y^2p - xyq = x(z-2y)$ *where* $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$. **[Summer 2017]**

Solution

$$P = y^2, \quad Q = -xy, \quad R = zx - 2xy$$

The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{zx - 2xy} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\,dx + y\,dy = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$x^2 + y^2 = c_1 \quad \text{where } 2c = c_1 \qquad \text{...(2)}$$

Taking 0, –2, 1 as multipliers,

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{xz - 2xy} = \frac{-2dy + dz}{2xy + xz - 2xy}$$

Taking the second and fourth fractions,

$$\frac{-2dy + dz}{xz} = \frac{dy}{-xy}$$

$$\frac{-2dy}{z} + \frac{dz}{z} = -\frac{dy}{y}$$

$$-2y\,dy + y\,dz = -z\,dy$$

$$z\,dy + y\,dz = 2y\,dy$$

$$d(yz) = 2y\,dy$$

Integrating,

$$yz = y^2 + c_2$$

$$yz - y^2 = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f(x^2 + y^2,\ yz - y^2) = 0$$

Example 11

Solve $z(z^2 + xy)(px - qy) = x^4$.

Solution

Rewriting the equation in the $Pp + Qq = R$ form,

$$xz(z^2 + xy)p - yz(z^2 + xy)q = x^4$$

$$P = xz(z^2 + xy), \quad Q = -yz(z^2 + xy), \quad R = x^4$$

The auxiliary equations are

$$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)}$$

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating,

$$\log x + \log y = \log c_1$$

$$\log xy = \log c_1$$

$$xy = c_1 \qquad \text{...(2)}$$

Taking the first and third fractions in Eq. (1),

$$\frac{dx}{xz(z^2 + xy)} = \frac{dz}{x^4}$$

$$x^3 dx = z(z^2 + xy)dz$$

Putting $xy = c_1$ from Eq. (2),

$$x^3 dx = z(z^2 + c_1)\,dz$$

$$x^3 dx - (z^3 + c_1 z)\,dz = 0$$

Integrating,

$$\frac{x^4}{4} - \frac{z^4}{4} - c_1 \frac{z^2}{2} = c$$

$$x^4 - z^4 - 2c_1 z^2 = 4c = c_2 \quad \text{where} \quad 4c = c_2$$

Putting $c_1 = xy$ from Eq. (2),

$$x^4 - z^4 - 2xyz^2 = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left(xy,\, x^4 - z^4 - 2xyz^2\right) = 0$$

Example 12

Solve $\quad x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$.

Solution

$$P = x(y^2+z), \quad Q = -y(x^2+z), \quad R = z(x^2-y^2)$$

The auxiliary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y^2+z)-(x^2+z)+(x^2-y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating,

$$\log x + \log y + \log z = \log c_1$$

$$\log xyz = \log c_1$$

$$xyz = c_1 \qquad \text{...(1)}$$

Taking $x, y, -1$ as multipliers,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

$$x\,dx + y\,dy - dz = 0$$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}-z=c$$

$$x^2+y^2-2z=2c=c_2 \qquad \text{...(2)}$$

From Eqs (1) and (2), the general solution is

$$f(xyz,\ x^2+y^2-2z)=0$$

Example 13

Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$. **[Winter 2017]**

Solution

$$P=x^2-yz, \qquad Q=y^2-zx, \qquad R=z^2-xy$$

The auxiliary equations are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

Each of these fractions is equal to

$$\frac{dx-dy}{(x^2-yz)-(y^2-zx)}=\frac{dy-dz}{(y^2-zx)-(z^2-xy)}=\frac{dz-dx}{(z^2-xy)-(x^2-yz)}$$

$$\frac{dx-dy}{(x^2-y^2)+z(x-y)}=\frac{dy-dz}{(y^2-z^2)+x(y-z)}=\frac{dz-dx}{(z^2-x^2)+y(z-x)}$$

$$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(y+z+x)}=\frac{dz-dx}{(z-x)(z+x+y)} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$$

$$\frac{d(x-y)}{x-y}-\frac{d(y-z)}{y-z}=0$$

Integrating,

$$\log(x-y)-\log(y-z)=\log c_1$$

$$\log\left(\frac{x-y}{y-z}\right)=\log c_1$$

$$\frac{x-y}{y-z}=c_1 \qquad \text{...(2)}$$

Taking the second and third fractions from Eq. (1),

$$\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}$$

$$\frac{d(y-z)}{y-z}-\frac{d(z-x)}{z-x}=0$$

Integrating,

$$\log(y-z)-\log(z-x)=\log c_2$$

$$\log\left(\frac{y-z}{z-x}\right)=\log c_2$$

$$\frac{y-z}{z-x}=c_2 \qquad ...(3)$$

From Eqs (2) and (3), the general solution is

$$f\left(\frac{x-y}{y-z},\ \frac{y-z}{z-x}\right)=0$$

Example 14

Solve $z-xp-yq=a\sqrt{x^2+y^2+z^2}$.

Solution

Rewriting the equation in the $Pp+Qq=R$ form,

$$xp+yq=z-a\sqrt{x^2+y^2+z^2}$$

$$P=x,\ Q=y,\ R=z-a\sqrt{x^2+y^2+z^2}$$

The auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-a\sqrt{x^2+y^2+z^2}}=\frac{xdx+ydy+zdz}{x^2+y^2+z^2-za\sqrt{x^2+y^2+z^2}} \qquad ...(1)$$

Let $x^2+y^2+z^2=u^2$ ...(2)

Differentiating Eq. (2),

$$2xdx+2ydy+2zdz=2udu$$

$$xdx+ydy+zdz=udu$$

Substituting in Eq. (1),

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-au}=\frac{udu}{u^2-azu}$$

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-au}=\frac{du}{u-az}=\frac{dz+du}{(z-au)+(u-az)}$$
$$=\frac{dz+du}{(z+u)-a(u+z)}=\frac{dz+du}{(z+u)(1-a)}$$

$$\therefore \qquad \frac{dx}{x}=\frac{dy}{y}=\frac{dz+du}{(1-a)(z+u)} \qquad ...(3)$$

Taking the first and second fractions in Eq. (3),

$$\frac{dx}{x}=\frac{dy}{y}$$
$$\frac{dx}{x}-\frac{dy}{y}=0$$

Integrating,

$$\log x-\log y=\log c_1$$
$$\log\frac{x}{y}=\log c_1$$
$$\frac{x}{y}=c_1 \qquad ...(4)$$

Taking the first and third fractions in Eq. (3),

$$\frac{dx}{x}=\frac{dz+du}{(1-a)(z+u)}$$
$$(1-a)\frac{dx}{x}-\frac{d(z+u)}{z+u}=0$$

Integrating,

$$(1-a)\log x-\log(z+u)=\log c_2$$
$$\log\left(\frac{x^{1-a}}{z+u}\right)=\log c_2$$
$$\frac{x^{1-a}}{z+u}=c_2$$
$$\frac{x^{1-a}}{z+\sqrt{x^2+y^2+z^2}}=c_2 \qquad ...(5)$$

From Eqs (4) and (5), the general solution is

$$f\left(\frac{x}{y},\quad \frac{x^{1-a}}{z+\sqrt{x^2+y^2+z^2}}\right)=0$$

EXERCISE 6.3

Solve the following:

1. $\dfrac{y^2zp}{x} + zxq = y^2$

$$\left[\textbf{Ans.:}\ f\left(x^3 - y^3,\ x^2 - z^2\right) = 0\right]$$

2. $p - q = \log(x + y)$

$$\left[\textbf{Ans.:}\ f\left[x + y,\ x\log(x + y) - z\right] = 0\right]$$

3. $xzp + yzq = xy$

$$\left[\textbf{Ans.:}\ f\left(\frac{x}{y},\ xy - z^2\right) = 0\right]$$

4. $\left(y^2 + z^2\right)p - xyq + zx = 0$

$$\left[\textbf{Ans.:}\ f\left(\frac{y}{z},\ x^2 + y^2 + z^2\right) = 0\right]$$

5. $p + 3q = 5z + \tan(y - 3x)$

$$\left[\textbf{Ans.:}\ f\left(y - 3x,\ e^{-5x}\left\{5z + \tan(y - 3x)\right\}\right) = 0\right]$$

6. $\left(x^2 - y^2 - z^2\right)p + 2xyq = 2xz$

$$\left[\textbf{Ans.:}\ f\left(x^2 + y^2 + z^2,\ \frac{y}{z}\right) = 0\right]$$

7. $(y + z)p + (z + x)q = x + y$

$$\left[\textbf{Ans.:}\ f\left(\frac{x - y}{y - z},\ \frac{y - z}{\sqrt{x + y + z}}\right) = 0\right]$$

8. $\left(z^2 - 2yz - y^2\right)p + (xy + zx)q = xy - zx$

$$\left[\textbf{Ans.:}\ f\left(x^2 + y^2 + z^2,\ y^2 - 2yz - z^2\right) = 0\right]$$

9. $\dfrac{(y - z)p}{yz} + \dfrac{(z - x)q}{zx} = \dfrac{x - y}{xy}$

$$\left[\textbf{Ans.:}\ f\left(x + y + z,\ xyz\right) = 0\right]$$

10. $x^2(y - z)p + (z - x)y^2q = z^2(x - y)$

$$\left[\textbf{Ans.:}\ f\left(xyz,\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0\right]$$

11. $p - 2q = 3x^2 \sin(y + 2x)$

$$\left[\textbf{Ans.:} \ f\left(2x + y,\ x^3 \sin(y + 2x) - z\right) = 0\right]$$

12. $p \tan x + q \tan y = \tan z$

$$\left[\textbf{Ans.:} \ f\left(\frac{\sin z}{\sin y}, \frac{\sin x}{\sin y}\right) = 0\right]$$

13. $(mz - ny)p + (nx - lz)q = ly - mx$

$$\left[\textbf{Ans.:} \ f\left(x^2 + y^2 + z^2,\ lx + my + nz\right)\right]$$

14. $z(p - q) = z^2 + (x + y)^2$

$$\left[\textbf{Ans.:} \ f\left[x + y,\ e^{2y}\left\{z^2 + (x + y)^2\right\}\right] = 0\right]$$

6.6 NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A partial differential equation of first order is said to be nonlinear if p and q have degree more than one.

The complete solution of a nonlinear equation is given by

$$f(x, y, z, a, b) = 0$$

where a and b are two arbitrary constants. Four standard forms of these equations are as follows:

6.6.1 Form I $f(p, q) = 0$

Let the equation be $\quad f(p, q) = 0 \qquad$ ...(6.11)

Assuming $p = a$, Eq. (6.11) reduces to

$$f(a, q) = 0$$

Solving for q,

$$q = \phi(a)$$

Now,

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{dy} dy \\ &= p\,dx + q\,dy \\ dz &= a\,dx + q\,dy \end{aligned}$$

Integrating,

$$\begin{aligned} z &= ax + qy + c \\ &= ax + \phi(a)y + c \end{aligned}$$

where a and c are arbitrary constants.

Hence, the complete solution is

$$z = ax + by + c$$

where $b = \phi(a)$, i.e., a and b satisfy the equation $f(a, b) = 0$.

Example 1

Solve $\sqrt{p} + \sqrt{q} = 1$. **[Winter 2014]**

Solution

The equation is of the form $f(p, q) = 0$.

$$f(p, q) = \sqrt{p} + \sqrt{q} - 1$$

The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation

$$f(a,b) = 0$$

$$\sqrt{a} + \sqrt{b} - 1 = 0$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1 - \sqrt{a})^2$$

Hence, the complete solution is

$$z = ax + (1 - \sqrt{a})^2 y + c$$

Example 2

Solve $p + q^2 = 1$. **[Winter 2012]**

Solution

The given equation is of the form $f(p, q) = 0$.

$$f(p, q) = p + q^2 - 1$$

The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a + b^2 - 1 = 0$$

$$b^2 = 1 - a$$

$$b = \sqrt{1 - a}$$

Hence, the complete solution is

$$z = ax + (\sqrt{1 - a})y + c$$

Example 3

Solve $p^2 + q^2 = 1.$ **[Summer 2018]**

Solution

The equation is of the form $f(p, q) = 0$.

$$f(p, q) = p^2 + q^2 - 1$$

The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a^2 + b^2 - 1 = 0$$

$$b = \sqrt{1 - a^2}$$

Hence, the complete solution is

$$z = ax + \sqrt{1 - a^2}\, y + c$$

Example 4

Solve $p^2 + q^2 = npq.$ **[Winter 2014]**

Solution

The equation is of the form $f(p, q) = 0$.

$$f(p, q) = p^2 + q^2 - npq$$

The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a^2 + b^2 - nab = 0$$

$$b^2 - nab + a^2 = 0$$

$$b = \frac{na \pm \sqrt{n^2 a^2 - 4a^2}}{2}$$

$$= \frac{na \pm a\sqrt{n^2 - 4}}{2}$$

$$= a\left(\frac{n \pm \sqrt{n^2 - 4}}{2}\right)$$

Hence, the complete solution is

$$z = ax + \frac{ay}{2}\left(n \pm \sqrt{n^2 - 4}\right) + c$$

Example 5

Solve $x^2p^2 + y^2q^2 = z^2$.

Solution

Rewriting the equation,

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{...(1)}$$

Let $\frac{dx}{x} = dX, \quad \frac{dy}{y} = dY, \quad \frac{dz}{z} = dZ$

$$\log x = X, \quad \log y = Y, \quad \log z = Z$$

Differentiating $\log z = Z$ partially w.r.t. x,

$$\frac{1}{z}\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial X}\cdot\frac{dX}{dx} = \frac{\partial Z}{\partial X}\cdot\frac{1}{x}$$

$$\frac{x}{z}\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial X}$$

Similarly, differentiating $\log z = Z$ partially w.r.t. y,

$$\frac{y}{z}\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial Y}$$

Substituting in Eq. (1),

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$$

$$P^2 + Q^2 = 1$$

The equation is of the form $f(P, Q) = 0$.

$$f(P, Q) = P^2 + Q^2 - 1$$

The complete solution is

$$Z = aX + bY + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a^2 + b^2 - 1 = 0$$

$$b = \sqrt{1 - a^2}$$

Hence, the complete solution is

$$Z = aX + \sqrt{1-a^2}\,Y + c$$

$$\log z = a\log x + \sqrt{1-a^2}\,\log y + c$$

Example 6

Solve $(x^2+y^2)(p^2+q^2) = 1.$

Solution

Let $x = r\cos\theta, \quad y = r\sin\theta$

$$r^2 = x^2 + y^2 \qquad\qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$2r\frac{\partial r}{\partial x} = 2x \qquad\qquad \frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad\qquad = -\frac{\sin\theta}{r}$$

and $$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \text{ and } \frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}$$

$$= \frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial\theta}\left(-\frac{\sin\theta}{r}\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial z}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$

$$= \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial\theta}\left(\frac{\cos\theta}{r}\right)$$

$$p^2 + q^2 = \left[\frac{\partial z}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}\right]^2 + \left[\frac{\partial z}{\partial r}\sin\theta + \frac{\cos\theta}{r}\frac{\partial z}{\partial\theta}\right]^2$$

$$= \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2$$

Substituting in the given equation,

$$(x^2+y^2)(p^2+q^2)=1$$

$$r^2\left[\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2\right]=1$$

$$\left(r\frac{\partial z}{\partial r}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=1 \qquad \text{...(1)}$$

Let $\qquad \dfrac{\mathrm{d}r}{r}=\mathrm{d}R$

$$\log r = R$$

Differentiating w.r.t. z,

$$\frac{1}{r}\frac{\partial r}{\partial z}=\frac{\partial R}{\partial z}$$

$$r\frac{\partial z}{\partial r}=\frac{\partial z}{\partial R}$$

Substituting in Eq. (1),

$$\left(\frac{\partial z}{\partial R}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=1$$

$$P^2+Q^2=1$$

The equation is of the form $f(P, Q) = 0$.

$$f(P, Q) = P^2 + Q^2 - 1$$

The complete solution is

$$z = aR + b\theta + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a^2+b^2-1=0$$

$$b=\sqrt{1-a^2}$$

Hence, the complete solution is

$$z = aR+\sqrt{1-a^2}\,\theta+c$$

$$= a\log r+\sqrt{1-a^2}\,\theta+c$$

$$= a\log\sqrt{x^2+y^2}+\sqrt{1-a^2}\,\tan^{-1}\left(\frac{y}{x}\right)+c$$

6.6.2 Form II $f(z, p, q) = 0$

Let the equation be $f(z, p, q) = 0$ ...(6.12)

Assuming $q = ap$, Eq. (6.12) reduces to

$$f(z, p, ap) = 0$$

Solving for p,

$$p = \phi(z)$$

Now,

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= p\,dx + q\,dy \\ &= p\,dx + ap\,dy \\ &= p(dx + a\,dy) \\ &= \phi(z)(dx + a\,dy) \end{aligned}$$

Integrating,

$$\int \frac{dz}{\phi(z)} = x + ay + b$$

which gives the complete solution of Eq. (6.12), where a and b are arbitrary constants.

Example 1

Solve $z^2(p^2z^2 + q^2) = 1.$

Solution

$$z^2(p^2z^2 + q^2) = 1 \qquad ...(1)$$

Putting $q = ap$, Eq. (1) reduces to

$$z^2(p^2z^2 + a^2p^2) = 1$$

$$p^2 = \frac{1}{z^2(z^2 + a^2)}$$

$$p = \frac{1}{z\sqrt{z^2 + a^2}}$$

Now,

$$\begin{aligned} dz &= p\,dx + q\,dy \\ &= p\,dx + ap\,dy \\ &= p(dx + a\,dy) \\ &= \frac{1}{z\sqrt{z^2 + a^2}}(dx + a\,dy) \end{aligned}$$

$$z\sqrt{z^2 + a^2}\,dz = dx + a\,dy$$

Integrating,

$$\int (z^2 + a^2)^{\frac{1}{2}} \frac{2z}{2} \mathrm{d}z = x + ay + b$$

$$\frac{1}{2} \frac{(z^2 + a^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} = x + ay + b \quad \left[\because \int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$(z^2 + a^2)^{\frac{3}{2}} = 3(x + ay + b)$$

$$(z^2 + a^2)^3 = 9(x + ay + b)^2$$

which gives the complete solution of the given equation.

Example 2

Solve $p(1 + q) = qz.$ **[Summer 2014]**

Solution

$$p(1 + q) = qz \qquad \text{...(1)}$$

Putting $q = ap$, Eq. (1) reduces to

$$p(1 + ap) = apz$$

$$1 + ap = az$$

$$p = \frac{az - 1}{a}$$

$$= z - \frac{1}{a}$$

Now,

$$\mathrm{d}z = p\,\mathrm{d}x + q\,\mathrm{d}y$$

$$= p\,\mathrm{d}x + ap\,\mathrm{d}y$$

$$= p\,(\mathrm{d}x + a\,\mathrm{d}y)$$

$$= p\,(\mathrm{d}x + a\,\mathrm{d}y)$$

$$= \left(z - \frac{1}{a}\right)(\mathrm{d}x + a\,\mathrm{d}y)$$

$$= \left(\frac{az - 1}{a}\right)(\mathrm{d}x + a\,\mathrm{d}y)$$

$$\frac{a\,\mathrm{d}z}{(az - 1)} = \mathrm{d}x + a\,\mathrm{d}y$$

Integrating,

$$\log\,(az - 1) = x + ay + b$$

which gives the complete solution of the given equation.

Example 3

Solve $q^2y^2 = z(z - px)$.

Solution

Rewriting the equation,

$$\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right) \qquad \text{...(1)}$$

Let $\dfrac{dx}{x} = dX$, $\qquad \dfrac{dy}{y} = dY$

$\log x = X$, $\qquad \log y = Y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{dX}{dx} = \frac{\partial z}{\partial X}\cdot\frac{1}{x}$$

$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} = P, \text{ say}$$

Also,

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{dY}{dy} = \frac{\partial z}{\partial Y}\cdot\frac{1}{y}$$

$$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} = Q, \text{ say}$$

Substituting in Eq. (1),

$$Q^2 = z(z - P) \qquad \text{...(2)}$$

The equation is in the form $f(z, P, Q) = 0$.

Putting $Q = a\,P$ in Eq. (2),

$$a^2P^2 = z(z - P)$$

$$a^2P^2 + zP - z^2 = 0$$

$$P = \frac{-z \pm \sqrt{z^2 + 4a^2z^2}}{2a^2}$$

$$= \frac{z}{2a^2}\left[-1 \pm \sqrt{1 + 4a^2}\right]$$

$$= Az$$

where

$$A = \frac{-1 \pm \sqrt{1 + 4a^2}}{2a^2}$$

Now,
$$\begin{aligned} dz &= \frac{\partial z}{\partial X}dX + \frac{\partial z}{\partial Y}dY \\ &= PdX + QdY \\ &= PdX + aPdY \\ &= P(dX + adY) \\ &= Az(dX + adY) \end{aligned}$$
$$\frac{dz}{Az} = dX + adY$$

Integrating,
$$\begin{aligned} \frac{1}{A}\log z &= X + aY + \log b \\ &= \log x + a\log y + \log b \end{aligned}$$
$$\log z^{\frac{1}{A}} = \log xy^a b$$
$$z^{\frac{1}{A}} = bxy^a$$

which gives the complete solution of the given equation,

where
$$A = \frac{-1 \pm \sqrt{1+4a^2}}{2a^2}$$

6.6.3 Form III $f(x, p) = g(y, q)$

Let the equation be $\quad f(x, p) = g(y, q) \qquad$ ...(6.13)

Let $\quad f(x, p) = a, \quad g(y, q) = a$

Solving these equations for p and q,

$$p = f_1(x), \quad q = g_1(y)$$

Now,
$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= pdx + qdy \\ &= f_1(x)dx + g_1(y)dy \end{aligned}$$

Integrating,
$$z = \int f_1(x)dx + \int g_1(y)dy + b$$

which gives the complete solution of Eq. (6.13).

Example 1

Solve $\quad yp = 2yx + \log q.$

Solution

Dividing the equation by y,

$$p = 2x + \frac{1}{y}\log q$$

$$p - 2x = \frac{1}{y}\log q$$

Let $\quad p - 2x = a, \qquad \frac{1}{y}\log q = a$

$$p = 2x + a, \qquad \log q = ay$$

$$q = e^{ay}$$

Now, $\quad dz = p\,dx + q\,dy$

$$= (2x + a)dx + e^{ay}dy$$

Integrating,

$$z = x^2 + ax + \frac{e^{ay}}{a} + b$$

$$az = ax^2 + a^2x + e^{ay} + ab$$

which gives the complete solution of the given equation.

Example 2

Solve $\quad p - x^2 = q + y^2.$ **[Summer 2015]**

Solution

Let $\quad p - x^2 = q + y^2 = a$

$$p = a + x^2 \quad \text{and} \quad q = a - y^2$$

Now, $\quad dz = p\,dx + q\,dy$

$$= (a + x^2)dx + (a - y^2)dy$$

Integrating,

$$z = \left(ax + \frac{x^3}{3}\right) + \left(ay - \frac{y^3}{3}\right) + b$$

which gives the complete solution of the given equation.

Example 3

Solve $\quad p^2 + q^2 = x + y.$ **[Summer 2014]**

Solution

Rewriting the given equation,

$$p^2 - x = y - q^2$$

Let $p^2 - x = a,$ $\quad y - q^2 = a$

$$p = \sqrt{x+a} \qquad q = \sqrt{u-a}$$

Now,
$$dz = p dx + q dy$$
$$= \sqrt{x+a}\, dx + \sqrt{y-a}\, dy$$

Integrating,
$$z = \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + b$$

which gives the complete solution of the given equation.

Example 4

Solve $p^2 - q^2 = x - y.$ **[Winter 2014; Summer 2018, 2017]**

Solution

$$p^2 - q^2 = x - y$$
$$p^2 - x = q^2 - y$$

Let $p^2 - x = a,$ $\quad q^2 - y = a$

$$p^2 = a + x, \qquad q^2 = a + y$$
$$p = \sqrt{x+a}, \qquad q = \sqrt{y+a}$$

Now,
$$dz = p dx + q\, dy$$
$$= \sqrt{x+a}\, dx + \sqrt{y+a}\, dy$$

Integrating,
$$z = \frac{2}{3}(x+a)^{\frac{2}{3}} + \frac{2}{3}(y+a)^{\frac{2}{3}} + b$$

which gives the complete solution of the given equation.

Example 5

Solve $zpy^2 = x(y^2 + z^2q^2).$

Solution

$$zpy^2 = x(y^2 + z^2q^2) \qquad \text{...(1)}$$

Let
$$z dz = dZ$$
$$\frac{z^2}{2} = Z \qquad \text{...(2)}$$

Differentiating Eq. (2) w.r.t. x,

$$z\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = P, \quad \text{say}$$

$$zp = P$$

Differentiating Eq.(2) w.r.t. y,

$$z\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial y} = Q, \quad \text{say}$$

$$zq = Q$$

Substituting in Eq. (1),

$$Py^2 = x(y^2 + Q^2)$$

$$\frac{P}{x} = \frac{Q^2 + y^2}{y^2}$$

The equation is in the form $f(x, P) = g(y, Q)$.

Let $\qquad \dfrac{P}{x} = a, \qquad \dfrac{Q^2 + y^2}{y^2} = a$

$$P = ax, \qquad Q = y\sqrt{a-1}$$

Now, $\qquad \mathrm{d}Z = \dfrac{\partial Z}{\partial x}\mathrm{d}x + \dfrac{\partial Z}{\partial y}\mathrm{d}y$

$$z\mathrm{d}z = P\mathrm{d}x + Q\mathrm{d}y$$

$$= ax\mathrm{d}x + y\sqrt{a-1}\,\mathrm{d}y$$

Integrating,

$$\frac{z^2}{2} = a\frac{x^2}{2} + \frac{y^2}{2}\sqrt{a-1} + \frac{b}{2}$$

$$z^2 = ax^2 + y^2\sqrt{a-1} + b$$

which gives the complete solution of the given equation.

Example 6

Solve $\quad p^2 + q^2 = z^2(x + y)$. **[Winter 2012]**

Solution

The given equation can be written as

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x + y \qquad \text{...(1)}$$

Let $\qquad Z = \log z$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{1}{z} \cdot p = \frac{p}{z}$$

and
$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{1}{z} \cdot q = \frac{q}{z}$$

Substituting $\frac{p}{z}$ and $\frac{q}{z}$ in Eq. (1),

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x + y$$

$$P^2 + Q^2 = x + y \qquad \text{where } P = \frac{\partial Z}{\partial x} \quad \text{and} \quad Q = \frac{\partial Z}{\partial y}$$

$$P^2 - x = y - Q^2$$

Let
$$P^2 - x = a, \qquad\qquad y - Q^2 = a$$
$$P = \sqrt{x + a}, \qquad\qquad Q = \sqrt{y - a}$$

Now,
$$\mathrm{d}Z = P\mathrm{d}x + Q\mathrm{d}y$$
$$= \sqrt{x+a}\ \mathrm{d}x + \sqrt{y-a}\ \mathrm{d}y$$

Integrating,

$$Z = \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + b$$

$$\log z = \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + b$$

which gives the complete solution of the given equation.

Example 7

Solve $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.

Solution

$$(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1 \qquad \text{...(1)}$$

Let $u = x + y, \quad v = x - y$

Considering z as a function of u and v (Fig. 6.3),

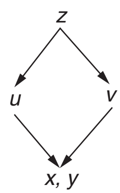

Fig. 6.3

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$p = \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1) = P + Q, \text{ say}$$

where $P = \dfrac{\partial z}{\partial u},\ Q = \dfrac{\partial z}{\partial v}$

Also,
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$
$$q = \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(-1) = P - Q$$

Substituting in Eq. (1),
$$u(2P)^2 + v(2Q)^2 = 1$$
$$4P^2u = 1 - 4Q^2v$$

The equation is in the form $f(u, P) = g(v, Q)$.

Let $\quad 4P^2u = a, \qquad 1 - 4Q^2v = a$
$$P = \frac{\sqrt{a}}{2\sqrt{u}}, \qquad Q = \frac{1}{2}\sqrt{\frac{1-a}{v}}$$

Now,
$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$
$$= P\,du + Q\,dv$$
$$= \frac{\sqrt{a}}{2\sqrt{u}}du + \frac{1}{2}\sqrt{\frac{1-a}{v}}dv$$

Integrating,
$$\int dz = \frac{\sqrt{a}}{2}\int u^{-\frac{1}{2}}du + \frac{\sqrt{1-a}}{2}\int v^{-\frac{1}{2}}dv + b$$
$$z = \frac{\sqrt{a}}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + \frac{\sqrt{1-a}}{2}\left(\frac{v^{\frac{1}{2}}}{\frac{1}{2}}\right) + b$$
$$= \sqrt{au} + \sqrt{1-a}\sqrt{v} + b$$
$$= \sqrt{a(x+y)} + \sqrt{(1-a)(x-y)} + b$$

which gives the complete solution of the given equation.

6.6.4 Form IV (Clairaut Equation)

Let the equation be $\quad z = px + qy + f(p, q)$

The complete solution of this equation is
$$z = ax + by + f(a, b) \qquad \text{...(6.14)}$$

which is obtained by replacing p by a and q by b in Eq. (6.14).

Example 1

Solve $z = px + qy - 2\sqrt{pq}$. **[Winter 2013]**

Solution

The given equation is of the form

$$z = px + qy + f(p, q)$$

Hence, the complete solution is obtained by replacing p by a and q by b in the given equation.

$$z = ax + by - 2\sqrt{ab}$$

Example 2

Solve $z = px + qy + p^2q^2$. **[Summer 2013]**

Solution

The given equation is of the form

$$z = px + qy + f(p, q)$$

Hence, the complete solution is obtained by replacing p by a and q by b in the given equation.

$$z = ax + by + a^2b^2$$

Example 3

Solve $z = px + qy + c\sqrt{1 + p^2 + q^2}$.

Solution

The given equation is of the form

$$z = px + qy + f(p, q)$$

Hence, the complete solution is obtained by replacing p by a and q by b in the given equation.

$$z = ax + by + c\sqrt{1 + a^2 + b^2}$$

Example 4

Solve $(p - q)(z - px - qy) = 1$.

Solution

Rewriting the given equation in Clairaut's form,

$$z - px - qy = \frac{1}{p - q}$$

$$z = px + qy + \frac{1}{p-q}$$

The given equation is of the form

$$z = px + qy + f(p, q)$$

Hence, the complete solution is obtained by replacing p by a and q by b in the given equation.

$$z = ax + by + \frac{1}{a-b}$$

EXERCISE 6.4

Find the complete solutions of the following equations:

Form I

1. $q = 3p^2$

$$\left[\textbf{Ans.:}\ z = ax + 3a^2 y + c\right]$$

2. $p^2 - q^2 = 4$

$$\left[\textbf{Ans.:}\ z = ax + y\sqrt{a^2 - 4} + c\right]$$

3. $p + q = pq$

$$\left[\textbf{Ans.:}\ z = ax + \frac{ay}{a-1} + c\right]$$

4. $p = e^q$

$$\left[\textbf{Ans.:}\ z = ax + y \log a + c\right]$$

5. $(y - x)(qy - px) = (p - q)^2$

$$\left[\textbf{Ans.:}\ z = b^2(x + y) + bxy + c\right]$$

Form II

1. $p(1 + q) = qz$

$$\left[\textbf{Ans.:}\ \log(az - 1) = x + ay + b\right]$$

2. $p^3 + q^3 = 27z$

$$\left[\textbf{Ans.:}\ (1 + a^3)z^2 = 8(x + ay + b)^3\right]$$

3. $p(1 + q^2) = q(z - k)$

$$\left[\textbf{Ans.:}\ 4a(z - k) = 4 + (x + ay + c)^2\right]$$

4. $z^2(p^2x^2 + q^2) = 1$

$$\left[\textbf{Ans.:}\ z^2\sqrt{1 + a^2} = \pm 2(\log x + ay) + b\right]$$

5. $pq = x^m y^n z^l$

$$\left[\textbf{Ans.:}\ \frac{z^{-\frac{l}{2}+1}}{-\frac{l}{2}+1} = \frac{1}{\sqrt{a}}\left(\frac{x^{m+1}}{m+1} + a\frac{y^{n+1}}{n+1}\right) + b\right]$$

Form III

1. $\sqrt{p} + \sqrt{q} = 2x$

$$\left[\textbf{Ans.:}\ z = \frac{1}{6}(a+2x)^3 + a^2 y + b\right]$$

2. $q = xyp^2$

$$\left[\textbf{Ans.:}\ 16ax = \left(2z - ay^2 - 2b\right)^2\right]$$

3. $z\left(p^2 - q^2\right) = x - y$

$$\left[\textbf{Ans.:}\ z^{\frac{3}{2}} = (x+a)^{\frac{3}{2}} + (y+a)^{\frac{3}{2}} + c\right]$$

4. $p + q = \sin x + \sin y$

$$\left[\textbf{Ans.:}\ z = ax - \cos x - \cos y - ay + b\right]$$

5. $y^2 q^2 - xp + 1 = 0$

$$\left[\textbf{Ans.:}\ z = \left(a^2 + 1\right)\log x + a\log y + b\right]$$

Form IV

1. $z = px + qy - p^2 q$

$$\left[\textbf{Ans.:}\ z = ax + by - a^2 b\right]$$

2. $z = px + qy - pq$

$$\left[\textbf{Ans.:}\ z = ax + by - ab\right]$$

3. $pqz = p^2\left(xq + p^2\right) + q^2\left(yp + q^2\right)$

$$\left[\textbf{Ans.:}\ z = ax + by + \left(\frac{a^3}{b} + \frac{b^3}{a}\right)\right]$$

4. $(px + qy - z)^2 = d\left(1 + p^2 + q^2\right)$

$$\left[\textbf{Ans.:}\ z = ax + by \pm \mathrm{d}\sqrt{1 + a^2 + b^2}\right]$$

5. $4xyz = pq + 2px^2 y + 2qxy^2$

$$\left[\textbf{Ans.:}\ z = ax^2 + by^2 + ab\right]$$

6.7 CHARPIT'S METHOD

This method is a general method to find the complete solution of a first-order nonlinear partial differential equation. This method is applied to solve those equations that cannot be reduced to any of the standard forms.

Let the given equation be $f(x, y, z, p, q) = 0$ ...(6.15)

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$
$$= p\,dx + q\,dy \qquad \text{...(6.16)}$$

To integrate Eq. (6.16), p and q must be in terms of x, y, and z. For this purpose, let us assume another relation in x, y, z, p, and q as

$$g(x, y, z, p, q) = 0 \qquad \text{...(6.17)}$$

p and q are obtained on solving Eqs (6.15) and (6.17). Substituting p and q in Eq. (6.16) and then integrating the equation, the complete solution of Eq. (6.15) is obtained.

To determine g, differentiating Eqs (6.15) and (6.17) w.r.t. x and y,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} = 0 \qquad \text{...(6.18)}$$

and

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}p + \frac{\partial g}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial x} = 0 \qquad \text{...(6.19)}$$

Also,

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y} = 0 \qquad \text{...(6.20)}$$

$$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}q + \frac{\partial g}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial y} = 0 \qquad \text{...(6.21)}$$

Eliminating $\frac{\partial p}{\partial x}$ from Eqs (6.18) and (6.19), by multiplying Eq. (6.18) with $\frac{\partial g}{\partial p}$ and Eq. (6.19) with $\frac{\partial f}{\partial p}$ and subtracting,

$$\left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial p} - \frac{\partial g}{\partial x}\cdot\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial p} - \frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\cdot\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \qquad \text{...(6.22)}$$

Similarly, eliminating $\frac{\partial q}{\partial y}$ from Eqs (6.20) and (6.21),

$$\left(\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial q} - \frac{\partial g}{\partial y}\cdot\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial q} - \frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\cdot\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \qquad \text{...(6.23)}$$

Adding Eqs (6.22) and (6.23) and using $\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial p}{\partial y}$,

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial g}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial y}=0$$

$$f_p\frac{\partial g}{\partial x}+f_q\frac{\partial g}{\partial y}+(pf_p+qf_q)\frac{\partial g}{\partial z}-(f_x+pf_z)\frac{\partial g}{\partial p}-(f_y+qf_z)\frac{\partial g}{\partial q}=0 \quad ...(6.24)$$

where $f_p=\frac{\partial f}{\partial p}$, $f_q=\frac{\partial f}{\partial q}$, $f_x=\frac{\partial f}{\partial x}$, $f_y=\frac{\partial f}{\partial y}$, $f_z=\frac{\partial f}{\partial z}$

Equation (6.24) is Lagrange's linear partial differential equation in g. Its subsidiary equations are

$$\frac{\mathrm{d}x}{f_p}=\frac{\mathrm{d}y}{f_q}=\frac{\mathrm{d}z}{pf_p+qf_q}=\frac{\mathrm{d}p}{-(f_x+pf_z)}=\frac{\mathrm{d}q}{-(f_y+qf_z)}=\frac{\mathrm{d}g}{0}$$

These equations are known as *Charpit's equations*. Solving these equations, p and q are obtained. The simplest of the relations should be taken to obtain p and q easily.

Example 1

Solve $px + qy = pq$. **[Summer 2016]**

Solution

$$px+qy=pq \quad ...(1)$$

Let $f(x, y, z, p, q) = px + qy - pq = 0$

The auxiliary equations are

$$\frac{\mathrm{d}x}{f_p}=\frac{\mathrm{d}y}{f_q}=\frac{\mathrm{d}z}{pf_p+qf_q}=\frac{\mathrm{d}p}{-(f_x+pf_z)}=\frac{\mathrm{d}q}{-(f_y+qf_z)}$$

$$\frac{\mathrm{d}x}{x-q}=\frac{\mathrm{d}y}{y-p}=\frac{\mathrm{d}z}{p(x-q)+q(y-p)}=\frac{\mathrm{d}p}{-p}=\frac{\mathrm{d}q}{-q} \quad ...(2)$$

Taking the last two fractions in Eq. (2),

$$\frac{\mathrm{d}p}{p}=\frac{\mathrm{d}q}{q}$$

Integrating,

$$\log p=\log q+\log a$$

$$p=qa \quad ...(3)$$

Putting $p = aq$ in Eq. (1),

$$aqx+qy=aq^2$$

$$q=\frac{y+ax}{a}$$

Putting q in Eq (3),

$$p = y + ax$$

Now,
$$dz = p\,dx + q\,dy$$
$$= (y+ax)dx + \left(\frac{y+ax}{a}\right)dy$$

$$a\,dz = (y+ax)(dy + a\,dx)$$

Integrating,

$$az = \frac{(y+ax)^2}{2} + b$$

which gives the complete solution of the given equation.

Example 2

Solve $p = (z + qy)^2$. **[Summer 2018]**

Solution

$$p = (z + qy)^2 \qquad \text{...(1)}$$

Let $f(x, y, z, p, q) = p - (z + qy)^2 = 0$

The auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\frac{dx}{1} = \frac{dy}{-2(z+qy)y} = \frac{dz}{p - 2qy(z+qy)} = \frac{dp}{2p(z+qy)} = \frac{dq}{4q(z+qy)} \qquad \text{...(2)}$$

Taking second and fourth fractions in Eq. (2),

$$\frac{dy}{-2(z+qy)y} = \frac{dp}{2p(z+qy)}$$

$$-\frac{dy}{y} = \frac{dp}{p}$$

Integrating,

$$-\log y = \log p + \log a$$

$$\log y^{-1} = \log pa$$

$$y^{-1} = pa$$

$$p = \frac{1}{ay}$$

Putting $p = \dfrac{1}{ay}$ in Eq. (1),

$$\frac{1}{ay} = (z+qy)^2$$

$$z+qy = \frac{1}{\sqrt{ay}}$$

$$qy = -z + \frac{1}{\sqrt{ay}}$$

$$q = -\frac{z}{y} + \frac{1}{y\sqrt{ay}}$$

Now, $$dz = p\,dx + q\,dy$$

$$= \frac{1}{ay}dx - \left(\frac{z}{y} - \frac{1}{y\sqrt{ay}}\right)dy$$

$$y\,dz = \frac{dx}{a} - z\,dy + \frac{1}{\sqrt{ay}}dy$$

$$y\,dz + z\,dy = \frac{dx}{a} + \frac{1}{\sqrt{a}}y^{-\frac{1}{2}}dy$$

$$d(yz) = \frac{dx}{a} + \frac{1}{\sqrt{a}}y^{-\frac{1}{2}}dy$$

Integrating,

$$yz = \frac{x}{a} + \frac{1}{\sqrt{a}}\frac{y^{-\frac{1}{2}}}{\left(\frac{1}{2}\right)} + b$$

$$z = \frac{x}{dy} + \frac{2}{\sqrt{ay}} + \frac{b}{y}$$

which gives the complete solution of the given equation.

Example 3

Solve $(x^2 - y^2)pq - xy(p^2 - q^2) - 1 = 0.$

Solution

$$(x^2 - y^2)pq - xy\,(p^2 - q^2) - 1 = 0 \qquad ...(1)$$

Let $$f(x, y, z, p, q) = (x^2 - y^2)pq - xy\,(p^2 - q^2) - 1$$

The auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\frac{dx}{(x^2-y^2)q-2pxy}=\frac{dy}{(x^2-y^2)p+2qxy}$$

$$=\frac{dz}{2pq(x^2-y^2)-2xy(p^2-q^2)}$$

$$=\frac{dp}{-2xpq+y(p^2-q^2)}$$

$$=\frac{dq}{2ypq+x(p^2-q^2)} \quad ...(2)$$

Taking p, q, x, y as multipliers for first, second, fourth, and fifth fractions respectively, in Eq. (2),

$$\text{Each fraction } = \frac{p\,dx+q\,dy+x\,dp+y\,dq}{0}$$

$$p\,dx+q\,dy+x\,dp+y\,dq=0$$

$$(x\,dp+p\,dx)+(y\,dq+q\,dy)=0$$

$$d(xp)+d(yq)=0$$

Integrating,

$$xp+yq=a$$

$$p=\frac{a-yq}{x} \quad ...(3)$$

Putting p in Eq. (1),

$$(x^2-y^2)\left(\frac{a-yq}{x}\right)q-xy\left[\left(\frac{a-yq}{x}\right)^2-q^2\right]-1=0$$

$$\left(\frac{a-yq}{x}\right)(x^2q-y^2q-ya+y^2q)+xyq^2-1=0$$

$$(a-yq)(x^2q-ya)+x^2yq^2-x=0$$

$$ax^2q-ya^2-x^2yq^2+y^2aq+x^2yq^2-x=0$$

$$(ax^2+ay^2)q=x+a^2y$$

$$q=\frac{x+a^2y}{a(x^2+y^2)}$$

Putting q in Eq. (3),

$$p=\frac{1}{x}\left[a-\frac{xy+a^2y^2}{a(x^2+y^2)}\right]$$

$$=\frac{1}{x}\left[\frac{a^2x^2+a^2y^2-xy-a^2y^2}{a(x^2+y^2)}\right]$$

$$= \frac{a^2x-y}{a(x^2+y^2)}$$

Now, $$\mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y$$

$$= \frac{a^2x-y}{a(x^2+y^2)}\mathrm{d}x + \frac{x+a^2y}{a(x^2+y^2)}\mathrm{d}y$$

$$= a\left(\frac{x\mathrm{d}x + y\mathrm{d}y}{x^2+y^2}\right) + \frac{x\mathrm{d}y - y\mathrm{d}x}{a(x^2+y^2)}$$

$$= a\mathrm{d}\left[\frac{1}{2}\log(x^2+y^2)\right] + \frac{1}{a}\mathrm{d}\left(\tan^{-1}\frac{y}{x}\right)$$

Integrating,

$$z = \frac{a}{2}\log(x^2+y^2) + \frac{1}{a}\tan^{-1}\left(\frac{y}{x}\right) + b$$

which gives the complete solution of the given equation.

EXERCISE 6.5

Apply Charpit's method to find the complete solutions of the following:

1. $2zx - px^2 - 2qxy + pq = 0$

$$\left[\textbf{Ans.:}\ z = ay + b\left(x^2 - a\right)\right]$$

2. $z^2\left(p^2z^2 + q^2\right) = 1$

$$\left[\textbf{Ans.:}\ \left(a^2z + 1\right)^3 = 9a^4\left(ax + y + b\right)^2\right]$$

3. $yzp^2 - q = 0$

$$\left[\textbf{Ans.:}\ z^2\left(a - y^2\right) = \left(x + b\right)^2\right]$$

4. $2z + p^2 + qy + 2y^2 = 0$

$$\left[\textbf{Ans.:}\ y^2\left[\left(x - a\right)^2 + y^2 + 2z\right] = b\right]$$

5. $p^2 - y^2q = y^2 - x^2$

$$\left[\textbf{Ans.:}\ z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{a^2}{y} - y + b\right]$$

6. $z^2 = pqxy$

$$\left[\textbf{Ans.}: z = bx^a y^{\frac{1}{a}}\right]$$

7. $qz - p^2y - q^2y = 0$

$$\left[\textbf{Ans.}: z^2 = a\left[y^2 + (x+b)^2\right]\right]$$

6.8 HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \text{...(6.25)}$$

where $a_0, a_1, \ldots, a_n$ are constants is known as a homogeneous linear partial differential equation of n^{th} order with constant coefficients. Since all the terms in the equation contain derivatives of the same order, it is known as a *homogeneous equation.*

Replacing $\frac{\partial}{\partial x}$ by D and $\frac{\partial}{\partial y}$ by D' in Eq. (6.25),

$$(a_0 D^n + a_1 D^{n-1} D' + \cdots + a_n D'^n) z = F(x, y)$$

$$f(D, D')z = F(x, y)$$

where $\quad f(D, D') = a_0 D^n + a_1 D^{n-1} D' + \cdots + a_n D'^n$

which is a linear differential operator.

As in the case of ordinary linear differential equations with constant coefficients, the complete solution of Eq. (6.25) is obtained in two parts, one as a Complementary Function (CF) and the other as a Particular Integral (PI).

The complementary function is the solution of the equation $f(D, D')\, z = 0$.

6.8.1 Rules to Obtain the Complementary Function

Let the given equation be $f(D, D')\, z = F(x, y)$ ...(6.26)

where $\quad f(D, D') = a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n$

Let $z = g(y + mx)$ be its complementary function.

Thus, $z = g(y + mx)$ is the solution of the equation.

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n) z = 0 \qquad \text{...(6.27)}$$

$$Dz = \frac{\partial z}{\partial x} = mg'(y+mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2 g''(y+mx)$$

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$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n g^{(n)}(y+mx)$$

and $$D'z = \frac{\partial z}{\partial y} = g'(y+mx)$$

$$D'^2 z = \frac{\partial^2 z}{\partial y^2} = g''(y+mx)$$

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$$D'^n z = \frac{\partial^n z}{\partial y^n} = g^{(n)}(y+mx)$$

Substituting in Eq. (6.27),

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n) g^{(n)}(y+mx) = 0$$

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0 \qquad \text{...(6.28)}$$

Equation (6.28) is known as the auxiliary equation.

Let $m_1, m_2, m_3, \ldots, m_n$ be the roots of Eq. (6.28).

Case I Roots of Auxiliary Equation are Distinct

If $m_1, m_2, m_3, \ldots, m_n$ are real and distinct then Eq. (6.27) reduces to

$$(D - m_1 D')(D - m_2 D') \ldots (D - m_n D') z = 0 \qquad \text{...(6.29)}$$

$$(D - m_1 D') z = 0$$

$$p - m_1 q = 0 \qquad \text{...(6.30)}$$

This is a Lagrange's linear equation. The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{0}$$

$$dy + m_1 x = 0, \; dz = 0$$

Integrating,

$$y + m_1 x = a, \qquad z = b$$

The solution of Eq. (6.29) is

$$z = \phi_1 (y + m_1 x)$$

Similarly, the solutions of the other factors of Eq. (6.29) are

$$z = \phi_2(y+m_2x),\ z = \phi_3(y+m_3x), \ldots, z = \phi_n(y+m_nx)$$

Hence, the complementary function of Eq. (6.26) is

$$\text{CF} = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \cdots + \phi_n(y+m_nx)$$

Case II Roots of Auxiliary Equation are Equal (Repeated)

Let the auxiliary equation have two equal roots as $m_1 = m_2 = m$.
Then Eq. (6.26) reduces to

$$(D - m D')^2 (D - m_3 D') \cdots (D - m_n D')z = 0 \qquad \text{...(6.31)}$$

$$(D - m D')^2 z = 0 \qquad \text{...(6.32)}$$

$$(D - mD')u = 0 \quad \text{where} \quad u = (D - mD')z$$

Since this equation is Lagrange's linear equation,

$$u = \phi(y+mx)$$
$$(D - mD')z = \phi(y+mx)$$
$$p - mq = \phi(y+mx)$$

The subsidiary equations are

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-m} = \frac{\mathrm{d}z}{\phi(y+mx)} \qquad \text{...(6.33)}$$

Taking the first and second fractions of Eq. (6.33),

$$-m\mathrm{d}x = \mathrm{d}y$$
$$\mathrm{d}y + m\,\mathrm{d}x = 0$$

Integrating,

$$y + m\,x = a \qquad \text{...(6.34)}$$

Taking the first and third fractions of Eq. (6.33),

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}z}{\phi(y+mx)} = \frac{\mathrm{d}z}{\phi(a)} \qquad \text{[Using Eq. (6.34)]}$$

$$dz = \phi\,(a)\,dx$$

Integrating,

$$z = x\phi(a) + b$$
$$= x\phi(y+mx) + f(y+mx)$$

Thus, the complete solution of Eq. (6.32) is

$$z = x\phi(y+mx) + f(y+mx)$$

The solutions of other factors of Eq. (6.31) are same as Case I.
Hence, the complementary function of Eq. (6.25) is

$$\text{CF} = f(y+mx) + x\phi(y+mx) + \phi_3(y+m_3x) + \cdots + \phi_n(y+m_nx)$$

In general, if n roots of an auxiliary equation are equal,

$$\text{CF} = \phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_3(y+mx) + \cdots + x^{n-1}\phi_n(y+mx)$$

Notes

(i) The auxiliary equation is obtained by replacing D with m and D' with 1 in the given differential equation.

(ii) If $F(x, y) = 0$, the particular integral = 0.

Example 1

Solve $\dfrac{\partial^2 z}{\partial x^2} = z$. **[Winter 2014]**

Solution

The equation can be written as

$$\frac{\partial^2 z}{\partial x^2} - z = 0$$

The auxiliary equation is

$$\begin{aligned} m^2 - 1 &= 0 \\ m &= \pm 1 \quad \text{(distinct)} \\ \text{CF} &= \phi_1(y + x) + \phi_2(y - x) \\ F(x, y) &= 0 \\ \text{PI} &= 0 \end{aligned}$$

Hence, the complete solution is

$$z = \phi_1(y + x) + \phi_2(y - x)$$

Example 2

Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = 0.$

Solution

The equation can be written as

$$(D^2 - DD' - 6D'^2)z = 0$$

The auxiliary equation is

$$\begin{aligned} m^2 - m - 6 &= 0 \\ m &= -2, 3 \qquad \text{(distinct)} \\ \text{CF} &= \phi_1(y - 2x) + \phi_2(y + 3x) \\ F(x, y) &= 0 \\ \text{PI} &= 0 \end{aligned}$$

Hence, the complete solution is

$$z = \phi_1(y - 2x) + \phi_2 (y + 3x)$$

Example 3

Solve $25r - 40s + 16t = 0.$

Solution

The equation can be written as

$$25\frac{\partial^2 z}{\partial x^2} - 40\frac{\partial^2 z}{\partial x \partial y} + 16\frac{\partial^2 z}{\partial y^2} = 0$$

$$(25D^2 - 40DD' + 16D'^2)z = 0$$

The auxiliary equation is

$$25m^2 - 40m + 16 = 0$$

$$m = \frac{4}{5}, \frac{4}{5} \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1\left(y + \frac{4}{5}x\right) + x\phi_2\left(y + \frac{4}{5}x\right)$$

$$= f_1(5y + 4x) + xf_2(5y + 4x)$$

$$F(x, y) = 0$$

$$\text{PI} = 0$$

Hence, the complete solution is

$$z = f_1(5y + 4x) + xf_2(5y + 4x)$$

Example 4

Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial^2 x \partial y} + 2\dfrac{\partial^3 z}{\partial y^3} = 0.$ **[Winter 2014]**

Solution

The auxiliary equation is

$$m^3 - 3m^2 + 2 = 0$$

$$m^2(m-1) - 2m(m-1) - 2(m-1) = 0$$

$$(m-1)(m^3 - 2m - 2) = 0$$

$$m = 1 \qquad \text{(distinct)}, m = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{3} = 1 \pm \sqrt{3} \qquad \text{(distinct)}$$

$$\text{CF} = \phi_1(y+x) + \phi_2[y+(1+\sqrt{3})x] + \phi_3[y+(1-\sqrt{3})x]$$
$$F(x, y) = 0$$
$$\text{PI} = 0$$

Hence, the complete solution is

$$z = \phi_1(y+x) + \phi_2\left[y+(1+\sqrt{3})x\right] + \phi_3\left[y+(1-\sqrt{3})x\right]$$

6.8.2 Rules to Obtain the Particular Integral

Let the differential equation be $f(D, D')\, z = F(x, y)$

Particular integral $\text{PI} = \dfrac{1}{f(D, D')} F(x, y)$

The particular integral depends on the form of $F(x, y)$. Different cases are as follows:

Case I $F(x, y) = e^{ax + by}$

$$\text{PI} = \frac{1}{f(D, D')} e^{ax+by}$$

Replacing D by a and D' by b,

$$\text{PI} = \frac{1}{f(a, b)} e^{ax+by}, \quad f(a, b) \neq 0$$

If $f(a, b) = 0$ then $m = \dfrac{a}{b}$ is a root of the auxiliary equation.

Let $m = \dfrac{a}{b}$ be a root repeated r times.

Then $\quad f(D, D') = \left(D - \dfrac{a}{b} D'\right)^r g(D, D')$

$$\text{PI} = \frac{1}{\left(D - \frac{a}{b} D'\right)^r g(D, D')} e^{ax+by}$$
$$= \frac{x^r}{r!} \frac{1}{g(a, b)} e^{ax+by}, \; g(a, b) \neq 0$$

Case II $F(x, y) = \sin(ax + by)$ **or** $\cos(ax + by)$

$$\text{PI} = \frac{1}{f(D^2, DD', D'^2)} \sin(ax + by)$$

Replacing D^2 by $-a^2$, D'^2 by $-b^2$ and DD' by $-ab$,

$$\text{PI} = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by), f(-a^2, -ab, -b^2) \neq 0$$

Case III $F(x, y) = x^m y^n$

$$\text{PI} = \frac{1}{f(D, D')} x^m y^n = \left[f(D, D')\right]^{-1} x^m y^n$$

$[f(D, D')]^{-1}$ is expanded using binomial expansion according to the following rules:

(i) If $n < m$, expand in powers of $\dfrac{D'}{D}$.

(ii) If $m < n$, expand in powers of $\dfrac{D}{D'}$.

Case IV If $F(x, y)$ is not in any of the previous three standard forms

$$\text{PI} = \frac{1}{f(D, D')} F(x, y)$$

Express $f(D, D')$ in linear factors of D and separate each factor of $\dfrac{1}{f(D, D')}$ using the partial fraction method. Operate each part on $F(x, y)$ considering $\dfrac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$, where c is replaced by $y + mx$ after integration.

Example 1

Solve $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{2\partial^2 z}{\partial y^2} = x + y$. **[Winter 2017; Summer 2015]**

Solution

The equation can be written as

$$(D^2 + 3DD' + 2D'^2)z = x + y$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m + 2)(m + 1) = 0$$

$$m = -2, -1 \qquad \text{(distinct)}$$

$$\text{CF} = \phi_1(y - 2x) + \phi_2(y - x)$$

$$\text{PI} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y)$$

$$= \frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1-\frac{3D'}{D}\right](x+y)$$

$$= \frac{1}{D^2}\left[(x+y)-\frac{3}{D}(1)\right]$$

$$= \frac{1}{D^2}\left[(x+y)-3x\right]$$

$$= \frac{1}{D^2}\left[-2x+y\right]$$

$$= \frac{1}{D}\left[-2\cdot\frac{x^2}{2}+xy\right]$$

$$= -\frac{1}{3}x^3+\frac{1}{2}x^2y$$

Hence, the complete solution is

$$z = \phi_1(y-2x)+\phi_2(y-x)-\frac{1}{3}x^3+\frac{1}{2}x^2y$$

Example 2

Solve $(D^2 + 10DD' + 25D'^2)z = e^{3x+2y}$. **[Winter 2014]**

Solution

The auxiliary equation is

$$m^2+10m+25=0$$

$$(m+5)^2=0$$

$$m=-5,-5 \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1(y-5x)+x\phi_2(y-5x)$$

$$\text{PI} = \frac{1}{D^2+10DD'+25D'^2}e^{3x+2y}$$

$$= \frac{1}{3^2+10(3)(2)+25(2)^2}e^{3x+2y}$$

$$= \frac{1}{169}e^{3x+2y}$$

Hence, the complete solution is

$$z = \phi_1(y-5x) + x\phi_2(y-5x) + \frac{1}{169}e^{3x+2y}$$

Example 3

Solve $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = e^{2x+3y}$. **[Summer 2018]**

Solution

The equation can be written as

$$(D^2 - 4DD' + 4D'^2)z = e^{2x+3y}$$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2 \quad \text{(repeated)}$$

$$\text{CF} = \phi_1(y+2x) + x\phi_2(y+2x)$$

$$\text{PI} = \frac{1}{D^2 - 4DD' + 4D'^2}e^{2x+3y}$$

$$= \frac{1}{2^2 - 4(2)(3) + 4(3)^2}e^{2x+3y}$$

$$= \frac{1}{16}e^{2x+3y}$$

Hence, the complete solution is

$$z = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{1}{16}e^{2x+3y}$$

Example 4

Solve $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$.

Solution

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1 \quad \text{(repeated)}$$

$$\text{CF} = \phi_1(y+x) + x\phi_2(y+x)$$

$$\text{PI} = \frac{1}{D^2 - 2DD' + D'^2} e^{x+2y} + \frac{1}{D^2 - 2DD' + D'^2} x^3$$

$$= \frac{1}{1^2 - 2(1)(2) + 2^2} e^{x+2y} + \frac{1}{(D - D')^2} x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2} x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left(1 + 2\frac{D'}{D} + 3\frac{D'^2}{D^2} + \cdots\right) x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left(x^3 + \frac{2}{D}D'x^3 + \frac{3}{D^2}D'^2x^3 + \cdots\right)$$

$$= e^{x+2y} + \frac{1}{D^2} x^3$$

$$= e^{x+2y} + \frac{1}{D}\int x^3 \mathrm{d}x$$

$$= e^{x+2y} + \int \frac{x^4}{4} \mathrm{d}x$$

$$= e^{x+2y} + \frac{1}{20} x^5$$

Hence, the complete solution is

$$z = \phi_1(y + x) + x\phi_2(y + x) + e^{x+2y} + \frac{1}{20} x^5$$

Example 5

Solve $4r + 12s + 9t = e^{3x - 2y}$.

Solution

The equation can be written as

$$(4D^2 + 12DD' + 9D'^2)z = e^{3x-2y}$$

The auxiliary equation is

$$4m^2 + 12m + 9 = 0$$

$$m = -\frac{3}{2}, -\frac{3}{2} \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1\left(y - \frac{3}{2}x\right) + x\phi_2\left(y - \frac{3}{2}x\right)$$

$$= f_1(2y - 3x) + x\, f_2(2y - 3x)$$

$$\text{PI} = \frac{1}{4D^2 + 12DD' + 9D'^2} e^{3x-2y}$$

$$= \frac{1}{4\left(D + \frac{3}{2}D'\right)^2} e^{3x-2y}$$

$$= \frac{1}{4}\frac{x^2}{2!} e^{3x-2y} \qquad \left[\because D + \frac{3}{2}D' = 0 \text{ at } D = 3, D' = -2\right]$$

$$= \frac{1}{8} x^2 e^{3x-2y}$$

Hence, the complete solution is

$$z = f_1(2y - 3x) + x\, f_2(2y - 3x) + \frac{1}{8} x^2 e^{3x-2y}$$

Aliter for PI

$$\text{PI} = \frac{1}{4D^2 + 12DD' + 9D'^2} e^{3x-2y}$$

Since the denominator is zero at $D = 3$ and $D' = -2$, differentiating the denominator w.r.t. D and premultiplying by x,

$$\text{PI} = x \frac{1}{8D + 12D'} e^{3x-2y}$$

$$= x^2 \frac{1}{8} e^{3x-2y} \qquad [\text{Differentiating again and premultiplying by } x]$$

$$= \frac{1}{8} x^2 e^{3x-2y}$$

Example 6

Solve $(D^2 - 2DD')\, z = \sin x \cos 2y.$

Solution

The auxiliary equation is

$$m^2 - 2m = 0$$

$$m = 0,\ 2 \text{ (distinct)}$$

$$\text{CF} = \phi_1(y) + \phi_2(y + 2x)$$

$$\text{PI} = \frac{1}{D^2 - 2DD'} (\sin x \cos 2y)$$

$$= \frac{1}{D^2 - 2DD'} \frac{1}{2} [\sin(x + 2y) + \sin(x - 2y)]$$

$$= \frac{1}{D^2 - 2DD'} \frac{1}{2} \sin(x+2y) + \frac{1}{D^2 - 2DD'} \frac{1}{2} \sin(x-2y)$$

$$= \frac{1}{2}\left[\frac{1}{-1^2 - 2\{-(1)(2)\}} \sin(x+2y) + \frac{1}{-1^2 - 2\{-(1)(-2)\}} \sin(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin(x+2y) - \frac{1}{5}\sin(x-2y)\right]$$

Hence, the complete solution is

$$z = \phi_1(y) + \phi_2(y+2x) + \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y)$$

Example 7

Solve $(D^2 + DD' - 6\,D'^2)\,z = \sin(2x+y)$.

Solution

The auxiliary equation is

$$m^2 + m - 6 = 0$$

$$m = -3, 2 \quad \text{(distinct)}$$

$$\text{CF} = \phi_1(y-3x) + \phi_2(y+2x)$$

$$\text{PI} = \frac{1}{D^2 + DD' - 6D'^2} \sin(2x+y)$$

Since the denominator is zero after replacing D^2 by -2^2, DD' by $-(2)(1)$, and D'^2 by -1^2, the general method needs to be applied.

$$\text{PI} = \frac{1}{(D+3D')(D-2D')} \sin(2x+y)$$

$$= \frac{1}{D+3D'}\left[\frac{1}{D-2D'}\sin(2x+y)\right]$$

$$= \frac{1}{D+3D'}\left[\int \sin\{2x + (c-2x)\}\,dx\right]$$

$$= \frac{1}{D+3D'}\left[\int \sin c\,dx\right]$$

$$= \frac{1}{D+3D'}[x \sin c]$$

$$= \frac{1}{D+3D'}[x\sin(y+2x)]$$

$$= \int x \sin[(c+3x)+2x]\,dx$$

$$= \int x \sin(5x + c)\,dx$$

$$= x\left[\frac{-\cos(5x+c)}{5}\right] - \left[\frac{-\sin(5x+c)}{25}\right]$$

$$= -\frac{1}{5}x\cos(5x + y - 3x) + \frac{1}{25}\sin(5x + y - 3x)$$

$$= -\frac{1}{5}x\cos(y + 2x) + \frac{1}{25}\sin(y + 2x)$$

Hence, the complete solution is

$$z = \phi_1(y - 3x) + \phi_2(y + 2x) - \frac{1}{5}x\cos(y + 2x) + \frac{1}{25}\sin(y + 2x)$$

Example 8

Solve $(D^2 - 2DD' + D'^2)z = \tan(y + x)$.

Solution

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1 \quad \text{(repeated)}$$

$$\text{CF} = \phi_1(y + x) + x\phi_2(y + x)$$

Since $F(x, y) = \tan(y + x)$ is not in any of the standard forms, the general method needs to be applied.

$$\text{PI} = \frac{1}{(D - D')^2}\tan(y + x)$$

$$= \frac{1}{(D - D')}\left[\frac{1}{D - D'}\tan(y + x)\right]$$

$$= \frac{1}{(D - D')}\left[\int \tan\{(c - x) + x\}\,dx\right]$$

$$= \frac{1}{D - D'}\left[\int \tan c\,dx\right]$$

$$= \frac{1}{D - D'}\left[x\tan c\right]$$

$$= \frac{1}{D - D'}\left[x\tan(y + x)\right]$$

$$= \int x\tan\{(c - x) + x\}\,dx$$

$$= \int x\tan c\,dx$$

$$= \frac{1}{2} x^2 \tan c$$

$$= \frac{1}{2} x^2 \tan (y + x)$$

Hence, the complete solution is

$$z = \phi_1 (y + x) + x\phi_2 (y + x) + \frac{1}{2} x^2 \tan(y + x)$$

Example 9

Solve $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x}$. **[Winter 2016]**

Solution

The equation can be written as

$$(D^3 - 2D^2 D')\, z = 2e^{2x}$$

The auxiliary equation is

$$m^3 - 2m^2 = 0$$

$$m^2(m - 2) = 0$$

$$m = 0, 0, 2 \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1(y) + x\phi_2(y) + \phi_3(y + 2x)$$

$$\text{PI} = \frac{1}{D^3 - 3D^2 D'} 2e^{2x}$$

$$= \frac{1}{(2)^3 - 2(2)^2 \cdot 0} 2e^{2x}$$

$$= \frac{2}{8} e^{2x}$$

$$= \frac{1}{4} e^{2x}$$

Hence, the complete solution is

$$z = \phi_1(y) + x\phi_2(y) + \phi_3(y + 2x) + \frac{1}{4} e^{2x}$$

Example 10

Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$. **[Winter 2017]**

Solution

The equation can be written as

$$(D^3 - 3D^2 D' + 4D'^3)\, z = e^{x+2y}$$

The auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$m = -1,\ m = 2, 2 \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1(y - x) + \phi_2(y + 2x) + x\phi_3(y + 2x)$$

$$\text{PI} = \frac{1}{D^3 - 3D^2 D' + 4D'^3}\, e^{x+2y}$$

$$= \frac{1}{(1)^3 - 3(1)^2(2) + 4(2)^3}\, e^{x+2y}$$

$$= \frac{1}{1 - 6 + 32}\, e^{x+2y}$$

$$= \frac{1}{27}\, e^{x+2y}$$

Hence, the complete solution is

$$z = \phi_1(y - x) + \phi_2(y + 2x) + x\phi_3(y + 2x)$$

EXERCISE 6.6

Solve the following:

1. $2\dfrac{\partial^2 z}{\partial x^2} + 5\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0$

$\left[\textbf{Ans.:}\ z = f_1(y - 2x) + f_2(2y - x)\right]$

2. $\dfrac{\partial^3 z}{\partial x^3} + 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 0$

$\left[\textbf{Ans.:}\ z = f_1(y) + f_2(y + 2x) + x f_3(y + 2x)\right]$

3. $\left(D^2 - 2DD' - 15D'^2\right)z = 12xy$

$$\left[\textbf{Ans.: } z = f_1(y+5x) + f_2(y-3x) + x^4 + 2x^3y\right]$$

4. $r - 2s + t = \sin(2x+3y)$

$$\left[\textbf{Ans.: } z = f_1(x+y) + x\,f_2(x+y) - \sin(2x+3y)\right]$$

5. $\left(2D^2 - 5DD' + 2D'^2\right)z = 5\sin(2x+y)$

$$\left[\textbf{Ans.: } z = f_1(2y+x) + f_2(y+2x) - \frac{5x}{3}\cos(2x+y)\right]$$

6. $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^3 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = \sin(2x+y)$

$$\left[\textbf{Ans.: } z = f_1(y) + f_2(y+2x) + xf_3(y+2x) - \frac{x^2}{4}\cos(2x+y)\right]$$

7. $\left(D^2 - DD' - 2D'^2\right)z = (y-1)e^x$

$$\left[\textbf{Ans.: } z = f_1(y+2x) + f_2(y-x) + ye^x\right]$$

8. $r + s - 6t = y\cos x$

$$\left[\textbf{Ans.: } z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x\right]$$

9. $\left(D^2 + 2DD' + D'^2\right)z = 2\cos y - x\sin y$

$$\left[\textbf{Ans.: } z = f_1(y-x) + xf_2(y-x) + x\sin y\right]$$

10. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y\sin x$

$$\left[\textbf{Ans.: } z = f_1(y-3x) + f_2(y+2x) - (y\sin x + \cos x)\right]$$

6.9 NONHOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

If in the equation $f(D, D')z = F(x, y)$ ...(6.35)

each term of $f(D, D')$ does not contain the derivatives of the same order then the equation is known as a nonhomogeneous equation.

To find the complementary function of Eq. (6.35), factorize $f(D, D')$ into the linear factors as $(D - mD' - c)$ and obtain the solution of the equation $(D - mD' - c)z = 0$.

$$(D - mD' - c)z = 0$$

$$p - mq = cz$$

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz} \qquad \text{...(6.36)}$$

Taking the first and second fractions from Eq. (6.36),

$$-m dx = dy$$

$$m dx + dy = 0$$

Integrating,

$$mx + y = a$$

Taking the first and third fractions from Eq. (6.36),

$$dx = \frac{dz}{cz}$$

$$\frac{dz}{z} = c dx$$

Integrating,

$$\log z = cx + \log b$$

$$\log \frac{z}{b} = cx$$

$$z = be^{cx}$$

Taking $\qquad b = \phi(a)$

$$z = e^{cx}\phi(a)$$

$$= e^{cx}\phi(y + mx)$$

Similarly, solutions corresponding to other factors can be obtained. All the solutions are added up to obtain the complementary function.

The methods to find the particular integral are same as those of homogeneous linear equations.

Example 1

Solve $\quad (D^2 - DD' + D' - 1)\, z = \cos(x + 2y) + e^y.$

Solution

$$D^2 - DD' + D' - 1 = (D - 1)(D - D' + 1)$$

(i) For the equation $(D - 1)\, z = 0$,

$m = 0, \ c = 1$

the solution is

$$z = e^x \phi_1 (y)$$

(ii) For the equation $(D - D' + 1) z = 0$,

$$m = 1, c = -1$$

the solution is

$$z = e^{-x} \phi_2 (y + x)$$

Hence, $\text{CF} = e^x \phi_1(y) + e^{-x} \phi_2 (y + x)$

$$\text{PI} = \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) + \frac{1}{D^2 - DD' + D' - 1}e^y$$

$$= \frac{1}{-1^2 - \{-(1)(2)\} + D' - 1}\cos(x+2y) + x\frac{1}{2D - D'}e^y$$

$$= \frac{1}{D'}\cos(x+2y) + x\frac{1}{2(0)-1}e^y$$

$$= \frac{D'}{D'^2}\cos(x+2y) - xe^y$$

$$= \frac{D'\cos(x+2y)}{-2^2} - xe^y$$

$$= \frac{-2\sin(x+2y)}{-4} - xe^y$$

$$= \frac{1}{2}\sin(x+2y) - xe^y$$

Hence, the complete solution is

$$z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + \frac{1}{2}\sin(x+2y) - xe^y.$$

EXERCISE 6.7

Solve the following:

1. $\left(D^2 - D'^2 + D - D'\right)z = 0$

$$\left[\textbf{Ans.}: z = f_1(y+x) + e^{-x} f_2(y-x)\right]$$

2. $\left(D - D' - 1\right)\left(D - D' - 2\right)z = e^{2x-y} + x$

$$\left[\textbf{Ans.}: z = e^x f_1(y+x) + e^{2x} f_2(y+z) + \frac{x}{2} + \frac{3}{4} + \frac{1}{2}e^{2x-y}\right]$$

3. $r - s + p = 1$

$$\left[\textbf{Ans.}: z = f_1(y) + e^{-x} f_2(y+x) + x\right]$$

4. $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$

$$\left[\textbf{Ans.}: z = e^x f_1(y) + e^{-x} f_2(y+x) + \frac{1}{2}\sin(x+2y) - xe^y\right]$$

5. $(D^2 - DD' - 2D)z = \sin(3x+4y) - e^{2x+y}$

$$\left[\textbf{Ans.}: z = f_1(y) + e^{2x} f_2(y+x) + \frac{1}{15}\sin(3x+4y) + \frac{2}{15}\cos(3x+3y) + \frac{1}{2}e^{2x+y}\right]$$

6.10 CLASSIFICATION OF SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

The general form of a nonhomogeneous second order partial differential equation in the function of two independent variables x, y is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = F(x, y) \qquad \text{...(6.37)}$$

Equation (6.37) is linear or quasi-linear accordingly as f is linear or nonlinear. Equation (6.37) is homogeneous if $F(x, y) = 0$.

Equation (6.37) is elliptic if $B^2 - 4AC < 0$, parabolic if $B^2 - 4AC = 0$ and hyperbolic if $B^2 - 4AC > 0$. Three fundamental types of second-order linear partial differential equations appear frequently in many applications as follows:

(i) The one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is parabolic.

(ii) The one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is hyperbolic.

(iii) The two-dimensional Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is elliptic.

Example 1

Classify the following partial differential equations as parabolic, hyperbolic, and elliptic.

(a) $\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial x^2} = 0$

(b) $4\dfrac{\partial^2 u}{\partial t^2} - 9\dfrac{\partial^2 u}{\partial x \partial t} + 5\dfrac{\partial^2 u}{\partial x^2} = 0$

(c) $2\dfrac{\partial^2 u}{\partial t^2} + 4\dfrac{\partial^2 u}{\partial x \partial t} + 3\dfrac{\partial^2 u}{\partial x^2} = 0$

Solution

(a) $A = 4, \quad B = 4, \quad C = 1$

$B^2 - 4AC = 16 - 16 = 0$

Hence, the partial differential equation is parabolic.

(b) $A = 5, \quad B = -9, \quad C = 4$

$B^2 - 4AC = 81 - 80 = 1$

Hence, the partial differential equation is hyperbolic.

(c) $A = 3, \quad B = 4, \quad C = 2$

$B^2 - 4AC = 16 - 24 = -8 < 0$

Hence, the partial differential equation is elliptic.

6.11 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

In many physical problems in electromagnetic theory, fluid mechanics, solid mechanics, heat transfer, etc., solutions of partial differential equations are required. These equations satisfy some specified conditions known as *boundary conditions*. The partial differential equation together with these boundary conditions, constitutes a *boundary-value problem*.

The method of separation of variables is an important tool to solve such boundary-value problems when the partial differential equation is linear and boundary conditions are homogeneous. Unlike ordinary differential equations, the general solution of a partial differential equation involves arbitrary functions which requires the knowledge of single and double Fourier series.

6.12 METHOD OF SEPARATION OF VARIABLES

Separation of variables is also known as the *Fourier method*. It is a powerful technique to solve partial differential equations. This method is explained with the help of the following examples.

Example 1

Solve $x\dfrac{\partial u}{\partial x} - 2y\dfrac{\partial u}{\partial y} = 0.$ **[Winter 2016; Summer 2013]**

Solution

Let the solution be

$$u(x, y) = X(x)\,Y(y) \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = X'Y, \qquad \frac{\partial u}{\partial y} = XY'$$

Substituting in the given equation,

$$xX'Y - 2yXY' = 0$$

$$\frac{xX'}{X} = \frac{2yY'}{y} = k, \quad \text{say}$$

$$\frac{xX'}{X} = k, \qquad \frac{2yY'}{Y} = k$$

Solving both the equations,

$$\log X = k\log x + \log c_1$$

$$\log X = \log x^k c_1$$

$$X = c_1 x^k$$

and

$$\log Y = \frac{k}{2}\log y + \log c_2$$

$$= \log y^{\frac{k}{2}} c_2$$

$$Y = c_2 y^{\frac{k}{2}}$$

Substituting these values in Eq. (1),

$$u(x, y) = c_1 x^k \cdot c_2 y^{\frac{k}{2}}$$

$$= Ax^k y^{\frac{k}{2}} \qquad \text{where} \quad A = c_1 c_2$$

Example 2

Solve $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 2(x + y)u.$ **[Winter 2014]**

Solution

Let the solution be

$$u(x, y) = X(x)\, Y(y) \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = X'Y, \qquad \frac{\partial u}{\partial y} = XY'$$

Substituting in the given equation,

$$X'Y + XY' = 2(x+y)XY$$

$$X'Y + XY' = 2xXY + 2yXY$$

$$X'Y - 2xXY = 2yXY - XY'$$

$$(X' - 2xX)Y = (2yY - Y')X$$

$$\frac{X' - 2xX}{X} = \frac{2yY - Y'}{Y} = k, \qquad \text{say}$$

$$\frac{X'}{X} - 2x = k, \qquad -\frac{Y'}{Y} + 2y = k$$

Solving both the equations,

$$\log X - x^2 = kx + c_1$$

$$\log X = x^2 + kx + c_1$$

$$X = e^{x^2+kx+c_1}$$

$$= e^{x^2+kx} e^{c_1}$$

$$= Ae^{x^2+kx}$$

and

$$-\log Y + y^2 = ky + c_2$$

$$\log Y - y^2 = -ky - c_2$$

$$\log Y = y^2 - ky - c_2$$

$$Y = e^{y^2-ky-c_2}$$

$$= e^{y^2-ky} e^{-c_2}$$

$$= Be^{y^2-ky}$$

Substituting these values in Eq. (1),

$$u(x, y) = Ae^{x^2+kx} Be^{y^2-ky}$$

$$= ABe^{x^2+kx+y^2-ky}$$

$$= Ce^{x^2+kx+y^2-ky}, \qquad \text{where } AB = C$$

Example 3

Solve $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0.$ **[Winter 2017; Summer 2014]**

Solution

Let the solution be

$$z(x, y) = X(x)\ Y(y) \qquad \text{...(1)}$$

$$\frac{\partial z}{\partial x} = X'Y, \qquad \frac{\partial z}{\partial y} = XY'$$

$$\frac{\partial^2 z}{\partial x^2} = X''Y$$

Substituting in the given equation,

$$X''Y - 2X'Y + XY' = 0$$

$$X''Y - 2X'Y = -XY'$$

$$(X'' - 2X')Y = -XY'$$

$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = k, \qquad \text{say}$$

$$\frac{X'' - 2X'}{X} = k, \qquad -\frac{Y'}{Y} = k$$

Solving both the equations,

$$X'' - 2X' - kX = 0$$

The auxiliary equation is

$$m^2 - 2m - k = 0$$

$$m = \frac{2 \pm \sqrt{4 + 4k}}{2}$$

$$= 1 \pm \sqrt{1 + k} \qquad \text{(distinct)}$$

$$X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$$

and

$$Y' = -ky$$

$$\frac{Y'}{Y} = -k$$

$$\log Y = -ky + c$$

$$Y = e^{-ky+c}$$

$$= e^{-ky} e^c$$

$$= c_3 e^{-ky}$$

Substituting these values in Eq. (1),

$$z(x,y) = \left[c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\right] c_3 e^{-ky}$$

$$= \left[Ae^{(1+\sqrt{1+k})x} + Be^{(1-\sqrt{1+k})x}\right] e^{-ky}$$

where $A = c_1c_3$ and $B = c_2c_3$

Example 4

Solve $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, *given that* $u(0, y) = 8e^{-3y}$.

Solution

Let the solution be

$$u\,(x, y) = X(x)\,Y(y) \qquad \text{... (1)}$$

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY'$$

Substituting in the given equation,

$$X'Y = 4XY'$$

$$\frac{X'}{X} = \frac{4Y'}{Y} = k, \text{ say}$$

$$\frac{X'}{X} = k, \quad \frac{4Y'}{Y} = k$$

Solving both the equations,

$$\log X = kx + \log c_1$$

$$\log \frac{X}{c_1} = kx$$

$$X = c_1 e^{kx}$$

and $\quad 4 \log Y = ky + \log c_2$

$$\log \frac{Y^4}{c_2} = ky$$

$$Y^4 = c_2 e^{ky}$$

$$Y = ce^{\frac{ky}{4}}, \quad \text{where } c = c_2^{\frac{1}{4}}$$

Substituting these values in Eq. (1),

$$u(x, y) = XY = c_1 c e^{k\left(x+\frac{y}{4}\right)} = Ae^{k\left(x+\frac{y}{4}\right)}, \text{ where } c_1 c = A \qquad \text{...(2)}$$

Given $\quad u(0, y) = 8\,e^{-3y}$

$$Ae^{k\left(0+\frac{y}{4}\right)} = 8e^{-3y}$$

Comparing both the sides,

$$A = 8, \quad \frac{k}{4} = -3, \quad k = -12$$

Hence, $$u(x, y) = 8\,e^{-12\left(x+\frac{y}{4}\right)}$$
$$= 8\,e^{-12x-3y}$$

Example 5

Using the method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \quad u(x,0) = 6e^{-3x}$$

[Summer 2017, 2015]

Solution

Let the solution be

$$u(x, t) = X(x)\,T(t) \tag{1}$$

$$\frac{\partial u}{\partial x} = X'(x)T(t), \qquad \frac{\partial u}{\partial t} = X(x)T'(t)$$

Substituting in the given equation,

$$XT = 2XT' + XT$$

$$X'T = (2T' + T)X$$

$$\frac{X'}{X} = \frac{2T'+T}{T} = k, \text{ say}$$

$$\frac{X'}{X} = k, \quad \frac{T'}{T} = \frac{1}{2}(k-1)$$

Solving both the equations,

$$\log X = kx + \log c_1$$

$$\log\frac{X}{c_1} = kx$$

$$X = c_1 e^{kx}$$

and $$\log T = \frac{1}{2}(k-1)\,t + \log c_2$$

$$\log\frac{T}{c_2} = \frac{1}{2}(k-1)t$$

$$T = c_2 e^{\frac{1}{2}(k-1)t}$$

Substituting these values in Eq. (1),

$$u(x, t) = XT$$

$$= c_1 e^{kx} \cdot c_2 e^{\frac{1}{2}(k-1)t} \qquad \text{...(2)}$$

Given $\quad u(x, 0) = 6e^{-3x}$ ...(3)

$$c_1 c_2 e^{kx} = 6e^{-3x} \qquad \text{[From Eq. (2)]}$$

Comparing both the sides,

$$c_1 c_2 = 6 \text{ and } k = -3$$

Hence, $\quad u(x, t) = 6e^{-3x} e^{-2t}$

$$= 6e^{-(3x + 2t)}$$

Example 6

Solve the equation $u_x = 2u_t + u$ given $u(x, 0) = 4e^{-4x}$, by the method of separation of variable. **[Summer 2016]**

Solution

Let the solution be

$$u(x, t) = X(x)\, T(t) \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = X'T, \qquad \frac{\partial u}{\partial t} = XT'$$

Substituting in the given equation,

$$X'T = 2XT' + XT$$

$$X'T = (2T' + T)X$$

$$\frac{X'}{X} = \frac{2T' + T}{T} = k, \text{ say}$$

$$\frac{X'}{X} = k, \qquad \frac{T'}{T} = \frac{1}{2}(k-1)$$

Solving both the equations,

$$\log X = kx + \log c_1$$

$$\frac{X}{c_1} = e^{kx}$$

$$X = c_1 e^{kx}$$

and $$\log T = \left(\frac{k-1}{2}\right)t + \log c_2$$

$$\frac{T}{c_2} = e^{\left(\frac{k-1}{2}\right)t}$$

$$T = c_2\, e^{\left(\frac{k-1}{2}\right)t}$$

Substituting these values in Eq. (1),

$$u(x, t) = c_1\, e^{kx}\; c_2\, e^{\left(\frac{k-1}{2}\right)t}$$

$$= c_1 c_2\, e^{kx+\left(\frac{k-1}{2}\right)t} \qquad \text{...(2)}$$

Given $$u(x, 0) = 4\, e^{-4x} \qquad \text{...(3)}$$

$$c_1 c_2\, e^{kx} = 4\, e^{-4x} \qquad \text{[From Eq. (2)]}$$

Comparing both the sides,

$$c_1 c_2 = 4 \quad \text{and} \quad k = -4$$

Hence, $$u(x, t) = 4\, e^{-4x+\left(\frac{-4-1}{2}\right)t}$$

$$u(x, t) = 4\, e^{-4x-\frac{5}{2}t} = 4\, e^{-\left(4x+\frac{5}{2}t\right)}$$

Example 7

Solve $2\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t} + u$ *subject to the condition* $u(x, 0) = 4e^{-3x}$.

[Winter 2012]

Solution

Let the solution be

$$u(x, t) = XT \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = X'T, \qquad \frac{\partial u}{\partial t} = XT'$$

Substituting in the given equation,

$$2X'T = XT' + XT = X(T' + T)$$

$$\frac{2X'}{X} = \frac{T'+T}{T} = k, \qquad \text{say}$$

$$\frac{X'}{X} = \frac{k}{2}, \qquad \frac{T'}{T} + 1 = k$$

$$\frac{T'}{T} = k - 1$$

Solving both the equations,

$$\log X = \frac{k}{2}x + \log c_2$$

$$\frac{X}{c_2} = e^{\frac{k}{2}x}$$

$$X = c_2 e^{\frac{k}{2}x}$$

and $\qquad \log T = (k-1)t + \log c_3$

$$\frac{T}{c_3} = e^{(k-1)t}$$

$$T = c_3 e^{(k-1)t}$$

Substituting these values in Eq. (1),

$$u(x,t) = c_2 e^{\frac{k}{2}x} c_3 e^{(k-1)t}$$

$$= c_2 c_3 e^{\frac{k}{2}x} e^{(k-1)t} \qquad \text{...(2)}$$

Given $\qquad u(x, 0) = 4e^{-3x}$ ...(3)

$$c_2\ c_3\ e^{\frac{k}{2}x} e^0 = 4e^{-3x} \qquad \text{[From Eq. (2)]}$$

Comparing both the sides,

$$c_2 c_3 = 4 \qquad \text{and} \qquad \frac{k}{2} = -3$$

$$k = -6$$

Hence, $\qquad u(x, t) = 4e^{-3x}e^{-7t}$

$$= 4e^{-(3x + 7t)}$$

Example 8

Solve $\quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u.$

Solution

Let the solution be

$$u(x, y) = X(x)\, Y(y)$$

$$\frac{\partial u}{\partial x} = X'Y, \qquad \frac{\partial u}{\partial y} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = X''Y$$

Substituting in the given equation,

$$X''Y = XY' + 2XY$$

Dividing by XY,

$$\frac{X''}{X} = \frac{Y'}{Y} + 2$$

$$\frac{X''}{X} - 2 = \frac{Y'}{Y} = k, \text{ say}$$

$$\frac{X'' - 2X}{X} = k, \quad \frac{Y'}{Y} = k$$

$$X'' - (k+2)X = 0 \quad \ldots(1), \qquad \frac{Y'}{Y} = k \quad \ldots(2)$$

To solve Eq. (1), the auxiliary equation is

$$m^2 - (k+2)m = 0$$

$$m = 0,\ k + 2$$

$$X = c_1 e^{0x} + c_2 e^{(k+2)x}$$

$$= c_1 + c_2 e^{(k+2)x}$$

The solution of Eq. (2) is

$$\log Y = ky + \log c_3$$

$$\log \frac{Y}{c_3} = ky$$

$$Y = c_3 e^{ky}$$

Hence, $u = XY = \left[c_1 + c_2 e^{(k+2)x}\right] c_3 e^{ky}$

$$= Ae^{ky} + Be^{k(x+y)+2x}, \text{ where } c_1 c_3 = A,\ c_2 c_3 = B$$

EXERCISE 6.8

Solve the following equations by the method of separation of variables:

1. $y^3 \dfrac{\partial z}{\partial x} + x^2 \dfrac{\partial z}{\partial y} = 0$

$$\left[\textbf{Ans.: } z = ce^{k\left(\frac{x^3}{3} - \frac{y^4}{4}\right)}\right]$$

2. $2xz_x - 3yz_y = 0$

$$\left[\textbf{Ans.: } z = Ax^{\frac{k}{2}} y^{\frac{k}{3}} \quad \text{where } A = c_1 c_2\right]$$

3. $4u_x + u_y = 3u$ with $u(0, y) = 3e^{-y} - e^{-5y}$

$$\left[\textbf{Ans.: } u = 3e^{x-y} - e^{2x-5y}\right]$$

4. $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$, given that $u = 0$ when $t = 0$ and $\dfrac{\partial u}{\partial t} = 0$ when $x = 0$.

Show that as t tends to ∞, u tends to $\sin x$.

$$\left[\textbf{Ans.: } u = \left[k \sin x + k(1 - e^t)\right]\left[-\frac{1}{k}e^{-t} + \frac{1}{k}\right] = \left[\sin x + (1 - e^t)\right]\left[1 - e^{-t}\right]\right]$$

5. $4u_x + u_y = 3u$ and $u(0, y) = e^{-5y}$

$$\left[\textbf{Ans.: } u = e^{2x-5y}\right]$$

6. $3u_x + 2u_y = 0$ with $u(x, 0) = 4e^{-x}$

$$\left[\textbf{Ans.: } u = 4e^{-\frac{(2x-3y)}{2}}\right]$$

6.13 ONE-DIMENSIONAL WAVE EQUATION

Consider an elastic string stretched to a length l along the x-*ax*is with its two fixed ends at $x = 0$ and $x = l$ (Fig. 6.4).

To obtain the deflection $y(x, t)$ at any point x and at any time $t > 0$, the following assumptions are made:

(i) The string is homogeneous with constant density ρ.

(ii) The string is perfectly elastic and offers no resistance to bending.

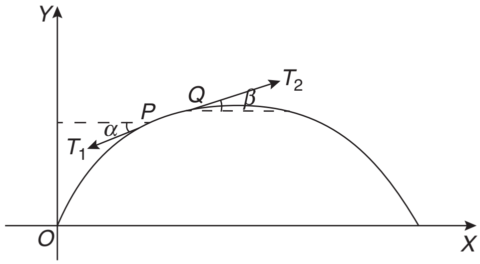

Fig. 6.4 ***One-dimensional wave equation***

(iii) The tension in the string is so large that the force due to weight of the string can be neglected. Consider the motion of the small portion PQ of length δx of the string (as shown in Fig. 6.4). Since the string produces no resistance to bending, the tensions T_1 and T_2 at points P and Q will act tangentially at P and Q respectively.

Assuming that the points on the string move only in the vertical direction, there is no motion in the horizontal direction.

Hence, the sum of the forces in the horizontal direction must be zero.

$$-T_1 \cos\alpha + T_2 \cos\beta = 0$$

$$T_1 \cos\alpha = T_2 \cos\beta = T \text{ (constant), say} \qquad \text{...(6.38)}$$

The forces acting vertically on the string are the vertical components of tension at points P and Q. Thus, the resultant vertical force acting on PQ is $T_2 \sin\beta - T_1 \sin\alpha$. By Newton's second law of motion,

Resultant force = Mass × Acceleration

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho\delta x)\left(\frac{\partial^2 y}{\partial t^2}\right) \qquad \text{...(6.39)}$$

$$\frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T} = \frac{(\rho\delta x)}{T}\left(\frac{\partial^2 y}{\partial t^2}\right)$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\delta x}{T}\left(\frac{\partial^2 y}{\partial t^2}\right) \qquad \text{[Using Eq. (6.38)]}$$

$$\tan\beta - \tan\alpha = \frac{\rho\delta x}{T}\frac{\partial^2 y}{\partial t^2} \qquad \text{...(6.40)}$$

Since tan α and tan β are the slopes of the curve at points P and Q respectively,

$$\tan\alpha = \left(\frac{\partial y}{\partial x}\right)_P = \left(\frac{\partial y}{\partial x}\right)_x$$

$$\tan\beta = \left(\frac{\partial y}{\partial x}\right)_Q = \left(\frac{\partial y}{\partial x}\right)_{x+\delta x}$$

Substituting in Eq. (6.40),

$$\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x = \frac{\rho\delta x}{T}\frac{\partial^2 y}{\partial t^2}$$

Dividing by δx and taking limit $\delta x \to 0$,

$$\lim_{\delta x\to 0}\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{where } c^2 = \frac{T}{\rho}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

This equation is known as the *one-dimensional wave equation.*

6.13.1 Solution of the One-Dimensional Wave Equation

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(6.41)}$$

Let $y = X(x)T(t)$ be a solution of Eq. (6.41).

$$\frac{\partial^2 y}{\partial t^2} = XT'', \qquad \frac{\partial^2 y}{\partial x^2} = X''T$$

Substituting in Eq. (6.41),

$$XT'' = c^2X''T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

Since X and T are only the functions of x and t respectively, this equation holds good if each term is a constant.

Let $\qquad \dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T} = k$, say

Considering $\frac{X''}{X} = k$, $\quad \frac{d^2 X}{dx^2} - kX = 0$ ...(6.42)

Considering $\frac{1}{c^2}\frac{T''}{T} = k$, $\frac{d^2 T}{dt^2} - kc^2 T = 0$ ...(6.43)

Solving Eqs (6.42) and (6.43), the following cases arise:

(i) When k is positive

Let $k = m^2$

$$\frac{d^2 X}{dx^2} - m^2 X = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} - m^2 c^2 T = 0$$

$$X = c_1 e^{mx} + c_2 e^{-mx} \quad \text{and} \quad T = c_3 e^{mct} + c_4 e^{-mct}$$

Hence, the solution of Eq. (6.41) is

$$y = (c_1 e^{mx} + c_2 e^{-mx})(c_3 e^{mct} + c_4 e^{-mct}) \qquad \text{...(6.44)}$$

(ii) When k is negative

Let $k = -m^2$

$$X = c_1 \cos mx + c_2 \sin mx \quad \text{and} \quad T = c_3 \cos mct + c_4 \sin mct$$

Hence, the solution of Eq. (6.41) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \text{...(6.45)}$$

(iii) When $k = 0$

$$X = c_1 x + c_2 \quad \text{and} \quad T = c_3 t + c_4$$

Hence, the solution of Eq. (6.41) is

$$y = (c_1 x + c_2)(c_3 t + c_4) \qquad \text{...(6.46)}$$

Out of these three solutions, we need to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with problems on vibrations, y must be a periodic function of x and t. Thus, the solution must involve trigonometric terms.

Hence, the solution is of the form given by Eq. (6.45).

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct)$$

Example 1

Find the solution of the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ *such that* $y = a \cos pt$ *when* $x = l$, *and* $y = 0$ *when* $x = 0$.

Solution

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

Since y is periodic, the solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $y = 0$

(ii) At $x = l$, $y = a \cos pt$

Putting the condition (i) in Eq. (2),

$$0 = c_1 (c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$\begin{aligned} y &= c_2 \sin mx (c_3 \cos mct + c_4 \sin mct) \\ &= c_2 c_3 \sin mx \cos mct + c_2 c_4 \sin mx \sin mct \end{aligned} \qquad \text{...(3)}$$

Putting the condition (ii) in Eq. (3),

$$a \cos pt = c_2 c_3 \sin ml \cos mct + c_2 c_4 \sin ml \sin mct$$

Equating coefficients of sine and cosine terms,

$$a = c_2 c_3 \sin ml \text{ , if } mc = p \quad \Rightarrow \quad c_2 c_3 = \frac{a}{\sin ml}, \text{ if } m = \frac{p}{c}$$

and $\quad 0 = c_2 c_4 \sin ml \quad \Rightarrow \quad c_4 = 0 \qquad [\because c_2 \neq 0, \text{ otherwise } y = 0]$

Substituting these values in Eq. (3),

$$\begin{aligned} y &= \frac{a}{\sin ml} \sin mx \cos mct \\ &= \frac{a}{\sin \frac{pl}{c}} \sin \frac{px}{c} \cos pt \end{aligned}$$

Example 2

A tightly stretched string with fixed end points $x = 0$ and $x = l$ in the shape defined by $y = kx(l - x)$, where k is a constant, is released from this position of rest. Find $y(x, t)$ if $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$.

Solution

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad ...(2)$$

The boundary conditions are

(i) At $x = 0$, $y = 0$, for all t, i.e., $y(0, t) = 0$

(ii) At $x = l$, $y = 0$, for all t, i.e., $y(l, t) = 0$

The initial conditions are

(iii) At $t = 0$, $y = kx(l - x)$, i.e., $y(x, 0) = kx(l - x)$

(iv) At $t = 0$, $\frac{\partial y}{\partial t} = 0$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$\therefore \quad c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \qquad ...(3)$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml(c_3 \cos mct + c_4 \sin mct)$$

$$\sin ml = 0 \qquad [\because c_2 \neq 0]$$

$$ml = n\pi, \quad n \text{ is an integer}$$

$$m = \frac{n\pi}{l}$$

Putting $m = \frac{n\pi}{l}$ in Eq. (3),

$$y = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right) \qquad ...(4)$$

$$\frac{\partial y}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left[-c_3\left(\sin\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) + c_4\left(\cos\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right)\right] \qquad ...(5)$$

Applying the condition (iv) in Eq. (5),

$$0 = c_2 \sin\frac{n\pi x}{l}\left(c_4 . \frac{n\pi c}{l}\right)$$

$$c_4 = 0 \qquad [\because c_2 \neq 0]$$

Putting $c_4 = 0$ in Eq. (4),

$$y = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$$= b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \text{ where } c_2 c_3 = b_n$$

Putting $n = 1, 2, 3, \ldots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \ldots(6)$$

Applying the condition (iii) in Eq. (6),

$$kx(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(7)$$

Equation (7) represents the Fourier half-range sine series.

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2k}{l}\left|(lx-x^2)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right|_0^l \\
&= \frac{2k}{l}\left[-\frac{2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right] \\
&= \frac{4kl^2}{n^3\pi^3}\left[-(-1)^n + 1\right]
\end{aligned}$$

Substituting b_n in Eq. (6), the solution is

$$\begin{aligned}
y(x,t) &= \frac{4kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \\
&= \frac{8kl^2}{\pi^3}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^3}\sin\frac{(2r-1)\pi x}{l}\cos\frac{(2r-1)\pi ct}{l}
\end{aligned}$$

$$\left[\begin{aligned} \because 1-(-1)^n &= 0 \quad \text{for } n \text{ even} \\ &= 2 \quad \text{for } n \text{ odd} \\ \text{Taking } n &= 2r-1 \end{aligned}\right]$$

Example 3

The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent times and show that the midpoint of the string always remains at rest.

Solution

Let A and C be the points of the trisection of the string OE of length l. Initially, the string is held in the form $OBDE$ in such a manner that

$$AB = CD = h, \text{ say (Fig. 6.5)}$$

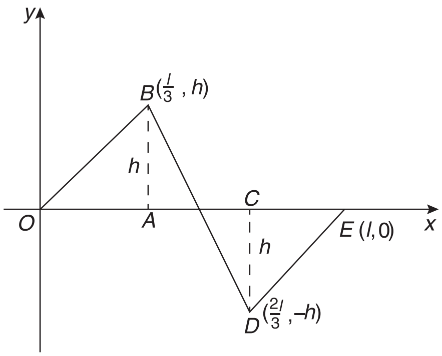

Fig. 6.5

The equation of the line OB is

$$y - 0 = \frac{h-0}{\frac{l}{3}-0}(x-0)$$

$$y = \frac{3h}{l}x$$

The equation of the line BD is

$$y - h = \frac{-h-h}{\frac{2l}{3}-\frac{l}{3}}\left(x-\frac{l}{3}\right)$$

$$= -\frac{6h}{l}\left(x-\frac{l}{3}\right)$$

$$= -\frac{6hx}{l} + 2h$$

$$y = 3h - \frac{6hx}{l}$$

$$= \frac{3h}{l}(l-2x)$$

The equation of the line DE is

$$y - 0 = \frac{-h-0}{\frac{2l}{3}-l}(x-l)$$

$$= \frac{3h}{l}(x-l)$$

The displacement y of any point of the string is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(1)}$$

The solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \quad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $y = 0$ for all t, i.e., $y(0, t) = 0$

(ii) At $x = l$, $y = 0$ for all t, i.e., $y(l, t) = 0$

Since initially ($t = 0$) the string rests in the form of *OBDE*, the initial conditions are

(iii) At $t = 0$,
$$\begin{aligned} y(x,0) &= \frac{3hx}{l}, && 0 \le x \le \frac{l}{3} \\ &= \frac{3h}{l}(l-2x), && \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3h}{l}(x-l), && \frac{2l}{3} \le x \le l \end{aligned}$$

(iv) At $t = 0$, $\dfrac{\partial y}{\partial t} = 0$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$
$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \quad \text{...(3)}$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml(c_3 \cos mct + c_4 \sin mct)$$

$$\sin ml = 0 \qquad [\because c_2 \neq 0]$$
$$ml = n\pi, \qquad n \text{ is an integer}$$
$$m = \frac{n\pi}{l}$$

Putting $m = \dfrac{n\pi}{l}$ in Eq. (3),

$$y = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right) \quad \text{...(4)}$$

$$\frac{\partial y}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left[-c_3\left(\sin\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) + c_4\left(\cos\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right)\right] \quad \text{...(5)}$$

Applying the condition (iv) in Eq. (5),

$$0 = c_2 \sin\frac{n\pi x}{l}\left(c_4 \frac{n\pi c}{l}\right)$$

$$c_4 = 0 \qquad [\because c_2 \neq 0]$$

Putting $c_4 = 0$ in Eq. (4),

$$\begin{aligned} y &= c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \\ &= b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \quad \text{where } c_2c_3 = b_n \end{aligned}$$

Putting $n = 1, 2, 3, \ldots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \text{...(6)}$$

Applying the condition (iii) in Eq. (6),

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \text{...(7)}$$

Equation (7) represents the Fourier half-range sine series.

$$\therefore b_n = \frac{2}{l}\int_0^l y\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{3}} \frac{3hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{3}}^{\frac{2l}{3}} \frac{3h}{l}(l-2x)\sin\frac{n\pi x}{l}\,dx + \int_{\frac{2l}{3}}^{l} \frac{3h}{l}(x-l)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6h}{l^2}\left[\left| x\left\{-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 1\left\{-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_0^{\frac{l}{3}} + \left|(l-2x)\left\{-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - (-2)\left\{-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_{\frac{l}{3}}^{\frac{2l}{3}}\right.$$

$$\left. + \left|(x-l)\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 1\left\{\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_{\frac{2l}{3}}^{l}\right]$$

$$= \frac{6h}{l^2}\left[-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin n\pi - \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$= \frac{6h}{l^2}\frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} + (-1)^n \sin\frac{n\pi}{3}\right] \quad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n \sin\frac{n\pi}{3}\right]$$

$$= \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3}\left[1 + (-1)^n\right]$$

Substituting in Eq. (6), the solution is

$$y(x,t) = \frac{18h}{\pi^2}\sum_{n=1}^{\infty}\sin\frac{n\pi}{3}\left[\frac{1+(-1)^n}{n^2}\right]\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$$= \frac{18h}{\pi^2}\sum_{r=1}^{\infty}\sin\frac{2r\pi}{3}\cdot\frac{2}{(2r)^2}\sin\frac{2r\pi x}{l}\cos\frac{2r\pi ct}{l} \quad \left[\begin{array}{ll}\because\ 1+(-1)^n = 0, & \text{for } n \text{ odd}\\ \qquad\qquad\quad = 2, & \text{for } n \text{ even}\\ \text{Taking } n = 2r & \end{array}\right]$$

$$= \frac{9h}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin\frac{2r\pi}{3}\sin\frac{2r\pi x}{l}\cos\frac{2r\pi ct}{l} \qquad \text{...(8)}$$

To find the displacement of the midpoint, putting $x = \frac{l}{2}$ in Eq. (8),

$$y\left(\frac{l}{2}, t\right) = 0 \qquad [\because \sin r\pi = 0 \text{ as } r \text{ is positive integer}]$$

Example 4

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in the equilibrium position. It is set vibrating by giving to each of its points Q, a velocity of $v_0 \sin^3 \frac{\pi x}{l}$. Find the displacement $y(x, t)$.

[Winter 2017]

Solution

The equation of the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $y = 0$ for all t, i.e., $y(0, t) = 0$

(ii) At $x = l$, $y = 0$ for all t, i.e., $y(l, t) = 0$

The initial conditions are

(iii) At $t = 0$, $\quad y = 0$, $\quad$ i.e., $\quad y(x, 0) = 0$

(iv) At $t = 0$, $\quad \dfrac{\partial y}{\partial t} = v_0 \sin^3 \dfrac{\pi x}{l}$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \qquad ...(3)$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml(c_3 \cos mct + c_4 \sin mct)$$

$$\sin ml = 0 \quad [\because c_2 \neq 0]$$

$$ml = n\pi, \qquad n \text{ is an integer}$$

$$m = \frac{n\pi}{l}$$

Putting $m = \dfrac{n\pi}{l}$ in Eq. (3),

$$y = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right) \qquad ...(4)$$

Applying the condition (iii) in Eq. (4),

$$0 = c_2 \sin\frac{n\pi x}{l}(c_3)$$

$$= c_2 c_3 \sin\frac{n\pi x}{l}$$

$$c_2 c_3 = 0$$

Putting $c_2 c_3 = 0$ in Eq. (4),

$$y = c_2 c_4 \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}$$

$$= b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}, \quad \text{where } c_2 c_4 = b_n$$

Putting $n = 1, 2, 3, ...$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l} \qquad ...(5)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\left(\cos\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) \qquad ...(6)$$

Applying the condition (iv) in Eq. (6),

$$v_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \sin\frac{n\pi x}{l}$$

$$\frac{v_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \sin\frac{n\pi x}{l} \qquad \left[\because \sin 3\pi = 3\sin\pi - 4\sin^3\pi\right]$$

$$= b_1 \frac{\pi c}{l}\sin\frac{\pi x}{l} + b_2\frac{2\pi c}{l}\sin\frac{2\pi x}{l} + b_3\frac{3\pi c}{l}\sin\frac{3\pi x}{l} + b_4\frac{4\pi c}{l}\sin\frac{4\pi x}{l} + \cdots$$

Comparing coefficients of sine terms on both sides,

$$\frac{3v_0}{4} = b_1\frac{\pi c}{l}, \quad 0 = b_2\frac{2\pi c}{l}, \quad -\frac{v_0}{4} = b_3\frac{3\pi c}{l}, \quad 0 = b_4\frac{4\pi c}{l}, \ldots.$$

$$b_1 = \frac{3lv_0}{4\pi c}, \quad b_2 = 0, \qquad b_3 = -\frac{lv_0}{12\pi c}, \quad b_4 = 0, \ldots., b_n = 0, \text{ for } n \ge 4$$

Substituting $b_1, b_2, b_3, b_4 \ldots$ in Eq. (5), the solution is

$$y(x,t) = \frac{3lv_0}{4\pi c}\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \frac{lv_0}{12\pi c}\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}$$

$$= \frac{lv_0}{12\pi c}\left(9\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}\right)$$

Example 5

Find the solution of the wave equation $u_{tt} = c^2 u_{xx}$ $\quad 0 \le x \le L$ *satisfying the condition:*

$$u(0, t) = u(L, t) = 0, \ u_t(x, 0) = 0, \ u(x, 0) = \frac{\pi x}{L} \qquad 0 \le x \le L$$

[Summer 2017]

Solution

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

The solution of Eq. (1) is

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \ldots(2)$$

The boundary conditions are

(i) At $x = 0, \quad u = 0$

(ii) At $x = L, \quad u = 0$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$u = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \qquad ...(3)$$

Putting the condition (ii) in Eq. (3),

$$c_2 \sin mL(c_3 \cos mct + c_4 \sin mct) = 0$$

$$\sin mL = 0$$

$$mL = n\pi, \ n \text{ is an integer}$$

$$m = \frac{n\pi}{L}$$

Putting $m = \frac{n\pi}{L}$ in Eq. (3),

$$u(x,t) = c_2 \sin\frac{n\pi x}{L}\left\{c_3 \cos\left(\frac{n\pi ct}{L}\right) + c_4 \sin\left(\frac{n\pi ct}{L}\right)\right\}$$

$$= \sin\frac{n\pi x}{L}\left\{b_n \cos\left(\frac{n\pi ct}{L}\right) + c_n \sin\left(\frac{n\pi ct}{L}\right)\right\} \qquad ...(4)$$

where $\quad c_2c_3 = b_n, \quad c_2c_4 = c_n$

The initial conditions are

(iii) At $t = 0$, $\quad \frac{\partial u}{\partial t} = 0$

(iv) At $t = 0$, $\quad u = \frac{\pi x}{L}, \quad 0 \le x \le L$

Differentiating Eq. (4) partially w.r.t. t,

$$\frac{\partial u}{\partial t} = \sin\frac{n\pi x}{L}\left\{-b_n \sin\left(\frac{n\pi ct}{L}\right)\left(\frac{n\pi c}{L}\right) + c_n \cos\left(\frac{n\pi ct}{L}\right)\left(\frac{n\pi c}{L}\right)\right\} \qquad ...(5)$$

Applying the condition (iii) in Eq. (5),

$$0 = c_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)$$

$$c_n = 0$$

Putting $c_n = 0$ in Eq. (4),

$$u = b_n \sin\frac{n\pi x}{L} \cos\left(\frac{n\pi ct}{L}\right)$$

Putting $n = 1, 2, 3 \cdots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) \qquad \text{...(6)}$$

Applying the condition (iv) in Eq. (6),

$$\frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad (7)$$

Equation (7) represents the Fourier half-range sine series.

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2\pi}{L^2}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2\pi}{L^2}\left|(x)\left(-\cos\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right) - (L)\left(-\sin\frac{n\pi x}{L}\right)\left(\frac{L^2}{n^2\pi^2}\right)\right|_0^L \\
&= \frac{2\pi}{L^2}\left|-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right|_0^L \\
&= \frac{2\pi}{L^2}\cdot\frac{-L^2}{n\pi}(-1)^n \\
&= -\frac{2}{n}(-1)^n \\
&= \frac{2(-1)^{n+1}}{n}
\end{aligned}$$

Substituting b_n in Eq. (6), the solution is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

EXERCISE 6.9

1. A string of length l is stretched and fastened to two fixed points. Find the solution of the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ when initial displacement $y(x,0) = b\sin\frac{\pi x}{l}$.

$\left[\textbf{Ans.:}\ y(x,t) = b\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}\right]$

2. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \dfrac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.

$$\left[\textbf{Ans.:}\ y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}\right]$$

3. An elastic string is stretched between two points at a distance l apart. In its equilibrium position, a point at a distance a $(a < l)$ from one end is displaced through a distance b transversely and then released from this position. Obtain $y(x, t)$, the vertical displacement if y satisfies the equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$.

$$\left[\textbf{Ans.:}\ y(x,t) = \frac{2bl^2}{a(l-a)\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}\right]$$

4. A tightly stretched violin string of length l fixed at both ends is plucked at $x = \dfrac{1}{3}$ and assumes initially the shape of a triangle of height a. Find the displacement y at any distance x and at any time t after the string is released from rest.

$$\left[\textbf{Ans.:}\ y(x,t) = \frac{9a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}\right]$$

5. If a string of length l is initially at rest in the equilibrium position and each of its points is given a velocity v such that

$$\begin{aligned} v &= c\,x, && 0 < x < \frac{l}{2} \\ &= c\,(lx), && \frac{l}{2} < x < l \end{aligned}$$

find the displacement $y(x, t)$ at any time t.

$$\left[\textbf{Ans.:}\ y(x,t) = \frac{4l^2c}{a\pi^3}\left\{\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} - \frac{1}{33}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l} + ----\right\}\right]$$

6. A string of length l is stretched and fastened to two fixed points. Find the solution of the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ when initial velocity $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = b\sin\dfrac{3\pi x}{l}\cos\dfrac{2\pi x}{l}$.

$$\left[\textbf{Ans.:}\ y(x,t) = \frac{lb}{2a\pi}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} + \frac{lb}{5a\pi}\sin\frac{5\pi x}{l}\sin\frac{5\pi at}{l}\right]$$

6.14 D' ALEMBERT'S SOLUTION OF THE WAVE EQUATION

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(6.47)}$$

Let $u = x + ct$, $v = x - ct$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2}$$

$$= \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly,

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting $\frac{\partial^2 y}{\partial t^2}$ and $\frac{\partial^2 y}{\partial x^2}$ in Eq. (6.47),

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \qquad \text{...(6.48)}$$

Integrating w.r.t. v,

$$\frac{\partial y}{\partial u} = f(u) \qquad \text{...(6.49)}$$

where $f(u)$ is an arbitrary function of u.

Integrating Eq. (6.49) w.r.t. u,

$$y = \int f(u)du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v.

$$y = \phi(u) + \psi(v)$$

where $\quad \phi(u) = \int f(u)\,du$

$$y(x,t) = \phi(x+ct) + \psi(x-ct) \qquad \text{...(6.50)}$$

This is the general solution of Eq. (6.47).

Assume the following conditions to determine ϕ and ψ,

Let at $t = 0$, $y(x,0) = f(x)$ and $\dfrac{\partial y}{\partial t} = 0$

Differentiating Eq. (6.50) w.r.t. t,

$$\frac{\partial y}{\partial t} = c\phi'(x+ct) - c\psi'(x-ct) \qquad \text{...(6.51)}$$

Putting $t = 0$ in Eq. (6.50),

$$f(x) = \phi(x) + \psi(x) \qquad \text{...(6.52)}$$

Putting $t = 0$ in Eq. (6.51),

$$0 = c\phi'(x) - c\psi'(x)$$

$$\phi'(x) = \psi'(x)$$

Integrating,

$$\phi(x) = \psi(x) + k$$

Putting $\phi\,(x)$ in Eq. (6.52),

$$f(x) = \psi(x) + k + \psi(x)$$

$$\psi(x) = \frac{1}{2}\left[f(x) - k\right]$$

$$\therefore \qquad \phi(x) = \frac{1}{2}\left[f(x) + k\right]$$

Replacing x by $(x + ct)$ in $\phi(x)$ and x by $(x - ct)$ in $\psi(x)$ and substituting in Eq. (6.50),

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + k\right] + \frac{1}{2}\left[f(x-ct) - k\right]$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

This is known as D'Alembert's solution of the wave equation (6.47).

Example 1

Using D'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection $f(x) = a\sin^2 \pi x$.

Solution

The equation of the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

By D' Alembert's method, the solution is

$$\begin{aligned}
y(x,t) &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] \\
&= \frac{1}{2}\left[a\sin^2 \pi(x+ct) + a\sin^2 \pi(x-ct)\right] \\
&= \frac{a}{2}\left[\frac{1-\cos 2\pi(x+ct)}{2} + \frac{1-\cos 2\pi(x-ct)}{2}\right] \\
&= \frac{a}{4}\left[2 - \left\{\cos 2\pi(x+ct) + \cos 2\pi(x-ct)\right\}\right] \\
&= \frac{a}{4}\left[2 - 2\cos 2\pi x \cos 2\pi ct\right] \\
&= \frac{a}{2}\left[1 - \cos 2\pi x \cos 2\pi ct\right]
\end{aligned}$$

6.15 ONE-DIMENSIONAL HEAT-FLOW EQUATION

Consider a homogeneous bar of uniform cross-sectional area A and density ρ placed along the x-axis with one end at the origin O (Fig. 6.6). Let us assume that the bar is insulated laterally and, therefore, heat flows only in the x-direction.

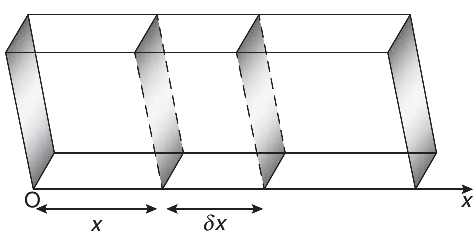

Fig. 6.6 ***One-dimensional heat flow***

Let $u(x, t)$ be the temperature at a distance x from the origin. If δu be the change in temperature in a slab of thickness δx of the bar then quantity of heat in this slab = $s\rho A\,\delta x\,\delta u$

where s is the specific heat of the bar.

The amount of heat crossing any section of the bar $= kA\left(\frac{\partial u}{\partial x}\right)\delta t$

where A = area of cross section of the bar

$\dfrac{\partial u}{\partial x}$ = temperature gradient at the section

δt = time of flow of heat

k = thermal conductivity of the material of the bar

Let Q_1 and Q_2 be the quantity of heat flowing into and flowing out of the slab respectively.

$$Q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \delta t \text{ and } Q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

The negative sign indicates that heat flows in the direction of decreasing temperature. The quantity of heat retained in the slab = $Q_1 - Q_2$

$$s\rho A\delta x\delta u = -kA\left(\frac{\partial u}{\partial x}\right)_x \delta t + kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

$$\frac{\delta u}{\delta t} = \frac{k}{s\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Taking limit $\delta x \to 0$ and $\delta t \to 0$,

$$\frac{\partial u}{\partial t} = \frac{k}{s\rho}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(6.53)}$$

where $\dfrac{k}{s\rho} = c^2$ is known as *diffusivity of the material of the bar*.

Equation (6.53) is known as the *one-dimensional heat-flow equation*.

6.15.1 Solution of the One-Dimensional Heat-Flow Equation

The one-dimensional heat-flow equation is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(6.54)}$$

Let $u = X(x) \cdot T(t)$ be a solution of Eq. (6.54),

$$\frac{\partial^2 u}{\partial x^2} = X''T, \qquad \frac{\partial u}{\partial t} = XT'$$

Substituting in Eq. (6.54),

$$XT' = c^2 X''T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k, \text{ say}$$

Considering $\frac{X''}{X} = k$, $\quad \frac{d^2X}{dx^2} - kX = 0$ ...(6.55)

Considering $\frac{1}{c^2}\frac{T'}{T} = k$, $\quad \frac{dT}{dt} - kc^2T = 0$ ...(6.56)

Solving Eqs (6.55) and (6.56), the following cases arise:

(i) When k is positive

Let $k = m^2$

$$\frac{d^2X}{dx^2} - m^2X = 0 \quad \text{and} \quad \frac{dT}{dt} - m^2c^2T = 0$$

$$X = c_1' e^{mx} + c_2' e^{-mx} \quad \text{and} \quad T = c_3' e^{m^2c^2t}$$

Hence, the solution of Eq. (6.54) is

$$u = \left(c_1' e^{mx} + c_2' e^{-mx} \right)\left(c_3' e^{m^2c^2t} \right) \qquad \text{...(6.57)}$$

(ii) When k is negative

Let $k = -m^2$

$$\frac{d^2X}{dx^2} + m^2X = 0 \quad \text{and} \quad \frac{dT}{dt} + m^2c^2T = 0$$

$$X = c_1' \cos mx + c_2' \sin mx \quad \text{and} \quad T = c_3' e^{-m^2c^2t}$$

Hence, the solution of Eq. (6.54) is

$$u = \left(c_1' \cos mx + c_2' \sin mx\right)\left(c_3' e^{-m^2c^2t}\right) \qquad \text{...(6.58)}$$

(iii) When $k = 0$

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

$$X = c_1' x + c_2' \quad \text{and} \quad T = c_3'$$

Hence, the solution of Eq. (6.54) is

$$u = (c_1' x + c_2')\, c_3' \qquad \text{...(6.59)}$$

Out of these three solutions, we need to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with problems of heat conduction, temperature u must decrease with the increase of time.

Hence, the solution is of the form given by Eq. (6.58),

$$u = (c_1 \cos mx + c_2 \sin mx)\, e^{-m^2c^2t}$$

where $c_1' c_3' = c_1$ and $c_2' c_3' = c_2$

- *Transient Solution* The solution is known as transient if u decreases as t increases.
- *Steady-state Condition* A condition is known as steady state if the dependent variables are independent of the time t.

One End Insulated

Example 1

The differential equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation subject to the following conditions:*

(i) u *is finite when* $t \to \infty$

(ii) $\frac{\partial u}{\partial x} = 0$ *when* $x = 0$ *for all values of* t

(iii) $u = 0$ *when* $x = l$ *for all values of* t

(iv) $u = u_0$ *when* $t = 0$ *for* $0 < x < l$

Solution

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

Since u is finite when $t \to \infty$, the solution of Eq. (1) is of the form

$$u = \left(c_1 \cos mx + c_2 \sin mx\right)e^{-km^2 t} \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = l$, $u = 0$ for all t, i.e., $u\,(l, t) = 0$

(ii) At $x = 0$, $\frac{\partial u}{\partial x} = 0$ for all t, i.e., $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$

The initial conditions are

(iii) At $t = 0$, $u = u_0$ for $0 < x < l$

Applying the condition (ii) in Eq. (2),

$$\frac{\partial u}{\partial x} = (-c_1 m \sin mx + c_2 m \cos mx)e^{-km^2 t}$$

$$0 = c_2 m e^{-km^2 t}$$

$$\therefore \qquad c_2 = 0$$

Putting $c_2 = 0$ in Eq. (2),

$$u = c_1 e^{-km^2 t} \cos mx \qquad \text{...(3)}$$

Applying the condition (i) in Eq. (3),

$$0 = c_1 e^{-km^2 t} \cos ml$$

$$\cos ml = 0 \qquad [\because c_1 \neq 0]$$

$$= \cos(2n+1)\frac{\pi}{2}, \ n \text{ is an integer.}$$

$$ml = (2n+1)\frac{\pi}{2}$$

$$m = (2n+1)\frac{\pi}{2l}$$

Putting $m = (2n+1)\dfrac{\pi}{2l}$ in Eq. (3),

$$u = c_1 \, \mathrm{e}^{-k(2n+1)^2 \frac{\pi^2}{4l^2} t} \cos(2n+1)\frac{\pi x}{2l}$$

Putting $n = 0, 1, 2, ...$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = \sum_{n=0}^{\infty} a_{2n+1} \, e^{-k(2n+1)^2 \frac{\pi^2 t}{4l^2}} \cos(2n+1)\frac{\pi x}{2l} \qquad ...(4)$$

Applying the condition (iii) to Eq. (4),

$$u(x,0) = \sum_{n=0}^{\infty} a_{2n+1} e^0 \cos(2n+1)\frac{\pi x}{2l}$$

$$u_0 = \sum_{n=0}^{\infty} a_{2n+1} \cos(2n+1)\frac{\pi x}{2l} \qquad ...(5)$$

Equation (5) represents the Fourier half-range cosine series.

$$a_{2n+1} = \frac{2}{l}\int_0^l u_0 \cos(2n+1)\frac{\pi x}{2l} dx$$

$$= \frac{2u_0}{l}\left[\frac{\sin(2n+1)\frac{\pi x}{2l}}{(2n+1)\frac{\pi}{2l}}\right]_0^l$$

$$= \frac{2u_0}{l} \cdot \frac{2l}{(2n+1)\pi} \sin(2n+1)\frac{\pi}{2}$$

$$= \frac{4u_0}{(2n+1)\pi} \sin(2n+1)\frac{\pi}{2}$$

Substituting a_{2n+1} in Eq. (4), the general solution is

$$u = \frac{4u_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin(2n+1)\frac{\pi}{2} \cos(2n+1)\frac{\pi x}{2l} e^{-k(2n+1)^2 \frac{\pi^2 t}{4l^2}}$$

Steady State and Zero-Boundary Conditions

Example 1

A laterally insulated bar of length l has its ends A and B maintained at 0°C *and* 100°C *respectively until steady-state conditions prevail. If the temperature at B is suddenly reduced to* 0°C *and kept so while that of A is maintained at* 0°C*, find the temperature at a distance x from A at any time t.*

Solution

The equation for heat conduction is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

In the steady-state condition at $t = 0$, u is independent of t.

$$\therefore \qquad \frac{\partial u}{\partial t} = 0$$

Thus, Eq. (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{...(2)}$$

Integrating Eq. (2) two times, its general solution is

$$u = ax + b \qquad \text{...(3)}$$

At $x = 0$, $u = 0$ and at $x = l$, $u = 100$

Applying these conditions to Eq. (3),

$$b = 0, \ a = \frac{100}{l}$$

Putting a and b in Eq. (3),

$$u = \frac{100}{l} x$$

Thus, the initial condition is

(i) At $t = 0$, $\qquad u = \dfrac{100}{l} x$

The boundary conditions are

(ii) At $x = 0$, $u = 0$ for all t, i.e., $u(0, t) = 0$

(iii) At $x = l$, $u = 0$ for all t, i.e., $u(l, t) = 0$

Since u decreases as t increases, the solution of Eq. (1) is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)e^{-c^2m^2t} \qquad \text{...(4)}$$

Applying the condition (ii) in Eq. (4),

$$0 = c_1 e^{-c^2m^2t}$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (4),

$$u = c_2 \sin mx \cdot e^{-c^2m^2t} \qquad \text{...(5)}$$

Applying the condition (iii) in Eq. (5),

$$0 = c_2 \sin ml \cdot e^{-c^2m^2t}$$

$$\sin ml = 0 \quad [\because c_2 \neq 0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$ml = n\pi$$

$$m = \frac{n\pi}{l}$$

Putting $m = \frac{n\pi}{l}$ in Eq. (5),

$$u = c_2 \sin \frac{n\pi x}{l} e^{-c^2 \frac{n^2\pi^2}{l^2}t}$$

Putting $n = 1, 2, 3, \ldots\ldots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2}t} \qquad \text{...(6)}$$

Applying the condition (i) to Eq. (6),

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^0$$

$$\frac{100}{l} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{...(7)}$$

Equation (7) represents the Fourier half-range sine series.

$$b_n = \frac{2}{l}\int_0^l \frac{100}{l} x \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left| x\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 1\left\{\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_0^l$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}\cos n\pi + \frac{l^2}{n^2\pi^2}\sin n\pi\right]$$

$$= -\frac{200}{n\pi}(-1)^n \qquad \left[\because \quad \cos n\pi = (-1)^n, \quad \sin n\pi = 0\right]$$

$$= \frac{200}{n\pi}(-1)^{n+1}$$

Substituting b_n in Eq. (6), the general solution is

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{c^2 n^2 \pi^2}{l^2}t}$$

Steady-State and Nonzero Boundary Conditions

Example 1

A bar AB of 10 *cm length has its ends A and B kept at* 30°*C and* 100°*C respectively, until steady-state condition is reached. Then the temperature at A is lowered to* 20°*C and that at B to* 40°*C and these temperatures are maintained. Find the subsequent temperature distribution in the bar.*

Solution

Let the equation for heat conduction be

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

In the steady-state condition at $t = 0$, u is independent of t.

$$\frac{\partial u}{\partial t} = 0$$

Thus, Eq. (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{...(2)}$$

Integrating Eq. (2) two times, its general solution is

$$u = ax + b \qquad ...(3)$$

At $x = 0$, $u = 30$ and at $x = 10$, $u = 100$
Applying these conditions in Eq. (3),

$$30 = b \text{ and } 100 = 10a + b = 10a + 30$$

$$a = 7$$

Putting a and b in Eq. (3),

$$u = 7x + 30$$

Thus, the initial condition is

(i) At $t = 0$, $u = 7x + 30$

The boundary conditions are

(ii) At $x = 0$, $u = 20$ for all t, i.e., $u(0, t) = 20$
(iii) At $x = 10$, $u = 40$ for all t, i.e., $u(10, t) = 40$

Since temperature at the end points is nonzero, these conditions are called nonhomogeneous.
To find the temperature distribution in the bar, assume the solution as

$$u(x,t) = u_s(x) + u_{tr}(x,t) \qquad ...(4)$$

where $u_s(x)$ is the steady-state solution and $u_{tr}(x)$ is the transient solution.

To determine $u_s(x)$, solve $\dfrac{\partial^2 u_s}{\partial x^2} = 0$

Its solution is

$$u_s = a_1 x + b_1 \qquad ...(5)$$

At $x = 0$, $u_s = 20$ and at $x = 10$, $u_s = 40$
Applying these conditions in Eq. (5),

$$20 = b_1 \text{ and } 40 = 10a_1 + b_1 = 10a_1 + 20$$

$$a_1 = 2$$

Thus, $$u_s = 2x + 20 \qquad ...(6)$$

Since $u_{tr}(x, t)$ satisfies the one-dimensional heat equation,

$$u_{tr}(x,t) = (c_1 \cos mx + c_2 \sin mx)e^{-c^2m^2t} \qquad ...(7)$$

Substituting Eqs (6) and (7) in Eq. (4),

$$u(x,t) = 2x + 20 + (c_1 \cos mx + c_2 \sin mx)e^{-c^2m^2t} \qquad ...(8)$$

Applying the condition (ii) in Eq. (8),

$$20 = 20 + c_1 e^{-c^2m^2t}$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (8),

$$u = 2x + 20 + c_2 \sin mx \cdot e^{-c^2m^2t} \qquad ...(9)$$

Applying the condition (iii) in Eq. (9),

$$40 = 20 + 20 + c_2 \sin 10m \cdot e^{-c^2 m^2 t}$$

$$0 = c_2 \sin 10m \cdot e^{-c^2 m^2 t}$$

$$\sin 10m = 0 \qquad [\because c_2 \neq 0]$$

$$= \sin n\pi, \ n \text{ is an integer}$$

$$m = \frac{n\pi}{10}$$

Putting $m = \frac{n\pi}{10}$ in Eq. (9),

$$u = 2x + 20 + c_2 \sin \frac{n\pi x}{10} \cdot e^{-c^2 \frac{n^2\pi^2}{100} t}$$

Putting $n = 1, 2, 3, ...$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \cdot e^{-\frac{c^2 n^2 \pi^2}{100} t} \qquad ...(10)$$

Applying the condition (i) in Eq. (10),

$$u(x,0) = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^0$$

$$7x + 30 = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}$$

$$5x + 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \qquad ...(11)$$

Equation (11) represents the Fourier half-range sine series.

$$b_n = \frac{2}{10} \int_0^{10} (5x + 10) \sin \frac{n\pi x}{10} \, dx$$

$$= \frac{1}{5} \left| (5x+10) \left\{ \frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right\} - 5 \left\{ \frac{-\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10} \right)^2} \right\} \right|_0^{10}$$

$$= \frac{1}{5} \left| -\frac{(5x+10)10}{n\pi} \cos \frac{n\pi x}{10} + \frac{5(100)}{n^2\pi^2} \sin \frac{n\pi x}{10} \right|_0^{10}$$

$$= \frac{1}{5}\left[-\frac{600}{n\pi}\cos n\pi + \frac{100}{n\pi}\right] \qquad [\because \sin n\pi = 0]$$

$$= \frac{20}{n\pi}\left[-6(-1)^n + 1\right]$$

Substituting b_n in Eq. (10), the general solution is

$$u(x,t) = 2x + 20 + \frac{20}{\pi}\sum\left[\frac{1-6(-1)^n}{n}\right]\sin\frac{n\pi x}{10}\cdot e^{-\frac{c^2n^2\pi^2}{100}t}$$

Both Ends Insulated

Example 1

The temperature at one end of a $50\,cm$ *long bar with insulated sides, is kept at* $0°C$ *and that the other end is kept at* $100°C$ *until steady-state condition prevails. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution

The equation for temperature distribution is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

In the steady-state condition at $t = 0$, u is independent of t.

$$\therefore \qquad \frac{\partial u}{\partial t} = 0$$

Thus, Eq. (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad ...(2)$$

Integrating Eq. (2) two times, its general solution is

$$u = ax + b \qquad ...(3)$$

At $x = 0$, $u = 0$ and at $x = 50$, $u = 100$

Applying these conditions in Eq. (3),

$$0 = b, \quad 100 = 50\,a + b = 50a$$
$$a = 2$$

Putting a and b in Eq. (3),

$$u = 2x$$

Thus, the initial condition is

(i) At $t = 0$, $u = 2x$

When the ends $x = 0$ and $x = 50$ of the bar are insulated, no heat can flow through them.

Thus, the boundary conditions are

(ii) At $x = 0$, $\frac{\partial u}{\partial x} = 0$ for all t, i.e., $\frac{\partial u(0,t)}{\partial x} = 0$

(iii) At $x = 50$, $\frac{\partial u}{\partial x} = 0$ for all t, i.e., $\frac{\partial u(50,t)}{\partial x} = 0$

The solution of Eq. (1) is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)e^{-c^2m^2t} \quad \text{...(4)}$$

$$\frac{\partial u}{\partial x} = (-c_1 m \sin mx + c_2 m \cos mx)e^{-c^2m^2t} \quad \text{...(5)}$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 m e^{-c^2m^2t}$$

$$c_2 = 0$$

Putting $c_2 = 0$ in Eq. (5),

$$\frac{\partial u}{\partial x} = -c_1 m \sin mx \cdot e^{-c^2m^2t} \quad \text{...(6)}$$

Applying the condition (iii) in Eq. (6),

$$0 = -c_1 m \sin 50m \cdot e^{-c^2m^2t}$$

$$\sin 50\,m = 0 \qquad [\because c_1 \neq 0, \text{ otherwise } u(x,t) = 0]$$

$$= \sin n\pi, \qquad n \text{ is an integer}$$

$$50m = n\pi$$

$$m = \frac{n\pi}{50}$$

Putting $m = \frac{n\pi}{50}$ and $c_2 = 0$ in Eq. (4),

$$u(x,t) = c_1 \cos\frac{n\pi x}{50} \cdot e^{-\frac{c^2n^2\pi^2}{2500}t} \quad \text{...(7)}$$

Putting $n = 0, 1, 2, ...$ in Eq. (7) and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{50} \cdot e^{-\frac{c^2n^2\pi^2}{2500}t}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi\, x}{50} \cdot e^{-\frac{c^2n^2\pi^2}{2500}t} \quad \text{...(8)}$$

Applying the condition (i) in Eq. (8),

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi\, x}{50} \cdot e^0$$

$$2x = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{50} \qquad \text{...(9)}$$

Equation (9) represents the Fourier half-range cosine series.

$$\begin{aligned}
a_0 &= \frac{1}{50}\int_0^{50} 2x dx \\
&= \frac{1}{25}\left|\frac{x^2}{2}\right|_0^{50} \\
&= 50
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{50}\int_0^{50} 2x\cdot\cos\frac{n\pi x}{50}\,\mathrm{d}x \\
&= \frac{2}{25}\left| x\cdot\frac{\sin\frac{n\pi x}{50}}{\left(\frac{n\pi}{50}\right)} - 1\left\{\frac{-\cos\frac{n\pi x}{50}}{\left(\frac{n\pi}{50}\right)^2}\right\}\right|_0^{50} \\
&= \frac{2}{25}\left[\left(\frac{50}{n\pi}\right)^2(\cos n\pi - \cos 0)\right] \\
&= \frac{200}{n^2\pi^2}\left[(-1)^n - 1\right]
\end{aligned}$$

Substituting a_0 and a_n in Eq. (8), the general solution is

$$\begin{aligned}
u(x,t) &= 50 + \frac{200}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{50}\cdot e^{-\frac{c^2 n^2 \pi^2}{2500}t} \\
&= 50 - \frac{400}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}\cos\frac{(2r-1)\pi x}{50}\cdot e^{-\frac{c^2(2r-1)^2\pi^2}{2500}t}
\end{aligned}$$

$$\left[\begin{aligned} \because (-1)^n - 1 &= 0, \quad \text{if } n \text{ is even} \\ &= -2, \quad \text{if } n \text{ is odd} \\ \text{Taking} \quad n &= 2r-1 \end{aligned}\right]$$

Zero Boundary Conditions

Example 1

Find the temperature in a laterally insulated bar of $2\,cm$ length whose ends are kept at zero temperature and the initial temperature is

$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$.

Solution

The equation for heat conduction is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

Since both the ends of the bar are at zero temperature, the solution of Eq. (1) is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)e^{-c^2m^2t} \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $u = 0$ for all t, i.e., $u(0, t) = 0$

(ii) At $x = 2$, $u = 0$ for all t, i.e., $u(2, t) = 0$

The initial conditions are

(iii) At $t = 0$, $u = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

Applying the condition (i) in Eq. (2),

$$0 = c_1\, e^{-c^2m^2t}$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$u = c_2 \sin mx \cdot e^{-c^2m^2t} \qquad \text{...(3)}$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin 2m \cdot e^{-c^2m^2t}$$

$$\sin 2m = 0 \qquad [\because c_2 \neq 0]$$

$$= \sin n\pi, \qquad n \text{ is an integer}$$

$$2m = n\pi$$

$$m = \frac{n\pi}{2}$$

Putting $m = \frac{n\pi}{2}$ in Eq. (3),

$$u = c_2 \sin\frac{n\pi x}{2} \cdot e^{-c^2\frac{n^2\pi^2}{4}t}$$

Putting $n = 1, 2, 3, \ldots\ldots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \cdot e^{-\frac{c^2n^2\pi^2}{4}t} \qquad \text{...(4)}$$

Applying the condition (iii) in Eq. (4),

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\cdot e^0$$

$$\sin\frac{\pi x}{2}+3\sin\frac{5\pi x}{2}=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$=b_1\sin\frac{\pi x}{2}+b_2\sin\pi x+b_3\sin\frac{3\pi x}{2}+b_4\sin 2\pi x+b_5\sin\frac{5\pi x}{2}+\ldots$$

Comparing coefficients on both sides,

$$b_1=1,\quad b_2=0,\quad b_3=0,\quad b_4=0,\quad b_5=3,\quad b_6=0\ldots,\quad b_n=0,\quad \text{for } n\ge 6$$

Substituting the values of b's in Eq. (4), the general solution is

$$u(x,\,t)=b_1\sin\frac{\pi x}{2}\cdot e^{-\frac{c^2\pi^2}{4}t}+b_5\sin\frac{5\pi x}{2}\cdot e^{-\frac{25c^2\pi^2}{4}t}$$

$$=\sin\frac{\pi x}{2}\cdot e^{-\frac{c^2\pi^2}{4}t}+3\sin\frac{5\pi x}{2}\cdot e^{-\frac{25c^2\pi^2}{4}t}$$

Example 2

A homogeneous rod of conducting material of 100 cm length has its ends kept at zero temperature and the temperature initially is

$$u(x,0)=\begin{cases} x & 0\le x\le 50 \\ 100-x & 50\le x\le 100\end{cases}$$

Find the temperature $u(x,\,t)$ at any time. **[Summer 2015]**

Solution

Let the equation for heat conduction be

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

The solution of the heat equation is

$$u(x,t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}e^{-\frac{c^2\pi^2 n^2 t}{l^2}} \qquad \ldots(2)$$

By the initial condition,

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}=f(x) \qquad \ldots(3)$$

which is a half-range sine series where $b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$ ...(4)

$$b_n = \frac{2}{l}\left[\int_0^{\frac{l}{2}} f(x)\sin\frac{nx\pi}{l}\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

Here, $l = 100 \quad \therefore \quad \frac{l}{2} = 50$

$$b_n = \frac{2}{l}\left[\int_0^{\frac{l}{2}} x\,\sin\frac{n\pi x}{l}\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} (l-x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{2}{l}\left[\left| x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{2}{l}\left[\left| -\frac{xl}{n\pi}\left(\cos\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi x}{l}\right)\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left| -\frac{l(l-x)}{n\pi}\left(\cos\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi x}{l}\right)\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4l}{n^2\pi^2}(-1)^{n+1} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{4l(-1)^r}{(2r-1)^2\pi^2} \qquad \text{where } r = 1, 2, 3, \ldots, n = 2r-1$$

Substituting the value of b_n in Eq. (2), the general solution is

$$u(x,t) = \frac{4l}{\pi^2}\sum_{r=1}^{\infty}\frac{(-1)^r}{(2r-1)^2}\sin\frac{(2r-1)\pi x}{l}\cdot e^{-\frac{c^2\pi^2 t}{l^2}(2r-1)^2}$$

Putting $l = 100$,

$$u(x,t) = \frac{400}{\pi^2}\sum_{r=1}^{\infty}\frac{(-1)^r}{(2r-1)^2}e^{-\left[\frac{(2r-1)c\pi}{100}\right]^2 t}\sin\frac{(2r-1)\pi x}{100}$$

Example 3

Using separable variable technique, find the acceptable general solution to the one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ *and find the solution satisfying the condition* $u(0, t) = u(\pi, t) = 0$ *for* $t > 0$ *and* $u(x, 0) = \pi - x$, $0 < x < \pi$. **[Winter 2015]**

Solution

The heat equation is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

Let the solution be

$$u(x, t) = X(x)\, T(t)$$

$$\frac{\partial u}{\partial t} = XT'$$

and

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting these values in Eq. (1),

$$XT' = c^2X''T$$

$$\frac{T'}{T} = c^2\frac{X''}{X}$$

$$\frac{T'}{c^2T} = \frac{X''}{X} = -k^2, \text{ say}$$

$$\frac{T'}{T} = -k^2c^2$$

$$\log T = -k^2c^2t$$

$$T = -e^{k^2c^2t}$$

$$= c_1e^{-k^2c^2t}$$

$$\frac{X''}{X} = -k^2$$

$$X'' + k^2X = 0$$

$$(D^2 + k^2)X = 0$$

A.E. $\quad m^2 + k^2 = 0$

$$m = \pm ik$$

$$X = c_2\cos(kx) + c_3\sin(kx)$$

Hence, the solution is

$$u(x, t) = X(x)\, T(t)$$

$$= c_1 e^{-k^2c^2t}\,[c_2 \cos(kx) + c_3 \sin(kx)]$$

$$= Ae^{-k^2c^2t}\cos(kx) + Be^{-k^2c^2t}\sin(kx) \qquad \text{...(2)}$$

where $A = c_1c_2$ and $B = c_1c_3$

Given $u(0, t) = 0$

$$A = 0$$

Putting $u(\pi, t) = 0$ in Eq. (2),

$$0 = Be^{-k^2c^2t}\sin(k\pi)$$

$$\sin k\pi = 0 \qquad (\quad B \neq 0)$$

$$\sin k\pi = \sin n\pi$$

$$k = n$$

Putting $k = n$ in Eq. (2),

$$u = Be^{-n^2c^2t}\sin nx$$

Putting $n = 1, 2, 3, \ldots$ and adding all these solutions by principle of superposition, the general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2c^2t}\sin nx \qquad \text{...(3)}$$

Applying the condition, $u(x, 0) = \pi - x \quad 0 < x < \pi$ in Eq. (3),

$$u(x, 0) = \pi - x = \sum_{n=1}^{\infty} b_n \sin nx\, e^0, \text{where } e^0 = 1$$

$$\pi - x = \sum_{n=1}^{\infty} b_n \sin nx,\ 0 < x < \pi$$

Equation (4) represents the Fourier half-range sine series.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin nx\, dx$$

$$= \frac{2}{\pi}\left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left|(x-\pi)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(-\pi)\cdot\frac{1}{n}\right] \qquad [\because \sin n\pi = \sin 0 = 0, \cos 0 = 1]$$

$$= \frac{2}{n}$$

Substituting b_n in Eq. (3), the solution is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2}{n} e^{-n^2c^2t} \sin nx$$

EXERCISE 6.10

1. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ and $u(l, t) = 0$ where $0 < x < 1, t > 0$.

$$\left[\textbf{Ans.}: u(x, t) = 3\sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x\right]$$

2. Find the transient-state temperature of a nonradiating rod of length π whose ends are kept at ice-cold temperature, the temperature of the rod being initially $(\pi x - x^2)$ at a distance x from an end.

$$\left[\textbf{Ans.}: u(x,t) = \frac{8}{\pi}\sum_{r=1}^{\infty} \frac{1}{(2r-1)^3} \sin(2r-1)\pi \cdot e^{-c^2(2r-1)^2 t}\right]$$

3. Solve the equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod of length l subject to the following conditions:

(i) u is finite for $t \to \infty$

(ii) $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ for all t

(iii) $u = lx - x^2$ for $t = 0$ between $x = 0$ and $x = l$

$$\left[\textbf{Ans.: } u(x,t) = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-\frac{4m^2\pi^2 k}{l^2}t}\right]$$

4. A bar *AB* of 20 cm length has its ends *A* and *B* kept at 30°C and 80°C until steady-state prevails. Then the temperatures at *A* and *B* are suddenly changed to 40°C and 60°C respectively. Find the temperature distribution of the rod.

$$\left[\textbf{Ans.: } u(x,t) = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1+2\cos n\pi}{n}\sin\frac{n\pi x}{20}\cdot e^{-\frac{n^2\pi^2c^2}{400}t}\right]$$

5. A rod of length *l* has its ends *A* and *B* kept at 0°C and 100°C respectively until steady-state condition prevails. Temperature at *A* is raised to 25°C and that of *B* is reduced to 75°C and kept so. Find the temperature distribution.

$$\left[\textbf{Ans.: } u(x,t) = \frac{50x}{l} + 25 - \frac{50}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}e^{-\frac{4c^2n^2\pi^2}{l^2}t}\right]$$

6. A 100 cm long bar, with insulated sides, has its ends kept at 0°C and 100°C until steady-state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.

$$\left[\textbf{Ans.: } u(x,t) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}e^{\frac{-c^2(2n-1)^2\pi^2}{l^2}t}\right]$$

7. A bar with insulated sides is initially at a temperature of 0°C throughout. The end $x = 0$ is kept at 0°C and heat is suddenly applied so that $\frac{\partial u}{\partial x} = 10$ at $x = l$ for all t. Find the temperature distribution.

$$\left[\textbf{Ans.: } u(x,t) = 10x - \frac{80l}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sin\left(\frac{2n+1}{2l}\right)\pi x\cdot e^{\frac{-c^2(2n+1)^2\pi^2}{l^2}t}\right]$$

8. Using D'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.

$$[\textbf{Ans.: } y(x,t) = k(\sin x\cos ct - \sin 2x\cos 2ct)]$$

6.16 TWO-DIMENSIONAL HEAT-FLOW EQUATION

Consider a homogeneous metal plate of uniform thickness h (cm), density ρ (g/cm^3), specific heat s (cal/g/deg), and thermal conductivity k (cal/cm deg). Assume that the faces of the plate are perfectly insulated so that no heat flows in the transversal direction to the plate. Hence, heat is allowed to flow only in the directions of the plane of the plate. Therefore, the flow is said to be two-dimensional. Let the plate be in the xy-plane and u be the temperature at any point of the plate. Since the faces of the plate are insulated, u depends only on x, y, and the time t.

Consider a small rectangular element $ABCD$ of the plate with vertices A (x, y), B $(x + \delta x, y)$, C $(x + \delta x, y + \delta y)$, and D $(x, y + \delta y)$ (Fig. 6.7).

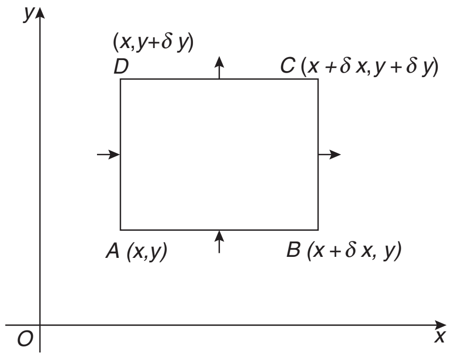

Fig 6.7 ***Two-dimensional heat flow***

The amount of heat, at time t, in the element is $Q = \rho\,\delta x\,\delta y\,hsu$

The rate of change of Q w. r. t. time is $\dfrac{dQ}{dt} = \rho\,\delta x\,\delta y\,hs\dfrac{\partial u}{\partial t}$ ...(6.60)

The amount of heat entering the element in 1 second from the side

$$AB = -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y$$

The amount of heat entering the element in 1 second from the side

$$AD = -kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_x$$

The amount of heat flowing out the element in 1 second from the side

$$CD = -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$$

The amount of heat flowing out the element in 1 second from the side

$$BC = -kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

The total rate of gain of heat by the element

$$= -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y - kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_x + kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= kh\,\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] + kh\,\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$= kh\,\delta x\delta y\left[\frac{\left(\dfrac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\dfrac{\partial u}{\partial y}\right)_{y}}{\delta y}\right]+kh\delta x\delta y\left[\frac{\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\dfrac{\partial u}{\partial x}\right)_{x}}{\delta x}\right] \qquad ...(6.61)$$

Equating Eqs (6.60) and (6.61),

$$\rho\delta x\delta yhs\frac{\partial u}{\partial t}=kh\,\delta x\delta y\left[\frac{\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\dfrac{\partial u}{\partial x}\right)_{x}}{\delta x}+\frac{\left(\dfrac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\dfrac{\partial u}{\partial y}\right)_{y}}{\delta y}\right]$$

Dividing both sides by $h\delta x\delta y$ and taking limit $\delta x\to 0,\ \delta y\to 0,$

$$\frac{\partial u}{\partial t}=\frac{k}{\rho s}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad ...(6.62)$$

where $\dfrac{k}{\rho s}=c^2$ is known as the *diffusivity of the material of the plate*.

Equation (6.62) is known as the *two-dimensional heat-flow equation* and gives the temperature distribution of the plate in the transient state.
In the steady state, u is independent of t.

$$\therefore \qquad \frac{\partial u}{\partial t}=0$$

Hence, Eq. (6.62) reduces to $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

which is Laplace's equation in two dimensions.

Solution of Laplace's Equation

Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad ...(6.63)$$

Let $u = X(x).Y(y)$ be a solution of Eq. (6.63).

$$\frac{\partial^2 u}{\partial x^2}=X''Y, \quad \frac{\partial^2 u}{\partial y^2}=XY''$$

Substituting in Eq (6.63),

$$X''Y + XY'' = 0$$

Dividing by XY,

$$\frac{X''}{X} = -\frac{Y''}{Y} = k, \text{ say}$$

Considering $\frac{X''}{X} = k$, $\quad \frac{d^2X}{dx^2} - kX = 0 \qquad \text{...(6.64)}$

Considering $-\frac{Y''}{Y} = k$, $\quad \frac{d^2Y}{dy^2} + kY = 0 \qquad \text{...(6.65)}$

Solving Eqs (6.64) and (6.65), the following cases arise.

(i) When *k* is positive

Let $k = m^2$

$$\frac{d^2X}{dx^2} - m^2X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + m^2Y = 0$$

$$X = c_1e^{mx} + c_2e^{-mx} \quad \text{and} \quad Y = c_3 \cos my + c_4 \sin my$$

Hence, the solution of Eq. (6.63) is

$$u = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my)$$

(ii) When *k* is negative

Let $k = -m^2$

$$\frac{d^2X}{dx^2} + m^2X = 0 \text{ and } \frac{d^2Y}{dy^2} - m^2Y = 0$$

$$X = c_1 \cos mx + c_2 \sin mx \text{ and } Y = c_3 e^{my} + c_4 e^{-my}$$

Hence, the solution of Eq. (6.63) is

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my})$$

(iii) When *k* = 0

$$\frac{d^2X}{dx^2} = 0 \text{ and } \frac{d^2Y}{dy^2} = 0$$

$$X = c_1x + c_2 \text{ and } Y = c_3y + c_4$$

Hence, the solution of Eq. (6.63) is

$$u = (c_1x + c_2)(c_3y + c_4)$$

Out of these three solutions, we need to choose that solution which is consistent with the physical nature of the problem.

Example 1

Find the steady-state temperature distribution in a thin plate bounded by the lines $x = 0$, $x = a$, $y = 0$, and $y \to \infty$ assuming that heat cannot escape from either surface. The sides $x = 0$, $x = a$, and $y \to \infty$ being kept at zero temperature and $y = 0$ is kept at $f(x)$.

Solution

In steady state, the heat equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

The three possible solutions of Eq. (1) are

$$u = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my) \qquad ...(2)$$

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my}) \qquad ...(3)$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \qquad ...(4)$$

The boundary conditions are

(i) At $x = 0$, $u = 0$, i.e., $u(0, y) = 0$
(ii) At $x = a$, $u = 0$, i.e., $u(a, y) = 0$
(iii) At $y \to \infty$, $u = 0$, i.e., $u(x, \infty) = 0$
(iv) At $y = 0$, $u = f(x)$, i.e., $u(x, 0) = f(x)$

Applying conditions (i) and (ii) in Eq. (2),

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 e^{ma} + c_2 e^{-ma} = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.

Hence, the solution by Eq. (2) is rejected.

Applying conditions (i) and (ii) in Eq. (4),

$$c_2 = 0 \quad \text{and} \quad c_1 a + c_2 = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.

Hence, the solution by Eq. (4) is rejected.

Thus, the suitable solution for the present problem is a solution by Eq. (3).

Applying the condition (i) in Eq. (3),

$$0 = c_1 (c_3 e^{my} + c_4 e^{-my})$$
$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (3),

$$u = c_2 \sin mx\,(c_3 e^{my} + c_4 e^{-my}) \qquad ...(5)$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 \sin ma\,(c_3 e^{my} + c_4 e^{-my})$$

$$\sin ma = 0 \quad [\because c_2 \neq 0, \text{ otherwise } u = 0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$ma = n\pi$$

$$m = \frac{n\pi}{a}$$

Putting $m = \frac{n\pi}{a}$ in Eq. (5),

$$u = c_2 \sin \frac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right) \qquad ...(6)$$

Applying the condition (iii) in Eq. (6) after rewriting as

$$u e^{-\frac{n\pi y}{a}} = c_2 \sin \frac{n\pi x}{a}\left(c_3 + c_4 e^{-\frac{2n\pi y}{a}}\right)$$

$$0 = c_2 \sin \frac{n\pi x}{a}(c_3) \quad [\because e^{-\infty} = 0]$$

$$c_3 = 0 \quad [\because c_2 \neq 0]$$

Putting $c_3 = 0$ in Eq. (6),

$$u = c_2 c_4 \sin \frac{n\pi x}{a} \cdot e^{-\frac{n\pi y}{a}} = b_n \sin \frac{n\pi x}{a} \cdot e^{-\frac{n\pi y}{a}}, \quad \text{where } c_2 c_4 = b_n$$

Putting $n = 1, 2, 3, \ldots..$ and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \cdot e^{-\frac{n\pi y}{a}} \qquad ...(7)$$

Applying the condition (iv) in Eq. (7),

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \cdot e^{0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \qquad ...(8)$$

Equation (8) represents the Fourier half-range sine series in (0, a).

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \qquad ...(9)$$

Hence, the required temperature distribution is given by Eq. (8), where b_n is given by Eq. (9).

Example 2

Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ *subject to the conditions*

$u(0, y) = u(l, y) = u(x, 0) = 0$, *and* $u(x, a) = \sin\frac{n\pi x}{l}$.

Solution

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad ...(1)$$

The three possible solutions of Eq. (1) are

$$u = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my) \qquad ...(2)$$

$$u = (c_1 \cos mx + c_2 \sin mx)\,(c_3 e^{my} + c_4 e^{-my}) \qquad ...(3)$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \qquad ...(4)$$

The boundary conditions are

(i) $u(0, y) = 0$, i.e., at $x = 0$, $u = 0$
(ii) $u(l, y) = 0$, i.e., at $x = l$, $u = 0$
(iii) $u(x, 0) = 0$, i.e., at $y = 0$, $u = 0$
(iv) $u(x, a) = \sin\frac{n\pi x}{l}$, i.e., at $y = a$, $u = \sin\frac{n\pi x}{l}$

Applying conditions (i) and (ii) in Eq. (2),

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 e^{ml} + c_2 e^{-ml} = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (2) is rejected.
Applying conditions (i) and (ii) in Eq. (4),

$$c_2 = 0 \quad \text{and} \quad c_1 l + c_2 = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (4) is rejected.

Thus, the suitable solution for the present problem is a solution by Eq. (3).
Applying the condition (i) in Eq. (3),

$$0 = c_1 (c_3 e^{my} + c_4 e^{-my})$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (3),

$$u = c_2 \sin mx (c_3 e^{my} + c_4 e^{-my}) \qquad \text{...(5)}$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 \sin ml (c_3 e^{my} + c_4 e^{-my})$$

$$\sin ml = 0 \qquad [\because c_2 \neq 0, \text{ otherwise } u = 0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$ml = n\pi$$

$$m = \frac{n\pi}{l}$$

Putting $m = \dfrac{n\pi}{l}$ in Eq. (5),

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} + c_4 e^{-\frac{n\pi y}{l}} \right) \qquad \text{...(6)}$$

Applying the condition (iii) in Eq. (6),

$$0 = c_2 \sin \frac{n\pi x}{l} (c_3 e^0 + c_4 e^0)$$

$$c_3 + c_4 = 0, \quad c_4 = -c_3$$

Putting $c_4 = -c_3$ in Eq. (6),

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} - c_3 e^{-\frac{n\pi y}{l}} \right)$$

$$u(x, y) = c_2 c_3 \sin \frac{n\pi x}{l} \cdot \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right)$$

$$= b_n \sin \frac{n\pi x}{l} \cdot 2 \sinh \frac{n\pi y}{l} \qquad \text{...(7)}$$

where $c_2 c_3 = b_n$

Applying the condition (iv) in Eq. (7),

$$u(x,a) = 2b_n \sin\frac{n\pi x}{l} \sinh\frac{n\pi a}{l}$$

$$\sin\frac{n\pi x}{l} = 2b_n \sin\frac{n\pi x}{l} \sinh\frac{n\pi a}{l}$$

$$b_n = \frac{1}{2\sinh\frac{n\pi a}{l}}$$

Substituting b_n in Eq. (7), the general solution of Eq. (1) is

$$u(x,y) = \frac{1}{2\sinh\frac{n\pi a}{l}} \sin\frac{n\pi x}{l} \sinh\frac{n\pi y}{l}$$

$$= \frac{\sinh\frac{n\pi y}{l}}{2\sinh\frac{n\pi a}{l}} \sin\frac{n\pi x}{l}$$

Example 3

A rectangular plate with insulated surface has a width of a cm and is so long as compared to its width that it may be considered of infinite length without introducing an appreciable error. If the two long edges $x = 0$ and $x = a$ as well as the one short edge are kept at $0°C$ and the temperature of the other short edge $y = 0$ is given by

$$u = kx, \qquad 0 \le x \le \frac{a}{2}$$

$$= k(a-x), \qquad \frac{a}{2} \le x \le a$$

find the temperature $u(x, y)$ at any point (x, y) of the plate in the steady state.

Solution

In the steady state, the heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$

The three possible solutions of Eq. (1) are

$$u = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my) \qquad ...(2)$$

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my}) \qquad ...(3)$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \qquad ...(4)$$

The boundary conditions are

(i) At $x = 0$, $u = 0$, i.e., $u(0, y) = 0$

(ii) At $x = a$, $u = 0$, i.e., $u(a, y) = 0$

(iii) At $y \to \infty$, $u = 0$, i.e., $u(x, \infty) = 0$

(iv) At $y = 0$, $u = kx$, $0 \le x \le \dfrac{a}{2}$

$= k(a - x)$, $\dfrac{a}{2} \le x \le a$

Applying conditions (i) and (ii) in Eq. (2),

$$c_1 + c_2 = 0 \text{ and } c_1 e^{ma} + c_2 e^{-ma} = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (2) is rejected.

Applying conditions (i) and (ii) in Eq. (4),

$$c_2 = 0 \text{ and } c_1 a + c_2 = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (4) is rejected.
Thus, the suitable solution for the present problem is a solution by Eq. (3).
Applying the condition (i) in Eq. (3),

$$0 = c_1 (c_3 e^{my} + c_2 e^{-my})$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (3),

$$u = c_2 \sin mx (c_3 e^{my} + c_4 e^{-my}) \qquad ...(5)$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 \sin ma (c_3 e^{my} + c_4 e^{-my})$$

$$\sin\ ma = 0 \quad [\because c_2 \ne 0, \text{ otherwise } u = 0]$$

$$= \sin n\pi \quad n \text{ is an integer}$$

$$ma = n\pi$$

$$m = \frac{n\pi}{a}$$

Putting $m = \frac{n\pi}{a}$ in Eq. (5),

$$u = c_2 \sin\frac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right) \quad ...(6)$$

Applying the condition (iii) in Eq. (6) after rewriting as

$$ue^{-\frac{n\pi y}{a}} = c_2 \sin\frac{n\pi x}{a}\left(c_3 + c_4 e^{-\frac{2n\pi y}{a}}\right)$$

$$0 = c_2 \sin\frac{n\pi x}{a}(c_3)$$

$$c_3 = 0 \quad [\because c_2 \neq 0]$$

Putting $c_3 = 0$ in Eq. (6),

$$u = c_2 c_4 \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}} = b_n \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}}, \text{ where } c_2 c_4 = b_n$$

Putting $n = 1, 2, 3, ...$ and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}} \quad ...(7)$$

Applying the condition (iv) in Eq. (7),

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \quad ... (8)$$

Equation (8) represents the Fourier half-range sine series in $(0, a)$.

$$b_n = \frac{2}{a}\int_0^a u(x,0)\sin\frac{n\pi x}{a}dx$$

$$= \frac{2}{a}\left[\int_0^{\frac{a}{2}} kx \cdot \sin\frac{n\pi x}{a}dx + \int_{\frac{a}{2}}^{a} k(a-x)\sin\frac{n\pi x}{a}dx\right]$$

$$= \frac{2k}{a}\left| x\cdot\frac{\left(-\cos\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)} - 1\left\{\frac{\left(-\sin\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)^2}\right\}\right|_0^{\frac{a}{2}}$$

$$+\frac{2k}{a}\left|(a-x)\frac{\left(-\cos\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)} - (-1)\frac{\left(-\sin\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)^2}\right|_{\frac{a}{2}}^{a}$$

$$= \frac{2k}{a}\left[\frac{-a^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{a^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4ka}{\pi^2 n^2}\sin\frac{n\pi}{2}$$

Substituting b_n in Eq. (7), the general solution is

$$u(x,y) = \frac{4ka}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}}$$

$$= \frac{4ka}{\pi^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\sin\frac{(2r+1)\pi x}{a}\cdot e^{-\frac{(2r+1)\pi y}{a}}$$

$$\left[\begin{array}{l} \because \sin\frac{n\pi}{2} = 0, \quad \text{if } n \text{ is even} \\ \qquad\qquad = 1 \text{ or } -1, \quad \text{if } n \text{ is odd} \\ \text{Putting } n = 2r+1, \\ \quad \sin\frac{n\pi}{2} = \sin\frac{(2r+1)\pi}{2} = \sin\left(\pi r + \frac{\pi}{2}\right) \\ \qquad\qquad\qquad = \cos\pi r = (-1)^r \end{array}\right]$$

EXERCISE 6.11

1. A rectangular plate with an insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in length without

introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x,0) = 100 \sin\frac{\pi x}{8}, \quad 0 < x < 8$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane of the plate is given by

$$u(x,y) = 100e^{-\frac{\pi y}{8}} \sin\frac{\pi x}{8}$$

2. The function $v(x,y)$ satisfies the Laplace's equation in rectangular coordinates (x, y) and for points within the rectangle $x = 0$, $x = a$, $y = 0$, $y = b$, it satisfies the conditions, $v(0,y) = v = (a,y) = v(x,b) = 0$ and $v(x,0) = x(a-x), 0 < x < a$. Show that $v(x, y)$ is given by

$$v(x,y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\dfrac{\sin(2n+1)\pi x}{a}}{(2n+1)^3}\frac{\dfrac{\sinh(2n+1)\pi(b-y)}{a}}{\sinh\dfrac{(2n+1)\pi b}{a}}$$

3. A long rectangular plate of width a cm with an insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0,y) = 0, v(a,y) = 0, v(x,\infty) = 0, v(x,0) = kx$. Show that steady-state temperature within the plate is

$$v(x,y) = \frac{2ak}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot e^{-\frac{n\pi y}{a}} \sin\frac{n\pi x}{a}$$

4. A rectangular plate with 6 cm wide insulated surfaces is so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x,0) = 90\sin\frac{\pi x}{6}$, $0 < x < 6$ while the two long edges $x = 0$ and $x = 6$ as well as the other short edge are maintained at 0°C, find the function $u(x, y)$ given the steady-state temperature at any point (x, y) of the plate.

$$\left[\textbf{Ans.:}\ u(x,y) = 90\left(\sin\frac{\pi x}{6}\right)e^{\frac{-\pi y}{6}}\right]$$

Points to Remember

Formation of Partial Differential Equations

Partial differential equations can be formed using the following methods:

1. By elimination of arbitrary constants in the equation of the type

$$f(x, y, z, a, b) = 0$$

where a and b are arbitrary constants.

2. By elimination of arbitrary functions in the equation of the type

$$z = f(u)$$

where u is a function of x, y, and z.

Linear Partial Differential Equations of First Order

A quasi-linear partial differential equation is represented as

$$P(x, y, z) \cdot p + Q(x, y, z) \cdot q = R(x, y, z)$$

This equation is known as *Lagrange's linear equation*.

If P and Q are independent of z, and R is linear in z then the equation is known as a *linear equation.*

The general solution of *Lagrange's linear equation* $Pp + Qq = R$ is given by

$$f(u, v) = 0$$

where f is an arbitrary function and u, v are functions of x, y, and z.

Nonlinear Partial Differential Equations of First Order

A partial differential equation of first order is said to be nonlinear if p and q have degree more than one.

The complete solution of a nonlinear equation is given by

$$f(x, y, z, a, b) = 0$$

where a and b are two arbitrary constants. Four standard forms of these equations are as follows:

Form I	$f(p, q) = 0$
Form II	$f(z, p, q) = 0$
Form III	$f(x, p) = g(y, q)$
Form IV (Clairaut equation)	$z = px + qy + f(p, q)$

Homogeneous Linear Partial Differential Equations with Constant Coefficients

An equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad ...(1)$$

where $a_0, a_1, \ldots, a_n$ are constants is known as a homogeneous linear partial differential equation of n^{th} order with constant coefficients.

The complete solution of Eq. (1) is obtained in two parts, one as a Complementary Function (CF) and the other as a Particular Integral (PI).

The complementary function is the solution of the equation $f(D, D')\, z = 0$.

1. Rules to Obtain the Complementary Function

Let the given equation be $f(D, D')\, z = F(x, y)$

where $\quad f(D, D') = a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n$

The auxiliary equation is

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0$$

Let $m_1, m_2, m_3, \ldots, m_n$ be the roots of auxiliary equation.

Case I Roots of Auxiliary Equation are Distinct

$$\text{CF} = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x)$$

Case II Roots of Auxiliary Equation are Equal (Repeated)

In general, if n roots of an auxiliary equation all are equal to m,

$$\text{CF} = \phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + \cdots + x^{n-1}\phi_n(y + mx)$$

2. Rules to Obtain the Particular Integral

Particular integral $\text{PI} = \dfrac{1}{f(D, D')} F(x, y)$

The particular integral depends on the form of $F(x, y)$.

Nonhomogeneous Linear Partial Differential Equations with Constant Coefficients

If in the equation $\qquad f(D, D')z = F(x, y)$

each term of $f(D, D')$ does not contain the derivatives of the same order then the equation is known as a nonhomogeneous equation.

Classification of Second Order Linear Partial Differential Equations

The general form of a nonhomogeneous second order partial differential equation in the function of two independent variables x, y is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = F(x, y) \qquad \text{...(2)}$$

Equation (2) is linear or quasi-linear accordingly as f is linear or nonlinear. Equation (2) is homogeneous if $F(x, y) = 0$.

Equation (2) is elliptic if $B^2 - 4AC < 0$, parabolic if $B^2 - 4AC = 0$ and hyperbolic if $B^2 - 4AC > 0$.

One-Dimensional Wave Equation

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct)$$

D' Alembert's Solution of the Wave Equation

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y(x, t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

One-Dimensional Heat-Flow Equation

The one-dimensional heat-flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The solution is

$$u = (c_1 \cos mx + c_2 \sin mx)\, e^{-m^2 c^2 t}$$

- *Transient Solution* The solution is known as transient if u decreases as t increases.
- *Steady-state Condition* A condition is known as steady state if the dependent variables are independent of the time t.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The equation $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4[(x-2)^2 + (y-3)^2]$ is of order ____ and degree _____

(a) 1, 2 (b) 2, 1 (c) 1, 1 (d) 1, 3

2. The solution of $(y - z)p + (z - x)q = x - y$ is **[Summer 2016]**

(a) $f(x + y + z) = xyz$ (b) $f(x^2 + y^2 + z^2) = xyz$

(c) $f(x^2 + y^2 + z^2, x^2y^2z^2) = 0$ (d) $f(x + y + z, x^2 + y^2 + z^2) = 0$

3. The solution of $p + q = z$ is

(a) $f(x + y, y + \log z) = 0$ (b) $f(xy, y \log z) = 0$

(b) $f(x - y, y - \log z) = 0$ (d) None of these

4. The solution $\frac{\partial^2 z}{\partial y^2} = \sin xy$ is

(a) $z = -x^2 \sin xy + yf(x) + \phi(x)$ (b) $z = x^2 \sin xy + yf(x) + \phi(x)$

(c) $z = x^2 \sin xy - yf(x) + \phi(x)$ (d) $z = x^2 \sin xy + yf(x) - \phi(x)$

5. The solution of $q = 3p^2$ is

(a) $z = ax + 3a^2y + c$ (b) $z = ax - 3a^2y + c$
(c) $z = ax^2 + 3ay + c$ (d) $z = ax^2 - 3ay + c$

6. The solution of $p(1 + q) = qz$ is

(a) $\log (az + 1) = x + ay + b$ (b) $\log (az - 1) = x + ay + b$
(c) $\log (az - 1) = x - ay + b$ (d) $\log (az - 1) = x + ay - b$

7. The solution of $q = xyp^2$ is

(a) $6\,ax = (2z - ay^2 - 2b^2)$ (b) $6\,ax = (2z + ay^2 + 2b^2)$
(c) $16\,ax = (2z - ay^2 - 2b^2)$ (d) $16\,ax = (2z + ay^2 + 2b^2)$

8. The solution of $z = px + qy - pq$ is

(a) $z = ax + by$ (b) $z = ax - by$
(c) $z = ax - by + ab$ (d) $z = ax + by - ab$

9. The order of the partial differential equation obtained by eliminating f from $z = f(x^2 + y^2)$ is

(a) 2 (b) 1 (c) 3 (d) None of these

10. By eliminating arbitrary constants from $z = ax + by + ab$, the partial differential equation formed is

(a) $z = px + qy + pq$ (b) $z = px + qy$
(c) $z = px - qy + pq$ (d) $z = px + qy - pq$

11. Particular integral of $(D^2 - D'^2)z = \cos (x + y)$ is

(a) $x \cos (x + y)$ (b) $\frac{x}{2}\cos(x+y)$
(c) $x \sin (x + y)$ (d) $\frac{x}{2}\sin(x+y)$

12. The solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is

(a) $z = (1 + x + x^2)\,f(y)$ (b) $z = (1 + y + y^2)\,f(x)$
(c) $z = f_1(x) + y\,f_2(x) + y^2\,f_3(x)$ (d) $z = f_1(y) + x\,f_2(y) + x^2\,f_3(y)$

13. The partial differential equation $\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial x^2} = 0$ is

(a) elliptic (b) hyperbolic (c) parabolic (d) None of these

14. The partial differential equation $y\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 0$ is elliptic if

(a) $x^2 = y^2$ (b) $x^2 < y^2$ (c) $x^2 + y^2 = 1$ (d) $x^2 + y^2 > 1$

15. The complementary function of $r - 7s + 6t = e^{x+y}$ is

(a) $f_1\,(y + x) + f_2\,(y + 6x)$ (b) $f_1\,(y - x) + f_2\,(y + 6x)$
(c) $f_1\,(y + x) + f_2\,(y - 6x)$ (d) $f_1\,(y - x) + f_2\,(y - 6x)$

16. The partial differential equation by eliminating the arbitrary function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ is

(a) $px^2 + qy = 2y^2$ (b) $px^2 - qy = 2y^2$

(c) $py^2 + qx = 2y^2$ (d) $py^2 - qy = 2x^2$

17. The partial differential equation by eliminating the arbitrary function from the relation $z = f(\sin x + \cos y)$ is

(a) $p \sin x + q \cos y = 0$ (b) $p \sin y + q \cos x = 0$

(c) $q \sin y + p \cos x = 0$ (d) $p \sin y - q \cos x = 0$

18. If $u = x^2 + t^2$ is a solution of $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, then $c =$

(a) 1 (b) 2 (c) 0 (d) 3

19. The particular integral of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$ is

(a) $\frac{1}{2} e^{x+2y}$ (b) $-\frac{x}{2} e^{x+2y}$

(d) xe^{x+2y} (d) $x^2 e^{x+2y}$

20. The solution of $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given $u(0, y) = 8e^{-3y}$, is

(a) $u = 8e^{12x+3y}$ (b) $u = -8e^{12x+3y}$

(c) $u = 8e^{-12x-3y}$ (d) $u = 8e^{-12x+3y}$

21. The solution of $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, given $z(x, 0) = 4e^{-3x}$, is

(a) $z = 4e^{3x+t}$ (b) $z = 4e^{3x-t}$

(c) $z = 4e^{-3x-t}$ (d) $z = 4e^{-3x+t}$

22. The partial differential equation $2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 6$ is **[Summer 2016]**

(a) elliptic (b) hyperbolic

(c) parabolic (d) None of these

23. The solution of $(D + D')z = \cos x$ is **[Summer 2016]**

(a) $\phi_1(y - x) + \cos x$ (b) $\phi_1(y + x) + \sin x$

(c) $\phi_1(y - x) + \tan x$ (d) $\phi_1(y - x) + \sin x$

24. The number of initial and boundary conditions required to solve the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ are **[Winter 2016]**

(a) 2, 1 (b) 1, 1 (c) 1, 2 (d) 2, 2

25. The solution to $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + u$ is **[Winter 2015]**

(a) $u(t, x) = 50e^{\frac{t-x}{2}}$ (b) $u(t, x) = 50e^{\frac{x-t}{2}}$

(c) $u(t, x) = 25e^{\frac{t-x}{2}}$ (d) $u(t, x) = 25e^{\frac{x-t}{2}}$

26. Which of the following is not an example of a first order differential equation of Clairout's form? **[Winter 2015]**

(a) $px + qy - 2\sqrt{pq}$ (b) $px + qy = p^2q^2$

(c) $p^2 + q^2 = z^2(x + y)$ (d) $px + qy + \frac{1}{p-q}$

27. By eliminating the arbitrary function from $z = f(x + at) + g(x - at)$, the partial differential equation formed is **[Winter 2016]**

(a) $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$ (b) $\frac{\partial^2 z}{\partial t^2} = \frac{1}{a^2} \frac{\partial^2 z}{\partial x^2}$

(c) $\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial t^2}$ (d) $\frac{\partial^2 z}{\partial t^2} = \frac{1}{a} \frac{\partial^2 z}{\partial x^2}$

28. The general solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is **[Summer 2017]**

(a) $\phi_1(y) + x\phi_2(y) + x^2\phi_3(y)$ (b) $\phi_1(y) + x\phi_2(y)$

(c) $\phi_1(y) + \phi_2(y) + x\phi_3(y)$ (d) $\phi_1(y) + \phi_2(y) + x^2\phi_3(y)$

29. The general solution of $p + q = z$ is **[Summer 2017]**

(a) $\log z = \frac{1}{a}(x + ay + b)$ (b) $\log z = \frac{1}{1+a}(x + ay + b)$

(c) $\log z = x + ay + b$ (d) $\log z = (1 + a)(x + ay + b)$

Answers

1. (a)	2. (d)	3. (c)	4. (a)	5. (a)	6. (b)	7. (c)	8. (d)
9. (b)	10. (a)	11. (d)	12. (d)	13. (c)	14. (b)	15. (a)	16. (a)
17. (b)	18. (a)	19. (b)	20. (c)	22. (a)	23. (a)	24. (d)	25. (c)
26. (a)	27. (c)	28. (a)	29. (b)				

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