Fourier series

Let $f:[-\pi,\pi]\to R$ be continuous and integrable in $[-\pi,\pi]$ or if f is unbounded on $[-\pi,\pi]$; the improper integral $\int_{-\pi}^{\pi} f(x)\,dx$ be absolutely convergent, then the trigonometric series;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called the Fourier series in $[-\pi,\pi]$

where a_0, b_n, a_n are Fourier coefficients.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

(*) Dirichlet's condition :-

→ Two conditions :-

(a) $f(x)$ is bounded periodic with a period 2π and integrable on $[-\pi,\pi]$ and the interval can be broken up into a finite number of open partial intervals in each of which $f(x)$ is monotonic [i.e. $f(x)$ is bounded periodic with period 2π and integrable on $[-\pi,\pi]$ and piecewise monotonic on $[-\pi,\pi]$].

(ii) $f(x)$ has a finite number of points of infinite discontinuity in the interval. When arbitrary small number of these points are excluded, $f(x)$ is bounded in the remainder of the interval and this can be broken up into a finite number of open partial intervals in each of which $f(x)$ is monotonic. Further the improper integral $\int_{-\pi}^{\pi} f(x)\,dx$

is to be absolutely convergent.

Important integrations :-

(i) $$\int_0^{2\pi} \sin nx\,dx = 0 = \int_0^{2\pi} \cos nx\,dx.$$

(ii) $$\int_0^{2\pi} \sin^2 nx\,dx = \int_0^{2\pi} \cos^2 nx\,dx = 0.$$

(iii) $$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases}$$

(iv) $$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases}$$

(v) $$\int_0^{2\pi} \sin nx \cos mx\,dx = \int_0^{2\pi} \cos nx \sin mx\,dx = 0.$$

The integrations above of limit ~~[0, 2π~~ $\int_0^{2\pi}$ are also applicable for $\int_{-\pi}^{\pi}$

(*) Discontinuous function :-

→ At points of discontinuity, Fourier series gives the value of $f(x)$ as the arithmetic mean of the left and right limits.

At the point of discontinuity, $x = c$.

At $x = c$; $f(x) = \frac{1}{2}\left[f(c-0) + f(c+0) \right]$.