Engineering Mathematics 19/08/2021

Homework

Q.1. Find the fourier series expansion of the periodic function with period 2π defined as.

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

Calculate the sum of the series at $x = \pm\pi$ and deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots$

Ans: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -1dx + \int_{0}^{\pi} dx\right]$$

$$= \frac{1}{\pi}[2\pi] = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left[-\int_{-\pi}^{0} \cos nx + \int_{0}^{\pi} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right] = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -1 \sin nx\, dx + \int_{0}^{\pi} 1.\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-\frac{\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n} - \frac{\cos n\pi}{n}\right) + \left(\frac{1}{n} - \frac{\cos n\pi}{n}\right)\right]$$

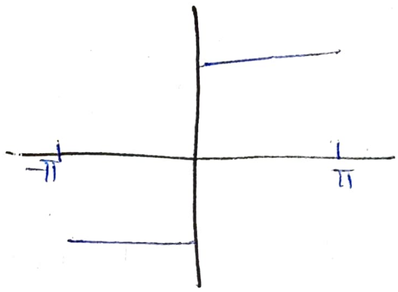
$$= \frac{2}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right]$$

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right] \sin nx$$

$$= \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} \cdots\right]$$

Now, f is discontinuous at $x = 0$

$f(0) = 1$, $f(\frac{\pi}{2}) = 1$



$$1 = \frac{4}{\pi}\left[1 + \left(\frac{-1}{3}\right) + \frac{1}{5} + \cdots\right]$$

$$\rightarrow \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots\cdots$$

Q. Obtain Fourier Series

$f(x) = x\sin x$, $x \in [-\pi, \pi)$

$f(x)' = x\cos x + \sin x$ $\quad > 0$ in $(0, \frac{\pi}{2})$

$\quad < 0$ in $(\frac{\pi}{2}, \pi)$

$f'(-\pi) = \pi$

$f(0) = 0$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x \, dx = \frac{2}{\pi}\int_0^{\pi} x\sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin(2x)\, dx$$

$$= \frac{1}{\pi}\left[\left(\frac{-x\cos 2x}{2}\right)_0^{\pi} + \int_0^{\pi} \frac{\cos 2x}{2}\, dx\right]$$

$$= \frac{1}{\pi} \times \frac{\pi(-1)}{2} = -\frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cdot\frac{1}{2}\left[\sin(x+nx) + \sin(x-n\pi)\right]dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\cdot\left[\sin(1+n)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{\pi}\left[\left(x\cdot\frac{-\cos(1+n)x}{1+n}\right)_0^{\pi} + \left(x\cdot\frac{-\cos(1-n)x}{1-n}\right)_0^{\pi}\right]$$

$$= -\left[\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]$$

$$= -\left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{1-n}}{1-n}\right] \qquad n \neq 1$$

$$= -(-1)^{1-n}\left[\frac{(-1)^{2n}}{1+n} + \frac{1}{1-n}\right]$$

$$= -(-1)^{1-n}\left[\frac{1}{n+1} + \frac{1}{1-n}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\sin nx\,dx \longrightarrow \text{odd function}$$

$$= 0$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x \sum_{n=2}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cos nx$$

$$= 1 - \frac{\cos x}{2} + 2 \sum_{n=2}^{\infty} \frac{(-1)^{1-n}}{n^2 - 1} \cos nx$$

$$= 1 - \frac{\cos x}{2} + 2 \left[-\frac{\cos 2x}{2^2 - 1} + \frac{\cos 3x}{3^2 - 1} - \frac{\cos 4x}{4^2 - 1} + \cdots \right]$$

Ans

—— x ——