CHAPTER 2

Fourier Series and Fourier Integral

Chapter Outline

2.1 INTRODUCTION

Fourier series is used in the analysis of periodic functions. Many of the phenomena studied in engineering and sciences are periodic in nature, e.g., current and voltage in an ac circuit. These periodic functions can be analyzed into their constituent components by a Fourier analysis. The Fourier series makes use of orthogonality relationships of the sine and cosine functions. It decomposes a periodic function into a sum of sine-cosine functions. The computation and study of Fourier series is known as *harmonic analysis*. It has many applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, etc.

2.2 PERIODIC FUNCTIONS

A function $f(x)$ is said to be periodic with period $T > 0$, if $f(x) = f(x + T)$ for all real x. The function $f(x)$ repeats itself after each interval of T. If $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \ldots$ then T is called the period of the function $f(x)$.
e.g. $\sin x$ is a periodic function with period 2π. Hence, $\sin x = \sin(x + 2\pi)$.

2.3 FOURIER SERIES

Representation of a function over a certain interval by a linear combination of mutually orthogonal functions is called *Fourier series representation*.

Convergence of the Fourier Series (Dirichlet's Conditions)

A function $f(x)$ can be represented by a complete set of orthogonal functions within the interval $(c, c + 2l)$. The Fourier series of the function $f(x)$ exists only if the following conditions are satisfied:

(i) $f(x)$ is periodic, i.e., $f(x) = f(x + 2l)$, where $2l$ is the period of the function $f(x)$.
(ii) $f(x)$ and its integrals are finite and single-valued.
(iii) $f(x)$ has a finite number of discontinuities, i.e., $f(x)$ is piecewise continuous in the interval $(c, c + 2l)$.
(iv) $f(x)$ has a finite number of maxima and minima.

These conditions are known as *Dirichlet's conditions*.

2.4 TRIGONOMETRIC FOURIER SERIES

We know that the set of functions $\sin\dfrac{n\pi x}{l}$ and $\cos\dfrac{n\pi x}{l}$ are orthogonal in the interval $(c, c + 2l)$ for any value of c, where $n = 1, 2, 3, \ldots$.

i.e.,
$$\begin{aligned}\int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,\mathrm{d}x &= 0 \qquad m \neq n\\ &= l \qquad m = n\end{aligned}$$

$$\begin{aligned}\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,\mathrm{d}x &= 0 \qquad m \neq n\\ &= l \qquad m = n\end{aligned}$$

$$\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,\mathrm{d}x = 0 \quad \text{for all } m, n$$

Hence, any function $f(x)$ can be represented in terms of these orthogonal functions in the interval $(c, c + 2l)$ for any value of c.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This series is known as a *trigonometric Fourier series* or simply, a *Fourier series*. For example, a square function can be constructed by adding orthogonal sine components (Fig. 2.1).

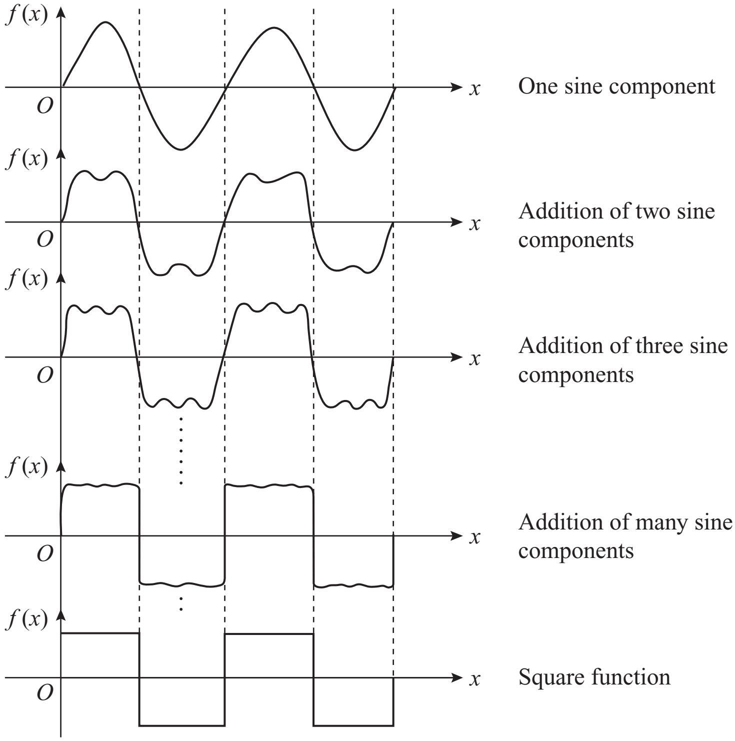


Fig. 2.1 ***Representation of a function in terms of sine components***

2.5 FOURIER SERIES OF FUNCTIONS OF ANY PERIOD

Let $f(x)$ be a periodic function with period $2l$ in the interval $(c, c+2l)$. Then the Fourier series of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(2.1)$$

Determination of a_0

Integrating both the sides of Eq. (2.1) w.r.t. x in the interval $(c,\ c+2l)$,

$$\int_c^{c+2l} f(x)\mathrm{d}x = a_0\int_c^{c+2l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\right)\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= a_0(c + 2l - c) + 0 + 0$$

$$= 2la_0$$

Hence,
$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\mathrm{d}x \qquad \ldots(2.2)$$

Determination of a_n

Multiplying both the sides of Eq. (2.1) by $\cos\frac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = a_0\int_c^{c+2l}\cos\frac{n\pi x}{l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= 0 + la_n + 0$$

$$= l\,a_n$$

Hence,

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \qquad \ldots(2.3)$$

Determination of b_n

Multiplying both the sides of Eq. (2.1) by $\sin\frac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = a_0\int_c^{c+2l}\sin\frac{n\pi x}{l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= 0 + 0 + lb_n$$

$$= l\,b_n$$

Hence,

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x \qquad \ldots(2.4)$$

The formulae (2.2), (2.3), and (2.4) are known as *Euler's formulae* which give the values of coefficients a_0, a_n, and b_n. These coefficients are known as *Fourier coefficients*.

Corollary 1 **When $c = 0$ and $2l = 2\pi$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

Corollary 2 **When $c = -\pi$ and $2l = 2\pi$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

Corollary 3 **When $c = 0$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_{0}^{2l} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_{0}^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Corollary 4 **When $c = -l$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\mathrm{d}x$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

Fourier Series Expansion with Period 2π

Example 1

Find the Fourier series of $f(x) = x$ in the interval $(0, 2\pi)$.

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|\frac{x^2}{2}\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left(\frac{4\pi^2}{2}\right) \\
&= \pi
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left(\frac{\cos 2n\pi}{n^2} - \frac{\cos 0}{n^2}\right) && [\because \sin 2n\pi = \sin 0 = 0] \\
&= 0 && [\because \cos 2n\pi = \cos 0 = 1]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[-2\pi\left(\frac{\cos 2n\pi}{n}\right)\right] && [\because \sin 2n\pi = \sin 0 = 0] \\
&= -\frac{2}{n} && [\because \cos 2n\pi = 1]
\end{aligned}$$

Hence,
$$f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$x = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

Example 2

Find the Fourier series of $f(x) = x^2$ *in the interval* $(0, 2\pi)$ *and, hence, deduce that* $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \cdots$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx \\
&= \frac{1}{2\pi}\left|\frac{x^3}{3}\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left(\frac{8\pi^3}{3}\right) \\
&= \frac{4\pi^2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx \\
&= \frac{1}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[4\pi\left(\frac{\cos 2n\pi}{n^2}\right)\right] && [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right) && [\because\ \cos 2n\pi = 1] \\
&= \frac{4}{n^2}
\end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[4\pi^2\left(-\frac{\cos 2n\pi}{n}\right) + 2\left(\frac{\cos 2n\pi}{n^3}\right) - 2\left(\frac{\cos 0}{n^3}\right)\right] \\
&= \frac{1}{\pi}\left(-\frac{4\pi^2}{n}\right) \qquad [\because\ \cos 2n\pi = \cos 0 = 1] \\
&= -\frac{4\pi}{n}
\end{aligned}
$$

Hence,
$$f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$
\begin{aligned}
x^2 &= \frac{4\pi^2}{3} + 4\left(\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right) \\
&\quad - 4\pi\left(\frac{1}{1}\sin x + \frac{1}{2}\sin x + \frac{1}{3}\sin 3x + \cdots\right) \qquad \text{... (1)}
\end{aligned}
$$

Putting $x = \pi$ in Eq. (1),

$$
\begin{aligned}
\pi^2 &= \frac{4\pi^2}{3} + 4\left(\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \cdots\right) + 0 \\
&= \frac{4\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\right) \\
\frac{\pi^2}{12} &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots
\end{aligned}
$$

Example 3

Find the Fourier series of $f(x) = \frac{1}{2}(\pi - x)$ *in the interval* $(0, 2\pi)$.

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ **[Winter 2013]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\mathrm{d}x \\
&= \frac{1}{4\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{2\pi} \\
&= \frac{1}{4\pi}(2\pi^2 - 2\pi^2) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left[-\frac{\cos 2n\pi}{n^2} + \frac{\cos 0}{n^2}\right] \\
&= 0 \qquad [\because\ \cos 2n\pi = \cos 0 = 1]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left[(-\pi)\left(-\frac{\cos 2n\pi}{n}\right) - \pi\left(-\frac{\cos 0}{n}\right)\right] \qquad [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= \frac{1}{2\pi}\left(\frac{\pi}{n} + \frac{\pi}{n}\right) \qquad [\because\ \cos 2n\pi = \cos 0 = 1] \\
&= \frac{1}{n}
\end{aligned}
$$

Hence, $f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$

$$
\begin{aligned}
\frac{1}{2}(\pi - x) &= \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x \\
&\quad + \frac{1}{6}\sin 6x + \frac{1}{7}\sin 7x + \cdots \qquad \text{...(1)}
\end{aligned}
$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\frac{1}{2}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin 2\pi + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{6}\sin 3\pi + \frac{1}{7}\sin\frac{7\pi}{2} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Example 4

Obtain the Fourier series of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ *in the interval* $0 \le x \le 2\pi$.

Hence, deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$ **[Winter 2014]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \mathrm{d}x \\
&= \frac{1}{8\pi}\left|\frac{(\pi - x)^3}{-3}\right|_0^{2\pi} \\
&= -\frac{1}{24\pi}(-\pi^3 - \pi^3) \\
&= \frac{\pi^2}{12}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx\,\mathrm{d}x
\end{aligned}$$

$$= \frac{1}{4\pi}\left|(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[2\pi\left(\frac{\cos 2n\pi}{n^2}\right) - \left\{-2\pi\left(\frac{\cos 0}{n^2}\right)\right\}\right] \quad [\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{1}{4\pi}\left(\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right) \quad [\because \cos 2n\pi = \cos 0 = 1]$$

$$= \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx\, dx$$

$$= \frac{1}{4\pi}\left|(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left\{\pi^2\left(-\frac{\cos 2n\pi}{n}\right) + \frac{2\cos 2n\pi}{n^3}\right\} - \left\{\pi^2\left(-\frac{\cos 0}{n}\right) + 2\left(\frac{\cos 0}{n^3}\right)\right\}\right]$$

$$[\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{1}{4\pi}\left(-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right) \quad [\because \cos 2n\pi = \cos 0 = 1]$$

$$= 0$$

Hence, $f(x) = \dfrac{\pi^2}{12} + \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots \qquad \text{....(1)}$$

Putting $x = \pi$ in Eq. (1),

$$0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Example 5

Find the Fourier series for $f(x) = e^{ax}$ in $(0, 2\pi)$, $a > 0$. **[Summer 2018]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ax}\mathrm{d}x$$
$$= \frac{1}{2\pi}\left|\frac{e^{ax}}{a}\right|_0^{2\pi}$$
$$= \frac{1}{2a\pi}(e^{2a\pi} - 1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right|_0^{2\pi}$$
$$= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}(a\cos 2n\pi) - \frac{a}{a^2+n^2}\right] \quad \left[\begin{array}{l}\because \sin 2n\pi = \sin 0 = 0 \\ \cos 0 = 1\end{array}\right]$$
$$= \frac{a}{\pi(a^2+n^2)}(e^{2a\pi} - 1) \quad [\because \cos 2n\pi = 1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{2\pi} a^{ax}\sin nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right|_0^{2\pi}$$
$$= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}(-n\cos 2n\pi) + \frac{n}{a^2+n^2}\right] \quad \left[\begin{array}{l}\because \sin 2n\pi = \sin 0 = 0 \\ \cos 0 = 1\end{array}\right]$$
$$= \frac{n}{\pi(a^2+n^2)}(1 - e^{2a\pi}) \quad [\because \cos 2n\pi = 1]$$

Hence,
$$f(x) = \frac{1}{2a\pi}(e^{2a\pi} - 1) + \frac{a(e^{2a\pi} - 1)}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}\cos nx + \frac{1 - e^{2a\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n}{a^2+n^2}\sin nx$$

Example 6

Find the Fourier series of $f(x)=\dfrac{3x^2-6x\pi+2\pi^2}{12}$ *in the interval* $(0, 2\pi)$

Hence, deduce that $\dfrac{\pi^2}{6}=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{3x^2-6x\pi+2\pi^2}{12}\mathrm{d}x \\
&= \frac{1}{24\pi}\left|3\left(\frac{x^3}{3}\right)-6\pi\left(\frac{x^2}{2}\right)+2\pi^2x\right|_0^{2\pi} \\
&= \frac{1}{24\pi}\left[3\left(\frac{8\pi^3}{3}\right)-6\pi\left(\frac{4\pi^2}{2}\right)+4\pi^3\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6x\pi+2\pi^2}{12}\right)\cos nx\,\mathrm{d}x \\
&= \frac{1}{12\pi}\left|(3x^2-6x\pi+2\pi^2)\left(\frac{\sin nx}{n}\right)-(6x-6\pi)\left(-\frac{\cos nx}{n^2}\right)+6\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi} \\
&= \frac{1}{12\pi}\left[(6\pi)\left(\frac{\cos 2n\pi}{n^2}\right)-(-6\pi)\left(\frac{\cos 0}{n^2}\right)\right] \qquad [\because\ \sin 2n\pi=\sin 0=0] \\
&= \frac{1}{12\pi}\left(\frac{6\pi}{n^2}+\frac{6\pi}{n^2}\right) \qquad [\because\ \cos 2n\pi=\cos 0=1] \\
&= \frac{1}{n^2}
\end{aligned}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6x\pi+2\pi^2}{12}\right)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{12\pi}\left|(3x^2-6x\pi+2\pi^2)\left(-\frac{\cos nx}{n}\right)-(6x-6\pi)\left(-\frac{\sin nx}{n^2}\right)+6\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[(12\pi^2-12\pi^2+2\pi^2)\left(-\frac{\cos 2n\pi}{n}\right)+6\left(\frac{\cos 2n\pi}{n^3}\right)-(2\pi)^2\left(-\frac{\cos 0}{n}\right)-6\left(\frac{\cos 0}{n^3}\right)\right] \quad [\because\ \sin 2n\pi=\sin 0=0]$$

$$= 0 \qquad [\because\ \cos 2n\pi = \cos 0 = 1]$$

Hence,
$$f(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\frac{3x^2-6x\pi+2\pi^2}{12}=\cos x+\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x+\cdots \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$\frac{\pi^2}{6}=\cos 0+\frac{1}{2^2}\cos 0+\frac{1}{3^2}\cos 0+\cdots$$

$$=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

Example 7

Find the Fourier series of $f(x) = e^{-x}$ *in the interval* $(0, 2\pi)$.

Hence, deduce that $\frac{\pi}{2}\frac{1}{\sinh\pi}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$. **[Summer 2014]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x$$

$$=\frac{1}{2\pi}\int_0^{2\pi}e^{-x}\mathrm{d}x$$

$$=\frac{1}{2\pi}\left|-e^{-x}\right|_0^{2\pi}$$

$$= \frac{-e^{-2\pi} + e^0}{2\pi}$$

$$= \frac{1 - e^{-2\pi}}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-\cos 2n\pi) - \frac{1}{n^2+1}(-\cos 0)\right] \qquad [\because\ \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi(n^2+1)}(1 - e^{-2\pi}) \qquad [\because\ \cos 2n\pi = \cos 0 = 1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\sin nx - n\cos nx)\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-n\cos 2n\pi) - \frac{1}{n^2+1}(-n\cos 0)\right] \qquad [\because\ \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{n}{\pi(n^2+1)}(1 - e^{-2\pi}) \qquad [\because\ \cos 2n\pi = \cos 0 = 1]$$

Hence, $$f(x) = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos nx + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx \qquad \dots (1)$$

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} \qquad [\because\ \cos n\pi = (-1)^n, \sin n\pi = 0]$$

$$e^{-\pi} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

$$= \frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}(1-e^{-2\pi})} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

Hence,
$$\frac{\pi}{2}\frac{1}{\sinh \pi} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

Example 8

Find the Fourier series of $f(x)=\sqrt{1-\cos x}$ *in the interval* $(0, 2\pi)$. *Hence, deduce that* $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sqrt{1-\cos x} = \sqrt{2}\sin\frac{x}{2}$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi}\left|-2\cos\frac{x}{2}\right|_0^{2\pi} \\
&= \frac{\sqrt{2}}{2\pi}(-2\cos\pi + 2\cos 0) \\
&= \frac{2\sqrt{2}}{\pi} \qquad [\because\ \cos\pi = -1, \cos 0 = 1]
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cos nx\,\mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{2n+1}{2}\right)x - \sin\left(\frac{2n-1}{2}\right)x\right]\mathrm{d}x
\end{aligned}$$

$$= \frac{\sqrt{2}}{2\pi}\left|-\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right)x + \frac{2}{2n-1}\cos\left(\frac{2n-1}{2}\right)x\right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{2n+1}\cos(2n\pi+\pi) + \frac{2\cos 0}{2n+1} + \frac{2}{2n-1}\cos(2n\pi-\pi) - \frac{2\cos 0}{2n-1}\right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{4}{2n+1} - \frac{4}{2n-1}\right] \qquad [\because\ \cos(2n+1)\pi = \cos(2n-1)\pi = -1, \cos 0 = 1]$$

$$= -\frac{4\sqrt{2}}{\pi}\frac{1}{4n^2-1}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx\,\mathrm{d}x$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{2n-1}{2}\right)x - \cos\left(\frac{2n+1}{2}\right)x\right]\mathrm{d}x$$

$$= \frac{\sqrt{2}}{2\pi}\left|\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right|_0^{2\pi}$$

$$= 0 \quad [\because\ \sin(2n-1)\pi = \sin(2n+1)\pi = \sin 0 = 0]$$

Hence,
$$f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

Example 9

Find the Fourier series of $f(x) = -1 \qquad 0 < x < \pi$

$$= 2 \qquad \pi < x < 2\pi$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\int_0^{\pi}(-1)\mathrm{d}x + \int_{\pi}^{2\pi} 2\mathrm{d}x\right] \\
&= \frac{1}{2\pi}\left[\left|-x\right|_0^{\pi} + \left|2x\right|_{\pi}^{2\pi}\right] \\
&= \frac{1}{2\pi}\left[(-\pi)+(4\pi-2\pi)\right] \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_0^{\pi}(-1)\cos nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 2\cos nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[-\left|\frac{\sin nx}{n}\right|_0^{\pi} + 2\left|\frac{\sin nx}{n}\right|_{\pi}^{2\pi}\right] \\
&= 0 \quad [\because\ \sin 2n\pi = \sin n\pi = \sin 0 = 0]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_0^{\pi}(-1)\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 2\sin nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[\left|\frac{\cos nx}{n}\right|_0^{\pi} + \left|-\frac{2\cos nx}{n}\right|_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\frac{\cos n\pi}{n} - \frac{\cos 0}{n} - \frac{2\cos 2n\pi}{n} + \frac{2\cos n\pi}{n}\right] \\
&= \frac{3}{n\pi}[(-1)^n - 1] \quad \left[\because\ \cos 2n\pi = \cos 0 = 1,\ \cos n\pi = (-1)^n\right]
\end{aligned}
$$

Hence,
$$
\begin{aligned}
f(x) &= \frac{1}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx \\
&= \frac{1}{2} + \frac{3}{\pi}\left(-2\sin x - \frac{2}{3}\sin 3x - \frac{2}{5}\sin 5x - \cdots\right) \\
&= \frac{1}{2} - \frac{6}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)
\end{aligned}
$$

Example 10

Find the Fourier series of $f(x) = x^2 \quad 0 < x < \pi$ **[Winter 2012]**

$\qquad\qquad\qquad\qquad\quad = 0 \quad \pi < x < 2\pi$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\int_0^{\pi} x^2\mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot \mathrm{d}x\right] \\
&= \frac{1}{2\pi}\left|\frac{x^3}{3}\right|_0^{\pi} \\
&= \frac{1}{2\pi}\left(\frac{\pi^3}{3}\right) \\
&= \frac{\pi^2}{6}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_0^{\pi} x^2\cos nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot\cos nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(2\pi\frac{\cos n\pi}{n^2}\right) \quad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{n^2}(-1)^n \quad \left[\because\ \cos n\pi = (-1)^n\right]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_0^{\pi} x^2\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot\sin nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left|x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{1}{\pi}\left[-\pi^2\left(\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) - \frac{2\cos 0}{n^3}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]
\end{aligned}$$

$$=\frac{1}{\pi}\left[-\pi^2\frac{(-1)^n}{n}+2\frac{(-1)^n}{n^3}-\frac{2}{n^3}\right]\quad\left[\because\ \cos n\pi=(-1)^n,\cos 0=1\right]$$

Hence, $f(x)=\dfrac{\pi^2}{6}+2\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos nx+\dfrac{1}{\pi}\sum\limits_{n=1}^{\infty}\left[\dfrac{-\pi^2(-1)^n}{n}+\dfrac{2(-1)^n}{n^3}-\dfrac{2}{n^3}\right]\sin nx$

Example 11

Expand f(x) in Fourier series in the interval $(0, 2\pi)$ *if*

$$f(x)=\begin{cases}-\pi & 0<x<\pi\\ x-\pi & \pi<x<2\pi\end{cases}$$

and hence, show that $\sum\limits_{n=0}^{\infty}\dfrac{1}{(2n+1)^2}=\dfrac{\pi^2}{8}$. **[Winter 2016; Summer 2018]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)\,\mathrm{d}x$$

$$=\frac{1}{2\pi}\left[\int_0^{\pi}(-\pi)\mathrm{d}x+\int_{\pi}^{2\pi}(x-\pi)\,\mathrm{d}x\right]$$

$$=\frac{1}{2\pi}\left[(-\pi)\left|x\right|_0^{\pi}+\left|\frac{x^2}{2}-\pi x\right|_{\pi}^{2\pi}\right]$$

$$=\frac{1}{2\pi}\left[-\pi^2+2\pi^2-2\pi^2-\frac{\pi^2}{2}+\pi^2\right]$$

$$=-\frac{\pi}{4}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,\mathrm{d}x$$

$$=\frac{1}{\pi}\left[\int_0^{\pi}(-\pi)\cos nx\,\mathrm{d}x+\int_{\pi}^{2\pi}(x-\pi)\cos nx\,\mathrm{d}x\right]$$

$$=\frac{1}{\pi}\left[(-\pi)\left|\frac{\sin nx}{n}\right|_0^{\pi}+\left|(x-\pi)\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right] \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] \qquad \begin{array}{l}[\because \sin 2n\pi = \sin n\pi = 0]\\ [\cos 2n\pi = 1, \cos n\pi = (-1)^n]\end{array}$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}(-\pi)\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi}(x-\pi)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|-\frac{\cos nx}{n}\right|_0^{\pi} + \left|(x-\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{(-1)^n}{n} - \frac{1}{n}\right\} + \left|-(x-\pi)\left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{(-1)^n}{n} - \frac{1}{n} - \frac{1}{n} [\because \cos 2n\pi = 1, \cos n\pi = (-1)^n, \sin 2n\pi = \sin n\pi = 0]$$

$$= \frac{(-1)^n}{n} - \frac{2}{n}$$

$$= \frac{1}{n}[(-1)^n - 2]$$

Hence, $$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}[(-1)^n - 2]\sin nx$$

$$= -\frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$
$$- 3\sin x - \frac{1}{2}\sin 2x - \sin 3x - \cdots$$

$$= -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} - \sum_{n=1}^{\infty}\left[\frac{2-(-1)^n}{n}\right]\sin nx \qquad ...(1)$$

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)}{(2n+1)^2} - 0$$

$$\frac{1}{2}\left[\lim_{x\to\pi^-} f(x)+\lim_{x\to\pi^+} f(x)\right]=-\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{1}{2}[-\pi+0]+\frac{\pi}{4}=-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi}{4}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$$

Example 12

Find the Fourier series of $f(x) = x + x^2$ in the interval $(-\pi, \pi)$, and hence, deduce that

(i) $$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$$

(ii) $$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$ **[Winter 2017, 2012]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$\begin{aligned}a_0&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\mathrm{d}x\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}(x+x^2)\mathrm{d}x\\&=\frac{1}{2\pi}\left|\frac{x^2}{2}+\frac{x^3}{3}\right|_{-\pi}^{\pi}\\&=\frac{1}{2\pi}\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)\\&=\frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right)\\&=\frac{\pi^2}{3}\end{aligned}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(\frac{\sin nx}{n}\right)-(1+2x)\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2\pi)\left(\frac{\cos n\pi}{n^2}\right)-(1-2\pi)\left\{\frac{\cos(-n\pi)}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[4\pi\left(\frac{\cos n\pi}{n^2}\right)\right] \qquad [\because\ \cos(-n\pi)=\cos(n\pi)]$$

$$= \frac{4(-1)^n}{n^2} \qquad [\because\ \cos n\pi=(-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(-\frac{\cos nx}{n}\right)-(1+2x)\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right)+2\left(\frac{\cos n\pi}{n^3}\right)+(-\pi+\pi^2)\left\{\frac{\cos(-n\pi)}{n}\right\}-2\left\{\frac{\cos(-n\pi)}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] \qquad [\because\ \cos(-n\pi)=\cos n\pi]$$

$$= \frac{-2(-1)^n}{n} \qquad [\because\ \cos n\pi=(-1)^n]$$

Hence,
$$f(x) = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$x+x^2 = \frac{\pi^2}{3}+4\left(-\frac{1}{1^2}\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\cdots\right)$$
$$-2\left(-\frac{1}{1}\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\cdots\right) \qquad \text{...(1)}$$

(i) Putting $x=0$ in Eq. (1),

$$0 = \frac{\pi^2}{3}+4\left(-\frac{1}{1^2}\cos 0+\frac{1}{2^2}\cos 0-\frac{1}{3^2}\cos 0+\cdots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots$$

(ii) Putting $x = \pi$ in Eq. (1),

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \cdots\right]$$

$$= \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (2)$$

Putting $x = -\pi$ in Eq. (1),

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos(-\pi) + \frac{1}{2^2}\cos(-2\pi) - \frac{1}{3^2}\cos(-3\pi) + \cdots\right]$$

$$= \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (3)$$

Adding Eqs (2) and (3),

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 13

Find the Fourier series expansion of the periodic function $f(x) = x - x^2$ in the interval $-\pi \le x \le \pi$ and show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

[Summer 2017]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$$= \frac{1}{2\pi}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{1}{2\pi}\left(-\frac{2\pi^3}{3}\right)$$

$$= -\frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|(x - x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left|(x - x^2)\left(\frac{\sin nx}{n}\right) + (1-2x)\left(\frac{\cos nx}{n^2}\right) + \frac{2\sin nx}{n^3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1-2\pi)\frac{(-1)^n}{n^2} - (1+2\pi)\frac{(-1)^n}{n^2}\right] \qquad \begin{bmatrix}\sin n\pi = \sin(-n\pi) = 0\\ \cos n\pi = 0\end{bmatrix}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{2\pi(-1)^n}{n^2} - \frac{(-1)^n}{n^2} - \frac{2\pi(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi(-1)^n}{n^2}\right]$$

$$= -\frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left|-(x - x^2)\left(\frac{\cos nx}{n}\right) + (1-2x)\left(\frac{\sin x}{n^2}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2 - \pi)\frac{(-1)^n}{n} - 2\frac{(-1)^n}{n^3} + (-\pi - \pi^2)\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3}\right]$$

$$[\because \cos n\pi = (-1)^n, \sin n\pi = \sin(-n\pi) = 0]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2(-1)^n}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi^2(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi(-1)^n}{n}\right]$$

$$= -\frac{2(-1)^n}{n}$$

Hence, $$f(x) = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$x - x^2 = -\frac{\pi^2}{3} - 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right)$$
$$-2\left(-\frac{1}{1}\sin nx + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right) \qquad ...(1)$$

Putting $x = 0$ in Eq. (1),

$$0 = -\frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right)$$

$$\frac{\pi^2}{3} = 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

Example 14

Find the Fourier series of $f(x) = x + |x|$ in the interval $-\pi < x < \pi$.

[Winter 2015, 2014]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (x + |x|)\,dx$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} x\,dx + \int_{-\pi}^{\pi} |x|\,dx\right]$$
$$= \frac{1}{2\pi}\left[0 + 2\int_{0}^{\pi} |x|\,dx\right] \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is even function} \atop = 0, \quad \text{if } f(x) \text{ is odd function}\right]$$
$$= \frac{1}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right)$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+|x|)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,\mathrm{d}x + \int_{-\pi}^{\pi} |x|\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[0 + 2\int_0^{\pi} |x|\cos nx\,\mathrm{d}x\right] \quad \left[\begin{array}{l}\because\ x\cos nx \text{ is odd function} \\ \text{and } |x|\cos nx \text{ is even function}\end{array}\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \quad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+|x|)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x + \int_{-\pi}^{\pi} |x|\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx\,\mathrm{d}x + 0\right] \quad \left[\begin{array}{l}\because\ x\sin nx \text{ is an even function} \\ |x|\sin x \text{ is an odd function}\end{array}\right]$$

$$= \frac{2}{\pi}\left|x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{2}{n}(-1)^n \quad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence, $$f(x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos nx-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$x+|x|=\frac{\pi}{2}+\frac{2}{\pi}\left[-\frac{2}{1^2}\cos x-\frac{2}{3^2}\cos 3x-\frac{2}{5^2}\cos 5x-\cdots\right]$$
$$-2\left[-\frac{1}{1}\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\cdots\right]$$

$$=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\cdots\right)$$
$$+2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots\right)$$

Example 15

Find the Fourier series of $f(x) = e^{ax}$ in the interval $(-\pi, \pi)$.

[Winter 2013]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,\mathrm{d}x$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ax}\,\mathrm{d}x$$
$$=\frac{1}{2\pi}\left|\frac{e^{ax}}{a}\right|_{-\pi}^{\pi}$$
$$=\frac{1}{2\pi a}(e^{a\pi}-e^{-a\pi})$$
$$=\frac{\sinh a\pi}{\pi a}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,\mathrm{d}x$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\cos n\pi) - \frac{e^{-a\pi}}{a^2+n^2}\{a\cos(-n\pi)\}\right] \quad [\because \ \sin n\pi = \sin(-n\pi) = 0]$$

$$= \frac{a\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi}) \quad [\because \ \cos(-n\pi) = \cos n\pi]$$

$$= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)} \quad [\because \ \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx \, dx$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{e^{-a\pi}}{a^2+n^2}\{n\cos(-n\pi)\}\right]$$

$$= -\frac{n\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi}) \quad [\because \ \cos(-n\pi) = \cos n\pi]$$

$$= -\frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)} \quad [\because \ \cos n\pi = (-1)^n]$$

Hence, $$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx - \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

$$= \frac{\sinh a\pi}{a\pi} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)$$

Example 16

Find the Fourier series of $f(x) = 0 \qquad -\pi < x < 0$

$= x \qquad 0 < x < \pi$ **[Summer 2013]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx\right]$$
$$= \frac{1}{2\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$
$$= \frac{1}{2\pi}\left(\frac{\pi^2}{2} - 0\right)$$
$$= \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$
$$= \frac{1}{\pi}\left|x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$
$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$
$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$
$$= \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$
$$= \frac{1}{\pi}\left|x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}$$
$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$
$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] \qquad \left[\because\ \cos n\pi = (-1)^n\right]$$
$$= -\frac{(-1)^n}{n}$$

Hence, $f(x) = \dfrac{\pi}{4} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n}{n^2} - \dfrac{1}{n^2}\right]\cos nx - \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin nx$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left(-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \cdots\right)$$
$$-\left(-\frac{1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right)$$
$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$
$$+\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

Example 17

Find the Fourier series of $f(x) = -\pi \qquad -\pi < x < 0$

$\qquad = x \qquad 0 < x < \pi$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots.$ **[Summer 2016, 2014]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)\mathrm{d}x + \int_0^{\pi} x\,\mathrm{d}x\right] \\
&= \frac{1}{2\pi}\left[\left|-\pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_0^{\pi}\right] \\
&= \frac{1}{2\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] \\
&= -\frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,\mathrm{d}x + \int_0^{\pi} x\cos nx\,\mathrm{d}x\right]
\end{aligned}$$

$$= \frac{1}{\pi}\left[-\pi \left| \frac{\sin nx}{n} \right|_{-\pi}^{0} + \left| x\left(\frac{\sin nx}{n} \right) - \left(-\frac{\cos nx}{n^2} \right) \right|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} \right] \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1 \right] \quad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx \,\mathrm{d}x + \int_{0}^{\pi} x\sin nx \,\mathrm{d}x \right]$$

$$= \frac{1}{\pi}\left[-\pi \left| -\frac{\cos nx}{n} \right|_{-\pi}^{0} + \left| x\left(-\frac{\cos nx}{n} \right) - \left(-\frac{\sin nx}{n^2} \right) \right|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\pi\left\{ \frac{\cos 0}{n} - \frac{\cos(-n\pi)}{n} \right\} + \pi\left(-\frac{\cos n\pi}{n} \right) \right] \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n}[1 - 2\cos n\pi] \quad [\because \cos 0 = 1, \cos(-n\pi) = \cos n\pi]$$

$$= \frac{1}{n}[1 - 2(-1)^n] \quad [\because \cos n\pi = (-1)^n]$$

Hence, $$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2} \right]\cos nx + \sum_{n=1}^{\infty}\left[\frac{1 - 2(-1)^n}{n} \right]\sin nx \qquad \ldots (1)$$

At $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x \to 0^-} f(x) + \lim_{x \to 0^+} f(x) \right] = \frac{-\pi + 0}{2} = -\frac{\pi}{2}$$

Putting $x = 0$ in Eq. (1),

$$f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2} \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 18

Find the Fourier series of
$$f(x) = \begin{cases} -x - \pi & -\pi < x < 0 \\ x + \pi & 0 < x < \pi \end{cases}$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-x-\pi)\mathrm{d}x + \int_{0}^{\pi}(x+\pi)\,\mathrm{d}x\right] \\
&= \frac{1}{2\pi}\left[\left|-\frac{x^2}{2} - \pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2} + \pi x\right|_{0}^{\pi}\right] \\
&= \frac{1}{2\pi}\left[\left(\frac{\pi^2}{2} - \pi^2\right) + \left(\frac{\pi^2}{2} + \pi^2\right)\right] \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\cos nx\,\mathrm{d}x + \int_{0}^{\pi}(x+\pi)\cos nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[\left|(-x-\pi)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\pi}^{0}\right. \\
&\qquad \left. + \left|(x+\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{0}^{\pi}\right] \\
&= \frac{1}{\pi}\left[\left\{-\frac{\cos 0}{n^2} + \frac{\cos(-n\pi)}{n^2}\right\} + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] \qquad \left[\because\ \begin{aligned}\sin n\pi &= \sin(-n\pi) \\ &= \sin 0 = 0\end{aligned}\right] \\
&= \frac{2}{\pi n^2}[(-1)^n - 1] \quad \left[\because\ \cos(-n\pi) = \cos n\pi = (-1)^n,\ \cos 0 = 1\right]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\sin nx\,\mathrm{d}x + \int_{0}^{\pi}(x+\pi)\sin nx\,\mathrm{d}x\right]
\end{aligned}$$

$$= \frac{1}{\pi}\left[\left|(-x-\pi)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\pi}^{0} + \left|(x+\pi)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{(-\pi)\left(-\frac{\cos 0}{n}\right)\right\}+\left\{(2\pi)\left(-\frac{\cos n\pi}{n}\right)+\pi\left(\frac{\cos 0}{n}\right)\right\}\right]$$

$$\left[\because \sin n\pi = \sin(-n\pi) = \sin 0 = 0\right]$$

$$= \frac{2}{n}[1-(-1)^n] \qquad \left[\because \cos 0 = 1, \cos(n\pi) = (-1)^n\right]$$

Hence, $f(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx + 2\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin nx$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right) + 4\left(\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Example 19

Find the Fourier series of $f(x) = 0 \qquad -\pi < x < 0$

$= \sin x \qquad 0 < x < \pi$

Hence, deduce that $\dfrac{1}{2} = \dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \ldots$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left|-\cos x\right|_{0}^{\pi}$$

$$= \frac{1}{2\pi}(-\cos\pi + \cos 0)$$

$$= \frac{1}{\pi} \quad [\because \cos\pi = -1, \cos 0 = 1]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,\mathrm{d}x + \int_{0}^{\pi}\sin x\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right|_{0}^{\pi}, \quad n \neq 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos 0}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{\cos 0}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right], \quad n \neq 1 \quad \left[\begin{array}{c}\because \cos(n+1)\pi = (-1)^{n+1} \\ \cos(n-1)\pi = (-1)^{n-1} \\ \cos 0 = 1\end{array}\right]$$

$$= -\frac{1}{\pi(n^2-1)}[1+(-1)^n], \quad n \neq 1$$

For $n = 1$,

$$a_1 = \frac{1}{\pi}\int_{0}^{\pi}\sin x\cos x\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\sin 2x\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2\pi}{2} + \frac{\cos 0}{2}\right]$$

$$= 0 \quad [\because \cos 2\pi = \cos 0 = 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin nx\,\mathrm{d}x + \int_{0}^{\pi}\sin x\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[\cos(n-1)x - \cos(n+1)x]\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right|_0^{\pi}, \quad n \neq 1$$

$$= 0, \quad n \neq 1 \quad [\because \sin(n-1)\pi = \sin(n+1)\pi = \sin 0 = 0]$$

For $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} (1 - \cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left| x - \frac{\sin 2x}{2} \right|_0^{\pi}$$

$$= \frac{1}{2\pi}(\pi) \quad [\because \sin 2\pi = \sin 0 = 0]$$

$$= \frac{1}{2}$$

Hence, $$f(x) = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \cdots\right) + \frac{1}{2}\sin x \quad \text{... (1)}$$

At $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x \to 0^-} f(x) + \lim_{x \to 0^+} f(x)\right] = 0$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\right)$$

$$\frac{1}{2} = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \ldots$$

Example 20

Find the Fourier series of $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x - \pi & 0 < x < \pi \end{cases}$

[Summer 2015]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)\,\mathrm{d}x + \int_{0}^{\pi}(x-\pi)\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left[(-\pi)\left|x\right|_{-\pi}^{0} + \left|\frac{x^2}{2} - \pi x\right|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[(-\pi)[-(-\pi)] + \left(\frac{\pi^2}{2} - \pi^2\right)\right]$$

$$= \frac{1}{2\pi}\left[-\pi^2 - \frac{\pi^2}{2}\right]$$

$$= \frac{1}{2\pi}\left(-\frac{3\pi^2}{2}\right)$$

$$= -\frac{3\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,\mathrm{d}x + \int_{0}^{\pi}(x-\pi)\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-\pi)(0) + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_{0}^{\pi}\right] \quad [\because \sin(-n\pi) = \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] \quad [\because \sin n\pi = \sin 0 = 0, \cos 0 = 1]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx \,\mathrm{d}x + \int_{0}^{\pi}(x-\pi)\sin nx \,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|-\frac{\cos nx}{n}\right|_{-\pi}^{0} + \left|(x-\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{1}{n} - \frac{\cos n\pi}{n}\right\} + \left|-(x-\pi)\left(\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right|_{0}^{\pi}\right] \qquad [\ \cos 0 = 1]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{1}{n} - \frac{\cos n\pi}{n}\right) + (-\pi)\left(\frac{1}{n}\right)\right] \qquad \left[\begin{array}{c}\because \sin n\pi = \sin 0 = 0\\ \cos 0 = 1\end{array}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi}{n}\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$= -\frac{(-1)^n}{n}$$

$$= \frac{(-1)^{n+1}}{n}$$

Hence, $f(x) = -\dfrac{3\pi}{4} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n^2}[(-1)^n - 1]\cos nx + \sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin nx$

$$= -\frac{3\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \cdots\right) + \left(\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

Example 21

Find the Fourier series of $f(x) = x \qquad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$$= \pi - x \qquad \frac{\pi}{2} < x < \frac{3\pi}{2}$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\mathrm{d}x\right] \\
&= \frac{1}{2\pi}\left[\left|\frac{x^2}{2}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right] \\
&= \frac{1}{2\pi}\left[\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) + \left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\cos nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right] \\
&= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right] \\
&= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin n\pi\sin\frac{n\pi}{2}\right]
\end{aligned}$$

$$\left[\begin{aligned} \because\ \sin A + \sin B &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \\ \cos A - \cos B &= 2\sin\frac{A+B}{2}\sin\frac{B-A}{2} \end{aligned}\right]$$

$$= 0 \qquad [\because\ \sin n\pi = 0]$$

$$b_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{3}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{3n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi \sin\frac{n\pi}{2} + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi n^2}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right] \qquad [\because \ \sin n\pi = 0]$$

Hence, $f(x) = \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\left[3\sin\dfrac{n\pi}{2} - \sin\dfrac{3n\pi}{2}\right]\sin nx$

Fourier Series Expansion with Period 2*l*

Example 22

Find the Fourier series of $f(x) = x^2$ *in the interval* (0, 4). *Hence, deduce that* $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$

Solution

The Fourier series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$
\begin{aligned}
a_0 &= \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x \\
&= \frac{1}{4}\int_0^4 x^2\mathrm{d}x \\
&= \frac{1}{4}\left|\frac{x^3}{3}\right|_0^4 \\
&= \frac{1}{4}\left(\frac{64}{3}\right) \\
&= \frac{16}{3}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \\
&= \frac{1}{2}\int_0^4 x^2\cos\frac{n\pi x}{2}\mathrm{d}x \\
&= \frac{1}{2}\left|x^2\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(2x)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)+2\left(-\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right)\right|_0^4 \\
&= \frac{1}{2}\left[8\left(\frac{4}{n^2\pi^2}\cos 2n\pi\right)\right] \qquad [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= \frac{1}{2}\left[8\left(\frac{4}{n^2\pi^2}\right)\right] \qquad [\because\ \cos 2n\pi = 1] \\
&= \frac{16}{n^2\pi^2}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x \\
&= \frac{1}{2}\int_0^4 x^2\sin\frac{n\pi x}{2}\mathrm{d}x \\
&= \frac{1}{2}\left|x^2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)-2x\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)+2\left(\frac{8}{n^3\pi^3}\cos\frac{n\pi x}{2}\right)\right|_0^4 \\
&= \frac{1}{2}\left[16\left(-\frac{2}{n\pi}\cos 2n\pi\right)+2\left(\frac{8}{n^3\pi^3}\cos 2n\pi\right)-2\left(\frac{8}{n^3\pi^3}\cos 0\right)\right] \\
&= \frac{1}{2}\left(-\frac{32}{n\pi}\right) \qquad [\because\ \cos 2n\pi = \cos 0 = 1] \\
&= -\frac{16}{n\pi}
\end{aligned}
$$

Hence,
$$f(x) = \frac{16}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

$$x^2 = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{2^2}\cos\pi x + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\right)$$
$$-\frac{16}{\pi}\left(\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\pi x + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right) \qquad ...(1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$-\frac{1}{3} = \frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (2)$$

Putting $x = 4$ in Eq. (1),

$$16 = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$\frac{2}{3} = \frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (3)$$

Adding Eqs (2) and (3),

$$\frac{1}{3} = \frac{2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 23

Find the Fourier series of $f(x) = 4 - x^2$ *in the interval* (0, 2). *Hence, deduce that* $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$
\begin{aligned}
a_0 &= \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^2 (4-x^2)\mathrm{d}x \\
&= \frac{1}{2}\left|4x-\frac{x^3}{3}\right|_0^2 \\
&= \frac{1}{2}\left(8-\frac{8}{3}\right) \\
&= \frac{8}{3}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \\
&= \int_0^2 (4-x^2)\cos n\pi x\,\mathrm{d}x \\
&= \left|(4-x^2)\left(\frac{\sin n\pi x}{n\pi}\right)-(-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)+(-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^2 \\
&= -4\left(\frac{\cos 2n\pi}{n^2\pi^2}\right) \qquad [\because \ \sin n\pi = \sin 0 = 0] \\
&= -\frac{4}{n^2\pi^2} \qquad [\because \ \cos 2n\pi = 1]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x \\
&= \int_0^2 (4-x^2)\sin n\pi x\,\mathrm{d}x \\
&= \left|(4-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right)-(-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)+(-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^2 \\
&= -2\left(\frac{\cos 2n\pi}{n^3\pi^3}\right)+4\left(\frac{\cos 0}{n\pi}\right)+2\left(\frac{\cos 0}{n^3\pi^3}\right) \qquad [\because \ \sin n\pi = \sin 0 = 0] \\
&= \frac{4}{n\pi} \qquad [\because \ \cos 2n\pi = \cos 0 = 1]
\end{aligned}
$$

Hence, $$f(x) = \frac{8}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x$$

$$
\begin{aligned}
4-x^2 &= \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \cdots\right) \\
&\quad + \frac{4}{\pi}\left(\frac{1}{1}\sin\pi x + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 2\pi x + \cdots\right) \qquad \text{...(1)}
\end{aligned}
$$

Putting $x = 0$ in Eq. (1),

$$4 = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{1}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (2)$$

Putting $x = 2$ in Eq. (1),

$$0 = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$-\frac{2}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (3)$$

Adding Eqs (2) and (3),

$$-\frac{1}{3} = -\frac{2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 24

Find the Fourier series of $f(x) = 2x - x^2$ *in the interval* (0, 3). *Hence, deduce that* $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ **[Summer 2016]**

Solution

The Fourier series of $f(x)$ with period $2l = 3$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x$$

$$= \frac{1}{3}\int_0^{3} (2x - x^2)\mathrm{d}x$$

$$= \frac{1}{3}\left|x^2 - \frac{x^3}{3}\right|_0^3$$

$$= \frac{1}{3}\left(9 - \frac{27}{3}\right)$$

$$= 0$$

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \\
&= \frac{2}{3}\int_0^{3}(2x-x^2)\cos\frac{2n\pi x}{3}\,\mathrm{d}x \\
&= \frac{2}{3}\left|(2x-x^2)\left(\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\right)-(2-2x)\left(-\frac{9}{4n^2\pi^2}\cos\frac{2n\pi x}{3}\right)\right. \\
&\qquad\qquad \left.+(-2)\left(-\frac{27}{8n^3\pi^3}\sin\frac{2n\pi x}{3}\right)\right|_0^3 \\
&= \frac{2}{3}\left[4\left(-\frac{9}{4n^2\pi^2}\cos 2n\pi\right)+2\left(-\frac{9}{4n^2\pi^2}\cos 0\right)\right] \quad [\because\ \sin 2n\pi=\sin 0=0] \\
&= \frac{2}{3}\left[\frac{9}{4n^2\pi^2}(-4-2)\right] \quad [\because\ \cos 2n\pi=\cos 0=1] \\
&= -\frac{9}{n^2\pi^2}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \\
&= \frac{2}{3}\int_0^{3}(2x-x^2)\sin\frac{2n\pi x}{3}\,\mathrm{d}x \\
&= \frac{2}{3}\left|(2x-x^2)\left(-\frac{3}{2n\pi}\cos\frac{2n\pi x}{3}\right)-(2-2x)\left(-\frac{9}{4n^2\pi^2}\sin\frac{2n\pi x}{3}\right)\right. \\
&\qquad\qquad \left.+(-2)\left(\frac{27}{8n^3\pi^3}\cos\frac{2n\pi x}{3}\right)\right|_0^3 \\
&= \frac{2}{3}\left[(-3)\left(-\frac{3}{2n\pi}\cos 2n\pi\right)-(2)\left(\frac{27}{8n^3\pi^3}\cos 2n\pi\right)\right. \\
&\qquad\qquad \left.+2\left(\frac{27}{8n^3\pi^3}\cos 0\right)\right] \quad [\because\ \sin 2n\pi=\sin 0=0] \\
&= \frac{2}{3}\left(\frac{9}{2n\pi}\right) \quad [\because\ \cos 2n\pi=\cos 0=1] \\
&= \frac{3}{n\pi}
\end{aligned}
$$

Hence, $\quad f(x) = -\dfrac{9}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3}+\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3}$

$$2x - x^2 = -\frac{9}{\pi^2}\left(\frac{1}{1^2}\cos\frac{2\pi x}{3} + \frac{1}{2^2}\cos\frac{4\pi x}{3} + \frac{1}{3^2}\cos\frac{6\pi x}{3} + \cdots\right)$$

$$+\frac{3}{\pi}\left(\frac{1}{1}\sin\frac{2\pi x}{3} + \frac{1}{2}\sin\frac{4\pi x}{3} + \frac{1}{3}\sin\frac{6\pi x}{3} + \cdots\right) \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = -\frac{9}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$0 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \qquad \ldots (2)$$

Putting $x = 3$ in Eq. (1),

$$-3 = -\frac{9}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \qquad \ldots (3)$$

Adding Eqs (2) and (3),

$$\frac{\pi^2}{3} = 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 25

For the function $f(x) = \begin{cases} x & 0 \le x \le 2 \\ 4 - x & 2 \le x \le 4 \end{cases}$, *find its Fourier series. Hence, show that* $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. **[Winter 2015]**

Solution

The Fourier series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,\mathrm{d}x$$

$$= \frac{1}{4}\int_0^4 f(x)\,\mathrm{d}x$$

$$= \frac{1}{4}\left[\int_0^2 x\,\mathrm{d}x + \int_2^4 (-x+4)\,\mathrm{d}x\right]$$

$$= \frac{1}{4}\left[\left|\frac{x^2}{2}\right|_0^2 + \left|\frac{-x^2}{2} + 4x\right|_2^4\right]$$

$$= \frac{1}{4}\left[(2-0) + \left\{(-8+16) - (-2+8)\right\}\right]$$

$$= \frac{1}{4}\left[2 + (8-6)\right]$$

$$= 1$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^2 x\cos\frac{n\pi x}{2}\,\mathrm{d}x + \int_2^4 (4-x)\cos\frac{n\pi x}{2}\,\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|(x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_0^2\right.$$

$$\left. + \left|(4-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[\left|\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right|_0^2\right.$$

$$\left. + \left|\frac{2(4-x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos(n\pi) - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos(2n\pi) + \frac{4}{n^2\pi^2}\cos(n\pi)\right]$$

$$= \frac{1}{2}\left[\frac{8}{n^2\pi^2}\cos n\pi - \frac{8}{n^2\pi^2}\right] \qquad [\because \cos 2n\pi = 1]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_0^4 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_0^2 x\sin\frac{n\pi x}{2}\,dx + \int_2^4 (4-x)\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left|(x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_0^2 + \left|(4-x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[\left|\frac{-2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right|_0^2 + \left|\frac{-2(4-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + \frac{4}{n\pi}\cos n\pi\right]$$

$$= 0$$

Hence, $$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\left\{(-1)^n - 1\right\}}{n^2}\right]\cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{2} \qquad \text{...(1)}$$

Putting $x = 2$ in Eq. (1),

$$2 = 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2}\right]\cos(2n+1)\pi$$

$$2 = 1 + \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$1 = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 26

Find the Fourier series of $f(x) = \pi x \qquad 0 < x < 1$

$\qquad\qquad\qquad\qquad\qquad = 0 \qquad 1 < x < 2$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\int_0^1 \pi x\,\mathrm{d}x + \int_1^2 0\cdot\mathrm{d}x\right)$$

$$= \frac{1}{2}\left|\frac{\pi x^2}{2}\right|_0^1$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x\cos n\pi x\,\mathrm{d}x + \int_1^2 0\cdot\cos n\pi x\,\mathrm{d}x$$

$$= \left|\pi x\left(\frac{\sin n\pi x}{n\pi}\right) - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= \left[\pi\left(\frac{\cos n\pi}{n^2\pi^2}\right) - \pi\left(\frac{\cos 0}{n^2\pi^2}\right)\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x\sin n\pi x\,\mathrm{d}x + \int_1^2 0\cdot\sin n\pi x\,\mathrm{d}x$$

$$= \left|\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= -\frac{\pi\cos n\pi}{n\pi} \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{(-1)^n}{n} \qquad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence,
$$f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left(-\frac{2}{1^2}\cos \pi x - \frac{2}{3^2}\cos 3\pi x - \cdots\right)$$

$$-\left(-\frac{1}{1}\sin \pi x + \frac{1}{2}\sin 2\pi x - \frac{1}{3}\sin 3\pi x + \cdots\right)$$

Example 27

Find the Fourier series of the periodic function with a period 2 of

$$f(x)=\pi \qquad 0 \le x \le 1$$
$$=\pi(2-x) \qquad 1 \le x \le 2$$

[Summer 2013]

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$=a_0+\sum_{n=1}^{\infty} a_n \cos n\pi x+\sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0=\frac{1}{2l}\int_0^{2l} f(x)\,\mathrm{d}x$$
$$=\frac{1}{2}\left[\int_0^1 \pi\,\mathrm{d}x+\int_1^2 \pi(2-x)\,\mathrm{d}x\right]$$
$$=\frac{\pi}{2}\left[\left|x\right|_0^1+\left|2x-\frac{x^2}{2}\right|_1^2\right]$$
$$=\frac{\pi}{2}\left[(1)+\left(4-2-2+\frac{1}{2}\right)\right]$$
$$=\frac{3\pi}{4}$$

$$a_n=\frac{1}{l}\int_0^{2l} f(x)\cos n\pi x\,\mathrm{d}x$$
$$=\int_0^1 \pi\cos n\pi x\,\mathrm{d}x+\int_1^2 \pi(2-x)\cos n\pi x\,\mathrm{d}x$$
$$=\left|\pi\left(\frac{\sin n\pi x}{n\pi}\right)\right|_0^1+\left|\pi(2-x)\left(\frac{\sin n\pi x}{n\pi}\right)-(-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2$$
$$=\left(-\frac{\cos 2n\pi}{n^2\pi}+\frac{\cos n\pi}{n^2\pi}\right) \qquad [\because\ \sin n\pi=\sin 0=0]$$
$$=\frac{1}{n^2\pi}\left[(-1)^n-1\right] \qquad \left[\because\ \cos n\pi=(-1)^n,\ \cos 2n\pi=1\right]$$

$$b_n=\frac{1}{l}\int_0^{2l} f(x)\sin n\pi x\,\mathrm{d}x$$
$$=\int_0^1 \pi\sin n\pi x\,\mathrm{d}x+\int_1^2 \pi(2-x)\sin n\pi x\,\mathrm{d}x$$

$$= \left|\pi\left(-\frac{\cos n\pi x}{n\pi}\right)\right|_0^1 + \left|\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left[\pi\left(-\frac{\cos n\pi}{n\pi}\right) + \pi\left(\frac{\cos 0}{n\pi}\right) + \pi\left(-\frac{\cos n\pi}{n\pi}\right)\right] \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \left(-\frac{2\cos n\pi}{n}\right) + \left(\frac{\cos 0}{n}\right)$$

$$= \frac{1-2(-1)^n}{n} \quad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $f(x) = \dfrac{3\pi}{4} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n - 1}{n^2}\right]\cos n\pi x + \sum_{n=1}^{\infty}\left[\dfrac{1-2(-1)^n}{n}\right]\sin n\pi x$

$$= \frac{3\pi}{4} - \frac{2}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$

$$+ \left(\frac{3}{1}\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{3}{3}\sin 3\pi x - \cdots\right)$$

Example 28

Find the Fourier series of

$$\begin{aligned} f(x) &= \pi x & 0 \le x < 1 \\ &= 0 & x = 1 \\ &= \pi(x-2) & 1 < x \le 2 \end{aligned}$$

Hence, deduce that $\dfrac{\pi}{4} = \dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \ldots$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(x-2)\,dx\right]$$

$$= \frac{1}{2}\left[\pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|\frac{x^2}{2} - 2x\right|_1^2\right]$$

$$= 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x\cos n\pi x\,\mathrm{d}x + \int_1^2 \pi(x-2)\cos n\pi x\,\mathrm{d}x$$

$$= \pi\left[\left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{\cos 0}{n^2\pi^2} + \frac{\cos 2n\pi}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= 0 \quad [\because\ \cos 0 = \cos 2n\pi = 1]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x\sin n\pi x\,\mathrm{d}x + \int_1^2 \pi(x-2)\sin n\pi x\,\mathrm{d}x$$

$$= \pi\left[\left|x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi}\right] \quad [\because\ \sin 2n\pi = \sin n\pi = \sin 0 = 0]$$

$$= -\frac{2(-1)^n}{n} \quad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence, $f(x) = 2\sum_{n=1}^{\infty}\left[-\frac{(-1)^n}{n}\right]\sin n\pi x$

$$= 2\left(\frac{1}{1}\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \frac{1}{4}\sin 4\pi x + \frac{1}{5}\sin 5\pi x - \cdots\right) \quad \ldots (1)$$

Putting $x = \frac{1}{2}$ in Eq. (1),

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{1}\sin\frac{\pi}{2} - \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} - \cdots\right)$$

$$\frac{\pi}{2} = 2\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \ldots\right)$$

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \ldots$$

Example 29

Find the Fourier series of $f(x) = x \qquad -1 < x < 0$

$= 2 \qquad 0 < x < 1$ **[Winter 2012]**

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{2}\left(\int_{-1}^{0} x\,dx + \int_{0}^{1} 2\,dx\right)$$

$$= \frac{1}{2}\left[\left|\frac{x^2}{2}\right|_{-1}^{0} + |2x|_0^1\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2} + 2\right]$$

$$= \frac{3}{4}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\left[\int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1} 2\cos n\pi x\,dx\right]$$

$$= \left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + 2\left|\frac{\sin n\pi x}{n\pi}\right|_0^1$$

$$= \frac{\cos 0}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2} \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi^2}\left[1 - (-1)^n\right] \quad \left[\because \cos 0 = 1, \cos n\pi = (-1)^n\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{1}\left[\int_{-1}^{0} x\sin n\pi x\,\mathrm{d}x + \int_{0}^{1} 2\sin n\pi x\,\mathrm{d}x\right]$$

$$= \left| x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + 2\left|-\frac{\cos n\pi x}{n\pi}\right|_{0}^{1}$$

$$= -\frac{\cos n\pi}{n\pi} - \frac{2\cos n\pi}{n\pi} + \frac{2\cos 0}{n\pi} \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n\pi}\left[-3(-1)^n + 2\right] \qquad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$= \frac{1}{n\pi}\left[2 - 3(-1)^n\right]$$

Hence,
$$f(x) = \frac{3}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos n\pi x + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{2-3(-1)^n}{n}\right]\sin n\pi x$$

$$= \frac{3}{4} + \frac{1}{\pi^2}\left(\frac{2}{1^2}\cos\pi x + \frac{2}{3^2}\cos 3\pi x + \frac{2}{5^2}\cos 5\pi x + \cdots\right)$$

$$+ \frac{1}{\pi}\left(\frac{5}{1}\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{5}{3}\sin 3\pi x - \frac{1}{4}\sin 4\pi x + \cdots\right)$$

$$= \frac{3}{4} + \frac{2}{\pi^2}\left(\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5nx + \cdots\right)$$

$$+ \frac{5}{\pi}\left(\sin\pi x + \frac{1}{3}\sin 3\pi x + \cdots\right) - \frac{1}{\pi}\left(\frac{1}{2}\sin 2\pi x + \frac{1}{4}\sin 4\pi x + \cdots\right)$$

Example 30

Find the Fourier series of $f(x) = 4 - x \qquad 3 < x < 4$

$\qquad\qquad\qquad\qquad = x - 4 \qquad 4 < x < 5$

Solution

The Fourier series of $f(x)$ with period $2l = 5 - 3 = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n\cos n\pi x + \sum_{n=1}^{\infty} b_n\sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\mathrm{d}x$$
$$= \frac{1}{2}\int_3^5 f(x)\mathrm{d}x$$
$$= \frac{1}{2}\left[\int_3^4 (4-x)\mathrm{d}x + \int_4^5 (x-4)\mathrm{d}x\right]$$
$$= \frac{1}{2}\left[\left|4x - \frac{x^2}{2}\right|_3^4 + \left|\frac{x^2}{2} - 4x\right|_4^5\right]$$
$$= \frac{1}{2}\left[\left\{\left(16 - \frac{16}{2}\right) - \left(12 - \frac{9}{2}\right)\right\} + \left\{\left(\frac{25}{2} - 20\right) - \left(\frac{16}{2} - 16\right)\right\}\right]$$
$$= \frac{1}{2}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$
$$= \int_3^4 (4-x)\cos n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\cos n\pi x\,\mathrm{d}x$$
$$= \left|(4-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_3^4 + \left|(x-4)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_4^5$$
$$= -\frac{1}{n^2\pi^2}(\cos 4n\pi - \cos 3n\pi) + \frac{1}{n^2\pi^2}(\cos 5n\pi - \cos 4n\pi)\ [\because\ \sin 3n\pi = \sin 5n\pi = 0]$$
$$= -\frac{1}{n^2\pi^2}[(-1)^{4n} - (-1)^{3n} - (-1)^{5n} + (-1)^{4n}]$$
$$= \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$
$$= \int_3^4 (4-x)\sin n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\sin n\pi x\,\mathrm{d}x$$
$$= \left|(4-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_3^4$$
$$+ \left|(x-4)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_4^5$$
$$= \frac{1}{n\pi}\cos 3n\pi - \frac{1}{n\pi}\cos 5n\pi \quad [\because\ \sin 4n\pi = \sin 3n\pi = \sin 5n\pi = 0]$$
$$= 0 \quad \left[\because\ \cos 3n\pi = \cos 5n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x$$
$$= \frac{1}{2} + \frac{2}{\pi^2}\left(-\frac{2}{1^2}\cos \pi x - \frac{2}{3^2}\cos 3\pi x - \frac{2}{5^2}\cos 5\pi x - \cdots\right)$$
$$= \frac{1}{2} - \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos \pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \cdots\right)$$

Example 31

Find the Fourier series of $$f(x) = 0 \qquad -5 < x < 0$$
$$= 3 \qquad 0 < x < 5$$

Solution

The Fourier series of $f(x)$ with period $2l = 10$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{5} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{5}$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\mathrm{d}x$$
$$= \frac{1}{10}\left(\int_{-5}^{0} 0\,\mathrm{d}x + \int_{0}^{5} 3\mathrm{d}x\right)$$
$$= \frac{1}{10}\left|3x\right|_0^5$$
$$= \frac{1}{10}(15)$$
$$= \frac{3}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$
$$= \frac{1}{5}\left(\int_{-5}^{0} 0\cdot\cos\frac{n\pi x}{5}\mathrm{d}x + \int_{0}^{5} 3\cos\frac{n\pi x}{5}\mathrm{d}x\right)$$
$$= \frac{3}{5}\left|\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right|_0^5$$
$$= 0 \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$
\begin{aligned}
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \\
&= \frac{1}{5}\left(\int_{-5}^{0} 0\cdot\sin\frac{n\pi x}{5}\,\mathrm{d}x + \int_{0}^{5} 3\sin\frac{n\pi x}{5}\,\mathrm{d}x\right) \\
&= \frac{3}{5}\left|\frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right)\right|_0^5 \\
&= \frac{3}{n\pi}[-\cos n\pi + \cos 0] \\
&= \frac{3}{n\pi}[1-(-1)^n] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}
$$

Hence, $$f(x) = \frac{3}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin\frac{n\pi x}{5}$$
$$= \frac{3}{2} + \frac{3}{\pi}\left(\frac{2}{1}\sin\frac{\pi x}{5} + \frac{2}{3}\sin\frac{3\pi x}{5} + \cdots\right)$$

Example 32

Find the Fourier series of $f(x) = x \qquad -1 < x < 0$

$\qquad\qquad\qquad\qquad\quad = x + 2 \qquad 0 < x < 1$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= \frac{1}{2l}\int_{-l}^{l} f(x)\,\mathrm{d}x \\
&= \frac{1}{2}\left[\int_{-1}^{0} x\mathrm{d}x + \int_{0}^{1} (x+2)\mathrm{d}x\right] \\
&= \frac{1}{2}\left[\left|\frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2} + 2x\right|_0^1\right] \\
&= \frac{1}{2}\left[-\frac{1}{2} + \left(\frac{1}{2} + 2\right)\right] \\
&= 1
\end{aligned}
$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \left[\int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1}(x+2)\cos n\pi x\,dx\right]$$

$$= \left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{0}^{1}$$

$$= \left[\left\{\frac{\cos 0}{n^2\pi^2} - \frac{\cos(-n\pi)}{n^2\pi^2}\right\} + \left\{\frac{\cos n\pi}{n^2\pi^2} - \frac{\cos 0}{n^2\pi^2}\right\}\right] \quad [\because\ \sin n\pi = \sin(-n\pi) = \sin 0 = 0]$$

$$= 0 \quad [\because\ \cos(-n\pi) = \cos n\pi]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \left[\int_{-1}^{0} x\sin n\pi x\,dx + \int_{0}^{1}(x+2)\sin n\pi x\,dx\right]$$

$$= \left|x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{0}^{1}$$

$$= \left[\left\{-\frac{\cos(-n\pi)}{n\pi}\right\} + \left\{3\left(-\frac{\cos n\pi}{n\pi}\right) - 2\left(-\frac{\cos 0}{n\pi}\right)\right\}\right] \quad \left[\begin{aligned}\because\ \sin n\pi &= \sin(-n\pi)\\ &= \sin 0 = 0\end{aligned}\right]$$

$$= \left[\frac{-(-1)^n}{n\pi} - \frac{3(-1)^n}{n\pi} + \frac{2}{n\pi}\right] \quad [\because\ \cos(-n\pi) = \cos n\pi = (-1)^n, \cos 0 = 1]$$

$$= \frac{2}{n\pi}[1 - 2(-1)^n]$$

Hence,
$$f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin n\pi x$$

$$= 1 + \frac{2}{\pi}\left(\frac{3}{1}\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{3}{3}\sin 3\pi x - \cdots\right)$$

EXERCISE 2.1

Find the Fourier series of the following functions:

1. $f(x) = e^x \qquad 0 < x < 2\pi$

$$\left[\textbf{Ans.:}\ \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx - n\sin nx}{n^2+1}\right]\right]$$

2. $f(x) = 1 \qquad 0 < x < \pi$

$\quad = 2 \qquad \pi < x < 2\pi$

Hence, deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

$$\left[\textbf{Ans.:}\ \frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx\right]$$

3. $f(x) = x \qquad 0 < x < \pi$

$\quad = 2\pi - x \qquad \pi < x < 2\pi$

$$\left[\textbf{Ans.:}\ \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx\right]$$

4. $f(x) = 1 \qquad -\pi < x \le 0$

$\quad = -2 \qquad 0 < x \le \pi$

$$\left[\textbf{Ans.:} -\frac{1}{2} - \frac{6}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}\right]$$

5. $f(x) = -x \qquad -\pi < x \le 0$

$\quad = 0 \qquad 0 < x \le \pi$

$$\left[\textbf{Ans.:}\ \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx\right]$$

6. $f(x) = \dfrac{1}{2} \qquad -\pi < x < 0$

$\quad = \dfrac{x}{\pi} \qquad 0 < x < \pi$

$$\left[\textbf{Ans.:}\ \frac{1}{2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx\right]$$

7. $f(x) = x - \pi \qquad -\pi < x < 0$

$\quad = \pi - x \qquad 0 < x < \pi$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

$$\left[\textbf{Ans.:} -\frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x + 4\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x\right]$$

8. $f(x) = \cos x \quad -\pi < x < 0$
$\quad = \sin x \quad 0 < x < \pi$

$$\left[\textbf{Ans.:} \frac{1}{\pi} + \frac{1}{2}(\cos x + \sin x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx\right]$$

9. $f(x) = 2 - \dfrac{x^2}{2} \qquad 0 \le x \le 2$

$$\left[\textbf{Ans.:} \frac{4}{3} - \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

10. $f(x) = \dfrac{1}{2}(\pi - x) \qquad 0 < x < 2$

$$\left[\textbf{Ans.:} (\pi - 1) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

11. $f(x) = 1 \qquad 0 < x < 1$
$\quad = 2 \qquad 1 < x < 2$

$$\left[\textbf{Ans.:} 3 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\pi x\right]$$

12. $f(x) = x \qquad 0 < x < 1$
$\quad = 0 \qquad 1 < x < 2$

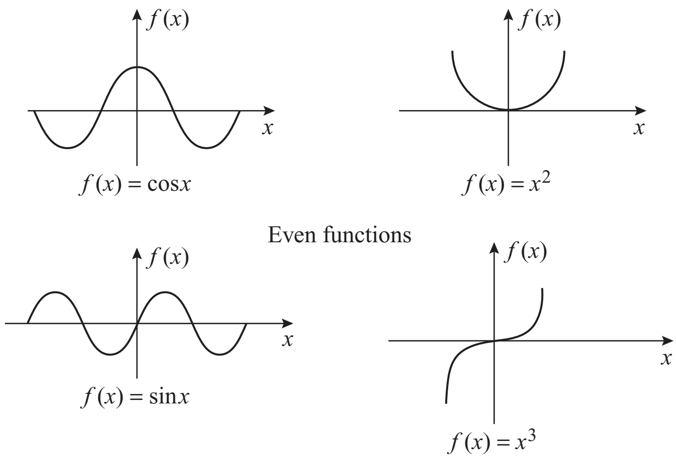
$$\left[\textbf{Ans.:} \frac{1}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin n\pi x\right]$$

13. $f(x) = 2 \qquad -2 < x < 0$
$\quad = x \qquad 0 < x < 2$

$$\left[\textbf{Ans.:} \frac{3}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\frac{\cos n\pi x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}\right]$$

2.6 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$ for all x, (Fig. 2.2).



Odd functions

Fig. 2.2 ***Even and odd functions***

Properties of Even and Odd Functions

(i) The product of two even functions is even.
(ii) The product of two odd functions is even.
(iii) The product of an even function and an odd function is odd.
(iv) The sum or difference of two even functions is even.
(v) The sum or difference of two odd functions is odd.
(vi) If $f(x)$ is even, $\int_{-l}^{l} f(x)\,\mathrm{d}x = 2\int_{0}^{l} f(x)\,\mathrm{d}x$
(viii) If $f(x)$ is odd, $\int_{-l}^{l} f(x)\,\mathrm{d}x = 0$

We know that the Fourier series of a function $f(x)$ in the interval $(-l, l)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$a_0 = \frac{1}{2l}\int_{l}^{l} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Case I When $f(x)$ is an even function, $\int_{-l}^{l} f(x)\,\mathrm{d}x = 2\int_{0}^{l} f(x)\,\mathrm{d}x$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

Since the product of two even functions is even,

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

Since the product of an even function and an odd function is odd,

$$b_n = 0$$

Corollary The Fourier series of an even function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where
$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

Case II When $f(x)$ is an odd function,

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Corollary The Fourier series of an odd function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x$$

Thus, the Fourier series of an even function consists entirely of cosine terms while the Fourier series of an odd function consists entirely of sine terms.

Example 1

Find the Fourier series of $f(x) = x$ in $-\pi < x < \pi$. **[Summer 2014]**

Solution

$$f(-x) = -x \qquad -\pi < -x < \pi$$
$$f(-x) = -f(x) \quad \pi > x > -\pi \quad \text{or} \quad -\pi < x < \pi$$

$f(x) = x$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\sin nx \, dx \\
&= \frac{2}{\pi}\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n}\right] \quad [\because \sin n\pi = \sin 0 = 0] \\
&= -\frac{2}{n}(-1)^n \quad \left[\because \cos n\pi = (-1)^n\right]
\end{aligned}$$

Hence, $$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx$$

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

Example 2

Find the Fourier series of $f(x) = x^2$ in the interval $(-\pi, \pi)$. Hence, deduce that $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots$ **[Summer 2016]**

Solution

$f(x) = x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x^2\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^3}{3}\right) \\
&= \frac{\pi^2}{3} \\
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{4}{n^2}\cos n\pi \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{4}{n^2}(-1)^n \qquad \left[\because\ \cos n\pi = (-1)^n\right]
\end{aligned}$$

Hence, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right) \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 3

Find the Fourier series of $f(x) = x^3$ in the interval $(-\pi, \pi)$.

Solution

$f(x) = x^3$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx \\
&= \frac{2}{\pi}\left| x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left(-\pi^3\frac{\cos n\pi}{n} + 6\pi\frac{\cos n\pi}{n^3}\right) \qquad [\because \sin n\pi = \sin 0 = 0] \\
&= 2(-1)^n\left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right) \qquad \left[\because \cos n\pi = (-1)^n\right]
\end{aligned}$$

Hence, $$f(x) = 2\sum_{n=1}^{\infty}(-1)^n\left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right)\sin nx$$

$$= -2\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx + 6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nx$$

$$x^3 = 2\pi^2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right) - 6\left(\sin x - \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x - \cdots\right)$$

Example 4

Find the Fourier series of $f(x) = \dfrac{\pi^2}{12} - \dfrac{x^2}{4}$ *in the interval* $(-\pi, \pi)$ *and deduce that* $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots.$

Solution

$f(x) = \dfrac{\pi^2}{12} - \dfrac{x^2}{4}$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)dx \\
&= \frac{1}{\pi}\left|\frac{\pi^2 x}{12} - \frac{x^3}{12}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^3}{12} - \frac{\pi^3}{12}\right) \\
&= 0 \\
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx\,dx \\
&= \frac{2}{\pi}\left|\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{x}{2}\right)\left(-\frac{\cos nx}{n^2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left(-\frac{\pi}{2n^2}\cos n\pi\right) \qquad [\because \sin n\pi = \sin 0 = 0] \\
&= \frac{-(-1)^n}{n^2} \qquad \left[\because \cos n\pi = (-1)^n\right]
\end{aligned}$$

Hence, $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{-(-1)^n}{n^2}\cos nx$

$$\frac{\pi^2}{12} - \frac{x^2}{4} = \frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 5

Find the Fourier series of $f(x) = \sin ax$ *in the interval* $(-\pi, \pi)$.

Solution

$$f(-x) = \sin a(-x) = -\sin ax$$
$$f(-x) = -f(x)$$

$f(x) = \sin ax$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} [\cos(n-a)x - \cos(n+a)x]\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right] \qquad [\because \sin 0 = 0] \\
&= \frac{1}{\pi}\left(\frac{\sin n\pi\cos a\pi - \sin a\pi\cos n\pi}{n-a} - \frac{\sin n\pi\cos a\pi + \sin a\pi\cos n\pi}{n+a}\right) \\
&= \frac{1}{\pi}\left[\frac{-(-1)^n \sin a\pi}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] \qquad \left[\because \sin n\pi = 0, \cos n\pi = (-1)^n\right] \\
&= \frac{-(-1)^n \sin a\pi}{\pi}\left(\frac{1}{n-a} + \frac{1}{n+a}\right) \\
&= \frac{2n(-1)^n \sin a\pi}{\pi(a^2 - n^2)}
\end{aligned}$$

Hence, $f(x) = \dfrac{2\sin a\pi}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{a^2 - n^2}\sin nx$

$$\sin ax = -\frac{2\sin a\pi}{\pi}\left[\frac{1}{a^2-1^2}\sin x - \frac{2}{a^2-2^2}\sin 2x + \frac{3}{a^2-3^2}\sin 3x - \cdots\right]$$

Example 6

Find the Fourier series of $f(x) = x \sin x$ in the interval $(-\pi, \pi)$. Hence, deduce that $\dfrac{\pi - 1}{4} = \dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \ldots$

Solution

$$f(-x) = -x \sin(-x)$$
$$= x \sin x$$
$$= f(x)$$

$f(x) = x \sin x$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} x \sin x\, \mathrm{d}x$$
$$= \frac{1}{\pi}\left| x(-\cos x) - (-\sin x)\right|_0^{\pi}$$
$$= \frac{1}{\pi}\left[\pi(-\cos\pi)\right] \quad [\because \sin\pi = \sin 0 = 0]$$
$$= 1 \quad [\because \cos\pi = -1]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, \mathrm{d}x$$
$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\, \mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left| x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - \left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right|_0^{\pi}, n \ne 1$$
$$= \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{n+1} + \pi\frac{\cos(n-1)\pi}{n-1}\right], n \ne 1 \quad [\because \sin(n+1)\pi = \sin(n-1)\pi = 0]$$
$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} \quad \left[\because \cos(n+1)\pi = \cos(n-1)\pi = -(-1)^n\right]$$
$$= \frac{-2(-1)^n}{n^2-1}$$
$$= \frac{2(-1)^{n+1}}{n^2-1},\ n \ne 1$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left| -x\frac{\cos 2x}{2} + \frac{\sin 2x}{4} \right|_0^{\pi}$$
$$= \frac{1}{\pi}\left(-\pi\frac{\cos 2\pi}{2} \right) \qquad [\because\ \sin 2\pi = \sin 0 = 0]$$
$$= -\frac{1}{2} \qquad [\because\ \cos 2\pi = 1]$$

Hence,
$$f(x) = 1 - \frac{1}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx$$
$$x\sin x = \frac{1}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx$$
$$= \frac{1}{2} - 2\left(\frac{1}{3}\cos 2x - \frac{1}{8}\cos 3x + \frac{1}{15}\cos 4x - \cdots \right) \qquad \ldots (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\frac{\pi}{2}\sin\frac{\pi}{2} = \frac{1}{2} + 2\left(-\frac{1}{3}\cos\pi + \frac{1}{8}\cos\frac{3\pi}{2} - \frac{1}{15}\cos 2\pi + \cdots \right)$$
$$\frac{\pi}{2} = \frac{1}{2} - \frac{2}{3}\cos\pi - \frac{2}{15}\cos 2\pi - \ldots$$
$$\frac{\pi - 1}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \ldots$$

Example 7

Find the Fourier series of $f(x) = \cosh ax$ *in the interval* $(-\pi, \pi)$.

Solution

$$f(-x) = \cosh a(-x)$$
$$= \cosh ax$$
$$= f(x)$$

$f(x) = \cosh ax$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \cosh ax\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \left(\frac{e^{ax}+e^{-ax}}{2}\right)\mathrm{d}x \\
&= \frac{1}{2\pi}\left|\frac{e^{ax}}{a}+\frac{e^{-ax}}{-a}\right|_0^{\pi} \\
&= \frac{1}{2\pi a}\left(e^{a\pi}-e^{-a\pi}\right) \\
&= \frac{\sinh a\pi}{\pi a}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \cosh ax\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \left(\frac{e^{ax}+e^{-ax}}{2}\right)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} (e^{ax}\cos nx+e^{-ax}\cos nx)\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx+n\sin nx)+\frac{e^{-ax}}{a^2+n^2}(-a\cos nx+n\sin nx)\right|_0^{\pi} \\
&= \frac{a\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi}-e^{-a\pi}) \quad [\because\ \sin n\pi=\sin 0=0] \\
&= \frac{a\cos n\pi}{\pi(a^2+n^2)}2\sinh a\pi \\
&= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)}
\end{aligned}
$$

Hence, $f(x)=\dfrac{\sinh a\pi}{\pi a}+\dfrac{2a}{\pi}\sinh a\pi\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx$

Example 8

Find the Fourier series of $f(x)=e^{-|x|}$ *in the interval* $(-\pi, \pi)$.

Solution

$$f(x) = e^{-|x|}$$
$$f(-x) = e^{-|-x|}$$
$$= e^{-|x|} = f(x)$$

$f(x) = e^{-|x|}$ is an even function.

Hence, $b_n = 0$

$$f(x) = e^{x} \qquad -\pi < x < 0$$
$$= e^{-x} \qquad 0 < x < \pi$$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} e^{-x}\mathrm{d}x$$
$$= \frac{1}{\pi}\left|-e^{-x}\right|_0^{\pi}$$
$$= \frac{1}{\pi}(1-e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x$$
$$= \frac{2}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right|_0^{\pi}$$
$$= \frac{2}{\pi(n^2+1)}\left[e^{-\pi}(-\cos n\pi)+\cos 0\right] \quad [\because \ \sin n\pi = \sin 0 = 0, \cos 0 = 1]$$
$$= \frac{2}{\pi(n^2+1)}[1-(-1)^n e^{-\pi}]$$

Hence, $f(x) = \dfrac{1}{\pi}(1-e^{-\pi}) + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n e^{-\pi}}{n^2+1}\right]\cos nx$

Example 9

Find the Fourier series of $f(x) = |\cos x|$ *in the interval* $(-\pi, \pi)$.

Solution

$f(x) = |\cos x|$ is an even function.

Hence, $b_n = 0$

$$f(x) = \cos x \qquad 0 < x < \frac{\pi}{2}$$

$$= -\cos x \qquad \frac{\pi}{2} < x < \pi$$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|\sin x\right|_0^{\frac{\pi}{2}} - \left|\sin x\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\sin\frac{\pi}{2} - \left(\sin\pi - \sin\frac{\pi}{2}\right)\right]$$

$$= \frac{2}{\pi} \qquad \left[\because\ \sin\frac{\pi}{2} = 1, \sin\pi = 0\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\cos nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \left\{\cos(n+1)x + \cos(n-1)x\right\}\mathrm{d}x - \int_{-\frac{\pi}{2}}^{\pi}\{\cos(n+1)x + \cos(n-1)x\}\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_0^{\frac{\pi}{2}} - \left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_{\frac{\pi}{2}}^{\pi}\right], \quad n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right], \quad n \neq 1 \qquad \left[\begin{array}{l}\because\ \sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right) = \cos\frac{n\pi}{2} \\ \quad\ \sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) = -\cos\frac{n\pi}{2}\end{array}\right]$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \quad n \neq 1$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos^2 x\,dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos^2 x)dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right)dx - \int_{\frac{\pi}{2}}^{\pi} \left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left|x + \frac{\sin 2x}{2}\right|_0^{\frac{\pi}{2}} - \left|x + \frac{\sin 2x}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] \quad [\because \ \sin\pi = \sin 2\pi = 0]$$

$$= 0$$

Hence, $$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\left(-\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x - \frac{1}{35}\cos 6x + \ldots\right)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x - \ldots\right)$$

Example 10

Find the Fourier series of $f(x) = |x|$ *in the interval* $[-\pi, \pi]$.

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$ **[Winter 2016]**

Solution

$$f(x) = |x| \qquad -\pi < x < \pi$$

i.e., $$f(x) = -x \qquad -\pi < x \le 0$$
$$= x \qquad 0 \le x < \pi$$

$f(x) = |x|$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|x\left(\frac{\sin nx}{n}\right)+\frac{\cos nx}{n^2}\right|_0^{\pi} \\
&= \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2}-\frac{\cos 0}{n^2}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{\pi n^2}[(-1)^n - 1] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}$$

Hence, $$f(x) = \frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cos nx$$

$$|x| = \frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\dots\right) \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$$

Example 11

Find the Fourier series of $f(x) = -k \qquad -\pi < x < 0$

$\qquad = k \qquad 0 < x < \pi$ **[Winter 2014]**

Solution

$$\begin{aligned} f(-x) &= -k \qquad -\pi < -x < 0 \quad \text{or} \quad 0 < x < \pi \\ &= k \qquad 0 < -x < \pi \quad \text{or} \quad -\pi < x < 0 \\ f(-x) &= -f(x) \end{aligned}$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, \mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} k\sin nx \, \mathrm{d}x \\ &= \frac{2k}{\pi}\left| -\frac{\cos nx}{n} \right|_0^{\pi} \\ &= \frac{2k}{n\pi}(-\cos n\pi + \cos 0) \\ &= \frac{2k}{n\pi}\left[-(-1)^n + 1\right] \quad \left[\because \ \cos n\pi = (-1)^n, \cos 0 = 1\right] \\ &= \frac{2k}{n\pi}\left[1-(-1)^n\right] \end{aligned}$$

Hence,

$$\begin{aligned} f(x) &= \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-(-1)^n\right]\sin nx \\ &= \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right) \end{aligned}$$

Example 12

Find the Fourier series of $f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$ **[Summer 2015]**

Solution

$$\begin{aligned} f(x) &= \pi - x \qquad 0 < x < \pi \\ f(-x) &= \pi + x \qquad -\pi < x < 0 \end{aligned}$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{n} \\
&= \frac{2}{\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - \frac{\cos nx}{n^2}\right|_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} + \frac{\cos 0}{n^2}\right] && [\because \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] && [\because \cos n\pi = (-1)^n, \cos 0 = 1] \\
&= \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]
\end{aligned}$$

Hence, $$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$

Example 13

Find the Fourier series of the function f(x) given by

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi} & 0 \le x \le \pi \end{cases}$$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$ **[Winter 2016]**

Solution

$$f(-x) = 1 + \frac{2(-x)}{\pi} \qquad -\pi \le -x \le 0$$

$$= 1 - \frac{2x}{\pi} \qquad 0 \le x \le \pi$$

$$f(-x) = 1 - \frac{2}{\pi}(-x) \qquad 0 \le x \le \pi$$

$$= 1 + \frac{2x}{\pi} \qquad -\pi \le x \le 0$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\mathrm{d}x$$

$$= \frac{1}{\pi}\left| x - \frac{2}{\pi}\cdot\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left| x - \frac{x^2}{\pi}\right|_0^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\frac{2}{\pi n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{\pi n^2}\cos n\pi + \frac{2}{\pi n^2}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2\pi^2}[1-\cos n\pi]$$

$$= \frac{4}{n^2\pi^2}[1-(-1)^n] \qquad [\because\ \cos n\pi = (-1)]$$

Hence, $$f(x) = 0 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[1-(-1)^n]\cos nx$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

$$= \frac{4\cdot 2}{\pi^2}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right) \qquad \ldots(1)$$

At $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x \to 0^-} f(x) + \lim_{x \to 0^+} f(x)\right]$$

$$f(0) = \frac{1}{2}[1+1] = \frac{2}{2} = 1$$

Putting $x = 0$ in Eq. (1),

$$f(0) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos(0) + \frac{1}{3^2}\cos(0) + \frac{1}{5^2}\cos(0) + \cdots\right]$$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 14

Find the Fourier series of $f(x) = \cos x \quad -\pi < x < 0$

$$= -\cos x \quad 0 < x < \pi$$

Solution

$$\begin{aligned} f(-x) &= \cos(-x) & -\pi < -x < 0 \\ &= -\cos(-x) & 0 < -x < \pi \\ f(-x) &= \cos x & 0 < x < \pi \\ &= -\cos x & -\pi < x < 0 \\ f(-x) &= -f(x) \end{aligned}$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx \\ &= \frac{2}{\pi}\int_0^{\pi} (-\cos x)\sin nx \, dx \\ &= -\frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x] dx \\ &= -\frac{1}{\pi}\left|-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right|_0^{\pi}, \; n \neq 1 \end{aligned}$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right], \; n \neq 1$$

$$= -\frac{1}{\pi}\left(\frac{1+\cos n\pi}{n+1} + \frac{1+\cos n\pi}{n-1}\right), \; n \neq 1 \; [\because \cos(n\pi+\pi) = \cos(n\pi-\pi) = -\cos n\pi]$$

$$= -\frac{2n}{\pi(n^2-1)}(1+\cos n\pi), \quad n \neq 1$$

$$= -\frac{2n}{\pi(n^2-1)}[1+(-1)^n], \; n \neq 1 \quad \left[\because \; \cos n\pi = (-1)^n\right]$$

For $n = 1$,

$$b_1 = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin x \, dx$$

$$= -\frac{1}{\pi}\int_0^{\pi}\sin 2x \, dx$$

$$= -\frac{1}{\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi}$$

$$= \frac{1}{2\pi}(\cos 2\pi - \cos 0)$$

$$= 0 \quad [\because \; \cos 2\pi = \cos 0 = 1]$$

Hence,

$$f(x) = -\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n}{n^2-1}[1+(-1)^n]\sin nx$$

$$= -\frac{2}{\pi}\left(\frac{2}{3}2\sin 2x + \frac{4}{15}2\sin 4x + \frac{6}{35}2\sin 6x + \cdots\right)$$

$$= -\frac{8}{\pi}\left(\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \cdots\right)$$

Example 15

Obtain the Fourier series of periodic function $f(x) = 2x$, where $-1 < x < 1$. **[Winter 2016]**

Solution

$$f(x) = 2x$$

$$f(-x) = -2x = -f(x)$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function $f(x)$ with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= 2\int_0^l 2x \sin n\pi x\,dx$$

$$= 2\left| 2x\left(-\frac{\cos n\pi x}{n\pi}\right) - (2)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= 2\left| -\frac{2x}{n\pi}\cos n\pi x + \frac{2}{n^2\pi^2}\sin n\pi x\right|_0^1$$

$$= 2\left(-\frac{2}{n\pi}\cos n\pi\right) \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4}{n\pi}(-1)^n \qquad \left[\because \cos n\pi = (-1)^n\right]$$

$$= (-1)^{n+1}\frac{4}{n\pi}$$

Hence, $$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{4}{n\pi}\sin n\pi x$$

Example 16

Find the Fourier series of $f(x) = 1 - x^2$ *in the interval* $(-1, 1)$.

Solution

$$f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x)$$

$f(x) = 1 - x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$
$$= \int_0^1 (1-x^2)\,\mathrm{d}x$$
$$= \left| x - \frac{x^3}{3} \right|_0^1$$
$$= 1 - \frac{1}{3}$$
$$= \frac{2}{3}$$
$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$
$$= 2\int_0^l (1-x^2)\cos n\pi x\,\mathrm{d}x$$
$$= 2\left| (1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right) \right|_0^1$$
$$= 2\left(-2\frac{\cos n\pi}{n^2\pi^2}\right) \quad [\because \ \sin n\pi = \sin 0 = 0]$$
$$= \frac{-4(-1)^n}{n^2\pi^2} \quad \left[\because \ \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$
$$1 - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\left(-\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x - \frac{1}{3^2}\cos 3\pi x + \cdots\right)$$

Example 17

Find the Fourier series of $f(x) = x^2 - 2$ *in* $-2 \le x \le 2$. **[Winter 2014]**

Solution

$f(x) = x^2 - 2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$\begin{aligned}
a_0 &= \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^2 (x^2-2)\,\mathrm{d}x \\
&= \frac{1}{2}\left|\frac{x^3}{3}-2x\right|_0^2 \\
&= \frac{1}{2}\left(\frac{8}{3}-4\right) \\
&= -\frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \\
&= \frac{2}{2}\int_0^2 (x^2-2)\cos\frac{n\pi x}{2}\,\mathrm{d}x \\
&= \left|(x^2-2)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(2x)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)+(2)\left(-\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right)\right|_0^2 \\
&= -(4)\left(-\frac{4}{n^2\pi^2}\cos n\pi\right) \\
&= \frac{16}{n^2\pi^2}\cos n\pi \qquad [\because \sin n\pi = \sin 0 = 0] \\
&= \frac{16}{n^2\pi^2}(-1)^n \qquad \left[\because \cos n\pi = (-1)^n\right]
\end{aligned}$$

Hence,
$$f(x) = -\frac{2}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n\cos\frac{n\pi x}{2}$$

$$x^2-2 = -\frac{2}{3}-\frac{16}{\pi^2}\left(\cos\frac{\pi x}{2}-\frac{1}{4}\cos\pi x+\frac{1}{9}\cos\frac{3\pi x}{2}-\cdots\right)$$

Example 18

Find the Fourier series of $f(x) = x|x|$ in the interval $(-1, 1)$.

Solution

$$\begin{aligned}
f(x) &= x|x| \\
f(-x) &= -x|-x| \\
&= -x|x| = -f(x)
\end{aligned}$$

$f(x) = x|x|$ is an odd function.
Hence, $a_0 = 0$ and $a_n = 0$

$$\begin{aligned} f(x) &= -x^2 \qquad -1 < x < 0 \\ &= x^2 \qquad\quad 0 < x < 1 \end{aligned}$$

The Fourier series of an odd function with period $2l = 2$ is given by

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty} b_n \sin n\pi x \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \\ &= 2\int_0^1 x^2 \sin n\pi x\,\mathrm{d}x \\ &= 2\left| x^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2x\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^1 \\ &= 2\left[-\frac{\cos n\pi}{n\pi} + \frac{2\cos n\pi}{n^3\pi^3} - \frac{2\cos 0}{n^3\pi^3}\right] \qquad [\because\ \sin n\pi = \sin 0 = 0] \\ &= 2\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right] \end{aligned}$$

Hence, $\quad f(x) = 2\sum_{n=1}^{\infty}\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]\sin n\pi x$

Example 19

Find the Fourier series of $f(x) = x - x^3$ in $-1 < x < 1$. **[Winter 2013]**

Solution

$$\begin{aligned} f(-x) &= -x + x^3 \qquad -1 < x < 1 \\ &= -(x - x^3) \\ &= -f(x) \end{aligned}$$

$f(x) = x - x^3$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period $2l = 2$ is given by

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty} b_n \sin n\pi x \end{aligned}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= 2\int_0^1 (x-x^3)\sin n\pi x\,\mathrm{d}x$$

$$= 2\left|(x-x^3)\left(-\cos\frac{n\pi x}{n\pi}\right)-(1-3x^2)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)+(-6x)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)-(-6)\left(\frac{\sin n\pi x}{n^4\pi^4}\right)\right|_0^1$$

$$= 2\left[-6\frac{\cos n\pi}{n^3\pi^3}\right] \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= -12\frac{(-1)^n}{n^3\pi^3} \quad \left[\because \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x$$

$$x - x^3 = \frac{12}{\pi^3}\left(\sin\pi x - \frac{\sin 2\pi x}{2^3}+\frac{\sin 3\pi x}{3^3}-\cdots\right)$$

Example 20

Find the Fourier series of $$f(x) = \frac{1}{2}+x \qquad -\frac{1}{2}<x<0$$
$$= \frac{1}{2}-x \qquad 0<x<\frac{1}{2}$$

Solution

$$f(-x) = \frac{1}{2}-x \qquad -\frac{1}{2}<-x<0 \text{ or } 0<x<\frac{1}{2}$$
$$= \frac{1}{2}+x \qquad 0<-x<\frac{1}{2} \text{ or } -\frac{1}{2}<x<0$$
$$f(-x) = f(x)$$

$f(x)$ is an even function.
Hence, $b_n = 0$
The Fourier series of an even function with period $2l = 1$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x$$

$$= \frac{1}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-x\right)\mathrm{d}x$$

$$= 2\left|\frac{x}{2}-\frac{x^2}{2}\right|_0^{\frac{1}{2}}$$

$$= 2\left(\frac{1}{4}-\frac{1}{8}\right)$$

$$= \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-x\right)\cos 2n\pi x\ \mathrm{d}x$$

$$= 4\left|\left(\frac{1}{2}-x\right)\left(\frac{\sin 2n\pi x}{2n\pi}\right)-(-1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right|_0^{\frac{1}{2}}$$

$$= 4\left[\left(-\frac{\cos n\pi}{4n^2\pi^2}+\frac{\cos 0}{4n^2\pi^2}\right)\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi^2}\left[1-(-1)^n\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = \frac{1}{4}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos\ 2n\pi x$$

EXERCISE 2.2

Find the Fourier series of the following functions:

1. $f(x) = \dfrac{x(\pi^2 - x^2)}{12} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\right]$$

2. $f\ (x) = \cos ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{\sin\ a\pi}{\pi}+\frac{2a\ \sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}\cos\ nx\right]$$

3. $f(x) = x\cos x \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} -\frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\sin nx\right]$$

4. $f(x) = |\sin x| \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx\right]$$

5. $f(x) = \sqrt{1-\cos\ x} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos\ nx\right]$$

6. $f(x) = \sinh ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{a^2+n^2}\sin nx\right]$$

7.
$$f(x) = \frac{-(\pi+x)}{2} \qquad -\pi < x < 0$$
$$= \frac{\pi - x}{2} \qquad 0 < x < \pi$$

$$\left[\textbf{Ans.:} \sum_{n=1}^{\infty}\frac{1}{n}\sin nx\right]$$

8.
$$f(x) = x + \frac{\pi}{2} \qquad -\pi < x < 0$$
$$= \frac{\pi}{2} - x \qquad 0 < x < \pi$$

$$\left[\textbf{Ans.:} \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx\right]$$

9. $f(x) = |x| \qquad -2 < x < 2$

$$\left[\textbf{Ans.:} 1 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos n\pi x\right]$$

10. $f(x) = a^2 - x^2 \qquad -a < x < a$

$$\left[\textbf{Ans.:} \frac{2a^2}{3} - \frac{4a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{a}\right]$$

11. $f(x) = \sin ax \qquad -l < x < l$

$$\left[\textbf{Ans.:} \sin\ al\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2-a^2l^2}\sin\frac{n\pi x}{l}\right]$$

2.7 HALF-RANGE FOURIER SERIES

Any arbitrary function $f(x)$ with period $2l$ which is defined in half of the interval $(0, l)$ can also be represented in terms of sine and cosine functions. A half-range expansion containing only cosine terms is known as a *half-range cosine series*. Similarly, a half-range expansion containing only sine terms is known as a *half-range sine series*.

To represent any function $f(x)$ in the half-range cosine series in the interval $(0, l)$, we extend the function by reflecting it in the vertical axis (i.e., y axis) so that $f(-x) = f(x)$. The extended function is an even function in $(-l, l)$ and is periodic with period $2l$. The half-range cosine series of such a function is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{l} \int_0^l f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\,\mathrm{d}x$$

Corollary If any function with period 2π is defined in the interval $(0, \pi)$ then the half-range cosine series of such a function is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where
$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,\mathrm{d}x$$

Similarly, to represent any function $f(x)$ in the half-range sine series in the interval $(0, l)$, we extend the function by reflecting it in the origin so that $f(-x) = -f(x)$. The extended function is an odd function in $(-l, l)$ and is periodic with period $2l$. The half-range sine series of such a function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where
$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\,\mathrm{d}x$$

Corollary If any function with period 2π is defined in the interval $(0, \pi)$ then the half-range sine series of such a function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,\mathrm{d}x$$

Example 1

Find the half-range cosine series of $f(x) = x$ *in the interval* $(0, \pi)$.

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} x\,dx \\
&= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \\
&= \frac{2}{\pi}\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \qquad [\because \ \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{\pi n^2}[(-1)^n - 1] \qquad \left[\because \ \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}$$

Hence,

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx$$

$$x = \frac{\pi}{2} + \frac{2}{\pi}\left(-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \cdots\right)$$

Example 2

Find the Fourier cosine series of $f(x) = x^2 \quad 0 < x < \pi$ **[Summer 2015]**

Solution

The Fourier cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2 \mathrm{d}x$$

$$a_0 = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{3}\right)$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) + 2x\left(\frac{\cos nx}{n^2}\right) - \left(\frac{2\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n^2}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4\cos n\pi}{n^2}$$

$$= \frac{4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-)^n]$$

Hence,
$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Example 3

Find the half-range sine series of $f(x) = x^2$ in the interval $(0, \pi)$.

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx \\
&= \frac{2}{\pi}\left| x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left[-\pi^2\left(\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) - 2\left(\frac{\cos 0}{n^3}\right)\right] \quad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right] \quad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}$$

Hence, $f(x) = \dfrac{2}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{-\pi^2(-1)^n}{n} + \dfrac{2(-1)^n}{n^3} - \dfrac{2}{n^3}\right]\sin nx$

Example 4

Find the half-range sine series of $f(x) = x^3$ in $0 \le x \le \pi$.

[Summer 2017]

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^3 \sin nx \, dx$$

$$= \frac{2}{\pi} \left| x^3 \left(-\frac{\cos nx}{x} \right) - (3x^2) \left(-\frac{\sin nx}{n^2} \right) + (6x) \left(\frac{\cos nx}{n^3} \right) - (6) \left(\frac{\sin nx}{n^4} \right) \right|_0^{\pi}$$

$$= \frac{2}{\pi} \left| -x^3 \left(\frac{\cos nx}{n} \right) + 3x^2 \left(\frac{\sin nx}{n^2} \right) + 6x \left(\frac{\cos nx}{n^3} \right) - 6 \left(\frac{\sin nx}{n^4} \right) \right|_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\pi^3 \frac{(-1)^n}{n} + 6\pi \frac{(-1)^n}{n^3} \right] \qquad \left[\begin{array}{l} \because \sin n\pi = \sin = 0 \\ \cos n\pi = (-1) \end{array} \right]$$

$$= \frac{2}{\pi} \cdot \pi \left[\frac{6}{n^3} - \frac{\pi^2}{n} \right] (-1)^n$$

$$= \frac{2(-1)^n}{n^3} (6 - n^2\pi^2)$$

Hence, $f(x) = 2 \sum_{n=1}^{\infty} \frac{1}{n^3} (-1)^n \, (6 - n^2\pi^2) \sin nx$

Example 5

Find the cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$.

[Winter 2017]

Solution

The cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\pi - x) \, dx$$

$$= \frac{1}{\pi} \left| \pi x - \frac{x^2}{2} \right|_0^{\pi}$$

$$= \frac{1}{\pi} \left(\pi^2 - \frac{\pi^2}{2} \right)$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right] \quad \left[\because \begin{array}{l}\sin n\pi = \sin 0 = 0\\ \cos 0 = 1\end{array}\right]$$

$$= \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right] \quad [\because \cos n\pi = (-1)^n]$$

Hence, $f(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$

Example 6

Find the half-range cosine series of $f(x) = x(\pi - x)$ *in the interval* $(0, \pi)$ *and, hence, deduce that*

(i) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ (ii) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots$

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x(\pi - x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\pi\frac{x^2}{2} - \frac{x^3}{3}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\pi\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right|_0^\pi$$

$$= \frac{2}{\pi}\left[(\pi - 2\pi)\frac{\cos n\pi}{n^2} - \frac{\pi\cos 0}{n^2}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{2}{n^2}[1 + (-1)^n] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n}{n^2}\right]\cos nx$$

$$x(\pi - x) = \frac{\pi^2}{6} - 2\left(\frac{2}{2^2}\cos 2x + \frac{2}{4^2}\cos 4x + \frac{2}{6^2}\cos 6x + \cdots\right) \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi^2}{6} - 4\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{6} - 2\left(\frac{2}{2^2}\cos\pi + \frac{2}{4^2}\cos 2\pi + \frac{2}{6^2}\cos 3\pi + \cdots\right)$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{6} - 4\left(-\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \ldots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 7

Find the Fourier sine series of $f(x) = e^x$ in $0 < x < \pi$. **[Summer 2015]**

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} e^x \sin nx \,dx$$

$$= \frac{2}{\pi}\left| \frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-n\cos n\pi) - \frac{e^0}{1+n^2}(-n\cos 0)\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-1)^n(-n) + n\cdot\frac{1}{1+n^2}\right] \qquad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

$$= \frac{2}{\pi}\cdot\frac{n}{(1+n^2)}[e^{\pi}(-1)^{n+1}+1]$$

Hence, $$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{(1+n^2)}[e^{\pi}(-1)^{n+1}+1]\sin nx$$

$$e^x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{1+n^2}[e^{\pi}(-1)^{n+1}+1]\sin nx$$

Example 8

Find the Fourier cosine series of $f(x) = e^{-x}$, where $0 \le x \le \pi$.

[Winter 2015]

Solution

The Fourier cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{-x}\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{-1}\right|_0^{\pi}$$

$$= -\frac{1}{\pi}\left[e^{-\pi} - e^{-0}\right]$$

$$= -\frac{1}{\pi}\left[e^{-\pi} - 1\right]$$

$$= \frac{1}{\pi}(1 - e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|\frac{e^{-x}}{1+n^2}\{(-1)\cos nx + n\sin x\}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left|\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{-\pi}}{1+n^2}(-\cos n\pi) - \frac{e^0}{1+n^2}(-1)\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi}\left[\frac{-e^{-\pi}(-1)^n}{1+n^2} + \frac{1}{1+n^2}\right] \qquad [\because\ \cos n\pi = (-1)^n]$$

$$= \frac{2}{\pi}\cdot\frac{1}{1+n^2}\left[1-(-1)^n e^{-\pi}\right]$$

Hence, $$f(x) = \frac{1}{\pi}(1-e^{-\pi}) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n e^{\pi}]}{1+n^2}\cos nx$$

Example 9

Find the half-range cosine series of $f(x) = \sin x$ *in the interval* $(0, \pi)$ *and hence, deduce that* $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

[Winter 2014; Summer 2018]

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \sin x\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|-\cos x\right|_0^{\pi} \\
&= \frac{1}{\pi}(-\cos \pi + \cos 0) \\
&= \frac{2}{\pi} \qquad [\because \ \cos \pi = -1, \cos 0 = 1]
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right|_0^{\pi}, \ n \neq 1 \\
&= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{\cos 0}{n+1} - \frac{\cos 0}{n-1} \right], \ n \neq 1 \\
&= -\frac{2}{\pi(n^2-1)}[1 + (-1)^n], \ n \neq 1 \\
&\left[\because \cos(n\pi + \pi) = \cos(n\pi - \pi) = -\cos n\pi = -(-1)^n, \cos 0 = 1\right]
\end{aligned}$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx$$
$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$
$$= \frac{1}{\pi}\left| -\frac{\cos 2x}{2} \right|_0^{\pi}$$
$$= \frac{1}{2\pi}(-\cos 2\pi + \cos 0)$$
$$= 0 \quad [\because \cos 2\pi = \cos 0 = 1]$$

Hence, $$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx$$

$$\sin x = \frac{2}{\pi} - \frac{2}{\pi}\left(\frac{2}{3}\cos 2x + \frac{2}{15}\cos 4x + \cdots\right) \qquad \ldots (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\sin\frac{\pi}{2} = \frac{2}{\pi} - \frac{2}{\pi}\left(\frac{2}{3}\cos \pi + \frac{2}{15}\cos 2\pi + \cdots\right)$$
$$1 = \frac{2}{\pi} - \frac{2}{\pi}\left(-\frac{2}{3} + \frac{2}{15} - \ldots\right)$$
$$1 = \frac{2}{\pi} + \frac{2}{\pi}\left[\left(1 - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots\right]$$
$$1 = \frac{2}{\pi}\left(2 - \frac{2}{3} + \frac{2}{5} - \ldots\right)$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

Example 10

For the function $f(x) = \cos 2x$, find the Fourier sine series over $(0, \pi)$.
[Winter 2015]

Solution

The Fourier sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos 2x\,\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\cos 2x\,\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} \left[\sin(2+n)x - \sin(2-n)x\right]\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\left(-\frac{\cos(n+2)x}{n+2}\right) - \left(-\frac{\cos(2-n)x}{2-n}\right)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(2\pi+n\pi)}{n+2} - \frac{\cos(2\pi-n\pi)}{n-2} + \frac{1}{n+2} + \frac{1}{n-2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos n\pi}{n+2} - \frac{\cos n\pi}{n-2} + \frac{1}{n+2} + \frac{1}{n-2}\right]$$

$$= \frac{1}{\pi}\left[(-1)^{n+1}\left\{\frac{1}{n+2} + \frac{1}{n-2}\right\} + \left\{\frac{1}{n+2} + \frac{1}{n-2}\right\}\right]$$

$$[\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+1} + 1\{(n-2+n+2)\}}{(n+2)\,(n-2)}\right]$$

$$= \frac{2n\left[(-1)^{n+1} + 1\right]}{n^2 - 4}, \quad \text{if} \quad n \neq 2$$

$$b_2 = \frac{1}{\pi}\int_0^{\pi} 2\cos 2x\;\sin 2x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 4x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{\cos 4x}{4}\right|_0^{\pi}$$

$$= \frac{1}{4\pi}(-1+1)$$
$$= 0$$

Hence, $$f(x) = 2\sum_{\substack{n=1 \\ n\neq 2}}^{\infty}\left[\frac{n\left\{(-1)^{n+1}+1\right\}}{n^2-4}\right]\sin nx$$

Example 11

Find the half-range cosine series of $f(x)$, where

$$f(x) = x \qquad 0 < x < \frac{\pi}{2}$$
$$= \pi - x \qquad \frac{\pi}{2} < x < \pi$$

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\,dx\right]$$
$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$
$$= \frac{1}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$
$$= \frac{\pi}{4}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left| x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\frac{\pi}{2}} + \left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2}\cdot\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{\cos 0}{n^2}\right) + \left(-\frac{\cos n\pi}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right)\right]$$

$$[\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right] \qquad \left[\because\ \cos 0 = 1, \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos nx$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{2^2}(-4)\cos 2x + \frac{1}{6^2}(-4)\cos 6x + \frac{1}{10^2}(-4)\cos 10x + \ldots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right)$$

Example 12

Find the half-range sine series of $f(x)$, where

$$f(x) = \frac{\pi}{3} \qquad 0 \le x < \frac{\pi}{3}$$

$$= 0 \qquad \frac{\pi}{3} \le x < \frac{2\pi}{3}$$

$$= -\frac{\pi}{3} \qquad \frac{2\pi}{3} \le x \le \pi$$

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{3}} \frac{\pi}{3}\sin nx\,\mathrm{d}x + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 0\cdot\sin nx\,\mathrm{d}x + \int_{\frac{2\pi}{3}}^{\pi}\left(-\frac{\pi}{3}\right)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{2}{3}\left[\left|-\frac{\cos nx}{n}\right|_0^{\frac{\pi}{3}} - \left|\frac{-\cos nx}{n}\right|_{\frac{2\pi}{3}}^{\pi}\right]$$

$$= \frac{2}{3n}\left[-\cos\frac{n\pi}{3} + \cos 0 + \cos n\pi - \cos\frac{2n\pi}{3}\right]$$

$$= \frac{2}{3n}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right] \quad \left[\begin{array}{l}\because\ \cos 0 = 1, \cos n\pi = (-1)^n \\ \cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\end{array}\right]$$

Hence,
$$f(x) = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right]\sin nx$$

Example 13

Find the half-range sine series of $f(x) = lx - x^2$ *in the interval* $(0, l)$ *and, hence, deduce that*

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots$$

Solution

The half-range sine series of $f(x)$ with period $2l$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\int_0^{l}(lx - x^2)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left|(lx - x^2)\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right] - (l - 2x)\frac{l^2}{n^2\pi^2}\left[-\sin\frac{n\pi x}{l}\right] + \left[(-2)\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]\right|_0^{l}$$

$$= \frac{2}{l}\left[-\frac{2l^3}{n^3\pi^3}(\cos n\pi - \cos 0)\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{4l^2}{n^3\pi^3}[1 - (-1)^n] \quad [\because\ \cos n\pi = (-1)^n]$$

Hence, $$f(x) = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin\frac{n\pi x}{l}$$

$$lx - x^2 = \frac{8l}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \cdots\right] \qquad \ldots (1)$$

Putting $x = \frac{l}{2}$ in Eq. (1),

$$\frac{l^2}{2} - \frac{l^2}{4} = \frac{8l^2}{\pi^3}\left(\frac{1}{1^3}\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3}\sin\frac{5\pi}{2} + \cdots\right)$$

$$\frac{l^2}{4} = \frac{8l^2}{\pi^3}\left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right)$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots$$

Example 14

Find the Fourier cosine series of $f(x) = x$ in $0 < x < l$. **[Winter 2013]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\begin{aligned}
a_0 &= \frac{1}{l}\int_0^l f(x)\,dx \\
&= \frac{1}{l}\int_0^l x\,dx \\
&= \frac{1}{l}\left|\frac{x^2}{2}\right|_0^l \\
&= \frac{1}{l}\left(\frac{l^2}{2}\right) \\
&= \frac{l}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \\
&= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx
\end{aligned}$$

$$= \frac{2}{l}\left| x\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left(\frac{l^2}{n^2\pi^2}\cos n\pi - \frac{l^2}{n^2\pi^2}\cos 0\right) \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2l}{n^2\pi^2}[(-1)^n - 1] \quad \left[\because \ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{l}$$

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{l}\right) + \cdots\right]$$

Example 15

Find the Fourier cosine series of $f(x) = x^2$ in $0 < x < l$. **[Summer 2013]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$= \frac{1}{l}\int_0^l x^2\mathrm{d}x$$

$$= \frac{1}{l}\left|\frac{x^3}{3}\right|_0^l$$

$$= \frac{1}{l}\left(\frac{l^3}{3}\right)$$

$$= \frac{1}{3}l^2$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left| x^2\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + 2\left(\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left[\frac{2l^3}{n^2\pi^2}\cos n\pi\right] \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n \quad \left[\because \ \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{1}{3}l^2 + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}$$

$$x^2 = \frac{1}{3}l^2 - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) - \frac{1}{2^2}\cos\left(\frac{2\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) - \cdots\right]$$

Example 16

Obtain the Fourier cosine series of the function $f(x) = e^x$ in the range $(0, l)$. **[Winter 2014]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$= \frac{1}{l}\int_0^l e^x\,\mathrm{d}x$$

$$= \frac{1}{l}\left|e^x\right|_0^l$$

$$= \frac{1}{l}(e^l - 1)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\int_0^l e^x\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left|\frac{e^x}{1+\frac{n^2\pi^2}{l^2}}\left\{\cos\left(\frac{n\pi x}{l}\right) + \frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)\right\}\right|_0^l$$

$$= \frac{2l}{l^2+n^2\pi^2}(e^l\cos n\pi - e^0\cos 0) \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2l}{l^2 + n^2\pi^2}[e^l(-1)^n - 1] \quad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

Hence, $f(x) = \frac{1}{l}(e^l - 1) + \sum_{n=1}^{\infty} \frac{2l}{l^2 + n^2\pi^2}\left[e^l(-1)^n - 1\right]\cos\frac{n\pi x}{l}$

Example 17

Find the half-range cosine series of $f(x)$, where

$$\begin{aligned} f(x) &= kx & 0 \le x \le \frac{l}{2} \\ &= k(l - x) & \frac{l}{2} \le x \le l \end{aligned}$$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$ **[Summer 2016]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\begin{aligned} a_0 &= \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x \\ &= \frac{1}{l}\left[\int_0^{\frac{l}{2}} kx\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} k(l - x)\,\mathrm{d}x\right] \\ &= \frac{1}{l}\left[k\left|\frac{x^2}{2}\right|_0^{\frac{l}{2}} + k\left|lx - \frac{x^2}{2}\right|_{\frac{l}{2}}^{l}\right] \\ &= \frac{k}{l}\left[\frac{l^2}{8} + \left(l^2 - \frac{l^2}{2}\right) - \left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right] \\ &= \frac{kl}{4} \end{aligned}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx\cos\frac{n\pi x}{l}\mathrm{d}x + \int_{\frac{l}{2}}^{l} k(l-x)\cos\frac{n\pi x}{l}\mathrm{d}x\right]$$

$$= \frac{2k}{l}\left[\left| x\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - \left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - (-1)\left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)_{\frac{l}{2}}^{l}\right|\right]$$

$$= \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - \left\{1+(-1)\right\}^n\right] \quad \left[\because \sin n\pi = 0, \cos n\pi = (-1)^n,\ \cos 0 = 1\right]$$

Hence,
$$f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - \left\{1+(-1)^n\right\}\right]\cos\frac{n\pi x}{l} \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{kl}{4} + \frac{2kl}{\pi^2}\left(-\frac{4}{2^2} - \frac{4}{6^2} - \frac{4}{10^2} - \ldots\right)$$

$$0 = \frac{kl}{4} - \frac{2kl}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \qquad \ldots (2)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 18

Find the half-range sine series of
$$f(x) = \frac{2x}{l} \qquad 0 \le x \le \frac{l}{2}$$
$$= \frac{2(l-x)}{l} \qquad \frac{l}{2} \le x \le l$$

Solution

The half-range sine series of $f(x)$ with period $2l$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} \frac{2x}{l}\sin\frac{n\pi x}{l}\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} \frac{2(l-x)}{l}\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{4}{l^2}\left[\left| x\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - \left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{4}{l^2}\frac{l^2}{n^2\pi^2}\left(2\sin\frac{n\pi}{2}\right)$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $f(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$

Example 19

Express $f(x) = x$ as a

(i) half-range sine series in $0 < x < 2$

(ii) half-range cosine series in $0 < x < 2$ **[Summer 2014]**

Solution

(i) The half-range sine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \left| x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_0^2$$

$$= 2\left(-\frac{2}{n\pi}\right)\cos n\pi \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4(-1)^n}{n\pi} \qquad \left[\because \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}$$

$$x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin \pi x + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

(ii) The half-range cosine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^2 x\,\mathrm{d}x$$

$$= \frac{1}{2}\left|\frac{x^2}{2}\right|_0^2$$

$$= \frac{1}{2}(2)$$

$$= 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \left| x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right|_0^2$$

$$= \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\cos 0 \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \qquad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{2}$$

$$x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

Example 20

Find the Fourier sine series of $f(x) = 2x$ in $0 < x < 1$. **[Summer 2015]**

Solution

The Fourier sine series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= 2\int_0^1 (2x)\sin n\pi x\,\mathrm{d}x$$

$$= 4\left|(x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= 4\left|-x\left(\frac{\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2}\right|_0^1$$

$$= 4\left[-\frac{\cos n\pi}{n\pi}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4(-1)^n}{n\pi} \qquad [\because \cos n\pi = (-1)^n]$$

$$= \frac{4}{n\pi}(-1)^{n+1}$$

Hence, $$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x$$

Example 21

Find the half-range cosine series of $f(x) = (x-1)^2$ in $0 < x < 1$.

[Summer 2015]

Solution

The half-range cosine series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$= \int_0^1 (x-1)^2\,\mathrm{d}x$$

$$= \left|\frac{(x-1)^3}{3}\right|_0^1$$

$$= \left[0 - \frac{(-1)}{3}\right]$$

$$= \frac{1}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\frac{\cos n\pi x}{l}\,\mathrm{d}x$$

$$= 2\int_0^1 (x-1)^2 \cos n\pi x\,\mathrm{d}x$$

$$= 2\left|(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= 2\left|(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1)\left(\frac{\cos n\pi x}{n^2\pi^2}\right) - 2\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= \frac{4\cos 0}{n^2\pi^2} \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2\pi^2} \qquad [\because \cos 0 = 1]$$

Hence, $$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

Example 22

Find the half-range sine series of $f(x) = x \qquad 0 < x < 1$

$= 2 - x \qquad 1 < x < 2$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

Solution

The half-range sine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 x\sin\frac{n\pi x}{2}\,\mathrm{d}x + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \left| x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_0^1$$

$$+ \left|(2-x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_1^2$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $f(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{2}\sin\frac{\pi x}{2} + \frac{1}{3^2}\sin\frac{3\pi}{2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi}{2}\sin\frac{5\pi x}{2} + \cdots\right]$$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \cdots\right] \qquad \ldots(1)$$

At $x = 1$,

$$f(1) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right] = \frac{1+(2-1)}{2} = 1$$

Putting $x = 1$ in Eq. (1),

$$f(1) = \frac{8}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{2} - \frac{1}{3^2}\sin\frac{3\pi}{2} + \frac{1}{5^2}\sin\frac{5\pi}{2} - \cdots\right)$$

$$1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 23

Find the half-range cosine series of $f(x) = 1 \quad 0 \le x \le 1$

$= x \quad 1 \le x \le 2$

Solution

The half-range cosine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^1 1\mathrm{d}x + \int_1^2 x\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|x\right|_0^1 + \left|\frac{x^2}{2}\right|_1^2\right]$$

$$= \frac{5}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \int_0^1 1\cdot\cos\frac{n\pi x}{2}\mathrm{d}x + \int_1^2 x\cos\frac{n\pi x}{2}\mathrm{d}x$$

$$= \left|\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right|_0^1 + \left[\left|x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right|_1^2\right]$$

$$= \left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right) + \left(\frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right] \quad [\because \cos n\pi = (-1)^n]$$

Hence,
$$f(x) = \frac{5}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\cos\frac{n\pi x}{2}$$

EXERCISE 2.3

1. Find the half-range cosine series of $f(x) = x \sin x$ in $0 < x < \pi$.

$$\left[\textbf{Ans.:}\ 1 - \frac{1}{2}\cos x + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos nx\right]$$

2. Find the half-range cosine series of $f(x) = (x-1)^2$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x\right]$$

3. Find the half-range cosine series of $f(x) = e^x$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ (e-1) + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}[e(-1)^n - 1]\cos n\pi x\right]$$

4. Find the half-range sine series of

$$\begin{aligned} f(x) &= x & 0 \le x \le 2 \\ &= 4 - x & 2 \le x \le 4 \end{aligned}$$

$$\left[\textbf{Ans.:}\ \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{4}\right]$$

5. Find the half-range sine and cosine series of $f(x) = x - x^2$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ \frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin(2n+1)\pi x,\quad \frac{1}{6} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\cos 2n\pi x\right]$$

6. Find the half-range sine and cosine series of $f(x) = a\left(1 - \frac{x}{l}\right)$ in $0 < x < l$.

$$\left[\textbf{Ans.:}\ \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l},\quad \frac{a}{2} + \frac{4a}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{l}\right]$$

7. Find the half-range sine series of $f(x) = \sin^2 x$ in $0 < x < \pi$.

$$\left[\textbf{Ans.:} -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)(2n+1)(2n+3)}\right]$$

8. Find the half-range sine series of

$$\begin{aligned} f(x) &= \frac{2x}{3} && 0 \le x \le \frac{\pi}{3} \\ &= \frac{\pi - x}{3} && \frac{\pi}{3} \le x \le \pi \end{aligned}$$

$$\left[\textbf{Ans.:} \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin nx\right]$$

9. Find the half-range sine series of

$$\begin{aligned} f(x) &= x && 0 \le x < 1 \\ &= 1 && 1 \le x < 2 \\ &= 3 - x && 2 \le x \le 3 \end{aligned}$$

$$\left[\textbf{Ans.:} \frac{6}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)\right]\sin\frac{n\pi x}{3}\right]$$

2.8 FOURIER INTEGRAL

Let $f(x)$ be a function which is piecewise continuous in every finite interval in $(-\infty, \infty)$ and absolutely integrable in $(-\infty, \infty)$.

We know that the Fourier series of the function $f(x)$ in any interval $(-l, l)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \text{...(2.5)}$$

where
$$\begin{aligned} a_0 &= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt \\ a_n &= \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt \end{aligned}$$

Substituting the values of a_0, a_n, and b_n in Eq. (2.5),

$$f(x)=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\sum_{n=1}^{\infty}\frac{1}{l}f(t)\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l}dt$$

$$+\sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l}f(t)\sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\,\mathrm{d}t$$

$$=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l}f(t)\left[\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l}+\sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\right]\mathrm{d}t$$

$$=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l}f(t)\cos\frac{n\pi}{l}(t-x)\,\mathrm{d}t$$

Putting $\omega_n=\dfrac{n\pi}{l}$ and $\Delta\omega_n=\omega_{n+1}-\omega_n=(n+1)\dfrac{\pi}{l}-\dfrac{n\pi}{l}=\dfrac{\pi}{l}$,

$$f(x)=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\frac{\Delta\omega_n}{\pi}\sum_{n=1}^{\infty}\int_{-l}^{l}f(t)\cos\omega_n\,(t-x)\,dt$$

$$=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{-l}^{l}f(t)\cos\,\omega_n(t-x)\,\mathrm{d}t\right]\Delta\omega_n \qquad \text{...(2.6)}$$

As $l\to\infty, \dfrac{1}{l}\to 0$ and $\Delta\omega_n=\dfrac{\pi}{l}\to 0$, the infinite series in Eq. (2.6) becomes an integral from 0 to ∞.

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}f(t)\cos\omega(t-x)\,\mathrm{d}t\right]\mathrm{d}\omega \quad [\because\ l\to\infty, \Delta\omega_n\to d\omega] \qquad \text{...(2.7)}$$

Equation (2.7) is called *the Fourier integral of f(x)*.

Expanding $\cos\omega(t-x)$ in Eq. (2.7),

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}f(t)(\cos\omega t\cos\omega x+\sin\omega t\sin\omega x)\,\mathrm{d}t\right]\mathrm{d}\omega$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\,\cos\omega t\mathrm{d}t\,\cos\omega x\,\mathrm{d}\omega+\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\sin\omega t\,\mathrm{d}t\sin\omega x\,\mathrm{d}\omega$$

$$=\int_{0}^{\infty}A(\omega)\cos\omega x\,\mathrm{d}\omega+\int_{0}^{\infty}B(\omega)\sin\omega x\,\mathrm{d}\omega \qquad \text{...(2.8)}$$

where $\quad A(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(t)\cos\omega t\,\mathrm{d}t$

and $\quad B(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(t)\sin\omega t\,\mathrm{d}t$

Fourier Cosine and Sine Integrals

When $f(x)$ is an even function,

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t \,\mathrm{d}t$$

$$B(\omega) = 0$$

The Fourier integral of an even function $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \,\mathrm{d}\omega \qquad \text{...(2.9)}$$

Equation (2.9) is called the Fourier cosine integral of $f(x)$.
When $f(x)$ is an odd function,

$$A(\omega) = 0$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t \,\mathrm{d}t$$

The Fourier integral of an odd function $f(x)$ is given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x \,\mathrm{d}\omega \qquad \text{...(2.10)}$$

Equation (2.10) is called the Fourier sine integral of $f(x)$.

Example 1

Using Fourier integral representation, show that

$$\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\,d\omega = 0 \qquad x < 0$$

$$= \frac{\pi}{2} \qquad x = 0$$

$$= \pi e^{-x} \qquad x > 0$$

[Winter 2014; Summer 2015]

Solution

Let

$$f(x) = 0 \qquad x < 0$$

$$= \frac{1}{2} \qquad x = 0$$

$$= e^{-x} \qquad x > 0$$

The Fourier integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \,\mathrm{d}\omega + \int_0^\infty B(\omega)\sin\omega x \,\mathrm{d}\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos\omega t \,\mathrm{d}t$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^0 0\cdot\cos\omega t \,\mathrm{d}\omega + \int_0^\infty e^{-t}\cos\omega t \,\mathrm{d}\omega\right]$$

$$= \frac{1}{\pi}\left|\frac{e^{-t}}{1+\omega^2}(-\cos\omega t + \omega\sin\omega t\right|_0^\infty$$

$$= \frac{1}{\pi(1+\omega^2)} \qquad [\because \quad \cos 0 = 1, \sin 0 = 0]$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t \; \mathrm{d}t$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0\cdot\sin\omega t \; \mathrm{d}\omega + \int_0^\infty e^{-t}\sin\omega t \; \mathrm{d}\omega\right]$$

$$= \frac{1}{\pi}\left|\frac{e^{-t}}{1+\omega^2}(-\sin\omega t - \omega\cos\omega t)\right|_0^\infty$$

$$= -\frac{1}{\pi(1+\omega^2)}(-\omega)$$

$$= \frac{\omega}{\pi(1+\omega^2)}$$

Hence, $$f(x) = \frac{1}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\cos\omega x \; \mathrm{d}\omega + \frac{1}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin\omega x \; \mathrm{d}\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} \; \mathrm{d}\omega$$

$$\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\mathrm{d}\omega = \pi f(x)$$

$$= \begin{cases} 0 & x < 0 \\ \dfrac{\pi}{2} & x = 0 \\ \pi e^{-x} & x > 0 \end{cases}$$

Example 2

Express the function $f(x) = 2 \quad |x| < 2$

$= 0 \quad |x| > 2$

as Fourier integral. **[Summer 2017, 2016]**

Solution

The function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \; \mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^2 2\cdot\cos\omega t\,\mathrm{d}t$$

$$= \frac{4}{\pi}\left|\frac{\sin\omega t}{\omega}\right|_0^2$$

$$= \frac{4}{\pi}\frac{\sin 2\omega}{\omega} \qquad [\because \sin 0 = 0]$$

Hence, $$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin 2\omega\cos\omega x}{\omega}\,\mathrm{d}\omega$$

$$\int_0^\infty \frac{\sin 2\omega\cos\omega x}{\omega}\,\mathrm{d}\omega = \frac{\pi}{4}f(x)$$

$$= \begin{cases} \dfrac{\pi}{2} & |x| < 2 \\ 0 & |x| > 2 \end{cases} \qquad \text{...(1)}$$

At $|x| = 2$, i.e., $x = \pm 2$, $f(x)$ is discontinuous.

At $x = 2$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to -2^-} f(x) + \lim_{x\to -2^+} f(x)\right]$$

$$= \frac{1}{2}[2+0]$$

$$= 1$$

At $x = -2$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to -2^-} f(x) + \lim_{x\to -2^+} f(x)\right]$$

$$= \frac{1}{2}[0+2]$$

$$= 1$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{\sin 2\omega\cos\omega x}{\omega}\,\mathrm{d}\omega = \begin{cases} \dfrac{\pi}{2} & |x| < 2 \\ \dfrac{\pi}{4} & |x| = 2 \\ 0 & |x| > 2 \end{cases}$$

Example 3

Find the Fourier integral representation of the function

$$f(x) = 1 \qquad |x| < 1$$
$$= 0 \qquad |x| > 1$$

Hence, evaluate (i) $\int_0^\infty \frac{\sin\omega \cos\omega x}{\omega} \mathrm{d}\omega$ (ii) $\int_0^\infty \frac{\sin\omega}{\omega} \mathrm{d}\omega$

[Winter 2016, 2014, 2013]

Solution

The function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \,\mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t \,\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^1 1\cdot\cos\omega t \,\mathrm{d}t$$

$$= \frac{2}{\pi}\left|\frac{\sin\omega t}{\omega}\right|_0^1$$

$$= \frac{2}{\pi}\frac{\sin\omega}{\omega} \qquad [\because \sin 0 = 0]$$

Hence, $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\mathrm{d}\omega$$

(i) $$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega} d\omega = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases} \qquad \text{...(1)}$$

At $|x| = 1$, i.e., $x = \pm 1$, $f(x)$ is discontinuous.

At $x = 1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right]$$

$$= \frac{1}{2}(1+0)$$

$$= \frac{1}{2}$$

At $x = -1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to -1^-} f(x) + \lim_{x\to -1^+} f(x)\right]$$
$$= \frac{1}{2}(0+1)$$
$$= \frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,\mathrm{d}\omega = \begin{cases} \dfrac{\pi}{2} & |x|<1 \\ \dfrac{\pi}{4} & |x|=1 \\ 0 & |x|>1 \end{cases}$$

(ii) Putting $x = 0$ in Eq. (1),

$$\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2}f(0) = \frac{\pi}{2} \qquad [\because\ f(0)=1]$$

Example 4

Find the Fourier integral representation of the function

$$f(x) = 1 - x^2 \qquad |x| \le 1$$
$$= 0 \qquad |x| > 1$$

Solution

$f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,\mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,\mathrm{d}t$$
$$= \frac{2}{\pi}\int_0^\infty (1-t^2)\cos\omega t\,\mathrm{d}t$$
$$= \frac{2}{\pi}\left|(1-t)^2\left(\frac{\sin\omega t}{\omega}\right) - (-2t)\left(-\frac{\cos\omega t}{\omega^2}\right) + (-2)\left(-\frac{\sin\omega t}{\omega^3}\right)\right|_0^1$$
$$= \frac{2}{\pi}\left(-\frac{2\cos\omega}{\omega^2} + \frac{2\sin\omega}{\omega^3}\right) \qquad [\because\ \sin 0 = 0]$$
$$= \frac{4}{\pi}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)$$

Hence, $$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\omega x \, \mathrm{d}\omega$$

Example 5

Find the Fourier cosine integral of $f(x) = e^{-kx}$, where $x > 0$, $k > 0$. **[Winter 2016]**

Solution

The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \, \mathrm{d}\omega$$

$$\begin{aligned}
A(\omega) &= \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t \, \mathrm{d}t \\
&= \frac{2}{\pi}\int_0^\infty e^{-kt}\cos\omega t \, \mathrm{d}t \\
&= \frac{2}{\pi}\left|\frac{e^{-kt}}{k^2+\omega^2}(-k\cos\omega t + \omega\sin\omega t\right|_0^\infty \\
&= \frac{2}{\pi}\left(\frac{k}{k^2+\omega^2}\right) \qquad [\because \cos 0 = 1, \sin 0 = 0]
\end{aligned}$$

Hence, $$f(x) = \frac{2k}{\pi}\int_0^\infty \frac{1}{k^2+\omega^2}\cos\omega x \, \mathrm{d}\omega$$

$$\begin{aligned}
\int_0^\infty \frac{\cos\omega x}{\omega^2+k^2}\mathrm{d}\omega &= \frac{\pi}{2k}f(x) \\
&= \frac{\pi}{2k}e^{-kx} \qquad x > 0, \qquad k > 0
\end{aligned}$$

Example 6

Find the Fourier cosine integral of $f(x) = \frac{\pi}{2}e^{-x}$, $x \geq 0$. **[Winter 2015]**

Solution

The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \, \mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\ \cos\omega t \, \mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{\pi}{2} e^{-t} \cos \omega t \, dt$$

$$= \int_0^{\infty} e^{-t} \cos \omega t \, dt$$

$$= \left| \frac{e^{-t}}{1+\omega^2}(-\cos\omega t + \omega \sin \omega t) \right|_0^{\infty}$$

$$= \frac{1}{1+\omega^2} \qquad [\because \cos 0 = 1, \sin 0 = 0]$$

Hence, $$f(x) = \int_0^{\infty} \frac{1}{1+\omega^2} \cos \omega x \, d\omega$$

$$\int_0^{\infty} \frac{\cos \omega x}{1+\omega^2} \, d\omega = f(x)$$

$$\int_0^{\infty} \frac{\cos \omega x}{1+\omega^2} \, d\omega = \frac{\pi}{2} e^{-x}$$

Example 7

Find the Fourier cosine integral of the function $f(x) = \cos x \quad |x| < \frac{\pi}{2}$

$$= 0 \qquad |x| > \frac{\pi}{2}$$

Solution

The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x \, d\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t) \cos \omega t \, dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos t \cos \omega t \, dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\cos(1+\omega)t + \cos(1-\omega)t\right] dt$$

$$= \frac{1}{\pi}\left|\frac{\sin(1+\omega)t}{1+\omega} + \frac{\sin(1-\omega)t}{1-\omega}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{\sin(1+\omega)\frac{\pi}{2}}{1+\omega} + \frac{\sin(1-\omega)\frac{\pi}{2}}{1-\omega}\right] \quad [\because\ \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\cos\left(\frac{\pi\omega}{2}\right)}{1+\omega} + \frac{\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega}\right]$$

$$= \frac{1}{\pi}\frac{2\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2}$$

Hence, $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2}\cos\omega x\ \mathrm{d}\omega$$

Example 8

Express the function $f(x) = 1 \qquad 0 \le x < \pi$

$\qquad\qquad\qquad\qquad = 0 \qquad x > \pi$

as a Fourier sine integral and hence, evaluate

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\ \mathrm{d}\omega$$ **[Winter 2017]**

Solution

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\ \mathrm{d}\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t\ \mathrm{d}t$$

$$= \frac{2}{\pi}\left[\int_0^\pi 1\cdot\sin\omega t\ \mathrm{d}t + \int_\pi^0 0\cdot\sin\omega t\ \mathrm{d}t\right]$$

$$= \frac{2}{\pi}\left|-\frac{\cos\omega t}{\omega}\right|_0^\pi$$

$$= \frac{2}{\pi}\left(\frac{-\cos\pi\omega + 1}{\omega}\right) \quad [\because\ \cos 0 = 1]$$

$$= \frac{2}{\pi}\left(\frac{1-\cos\pi\omega}{\omega}\right)$$

Hence,
$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x \, d\omega$$

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x \, d\omega = \frac{\pi}{2} f(x)$$
$$= \begin{cases} \frac{\pi}{2} & 0 \le x < \pi \\ 0 & x > \pi \end{cases} \quad \text{...(1)}$$

At $x = \pi$, $f(x)$ is discontinuous.

$$f(x) = \frac{1}{2}\left[\lim_{x\to\pi^-} f(x) + \lim_{x\to\pi^+} f(x)\right]$$
$$= \frac{1}{2}(1+0)$$
$$= \frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x \, d\omega = \begin{cases} \frac{\pi}{2} & 0 \le x < \pi \\ \frac{\pi}{4} & x = \pi \\ 0 & x > \pi \end{cases}$$

Example 9

Express the function $f(x) = \sin x \qquad 0 \le x \le \pi$

$= 0 \qquad x > \pi$

as a Fourier sine integral and show that

$$\int_0^\infty \frac{\sin\omega x \sin\pi\omega}{1-\omega^2} d\omega = \frac{\pi}{2}\sin x \qquad 0 \le x \le \pi$$

Solution

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x \, d\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t \, dt$$
$$= \frac{2}{\pi}\left[\int_0^\pi \sin t \sin\omega t \, dt + \int_\pi^\infty 0\cdot\sin\omega t \, dt\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\cos(\omega-1)t - \cos(\omega+1)t\right]\mathrm{d}t$$

$$= \frac{1}{\pi}\left|\frac{\sin(\omega-1)t}{\omega-1} - \frac{\sin(\omega+1)t}{\omega+1}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(\omega-1)\pi}{\omega-1} - \frac{\sin(\omega+1)\pi}{\omega+1}\right] \qquad [\because\ \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[-\frac{\sin\pi\omega}{\omega-1} + \frac{\sin\pi\omega}{\omega+1}\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\sin\pi\omega}{\omega^2-1}\right)$$

$$= \frac{2}{\pi}\left(\frac{\sin\pi\omega}{1-\omega^2}\right)$$

Hence, $$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\pi\omega}{1-\omega^2}\sin\omega x\, d\omega, \qquad \omega \neq 1$$

$$\int_0^{\infty}\frac{\sin\omega x\sin\pi\omega}{1-\omega^2}\mathrm{d}\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \dfrac{\pi}{2}\sin x & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$$

Example 10

Find the Fourier sine integral of $f(x) = e^{-bx}$.

Hence, show that $\dfrac{\pi}{2}e^{-bx} = \displaystyle\int_0^{\infty}\frac{\omega\sin\omega x}{b^2+\omega^2}\mathrm{d}\omega$

Solution

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int_0^{\infty} B(\omega)\sin\omega x\,\mathrm{d}\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t)\sin\omega t\,\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^{\infty} e^{-bt}\sin\omega t\,\mathrm{d}t$$

$$= \frac{2}{\pi}\left|\frac{e^{-bt}}{b^2+\omega^2}(-b\sin\omega t - \omega\cos\omega t)\right|_0^{\infty}$$

$$= \frac{2}{\pi}\left(\frac{\omega}{b^2+\omega^2}\right) \quad [\because \ \cos 0 = 1, \sin 0 = 0]$$

Hence, $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\omega \sin \omega x}{b^2+\omega^2}\,\mathrm{d}\omega$$

$$\int_0^\infty \frac{\omega \sin \omega x}{b^2+\omega^2}\,\mathrm{d}\omega = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-bx}$$

Example 11

Show that $\int\limits_0^\infty \frac{\lambda^3 \sin \lambda x}{\lambda^4+4}\,\mathrm{d}\lambda = \frac{\pi}{2} e^{-x} \cos x$, *where* $x > 0$. **[Winter 2015]**

Solution

$$f(x) = \frac{\pi}{2} e^{-x} \cos x, \quad x > 0$$

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int\limits_0^\infty B(\lambda) \sin \lambda x \,\mathrm{d}\lambda$$

$$B(\lambda) = \frac{2}{\pi}\int\limits_0^\infty f(x) \sin \lambda x \,\mathrm{d}x$$

$$= \frac{2}{\pi}\int\limits_0^\infty \frac{\pi}{2} e^{-x} \cos x \sin \lambda x \,\mathrm{d}x$$

$$= \int\limits_0^\infty e^{-x} \cos x \sin \lambda x \,\mathrm{d}x$$

$$= \frac{1}{2}\int\limits_0^\infty e^{-x} (2\cos x \sin \lambda x)\,\mathrm{d}x$$

$$= \frac{1}{2}\int\limits_0^\infty e^{-x} \left[\sin(\lambda+1)x + \sin(\lambda-1)x\right]\mathrm{d}x$$

$$= \frac{1}{2}\left[\int\limits_0^\infty e^{-x} \sin(\lambda+1)x \,\mathrm{d}x + \int\limits_0^\infty e^{-x} \sin(\lambda-1)x \,\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|\frac{e^{-x}}{1+(\lambda+1)^2}\{-\sin(\lambda+1)x-(\lambda+1)\cos(\lambda+1)x\}\right|_0^\infty\right.$$

$$\left.+\left|\frac{e^{-x}}{1+(\lambda-1)^2}\{-\sin(\lambda-1)x-(\lambda-1)\cos(\lambda-1)x\}\right|_0^\infty\right], \quad (x>0)$$

$$= \frac{1}{2}\left[\frac{(\lambda+1)}{1+(\lambda+1)^2}+\frac{(\lambda-1)}{1+(\lambda-1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{\lambda+1}{\lambda^2+2\lambda+2}+\frac{\lambda-1}{\lambda^2-2\lambda+2}\right]$$

$$= \frac{1}{2}\left[\frac{(\lambda+1)(\lambda^2-2\lambda+2)+(\lambda-1)(\lambda^2+2\lambda+2)}{(\lambda^2+2\lambda+2)(\lambda^2-2\lambda+2)}\right]$$

$$= \frac{1}{2}\left[\frac{\lambda^3-2\lambda^2+2\lambda+\lambda^2-2\lambda+2+\lambda^3+2\lambda^2+2\lambda-\lambda^2-2\lambda-2}{\lambda^4+4}\right]$$

$$= \frac{1}{2}\left[\frac{2\lambda^3}{\lambda^4+4}\right]$$

$$= \frac{\lambda^3}{\lambda^4+4}$$

Hence, $$f(x) = \int_0^\infty \frac{\lambda^3}{\lambda^4+4}\sin\lambda x\, d\lambda$$

$$\int_0^\infty \frac{\lambda^3}{\lambda^4+4}\sin\lambda x\, d\lambda = \frac{\pi}{2}e^{-x}\cos x$$

EXERCISE 2.4

1. Find the Fourier integral representations of the following functions:

(i) $f(x) = x \qquad |x|<1$
$\qquad = 0 \qquad |x|>1$

(ii) $f(x) = -e^{ax} \qquad x<0$
$\qquad = e^{-ax} \qquad x>0$

$$\left[\textbf{Ans.: (i)} \int_{-\infty}^{\infty}\frac{\sin\omega-\omega\cos\omega}{i\pi\omega^2}e^{i\omega x}\,d\omega \quad \text{(ii)} \frac{2}{\pi}\int_0^\infty \sin\omega x\frac{\omega}{a^2+\omega^2}\,d\omega\right]$$

2. Find the Fourier sine integral of $f(x) = e^{-ax} - e^{-bx}$.

$$\left[\textbf{Ans.}: \frac{2}{\pi}\int_0^{\infty} \frac{(b^2-a^2)\omega\sin\omega x}{(a^2+\omega^2)(b^2+\omega^2)}\, d\omega\right]$$

3. Find the Fourier cosine integral of $f(x) = e^{-x}\cos x$.

$$\left[\textbf{Ans.}: \frac{2}{\pi}\int_0^{\infty} \frac{\omega^2+2}{\omega^4+4}\cos\omega x\, d\omega\right]$$

4. Express the function

$$\begin{aligned} f(x) &= \frac{\pi}{2} \quad 0 < x < \pi \\ &= 0 \quad x < \pi \end{aligned}$$

as the Fourier sine integral and show that

$$\int_0^{\infty} \frac{1-\cos\pi\omega}{\omega}\sin\omega x\, d\omega = \frac{\pi}{2}$$

$$\left[\textbf{Ans.}: \int_0^{-1} \frac{1-\cos\pi\omega}{\omega}\sin\omega x\, d\omega\right]$$

Points to Remember

Fourier Series in the Interval $(0, 2\pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Fourier Series in the Interval $(c, c + 2l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

where $2l$ is the length of the interval.

Fourier Series of Even Function in the Interval $(-\pi, \pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

Fourier Series of Even Function in the interval $(-l, l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = 0$$

Fourier Series of Odd Function in the Interval $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Fourier Series of Odd Function in the Interval $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Half-Range Cosine Series in the Interval (0, *π*)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

Half-Range Cosine Series in the Interval (0, *l*)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l} \int_0^{l} f(x)\,dx$$

$$a_n = \frac{2}{l} \int_0^{l} f(x) \cos \frac{n\pi x}{l}\,dx$$

Half-Range Sine Series in the Interval (0, *π*)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx$$

Half-Range Sine Series in the Interval (0, *l*)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^{l} f(x) \sin \frac{n\pi x}{l}\,dx$$

Fourier Integral Theorem

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x\,d\omega + \int_0^{\infty} B(\omega) \sin \omega x\,d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t\,dt$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t\,dt$$

Fourier Cosine Integral

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x\,d\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t \, dt$$

$$B(\omega) = 0$$

Fourier Sine Integral

$$f(x) = \int_0^\infty B(\omega)\sin\omega x \, d\omega$$

$$A(\omega) = 0$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t \, dt$$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. If $f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$

then $f(x)$ is a/an _______ function in (–1 , 1).

(a) even (b) odd (c) constant (d) none of these

2. If $f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

then $f(x)$ is a/an _____ function in ($-\pi$, π).

(a) even (b) odd (c) constant (d) none of these

3. If $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi} & 0 \le x \le \pi \end{cases}$

then $f(x)$ is a/an _____ function in ($-\pi$, π).

(a) even (b) odd (c) constant (d) none of these

4. The Fourier series expansion of $f(x) = \begin{cases} -x^2 & -\pi < x \le 0 \\ x^2 & 0 \le x \le \pi \end{cases}$ contains no ______ terms.

(a) sine (b) cosine (c) constant (d) none of these

5. The Fourier series expansion of $f(x) = \begin{cases} -x & -4 \le x \le 0 \\ x & 0 \le x \le 4 \end{cases}$ contains no ______ terms.

(a) sine (b) cosine (c) constant (d) none of these

6. If $f(x)$ is an even function in $(-\pi, \pi)$, then the graph of $f(x)$ is symmetrical about the _______.

(a) x-axis (b) y-axis (c) origin (d) none of these

7. If $f(x)$ is an odd function in $(-l, l)$, then the graph of $f(x)$ is symmetrical about the _______

(a) x-axis (b) y-axis (c) origin (d) none of these

8. If $f(x)$ is an even function in the interval $(-l, l)$, then the value of b_n is

(a) $\dfrac{\pi}{2}$ (b) π (c) 1 (d) 0

9. If $f(x)$ is an odd function in $(-l, l)$, then the values of a_0 and a_1 are

(a) 0, 0 (b) π, π (c) $\dfrac{\pi}{2}, \pi$ (d) 1, 1

10. If $f(x) = x$ in $(-\pi, \pi)$, then the Fourier coefficient a_2 is

(a) π (b) 0 (c) 1 (d) -1

11. If $f(x) = \cos x$ in $(-\pi, \pi)$, then the Fourier coefficient b_n is

(a) 0 (b) π (c) 1 (d) none of these

12. In the Fourier series expansion of $f(x) = x \sin x$ in $(-\pi, \pi)$, the _____ terms are absent.

(a) sine (b) cosine (c) constant (d) none of these

13. If $f(x) = x \cos x$ in $(-\pi, \pi)$, then b_1 is

(a) 0 (b) π (c) 1 (d) none of these

14. Which of the following is neither an even function nor an odd function?

(a) $x \sin x$ (b) x^2 (c) e^{-x} (d) $x \cos x$

15. Fundamental period of $\sin 2x$ is

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) 2π (d) π

16. Fundamental period of $\tan 3x$ is

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$ (c) π (d) $\dfrac{\pi}{4}$

17. If $f(x + nT) = f(x)$ where n is any integer, then the fundamental period of $f(x)$ is

(a) $2T$ (b) $\dfrac{T}{2}$ (c) T (d) $3T$

18. For half-range sine series of $f(x) = \cos x$, $0 \le x \le \pi$ and period 2π, Fourier series is represented by $\sum_{n=1}^{\infty} b_n \sin nx$, then Fourier coefficient b_1 is

(a) $\dfrac{1}{\pi}$ (b) 0 (c) $\dfrac{2}{\pi}$ (d) $-\dfrac{2}{\pi}$

19. A function $f(x)$ is said to be periodic of period T if

(a) $f(x+T) = f(x)$ for all x (b) $f(x+T) = f(T)$ for all x

(c) $f(-x) = f(x)$ for all x (d) $f(-x) = -f(x)$ for all x

20. Fourier series representation of a periodic function $f(x)$ with period 2π which satisfies Dirichlet's conditions is

(a) $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

(b) $a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$

(c) $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx)(b_n \sin nx)$

(d) $a_0 + a_n \cos nx + b_n \sin nx$

21. A function $f(x)$ is said to be even if

(a) $f(-x) = f(x)$ (b) $f(-x) = -f(x)$

(c) $f(x + 2\pi) = f(x)$ (d) $f(-x) = [f(x)]^2$

22. A function $f(x)$ is said to be odd if

(a) $f(-x) = f(x)$ (b) $f(-x) = -f(x)$

(c) $f(x + 2\pi) = f(x)$ (d) $f(-x) = [f(x)]^2$

23. Which of the following is an odd function?

(a) $\sin x$ (b) $e^x + e^{-x}$ (c) $e^{|x|}$ (d) $\pi^2 - x^2$

24. Which of the following is an even function?

(a) $\sin x$ (b) $e^x - e^{-x}$ (c) $x \cos x$ (d) $\cos x$

25. For an even function $f(x)$ defined in the interval $-\pi \le x \le \pi$ and $f(x+2\pi) = f(x)$, the Fourier series is

(a) $\sum_{n=1}^{\infty} b_n \sin x$ (b) $a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

(c) $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (d) $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

26. For an odd function $f(x)$ defined in the interval $-\pi \le x \le \pi$ and $f(x + 2\pi) = f(x)$, the Fourier series is

(a) $\sum_{n=1}^{\infty} b_n \sin nx$ (b) $a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

(c) $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (d) $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

27. Half-range Fourier cosine series for $f(x)$ defined in the interval $(0, \pi)$ is

(a) $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (b) $a_n + \sum_{n=1}^{\infty} a_n \cos \frac{nx}{l}$

(c) $\sum_{n=1}^{\infty} a_n \cos nx$ (d) $a_n + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

28. Half-range Fourier sine series for $f(x)$ defined in the interval $(0, \pi)$ is

(a) $\sum_{n=1}^{\infty} b_n \sin nx$ (b) $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

(b) $a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$ (d) $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

29. The Fourier series of an odd periodic function contains only
(a) odd harmonics (b) even harmonics
(c) cosine terms (d) sine terms

30. The trigonometric Fourier series of an even function does not have
(a) constant (b) cosine terms (c) sine terms (d) odd harmonic terms

31. For the function $f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$, the value of a_0 in the Fourier series expansion will be

(a) k (b) $2k$ (c) 0 (d) $-k$

32. The value of a_0 in Fourier series expansion of $f(x) = x^2$, $-1 < x < 1$ is

(a) $\frac{1}{3}$ (b) 3 (c) $\frac{1}{2}$ (d) 1

Answers

1. (b)	2. (a)	3. (a)	4. (b)	5. (a)	6. (b)	7. (c)	8. (d)
9. (a)	10. (b)	11. (a)	12. (a)	13. (a)	14. (c)	15. (d)	16. (b)
17. (c)	18. (b)	19. (a)	20. (a)	21. (a)	22. (b)	23. (a)	24. (d)
25. (c)	26. (a)	27. (a)	28. (a)	29. (d)	30. (c)	31. (c)	32. (a)