Day 16 Class

Various types of matrix

- Adjacency Matrix
- Incidence Matrix
- Reduced Incidence Matrix
- Sub Matrix
- Circuit Matrix
- Fundamental Circuit Matrix
- Cut Set Matrix
- Path Matrix

CIRCUIT MATRIX

Let the number of different circuits in a graph G be q and the number of edges in G be e. Then a *circuit matrix* $\mathsf{B} = [b_{ij}]$ of G is a q by e, (0, 1)-matrix defined as follows:

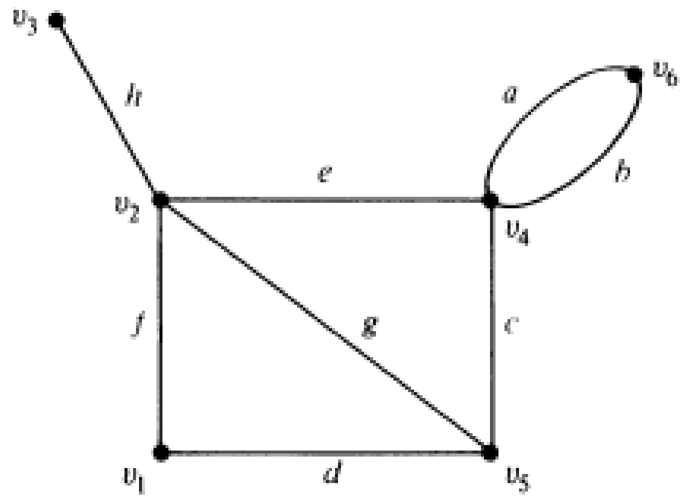
$$b_{ij} = 1, \qquad \text{if } i\text{th circuit includes } j\text{th edge, and}$$
$$= 0, \qquad \text{otherwise.}$$

To emphasize the fact that B is a circuit matrix of graph G, the circuit matrix may also be written as $\mathsf{B}(G)$.

The graph in Fig. 7-1(a) has four different circuits, $\{a, b\}$, $\{c, e, g\}$, $\{d, f, g\}$, and $\{c, d, f, e\}$. Therefore, its circuit matrix is a 4 by 8, (0, 1)-matrix as shown:

CIRCUIT MATRIX

$$\mathsf{B}(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array},$$



Observations:

1. A column of all zeros corresponds to a noncircuit edge (i.e., an edge that does not belong to any circuit).
2. Each row of $B(G)$ is a circuit vector.
3. Unlike the incidence matrix, a circuit matrix is capable of representing a self-loop—the corresponding row will have a single 1.
4. The number of 1's in a row is equal to the number of edges in the corresponding circuit.
5. If graph G is separable (or disconnected) and consists of two blocks (or components) g_1 and g_2, the circuit matrix $B(G)$ can be written in a block-diagonal form as

$$B(G) = \left[\begin{array}{c|c} B(g_1) & 0 \\ \hline 0 & B(g_2) \end{array}\right],$$

   where $B(g_1)$ and $B(g_2)$ are the circuit matrices of g_1 and g_2. This observation results from the fact that circuits in g_1 have no edges belonging to g_2, and vice versa (Problem 4-14).
6. Permutation of any two rows or columns in a circuit matrix simply corresponds to relabeling the circuits and edges.

Contd..

THEOREM 7-4

Let B and A be, respectively, the circuit matrix and the incidence matrix (of a self-loop-free graph) whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row A; that is,

$$A \cdot B^T = B \cdot A^T = 0 \qquad (\text{mod } 2), \tag{7-4}$$

where superscript T denotes the transposed matrix.

Contd..

$$A \cdot B^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \pmod 2.$$

FUNDAMENTAL CIRCUIT MATRIX

A submatrix (of a circuit matrix) in which all rows correspond to a set of fundamental circuits is called a *fundamental circuit matrix* B_f. A graph and its fundamental circuit matrix with respect to a spanning tree (indicated by heavy lines) are shown in Fig. 7-2.

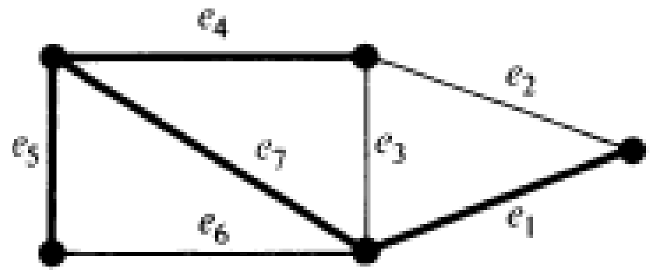
As in matrices A and B, permutations of rows (and/or of columns) do not affect B_f. If n is the number of vertices and e the number of edges in a connected graph, then B_f is an $(e - n + 1)$ by e matrix, because the number of fundamental circuits is $e - n + 1$, each fundamental circuit being produced by one chord.

Contd..

$$
\begin{array}{ccc:cccc}
e_2 & e_3 & e_6 & e_1 & e_4 & e_5 & e_7
\end{array}
$$

$$
\left[\begin{array}{ccc:cccc}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}\right]
$$

(b)



(a)

Fig. 7-2 Graph and its fundamental circuit matrix (with respect to the spanning tree shown in heavy lines).

Contd..

A matrix B_f thus arranged can be written as

$$B_f = [I_\mu \mid B_t], \qquad (7\text{-}5)$$

where I_μ is an identity matrix of order $\mu = e - n + 1$, and B_t is the remaining μ by $(n - 1)$ submatrix, corresponding to the branches of the spanning tree.

From Eq. (7-5) it is clear that the

$$\text{rank of } B_f = \mu = e - n + 1.$$

Since B_f is a submatrix of the circuit matrix B, the

$$\text{rank of } B \geq e - n + 1.$$

CIRCUIT MATRIX IN DIGRAPH

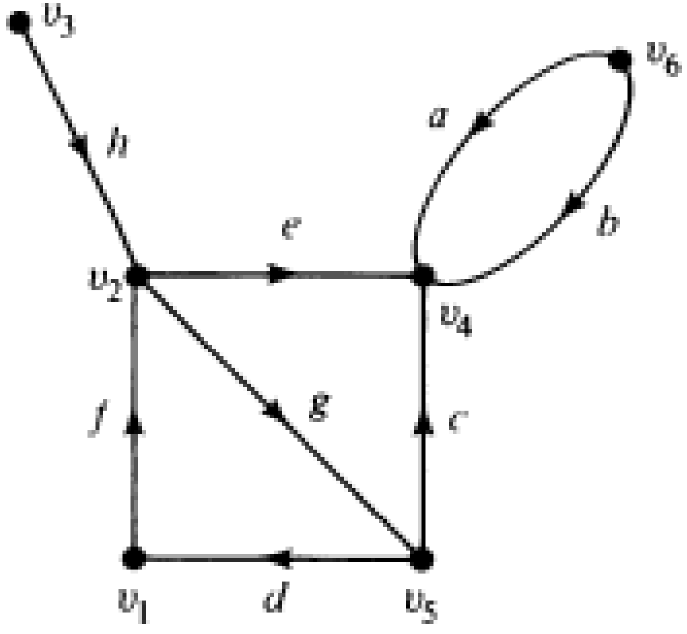
Circuit Matrix of a Digraph: Let G be a digraph with e edges and q circuits (directed circuits or semicircuits). An arbitrary orientation (clockwise or counterclockwise) is assigned to each of the q circuits. Then a circuit matrix $\mathsf{B} = [b_{ij}]$ of the digraph G is a q by e matrix defined as

$b_{ij} = 1$, if ith circuit includes jth edge, and the orientations of the edge and circuit coincide,

$= -1$, if ith circuit includes jth edge, but the orientations of the two are opposite,

$= 0$, if ith circuit does not include the jth edge.

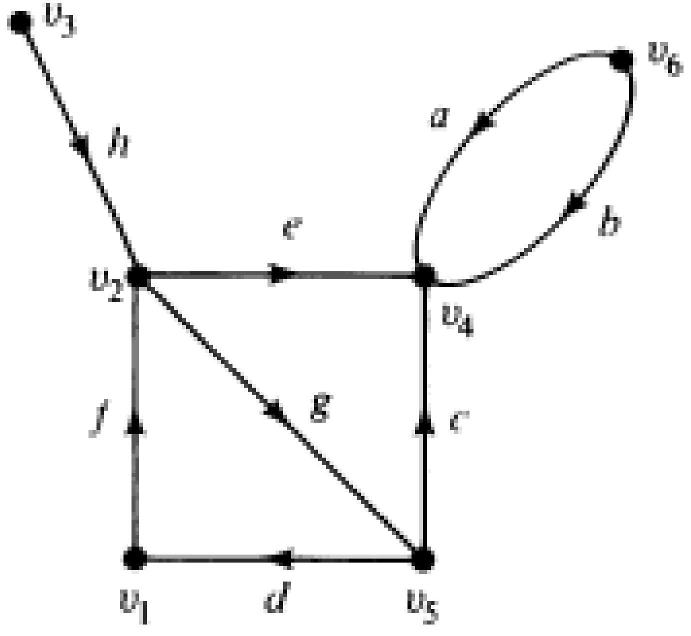
Contd..



(a)

$$\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \left[\begin{array}{rrrrrrrr} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

FUNDAMENTAL CIRCUIT MATRIX IN DIGRAPH



(a)

Fundamental Circuit Matrix: The μ fundamental circuits each made by a chord (with respect to some specified spanning tree) define a fundamental circuit matrix B_f for a digraph. The orientation assigned to each of the fundamental circuits is chosen to coincide with that of the chord. Therefore, B_f, a μ by e matrix, can be expressed exactly in the same form as in the case of an undirected graph in Section 7-4:

$$B_f = [I_\mu \,\vdots\, B_t],$$

where I_μ is the identity matrix of order μ, and the columns of B_t correspond to the edges in a spanning tree. This is illustrated in Fig. 9-18.

$$B_f = \begin{array}{c} \begin{array}{cccccccc} b & d & g & a & c & e & f & h \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

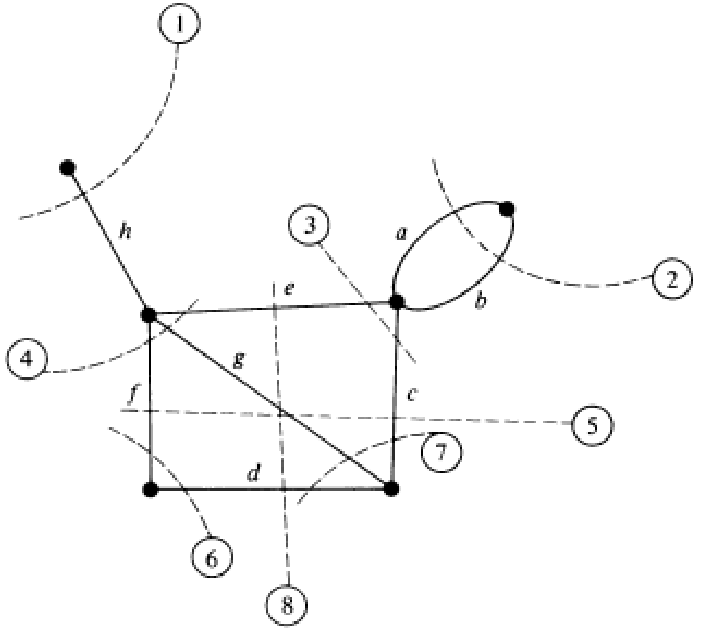
CUT SET MATRIX

Analogous to a circuit matrix, we can define a *cut-set matrix* $\mathbf{C} = [c_{ij}]$ in which the rows correspond to the cut-sets and the columns to the edges of the graph, as follows:

$$
\begin{aligned}
c_{ij} &= 1, \quad \text{if } i\text{th cut-set contains } j\text{th edge, and} \\
&= 0, \quad \text{otherwise.}
\end{aligned}
$$

CUT SET MATRIX

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



Question

What is a fundamental cut-set matrix?

PATH MATRIX

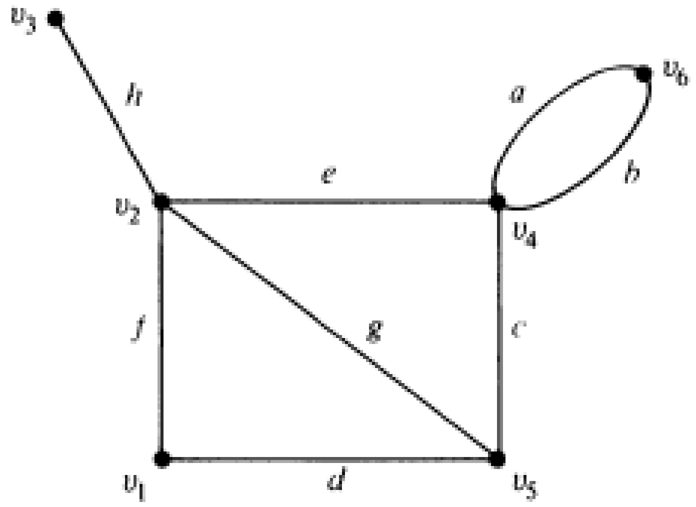
Another (0, 1)-matrix often convenient to use in communication and transportation networks is the *path matrix*. A path matrix is defined for a specific pair of vertices in a graph, say (x, y), and is written as $\mathsf{P}(x, y)$. The rows in $\mathsf{P}(x, y)$ correspond to different paths between vertices x and y, and the columns correspond to the edges in G. That is, the path matrix for (x, y) vertices is $\mathsf{P}(x, y) = [p_{ij}]$, where

$$p_{ij} = 1, \quad \text{if } j\text{th edge lies in } i\text{th path, and}$$
$$= 0, \quad \text{otherwise.}$$

As an illustration, consider all paths between vertices v_3 and v_4 in Fig. 7-1(a). There are three different paths; $\{h, e\}$, $\{h, g, c\}$, and $\{h, f, d, c\}$. Let us number them 1, 2, and 3, respectively. Then we get the 3 by 8 path matrix $\mathsf{P}(v_3, v_4)$:

PATH MATRIX

$$P(v_3, v_4) = \begin{array}{c c} & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$



Observations

1. A column of all 0's corresponds to an edge that does not lie in any path between x and y.
2. A column of all 1's corresponds to an edge that lies in every path between x and y.
3. There is no row with all 0's.
4. The ring sum of any two rows in $P(x, y)$ corresponds to a circuit or an edge-disjoint union of circuits.

Theorem 7-7

If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix $P(x, y)$, then the product (mod 2)

$$A \cdot P^T(x, y) = M,$$

where the matrix M has 1's in two rows x and y, and the rest of the $n - 2$ rows are all 0's.

Proof: The proof is left as an exercise for the reader (Problem 7-14).

As an example, multiply the incidence matrix in Fig. 7-1 to the transposed $P(v_3, v_4)$, just discussed.

$$A \cdot P^T(v_3, v_4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{(mod 2).}$$