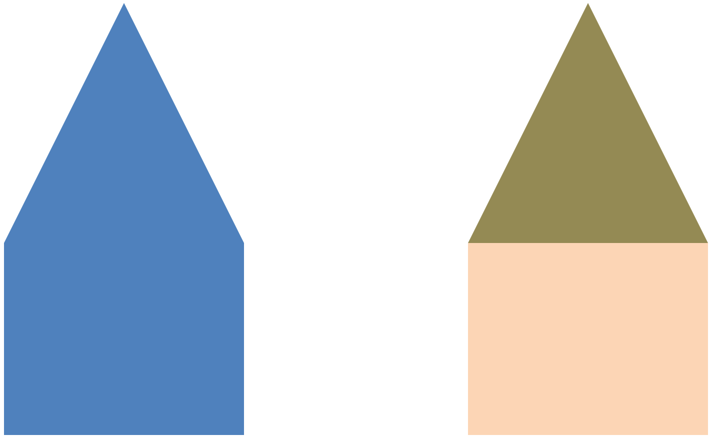
Day 17 Class

GRAPH COLORING

If a graph G is given and it is required to be colored in such a way that no two Adjacent vertices have the same color, then what is the minimum number of color required to color G properly?



CHROMATIC NUMBER

- Painting all vertices of a graph with colors such that no two adjacent vertices have the same color is called proper coloring of a graph. A graph in which every vertex has been assigned a color according to a proper coloring is called properly colored graph.

PROPER COLORINGS OF A GRAPH

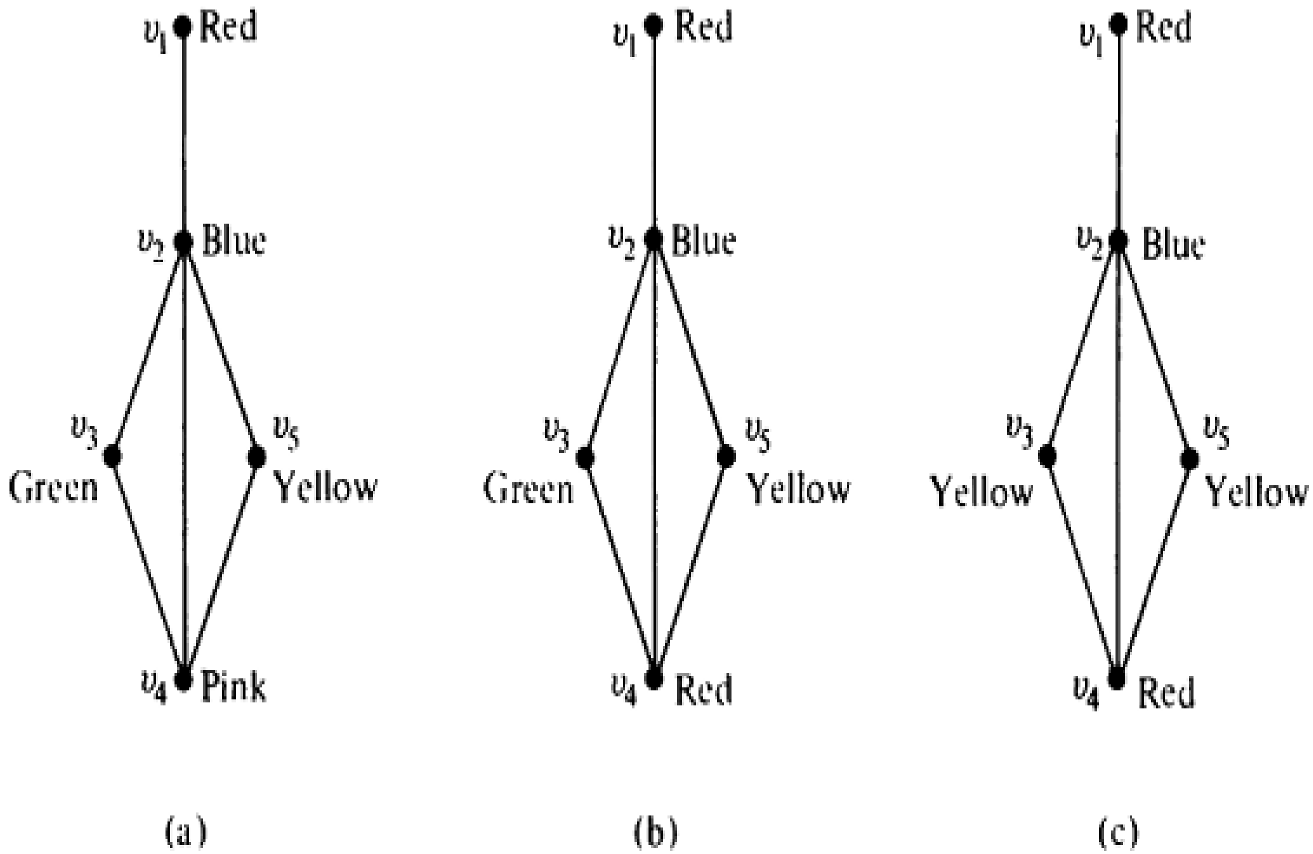


Fig. 8-1 Proper colorings of a graph.

- A graph that requires k different colors for proper coloring is called k-chromatic graph.
- K is called the chromatic number of G. In the above example the chromatic number is 3 .

Observations:

1. A graph consisting of only isolated vertices is 1-chromatic.
2. A graph with one or more edges (not a self-loop, of course) is at least *2-chromatic* (also called *bichromatic*).
3. A complete graph of n vertices is n-chromatic, as all its vertices are adjacent. Hence a graph containing a complete graph of r vertices is at least r-chromatic. For instance, every graph having a triangle is at least 3-chromatic.
4. A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.

THEOREM 8-1

Every tree with two or more vertices is 2-chromatic.

Though a tree is 2-chromatic, not every 2-chromatic graph is a tree. (The utilities graph, for instance, is not a tree.) What then is the characterization

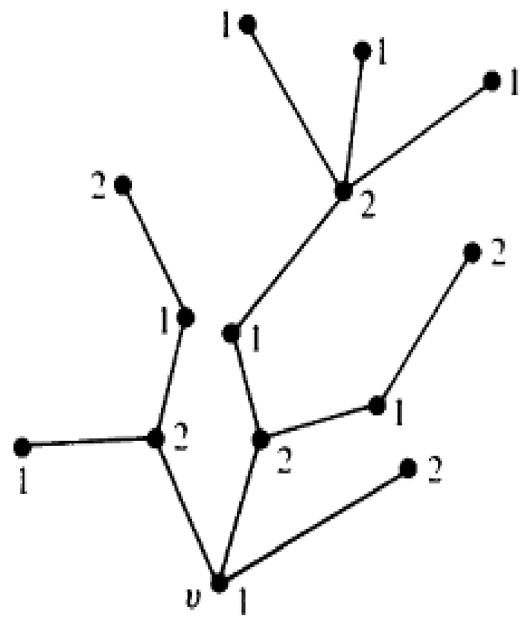


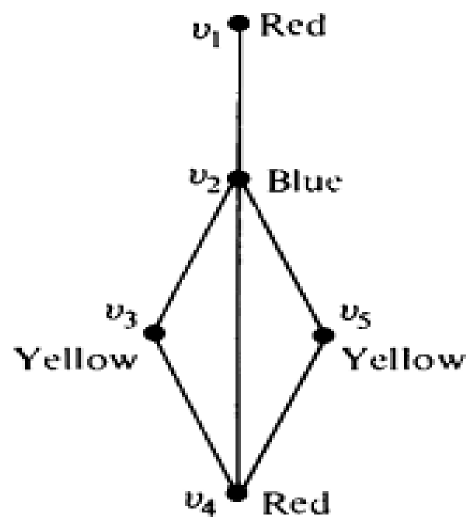
Fig. 8-2 Proper coloring of a tree.

Independent Set

A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets. For example, the coloring in Fig. 8-1(c) produces the partitioning

$$\{v_1, v_4\}, \quad \{v_2\}, \quad \text{and} \quad \{v_3, v_5\}.$$

No two vertices in any of these three subsets are adjacent. Such a subset of vertices is called an independent set; more formally:



(c)

Contd..

A set of vertices in a graph is said to be an *independent set* of vertices or simply an *independent set* (or an *internally stable set*) if no two vertices in the set are adjacent. For example, in Fig. 8-3, $\{a, c, d\}$ is an independent set. A single vertex in any graph constitutes an independent set.

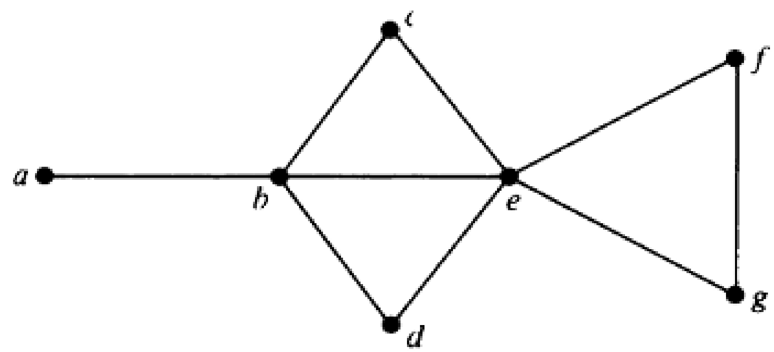


Fig 8.3

A *maximal independent set* (or *maximal internally stable set*) is an independent set to which no other vertex can be added without destroying its independence property. The set $\{a, c, d, f\}$ in Fig. 8-3 is a maximal independent set. The set $\{b, f\}$ is another maximal independent set.

Contd..

The number of vertices in the largest independent set of a graph G is called the *independence number* (or *coefficient of internal stability*), $\beta(G)$.

Consider a κ-chromatic graph G of n vertices properly colored with κ different colors. Since the largest number of vertices in G with the same color cannot exceed the independence number $\beta(G)$, we have the inequality

$$\beta(G) \geq \frac{n}{\kappa}\cdot$$

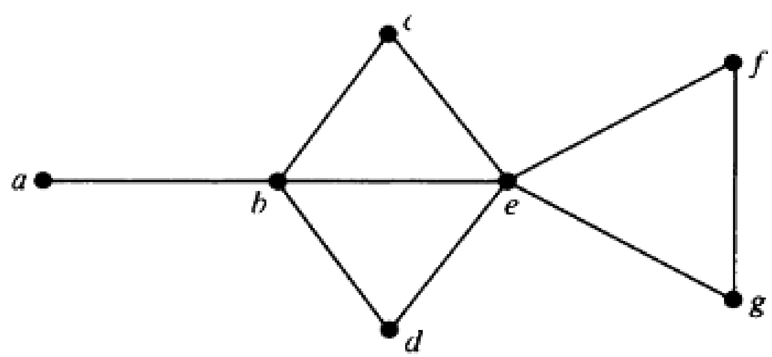


Fig 8.3

Dominating Set

A *dominating set* (or an *externally stable set*) in a graph G is a set of vertices that dominates every vertex v in G in the following sense: Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set. For instance, the vertex set $\{b, g\}$ is a dominating set in Fig. 8-3. So is the set $\{a, b, c, d, f\}$ a dominating set. A dominating set need not be independent. For example, the set of all its vertices is trivially a dominating set in every graph.

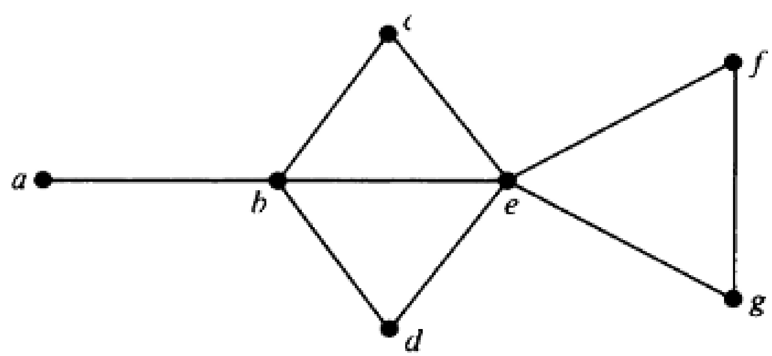


Fig 8.3

Contd..

A *minimal dominating set* is a dominating set from which no vertex can be removed without destroying its dominance property. For example, in Fig. 8-3, $\{b, e\}$ is a minimal dominating set. And so is $\{a, c, d, f\}$.

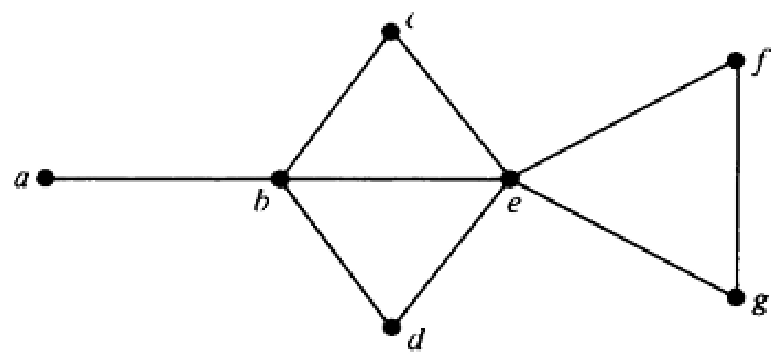


Fig 8.3

The number of vertices in the smallest minimal dominating set of a graph G is called the *domination number*, $\alpha(G)$.

In any graph G,

$$\alpha(G) \leq \beta(G).$$

Chromatic Polynomial

- A given graph of n vertices can be colored in many different colors using sufficiently large number of colors. This property of graph can be expressed elegantly by using a polynomial called the chromatic polynomial of G.

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors.

Contd..

Let c_i be the different ways of properly coloring G using exactly i different colors. Since i colors can be chosen out of λ colors in

$$\binom{\lambda}{i} \qquad \text{different ways,}$$

there are $c_i\binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors.

Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms; that is,

$$\begin{aligned} P_n(\lambda) &= \sum_{i=1}^{n} c_i\binom{\lambda}{i} \\ &= c_1\frac{\lambda}{1!} + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots \\ &\quad + c_n\frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!}. \end{aligned}$$

EXAMPLE- FINDING THE CHROMATIC POLYNOMIAL OF THE FOLLOWING GRAPH

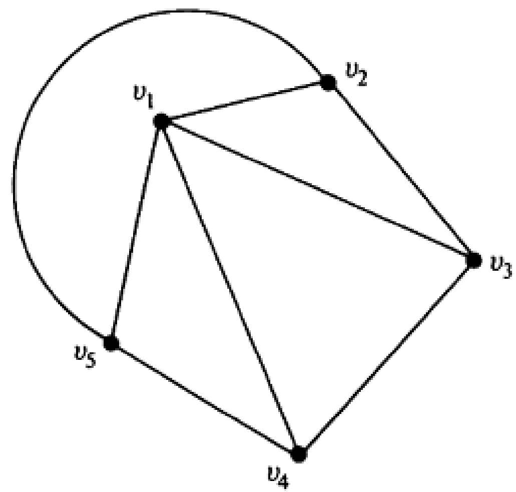


Fig. 8-4 A 3-chromatic graph.

A graph with n vertices and using n different colors can be properly colored in $n!$ ways; that is,

$$c_n = n!.$$

Since the graph in Fig. 8-4 has a triangle, it will require at least three different colors for proper coloring. Therefore,

$$c_1 = c_2 = 0 \quad \text{and} \quad c_5 = 5!.$$

Moreover, to evaluate c_3, suppose that we have three colors x, y, and z. These three colors can be assigned properly to vertices v_1, v_2, and v_3 in $3! = 6$ different ways. Having done that, we have no more choices left, because vertex v_5 must have the same color as v_3, and v_4 must have the same color as v_2. Therefore,

$$c_3 = 6.$$

Similarly, with four colors, v_1, v_2, and v_3 can be properly colored in $4 \cdot 6 = 24$ different ways. The fourth color can be assigned to v_4 or v_5, thus providing two choices. The fifth vertex provides no additional choice. Therefore,

$$c_4 = 24 \cdot 2 = 48.$$

Substituting these coefficients in $P_5(\lambda)$, we get, for the graph in Fig. 8-4,

$$\begin{aligned} P_5(\lambda) = {} & \lambda(\lambda - 1)(\lambda - 2) + 2\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) \\ & + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) \\ = {} & \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 5\lambda + 7). \end{aligned}$$

The presence of factors $\lambda - 1$ and $\lambda - 2$ indicates that G is at least 3-chromatic.