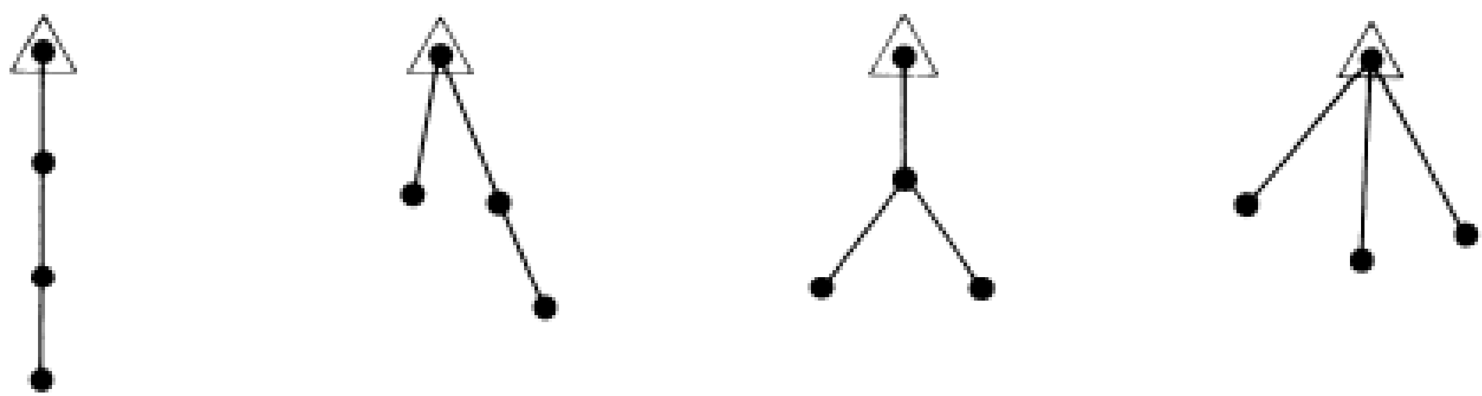
DAY 7

Rooted Tree

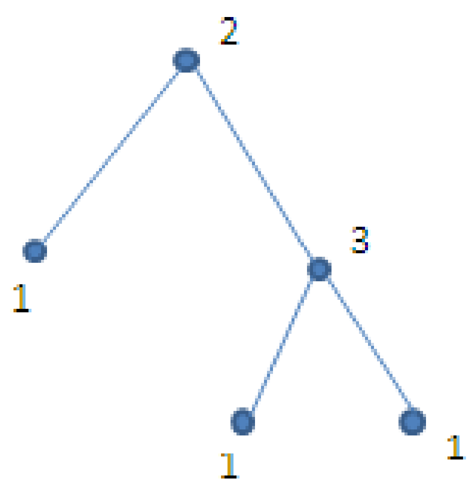
A tree in which one vertex is distinguished from all other vertices is called a Rooted Tree and that specific vertex is called the Root.



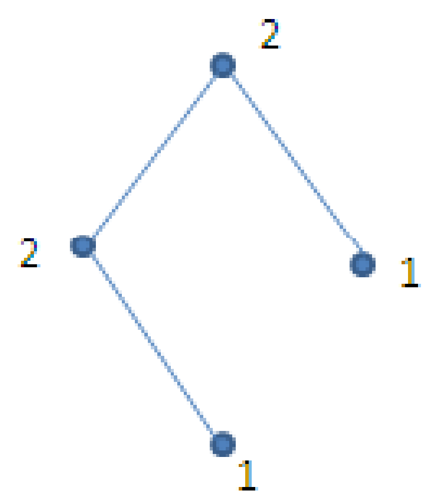
Rooted tree with four vertices

Binary Rooted Tree

It is a tree in which there is exactly one vertex of degree two and the remaining vertices are of either degree 1 or 3.



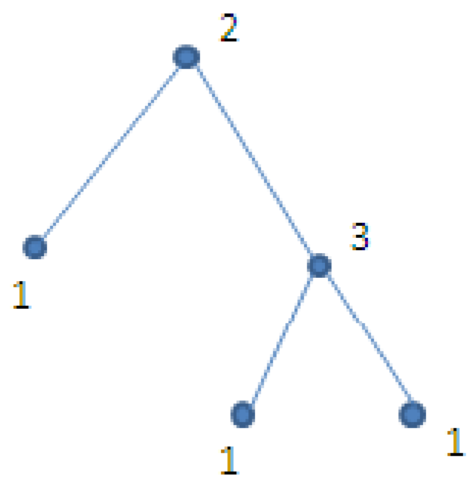
(a)



(b)

Properties

1. The number of vertices in a binary tree is always odd.



(a)

There is exactly one vertex with d(2).

So, there will be (n-1) vertices with either d(1) or d(3).

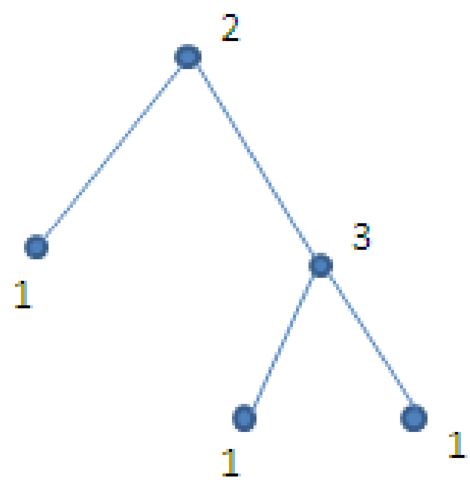
We know, there are even number of odd degree vertices in a graph.

That means, (n-1) is even.

Therefore, n is odd.

Properties

2. The number of pendant vertices in a binary tree is $(n+1)/2$.



(a)

Let, there are p no. of pendant vertices in the tree.

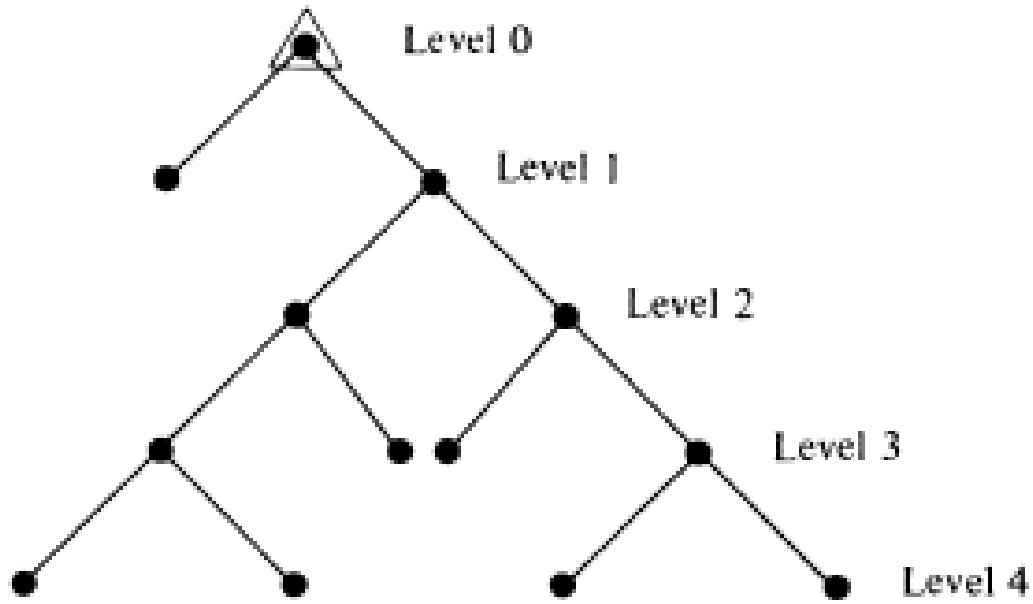
There is exactly one $d(2)$ vertex.

Therefore, there are $(n-p-1)$ no. of vertices having $d(3)$

As per handshaking lemma,
$\{(P*1)+(1*2)+(n-p-1)*3\} = 2*(n-1)$

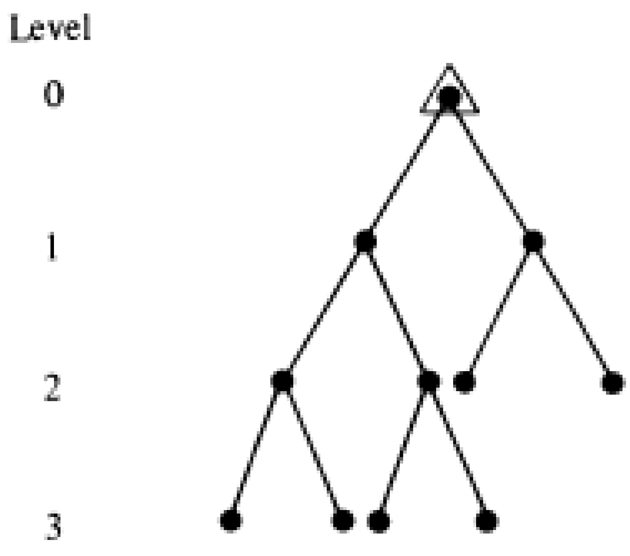
So, $p = (n+1)/2$
And no. of $d(3)$ vertex is $= (n-1)/2$

Level of tree



The maximum level of a binary tree is known as its “height”

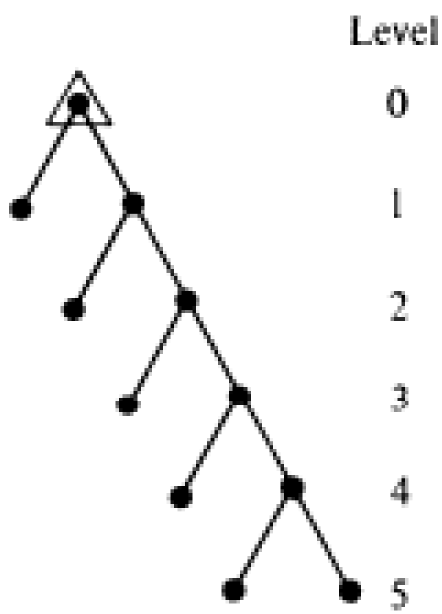
Contd..



$$\min l_{\max} = \lceil \log_2 (n+1) - 1 \rceil,$$

$$\min l_{\max} = \lceil (\log_2 12) - 1 \rceil$$

Contd..



$$\max l_{\max} = \frac{11-1}{2} = 5$$

$$\max l_{\max} = \frac{n-1}{2}.$$

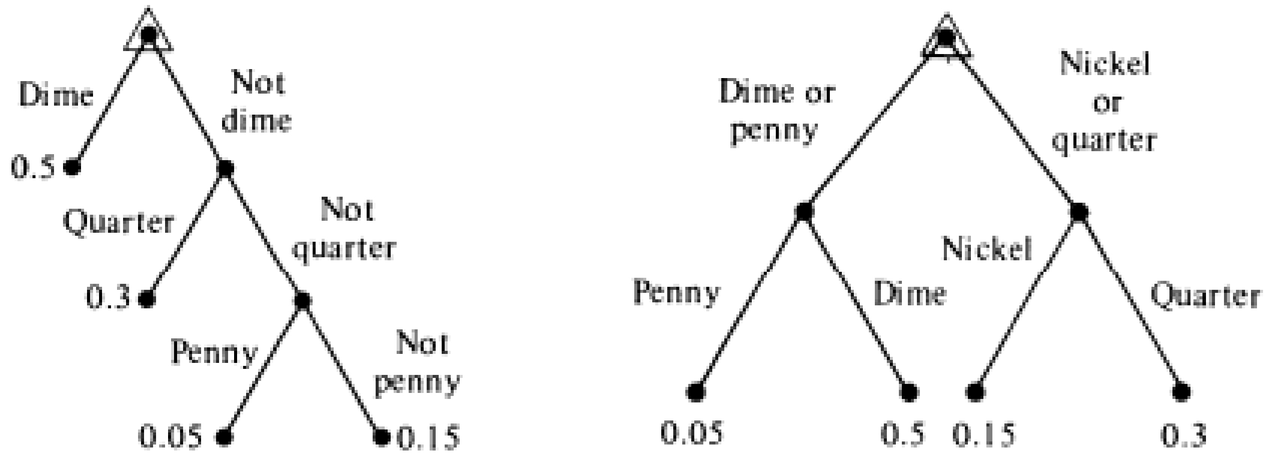
Weighted path length

Weighted Path Length: In some applications, every pendant vertex v_j of a binary tree has associated with it a positive real number w_j. Given w_1, $w_2, \ldots, w_m$ the problem is to construct a binary tree (with m pendant vertices) that minimizes

$$\sum w_j l_j,$$

where l_j is the level of pendant vertex v_j, and the sum is taken over all pendant vertices.

Contd..



$\Sigma w_i \cdot l_i = 1.7$

(a)

$\Sigma w_i \cdot l_i = 2$

(b)

(0.5x1)+(0.3X2)+(0.05X3)+(0.15X3) = 1.7

(0.05x2)+(0.5X2)+(0.15X2)+(0.3X2) = 2

Counting Tree

In 1857, Arthur Cayley discovered trees while he was trying to count the number of structural isomers of the saturated hydrocarbons (or paraffin series) C_kH_{2k+2}. He used a connected graph to represent the C_kH_{2k+2} molecule. Corresponding to their chemical valencies, a carbon atom was represented by a vertex of degree four and a hydrogen atom by a vertex of degree one (pendant vertices). The total number of vertices in such a graph is

$$n = 3k + 2,$$

and the total number of edges is

$$\begin{aligned} e &= \frac{1}{2}(\text{sum of degrees}) = \frac{1}{2}(4k + 2k + 2) \\ &= 3k + 1. \end{aligned}$$

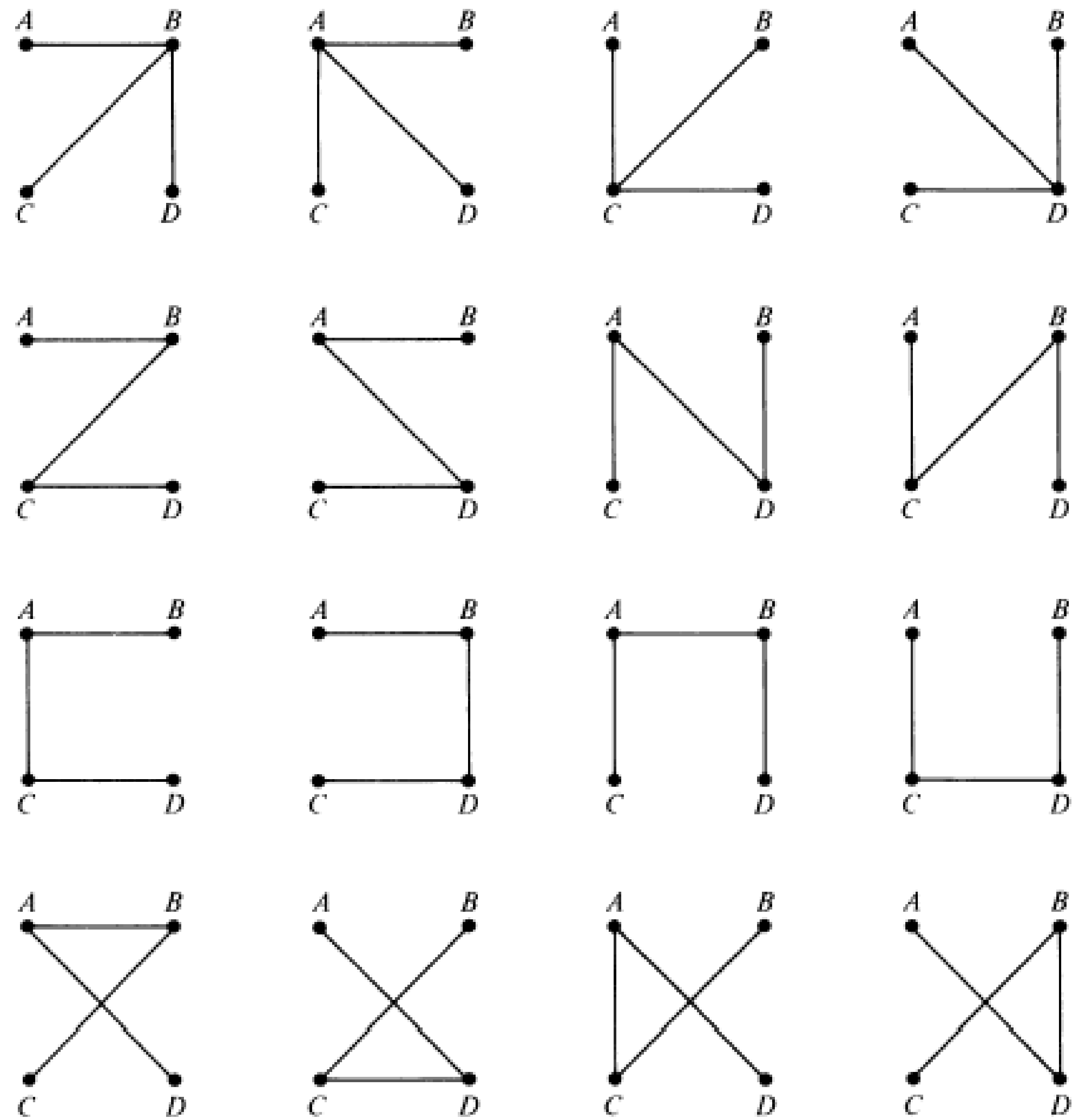
Since the graph is connected and the number of edges is one less than the number of vertices, it is a tree.

Contd..

What is the number of different labeled trees one can construct with n distinct vertices?

$\mathbf{n^{n-2}}$

For n=4 total no. of trees = 16


A B C D
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Theorem 3-10

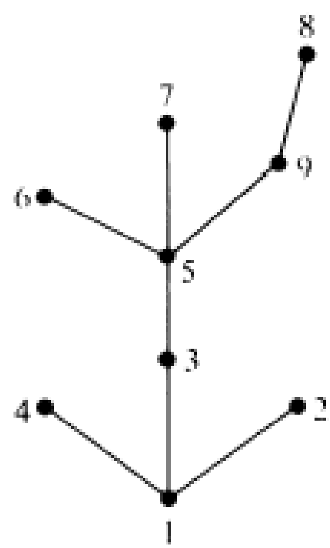
There are n^{n-2} labeled trees with n vertices ($n \geq 2$).

Proof of Theorem 3-10: Let the n vertices of a tree T be labeled $1, 2, 3, \ldots, n$. Remove the pendant vertex (and the edge incident on it) having the smallest label, which is, say, a_1. Suppose that b_1 was the vertex adjacent to a_1. Among the remaining $n - 1$ vertices let a_2 be the pendant vertex with the smallest label, and b_2 be the vertex adjacent to a_2. Remove the edge (a_2, b_2). This operation is repeated on the remaining $n - 2$ vertices, and then on $n - 3$ vertices, and so on. The process is terminated after $n - 2$ steps, when only two vertices are left. The tree T defines the sequence

$$(b_1, b_2, \ldots, b_{n-2}) \tag{10-3}$$

uniquely. For example, for the tree in Fig. 10-1 the sequence is (1, 1, 3, 5, 5, 5, 9). Note that a vertex i appears in sequence (10-3) if and only if it is not pendant (see Problem 10-2).

Conversely, given a sequence (10-3) of $n - 2$ labels, an n-vertex tree can be



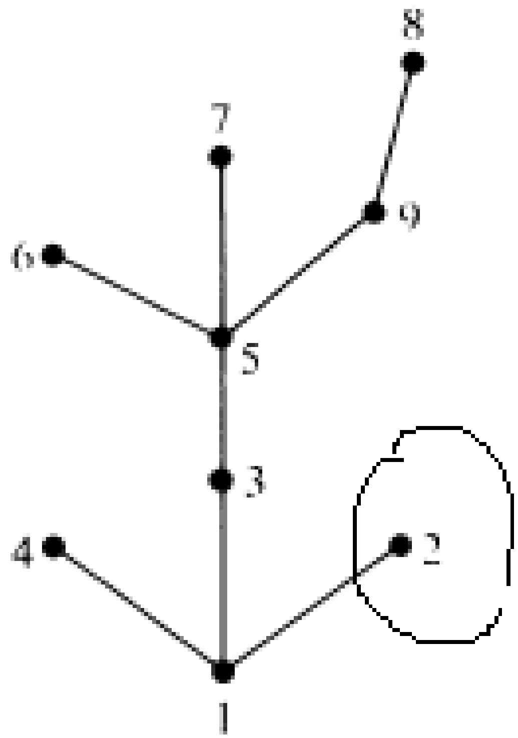


Fig. 10-1 Nine-vertex labeled tree, which yields sequence (1,).

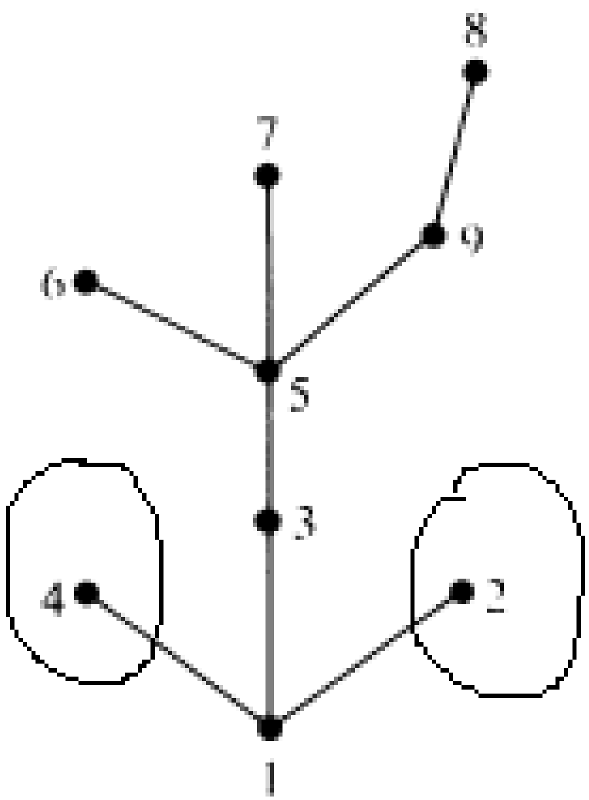


Fig. 10-1 Nine-vertex labeled tree, which yields sequence (1, 1,).

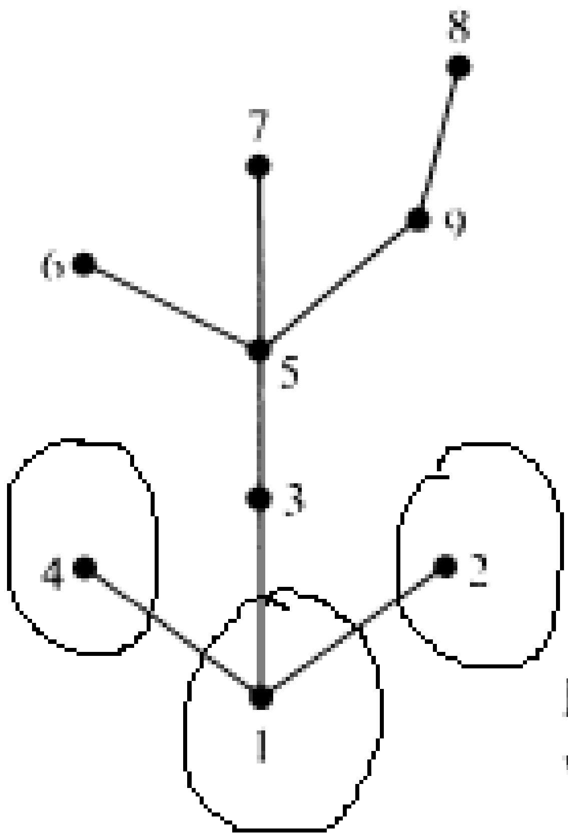


Fig. 10-1 Nine-vertex labeled tree, which yields sequence (1, 1, 3,).

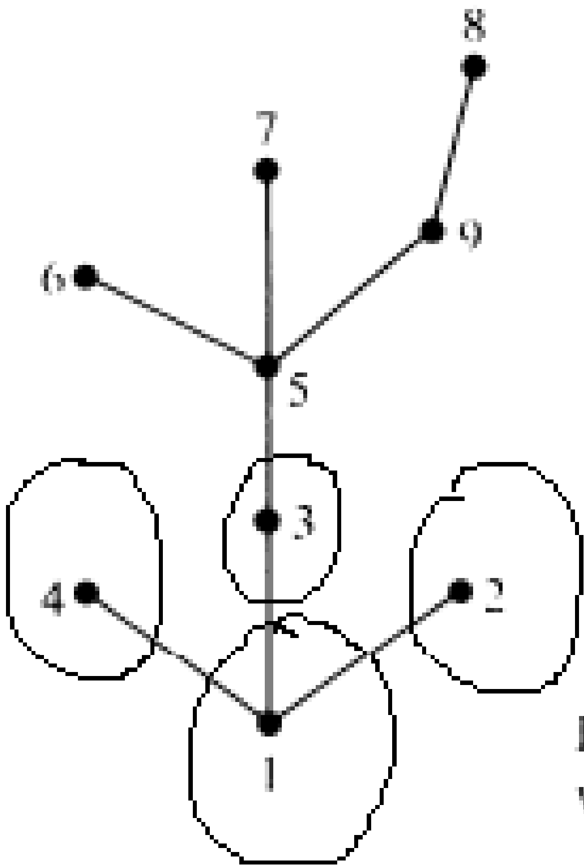


Fig. 10-1 Nine-vertex labeled tree, which yields sequence (1, 1, 3, 5,).

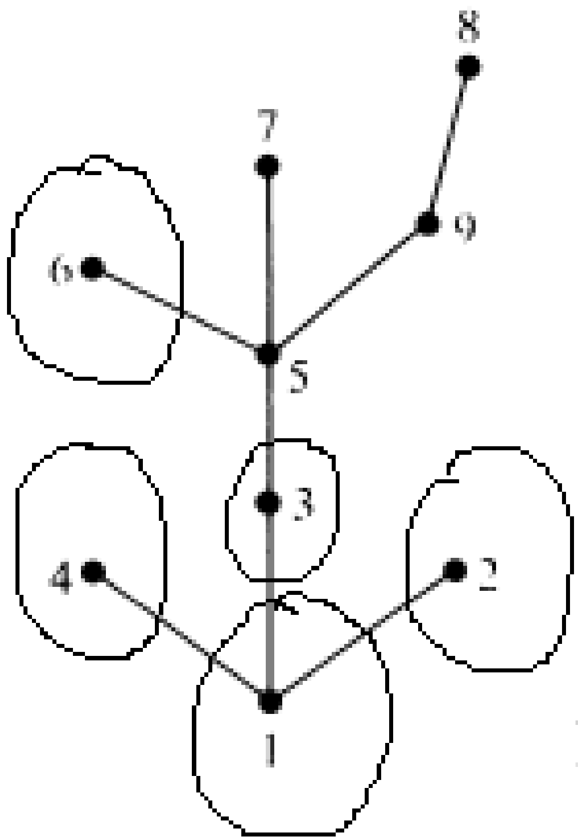


Fig. 10-1 Nine-vertex labeled tree, which yields sequence (1, 1, 3, 5, 5,).

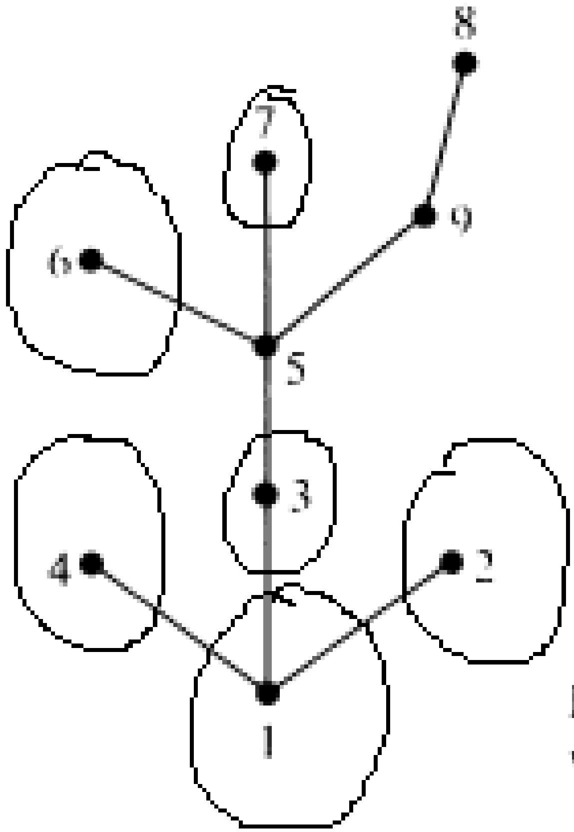


Fig. 10-1 Nine-vertex labeled tree, which yields sequence (1, 1, 3, 5, 5, 5,).

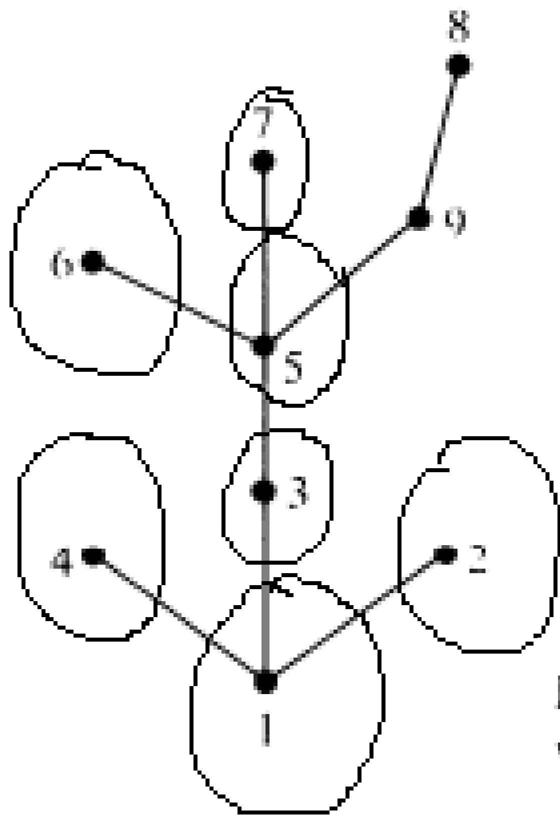
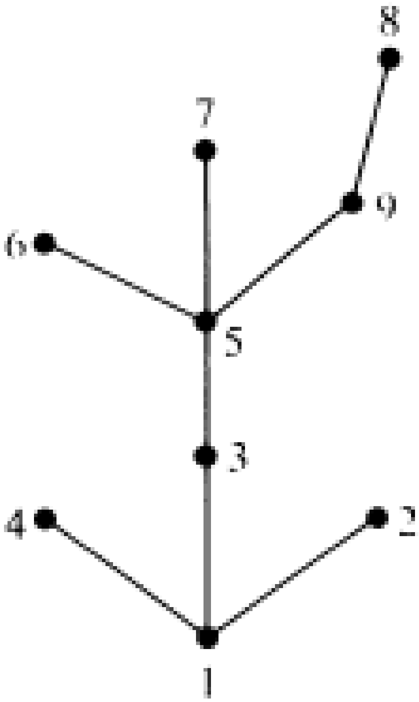


Fig. 10-1 Nine-vertex labeled tree, which yields sequence (1, 1, 3, 5, 5, 5, 9).



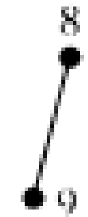
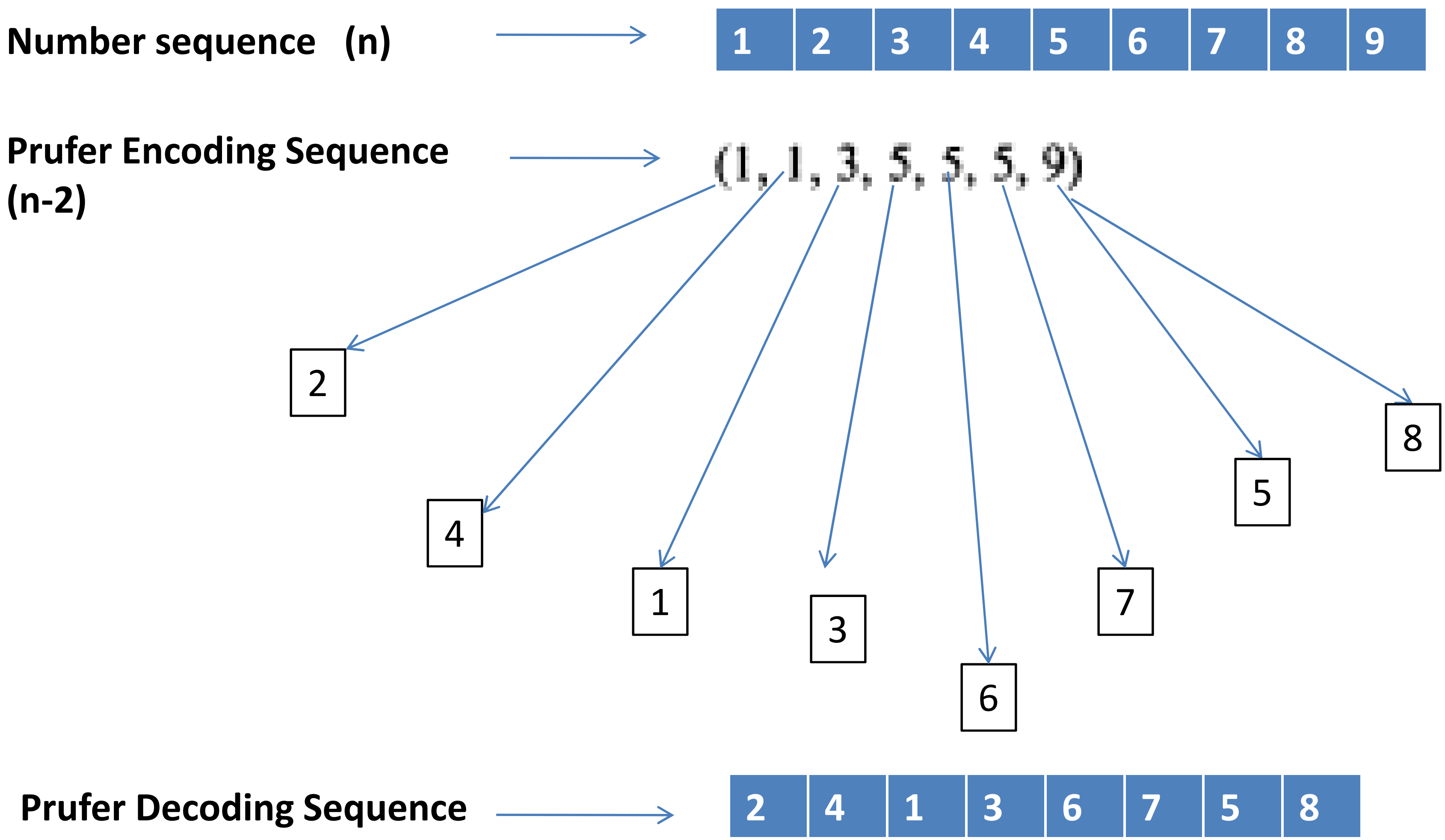


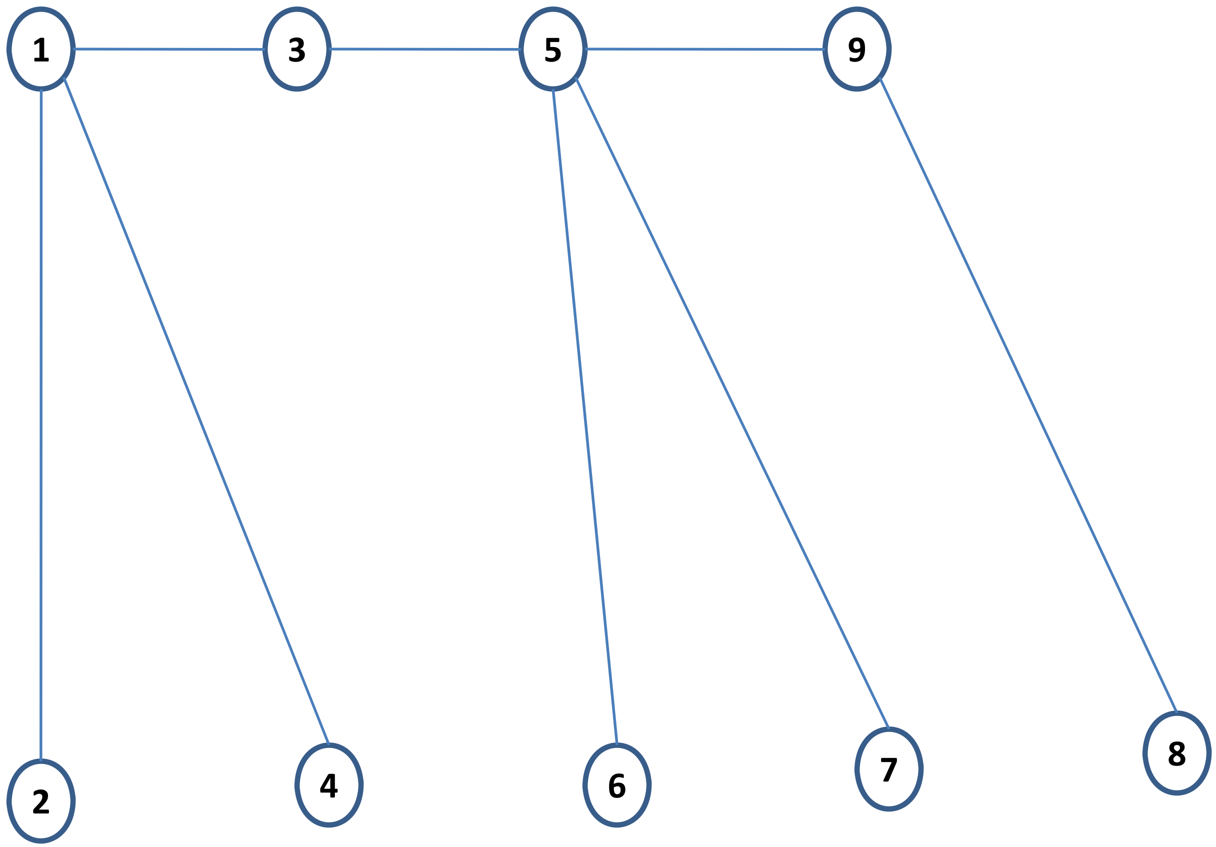
Fig. 10-1 Nine-vertex labeled tree, which yields sequence (1, 1, 3, 5, 5, 5, 9).

This is the Prufer Encoding Sequence


Number sequence (n)
1 2 3 4 5 6 7 8 9
Prufer Encoding Sequence (n-2)
(1, 1, 3, 5, 5, 5, 9)
2
4
1
3
6
7
5
8
Prufer Decoding Sequence
2 4 1 3 6 7 5 8

Prufer Encoding Sequence

1	1	3	5	5	5	9



Prufer Decoding Sequence

2	4	1	3	6	7	5	8

constructed uniquely, as follows: Determine the first number in the sequence

$$1, 2, 3, \ldots, n \tag{10-4}$$

that does not appear in sequence (10-3). This number clearly is a_1. And thus the edge (a_1, b_1) is defined. Remove b_1 from sequence (10-3) and a_1 from (10-4). In the remaining sequence of (10-4) find the first number that does not appear in the remainder of (10-3). This would be a_2, and thus the edge (a_2, b_2) is defined. The construction is continued till the sequence (10-3) has no element left. Finally, the last two vertices remaining in (10-4) are joined. For example, given a sequence

$$(4, 4, 3, 1, 1),$$

we can construct a seven-vertex tree as follows: (2, 4) is the first edge. The second is (5, 4). Next, (4, 3). Then (3, 1), (6, 1), and finally (7, 1), as shown in Fig. 10-2.

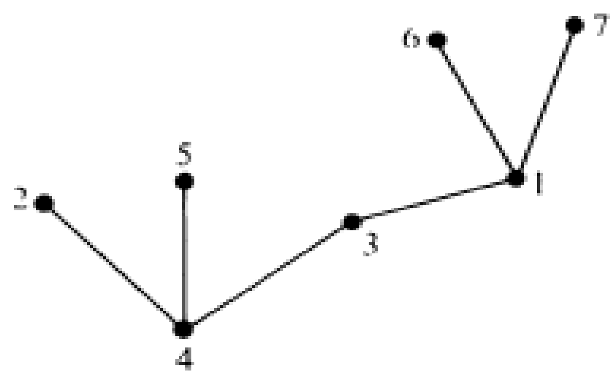


Fig. 10-2 Tree constructed from sequence (4, 4, 3, 1, 1).

For each of the $n - 2$ elements in sequence (10-3) we can choose any one of n numbers, thus forming

$$n^{n-2} \tag{10-5}$$

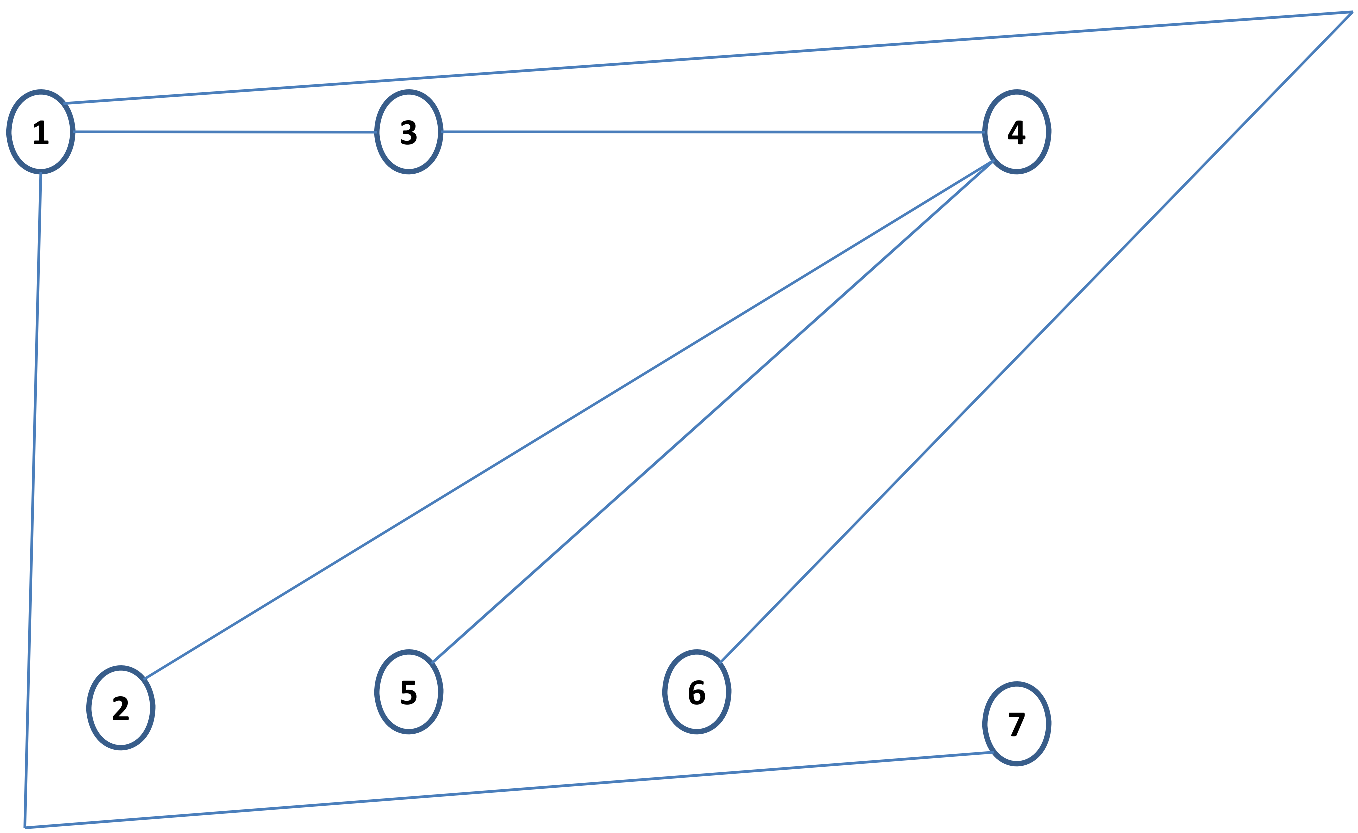
$(n - 2)$-tuples, each defining a distinct labeled tree of n vertices. And since each tree defines one of these sequences uniquely, there is a one-to-one correspondence between the trees and the n^{n-2} sequences. Hence the theorem. ■

Number sequence

1	2	3	4	5	6	7

Prufer Encoding Sequence

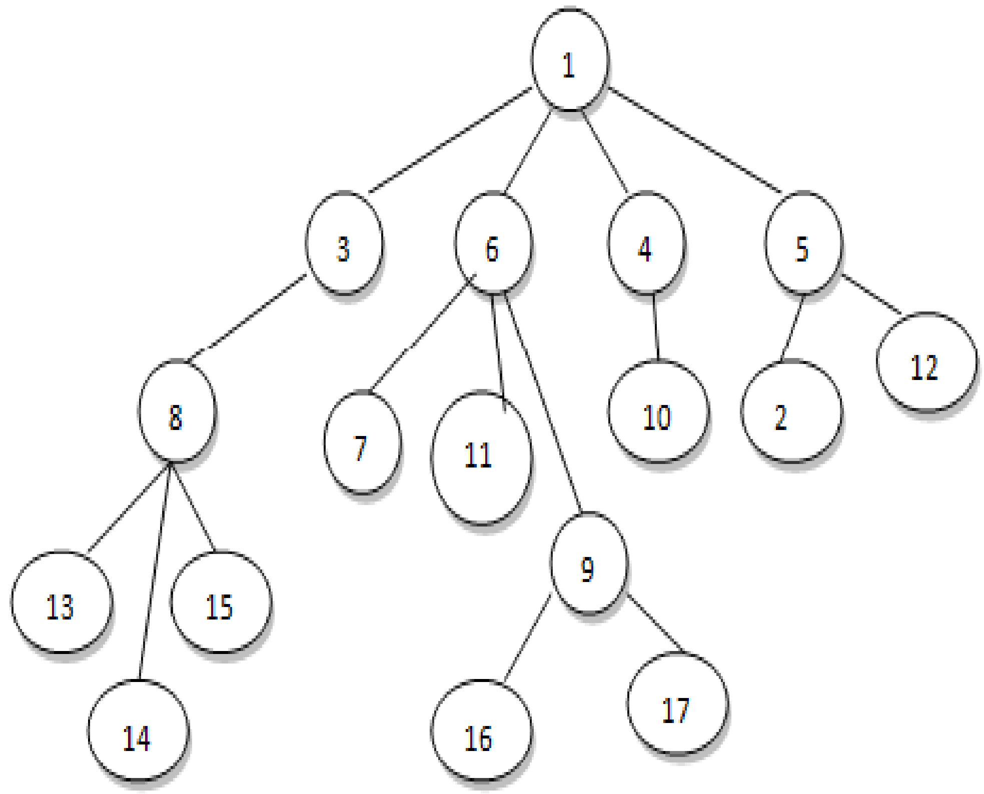
4	4	3	1	1



Prufer Decoding Sequence

2	5	4	3	6	7

Solve



From the given tree, find the Prufer encoding and decoding sequences and from the sequence rebuild the tree.

Solve

From the given sequence, find out the Prufer decoding sequence and reconstruct the original tree.

(i) (3, 3, 11, 11, 8, 8, 8, 9, 10, 12)

(ii) (3, 3, 4, 1, 5, 5, 1, 6, 6, 6, 2, 1, 3, 9, 9)