9

Partial Differential Equations

9.1. PARTIAL DIFFERENTIAL EQUATIONS are those equations which contain partial differential coefficients, independent variables and dependent variables.

The independent variables will be denoted by x and y and the dependent variable by z. The partial differential coefficients are denoted as follows:

$$\frac{\partial z}{\partial x} = p, \qquad \frac{\partial z}{\partial y} = q.$$

$$\frac{\partial^2 z}{\partial x^2} = r, \qquad \frac{\partial^2 z}{\partial x \partial y} = s, \qquad \frac{\partial^2 z}{\partial y^2} = t$$

9.2. ORDER of a partial differential equation is the same as that of the order of the highest differential coefficient in it.

9.3 CLASSIFICATION

Consider the equation. $A\dfrac{\partial^2 u}{\partial x^2} + B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} + F(x, y, u, p, q) = 0$ ... (1)

Where A, B, C may be constants or functions of x and y. Now the equation (1) is

1. Parabolic; if $B^2 - 4AC = 0$

2. Elliptic; if $B^2 - 4AC < 0$

3. Hyperbolic; if $B^2 - 4AC > 0$

9.4 METHOD OF FORMING PARTIAL DIFFERENTIAL EQUATIONS

A partial differential equation is formed by two methods.

(i) By eliminating arbitrary constants.

(ii) By eliminating arbitrary functions.

(*i*) **Method of elimination of arbitrary constants**

Example 1. *Form a partial differential equation from*

$$x^2 + y^2 + (z - c)^2 = a^2.$$

Solution. $x^2 + y^2 + (z - c)^2 = a^2$ ...(1)

(1) contains two arbitrary constants a and c.

Differentiating (1) partially w.r.t. x we get

$$2x + 2(z - c)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \qquad x + (z - c)\,p = 0 \qquad ...(2)$$

Differentiating (1) partially w.r.t. y we get

$$2y + 2(z - c)\frac{\partial z}{\partial y} = 0$$

$$y + (z - c)\, q = 0 \qquad ...(3)$$

Let us eliminate c from (2) and (3)

From (2) $\qquad (z - c) = -\dfrac{x}{p}$

Putting this value of $z - c$ in (3), we get $y - \dfrac{x}{p} q = 0$

or $\qquad yp - xq = 0$ **Ans.**

(*ii*) **Method of elimination of arbitrary functions**

Example 2. *Form the partial differential equation from*

$$z = f(x^2 - y^2)$$

Solution. $\qquad z = f(x^2 - y^2) \qquad ...(1)$

Differentiating (1) w.r.t x and y

$$P = \frac{\partial z}{\partial x} = f'(x^2 - y^2)\, 2x \qquad ...(2)$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)\,(-2y) \qquad ...(3)$$

Dividing (2) by (3) we get $\dfrac{p}{q} = \dfrac{-x}{y}$ or $py = -qx$

or $\qquad yp + xq = 0$ **Ans.**

EXERCISE 9.1

Form the partial differential equation

1. $z = (x + a)(y + b)$ **Ans.** $pq = z$
2. $(x - h)^2 + (y - k)^2 + z^2 = a^2$ **Ans.** $z^2(p^2 + q^2 + 1) = a^2$
3. $2z = (a\,x + y)^2 + b$ **Ans.** $p\,x + q\,y = q^2$
4. $ax^2 + by^2 + z^2 = 1$ **Ans.** $z(px + qy) = z^2 - 1$
5. $x^2 + y^2 = (z - c)^2 \tan^2 \alpha$ **Ans.** $yp - xq = 0$
6. $z = f(x^2 + y^2)$ **Ans.** $yp - xq = \mathbf{0}$
7. $2z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ (*A.M.I.E., Winter 2001*) **Ans.** $2z = xp + yq$
8. $f(x + y + z,\ x^2 + y^2 + z^2) = 0$ **Ans.** $(y - z)\,p + (z - x)\,q = x - y$

9.5 SOLUTION OF EQUATION BY DIRECT INTEGRATION

Example 3. *Solve* $\qquad \dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

Solution. $\qquad \dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

Integrating w.r.t. 'x', we get $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{2}\sin(2x + 3y) + f(y)$

Integrating w.r.t. x, we get $\dfrac{\partial z}{\partial y} = -\dfrac{1}{4}\cos(2x + 3y) + x \int f(y)\, dy + g(y)$

$$= -\frac{1}{4}\cos(2x+3y) + x\phi(y) + g(y)$$

Integrating w.r.t. 'y' we get

$$z = \frac{1}{12}\sin(2x+3y) + x\int\phi(y)\,dy + \int g(y)\,dy$$

$$z = -\frac{1}{12}\sin(2x+3y) + x\phi_1(y) + \phi_2(y)$$ **Ans.**

Example 4. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 y$

subject to the condition $z(x, 0) = x^2$ and $z(1, y) = \cos y$.

Solution. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = x^2 y$

On integrating w.r.t. x, we obtain $\dfrac{\partial z}{\partial y} = \dfrac{x^3}{3}y + f(y)$

Integrating w.r.t. y, we obtain $z = \dfrac{x^3}{3}\cdot\dfrac{y^2}{2} + \int f(y)\,dy + g(x)$

$$\left[F(y) = \int f(y)\,dy\right]$$

or $$z = \frac{x^3 y^2}{6} + F(y) + g(x) \quad \ldots (1)$$

Condition 1: Putting $z = x^2$ and $y = 0$ in (1), we get

$$x^2 = 0 + F(0) + g(x)$$

Putting the value of $g(x)$ in (1), we get $z = \dfrac{x^3 y^2}{6} + F(y) + x^2 - F(0)$ ...(2)

Condition 2: $z(1, y) = \cos y$

Putting $x = 1$ and $z = \cos y$ in (2), we get

$$\cos y = \frac{y^2}{6} + F(y) + 1 - F(0)$$

Putting the value of $F(y)$ in (2), we obtain

$$z = \frac{1}{6}x^3y^2 + \cos y - \frac{1}{6}y^2 - 1 + F(0) + x^2 - F(0)$$

or $$z = \frac{1}{6}x^3y^2 + \cos y - \frac{1}{6}y^2 - 1 + x^2$$ **Ans.**

Example 5. *Solve* $\dfrac{\partial^2 z}{\partial y^2} = z$, *if $y = 0$, $z = e^x$ and* $\dfrac{\partial z}{\partial y} = e^{-x}$

Solution. If z is a function of y alone, then

$$z = \sinh y \cdot f(x) + \cosh y \cdot \phi(x) \quad \ldots (1)$$

$$\frac{\partial^2 z}{\partial y^2} = z \quad \Rightarrow \quad (D^2 - 1)z = 0 \quad \Rightarrow \quad m = \pm 1$$

$$\Rightarrow \quad z = A e^{y} + B e^{-y} = A \sinh y + B \cosh y$$

$$= f(x)\sinh y + \phi(x) \cdot \cosh y$$

On putting $y = 0$ and $z = e^x$ in (1), we obtain

$$e^x = \phi(x)$$

(1) becomes $$z = \sinh y \,.\, f(x) + \cosh y \,.\, e^x \qquad ...(2)$$

On differentiating (2) w.r.t. y, we get

$$\frac{\partial z}{\partial y} = \cosh y \cdot f(x) + \sinh y \cdot e^x \qquad ...(3)$$

On putting $y = 0$ and $\frac{\partial z}{\partial y} = e^{-x}$ in (3), we obtain

$$e^{-x} = f(x)$$

(2) becomes, $$z = e^{-x} \sinh y + e^x \cosh y$$ **Ans.**

EXERCISE 9.2

Solve the following:

1. $\frac{\partial^2 z}{\partial x \partial y} = xy^2$ **Ans.** $z = \frac{x^2 y^3}{6} + f(y) + \phi(x)$
2. $\frac{\partial^2 z}{\partial x \partial y} = e^y \cos x$ **Ans.** $z = e^y \sin x + f(y) + \phi(x)$
3. $\frac{\partial^2 z}{\partial x \partial y} = \frac{y}{x} + 2$ **Ans.** $z = \frac{y^2}{2} \log x + 2xy + f(v) + \phi(x)$
4. $\frac{\partial^2 z}{\partial x^2} = a^2 z$, when $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$ **Ans.** $z = \sin x + e^y \cos x$
5. $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ if $\frac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$, and $z = 0$ when y is an odd multiple of $\frac{\pi}{2}$.

   Ans. $z = \cos x \cos y + \cos y$
6. The partial differential equation $y \frac{\partial^2 u}{\partial x^2} + 2x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y} = 0$ is elliptic if

   (*a*) $x^2 = y^2$ (*b*) $x^2 < y^2$ (*c*) $x^2 + y^2 > 1$ (*d*) $x^2 + y^2 = 1$

   (*A.M.I.E.T.E., Dec. 2004*) **Ans.** (*b*)

9.6 LAGRANGE'S LINEAR EQUATION IS AN EQUATION OF THE TYPE

$$Pp + Qq = R$$

where P, Q, R are the functions of x, y, z and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$

Solution. $$Pp + Qq = R \qquad ...(1)$$

This form of the equation is obtained by eliminating an arbitrary function f from

$$f(u, v) = 0 \qquad ...(2)$$

where u, v are functions of x, y, z.

Differentiating (2) partially w.r.t. to x and y.

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0 \quad ...(3) \quad \text{and} \quad \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0 \quad ...(4)$$

Let us eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4).

From (3), $\dfrac{\partial f}{\partial u}\left[\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial z}p\right]=-\dfrac{\partial f}{\partial v}\left[\dfrac{\partial v}{\partial x}+\dfrac{\partial v}{\partial z}p\right]$ ...(5)

From (4), $\dfrac{\partial f}{\partial u}\left[\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}q\right]=-\dfrac{\partial f}{\partial v}\left[\dfrac{\partial v}{\partial y}+\dfrac{\partial v}{\partial z}q\right]$ ...(6)

Dividing (5) by (6), we get $\dfrac{\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial z}.p}{\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}.q}=\dfrac{\dfrac{\partial v}{\partial x}+\dfrac{\partial v}{\partial z}.p}{\dfrac{\partial v}{\partial y}+\dfrac{\partial v}{\partial z}.q}$

or $\left[\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial z}.p\right]\left[\dfrac{\partial v}{\partial y}+\dfrac{\partial v}{\partial z}q\right]=\left[\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}.q\right]\left[\dfrac{\partial v}{\partial y}+\dfrac{\partial v}{\partial z}.p\right]$

$$\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}.q+\frac{\partial u}{\partial z}\times p\times\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial z}.pq$$

or
$$=\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}.p+\frac{\partial u}{\partial z}.q\times\frac{\partial v}{\partial x}+\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial z}.pq$$

$$\left[\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial y}\right]p+\left[\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}\right]q=\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x} \quad ...(7)$$

If (1) and (7) are the same, then the coefficients of p, q are equal.

$$P=\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial y}$$
$$Q=\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z} \quad ...(8)$$
$$R=\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}$$

Now suppose $u=c_1$ and $v=c_2$ are two solutions, where a, b are constants.

Differentiating $u=c_1$ and $v=c_2$

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0 \quad ...(9)$$

and
$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0 \quad ...(10)$$

Solving (9) and (10), we get

$$\frac{dx}{\dfrac{\partial u}{\partial y}\times\dfrac{\partial v}{\partial z}-\dfrac{\partial u}{\partial z}\times\dfrac{\partial v}{\partial y}}=\frac{dy}{\dfrac{\partial u}{\partial z}\times\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial x}\times\dfrac{\partial v}{\partial z}}=\frac{dz}{\dfrac{\partial u}{\partial x}\times\dfrac{\partial v}{\partial y}-\dfrac{\partial u}{\partial y}\times\dfrac{\partial v}{\partial x}} \quad ...(11)$$

From (8) and (11) $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$

Solutions of these equations are $u = c_1$ and $v = C_2$

$\therefore$ *$f(u, v) = 0$ is the required solution of (1).*

9.7 WORKING RULE

First step. Write down the auxiliary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Second step. Solve the above auxiliary equations.

Let the two solutions be $u = c_1$ and $v = c_2$.

Third step. Then $f(u, v) = 0$ or $u = \phi(v)$ is the required solution of

$$Pp + Qq = R.$$

Example 6. *Solve the following partial differential equation*

$$yq - xp = z. \qquad \text{where } p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}.$$

Solution. $yq - xp = z$

Here the auxiliary equations are

$$\Rightarrow \quad \frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \quad -\log x = \log y - \log a \qquad \text{(From first two equations)}$$

$$\Rightarrow \quad xy = a \qquad ...(1)$$

$$\Rightarrow \quad \log y = \log z + \log b \qquad \text{(From last two equations)}$$

$$\frac{y}{z} = b \qquad ...(2)$$

From (1) and (2)

Hence the solution is $f\left(xy, \frac{y}{z}\right) = 0$ **Ans.**

Example 7. *Solve* $y^2p - xyq = x(z - 2y)$ (*A.M.I.E., Summer 2001*)

Solution. $y^2p - xyq = x(z - 2y)$

The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \qquad ...(1)$$

Considering first two members of the equations

$$\frac{dx}{y} = \frac{dy}{-x} \qquad \Rightarrow x\,dx = -y\,dy$$

Integrating $\frac{x^2}{2} = -\frac{y^2}{2} + \frac{C_1}{2} \qquad \Rightarrow x^2 + y^2 = C_1 \qquad ...(2)$

From last two equations of (1)

$$-\frac{dy}{y} = \frac{dz}{z-2y}$$

$$\Rightarrow \quad -zdy + 2y\,dy = ydz \qquad \Rightarrow \quad 2y\,dy = y\,dz + z\,dy$$

On integration, we get

$$y^2 = yz + C_2$$

$$y^2 - yz = C_2 \qquad ...(3)$$

From (2) and (3)

$$x^2 + y^2 = f(y^2 - yz)$$ **Ans.**

Example 8. *Solve* $(x^2-yz)\,p + (y^2-zx)\,q = z^2-xy$ (*A.M.I.E., Summer 2001*)

Solution. $(x^2-yz)p + (y^2-zx)q = z^2-xy$ ...(1)

The auxiliary equations are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

or
$$\frac{dx-dy}{x^2-yz-y^2+zx}=\frac{dy-dz}{y^2-zx-z^2+xy}=\frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(x+y+z)(y-z)}=\frac{dz-dx}{(x+y+z)(z-x)}$$

$$\frac{dx-dy}{(x-y)}=\frac{dy-dz}{(y-z)}=\frac{dz-dx}{(z-x)} \qquad ...(2)$$

Intergrating first members of (2), we have

$$\log(x-y)=\log(y-z)+\log c_1$$

$$\log\frac{x-y}{y-z}=\log c_1 \quad \text{or} \quad \frac{x-y}{y-z}=c_1$$

Similarly from last two members of (2) ,we have

$$\frac{y-z}{z-x}=c_2$$

The required solution is

$$f\left[\frac{x-y}{y-z},\frac{y-z}{z-x}\right]=0$$ **Ans.**

9.8 METHOD OF MULTIPLIERS

Let the auxiliary equations be

$$\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}$$

l, m, n may be constants or functions of x, y, z then we have

$$\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lp+mQ+nR}$$

$1, m, n$ are chosen in such a way that

$$1P+mQ+nR=0$$

Thus $$ldx+mdy+ndz=0$$

Solve this differential equation, if the solution is $u=c_1$.

Similarly, choose another set of multipliers (l_1, m_1, n_1) and if the second solution is $v=C_2$.

$\therefore$ Required solution is $f(u, v)=0$.

Example 9. *Solve*

$$(mz-ny)\frac{\partial z}{\partial x}+(nx-lz)\frac{\partial z}{\partial y}=ly-mx$$ (*A.M.I.E. Winter 2001*)

Solution. $$(mz-ny)\frac{\partial z}{\partial x}+(nx-lz)\frac{\partial z}{\partial y}=ly-mx$$

Here, the auxiliary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using multipliers x, y, z we get

$$\text{Each fraction} = \frac{x\,dx+y\,dy+z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$\therefore \quad \Rightarrow \qquad x\,dx + y\,dy + z\,dz = 0$

which on integration gives $x^2+y^2+z^2=c_1$ ...(1)

Again using multipliers, $l, m, n,$ we get

$$\text{each fraction} = \frac{l\,dx+m\,dy+n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=\frac{l\,dx+m\,dy+n\,dz}{0}$$

$\therefore \quad \Rightarrow \qquad ldx + m\,dy + n\,dz = 0$

which, on integration gives.

$$lx+m\,y+n\,z=c_2 \qquad \text{...(2)}$$

Hence from (1) and (2), the required solution is $x^2+y^2+z^2=f(lx+my+nz)$ **Ans.**

Example 10. *Find the general solution of*

$$x\,(z^2-y^2)\frac{\partial z}{\partial x}+y\,(x^2-z^2)\frac{\partial z}{\partial y}=z\,(y^2-x^2)$$

Solution. $x\,(z^2-y^2)\dfrac{\partial z}{\partial x}+y\,(x^2-z^2)\dfrac{\partial z}{\partial y}=z\,(y^2-x^2)$

The auxiliary simultaneous equations are

$$\frac{dx}{x(z^2-y^2)}=\frac{dy}{y(x^2-z^2)}=\frac{dz}{z(y^2-x^2)} \qquad \text{...(1)}$$

Using multipliers x, y, z we get

Each term of (1) is equal to

$$\frac{x\,dx+y\,dy+z\,dz}{x^2(z^2-y^2)+y^2(x^2-z^2)+z^2(y^2-x^2)}=\frac{xdx+ydy+zdz}{0}$$

$\Rightarrow \qquad x\,dx + y\,dy + z\,dz = 0$

On integration $\qquad x^2+y^2+z^2= C_1$ ...(2)

Again (1) can be written as

$$\frac{\frac{dx}{x}}{z^2-y^2}=\frac{\frac{dy}{y}}{x^2-z^2}=\frac{\frac{dz}{z}}{y^2-x^2}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{(z^2-y^2)+(x^2-z^2)+(y^2-x^2)}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

$\Rightarrow \quad \log x + \log y + \log z = \log C_2$

$\Rightarrow \quad \log xyz = \log C_2 \qquad \Rightarrow \qquad xyz = C_2$ ...(3)

From (2) and (3), the general solution is $xyz=f(x^2+y^2+z^2)$ **Ans.**

Example 11. *Solve the partial differential equation*

$$\frac{y-z}{yz}p=\frac{z-x}{zx}q=\frac{x-y}{xy}$$ (*A.M.I.E., Winter 2001*)

Solution. $$\frac{y-z}{yz}p=\frac{z-x}{zx}q=\frac{x-y}{xy}$$

Multiplying by xyz, we get

$$x(y-z)p+y(z-x)q=z(x-y)$$

$$\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}=\frac{dz+dy+dz}{x(y-z)+y(z-x)+z(x-y)} \quad \ldots (1)$$

$$=\frac{dx+dy+dz}{0}$$

$$\therefore \quad dx+dy+dz=0$$

Which on integration gives

$$x+y+z=a \quad \ldots (2)$$

Again (1) can be written

$$\frac{\frac{dx}{x}}{y-z}=\frac{\frac{dy}{y}}{z-x}=\frac{\frac{dz}{z}}{x-y}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{(y-z)+(z-x)+(x-y)}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

or $$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

On integration we get

$$\log x+\log y+\log z=\log b \Rightarrow \log xyz=\log b \Rightarrow xyz=b \quad \ldots (3)$$

From (2) and (3) the general solution is

$$xyz .= f(x+y+z)$$ **Ans.**

Example 12. *Solve* $(x^2-y^2-z^2)\,p+2xy\,q=2xz$. (*A.M.I.E., Summer, 2004, 2000*)

Solution. $(x^2-y^2-z^2)\,p+2xyq=2xz$

Here the auxiliary equations are

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz} \quad \ldots (1)$$

From the last two members of (1) we have dz

$$\frac{dy}{y}=\frac{dz}{z}$$

which on integration gives

$$\log y=\log z+\log a \quad \text{or} \quad \log\frac{y}{z}=\log a$$

or $$\frac{y}{z}=a \quad \ldots (2)$$

Using multipliers x, y, z we have

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}=\frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$$

$$\frac{2x\,dx+2y\,dy+2z\,dz}{\left(x^2+y^2+z^2\right)}=\frac{dz}{z}$$

which on integration gives

$$\log(x^2+y^2+z^2)=\log z+\log b$$

$$\frac{x^2+y^2+z^2}{z}=b \qquad ...(3)$$

Hence from (2) and (3), the required solution is

$$x^2+y^2+z^2=z\,f\left(\frac{y}{z}\right)$$ **Ans.**

Example 13. *Solve the differential equation*

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z.$$

Solution. $$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z. \qquad ...(1)$$

The auxiliary equations of (1) are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z} \qquad ...(2)$$

Take first two members of (2) and integrate them

$$-\frac{1}{x}=-\frac{1}{y}+c$$

$$\frac{1}{x}-\frac{1}{y}=c_1 \qquad ...(3)$$

(2) can be written as $$\frac{\frac{dx}{x}}{x}=\frac{\frac{dy}{y}}{y}+\frac{\frac{dz}{z}}{x+y}=\frac{\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}}{(x+y)-(x+y)}$$

or $$\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}=0$$

On integration we get

or $$\log x+\log y-\log z=\log c_2$$

or $$\log\frac{xy}{z}=\log c_2 \qquad \text{or} \qquad \frac{xy}{z}=c_2 \qquad ...(4)$$

From (3) and (4) we have

$$f\left[\frac{1}{x}-\frac{1}{y},\frac{xy}{z}\right]=0$$ **Ans.**

Example 14. *Find the general solution of*

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=xyt$$

Solution. The auxiliary equations are $$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{t}=\frac{dz}{xyt} \qquad ...(1)$$

Taking the first two members and integrating, we get

$$\log x=\log y+\log a$$

$\Rightarrow \quad \log x = \log\ ay \quad \Rightarrow \quad x = ay \quad \Rightarrow \quad y/x = a$ ...(2)

Similarly, from the 2nd and 3rd members

$$\frac{t}{y} = b \qquad ...(3)$$

Multiplying the equations (1) by xyt, we get

$$dz = \frac{tydx}{1} = \frac{txdy}{1} = \frac{xydt}{1} = \frac{tydx + txdy + xydt}{3}$$

Integrating,

$$z = \frac{1}{3}xyt + c \quad or \quad z - \frac{1}{3}xyt = c \qquad ...(4)$$

From (2), (3) and (4) the solution is

$$z - \frac{1}{3}xyt = f\left(\frac{y}{x}\right) + \phi\left(\frac{t}{y}\right) \qquad \textbf{Ans.}$$

Example 15. *Solve* $(y + z)p - (x + z)\, q = x - y$

Solution. $\quad (y+z)p - (x+z)q = x - y$ ... (1)

$\therefore$ The auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} \qquad ...(2)$$

$$\Rightarrow \quad \frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{dx+dy+dz}{y+z-(x+z)+x-y}$$

$$\Rightarrow \quad \frac{dz}{x-y} = \frac{dx+dy+dz}{0}$$

Thus, we have $\quad dx + dy + dz = 0$

which on integration gives $\quad x + y + z = c_1,$ ...(3)

Let us use multipliers $(x, y, -z)$ for (2)

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{x\,dx + y\,dy + zdz}{x(y+z) - y(x+z) - z(x-y)}$$

or
$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{x\,dx + y\,dy - z\,dz}{0}$$

Integrating $\quad x\,dx + y\,dy - z\,dz = 0$, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c_2$$

or
$$x^2 + y^2 - z^2 = 2c_2 \qquad ...(4)$$

From (3) and (4), we get the required solution

$$f(x + y + z,\ x^2 + y^2 - z^2) = 0 \qquad \textbf{Ans.}$$

Example 16. *Solve* $\quad zp + yq = x$

Solution. $\quad zp + yq = x$ ...(1)

The auxiliary equations are $\dfrac{dx}{z} = \dfrac{dy}{y} = \dfrac{dz}{x}$

$\qquad (i) \quad (ii) \quad (iii)$

From (*i*) and (*ii*) $\quad \dfrac{dx}{z} = \dfrac{dz}{x}$ or $x.dx = z.\, dz$

$$\Rightarrow \quad \frac{x^2}{2} = \frac{z^2}{2} - \frac{c_1}{2} \quad \text{or} \quad x^2 = z^2 - c_1 \qquad ...(2)$$

$$\Rightarrow \quad z = \sqrt{x^2 + c_1}$$

Putting the value of z in (1)

$$\frac{dx}{\sqrt{x^2 + c_1}} = \frac{dy}{y}$$

$$\sinh^{-1}\frac{x}{\sqrt{c_1}} = \log y + c_2 \quad \text{or} \quad \sinh^{-1}\frac{x}{\sqrt{c_1}} - \log y = c_2 \qquad ...(3)$$

From (2) and (3), the required solution is

$$f\left(z^2 - x^2\right) = \sinh^{-1}\frac{x}{\sqrt{c_1}} - \log y \qquad \textbf{Ans.}$$

Example 17. *Solve* $px\,(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$. (*A.M.I.E., Summer 2000*)

Solution. $\quad px\,(z - 2y^2) = (z - qy)\,(z - y^2 - 2x^3) \qquad ...(1)$

$$\Rightarrow \quad px\,(z - 2y^2) + qy\,(z - y^2 - 2x^3) = z\,(z - y^2 - 2x^3)$$

Here the auxiliary equations are

$$\frac{dx}{x\left(z - 2y^2\right)} = \frac{dy}{y\left(z - y^2 - 2x^3\right)} = \frac{dz}{z\left(z - y^2 - 2x^3\right)} \qquad ...(2)$$

From the last two members of (2) we have

$$\frac{dy}{y} = \frac{dz}{z}$$

which gives on integration

$\log y = \log z + \log a$ or $y = a\,z \qquad ...(3)$

From the first and third members of (2) we have

$$\frac{dx}{x(z - 2y^2)} = \frac{dz}{z(z - y^2 - 2x^3)} \qquad \text{Put } y = az$$

$$\Rightarrow \quad \frac{dx}{x\left(z - 2a^2z^2\right)} = \frac{dz}{z\left(z - a^2z^2 - 2x^3\right)}$$

$$\frac{dx}{x\left(1 - 2a^2z\right)} = \frac{dz}{z - a^2z^2 - 2x^3}$$

$$\Rightarrow \quad z\,dx - a^2z^2dx - 2x^3dx = xdz - 2a^2xz\,dz$$

$$\Rightarrow \quad (\,xdz - zdx) - a^2\,(2xz\,dz - z^2\,dx) + 2x^3dx = 0$$

On integrating, we have

$$\frac{z}{x} - a^2\frac{z^2}{x} + x^2 = b \qquad ...(4)$$

From (3) and (4), we have

$$\frac{y}{z} = f\left(\frac{z}{x} - \frac{a^2z^2}{x} + x^2\right) \qquad \textbf{Ans.}$$

EXERCISE 9.3

Solve the following partial differential equations :

1. $p \tan x + q \tan y = \tan z$ **Ans.** $f\left(\dfrac{\sin x}{\sin y}, \dfrac{\sin y}{\sin z}\right) = 0$

2. $y\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z^2 + 1$ (*AMIE. Winter 2002*) **Ans.** $f(x - y) = \log y - \tan^{-1} z$

3. $(y - z)p + (x - y)q = z - x$ **Ans.** $f(x + y + z, x^2 + 2yz) = 0$

4. $(y + zx)p - (x + yz)q = x^2 - y^2$ **Ans.** $f(x^2 + y^2 - z^2) = (x - y)^2 - (z + 1)^2$

5. $zx\dfrac{\partial z}{\partial x} - zy\dfrac{\partial z}{\partial y} = y^2 - x^2$ **Ans.** $f(x^2 + y^2 + z^2, xy) = 0$

6. $pz - qz = z^2 + (x + y)^2$ **Ans.** $[z^2 + (x + y)^2]e^{-2x} = f(x + y)$

7. $p + q + 2xz = 0$ **Ans.** $f(x - y) = x^2 + \log z$

8. $x^2p + y^2q + z^2 = 0$ **Ans.** $f\left(\dfrac{1}{y} - \dfrac{1}{x}, \dfrac{1}{y} + \dfrac{1}{z}\right) = 0$

9. $(x^2 + y^2)p + 2xyq = (x + y)z$ **Ans.** $f\left(\dfrac{x+y}{z}, \dfrac{2y}{x^2 - y^2},\right) = 0$

10. $\dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} = 2x - e^y + 1$ **Ans.** $f(2x + y) = z - \dfrac{(2x+1)^2}{4} - \dfrac{e^y}{2}$

11. $p + 3q = 5z + \tan(y - 3x)$ **Ans.** $f(y - 3x) = \dfrac{e^{5x}}{5z + \tan(y - 3x)}$

12. $xp - yq + x^2 - y^2 = 0$ **Ans.** $f(xy) = \dfrac{x^2}{2} + \dfrac{y^2}{2} + z$

13. $(x+y)\left(\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\right) = z - 1$ **Ans.** $f(x - y) = \dfrac{x + y}{(z-1)^2}$

14. $(x^3 + 3xy^2)\dfrac{\partial z}{\partial x} + (y^3 + 3x^2y)\dfrac{\partial z}{\partial y} = 2(x^2 + y^2)z$ **Ans.** $f\left(\dfrac{xy}{z^2}, (x - y)^{-2} - (x + y)^{-2}\right) = 0$

15. $(z^2 - 2yz - y^2)P + (xy + zx)q = xy - zx$ **Ans.** $(x^2 + y^2 + z^2) = f(y^2 - 2yz - z^2)$

16. Find the solution of the equation $\dfrac{x\partial z}{\partial y} - \dfrac{y\partial z}{\partial x} = 0,$ which passes through the curve $z = 1$, $x^2 + y^2 = 4$ **Ans.** $f(x^2 + y^2 - 4, z - 1) = 0$

17. $2x(y + z^2)p + y(2y + z^2)q = z^3$ (*AMIE Winter 2003*)

18. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$ **Ans.** $u = ue^{-x+\frac{3y}{2}}$

19. $4\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 3u$, when $t = 0$, $u = 3e^{-x} - e^{-5x}$ **Ans.** $u = 3e^{-x+t} - 3e^{-5x+2t}$

9.9 PARTIAL DIFFERENTIAL EQUATIONS NON-LINEAR IN p AND q.

We give below the methods of solving non-linear partial differential equations in certain standard form only.

Type I. *Equation of the Type* $f(p, q) = 0$ *i.e.*, equations containing p and q only.

Method. Let the required solution be

$$z = ax + by + c \qquad ...(1)$$

$$\therefore \quad \frac{\partial z}{\partial x} = a, \quad \frac{\partial z}{\partial y} = b.$$

On putting these values in $f(p, q) = 0$
we get $f(a, b) = 0$,
From this, find the value of b in terms of a and substitute the value of b in (1), that will be the required solution.

Example 18. *Solve* $p^2 + q^2 = 1$ ...(1)

Solution. Let $z = ax + by + c$ ...(2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = a, \qquad q = \frac{\partial z}{\partial y} = b$$

On substituting the values of p and q in (1), we have

$$\therefore \quad a^2 + b^2 = 1 \text{ or } \quad b = \sqrt{1 - a^2}$$

Putting the value of b in (2), we get $z = ax + \sqrt{1 - a^2}\, y + c$

This is the required solution. **Ans.**

Example 19. *Solve* $x^2p^2 + y^2 q^2 = z^2$. (*RGPV, Bhopal, Feb. 2008*)

Solution. This equation can be transformed in the above type.

$$\frac{x^2}{z^2}p^2 + \frac{y^2}{z^2}q^2 = 1$$

$$\Rightarrow \quad \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{Z}\frac{\partial z}{\partial y}\right)^2 = 1 \Rightarrow \left(\frac{\frac{\partial z}{z}}{\frac{\partial x}{x}}\right)^2 + \left(\frac{\frac{\partial z}{z}}{\frac{\partial y}{y}}\right)^2 = 1 \qquad ...(1)$$

Let $\quad \frac{\partial z}{z} = \partial Z, \qquad \frac{\partial x}{x} = \partial X, \qquad \frac{\partial y}{y} = \partial Y,$

$\therefore \quad \log z = Z, \quad \log x = X, \quad \log y = Y$

$\therefore$ (1) can be written as

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \qquad ...(2)$$

$$\Rightarrow \quad P^2 + Q^2 = 1$$

Let the required solution be

$$Z = aX + bY + c$$

$$P = \frac{\partial Z}{\partial X} = a, \; Q = \frac{\partial Z}{\partial Y} = b$$

From (2) we have

$$a^2 + b^2 = 1 \text{ or } b = \sqrt{1 - a^2}$$

$$Z = a\,X + \sqrt{1 - a^2}\; Y + c$$

$$\log z = a \log x + \sqrt{1 - a^2}\; \log y + c$$ **Ans.**

EXERCISE 9.4

Solve the following partial differential equations

1. $pq = 1$ **Ans.** $z = ax + \frac{1}{a}\, y + c$ **2.** $\sqrt{p} + \sqrt{q} = 1$ **Ans.** $z = ax + (1 - \sqrt{a}\,)^2 y + c$

3. $p^2 - q^2 = 1$ **Ans.** $z = ax - \sqrt{(a^2 - 1)}\; y + c$ **4.** $pq + p + q = 0$ **Ans.** $z = ax - \frac{a}{1 + a}\; y + c$

Type II. *Equation of the type*

$$z = px + qy + f(p,q)$$

Its solution is $z = ax + by + f(a, b)$

Example 20. *Solve* $z = px + qy + p^2 + q^2$

Solution. $z = px + qy + p^2 + q^2$ $\quad p = a, q = b$

Its solution is $z = ax+by+a^2 + b^2$ **Ans.**

Example 21. *Solve* $z = px + qy + 2\sqrt{pq}$

Solution. $z = px+ qy + 2\sqrt{pq}$

Its solution is $z = ax+by+2\sqrt{ab}$ **Ans.**

Type III. Equation of the type f(z, p, q) = 0 equations not containing x and y.

Let z be a function of u where

$$u = x + ay.$$

$$\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial u}{\partial y} = a$$

Then

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y} = \frac{dz}{du}(a)$$

On putting the values of p and q in the given equation $f(z, p, q) = 0$, it becomes

$f\left(z, \frac{dz}{dy}, a\frac{dz}{du}\right) = 0$ which is an ordinary differential equation of the first order.

Rule. Assume $u = x + ay$; replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ in the given equation and then solve the ordinary differential equation obtained.

Example 22. *Solve*

$$p(1 + q) = qz$$

Solution. $p(1 + q) = qz$ ... (1)

Let $u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

(1) becomes

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z \quad \text{or} \quad 1 + a\frac{\partial z}{\partial u} = az$$

$$\Rightarrow \quad a\frac{dz}{du} = az - 1 \Rightarrow du = \frac{a\,dz}{az - 1}$$

Integrating, we get

$$u = \log(az - 1) + \log c$$

$$x + ay = \log c\,(az - 1)$$ **Ans.**

Example 23. *Solve* $p(1+q^2) = q(z-a)$.

Solution. Let $u = x + by$

So that $$p = \frac{dz}{du} \quad \text{and} \quad q = b\frac{dz}{du}$$

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du}\left[1+b^2\left(\frac{dz}{du}\right)^2\right] = b\frac{dz}{du}(z-a)$$

$$1+b^2\left(\frac{dz}{du}\right)^2 = b(z-a) \quad \text{or} \quad b^2\left(\frac{dz}{du}\right)^2 = bz-ab-1$$

$$\frac{dz}{du} = \frac{1}{b}\sqrt{bz-ab-1}$$

$$\int\frac{b\,dz}{\sqrt{bz-ab-1}} = \int du + c$$

$$2\sqrt{bz-ab-1} = u+c$$

$$4(bz-ab-1) = (u+c)^2$$

$$4(bz-ab-1) = (x+by+c)^2$$ **Ans.**

Example 24. *Solve* $z^2(p^2x^2+q^2) = 1$ ...(1)

Solution. $z^2(p^2x^2+q^2) = 1$

$$\Rightarrow \quad z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = 1 \quad \Rightarrow \quad z^2\left[\left(\frac{\partial z}{\frac{\partial x}{x}}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = 1$$

$$\Rightarrow \quad z^2\left[\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = 1 \qquad ...(2)$$

where $\frac{\partial x}{x} = \partial X$ or $\log x = X$

Let $$u = X + ay$$

$$\frac{\partial z}{\partial X} = \frac{dz}{du} \quad \text{and} \quad \frac{\partial z}{\partial y} = a\frac{dz}{du}$$

Then (2) becomes

$$z^2\left[\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2\right] = 1 \Rightarrow \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2}$$

$$\Rightarrow \quad \left(\frac{dz}{du}\right)^2 = \frac{1}{z^2(1+a^2)} \Rightarrow \frac{dz}{du} = \frac{1}{z\sqrt{1+a^2}} \Rightarrow z\,dz = \frac{du}{\sqrt{1+a^2}}$$

$$\Rightarrow \quad \int z\,dz = \int\frac{du}{\sqrt{1+a^2}} + c \quad \text{or} \quad \frac{z^2}{2} = \frac{u}{\sqrt{1+a^2}} + c$$

$$\sqrt{1+a^2}\,\frac{z^2}{2} = u + c\sqrt{1+a^2}$$

$$= X + ay + c\sqrt{1+a^2}$$

$$= \log x + ay + c\sqrt{1+a^2}$$ **Ans.**

EXERCISE 9.5

Solve

1. $z^2(p^2+q^2+1)=1$ **Ans.** $(1-z^2)^{\frac{1}{2}} = -\dfrac{x+ay}{\sqrt{1+a^2}}+c$

2. $1+q^2=q(z-a)$ **Ans.** $\dfrac{x+by}{b}+\dfrac{1}{4}(z-a)^2=\dfrac{1}{4}(z-a)\sqrt{(z-a)^2-2^2}+4\cosh^{-1}\left(\dfrac{z-a}{2}\right)$

3. $x^2p^2+y^2q^2=z$ **Ans.** $2\sqrt{z}=\dfrac{\log x+a\log y}{\sqrt{1+a^2}}+c$

Type IV. *Equation of the type* $f_1(x,p)=f_2(y,q)$

In these equations, z is absent and the terms containing x and p can be written on one side and the terms containing y and q can be written on the other side.

Method. Let $f_1(x,p)=f_2(y,q)=a$

$f_1(x,p)=a$, solve it for p. Let $p=F_1(x)$

$f_2(y,q)=a$, solve it for q. Let $q=F_2(y)$

Since $$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \Rightarrow dz=p\,dx+q\,dy$$

$$\Rightarrow dz=F_1(x)\,dx+F_2(y)\,dy \Rightarrow z=\int F_1(x)\,dx+\int F_2(y)\,dy+c$$

Example 25. *Solve* $p-x^2=q+y^2$.

Solution. $p-x^2=q+y^2=c$ (say)

i.e. $p=x^2+c$ and $q=c-y^2$

Putting these values of p and q in

$$dz=pdx+qdy = (x^2+c)\,dx+(c-y^2)\,dy$$

$$z=\left(\frac{x^3}{x}+cx\right)+\left(cy-\frac{y^3}{3}\right)+c_1$$ **Ans.**

Example 26. *Solve* $p^2+q^2=z^2(x+y)$.

Solution. $p^2+q^2=z^2(x+y) \Rightarrow \left(\dfrac{p}{z}\right)^2+\left(\dfrac{q}{z}\right)^2=(x+y)$

$$\Rightarrow \left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2+\left(\frac{1}{z}\frac{\partial z}{\partial y}\right)^2=x+y \Rightarrow \left(\frac{\frac{\partial z}{z}}{\partial x}\right)+\left(\frac{\frac{\partial z}{z}}{\partial y}\right)^2=x+y$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=x+y \quad \text{where } \frac{\partial z}{z}=\partial Z \text{ or } \log z=Z$$

$\Rightarrow$ $p^2+Q^2=x+y$ $\Rightarrow$ $p^2-x=y-Q^2=a$

$P^2-x=a$ $\Rightarrow$ $P=\sqrt{a+x}$

$y-Q^2=a$ $\Rightarrow$ $Q=\sqrt{y-a}$

Therefore, the equation $dZ=\dfrac{\partial Z}{\partial x}dx+\dfrac{\partial Z}{\partial y}dy$

$dZ=Pdx+Qdy$ gives

$$dZ=\sqrt{a+x}\,dx+\sqrt{y-a}\,dy$$

$$z = \int \sqrt{a+x}\,dx + \int \sqrt{y-a}\,dy + c$$

$\Rightarrow$ $$\log z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c$$ **Ans.**

EXERCISE 9.6

Solve

1. $q - p + x - y = 0$ **Ans.** $2z = (x+a)^2 + (y+a)^2 + b$

2. $\sqrt{p} + \sqrt{q} = 2x$ **Ans.** $z = \frac{1}{6}(2x - a)^3 + a^2 y + b$

3. $q = xp + p^2$ **Ans.** $z = -\frac{x^2}{4} + \left\{\frac{x\sqrt{x^2+4a}}{4} + a\log\left(x + \sqrt{x^2+4a}\right)\right\} + ay + b$

4. $z^2(p^2 + q^2) = x^2 + y^2$

Ans. $z^2 = x\sqrt{x^2+a} + a\log(x + \sqrt{x^2+a}) + y\sqrt{y^2-a} - a\log(y + \sqrt{y^2-a}) + 2b$

5. $z(p^2 + q^2) = x - y$ **Ans.** $z^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + b$

6. $p^2 - q^2 = x - y$ **Ans.** $z = \frac{2}{3}(x+c)^{3/2} + \frac{2}{3}(y+c)^{3/2} + c_1$

7. $(p^2 + q^2)y = qz$ **Ans.** $z^2 = (cx + a)^2 + c^2y^2$

8. Tick √ the correct answer.

(*a*) The partial differential equation from $z = (a+x)^2 + y$ is

(*i*) $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$ (*ii*) $z = \frac{1}{4}\left(\frac{\partial z}{\partial y}\right)^2 + y$ (*iii*) $z = \left(\frac{\partial z}{\partial x}\right)^2 + y$ (*iv*) $z = \left(\frac{\partial z}{\partial y}\right)^2 + y$

(*b*) The solution of $xp + yq = z$ is

(*i*) $f(x, y) = 0$ (*ii*) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$ (*iii*) $f(xy, yz) = 0$ (*iv*) $f(x^2, y^2) = 0$

(*c*) The solution of $p + q = z$ is

(*i*) $f(x+y, y+\log z) = 0$ (*ii*) $f(xy, y\log z) = 0$

(*iii*) $f(x-y, y-\log z) = 0$ (*iv*) None of these

(*d*) The solution of $(y-z)p + (z-x)q = x - y$ is

(*i*) $f(x+y+z) = xyz$ (*ii*) $f(x^2+y^2+z^2) = xyz$

(*iii*) $f(x^2+y^2+z^2, x^2y^2z^2) = 0$ (*iv*) $f(x+y+z) = x^2+y^2+z^2$

Ans. (*a*) (*i*), (*b*) (*ii*), (*c*) (*iii*), (*d*), (*iv*)

9.10 CHARPIT'S METHOD

General method for solving partial differential equation with two independent variables.

Solution. Let the general partial differential equation be

$$f(x, y, z, p, q) = 0 \qquad \text{... (1)}$$

Since z depends on x, y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = p\,dx + q\,dy \qquad \text{... (2)}$$

The main aim in Charpits method is to find another relation between the variables x, y, z and p, q. Let the relation be

$$\phi(x, y, z, p, q) = 0 \qquad \text{...(3)}$$

On solving (1) and (3), we get the values of p and q.

These values of p and q when substituted in (2), it becomes integrable.

To determine ϕ, (1) and (3) are differentiated w.r.t. x and y giving

$$\left.\begin{aligned}\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0\\ \frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial z}p+\frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x}=0\end{aligned}\right\}\text{ w.r.t. x, (First pair)}$$

$$\left.\begin{aligned}\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0\\ \frac{\partial \phi}{\partial y}+\frac{\partial \phi}{\partial z}q+\frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y}=0\end{aligned}\right\}\text{ w.r.t. } y\text{, (Second pair)}$$

Eliminating $\frac{\partial p}{\partial x}$ between the equation of first pair, we have

$$-\frac{\partial p}{\partial x}=\frac{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}}{\frac{\partial f}{\partial p}}=\frac{\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial z}p+\frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x}}{\frac{\partial \phi}{\partial p}}$$

or $$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p}-\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p}-\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right)+\frac{\partial q}{\partial y}\left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p}-\frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)=0 \qquad ...(4)$$

On eliminating $\frac{\partial q}{\partial y}$ between the equations of second pair, we have

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q}-\frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial q}-\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right)+\frac{\partial q}{\partial y}\left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q}-\frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)=0 \qquad ...(5)$$

Adding (4) and (5) and keeping in view the relation on, the terms of the last brackets of (4) and(5) cancel. On rearranging, we get

$$\frac{\partial \phi}{\partial f}\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)+\frac{\partial \phi}{\partial q}\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)+\frac{\partial \phi}{\partial z}\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y}=0$$

or $$\left(-\frac{\partial f}{\partial p}\right)\left(\frac{\partial \phi}{\partial x}\right)+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z}+\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q}=0 \qquad ...(6)$$

Equation (6) is a Lagrange's linear equation of the first order with x, y, z, p, q as independent variables and 4) as dependent variable. Its subsidiary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}--q\frac{\partial f}{\partial q}}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{\partial \phi}{0} \qquad ...(7)$$

(Commit to memory)

Any of the integrals of (7) satisfies (6). Such an integral involving p or q or both may be taken as assumed relation (3). However, we should choose the simplest integral involving p and q derived from (7). This relation and equation (1) gives the values of p and q. The values of p and q are substituted in (2). On integration new eq. (2) gives the solution of (1).

Example 27. *Solve* $px + qy = pq$

Solution. $f(x, y, z, p, q) = 0$ is $px + qy - pq = 0$ ...(1)

$$\frac{\partial f}{\partial x} = p,\quad \frac{\partial f}{\partial y} = q,\quad \frac{\partial f}{\partial z} = 0,\quad \frac{\partial f}{\partial p} = x-q,\quad \frac{\partial f}{\partial q} = y-p$$

Charpits' equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = -\frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - -q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\partial\phi}{0}$$

$$\frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p} = \frac{dq}{q} = \frac{d\phi}{0}$$

We have to choose the simplest integral involving p and q

$\Rightarrow$ $$\frac{dp}{p} = \frac{dq}{q} \text{ or } \log p = \log q + \log a \Rightarrow p = aq$$

Putting for p in the given equation (1), we get

$$q\,(ax+y) = aq^2 \qquad \therefore\quad q = \frac{y+ax}{a}$$

$\therefore$ $$p = aq = y+ax$$

Now $$dz = pdx + qdy \qquad \ldots(2)$$

Putting for p and q in (2), we get

$$dz = (y+ax)\,dx + \frac{y+ax}{a}\,dy$$

$$adz = (y+ax) + (y+a\,x)\,dy$$

$$adz = (y+ax)\,(adx+dy)$$

Integrating $$az = \frac{(y+ax)^2}{2} + b$$ **Ans.**

Example 28. *Solve* $(p^2+q^2)y = qz.$ ... (1)

Solution. $f(x, y, z, p, q) = 0$ is $(p^2+q^2)y - qz = 0$

$$\frac{\partial f}{\partial x} = 0,\quad \frac{\partial f}{\partial x} = p^2+q^2,\quad \frac{\partial f}{\partial z} = -q,\quad \frac{\partial f}{\partial p} = 2py,\quad \frac{\partial f}{\partial q} = 2qy - z$$

Now Charpits equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$\Rightarrow$ $$\frac{dx}{-2py} = \frac{dy}{-2q+z} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2} = \frac{d\phi}{0}$$

We have to choose the simplest integral involving p and q.

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow \frac{dp}{q} = \frac{dq}{p} \Rightarrow pdp + qdp = 0$$

Integrating $p^2 + q^2 = a^2$(say)

Putting for $p^2 + q^2$ in the equation (1), we get

$$a^2 y = qz \Rightarrow q = \frac{a^2 y}{z} \text{ so } p = \sqrt{a^2 - q^2} = \sqrt{a^2 - \frac{a^4 y^2}{z^2}}$$

$$p = \frac{a}{z}\sqrt{z^2 - a^2 y^2}$$

Now $$dz = pdx + qdy \qquad ...(2)$$

Putting for p and q in (2), we get,

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2 y}{z}dy \quad \Rightarrow \quad zdz = a\sqrt{z^2 - a^2y^2} + a^2\,y\,dy$$

$$\frac{zdz - a^2\,y\,dy}{\sqrt{z^2 - a^2y^2}} = a.dx$$

Integrating, we get, $\frac{1}{2}\frac{2}{1}\sqrt{z^2 - a^2y^2} = ax + b$

On squaring, $z^2 - a^2\,y^2 = (ax + b)^2$ **Ans.**

EXERCISE 9.7

Solve the following:

1. $z = p \cdot q$ **Ans.** $2\sqrt{az} = ax + y + \sqrt{ab}$
2. $(p + q)(px + qy) - 1 = 0$ **Ans.** $z\sqrt{(1+a)} = 2\sqrt{(ax+y)} + b$
3. $z = px + gy + p^2 + q^2$ **Ans.** $z = ax + by + a^2 + b^2$
4. $z = p^2x + q^2\,y$ **Ans.** $(1 + a)\,z = \left[\sqrt{ax} + \sqrt{(b+y)}\right]^2$
5. $z^2 = pq\,xy$ **Ans.** $z = ax^b\,y^{1/b}$
6. $px + pq + qy = yz$ **Ans.** $\log(z - ax) = y - a\log(a + y) + b$
7. $q + xp = p^2$ **Ans.** $z = ax\,e^{-y} - \frac{1}{2}a^2e^{-2y} + b$

9.11 LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS OF nTH ORDER WITH CONSTANT COEFFICIENTS

An equation of the type

$$a_0\frac{\partial^n z}{\partial x^n} + a_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + ... + a_n\frac{\partial^n z}{\partial y^n} = F(x, y) \qquad ... (1)$$

is called a homogeneous linear partial differential equation of nth order with constant coefficients. It is called homogeneous because all the terms contain derivatives of the same order.

Putting $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$, (1) becomes

$$(a_0 + D^n + a_1D^{n-1}\,D' + ... + a_n\,D'^{\,n})z = F(x,y)$$

or $$f(D, D')\,z = F(x, y)$$

9.12 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation

$$a_0\frac{\partial^2 z}{\partial x^2} + a_1\frac{\partial^2 z}{\partial x\partial y} + a_2\frac{\partial^2 z}{\partial y^2} = 0 \quad \text{or} \quad (a_0D^2 + a_1DD' + a_2\,D'^2)\,z = 0$$

1st step : Put $D = m$ and $D' = 1$

$$a_0\,m^2 + a_1m + a_2 = 0$$

This is the auxiliary equation.

2nd step : Solve the auxiliary equation.

Case 1. If the roots of the auxiliary equation are real and different; say m_1, m_2

Then C.F. $= f_1(y + m_1\,x) + f_2\,(y + m_2\,x)$.

Case 2. If the roots are equal; say m

Then C.F. $= f_1(y+mx) + xf_2(y+mx)$

Example 29. *Solve* $(D^3 - 4D^2 D' + 3D D'^2)z = 0$.

Solution. $(D^3 - 4D^2 D' + 3D D'^2)z = 0$ $[D = m, D' = 1]$

Its auxiliary equation is

$$m^3 - 4m^2 + 3m = 0 \quad \Rightarrow \quad m(m^2 - 4m + 3) = 0$$

$$m(m-1)(m-3) = 0 \quad \Rightarrow \quad m = 0, 1, 3$$

The required solution is $z = f_1(y) + f_2(y+x) + f_3(y+3x)$ **Ans.**

Example 30. *Solve* $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = 0$

Solution. $(D^2 - 4D D' + 4D'^2)z = 0$

Its auxiliary equation is $[D = m, D' = 1]$

$$m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m-2)^2 = 0 \Rightarrow m = 2, 2$$

The required solution is $z = f_1(y+2x) + x f_2(y+2x)$ **Ans.**

EXERCISE 9.8

Solve the following equations :

1. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{4\partial^2 z}{\partial x \partial y} - 5\dfrac{\partial^2 z}{\partial y^2} = 0$ **Ans.** $z = f_1(y+x) + f_2(y-5x)$
2. $2\dfrac{\partial^2 z}{\partial x^2} + 5\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0$ **Ans.** $z = f_1(2y-x) + f_2(y-2x)$
3. $(D^3 - 6D^2 D' + 11D D'^2 - 6D'^3)z =$ **Ans.** $z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$
4. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$ **Ans.** $z = fi(y+x) + xf_2(y+x)$
5. $(D^3 - 6.D^2 D' + 12D D'^2 - 8D'^3)z = 0$ **Ans.** $z = f_1(y+2x) + xf_2(y+2x) + x^2 f_3(y+2x)$
6. $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^4 z}{\partial y^4} = 0$ **Ans.** $z = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$
7. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, when $u = \sin y$, $x = 0$ for all y and $u \to 0$ when $x \to \infty$.

   Ans. $u = f_1(y+ix) + f_2(y-ix)$

9.13. RULES FOR FINDING THE PARTICULAR INTEGRAL

Given partial differential equation is

$$f(D, D')z = F(x, y)$$

$$P.I. = \frac{1}{f(D, D')} F(x, y)$$

(*i*) When $F(x,y) = e^{ax+by}$

$$P.I. = \frac{1}{f(D, D')} e^{ax+by} = \frac{e^{ax+by}}{f(a, b)} \qquad \text{[Put } D = a, D' = b]$$

(*ii*) When F(x,y) = sin($ax + by$) or cos ($ax + by$)

$$P.I. = \frac{1}{f\left(D^2, DD', D'^2\right)} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$= \frac{\sin(ax+by) \, or \cos(ax+by)}{f\left(-a^2, -ab, -b^2\right)} \qquad \left[\begin{array}{l}\text{Put } \; D^2 = -a^2 \\ \quad DD' = -ab, \; D'^2 = -b^2\end{array}\right]$$

(*iii*) When F (x,y) = $x^m y^n$

$$P.I. = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

Expand $[f(D, D')]^{-1}$ in ascending power of D or D' and operate on $x^m y^n$ term by term.

(*iv*) When = Any function $F(x, y)$

$$P.I. = \frac{1}{f(D, D')} F(x, y)$$

Resolve $\frac{1}{f(D, D')}$ into partial fractions

Considering $f(D, D')$ as a function of D alone

$$P.I. = \frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$$

where c is replaced by $y + mx$ after integration.

Case 1. *When R.H. S.* = e^{ax+by}

Example 31. *Solve* : $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

Solution. $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

Given equation in symbolic form is

$$(D^3 - 3D^2 D' + 4D'^3)z = e^{x+2y}$$

Its A.E. is $m^3 - 3m^2 + 4 = 0$ whence, $m = -1, 2, 2$.

$$\text{C.F.} = f_1(y - x) + f_2(y + 2x) + x f_3(y + 2x)$$

$$P.I. = \frac{1}{D^3 - 3D^2 D' + 4D'^3} e^{x+2y}$$

Put $D = 1, D' = 2$ $\qquad = \frac{1}{1 - 6 + 32} e^{x+2y} = \frac{e^{x+2y}}{27}$

Hence complete solution is

$$z = f_1(y - x) + f_2(y + 2x) + x f_3(y + 2x) + \frac{e^{x+2y}}{27} \qquad \textbf{Ans.}$$

EXERCISE 9.9

Solve the following equations:

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x+2y}$ **Ans.** $z = f_1(y + x) + f_2(y - x) - \frac{e^{x+2y}}{3}$

2. $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$ **Ans.** $z = f_1(y + 2x) + f_2(y + 3x) + \frac{1}{2} e^{x+y}$

3. $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = e^{2x+y}$ **Ans.** $z = f_1(y+2x) + xf_2(y+2x) + \dfrac{x^2}{2}e^{2x+y}$

4. $\dfrac{\partial^2 z}{\partial x^2} - 7\dfrac{\partial^2 z}{\partial x \partial y} + 12\dfrac{\partial^2 z}{\partial y^2} = e^{x-y}$ **Ans.** $z = f_1(y+3x) + f_2(y+4x) + \dfrac{1}{20}e^{x-y}$

5. $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x-y}$ **Ans.** $z = f_1(y) + x f_2(y) + f_3(y+2x) + \dfrac{1}{8}e^{2x-y}$

6. $(D^2 - 2DD' + D'^2)z = e^{x+2y}$ **Ans.** $z = f_1(y+x) + xf_2(y+x) + e^{x+2y}$

7. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} - 2\dfrac{\partial z}{\partial x} + 2\dfrac{\partial z}{\partial y} = e^{2x+3y}$ **Ans.** $z = f_1(y+x) + e^{2x}f_2(y-x) - \dfrac{1}{3}e^{2x+3y}$

8. $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x \partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = \exp(3x-2y)$ **Ans.** $z = f_1(y+2x) + f_2(y+3x) + \dfrac{1}{63}e^{3x-2y}$

Case II. When R.H.S. = sin $(ax + by)$ or cos $(ax + by)$

Example 32. *Solve* $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 2\sin(3x+2y)$

Solution. $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 2\sin(3x+2y)$

Putting $\dfrac{\partial}{\partial x} = D, \quad \dfrac{\partial}{\partial y} = D'$

$$D^3 z - 4D^2 D' z + 4DD'^2 z = 2\sin(3x+2y)$$

A.E. is $D^3 - 4D^2 D' + 4DD'^2 = 0 \Rightarrow D(D^2 - 4DD' + 4D'^2) = 0$

Put $D = m,\ D' = 1$

$$m(m^2 - 4m + 4) = 0 \Rightarrow m(m-2)^2 = 0 \Rightarrow m = 0, 2, 2$$

C.F. is $f_1(y) + f_2(y+2x) + xf_3(y+2x)$

$$P.I. = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 2\sin(3x+2y) = 2.\frac{1}{D(D^2 - 4DD' + 4D'^2)}\sin(3x+2y)$$

$$= 2.\frac{1}{D[-9 - 4(-6) + 4(-4)]}\sin(3x+2y) = -\frac{2}{D}\sin(3x+2y)$$

$$= -\frac{2}{3}[-\cos(3x+2y)] = \frac{2}{3}\cos(3x+2y)$$

General solution is

$$z = f_1(y) + f_2(y+2x) + xf_3(y+2x) + \frac{2}{3}\cos(3x+2y)$$ **Ans.**

Example 33. *Solve* $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

Solution. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

The given equation can be written in the form

$$(D^2 - DD')z = \sin x \cos 2y \qquad \text{where } D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$$

Writing $D = m$ and $D' = 1$, the auxiliary equation is

$$m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0, 1$$

$$C.F. = f_1(y) + f_2(y+x)$$

$$P.I. = \frac{1}{D^2 - DD'} \sin x \cos 2y = \frac{1}{D^2 - DD'} \frac{1}{2} [\sin(x+2y) + \sin(x-2y)]$$

$$= \frac{1}{2} \frac{1}{D^2 - DD'} \sin(x+2y) + \frac{1}{2} \frac{1}{D^2 - DD'} \sin(x-2y)$$

Put $D^2 = -1, DD' = -2$ in the first integral and $D^2 = -1, DD' = 2$ in the second integral.

$$\text{P.I.} = \frac{1}{2} \frac{\sin(x+2y)}{-1-(-2)} + \frac{1}{2} \frac{\sin(x-2y)}{-1-(2)} = \frac{1}{2} \sin(x+2y) - \frac{1}{6} \sin(x-2y)$$

Hence the complete solution is z = C.F. + P.I.

i.e. $$z = f_1(y) + f_2(y+x) + \frac{1}{2} \sin(x+2y) - \frac{1}{6} \sin(x-2y)$$ **Ans.**

Example 34. *Solve* $(D^2 + DD' - 6D'^2) z = \cos(2x + y)$

Solution. $(D^2 + DD' - 6D'^2) z = \cos(2x + y)$

A.E. is $m^2 + m - 6 = 0 \Rightarrow m = 2, -3$

$$\text{C.F.} = f_1(y + 2x) + f_2(y - 3x)$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D^{2'}} \cos(2x + y)$$

$$D^2 + DD' - 6D'^2 = -4 - 2 - 6(-1) = 0$$

$\therefore$ It is a case of failure.

Now $$P.I. = \frac{1}{D^2 + DD' - 6D^{2'}} \cos(2x+y)$$ (Case IV)

$$= x \frac{1}{2D + D'} \cos(2x+y) = x \frac{D}{2D^2 + DD'} \cos(2x+y)$$

$$= x \frac{D}{2(-4) - 2} \cos(2x+y) = -\frac{x}{10} D \cos(2x+y)$$

$$= 2 \frac{x}{10} \sin(2x+y) = \frac{x}{5} \sin(2x+y)$$

$$z = f_1(y + 2x) + f_2(y - 3x) + \frac{x}{5} \sin(2x + y)$$ **Ans.**

Example 35. *Solve the equation*

$$(D^3 - 7DD'^2 - 6D'^3) z = \sin(x + 2y) + e^{2x+y}.$$

Solution $(D^3 - 7DD'^2 - 6D'^3) z = \sin(x + 2y) + e^{2x+y}$ ...(1)

Its auxiliary equation is

$$m^3 - 7m - 6 = 0 \Rightarrow (m+1)(m+2)(m-3) = 0 \Rightarrow m = -1, -2, 3$$

$$\text{C.F.} = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$$

$$\text{P.I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \left[\sin(x+2y) + e^{2x+y}\right]$$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x+2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3}e^{2x+y}$$

$$= \frac{1}{D^2.D - 7DD'^2 - 6D'^2D'}\sin(x+2y) + \frac{e^{2x+y}}{(2)^3 - 7(2)(1)^2 - 6(1)^3}$$

Put $D^2 = -1,\ D'^2 = -2^2$

$$= \frac{1}{-D - 7D(-4) - 6(-4)D'}\sin(x+2y) + \frac{e^{2x+y}}{8-14-6}$$

$$= \frac{1}{27D + 24D'}\sin(x+2y) - \frac{1}{12}e^{2x+y} = \frac{1}{3}\frac{1}{9D+8D'}\sin(x+2y) - \frac{1}{12}e^{2x+y}$$

$$= \frac{1}{3}\frac{D}{9D^2 + 8DD'}\sin(x+2y) - \frac{1}{12}e^{2x+y} = \frac{1}{3}\frac{D}{9(-1)+8(-2)}\sin(x+2y) - \frac{1}{12}e^{2x+y}$$

$$= -\frac{1}{75}D\sin(x+2y) - \frac{1}{12}e^{2x+y} = -\frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$$

Hence the complete solution is

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$$ **Ans.**

EXERCISE 9.10

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$ **Ans.** $z = f_1(y+x) + xf_2(y+x) - \sin x$

2. $[2D^2 - 5DD' + 2D'^2]\ z = 5\sin(2x+y)$. **Ans.** $z = f_1(y+2x) + f_2(2y+x) - \frac{5}{3}x\cos(2x+y)$

3. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos(x+2y)$ **Ans.** $z = f_1(y) + f_2(y+x) + \cos(x+2y)$

4. $(D^2 - DD')\,z = \cos x \cos 2y$ **Ans.** $z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$

5. $(D^2 + 2D'D + D'^2)\,z = \sin(x+2y)$ **Ans.** $z = f_1(y-x) + xf_2(y-x) - \frac{1}{9}\sin(x+2y)$

6. $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + \sin(x+2y)$

Ans. $z = c_1 f(y+x) + f_2(y+2x) + \frac{1}{4}e^{2x+3y} - \frac{1}{15}\sin(x-2y)$

Case III. When R.H.S. $= x^m y^n$

Example 36. *Find the general integral of the equation*

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$$

Solution. $$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$$

with $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$, the given equation can be written in the form

$$(D^2 + 3DD' + 2D'^2)\,z = x + y$$

Writing $D = m$ and $D' = 1$, the auxiliary equation is

$$m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$$

$\therefore \qquad$ C.F. $= f_1(y-x) + f_2(y-2x)$

$$\text{P.I.} = \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{D^2}\left(1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)^{-1}(x+y) = \frac{1}{D^2}\left(1-\frac{3D'}{D}....\right)(x+y)$$

$$= \frac{1}{D^2}\left[x+y-3\frac{1}{D}(1)\right] = \frac{1}{D^2}[x+y-3x]$$

$$= \frac{1}{D^2}[y-2x] = \frac{x^2}{2}y - \frac{x^3}{3}$$

Hence the complete solution is $\qquad z = f_1(y-x) + f_2(y-2x) + \dfrac{x^2 y}{2} - \dfrac{x^3}{3}$ **Ans.**

Example 37. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = x + y$

Solution. With $D = \dfrac{\partial}{\partial x}, D' = \dfrac{\partial}{\partial y}$, the given equation can be written in the form

$$\left(D^2 + DD' - 6D'^2\right)z = x + y$$

Writing $D = m$ and $D' = 1$, the auxiliary equation is $m^2 + m - 6 = 0$

$\Rightarrow \qquad (m+3)(m-2) = 0 \quad \Rightarrow \quad m = -3, 2$

$\therefore \qquad$ C.F. $= f_1(y-3x) + f_2(y+2x)$

$\therefore \qquad$ P.I. $= \dfrac{1}{D^2+DD'-6D^2}(x+y)$

$$= \frac{1}{D^2}\left(1+\frac{D'}{D}-\frac{6D'^2}{D^2}\right)^{-1}(x+y) = \frac{1}{D^2}\left[1-\frac{D'}{D}+....\right](x+y)$$

$$= \frac{1}{D^2}\left(x+y-\frac{1}{D}(1)\right) = \frac{1}{D^2}(x+y-x) = \frac{1}{D^2}y = \frac{yx^2}{2}$$

The complete solution is

$$z = f_1(y-3x) + f_2(y+2x) + \frac{yx^2}{2} \qquad \textbf{Ans.}$$

Example 38. *Solve* $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$ (*A.M.I.E., Summer 2004, 2001*)

Solution. $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$

$\Rightarrow \qquad (D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$

Its auxiliary equation is

$$m^3 - 2m^2 = 0$$

$\Rightarrow \qquad m^2(m-2) = 0$

$\Rightarrow \qquad m = 0, 0, 2.$

C.F. $= f_1(y) + xf_2(y) + f_3(y+2x)$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2D'}\left(2e^{2x} + 3x^2 y\right)$$

$$= \frac{1}{D^3 - 2D^2 D'} 2e^{2x} + \frac{1}{D^3 - 2D^2 D'} 3x^2 y$$

$$= 2\frac{e^{2x}}{(2)^3 - 2(2)^2(0)} + 3.\frac{1}{D^3\left(1 - \frac{2D'}{D}\right)} x^2 y = \frac{2e^{2x}}{8} + \frac{3}{D^3}\left(1 - \frac{2D'}{D}\right)^{-1} x^2 y$$

$$= \frac{e^{2x}}{4} + \frac{3}{D^3}\left(1 + \frac{2D'}{D} ...\right) x^2 y = \frac{e^{2x}}{4} + \frac{3}{D^3}\left[x^2 y + \frac{2}{D} x^2\right] = \frac{e^{2x}}{4} + \frac{3}{D^3}\left(x^2 y + \frac{2x^3}{3}\right)$$

$$= \frac{e^{2x}}{4} + 3y\frac{1}{D^3} x^2 + \frac{2}{D^3} x^3 = \frac{e^{2x}}{4} + 3y\frac{x^5}{3.4.5} + 2\frac{x^6}{4.5.6} = \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

$$= \frac{1}{60}\left(15e^{2x} + 3x^5 y + x^6\right)$$

Hence the complete solution is

$$z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5 y + x^6) \qquad \textbf{Ans.}$$

EXERCISE 9.11

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$ **Ans.** $z = f_1(y - x) + f_2(y + x) + \frac{x^3}{6} - \frac{x^2 y}{2}$

2. $\frac{\partial^2 z}{\partial x^2} + \frac{3\partial^2 z}{\partial x \partial y} + \frac{2\partial^2 z}{\partial y^2} = 12xy$ (*A.M.I.E., Winter 2001*)

Ans. $z = f_1(y - x) + f_2(y - 2x) + 2x^3 y - \frac{3x^4}{2}$

3. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = xy$ **Ans.** $z = f_1(y - 2x) + f_2(y + 3x) + \frac{x^3 y}{6} + \frac{x^4}{24}$

4. $r + 2s + t = 2(y - x) + \sin(x - y)$ **Ans.** $z = f_1(y - x) + xf_2(y - x) + x^2 y - x^3 + \frac{x^2}{2}\sin(x - y)$

5. $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = x^2$ **Ans.** $z = f_1(y + ax) + f_2(y - ax) + \frac{x^4}{12}$

6. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$ **Ans.** $z = f_1(y + x) + xf_2(y + x) + \frac{x^4}{12} + \frac{x^2 y}{2} + \frac{x^3}{3}$

7. $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} - 4\frac{\partial^2 z}{\partial y^2} = x + \sin y$ **Ans.** $z = f_1(y + x) + f_2(y - 4x) + \frac{x^3}{6} + \frac{1}{4}\sin y$

8. $(D^3 - 3D^2 D')z = x^2 y$ **Ans.** $z = f_1(y) + x f_2(y) + f_3(y + 3x) + \frac{x^5 y}{60} + \frac{x^6}{120}$

Case IV. When R.H.S. = Any function

Example 39. *Solve* $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$

Solution. $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$

A.E. is $D^2 - DD' - 2D'^2 = 0 \quad \Rightarrow \quad m^2 - m - 2 = 0$

$\Rightarrow \quad (m - 2)(m + 1) = 0 \quad \Rightarrow \quad m = 2, -1$

$$\text{C.F.} = f_1(y + 2x) + f_2(y - x)$$

$$\text{P.I.} = \frac{1}{D^2 - DD' - 2D'^2}(y - 1)e^x$$

$$= \frac{1}{(D+D')(D-2D')}(y-1)e^x = \frac{1}{D+D'}\int\left[(c-2x-1)e^x dx\right] \qquad \text{[Put } y = c-2x\text{]}$$

$$= \frac{1}{D+D'}\left[(c-2x-1)e^x + 2e^x\right]$$

$$= \frac{1}{D+D'}\left[ce^x - 2xe^x + e^x\right] \qquad \text{[Put } c = y+2x\text{]}$$

$$= \frac{1}{D+D'}\left[(y+2x)e^x - 2xe^x + e^x\right] = \frac{1}{D+D'}\left[ye^x + e^x\right]$$

$$= \int\left[(c+x)e^x + e^x\right]dx \qquad \text{[Put } y = c+x\text{]}$$

$$= (c+x)e^x - e^x + e^x$$

$$= ce^x + xe^x = (y-x)e^x + xe^x \qquad \text{[Put } c = y-x\text{]}$$

$$= y\,e^x$$

Hence complete solution is $z = f_1(y+2x) + f_2(y-x) + ye^x$ **Ans.**

Example 40. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y\cos x$

Solution. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y\cos x$

$$(D^2 + DD' - 6D'^2) = y\cos x$$

Its auxiliary equation is $m^2 + m - 6 = 0$

$$(m+3)(m-2) = 0$$

$$m = 2, -3$$

$$\text{C.F.} = f_1(y+2x) + f_2(y-3x)$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\,y\cos x = \frac{1}{(D-2D')(D+3D')}\,y\cos x$$

$$= \frac{1}{D-2D'}\int(c+3x)\cos x\,dx \qquad \text{Put } y = c+3x$$

$$= \frac{1}{D-2D'}[(c+3x)\sin x + 3\cos x] = \frac{1}{D-2D'}[y\sin x + 3\cos x] \qquad \text{Put } c+3x = y$$

$$= \int[(c-2x)\sin x + 3\cos x]\,dx \qquad \text{Put } y = c-2x$$

$$= (c-2x)(-\cos x) - 2\sin x + 3\sin x = -y\cos x + \sin x \qquad \text{Put } c-2x = y$$

Hence the complete solution is

$$z = f_1(y+2x) + f_2(y-3x) + \sin x - y\cos x$$ **Ans.**

EXERCISE 9.12

Solve the following equations:

1. $(D - D')(D + 2D')\,z = (y+1)\,e^x$ **Ans.** $z = f_1(y+x) + f_2(y-2x) + y\,e^x$

2. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = \tan^3 x\tan y - \tan x\tan^3 y$ **Ans.** $z = f_1(y+x) + f_2(x-y) + \dfrac{1}{2}\tan x\tan y$

3. $(D^2 - DD' - 2D'^2)\,z = (2x^2 + xy - y^2)\sin xy - \cos xy$ **Ans.** $z = f_1(y+2x) + f_2(y-x) + \sin xy$

4. Tick √ the correct answer :

(*a*) The solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is

(*i*) $z = f_1(y) + xf_2(y) + x^2 f_3(y)$ (*ii*) $z = (1 + x + x^2) f(y)$

(*iii*) $z = f_1(x) + yf_2(x) + y^2 f_3(x)$ (*iv*) $z = (1 + y + y^2) f(x)$

(*b*) The solution of $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ is

(*i*) $z = f_1(y + x) + f_1(y - x)$ (*ii*) $z = f_1(y + x) + f_2(y - x)$

(*iii*) $z = f_2(y + x) + f_2(y - x)$ (*iv*) $z = f(x^2 - y^2)$

(*c*) Particular integral of $(2D^2 - 3DD' + D'^2) z = e^{x+2y}$ is

(*i*) xe^{x+2y} (*ii*) $\frac{1}{2} e^{x+2y}$ (*iii*) $-\frac{x}{2} e^{x+2y}$ (*iv*) $\frac{x^2}{2} e^{x+2y}$

(*d*) Particular integral of $(D^2 - D'^2) z = \cos(x + y)$ is

(*i*) $\frac{x}{2} \cos(x + y)$ (*ii*) $x \sin(x + y)$ (*iii*) $x \cos(x + y)$ (*iv*) $\frac{x}{2} \sin(x + y)$

Ans. *(a) (i), (b) (ii), (c) (iii), (d) (iv).*

9.14 NON-HOMOGENEOUS LINEAR EQUATIONS

The linear differential equations which are not homogeneous are called Non-homogeneous Linear Equations.

For example,

$$3\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} + 5\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = 0$$

$$f(D, D') = f_1(x, y)$$

Its solution, $z = \text{C.F.} + \text{P.I.}$

Complementary Function: Let the non-homogeneous equation be

$$(D - mD' - a) z = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} - az = 0$$

$$p - mq = az$$

The Lagrange's subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}$$

From first two relations we have, $-mdx = dy$

$$dy + mdx = 0 \Rightarrow y + mx = c_1 \qquad \text{... (1)}$$

and from first and third relation, $dx = \frac{dz}{az} \Rightarrow x = \frac{1}{a} \log z + c_2 \Rightarrow z = c_3 e^{ax}$ ... (2)

From (1) and (2), we have $z = e^{ax} \phi(y + mx)$

Similarly the solution of $(D - m D' - a)^2 Z = 0$ is

$$z = e^{ax}\phi_1(y + mx) + xe^{ax}\phi_2(y + mx)$$

Example 41. *Solve* $(D + D' - 2)(D + 4D' - 3) z = 0$

Solution. The equation can be rewritten as $\left\{D - (-D)' - 2\right\}\left\{D - (-4D') - 3\right\} z = 0$

Hence the solution is

$$z = e^{2x}\phi_1(y - mx) + e^{3x}\phi_2(y - 4mx)$$ **Ans.**

Example 42. *Solve* $(D + 3D' + 4)^2 z = 0$

Solution. The equation is rewritten as

$$[D - (-3D') - (-4)]^2 z = 0$$

Hence the solution is

$$z = e^{-4x}\phi_1(y - 3x) + x\,e^{-4x}\phi_2(y - 3x)$$ **Ans.**

Example 43. *Solve* $r + 2s + t + 2p + 2q + z = 0$

Solution. The equation is rewritten as

$$(D^2 + 2DD' + D^2 + 2D + 2D' + 1)\,z = 0$$

$$\Rightarrow \quad [(D + D')^2 + 2(D + D') + 1]\,z = 0$$

$$\Rightarrow \quad (D + D' + 1)^2 z = 0$$

$$\Rightarrow \quad [D - (-D') - (-1)]^2 z = 0$$

Hence the solution is

$$z = e^{-x}\phi_1 y(y - x) + xe^{-x}\phi_2(y - x)$$

Example 44. *Solve* $r - t + p - q = 0$

Solution. The equation is rewritten as

$$(D^2 - D'^2 + D - D')\,z = 0$$

$$\Rightarrow \quad [(D - D')(D + D') + 1(D - D')]\,z = 0$$

$$\Rightarrow \quad (D - D')(D + D' + 1)z = 0$$

Hence the solution is

$$z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$$ **Ans.**

Particular Integral

Case 1. $\dfrac{1}{F(D, D')}e^{ax+by} = \dfrac{1}{F(a, b)}e^{ax+by}$

Example 45. Solve $(D - D' - 2)(D - D' - 3)\,z = e^{3x-2y}$

The complementary function is

$$e^{2x}\phi_1(y + x) + e^{3x}\phi_2(y + x)$$

$$\text{P.I.} = \frac{1}{(D - D' - 2)(D - D' - 3)}e^{3x-2y} = \frac{1}{[3 - (-2) - 2][3 - (-2) - 3]}e^{3x-2y} = \frac{1}{6}e^{3x-2y}$$

Hence the complete solution is

$$z = e^{2x}\phi_1(y + x) + e^{3x}\phi_2(y + x) + \frac{1}{6}e^{3x-2y}$$ **Ans.**

Case 2. $\dfrac{1}{F(D^2, DD', D'^2)}\sin(ax + by) = \dfrac{1}{F(-a^2, -ab, -b^2)}\sin(ax + by)$

Example 46. *Solve* $(D + 1)(D + D' - 1)\,z = \sin(x + 2y)$

Solution. $\text{C.F.} = e^{-x}\phi(y) + e^{-x}\phi_2(y - x)$

$$\text{P.I.} = \frac{1}{(D + 1)(D + D' - 1)}\sin(x + 2y) = \frac{1}{D^2 + DD' + D' - 1}\sin(x + 2y)$$

$$= \frac{1}{-1+(-2)+D'-1}\sin(x+2y) = \frac{1}{D'-4}\sin(x+2y)$$

$$= \frac{D'+4}{(D'^2-16)}\sin(x+2y) = \frac{D'+4}{(-4-16)}\sin(x+2y)$$

$$= -\frac{1}{20}(D'+4)\sin(x+2y) = -\frac{1}{20}\left[D'\sin(x+2y)+4\sin(x+2y)\right]$$

$$= -\frac{1}{20}\left[2\cos(x+2y)+4\sin(x+2y)\right]$$

Hence, the solution is $z = e^{-x}\phi\ (y) + e^{-x}\phi_2\,(y-x) - \frac{1}{10}\,[\cos\,(x+2y) + 2\,\sin(x+2y)]$ **Ans.**

Case 3. $\frac{1}{F(D,D')}x^m y^n = \left[F(D,D')\right]^{-1} x^m y^n$

Example 47. *Solve* $[D^2 - D^{2'} + D + 3\,D' - 2]\,z = x^2 y$

Solution. $(D - D' + 2)\,(D + D' - 1)\ z = 0$

C.F. $= e^{-2x}\,\phi_1(y+x) + e^x\,\phi_2\,(y-x)$

$$\text{P.I.} = \frac{1}{(D-D'+2)(D+D'-1)}x^2 y$$

$$= \frac{1}{D^2-D'^2+D+3D'-2}x^2 y = -\frac{1}{2}\frac{1}{1-\frac{3D'}{2}-\frac{D}{2}+\frac{D'^2}{2}-\frac{D^2}{2}}x^2 y$$

$$= -\frac{1}{2}\left[1-\frac{1}{2}\left(3D'+D-D'^2\right)+D^2\right]^{-1} x^2 y$$

$$= -\frac{1}{2}\left[1+\frac{1}{2}\left(3D'+D-D'^2+D^2\right)+\frac{1}{4}\left(3D'+D-D'^2+D^2\right)^2 + \frac{1}{8}\left(3D'+D-D'^2+D^2\right)^3 + \right]x^2 y$$

$$= -\frac{1}{2}\left[1+\frac{1}{2}\left(3D'+D-D'^2+D^2\right)+\frac{1}{4}\left(9D'^2+D^2+6DD'+6D^2D'\right) + \frac{1}{8}\left(9D^2D'\right)+\ldots\right]x^2 y$$

$$= -\frac{1}{2}\left[x^2y+\frac{1}{2}\left(3x^2+2xy-0+2y\right)+\frac{1}{4}\left(0+2y+12x+12\right)+\frac{1}{8}(18)\right]$$

$$= -\frac{1}{2}\left[x^2y+\frac{3x^2}{2}+xy+y+\frac{y}{2}+3x+3+\frac{9}{4}\right] = -\frac{1}{2}\left(x^2y+\frac{3x^2}{2}+xy+\frac{3y}{2}+3x+\frac{21}{4}\right)$$

Hence the complete solution is

$$z = e^{-2x}\phi_1(y+x)+e^x\phi_2(y-x)-\frac{1}{2}\left(x^2y+\frac{3x^2}{2}+xy+\frac{3y}{2}+3x+\frac{21}{4}\right)$$ **Ans.**

Case 4. $\frac{1}{F(D,D')}\left[e^{ax+by}\phi(x,y)\right] = e^{ax+by}\frac{1}{F(D+a,D'+b)}\phi(x,y)$

Example 48. *Solve* $(D - 3D' - 2)^2 z = 2e^{2x} \sin(y + 3x)$

Solution. A.E.is $(D - 3D' - 2)^2 = 0$

C.F. $= e^{2x}\phi_1(y + 3x) + x\,e^{2x}\phi_2(y + 3x)$

$$\text{P.I.} = \frac{1}{(D-3D'-2)^2} 2e^{2x} \cdot \sin(y+3x)$$

$$= 2e^{2x} \frac{1}{(D+2-3D'-2)^2} \sin(y+3x) = 2e^{2x} \frac{1}{(D-3D')^2} \sin(y+3x)$$

$$= 2e^{2x} \cdot x \frac{1}{2(D-3D')} \sin(y+3x) \text{ (As denominator becomes zero)}$$

$$= 2x^2 e^{2x} \frac{1}{2} \sin(y+3x) \qquad \text{(Again differentiate)}$$

$$= x^2 e^{2x} \sin(y + 3x)$$

Hence the complete solution is

$$z = e^{2x}\phi(y + 3x) + x\,e^{2x}\phi_2(y + 3x) + .x^2 e^{2x} \sin(y + 3x)$$ **Ans.**

Example 49. *Solve* $(D^2 + DD' - 6D'^2)z = x^2 \sin(x + y)$

Solution. $(D^2 + DD' - 6D'^2)z = x^2 \sin(x + y)$

For complementary function

$$(D^2 + DD' - 6D'^2) = 0 \quad \Rightarrow \quad (D - 2D')(D + 3D') = 0$$

$$\text{C.F.} = \phi_1(y + 2x) + \phi_2(y - 3x)$$

$$\text{P.I.} = \frac{1}{D - DD' - 6D'^2} x^2 \sin(x+y)$$

$$= \text{Imaginary part of} \frac{1}{D^2 - DD' - 6D'^2} x^2 [\cos(x + y) + i \sin(x + y)]$$

$$= \text{Imaginary part of } \frac{1}{D^2 - DD' - 6D'^2} x^2 e^{i(x+y)} = \text{Imaginary part of } e^{iy} \frac{1}{D^2 - Di - 6(i)^2} x^2 e^{ix}$$

$$= \text{Imaginary part of } e^{i(x+y)} \frac{1}{(D+i)^2 + (D+i)i + 6} x^2$$

$$= \text{Imaginary part of } e^{i(x+y)} \frac{1}{D^2 + 3iD + 4} x^2 \quad = \text{Imaginary part of} \frac{e^{i(x+y)}}{4} \frac{1}{1 + \frac{3iD}{4} + \frac{D^2}{4}} x^2$$

$$= \text{Imaginary part of } \frac{e^{i(x+y)}}{4} \left[1 + \frac{3iD}{4} + \frac{D^2}{4}\right]^{-1} x^2$$

$$= \text{Imaginary part of } \frac{e^{i(x+y)}}{4} \left[1 - \frac{3iD}{4} - \frac{D^2}{4} - \frac{9D^2}{16} \ldots\right] x^2$$

$$= \text{Imaginary part of} \frac{e^{i(x+y)}}{4} \left[x^2 - \frac{3ix}{2} - \frac{2}{4} - \frac{9}{16}(2)\right]$$

$$= \text{Imaginary part of} \frac{1}{4}[\cos(x+y) + i\sin(x+y)]\left[x^2 - \frac{3ix}{2} - \frac{13}{8}\right]$$

$$= \frac{1}{4}\left[\sin(x+y)\left(x^2-\frac{13}{8}\right)-\frac{3}{2}x\cos(x+y)\right]=\frac{1}{4}\sin(x+y)\left(x^2-\frac{13}{8}\right)-\frac{3x}{8}\cos(x+y)$$

Hence, the complete solution is

$$z = \phi_1(y+2x)+\phi_2(y-3x)+\frac{1}{4}\sin(x+y)\left(x^2-\frac{13}{8}\right)-\frac{3x}{8}\cos(x+y) \qquad \textbf{Ans.}$$

EXERCISE 9.13

Solve the following equations:

1. $(D^2+2DD'+D'^2-2D-2D')z=0.$ **Ans.** $z=f_1(x-y)+e^{2x}f_2(x-y)$

2. $(D^2-D'^2-3D+3D')z=e^{x-2y}$ **Ans.** $z=\phi_1(y+x)+e^{3x}\phi_2(y-x)-\frac{1}{12}e^{x-2y}$

3. $(D-D'-1)(D+D'-2)z=e^{2x-y}$ **Ans.** $z=e^x\phi_1(x+y)+e^{2x}\phi_2(y-x)-\frac{1}{2}e^{2x-y}$

4. $(D^2-D'^2-3D+3D')z=e^{x+2y}$ **Ans.** $z=\phi_1(y+x)+e^{3x}\phi_1(x-y)-xe^{x+2y}$

5. $(D+D')(D+D'-2)z=\sin(x+2y)$

Ans. $z=\phi_1(y-x)+e^{2x}(y-x)+\frac{1}{117}[6\cos(x+2y)-9\sin(x+2y)]$

6. $(D^2-DD'-2D)z=\cos(3x+4y)$

Ans. $z=\phi_1(y)+e^{2x}\phi_2(y+x)+\frac{1}{15}[\cos(3x+4y)-2\sin(3x+4y)]$

7. $(DD'+D-D'-1)z=xy$ **Ans.** $z=e^{-y}\phi_1(x)+e^x\phi_2(y)-(xy+y-x-1)$

8. $(D+D'-1)(D+2D'-3)z=4+3x+6y$ **Ans.** $z=e^x\phi_1(x-y)+e^{3x}\phi_2(2x-y)+6+x+2y$

9. $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}-3\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}=xy+e^{x+2y}$ (*UP. HI Semester, Summer 2002*)

Ans. $z=f_1(y+x)+e^{3x}f_2(y-x)-\frac{1}{3}\left(\frac{x^2y}{3}+\frac{x^3}{6}+\frac{x^2}{3}+\frac{xy}{3}+\frac{2x}{9}\right)-xe^{2x-y}$

10. $(D-D'-1)(D-D'-2)z-e^{2x-y}$ **Ans.** $z=e^xf_1(y+x)+e^{2x}f_2(y+x)+\frac{1}{2}e^{2x-y}$

11. $D(D+D'-1)(D+3D'-2)z=x^2-4xy+2y^2$

Ans. $z=\phi_1(y)+e^x\phi_2(x-y)+e^{2x}\phi_3(3x-y)+\frac{1}{2}\left[\frac{x^3}{3}-2x^2y+2xy^2-\frac{7}{2}x^2+4xy+\frac{x}{2}\right]$

12. $(D-D'+2)(D+D'-1)z=e^{x-y}-x^2y$

Ans. $z=e^{2y}\phi_1(x+y)e^x\phi_2(x-y)-\frac{e^{x-y}}{4}+\frac{1}{2}\left[x^2y+xy+\frac{3x^2}{2}+\frac{3}{2}y+3x+\frac{21}{4}\right]$

13. $(D^2-DD'-2D'^2+2D'+2D)z=e^{2x+3y}+\sin(2x+y)+xy$

Ans. $z=\phi_1(x-y)+e^y\phi_2(2x+y)-\frac{1}{10}e^{2x+3y}-\frac{1}{6}\cos(2x+y)+\frac{x}{24}(6xy-6y+9x-2x^2-12)$

9.15 MONGE'S METHOD (Non linear equation of the second order)

Let the equation be $Rr+Ss+Tt=V$ ... (1)

where R, S, T, V are functions of x, y, z, p and q. $r=\frac{\partial^2 f}{\partial x^2}$, $s=\frac{\partial^2 f}{\partial x\partial y}$, $t=\frac{\partial^2 f}{\partial y^2}$

We have $$dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy=rdx+sdy \qquad ...(2)$$

and $$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + tdy \qquad ...(3)$$

From (2) and (3), we have $r = \dfrac{dp - sdy}{dx}$ and $t = \dfrac{dq - sdx}{dy}$

Putting the value of r and t in (1), we get

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V$$

$$\Rightarrow \quad R\,dp\,dy + T\,dq\,dx - V\,dx\,dy - s(Rdy^2 - S\,dx\,dy + Tdx^2) = 0 \qquad ...(4)$$

Equation (4) is satisfied if

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \qquad ...(5)$$

$$Rdy^2 - S\,dx\,dy + Tdx^2 = 0 \qquad ...(6)$$

Equations (5) and (6) are called **Monge's equations.**

Since (6) can be factorised into two equations.

$$dy - m_1\,dx = 0 \text{ and } dy - m_2\,dx = 0$$

Now combine $dy - m_1\,dx = 0$ and equation (5). If need be, we may also use the relation $dz = p \cdot dx + q \cdot dy$ while solving (5) and (6). The solution leads to two integrals

$$u\,(x, y, z, p, q) = a \text{ and } V(x, y, z, p, q) = b$$

Then we get a relation between u and v. $V = f_1(u)$ ...(7)

Equation (7) is further integrated by methods of first order equations.

Note. If the intermediate solution is of the form $Pr + Qq = R$, then we use lagrange's equation.

Example 50. *Solve* $r = a^2 t$.

Solution. We have $dp = rdx + sdy$ and $dq = sdx + tdy$ which gives

$$r = \frac{dp - sdy}{dx} \text{ and } t = \frac{dq - sdx}{dy}$$

Putting these values of r and t in $r = a^2 t$, we get $\dfrac{dp - sdy}{dx} = a^2\dfrac{dq - sdx}{dy}$

$$\Rightarrow \quad dpdy - a^2dxdq - s\,(dy^2 - a^2dx^2) = 0$$

Thus, the Monges' equations are

$$dp\,dy - a^2\,dx\,dq = 0 \qquad ...(1)$$

$$dy^2 - a^2\,dx^2 = 0 \qquad ...(2)$$

(2) can be resolved into factors

$$dy - adx = 0 \qquad ...(3)$$

and $$dy + adx = 0 \qquad ...(4)$$

Combining (3) with (1), we get

$$dp\,(adx) - a^2\,dx\,dq = 0 \text{ or } dp - adq = 0 \qquad ...(5)$$

(3) and (5) on integration give respectively

$$\left.\begin{aligned} y - ax &= A \\ \text{and } p - aq &= B \end{aligned}\right] \Rightarrow p - aq = f_1(y - ax) \qquad ...(6)$$

Similarly combining (4) and (1)

$$p + aq = f_2\,(y + ax) \qquad ...(7)$$

Adding and subtracting (6) and (7), we get

$$p=\frac{1}{2}\left[f_1(y-ax)+f_2(y+ax)\right],q=\frac{1}{2a}\left[f_2(y+ax)-f_1(y-ax)\right]$$

Substituting these values in $dz = p\,dx + q\,dy$

$$dz=\frac{1}{2}\left[f_1(y-ax)+f_2(y+ax)\right]dx+\frac{1}{2a}\left[f_2(y+ax)-f_1(y-ax)\right]dy$$

$$dz=\frac{1}{2a}(dy+adx)f_2(y+ax)-\frac{1}{2a}(dy-a\,dx)f_1(y-ax)$$

Integrating, $$z=\frac{1}{2a}\phi_1(y+ax)-\frac{1}{2a}\phi_2(y-a\,x)$$

$\Rightarrow$ $$z=F_1(y+ax)+F_2(y-a\,x)$$ **Ans.**

Example 51. *Solve* $r - t\cos^2 x + p\tan x = 0$

Solution. $$r=\frac{dp-sdy}{dx} \text{ and } t=\frac{dq-sdx}{dy}$$

Putting for r and t in the given equation, we get

$$\frac{dp-sdy}{dx}-\frac{dq-sdx}{dy}cos^2 x+p\tan x=0$$

$\Rightarrow$ $$dp\,dy - sdy^2 - dx\,dq\cos^2 x + sdx^2\cos^2 x + p\,dx\,dy\tan x = 0$$

$\Rightarrow$ $$dp\,dy - dx\,dq\cos^2 x + p\,dx\,dy\tan x - s(dy^2 - dx^2\cos^2 x) = 0$$

Monge's equations are

$$dp\,dy - dx\,dq\cos^2 x + p\,dx\,dy\tan x = 0 \quad \text{...(1)}$$

$$dy^2 - dx^2\cos^2 x = 0 \quad \text{...(2)}$$

Eq. (2) is factorised $(dy + dx\cos x)(dy - dx\cos x) = 0$

$$dy - dx\cos x = 0 \quad \text{...(3)}$$

$$dy + dx\cos x = 0 \quad \text{...(4)}$$

Integrating (3) and (4), we get

$$y - \sin x = A \quad \text{...(5)}$$

$$y + \sin x = B \quad \text{...(6)}$$

Combining (3) and (1), we get

$$dp - dq \cdot \cos x + p\tan x d x = 0$$

$\Rightarrow$ $$(dp\sec x + p\sec x\tan x\,dx) - dq = 0$$

Integrating $$p\sec x - q = B \quad \text{...(7)}$$

Combining (5) and (7), we have

$$p\sec x - q = f_1(y - \sin x) \quad \text{...(8)}$$

On combining (6) and (7), we get

$$p\sec x + q = f_2(y + \sin x) \quad \text{...(9)}$$

From (5) and (9)

$$p=\frac{1}{2}\cos x[f_1(y-\sin x)+f_2(y+\sin x)] \quad \text{and} \quad q=\frac{1}{2}[f_2(y+\sin x)-f_1(y-\sin x)]$$

Putting for p and q in $dz = pdx + qdy$, we get

$$dz=\frac{1}{2}\cos x[f_1(y-\sin x)+f_2(y+\sin x)]\,dx+\frac{1}{2}[f_2(y+\sin x)-f_1(y-\sin x)]\,dy$$

$\Rightarrow$ $$dz=\frac{1}{2}f_2(y+\sin x)[dy+\cos x\,dx]-\frac{1}{2}f_1(y-\sin x)[dy-\cos x\,dx]$$

Integrating we get $$z=\frac{1}{2}F_2(y+\sin x)+F_1(y-\sin x)$$ **Ans.**

EXERCISE 9.14

Solve

1. $r+(a+b)s+abt=xy$ **Ans.** $z=\frac{1}{6}x^3y-(a+b)\frac{x^4}{24}+F_1(y-ax)+F_2(y-bx)$

2. $y^2r-2ys+t=p+6y$ **Ans.** $z=y^3-yF_1(y^2+2x)+F_2(y^2+2x)$

3. $xy(t-r)+(x^2-y^2)(s-2)=py-qx$ **Ans.** $z=xy+F_1(x^2+y^2)+F_2\left(\frac{y}{x}\right)$

4. $(1+q)^2r-2(1+p+q+pq)s+(1+p)^2t=0$ **Ans.** $z=F_1(x+y+z)+xF_2(x+y+z)$

5. $t-r\sec^4 y=2q\tan y$ **Ans.** $z=F_1(x-\tan y)+F_2(x+\tan y)$

6. $(q+1)s=(p+1)t$ **Ans.** $z=f_1(x)+f_2(x+y+z)$

7. $(r-s)y+(s-t)x+q-p=0$ **Ans.** $z=f_1(x+y)+f_2(x^2-y^2)$

Partial Differential Equations in Practical Problems

9.16 INTRODUCTION

In practical problems, the following types of equations are generally used

(*i*) *Wave equation* : $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$

(*ii*) *One-dimensional heat flow* : $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$

(*iii*) *Two-dimensional heat flow* : $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

(*iv*) *Radio equations* : $-\frac{\partial V}{\partial x}=L\frac{\partial I}{\partial t},-\frac{\partial I}{\partial x}=C\frac{\partial V}{\partial t}$

9.17 METHOD OF SEPARATION OF VARIABLES

In this method, we assume that the dependent variable is the product of two functions, each of which involves only one of the independent variables. So two ordinary differential equations are formed.

Example 1. *Using the method of separation of variables, solve*

$$\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u$$

where $u(x,0)=6e^{-3x}$ (*A.M.I.E.T.E., Summer 2002*)

Solution.
$$\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u \qquad \ldots(1)$$

Let
$$u=X(x).T(t) \qquad \ldots(2)$$

where X is a function of x only and T is a function of t only.

Putting the value of u in (1), we get

$$\frac{\partial(X.T)}{\partial x}=2\frac{\partial}{\partial t}(X.T)+X.T$$

$$T\frac{dX}{dx}=2X\frac{dT}{dt}+X.T \Rightarrow TX'=2X.T'+X.T \Rightarrow T.\frac{X'}{X}=2\frac{T'}{T}+1=c\text{ (say)}$$

(a) $\frac{X'}{X} = c \Rightarrow \frac{1}{X}\frac{dX}{dx} = c \Rightarrow \frac{dX}{X} = c\,dx$

On integration $\log X = cx + \log a$. $\Rightarrow \log\frac{X}{a} = cx \Rightarrow \frac{X}{a} = e^{cx} \Rightarrow X = ae^{cx}$

(b) $\frac{2T'}{T} + 1 = c \Rightarrow \frac{T'}{T} = \frac{1}{2}(c-1) \Rightarrow \frac{1}{T}\frac{dT}{dt} = \frac{1}{2}(c-1) \Rightarrow \frac{dT}{T} = \frac{1}{2}(c-1)dt$

On integration $\log T = \frac{1}{2}(c-1)\,t + \log b \Rightarrow \log\frac{T}{b} = \frac{1}{2}(c-1)t$

$\Rightarrow \frac{T}{b} = e^{\frac{1}{2}(c-1)t} \Rightarrow T = be^{\frac{1}{2}(c-1)t}$

Putting the value of X and T in (2), we have

$$u = ae^{cx}.be^{\frac{1}{2}(c-1)t}$$

$$\Rightarrow \quad u = ab\,e^{cx+\frac{1}{2}(c-1)t} \qquad ...(3)$$

$$\Rightarrow \quad u(x,0) = ab\,e^{cx}$$

But $\quad u(x,0) = 6e^{-3x}$

i.e. $\quad ab\,e^{cx} = 6e^{-3x} \Rightarrow ab = 6$ and $c = -3$

Putting the value of ab and c in (3), we have

$$u = 6e^{-3x+\frac{1}{2}(-3-1)t}$$

$$u = 6e^{-3x-2t}$$

which is the required solution. **Ans.**

Example 2. *Use the method of separation of variables to solve the equation :*

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$

given that $v = 0$ *when* $t \to \infty$, *as well as* $v = 0$ *at* $x = 0$ *and* $x = l$.

Solution. $$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \qquad ...(1)$$

Let us assume that $v = XT$ where X is a function of x only and T that of t only.

$$\frac{\partial v}{\partial t} = X\frac{dT}{dt} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = T\frac{d^2X}{dx^2}$$

Substituting these values in (1), we get

$$X\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

Let each side of (2) be equal to a constant $(-p^2)$

$$\Rightarrow \quad \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \qquad ...(2)$$

$$\frac{1}{T}\frac{dT}{dt} = -p^2 \Rightarrow \frac{dT}{dt} + p^2T = 0 \qquad ...(3)$$

and $$\frac{1}{X}\frac{d^2T}{dx^2} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0 \qquad ...(4)$$

Solving (3) and (4), we have

$$T = C_1e^{-p^2t}$$

$$X = C_2 \cos px + C_3 \sin px \qquad ...(5)$$

$$\therefore \qquad v = C_1 e^{-p^2 t}(C_2 \cos px + C_3 \sin px)$$

Putting $x = 0$, $v = 0$ in (5), we get

$0 = C_1 e^{-p^2 t} C_2 \qquad \therefore C_2 = 0$, since $C_1 \neq 0$

On putting the value of C_2 in (5), we get

$$v = C_1 e^{-p^2 t} C_3 \sin px \qquad ..(6)$$

Again putting $x = l$, $v = 0$ in (6), we get

$$0 = C_1 e^{-p^2 t} . C_3 \sin pl$$

Since C_3 cannot be zero.

$\therefore \sin pl = 0 = \sin n\pi \;\therefore\qquad p = \dfrac{n\pi}{l}$, n is any integer.

On putting the value of p in (6) it becomes

$$v = C_1 C_3 e^{-\frac{n^2\pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$$

Hence $\qquad v = b_n e^{\frac{-n^2\pi^2 t}{l^2}} \sin \dfrac{n\pi x}{l} \qquad$ where $b_n = C_1 C_3$

This equation satisfies the given condition for all integral values of n. Hence taking $n = 1, 2, 3, \ldots\ldots\ldots\ldots,$ *the most general solution is*

$$v = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2 t}{l^2}} \frac{\sin n\pi x}{l} \qquad \textbf{Ans.}$$

Exercise 9.15

Using the method of separation of variables, find the solution of the following equations

1. $2x\dfrac{\partial z}{\partial x} - 3y\dfrac{\partial z}{\partial y} = 0$ **Ans.** $z = c\,x^{\frac{k}{2}} y^{\frac{k}{3}}$

2. $\dfrac{\partial u}{\partial x} + u = \dfrac{\partial u}{\partial t}$ if $u = 4e^{-3x}$, when $t = 0$ **Ans.** $u = 4e^{-3x-2t}$

3. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, and $u = e^{-5y}$, when $x = 0$. **Ans.** $u = e^{2x-5y}$

4. $4\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 3u, u = 3e^{-x} - e^{-5x}$ at $t = 0$ (*A.M.I.E.T.E, Winter 2002, 2000*) **Ans.** $u = 3e^{t-x} - e^{2t-5x}$

5. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0;\; u(x,0) = 4e^{-x}$ (*A.M.I.E.T.E, Summer, 2000*) **Ans.** $u = 4e^{-x+3/2y}$

6. $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 2(x+y)u$ **Ans.** $u = ce^{x^2+y^2+k(x-y)}$

7. $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ If $u(x,0) = 4x - \dfrac{1}{2}x^2$ **Ans.** $u = \left(4x - \dfrac{x^2}{2}\right)e^{-p^2 t}$

8. $\dfrac{\partial^2 u}{\partial x^2} = 2\dfrac{\partial u}{\partial t}$ if $u(x,0) = x(4-x)$ **Ans.** $u = x(4-x)e^{-\frac{p^2 t}{2}}$

9. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ if $u(x,0) = 2x$ when $0 \le x \le \dfrac{l}{2}$

$$= 2(l-x) \text{ when } \frac{l}{2} \le x \le l$$

Ans. $u = 2xe^{-h^2 t}$ for $0 \le x \le \dfrac{l}{2}, u = 2(l-x)e^{-h^2 t}$ for $\dfrac{l}{2} \le x \le l$

10. $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$ if $u(x,0)=\sin\pi x$ **Ans.** $u=\sin\pi x.e^{-p^2 t}$

11. $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}$ if $u(x,0)=x^2(25-x^2)$ **Ans.** $u=x^2(25-x^2)e^{-p^2 t}$

12. $x^2 u_{xx}+3y^2 u=0$

13. $\dfrac{\partial^2 z}{\partial x^2}-2\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=0$ **Ans.** $z=c_1 e^{[1+\sqrt{(1-p)^2}]x+p^2 y}+c_2 e^{[1-\sqrt{1-p^2}]x+p^2 y}$

14. $\dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2}$ If $u(x,0)=\dfrac{1}{2}x\,(1-x)$ **Ans.** $u=\dfrac{x}{2}(1-x)\cos pt+c_2\sin pt(c_3\cos px+c_4\sin px)$

15. $16\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}$ if $u(x,0)=x^2(5-x)$ **Ans.** $u=x^2(5-x)\cos pt+c_4\sin pt\left(c_1\cos\dfrac{px}{4}+c_2\sin\dfrac{px}{4}\right)$

16. $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial y}+2u$ if $u=0,$ **Ans.** $u=(c_1\cos px+c_2\sin px)\,c_3\,e^{-(p^2+2)y}$

17. $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$ **Ans.** $u=A\,e^{1/2(x^2-y^2)k}$

18. $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial y},$ **Ans.** $u=Ae^{k(x+y)}$

19. $\dfrac{\partial u}{\partial x}=2\dfrac{\partial u}{\partial y}+u,\quad u(x,0)=4e^{-3x}$ (*A.M.I.E.T.E., Summer 2001*) **Ans.** $u=4e^{-(3x+2y)}$

20. $2\dfrac{\partial u}{\partial x}+3\dfrac{\partial u}{\partial y}+5u=0,\ u(0,y)=2e^{-y}$ **Ans.** $u=2e^{-x-y}$

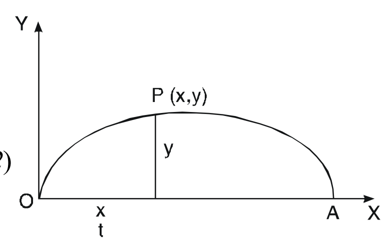
21. $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=3u$ and $u=e^{-5y}$ when $x=0$ **Ans.** $u=e^{8x-5y}$

22. $\dfrac{\partial u}{\partial x}-2\dfrac{\partial u}{\partial y}=u$, given that $u(x,0)=3e^{-5x}+2e^{-3x}$ **Ans.** $u=3e^{-5x-3y}+2e^{-3x-2y}$

9.18 EQUATION OF VIBRATING STRING

Consider an elastic string tightly stretched between two points O and A. Let O be the origin and OA as x-axis. On giving a small displacement to the string, perpendicular to its length (parallel to the y-axis). Let y be the displacement at the point $P(x, y)$ at any time. The wave equation.

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$$



Example 3. *Obtain the solution of the wave equation*

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$$ (*A.M.I.E.T.E., Summer 2002*)

using the method of separation of variables.

Solution. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Let $y = XT$ where X is a function of x only and $\boldsymbol{T}$ is a function of $\boldsymbol{t}$ only.

$$\frac{\partial y}{\partial t} = X\frac{dT}{dt} \quad \text{and} \quad \frac{\partial y}{\partial x} = T\frac{dX}{dx}$$

Since T and X are functions of a single variable only.

$$\frac{\partial^2 y}{\partial t^2} = X\frac{d^2T}{dt^2} \qquad \textit{and} \quad \frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{dx^2}$$

Substituting these values in the given equation, we get

$$X\frac{d^2T}{dt^2} = c^2T\frac{d^2X}{dx^2}$$

By separating the variables, we get

$$\frac{\dfrac{d^2T}{dt^2}}{c^2T} = \frac{\dfrac{d^2X}{dx^2}}{X} = k \qquad \text{(say).}$$

(Each side is constant, since the variables x and y are independent).

$$\therefore \qquad \frac{d^2T}{dt^2} - k\,c^2T = 0 \qquad \text{and} \qquad \frac{d^2X}{dx^2} - kX = 0$$

Auxiliary equations are

$$m^2 - kc^2 = 0 \;\Rightarrow\; m = \pm c\sqrt{k} \quad \textit{and} \quad m^2 - k = 0 \;\Rightarrow\; m = \pm\sqrt{k}$$

Case 1. If $k > 0$.

$$T = C_1 e^{c\sqrt{k}t} + C_2 e^{-c\sqrt{k}t}$$
$$X = C_3 e^{c\sqrt{k}x} + C_4 e^{-\sqrt{k}x}$$

Case 2. If $k < 0$.

$$T = C_5 \cos c\sqrt{k}t + C_6 \sin c\sqrt{k}t$$
$$X = C_7 \cos\sqrt{k}x + C_8 \sin\sqrt{k}x$$

Case 3. If $k = 0$.

$$T = C_9 t + C_{10}$$
$$X = C_{11}x + C_{12}$$

These are the three cases depending upon the particular problems. Here we are dealing with wave motion ($k < 0$).

$$y = TX$$
$$y = \left(C_5 \cos c\sqrt{k}t + C_6 \sin c\sqrt{k}t\right) \times \left(C_7 \cos\sqrt{k}x + C_8 \sin\sqrt{k}x\right) \qquad \textbf{Ans.}$$

Example 4. *Find the solution of the wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

such that $y = P_o \cos pt$, (P_o is a constant) when $x = l$ and $y = 0$ when $x = 0$.

Solution. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Its solution is as given in Example 3 on page 710.

$$y=\left(c_1\cos c\sqrt{kt}+c_2\sin c\sqrt{kt}\right)\left(c_3\cos\sqrt{kx}+c_4\sin\sqrt{kx}\right) \qquad ...(2)$$

Put $y=0$, when $x=0$

$$0=\left(c_1\cos c\sqrt{kt}+c_2\sin c\sqrt{kt}\right)c_3 \quad \Rightarrow c_3=0$$

(2) is reduced to

$$y=\left(c_1\cos c\sqrt{kt}+c_2\sin c\sqrt{kt}\right)c_4\sin\sqrt{kx}$$

$$y=c_1c_4\cos c\sqrt{kt}\sin\sqrt{kx}+c_2c_4\sin c\sqrt{kt}\sin\sqrt{kx} \qquad ...(3)$$

put $y=P_0\cos pt$ *when* $x=l$

$$P_0\cos pt=c_1c_4\cos c\sqrt{kt}\sin\sqrt{kl}+c_2c_4\sin c\sqrt{kt}\sin\sqrt{kl}$$

Equating the coefficient of sin and cos on both sides

$$P_0=c_1c_4\sin\sqrt{kl} \quad \Rightarrow c_1c_4=\frac{P_0}{\sin\sqrt{k}\,l}$$

$$0=c_2c_4\sin\sqrt{kl} \quad \Rightarrow c_2=0$$

And $p=c\sqrt{k}$ $\qquad \Rightarrow \dfrac{p}{c}=\sqrt{k}$

(3) becomes $\qquad y=\dfrac{P_0}{\sin\sqrt{kl}}\cos pt\sin\dfrac{p}{c}x$

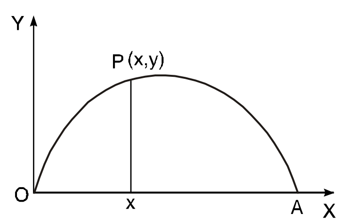
$$y=\frac{P_0}{\sin\frac{p}{c}l}\cos pt\sin\frac{p}{c}x \qquad \textbf{Ans.}$$

Example 5. *A string is stretched and fastened to two points l apart Motion is started by displacing the string in the form $y = a\sin\dfrac{\pi x}{l}$ from which it is released at a time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$$y(x,t)=a\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right) \qquad (\textit{A.M.I.E.T.E., Winter 2003, A.M.I.E., Winter 2001})$$

Solution. The vibration of the string is given by:

$$\frac{\partial^2 y}{\partial t^2}=C^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$



As the end points of the string are fixed, for all time,

$$y(0,t)=0 \qquad ...(2)$$

and $\qquad y(l,t)=0 \qquad ...(3)$

Since the initial transverse velocity of any point of the string is zero, therefore,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0}=0 \qquad ...(4)$$

Also $\qquad y(x,0)=a\sin\dfrac{\pi x}{l} \qquad ...(5)$

Now we have to solve (1), subject to the above boundary conditions. Since the vibration of the string is periodic, therefore, the solution of (1) is of the form

$$y(x,t)=(C_1\cos px+C_2\sin px)(C_3\cos Cpt+C_4\sin Cpt) \qquad ...(6)$$

By (2) $\qquad y(0,t)=C_1\left(C_3\cos C\,pt+C_4\sin C\,pt\right)=0$

For this to be true for all time, $C_1 = 0$.

Hence $\quad y(x,t) = C_2 \sin px\left(C_3 \cos Cpt + C_4 \sin Cpt\right)$ ...(7)

and $\quad \dfrac{\partial y}{\partial t} = C_2 \sin px\left[C_3\left(-Cp\sin Cpt\right) + C_4\left(Cp\cos Cpt\right)\right]$

$\therefore$ By (4) $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px\left(C_4 Cp\right) = 0$

Whence $\quad C_2 C_4 Cp = 0$

If $C_2 = 0$, (7) will lead to the trivial solution $y\ (x,\ t) = 0$.

$\therefore$ the only possibility is that $C_4 = 0$

Thus (7) becomes

$$y\,(x,\ t\,) = C_2 C_3 \sin px \cos Cpt \qquad ...(8)$$

If $x = l$ then $y = 0, 0 = C_2 C_3 \sin\ p\ l \cos Cpt$, for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin p\ l = 0\ \therefore\ p\ l = n\pi$

i.e. $\quad p = \dfrac{n\pi}{l}$,where n is an integer.

Hence (8) reduces to

$$y(x,t) = C_2 C_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi Ct}{l} \qquad ...(9)$$

Finally imposing the last condition (5), we have

$$y(x,0) = C_2 C_3 \sin\frac{n\pi x}{l} = a\sin\frac{\pi x}{l}$$

which will be satisfied by taking $C_2 C_3 = a\ and\ n = 1$

Hence the required solution is

$$y(x,t) = a\sin\frac{\pi x}{l} = \cos\frac{\pi C\,t}{l}$$ **Proved.**

Example 6. *The vibrations of an elastic string is governed by the partial differential equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The length of the string is π *and the ends are fixed .The initial velocity is zero and the initial deflection is u (x, 0) = 2(sin x +sin 3x).Find the deflection u (x, t)of the vibrating string for* $t \geq 0$.

Solution. $\quad \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$

$$\Rightarrow \quad u = \left(c_1 \cos pt + c_2 \sin pt\right)\left(c_3 \cos px + c_4 \sin px\right) \qquad ...(1)$$

On putting $x = 0, u = 0$ in (1),we get

$$0 = \left(c_1 \cos pt + c_2 \sin pt\right)c_3 \Rightarrow c_3 = 0$$

On putting $c_3 = 0$ in (1),it reduces

$$u = \left(c_1 \cos pt + c_2 \sin pt\right)c_4 \sin px \qquad ...(2)$$

On putting $x = \pi$ and $u = 0$ in (2), we have

$$0 = (c_1 \cos pt + c_2 \sin pt) c_4 \sin p\pi$$

$\sin p\pi = 0 = \sin n\pi \qquad n = 1, 2, 3, 4...$

$\therefore \quad p\pi = n\pi \quad$ or $\quad p = n$

On substituting the value of p in (2), we get

$$u = (c_1 \cos nt + c_2 \sin nt) c_4 \sin nx \qquad ...(3)$$

On differentiating (3) w.r.t. "t", we get

$$\frac{du}{dt} = (-c_1 n \sin nt + c_2 n \cos nt) c_4 \sin nx \qquad ...(4)$$

On putting $\frac{du}{dt} = 0$, $t = 0$ in (4) we have

$$0 = (c_2 n)(c_4 \sin nx) \Rightarrow c_2 = 0$$

On putting $c_2 = 0$, (3) becomes

$$u = (c_1 \cos nt)(c_4 \sin nx)$$

$$u = c_1 c_4 \cos nt \sin nx \qquad ...(5)$$

given $u(x,0) = 2\,(\sin x + \sin 3x)$

On putting $t = 0$ in (5), we have

$$u = (x,0) = c_1 c_4 \sin nx$$

$$2\,(\sin x + \sin 3x) = c_1 c_4 \sin nx$$

$$4 \sin 2x \cos x = c_1 c_4 \sin nx$$

$$c_1 c_4 = 4 \cos x \qquad 2 = n$$

On substituting the value of $c_1 c_4$ and $n = 2$,(5) becomes

$$u(x,t) = 4 \cos x \cos 2t \sin 2x$$ **Ans**

Example 7. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement $y\,(x,t)$.*

Solution. Let the equation to the vibrating string be

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Here the initial conditions are

$$y(0,t) = 0, \; y(l,t) = 0$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t = 0, \; y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$$

The solution of (1) is of the form

$$y = (c_1 \cos px + c_4 \sin px)\,(c_3 \cos pct + c_4 \sin pct) \qquad ...(2)$$

Now $y\,(0,t) = 0$ gives

$$0 = c_1(c_3 \cos pct + c_4 \sin pct) \Rightarrow c_1 = 0$$

Hence (2) becomes

$$y = c_2 \sin px (c_3 \cos pct + c_4 \sin pct) \qquad ...(3)$$

$y(l,t) = 0$ gives

$$0 = c_2 \sin pl\left(c_3 \cos pct + c_4 \sin pct\right)$$

$\therefore$ $\sin pl = 0 = \sin n\pi$ or $pl = n\pi$, or $p = \dfrac{n\pi}{l}$ where $n = 0, 1,2,3......$

On putting the value of p *in* (3), we get

$$y = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right) \qquad ...(4)$$

Now $$\frac{\partial y}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(-\frac{n\pi c}{l} c_3 \sin\frac{n\pi c}{l} t + c_4 \frac{n\pi c}{l} \cos\frac{n\pi ct}{l}\right)$$

Since $\dfrac{\partial y}{\partial t} = 0$ when $t = 0$, we have

$$0 = c_2 \sin\frac{n\pi x}{l} c_4 \frac{n\pi c}{l} \qquad \Rightarrow c_4 = 0$$

Now (4) reduces to

$$y = c_2 c_3 \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

$$\Rightarrow \quad y = b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c}{l} t \qquad (b_n = c_2 c_3)$$

$$\Rightarrow \quad y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c}{l} t \qquad ...(5)$$

But $$y(x,0) = y_0 \sin^3\frac{\pi x}{l} = \frac{y_0}{4}\left(3 \sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) \qquad \text{(given)} \quad ...(6)$$

On putting $t = 0$ in (5), we get, $$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(7)$$

From (6) and (7), we have

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \frac{y_0}{4}\left(3 \sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

$$y = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} = \frac{3y_0}{4} \sin\frac{\pi x}{l} - \frac{y_0}{4} \sin\frac{3\pi x}{l}$$

Comparing the coefficients, we have

$$b_1 = \frac{3y_0}{4}, \qquad b_3 = -\frac{y_0}{4}$$

and all others b's are zero.

Hence (5) becomes

$$y = \frac{y_0}{4}\left(3\sin\frac{\pi x}{1} \cos\frac{c\pi t}{l} - \sin\frac{3\pi x}{l} \cos\frac{3c\pi t}{l}\right) \qquad \textbf{Ans.}$$

Example 8. *Solve the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

under the condition:

$$u = 0 \text{ when } x = 0 \text{ and } x = \pi$$

$$\frac{\partial u}{\partial t} = 0 \text{ when } t = 0 \text{ and } u(x,0) = x, 0 < x < \pi.$$

Solution. The solution is of the form

$$u(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos a\,pt + c_4 \sin a\,pt) \qquad ...(1)$$

Since $u(0, t) = 0$.

$$0 = c_1\left(c_3 \cos a\,pt + c_4 \sin a\,pt\right) \Rightarrow \quad c_1 = 0$$

Then (1) becomes

$$u(x,\ t) = c_2 \sin px\,(c_3 \cos a\,pt + c_4 \sin a\,pt) \qquad ...(2)$$

$$u(\pi, t) = 0$$

$$0 = c_2 \sin p\pi\,(c_3 \cos a\,pt + c_4 \sin a\,pt) \Rightarrow \sin p\,\pi = 0 = \sin n\,\pi \text{ or } p = n$$

Thus $u(x,\ t) = c_2 \sin nx\,(c_3 \cos a\,nt + c_4 \sin a\,n\,t)$

$$u(x,\ t) = \sin n\,x\,(b_1 \cos a\,nt + b_2 \sin a\,n\,t) \qquad ...(3)$$

Now $\dfrac{\partial u}{\partial t} = \sin nx\ (-\,ab_1\,n \sin ant + ab_2 n \cos ant)$

As $\dfrac{\partial u}{\partial t} = 0$ when $t = 0$ we have

$$0 = \sin nx(ab_2 n) \qquad \Rightarrow \qquad b_2 = 0$$

$$u(x,t) = \sin nx(b_1 \cos ant) \text{ or } u(x,\ t) = b_n \sin nx \cos a\,nt \qquad ...(4)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \cos ant \qquad ...(5)$$

On putting $u(x,\ 0) = x$, we have

$$x = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} \qquad ..(6)$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

Hence, the required solution is

$$u(x,t) = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx \cos nat \qquad \textbf{Ans.}$$

Example 9. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k\,(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.*

(*A.M.I.E.T.E., Dec. 2006, Summer 2000*)

Solution. The vibration of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

As the end points of the string are fixed for all time,

$$y(0, t) = 0 \qquad ...(2)$$

and $$y(l, t) = 0 \qquad ...(3)$$

since the initial transverse velocity of any point of the string is zero, therefore,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad ...(4)$$

and $$y(x,\ 0) = k\,(l\,x - x^2) \qquad ...(5)$$

Solution of (1) is $\quad y = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos c\,pt + c_4 \sin cpt) \qquad ...(6)$

By (2), $y(0, t) = 0$

$$0 = c_1 (c_3 \cos c\,pt + c_4 \sin c\,pt) \qquad \therefore \qquad c_1 = 0$$

Hence (6) becomes $\quad y = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \qquad \text{...(7)}$

$$\frac{\partial y}{\partial t} = c_2 \sin px(-c_3 cp \sin cpt + c_4 cp \cos cpt)$$

By (4) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$

$$0 = c_2 \sin px(c_4 cp) \qquad \Rightarrow \qquad c_4 = 0 \qquad (\text{since } c_2 \neq 0)$$

Hence (7) is reduced to

$$y = c_2 \sin px(c_3 \cos cpt)$$

$$y = c_2 c_3 \sin px \cos cpt \qquad \text{...(8)}$$

$$y(l, t) = 0$$

On putting $x = l$ in equation (8), we get

$$0 = c_2 c_3 \sin pl \cos cpt \qquad \Rightarrow \qquad 0 = \sin pl$$

$\Rightarrow \qquad \sin n\pi = \sin pl \qquad$ or $\qquad pl = n\pi, \quad p = \dfrac{n\pi}{l} \qquad$ where $n = 1, 2, 3....$

On putting $p = \dfrac{n\pi}{1}$, equation (8) becomes $\qquad (c_2 c_3 = b_n)$

$$\Rightarrow \qquad y = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} t$$

We can have any number of solutions by taking different integral values of n and the complete solution will be the sum of these solutions. Thus,

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c}{l} t \qquad \text{... (9)}$$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$lx - x^2 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{[Using (5)]} \qquad \text{...(10)}$$

Now it is clear that (10) represents the expansion of $f(x)$ in the form of a Fourier sine series and consequently

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \qquad \text{...(11)}$$

$$b_n = \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[(lx - x^2)\left(-\cos \frac{n\pi x}{l}\right) \frac{l}{n\pi} - (l - 2x)\left(-\sin \frac{n\pi x}{l}\right) \frac{l^2}{n^2\pi^2} + (-2)\left(\cos \frac{n\pi x}{l}\right) \frac{l^3}{n^3\pi^3} \right]_0^l$$

$$= \frac{2}{l} \left[(-1)^{n+1} \frac{2l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3} \right] = \frac{8l^2}{n^3\pi^3}, \quad \text{when } n \text{ is odd}$$

$$= 0, \qquad \text{when } n \text{ is even}$$

Putting the value of b_n in (9), we get

$$y = \sum \frac{8l^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi c}{l} t \text{ when } n \text{ is odd.} \qquad \textbf{Ans.}$$

9.19. SOLUTION OF WAVE EQUATION BY D'ALMBERT'S METHOD

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

Let us introduce the two new independent variables $u = x + ct,\quad v = x - ct$

So that y becomes a function of u and v

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u}(1) + \frac{\partial y}{\partial v}(1) = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \qquad \ldots(2)$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial y}{\partial u}c + \frac{\partial y}{\partial v}(-c) = c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \qquad \left[\because \frac{\partial u}{\partial t} = c, \frac{\partial v}{\partial t} = -c\right]$$

$$\frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$= c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) \qquad \ldots(3)$$

Substituting the values of $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ from (2) and (3) in (1), we get

$$c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) = c^2\left(\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) \text{ or } \frac{\partial^2 y}{\partial u \partial v} = 0 \qquad \ldots(4)$$

Integrating (4) *w.r.t v,* we get $\frac{\partial y}{\partial u} = f(u)$ ... (5)

where $f(u)$ is constant in respect of v. Again integrating (5) *w. r.t 'u,'* we get

$$y = \int f(u)\,du + \psi(v)$$

where $\psi(v)$ is constant in respect of u

$$y = \phi(u) + \psi(v) \text{ where } \phi(u) = \int f(u)\,du$$

$$\Rightarrow \quad y(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \ldots(6)$$

This is D'Almberts solution of wave equations (1)

To determine ϕ, ψ let us apply initial conditions, $y(x,0) = f(x)$ and $\frac{\partial y}{\partial t} = 0$ when $t = 0$.

Differentiating (6) w. r.t. *"t"*, we get

$$\frac{\partial y}{\partial t} = c\phi'(x+ct) - c\psi'(x-ct) \quad ...(7)$$

Putting $\frac{\partial y}{\partial t} = 0,$ and $t = 0$ in (7) we get $\quad 0 = c\phi'(x) - c\psi'(x)$

$$\Rightarrow \quad \phi'(x) = \psi'(x) \quad \Rightarrow \quad \phi(x) = \psi(x) + b$$

Again substituting $y = f(x)$ and $t = 0$ in (6) we get

$$f(x) = \phi(x) + \psi(x) \quad \Rightarrow f(x) = [\psi(x) + b] + \psi(x)$$

$$\Rightarrow \quad f(x) = 2\psi(x) + b$$

so that $\psi(x) = \frac{1}{2}[f(x) - b]$ and $\phi(x) = \frac{1}{2}[f(x) + b]$

On putting the values of $\phi(x+ct)$ and $\psi(x-ct)$ in (6), we get

$$y(x,t) = \frac{1}{2}[f(x+ct) + b] + \frac{1}{2}[f(x-ct) - b]$$

$$\Rightarrow \quad y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] \quad \textbf{Ans.}$$

EXERCISE 9.16

1. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

under the conditions $u = 0$, when $x = 0$ and $x = \pi$

$\frac{\partial u}{\partial t} = 0$ when $t = 0$ and $u(x, 0) = x, 0 < x < \pi.$ **Ans.** $u = \frac{2\sum(-1)^{n+1}}{n} \sin nx \cos nct$

2. Using the transformations $v = x + ct$ and $z = x - ct$, solve the following :

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2};$ $\quad \frac{\partial u}{\partial t}(x,0) = 0;$ $\quad u(x,0) = f(x).$ **Ans.** $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$

3. A string of length l is initially at rest in equilibrium position and each of its points given velocity,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$$

Find the displacement $y(x, t)$. (*A.M.I.E.T.E., Summer 2001*) **Ans.** $y(x,t) = \sum b_n \sin \frac{n\pi x}{l} \sin \frac{nc\pi}{l} t$

4. Find the solution of the equation of a vibrating string of length l satisfying the initial conditions :

$$y = f(x) \text{ when } t = 0, \text{ and } \frac{\partial y}{\partial t} = g(x) \text{ when } t = 0$$

It is assumed that the equation of a vibrating string is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

Ans. $y(x,t) = \sum_{n=1}^{\infty} (b_n \cos \frac{n\pi at}{l} + c_n \sin \frac{n\pi at}{l}) \sin \frac{n\pi}{l} x$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, C_n = \frac{2}{cn\pi} \int_0^l g(x) \sin \frac{n\pi x}{l} dx$

5. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.

Ans. $y = \frac{8\lambda l^3}{c\pi^4} \sum_{(n=1)}^{n} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)\pi ct}{l}$

6. A tightly stretched string of length l fastened at both ends, is disturbed from the position of equilibrium by imparting to each of its points an initial velocity of magnitude $f(x)$. Show that the solution of the problem is

$$u(x,t)=\frac{2}{l}\sum_{n=1}^{\infty}\left[\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx\right]\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

7. A tightly stretched string with fixed end points $x=0$ *and* $x=\pi$ is initially at rest in its equilibrium position.If it is set vibrating by giving each point a velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0}=0.03\sin x-0.04\sin 3x$ then find the displacement $y(x, t)$ at any point of the string at any time t.

8. Find the solution of the equation $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ which satisfies the conditions.

$u(0,t)=0, u(l,t)=0, u(x,0)=\phi(x), u_t(x,0)=0$ **Ans.** $u(x,t)=\sum_{n=1}^{\infty} b_n\cos\frac{n\pi at}{l}\sin\frac{n\pi x}{l}$

9. Find the solution of the equation $\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ subject to the boundary conditions.

$y(0,t)=0,\quad y(l,t)=0,\quad y(x,0)=\phi(x),\quad \frac{\partial y}{\partial t}(x,0)=\psi(x)$

Ans. $y=\phi(x).\cos\frac{n\pi ct}{l}+\frac{l\psi(x)}{n\pi c}.\frac{\sin\frac{n\pi ct}{l}}{\sin\frac{n\pi x}{l}}$

10. The vibrations of an elastic string of length l are governed by the one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$. The string is fixed at the ends.

$u(0,t)=0=u(l,t)$ for all t. The initial deflection is

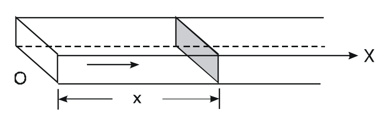
$u(x,0)=x;\ 0<x<l/2,\quad u(x,0)=l-x;\ \frac{l}{2}\le x\le l$

and the initial velocity is zero. Find the deflection of the string at any instant of time.

(*A.M.I.E.T.E., Summer 2001, ,A.M.I.E. Summer 2001*) **Ans.** $\frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$

9.20 ONE DIMENSIONAL HEAT FLOW

Let heat flow along a bar of uniform cross-section, in the direction perpendicular to the cross-section. Take one end of the bar as origin and the direction of heat flow is along x-axis.



Let the temperature of the bar at any time t at a point x distance from the origin be $u(x,t)$. Then the equation of one dimensional heat flow is $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$

Example 10. *A rod of length l with insulated sides is initially at a uniform temperature u. Its ends are suddenly cooled to O°C and are kept at that temperature. Prove that the temperature function u (x, t) is given by*

$$u(x,t)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}.e^{-\frac{c^2\pi^2n^2t}{l^2}}$$

where b_n is determined from the equation.

$$U_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Solution. Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Let us assume that $u = XT$, where X is a function of x alone and T that of t alone.

$$\therefore \quad \frac{\partial u}{\partial t} = X \frac{dT}{dt} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = T \frac{d^2 X}{dx^2}$$

Substituting these values in (1), we get $X\dfrac{dT}{dt} = c^2 T \dfrac{d^2 X}{dx^2}$

i.e.
$$\frac{1}{c^2 T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} \qquad ...(2)$$

Let each side be equal to a constant $(-p^2)$.

$$\frac{1}{c^2 T} \frac{dT}{dt} = -p^2 \quad \Rightarrow \quad \frac{dT}{dt} + p^2 c^2 T = 0 \qquad ...(3)$$

and
$$\frac{1}{X} \frac{d^2 X}{dx^2} = -p^2 \quad \Rightarrow \quad \frac{d^2 X}{dx^2} + p^2 X = 0 \qquad ...(4)$$

Solving (3) and (4) we have

$$T = c_1 e^{-p^2 c^2 t} \quad \text{and} \quad X = c_2 \cos px + c_3 \sin px$$

$$\therefore \quad u = c_1 e^{-p^2 c^2 t} (c_2 \cos px + c_3 \sin px) \qquad ...(5)$$

Putting $x = 0$, $u = 0$ in (5), we get

$$0 = c_1 e^{-p^2 c^2 t} (c_2) \quad \Rightarrow \quad c_2 = 0 \text{ since } c_1 \neq 0$$

(5) becomes
$$u = c_1 e^{-p^2 c^2 t} c_3 \sin px \qquad ...(6)$$

Again putting $x = l$, $u = 0$ in (6), we get

$$0 = c_1 e^{-p^2 c^2 t} . c_3 \sin pl \Rightarrow \quad \sin pl = 0 = \sin n\pi$$

$$\Rightarrow \quad pl = n\pi \quad \Rightarrow \quad p = \frac{n\pi}{l}, n \text{ is any integer}$$

Hence (6) becomes $u = c_1 c_3 e^{\frac{-n^2 c^2 \pi^2}{l^2} t} \sin \dfrac{n\pi x}{l} = b_n e^{-\frac{n^2 c^2 \pi^2 t}{l^2}} \sin \dfrac{n\pi}{l} x, \qquad b_n = c_1 c_3$

This equation satisfies the given conditions for all integral values of n. Hence taking $n = 1, 2, 3, \ldots\ldots$, *the most general solution is*

$$u = \sum_{n=1}^{\infty} b_n e^{\frac{-n^2 c^2 \pi^2 t}{l^2}} \sin \frac{n\pi}{l} x$$

By initial conditions $u = U_0$ when $t = 0$

$$U_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Proved.

Example 11. *Find the solution of*

$$\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$$

for which $u(0, t) = u(l, t) = 0$, $u(x, 0) = \sin \frac{\pi x}{l}$ *by method of variable separable.*

Solution. $$\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t} \qquad \text{... (1)}$$

In example 10 the given equation was

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t} \qquad \text{... (2)}$$

On comparing (1) and (2) we get $h^2 = \frac{1}{c^2}$

Thus solution of (1) is

$$u = (c_2 \cos px + c_3 \sin px)\, c_1\, e^{-\frac{p^2 t}{h^2}} \qquad \text{[Using (5) of example (10)]} \qquad \text{...(3)}$$

On putting $x = 0$, $u = 0$ in (3), we get

$$0 = c_1 c_2 e^{-\frac{p^2 t}{h^2}} \qquad c_1 \neq 0 \Rightarrow c_2 = 0$$

(3) is reduced to

$$u = c_3 \sin px c_1 e^{-\frac{p^2 t}{h^2}} \qquad \text{...(4)}$$

On putting $x = l$ and $u = 0$ in (4), we get

$$0 = c_3 \sin pl c_1 e^{-\frac{p^2 t}{h^2}}$$

$$c_3 \neq 0,\ c_1 \neq 0 \qquad [\therefore \sin pl = 0 = \sin n\pi \Rightarrow p = \frac{n\pi}{l}]$$

Now (4) is reduced to

$$u = c_1 c_3 \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{h^2 l^2}} \qquad \text{...(5)}$$

On putting $t = 0$, $u = \sin \frac{\pi x}{l}$ in (5) we get

$$\sin \frac{\pi x}{l} = c_4 \sin \frac{n\pi x}{l} \qquad [\text{ put } c_1 c_3 = c_4]$$

This equation will be satisfied if

$$n = 1 \text{ and } c_4 = 1$$

On putting the values of c_4 and n in (5), we have

$$u = \sin \frac{\pi x}{l} e^{-\frac{\pi^2 t}{h^2 l^2}}$$ **Ans.**

Example 12. *The ends A and B of a rod 20 cm long have the temperatures at 30° C and at 80°C until steady state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.*

Solution. The initial temperature distribution in the rod is

$$u = 30 + \frac{50}{20}x \qquad i.e., u = 30 + \frac{5}{2}x$$

and the final distribution (*i.e.* in steady state)is

$$u = 40 + \frac{20}{20}x = 40 + x$$

To get u in the intermediate period, reckoning time from the instant when the end temperature were changed, we assumed

$$u = u_1(x, t) + u_2(x)$$

where $u_2(x)$ is the steady state temperature distribution in the rod (*i.e.* temperature after a sufficiently long time) and $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

Thus $\qquad u_2(x) = 40 + x$

Now $u_1(x, t)$ satisfies the one-dimensional heat-flow equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Hence u is of the form

$$u = 40 + x + \sum (a_k \cos kx + b_k \sin kx) e^{-c^2 k^2 t}$$

Since $\qquad u = 40°$, when $x = 0$ and $u = 60°$, when $x = 20$, we get

$$a_k = 0, k = \frac{n\pi}{20}$$

Hence
$$u = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-c^2\left(\frac{n\pi}{20}\right)^2 t} \qquad ...(1)$$

Using the initial condition *i.e.*,

$$u = 30 + \frac{5}{2}x \text{ when } t = 0, \text{ we get}$$

$$30 + \frac{5}{2}x = 40 + x + \sum b_n \sin \frac{n\pi}{20}x \quad \Rightarrow \quad \frac{3}{2}x - 10 = \sum b_n \sin \frac{n\pi x}{20}$$

Hence
$$b_n = \frac{2}{20}\int_0^{20} \left(\frac{3}{2}x - 10\right) \sin \frac{n\pi x}{20} dx$$

$$= \frac{1}{10}\left[\left(\frac{3x}{2} - 10\right)\left(-\frac{20}{n\pi}\cos\frac{n\pi x}{20}\right) - \frac{3}{2}\left(\frac{-400}{n^2\pi^2}\sin\frac{n\pi x}{20}\right)\right]_0^{20}$$

$$= \frac{1}{10}\left[-20\left(\frac{20}{n\pi}\right)(-1)^n - (-10)\left(\frac{20}{n\pi}\right)\right] = -\frac{20}{n\pi}\left[2(-1)^n + 1\right]$$

Putting this value of b_n in (1), we get

$$\therefore \qquad u = 40 + x - \frac{20}{\pi}\sum \left[\left(\frac{2(-1)^n}{n}\right) \sin \frac{n\pi x}{20} . e^{-\left(\frac{cn\pi}{20}\right)^2 t}\right] \qquad \textbf{Ans.}$$

EXERCISE 9.17

1. Solve the following boundary value problem which arises in the heat conduction in a rod :

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(l, t) = 0 \qquad \textit{(A.M.I.E.T.E., Summer 2002)}$$

$$u(x, 0) = 100\frac{x}{l} \qquad \textbf{Ans.} \quad u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2}{l^2}t}$$

2. Determine the solution of one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0, t) = 0$, $u(l, t) = 0$ $(t > 0)$ and the initial condition $u(x, 0) = x$, l being the length of the bar.

Ans. $y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2l}{n\pi} \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2c^2t}{l^2}}$

3. Solve the non-homogeneous heat conduction equation $u = a^2 u_{xx} + \sin 3\pi x$ subject to the following conditions :

$$u(x, 0) = \sin 2\pi x, u(0, t) = u(l, t) = 0$$

4. Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$, given that (*i*) $u = 0$ when $x = 0$ and $x = l$ for all t

(*ii*) $u = 3\sin \frac{\pi x}{l}$, when $t = 0$ for all x, $0 < x < l$. **Ans.** $u = 3\sin \frac{\pi x}{l} e^{\frac{-a^2\pi^2 t}{l^2}}$

5. (*a*) Find by the method of separation of variables the solution of $U(x, t)$ of the boundary value problem

$$\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2}, \quad t > 0, 0 < \alpha < 2$$

$$U(0, t) = 0, \quad U(2, t) = 0$$

$$U(x, 0) = x, \quad 0 < x < 2$$

Ans. $U = \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{2}}{\sin \frac{n\pi}{2}} e^{-\frac{3n^2\pi^2 t}{4}}$

(*b*) The ends A and B of a rod 30 cm long have their temperatures kept at 20°C and 80°C respectively until steady-state conditions prevail. The temperature of the end B is suddenly reduced to 60°C and kept so while at the end A temperature is raised to 40°C. Find temperature distribution in the rod at time *t*. (*A.M.I.E.T.E., Winter 2002*)

6. The equation $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar, given that $u = u_0 \sin nt$ when $x = 0$, for all values of t and $u = 0$ when x is very large.

Without radiation, show that if $u = Ae^{-gx} \sin(nt - gx)$, where A, g and n are positive constants, then $g = \sqrt{\frac{n}{2\mu}}$

7. An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steadystate condition prevails. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t, solve the equation of heat

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

by the method of separation of variables and obtain the solution. (*A.M.I.E., Summer 2004*)

Ans. $u(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} . e^{-\frac{c^2n^2\pi^2}{l^2} \cdot t}$

8. Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ under the conditions

$$u'(0, t) = 0 \quad t > 0$$

$$u'(\pi, t) = 0$$

$$u(x, 0) = x^2, \quad 0 < x < \pi$$

Ans. $u(x,t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx e^{-a^2n^2t}$

9. A rod of length l has its lateral surface insulated and is so thin that heat flow in the rod can be regarded as one dimensional. Initially the rod is at the temperature 100 throughout. At $t = 0$ the temperaure at the left end of the rod is suddenly reduced to 50 and maintained thereafter at this value, while the right end is maintained at 100. Let $u(x, t)$ be the temperature at point x in the rod at any subsequent time t.

(*i*) Write down the appropriate partial differential equation for $u(x, t)$ with initial and boundary conditions.

(*ii*) Solve the differential equation in (*i*) above using method of separation of variables and show that

$$u(x,t) = 50 + \frac{50x}{l} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}\exp\frac{-n^2\pi^2 t}{a^2 l^2}$$

Where a^2 is the constant involved in the partial differential equation. (*A.M.I.E.T.E., Dec. 2004*)

10. A uniform rod of length a whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature, the other end remains thermally insulated. Find the temperature $\theta(x, t)$.

$$\textbf{Ans. } \theta(x,t) = \frac{4\theta}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{(2n+1)\pi x}{2a}}{2n+1}e^{-\frac{(2n+1)^2\pi^2 c^2 t}{4a^2}}$$

11. Solve $\frac{\partial U}{\partial t} = a^2\frac{\partial^2 U}{\partial x^2}$ under the conditions

(*i*) $U \neq \infty$ if $t \to \infty$; (*ii*) $U(0,t) = U(\pi,t) = 0$; (*iii*) $U(x,0) = \pi x - x^2$

$$\textbf{Ans. } u = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}e^{-a^2}(2n-1)^2 t$$

12. The temperature distribution in a bar of length π, which is prefectly insulated at the ends $x = 0$ and $x = \pi$ is governed by the partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. Assuming the initial temperature as $u(x,0) = f(x) = \cos 2x$, find the temprature distribution at any instant of time.

Ans. $u = e^{4t}\cos 2x$

13. The heat flow in a bar of length 10 cm of homogeneous material is governed by the partial differential equation $u_t = c^2 u_{xx}$. The ends of the bar are kept at temperature 0°C, and the initial temperature is $f(x) = x(10 - x)$. Find the temperature in the bar at any instant of time.

14. Find the temperature $u(x, t)$ in a bar of length π which is perfectly insulated everywhere including the ends $x = 0$ and $x = \pi$. This leads to the conditions $\frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(\pi,t) = 0$. Further the initial conditions are as given below:

$$u(x,0) = \begin{cases} x, 0 < x < \pi/2 \\ \pi x, \pi/2 \le x < \pi \end{cases}$$

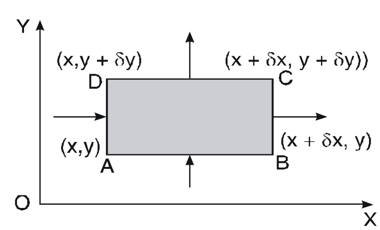
Find the solution by the separation of variables.

9.21 TWO DIMENSIONAL HEAT FLOW

Consider the heat flow in a metal plate of uniform thickness, in the directions parallel to length and breadth of the plate. There is no heat flow along the normal to the plane of the rectangle.

Let $u(x, y)$ be the temperature at any point (x, y) of the plate at time t is given by

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \ldots(1)$$



In the steady state, u does not change with t.

$$\therefore \quad \frac{\partial u}{\partial t} = 0$$

(1) becomes $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

This is called Laplace equation in two dimensions.

Example 13. *Solve* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ *which satisfies the conditions*

$$u(0,y)=u(l,y)=u(x,0)=0$$

and $u(x, a)=\sin\frac{n\pi x}{l}$ (*A.M.I.E.T.E., Winter 2000*)

Solution. $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{... (1)}$$

Let $$u=X(x)\,.\,Y(y) \qquad \text{...(2)}$$

Putting the values of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ in (1), we have

$$X''Y+XY''=0$$

$$\Rightarrow \qquad \frac{X''}{X}=-\frac{Y''}{Y}=-p^2 \qquad \text{(say)}$$

$$\therefore \qquad X''=-p^2X \quad \Rightarrow \quad X''+p^2X=0 \qquad \text{...(3)}$$

and $$Y''=p^2Y \quad \Rightarrow \quad Y''-p^2Y=0 \qquad \text{...(4)}$$

A.E. of (3) is $m^2+p^2=0 \quad \Rightarrow \quad m=\pm ip$

$$\therefore \qquad X=c_1\cos px+c_2\sin px$$

A.E. of (4) is $m^2-p^2=0 \quad \Rightarrow \quad m=\pm p$

$$\therefore \qquad Y=c_3e^{py}+c_4e^{-py}$$

Putting the values of X and Y in (2), we have

$$u=(c_1\cos px+c_2\sin px)(c_3e^{py}+c_4e^{-py}) \qquad \text{...(5)}$$

Putting $x=0, u=0$ in (5), we have

$$0=c_1(c_3e^{py}+c_4e^{-py})$$

$$\therefore \qquad c_1=0$$

(5) is reduaced to $u=c_2\sin px(c_3e^{py}+c_4e^{-py})$ ... (6)

On putting $x=l,\ u=0$, we have

$$0=c_2\sin pl(c_3e^{py}+c_4e^{-py})$$

$$c_2\neq 0 \quad \therefore \ \sin p\,l=0=\sin n\,\pi$$

$$\Rightarrow \qquad pl=n\pi \quad \Rightarrow \quad p=\frac{n\pi}{l}$$

Now (6) becomes

$$u=c_2\sin\frac{n\pi x}{l}\left(c_3e^{\frac{n\pi y}{l}}+c_4e^{\frac{-n\pi y}{l}}\right) \qquad \text{... (7)}$$

On putting $u=0$ and $y=0$ in (7), we have

$$0=c_2\sin\frac{n\pi x}{l}(c_3+c_4)$$

$$\therefore \qquad c_3+c_4=0 \quad \Rightarrow \quad c_3=-c_4$$

(7) becomes $$u=c_2c_3\sin\frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}}-e^{\frac{-n\pi y}{l}}\right) \qquad \text{...(8)}$$

On putting $y = a$ and $u = \sin\frac{n\pi x}{l}$ in (8), we get

$$\sin\frac{n\pi x}{l} = c_2 c_3 \sin\frac{n\pi x}{l}\left(e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}\right) \qquad i.e. \qquad c_2 c_3 = \frac{1}{e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}}$$

Putting this value in (8), we have

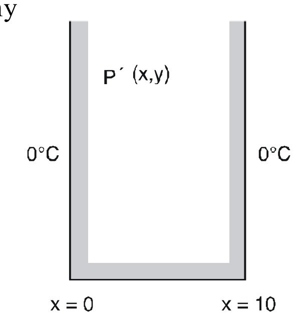
$$u = \sin\frac{n\pi x}{l}\,\frac{e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}}}{e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}} \quad \Rightarrow \quad u = \sin\frac{n\pi x}{l}\,\frac{\sinh\frac{n\pi y}{l}}{\sinh\frac{n\pi a}{l}} \qquad \textbf{Ans.}$$

Example 14. *A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge y = 0 is given by*

$$u(x, 0) = 20x, \quad 0 < x \le 5$$
$$= 20(10 - x), \quad 5 < x < 10$$

while the two long edges x = 0 and x = 10 as well as the other short edges are kept at 0°C. Find the steady state temperature at any point (x, y) *of the plate.*

Solution. In the steady state, the temperature $u(x, y)$ at any point $p(x, y)$ satisfy the equation



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$

The boundary conditions are

$$u(0, y) = 0 \text{ for all values of } y \qquad \text{...(2)}$$
$$u(10, y) = 0 \text{ for all values of } y \qquad \text{...(3)}$$
$$u(x, \infty) = 0 \text{ for all values of } x \qquad \text{...(4)}$$
$$u(x, 0) = 20x \qquad 0 < x \le 5$$
$$= 20(10 - x) \qquad 5 < x < 10 \qquad \text{...(5)}$$

Now three possible solutions of (1) are

$$u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \qquad \text{...(6)}$$
$$u = (C_5 \cos px + C_6 \sin px)(C_7 e^{px} + C_8 e^{-py}) \qquad \text{...(7)}$$
$$u = (C_9 x + C_{10})(C_{11} y + C_{12}) \qquad \text{...(8)}$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (6) and (8) cannot satisfy the condition (2), (3)and (4). Thus, only possible solution is (7) *i.e.*, of the form.

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \qquad \text{...(9)}$$

By (2) $\quad u(0, y) = C_1(C_3 e^{py} + C_4 e^{-py}) = 0 \quad$ for all values of y

$\therefore \qquad C_1 = 0$

$\therefore$ (9) reduces to $u(x, y) = C_2 \sin px\,(C_3 e^{py} + C_4 e^{-py}) \qquad \text{...(10)}$

By (3) $u(10, y) = C_2 \sin 10p\,(C_3 e^{py} + C_4 e^{-py}) = 0 \qquad C_2 \neq 0$

$\therefore \quad \sin 10p = 0 \quad \Rightarrow \quad 10p = n\pi \quad \Rightarrow \quad p = \frac{n\pi}{10}$

Also to satisfy the condition (4) *i.e.*, $u = 0$ as $y \to \infty$

$$C_3 = 0$$

Hence (10) takes the form $u(x, y) = C_2 C_4 \sin px . e^{-py}$

$\Rightarrow \quad u(x, y) = b_n \sin px . e^{-py}$ where $b_n = C_2 C_4$

$\therefore$ The most general solution that satisfies (2), (3) & (4) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin pxe^{-py} \qquad ...(5)$$

Putting $y = 0$, $\quad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin px \quad$ where $p = \frac{n\pi}{10}$

This requires the expansion of u in Fourier series in the interval $x = 0$ and $x = 5$ and from $x = 5$ to $x = 10$.

$$b_n = \frac{2}{10}\int_0^5 20x \sin px dx + \frac{2}{10}\int_5^{10} 20\,(10 - x) \sin px dx$$

$$b_n = 4\int_0^5 x \sin px dx + 4\int_5^{10} (10 - x) \sin px dx$$

$$= 4\left[x\left(\frac{-\cos px}{p}\right) - (1)\left(\frac{-\sin px}{p^2}\right)\right]_0^5 + 4\left[(10 - x)\left(\frac{-\cos px}{p}\right) - (-1)\left(\frac{-\sin px}{p^2}\right)\right]_5^{10}$$

$$= 4\left[\frac{-5\cos 5p}{p} + \frac{\sin 5p}{p^2}\right] + 4\left[0 - \frac{\sin 10p}{p^2} + \frac{5\cos 5p}{p} + \frac{\sin 5p}{p^2}\right]$$

$$= 4\left[\frac{2\sin 5p}{p^2} - \frac{\sin 10p}{p^2}\right] \qquad \left(p = \frac{n\pi}{10}\right)$$

$$= 4\left[\frac{2\sin 5.\frac{n\pi}{10}}{\frac{n^2\pi^2}{100}} - \frac{\sin 10.\frac{n\pi}{10}}{\frac{n^2\pi^2}{100}}\right] = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{400}{n^2\pi^2}\sin n\pi$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} = 0 \text{ if n is even.} = \pm\frac{800}{n^2\pi^2} \text{ if n is odd.} \quad \text{or} \quad b_n = \frac{(-1)^{n+1} 800}{(2n-1)^2\pi^2}$$

On putting the value of b_n in (5) the temperature at any point (x, y) is given by

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{10} e^{-\frac{(2n-1)\pi y}{10}} \qquad \textbf{Ans.}$$

Exercise 9.18

1. Find by the method of separation of variables, a particular solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

that tends to zero as x tends to infinity and is equal to $2 \cos y$ when $x = 0$ **Ans.** $u = 2e^{-x} \cos y$

2. Solve the equation : $\quad u_{xx} + u_{yy} = 0$

$$u = (0, y) = u(\pi, y) = 0 \text{ for all } y,$$

$$u(x, 0) = k, \qquad 0 < x < \pi$$

$\lim_{y\to\infty} u(x, y) = 0 \qquad 0 < x < \pi$ (*A.M.I.E.T.E., Summer 2003*)

Ans. $u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx e^{-ny}, \quad k = \sum_{n=1}^{\infty} b_n \sin nx$

3. Find the solution of Laplace's equation $\nabla^2 \psi = 0$ in cartesian coordinates in the region $0 \le x \le a, 0 \le y \le b_0$ to satisfying the conditions $y = 0$ on $x = 0$, $x = a$, $y = 0$, $y = b$ and $\psi = x(a - x), 0 < x < a$. (*A.M.I.E.T.E., Winter 2001*)

$$\textbf{Ans. } \psi = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \frac{\sin \frac{\pi x}{a} \sinh \frac{(2n+1)\pi y}{a}}{\sinh \frac{(2n+1)\pi b}{a}}$$

4. An infinitely long uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.

(*A.M.I.E.T.E.,Dec. 2005*) **Ans.** $u(x, y) = \frac{4u_0}{\pi}\left[e^{-y} \sin x + \frac{1}{3} e^{-3y} \sin 3x + \frac{1}{5} e^{-5y} \sin 5x + \ldots\right]$

5. Solve $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, given that
(*i*) $V = 0$ when $x = 0$ and $x = c$ (*ii*) $V \to 0$ as $y \to \infty$; (*iii*) $V = V_0$ when $y = 0$.

Ans. $V(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} e^{-\frac{n\pi y}{c}}, V_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$

6. The steady state temperature distribution in a thin plate bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = \infty$, is governed by the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Obtain the steady state temperature distribution under the conditions

$$u(0, y) = 0, \qquad u(a, y) = 0, \qquad u(x, \infty) = 0$$
$$u(x, 0) = x, \qquad 0 \le x \le a/2$$
$$= a - x \qquad a/2 \le x \le a$$

7. The points of trisection of a tightly stretched string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.

9.22. LAPLACE EQUATION IN POLAR CO-ORDINATES

Example 15. *Solve* $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$
by the method of separation of variables.

Solution. $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \Rightarrow \quad r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{dr} + \frac{\partial^2 u}{\partial \theta^2} = 0$ ...(1)

Let $u = R(r). T(\theta)$

$$\frac{\partial u}{\partial r} = \frac{dR}{dr}.T(\theta) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial r^2} = \frac{d^2 R}{dr^2}.T(\theta)$$

$$\frac{\partial u}{\partial \theta} = R(r).\frac{dT}{d\theta} \quad \text{and} \quad \frac{\partial^2 u}{\partial \theta^2} = R(r).\frac{d^2T}{d\theta^2}$$

Putting the values of $\frac{\partial^2 u}{\partial r^2}$, $\frac{\partial u}{\partial r}$ and $\frac{\partial^2 u}{\partial \theta^2}$ in (1), we get

$$r^2.\frac{d^2R}{dr^2}.T(\theta) + r\frac{dR}{dr}.T(\theta) + R(r)\frac{d^2T}{d\theta^2} = 0$$

$$\left(r^2\frac{d^2R}{dr^2} + r.\frac{dR}{dr}\right).T + R\frac{d^2T}{d\theta^2} = 0$$

$$\frac{r^2\frac{d^2R}{dr^2} + r.\frac{dR}{dr}}{R} = -\frac{1}{T}\frac{d^2T}{d\theta^2} = h \qquad \text{(say)}$$

$r^2\frac{d^2R}{dr^2} + r.\frac{dR}{dr} - hR = 0$	$\frac{d^2T}{d\theta^2} + hT = 0$
Put $r = e^z$	$(D^2 + h)T = 0$
$(D(D-1) + D - h)R = 0$	$D^2 + h = 0$ or $D = \pm i\sqrt{h}$
$D^2 - h = 0 \to D = \pm\sqrt{h}$	$T = c_3\cos(\sqrt{h}\theta) + c_4\sin(\sqrt{h}\theta)$
$R = c_1e^{\sqrt{h}z} + c_2e^{-\sqrt{h}z}$	
$R = c_1r^{\sqrt{h}} + c_2r^{-\sqrt{h}}$	

$$u = (c_1r^{\sqrt{h}} + c_2r^{-\sqrt{h}})[c_3 \cos(\sqrt{h}\theta) + c_4 \sin(\sqrt{h}\theta)] \qquad ...(2)$$

Case 1. If $h = k^2$

(2) becomes $u = (c_1r^k + c_2r^{-k})[c_3\cos(k\theta) + c_4\sin(k\theta)]$

Case 2. If $h = 0$

(2) becomes $u = (c_5 + z\,c_6)\,(c_7 + \theta c_8)$

$= [C_5 + (\log\, r)\, c_6]\,[c_7 + \theta c_8]$

Case 3. If $h = -p^2$

(2) becomes $u = (c_9\cos pz + c_{10}\sin pz)(c_{11}e^{p\theta} + c_{12}e^{-p\theta})$

Then there are three possible solutions

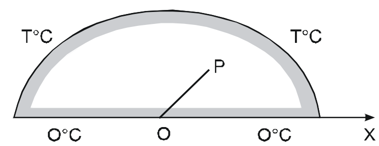
$$u = (c_1r^k + c_2r^{-k})[c_3\cos(k\theta) + c_4\sin(k\theta)]$$

$$u = (c_5 + c_6\log r)(c_7 + c_8\theta)$$

$$u = [c_9\cos(p\log r) + c_{10}\sin(p\log r)][c_{11}e^{p\theta} + c_{12}e^{-p\theta}] \qquad \textbf{Ans.}$$

Example 16. *The diameter of a semicircular plate of radius a is kept at 0°C and the temperature at the semicircular boundary is T° C. Find the steady state temperature in the plate.*

Solution. Let the centre O of the semicircular plate be the pole and the bounding diameter be as the initial line. Let $u\,(r, \theta)$ be the steady state temperature at any point $p\,(r, \theta)$ and u satisfies the equation



$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(1)$$

The boundary conditions are

(i) $u(r,0)=0$ $0 \le r \le a$

(ii) $u(r,\pi)=0$ $0 \le r \le a$

(iii) $u(a,\theta)=T.$

From conditions (ii) and (iii), we have $u \to 0$ *as* $r \to 0$. Hence the appropriate solution of (i) is as solved in example 15.

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \qquad \ldots(2)$$

Putting $u\ (r, 0) = 0$ in (2), we get

$$0 = (c_1 r^p + c_2 r^{-p})\, c_3 \to c_3 = 0$$

(2) becomes

$$u = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\theta \qquad \ldots(3)$$

Putting $u(r,\pi)=0$ in (3), we get

$$0 = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\pi \Rightarrow \sin p\pi = 0 = \sin n\pi$$

$$\Rightarrow \qquad p\pi = n\pi \qquad \Rightarrow \qquad p = n$$

(3) becomes, on putting $p = n$

$$u = (c_1 r^n + c_2 r^{-n}) c_4 \sin n\theta \qquad \ldots(4)$$

Since, $u = 0$ when $r = 0$

$$0 = c_2$$

(4) becomes, $u = c_1 c_4 r^n \sin n\theta$

The most general solution of (1) is

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad \ldots(5)$$

Putting $r = a$ and $u = T$ *in* (5), we have

$$T = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

By Fourier half range series, we get

$$b_n\, a^n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta\, d\theta = \frac{2}{\pi} T \left(\frac{-\cos n\theta}{n}\right)_0^{\pi} = \frac{2T}{n\pi}[-(-1)^n + 1]$$

$b_n a^n = 0,$ When n is even.

$b_n a^n = \dfrac{4T}{n\pi},$ When n is odd.

$$\Rightarrow \qquad b_n = \frac{4T}{n\pi a^n}$$

Hence, (5) becomes

$$u(r,\theta) = \frac{4T}{\pi}\left[\frac{r/a}{1}\sin\theta + \frac{(r/a)^3}{3}\sin 3\theta + \frac{(r/a)^5}{5}\sin 5\theta + \ldots\right]$$

Ans.

Exercise 9.19

1. Solve the steady-state temperature equation

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}=0;\ 10\le r\le 20, 0\le\theta\le 2\pi$$

subject to the following conditions:

$$T(10,\theta)=15\cos\theta \text{ and } T\ (20,\theta)=30\sin\theta$$

Ans. $T(r,\theta)=\frac{4T}{\pi}\left[\frac{r}{a}\sin\theta+\frac{1}{3}\left(\frac{r}{a}\right)^3\sin 3\theta+.....\right]$

2. A semi-circular plate of radius a has its circumference kept at temperature $u\ (a,\theta)=k\theta\ (\pi-\theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r,\theta)$ of the plate assuming the lateral surfaces of the plate to be insulated.

Ans. $u(r,\theta)=\frac{8k}{\pi}\sum_1^{\infty}\left(\frac{r}{a}\right)^{2n-1}\frac{\sin(2n-1)\theta}{(2n-1)^3}$

3. Find the steady state temperature in a circular plate of radius a which has one half of its circumference at 0°C and the other half at 60°C.

Ans. $u(r,\theta)=50-\frac{200}{n}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta.$

9.23 TRANSMISSION LINE EQUATIONS

$$\frac{\partial^2 V}{\partial x^2}=RC\frac{\partial V}{\partial t}$$

$$\frac{\partial^2 i}{\partial x^2}=RC\frac{\partial i}{\partial t}$$

are called telegraph equations,

where V = potential, i = current, C = capacitance, L = inductance

$$\frac{\partial^2 V}{\partial x^2}=LC\frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 i}{\partial x^2}=LC\frac{\partial^2 i}{\partial t^2}$$

are called radio equations.

Example 17. *Find the current i and voltage v in a transmission line of length l, t seconds after the ends are suddenly grounded given that* $i(x,0)=i_0, v(x,0)=v_0\sin\left(\frac{\pi x}{l}\right)$ *and that R and G are negligible.*

Solution. $$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2}$$

Let $v = XT$ where X and T are the functions of x and t respectively.

$$\frac{\partial^2 v}{\partial x^2}=T\frac{d^2X}{dx^2} \quad\text{and}\quad \frac{\partial^2 v}{\partial t^2}=X\frac{d^2T}{dt^2}$$

$$T\frac{d^2X}{dx^2}=LCX\frac{d^2T}{dt^2}$$

$$\frac{\frac{d^2X}{dx^2}}{X} = LC\frac{\frac{d^2T}{dt^2}}{T} = -p^2 \qquad \text{say}$$

Since the initial conditions suggest the values of v and i are periodic functions,

$\therefore$
$$X = c_1 \cos px + c_2 \sin px$$
$$T = c_3 \cos\frac{pt}{\sqrt{LC}} + c_4 \sin\frac{pt}{\sqrt{LC}}$$
$$v = XT$$

$\Rightarrow$
$$v = (c_1 \cos px + c_2 \sin px)(c_3 \cos\frac{pt}{\sqrt{LC}} + c_4 \sin\frac{pt}{\sqrt{LC}}) \qquad ...(1)$$

When $t = 0$, $v = v_0 \sin\frac{\pi x}{l}$

$$v_0 \sin\frac{\pi x}{l} = (c_1 \cos px + c_2 \sin px)c_3 \qquad ...(2)$$

On equating the coefficients, we get

$$c_1c_3 = 0 \quad \Rightarrow \quad c_1 = 0 \qquad \text{and} \qquad c_2c_3 = v_0, \quad p = \frac{\pi}{l}$$

(1) becomes

$$v = \sin\frac{\pi x}{l}\left[v_0 \cos\frac{pt}{\sqrt{LC}} + c_2c_4 \sin\frac{pt}{\sqrt{LC}}\right] \qquad ...(3)$$

Now when $t = 0, i = i_0$ (constant)

Hence
$$\frac{\partial i}{\partial x} = 0$$
$$\frac{\partial i}{\partial x} = \frac{-c\partial v}{\partial t} \qquad \therefore \frac{\partial v}{\partial t} = 0 \ \text{ when } t = 0$$

Now
$$\frac{\partial v}{\partial t} = \sin\frac{\pi x}{l}\left(\frac{p}{\sqrt{LC}}\right)\left[-v_0 \sin\frac{pt}{\sqrt{LC}} + c_2c_4 \cos\frac{pt}{\sqrt{LC}}\right] \qquad ...(4)$$

On putting $\frac{\partial v}{\partial t} = 0$ and $t = 0$ in (4), we get $c_2c_4 = 0 \Rightarrow c_4 = 0$

(3) is reduced to $v = v_0 \sin\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}}$

$\Rightarrow$
$$\frac{\partial v}{\partial x} = \frac{x}{l}v_0 \cos\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}} = -L\frac{\partial i}{\partial t} \qquad ...(5)$$

and
$$\frac{\partial v}{\partial t} = -\frac{v_0\pi}{l\sqrt{LC}}\sin\frac{\pi x}{l}\sin\frac{\pi t}{t\sqrt{LC}} = \frac{-1}{C}\frac{\partial \iota}{\partial x} \qquad ...(6)$$

Integrating (5) and (6), we get

$$i = -v_0\sqrt{\frac{C}{L}}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{LC}} + f(x)$$

$$i = -v_0\sqrt{\frac{C}{L}}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{LC}} + F(t)$$

$\therefore$ $f(x)$ and $F(t)$ must be constant only, since $i = i_0$ when $t = 0$

$\therefore$ Constant $= i_0 = f(x)$

Hence
$$i = i_0 - v_0\sqrt{\frac{C}{L}}\cos\frac{\pi x}{l}\sin\frac{\pi t}{\sqrt{LC}}. \qquad \textbf{Ans.}$$

Exercise 9.20

1. A transmission line 1,000 miles long is initially under steady state condition with potential 1,300 volts at the sending end ($x = 0$) and 1,200 volts at the receiving end ($x = 1000$). The terminal end of the line is suddenly grounded but the potential at the source is kept at 1,300 volts. Assuming the inductance and leakage to neglible, find the potential v (x, t), if it atisfies the equation $v_t = \left(\frac{1}{RC}\right) v_{xx}$.

Ans. $v(x, y) = \sum_{1}^{\infty} b_n \sin nx\, e^{-ny}$ *and* $k = \sum_{1}^{\infty} b_n \sin nx$

2. Obtain a solution of the telegraph equation

$$\frac{\partial^2 e}{\partial x^2} = RC \frac{\partial e}{\partial t}$$

suitable for the case when e decays with the time and when there is steady fall of potential from e_0 to 0 along the line of length l initially and the sending end is suddenly earthed.

Ans. $e(x, t) = \frac{2e_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} e \frac{-n^2\pi^2 l}{CRt^2}$

3. Fill in the blanks :

(*a*) The general solution of the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 0$ is ....

(*b*) The general solution of the equation $\frac{\partial^2 z}{\partial x \partial y} = 0$ is ....

(*c*) The solution of $z\,(x,y)$ of the equation $\frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 0$ is ....

(*d*) The solution of $3x \frac{\partial z}{\partial x} - 5y \frac{\partial z}{\partial y} = 0$ is ....

(*e*) The solution of $\frac{\partial^2 z}{\partial x^2} = \sin(x\,y)$ is ....

(*f*) The solution of $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ if $u\,(0, t) = u\,(3, t) = 0$ and $u(x, 0) = 5 \sin 4\pi x - 3 \sin 8\pi\, x$ is......................

(*g*) If the unknown function in a differential equation depends on more than one independent variables, then the differential equation is said to be (*A.M.I.E., Winter 2001*)

Ans. (*a*) $(C_1 \cos px + C_2 \sin px)(C_3 \cos pt + C_4 \sin pt)$

(*b*) $f_1(x) + f_2(y)$

(*c*) $f(x + \log y, z) = 0$

(*d*) $f(x^5 y^3, z) = 0$

(*e*) $-\frac{1}{y^2} \sin\,(xy) + x\, f_1(y) + f_2(y)$

(*f*) $(5 \sin 4\pi x e^{-32x^2 t} - 3 \sin 8\pi x\, e^{-128\pi n^2 t})$

(*g*) Partial Differential Equation