Eleven

Partial Differential Equations

CHAPTER OUTLINE

11.1 INTRODUCTION

A Partial Differential Equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables. These equations are used to describe phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics.

11.2 PARTIAL DIFFERENTIAL EQUATIONS

A differential equation containing one or more partial derivatives is known as a *partial differential equation*. Partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are denoted by p, q, r, s, t respectively. The order of a partial differential equation is the order of the highest-order partial derivative present in the equation. The degree of a partial differential equation is the power of the highest order partial derivative present in the equation.

11.3 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

The partial differential equation can be formed using the following methods:

11.3.1 By Elimination of Arbitrary Constants

Let

$$f(x, y, z, a, b) = 0 \qquad ...(11.1)$$

be an equation, where a and b are arbitrary constants (Fig. 11.1).

Differentiating Eq. (11.1) partially w.r.t. x,

$$\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=0$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot p=0 \qquad ...(11.2)$$

Differentiating Eq. (11.1) partially w.r.t. y,

$$\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}=0$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot q=0 \qquad ...(11.3)$$

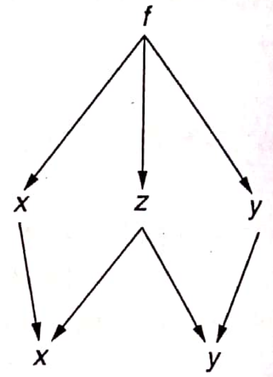

Fig. 11.1 Composite function

By eliminating a, b from Eqs (11.1), (11.2), and (11.3), a partial differential equation of first order is obtained.

Note If the number of arbitrary constants is more than the number of independent variables in Eq.(11.1) then the partial differential equation obtained is of higher order or higher degree (more than one).

EXAMPLE 11.1 *Form a partial differential equation by eliminating the arbitrary constants from the equation $z = ax^2 + by^2$.*

Solution:

$$z = ax^2 + by^2 \qquad ...(1)$$

Differenting Eq. (1) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x}=2ax, \qquad \frac{\partial z}{\partial y}=2by$$

$$p=2ax, \qquad q=2by$$

$$a=\frac{p}{2x}, \qquad b=\frac{q}{2y}$$

Substituting a and b in Eq. (1),

$$z=\frac{p}{2x}x^2+\frac{q}{2y}y^2$$

$$2z=px+qy$$

which is a partial differential equation of first order.

EXAMPLE 11.2 *Form a partial differential equation by eliminating the arbitrary constants from the equation* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Solution:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0, \qquad \frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

$$\frac{x}{a^2}+\frac{z}{c^2}p=0 \quad \text{...(2)}, \qquad \frac{y}{b^2}+\frac{z}{c^2}q=0 \qquad \text{...(3)}$$

Differentiating Eq. (2) partially w.r.t. x,

$$\frac{1}{a^2}+\frac{p}{c^2}\frac{\partial z}{\partial x}+\frac{z}{c^2}\frac{\partial p}{\partial x}=0$$

$$\frac{c^2}{a^2}+p^2+zr=0 \qquad \left[\because \frac{\partial p}{\partial x}=\frac{\partial^2 z}{\partial x^2}=r\right]$$

Substituting $\frac{c^2}{a^2}=-\frac{zp}{x}$ from Eq. (2),

$$-\frac{zp}{x}+p^2+zr=0$$

$$-zp+xp^2+xzr=0$$

which is a partial differential equation of second order.
Similarly, differentiating Eq. (3) partially w.r.t. y,

$$\frac{1}{b^2}+\frac{q}{c^2}\frac{\partial z}{\partial y}+\frac{z}{c^2}\frac{\partial q}{\partial y}=0$$

$$\frac{c^2}{b^2}+q^2+zt=0 \qquad \left[\because \frac{\partial q}{\partial y}=\frac{\partial^2 z}{\partial y^2}=t\right]$$

Substituting $\frac{c^2}{b^2}=-\frac{zq}{y}$ from Eq. (3),

$$-\frac{zq}{y}+q^2+zt=0$$

$$-zq+yq^2+yzt=0$$

which is also a partial differential equation of order two. Hence, two partial differential equations of order two are obtained.

EXAMPLE 11.3 *Find the differential equation of all planes which are at a constant distance 'a' from the origin.*

Solution: The equation of the plane in normal form is

$$lx + my + nz = a \qquad ...(1)$$

where l, m, and n are the direction cosines of the normal from the origin to the plane.

$$l^2 + m^2 + n^2 = 1 \qquad ...(2)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$l + n\frac{\partial z}{\partial x} = 0, \qquad m + n\frac{\partial z}{\partial y} = 0$$

$$l + np = 0, \qquad m + nq = 0$$

$$l = -np \quad ...(3) \qquad m = -nq \quad ...(4)$$

Substituting l and m in Eq. (2),

$$n^2p^2 + n^2q^2 + n^2 = 1$$

$$n^2\left(p^2 + q^2 + 1\right) = 1$$

$$n = \frac{1}{\sqrt{1 + p^2 + q^2}}$$

Substituting l and m from Eqs (3) and (4) in Eq. (1),

$$-npx - nqy + nz = a$$

$$px + qy - z = -\frac{a}{n}$$

$$px + qy - z = -a\sqrt{1 + p^2 + q^2}$$

$$(px + qy - z)^2 = a^2\left(1 + p^2 + q^2\right)$$

which is a partial differential equation of order one and degree two.

11.3.2 By Elimination of Arbitrary Functions

(a) Let the given equation be $z = f(u)$ ...(11.4)
where u is a function of x, y, and z (Fig. 11.2).
Differentiating Eq. (11.4) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x} \qquad ...(11.5)$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y} \qquad ...(11.6)$$

By eliminating the arbitrary function f from Eqs (11.4), (11.5), and (11.6), a partial differential equation of first order is obtained.

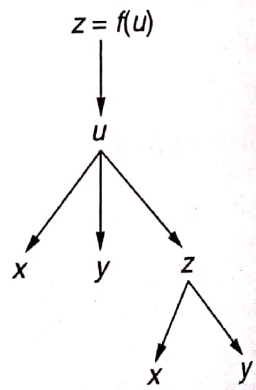

Fig. 11.2 Chain rule

(b) Let the given equation be $F(u, v) = 0$ ...(11.7)
where u and v are functions of x, y, and z.

Differentiating Eq. (11.7) partially w.r.t. x and y,

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0 \quad ...(11.8)$$

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0 \quad ...(11.9)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from Eqs (11.8) and (11.9),

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y} \end{vmatrix}=0$$

Expanding this determinant, a partial differential equation of first order is obtained.

EXAMPLE 11.4 *Form a partial differential equation by eliminating the arbitrary function from the equation* $z = e^{my}\phi(x-y)$.

Solution: $$z = e^{my}\phi(x-y) \quad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = e^{my}\phi'(x-y) \quad ...(2)$$

$$p = e^{my}\phi'(x-y)$$

and
$$\frac{\partial z}{\partial y} = me^{my}\phi(x-y)+\left[e^{my}\phi'(x-y)\right](-1) = me^{my}\phi(x-y)-e^{my}\phi'(x-y)$$

$$q = me^{my}\phi(x-y)-e^{my}\phi'(x-y) = mz-p \quad \text{[Using Eqs (1) and (2)]}$$

$$p+q = mz$$

which is a partial differential equation of first order.

EXAMPLE 11.5 *Form a partial differential equation by eliminating the arbitrary functions from the equation* $z = f(x+ay)+\phi(x-ay)$.

Solution: $$z = f(x+ay)+\phi(x-ay) \quad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = f'(x+ay)+\phi'(x-ay) \quad ...(2)$$

and $$\frac{\partial z}{\partial y} = f'(x+ay)\cdot a + \phi'(x-ay)\cdot(-a) = af'(x+ay) - a\phi'(x-ay) \quad ...(3)$$

Differentiating Eq. (2) partially w.r.t. x and Eq. (3) partially w.r.t. y,

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \quad ...(4)$$

and $$\frac{\partial^2 z}{\partial y^2} = a\ f''(x+ay)\cdot a - a\phi''(x-ay)\cdot(-a)$$

$$= a^2\ f''(x+ay) + a^2\phi''(x-ay) = a^2\left[f''(x+ay) + \phi''(x-ay)\right]$$

$$= a^2 \frac{\partial^2 z}{\partial x^2} \qquad \text{[Using Eq. (4)]}$$

$$t = a^2 r$$

which is a partial differential equation of second order.

EXAMPLE 11.6 *Form a partial differential equation by eliminating the arbitrary function from the equation* $f\left(x+y+z,\ \ x^2+y^2-z^2\right)=0$.

Solution: $$f\left(x+y+z,\ \ x^2+y^2-z^2\right)=0$$

Let $$u = x+y+z, \qquad v = x^2+y^2-z^2$$

$$f(u, v) = 0 \quad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

$$\frac{\partial f}{\partial u}\left[1 + 1\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial f}{\partial v}\left[2x - 2z\frac{\partial z}{\partial x}\right] = 0$$

$$(1+p)\frac{\partial f}{\partial u} + 2(x-zp)\frac{\partial f}{\partial v} = 0 \quad ...(2)$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right] = 0$$

$$\frac{\partial f}{\partial u}\left[1 + 1\cdot\frac{\partial z}{\partial y}\right] + \frac{\partial f}{\partial v}\left[2y - 2z\frac{\partial z}{\partial y}\right] = 0$$

$$(1+q)\frac{\partial f}{\partial u} + 2(y-zq)\frac{\partial f}{\partial v} = 0 \quad ...(3)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from Eqs (2) and (3),

$$\begin{vmatrix} 1+p & 2(x-zp) \\ 1+q & 2(y-zq) \end{vmatrix} = 0$$

$$(1+p)2(y-zq)-(1+q)2(x-zp)=0$$

$$y-zq+yp-zpq-x+zp-xq+zpq=0$$

$$(y+z)p-(x+z)q=x-y$$

which is a partial differential equation of first order.

EXERCISE 11.1

I. Form partial differential equations by eliminating the arbitrary constants.

1. $z = ax + by + ab$

 [**Ans.:** $z = px + qy + pq$]

2. $z = axe^{y} + \frac{1}{2}a^2e^{2y} + b$

 [**Ans.:** $q = xp + p^2$]

3. $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$

 [**Ans.:** $p^2 + q^2 = \tan^2 \alpha$]

4. $(x-h)^2 + (y-k)^2 + z^2 = c^2$

 [**Ans.:** $z^2(p^2+q^2+1) = c^2$]

II. Form partial differential equations by eliminating the arbitrary functions.

1. $z = f\left(\frac{y}{x}\right)$

 [**Ans.:** $xp + yq = 0$]

2. $z = (x+y)\phi(x^2-y^2)$

 [**Ans.:** $yp'' + xq = z$]

3. $z = y^2 + 2f\left(\frac{1}{x} + \log_e y\right)$

 [**Ans.:** $x^2p + yq = 2y^2$]

4. $z = x + y + f(xy)$

 [**Ans.:** $xp - yq = x - y$]

5. $z = f(x) + e^{y}g(x)$

 [**Ans.:** $t = q$]

6. $z = f(x+y)\cdot g(x-y)$

 [**Ans.:** $(r-t)z = (p+q)(p-q)$]

7. $f(xy+z^2, x+y+z) = 0$

 [**Ans.:** $(2z-x)p + (y-2z)q = x - y$]

8. $f(x^2+y^2+z^2,\ z^2-2xy) = 0$

 [**Ans.:** $(x+y)[z(q-p)+(y-x)] = 0$]

11.4 LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A partial differential equation of first order is said to be linear if the dependent variable and its derivatives are of degree one and the products of the dependent variable and its derivatives do not appear in the equation.

The equation is said to be quasi-linear if the degree of highest-order derivative is one and the products of the highest-order partial derivatives are not present. A quasi-linear partial differential equation is represented as

$$P(x,y,z)\cdot p+Q(x,y,z)\cdot q=R(x,y,z)$$

This equation is known as *Lagrange's linear equation*.
If P and Q are independent of z, and R is linear in z then the equation is known as a *linear equation.*

The general solution of lagrange's linear equation $Pp + Qq = R$ is given by

$$f(u, v) = 0$$

where f is an arbitrary function and u, v are functions of x, y, and z.

Working Rule for Solving Lagrange's Linear Equation

1. Write the given differential equation in the standard form $Pp + Qq = R$
2. Form the Lagrange's auxiliary (subsidiary) equation

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \text{...(11.10)}$$

3. Solve the simultaneous equations in Eq. (11.10) to obtain its two independent solutions as $u=c_1$, $v=c_2$.
4. Write the general solution of the given equation as

$$f(u, v) = 0 \text{ or } u = \phi(v)$$

EXAMPLE 11.7 *Solve* $yzp - xzq = xy$.

Solution: $P=yz,\quad Q=-xz,\quad R=xy$

The auxiliary equations are

$$\frac{dx}{yz}=\frac{dy}{-xz}=\frac{dz}{xy} \qquad \text{...(1)}$$

Taking first and second fractions in Eq. (1),

$$\frac{dx}{yz}=\frac{dy}{-xz}$$

$$xdx + ydy = 0$$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}=c$$

$$x^2+y^2=c_1 \qquad \text{...(2)}$$

where $c_1 = 2c$

Taking second and third fractions in Eq. (1),

$$\frac{dy}{-xz} = \frac{dz}{xy}$$

$$ydy + zdz = 0$$

Integrating,

$$\frac{y^2}{2} + \frac{z^2}{2} = c'$$

$$y^2 + z^2 = c_2 \quad \text{...(3)}$$

where $c_2 = 2c'$

From Eqs (2) and (3), the general solution is

$$f(x^2 + y^2, y^2 + z^2) = 0$$

EXAMPLE 11.8 *Solve* $z(z^2 + xy)(px - qy) = x^4$.

Solution: Rewriting the equation in the $Pp + Qq = R$ form,

$$xz(z^2 + xy)p - yz(z^2 + xy)q = x^4$$

$$P = xz(z^2 + xy), \quad Q = -yz(z^2 + xy), \quad R = x^4$$

The auxiliary equations are

$$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4} \quad \text{...(1)}$$

Taking first and second fractions from Eq. (1),

$$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)}$$

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating,

$$\log x + \log y = \log c_1$$

$$\log xy = \log c_1$$

$$xy = c_1 \quad \text{...(2)}$$

Taking first and third fractions in Eq. (1),

$$\frac{dx}{xz(z^2 + xy)} = \frac{dz}{x^4}$$

$$x^3 dx = z(z^2 + xy)dz$$

Putting $xy = c_1$ from Eq. (2),

$$x^3 dx = z\left(z^2 + c_1\right)dz$$

$$x^3 dx - \left(z^3 + c_1 z\right)dz = 0$$

Integrating,

$$\frac{x^4}{4} - \frac{z^4}{4} - c_1\frac{z^2}{2} = c$$

$$x^4 - z^4 - 2c_1\ z^2 = 4c = c_2,\ \text{where}\ \ 4c = c_2$$

Putting $c_1 = xy$ from Eq. (2),

$$x^4 - z^4 - 2xyz^2 = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left(xy,\ x^4 - z^4 - 2xyz^2\right) = 0$$

EXAMPLE 11.9 *Solve* $x\left(y^2 + z\right)p - y\left(x^2 + z\right)q = z\left(x^2 - y^2\right)$.

Solution: $P = x\left(y^2 + z\right),\quad Q = -y\left(x^2 + z\right),\quad R = z\left(x^2 - y^2\right)$

The auxiliary equation is

$$\frac{dx}{x\left(y^2 + z\right)} = \frac{dy}{-y\left(x^2 + z\right)} = \frac{dz}{z\left(x^2 - y^2\right)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\frac{dx}{x\left(y^2 + z\right)} = \frac{dy}{-y\left(x^2 + z\right)} = \frac{dz}{z\left(x^2 - y^2\right)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\left(y^2 + z\right) - \left(x^2 + z\right) + \left(x^2 - y^2\right)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating,

$$\log x + \log y + \log z = \log c_1$$

$$\log xyz = \log c_1$$

$$xyz = c_1 \qquad \text{...(1)}$$

Taking $x, y, -1$ as multipliers,

$$\frac{dx}{x\left(y^2 + z\right)} = \frac{dy}{-y\left(x^2 + z\right)} = \frac{dz}{z\left(x^2 - y^2\right)} = \frac{xdx + ydy - dz}{x^2\left(y^2 + z\right) - y^2\left(x^2 + z\right) - z\left(x^2 - y^2\right)} = \frac{xdx + ydy - dz}{0}$$

$$xdx + ydy - dz = 0$$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}-z=c$$

$$x^2+y^2-2z=2c=c_2 \qquad \text{...(2)}$$

From Eqs (1) and (2), the general solution is

$$f\left(xyz,\ x^2+y^2-2z\right)=0$$

EXAMPLE 11.10 *Solve* $\left(x^2-yz\right)p+\left(y^2-zx\right)q=z^2-xy.$

Solution: $P=x^2-yz,\quad Q=y^2-zx,\quad R=z^2-xy$

The auxiliary equation is

$$\frac{\mathrm{d}x}{x^2-yz}=\frac{\mathrm{d}y}{y^2-zx}=\frac{\mathrm{d}z}{z^2-xy}$$

Each of these fractions is equal to

$$\frac{\mathrm{d}x-\mathrm{d}y}{\left(x^2-yz\right)-\left(y^2-zx\right)}=\frac{\mathrm{d}y-\mathrm{d}z}{\left(y^2-zx\right)-\left(z^2-xy\right)}=\frac{\mathrm{d}z-\mathrm{d}x}{\left(z^2-xy\right)-\left(x^2-yz\right)}$$

$$\frac{\mathrm{d}x-\mathrm{d}y}{\left(x^2-y^2\right)+z\left(x-y\right)}=\frac{\mathrm{d}y-\mathrm{d}z}{\left(y^2-z^2\right)+x\left(y-z\right)}=\frac{\mathrm{d}z-\mathrm{d}x}{\left(z^2-x^2\right)+y\left(z-x\right)}$$

$$\frac{\mathrm{d}x-\mathrm{d}y}{\left(x-y\right)\left(x+y+z\right)}=\frac{\mathrm{d}y-\mathrm{d}z}{\left(y-z\right)\left(y+z+x\right)}=\frac{\mathrm{d}z-\mathrm{d}x}{\left(z-x\right)\left(z+x+y\right)} \qquad \text{...(1)}$$

Taking first and second fractions from Eq. (1),

$$\frac{\mathrm{d}x-\mathrm{d}y}{x-y}=\frac{\mathrm{d}y-\mathrm{d}z}{y-z}$$

$$\frac{\mathrm{d}\left(x-y\right)}{x-y}-\frac{\mathrm{d}\left(y-z\right)}{y-z}=0$$

Integrating,

$$\log\left(x-y\right)-\log\left(y-z\right)=\log c_1$$

$$\log\left(\frac{x-y}{y-z}\right)=\log c_1$$

$$\frac{x-y}{y-z}=c_1 \qquad \text{...(2)}$$

Taking second and third fractions from Eq. (1),

$$\frac{dy - dz}{y - z} = \frac{dz - dx}{z - x}$$

$$\frac{d(y - z)}{y - z} - \frac{d(z - x)}{z - x} = 0$$

Integrating,

$$\log(y - z) - \log(z - x) = \log c_2$$

$$\log\left(\frac{y - z}{z - x}\right) = \log c_2$$

$$\frac{y - z}{z - x} = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0$$

EXAMPLE 11.11 *Solve* $z - xp - yq = a\sqrt{x^2 + y^2 + z^2}$.

Solution: Rewriting the equation in $Pp + Qq = R$ form,

$$xp + yq = z - a\sqrt{x^2 + y^2 + z^2}$$

$$P = x,\ Q = y,\ R = z - a\sqrt{x^2 + y^2 + z^2}$$

The auxiliary equation is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}} = \frac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - za\sqrt{x^2 + y^2 + z^2}} \qquad \text{...(1)}$$

Let $x^2 + y^2 + z^2 = u^2$ ...(2)

Differentiating Eq. (2),

$$2xdx + 2ydy + 2zdz = 2udu$$

$$xdx + ydy + zdz = udu$$

Substituting in Eq. (1),

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - au} = \frac{udu}{u^2 - azu}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - au} = \frac{du}{u - az} = \frac{dz + du}{(z - au) + (u - az)}$$

$$= \frac{dz + du}{(z+u) - a(u+z)} = \frac{dz + du}{(z+u)(1-a)}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz + du}{(1-a)(z+u)} \qquad \text{...(3)}$$

Taking first and second fractions in Eq. (3),

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating,

$$\log x - \log y = \log c_1$$

$$\log \frac{x}{y} = \log c_1$$

$$\frac{x}{y} = c_1 \qquad \text{...(4)}$$

Taking first and third fractions in Eq (3),

$$\frac{dx}{x} = \frac{dz + du}{(1-a)(z+u)}$$

$$(1-a)\frac{dx}{x} - \frac{d(z+u)}{z+u} = 0$$

Integrating,

$$(1-a)\log x - \log(z+u) = \log c_2$$

$$\log\left(\frac{x^{1-a}}{z+u}\right) = \log c_2$$

$$\frac{x^{1-a}}{z+u} = c_2$$

$$\frac{x^{1-a}}{z + \sqrt{x^2 + y^2 + z^2}} = c_2 \qquad \text{...(5)}$$

From Eqs (4) and (5), the general solution is

$$f\left(\frac{x}{y}, \frac{x^{1-a}}{z + \sqrt{x^2 + y^2 + z^2}}\right) = 0$$

EXERCISE 11.2

Solve the following:

1. $\dfrac{y^2zp}{x}+zxq=y^2$

$\left[\textbf{Ans.: } f\left(x^3-y^3,\ x^2-y^2\right)=0\right]$

2. $p-q=\log(x+y)$

$\left[\textbf{Ans.: } f\left[x+y,\ x\log(x+y)-z\right]=0\right]$

3. $xzp+yzq=xy$

$\left[\textbf{Ans.: } f\left(\dfrac{x}{y},\ xy-z^2\right)=0\right]$

4. $\left(y^2+z^2\right)p-xyq+zx=0$

$\left[\textbf{Ans.: } f\left(\dfrac{y}{z},\ x^2+y^2+z^2\right)=0\right]$

5. $p+3q=5z+\tan(y-3x)$

$\left[\textbf{Ans.: } f\left(y-3x,\ e^{-5x}\left\{5z+\tan(y-3x)\right\}\right)=0\right]$

6. $\left(x^2-y^2-z^2\right)p+2xyq=2xz$

$\left[\textbf{Ans.: } f\left(x^2+y^2+z^2,\ \dfrac{y}{z}\right)=0\right]$

7. $(y+z)p+(z+x)q=x+y$

$\left[\textbf{Ans.: } f\left(\dfrac{x-y}{y-z},\ \dfrac{y-z}{\sqrt{x+y+z}}\right)=0\right]$

8. $\left(z^2-2yz-y^2\right)p+(xy+zx)q=xy-zx$

$\left[\textbf{Ans.: } f\left(x^2+y^2+z^2,\ y^2-2yz-z^2\right)=0\right]$

9. $\dfrac{(y-z)p}{yz}+\dfrac{(z-x)q}{zx}=\dfrac{x-y}{xy}$

$\left[\textbf{Ans.: } f\left(x+y+z,\ xyz\right)=0\right]$

10. $x^2(y-z)p+(z-x)y^2q=z^2(x-y)$

$\left[\textbf{Ans.: } f\left(xyz,\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)=0\right]$

11. $p-2q=3x^2\sin(y+2x)$

$\left[\textbf{Ans.: } f\left(2x+y,\ x^3\sin(y+2x)-z\right)=0\right]$

12. $p\tan x+q\tan y=\tan z$

$\left[\textbf{Ans.: } f\left(\dfrac{\sin z}{\sin y},\ \dfrac{\sin x}{\sin y}\right)=0\right]$

13. $(mz-ny)p+(nx-lz)q=ly-mx$

$\left[\textbf{Ans.: } f\left(x^2+y^2+z^2,\ lx+my+nz\right)\right]$

14. $z(p-q)=z^2+(x+y)^2$

$\left[\textbf{Ans.: } f\left[x+y,\ e^{2y}\left\{z^2+(x+y)^2\right\}\right]=0\right]$

11.5 NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A partial differential equation of first order is said to be nonlinear if p and q have degree more than one. The complete solution of a nonlinear equation is given by

$$f(x,\ y,\ z,\ a,\ b)=0$$

where a and b are two arbitrary constants. Four standard forms of these equations are as follows:

11.5.1 Form I $f(p, q) = 0$

Let the equation be

$$f(p, q) = 0 \quad \text{...(11.11)}$$

Assuming $p = a$, Eq (11.11) reduces to

$$f(a, q) = 0$$

Solving for q,

$$q = \phi(a)$$

Now,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{dy}dy = pdx + qdy$$

$$dz = adx + qdy$$

Integrating,

$$z = ax + qy + c = ax + \phi(a)y + c$$

where a and c are arbitrary constants.

Hence, the complete solution is

$$z = ax + by + c$$

where $b = \phi(a)$, i.e., a and b satisfy the equation $f(a, b) = 0$.

EXAMPLE 11.12 *Solve* $p^2 + q^2 = 1$.

Solution: The equation is of the form $f(p, q) = 0$

$$f(p, q) = p^2 + q^2 - 1$$

The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation

$$f(a,b) = 0$$

$$a^2 + b^2 - 1 = 0$$

$$b = \sqrt{1 - a^2}$$

Hence, the complete solution is

$$z = ax + \sqrt{1 - a^2}\,y + c$$

EXAMPLE 11.13 *Solve* $x^2p^2 + y^2q^2 = z^2$.

Solution: Rewriting the equation,

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{...(1)}$$

Let

$$\frac{dx}{x} = dX \quad , \quad \frac{dy}{y} = dY \; , \quad \frac{dz}{z} = dZ$$

$$\log x = X \quad , \quad \log y = Y \quad , \quad \log z = Z$$

Differentiating $\log z = Z$ partially w.r.t. x,

$$\frac{1}{z}\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial X}\cdot\frac{\partial X}{\partial x} = \frac{\partial Z}{\partial X}\cdot\frac{1}{x}$$

$$\frac{x}{z}\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial X}$$

Similarly, differentiating $\log z = Z$ partially w.r.t. y,

$$\frac{y}{z}\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial Y}$$

Substituting in Eq. (1),

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$$

$$P^2 + Q^2 = 1$$

The equation is of the form $f(P, Q) = 0$.

$$f(P, Q) = P^2 + Q^2 - 1$$

The complete solution is

$$Z = aX + bY + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a^2 + b^2 - 1 = 0$$

$$b = \sqrt{1 - a^2}$$

Hence, the complete solution is

$$Z = aX + \sqrt{1-a^2}\,Y + c$$

$$\log z = a\log x + \sqrt{1-a^2}\,\log y + c$$

EXAMPLE 11.14 *Solve* $(x^2 + y^2)(p^2 + q^2) = 1$.

Solution: Let $x = r\cos\theta$, $y = r\sin\theta$

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$2r\frac{\partial r}{\partial x} = 2x, \qquad \frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \qquad = -\frac{\sin\theta}{r}$$

and $$\frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta \quad \text{and} \quad \frac{\partial \theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}$$

and $$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\frac{\partial z}{\partial r}\cos\theta+\frac{\partial z}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}=\frac{\partial z}{\partial r}\sin\theta+\frac{\partial z}{\partial \theta}\left(\frac{\cos\theta}{r}\right)$$

$$p^2+q^2=\left[\frac{\partial z}{\partial r}\cos\theta-\frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right]^2+\left[\frac{\partial z}{\partial r}\sin\theta+\frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right]^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Substituting in the given equation,

$$(x^2+y^2)(p^2+q^2)=1$$

$$r^2\left[\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2\right]=1$$

$$\left(r\frac{\partial z}{\partial r}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=1 \qquad ...(1)$$

Let $$\frac{dr}{r}=dR$$

$$\log r=R$$

Differentiating partially w.r.t. z,

$$\frac{1}{r}\frac{\partial r}{\partial z}=\frac{\partial R}{\partial z}$$

$$r\frac{\partial z}{\partial r}=\frac{\partial z}{\partial R}$$

Substituting in Eq. (1),

$$\left(\frac{\partial z}{\partial R}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=1$$

$$P^2+Q^2=1$$

The equation is of the form $f(P, Q)=0$.

$$f(P, Q)=P^2+Q^2-1$$

The complete solution is

$$z=aR+b\theta+c$$

where a, b satisfy the equation

$$f(a,b)=0$$

$$a^2+b^2-1=0$$

$$b=\sqrt{1-a^2}$$

Hence, the complete solution is

$$z = aR + \sqrt{1-a^2}\,\theta + c = a\log r + \sqrt{1-a^2}\,\theta + c = a\log\sqrt{x^2+y^2} + \sqrt{1-a^2}\tan^{-1}\left(\frac{y}{x}\right) + c$$

11.5.2 Form II $f(z, p, q) = 0$

Let the equation be

$$f(z, p, q) = 0 \qquad \text{...(11.12)}$$

Assuming $q = ap$, Eq. (11.12) reduces to

$$f(z, p, ap) = 0$$

Solving for p,

$$p = \phi(z)$$

Now, $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy = pdx + apdy = p(dx + ady) = \phi(z)(dx + ady)$

Integrating,

$$\int \frac{dz}{\phi(z)} = x + ay + b$$

which gives the complete solution of Eq. (11.12), where a and b are arbitrary constants.

EXAMPLE 11.15 *Solve $z^2(p^2z^2 + q^2) = 1$.*

Solution:

$$z^2(p^2z^2 + q^2) = 1 \qquad \text{...(1)}$$

Putting $q = ap$, Eq. (1) reduces to

$$z^2(p^2z^2 + a^2p^2) = 1$$

$$p^2 = \frac{1}{z^2(z^2+a^2)}$$

$$p = \frac{1}{z\sqrt{z^2+a^2}}$$

Now,
$$dz = pdx + qdy = pdx + apdy = p(dx + ady) = \frac{1}{z\sqrt{z^2+a^2}}(dx + ady)$$

$$z\sqrt{z^2+a^2}\,dz = dx + ady$$

Integrating,

$$\int \left(z^2 + a^2\right)^{\frac{1}{2}} \frac{2z}{2} \mathrm{d}z = x + ay + b$$

$$\frac{1}{2} \frac{\left(z^2 + a^2\right)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} = x + ay + b \qquad \left[\because \int \left[f(x)\right]^n f'(x)\mathrm{d}x = \frac{\left[f(x)\right]^{n+1}}{n+1}\right]$$

$$\left(z^2 + a^2\right)^{\frac{3}{2}} = 3\left(x + ay + b\right)$$

$$\left(z^2 + a^2\right)^3 = 9\left(x + ay + b\right)^2$$

which gives the complete solution of the given equation.

EXAMPLE 11.16 *Solve* $q^2y^2 = z(z - px)$.

Solution: Rewriting the equation,

$$\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right) \qquad ...(1)$$

Let

$$\frac{\mathrm{d}x}{x} = \mathrm{d}X, \qquad \frac{\mathrm{d}y}{y} = \mathrm{d}Y$$

$$\log x = X, \qquad \log y = Y$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{dX}{dx} = \frac{\partial z}{\partial X} \cdot \frac{1}{x}$$

$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} = P, \text{ say}$$

Also,

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{dY}{dy} = \frac{\partial z}{\partial Y} \cdot \frac{1}{y}$$

$$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} = Q, \text{ say}$$

Substituting in Eq. (1),

$$Q^2 = z(z - P) \qquad ...(2)$$

The equation is of the form $f(z, P, Q) = 0$.

Putting $Q = a\,P$ in Eq. (2),

$$a^2P^2 = z\left(z - P\right)$$

$$a^2P^2 + zP - z^2 = 0$$

$$P = \frac{-z \pm \sqrt{z^2 + 4a^2z^2}}{2a^2} = \frac{z}{2a^2}\left[-1 \pm \sqrt{1 + 4a^2}\right] = Az, \quad \text{where} \quad A = \frac{-1 \pm \sqrt{1 + 4a^2}}{2a^2}$$

Now, $$dz = \frac{\partial z}{\partial X} dX + \frac{\partial z}{\partial Y} dY = PdX + QdY = PdX + aPdY = P(dX + adY) = Az(dX + adY)$$

$$\frac{dz}{Az} = dX + adY$$

Integrating,

$$\frac{1}{A}\log z = X + aY + \log b = \log x + a\log y + \log b$$

$$\log z^{\frac{1}{A}} = \log xy^a b$$

$$z^{\frac{1}{A}} = bxy^a$$

which gives the complete solution of the given equation.

11.5.3 Form III $f(x, p) = g(y, q)$

Let the equation be $$f(x, p) = g(y, q) \qquad \text{...(11.13)}$$

Let $$f(x, p) = a, \quad g(y, q) = a$$

Solving these equations for p and q,

$$p = f_1(x), \quad q = g_1(y)$$

Now, $$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = pdx + qdy = f_1(x)dx + g_1(y)dy$$

Integrating,

$$z = \int f_1(x)dx + \int g_1(y)dy + b$$

which gives the complete solution of Eq. (11.13).

EXAMPLE 11.17 *Solve* $yp = 2yx + \log q$.

Solution: Dividing the equation by y,

$$p = 2x + \frac{1}{y}\log q$$

$$(p - 2x) = \frac{1}{y}\log q$$

Let

$$p - 2x = a, \qquad \frac{1}{y}\log q = a$$

$$p = 2x + a, \qquad \log q = ay$$

$$q = e^{ay}$$

Now,

$$dz = p\,dx + q\,dy = (2x + a)\,dx + e^{ay}\,dy$$

Integrating,

$$z = x^2 + ax + \frac{e^{ay}}{a} + b$$

$$az = ax^2 + a^2x + e^{ay} + ab$$

which gives the complete solution of the given equation.

EXAMPLE 11.18 *Solve* $zpy^2 = x(y^2 + z^2q^2)$.

Solution:

$$zpy^2 = x(y^2 + z^2q^2) \qquad \text{...(1)}$$

Let

$$z\,dz = dZ$$

$$\frac{z^2}{2} = Z \qquad \text{...(2)}$$

Differentiating Eq. (2) partially w.r.t. x,

$$z\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = P, \quad \text{say}$$

$$zp = P$$

and

$$z\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial y} = Q, \quad \text{say}$$

$$zq = Q$$

Substituting in Eq. (1),

$$Py^2 = x\left(y^2 + Q^2\right)$$

$$\frac{P}{x} = \frac{Q^2 + y^2}{y^2}$$

The equation is in the form $f(x, P) = g(y, Q)$.

Let

$$\frac{P}{x} = a, \qquad \frac{Q^2 + y^2}{y^2} = a$$

$$P = ax, \qquad Q = y\sqrt{a-1}$$

Now,
$$dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$$
$$zdz = Pdx + Qdy = axdx + y\sqrt{a-1}\,dy$$

Integrating,
$$\frac{z^2}{2} = a\frac{x^2}{2} + \frac{y^2}{2}\sqrt{a-1} + \frac{b}{2}$$
$$z^2 = ax^2 + y^2\sqrt{a-1} + b$$

which gives the complete solution of the given equation.

EXAMPLE 11.19 *Solve* $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$.

Solution:
$$(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1 \qquad ...(1)$$

Let
$$u = x+y \quad , \quad v = x-y$$

Considering z as a function of u and v (Fig. 11.3),

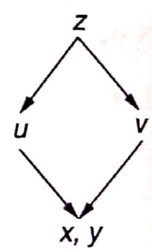

Fig. 11.3

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$p = \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1) = P + Q, \text{ say}$$

where $P = \frac{\partial z}{\partial u}, Q = \frac{\partial z}{\partial v}$

Also,
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$
$$q = \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(-1) = P - Q$$

Substituting in Eq. (1),
$$u(2P)^2 + v(2Q)^2 = 1$$
$$4P^2u = 1 - 4Q^2v$$

The equation is in the form $f(u, P) = g(v, Q)$.

Let $\quad 4P^2u = a, \qquad 1 - 4Q^2v = a$
$$P = \frac{\sqrt{a}}{2\sqrt{u}}, \qquad Q = \frac{1}{2}\sqrt{\frac{1-a}{v}}$$

Now, $$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv = Pdu + Qdv = \frac{\sqrt{a}}{2\sqrt{u}}du + \frac{1}{2}\sqrt{\frac{1-a}{v}}dv$$

Integrating,

$$\int dz = \frac{\sqrt{a}}{2}\int u^{-\frac{1}{2}}du + \frac{\sqrt{1-a}}{2}\int v^{-\frac{1}{2}}dv + b$$

$$z = \frac{\sqrt{a}}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + \frac{\sqrt{1-a}}{2}\left(\frac{v^{\frac{1}{2}}}{\frac{1}{2}}\right) + b = \sqrt{au} + \sqrt{1-a}\sqrt{v} + b = \sqrt{a(x+y)} + \sqrt{(1-a)(x-y)} + b$$

which gives the complete solution of the given equation.

11.5.4 Form IV (Clairaut's Equation)

Let the equation be $$z = px + qy + f(p, q)$$

The complete solution of this equation is

$$z = ax + by + f(a, b) \qquad ...(11.14)$$

which is obtained by replacing p by a and q by b in Eq. (11.14).

EXAMPLE 11.20 *Solve* $z = px + qy + c\sqrt{1+p^2+q^2}$.

Solution: The complete solution of this equation is obtained by replacing p by a and q by b in the given equation.

$$z = ax + by + c\sqrt{1+a^2+b^2}$$

EXAMPLE 11.21 *Solve* $(p-q)(z-px-qy) = 1$.

Solution: Rewriting the given equation in Clairaut's form,

$$z - px - qy = \frac{1}{p-q}$$

$$z = px + qy + \frac{1}{p-q}$$

The complete solution is

$$z = ax + by + \frac{1}{a-b}$$

EXERCISE 11.3

Find the complete solutions of the following equations:

Form I

1. $q=3p^2$

$$\left[\textbf{Ans.: } z=ax+3a^2y+c\right]$$

2. $p^2-q^2=4$

$$\left[\textbf{Ans.: } z=ax+y\sqrt{a^2-4}+c\right]$$

3. $p+q=pq$

$$\left[\textbf{Ans.: } z=ax+\frac{ay}{a-1}+c\right]$$

4. $p=e^q$

$$[\textbf{Ans.: } z=ax+y\log a+c]$$

5. $(y-x)(qy-px)=(p-q)^2$

$$\left[\textbf{Ans.: } z=b^2(x+y)+bxy+c\right]$$

Form II

1. $p(1+q)=qz$

$$\left[\textbf{Ans.: } \log(az-1)=x+ay+b\right]$$

2. $p^3+q^3=27z$

$$\left[\textbf{Ans.: } (1+a^3)z^2=8(x+ay+b)^3\right]$$

3. $p(1+q^2)=q(z-k)$

$$\left[\textbf{Ans.: } 4a(z-k)=4+(x+ay+c)^2\right]$$

4. $z^2(p^2x^2+q^2)=1$

$$\left[\textbf{Ans.: } z^2\sqrt{1+a^2}=\pm 2(\log x+ay)+b\right]$$

5. $pq=x^my^nz^l$

$$\left[\textbf{Ans.: } \frac{z^{-\frac{l}{2}+1}}{-\frac{l}{2}+1}=\frac{1}{\sqrt{a}}\left(\frac{x^{m+1}}{m+1}+a\frac{y^{n+1}}{n+1}\right)+b\right]$$

Form III

1. $\sqrt{p}+\sqrt{q}=2x$

$$\left[\textbf{Ans.: } z=\frac{1}{6}(a+2x)^3+a^2y+b\right]$$

2. $q=xyp^2$

$$\left[\textbf{Ans.: } 16ax=\left(2z-ay^2-2b\right)^2\right]$$

3. $z(p^2-q^2)=x-y$

$$\left[\textbf{Ans.: } z^{\frac{3}{2}}=(x+a)^{\frac{3}{2}}+(y+a)^{\frac{3}{2}}+c\right]$$

4. $p+q=\sin x+\sin y$

$$[\textbf{Ans.: } z=ax-\cos x-\cos y-ay+b]$$

5. $y^2q^2-xp+1=0$

$$\left[\textbf{Ans.: } z=(a^2+1)\log x+a\log y+b\right]$$

Form IV

1. $z=px+qy-p^2q$

$$\left[\textbf{Ans.: } z=ax+by-a^2b\right]$$

2. $z=px+qy-pq$

$$[\textbf{Ans.: } z=ax+by-ab]$$

3. $pqz=p^2(xq+p^2)+q^2(yp+q^2)$

$$\left[\textbf{Ans.: } z=ax+by+\left(\frac{a^3}{b}+\frac{b^3}{a}\right)\right]$$

4. $(px+qy-z)^2=d(1+p^2+q^2)$

$$\left[\textbf{Ans.: } z=ax+by\pm d\sqrt{1+a^2+b^2}\right]$$

5. $4xyz=pq+2px^2y+2qxy^2$

$$\left[\textbf{Ans.: } z=ax^2+by^2+ab\right]$$

11.6 CHARPIT'S METHOD

This method is a general method to find the complete solution of a first-order nonlinear partial differential equation. This method is applied to solve those equations that cannot be reduced to any of the standard forms.

Let the given equation be

$$f(x, y, z, p, q) = 0 \qquad \text{...(11.15)}$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p dx + q dy \qquad \text{...(11.16)}$$

To integrate Eq. (11.16), p and q must be in terms of x, y, and z. For this purpose, let us assume another relation in x, y, z, p, and q as

$$g(x, y, z, p, q) = 0 \qquad \text{...(11.17)}$$

p and q are obtained on solving Eqs (11.15) and (11.17). Substituting p and q in Eq. (11.16) and then integrating the equation, the complete solution of Eq. (11.15) is obtained.

To determine g, differentiating Eqs (11.15) and (11.17) partially w.r.t. x and y,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} = 0 \qquad \text{...(11.18)}$$

and

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}p + \frac{\partial g}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial x} = 0 \qquad \text{...(11.19)}$$

Also,

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y} = 0 \qquad \text{...(11.20)}$$

$$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}q + \frac{\partial g}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial y} = 0 \qquad \text{...(11.21)}$$

Eliminating $\frac{\partial p}{\partial x}$ from Eqs (11.18) and (11.19), by multiplying Eq. (11.18) with $\frac{\partial g}{\partial p}$ and Eq. (11.19) with $\frac{\partial f}{\partial p}$ and subtracting,

$$\left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial p} - \frac{\partial g}{\partial x}\cdot\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial p} - \frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\cdot\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \qquad \text{...(11.22)}$$

Similarly, eliminating $\frac{\partial q}{\partial y}$ from Eqs (11.20) and (11.21),

$$\left(\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial q} - \frac{\partial g}{\partial y}\cdot\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial q} - \frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\cdot\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \qquad \text{...(11.23)}$$

Adding Eqs (11.22) and (11.23) and using $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$,

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial g}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial y}=0$$

$$f_p\frac{\partial g}{\partial x}+f_q\frac{\partial g}{\partial y}+\left(pf_p+qf_q\right)\frac{\partial g}{\partial z}-\left(f_x+pf_z\right)\frac{\partial g}{\partial p}-\left(f_y+qf_z\right)\frac{\partial g}{\partial q}=0 \qquad \text{...(11.24)}$$

where $$f_p=\frac{\partial f}{\partial p},\ f_q=\frac{\partial f}{\partial q},\ f_x=\frac{\partial f}{\partial x},\ f_y=\frac{\partial f}{\partial y},\ f_z=\frac{\partial f}{\partial z}$$

Equation (11.24) is Lagrange's linear partial differential equation in g. Its subsidiary equations are

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-\left(f_x+pf_z\right)}=\frac{dq}{-\left(f_y+qf_z\right)}=\frac{dg}{0}$$

These equations are known as *Charpit's equation*. Solving these equations, p and q are obtained. The simplest of the relations should be taken to obtain p and q easily.

HISTORICAL DATA

Not much is known about **Paul Charpit de Villecourt**, a Frenchman; he is often described as a 'young mathematician'. He died in 1784, probably 'young'—his date of birth is not available. He submitted a paper to the French Royal Academy of Sciences in 1784, but it has never been published. In it, he described his method for solving PDEs. His work was known to a very small group of mathematicians in France, his methods first being presented by Lacroix in 1814. His work is, ultimately, a recasting and alternative interpretation of Lagrange's.

EXAMPLE 11.22 *Solve* $px+qy=pq$.

Solution:

$$px+qy=pq \qquad \text{...(1)}$$

Let $$f(x,y,z,p,q)=px+qy-pq=0$$

The auxiliary equations are

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-\left(f_x+pf_z\right)}=\frac{dq}{-\left(f_y+qf_z\right)}$$

$$\frac{dx}{x-q}=\frac{dy}{y-p}=\frac{dz}{p(x-q)+q(y-p)}=\frac{dp}{-p}=\frac{dq}{-q} \qquad \text{...(2)}$$

Taking the last two fractions in Eq. (2),

$$\frac{dp}{p}=\frac{dq}{q}$$

Integrating,

$$\log p = \log q + \log a$$

$$p = qa \qquad \text{...(3)}$$

Putting $p = aq$ in Eq. (1),

$$aqx + qy = aq^2$$

$$q = \frac{y + ax}{a}$$

Putting q in Eq (3),

$$p = y + ax$$

Now,

$$dz = pdx + qdy = (y + ax)dx + \left(\frac{y + ax}{a}\right)dy$$

$$adz = (y + ax)(dy + adx)$$

Integrating,

$$az = \frac{(y + ax)^2}{2} + b$$

which gives the complete solution of the given equation.

EXAMPLE 11.23 *Solve* $(x^2 - y^2)pq - xy(p^2 - q^2) - 1 = 0.$

Solution:

$$(x^2 - y^2)pq - xy\,(p^2 - q^2) - 1 = 0 \qquad \text{...(1)}$$

Let

$$f(x, y, z, p, q) = (x^2 - y^2)pq - xy\,(p^2 - q^2) - 1$$

The auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\frac{dx}{(x^2 - y^2)q - 2pxy} = \frac{dy}{(x^2 - y^2)p + 2qxy} = \frac{dz}{2pq(x^2 - y^2) - 2xy(p^2 - q^2)}$$

$$= \frac{dp}{-2xpq + y(p^2 - q^2)} = \frac{dq}{2ypq + x(p^2 - q^2)} \qquad \text{...(2)}$$

Taking p, q, x, y as multipliers for first, second, fourth, and fifth fractions respectively, in Eq. (2),

each fraction $= \dfrac{pdx + qdy + xdp + ydq}{0}$

$$pdx + qdy + xdp + ydq = 0$$
$$(xdp + pdx) + (ydq + qdy) = 0$$
$$d(xp) + d(yq) = 0$$

Integrating,

$$xp + yq = a$$
$$p = \frac{a - yq}{x} \qquad \text{...(3)}$$

Putting p in Eq. (1),

$$(x^2 - y^2)\left(\frac{a - yq}{x}\right)q - xy\left[\left(\frac{a - yq}{x}\right)^2 - q^2\right] - 1 = 0$$
$$\left(\frac{a - yq}{x}\right)(x^2q - y^2q - ya + y^2q) + xyq^2 - 1 = 0$$
$$(a - yq)(x^2q - ya) + x^2yq^2 - x = 0$$
$$ax^2q - ya^2 - x^2yq^2 + y^2aq + x^2yq^2 - x = 0$$
$$(ax^2 + ay^2)q = x + a^2y$$
$$q = \frac{x + a^2y}{a(x^2 + y^2)}$$

Putting q in Eq. (3),

$$p = \frac{1}{x}\left[a - \frac{xy + a^2y^2}{a(x^2 + y^2)}\right] = \frac{1}{x}\left[\frac{a^2x^2 + a^2y^2 - xy - a^2y^2}{a(x^2 + y^2)}\right] = \frac{a^2x - y}{a(x^2 + y^2)}$$

Now, $\quad dz = pdx + qdy = \dfrac{a^2x - y}{a(x^2 + y^2)}dx + \dfrac{x + a^2y}{a(x^2 + y^2)}dy = a\left(\dfrac{xdx + ydy}{x^2 + y^2}\right) + \dfrac{xdy - ydx}{a(x^2 + y^2)}$

$$= ad\left[\frac{1}{2}\log(x^2 + y^2)\right] + \frac{1}{a}d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Integrating,

$$z = \frac{a}{2}\log(x^2 + y^2) + \frac{1}{a}\tan^{-1}\left(\frac{y}{x}\right) + b$$

which gives the complete solution of the given equation.

EXERCISE 11.4

Apply Charpit's method to find the complete solutions of the following:

1. $2zx - px^2 - 2qxy + pq = 0$

   $\left[\textbf{Ans.:}\ z = ay + b\left(x^2 - a\right)\right]$

2. $z^2\left(p^2z^2 + q^2\right) = 1$

   $\left[\textbf{Ans.:}\ \left(a^2z + 1\right)^3 = 9a^4\left(ax + y + b\right)^2\right]$

3. $yzp^2 - q = 0$

   $\left[\textbf{Ans.:}\ z^2\left(a - y^2\right) = \left(x + b\right)^2\right]$

4. $2z + p^2 + qy + 2y^2 = 0$

   $\left[\textbf{Ans.:}\ y^2\left[\left(x - a\right)^2 + y^2 + 2z\right] = b\right]$

5. $p^2 - y^2q = y^2 - x^2$

   $\left[\textbf{Ans.:}\ z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{a^2}{y} - y + b\right]$

6. $z^2 = pqxy$

   $\left[\textbf{Ans.:}\ z = bx^a y^{\frac{1}{a}}\right]$

7. $qz - p^2y - q^2y = 0$

   $\left[\textbf{Ans.:}\ z^2 = a\left[y^2 + \left(x + b\right)^2\right]\right]$

11.7 HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$a_0\frac{\partial^n z}{\partial x^n} + a_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + a_n\frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \ldots(11.25)$$

where $a_0, a_1, \ldots, a_n$ are constants is known as a homogeneous linear partial differential equation of n^{th} order with constant coefficients. Since all the terms in the equation contain derivatives of the same order, it is known as a *homogeneous equation.*

Replacing $\frac{\partial}{\partial x}$ by D and $\frac{\partial}{\partial y}$ by D' in Eq. (11.25),

$$\left(a_0D^n + a_1D^{n-1}D' + \cdots + a_nD'^n\right)z = F(x, y)$$

$$f(D, D')z = F(x, y)$$

where

$$f(D, D') = a_0D^n + a_1D^{n-1}D' + \cdots + a_nD'^n$$

which is a linear differential operator.

As in the case of ordinary linear differential equations with constant coefficients, the complete solution of Eq. (11.25) is obtained in two parts, one as a Complementary Function (CF) and the other as a Particular Integral (PI).

The complementary function is the solution of the equation $f(D, D')\,z = 0$.

11.7.1 Rules to Obtain the Complementary Function

Let the given equation be $f(D,D')z = F(x,y)$ ...(11.26)

where $$f(D,D') = a_0 D^n + a_1 D^{n-1}D' + a_2 D^{n-2}D'^2 + \cdots + a_n D'^n$$

Let $z = g(y + mx)$ be its complementary function.

Thus, $z = g(y + mx)$ is the solution of the equation

$$\left(a_0 D^n + a_1 D^{n-1}D' + a_2 D^{n-2}D'^2 + \cdots + a_n D'^n\right) z = 0 \qquad \text{...(11.27)}$$

$$Dz = \frac{\partial z}{\partial x} = mg'(y+mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2 g''(y+mx)$$

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$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n g^{(n)}(y+mx)$$

and $$D'z = \frac{\partial z}{\partial y} = g'(y+mx)$$

$$D'^2 z = \frac{\partial^2 z}{\partial y^2} = g''(y+mx)$$

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$$D'^n z = \frac{\partial^n z}{\partial y^n} = g^{(n)}(y+mx)$$

Substituting in Eq. (11.27),

$$\left(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n\right) g^{(n)}(y+mx) = 0$$

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0 \qquad \text{...(11.28)}$$

Equation (11.28) is known as the auxiliary equation.

Let $m_1, m_2, m_3, \ldots, m_n$ be the roots of Eq. (11.28).

Case I Roots of Auxiliary Equation are Distinct

If $m_1, m_2, m_3, \ldots, m_n$ are real and distinct then Eq. (11.27) reduces to

$$(D - m_1 D')(D - m_2 D') \ldots (D - m_n D')z = 0 \qquad \text{...(11.29)}$$

$$(D - m_1 D')z = 0$$

$$p - m_1 q = 0 \qquad \text{...(11.30)}$$

This is a Lagrange's linear equation. The subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-m_1}=\frac{dz}{0}$$

$$dy+m_1\,x=0, \qquad dz=0$$

Integrating,

$$y+m_1x=a, \qquad z=b$$

The solution of Eq. (11.29) is

$$z=\phi_1\,(y+m_1\,x)$$

Similarly, solutions of the other factors of Eq. (11.29) are

$$z=\phi_2\left(y+m_2x\right),\ z=\phi_3\left(y+m_3x\right),\ldots,z=\phi_n\left(y+m_nx\right)$$

Hence, the complementary function of Eq. (11.26) is

$$\text{CF}=\phi_1\left(y+m_1x\right)+\phi_2\left(y+m_2x\right)+\cdots+\phi_n\left(y+m_nx\right)$$

Case II Roots of Auxiliary Equation are Equal (Repeated)

Let the auxiliary equation have two equal roots as $m_1=m_2=m$.

Then Eq. (11.26) reduces to

$$\left(D-mD'\right)^2\left(D-m_3\,D'\right)\cdots\left(D-m_n\,D'\right)z=0 \qquad \ldots(11.31)$$

$$\left(D-mD'\right)^2 z=0 \qquad \ldots(11.32)$$

$$\left(D-mD'\right)u=0, \quad \text{where} \quad u=\left(D-mD'\right)z$$

Since this equation is Lagrange's linear equation,

$$u=\phi\left(y+mx\right)$$

$$\left(D-mD'\right)z=\phi\left(y+mx\right)$$

$$p-mq=\phi\left(y+mx\right)$$

The subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{\phi\left(y+mx\right)} \qquad \ldots(11.33)$$

Taking first and second fractions of Eq. (11.33),

$$-m\,dx=dy$$

$$dy+m\,dx=0$$

Integrating,

$$y+mx=a \qquad \text{...(11.34)}$$

Taking first and third fractions of Eq. (11.33),

$$\frac{dx}{1}=\frac{dz}{\phi(y+mx)}=\frac{dz}{\phi(a)} \qquad \left[\text{Using Eq. (11.34)}\right]$$

$$dz=\phi(a)\,dx$$

Integrating,

$$z=x\phi(a)+b=x\phi(y+mx)+f(y+mx)$$

Thus, the complete solution of Eq. (11.32) is

$$z=x\phi(y+mx)+f(y+mx)$$

The solution of other factors of Eq. (11.31) are same as Case I.

Hence, the complementary function of Eq. (11.25) is

$$\text{CF} = f(y+mx)+x\phi(y+mx)+\phi_3(y+m_3x)+\cdots+\phi_n(y+m_nx)$$

In general, if n roots of an auxiliary equations are equal,

$$\text{CF} = \phi_1(y+mx)+x\phi_2(y+mx)+x^2\phi_3(y+mx)+\cdots+x^{n-1}\phi_n(y+mx)$$

Note

(i) The auxiliary equation is obtained by replacing D with m and D' with 1 in the given differential equation.

(ii) If $F(x, y)=0$, the particular integral $=0$.

EXAMPLE 11.24 *Solve* $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial x\partial y}-6\dfrac{\partial^2 z}{\partial y^2}=0.$

Solution: The equation can be written as

$$\left(D^2-DD'-6D'^2\right)z=0$$

The auxiliary equation is

$$m^2-m-6=0$$

$$m=-2,\ 3 \qquad \text{(distinct)}$$

$$\text{CF}=\phi_1(y-2x)+\phi_2(y+3x)$$

$$F(x, y) = 0$$
$$\text{PI} = 0$$

Hence, the complete solution is

$$z = \phi_1(y - 2x) + \phi_2\,(y + 3x)$$

EXAMPLE 11.25 *Solve* $25r - 40s + 16t = 0$.

Solution: The equation can be written as

$$25\frac{\partial^2 z}{\partial x^2} - 40\frac{\partial^2 z}{\partial x \partial y} + 16\frac{\partial^2 z}{\partial y^2} = 0$$
$$\left(25D^2 - 40DD' + 16D'^2\right)z = 0$$

The auxiliary equation is

$$25m^2 - 40m + 16 = 0$$
$$m = \frac{4}{5}, \frac{4}{5} \quad \text{(repeated)}$$
$$\text{CF} = \phi_1\left(y + \frac{4}{5}x\right) + x\phi_2\left(y + \frac{4}{5}x\right) = f_1(5y + 4x) + xf_2(5y + 4x)$$
$$F(x, y) = 0$$
$$\text{PI} = 0$$

Hence, the complete solution is

$$z = f_1(5y + 4x) + xf_2(5y + 4x)$$

11.7.2 Rules to Obtain the Particular Integral

Let the differential equation be $f(D, D')z = F(x, y)$

Particular integral $\qquad \text{PI} = \dfrac{1}{f(D, D')}F(x, y)$

The particular integral depends on the form of $F(x, y)$. Different cases are discussed as follows:

Case I $F(x, y) = e^{ax+by}$

$$\text{PI} = \frac{1}{f(D, D')}e^{ax+by}$$

Replacing D by a and D' by b,

$$\text{PI} = \frac{1}{f(a,b)} e^{ax+by}, \quad f(a,b) \neq 0$$

If $f(a, b) = 0$ then $m = \frac{a}{b}$ is a root of the auxiliary equation.

Let $m = \frac{a}{b}$ be a root repeated r times.

Then $$f(D,D') = \left(D - \frac{a}{b}D'\right)^r g(D,D')$$

$$\text{PI} = \frac{1}{\left(D - \frac{a}{b}D'\right)^r g(D,D')} e^{ax+by} = \frac{x^r}{r!} \frac{1}{g(a,b)} e^{ax+by}, \; g(a,b) \neq 0$$

Case II **$F(x, y) = \sin(ax + by)$ or $\cos(ax + by)$**

$$\text{PI} = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$$

Replacing D^2 by $-a^2$, D'^2 by $-b^2$ and DD' by $-ab$,

$$\text{PI} = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by), f(-a^2, -ab, -b^2) \neq 0$$

Case III **$F(x, y) = x^m y^n$**

$$\text{PI} = \frac{1}{f(D,D')} x^m y^n = [f(D,D')]^{-1} x^m y^n$$

$[f(D, D')]^{-1}$ is expanded using binomial expansion according to the following rules:

(i) if $n < m$, expand in powers of $\frac{D'}{D}$.

(ii) if $m < n$, expand in powers of $\frac{D}{D'}$.

Case IV If $F(x, y)$ is not in any of the previous three standard forms,

$$\text{PI} = \frac{1}{f(D,D')} F(x,y)$$

Express $f(D, D')$ in linear factors of D and separate each factor of $\frac{1}{f(D,D')}$ using partial fraction method. Operate each part on $F(x, y)$ considering $\frac{1}{D - mD'} F(x,y) = \int F(x, c - mx)\,dx$, where c is replaced by $y + mx$ after integration.

EXAMPLE 11.26 *Solve* $\left(D^2-2DD'+D'^2\right)z=e^{x+2y}+x^3$.

Solution: The auxiliary equation is

$$m^2-2m+1=0$$
$$m=1,1 \quad \text{(repeated)}$$

$$\text{CF}=\phi_1(y+x)+x\phi_2(y+x)$$

$$\text{PI}=\frac{1}{D^2-2DD'+D'^2}e^{x+2y}+\frac{1}{D^2-2DD'+D'^2}x^3$$
$$=\frac{1}{1^2-2(1)(2)+2^2}e^{x+2y}+\frac{1}{(D-D')^2}x^3$$
$$=e^{x+2y}+\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x^3$$
$$=e^{x+2y}+\frac{1}{D^2}\left(1+2\frac{D'}{D}+3\frac{D'^2}{D^2}+\cdots\right)x^3$$
$$=e^{x+2y}+\frac{1}{D^2}\left(x^3+\frac{2}{D}D'x^3+\frac{3}{D^2}D'^2x^3+\cdots\right)$$
$$=e^{x+2y}+\frac{1}{D^2}x^3=e^{x+2y}+\frac{1}{D}\int x^3dx$$
$$=e^{x+2y}+\int\frac{x^4}{4}dx=e^{x+2y}+\frac{x^5}{20}$$

EXAMPLE 11.27 *Solve* $4r+12s+9t=e^{3x-2y}$.

Solution: The equation can be written as

$$\left(4D^2+12DD'+9D'^2\right)z=e^{3x-2y}$$

The auxiliary equation is

$$4m^2+12m+9=0$$
$$m=-\frac{3}{2},-\frac{3}{2} \quad \text{(repeated)}$$

$$\text{CF}=\phi_1\left(y-\frac{3}{2}x\right)+x\phi_2\left(y-\frac{3}{2}x\right)=f_1(2y-3x)+x\,f_2(2y-3x)$$

$$\text{CF} = \phi_1\left(y-\frac{3}{2}x\right)+x\phi_2\left(y-\frac{3}{2}x\right) = f_1(2y-3x)+x\,f_2(2y-3x)$$

$$\text{PI} = \frac{1}{4D^2+12DD'+9D'^2}e^{3x-2y} = \frac{1}{4\left(D+\frac{3}{2}D'\right)^2}e^{3x-2y}$$

$$= \frac{1}{4}\frac{x^2}{2!}e^{3x-2y} \qquad \left[\because D+\frac{3}{2}D'=0 \text{ at } D=3, D'=-2\right]$$

$$= \frac{x^2}{8}e^{3x-2y}$$

Hence, the complete solution is

$$z = f_1(2y-3x)+x\,f_2(2y-3x)+\frac{x^2}{8}e^{3x-2y}$$

Aliter for PI

$$\text{PI} = \frac{1}{4D^2+12DD'+9D'^2}e^{3x-2y}$$

Since the denominator is zero at $D = 3$ and $D' = -2$, differentiating the denominator w. r. t. D and premultiplying by x,

$$\text{PI} = x\frac{1}{8D+12D'}e^{3x-2y}$$

$$= x^2\frac{1}{8}e^{3x-2y} \qquad \text{[Differentiating again and premultiplying by } x\text{]}$$

$$= \frac{x^2}{8}e^{3x-2y}$$

EXAMPLE 11.28 *Solve* $(D^2 - 2DD')\,z = \sin x \cos 2y$.

Solution: The auxiliary equation is

$$m^2 - 2m = 0$$

$$m = 0,\ 2 \text{ (distinct)}$$

$$\text{CF} = \phi_1(y)+\phi_2(y+2x)$$

$$F(x,y) = \sin x\cos 2y = \frac{1}{2}\left[\sin(x+2y)+\sin(x-2y)\right]$$

$$\text{PI} = \frac{1}{D^2-2DD'}\frac{1}{2}\sin(x+2y)+\frac{1}{D^2-2DD'}\frac{1}{2}\sin(x-2y)$$

$$=\frac{1}{2}\left[\frac{1}{-1^2-2\{-(1)(2)\}}\sin(x+2y)+\frac{1}{-1^2-2\{-(1)(-2)\}}\sin(x-2y)\right]$$

$$=\frac{1}{2}\left[\frac{1}{3}\sin(x+2y)-\frac{1}{5}\sin(x-2y)\right]$$

Hence, the complete solution is

$$z=\phi_1(y)+\phi_2(y+2x)+\frac{1}{6}\sin(x+2y)-\frac{1}{10}\sin(x-2y)$$

EXAMPLE 11.29 *Solve* $(D^2 + DD' - 6D'^2)\,z = \sin(2x+y)$.

Solution: The auxiliary equation is

$$m^2+m-6=0$$

$$m=-3,2$$

$$\text{CF} = \phi_1(y-3x)+\phi_2(y+2x)$$

$$\text{PI} = \frac{1}{D^2+DD'-6D'^2}\sin(2x+y)$$

Since the denominator is zero after replacing D^2 by -2^2, DD' by $-(2)(1)$, and D'^2 by -1^2, the general method needs to be applied.

$$\text{PI} = \frac{1}{(D+3D')(D-2D')}\sin(2x+y)$$

$$=\frac{1}{D+3D'}\left[\frac{1}{D-2D'}\sin(2x+y)\right]=\frac{1}{D+3D'}\left[\int\sin\{2x+(c-2x)\}\mathrm{d}x\right]=\frac{1}{D+3D'}\left[\int\sin c\,\mathrm{d}x\right]$$

$$=\frac{1}{D+3D'}[x\sin c]=\frac{1}{D+3D'}\left[x\sin(y+2x)\right]=\int x\sin[(c+3x)+2x]\mathrm{d}x$$

$$=\int x\sin(5x+c)\mathrm{d}x=x\left[\frac{-\cos(5x+c)}{5}\right]-\left[\frac{-\sin(5x+c)}{25}\right]$$

$$=-\frac{x}{5}\cos(5x+y-3x)+\frac{1}{25}\sin(5x+y-3x)=-\frac{x}{5}\cos(y+2x)+\frac{1}{25}\sin(y+2x)$$

Hence, the complete solution is

$$z=\phi_1(y-3x)+\phi_2(y+2x)-\frac{x}{5}\cos(y+2x)+\frac{1}{25}\sin(y+2x)$$

EXAMPLE 11.30 *Solve* $(D^2 - 2DD' + D'^2)\, z = \tan(y + x)$.

Solution: The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1 \quad \text{(repeated)}$$

$$\text{CF} = \phi_1(y+x) + x\phi_2(y+x)$$

Since $F(x, y) = \tan(y + x)$ is not in any of the standard forms, the general method needs to be applied.

$$\text{PI} = \frac{1}{(D-D')^2}\tan(y+x) = \frac{1}{(D-D')}\left[\frac{1}{D-D'}\tan(y+x)\right]$$

$$= \frac{1}{(D-D')}\left[\int \tan\{(c-x)+x\}\,dx\right]$$

$$= \frac{1}{D-D'}\left[\int \tan c\,dx\right] = \frac{1}{D-D'}[x\tan c] = \frac{1}{D-D'}[x\tan(y+x)]$$

$$= \int x\tan\{(c-x)+x\}\,dx = \int x\tan c\,dx = \frac{x^2}{2}\tan c = \frac{x^2}{2}\tan(y+x)$$

Hence, the complete solution is

$$z = \phi_1(y+x) + x\phi_2(y+x) + \frac{x^2}{2}\tan(y+x)$$

EXERCISE 11.5

Solve the following:

1. $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

   $\left[\textbf{Ans.: } z = f_1(y-2x) + f_2(2y-x)\right]$

2. $\frac{\partial^3 z}{\partial x^3} + 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 0$

   $\left[\textbf{Ans.: } z = f_1(y) + f_2(y+2x) + xf_3(y+2x)\right]$

3. $\left(D^2 - 2DD' - 15D'^2\right)z = 12xy$

   $\left[\textbf{Ans.: } z = f_1(y+5x) + f_2(y-3x) + x^4 + 2x^3y\right]$

4. $r - 2s + t = \sin(2x+3y)$

   $\left[\textbf{Ans.: } z = f_1(x+y) + xf_2(x+y) - \sin(2x+3y)\right]$

5. $\left(2D^2 - 5DD' + 2D'^2\right)z = 5\sin(2x+y)$

   $\left[\textbf{Ans.: } z = f_1(2y+x) + f_2(y+2x) - \frac{5x}{3}\cos(2x+y)\right]$

6. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^3 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = \sin(2x+y)$

   $\left[\textbf{Ans.: } z = f_1(y) + f_2(y+2x) + xf_3(y+2x) - \frac{x^2}{4}\cos(2x+y)\right]$

7. $\left(D^2 - DD' - 2D'^2\right)z = (y-1)e^x$

   $\left[\textbf{Ans.: } z = f_1(y+2x) + f_2(y-x) + ye^x\right]$

8. $r+s-6t = y\cos x$

$\left[\textbf{Ans.:} z = f_1(y-3x)+f_2(y+2x)-y\cos x+\sin x\right]$

9. $(D^2+2DD'+D'^2)z = 2\cos y - x\sin y$

$\left[\textbf{Ans.:}\ z = f_1(y-x)+xf_2(y-x)+x\sin y\right]$

10. $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial x\partial y}-6\dfrac{\partial^2 z}{\partial y^2} = y\sin x$

$\left[\textbf{Ans.:}\ z = f_1(y-3x)+f_2(y+2x) -(y\sin x+\cos x)\right]$

11.8 NONHOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In the equation

$$f(D,D')z = F(x,y) \qquad \text{...(11.35)}$$

if each term of $f(D, D')$ does not contain the derivatives of the same order then the equation is known as a nonhomogeneous equation.

To find the complementary function of Eq. (11.35), factorize $f(D, D')$ into the linear factors as $(D - mD' - c)$ and obtain the solution of the equation $(D - mD' - c)z = 0$.

$$(D-mD'-c)z = 0$$

$$p - mq = cz$$

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz} \qquad \text{...(11.36)}$$

Taking first and second fractions from Eq. (11.36),

$$-mdx = dy$$

$$mdx + dy = 0$$

Integrating,

$$mx + y = a$$

Taking first and third fractions from Eq. (11.36),

$$dx = \frac{dz}{cz}$$

$$\frac{dz}{z} = cdx$$

Integrating,

$$\log z = cx + \log b$$

$$\log \frac{z}{b} = cx$$

$$z = be^{cx}$$

Taking

$$b = \phi(a)$$

$$z = e^{cx}\phi(a) = e^{cx}\phi(y+mx)$$

Similarly, solutions corresponding to other factors can be obtained. All the solutions are added up to obtain the complementary function.

The methods to find the particular integral are same as those of homogeneous linear equations.

EXAMPLE 11.31 *Solve* $(D^2 - DD' + D' - 1)\, z = \cos(x + 2y) + e^y$.

Solution:

$$D^2 - DD' + D' - 1 = (D-1)(D-D'+1)$$

(i) For the equation $(D-1)\, z = 0$,

$$m = 0, \; c = 1$$

the solution is

$$z = e^x \phi_1(y)$$

(ii) For the equation $(D - D' + 1)\, z = 0$,

$$m = 1, \; c = -1$$

the solution is

$$z = e^{-x} \phi_2(y+x)$$

Hence,

$$\text{CF} = e^x \phi_1(y) + e^{-x} \phi_2(y+x)$$

$$\text{PI} = \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) + \frac{1}{D^2 - DD' + D' - 1}e^y$$

$$= \frac{1}{-1^2 - \{-(1)(2)\} + D' - 1}\cos(x+2y) + x \cdot \frac{1}{2D - D'}e^y$$

$$= \frac{1}{D'}\cos(x+2y) + x \cdot \frac{1}{2(0)-1}e^y = \frac{D'}{D'^2}\cos(x+2y) - xe^y = \frac{D'\cos(x+2y)}{-2^2} - xe^y$$

$$= \frac{-2\sin(x+2y)}{-4} - xe^y = \frac{1}{2}\sin(x+2y) - xe^y$$

Hence, the complete solution is

$$z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + \frac{1}{2}\sin(x+2y) - xe^y$$

EXERCISE 11.6

Solve the following:

1. $(D^2 - D'^2 + D - D')z = 0$

   $\left[\textbf{Ans.: } z = f_1(y+x) + e^{-x} f_2(y-x)\right]$

2. $(D - D' - 1)(D - D' - 2)z = e^{2x-y} + x$

   $\left[\begin{array}{l}\textbf{Ans.: } z = e^{x} f_1(y+x) + e^{2x} f_2(y+z) \\ \qquad + \dfrac{x}{2} + \dfrac{3}{4} + \dfrac{1}{2}e^{2x-y}\end{array}\right]$

3. $r - s + p = 1$

   $\left[\textbf{Ans.: } z = f_1(y) + e^{-x} f_2(y+x) + x\right]$

4. $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^{y}$

   $\left[\begin{array}{l}\textbf{Ans.: } z = e^{x} f_1(y) + e^{-x} f_2(y+x) \\ \qquad + \dfrac{1}{2}\sin(x+2y) - xe^{y}\end{array}\right]$

5. $(D^2 - DD' - 2D)z = \sin(3x+4y) - e^{2x+y}$

   $\left[\begin{array}{l}\textbf{Ans.: } z = f_1(y) + e^{2x} f_2(y+x) + \dfrac{1}{15}\sin(3x+4y) \\ \qquad + \dfrac{2}{15}\cos(3x+3y) + \dfrac{1}{2}e^{2x+y}\end{array}\right]$

11.9 NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

Monge's Method

This method is applied to solve the equation

$$Rr + Ss + Tt = V \qquad \text{...(11.37)}$$

where R, S, T, V are functions of x, y, z, p, and q.

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = r dx + s dy$$

$$r = \frac{dp - s dy}{dx}$$

and

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = s dx + t dy$$

$$t = \frac{dq - s dx}{dy}$$

Substituting r and t in Eq. (11.37),

$$R\left(\frac{dp - s dy}{dx}\right) + Ss + T\left(\frac{dq - s dx}{dy}\right) = V$$

$$R dy(dp - s dy) + Ss dx dy + T dx(dq - s dx) = V dx dy$$

$$(R dp dy + T dq dx - V dx dy) - s(R dy^2 - S dx dy + T dx^2) = 0 \qquad \text{...(11.38)}$$

Let us consider a relation among x, y, z, p, q such that

$$Rdy^2 - Sdxdy + Tdx^2 = 0 \quad ...(11.39)$$

$$Rdpdy + Tdqdx - Vdxdy = 0 \quad ...(11.40)$$

Equations (11.39) and (11.40) are known as *Monge's subsidiary equations*.

Let Eq. (11.39) be factorized as

$$dy - m_1 dx = 0 \quad ...(11.41)$$

$$dy - m_2 dx = 0 \quad ...(11.42)$$

From Eqs (11.40) and (11.41), two integrals are obtained as

$$u_1 = a \quad \text{and} \quad v_1 = b, \text{ say}$$

Then

$$u_1 = f_1(v_1) \quad ...(11.43)$$

is the solution and is called an *intermediate integral*.

Similarly, from Eqs (11.40) and (11.42), another intermediate integral is obtained as

$$u_2 = f_2(v_2) \quad ...(11.44)$$

From Eqs (11.43) and (11.44), the values of p and q are obtained in terms of x and y.

Substituting p and q in the equation

$$dz = pdx + qdy$$

and integrating the equation, the complete solution of Eq. (11.37) is obtained.

HISTORICAL DATA

Gaspard Monge, Comte de Péluse (1746–1818), was a French mathematician, the inventor of descriptive geometry (the mathematical basis of technical drawing), and the father of differential geometry. During the French Revolution, he served as the Minister of the Marine, and was involved in the reform of the French educational system, helping to found the École Polytechnique.

Monge's name is one of the 72 names inscribed on the base of the Eiffel Tower.

EXAMPLE 11.32 *Solve* $y^2r + 2xys + x^2t + px + qy = 0$.

Solution: The given equation is in the form $Rr + Ss + Tt = V$.

Monge's subsidiary equations are

$$Rdy^2 - Sdxdy + Tdx^2 = 0$$

$$Rdpdy + Tdqdx - Vdxdy = 0$$

$$R = y^2,\ S = 2xy,\ T = x^2,\ V = -(px + qy)$$

$$y^2 dy^2 - 2xy dx dy + x^2 dx^2 = 0 \qquad \text{...(1)}$$

$$y^2 dp dy + x^2 dq dx + (px + qy) dx dy = 0 \qquad \text{...(2)}$$

Equation (1) can be factorized as

$$(x dx - y dy)^2 = 0$$

$$x dx - y dy = 0 \qquad \text{...(3)}$$

Integrating,

$$\frac{x^2}{2} - \frac{y^2}{2} = a_1$$

$$x^2 - y^2 = a \qquad \text{...(4)}$$

where $a = 2a_1$

Putting $y dy = x dx$ from Eq. (3) in Eq. (2),

$$xy dx dp + x^2 dq dx + px dx dy + q dx (x dx) = 0$$

$$y dp + x dq + p dy + q dx = 0$$

$$(y dp + p dy) + (x dq + q dx) = 0$$

$$d(yp) + d(xq) = 0$$

Integrating,

$$yp + xq = b = f(a) \qquad \text{...(5)}$$

From Eqs (4) and (5), the intermediate integral is

$$yp + xq = f(y^2 - x^2)$$

This is a Lagrange's linear equation.

The subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{f(x^2 - y^2)} \qquad \text{...(6)}$$

Taking second and third fractions from Eq. (6),

$$\frac{dy}{x} = \frac{dz}{f(x^2 - y^2)}$$

$$\frac{dy}{\sqrt{a + y^2}} = \frac{dz}{f(a)} \quad [\text{Using Eq. (4)}]$$

$$dz = \frac{f(a) dy}{\sqrt{a + y^2}}$$

Integrating,

$$z = f(a)\log\left[y+\sqrt{a+y^2}\right]+c = f(x^2-y^2)\log\left[y+\sqrt{a+y^2}\right]+\phi(x^2-y^2)$$

EXAMPLE 11.33 *Solve* $t - r\sec^4 y = 2q\tan y$.

Solution: The given equation can be rewritten in the form

$$Rr + Ss + Tt = V \text{ as}$$
$$-\sec^4 y r + (0)s + t = 2q\tan y$$

Monge's subsidiary equations are

$$Rdy^2 - Sdxdy + Tdx^2 = 0$$
$$Rdpdy + Tdqdx - Vdxdy = 0$$
$$R = -\sec^4 y,\ S = 0,\ T = 1,\ V = 2q\tan y$$
$$-\sec^4 y\, dy^2 + dx^2 = 0 \quad \text{...(1)}$$
$$-\sec^4 y\, dp\, dy + dqdx - 2q\tan y\, dx\, dy = 0 \quad \text{...(2)}$$

Equation(1) can be factorized as

$$(dx + \sec^2 y\, dy)(dx - \sec^2 y\, dy) = 0$$
$$dx + \sec^2 y\, dy = 0 \quad \text{...(3)}$$
$$dx - \sec^2 y\, dy = 0 \quad \text{...(4)}$$

Integrating,

$$x + \tan y = a \quad \text{...(5)}$$
$$x - \tan y = b \quad \text{...(6)}$$

Putting $\sec^2 y\, dy = -dx$ from Eq. (3) in Eq. (2),

$$\sec^2 y\, dpdx + dqdx - 2q\tan y\, dxdy = 0$$
$$\sec^2 y dp + dq - 2q\tan y dy = 0$$
$$dp + dq\cos^2 y - 2q\sin y\cos y dy = 0$$
$$dp + d(q\cos^2 y) = 0$$

Integrating,

$$p + q\cos^2 y = c_1 = f_1(x + \tan y) \quad \text{...(7)}$$

Putting $\sec^2 y\, dy = dx$ from Eq. (4) in Eq. (2),

$$-\sec^2 y dpdx + dqdx - 2q\tan y dxdy = 0$$
$$-\sec^2 y dp + dq - 2q\tan y dy = 0$$
$$dp - \left(\cos^2 y dq - 2q\sin y\cos y dy\right) = 0$$
$$dp - d\left(q\cos^2 y\right) = 0$$

Integrating,

$$p - q\cos^2 y = c_2 = f_2(x - \tan y) \quad ...(8)$$

Adding Eqs (7) and (8),

$$p = \frac{1}{2}\left[f_1(x+\tan y) + f_2(x - \tan y)\right]$$

Subtracting Eq. (8) from Eq. (7),

$$q = \frac{1}{2}\sec^2 y\left[f_1(x+\tan y) - f_2(x-\tan y)\right]$$

Substituting p and q in the equation,

$$\begin{aligned} dz &= p\,dx + q\,dy \\ &= \frac{1}{2}\left[f_1(x+\tan y) + f_2(x-\tan y)\right]dx + \frac{1}{2}\sec^2 y\left[f_1(x+\tan y) - f_2(x-\tan y)\right]dy \\ &= \frac{1}{2}\left[f_1(x+\tan y)(dx + \sec^2 y\,dy) + f_2(x-\tan y)(dx - \sec^2 y\,dy)\right] \end{aligned}$$

Integrating,

$$z = \phi_1(x+\tan y) + \phi_2(x - \tan y)$$

EXERCISE 11.7

Solve the following:

1. $r = a^2 t$

$\left[\textbf{Ans.: } z = \phi_1(y+ax) + \phi_2(y-ax)\right]$

2. $(q+1)s = (p+1)t$

$\left[\textbf{Ans.: } z = \phi_1(x+y+z) + \phi_2(x)\right]$

3. $x^2 r - 2xs + t + q = 0$

$\left[\textbf{Ans.: } z = \phi_1(y+\log x) + x\phi_2(y+\log x)\right]$

4. $(r-t)xy - s(x^2-y^2) = qx - py$

$\left[\textbf{Ans.: } z = \phi_1(x^2+y^2) + \phi_2\left(\frac{y}{x}\right)\right]$

5. $r - 2s + t = \sin(2x+3y)$

$\left[\begin{array}{l}\textbf{Ans.: } z = \phi_1(x+y) + x\phi_2(x+y) \\ \qquad - \sin(2x+3y)\end{array}\right]$

6. $r - t\cos^2 x + p\tan x = 0$

$\left[\textbf{Ans.: } z = \phi_1(y - \sin x) + \phi_2(y + \sin x)\right]$

7. $x(r + 2xs + x^2 t) = p + 2x^3$

$\left[\begin{array}{l}\textbf{Ans.: } z = \phi_1(x^2 - 2y) \\ \qquad + \frac{x^2}{2}\phi_2(x^2-2y) + \frac{x^4}{4}\end{array}\right]$

8. $y^2 r - 2ys + t - p - 6y = 0$

$\left[\begin{array}{l}\textbf{Ans.: } z = y^3 - y\phi_1(y^2+2x) \\ \qquad + \phi_2(y^2+2x)\end{array}\right]$

9. $2x^2 r - 5xys + 2y^2 t = -2(px + qy)$

$\left[\textbf{Ans.: } z = f(yx^2) + g(xy^2)\right]$

10. $(e^x - 1)(qr - ps) = pqe^x$

$\left[\textbf{Ans.: } x = \phi_1(y) - \phi_2(z) + e^x\right]$

11.10 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

In many physical problems in electromagnetic theory, fluid mechanics, solid mechanics, heat transfer, etc., solutions of partial differential equations are required. These equations satisfy some specified conditions known as *boundary conditions*. The partial differential equation together with these boundary conditions, constitutes a *boundary-value problem*.

The method of separation of variables is an important tool to solve such boundary-value problems when the partial differential equation is linear and boundary conditions are homogeneous. Unlike ordinary differential equations, the general solution of a partial differential equation involves arbitrary functions which requires the knowledge of single and double Fourier series.

11.11 METHOD OF SEPARATION OF VARIABLES

Separation of variables is also known as the *Fourier method*. It is a powerful technique to solve partial differential equations. This method is explained with the help of the following examples.

EXAMPLE 11.34 *Solve* $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, *given that* $u(0, y) = 8e^{-3y}$.

Solution: Let the solution be $u(x, y) = X(x)\, Y(y)$

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY'$$

Substituting in the given equation,

$$X'Y = 4XY'$$

$$\frac{X'}{X} = \frac{4Y'}{Y} = k, \text{ say}$$

$$\frac{X'}{X} = k, \quad \frac{4Y'}{Y} = k$$

Solving both the equations,

$$\log X = kx + \log c_1, \qquad 4\log Y = ky + \log c_2$$

$$\log\frac{X}{c_1} = kx, \qquad \log\frac{Y^4}{c_2} = ky$$

$$X = c_1 e^{kx}, \qquad Y^4 = c_2 e^{ky}$$

$$Y = ce^{\frac{ky}{4}}, \quad \text{where } c = c_2^{\frac{1}{4}}$$

Thus, $$u(x, y) = XY = c_1 c e^{k\left(x+\frac{y}{4}\right)} = A\, e^{k\left(x+\frac{y}{4}\right)}, \text{ where } c_1 c = A$$

Given $$u(0, y) = 8e^{-3y}$$

$$Ae^{k\left(0+\frac{y}{4}\right)} = 8e^{-3y}$$

Comparing both the sides,

$$A = 8, \quad \frac{k}{4} = -3, \quad k = -12$$

Hence,
$$u = 8\,e^{-12\left(x+\frac{y}{4}\right)} = 8\,e^{-12x-3y}$$

EXAMPLE 11.35 *Solve* $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u.$

Solution: Let the solution be $u(x, y) = X(x)\, Y(y)$

$$\frac{\partial u}{\partial x} = X'Y, \qquad \frac{\partial u}{\partial y} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = X''Y$$

Substituting in the given equation,

$$X''Y = XY' + 2XY$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2$$

$$\frac{X''}{X} - 2 = \frac{Y'}{Y} = k, \text{ say}$$

$$\frac{X'' - 2X}{X} = k, \quad \frac{Y'}{Y} = k$$

$$X'' - (k+2)X = 0 \;\ldots(1), \qquad \frac{Y'}{Y} = k \;\ldots(2)$$

To solve Eq. (1), the auxiliary equation is

$$m^2 - (k+2)m = 0$$

$$m = 0, \ k+2$$

$$X = c_1 e^{0x} + c_2 e^{(k+2)x} = c_1 + c_2 e^{(k+2)x}$$

The solution of Eq. (2) is

$$\log Y = ky + \log c_3$$

$$\log \frac{Y}{c_3} = ky$$

$$Y = c_3 e^{ky}$$

Thus, $u = XY = \left[c_1 + c_2 e^{(k+2)x}\right] c_3 e^{ky} = Ae^{ky} + Be^{k(x+y)+2x}$, where $c_1c_3 = A$, $c_2c_3 = B$

11.12 ONE-DIMENSIONAL WAVE EQUATION

Consider an elastic string stretched to a length l along the x-axis with its two fixed ends at $x = 0$ and $x = l$ (Fig. 11.4).

To obtain the deflection $y(x, t)$ at any point x and at any time $t > 0$, the following assumptions are made:

(i) The string is homogeneous with constant density ρ.

(ii) The string is perfectly elastic and offers no resistance to bending.

(iii) The tension in the string is so large that the force due to weight of the string can be neglected.

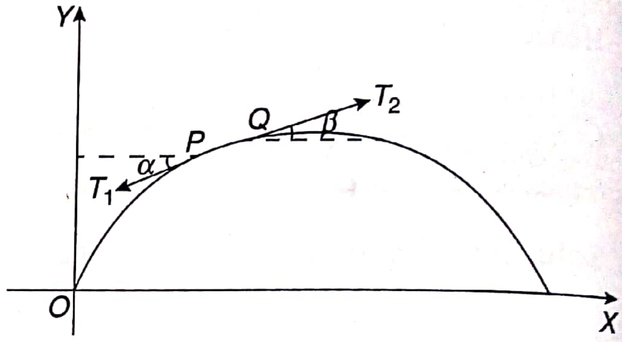

Fig. 11.4 One dimensional wave equation

Consider the motion of the small portion PQ of length δx of the string (Fig. 11.4). Since the string produces no resistance to bending, the tensions T_1 and T_2 at points P and Q will act tangentially at P and Q respectively.

Assuming that the points on the string move only in the vertical direction, there is no motion in the horizontal direction. Hence, the sum of the forces in the horizontal direction must be zero.

$$-T_1 \cos\alpha + T_2 \cos\beta = 0$$

$$T_1 \cos\alpha = T_2 \cos\beta = T \text{ (constant), say} \qquad \text{...(11.45)}$$

The forces acting vertically on the string are the vertical components of tension at points P and Q. Thus, the resultant vertical force acting on PQ is $T_2 \sin\beta - T_1 \sin\alpha$. By Newton's second law of motion,

$$\text{Resultant force} = \text{Mass} \times \text{Acceleration}$$

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\delta x\left(\frac{\partial^2 y}{\partial t^2}\right) \qquad \text{...(11.46)}$$

$$\frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T} = \frac{\rho\delta x}{T}\left(\frac{\partial^2 y}{\partial t^2}\right)$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\delta x}{T}\left(\frac{\partial^2 y}{\partial t^2}\right) \qquad \text{[Using Eq. (11.45)]}$$

$$\tan\beta - \tan\alpha = \frac{\rho\delta x}{T}\frac{\partial^2 y}{\partial t^2} \qquad \text{...(11.47)}$$

Since $\tan\alpha$ and $\tan\beta$ are the slopes of the curve at points P and Q respectively,

$$\tan\alpha = \left(\frac{\partial y}{\partial x}\right)_P = \left(\frac{\partial y}{\partial x}\right)_x$$

$$\tan\beta = \left(\frac{\partial y}{\partial x}\right)_Q = \left(\frac{\partial y}{\partial x}\right)_{x+\delta x}$$

Substituting in Eq. (11.47),

$$\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x = \frac{\rho\delta x}{T}\frac{\partial^2 y}{\partial t^2}$$

Dividing by δx and taking limit $\delta x \to 0$,

$$\lim_{\delta x\to 0}\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x} = \frac{\rho\partial^2 y}{T\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \quad \text{where } c^2 = \frac{T}{\rho}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

This equation is known as the *one-dimensional wave equation.*

Solution of the One-Dimensional Wave Equation

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(11.48)}$$

Let $y = X(x)T(t)$ be a solution of Eq. (11.48).

$$\frac{\partial^2 y}{\partial t^2} = XT'', \quad \frac{\partial^2 y}{\partial x^2} = X''T$$

Substituting in Eq. (11.48),

$$XT'' = c^2X''T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

Since X and T are only the functions of x and t respectively, this equation holds good if each term is a constant.

Let
$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k, \text{ say}$$

Considering $\frac{X''}{X} = k$, $\frac{d^2X}{dx^2} - kX = 0$ ...(11.49)

Considering $\frac{1}{c^2}\frac{T''}{T} = k$, $\frac{d^2T}{dt^2} - kc^2T = 0$ ...(11.50)

Solving Eqs (11.49) and (11.50), the following cases arise:

(i) When k is positive

Let $k = m^2$

$$\frac{d^2X}{dx^2} - m^2X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} - m^2c^2T = 0$$

$$X = c_1e^{mx} + c_2e^{-mx} \quad \text{and} \quad T = c_3e^{mct} + c_4e^{-mct}$$

Hence, the solution of Eq. (11.48) is

$$y = (c_1e^{mx} + c_2e^{-mx})(c_3e^{mct} + c_4e^{-mct}) \qquad \text{...(11.51)}$$

(ii) When k is negative

Let $k = -m^2$

$$X = c_1 \cos mx + c_2 \sin mx \quad \text{and} \quad T = c_3 \cos mct + c_4 \sin mct$$

Hence, the solution of Eq. (11.48) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \text{...(11.52)}$$

(iii) When $k = 0$

$$X = c_1x + c_2 \quad \text{and} \quad T = c_3t + c_4$$

Hence, the solution of Eq. (11.48) is

$$y = (c_1x + c_2)(c_3t + c_4) \qquad \text{...(11.53)}$$

Out of these three solutions, a solution is chosen which is consistent with the physical nature of the problem. y must be a periodic function of x and t. Thus, the solution must involve trigonometric terms.

Hence, the solution is of the form given by Eq. (11.52).

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct)$$

EXAMPLE 11.36 *Find the solution of the wave equation* $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ *such that* $y = a \cos pt$ *when* $x = l$, *and* $y = 0$ *when* $x = 0$.

Solution:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

Since y is periodic, the solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $y = 0$
(ii) At $x = l$, $y = a \cos pt$

Putting the condition (i) in Eq. (2),

$$0 = c_1 (c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) = c_2c_3 \sin mx \cos mct + c_2c_4 \sin mx \sin mct \qquad \text{...(3)}$$

Putting the condition (ii) in Eq. (3),

$$a \cos pt = c_2c_3 \sin ml \cos mct + c_2c_4 \sin ml \sin mct$$

Equating coefficients of sine and cosine terms,

$$a = c_2c_3 \sin ml, \text{ if } mc = p \quad \Rightarrow \quad c_2c_3 = \frac{a}{\sin ml}, \text{ if } m = \frac{p}{c}$$

and

$$0 = c_2c_4 \sin ml \quad \Rightarrow \quad c_4 = 0 \qquad [\because c_2 \neq 0 \text{ otherwise } y = 0]$$

Substituting these values in Eq. (3),

$$y = \frac{a}{\sin ml} \sin mx \cos mct = \frac{a}{\sin \frac{pl}{c}} \sin \frac{px}{c} \cos pt$$

EXAMPLE 11.37 *A tightly stretched string with fixed end points $x = 0$ and $x = l$ in the shape defined by $y = kx\,(l - x)$, where k is a constant, is released from this position of rest. Find $y(x, t)$ if $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.*

Solution:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $y = 0$, for all t, i.e., $y\,(0, t) = 0$
(ii) At $x = l$, $y = 0$, for all t, i.e., $y\,(l, t) = 0$

The initial conditions are

(iii) At $t = 0$, $y = kx(l - x)$ i.e., $y(x, 0) = kx(l - x)$

(iv) At $t = 0$, $\frac{\partial y}{\partial t} = 0$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \quad ...(3)$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml(c_3 \cos mct + c_4 \sin mct)$$

$$\sin ml = 0 \quad [\because c_2 \neq 0]$$

$$ml = n\pi, \quad n \text{ is an integer}$$

$$m = \frac{n\pi}{l}$$

Putting $m = \frac{n\pi}{l}$ in Eq. (3),

$$y = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right) \quad ...(4)$$

$$\frac{\partial y}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left[-c_3\left(\sin\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) + c_4\left(\cos\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right)\right] \quad ...(5)$$

Applying the condition (iv) in Eq. (5),

$$0 = c_2 \sin\frac{n\pi x}{l}\left(c_4 \frac{n\pi c}{l}\right)$$

$$c_4 = 0 \quad [\because c_2 \neq 0]$$

Putting $c_4 = 0$ in Eq. (4),

$$y = c_2c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} = b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \text{ where } c_2c_3 = b_n$$

Putting $n = 1, 2, 3, ...$ and adding all these solutions by principle of superposition, the general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad ...(6)$$

Applying the condition (iii) in Eq. (6),

$$kx(l-x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{...(7)}$$

Equation (7) represents the Fourier half-range sine series.

$$b_n=\frac{2}{l}\int_0^l k\left(lx-x^2\right)\sin\frac{n\pi x}{l}\,dx=\frac{2k}{l}\left|\left(lx-x^2\right)\left(2\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(l-2x)\left(2\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+(-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right|_0^l$$

$$=\frac{2k}{l}\left[-\frac{2l^3}{n^3\pi^3}\cos n\pi+\frac{2l^3}{n^3\pi^3}\right]=\frac{4kl^2}{n^3\pi^3}\left[-(-1)^n+1\right]\left[\because \cos n\pi=(-1)^n\right]$$

Substituting b_n in Eq. (6), the solution is

$$y(x,t)=\frac{4kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$$=\frac{8kl^2}{\pi^3}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^3}\sin\frac{(2r-1)\pi x}{l}\cos\frac{(2r-1)\pi ct}{l}$$

$$\left[\begin{aligned}&\because 1-(-1)^n=0, &&\text{for } n \text{ even}\\ &\qquad\qquad\quad =2, &&\text{for } n \text{ odd}\\ &\text{Taking } n=2r-1\end{aligned}\right]$$

EXAMPLE 11.38 *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent times and show that the midpoint of the string always remains at rest.*

Solution: Let A and C be the points of the trisection of the string OE of length l (Fig. 11.5). Initially the string is held in the form $OBDE$ in such a manner that $AB = CD = h$, say

The equation of the line OB is

$$y-0=\frac{h-0}{\frac{l}{3}-0}(x-0)$$

$$y=\frac{3h}{l}x$$

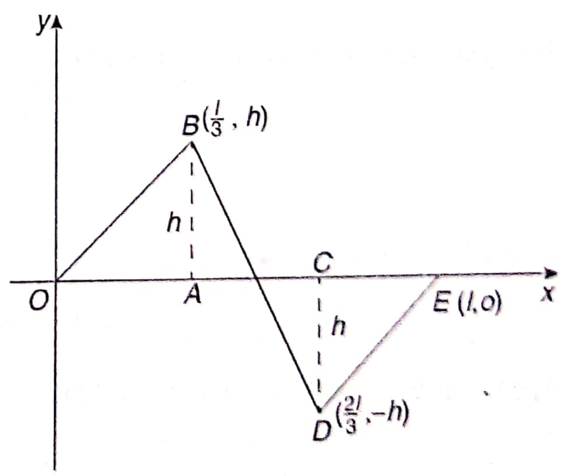

Fig. 11.5

The equation of the line BD is

$$y-h=\frac{-h-h}{\frac{2l}{3}-\frac{l}{3}}\left(x-\frac{l}{3}\right)=-\frac{6h}{l}\left(x-\frac{l}{3}\right)=-\frac{6hx}{l}+2h$$

$$y = 3h - \frac{6hx}{l} = \frac{3h}{l}(l - 2x)$$

The equation of the line DE is

$$y - 0 = \frac{-h - 0}{\frac{2l}{3} - l}(x - l) = \frac{3h}{l}(x - l)$$

The displacement y of any point of the string is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(1)}$$

The solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \quad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $y = 0$ for all t, i.e., $y(0, t) = 0$

(ii) At $x = l$, $y = 0$ for all t, i.e., $y(l, t) = 0$

Since initially ($t = 0$) the string rests in the form of $OBDE$, the initial conditions are

(iii) At $t = 0$,

$$\begin{aligned} y(x,0) &= \frac{3hx}{l}, & 0 \le x \le \frac{l}{3} \\ &= \frac{3h}{l}(l - 2x), & \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3h}{l}(x - l), & \frac{2l}{3} \le x \le l \end{aligned}$$

(iv) At $t = 0$, $\frac{\partial y}{\partial t} = 0$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \quad \text{...(3)}$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml(c_3 \cos mct + c_4 \sin mct)$$

$$\sin ml = 0 \quad [\because c_2 \ne 0]$$

$$ml = n\pi, \quad n \text{ is an integer}$$

$$m = \frac{n\pi}{l}$$

Putting $m=\frac{n\pi}{l}$ in Eq. (3),

$$y=c_2\sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi ct}{l}+c_4\sin\frac{n\pi ct}{l}\right) \quad ...(4)$$

$$\frac{\partial y}{\partial t}=c_2\sin\frac{n\pi x}{l}\left[-c_3\left(\sin\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right)+c_4\left(\cos\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right)\right] \quad ...(5)$$

Applying the condition (iv) in Eq. (5),

$$0=c_2\sin\frac{n\pi x}{l}\left(c_4\frac{n\pi c}{l}\right)$$

$$c_4=0 \quad [\because c_2\neq 0]$$

Putting $c_4=0$ in Eq. (4),

$$y=c_2c_3\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}=b_n\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \quad \text{where } c_2c_3=b_n$$

Putting n = 1, 2, 3, ... and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$y(x,t)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad ...(6)$$

Applying the condition (iii) in Eq. (6),

$$y(x,0)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \quad ...(7)$$

Equation (7) represents the Fourier half-range sine series.

$$b_n=\frac{2}{l}\int_0^l y\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}\left[\int_0^{\frac{l}{3}}\frac{3hx}{l}\sin\frac{n\pi x}{l}\,dx+\int_{\frac{l}{3}}^{\frac{2l}{3}}\frac{3h}{l}(l-2x)\sin\frac{n\pi x}{l}\,dx+\int_{\frac{2l}{3}}^{l}\frac{3h}{l}(x-l)\sin\frac{n\pi x}{l}\,dx\right]$$

$$=\frac{6h}{l^2}\left[\left|x\left\{2\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}-1\left\{2\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_0^{\frac{l}{3}}+\left|(l-2x)\left\{2\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}-(-2)\left\{2\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_{\frac{l}{3}}^{\frac{2l}{3}}\right.$$

$$\left.+\left|(x-l)\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}-1\left\{\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_{\frac{2l}{3}}^{l}\right]$$

$$= \frac{6h}{l^2}\left[-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}+\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}-\frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}+\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right.$$
$$\left.+\frac{l^2}{n^2\pi^2}\sin n\pi-\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}-\frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$= \frac{6h}{l^2}\frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right)$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3}+(-1)^n\sin\frac{n\pi}{3}\right] \qquad \left[\because \sin\frac{2n\pi}{3}=\sin\left(n\pi-\frac{n\pi}{3}\right)=-(-1)^n\sin\frac{n\pi}{3}\right]$$

$$= \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3}\left[1+(-1)^n\right]$$

Substituting in Eq. (6), the solution is

$$y(x,t)=\frac{18h}{\pi^2}\sum_{n=1}^{\infty}\sin\frac{n\pi}{3}\left[\frac{1+(-1)^n}{n^2}\right]\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \left[\begin{array}{ll}\because 1+(-1)^n=0, & \text{for } n \text{ odd}\\ \qquad\qquad =2, & \text{for } n \text{ even}\\ \text{Taking } n=2r & \end{array}\right]$$

$$=\frac{18h}{\pi^2}\sum_{r=1}^{\infty}\sin\frac{2r\pi}{3}\frac{2}{(2r)^2}\sin\frac{2r\pi x}{l}\cos\frac{2r\pi ct}{l}$$

$$=\frac{9h}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin\frac{2r\pi}{3}\sin\frac{2r\pi x}{l}\cos\frac{2r\pi ct}{l} \qquad \text{...(8)}$$

To find the displacement at the midpoint, putting $x=\frac{l}{2}$ in Eq. (8),

$$y\left(\frac{l}{2},t\right)=0 \qquad [\because \sin r\pi=0 \text{ as } r \text{ is positive integer}]$$

EXAMPLE 11.39 *A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in the equilibrium position. It is set vibrating by giving to each of its points a velocity of $v_0\sin^3\frac{\pi x}{l}$. Find the displacement $y(x, t)$.*

Solution: The equation of the vibrating string is

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of Eq. (1) is

$$y=(c_1\cos mx+c_2\sin mx)(c_3\cos mct+c_4\sin mct) \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $y = 0$ for all t, i.e., $y(0, t) = 0$

(ii) At $x = l$, $y = 0$ for all t, i.e., $y(l, t) = 0$

The initial conditions are

(iii) At $t = 0$, $y = 0$, i.e., $y(x, 0) = 0$

(iv) At $t = 0$, $\frac{\partial y}{\partial t} = v_0 \sin^3 \frac{\pi x}{l}$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \qquad ...(3)$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml(c_3 \cos mct + c_4 \sin mct)$$

$$\sin ml = 0 \quad [\because c_2 \neq 0]$$

$$ml = n\pi, \qquad n \text{ is an integer}$$

$$m = \frac{n\pi}{l}$$

Putting $m = \frac{n\pi}{l}$ in Eq. (3),

$$y = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \qquad ...(4)$$

Applying the condition (iii) in Eq. (4),

$$0 = c_2 \sin \frac{n\pi x}{l}(c_3) = c_2 c_3 \sin \frac{n\pi x}{l}$$

$$c_2 c_3 = 0$$

Putting $c_2 c_3 = 0$ in Eq. (4),

$$y = c_2 c_4 \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}, \quad \text{where } c_2 c_4 = b_n$$

Putting $n=1,2,3,\dots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$y(x,t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l} \quad \dots(5)$$

$$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\left(\cos\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) \quad \dots(6)$$

Applying the condition (iv) in Eq. (6),

$$v_0\sin^3\frac{\pi x}{l}=\sum_{n=1}^{\infty} b_n\frac{n\pi c}{l}\sin\frac{n\pi x}{l}$$

$$\frac{v_0}{4}\left(3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right)=\sum_{n=1}^{\infty} b_n\frac{n\pi c}{l}\sin\frac{n\pi x}{l} \qquad \left[\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta\right]$$

$$=b_1\frac{\pi c}{l}\sin\frac{\pi x}{l}+b_2\frac{2\pi c}{l}\sin\frac{2\pi x}{l}+b_3\frac{3\pi c}{l}\sin\frac{3\pi x}{l}+b_4\frac{4\pi c}{l}\sin\frac{4\pi x}{l}+\dots$$

Comparing coefficients of sine terms on both the sides,

$$\frac{3v_0}{4}=b_1\frac{\pi c}{l},\quad 0=b_2\frac{2\pi c}{l},\quad -\frac{v_0}{4}=b_3\frac{3\pi c}{l},\quad 0=b_4\frac{4\pi c}{l},\dots$$

$$b_1=\frac{3lv_0}{4\pi c},\quad b_2=0,\ b_3=-\frac{lv_0}{12\pi c},\quad b_4=0,\dots,\ b_n=0, \text{ for } n\ge 4$$

Substituting $b_1, b_2, b_3, b_4 \dots$ in Eq. (5), the solution is

$$y(x,t)=\frac{3lv_0}{4\pi c}\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l}-\frac{lv_0}{12\pi c}\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}=\frac{lv_0}{12\pi c}\left(9\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l}-\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}\right)$$

EXERCISE 11.8

1. A string of length l is stretched and fastened to two fixed points. Find the solution of the wave equation $\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$ when initial displacement $y(x,0)=b\sin\frac{\pi x}{l}$

   $\left[\textbf{Ans.: } y(x,t)=b\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}\right]$

2. A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y=y_0\sin^3\frac{\pi x}{l}$. If it is

released from rest from this position, find the displacement $y(x, t)$.

$$\left[\textbf{Ans.: } y(x,t)=\frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}-\frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}\right]$$

3. An elastic string is stretched between two points at a distance l apart. In its equilibrium position, a point at a distance a $(a < l)$ from one end is displaced through a distance b transversely and then released from this position. Obtain $y(x, t)$, the vertical displacement if y satisfies the equation $\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$.

$$\left[\textbf{Ans.: } y(x,t)=\frac{2bl^2}{a(l-a)\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}\right]$$

4. A tightly stretched violin string of length l fixed at both ends is plucked at $x=\frac{1}{3}$ and assumes initially the shape of a triangle of height a. Find the displacement y at any distance x and at any time t after the string is released from rest.

$$\left[\textbf{Ans.: } y(x,t)=\frac{9a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}\right]$$

5. If a string of length l is initially at rest in the equilibrium position and each of its points is given a velocity v such that

$$v=cx,\qquad 0<x<\frac{l}{2}$$
$$=c\,(lx),\qquad \frac{l}{2}<x<l$$

find the displacement $y(x, t)$ at any time t.

$$\left[\textbf{Ans.: } y(x,t)=\frac{4l^2c}{a\pi^3}\left\{\sin\frac{\pi x}{l}\sin\frac{\pi at}{l}-\frac{1}{33}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l}+\cdots\right\}\right]$$

6. A string of length l is stretched and fastened to two fixed points. Find the solution of the wave equation $\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$ when initial velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0}=b\sin\frac{3\pi x}{l}\cos\frac{2\pi x}{l}$.

$$\left[\textbf{Ans.: } y(x,t)=\frac{lb}{2a\pi}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l}+\frac{lb}{5a\pi}\sin\frac{5\pi x}{l}\sin\frac{5\pi at}{l}\right]$$

7. Using D' Alembert's method, find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.

$$\left[\textbf{Ans.: } y(x,t)=k(\sin x\cos ct-\sin 2x\cos 2ct)\right]$$

11.13 D' ALEMBERT'S SOLUTION OF THE WAVE EQUATION

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(11.54)$$

Let $u = x + ct$, $v = x - ct$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly, $$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting $\frac{\partial^2 y}{\partial t^2}$ and $\frac{\partial^2 y}{\partial x^2}$ in Eq. (11.54),

$$\frac{\partial^2 y}{\partial u \partial v} = 0$$

Integrating w.r.t. v,

$$\frac{\partial y}{\partial u} = f(u) \qquad \text{...(11.55)}$$

where $f(u)$ is an arbitrary function of u.

Integrating Eq. (11.55) w.r.t. u,

$$y = \int f(u)\mathrm{d}u + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v.

$$y = \phi(u) + \psi(v), \qquad \text{where } \phi(u) = \int f(u)\, du$$

$$y(x,t) = \phi(x+ct) + \psi(x-ct) \qquad \text{...(11.56)}$$

This is the general solution of Eq. (11.54).
Assume the following conditions to determine ϕ and ψ:

Let at $t = 0$, $y(x,0) = f(x)$ and $\frac{\partial y}{\partial t} = 0$

Differentiating Eq. (11.56) w.r.t. t,

$$\frac{\partial y}{\partial t} = c\phi'(x+ct) - c\psi'(x-ct) \qquad \text{...(11.57)}$$

Putting $t = 0$ in Eq. (11.56),

$$f(x) = \phi(x) + \psi(x) \qquad \text{...(11.58)}$$

Putting $t = 0$ in Eq. (11.57),

$$0 = c\phi'(x) - c\psi'(x)$$

$$\phi'(x) = \psi'(x)$$

Integrating,

$$\phi(x) = \psi(x) + k$$

Putting $\phi(x)$ in Eq. (11.58),

$$f(x) = \psi(x) + k + \psi(x)$$

$$\psi(x) = \frac{1}{2}[f(x) - k]$$

$$\phi(x) = \frac{1}{2}[f(x) + k]$$

Replacing x by $(x + ct)$ in $\phi(x)$ and x by $(x - ct)$ in $\psi(x)$ and substituting in Eq. (11.56),

$$y(x,t) = \frac{1}{2}[f(x+ct) + k] + \frac{1}{2}[f(x-ct) - k] = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

This is known as D' Alembert's solution of the wave equation (11.54).

EXAMPLE 11.40 *Using D'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection $f(x) = a\sin^2 \pi x$.*

Solution: The equation of the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

By D' Alembert's method, the solution is

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] = \frac{1}{2}\left[a\sin^2 \pi(x+ct) + a\sin^2 \pi(x-ct)\right]$$

$$= \frac{a}{2}\left[\frac{1-\cos 2\pi(x+ct)}{2} + \frac{1-\cos 2\pi(x-ct)}{2}\right] = \frac{a}{4}\left[2 - \{\cos 2\pi(x+ct) + \cos 2\pi(x-ct)\}\right]$$

$$= \frac{a}{4}[2 - 2\cos 2\pi x \cos 2\pi ct] = \frac{a}{2}[1 - \cos 2\pi x \cos 2\pi ct]$$

11.14 ONE-DIMENSIONAL HEAT FLOW EQUATION

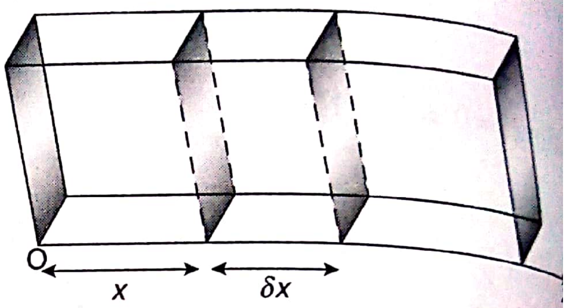

Fig. 11.6 One-dimensional heat flow

Consider a homogeneous bar of uniform cross-sectional area A and density ρ placed along the x-axis with one end at the origin O (Fig. 11.6). Let us assume that the bar is insulated laterally and, therefore, heat flows only in the x-direction.

Let $u(x, t)$ be the temperature at a distance x from the origin. If δu be the change in temperature in a slab of thickness δx of the bar then

quantity of heat in this slab $= s\rho A\delta x\delta u,$

where s is the specific heat of the bar.

The amount of heat crossing any section of the bar $= kA\left(\frac{\partial u}{\partial x}\right)\delta t,$

where A = area of cross section of the bar

$\frac{\partial u}{\partial x}$ = temperature gradient at the section

δt = time of flow of heat

k = thermal conductivity of the material of the bar

Let Q_1 and Q_2 be the quantity of heat flowing into and flowing out of the slab respectively.

$$Q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \delta t \text{ and } Q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

The negative sign indicates that heat flows in the direction of decreasing temperature.

The quantity of heat retained in the slab $= Q_1 - Q_2$

$$s\rho A\delta x\delta u = -kA\left(\frac{\partial u}{\partial x}\right)_x \delta t + kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

$$\frac{\delta u}{\delta t} = \frac{k}{s\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Taking limit $\delta x \to 0$ and $\delta t \to 0$,

$$\frac{\partial u}{\partial t} = \frac{k}{s\rho}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{...(11.59)}$$

where $\frac{k}{s\rho} = c^2$ is known as *diffusivity of the material of the bar.*

Equation (11.59) is known as the *one-dimensional heat flow equation.*

11.14.1 Solution of the One-Dimensional Heat Flow Equation

The one-dimensional heat-flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(11.60)$$

Let $u = X(x)\,T(t)$ be a solution of Eq. (11.60).

$$\frac{\partial^2 u}{\partial x^2} = X''T, \qquad \frac{\partial u}{\partial t} = XT'$$

Substituting in Eq. (11.60),

$$XT' = c^2 X''T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k, \text{ say}$$

Considering $\frac{X''}{X} = k$, $$\frac{d^2X}{dx^2} - kX = 0 \qquad ...(11.61)$$

Considering $\frac{1}{c^2}\frac{T'}{T} = k$, $$\frac{dT}{dt} - kc^2T = 0 \qquad ...(11.62)$$

Solving Eqs (11.61) and (11.62), the following cases arise:

(i) When *k* is positive

Let $k = m^2$

$$\frac{d^2X}{dx^2} - m^2X = 0 \quad \text{and} \quad \frac{dT}{dt} - m^2c^2T = 0$$

$$X = c_1' e^{mx} + c_2' e^{-mx} \quad \text{and} \quad T = c_3' e^{m^2c^2t}$$

Hence, the solution of Eq. (11.60) is

$$u = \left(c_1' e^{mx} + c_2' e^{-mx}\right)\left(c_3' e^{m^2c^2t}\right) \qquad ...(11.63)$$

(ii) When *k* is negative

Let $k = -m^2$

$$\frac{d^2X}{dx^2} + m^2X = 0 \quad \text{and} \quad \frac{dT}{dt} + m^2c^2T = 0$$

$$X = c_1' \cos mx + c_2' \sin mx \quad \text{and} \quad T = c_3' e^{-m^2c^2t}$$

Hence, the solution of Eq. (11.60) is

$$u = \left(c_1' \cos mx + c_2' \sin mx\right)\left(c_3' e^{-m^2c^2t}\right) \qquad ...(11.64)$$

(iii) When $k = 0$

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

$$X = c_1'x + c_2' \quad \text{and} \quad T = c_3'$$

Hence, the solution of Eq. (11.60) is

$$u = (c_1'x + c_2')\, c_3' \qquad \text{...(11.65)}$$

Out of these three solutions, a solution is chosen which is consistent with the physical nature of the problem. Temperature u must decrease with the increase of time.

Hence, the solution is of the form given by Eq. (11.64)

$$u = (c_1 \cos mx + c_2 \sin mx)\, e^{-m^2c^2t}$$

where $c_1'c_3' = c_1$ and $c_2'c_3' = c_2$

- **Transient Solution:** The solution is known as transient if u decreases as t increases.
- **Steady-state Condition:** A condition is known as steady state if the dependent variables are independent of the time t.

One End Insulated

EXAMPLE 11.41 *Solve the differential equation* $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation subject to the following conditions:*

(i) *u is finite when $t \to \infty$*

(ii) $\frac{\partial u}{\partial x} = 0$ *when $x = 0$ for all values of t*

(iii) *$u = 0$ when $x = l$ for all values of t*

(iv) *$u = u_0$ when $t = 0$ for $0 < x < l$*

Solution:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

Since u is finite when $t \to \infty$, the solution of Eq. (1) is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)\, e^{-km^2t} \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = l$, $\quad u = 0 \quad$ for all t, $\quad$ i.e., $u(l, t) = 0$

(ii) At $x = 0$, $\quad \frac{\partial u}{\partial x} = 0 \quad$ for all t, $\quad$ i.e., $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$

The initial conditions are

(iii) At $t = 0$, $u = u_0$ for $0 < x < l$

Applying the condition (ii) in Eq. (2),

$$\frac{\partial u}{\partial x} = \left(-c_1 m \sin mx + c_2 m \cos mx\right) e^{-km^2 t}$$

$$0 = c_2 m e^{-km^2 t}.$$

$$c_2 = 0$$

Putting $c_2 = 0$ in Eq. (2),

$$u = c_1 e^{-km^2 t} \cos mx \qquad \text{...(3)}$$

Applying the condition (i) in Eq. (3),

$$0 = c_1 e^{-km^2 t} \cos ml$$

$$\cos ml = 0 \qquad [\because c_1 \neq 0]$$

$$= \cos(2n+1)\frac{\pi}{2}, \ n \text{ is an integer}$$

$$ml = (2n+1)\frac{\pi}{2}$$

$$m = (2n+1)\frac{\pi}{2l}$$

Putting $m = (2n+1)\dfrac{\pi}{2l}$ in Eq. (3),

$$u = c_1 e^{-k(2n+1)^2 \frac{\pi^2}{4l^2} t} \cos(2n+1)\frac{\pi x}{2l}$$

Putting $n = 0, 1, 2, ...$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = \sum_{n=0}^{\infty} a_{2n+1} e^{-k(2n+1)^2 \frac{\pi^2 t}{4l^2}} \cos(2n+1)\frac{\pi x}{2l} \qquad \text{...(4)}$$

Applying the condition (iii) in Eq. (4),

$$u(x,0) = \sum_{n=0}^{\infty} a_{2n+1} e^0 \cos(2n+1)\frac{\pi x}{2l}$$

$$u_0 = \sum_{n=0}^{\infty} a_{2n+1} \cos(2n+1)\frac{\pi x}{2l} \qquad \text{...(5)}$$

Equation (5) represents the Fourier half-range cosine series.

$$a_{2n+1} = \frac{2}{l}\int_0^l u_0 \cos(2n+1)\frac{\pi x}{2l}\,dx = \frac{2u_0}{l}\left|\frac{\sin(2n+1)\frac{\pi x}{2l}}{(2n+1)\frac{\pi}{2l}}\right|_0^l$$

$$= \frac{2u_0}{l}\frac{2l}{(2n+1)\pi}\sin(2n+1)\frac{\pi}{2} = \frac{4u_0}{(2n+1)\pi}\sin(2n+1)\frac{\pi}{2}$$

Substituting a_{2n+1} in Eq. (4), the general solution is

$$u = \frac{4u_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)\frac{\pi}{2}\cos(2n+1)\frac{\pi x}{2l}e^{-k(2n+1)^2\frac{\pi^2 t}{4l^2}}$$

Steady State and Zero-Boundary Conditions

EXAMPLE 11.42 *A laterally insulated bar of length l has its ends A and B maintained at 0°C and 100°C respectively until steady-state conditions prevail. If the temperature at B is suddenly reduced to 0°C and kept so while that of A is maintained at 0°C, find the temperature at a distance x from A at any time t.*

Solution: The equation for heat conduction is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

In the steady-state condition at $t = 0$, u is independent of t.

$$\frac{\partial u}{\partial t} = 0$$

Thus, Eq. (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{...(2)}$$

Integrating Eq. (2) twice, its general solution is

$$u = ax + b \qquad \text{...(3)}$$

At $x = 0$, $u = 0$ and at $x = l$, $u = 100$

Applying these conditions to Eq. (3),

$$b = 0,\ a = \frac{100}{l}$$

Putting a and b in Eq. (3),

$$u = \frac{100}{l}x$$

Thus, the initial condition is

(i) At $t = 0$, $\quad u = \dfrac{100}{l}x$

The boundary conditions are

(ii) At $x = 0$, $\quad u = 0 \quad$ for all t, $\quad$ i.e., $\quad u(0,t) = 0$

(iii) At $x = l$, $\quad u = 0 \quad$ for all t, $\quad$ i.e., $\quad u(l,t) = 0$

Since u decreases as t increases, the solution of Eq. (1) is of the form

$$u = \left(c_1 \cos mx + c_2 \sin mx\right)e^{-c^2m^2t} \qquad \text{...(4)}$$

Applying the condition (ii) in Eq. (4),

$$0 = c_1 e^{-c^2m^2t}$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (4),

$$u = c_2 \sin mx \cdot e^{-c^2m^2t} \qquad \text{...(5)}$$

Applying the condition (iii) in Eq. (5),

$$0 = c_2 \sin ml \cdot e^{-c^2m^2t}$$

$$\sin ml = 0 \qquad [\because c_2 \neq 0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$ml = n\pi$$

$$m = \frac{n\pi}{l}$$

Putting $m = \dfrac{n\pi}{l}$ in Eq. (5),

$$u = c_2 \sin\frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2}t}$$

Putting $n = 1, 2, 3, \ldots\ldots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2}t} \qquad \text{...(6)}$$

Applying the condition (i) in Eq. (6),

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^0$$

$$\frac{100}{l}x=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \text{...(7)}$$

Equation (7) represents the Fourier half-range sine series.

$$b_n=\frac{2}{l}\int_0^l \frac{100}{l}x\sin\frac{n\pi x}{l}\,dx=\frac{200}{l^2}\left|x\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}-1\left\{\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_0^l$$

$$=\frac{200}{l^2}\left[-\frac{l^2}{n\pi}\cos n\pi+\frac{l^2}{n^2\pi^2}\sin n\pi\right]$$

$$=-\frac{200}{n\pi}(-1)^n \qquad \left[\because \cos n\pi=(-1)^n,\quad \sin n\pi=0\right]$$

$$=\frac{200}{n\pi}(-1)^{n+1}$$

Substituting b_n in Eq. (6), the general solution is

$$u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-\frac{c^2n^2\pi^2}{l^2}t}$$

Steady-State and Nonzero Boundary Conditions

EXAMPLE 11.43 *A bar AB of 10 cm length has its ends A and B kept at 30°C and 100°C respectively, until steady-state condition is reached. Then the temperature at A is lowered to 20°C and that at B to 40°C and these temperatures are maintained. Find the subsequent temperature distribution in the bar.*

Solution: The equation for heat conduction is

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

In the steady-state condition at $t=0$, u is independent of t.

$$\frac{\partial u}{\partial t}=0$$

Thus, Eq. (1) reduces to

$$\frac{\partial^2 u}{\partial x^2}=0 \qquad \text{...(2)}$$

Integrating Eq. (2) twice, its general solution is

$$u = ax + b \quad \text{...(3)}$$

At $x = 0$, $u = 30$ and at $x = 10$, $u = 100$

Applying these conditions in Eq. (3),

$$30 = b \text{ and } 100 = 10a + b = 10a + 30$$

$$a = 7$$

Putting a and b in Eq. (3),

$$u = 7x + 30$$

Thus, the initial condition is

(i) At $t = 0$, $u = 7x + 30$

The boundary conditions are

(ii) At $x = 0$, $u = 20$ for all t, i.e., $u(0, t) = 20$

(iii) At $x = 10$, $u = 40$ for all t, i.e., $u(10, t) = 40$

Since temperature at the end points is nonzero, these conditions are called nonhomogeneous. To find the temperature distribution in the bar, assume the solution as

$$u(x,t) = u_s(x) + u_{tr}(x,t) \quad \text{...(4)}$$

where $u_s(x)$ is the steady-state solution and u_{tr} is the transient solution.

To determine $u_s(x)$, solve $\dfrac{\partial^2 u_s}{\partial x^2} = 0$

Its solution is

$$u_s = a_1 x + b_1 \quad \text{...(5)}$$

At $x = 0$, $u_s = 20$ and at $x = 10$, $u_s = 40$

Applying these conditions in Eq. (5),

$$20 = b_1 \text{ and } 40 = 10a_1 + b_1 = 10a_1 + 20$$

$$a_1 = 2$$

Thus,

$$u_s = 2x + 20 \quad \text{...(6)}$$

Since $u_{tr}(x,t)$ satisfies the one-dimensional heat equation,

$$u_{tr}(x,t) = (c_1 \cos mx + c_2 \sin mx)e^{-c^2 m^2 t} \quad \text{...(7)}$$

Substituting Eqs (6) and (7) in Eq. (4),

$$u(x,t) = 2x + 20 + (c_1 \cos mx + c_2 \sin mx)e^{-c^2 m^2 t} \quad \text{...(8)}$$

Applying the condition (ii) in Eq (8),

$$20 = 20 + c_1 e^{-c^2 m^2 t}$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (8),

$$u(x,t) = 2x + 20 + c_2 \sin mx \cdot e^{-c^2 m^2 t} \quad \text{...(9)}$$

Applying the condition (iii) in Eq. (9),

$$40 = 20 + 20 + c_2 \sin 10m \cdot e^{-c^2 m^2 t}$$

$$0 = c_2 \sin 10m \cdot e^{-c^2 m^2 t}$$

$$\sin 10m = 0 \quad [\because c_2 \neq 0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$m = \frac{n\pi}{10}$$

Putting $m = \dfrac{n\pi}{10}$ in Eq. (9),

$$u(x,t) = 2x + 20 + c_2 \sin \frac{n\pi x}{10} e^{-c^2 \frac{n^2\pi^2}{100} t}$$

Putting $n = 1, 2, 3, \ldots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^{-\frac{c^2 n^2 \pi^2}{100} t} \quad \text{...(10)}$$

Applying the condition (i) in Eq. (10),

$$u(x,0) = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^0$$

$$7x + 30 = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}$$

$$5x + 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \quad \text{...(11)}$$

Equation (11) represents the Fourier half-range sine series.

$$b_n = \frac{2}{10} \int_0^{10} (5x+10) \sin \frac{n\pi x}{10} dx = \frac{1}{5} \left| (5x+10) \left\{ \frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right\} - 5 \left\{ \frac{-\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2} \right\} \right|_0^{10}$$

$$= \frac{1}{5}\left| -\frac{(5x+10)10}{n\pi}\cos\frac{n\pi x}{10} + \frac{5(100)}{n^2\pi^2}\sin\frac{n\pi x}{10} \right|_0^{10}$$

$$= \frac{1}{5}\left[-\frac{600}{n\pi}\cos n\pi + \frac{100}{n\pi} \right] \qquad [\because \sin n\pi = 0]$$

$$= \frac{20}{n\pi}\left[-6(-1)^n + 1 \right]$$

Substituting b_n in Eq. (10), the general solution is

$$u(x,t) = 2x + 20 + \frac{20}{\pi}\sum\left[\frac{1-6(-1)^n}{n}\right]\sin\frac{n\pi x}{10}e^{-\frac{c^2n^2\pi^2}{100}t}$$

Both Ends Insulated

EXAMPLE 11.44 *The temperature at one end of a 50 cm long bar with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state condition prevails. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution: The equation for temperature distribution is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

In the steady-state condition at $t = 0$, u is independent of t.

$$\frac{\partial u}{\partial t} = 0$$

Thus, Eq. (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{...(2)}$$

Integrating Eq. (2) twice, its general solution is

$$u = ax + b \qquad \text{...(3)}$$

At $x = 0$, $u = 0$ and at $x = 50$, $u = 100$

Applying these conditions in Eq. (3),

$$0 = b, \quad 100 = 50a + b = 50a$$
$$a = 2$$

Putting a and b in Eq. (3),

$$u = 2x$$

Thus, the initial condition is

(i) At $t = 0$, $u = 2x$

When the ends $x = 0$ and $x = 50$ of the bar are insulated, no heat can flow through them. Thus, the boundary conditions are

(ii) At $x = 0$, $\frac{\partial u}{\partial x} = 0$ for all t, i.e., $\frac{\partial u(0,t)}{\partial x} = 0$

(iii) At $x = 50$, $\frac{\partial u}{\partial x} = 0$ for all t, i.e., $\frac{\partial u(50,t)}{\partial x} = 0$

The solution of Eq. (1) is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)e^{-c^2m^2t} \qquad \text{...(4)}$$

$$\frac{\partial u}{\partial x} = (-c_1 m \sin mx + c_2 m \cos mx)e^{-c^2m^2t} \qquad \text{...(5)}$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 m e^{-c^2m^2t}$$

$$c_2 = 0$$

Putting $c_2 = 0$ in Eq. (5),

$$\frac{\partial u}{\partial x} = -c_1 m \sin mx \cdot e^{-c^2m^2t} \qquad \text{...(6)}$$

Applying the condition (iii) in Eq. (6),

$$0 = -c_1 m \sin 50m \cdot e^{-c^2m^2t}$$

$$\sin 50m = 0 \qquad [\because c_1 \neq 0, \text{ otherwise } u(x,t) = 0]$$

$$= \sin n\pi, \qquad n \text{ is an integer}$$

$$50m = n\pi$$

$$m = \frac{n\pi}{50}$$

Putting $m = \frac{n\pi}{50}$ and $c_2 = 0$ in Eq. (4),

$$u(x,t) = c_1 \cos \frac{n\pi x}{50} e^{-\frac{c^2n^2\pi^2}{2500}t} \qquad \text{...(7)}$$

Putting n = 0, 1, 2, ... in Eq. (7) and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{50} e^{-\frac{c^2n^2\pi^2}{2500}t} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{50} e^{-\frac{c^2n^2\pi^2}{2500}t} \qquad \text{...(8)}$$

Applying the condition (i) in Eq. (8),

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{50} e^0$$

$$2x = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{50} \qquad \text{...(9)}$$

Equation (9) represents the Fourier half-range cosine series.

$$a_0 = \frac{1}{50}\int_0^{50} 2x dx = \frac{1}{25}\left|\frac{x^2}{2}\right|_0^{50} = 50$$

$$a_n = \frac{2}{50}\int_0^{50} 2x \cos\frac{n\pi x}{50} dx = \frac{2}{25}\left| x\cdot\frac{\sin\frac{n\pi x}{50}}{\left(\frac{n\pi}{50}\right)} - 1\left\{\frac{-\cos\frac{n\pi x}{50}}{\left(\frac{n\pi}{50}\right)^2}\right\}\right|_0^{50}$$

$$= \frac{2}{25}\left[\left(\frac{50}{n\pi}\right)^2(\cos n\pi - \cos 0)\right] = \frac{200}{n^2\pi^2}\left[(-1)^n - 1\right].$$

Substituting a_0 and a_n in Eq. (8), the general solution is

$$u(x,t) = 50 + \frac{200}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{50} e^{-\frac{c^2 n^2 \pi^2}{2500}t}$$

$$= 50 - \frac{400}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}\cos\frac{(2r-1)\pi x}{50} e^{-\frac{c^2(2r-1)^2\pi^2}{2500}t} \qquad \left[\begin{array}{l} \because (-1)^n - 1 = 0, \text{ if } n \text{ is even} \\ \qquad\qquad\; = -2, \text{ if } n \text{ is odd} \\ \text{Taking } n = 2r-1 \end{array}\right]$$

Zero Boundary Conditions

EXAMPLE 11.45 *Find the temperature in a laterally insulated bar of 2 cm length whose ends are kept at zero temperature and the initial temperature is* $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$.

Solution: The equation for heat conduction is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

Since both the ends of the bar are at zero temperature, the solution of Eq. (1) is of the form

$$u=(c_1 \cos mx+c_2 \sin mx)e^{-c^2m^2t} \quad \ldots(2)$$

The boundary conditions are

(i) At $x = 0$, $u = 0$ for all t, i.e., $u(0, t) = 0$

(ii) At $x = 2$, $u = 0$ for all t, i.e., $u(2, t) = 0$

The initial conditions are

(iii) At $t = 0$, $u = \sin\frac{\pi x}{2}+3\sin\frac{5\pi x}{2}$

Applying the condition (i) in Eq. (2),

$$0= c_1 e^{-c^2m^2t}$$

$$c_1 =0$$

Putting $c_1 =0$ in Eq. (2),

$$u= c_2 \sin mx \cdot e^{-c^2m^2t} \quad \ldots(3)$$

Applying the condition (ii) in Eq. (3),

$$0= c_2 \sin 2m \cdot e^{-c^2m^2t}$$

$$\sin 2m=0 \qquad [\because c_2 \neq 0]$$

$$=\sin n\pi, \qquad n \text{ is an integer}$$

$$2m=n\pi$$

$$m=\frac{n\pi}{2}$$

Putting $m=\frac{n\pi}{2}$ in Eq. (3),

$$u=c_2 \sin\frac{n\pi x}{2} e^{-c^2\frac{n^2\pi^2}{4}t}$$

Putting n = 1, 2, 3, ... and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} e^{-\frac{c^2n^2\pi^2}{4}t} \quad \ldots(4)$$

Applying the condition (iii) in Eq. (4),

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} e^0$$

$$\sin\frac{\pi x}{2}+3\sin\frac{5\pi x}{2}=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$=b_1 \sin\frac{\pi x}{2}+b_2 \sin \pi x+b_3 \sin\frac{3\pi x}{2}+b_4 \sin 2\pi x +b_5 \sin\frac{5\pi x}{2}+\ldots\ldots$$

Comparing coefficients on both sides,

$$b_1=1,\ b_2=0,\ b_3=0,\ b_4=0,\ b_5=3,\ b_6=0.....,\ b_n=0,\ \text{for } n\geq 6$$

Substituting the values of b's in Eq. (4), the general solution is

$$u(x,t)=b_1\sin\frac{\pi x}{2}e^{-\frac{c^2\pi^2}{4}t}+b_5\sin\frac{5\pi x}{2}e^{-\frac{25c^2\pi^2}{4}t}=\sin\frac{\pi x}{2}e^{-\frac{c^2\pi^2}{4}t}+3\sin\frac{5\pi x}{2}e^{-\frac{25c^2\pi^2}{4}t}$$

EXERCISE 11.9

1. Solve the equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(x,0)=3\sin n\pi x$, $u(0,t)=0$ and $u(1,t)=0$, where $0<x<1, t>0$.

$$\left[\textbf{Ans.: } u(x,t)=3\sum_{n=1}^{\infty}e^{-n^2\pi^2 t}\sin n\pi x\right]$$

2. Find the transient state temperature of a nonradiating rod of length π whose ends are kept at ice-cold temperature, the temperature of the rod being initially $(\pi x-x^2)$ at a distance x from an end.

$$\left[\textbf{Ans.: } u(x,t)=\frac{8}{\pi}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^3}\sin(2r-1)\pi\cdot e^{-c^2(2r-1)^2 t}\right]$$

3. Solve the equation $\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod of length l subject to the following conditions:

(i) u is finite for $t\to\infty$

(ii) $\frac{\partial u}{\partial x}=0$ for $x=0$ and $x=l$ for all t

(iii) $u=lx-x^2$ for $t=0$ between $x=0$ and $x=l$

$$\left[\textbf{Ans.: } u(x,t)=\frac{l^2}{6}-\frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-\frac{4m^2\pi^2 k}{l^2}t}\right]$$

4. A bar AB of 20 cm length has its ends A and B kept at 30°C and 80°C until steady–state prevails. Then the temperatures at A and B are suddenly changed to 40°C and 60°C respectively. Find the temperature distribution of the rod.

$$\left[\textbf{Ans.: } u(x,t)=40+x-\frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1+2\cos n\pi}{n}\sin\frac{n\pi x}{20}\cdot e^{-\frac{n^2\pi^2c^2}{400}t}\right]$$

5. A rod of length l has its ends A and B kept at 0°C and 100°C respectively until steady-state condition prevails. Temperature at A is raised to 25°C and that of B is reduced to 75°C and kept so. Find the temperature distribution.

$$\left[\textbf{Ans.: } u(x,t)=\frac{50x}{l}+25-\frac{50}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}e^{-\frac{4c^2n^2\pi^2}{l^2}t}\right]$$

6. A 100 cm long bar, with insulated sides, has its ends kept at 0°C and 100°C until steady-state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.

$$\left[\textbf{Ans.:}\ u(x,t)=\frac{l}{2}-\frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}e^{\frac{-c^2(2n-1)^2\pi^2}{l^2}t}\right]$$

7. A bar with insulated sides is initially at temperature 0°C throughout. The end $x = 0$ is kept at 0°C and heat is suddenly applied so that $\frac{\partial u}{\partial x}=10$ at $x = l$ for all t. Find the temperature distribution.

$$\left[\textbf{Ans.:}\ u(x,t)=10x-\frac{80l}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sin\left(\frac{2n+1}{2l}\right)\pi x\cdot e^{\frac{-c^2(2n+1)^2\pi^2}{l^2}t}\right]$$

11.15 TWO-DIMENSIONAL HEAT FLOW EQUATION

Consider a homogeneous metal plate of uniform thickness h (cm), density ρ (g/cm^3), specific heat s (cal/g/deg), and thermal conductivity k (cal/cm deg). Assume that the faces of the plate are perfectly insulated so that no heat flows in the transversal direction to the plate. Hence, heat is allowed to flow only in the directions of the plane of the plate. Therefore, the flow is said to be two-dimensional. Let the plate be in the xy-plane and u be the temperature at any point of the plate. Since the faces of the plate are insulated, u depends only on x, y, and the time t.

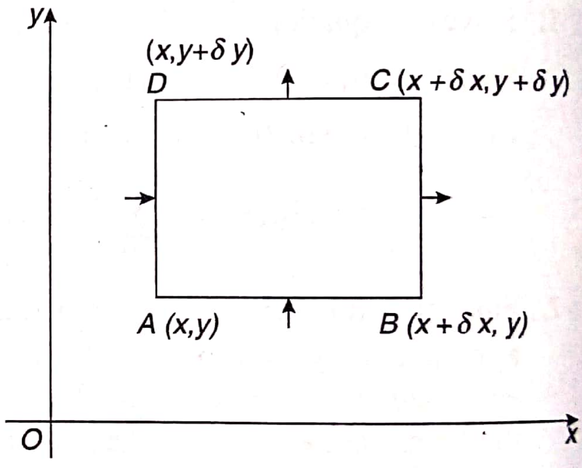

Fig. 11.7 Two-dimensional heat flow

Consider a small rectangular element $ABCD$ of the plate with vertices A (x, y), B $(x + \delta x, y)$, C $(x + \delta x, y + \delta y)$, and D $(x, y + \delta y)$ (Fig. 11.7).

The amount of heat, at time t, in the element is $Q = \rho\,\delta x\,\delta y\,hsu$

The rate of change of Q w. r. t. time is $\frac{dQ}{dt} = \rho\,\delta x\,\delta y\,hs\frac{\partial u}{\partial t}$ ...(11.66)

The amount of heat entering the element in 1 second from the side $AB = -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y$

The amount of heat entering the element in 1 second from the side $AD = -kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_x$

The amount of heat flowing out the element in 1 second from the side $CD = -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$

The amount of heat flowing out the element in 1 second from the side $BC = -kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

The total rate of gain of heat by the element

$$= -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y - kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_x + kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= kh\,\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] + kh\,\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$= kh\,\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] + kh\,\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right] \qquad \text{...(11.67)}$$

Equating Eqs (11.66) and (11.67),

$$\rho\,\delta x\,\delta y\,hs\frac{\partial u}{\partial t} = kh\,\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right]$$

Dividing both sides by $h\,\delta x\,\delta y$ and taking limit $\delta x \to 0,\ \delta y \to 0$,

$$\frac{\partial u}{\partial t} = \frac{k}{\rho s}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{...(11.68)}$$

where $\frac{k}{\rho s} = c^2$ is known as the *diffusivity of the material of the plate.*

Equation (11.68) is known as the *two-dimensional heat flow equation* and gives the temperature distribution of the plate in the transient state.

In the steady state, u is independent of t.

$$\frac{\partial u}{\partial t} = 0$$

Hence, Eq. (11.68) reduces to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

which is Laplace's equation in two dimensions.

Solution of Laplace's Equation

Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{...(11.69)}$$

Let $u = X(x)Y(y)$ be a solution of Eq. (11.69).

$$\frac{\partial^2 u}{\partial x^2}=X''Y, \quad \frac{\partial^2 u}{\partial y^2}=XY''$$

Substituting in Eq (11.69),

$$X''Y+XY''=0$$

$$\frac{X''}{X}=-\frac{Y''}{Y}=k, \text{ say}$$

Considering $\quad \frac{X''}{X}=k, \quad \frac{d^2X}{dx^2}-kX=0 \qquad \text{...(11.70)}$

Considering $\quad -\frac{Y''}{Y}=k, \quad \frac{d^2Y}{dy^2}+kY=0 \qquad \text{...(11.71)}$

Solving Eqs (11.70) and (11.71), the following cases arise:

(i) When k is positive

Let $k = m^2$

$$\frac{d^2X}{dx^2}-m^2X=0 \text{ and } \frac{d^2Y}{dy^2}+m^2Y=0$$

$$X=c_1e^{mx}+c_2e^{-mx} \text{ and } Y=c_3\cos my+c_4\sin my$$

Hence, the solution of Eq. (11.69) is

$$u=\left(c_1e^{mx}+c_2e^{-mx}\right)\left(c_3\cos my+c_4\sin my\right)$$

(ii) When k is negative

Let $k = -m^2$

$$\frac{d^2X}{dx^2}+m^2X=0 \text{ and } \frac{d^2Y}{dy^2}-m^2Y=0$$

$$X=c_1\cos mx+c_2\sin mx \text{ and } Y=c_3e^{my}+c_4e^{-my}$$

Hence, the solution of Eq. (11.69) is

$$u = \left(c_1 \cos mx + c_2 \sin mx\right)\left(c_3 e^{my} + c_4 e^{-my}\right)$$

(iii) When $k = 0$

$$\frac{d^2X}{dx^2} = 0 \text{ and } \frac{d^2Y}{dy^2} = 0$$

$$X = c_1 x + c_2 \text{ and } Y = c_3 y + c_4$$

Hence, the solution of Eq. (11.69) is

$$u = \left(c_1 x + c_2\right)\left(c_3 y + c_4\right)$$

Out of these three solutions, a solution is chosen which is consistent with the physical nature of the problem.

EXAMPLE 11.46 *Find the steady-state temperature distribution in a thin plate bounded by the lines $x = 0$, $x = a$, $y = 0$, and $y \to \infty$ assuming that heat cannot escape from either surface. The sides $x = 0$, $x = a$, and $y = \infty$ being kept at zero temperature and $y = 0$ is kept at $f(x)$.*

Solution: In steady state, the heat equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$

The three possible solutions of Eq. (1) are

$$u = \left(c_1 e^{mx} + c_2 e^{-mx}\right)\left(c_3 \cos my + c_4 \sin my\right) \qquad \text{...(2)}$$

$$u = \left(c_1 \cos mx + c_2 \sin mx\right)\left(c_3 e^{my} + c_4 e^{-my}\right) \qquad \text{...(3)}$$

$$u = \left(c_1 x + c_2\right)\left(c_3 y + c_4\right) \qquad \text{...(4)}$$

The boundary conditions are

(i) At $x = 0$, $u = 0$, i.e., $u(0, y) = 0$
(ii) At $x = a$, $u = 0$, i.e., $u(a, y) = 0$
(iii) At $y \to \infty$, $u = 0$, i.e., $u(x, \infty) = 0$
(iv) At $y = 0$, $u = f(x)$, i.e., $u(x, 0) = f(x)$

Applying conditions (i) and (ii) in Eq. (2),

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 e^{ma} + c_2 e^{-ma} = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.

Hence, the solution by Eq. (2) is rejected.
Applying conditions (i) and (ii) in Eq. (4),

$$c_2=0 \quad \text{and} \quad c_1 a+c_2=0$$

Solving these equations,

$$c_1=0,\ c_2=0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (4) is rejected.
Thus, the suitable solution for the present problem is a solution by Eq. (3).
Applying the condition (i) in Eq. (3),

$$0 = c_1\left(c_3 e^{my}+c_4 e^{-my}\right)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (3),

$$u=c_2 \sin mx\left(c_3 e^{my}+c_4 e^{-my}\right) \qquad \text{...(5)}$$

Applying the condition (ii) in Eq. (5),

$$0=c_2 \sin ma\left(c_3 e^{my}+c_4 e^{-my}\right)$$

$$\sin ma=0 \qquad \left[\because c_2 \neq 0, \text{ otherwise } u=0\right]$$

$$=\sin n\pi, \ n \text{ is an integer}$$

$$ma=n\pi$$

$$m=\frac{n\pi}{a}$$

Putting $m=\dfrac{n\pi}{a}$ in Eq. (5),

$$u=c_2 \sin\frac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}}+c_4 e^{-\frac{n\pi y}{a}}\right) \qquad \text{...(6)}$$

Applying the condition (iii) in Eq. (6) after rewriting as

$$u e^{-\frac{n\pi y}{a}}=c_2 \sin\frac{n\pi x}{a}\left(c_3+c_4 e^{-\frac{2n\pi y}{a}}\right)$$

$$0=c_2 \sin\frac{n\pi x}{a}(c_3) \qquad \left[\because e^{-\infty}=0\right]$$

$$c_3=0 \qquad \left[\because c_2 \neq 0\right]$$

Putting $c_3 = 0$ in Eq. (6),

$$u= c_2 c_4 \sin\frac{n\pi x}{a} e^{-\frac{n\pi y}{a}}=b_n \sin\frac{n\pi x}{a} e^{-\frac{n\pi y}{a}}, \quad \text{where } c_2 c_4=b_n$$

Putting $n = 1, 2, 3, \ldots\ldots$ and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} e^{-\frac{n\pi y}{a}} \qquad \ldots(7)$$

Applying the condition (iv) in Eq. (7),

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} e^0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \qquad \ldots(8)$$

Equation (8) represents the Fourier half-range sine series in $(0, a)$.

$$b_n = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a} dx \qquad \ldots(9)$$

Hence, the required temperature distribution is given by Eq. (8), where b_n is given by Eq. (9).

EXAMPLE 11.47 *Solve the Laplace equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *subject to the conditions* $u(0, y) = u(l, y) = u(x, 0) = 0$, *and* $u(x, a) = \sin\frac{n\pi x}{l}$.

Solution:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(1)$$

The three possible solutions of Eq. (1) are

$$u = \left(c_1 e^{mx} + c_2 e^{-mx}\right)\left(c_3 \cos my + c_4 \sin my\right) \qquad \ldots(2)$$

$$u = \left(c_1 \cos mx + c_2 \sin mx\right)\left(c_3 e^{my} + c_4 e^{-my}\right) \qquad \ldots(3)$$

$$u = \left(c_1 x + c_2\right)\left(c_3 y + c_4\right) \qquad \ldots(4)$$

The boundary conditions are

(i) $u(0, y) = 0$, i.e., at $x = 0,\ u = 0$
(ii) $u(l, y) = 0$, i.e., at $x = l,\ u = 0$
(iii) $u(x, 0) = 0$, i.e., at $y = 0,\ u = 0$
(iv) $u(x, a) = \sin\frac{n\pi x}{l}$, i.e., at $y = a,\ u = \sin\frac{n\pi x}{l}$

Applying conditions (i) and (ii) in Eq. (2),

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 e^{ml} + c_2 e^{-ml} = 0$$

Solving these equations,

$$c_1 = 0,\ c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (2) is rejected.
Applying conditions (i) and (ii) in Eq. (4),

$$c_2 = 0 \text{ and } c_1 l + c_2 = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (4) is rejected.
Thus, the suitable solution for the present problem is a solution by Eq. (3).
Applying the condition (i) in Eq. (3),

$$0 = c_1 \left(c_3 e^{my} + c_4 e^{-my}\right)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (3),

$$u = c_2 \sin mx \left(c_3 e^{my} + c_4 e^{-my}\right) \qquad \text{...(5)}$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 \sin ml \left(c_3 e^{my} + c_4 e^{-my}\right)$$

$$\sin ml = 0 \qquad \left[\because c_2 \neq 0, \text{otherwise } u = 0\right]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$ml = n\pi$$

$$m = \frac{n\pi}{l}$$

Putting $m = \frac{n\pi}{l}$ in Eq. (5),

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} + c_4 e^{-\frac{n\pi y}{l}} \right) \qquad \text{...(6)}$$

Applying the condition (iii) in Eq. (6),

$$0 = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^0 + c_4 e^0\right)$$

$$c_3 + c_4 = 0, \quad c_4 = -c_3$$

Putting $c_4 = -c_3$ in Eq. (6),

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} - c_3 e^{-\frac{n\pi y}{l}} \right)$$

$$u(x,y)=c_2\,c_3\sin\frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}}-e^{-\frac{n\pi y}{l}}\right)=b_n\sin\frac{n\pi x}{l}\,2\sinh\frac{n\pi y}{l} \qquad ...(7)$$

where $c_2\,c_3=b_n$

Applying the condition (iv) in Eq. (7),

$$u(x,a)=2b_n\sin\frac{n\pi x}{l}\sinh\frac{n\pi a}{l}$$

$$\sin\frac{n\pi x}{l}=2b_n\sin\frac{n\pi x}{l}\sinh\frac{n\pi a}{l}$$

$$b_n=\frac{1}{2\sinh\frac{n\pi a}{l}}$$

Substituting b_n in Eq. (7), the general solution of Eq. (1) is

$$u(x,y)=\frac{1}{2\sinh\frac{n\pi a}{l}}\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l}=\frac{\sinh\frac{n\pi y}{l}}{2\sinh\frac{n\pi a}{l}}\sin\frac{n\pi x}{l}$$

EXAMPLE 11.48 *A rectangular plate with insulated surface has 'a' width of a cm and is so long as compared to its width that it may be considered of infinite length without introducing an appreciable error. If the two long edges $x=0$ and $x=a$ as well as the one short edge are kept at 0°C and the temperature of the other short edge $y=0$ is given by*

$$u=kx,\qquad 0\le x\le\frac{a}{2}$$

$$=k(a-x),\qquad \frac{a}{2}\le x\le a$$

find the temperature $u(x,y)$ at any point (x,y) of the plate in the steady state.

Solution: In the steady state, the heat equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad ...(1)$$

The three possible solutions of Eq. (1) are

$$u=(c_1e^{mx}+c_2e^{-mx})(c_3\cos my+c_4\sin my) \qquad ...(2)$$

$$u=(c_1\cos mx+c_2\sin mx)(c_3e^{my}+c_4e^{-my}) \qquad ...(3)$$

$$u=(c_1x+c_2)(c_3y+c_4) \qquad ...(4)$$

The boundary conditions are

(i) At $x = 0$, $u = 0$, i.e., $u(0, y) = 0$

(ii) At $x = a$, $u = 0$, i.e., $u(a, y) = 0$

(iii) At $y \to \infty$, $u = 0$, i.e., $u(x, \infty) = 0$

(iv) At $y = 0$, $u = kx$, $0 \le x \le \frac{a}{2}$

$= k(a - x)$, $\frac{a}{2} \le x \le a$

Applying conditions (i) and (ii) in Eq. (2),

$$c_1 + c_2 = 0 \text{ and } c_1 e^{ma} + c_2 e^{-ma} = 0$$

Solving these equations,

$$c_1 = 0, \; c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (2) is rejected.

Applying conditions (i) and (ii) in Eq. (4),

$$c_2 = 0 \text{ and } c_1 a + c_2 = 0$$

Solving these equations,

$$c_1 = 0, \; c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (4) is rejected.
Thus, the suitable solution for the present problem is a solution by Eq. (3).
Applying the condition (i) in Eq. (3),

$$0 = c_1 \left(c_3 e^{my} + c_2 e^{-my}\right)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (3),

$$u = c_2 \sin mx \left(c_3 e^{my} + c_4 e^{-my}\right) \qquad \text{...(5)}$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 \sin ma \left(c_3 e^{my} + c_4 e^{-my}\right)$$

$$\sin ma = 0 \qquad [\because c_2 \ne 0, \text{ otherwise } u = 0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$ma = n\pi$$

$$m = \frac{n\pi}{a}$$

Putting $m=\frac{n\pi}{a}$ in Eq. (5),

$$u=c_2\sin\frac{n\pi x}{a}\left(c_3e^{\frac{n\pi y}{a}}+c_4e^{-\frac{n\pi y}{a}}\right) \quad ...(6)$$

Applying the condition (iii) in Eq. (6) after rewriting as

$$ue^{-\frac{n\pi y}{a}}=c_2\sin\frac{n\pi x}{a}\left(c_3+c_4e^{-\frac{2n\pi y}{a}}\right)$$

$$0=c_2\sin\frac{n\pi x}{a}(c_3)$$

$$c_3=0 \quad [\because c_2\neq 0]$$

Putting $c_3=0$ in Eq. (6),

$$u=c_2c_4\sin\frac{n\pi x}{a}e^{-\frac{n\pi y}{a}}=b_n\sin\frac{n\pi x}{a}e^{-\frac{n\pi y}{a}}, \text{ where } c_2c_4=b_n$$

Putting $n=1, 2, 3, \ldots$ and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,y)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{a}e^{-\frac{n\pi y}{a}} \quad ...(7)$$

Applying the condition (iv) in Eq. (7),

$$u(x,0)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{a} \quad ...(8)$$

Equation (8) represents the Fourier half-range sine series in $(0, a)$.

$$b_n=\frac{2}{a}\int_0^a u(x,0)\sin\frac{n\pi x}{a}dx=\frac{2}{a}\left[\int_0^{\frac{a}{2}}kx\sin\frac{n\pi x}{a}dx+\int_{\frac{a}{2}}^{a}k(a-x)\sin\frac{n\pi x}{a}dx\right]$$

$$=\frac{2k}{a}\left|x\cdot\frac{\left(-\cos\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)}-1\left\{\frac{\left(-\sin\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)^2}\right\}\right|_0^{\frac{a}{2}}+\frac{2k}{a}\left|(a-x)\frac{\left(-\cos\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)}-(-1)\frac{\left(-\sin\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)^2}\right|_{\frac{a}{2}}^{a}$$

$$=\frac{2k}{a}\left[\frac{-a^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{a^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]=\frac{4ka}{\pi^2n^2}\sin\frac{n\pi}{2}$$

Substituting b_n in Eq. (7), the general solution is

$$u(x,y)=\frac{4ka}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{a}e^{-\frac{n\pi y}{a}}$$

$$=\frac{4ka}{\pi^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\sin\frac{(2r+1)\pi x}{a}e^{-\frac{(2r+1)\pi y}{a}}$$

$$\left[\begin{aligned}&\because \sin\frac{n\pi}{2}=0, \quad \text{if } n \text{ is even}\\ &\qquad = 1 \text{ or } -1, \quad \text{if } n \text{ is odd}\\ &\text{Putting } n=2r+1,\\ &\sin\frac{n\pi}{2}=\sin\frac{(2r+1)\pi}{2}=\sin\left(\pi r+\frac{\pi}{2}\right)\\ &\qquad=\cos\pi r=(-1)^r\end{aligned}\right]$$

EXERCISE 11.10

1. A rectangular plate with an insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x,0) = 100\sin\frac{\pi x}{8}, \quad 0<x<8$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane of the plate is given by

$$u(x,y)=100e^{-\frac{\pi y}{8}}\sin\frac{\pi x}{8}$$

2. The function $v(x,y)$ satisfies the Laplace's equation in rectangular coordinates (x, y) and for points within the rectangle $x = 0$, $x = a$, $y = 0$, $y = b$, it satisfies the conditions $v(0,y) = v(a,y)=v(x,b)=0$ and $v(x,0) = x(a-x), 0<x<a$. Show that $v(x,y)$ is given by

$$v(x,y)=\frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\sin\frac{(2n+1)\pi x}{a}}{(2n+1)^3}\,\frac{\sinh\frac{(2n+1)\pi(b-y)}{a}}{\sinh\frac{(2n+1)\pi b}{a}}$$

3. A long rectangular plate of width a cm with an insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0,y)=0, v(a,y)=0, v(x,\infty)=0$, $v(x,0)=kx$. Show that steady-state temperature within the plate is

$$v(x,y)=\frac{2ak}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{n\pi y}{a}}\sin\frac{n\pi x}{a}$$

4. A rectangular plate with 6 cm wide insu-lated surfaces is so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x,0) = 90\sin\frac{\pi x}{6}$, $0<x<6$ while the two long edges $x = 0$ and $x = 6$ as well as the other short edge are maintained at 0°C, find the function $u(x, y)$ given the steady-state temperature at any point (x, y) of the plate.

$$\left[\textbf{Ans.: } u(x,y)=90\sin\frac{\pi x}{6}e^{-\frac{\pi y}{6}}\right]$$

11.16 LAPLACE'S EQUATION IN POLAR COORDINATES

The two-dimensional Laplace equation in Cartesian coordinates is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

By the transformation $x=r\cos\theta,\ y=r\sin\theta$, this equation is transformed to

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

which is known as *Laplace's equation in polar coordinates.*

Solution of Laplace's Equation in Polar Coordinates

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \qquad ...(11.72)$$

Let $u=R(r)\,F(\theta)$ be a solution of Eq. (11.72), where R is a function of r alone and F is a function of θ alone.

Differentiating $u=R(r)\,F(\theta)$ w.r.t. r and θ twice,

$$\frac{\partial u}{\partial r}=R'F\ ,\quad \frac{\partial u}{\partial \theta}=RF'$$

$$\frac{\partial^2 u}{\partial r^2}=R''F\ ,\quad \frac{\partial^2 u}{\partial \theta^2}=RF''$$

Substituting in Eq. (11.72),

$$R''F+\frac{1}{r}R'F+\frac{1}{r^2}RF''=0$$

$$r^2R''F+rR'F+RF''=0$$

$$F(r^2R''+rR')+RF''=0$$

$$\frac{r^2R''+rR'}{R}=-\frac{F''}{F} \qquad ...(11.73)$$

Since R is a function of r alone and F is a function of θ alone, and r and θ are independent variables, Eq. (11.73) holds good only if each side is equal to a constant.

$$\frac{r^2R''+rR'}{R}=-\frac{F''}{F}=k\ \text{(constant)}$$

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-kR=0 \qquad ...(11.74)$$

and

$$\frac{d^2F}{d\theta^2}+kF=0 \qquad ....(11.75)$$

Equation (11.74) is a Cauchy's homogeneous linear differential equation.
Putting $r = e^z$, Eq. (11.74) reduces to

$$D(D-1)R + DR - kR = 0, \text{ where } D \equiv \frac{d}{dz}$$

$$(D^2 - k)R = 0$$

$$\frac{d^2R}{dz^2} - kR = 0 \qquad \text{....(11.76)}$$

Solving Eqs (11.75) and (11.76), the following cases arise:

(i) When k is positive

Let $k = m^2$

$$\frac{d^2R}{dz^2} - m^2R = 0 \quad \text{and} \quad \frac{d^2F}{d\theta^2} + m^2F = 0$$

$$R = c_1 e^{mz} + c_2 e^{-mz} \quad \text{and} \quad F = c_3 \cos m\theta + c_4 \sin m\theta$$

$$= c_1 r^m + c_2 r^{-m}$$

Hence, the solution of Eq. (11.72) is

$$u = \left(c_1 r^m + c_2 r^{-m}\right)\left(c_3 \cos m\theta + c_4 \sin m\theta\right)$$

(ii) When k is negative

Let $k = -m^2$

$$\frac{d^2R}{dz^2} + m^2R = 0 \quad \text{and} \quad \frac{d^2F}{d\theta^2} - m^2F = 0$$

$$R = c_1 \cos mz + c_2 \sin mz \quad \text{and} \quad F = c_3 e^{m\theta} + c_4 e^{-m\theta}$$

$$= c_1 \cos(m \log r) + c_2 \sin(m \log r)$$

Hence, the solution of Eq. (11.72) is

$$u = [c_1 \cos(m \log r) + c_2 \sin(m \log r)](c_3 e^{m\theta} + c_4 e^{-m\theta})$$

(iii) When $k = 0$

$$\frac{d^2R}{dz^2} = 0 \quad \text{and} \quad \frac{d^2F}{d\theta^2} = 0$$

$$R = c_1 z + c_2 \quad \text{and} \quad F = c_3\theta + c_4$$

$$= c_1 \log r + c_2$$

Hence, the solution of Eq. (11.72) is

$$u = (c_1 \log r + c_2)(c_3 \theta + c_4)$$

Out of these three solutions, a solution is chosen which is consistent with the physical nature of the problem.

EXAMPLE 11.49 *Obtain steady-state temperature distribution in a semicircular plate of radius 'a', insulated on both faces, with its circular boundary kept at a constant temperature u_0 and boundary diameter kept at zero temperature.*

Solution: In steady state, the polar form of heat equation in two dimensions is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad \text{...(1)}$$

The three possible solutions of Eq. (1) are

$$u=(c_1 r^m + c_2 r^{-m})(c_3 \cos m\theta + c_4 \sin m\theta) \quad \text{...(2)}$$

$$u=[c_1 \cos(m \log r)+c_2 \sin(m \log r)](c_3 e^{m\theta}+c_4 e^{-m\theta}) \quad \text{...(3)}$$

$$u=(c_1 \log r + c_2)(c_3\theta + c_4) \quad \text{...(4)}$$

Consider the centre of the plate as the pole and bounding diameter as the initial line (Fig. 11.8). The boundary conditions are

(i) $u(r,0)=0, \quad 0 \le r \le a$

(ii) $u(r,\pi)=0, \quad 0 \le r \le a$

(iii) $u(a,\theta)=u_0$

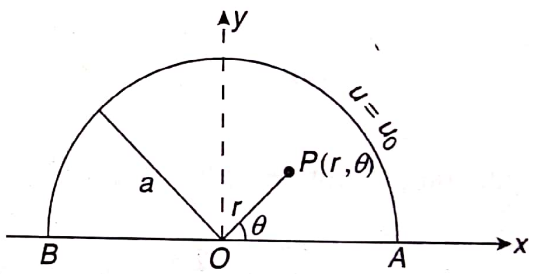

Fig. 11.8

Since conditions (i) and (ii) [at $r=0, u=0$] cannot be applied to solutions by Eqs (3) and (4), these solutions are rejected.

Thus, the solution by Eq. (2) is the suitable solution for the present problem.

Applying the condition (i) in Eq. (2),

$$0=(c_1 r^m + c_2 r^{-m})c_3$$

$$c_3 = 0$$

Putting $c_3 = 0$ in Eq. (2),

$$u=(c_1 r^m + c_2 r^{-m})c_4 \sin m\theta \quad \text{...(5)}$$

Applying the condition (ii) in Eq. (5),

$$0=(c_1 r^m + c_2 r^{-m})c_4 \sin m\pi$$

$$\sin m\pi = 0 \qquad [\because c_4 \ne 0, \text{ otherwise } u=0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$m\pi = n\pi$$

$$m = n$$

Putting $m=n$ in Eq. (5),

$$u=(c_1 r^n + c_2 r^{-n})c_4 \sin n\theta \quad \text{...(6)}$$

At $r = 0, u = 0$

$$ur^n = (c_1 r^{2n} + c_2) c_4 \sin n\theta$$

$$0 = (c_2) c_4 \sin n\theta$$

$$c_2 = 0 \qquad [\because c_4 \neq 0]$$

Putting $c_2 = 0$ in Eq. (6),

$$u = c_1 c_4 r^n \sin n\theta = b_n r^n \sin n\theta \text{, where } c_1 c_4 = b_n$$

Putting $n = 1, 2, 3, \ldots$ and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad \ldots(7)$$

Applying the condition (iii) in Eq. (7),

$$u_0 = \sum_{n=1}^{\infty} b_n a^n \sin n\theta \qquad \ldots(8)$$

Equation (8) represents the Fourier half-range sine series in $(0, \pi)$

$$b_n a^n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin n\theta \; d\theta$$

$$b_n = \frac{2u_0}{\pi a^n}\left|\frac{-\cos n\theta}{n}\right|_0^{\pi} = \frac{2u_0}{\pi n a^n}[-\cos n\pi + \cos 0] = \frac{2u_0}{\pi n a^n}[-(-1)^n + 1]$$

Substituting b_n in Eq. (7),

$$u = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{na^n}\right] r^n \sin n\theta \qquad \left[\begin{aligned} &\because 1-(-1)^n = 0, \; n \text{ is even} \\ &\qquad\qquad\quad = 2, \; n \text{ is odd} \\ &\text{Let } n = 2r+1 \end{aligned}\right]$$

$$= \frac{4u_0}{\pi}\sum_{r=0}^{\infty}\left(\frac{r}{a}\right)^{2r+1} \frac{\sin(2r+1)\theta}{(2r+1)}$$

EXAMPLE 11.50 *A plate in the shape of a truncated quadrant of a circle, is bounded by $r = a$, $r = b$, and $\theta = 0$, $\theta = \frac{\pi}{2}$. It has its faces insulated and heat flows in plane curves. It is kept at 0°C temperature along three of the edges while along the edge $r = a$, it is kept at the temperature $\theta\left(\frac{\pi}{2} - \theta\right)$. Determine the steady-state temperature distribution.*

Solution: The steady-state polar form of the heat equation in two dimensions is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \qquad ...(1)$$

The three possible solutions of Eq. (1) are

$$u=(c_1 r^m + c_2 r^{-m})(c_3\cos m\theta + c_4 \sin m\theta) \qquad ...(2)$$

$$u=[c_1\cos(m\log r)+c_2\sin(m\log r)](c_3 e^{m\theta}+c_4 e^{-m\theta}) \qquad ...(3)$$

$$u=(c_1\log r + c_2)(c_3\theta + c_4) \qquad ...(4)$$

In Fig. 11.9, the shaded area represents the plate.
The boundary conditions are

(i) $u(r,0)=0$

(ii) $u\left(r,\frac{\pi}{2}\right)=0$

(iii) $u(b,\theta)=0$

(iv) $u(a,\theta)=\theta\left(\frac{\pi}{2}-\theta\right)$

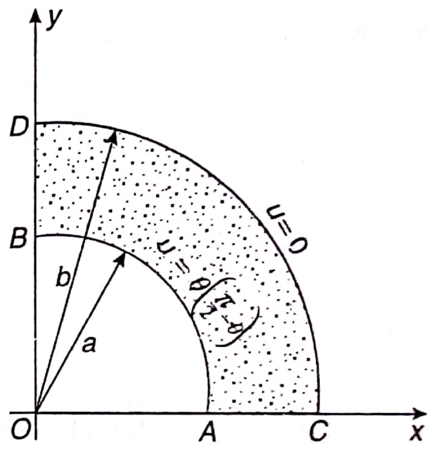

Fig. 11.9

When conditions (i) and (ii) are applied to Eqs (3) and (4), a trivial solution ($u=0$) of Eq. (1) is obtained.

Hence, the solutions by Eqs (3) and (4) are rejected.
Thus, the solution by Eq. (2) is the suitable solution for the present problem.
Applying the condition (i) in Eq. (2),

$$0=(c_1 r^m + c_2 r^{-m})c_3$$

$$c_3 = 0$$

Putting $c_3=0$ in Eq. (2),

$$u=(c_1 r^m + c_2 r^{-m})c_4 \sin m\theta \qquad ...(5)$$

Applying the condition (ii) in Eq. (5),

$$0=(c_1 r^m + c_2 r^{-m})c_4 \sin\frac{m\pi}{2}$$

$$\sin\frac{m\pi}{2}=0 \quad [\because c_4 \neq 0, \text{ otherwise } u=0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$\frac{m\pi}{2}=n\pi$$

$$m=2n$$

Putting $m = 2n$ in Eq. (5),

$$u = (c_1 r^{2n} + c_2 r^{-2n}) c_4 \sin 2n\theta \qquad \text{...(6)}$$

Applying the condition (iii) in Eq. (6),

$$0 = (c_1 b^{2n} + c_2 b^{-2n}) c_4 \sin 2n\theta$$

$$c_1 b^{2n} + c_2 b^{-2n} = 0$$

$$c_2 = -c_1 b^{4n}$$

Putting $c_2 = -c_1 b^{4n}$ in Eq. (6),

$$u = c_1 c_4 \sin 2n\theta\,(r^{2n} - b^{4n} r^{-2n}) = A_n \sin 2n\theta\,(r^{2n} - b^{4n} r^{-2n}), \qquad \text{where } c_1 c_4 = A_n$$

Putting $n = 1, 2, 3, \ldots..$ and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u = \sum_{n=1}^{\infty} A_n \sin 2n\theta\,(r^{2n} - b^{4n} r^{-2n}) \qquad \text{...(8)}$$

Applying the condition (iv) in Eq. (8),

$$\theta\left(\frac{\pi}{2} - \theta\right) = \sum_{n=1}^{\infty} A_n \sin 2n\theta\,(a^{2n} - b^{4n} a^{-2n}) = \sum_{n=1}^{\infty} B_n \sin 2n\theta \qquad \text{...(9)}$$

where $B_n = A_n (a^{2n} - b^{4n} a^{-2n})$

Equation (9) represents the Fourier half-range sine series in $\left(0, \frac{\pi}{2}\right)$.

$$B_n = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \theta\left(\frac{\pi}{2} - \theta\right) \sin 2n\theta \, d\theta$$

$$= \frac{4}{\pi} \left| \left(\frac{\pi\theta}{2} - \theta^2\right)\left(-\frac{\cos 2n\theta}{2n}\right) - \left(\frac{\pi}{2} - 2\theta\right)\left(-\frac{\sin 2n\theta}{4n^2}\right) + (-2)\left(\frac{\cos 2n\theta}{8n^3}\right) \right|_0^{\frac{\pi}{2}}$$

$$= \frac{4}{\pi}\left[\left(-\frac{\cos n\pi}{4n^3}\right) + \left(\frac{\cos 0}{4n^3}\right)\right] = \frac{1}{\pi n^3}[-(-1)^n + 1]$$

$$A_n (a^{2n} - b^{4n} a^{-2n}) = \frac{[1 - (-1)^n]}{\pi n^3}$$

$$A_n = \frac{[1 - (-1)^n]}{\pi n^3 (a^{2n} - b^{4n} a^{-2n})}$$

Substituting A_n is Eq. (8),

$$u = \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{[1-(-1)^n]}{n^3(a^{2n}-b^{4n}a^{-2n})}\right](r^{2n}-b^{4n}r^{-2n})\sin 2n\theta$$

$$= \frac{2}{\pi}\sum_{s=0}^{\infty}\frac{r^{2(2s+1)}-b^{4(2s+1)}r^{-2(2s+1)}}{[a^{2(2s+1)}-b^{4(2s+1)}a^{-2(2s+1)}]}\frac{\sin 2(2s+1)\theta}{(2s+1)^3} \qquad \left[\begin{aligned} \because 1-(-1)^n &= 2, \; n \text{ is odd} \\ &= 0, \; n \text{ is even} \\ \text{Let } n &= 2s+1 \end{aligned}\right]$$

$$= \frac{2}{\pi}\sum_{s=0}^{\infty}\left(\frac{a}{r}\right)^{2(2s+1)}\frac{r^{4(2s+1)}-b^{4(2s+1)}}{a^{4(2s+1)}-b^{4(2s+1)}}\frac{\sin 2(2s+1)\theta}{(2s+1)^3}$$

EXERCISE 11.11

1. Solve the differential equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

subject to the boundary conditions

(i) u is finite when $r \to 0$

(ii) $u = \sum c_n \cos n\theta$ when $r = a$

$$\left[\textbf{Ans.: } u(r,\theta) = \sum c_n \left(\frac{r}{a}\right)^n \cos n\theta\right]$$

2. A semicircular plate of radius a has its circumference kept at the temperature $u(a,\theta) = k\theta(\pi-\theta)$, while the boundary diameter is kept at zero temperature. Find the steady-state temperature distribution of the plate assuming the lateral surfaces of the plate to be insulated.

$$\left[\textbf{Ans.: } u(r,\theta) = \frac{8k}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{2n-1}\frac{\sin(2n-1)\theta}{(2n-1)^3}\right]$$

3. The bounding diameter of a semicircular plate of radius a cm is kept at 0°C and the temperature along the semicircular boundary is given by

$$u(a,\theta) = 50\theta, \qquad 0 < \theta \le \frac{\pi}{2}$$

$$= 50(\pi-\theta), \qquad \frac{\pi}{2} < \theta < \pi$$

Find the steady-state temperature function $u(r,\theta)$.

$$\left[\textbf{Ans.: } u(r,\theta) = \frac{200}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\left(\frac{r}{a}\right)^{2m-1}\sin(2m-1)\theta\right]$$

4. Find the steady-state temperature in a circular plate of radius a which has one-half of its circumference at 0°C and the other half at 60°C.

$$\left[\textbf{Ans.: } u(r,\theta) = 50 - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta\right]$$

11.17 TWO-DIMENSIONAL WAVE EQUATION (VIBRATING MEMBRANE)

Consider a tightly stretched membrane (e.g., the membrane of a drum). Let the membrane be distorted and at time $t = 0$, it is released and allowed to vibrate. Let $A'B'C'D'$ be an element of the membrane at time t and $ABCD$ be its projection on the xy-plane (Fig. 11.10). To obtain the deflection $u(x,y,t)$ at any point (x, y) at $t > 0$, the following assumptions are made.

(i) The membrane is homogeneous with constant density ρ.
(ii) The entire motion takes place in a direction perpendicular to the xy-plane.
(iii) The tension T per unit length is the same in all directions at every point.
(iv) The deflection u is small as compared to the size of the membrane. Therefore, all angles of inclination are small.
(v) The slopes $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are so small that their higher powers can be neglected.

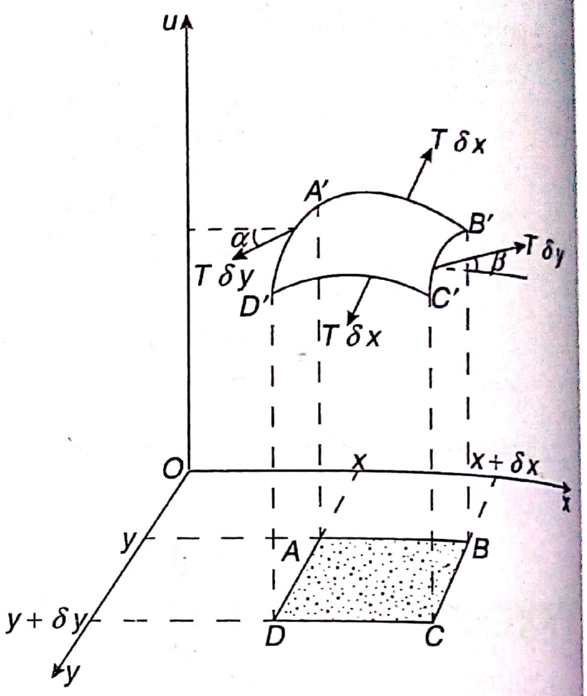

Fig. 11.10 Two-dimensional wave equation

Consider the motion of the element $A'B'C'D'$ with sides δx and δy. The forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Let the forces $T\delta y$ on edges $B'C'$ and $A'D'$ of length δy act at angles α and β to the horizontal.

Thus, the vertical component of forces $T\delta y = T\delta y \sin\beta - T\delta y \sin\alpha$

$$= T\delta y(\sin\beta - \sin\alpha)$$
$$= T\delta y(\tan\beta - \tan\alpha)$$
$$= T\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$\left[\because \alpha \text{ and } \beta \text{ are small and } \lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha} = \lim_{\alpha\to 0}\frac{\tan\alpha}{\alpha} = 1,\ \sin\alpha = \alpha = \tan\alpha,\ \sin\beta = \beta = \tan\beta\right]$

Similarly, the vertical component of forces $T\delta x = T\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right]$

The equation of motion of the element $A'B'C'D'$ is

Sum of forces in vertical direction = Mass × Acceleration in vertical direction

$$T\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] + T\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] = \rho\,\delta x\,\delta y\frac{\partial^2 u}{\partial t^2}$$

$$T\left[\frac{\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]}{\delta x} + \frac{\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right]}{\delta y}\right] = \rho\frac{\partial^2 u}{\partial t^2}$$

Taking limit $\delta x \to 0,\quad \delta y \to 0$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2 = \dfrac{T}{\rho}$.

This equation is known as the *two-dimensional wave equation.*

11.17.1 Solution of Two-dimensional Wave Equation (Rectangular Membrane)

Consider a thin vibrating rectangular membrane bounded by the lines $x = 0,\ x = a,\ y = 0,\ y = b$ in the xy-plane. Assume the entire boundary of the membrane is held fixed for all $t \ge 0$ and the membrane starts from rest from the initial position $u = f(x, y)$.
The two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{...(11.77)}$$

The boundary conditions are

(i) $u(0, y, t) = 0$ (ii) $u(a, y, t) = 0$

(iii) $u(x, 0, t) = 0$ (iv) $u(x, b, t) = 0$

The initial conditions are

(v) $u(x, y, 0) = f(x, y)$ (vi) $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = 0$

Let $u = X(x)\, Y(y)\, T(t)$ be a solution of Eq. (11.77).

$$\frac{\partial^2 u}{\partial t^2} = XYT'', \qquad \frac{\partial^2 u}{\partial x^2} = X''YT, \qquad \frac{\partial^2 u}{\partial y^2} = XY''T$$

Substituting in Eq. (11.77),

$$XYT'' = c^2(X''YT + XY''T)$$

$$\frac{T''}{T} = c^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right)$$

Since $X, Y,$ and T are only the functions of $x, y,$ and t respectively and x, y, t are independent variables, the above equation holds good if each term is a constant.

Let $\dfrac{X''}{X} = -\lambda^2,\ \dfrac{Y''}{Y} = -\mu^2$ [Since for positive constants and zero, a trivial solution ($u = 0$) is obtained, the negative constants are considered]

$$\frac{T''}{T} = c^2\left(-\lambda^2 - \mu^2\right)$$

$$X'' + \lambda^2 X = 0, \quad Y'' + \mu^2 Y = 0, \quad T'' + c^2(\lambda^2 + \mu^2)T = 0$$

Solving these equations,

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x$$

$$Y = c_3 \cos \mu y + c_4 \sin \mu y$$

$$T = c_5 \cos\left(c\sqrt{\lambda^2 + \mu^2}\right)t + c_6 \sin\left(c\sqrt{\lambda^2 + \mu^2}\right)t$$

Thus, the solution of Eq. (11.77) is

$$u=(c_1\cos\lambda x+c_2\sin\lambda x)(c_3\cos\mu y+c_4\sin\mu y)\left[c_5\cos\left(c\sqrt{\lambda^2+\mu^2}\right)t+c_6\sin\left(c\sqrt{\lambda^2+\mu^2}\right)t\right] \quad ...(11.78)$$

Applying the condition (i) in Eq. (11.78),

$$0=c_1(c_3\cos\mu y+c_4\sin\mu y)\left[c_5\cos\left(c\sqrt{\lambda^2+\mu^2}\right)t+c_6\sin\left(c\sqrt{\lambda^2+\mu^2}\right)t\right]$$

$$c_1=0$$

Putting $c_1=0$ in Eq. (11.78),

$$u=c_2\sin\lambda x\,(c_3\cos\mu y+c_4\sin\mu y)\left[c_5\cos\left(c\sqrt{\lambda^2+\mu^2}\right)t+c_6\sin\left(c\sqrt{\lambda^2+\mu^2}\right)t\right] \quad ...(11.79)$$

Applying the condition (ii) in Eq. (11.79),

$$0=c_2\sin\lambda a\,(c_3\cos\mu y+c_4\sin\mu y)\left[c_5\cos\left(c\sqrt{\lambda^2+\mu^2}\right)t+c_6\sin\left(c\sqrt{\lambda^2+\mu^2}\right)t\right]$$

$$\sin\lambda a=0 \quad [\because c_2\neq 0, \text{ otherwise } u=0]$$

$$=\sin n\pi, \quad n \text{ is an integer}$$

$$\lambda a=n\pi$$

$$\lambda=\frac{n\pi}{a}$$

Similarly, applying the conditions (iii) and (iv),

$$c_3=0, \quad \mu=\frac{m\pi}{b}, \quad m \text{ is an integer}$$

Substituting these values in Eq. (11.79),

$$u=\left(c_2\sin\frac{n\pi x}{a}\right)\left(c_4\sin\frac{m\pi y}{b}\right)[c_5\cos p_{mn}t+c_6\sin p_{mn}t] \quad ...(11.80)$$

where $p_{mn}=\pi c\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}$

$$u=\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}(A_{mn}\cos p_{mn}t+B_{mn}\sin p_{mn}t)$$

where $c_2c_4c_5=A_{mn}$, $c_2c_4c_6=B_{mn}$

Putting $n=1,2,3$........... and $m=1,2,3$........... in Eq. (11.80) and adding all these solutions by the principle of superposition, the general solution of Eq. (11.77) is

$$u=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}(A_{mn}\cos p_{mn}t+B_{mn}\sin p_{mn}t) \quad ...(11.81)$$

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}(-A_{mn}p_{mn}\sin p_{mn}t+B_{mn}p_{mn}\cos p_{mn}t) \quad ...(11.82)$$

Applying the condition (vi) in Eq. (11.82),

$$0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \left(B_{mn}\ p_{mn}\right)$$

$$B_{mn} = 0$$

Putting $B_{mn} = 0$ in Eq. (11.82),

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \cos p_{mn}\, t \qquad \text{...(11.83)}$$

Applying the condition (v) in Eq. (11.83),

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \qquad \text{...(11.84)}$$

Equation (11.84) represents double Fourier series,

where $$A_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} f(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\, dy\, dx$$

This is called *generalized Euler's formula.*
Putting the value of A_{mn} in Eq. (11.83), the solution of Eq. (11.77) is obtained.

Corollary If the initial displacement is zero and initial velocity is $g(x, y)$, i.e.,

$$u(x, y, 0) = 0 \quad \text{and} \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x, y)$$

Then $A_{mn} = 0$ and the solution reduces to

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sin p_{mn} t$$

where $$B_{mn} = \frac{4}{abp_{mn}} \int_0^a \int_0^b g(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\, dy\, dx$$

EXAMPLE 11.51 *Find the deflection $u(x, y, t)$ of a rectangular membrane $0 < x < a$, $0 < y < b$ given that its entire boundary is fixed, initial velocity is zero, and initial deflection $f(x, y) = k\,xy(a - x)(b - y)$.*

Solution: Since the boundary of the membrane is fixed and it starts from rest, the solution is given by Eq. (11.83).

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \cos p_{mn}\, t$$

where $$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dy\,dx$$

$$= \frac{4}{ab}\int_0^a\int_0^b kxy(a-x)(b-y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dy\,dx$$

$$= \frac{4k}{ab}\left[\int_0^a (ax-x^2)\sin\frac{n\pi x}{a}\,dx\int_0^b (by-y^2)\sin\frac{m\pi y}{b}\,dy\right]$$

$$= \frac{4k}{ab}\left|(ax-x^2)\left\{\frac{-\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right\}-(a-2x)\left\{\frac{-\sin\frac{n\pi x}{a}}{\left(\frac{n\pi}{a}\right)^2}\right\}+(-2)\left\{\frac{\cos\frac{n\pi x}{a}}{\left(\frac{n\pi}{a}\right)^3}\right\}\right|_0^a \cdot$$

$$\left|(by-y^2)\left\{\frac{-\cos\frac{m\pi y}{b}}{\frac{m\pi}{b}}\right\}-(b-2y)\left\{\frac{-\sin\frac{m\pi y}{b}}{\left(\frac{m\pi}{b}\right)^2}\right\}+(-2)\left\{\frac{\cos\frac{m\pi y}{b}}{\left(\frac{m\pi}{b}\right)^3}\right\}\right|_0^b$$

$$= \frac{4k}{ab}\left[-\frac{2a^3}{n^3\pi^3}(\cos n\pi-1)\right]\left[-\frac{2b^3}{m^3\pi^3}(\cos m\pi-1)\right] = \frac{16ka^2b^2}{\pi^6 m^3 n^3}\left[(-1)^n-1\right]\left[(-1)^m-1\right]$$

Hence, the required deflection is

$$u(x,y,t) = \frac{16ka^2b^2}{\pi^6}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left[(-1)^n-1\right]\left[(-1)^m-1\right]}{m^3n^3}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\cos p_{mn}t$$

where $p_{mn} = \pi c\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}$

EXAMPLE 11.52 *Determine the displacement function $u(x,y,t)$ of a rectangular membrane $0<x<a,\ 0<y<b$ with the entire boundary fixed and initial conditions $u(x,y,0)=f(x,y)=0$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x,y) = 1.$*

Solution: Since the boundary of the membrane is fixed and initial displacement is zero, the solution is given by the corollary

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin p_{mn}t$$

where $B_{mn} = \dfrac{4}{abp_{mn}} \int_0^a \int_0^b g(x,y) \sin\dfrac{n\pi x}{a} \sin\dfrac{m\pi y}{b}\, dy\, dx = \dfrac{4}{abp_{mn}} \int_0^a \int_0^b (1) \sin\dfrac{n\pi x}{a} \sin\dfrac{m\pi y}{b}\, dy\, dx$

$$= \frac{4}{abp_{mn}} \int_0^a \sin\frac{n\pi x}{a}\, dx \int_0^b \sin\frac{m\pi y}{b}\, dy = \frac{4}{abp_{mn}} \left| \frac{-\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}} \right|_0^a \left| \frac{-\cos\frac{m\pi y}{b}}{\frac{m\pi}{b}} \right|_0^b$$

$$= \frac{4}{\pi^2 mnp_{mn}}(-\cos n\pi + 1)(-\cos m\pi + 1) = \frac{4}{\pi^2 mnp_{mn}}[1-(-1)^n][1-(-1)^m]$$

Hence, the required deflection is

$$u(x,y,t) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\left[1-(-1)^n\right]\left[1-(-1)^m\right]}{mn\, p_{mn}} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sin p_{mn} t$$

where
$$p_{mn} = \pi c \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

11.17.2 Vibrations of a Circular Membrane

Consider a tightly stretched thin vibrating circular membrane of radius a offering no resistance to bending. Let the membrane start with velocity $g(r)$ from the initial position $u = f(r)$.
The two-dimensional wave equation for the vibration of the membrane in the polar form is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

Assume that the membrane is radially symmetric (or circularly symmetric or rotationally symmetric). Then the displacement u does not depend on θ and, therefore, the above equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right), \quad 0 < r < a \qquad ...(11.85)$$

The boundary conditions are

(i) $u(a,t) = 0$

The initial conditions are

(ii) $u(r,0) = f(r), \quad 0 < r < a$

(iii) $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = g(r), \ 0 < r < a$

Let $u = R(r)T(t)$ be a solution of Eq. (11.85), where R is a function of r alone and T is a function of t alone.

Differentiating $u = R(r)\,T(t)$ w.r.t. r and t twice,

$$\frac{\partial u}{\partial r} = R'T, \qquad \frac{\partial u}{\partial t} = RT'$$

$$\frac{\partial^2 u}{\partial r^2} = R''T, \qquad \frac{\partial^2 u}{\partial t^2} = RT''$$

Substituting in Eq. (11.85),

$$RT'' = c^2\left(R''T + \frac{1}{r}R'T\right)$$

$$\frac{T''}{c^2T} = \frac{R''}{R} + \frac{R'}{rR} \qquad \text{...(11.86)}$$

Since R is a function of r alone and T is a function of t alone, and r and t are independent variables, Eq. (11.86) holds good only if each side is equal to a constant.

$$\frac{T''}{c^2T} = \frac{R''}{R} + \frac{R'}{rR} = -k^2, \text{ say}$$

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} + k^2 = 0 \text{ and } \frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + k^2 = 0$$

$$\frac{d^2T}{dt^2} + k^2c^2T = 0 \qquad \text{...(11.87)}$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + k^2R = 0 \qquad \text{...(11.88)}$$

Putting $s = kr$ in Eq. (11.88), the equation transforms to $\dfrac{d^2R}{ds^2} + \dfrac{1}{s}\dfrac{dR}{ds} + R = 0$ ...(11.89)

Equation (11.89) is Bessel's equation and its general solution is

$$R = AJ_0(s) + BY_0(s),$$

where J_0 and Y_0 are Bessel's functions of the first and second kind of order zero.
Since the deflection of the membrane is always finite, $B = 0$

Taking $A = 1$, the solution reduces to

$$R(r) = J_0(s) = J_0(kr)$$

On the boundary of the membrane,

$$u(a,t) = 0$$

$$R(a)\,T(t) = 0$$

$$R(a) = 0$$

$$J_0(ka) = 0$$

$$k = \alpha_m, \quad m = 1,2,3,\ldots$$

Hence, the solutions of Eq. (11.88) are

$$R(r) = J_0(\alpha_m r), \quad m = 1, 2, 3, \ldots$$

The corresponding solutions of Eq. (11.87) are

$$T(t) = A_m \cos ckt + B_m \sin ckt = A_m \cos c\alpha_m t + B_m \sin c\alpha_m t$$

Hence, the solution of Eq. (11.85) is

$$u(r,t) = \left(A_m \cos c\alpha_m t + B_m \sin c\alpha_m t\right) J_0(\alpha_m r)$$

Putting $m = 1, 2, 3, \ldots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (11.85) is

$$u(r,t) = \sum_{m=1}^{\infty} \left(A_m \cos c\alpha_m t + B_m \sin c\alpha_m t\right) J_0(\alpha_m r) \qquad \ldots(11.90)$$

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty} \left(-A_m c\alpha_m \sin c\alpha_m t + B_m c\alpha_m \cos c\alpha_m t\right) J_0(\alpha_m r) \qquad \ldots(11.91)$$

Applying the condition (ii) in Eq. (11.90),

$$u(r,0) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r)$$

$$f(r) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) \qquad \ldots(11.92)$$

Applying the condition (iii) in Eq. (11.91),

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{m=1}^{\infty} B_m c\alpha_m J_0(\alpha_m r)$$

$$g(r) = \sum_{m=1}^{\infty} B_m \alpha_m c J_0(\alpha_m r) \qquad \ldots(11.93)$$

Equations (11.92) and (11.93) are the Fourier–Bessel series representation of $f(r)$ and $g(r)$ in $0 < r < a$ respectively, where A_m and B_m are given by

$$A_m = \frac{2}{a^2 \left[J_1(a\alpha_m)\right]^2} \int_0^a r f(r) J_0(\alpha_m r)\, dr \qquad \ldots(11.94)$$

$$B_m = \frac{2}{c\alpha_m a^2 \left[J_1(a\alpha_m)\right]^2} \int_0^a r g(r) J_0(\alpha_m r)\, dr \qquad \ldots(11.95)$$

Hence, the displacement function $u(r,t)$ is given by Eq. (11.90), where A_m and B_m are determined by Eqs (11.94) and (11.95) respectively.

EXAMPLE 11.53 *Determine the displacement $u(r,t)$ of a circular membrane tightly stretched and fixed to a circular frame of radius a with initial displacement $f(r) = 3J_0(r\alpha_1) + J_0(r\alpha_3)$ and initial velocity $g(r) = J_0(r\alpha_2)$.*

Solution: The displacement $u(r,t)$ is given by

$$u(r,t) = \sum_{m=1}^{\infty} (A_m \cos c\alpha_m t + B_m \sin c\alpha_m t)\, J_0(\alpha_m r)$$

$$A_m = \frac{2}{a^2 [J_1(a\alpha_m)]^2} \int_0^a r[3J_0(r\alpha_1) + J_0(r\alpha_3)]\, J_0(\alpha_m r)dr$$

Using the orthogonality relation,

$$A_1 = 3,\ A_2 = 0,\ A_3 = 1, \text{and } A_m = 0 \text{ for } m > 3$$

$$B_m = \frac{2}{c\alpha_m a^2 [J_1(a\alpha_m)]^2} \int_0^a r\, J_0(r\alpha_2) J_0(\alpha_m r)dr$$

Using the orthogonality relation,

$$B_1 = 0, B_2 = \frac{1}{c\alpha_2},\ B_m = 0,\ \textit{for } m > 2$$

Hence, the required displacement function is

$$u(r,t) = (3\cos c\alpha_1 t)\, J_0(\alpha_1 r) + \frac{1}{c\alpha_2}(\sin c\alpha_2 t)\, J_0(\alpha_2 r) + (\cos c\alpha_3 t) J_0(\alpha_3 r)$$

EXERCISE 11.12

1. Find the deflection $u(x,y,t)$ of the square membrane with $a = b = 1$ and $c = 1$ if the initial velocity is zero and the initial deflection $f(x,y) = k \sin \pi x \sin 2\pi y$.

$\left[\textbf{Ans.:}\ u(x,y,t) = k\cos(\pi\sqrt{5}\, t) \sin \pi x \sin 2\pi y\right]$

2. Find the deflection $u(x,y,t)$ of the rectangular membrane $0 < x < a,\ 0 < y < b$ given that its entire boundary is fixed, and the initial conditions are $u(x,y,0) = 0$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x,y)$

$$\left[\begin{aligned} &\textbf{Ans.:}\ u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} \cos\frac{n\pi x}{a} \cdot \cos\frac{m\pi y}{b} \phi_{mn}(t) \\ &\text{where } \phi_{mn}(t) = t, \text{ if } n = 0,\ m = 0 \\ &\qquad = \sin p_{mn}(t), \text{ otherwise} \\ &A_{mn}\phi'_{mn}(0) = \frac{\iint g(x,y)\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\, dx\, dy}{\iint \cos^2\left(\frac{n\pi x}{a}\right)\cos^2\left(\frac{m\pi y}{b}\right) dx\, dy} \end{aligned}\right]$$

3. Find the deflection $u(r,t)$ of a unit circular membrane $(a=1)$ if its boundary is fixed and initial conditions are $u(r,0)=f(r), \left(\frac{\partial u}{\partial t}\right)_{t=0}=0.$

$$\left[\begin{array}{l}\textbf{Ans.: } u(r,t)=\sum_{m=1}^{\infty} A_m \cos(c\alpha_m t)\cdot J_0(\alpha_m r) \\ \text{where } A_m=\dfrac{2}{[J_1(\alpha_m)]^2}\displaystyle\int_0^1 rf(r)\, J_0(\alpha_m r)\,dr\end{array}\right]$$

4. Determine the displacement $u(r,t)$ of a unit circular membrane (with $c=1$) with initial displacement zero and initial velocity $g(r)=k(1-r^2)$.

$$\left[\begin{array}{r}\textbf{Ans.: } u(r,t)=4k\displaystyle\sum_{m=1}^{\infty}\dfrac{J_2(\alpha_m)}{\lambda_m^3[J_1(\alpha_m)]^2} \\ \sin(\alpha_m t)\cdot J_0(\alpha_m r)\end{array}\right]$$

11.18 TRANSMISSION LINE

A transmission line is used to transfer electric energy from generating points to the distribution system. All transmission lines in a power system exhibit the electrical properties of resistance R, inductance L, capacitance C, and conductance G. Consider a cable of length l km carrying a current with resistance R ohm/km, inductance L henries/km, capacitance C farads/km, and leakage conductance G mhos/km.

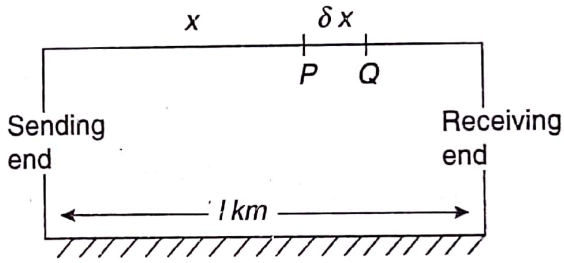

Fig. 11.11 Transmission line

Let the instantaneous voltage and current at any point P, distance x km from the sending end, and at time t second be $v(x,t)$ and $i(x,t)$ respectively (Fig. 11.11). Let Q be a point at a distance δx from P.

Applying Kirchhoff's voltage law to a small portion PQ of length δx,

Voltage drop across segment δx = Voltage drop due to resistance + Voltage drop due to inductance

$$-\delta v=Ri\,\delta x+L\frac{\partial i}{\partial t}\delta x$$

Dividing by δx and taking limit as $\delta x\to 0$,

$$-\frac{\partial v}{\partial x}=Ri+L\frac{\partial i}{\partial t} \qquad ...(11.96)$$

This equation is known as the *first transmission-line equation.*

Similarly, applying Kirchhoff's current law,

Current loss between P and Q = Current loss due to capacitance + Current loss due to leakage conductance

$$-\delta i=C\frac{\partial v}{\partial t}\delta x+Gv\,\delta x$$

Dividing by δx and taking limit as $\delta x \to 0$,

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv \qquad \text{...(11.97)}$$

This equation is known as the *second transmission-line equation*.
Rewriting Eq. (11.96) and (11.97),

$$\left(R + L\frac{\partial}{\partial t}\right)i + \frac{\partial v}{\partial x} = 0 \qquad \text{...(11.98)}$$

$$\left(G + C\frac{\partial}{\partial t}\right)v + \frac{\partial i}{\partial x} = 0 \qquad \text{...(11.99)}$$

Operating Eq. (11.98) by $\frac{\partial}{\partial x}$ and Eq. (11.99) by $\left(R + L\frac{\partial}{\partial t}\right)$ and subtracting,

$$\frac{\partial^2 v}{\partial x^2} - \left(R + L\frac{\partial}{\partial t}\right)\left(C\frac{\partial}{\partial t} + G\right)v = 0$$

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (LG + RC)\frac{\partial v}{\partial t} + RGV \qquad \text{...(11.100)}$$

Operating Eq. (11.98) by $\left(C\frac{\partial}{\partial t} + G\right)$ and Eq. (11.99) by $\frac{\partial}{\partial x}$ and subtracting,

$$\left(C\frac{\partial}{\partial t} + G\right)\left(R + L\frac{\partial}{\partial t}\right)i - \frac{\partial^2 i}{\partial x^2} = 0$$

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial x^2} + (LG + RC)\frac{\partial i}{\partial t} + RGi \qquad \text{...(11.101)}$$

Equations (11.100) and (11.101) are known as *telephone equations*. Equation (11.101) is same as Eq. (11.100) when v is replaced by i.

Case I If $L = G = 0$, Eq. (11.100) and (11.101) reduce to

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \qquad \text{...(11.102)}$$

$$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t} \qquad \text{...(11.103)}$$

Equations (11.102) and (11.103) are known as telegraph equations. Equations (11.102) and (11.103) are similar to the one-dimensional heat equation.

Case II If $R = G = 0$, Eq. (11.100) and (11.101) reduce to

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \qquad \text{...(11.104)}$$

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \qquad \text{...(11.105)}$$

Equations (11.104) and (11.105) are known as *radio equations*. Equations (11.104) and (11.105) are similar to the one-dimensional wave equation.
Rewriting Eq. (11.104),

$$\frac{\partial^2 v}{\partial t^2}=k^2\frac{\partial^2 v}{\partial x^2},\quad \text{where } k^2=\frac{1}{LC}$$

Its general solution is

$$v(x,t)=f_1(x+kt)+f_2(x-kt)$$

Simliarly, from Eq. (11.105),

$$i(x,t)=F_1(x+kt)+F_2(x-kt)$$

Hence, at any point along the lossless transmission line, the voltage $v(x,t)$ and the current $i(x,t)$ can be considered as the sum of an incident wave and a reflected wave.

EXAMPLE 11.54 *Neglecting R and G, find the voltage $v(x,t)$ in a transmission line of length l,t seconds after the ends are suddenly grounded. The initial conditions are $v(x,0)=v_0\sin\frac{\pi x}{l}$ and $i(x,0)=i_0$.*

Solution: Since $R=G=0$, the equation of the transmission line is

$$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2} \qquad ...(1)$$

The solution of Eq. (1) is

$$v=(c_1\cos mx+c_2\sin mx)\left(c_3\cos\frac{m}{\sqrt{LC}}t+c_4\sin\frac{m}{\sqrt{LC}}t\right) \qquad ...(2)$$

The ends of the transmission line are grounded.
The boundary conditions are

(i) At $x=0$, $v(0,t)=0$

(ii) At $x=l$, $v(l,t)=0$

The initial conditions are

(iii) At $t=0$, $v(x,0)=v_0\sin\frac{\pi x}{l}$

(iv) At $t=0$, $i(x,0)=i_0$, $\left.\frac{\partial v}{\partial t}\right|_{(x,0)}=-\frac{1}{C}\frac{\partial i(x,0)}{\partial x}=-\frac{1}{C}\frac{\partial i_0}{\partial x}=0$

Applying the condition (i) in Eq. (2),

$$0=c_1\left(c_3\cos\frac{m}{\sqrt{LC}}t+c_4\sin\frac{m}{\sqrt{LC}}t\right)$$

$$c_1=0$$

Putting $c_1=0$ in Eq. (2),

$$v=c_2\sin mx\left(c_3\cos\frac{m}{\sqrt{LC}}t+c_4\sin\frac{m}{\sqrt{LC}}t\right) \qquad ...(3)$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml\left(c_3 \cos\frac{m}{\sqrt{LC}}t + c_4 \sin\frac{m}{\sqrt{LC}}t\right)$$

$$\sin ml = 0 \qquad [\because c_2 \neq 0]$$

$$ml = n\pi, \quad n \text{ is an integer}$$

$$m = \frac{n\pi}{l}$$

Putting $m = \dfrac{n\pi}{l}$ in Eq. (3),

$$v = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi}{l\sqrt{LC}}t + c_4 \sin\frac{n\pi}{l\sqrt{LC}}t\right) \qquad ...(4)$$

$$\frac{\partial v}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left[-c_3\left(\sin\frac{n\pi}{l\sqrt{LC}}t\right)\left(\frac{n\pi}{l\sqrt{LC}}\right) + c_4\left(\cos\frac{n\pi}{l\sqrt{LC}}t\right)\left(\frac{n\pi}{l\sqrt{LC}}\right)\right] \qquad ...(5)$$

Applying the condition (iv) in Eq. (5),

$$0 = c_2 \sin\frac{n\pi x}{l}\left(c_4\frac{n\pi}{l\sqrt{LC}}\right)$$

$$c_4 = 0 \qquad [\because c_2 \neq 0]$$

Putting $c_4 = 0$ in Eq. (4),

$$v = c_2\, c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi}{l\sqrt{LC}}t = b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi}{l\sqrt{LC}}t, \quad \text{where } c_2\, c_3 = b_n$$

Putting $n = 1, 2, 3, \ldots$ and adding all these solutions by the principle of superposition, the general sloution of Eq. (1) is

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi}{l\sqrt{LC}}t \qquad ...(6)$$

Applying the condition (iii) in Eq. (6),

$$v_0 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(7)$$

Equation (7) represents the Fourier half-range sine series.

$$b_1 = v_0 \text{ and } b_n = 0 \text{ for } n > 1.$$

Hence, the solution is

$$v(x,t) = v_0 \sin\frac{\pi x}{l}\cos\frac{\pi}{l\sqrt{LC}}t$$

EXERCISE 11.13

1. Determine the voltage $v(x,t)$ in a transmission line of length l, t seconds after the ends were suddenly grounded. Assume that R and G are negligible and initial conditions are

$$v(x,0) = a_1 \sin\frac{\pi x}{l} + a_5 \sin\frac{5\pi x}{l} \text{ and } i(x,0) = i_0$$

$$\left[\textbf{Ans.: } v(x,t) = a_1 \sin\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}} + a_5 \sin\frac{5\pi}{l}\cos\frac{5\pi t}{l\sqrt{LC}}\right]$$

2. In the case of a submarine cable, assuming $L = C = 0$, find the voltage and current given that

$$v(0) = v_0 \text{ and } i(0) = i_0$$

$$\left[\begin{aligned}&\textbf{Ans.: } v(x) = v_0 \cosh \alpha x - l_0\ z_0 \sinh \alpha x,\\ &i(x) = i_0 \cosh \alpha x - \frac{v_0}{z_0}\sin \alpha x,\\ &\text{where } \alpha = \sqrt{GR}, z_0 = \sqrt{\frac{R}{G}}\end{aligned}\right]$$

3. Obtain the solution of the telegraph equation

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$$

suitable for the case when v decreases with time and when there is a steady decrease in potential from v_0 to 0 along the line of length l initially and the sending end is suddenly grounded.

$$\left[\textbf{Ans.: } v(x,t) = \frac{2v_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}e^{-\frac{n^2\pi^2 t}{l^2 RC}}\right]$$

4. Assuming $L = 0$, $G = 0$, find the voltage $v(x,t)$ in a transmission line of 1000 km length which is initially under steady-state conditions with 1300 volts at the source end and 1200 volts at the receiving ends. The receving end of the line is suddenly grounded while the voltage at the source end is maintained at 1300 volts.

$$\left[\begin{aligned}&\textbf{Ans.: } v(x,t) = 1300 - 1.3x + \frac{2400}{\pi}\\ &\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{1000}e^{-\frac{n^2\pi^2 t}{l^2 RC}}\end{aligned}\right]$$

11.19 LAPLACE'S EQUATION IN THREE DIMENSIONS

In steady state, the three-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

reduces to $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

which is known as the *three-dimensional Laplace's equation.*

Solution of Three-Dimensional Laplace's Equation

1. Cartesian Form

Laplace's equation in Cartesian form is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad \text{...(11.106)}$$

Let $u = X(x)Y(y)Z(z)$ be a solution of Eq. (11.106).

$$\frac{\partial^2 u}{\partial x^2}=X''YZ, \quad \frac{\partial^2 u}{\partial y^2}=XY''Z, \quad \frac{\partial^2 u}{\partial z^2}=XYZ''$$

Substituting in Eq. (11.106),

$$X''YZ+XY''Z+XYZ''=0$$

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$$

Since $X, Y,$ and Z are only the functions of x, y, and z respectively, and x, y, z are independent variables, the above equation holds good if each term is a constant.

Case I Let

$$\frac{X''}{X}=\lambda^2, \quad \frac{Y''}{Y}=\mu^2,$$

$$\frac{Z''}{Z}=-(\lambda^2+\mu^2)$$

$$X''-\lambda^2 X=0, \quad Y''-\mu^2 Y=0, \quad Z''+(\lambda^2+\mu^2)Z=0$$

Solving these equations,

$$X=c_1 e^{\lambda x}+c_2 e^{-\lambda x}, \qquad Y=c_3 e^{\mu y}+c_4 e^{-\mu y}$$

$$Z=c_5 \cos\left(\sqrt{\lambda^2+\mu^2}\right)z+c_6 \sin\left(\sqrt{\lambda^2+\mu^2}\right)z$$

Thus, the solution of Eq. (11.106) is

$$u=\left(c_1 e^{\lambda x}+c_2 e^{-\lambda x}\right)\left(c_3 e^{\mu y}+c_4 e^{-\mu y}\right)\left[c_5 \cos\left(\sqrt{\lambda^2+\mu^2}\right)z+c_6 \sin\left(\sqrt{\lambda^2+\mu^2}\right)z\right]$$

Case II Let

$$\frac{X''}{X}=-\lambda^2, \quad \frac{Y''}{Y}=-\mu^2$$

$$\frac{Z''}{Z}=\lambda^2+\mu^2$$

$$X''+\lambda^2 X=0, \quad Y''+\mu^2 Y=0, \quad Z''-(\lambda^2+\mu^2)Z=0$$

Solving these equations,

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x, \quad Y = c_3 \cos \mu y + c_4 \sin \mu y$$

$$Z = c_5 e^{\left(\sqrt{\lambda^2+\mu^2}\right)z} + c_6 e^{-\left(\sqrt{\lambda^2+\mu^2}\right)z}$$

Thus, the solution of Eq. (11.106) is

$$u = \left(c_1 \cos \lambda x + c_2 \sin \lambda x\right)\left(c_3 \cos \mu y + c_4 \sin \mu y\right)\left[c_5 e^{\left(\sqrt{\lambda^2+\mu^2}\right)z} + c_6 e^{-\left(\sqrt{\lambda^2+\mu^2}\right)z}\right]$$

Case III Let

$$\frac{X''}{X} = 0, \quad \frac{Y''}{Y} = 0$$

$$\frac{Z''}{Z} = 0$$

$$X'' = 0, \quad Y'' = 0, \quad Z'' = 0$$

Solving these equations,

$$X = c_1 x + c_2, \quad Y = c_3 y + c_4, \quad Z = c_5 z + c_6$$

Thus, the solution of Eq. (11.106) is

$$u = \left(c_1 x + c_2\right)\left(c_3 y + c_4\right)\left(c_5 z + c_6\right)$$

2. Cylindrical Form

Laplace's equation in cylindrical form is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{...(11.107)}$$

Let $u = R(r)\,F(\theta)\,Z(z)$ be a solution of Eq. (11.107).

$$\frac{\partial u}{\partial r} = R' FZ, \quad \frac{\partial^2 u}{\partial r^2} = R''FZ, \quad \frac{\partial^2 u}{\partial \theta^2} = RF''Z, \quad \frac{\partial^2 u}{\partial z^2} = RFZ''$$

Substituting in Eq. (11.107),

$$R''FZ + \frac{1}{r}R'FZ + \frac{1}{r^2}RF''Z + RFZ'' = 0$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{F''}{F} + \frac{Z''}{Z} = 0$$

$$\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{1}{r^2}\frac{F''}{F} = -\frac{Z''}{Z}$$

Since the right-hand side depends only on z, the above equation holds good if each side is a constant.

Let $$\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\frac{1}{r^2}\frac{F''}{F}=-\frac{Z''}{Z}=-m^2$$

Considering $-\frac{Z''}{Z}=-m^2,\quad Z''-m^2Z=0$

Solving,

$$z=c_1e^{mz}+c_2e^{-mz} \qquad \text{...(11.108)}$$

Considering $\frac{1}{R}\left(R''+\frac{1}{r}R'\right)+\frac{1}{r^2}\frac{F''}{F}=-m^2,\quad \frac{r^2R''}{R}+\frac{rR'}{R}+m^2r^2=-\frac{F''}{F}$

Since r and θ are independent, the above equation holds good if each side is a constant.

Let $$\frac{r^2R''}{R}+\frac{r^2R'}{R}+m^2r^2=-\frac{F''}{F}=n^2$$

Considering $-\frac{F''}{F}=n^2,\qquad F''+n^2F=0$

Solving,

$$F=c_3\cos n\theta+c_4\sin n\theta$$

Considering $\frac{r^2R''}{R}+\frac{rR'}{R}+m^2r^2=n^2,\qquad r^2R''+rR'+(m^2r^2-n^2)R=0$

This is Bessel's equation and its solution is

$$R=c_5J_n(mr)+c_6Y_n(mr)$$

Thus, the solution of Eq. (11.107) is

$$u=\left(c_1e^{mz}+c_2e^{-mz}\right)(c_3\cos n\theta+c_4\sin n\theta)\left[c_5J_n(mr)+c_6Y_n(mr)\right]$$

3. Spherical Form

Laplace's equation in spherical form is

$$\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=0 \qquad \text{...(11.109)}$$

Let $u=R(r)F(\theta)H(\phi)$ be a solution of Eq. (11.109).

$$\frac{\partial u}{\partial r}=R'FH,\quad \frac{\partial^2u}{\partial r^2}=R''FH,\quad \frac{\partial u}{\partial\theta}=RF'H,\quad \frac{\partial^2u}{\partial\theta^2}=RF''H,\quad \frac{\partial^2u}{\partial\phi^2}=RFH''$$

Substituting in Eq. (11.109),

$$R''FH+\frac{2}{r}R'FH+\frac{1}{r^2}RF''H+\frac{\cot\theta}{r^2}RF'H+\frac{1}{r^2\sin^2\theta}RFH''=0$$

$$\frac{R''}{R}+\frac{2}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{F''}{F}+\frac{\cot\theta}{r^2}\frac{F'}{F}+\frac{1}{r^2\sin^2\theta}\frac{H''}{H}=0$$

$$\left[\frac{1}{R}\left(R''+\frac{2}{r}R'\right)+\frac{1}{r^2F}\left(F''+F'\cot\theta\right)\right]r^2\sin^2\theta=-\frac{H''}{H}$$

Since the right-hand side depends only on ϕ, the above equation holds good if each side is a constant.

Let $$\left[\frac{1}{R}\left(R''+\frac{2}{r}R'\right)+\frac{1}{r^2F}\left(F''+F'\cot\theta\right)\right]r^2\sin^2\theta=-\frac{H''}{H}=m^2,\text{ say}$$

Considering $$-\frac{H''}{H}=m^2,\ H''+m^2H=0$$

Solving,

$$H=c_1\cos m\phi+c_2\sin m\phi$$

Considering $$\left[\frac{1}{R}\left(R''+\frac{2}{r}R'\right)+\frac{1}{r^2F}\left(F''+F'\cot\theta\right)\right]r^2\sin^2\theta=m^2$$

$$\frac{1}{R}\left(r^2R''+2rR'\right)+\frac{1}{F}\left(F''+F'\cot\theta\right)=\frac{m^2}{\sin^2\theta}$$

$$\frac{1}{F}\left(F''+F'\cot\theta\right)-\frac{m^2}{\sin^2\theta}=-\frac{1}{R}\left(r^2R''+2rR'\right)$$

Since r and θ are independent, the above equation holds good if each side is a constant.

Let $$\frac{1}{F}\left(F''+F'\cot\theta\right)-\frac{m^2}{\sin^2\theta}=-\frac{1}{R}\left(r^2R''+2rR'\right)=-n(n+1) \quad \text{...(11.110)}$$

From Eq. (11.110), considering $\frac{1}{F}\left(F''+F'\cot\theta\right)-\frac{m^2}{\sin^2\theta}=-n(n+1)$

$$F''+F'\cot\theta+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]F=0 \quad \text{...(11.111)}$$

This equation is the *associated Legendre's equation.*
Putting $\cos\theta=x$, Eq. (11.111) reduces to

$$(1-x^2)\frac{d^2F}{dx^2}-2x\frac{dF}{dx}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]F=0 \quad \text{...(11.112)}$$

The solution of Eq. (11.112) is

$$F = c_3 P_n^m(\cos\theta) + c_4 Q_n^m(\cos\theta)$$

From Eq. (11.110), considering $-\frac{1}{R}\left(r^2R'' + 2rR'\right) = -n(n+1)$

$$\frac{1}{R}\left(r^2R'' + 2rR'\right) = n(n+1) \qquad \text{...(11.113)}$$

Putting $R = r^k$ in Eq. (11.113),

$$\frac{1}{r^k}\left[r^2k(k-1)r^{k-2} + 2rkr^{k-1}\right] = n(n+1)$$

$$k^2 - k + 2k - n(n+1) = 0$$

$$k^2 + k - n(n+1) = 0$$

$$k = n \text{ or } k = -(n+1)$$

$$R = c_5 r^n + c_6 r^{-(n+1)}$$

Hence, by superposition principle, the general solution of Eq. (11.113) is

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(c_1\cos m\phi + c_2\sin m\phi)\left[c_3P_n^m(\cos\theta) + c_4Q_n^m(\cos\theta)\right]\left(c_5r^n + c_6r^{-(n+1)}\right)$$

Any solution of Eq. (11.109) of the form $R(r)F(\theta)H(\phi)$ is known as *spherical harmonic* and a solution of the form $F(\theta)H(\phi)$ is known as *surface harmonic.*

EXAMPLE 11.55 *Find the potential in the exterior of a sphere of unit radius when the potential on the surface is $u(1,\theta) = \cos 2\theta$.*

Solution: Let the centre of the sphere be the origin. Since the potential is independent of ϕ, Laplace's equation in spherical coordinates reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad \text{...(1)}$$

On physical grounds, the potential at infinity is zero. The boundary conditions are

(i) $\lim_{r\to\infty} u(r,\theta) = 0$

(ii) $u(1,\theta) = \cos 2\theta$

Let $u = R(r)F(\theta)$

$$\frac{\partial u}{\partial r} = R'F, \quad \frac{\partial^2 u}{\partial r^2} = R''F, \quad \frac{\partial u}{\partial \theta} = RF', \quad \frac{\partial^2 u}{\partial \theta^2} = RF''$$

Substituting in Eq. (1),

$$R''F + \frac{2}{r}R'F + \frac{1}{r^2}RF'' + \frac{\cot\theta}{r^2}RF' = 0$$

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{F''}{F} + \frac{\cot\theta}{r^2}\frac{F'}{F} = 0$$

$$\frac{1}{F}\left(F'' + F'\cot\theta\right) = -\frac{1}{R}\left(r^2R'' + 2rR'\right)$$

Since r and θ are independent variables, the above equation holds good if each side is a constant.

Let $\frac{1}{F}\left(F'' + F'\cot\theta\right) = -\frac{1}{R}\left(r^2R'' + 2rR'\right) = -n(n+1)$, say ...(2)

From Eq. (2), considering $\frac{1}{F}\left(F'' + F'\cot\theta\right) = -n(n+1)$

$$F'' + F'\cot\theta + n(n+1)F = 0 \qquad \text{...(3)}$$

Putting $\cos\theta = x$, Eq. (3) reduces to

$$(1-x^2)\frac{d^2F}{dx^2} - 2x\frac{dF}{dx} + n(n+1)F = 0$$

which is Legendre's equation. Its solution is

$$F(\theta) = c_1P_n(\cos\theta) + c_2Q_n(\cos\theta)$$

Since $F(\theta)$ is to be finite at $\theta = 0$, $c_2 = 0$,

$$F(\theta) = c_1P_n(\cos\theta) \qquad \text{...(4)}$$

From Eq. (2), considering

$$-\frac{1}{R}\left(r^2R'' + 2rR'\right) = -n(n+1)$$

$$\frac{1}{R}\left(r^2R'' + 2rR'\right) = n(n+1) \qquad \text{...(5)}$$

Putting $R = r^k$ in Eq. (5),

$$\frac{1}{r^k}\left[r^2k(k-1)r^{k-2} + 2rkr^{k-1}\right] = n(n+1)$$

$$k^2 + k - n(n+1) = 0$$

$$k = n \text{ or } k = -(n+1)$$

$$R(r) = c_3 r^n + c_4 r^{-(n+1)}$$

To satisfy the boundary condition (i),

$$c_3 = 0 \quad [\text{otherwise } u \to \infty]$$

$$R(r) = c_4 r^{-(n+1)}$$

Hence, the solution of Eq. (1) is

$$u(r,\theta) = c_4 r^{-(n+1)} c_1 P_n(\cos\theta) = c_n r^{-(n+1)} P_n(\cos\theta)$$

Putting $n = 1,2,3,\ldots$ and adding all these solution by the principle of superposition, where $c_1 c_4 = c_n$, the general solution of Eq. (1) is

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n r^{-(n+1)} P_n(\cos\theta) = \sum_{n=0}^{\infty} c_n r^{-(n+1)} P_n(\cos\theta) \qquad \ldots(6)$$

Applying the condition (ii) in Eq. (6),

$$u(1,\theta) = \sum_{n=0}^{\infty} c_n 1^{-(n+1)} P_n(\cos\theta)$$

$$\cos 2\theta = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$$

$$2\cos^2\theta - 1 = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$$

$$2x^2 - 1 = \sum_{n=0}^{\infty} c_n P_n(x) \qquad [\because \cos\theta = x]$$

which represents Legendre series expansion of $(2x^2 - 1)$.

$$\text{where} \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} (2x^2 - 1) P_n(x)\,dx \qquad \ldots(7)$$

Putting $n=0$ in Eq. (7),

$$c_0 = \frac{1}{2}\int_{-1}^{1} (2x^2 - 1) P_0(x)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} (2x^2 - 1)\cdot 1\,dx \qquad [\because P_0(x) = 1]$$

$$= \frac{1}{2}\left|2\frac{x^3}{3} - x\right|_{-1}^{1} = -\frac{1}{3}$$

Putting $n = 1$ in Eq. (7),

$$c_1 = \frac{3}{2}\int_{-1}^{1}(2x^2-1)P_1(x)\,dx$$

$$= \frac{3}{2}\int_{-1}^{1}(2x^2-1)\,x\,dx \qquad [\because P_1(x) = x]$$

$$= \frac{3}{2}\left|2\frac{x^4}{4}-\frac{x^2}{2}\right|_{-1}^{1} = 0$$

Putting the $n = 2$ in Eq. (7),

$$c_2 = \frac{5}{2}\int_{-1}^{1}(2x^2-1)P_2(x)\,dx$$

$$= \frac{5}{2}\int_{-1}^{1}(2x^2-1)\left(\frac{3x^2-1}{2}\right)dx \qquad \left[\because P_2(x) = \frac{3x^2-1}{2}\right]$$

$$= \frac{5}{4}\int_{-1}^{1}(6x^4-5x^2+1)\,dx = \frac{5}{4}\left|6\frac{x^5}{5}-5\frac{x^3}{3}+x\right|_{-1}^{1} = \frac{4}{3}$$

Since $\int_{-1}^{1} x^m P_n(x)\,dx = 0, \quad \text{if } m < n$

and $\int_{-1}^{1} P_n(x)\,dx = 0, \quad \text{if } n \geq 1$

where $P_n(x)$ is a polynomial of degree n.

$$c_3 = 0,\ c_4 = 0,\ \ldots\ldots\ldots\ldots,\ c_n = 0 \text{ for } n \geq 3$$

Putting the values of c_n's in Eq. (6),

$$u(r,\theta) = c_0 P_0(\cos\theta)r^{-1} + c_1P_1(\cos\theta)r^{-2} + c_2P_2(\cos\theta)r^{-3}$$

$$= -\frac{1}{3r} + 0 + \frac{4}{3r^3}\left(\frac{3\cos^2\theta-1}{2}\right) = \frac{-r^2+2\,(3\cos^2\theta-1)}{3r^3}$$

which is the required potential.

EXERCISE 11.14

1. Find the potential in the interior of a sphere of unit radius if the potential on the surface is $\cos^2\theta$.

$$\left[\textbf{Ans.:}\ u(r,\theta) = \frac{1}{3} + r^2\left(\cos^2\theta - \frac{1}{3}\right)\right]$$

2. Determine the potential outside and inside a spherical surface which is kept at a fixed distribution of electrical potential of the form $u = F(\theta)$. It is assumed that the space inside and outside the surface is free of charges.

$$\left[\textbf{Ans.:(i)}\, u(r,\theta) = \sum_{n=1}^{\infty} \frac{c_n}{r^{n+1}} P_n(\cos\theta)\right.$$

where

$$c_n = \frac{2n+1}{2}\, a^{n+1} \int_0^{\pi} F(\theta)\, P_n(\cos\theta)\cdot\sin\theta\, d\theta$$

$$\textbf{(ii)}\; u(r,\theta) = \sum_{n=0}^{\infty} b_n r^n P_n(\cos\theta),$$

$$\left.\text{where } b_n = \frac{2n+1}{2a^n} \int_0^{\pi} F(\theta) P_n(\cos\theta)\cdot\sin\theta\, d\theta\right]$$

3. Find the steady-state temperature in a uniform solid sphere of radius a when its surface is maintained at the temperature $F(\theta)$.

$$\left[\textbf{Ans.:}\; u(r,0) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta)\right.$$

$$\left.\text{where } c_n = \frac{2n+1}{2a^n} \int_0^{\pi} F(\theta) P_n(\cos\theta)\sin\theta\, d\theta\right]$$

4. Find the steady-state temperature in a uniform solid sphere of radius unity when one half of the surface is kept at constant temperature of 0°C and the other half at a constant temperature of 1°C.

$$\left[\textbf{Ans.:}\, u(r,0) = \frac{1}{2} + \frac{3}{4} r P_3(\cos\theta) - \frac{7}{16} r^3 P_3(\cos\theta) + \frac{11}{32} r^5 P_5(\cos\theta) + \ldots\right]$$

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