11

Partial Differential Equations

11.1 INTRODUCTION

A relation between the variables (including the dependent one) and the partial differential coefficients of the dependent variable with the two or more independent variables is called a partial differential equation (p.d.e.)

For example:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + xy \qquad \ldots(1)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(2)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = u \qquad \ldots(3)$$

with

$$\left.\begin{array}{llll} \dfrac{\partial u}{\partial x} = p, & \dfrac{\partial u}{\partial y} = q & & \\ \dfrac{\partial^2 u}{\partial x^2} = r, & \dfrac{\partial^2 u}{\partial x \partial y} = s, & \dfrac{\partial^2 u}{\partial y^2} = t & \\ \ldots & \ldots & \ldots & \text{etc} \end{array}\right\} \qquad \ldots(4)$$

as standard notations for partial differentiation coefficients.

The **order** of a partial differential equation is the order of the highest order differential coefficient occuring in the equation and the **degree** of the partial differential equation is the degree of the highest order differential coefficient occurring in the equation.

For example, equation (1) is of Ist order Ist degree, equation (2) is of 2nd order Ist degree whereas equation (3) is of 2nd order 3rd degree.

If each term of the equation contains either the dependent variable or one of its derivatives, it is said to be **homogeneous,** otherwise, **non-homogeneous**.

For example, equation (2) is homogeneous, whereas equation (1) is non-homogeneous.

The partial differential equation is said to be **linear** if the differential co-efficients occurring in it are of the Ist order only or in other word if in each of the term, the differential co-efficients are not in square or higher powers or their product, otherwise, **non-linear**.

e.g. $x^2p + y^2q = z$ is a linear in z and of first order

Further, a p.d.e. is said to be quasi-linear if degree of highest order derivative is one, no product of partial derivatives are present

e.g. $z - z_{xx} + (z_y)^2 = 0$ is a quasi-linear 2nd order.

11.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

These equations are formed either by the elimination of arbitrary constants or by the elimination of the arbitrary functions from a relation with one dependent variable and the rest two or more independent variables.

Observations: When p.d.e. formed by elimination of arbitrary constants

1. If the number of arbitrary constants are more than the number of independent variables in the given relations, the p.d.e. obtained by elimination will be of 2nd or higher order.
2. If the number of arbitrary constants equals the number of independent variables in the given relation, the p.d.e. obtained by elimination will be of order one.

Observations: When p.d.e. formed by elimination of arbitrary functions. When n is the number of arbitrary functions, we may get several p.d.e., but out of which generally one with two least order is selected.

e.g. $z = x f(y) + y g(x)$ involves two arbitrary functions, f and g. Here $\dfrac{\partial^4 z}{\partial x^2 \partial y^2} = 0$ ...(*i*)

and $xys = xp + yq - z$ (second order) ...(*ii*)

are the two p.d.e. are obtained by elimination of the arbitrary functions. However, 2nd equation being in lower in order to 1st is the desired p.d.e.

Example 1: Form a partial differential equation by eliminating *a, b, c* from the relation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ [NIT Kurukshetra, 2003; KUK, 2000]

Solution: Clearly in the given equation a, b, c are three arbitrary constants and z is a dependent variable, depending on x and y.

We can write the given relations as:

$$f(x, y, z) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0 \qquad \ldots(1)$$

then differentiating (1) partially with respect to x and y respectively, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0, \quad \left(\text{Keeping } \frac{\partial y}{\partial x} = 0\right)$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0, \quad \left(\text{Keeping } \frac{\partial x}{\partial y} = 0\right)$$

or

$$\frac{2x}{a^2} + \frac{2z}{c^2}\cdot\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad c^2x + a^2zp = 0 \qquad \ldots(2)$$

and $$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad c^2y + b^2zq = 0 \qquad \ldots(3)$$

Again differentiating (2) with respect to x, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2 z\frac{\partial^2 z}{\partial x^2} = 0$$

On substituting $\frac{c^2}{a^2} = -\frac{z}{x}\frac{\partial z}{\partial x}$ from (2) in above equation, we get

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

or $$xz \cdot \frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0 \qquad \ldots(4)$$

Similarly, differentiating (3) partially with respect to y and substituting the value of $\frac{c^2}{b^2}$ from (3) in the resultant equation, we have

$$yz\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \qquad \ldots(5)$$

Thus equations (4) and (5) are 'partial differential equations' of first degree and second order.

Example 2: Form partial differential equation from $z = x\, f_1(x + t) + f_2(x + t)$.

Solution: Clearly z is a function of x and t

$$p = \frac{\partial z}{\partial x} = f_1(x+t) + x\, f_1'(x+t) + f_2'(x+t)$$

$$q = \frac{\partial z}{\partial t} = x\, f_1'(x+t) + f_2'(x+t)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f_1'(x+t) + x\, f_1''(x+t) + f_1'(x+t) + f_2''(x+t)$$

$$= 2\, f_1'(x + t) + x\, f_1''(x + t) + f_2''(x + t)$$

$$s = \frac{\partial}{\partial t}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x \partial t} = f_1'(x+t) + x\, f_1''(x+t) + f_2''(x+t)$$

$$t = \frac{\partial^2 z}{\partial t^2} = x\, f_1''(x+t) + f_2''(x+t)$$

Now $$(r + t) = 2\, f_1'(x + t) + 2x\, f_1''(x + t) + 2\, f_2''(x + t) = 2s$$

or $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} - 2\frac{\partial^2 z}{\partial x \partial t} = 0$$

Example 3: Form the partial differential equation by eliminating the arbitrary function, $F(x + y + z, x^2 + y^2 + z^2) = 0$ [KUK, 2004-05, 2003-04]

Solution: Let $F\left((x+y+z),(x^2+y^2+z^2)\right)=0$ be $F(u,v)=0$ …(1)

where $u = (x + y + z)$ and $v = (x^2 + y^2 + z^2)$ …(2)

Clearly $F(u, v) = 0$ is an implicit function.

$$\therefore \quad \left.\begin{aligned} 0 = F_x &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \quad \ldots(i) \\ 0 = F_y &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y} \quad \ldots(ii) \end{aligned}\right\} \quad \ldots(3)$$

whereas $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=(1+p) \quad \ldots(4)$$

$$\left(\text{since } \frac{\partial y}{\partial x}=0=\frac{\partial x}{\partial y} \text{ as } x \text{ and } y \text{ are two independent variables}\right)$$

and $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}=(1+q) \quad \ldots(5)$$

Similarly, $$\frac{\partial v}{\partial x}=(2x+2zp) \quad \ldots(6)$$

$$\frac{\partial v}{\partial y}=(2y+2zq) \quad \ldots(7)$$

Thus, on substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in equation (3), we get

$$\left.\begin{aligned} 0 &= \frac{\partial F}{\partial u}(1+p)+\frac{\partial F}{\partial v}(2x+2pz) \quad \ldots(i) \\ 0 &= \frac{\partial F}{\partial u}(1+q)+\frac{\partial F}{\partial u}(2y+2qz) \quad \ldots(ii) \end{aligned}\right\} \quad \ldots(8)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, we get

$$\begin{vmatrix} (1+p) & (2x+2pz) \\ (1+q) & (2y+2qz) \end{vmatrix}=0 \quad \Rightarrow \quad p(y-z)+q(z-x)=(x-y)$$

which is the desired p.d.e.

Example 4: Form the partial differential equation (by eliminating the arbitrary function) from: $F(xy + z^2, x + y + z) = 0$. [NIT Kurukshetra, 2007; KUK, 2002-03]

Solution: Let $F(xy + z^2, x + y + z) = 0$ be $F(u, v) = 0$ …(1)

where $u = xy + z^2$

and $v = x + y + z$ …(2)

Clearly $F(u, v) = 0$ is an implicit relation, so that

$$\left.\begin{aligned} 0 = F_x &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \quad \ldots(i) \\ 0 = F_y &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} \quad \ldots(ii) \end{aligned}\right\} \qquad \ldots(3)$$

whereas $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) = (y + 2zp) \qquad \ldots(4)$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) = (x + 2zp) \qquad \ldots(5)$$

Similarly, $$\frac{\partial v}{\partial x} = (1 + p) \qquad \ldots(6)$$

$$\frac{\partial v}{\partial y} = (1 + q) \qquad \ldots(7)$$

On substituting the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in equation (3), we get

$$\left.\begin{aligned} 0 &= \frac{\partial F}{\partial u}(y + 2pz) + \frac{\partial F}{\partial v}(1 + p) \\ 0 &= \frac{\partial F}{\partial u}(x + 2qz) + \frac{\partial F}{\partial v}(1 + q) \end{aligned}\right\} \qquad \ldots(8)$$

On eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, we get

$$\begin{vmatrix} y + 2pz & 1 + p \\ x + 2qz & 1 + q \end{vmatrix} = 0$$

$\Rightarrow$ $p(2z - x) - (2z - y)q = (x - y)$ the desired partial differentiation equation.

Example 5: Form partial differential equation from the relation

(*i*) $\boldsymbol{z = y^2 + 2f\left(\frac{1}{x} + \log y\right)}$ **(*ii*)** $\boldsymbol{z = f_1(x + iy) + f_2(x - iy)}$**.**

Solution: (*i*) $$z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \qquad \ldots(1)$$

$\therefore$ $$\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\cdot\left(-\frac{1}{x^2}\right) \qquad \ldots(2)$$

and $$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\cdot\left(\frac{1}{y}\right) \qquad \ldots(3)$$

On eliminating of $2f'\left(\frac{1}{x} + \log y\right)$, we get (3) as

$$\frac{\partial z}{\partial y} = 2y + \left[-x^2\frac{\partial z}{\partial x}\right]\frac{1}{y}$$

$\Rightarrow \qquad yq = 2y^2 - x^2p;$ when $\dfrac{\partial z}{\partial x} = p$ and $\dfrac{\partial z}{\partial y} = q$.

(*ii*) Given $\qquad z = f_1(x + iy) + f_2(x - iy) \qquad \ldots(1)$

$$\frac{\partial z}{\partial x} = f_1'(x+iy) + f_2'(x-iy) \qquad \ldots(2)$$

$$\frac{\partial z}{\partial y} = i\, f_1'(x+iy) - i\, f_2'(x-iy) \qquad \ldots(3)$$

Similarly $\qquad \dfrac{\partial^2 z}{\partial x^2} = f_1''(x+iy) + f_2''(x-iy) \qquad \ldots(4)$

$$\frac{\partial^2 z}{\partial y^2} = i^2 f_1''(x+iy) + i^2 f_2''(x-iy) \qquad \ldots(5)$$

$\therefore \qquad \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0;$ where $i^2 = -1$

Example 6: Form partial differential equations from the solutions

(*i*) $z = f(x) + e^y g(x)$

(*ii*) $z = \dfrac{1}{r}[F(r - at) + F(r + at)]$ **[NIT Kurukshetra, 2008]**

Solution: (*i*): Given $z = f(x) + e^y g(x)$

$\therefore \qquad \dfrac{\partial z}{\partial y} = e^y g(x)$, Keeping $g(x)$ as constant.

and $\qquad \dfrac{\partial^2 z}{\partial y^2} = e^y g(x)$, (On differentiating again with respect to y)

Thus $\qquad \dfrac{\partial z}{\partial y} = \dfrac{\partial^2 z}{\partial y^2}$

(*ii*) Given $\qquad z = \dfrac{1}{r}[F(r - at) + F(r + at)] \qquad \ldots(1)$

$$\frac{\partial z}{\partial t} = \frac{1}{r}\left[F'(r-at)\cdot -a + F'(r+at)\cdot a\right] \qquad \ldots(2)$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{a^2}{r}\left[F''(r-at) + F''(r+at)\right] \qquad \ldots(3)$$

$$\frac{\partial z}{\partial r} = \frac{1}{r}\left[F'(r-at) + F'(r+at)\right] - \frac{1}{r^2}\left[F(r-at) + F(r+at)\right] \qquad \ldots(4)$$

$$\Rightarrow \qquad \frac{\partial z}{\partial r} = \frac{1}{r}\left[F'(r-at) + F'(r+at)\right] - \frac{z}{r}$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{r}\left[F''(r-at)+F''(r+at)\right] - \frac{1}{r^2}\left[F'(r-at)+F'(r+at)\right]$$

$$-\frac{1}{r^2}\left[F'(r-at)+F'(r+at)\right] + \frac{2}{r^3}\left[F(r-at)+F(r+at)\right]$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial r^2} = \frac{1}{r}\left[F''(r-at)+F''(r+at)\right] - \frac{2}{r^2}\left[F'(r-at)+F'(r+at)\right] + \frac{2}{r^3}\left[F(r-at)+F(r+at)\right] \quad \ldots(5)$$

On using (1), (3), (4) in (5), we get

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} - \frac{2}{r}\left[\frac{\partial z}{\partial r} + \frac{z}{r}\right] + \frac{2}{r^2} z$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{2}{r}\frac{\partial z}{\partial r} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} \quad \text{or} \quad \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right) = \frac{\partial^2 z}{\partial t^2}$$ is the desired p.d.e.

Example 7: (*i*) Find the differential equation of all planes which are at a constant distance '*a*' from the origin. [NIT Kurukshetra, 2006]

(*ii*) Find the differential equation of all spheres whose centre lies on the *z*-axis.

(*iii*) Find the differential equation of all spheres of radius '*d*' units having their centres in the xy-plane.

Solution: (*i*) Equation of all planes is

$$\alpha x + \beta y + \gamma z + \delta = 0 \quad \ldots(1)$$

Now perpendicular distance of $P(0, 0, 0)$ from the plane (1) is given equal to 'a', i.e.

$$\frac{\alpha \cdot 0 + \beta \cdot 0 + \gamma \cdot 0 + \delta}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}} = a$$

$$\Rightarrow \quad \delta = a\sqrt{\alpha^2 + \beta^2 + \gamma^2} \quad \ldots(2)$$

Now on substituting the value of δ in equation (1),

$$\alpha x + \beta y + \gamma z + a\sqrt{\alpha^2 + \beta^2 + \gamma^2} = 0 \quad \ldots(3)$$

Taking partial derivative of equation (3) with respect to x,

$$\alpha + \gamma \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \alpha + \gamma p = 0 \quad \ldots(4)$$

Likewise, $$\beta + \gamma \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \beta + \gamma q = 0 \quad \ldots(5)$$

On substituting values of α and β in terms of γ in equation (3), we get

$$-\gamma px - \gamma qy + \gamma z + a\sqrt{\gamma^2 p^2 + \gamma^2 q^2 + \gamma^2} = 0$$

$$\Rightarrow \quad z = px + qy - a\sqrt{1 + p^2 + q^2},$$ the desired partial differential equation.

(*ii*) Equation of spheres whose centre lies on z-axis is given by

$$x^2 + y^2 + (z - c)^2 = d^2 \quad \ldots(1)$$

(This represents a surface of revolution with axis OZ.)

First differentiating (1) partially with respect to x, we get

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0 \qquad \ldots(2)$$

Likewise differentiating partially (1) with respect to y, we get

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0 \qquad \ldots(3)$$

Now on eliminating $(z – c)$ from equations (2) and (3),

$qx – py = 0$, the desired p.d.e.

(*iii*) Equation of all the spheres of radius 'd' whose centre lies in xy plane is given by

$$(x – a)^2 + (y – b)^2 + z^2 = d^2 \qquad \ldots(1)$$

On differentiating (1) partially with respect to x,

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0 \qquad \ldots(2)$$

Likewise $\quad 2(y-b) + 2z\frac{\partial z}{\partial y} = 0 \qquad \ldots(3)$

Now on substituting values of $(x – a)$ and $(y – b)$ in equation (1) from equations (2) and (3) respectively, we get

$$\left(-z\frac{\partial z}{\partial x}\right)^2 + \left(-z\frac{\partial z}{\partial y}\right)^2 + z^2 = d^2$$

i.e. $\quad p^2z^2 + q^2z^2 + z^2 = d^2 \quad$ or $\quad z^2(p^2 + q^2 + 1) = d^2$

Note: Equation $(x – a)^2 + (y – b)^2 + z^2 = d^2$ represents a paraboloid of revolution with vertex at $(a, b, 0)$.

ASSIGNMENT 1

1. Form partial differential equations from the relations:

(*i*) $z = f\left(\frac{x}{y}\right)$ (*ii*) $z = e^{my}\,\phi(x – y)$ (*iii*) $z = axe^y + \frac{1}{2}a^2e^{2y} + b$

2. Form the partial differential equation (by eliminating the arbitrary function)

(*i*) $xyz = \phi(x + y + z)$ (*ii*) $z = f_1(x)\, f_2(y)$

3. Eliminate arbitrary constants a and b from the following relations:

(*i*) $z = ax + by + a^2 + b^2$ (*ii*) $z = axy + b$

(*iii*) $z = ae^{-b^2t}\cos bx$ (*iv*) $ax^2 + by^2 + cz^2 = 1$

4. If $z = f(x + ct) + \phi(x – ct)$, prove that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$ **[J&K, 2001; KUK, 2008, 2009]**

11.3 ABOUT SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

A solution of partial differential equation (p.d.e.) in some region R of the space of the independent variable is a function that has the partial derivatives appearing in the equation is some domain containing R and satisfies the equation everywhere in R (often one merely requires that the function is continuous on the boundary of R, has those derivatives in the interior of R, and satisfies the equation in the interior of R). In general, the totality of solutions of partial differential equation is very large.

e.g. (*i*) $u = x^2 - y^2$, (*iii*) $u = e^x \cos y$ (iii) $u = \log(x^2 - y^2)$ are three p.d.e. entirely different from each other, still are the solution of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, as you may verify. We shall see for unique solution of a p.d.e. corresponding to the given physical problem, will be obtainable by the use of additional conditions arising from the problem, for instance, the condition that the solution u assumes the given values on the boundary of the region considered (boundary conditions) or, when time t is one of the variables,

that u or $u_t = \dfrac{\partial u}{\partial t}$ or both prescribed at $t = 0$ (initial condition)

We categorize the solution in the following sub-heads:

1. Complete Solution (Complete Integral)

If we can obtain the relation $F(x, y, z, a, b) = 0$ which contains as many as arbitrary constants (viz., a and b) as there are independent variables in the partial differentiation equation $f(x, y, z, p, q) = 0$ is known as 'Complete solution'.

2. Particular Solution (Particular Integral)

Particular solution is obtained by giving particular values to the arbitrary constants or the arbitrary function in the complete solution.

3. General Solution (General Integral)

If in the solution $F(x, y, z, a, b) = 0$, we put $b = \phi(a)$ and obtain the envelop of the family of surfaces $F(x, y, z, a, \phi(a)) = 0$, we had a solution containing arbitrary function ϕ. This is called the general solution.

4. Singular Solution (Singular Integral)

The envelop of family of surfaces $F(x, y, z, a, b) = 0$ obtained by elimination of arbitrary constants a and b from $F(x, y, z, a, b) = 0$ and $\dfrac{\partial F}{\partial a} = 0 = \dfrac{\partial F}{\partial b}$, is called singular solution.

Remarks: A partial differential equation is said to be fully solved only if all the three types of integrals *viz.*, complete integral, general integral and singular integrals are obtained.

Example 8: Show that if U_1 and U_2 be two solutions of linear homogeneous equation $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2} = a\dfrac{\partial^2 U}{\partial t^2} + b\dfrac{\partial U}{\partial t}$, then $C_1U_1 + C_2U_2$ is also a solution.

Extend this result to a linear combination of n independent solutions. Will this result be true if $n \to \infty$.

Solution: As U_1 and U_2 are solutions of the given equation, therefore

$$\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2} + \frac{\partial^2 U_1}{\partial z^2} = a\frac{\partial^2 U_1}{\partial t^2} + b\frac{\partial U_1}{\partial t} \qquad \ldots(1)$$

and $$\frac{\partial^2 U_2}{\partial x^2}+\frac{\partial^2 U_2}{\partial y^2}+\frac{\partial^2 U_2}{\partial z^2}=a\frac{\partial^2 U_2}{\partial t^2}+b\frac{\partial U_2}{\partial t} \qquad \ldots(2)$$

Multiplying (1) by C_1 and (2) by C_2 and adding the two, we get

$$\frac{\partial^2}{\partial x^2}(C_1U_1+C_2U_2)+\frac{\partial^2}{\partial y^2}(C_1U_1+C_2U_2)+\frac{\partial^2}{\partial z^2}(C_1U_1+C_2U_2)$$

$$= a\frac{\partial^2}{\partial t^2}(C_1U_1+C_2U_2)+b\frac{\partial}{\partial t}(C_1U_1+C_2U_2) \qquad \ldots(3)$$

Thus $(C_1U_1 + C_2U_2)$ is also a solution of the given p.d.e.

Generalisation: If $U_1, U_2, \ldots, U_n$ are n independent solutions, then $C_1U_1 + C_2U_2 + \ldots + C_nU_n$ is also a solution.

Example 9: Verify that e^{-n^2t} sin nx is a solution of the heat equation $\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}$. Hence show that $\sum_{n=1}^{p} C_n e^{-n^2t}\sin nx$; where $C_1, C_2, \ldots, C_p$ are all arbitrary constants, is a solution of this equation satisfying the boundary conditions $U(0, t) = 0$ and $U(\pi, t) = 0$.

Solution: Take $U=e^{-n^2t}\sin nx$, then we need to prove that U satisfies the heat equation.

Now $$\frac{\partial U}{\partial x}=e^{-n^2t}n\cdot\cos nx \qquad \ldots(1)$$

$\Rightarrow$ $$\frac{\partial^2 U}{\partial x^2}=e^{-n^2t}(-n^2\sin nx) \qquad \ldots(2)$$

and $$\frac{\partial U}{\partial t}=e^{-n^2t}(-n^2)\sin nx \qquad \ldots(3)$$

Now (2) and (3), we have

$$\frac{\partial^2 U}{\partial x^2}=\frac{\partial U}{\partial t}$$

Thus $U=e^{-n^2t}$ sinnx is a solution of the given solution.

Let $n = 1, 2, \ldots, p$ in $U=e^{-n^2t}$, then we get p different solutions. Hence by principle of superposition, we have

$$U=C_1e^{-t}\sin x+C_2e^{-2^2t}\sin 2x+\ldots+C_pe^{-p^2t}\sin px \qquad \ldots(4)$$

or $$U(x,t)=\sum_{n=1}^{p}C_n e^{-n^2t}\sin nx \qquad \ldots(5)$$

is also a solution.

Further, $U(0, t) = 0 = U(p, t)$, since $\sin np = 0$ for all integer values of n.

ASSIGNMENT 2

1. Show that $U = f(x^2 - y^2)$ is a solution of $y\frac{\partial U}{\partial x} + x\frac{\partial U}{\partial y} = 0$.

2. Verify that $e^{-n^2t}\sin\left(\frac{nx}{C}\right)$ is a solution of the heat equation $\frac{\partial U}{\partial t} = C^2\frac{\partial^2 U}{\partial x^2}$

   Hence show that $\sum_{n=1}^{N} a_n e^{-n^2t}\sin\left(\frac{nx}{C}\right)$, where $a_1, a_2, \ldots$, are arbitrary constants, is also a solution satisfying the boundary conditions $U(0, t) = 0 = U(\pi C, t)$.

11.4 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

Partial differential equations occuring with only one partial derivative can be solved directly by integration. However, in such cases, we must use arbitrary function of variable in place of constant of integration.

Example 10: Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ given that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Solution: If z were a function of x alone, the solution would have been

$$z = C_1\cos x + C_2\sin x \qquad \ldots(1)$$

where C_1 and C_2 are arbitrary constants.

Since here z is a function of both x and y, therefore, C_1 and C_2 can be chosen arbitrary functions of y.

Whence the solution of the given equation is

$$z = f_1(y)\sin x + f_2(y)\cos x \qquad \ldots(2)$$

$$\Rightarrow \qquad \frac{\partial z}{\partial x} = f_1(y)\cos x - f_2(y)\sin x \qquad \ldots(3)$$

When $x = 0$, $z = e^y \Rightarrow e^y = f_2(y)$ ...(4)

Also $x = 0$, $\frac{\partial z}{\partial x} = 1 \Rightarrow 1 = f_1(y)$ ...(5)

Hence the required solution is $z = \sin x + e^y\cos x$.

Example 11: Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, given that $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$, and $z = 0$, when y is an odd multiple of $\frac{\pi}{2}$.

Solution: On integrating the given equation, $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ with respect to x keeping y constant,

$$\frac{\partial z}{\partial y} = -\cos x \sin y + \phi(y) \qquad \ldots(1)$$

Given $x = 0$, $\frac{\partial z}{\partial y} = -2\sin y$ implies $-2\sin y = -\sin y + \phi(y)$

or $$\phi(y) = -\sin y \qquad \ldots(2)$$

Thus (1) becomes $\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$ ...(3)

Now, on integrating (3) with respect to y, we get

$$z = \cos x \cos y + \cos y + \psi(x) \quad \text{...(4)}$$

Clearly, when y is an odd multiple of $\frac{\pi}{2}$, $z = 0 \Rightarrow \psi(x) = 0$

$\therefore$ $z = (1 + \cos x)\cos y$ is the required solution.

Example 12: Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$.

Solution: Given $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$ ...(1)

On integrating (1) with respect to x, keeping y constant, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x + 3y)}{2} + f(y) \quad \text{...(2)}$$

Again integrating (2) with respect to x keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\frac{\cos(2x + 3y)}{4} + x\, f(y) + \phi(y) \quad \text{...(3)}$$

Now on integrating (3) with respect to y keeping x constant, we get

$$z = -\frac{\sin(2x + 3y)}{12} + x \int f(y)\, dy + \int \phi(y) dy + \gamma(x)$$

or

$$z = -\frac{\sin(2x + 3y)}{12} + x\, \alpha(y) + \beta(y) + \gamma(x) \quad \text{...(4)}$$

Example 13: Solve $\log\left[\frac{\partial^2 z}{\partial x \partial y}\right] = (x + y)$.

Solution: Given $\log\left[\frac{\partial^2 z}{\partial x \partial y}\right] = (x + y)$ or $e^{\log\left[\frac{\partial^2 z}{\partial x \partial y}\right]} = e^{(x+y)}$

$\Rightarrow$ $\frac{\partial^2 z}{\partial x \partial y} = e^{(x+y)}$ ...(1)

On integrating (1) with respect to x keeping y constant, we get

$$\frac{\partial z}{\partial y} = e^{(x+y)} + f(y) \quad \text{...(2)}$$

where $f(y)$ is an arbitrary constant.

Now, integrating (2) again with respect to y, keeping x constant

$$z = e^{(x+y)} + x\, f(y) + \phi(x) \quad \text{...(3)}$$

Example 14: Solve $\dfrac{\partial^2 z}{\partial y^2} = z$, given that when $y = 0$; $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$.

[NIT Kurukshetra, 2010]

Solution: In the equation $\dfrac{\partial^2 z}{\partial y^2} = z$, if we treat z as pure function of y only, we could solve it like an ordinary differential equation with auxiliary equation as:

$$D^2 = 1 \quad \text{i.e.,} \quad D = \pm 1$$

so that
$$z = Ae^y + Be^{-y} \qquad \ldots(2)$$

Here z is a function of both x and y, since we are dealing in partial differential equations. Thus in $z = Ae^y + Be^{-y}$, A and B are arbitrary constants, but are like $A = \phi(x)$ and $B = \psi(x)$. Whence

$$z = \phi(x)\, e^y + \psi(x) e^{-y} \qquad \ldots(3)$$

Now, for $y = 0$, $z = e^x \Rightarrow e^x = \phi(x)\, e^0 + \psi(x)\, e^0$

i.e.
$$e^x = \phi(x) + \psi(x) \qquad \ldots(4)$$

Again for $y = 0$, $\dfrac{\partial z}{\partial y} = e^{-x}$ i.e., from equation (3), we get

$$e^{-x} = \left[\phi(x)\, e^y - \psi(x) e^{-y}\right]_{y=0}$$

$$\Rightarrow \qquad e^{-x} = \phi(x)\, e^0 - \psi(x)\frac{1}{e^{\circ}} = \phi(x) - \psi(x) \qquad \ldots(5)$$

Now, on solving equations (4) and (5) for $\phi(x)$ and $\psi(x)$, we get

$$\left.\begin{aligned} \phi(x) &= \frac{e^x + e^{-x}}{2} = \cosh x \\ \text{and} \quad \psi(x) &= \frac{e^x - e^{-x}}{2} = \sinh x \end{aligned}\right\}$$

Therefore, $z = (\cosh x\, e^y + \sinh x\, e^{-y})$ is the desired solution.

ASSIGNMENT 3

1. Solve $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$ **2.** Solve $\dfrac{\partial^2 z}{\partial y^2} = \sin(xy)$

3. Solve p.d.e. $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$, given that when $x = 0$, $\dfrac{\partial z}{\partial x} = a \sin y$ and $\dfrac{\partial z}{\partial y} = 0$.

4. $\dfrac{\partial^2 z}{\partial x \partial t} = e^{-t} \cos x$

5. $xys = 1$ $\left[\text{Hint: Rewrite as } \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{xy}\right]$

6. $\operatorname{Log} s = x + y$ $\left[\text{Hint: Rewrite as, } \dfrac{\partial^2 z}{\partial x \partial y} = e^{x+y}\right]$

11.5 LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A differential equation involving only first order partial differential coefficients p and q is called partial differential equation of first order. Further, if the degrees of p and q are unity only then it is termed as linear p.d.e. of first order. If each term of such an equation contains either the dependent variable or one of the derivatives, the equation is said to be homogeneous, otherwise non-homogeneous.

Some important partial differential equations of second order are as follows:

1. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, one dimensional wave equation (hyperbolic)

2. $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, one dimensional heat equation (parabolic)

3. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, two dimensional Laplace equation (elliptic)

4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$, two dimensional Poisson equation

5. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, two dimensional wave equation

6. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$, three dimensional Laplace equation

Here c is a constant, t is time, x, y, z are Cartesian co-ordinates Equations other than (4), all are homogeneous.

Lagrange's Linear Equation

Ist order linear partial differential equation in its standard form

$$Pp + Qq = R \qquad \text{...(1)}$$

where P, Q, R are functions of x, y, z is called Lagrange's Linear Equation. This equation is obtained by eliminating arbitrary function f from

$$f(u, v) = 0 \qquad \text{...(2)}$$

where u, v are functions of x, y, z.

Here we show that its solution depends on the solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{...(3)}$$

Differentiating (2) partially with respect to x and y respectively, we get

$$\left.\begin{aligned} \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \\ \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \end{aligned}\right\}, \quad \text{(as (2) is an implicit relation)}$$

More precisely,

$$\left.\begin{aligned}\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0\\ \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0\end{aligned}\right\}, \qquad \ldots(4)$$

(as $\frac{\partial f}{\partial x}=0=\frac{\partial x}{\partial y}$, x and y being two independent variables.)

From above equations, on eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, we have

$$\begin{vmatrix}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\end{vmatrix}=0 \qquad \ldots(5)$$

implying $$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \ldots(6)$$

which is the same as equation (1) with

$$P=\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)$$

$$Q=\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)$$

$$R=\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

Now in order to find u and v, let $u = a$ and $v = b$, where a and b are two arbitrary constants, so that

$$\left.\begin{aligned}0=du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz\\ \text{and}\quad 0=dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz\end{aligned}\right\} \qquad \ldots(7)$$

From above simultaneous equations, we get

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

or $$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Solution of above differential equation are $u = a$ and $v = b$.

whence the solution of Lagrange's Linear equation $Pp + Qq = R$ is

$$f(u, v) = 0 \quad \text{or} \quad f(a, b) = 0.$$

Working Rule for Solving Lagrange's Equations

(*i*) Corresponding to Lagrange's Equation (linear partial differential equation)

$$Pp + Qq = R.$$

Form the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(*ii*) Solve these auxiliary equations by the method of grouping or the method of the multiplier or both for getting two independent integrals, say, $u = a$ and $v = b$. Then the general integral of the given equation will be $f(u, v) = 0$ or $u = f(v)$, where f is an arbitrary function.

***Note*:** In case of linear equation with n independent variables, say, $P_1p_1 + P_2p_2 + \dots + P_np_n = R$,

where $\quad p_j = \dfrac{\partial z}{\partial x_j}, \quad j = 1, 2, \dots, n; \quad P_1, P_2, \dots, P_n$ and R are functions of $x_1, x_2, \dots, x_n$ and z.

The subsidiary equation is

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

and solution is $f(u_1, u_2, \dots, u_n) = 0$ where $u_1 =$ const., $u_2 =$ const. so on. $u_n =$ constant, are the solutions of the subsidiary equations.

Geometrical Interpretation of Lagrange's Equation

Lagrange's linear equation

$$Pp + Qq = R \qquad \dots(1)$$

may be written as

$$Pp + Qq + (-1)R = 0$$

Let the solution of (1) be

$$f(x, y, z) = 0 \qquad \dots(2)$$

representing a surface, the normal to which at any point has direction cosines proportional to

$$\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z}$$

or
$$\frac{-\dfrac{\partial f}{\partial x}}{-\dfrac{\partial f}{\partial z}} : \frac{-\dfrac{\partial f}{\partial y}}{-\dfrac{\partial f}{\partial z}} : -1$$

or
$$\frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} : -1 \quad \text{or} \quad p : q : -1$$

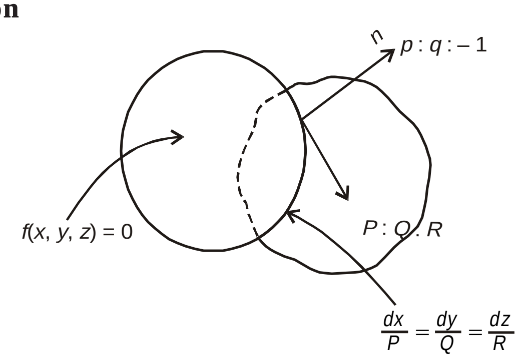

Fig. 11.1

Further, the simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots(3)$$

represent a family of curves such that the tangent to which at any point has the direction cosines proportional to P, Q, R and that $f(u, v) = 0$ represents a surface through such curves where u = constant, v = constant (say a and b respectively), are two particular integrals of (3).

Hence, the geometrical interpretation of equation (1) is that "the normal to the surface (2) is perpendicular to a line (say $u = a$ or $v = b = f(a)$) whose direction cosines are proportional to P, Q, R and so that the sum of their respective product is $Pp + Qq + R(-1) = 0$ or $Pp + Qq = R$.

Or in other words, the equation (1) states that normal to the surface (2) at any point is perpendicular to the members of the family (3) through that point and which is true for every point on the surface (1).

Thus, the equation (1), $Pp + Qq = R$ and the equation (3), $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ defines the same set of surfaces and hence equivalent.

Example 15: Solve the following equations

(*i*) $\boldsymbol{(z^2 - 2yz - y^2)p + (xy + xz)q = (xy - zx)}$

(*ii*) $\boldsymbol{p \tan x + q \tan y = \tan z}$ **(KUK, 2000)**

(*iii*) $\boldsymbol{px - qy = (y^2 - x^2)}$

(*iv*) $\boldsymbol{y^2 p - xyq = x(z - 2y)}$ **(KUK, 2008)**

Solution:

(*i*) The subsidiary equations are given by

$$\underset{\text{I}}{\frac{dx}{(z^2 - 2yz - y^2)}} = \underset{\text{II}}{\frac{dy}{(xy + xz)}} = \underset{\text{III}}{\frac{dz}{(xy - zx)}} = \underset{\text{IV}}{\frac{x\,dx + y\,dy + z\,dz}{0}} \qquad \ldots(1)$$

Taking $$\frac{dy}{(y+z)} = \frac{dz}{(y-z)}$$

Which on simplification results to

$$(y\,dy - z\,dz) + (z\,dy + y\,dz) = 0$$

$$\Rightarrow \quad \frac{y\,dy}{2} - \frac{z\,dz}{2} + d(yz) = 0$$

On integrating, we have

$$(y^2 - z^2 - 2yz) = C_1 \qquad \ldots(2)$$

Also from (1), we have

$$\frac{dx}{(z^2 - 2xz - y^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

i.e. $\quad x\,dx + y\,dy + z\,dz = 0 \quad \Rightarrow \quad x^2 + y^2 + z^2 = C_2 \qquad \ldots(3)$

$\therefore$ The desired solution is

$$f(C_1, C_2) = 0 \quad \text{or} \quad f(y_2 - z_2 - 2yz,\ x^2 + y^2 + z^2) = 0$$

(*ii*) $$\underset{\text{I}}{\frac{dx}{\tan x}} = \underset{\text{II}}{\frac{dy}{\tan y}} = \underset{\text{III}}{\frac{dz}{\tan z}}$$

On taking I & II, $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y}$

$\Rightarrow \quad \int \cot x\, dx = \int \cot y\, dy \quad$ or $\quad \log \sin x = \log \sin y - \log C_1$

or $$C_1 = \left(\frac{\sin y}{\sin x}\right) \qquad \ldots(1)$$

Likewise taking II & III, $\dfrac{dy}{\tan y} = \dfrac{dz}{\tan z} \quad \Rightarrow \int \cot y\, dy = \int \cot z\, dz$

or $\quad \log \sin y = \log \sin z - \log C_2$

$\Rightarrow$ $$C_2 = \left(\frac{\sin z}{\sin y}\right) \qquad \ldots(2)$$

$\therefore$ The desired solution is $f(C_1, C_2) = 0 = f\left(\dfrac{\sin y}{\sin x}, \dfrac{\sin z}{\sin y}\right)$

(*iii*) The subsidiary equations are

$$\underset{\text{I}}{\frac{dx}{x}} = \underset{\text{II}}{\frac{dy}{-y}} = \underset{\text{III}}{\frac{dz}{y^2 - x^2}} = \underset{\text{IV}}{\frac{x\,dx + y\,dy + dz}{0}}$$

On taking I & II, we have $\dfrac{dx}{x} = \dfrac{dy}{-y} \quad \Rightarrow \quad \log x = -\log y + \log C_1$

$\Rightarrow \quad \log(xy) = \log C_1 \quad \Rightarrow \quad xy = C_1 \qquad \ldots(1)$

Taking I & IV, $\dfrac{dx}{x} = \dfrac{(x\,dx + y\,dy + dz)}{0}$

or $\quad x\,dx + y\,dy + dz = 0 \quad \Rightarrow \quad x^2 + y^2 + 2z = C_2 \qquad \ldots(2)$

$\therefore \quad f(C_1, C_2) = 0 \quad$ or $\quad f(xy,\ x^2 + y^2 + 2z) = 0$, the desired solution.

(*iv*) Here subsidiary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z - 2y)} \qquad \ldots(1)$$

On taking I & II, $\dfrac{dx}{y^2} = \dfrac{dy}{-xy}$ $\Rightarrow$ $x\,dx + y\,dy = 0$, i.e. $(x^2 + y^2) = C_1$ …(2)

Likewise, on taking

$$\frac{dy}{-xy} = \frac{dz}{xz - 2xy} = \frac{dz - dy}{xz - xy}$$

i.e., $$\frac{dy}{-y} = \frac{dz - dy}{z - y} \qquad \text{…(3)}$$

On simplifying, we get

$$z\,dy + y\,dz = 2y\,dy, \quad \text{i.e.} \quad d(yz) = d(y^2) \quad \text{or} \quad y^2 - yz = C_2 \qquad \text{…(4)}$$

Alternately, $\dfrac{dy}{-y} = \dfrac{dz - dy}{z - y}$ is of the form $\dfrac{f'(x)}{f(x)}$

$\therefore$ $-\log y = \log(z - y) - \log C_2$ or $y(z - y) = C_2$ …(5)

Hence the solution is $f(x^2 + y^2, y^2 - yz) = 0$

Example 16: Solve Ist order linear partial differential equation $\boldsymbol{p(x^2 - y^2 - z^2) + q(2xy) = 2xz}$

Solution: Here we have,

$$\underset{\text{I}}{\frac{dx}{x^2 - y^2 - z^2}} = \underset{\text{II}}{\frac{dy}{2xy}} = \underset{\text{III}}{\frac{dz}{2xz}} = \underset{\text{IV}}{\frac{x\,dx + y\,dy + z\,dz}{x(x^2 - y^2 - z^2) + y(2xy) + z(2xz)}}$$

Taking II and III, $\dfrac{dy}{y} = \dfrac{dz}{z}$ $\Rightarrow$ $\dfrac{y}{z} = C_1$ …(1)

Taking III and IV, $\dfrac{dz}{2xz} = \dfrac{(x\,dx + y\,dy + z\,dz)}{x(x^2 + y^2 + z^2)}$

$\Rightarrow$ $$\frac{dz}{z} = \frac{d(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)}, \quad \left(\text{which is of the form } \frac{f'(x)}{f(x)}\right)$$

$\Rightarrow$ $\log z + \log C_2 = \log(x^2 + y^2 + z^2)$

$\Rightarrow$ $$C_2 = \frac{(x^2 + y^2 + z^2)}{z} \qquad \text{…(2)}$$

Hence the desired solution is $f(C_1, C_2) = 0$

or $$f\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0$$

Example 17: Solve the following Ist order linear partial differential equations

(*i*) $y^2zp + x^2zq = y^2x$

(*ii*) $p - q = \log(x + y)$ **[NIT Kurukshetra, 2007]**

(*iii*) $pyz + qzx = xy$

Solution:

(*i*) Here subsidiary equations are

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$$

On taking I and II, we get

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} \quad \text{or} \quad x^2dx = y^2dy \quad \text{or} \quad (x^3 - y^3) = C_1 \qquad \ldots(1)$$

Likewise, taking I & III, we get

$$\frac{dx}{y^2z} = \frac{dy}{y^2z} \quad \text{or} \quad x\,dx = z\,dz \quad \Rightarrow \quad (x^2 - y^2) = C_2 \qquad \ldots(2)$$

Hence the desired solution is $f(x^3 - y^3, x^2 - y^2) = 0$.

(*ii*) The subsidiary equations are $\underset{\text{I}}{\dfrac{dx}{1}} = \underset{\text{II}}{\dfrac{dy}{-1}} = \underset{\text{III}}{\dfrac{dz}{\log(x+y)}}$

On taking I and II,

$$\frac{dx}{1} = \frac{dy}{-1} \quad \Rightarrow \quad (x + y) = a \ \text{(say)} \qquad \ldots(1)$$

Now on taking I & III, we get

$$\frac{dx}{1} = \frac{dz}{\log(x+y)} \quad \Rightarrow \quad \frac{dx}{1} = \frac{dz}{\log a} \quad \Rightarrow \quad (\log a)\cdot dx = dz$$

On integrating, $(\log a)x = z + b$ (say) ...(2)

$\Rightarrow$ $x \cdot \log(x + y) = z + b$ On using (1)

$\Rightarrow$ $x \log(x + y) = z + \phi(a)$ as $b = \phi(a)$

$\Rightarrow$ $x \log(x + y) - z = \phi(x + y)$ ...(3)

(*iii*) Here the subsidiary equations are

$$\underset{\text{I}}{\frac{dx}{yz}} = \underset{\text{II}}{\frac{dy}{zx}} = \underset{\text{III}}{\frac{dz}{xy}}$$

On taking I and II, we get

$$\frac{dx}{yz} = \frac{dy}{zx} \quad \Rightarrow \quad x\,dx = y\,dy \quad \Rightarrow \quad (x^2 - y^2) = C_1 \qquad \ldots(1)$$

On taking II and III, we get

$$\frac{dy}{zx} = \frac{dz}{xy} \quad \Rightarrow \quad y\,dy = z\,dz \quad \Rightarrow \quad (y^2 - z^2) = C_2 \qquad \ldots(2)$$

Hence the desired solution is $f(C_1, C_2) = 0$ or $f(x^2 - y^2, y^2 - z^2) = 0$.

Example 18: Solve the following Lagrange's Linear partial differential equations

(*i*) $(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)$ **[KUK, 2009]**

(*ii*) $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$ **[KUK, 2004-05]**

(*iii*) $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$ **[NIT Jalandhar, 2006; KUK, 2003-04]**

Solution:

(*i*) Here the auxiliary equations are

$$\underset{\text{I}}{\frac{dx}{x^2 - yz}} = \underset{\text{II}}{\frac{dy}{y^2 - zx}} = \underset{\text{III}}{\frac{dz}{z^2 - xy}}$$

or $$\frac{dx}{(x^2 - yz)} = \frac{dy}{(y^2 - zx)} = \frac{dz}{(z^2 - xy)}$$

$$= \underset{\text{A}}{\frac{dx - dy}{(x^2 - yz) - (y^2 - zx)}} = \underset{\text{B}}{\frac{dy - dz}{(y^2 - zx) - (z^2 - xy)}} = \underset{\text{C}}{\frac{dz - dx}{(z^2 - xy) - (x^2 - yz)}}$$

On taking expressions A and B,

$$\frac{dx - dy}{(x^2 - yz) - (y^2 - zx)} = \frac{dy - dz}{(y^2 - zx) - (z^2 - xy)}$$

On simplification we get,

$$\frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)}$$

or $\dfrac{dx - dy}{(x - y)} = \dfrac{dy - dz}{(y - z)}$, which is of the form $\dfrac{f'(x)}{f(x)}$

On integration,

$$\log(x - y) = \log(y - z) + \log C_1 \quad \Rightarrow \quad C_1 = \left(\frac{x - y}{y - z}\right) \qquad \ldots(1)$$

Likewise, on taking II and III, $\dfrac{dy - dz}{(y - z)} = \dfrac{dz - dx}{(z - x)}$

or $$\log(y - z) = \log(z - x) + \log C_2 \quad \Rightarrow \quad C_2 = \left(\frac{y - z}{z - x}\right) \qquad \ldots(2)$$

Hence, the desired solution of the given p.d.e. is $f\left(\dfrac{x - y}{y - z}, \dfrac{y - z}{z - x}\right) = 0.$

(*ii*) Here, the subsidiary equations are

$$\underset{\text{I}}{\frac{dx}{x^2(y-z)}} = \underset{\text{II}}{\frac{dy}{y^2(z-x)}} = \underset{\text{III}}{\frac{dz}{z^2(x-y)}} \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x^2(y-z) + \frac{1}{y}y^2(z-x) + \frac{1}{z}z^2(x-y)}$$

$$\Rightarrow \quad \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\Rightarrow \quad \log xyz = \log C_1 \quad \Rightarrow \quad xyz = C_1 \qquad \ldots(2)$$

Likewise, (1) also becomes

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{(y-z)+(z-x)+(x-y)}$$

or $$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

On integration of each term with respective variable, we get

$$\Rightarrow \quad -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -\frac{1}{C_2} \quad \text{or} \quad C_2 = \left(x^{-1} + y^{-1} + z^{-1}\right)$$

$\therefore$ The desired solution, $f(xyz, x^{-1} + y^{-1} + z^{-1}) = 0$.

(*iii*) The subsidiary equations are

$$\underset{\text{I}}{\frac{dx}{x(y^2-z^2)}} = \underset{\text{II}}{\frac{dy}{y(z^2-x^2)}} = \underset{\text{III}}{\frac{dz}{z(x^2-y^2)}} = \underset{\text{IV}}{\frac{x\,dx + y\,dy + z\,dz}{0}} = \underset{\text{V}}{\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}} = \beta \qquad \ldots(1)$$

From above, expression IV gives

$$x\,dx + y\,dy + z\,dz = 0$$

$$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c \quad \Rightarrow \quad x^2 + y^2 + z^2 = \text{C}_1 \qquad \ldots(2)$$

From expression V, we get

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\log x + \log y + \log z = \log C_2 \quad \Rightarrow \quad xyz = C_2 \qquad \ldots(3)$$

Hence, the desired solution is $f(x^2 + y^2 + z^2, xyz) = 0$.

Example 19: Solve the following partial differential equations

(*i*) $x(y - z)p + y(z - x)q = z(x - y)$ **[KUK, 2002-03]**

(*ii*) $(y + z)p + (z + x)q = (x + y)$.

Solution: (*i*) Here the subsidiary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

On using the multiplier $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get

$$-\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} = \frac{dx+dy+dz}{0}$$

I II III IV V

From expression IV, we get $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$$\log xyz = \log C_1 \quad \Rightarrow \quad xyz = C_1$$

From expression V, $dx + dy + dz = 0 \quad \Rightarrow \quad (x + y + z) = C_2$

Hence the desired solution is $f(xyz, x + y + z) = 0$.

(*ii*) Here in this case, the auxiliary equations are

$$\frac{dx}{(y+z)} = \frac{dy}{(z+x)} = \frac{dz}{(x+y)} \qquad \ldots(1)$$

The relation (1) is extended to,

$$\frac{dx}{(y+z)} = \frac{dy}{(z+x)} = \frac{dz}{(x+y)} = \frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)}$$

I II III IV V VI

From IV and V, we get

$$\frac{(dx+dy+dz)}{2(x+y+z)} = \frac{(dx-dy)}{-(x-y)}, \quad \text{which is of the form } \frac{f'(x)}{f(x)}$$

$$\frac{1}{2}\log(x+y+z)+\log(x-y)=\log C$$

$$\log C_1 = \log(x+y+z)(x-y)^2 \quad \text{or} \quad C_1 = (x+y+z)(x-y)^2 \qquad \ldots(1)$$

From expressions V and VI,

$$\frac{(dx-dy)}{(x-y)}=\frac{(dy-dz)}{(y-z)}$$

$$\Rightarrow \quad \log(x-y) = \log(y-z) + \log C_2$$

$$\Rightarrow \quad \log\left(\frac{x-y}{y-z}\right)=\log C_2 \quad \text{or} \quad \left(\frac{x-y}{y-z}\right)=C_2 \qquad \ldots(2)$$

Hence, the desired solution is $f\left((x+y+z)(x-y)^2, \left(\frac{x-y}{y-z}\right)\right)=0.$

Example 20: Solve the linear partial differential equation

(*i*) $x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 = -x_1x_2x_3$ **(*ii*)** $-p_1 + p_2 + p_3 = 1$

(*iii*) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=xyz$ **[CDLU, 2004]**

Solution: (*i*) Comparing the given equation

$$x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 = -x_1x_2x_3 \qquad \ldots(1)$$

with the equation

$$P_1p_1 + P_2p_2 + P_3p_3 + \ldots + P_np_n = R \qquad \ldots(2)$$

we get the auxiliary equations as:

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=\frac{dx_3}{P_3}=\ldots=\frac{dx_n}{P_n}=R \qquad \ldots(3)$$

$$\therefore \quad \frac{dx_1}{x_2x_3}=\frac{dx_2}{x_3x_1}=\frac{dx_3}{x_1x_2}=\frac{dz}{-x_1x_2x_3} \qquad \ldots(4)$$

On taking I and IV, we get

$$x_1dx_1 + dz = 0 \quad \Rightarrow \quad x_1^2 + 2z = C_1 \qquad \ldots(5)$$

Likewise, from I & II and I & III, we get

$$x_1^2 - x_2^2 = C_2 \quad \text{and} \quad x_1^2 - x_3^2 = C_3 \qquad \ldots(6)$$

Hence the general integral is $f(x_1^2 + 2z,\ x_1^2 - x_2^2,\ x_1^2 - x_3^2) = 0.$

(*ii*) As explained above, the corresponding auxiliary equations in this case are

$$\underset{\text{I}}{\frac{dx_1}{-1}}=\underset{\text{II}}{\frac{dx_2}{1}}=\underset{\text{III}}{\frac{dx_3}{1}}=\underset{\text{IV}}{\frac{dz}{1}} \qquad \ldots(1)$$

Taking I and IV,

$$dx_1 + dz = 0 \quad \Rightarrow \quad x_1 + z = C_1 \qquad \ldots(2)$$

Likewise, from I and II and I and III, we get

$$x_1 + x_2 = C_2 \qquad \ldots(3)$$

and $$x_1 + x_3 = C_3 \qquad \ldots(4)$$

Hence, the desired solution is $f(x_1 + z, x_1 + x_2, x_1 + x_3) = 0$.

(*iii*) Here in this case when u is a function of three independent variables x, y and z, the desired auxiliary equations are

$$\underset{\text{I}}{\frac{dx}{x}} = \underset{\text{II}}{\frac{dy}{y}} = \underset{\text{III}}{\frac{dz}{z}} = \underset{\text{IV}}{\frac{du}{xyz}} \qquad \ldots(1)$$

On taking I and II, $\frac{dx}{x} = \frac{dy}{y}$

$$\Rightarrow \qquad \log x = \log y + \log C \quad \Rightarrow \quad \frac{x}{y} = C_1 \qquad \ldots(2)$$

Similarly taking II and III, we get $\frac{y}{z} = C_2$ ...(3)

Again $$\frac{yz\,dx + zx\,dy + xy\,dz}{3xyz} = \frac{du}{xyz}$$

$$\Rightarrow \quad yz\,dx + zx\,dy + xy\,dz = 3du \quad \text{or} \quad d(xyz) = 3du \quad \text{or} \quad xyz - 3u = C_3 \qquad \ldots(4)$$

Hence, the desired solution is $f\left(\frac{x}{y}, \frac{y}{z}, xyz - 3u\right)$.

Example 21: Solve $p + 5q = 9z + \tan(y - 5x)$.

Solution: Here the auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{5} = \frac{dz}{9z + \tan(y - 5x)} \qquad \ldots(1)$$

Taking I and II, we get

$$\frac{dx}{1} = \frac{dy}{5} \quad \Rightarrow \quad (y - 5x) = C_1 \qquad \ldots(2)$$

On taking I and III, we get

$$\frac{dx}{1} = \frac{dz}{9z + \tan C_1}$$

$$\therefore \qquad x = \frac{\log(9z + \tan C_1)}{9} - \log C_2$$

$$9x = \log(9z + \tan(y - 5x)) - \log C_2$$

$$\Rightarrow \qquad \log C_2 = \log[9z + \tan(y - 5x)] - 9x$$

$$\therefore \qquad C_2 = e^{-9x}[9z + \tan(y - 5x)] \qquad \ldots(3)$$

Hence, the desired solution is

$$f\left(e^{-9x}\left[9z + \tan(y - 5x)\right],\ y - 5x\right) = 0.$$

Example 22: Solve $p\cos(x + y) + q\sin(x + y) = z$.

Solution: The subsidiary equations are

$$\underset{\text{I}}{\frac{dx}{\cos(x+y)}} = \underset{\text{II}}{\frac{dy}{\sin(x+y)}} = \underset{\text{III}}{\frac{dz}{z}} = \underset{\text{IV}}{\frac{(dx+dy)}{\cos(x+y)+\sin(x+y)}} = \underset{\text{V}}{\frac{dx-dy}{\cos(x+y)-\sin(x+y)}}$$

Let $\qquad (x + y) = u \quad \Rightarrow \quad (dx + dy) = du$

From III and IV, $\dfrac{dz}{z} = \dfrac{du}{\cos u + \sin u} = \dfrac{du}{\sqrt{2}\sin\left(u + \frac{\pi}{4}\right)}$, since $\sin\left(u + \dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}(\sin u + \cos u)$

$$\Rightarrow \qquad \log z = \frac{1}{\sqrt{2}}\log[\operatorname{cosec} U - \cot U] + \log C_1, \qquad U = u + \frac{\pi}{4}$$

$$\Rightarrow \qquad \log\frac{z}{C_1} = \frac{1}{\sqrt{2}}\log\left[\frac{1}{\sin U} - \frac{\cos U}{\sin U}\right],$$

$$\Rightarrow \qquad \log\frac{z}{C_1} = \log\left[\frac{1 - \cos U}{\sin U}\right] = \log\left[\frac{2\sin^2\frac{U}{2}}{2\sin\frac{U}{2}\cos\frac{U}{2}}\right]$$

$$\Rightarrow \qquad \log\frac{z}{C_1} = \frac{1}{\sqrt{2}}\log\left[\tan\frac{U}{2}\right]$$

$$\Rightarrow \qquad \sqrt{2}\log\frac{z}{C_1} = \log\left[\tan\left(\frac{u}{2} + \frac{\pi}{8}\right)\right], \text{ as } U = u + \frac{\pi}{4}$$

$$\Rightarrow \qquad \frac{z^{\sqrt{2}}}{\tan\left[\frac{u}{2} + \frac{\pi}{8}\right]} = C_3$$

$$\Rightarrow \qquad C_3 = z^{\sqrt{2}}\cot\left[\frac{u}{2} + \frac{\pi}{8}\right] = z^{\sqrt{2}} \cdot \cot\left[\frac{x+y}{2} + \frac{\pi}{8}\right]$$

Further, $\dfrac{dx + dy}{\cos(x+y) + \sin(x+y)} = \dfrac{dx - dy}{\cos(x+y) - \sin(x+y)}$

$$\Rightarrow \qquad \int \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}(dx+dy) = \int (dx-dy)$$

Here LHS is comparable to $\int \frac{f'(x)}{f(x)}dx = \log f(x)$

$\therefore \qquad \log[\cos(x + y) + \sin(x+ y)] = (x - y) + \log C_2$

Rewrite as $\quad \log[\cos(x + y) + \sin(x + y)] = \log e^{(x-y)} + \log C_2$

$\Rightarrow \qquad e^{-(x-y)}[\cos(x + y) + \sin(x + y)] = C_2.$

Hence $f\left(z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right), e^{(y-x)}\{\cos(x+y)+\sin(x+y)\}\right) = 0$

Example 23: Solve the equation $\boldsymbol{z - xp - yq = a\sqrt{x^2+y^2+z^2}}$.

Solution: The subsidiary equations are as follows:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}} = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2) - az\sqrt{x^2+y^2+z^2}}$$

Putting $(x^2 + y^2 + z^2) = u^2$ so that $(x dx + y dy + z dz) = u\, du$

$$\therefore \qquad \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}} = \frac{u\,du}{u^2 - azu} = \frac{du}{u - az}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - au} = \frac{du}{u - az} = \frac{du + dz}{(1-a)(u+z)}$$

Taking $\quad \frac{dx}{x} = \frac{du+dz}{(1-a)(u+z)} \quad$ or $\quad (1-a)\frac{dx}{x} = \frac{du+dz}{(u+z)}$

Integrating $\quad (1 - a)\log x = \log(u + z) + \log C_1$

$$\Rightarrow \qquad x^{(1-a)} = C_1(u+z) = C_1\left\{z + \sqrt{x^2+y^2+z^2}\right\} \qquad \ldots(1)$$

Again, $\quad \frac{dx}{x} = \frac{dy}{y}$

$$\Rightarrow \qquad \log x = \log y + \log C_2 \quad \text{or} \quad \frac{x}{y} = C_2 \qquad \ldots(2)$$

Therefore, general solution is $f(C_1, C_2) = 0$

$$\Rightarrow \qquad x^{(1-a)} = \left\{z + \sqrt{x^2+y^2+z^2}\right\}\phi\left(\frac{x}{y}\right)$$

Example 24: Solve the Lagrange's Linear differential equation

$\boldsymbol{px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)}$ **[KUK, 2007, 2010]**

Solution: On rewriting the given equation, we have

$$px(z - 2y^2) + qy(z - y^2 - 2x^3) = z(z - y^2 - 2x^3)$$

On comparing with $Pp + Qq = R$, we have

$$\left.\begin{aligned} P &= x(z - 2y^2) \\ Q &= y(z - y^2 - 2x^3) \\ R &= z(z - y^2 - 2x^3) \end{aligned}\right\} \qquad \ldots(1)$$

Here subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or
$$\underset{\text{I}}{\frac{dx}{x(z-2y^2)}} = \underset{\text{II}}{\frac{dy}{y(z-y^2-2x^3)}} = \underset{\text{III}}{\frac{dz}{z(z-y^2-2x^3)}} \qquad \ldots(2)$$

On considering II and III, we get

$$\frac{dy}{y} = \frac{dz}{z} \quad \text{or} \quad y = az \qquad \ldots(3)$$

Now, consider I and III and make use of expression (3), we have

$$\frac{dx}{x(z-2a^2z^2)} = \frac{dz}{z(z-a^2z^2-2x^3)}$$

or
$$\frac{dx}{x(1-2a^2z)} = \frac{dz}{(z-a^2z^2-2x^3)}$$

$\Rightarrow \quad (x\,dz - z\,dx) + 2x^3\,dx = 2a^2\,xz\,dz - a^2\,z^2\,dx$

Divide throughout by x^2,

$$\frac{(x\,dz - z\,dx)}{x^2} + 2x\,dx = \frac{a^2(2xz\,dz - z^2\,dx)}{x^2}$$

$$d\left(\frac{z}{x}\right) + d(x^2) = a^2 d\left(\frac{z^2}{x}\right)$$

$$\left(\frac{z}{x} + x^2 - a^2\frac{z^2}{x}\right) = b_2 \quad \text{or} \quad \frac{z}{x} + x^2 - \frac{y^2}{x} = b$$

Hence the required solution is, $f\left(\frac{y}{z}, \frac{z}{x} + x^2 - \frac{y^2}{x}\right) = 0$

Example 25: Solve Lagrange's Linear differential equation

(*i*) $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z - a\sqrt{x^2+y^2+z^2}}$ **(*ii*)** $p(y^2 + z^2 + yz) + q(z^2 + zx + x^2) = (x^2 + xy + y^2)$.

Solution: The corresponding subsidiary equations are

(*i*) $$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-a\sqrt{x^2+y^2+z^2}}=\frac{x\,dx+y\,dy+z\,dz}{(x^2+y^2+z^2)-az\sqrt{x^2+y^2+z^2}} \qquad \ldots(1)$$

Let $x^2+y^2+z^2=t^2$ so that $(x\,dx+y\,dy+z\,dz)=t\,dt$ ...(2)

On using (2), relation (1) becomes

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-at}=\frac{t\,dt}{t^2-azt} \qquad \ldots(3)$$

Now from (3), we have

$$\frac{dx}{x}=\frac{dy}{y} \Rightarrow \log x=\log C_1 y \Rightarrow \frac{x}{y}=C_1 \qquad \ldots(4)$$

and $$\frac{dx}{x}=\frac{dz}{z-at}=\frac{dt}{t-az}=\frac{dt+dz}{(t-az)+(z-at)}$$

$$\Rightarrow \frac{(dt+dz)}{(1-a)(t+z)}=\frac{dx}{x}$$

$$\Rightarrow \log(t+z)=(1-a)\log x+\log C_2$$

$$\Rightarrow \frac{(t+z)}{x^{1-a}}=C_2 \Rightarrow \frac{\sqrt{x^2+y^2+z^2}+z}{x^{1-a}}=C_2$$

Here the solution is $f(C_1, C_2)=0=f\left(\frac{x}{y},\frac{\sqrt{x^2+y^2+z^2}+z}{x^{1-a}}\right)$

(*ii*) The subsidiary equations are

$$\frac{dx}{y^2+yz+z^2}=\frac{dy}{z^2+zx+x^2}=\frac{dz}{x^2+xy+y^2}=\frac{(dy-dx)}{(x-y)(x+y+z)}=\frac{(dz-dy)}{(y-z)(x+y+z)}$$

$$\Rightarrow \frac{(dy-dx)}{(x-y)}=\frac{(dz-dy)}{(y-z)}$$

$$\Rightarrow \log(y-x)=\log(y-z)+\log C_1$$

$$\Rightarrow (y-x)=C_1(y-z) \qquad \ldots(1)$$

Similarly $$\frac{dy-dx}{(x-y)(x+y+z)}=\frac{dz-dx}{(x-y)(x+y+z)}$$

$$\Rightarrow \frac{(dy-dx)}{(x-y)}=\frac{(dz-dx)}{(x-y)}$$

$$\Rightarrow \log(x-y)=\log(x-z)+\log C_2$$

$$\Rightarrow (x-y)=C_2(x-z) \qquad \ldots(2)$$

$\therefore$ The desired solution is $f(C_1, C_2)=0$ or $f\left(\frac{x-y}{x-z},\frac{y-x}{y-z}\right)=0.$

Example 26: $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=au+\dfrac{xy}{z}$.

Solution: Here the subsidiary equations are

$$\underset{\text{I}}{\frac{dx}{x}}=\underset{\text{II}}{\frac{dy}{y}}=\underset{\text{III}}{\frac{dz}{z}}=\underset{\text{IV}}{\frac{du}{au+\frac{xy}{z}}} \qquad \ldots(1)$$

Taking I and II, we get

$$\frac{dx}{x}=\frac{dy}{y} \Rightarrow \frac{y}{x}=C_1 \qquad \ldots(2)$$

Likewise from I & III, we have

$$\frac{z}{x}=C_2 \qquad \ldots(3)$$

Again from I and IV, we get

$$\frac{dx}{x}=\frac{du}{au+\frac{xy}{z}} \Rightarrow \frac{du}{dx}=\frac{au+\frac{xy}{z}}{x}$$

$$\Rightarrow \quad \frac{du}{dx}-\frac{a}{x}u=\frac{y}{z} \quad \text{or} \quad \frac{du}{dx}-\frac{a}{x}u=\frac{C_1}{C_2} \quad \text{(on using (2) and (3))} \qquad \ldots(4)$$

Which is an ordinary linear differential equation with Integrating Factor,

$$\text{I.F.}=e^{-\int\frac{a}{x}dx}=e^{-a\log x}=\frac{1}{x^a} \qquad \ldots(5)$$

$$\therefore \quad u\cdot\frac{1}{x^a}=\frac{C_1}{C_2}\int\frac{1}{x^a}dx+C_3$$

or $$ux^{-a}=\frac{y}{z}\cdot\frac{x^{-a+1}}{-a+1}+C_3$$

or $$ux^{-a}-\frac{y}{z}\cdot\frac{x^{1-a}}{1-a}=C_3 \qquad \ldots(6)$$

Hence the general integral, $f\left(\dfrac{y}{x},\ \dfrac{z}{x},\ ux^{-a}-\dfrac{y}{z}\cdot\dfrac{x^{1-a}}{1-a}\right)=0$

ASSIGNMENT 4

1. Solve $z(xp-yq)=(y^2-x^2)$
2. Solve $(y^3x-2x^4)+(2y^4-x^3y)q=9z(x^3-y^3)$
3. Solve Lagrange's equation $\dfrac{y^2z}{x}p+zxq=y^2$.

4. Solve linear partial differential equation $(mz - ny)p + (nx - lz)q = (ly - mx)$.

5. Find the surface whose tangent planes cut of an intercept of constant length k from the z-axis.

[Hint: Equation of the tangent plane at (x, y, z) is $(Z - z) = (X - x)p + (Y - y)q$.]

6. Solve $$(y + z + u)\frac{\partial u}{\partial x} + (z + x + u)\frac{\partial u}{\partial y} + (x + y + u)\frac{\partial u}{\partial z} = (x + y + z).$$

11.6 NON-LINEAR EQUATIONS OF FIRST ORDER

As already defined, when p and q occur other than in the first degree, the equation is a non-linear one and its general solution contains only two arbitrary constants (viz equal to the number of independent variable *i.e.*, x and y). These equations are discussed under following four standard forms.

I. When Equation Contains p and q Only (i.e., no x, y, z.) – First Standard Form

Let the equation be

$$f(p, q) = 0 \qquad \ldots(1)$$

The obvious complete solution for this equation is

$$z = ax + by + c \qquad \ldots(2)$$

viz replacement of p, q by two arbitrary constants a, b respectively as

$$p = \frac{\partial z}{\partial x} = a \quad \text{and} \quad q = \frac{\partial z}{\partial y} = b \qquad \ldots(3)$$

whence a and b are related by the relation

$$f(a, b) = 0 \qquad \ldots(4)$$

Further (4) gives $b = f(a)$ and with this, the complete solution (2) may be written as

$$z = ax + f(a)y + c \qquad \ldots(5)$$

Example 27: Solve (*i*) $pq + p + q = 0$ (*ii*) $p^3 - q^3 = 0$.

Solution: (*i*) As the given equation falls under the Ist category *i.e.*, $f(p, q) = 0$

whence $f(a, b) = ab + a + b = 0 \qquad \ldots(1)$

$$\Rightarrow \quad (a + 1)b + a = 0 \quad \text{or} \quad b = -\left(\frac{a}{a+1}\right) \qquad \ldots(2)$$

Hence the desired solution,

$$z = ax + by + C$$

$$z = ax + f(a)y + C, \text{ where } b = \mathrm{f}(a) \qquad \ldots(3)$$

$$z = ax - \frac{a}{(a+1)}y + C$$

(*ii*) Here, $f(a, b) = (a^3 - b^3) = 0 \quad \Rightarrow \quad (a - b)(a^2 + b^2 + ab) = 0$

i.e. either $a = b$ or $a^2 + b^2 + ab = 0$

On using, $b = a$ in the obvious solution, $z = ax + by + C$

we get $z = a(x + y) + C$.

Example 28: Solve $x^2p^2 + y^2q^2 = z^2$.

Solution: On rewriting the given equation as

$$\left(\frac{x}{z}\cdot\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \ldots(1)$$

Now let $\dfrac{dx}{x} = dX, \quad \dfrac{dy}{y} = dY, \quad \dfrac{dz}{z} = dZ.$ ...(2)

so that $X = \log x, \quad Y = \log y, \quad Z = \log z$ ...(3)

Further (2) implies

$$\frac{\partial Z}{\partial X} = \frac{x}{z}\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial Z}{\partial Y} = \frac{y}{z}\cdot\frac{\partial z}{\partial y} \qquad \ldots(4)$$

On using (4), (1) becomes

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1 \qquad \ldots(5)$$

where $f(p, q) = 0$ has complete integral as

$$Z = aX + bY + C$$

Here, $f(a, b) = 0 \;\Rightarrow\; a^2 + b^2 = 1$ or $b = \pm\sqrt{1-a^2}$ ...(6)

$\therefore$ $$Z = aX \pm \sqrt{1-a^2}\,Y + \log C$$

or $$\log z = a\log x \pm \sqrt{1-a^2}\log y + \log C, \quad \text{Using (3)}$$

$\Rightarrow$ $z = Cx^a \cdot y^{\sqrt{1-a^2}}$ is the required solution.

II. Equation Containing *p*, *q* and *z* (*i.e.*, no *x* and *y*) – Second Standard Form

The equation is of the form

$$f(p, q, z) = 0 \qquad \ldots(1)$$

Let us assume its solution be

$$z = \phi(u) \quad \text{where} \quad u = x + ay \qquad \ldots(2)$$

with $$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du} \qquad \ldots(3)$$

and $$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Whence the equation (1) reduces to

$$f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0 \qquad \ldots(4)$$

which is clearly a relation in z, $\frac{dz}{du}$ i.e. a first order ordinary linear differential equation, and hence solved by variable seperable method.

Note: Sometimes the equation in its given form is not of the form $f(p, q, z)$ but after certain transformation or substitution it reduces to $f(p, q, z) = 0$.

Example 29: Obtain the complete solution of following equations:

(*i*) $z = p^2 + q^2$ (*ii*) $p(1 + q^2) = q(z - a)$.

Solution: (*i*) Let $u = x + ay$, so that
$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}, \\ q &= \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du} \end{aligned}\right\}$$

With the above values of p and q, the given equation reduces to

$$z = \left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2$$

$$\Rightarrow \quad z = (1+a)^2\left(\frac{dz}{du}\right)^2 \quad \text{or} \quad du = (1+a)^{1/2}\frac{1}{z^{\frac{1}{2}}}dz$$

$\Rightarrow \quad u + b = 2(1 + a^2)^{1/2} z^{1/2}$, b is an arbitary constant

$\Rightarrow \quad (x + ay + b)^2 = 4(1 + a^2)z$

which is the desired solution.

(*ii*) With $u = x + by$ and $p = \frac{dz}{du}$, $q = b\frac{dz}{du}$, the given equation becomes

$$\frac{dz}{du}\left(1+\left(\frac{dz}{du}\right)^2\right) = b\left(\frac{dz}{du}\right)(z-a)$$

$$\Rightarrow \quad 1+\left(\frac{dz}{du}\right)^2 = b(z-a)$$

$$\Rightarrow \quad \left(\frac{dz}{du}\right)^2 = b(z-a)-1$$

$$\Rightarrow \quad \frac{dz}{\sqrt{b(z-a)-1}} = \pm\, du$$

$$\Rightarrow \quad 2\cdot\frac{\sqrt{bz-ba-1}}{b} = \pm\,(u+c)$$

$\Rightarrow \quad \sqrt{bz-ab-1} = \pm\frac{b}{2}(x+by+c)$ is the desired solution.

Example 30: Solve $z^2(p^2 + q^2 + 1) = C^2$.

Solution: Let $u = x + ay$ so that
$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du} \\ q &= \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du} \end{aligned}\right\}$$

Now given equation reduces to

$$z^2\left[\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 + 1\right] = C^2$$

$$\Rightarrow \quad z^2(1+a^2)\left(\frac{dz}{du}\right)^2 = (C^2 - z^2)$$

$$\Rightarrow \quad (1+a^2)^{\frac{1}{2}}\frac{dz}{du} = \pm\frac{\sqrt{C^2 - z^2}}{z}$$

or
$$\frac{z}{\sqrt{C^2 - z^2}}dz = \pm\frac{du}{\sqrt{1+a^2}}$$

On integrating both sides, we get

$$-\sqrt{(C^2 - z^2)} = \pm\frac{u}{\sqrt{1+a^2}} + b \quad \text{or} \quad \sqrt{C^2 - z^2} = \mp\frac{(x+ay)}{\sqrt{1+a^2}} - b$$

Alternately: Put $z\,dz = dZ$ …(1)

$$\Rightarrow \quad \frac{z^2}{2} = Z \quad \text{or} \quad z^2 = 2Z \qquad \text{…(2)}$$

Now
$$\frac{dZ}{dx} = \frac{dZ}{dz}\frac{dz}{dx} = zp \quad \text{(using (1))} \qquad \text{…(3)}$$
$$\frac{dZ}{dy} = \frac{dZ}{dz}\frac{dz}{dx} = zq$$

Therefore, the given equation reduces to

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + 2Z = C^2, \qquad \text{…(4)}$$

which is clearly of the form $f(p, q, z) = 0$

Now, let $Z = f(x + ay) = f(u)$, when $u = x + ay$

and
$$\left.\begin{aligned} \frac{dZ}{dx} &= \frac{dZ}{du}\frac{du}{\partial x} = \frac{dZ}{du}\cdot 1 = \frac{dZ}{du} \\ \frac{dZ}{dy} &= \frac{dZ}{du}\frac{du}{dy} = \frac{dZ}{du}\cdot a = a\frac{dZ}{du} \end{aligned}\right\} \qquad \text{…(5)}$$

Therefore (4) reduces to

$$\left(\frac{dZ}{du}\right)^2(1+a^2)+2Z=C^2$$

$$\Rightarrow \quad (1+a^2)^{1/2}\frac{dZ}{du}=(C^2-2Z)^{1/2}$$

$$\Rightarrow \quad (1+a^2)^{1/2}\frac{dZ}{(C^2-2Z)^{1/2}}=du$$

$$\Rightarrow \quad -\sqrt{1+a^2}\sqrt{C^2-2Z}=u+b$$

$\Rightarrow$ $(1 + a^2)(C^2 - 2z) = (x + ay + b^2)$ which is the desired solution.

Example 31: Solve $p(p^2 + 1) + (b - z)q = 0$. **[KUK, 2005]**

Solution: The equation $p(p^2 + 1) + (b - z)q = 0$ falls under non-linear partial differential equation of the type $f(p, q, z) = 0$

Take $\quad u = (x + ay)$, so that $\dfrac{\partial u}{\partial x}=1,\ \dfrac{\partial u}{\partial y}=a$ …(1)

therefore $\quad p=\dfrac{\partial z}{\partial x}=\dfrac{dz}{du}\dfrac{\partial u}{\partial x}=\dfrac{dz}{du}$

and $\quad q=\dfrac{\partial z}{\partial y}=\dfrac{dz}{du}\dfrac{\partial u}{\partial y}=\dfrac{dz}{du}a$ …(2)

Whence the given equation reduces to

$$\frac{dz}{du}\left\{\left(\frac{dz}{du}\right)^2+1\right\}+a(b-z)\frac{dz}{du}=0$$

or $$\underbrace{\frac{dz}{du}}_{\text{I}}\underbrace{\left[\left(\frac{dz}{du}\right)^2+1+a(b-z)\right]}_{\text{II}}=0 \qquad \ldots(3)$$

From II, $\quad \dfrac{dz}{du}=\sqrt{a(z-b)-1}$ or $\dfrac{dz}{\sqrt{a(z-b)-1}}=du$

$$\Rightarrow \quad \frac{\sqrt{a(z-b)-1}}{a\cdot\frac{1}{2}}=u+c$$

$\Rightarrow$ $2[a(z - b) - 1]^{\frac{1}{2}} = (au + c)$

$\Rightarrow$ $4[a(z - b) - 1] = [a(x + ay) + c]^2$

Example 32: Solve $z^2 = 1 + p^2 + q^2$.

Solution: Let $u = (x + ay)$ so that
$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}, \\ q &= \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du} \end{aligned}\right\}$$

With the above values of p and q, the given equation reduces to

$$z^2 = 1 + \left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2$$

or
$$(z^2 - 1) = (1 + a^2)\left(\frac{dz}{du}\right)^2$$

$\Rightarrow$
$$\sqrt{1+a^2}\,\frac{dz}{du} = \sqrt{z^2 - 1}$$

$\Rightarrow$
$$\int \frac{dz}{\sqrt{z^2-1}} = \frac{1}{\sqrt{1+a^2}}\int du$$

$\Rightarrow$
$$\cosh^{-1} z = \frac{u + c}{\sqrt{1+a^2}}$$

or
$$z = \cosh\left(\frac{x + ay + c}{\sqrt{1+a^2}}\right).$$

III. Variable Separable Form or $f_1(x, p) = f_2(y, q)$ – Third Standard Form

Let each side of this equation be equal to an arbitrary constant *i.e.*,

$$f_1(x, p) = f_2(y, q) = a, \text{ (say)}$$

Solve above relations for p and q,

$$p = F_1(x) \quad \text{and} \quad q = F_2(y),$$

then
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \text{ becomes}$$

$$dz = F_1(x)dx + F_2(y)dy$$

which on integration results in

$$z = \int F_1(x)\,dx + \int F_2(y)\,dy \text{ as the required complete solution.}$$

Example 33: Solve (*i*) $yp + xq + pq = 0$ [MDU, 2009] (*ii*) $yp = 2xy + \log q$.

Solution: (*i*) The given equation can be written like

$$yp + xq = -pq \quad \Rightarrow \quad \frac{y}{q} + \frac{x}{p} = -1$$

or
$$\left(\frac{y}{q}\right) = \left(-1 - \frac{x}{p}\right) = a \text{ (say)} \qquad \ldots(1)$$

So this clearly falls under category $f_1(x, p) = f_2(y, q)$

$$\text{whence} \quad \left.\begin{aligned} \frac{y}{q} &= a, \\ -1 - \frac{x}{p} &= a \end{aligned}\right\} \Rightarrow \left.\begin{aligned} q &= \frac{y}{a}, \\ p &= \left(\frac{x}{-1-a}\right) \end{aligned}\right\} \qquad \ldots(2)$$

Now we know that for $z(x, y)$,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \text{or} \quad dz = \left(\frac{x}{-1-a}\right)dx + \frac{y}{a}dy$$

$$\Rightarrow \qquad z = \frac{x^2}{2(-1-a)} + \frac{y^2}{2a} + c \quad \Rightarrow \quad 2z = -\frac{x^2}{(-1+a)} + \frac{y^2}{a} + b$$

(*ii*) The given equation can be written as

$$p = 2x + \frac{1}{y}\log q$$

$$\Rightarrow \qquad (p - 2x) = \frac{1}{y}\log q = a \qquad \text{(say)} \qquad \ldots(1)$$

which is clearly of the form $f_1(x, p) = f_2(y, q)$

$$\therefore \qquad \left.\begin{aligned} p - 2x &= a, \\ \frac{1}{y}\log q &= a \end{aligned}\right\} \Rightarrow \left.\begin{aligned} p &= (a + 2x), \\ q &= e^{ay} \end{aligned}\right\} \qquad \ldots(2)$$

Whence $\quad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad$ becomes

$$dz = (a + x)dx + e^{ay}\,dy$$

or
$$z = ax + x^2 + \frac{1}{a}e^{ay} + b$$

or $\quad az = ax^2 + a^2x + e^{ay} + ab,\;$ the desired solution of given equation.

Example 34: Solve (*i*) $p + q = \sin x + \sin y$, (*ii*) $\sqrt{p} + \sqrt{q} = x + y$

Solution: (*i*) Rewrite $p + q = \sin x + \sin y$ as:

$$(p - \sin x) = (\sin y - q) = a \text{ (say)}, \qquad \ldots(1)$$

where a is an arbitrary constant.
Now from (1), we have

$$\left.\begin{aligned} p - \sin x &= a, \\ \sin y - q &= a \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \frac{\partial z}{\partial x} &= (a + \sin x), \\ \frac{\partial z}{\partial y} &= (\sin y - a) \end{aligned}\right\} \qquad \ldots(2)$$

Now we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ …(3)

On using (2), $dz = (a + \sin x)\,dx + (\sin y - a)\,dy$

or $z = a(x - y) - (\cos x + \cos y + b)$

as the desired solution.

(*ii*) The given equation can be written as:

$$\sqrt{p} - x = y - \sqrt{q} = a\,(\text{say})$$

$$\Rightarrow \quad \left.\begin{aligned}\sqrt{p} - x = a\\ y - \sqrt{q} = a\end{aligned}\right\} \Rightarrow \left.\begin{aligned}a + x = \sqrt{p}\\ (y - a) = \sqrt{q}\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\frac{\partial z}{\partial x} = (a + x)^2\\ \frac{\partial z}{\partial y} = (y - a)^2\end{aligned}\right\} \qquad \dots(4)$$

Now $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy,$ on using (4) becomes

$$dz = (a + x)^2\,dx + (y - a)^2\,dy$$

$$\Rightarrow \quad 3z = (a + x)^3 + (y - a)^3 + b$$

which is the desired solution.

Example 35: Solve $z(p^2 - q^2) = (x - y)$.

Solution: The given equation can be written as:

$$\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = (x - y) \qquad \dots(1)$$

Putting $\sqrt{z}\,dz = dZ$ so that $Z = \frac{2}{3}z^{\frac{3}{2}}$ …(2)

Thus the given equation reduces to $\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = (x - y)$

or $(P^2 - Q^2) = (x - y)$; where $P = \frac{\partial Z}{\partial x},\ Q = \frac{\partial Z}{\partial y}$

or $(P^2 - x) = (Q^2 - y) = a$ (say), Third Standard Form …(3)

so that $\left.\begin{aligned}(P^2 - x) = a\\ (Q^2 - y) = a\end{aligned}\right\} \Rightarrow \begin{aligned}P = \sqrt{a + x}\\ Q = \sqrt{a + y}\end{aligned}$ …(4)

whence $dZ = P\,dx + Q\,dy$

i.e., $$dZ = \sqrt{a+x}\,dx + \sqrt{a+y}\,dy \qquad \ldots(5)$$

or $$Z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(a+y) + b$$

or $$z^{3/2} = (a + x)^{3/2} + (a + y)^{3/2} + c,$$

where a and c are two arbitrary constants.

IV. Fourth Standard Form: Equation of the Form $z = px + qy + f(p, q)$

The solution of this equation is

$$z = ax + by + f(a, b)$$

which is obtained by replacing p and q by arbitrary constants a and b respectively in the given equation. $\left(\text{i.e., } p = \frac{\partial z}{\partial x} = a \text{ and } q = \frac{\partial z}{\partial y} = b \text{ for } z(x, y)\right)$

This equation is analogous to Clairut's ordinary differential equation $y = px + f(p)$, where $p = \frac{dy}{dx}$ and has solution $y = ax + f(a)$, i.e. replace p by 'a'.

Example 36: Solve $z = px + qy - 2\sqrt{pq}$.

Solution: As the given equation is Fourth standard Form

Whence complete integral of the given equation is

$$z = ax + by - 2\sqrt{ab} \quad \text{for} \quad z(x, y) \qquad \ldots(1)$$

For singular Integral, differentiate (1) partially with respect to a and b, we have

$$\left.\begin{aligned} 0 &= x - \frac{2}{\sqrt{ab}} \cdot b, \\ 0 &= y - \frac{2}{\sqrt{ab}} \cdot a \end{aligned}\right\} \quad \Rightarrow \quad \left.\begin{aligned} x &= \sqrt{\frac{b}{a}}, \\ y &= \sqrt{\frac{a}{b}} \end{aligned}\right\} \qquad \ldots(2)$$

On eliminating a and b, the singular solution is $xy = 1$.

Miscellaneous Problems

Example 37: Obtain the complete solution of the equation
$(x - y)(px - qy) = (p - q)^2$.

Solution: Let $(x + y) = u$ and $xy = v$

so that

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \left(\frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot y\right) \quad \text{as} \quad \frac{\partial u}{\partial x} = 1,\ \frac{\partial v}{\partial x} = y$$

Similarly $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}$

$$= \left(\frac{\partial z}{\partial u}1 + \frac{\partial z}{\partial v}x\right) \quad \text{as} \quad \frac{\partial u}{\partial y} = 1, \ \frac{\partial v}{\partial y} = x$$

Now on substituting the values of $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in the given equation, we get

$$(x-y)\left(x\left(\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right) - y\left(\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)\right) = \left[\left(\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right) - \left(\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)\right]^2$$

$$\Rightarrow \qquad (x-y)\left(x\frac{\partial z}{\partial u} + xy\frac{\partial z}{\partial v} - y\frac{\partial z}{\partial u} - xy\frac{\partial z}{\partial v}\right) = \left[\left(y\frac{\partial z}{\partial v} - x\frac{\partial z}{\partial v}\right)\right]^2$$

$$\Rightarrow \qquad (x-y)(x-y)\frac{\partial z}{\partial u} = (x-y)^2\left(\frac{\partial z}{\partial v}\right)^2$$

$$\Rightarrow \qquad \frac{\partial z}{\partial u} = \left(\frac{\partial z}{\partial v}\right)^2$$

$$\Rightarrow \qquad P - Q^2 = 0 \quad \text{where} \quad P = \frac{\partial z}{\partial u}, \quad Q = \frac{\partial z}{\partial v}$$

This relationship is comparable to Ist category, where $f(p, q) = 0$

Therefore in finding its solution, replace P by a and Q by b in it i.e.

$$a - b^2 = 0 \quad \text{or} \quad a = b^2$$

Whence $\qquad z = au + bv + c$

$$z = a(x + y) + bxy + c$$

$$z = b^2(x + y) + b(xy) + c.$$

Example 38: Solve $pq = x^m y^n z^l$.

Solution: In the given equation, on putting

$$\frac{x^{m+1}}{m+1} = X, \qquad \frac{y^{n+1}}{n+1} = Y \qquad \ldots(1)$$

we get

$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = x^m\frac{\partial z}{\partial X} \\ \text{and} \quad q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = y^n\frac{\partial z}{\partial Y}\end{aligned}\right\} \qquad \ldots(2)$$

On substituting (2), the given equation reduces to

$$\frac{\partial z}{\partial X}\frac{\partial z}{\partial Y} = z^l \quad \text{i.e.} \quad P \cdot Q = z^l \qquad \ldots(3)$$

which is of the form $f(z, p, q) = 0$

$\therefore$ Putting $z = f(u)$ where $u = (X + aY)$

so that
$$\frac{\partial z}{\partial Y} = \frac{dz}{du}\frac{\partial u}{\partial X} = \frac{dz}{du} \qquad \ldots(4)$$

and
$$\frac{\partial z}{\partial Y} = \frac{dz}{du}\frac{\partial u}{\partial Y} = a\frac{dz}{du}$$

Equation (3) becomes,

$$a\left(\frac{dz}{du}\right)^2 = z^l \quad \text{or} \quad z^{-\frac{l}{2}}dz = \frac{1}{\sqrt{a}}du$$

which on integration implies

$$\frac{z^{-\frac{l}{2}+1}}{-\frac{l}{2}+1} = \frac{u}{\sqrt{a}} + b$$

or
$$\frac{z^{-\frac{l}{2}+1}}{-\frac{l}{2}+1} = \frac{1}{\sqrt{a}}\left(\frac{x^{m+1}}{m+1} + a\frac{y^{n+1}}{n+1}\right) + b \qquad \ldots(5)$$

Example 39: Solve $q^2y^2 = z(z - px)$.

Solution: On rewriting the given equation as

$$\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right) \qquad \ldots(1)$$

Let
$$\left.\begin{aligned}\frac{dx}{x} &= dX\\ \frac{dy}{y} &= dY\end{aligned}\right\} \text{ so that } \begin{aligned}X &= \log x\\ Y &= \log y\end{aligned} \qquad \ldots(2)$$

Therefore,
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y}\frac{1}{y}$$

$$\Rightarrow \quad \left.\begin{aligned}y\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial Y}\\ x\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial X}\end{aligned}\right\} \qquad \ldots(3)$$

Likewise, on using (3), (1) becomes

$$\left(\frac{\partial z}{\partial Y}\right)^2 = z\left(z - \frac{\partial z}{\partial X}\right) \qquad \ldots(4)$$

which is of the form $f(p, q, z) = 0$.

The obvious solution for this is

$$z = f(u),\quad u = X + aY \qquad \ldots(5)$$

Therefore
$$\left.\begin{aligned}\frac{\partial z}{\partial X} &= \frac{dz}{du}\frac{\partial u}{\partial X} = \frac{dz}{du}\cdot 1,\\ \frac{\partial z}{\partial Y} &= \frac{dz}{du}\frac{\partial u}{\partial Y} = \frac{dz}{du}a\end{aligned}\right\} \qquad \ldots(6)$$

On using (6), (4) becomes

$$a^2\left(\frac{dz}{du}\right)^2 + z\frac{dz}{du} - z^2 = 0$$

which is a quadratic in $\frac{dz}{du}$ and therefore,

$$\frac{dz}{du} = \frac{-z \pm \sqrt{z^2 + 4a^2z^2}}{2a^2} = \frac{z}{2a^2}\left[-1 \pm \sqrt{1+4a^2}\right]$$

$$\Rightarrow \quad 2a^2\frac{dz}{z} = \left[-1 \pm \sqrt{1+4a^2}\right]du$$

$$\Rightarrow \quad 2a^2 \log z = \left[-1 \pm \sqrt{1+4a^2}\right](u + \log c)$$

$$\Rightarrow \quad 2a^2 \log z = \left[-1 \pm \sqrt{1+4a^2}\right](X + aY + \log c) \quad \text{as} \quad u = X + aY$$

$$\Rightarrow \quad 2a^2 \log z = \left[-1 \pm \sqrt{1+4a^2}\right](\log x + a\log y + \log c)$$

$$\Rightarrow \quad \log z^{2a^2} = \left[-1 \pm \sqrt{1+4a^2}\right](\log xy^a c)$$

$$\Rightarrow \quad z^{2a^2} = \left(xy^a c\right)^{-1 \pm \sqrt{1+4a^2}} \quad \text{as the desired solution.}$$

Example 40: Solve $z^2(p^2 + q^2) = (x^2 + y^2)$.

Solution: On re-writing the above equation as

$$\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = (x^2 + y^2) \qquad \ldots(1)$$

Let $z\,dz = dZ$ so that $Z = \frac{z^2}{2}$ …(2)

Now
$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial x} = P \text{ (say)} \qquad \ldots(3)$$

and
$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y} = z\frac{\partial z}{\partial y} = Q \text{ (say)}$$

Whence the given equation reduces to $P^2 + Q^2 = x^2 + y^2$

$$\Rightarrow \quad (P^2 - x^2) = (y^2 - Q^2) = a \text{ (say)} \qquad \ldots(4)$$

which clearly falls under the category

$$f_1(x, p) = f_2(y, q) = a \text{ (say)}$$

$\therefore$ From (4), we get

and $$\left.\begin{array}{l} P^2 - x^2 = a \\ y^2 - Q^2 = a \end{array}\right\} \Rightarrow \left.\begin{array}{l} P^2 = a + x^2 \\ Q^2 = y^2 - a \end{array}\right\} \qquad \ldots(5)$$

$\therefore$ $$dZ = P\,dx + Q\,dy,$$

$\Rightarrow$ $$dZ = \sqrt{x^2 + a}\,dx + \sqrt{y^2 - a}\,dy$$

$\Rightarrow$ $$Z = \frac{x}{2}\sqrt{x^2 + a} + \frac{a}{2}\log\left[x + \sqrt{x^2 + a}\right]$$

$$+ \frac{y}{2}\sqrt{y^2 - a} - \frac{a}{2}\log\left[y + \sqrt{y^2 - a}\right] + b$$

$\Rightarrow$ $$z^2 = \left(x\sqrt{x^2 + a} + y\sqrt{y^2 - a}\right) + a\log\left(\frac{x + \sqrt{x^2 + a}}{y + \sqrt{y^2 - a}}\right) + 2b$$

which is the desired solution.

Example 41: Solve $\boldsymbol{z = px + qy + c\sqrt{(1 + p^2 + q^2)}}$.

Solution: Clearly this equation is of the form $z = px + qy + f(p, q)$ analogous to Clairaut's form.

$\therefore$ Complete solution is

$$z = ax + by + c\sqrt{(1 + a^2 + b^2)} \qquad \ldots(1)$$

Singular Integral: Differentiating (1) partially with respect to 'a' and 'b' respectively,

$$0 = x + \frac{ac}{\sqrt{(1 + a^2 + b^2)}} \qquad \ldots(2)$$

$$0 = y + \frac{bc}{\sqrt{(1 + a^2 + b^2)}} \qquad \ldots(3)$$

$\therefore$ $$(x^2 + y^2) = \frac{c^2(a^2 + b^2)}{(1 + a^2 + b^2)}$$

$$(x^2 + y^2) = \frac{c^2 + c^2(a^2 + b^2) - c^2}{(1 + a^2 + b^2)}$$

$$(x^2 + y^2) = \frac{c^2 + (1 + a^2 + b^2) - c^2}{(1 + a^2 + b^2)}$$

or $$(c^2 - x^2 - y^2) = \frac{c^2}{(1 + a^2 + b^2)}$$

or $$\frac{(1+a^2+b^2)}{c^2} = \frac{1}{(c^2-x^2-y^2)} \qquad \ldots(4)$$

Also from (2) and (3), $a = -\dfrac{x\sqrt{1+a^2+b^2}}{c} = \dfrac{x}{\sqrt{c^2-x^2-y^2}}$

$$b = \frac{-y\sqrt{1+a^2+b^2}}{c} = -\frac{y}{\sqrt{c^2-x^2-y^2}} \quad \text{(using (4))}$$

Putting these values of 'a' and 'b' in (1), the singular integral is

$$z = \frac{-x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{-c^2-x^2-y^2}{\sqrt{c^2-x^2-y^2}} = \sqrt{c^2-x^2-y^2}$$

or $(x^2 + y^2 + z^2) = c^2$.

ASSIGNMENT 5

Solve **1.** $q^2 = z^2p^2(1 - p^2)$ **2.** $z = px + qy + p^2 q^2$

3. $q(p - \cos x) = \cos y$ **4.** $(pa - p - q)(z - px - qy) = pq.$

11.7 CHARPIT'S METHOD

It is a general method due to Charpit for solving non-linear equations of first order. When it is difficult to solve such equations under any of the standard forms (as discussed in previous article) then this method is employed to find the complete integrals.

Let the given equation be $f(x, y, z, p, q) = 0$ …(1)

If we succeed to find another relation

$$F(x, y, z, p, q) = 0 \qquad \ldots(2)$$

satisfied by p and q, then we can solve equation (1) and (2) for p and q

Since z consists of two independent variables x and y.

$$\therefore \qquad dz = p\,dx + q\,dy,$$

For determining F, differentiate (1) and (2) with respect to x and y respectively giving

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \\ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \end{aligned}\right\} \qquad \ldots(4)$$

$$\left.\begin{aligned}\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0\\ \frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0\end{aligned}\right\}\qquad \ldots(5)$$

Eliminating $\frac{\partial p}{\partial x}$ from the first pair viz (4), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p+\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x}=0 \qquad \ldots(6)$$

Similarly on eliminating $\frac{\partial q}{\partial y}$ from the second pair viz (5), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q+\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y}=0 \qquad \ldots(7)$$

Since $\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial p}{\partial y}$,

whence the last terms in (6) and (7) are the same with opposite signs. Adding (6) and (7), we get

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0 \qquad \ldots(8)$$

Clearly this is Lagrange's equation (linear equation of first order) with x, y, z, p, q as independent variables and F as dependent variable. Thus, identical to Article 4.4, its solution will depend on solution of the subsidiary equations

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0} \qquad \ldots(10)$$

An integral of the above equations (10) which involves p or q or both may be taken as assumed relation (2). The more simple the integrals involving p or q or both derived from (10), the more easy to solve them for p and q and the given equation (1).

Example 42: Using Charpit's method find complete integral of $pxy + pq + qy = yz$. [KUK, 2002-03]

Solution: Here $f(x, y, z, p, q) = pxy + pq + qy - yz = 0$ …(1)
then by Charpit's method, the auxiliary equation is

$$\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dz}{-pf_p-qf_q}=\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{d\phi}{0}$$

or $$\frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)} = \frac{dz}{-p(xy+q)-q(p+y)} = \frac{dp}{py+p(-y)} = dz$$

$$= \frac{dq}{(px+q-z)+q(-y)}$$

implying $dp = 0$ or $p = a$ ...(2)

Putting $p = a$ in (1), $axy + aq + qy = yz$

or $$q(a+y) = y(z-ax)$$

$\therefore$ $$q = \frac{y(z-ax)}{(a+y)} \qquad \text{...(3)}$$

Also we know that for $z(x, y)$,

$$dz = p\,dx + q\,dy \qquad \text{...(4)}$$

On substituting the values of p and q from (2) and (3),

$$dz = a\,dx + \frac{y(z-ax)}{(a+y)}dy$$

or $$\frac{dz - a\,dx}{(z-ax)} = \left(1 - \frac{a}{a+y}\right)dy$$

Integrating,

$$\log(z-ax) = y - a\log(a+y) + \log b$$

or $$(z-ax) = b\,e^{y}(y+a)^{-a}.$$

Example 43: Find complete integral of the equation $p^2x + q^2y = z$. [NIT Kurukshetra, 2008]

Solution: Here $f(x, y, z, p, q) = p^2x + q^2y - z = 0$ ...(1)

By Charpit's method, auxiliary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{d\phi}{0}$$

i.e. $$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dp}{-p+p^2} = \frac{dq}{-q+q^2}$$

From above, we have

$$\frac{(p^2\,dx + 2px\,dp)}{p^2x} = \frac{(q^2\,dy + 2py\,dq)}{q^2y}$$

or $$\frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$$

On integration $x = aq^2y$ ...(2)

where a is an arbitrary constant.

Putting value of $p^2 x$ from (2) in (1),

$$aq^2y + q^2y = z \quad \text{or} \quad q = \left\{\frac{z}{(1+a)y}\right\}^{1/2} \qquad \ldots(3)$$

$\therefore$ From (1), $$p = \left\{\frac{az}{(1+a)x}\right\}^{1/2} \qquad \ldots(4)$$

Thus, $$dz = p\,dx + d\,dy = \left\{\frac{az}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy$$

or $$\sqrt{(1+a)}\,\frac{dz}{z^{1/2}} = \sqrt{a}\,\frac{dx}{x^{1/2}} + \frac{dy}{y^{1/2}}$$

or $$\sqrt{(1+a)\,z} = \sqrt{ax} + \sqrt{y} + b$$

which is required complete integral.

Example 44: Solve the equation $z^2 = pqxy$. **[KUK, 2003-04, 2010]**

Solution: $z^2 = pqxy$ may be written as

$$z^2 - pqxy = 0 \qquad \ldots(1)$$

then by Charpit's method of solving non-linear equations, we have auxiliary equations as:

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{pqxy + qpxy} = \frac{dp}{-pqy + p2z} = \frac{dq}{-pqx + q2z}$$

From above, we get

$$= \frac{x\,dp + p\,dx}{(-pqxy + 2pzx) + pqxy} = \frac{y\,dq + q\,dy}{(-pqxy + 2qzy) + qpxy}$$

i.e. $$\frac{x\,dp + p\,dx}{2pzx} = \frac{y\,dq + q\,dy}{2qzy} \qquad \ldots(2)$$

On integrating both sides,

$$\int \frac{(x\,dp + p\,dx)}{px} = \int \frac{(y\,dq + q\,dy)}{qy}$$

$\Rightarrow$ $$\log px = \log qyc \quad \text{or} \quad px = qyc$$

$\Rightarrow$ $$p = cq\frac{y}{x} \qquad \ldots(3)$$

Substituting this in equation (1), we get

$$z^2 - \left(cq\frac{y}{x}\right)q\,dx = 0 \quad \text{or} \quad q^2 = \frac{1}{c}\left(\frac{z}{y}\right)^2$$

or $$q = b\left(\frac{z}{y}\right) \qquad \ldots(4)$$

Again, putting this value of q in (3) for getting p,

$$z^2 - p\left(b\frac{z}{y}\right)xy = 0$$

$$z^2 - bxzp = 0 \quad \text{or} \quad p = \frac{1}{b}\left(\frac{z}{x}\right) \qquad \ldots(5)$$

Now $$dz = p\,dx + q\,dy = \frac{1}{b}\frac{z}{x}dx + b\frac{z}{y}dy$$

or $$\frac{1}{2}dz = \frac{1}{b}\frac{dx}{x} + b\frac{dy}{y}$$

or $$\log z = \log x^{1/b} + \log y^{b} + \log a$$

or $$z = ax^{1/b}y^{b} = ax^{c^{1/2}} \times y^{c^{-1/2}} \qquad \text{where } b^2 = \frac{1}{c}$$

Example 45: Solve the equation $q + xp = p^2$.

Solution: The given equation is

$$f(x, y, z, p, q) = (q + xp - p^2) = 0 \qquad \ldots(1)$$

Its subsidiary equations are given by

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_q + pf_z} = \frac{dq}{f_y + qf_z} = \frac{d\phi}{0}$$

or $$\frac{dx}{2p - x} = \frac{dy}{-1} = \frac{dz}{2p^2 - xp - q} = \frac{dp}{p + 0} = \frac{dq}{0} = \frac{d\phi}{0}$$

Taking $dq = 0 \Rightarrow q = c$ (constant) $\qquad \ldots(2)$

$\therefore$ $p^2 - xp - q = 0$ becomes $p^2 - xp - c = 0$

i.e. $$p = \frac{x \pm \sqrt{x^2 + 4c}}{2} \qquad \ldots(3)$$

Thus, $dz = (p\,dx + q\,dy)$

$$dz = \frac{x \pm \sqrt{x^2 + 4c}}{2}dx + c\,dy, \text{ using (2) and (3)}$$

$\Rightarrow$ $$z = \int\left(\frac{x}{2} \pm \frac{1}{2}\sqrt{x^2 + (2c^{1/2})^2}\right)dx + c\int dy + d$$

$$z = \frac{x^2}{4} \pm \left\{ \frac{x}{2}\sqrt{x^2+4c} + \frac{2c^{1/2}}{2}\sinh^{-1}\frac{x}{2c^{1/2}} \right\} + cy + d \qquad \ldots(4)$$

$$\left[\because \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a}{2}\sinh^{-1}\frac{x}{a} \right]$$

Alternatively: By another possible combination,

Also $\dfrac{dp}{p} = \dfrac{dy}{-1} \Rightarrow \log p = -y + \log b$ or $p = be^{-y}$ …(5)

Implying $q = p^2 - xp = b^2 e^{-2y} - x be^{-y}$, on using the above value of p. …(6)

$\therefore$ $dz = p\,dx + q\,dy$, using (5) and (6)

$dz = be^{-y}\,dx + (b^2 e^{-2y} - x\,be^{-y})\,dy$

$dz = b(e^{-y}\,dx - xe^{-y}\,dy) + b^2 e^{-2y}\,dy$

$$dz = b \cdot d\left(xe^{-y}\right) - \frac{b^2}{2} \cdot d\left(e^{-2y}\right)$$

Implying, $$z = b\,xe^{-y} - \frac{b^2}{2}e^{-2y} + a$$

Example 46: Solve $p(p^2 + 1) + (b - z)q = 0$.

Solution: Subsidiary equations under Charpit Method are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dF}{0} \qquad \ldots(1)$$

where
$$\left.\begin{aligned} f_p &= 3p^2 + 1 \\ f_q &= (b - z) \\ f_x &= 0 \\ f_y &= 0 \\ f_z &= -q \end{aligned}\right\} \qquad \ldots(2)$$

On using (2), equation (1) becomes

$$\frac{dx}{-(3p^2+1)} = \frac{dy}{-(b-z)} = \frac{dz}{-p(3p^2+1) - q(b-z)} = \frac{dp}{-pq} = \frac{dq}{-q^2} = \frac{dF}{0}$$

$\Rightarrow$ $p = aq$ …(3)

On using (3) in the given equation,

$aq(a^2q^2 + 1) + (b - z)q = 0$

$\Rightarrow$ $q[q^2 a^3 + \{-z + (a + b)\}] = 0$

$\Rightarrow$ either $q = 0$ or $q^2a^3 = (z - (a + b))$

or $$q = \frac{\sqrt{z-(a+b)}}{a\sqrt{a}},\quad p = \frac{\sqrt{z-(a+b)}}{\sqrt{a}} \qquad \ldots(4)$$

$\therefore$ $$dz = pdx + qdy$$

$\Rightarrow$ $$dz = \frac{\sqrt{z-(a+b)}}{\sqrt{a}}dx + \frac{\sqrt{z-(a+b)}}{a\sqrt{a}}dy$$

or $$\frac{dz}{\sqrt{z-(a+b)}} = \frac{dx}{\sqrt{a}} + \frac{dy}{a\sqrt{a}}$$

On integrating, we get

$$\frac{(z-(a+b))^{1/2}}{1/2} = \frac{x}{\sqrt{a}} + \frac{y}{a\sqrt{a}} + b$$

$$2\sqrt{z-(a+b)} = [(ax+y)+C]\,/\,a\sqrt{a}$$

Example 47: Solve 1 + p^2 = qz *by* Charpit's method.

Solution: Given $(1 + p^2 - qz) = 0$ …(1)

Here subsidiary equations are

or $$\frac{dx}{-2p} = \frac{dy}{+z} = \frac{dz}{-2p^2+qz} = \frac{dp}{0+p(-q)} = \frac{dq}{0+q(-q)}$$

Taking $$\frac{dp}{-pq} = \frac{dq}{-q^2}$$

or $$\log p = \log qa$$

$\Rightarrow$ $$p = qa \qquad \ldots(2)$$

Putting this value of p in (1), we have

$$a^2q^2 - zq + 1 = 0 \qquad \ldots(3)$$

which is a quadratic equation in q and on solving,

$$q = \frac{z \pm \sqrt{z^2-4a^2}}{2a^2},\quad p = \frac{1}{2a}\left[z \pm \sqrt{z^2-4a^2}\right]$$

Now $$dz = p\,dx + q\,dy$$

$\Rightarrow$ $$dz = a\left(\frac{z \pm \sqrt{z^2-4a^2}}{2a^2}\right)dx + \frac{\left(z \pm \sqrt{z^2-4a^2}\right)}{2a^2}dy$$

$\Rightarrow$ $$2a^2\frac{dz}{\left(z \pm \sqrt{z^2-4a^2}\right)} = (a\,dx + dy)$$

$$\Rightarrow \qquad 2a^2 \cdot \frac{\left(z \mp \sqrt{z^2 - 4a^2}\right)}{\left(z \pm \sqrt{z^2 - 4a^2}\right)\left(z \mp \sqrt{z^2 - 4a^2}\right)} dz = (a\,dx + dy) \text{ rationalization}$$

$$\Rightarrow \qquad 2a^2 \frac{\left(z \mp \sqrt{z^2 - 4a^2}\right)}{4a^2} dz = (a\,dx + dy) \quad \left[\text{taking} \pm \text{both signs for the term } \left(z \pm \sqrt{z^2 - 4a^1}\right)\right]$$

$$\Rightarrow \qquad \left[z \mp \sqrt{z^2 - 4a^2}\right] dz = 2(a\,dx + dy)$$

On integration (taking ± both signs for the term $\left(z \mp \sqrt{z^2 - 4a^2}\right)$

$$\frac{z^2}{2} \pm \left\{\frac{z}{2}\sqrt{z^2 - 4a^2} - 2a^2 \log\left(z + \sqrt{z^2 - 4a^2}\right)\right\} = 2(ax + y) + b.$$

Example 48: Solve the equation $p(q^2 + 1) + (b - z)q = 0$.

Solution: Here subsidiary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dF}{0}$$

$$\Rightarrow \qquad \frac{dx}{-(q^2 + 1)} = \frac{dy}{-2pq + z} = \frac{dz}{-[p(q^2 + 1) + q(2pq - z)]} = \frac{dp}{-pq} = \frac{dq}{-q^2} = \frac{dF}{0} \qquad \ldots(1)$$

$$\Rightarrow \qquad \frac{dp}{-pq} = \frac{dq}{-q^2}$$

or $\qquad \log p = \log q + \log c$

$\Rightarrow \qquad p = cq \qquad \ldots(2)$

On using (2), given equation becomes

$\qquad cq(q^2 + 1) + (b - z)q = 0$

or $\qquad q[cq^2 + c + b - z] = 0$

$\Rightarrow$ either $\qquad cq^2 + (b + c - z) = 0 \quad$ or $\quad q = 0 \qquad \ldots(3)$

Considering $cq^2 + (b + c - z) = 0$, we get

$$q = \pm\sqrt{\frac{z - (b + c)}{c}}, \quad p = \pm c\sqrt{\frac{z - (b + c)}{c}} \qquad \ldots(4)$$

Now $\qquad dz = p\,dx + q\,dy$

$$\Rightarrow \qquad dz = c\sqrt{\frac{z - (b + c)}{c}}dx + \sqrt{\frac{z - (b + c)}{c}}dy$$

or $\qquad \sqrt{c}\dfrac{dz}{\sqrt{z - (b + c)}} = (c\,dx + dy)$

$$\Rightarrow \qquad \sqrt{c}\,\frac{\sqrt{z-(b+c)}}{1/2} = (cx+y+a)$$

or $\quad 4(cz - bc - c^2) = (cx + y + a)^2.$

ASSIGNMENT 6

Solve

1. $(p^2 + q^2)y = qz$
2. $2zx - px^2 - 2pxy + pq = 0$
3. $2(z + xp + yp) = yp^2$
4. $x^2p^2 + y^2p^2 = z.$M [Hint: Put $X = \log x$, $Y = \log y$, $Z = \sqrt{z}$]
5. $z^2(p^2 + q^2) = x^2 + y^2.$ [Hint: Put $z^2 = Z$]

11.7 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$(D^n + k_1 D^{n-1} D'^1 + k_2 D^{n-2} D'^2 + \ldots + k_n D'^n)z = F(x, y) \qquad \ldots(1)$$

where $D = \dfrac{\partial}{\partial x}$ and $D' = \dfrac{\partial}{\partial y}$ and $k_1, \ldots, k_n$ all constants, is termed as a homogenous linear partial differential equation of nth order with constant coefficients.

Alike to an ordinary linear differential equation, we can rewrite (1) as

$$f(D, D')Z = F(x, y) \qquad \ldots(2)$$

and its complete solution consists of two parts: one is called complementary function and the other is called particular integral.

Complementary function is the solution of the equation

$$f(D, D')Z = 0 \qquad \ldots(3)$$

and particular integral is the particular solution of $f(D, D')Z = F(x, y)$ obtained by giving particular values to the arbitrary constants in the general solution due to $F(x, y)$.

To Find Complementary Functions

Take a simple case of 2nd order homogenous linear equation for finding complementary function and then extend it to higher order.

Let
$$\frac{\partial^2 z}{\partial x^2} + K_1\frac{\partial^2 z}{\partial x \partial y} + K_2\frac{\partial^2 z}{\partial y^2} = 0 \qquad \ldots(1)$$

be a second order equation which can be written in its simplified form as

$$(D^2 + K_1 DD' + K_2 D'^2) = 0 \qquad \ldots(2)$$

with its auxiliary equation as

$$(D^2 + K_1 DD' + K_2 D'^2) = 0 \qquad \ldots(3)$$

giving $D/D' = m_1, m_2$ (say) as two of its roots.

Case I: When the roots are real and distinct

Equation (2) may be written as

$$(D - m_1 D') + (D - m_2 D')z = 0 \qquad \ldots(4)$$

The solution of $(D - m_1 D')z = 0$ will satisfy equation (4),

Now $(D - m, D')z = 0$ *i.e.*, $p - m_1 q = 0$,

Lagrange's linear equation with auxiliary equation, is

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{0}$$

giving $(y + m_1 x) = a$ and $z = b$

$\therefore$ $z = \phi(y + m_1 x)$

Similarly (4) will also be satisfied by (4)

$(D - m_2 D')z = 0$ *i.e.*, $z = \psi(y + m_2 x)$

Hence in this case, the complete solution of (2) is

$$z = \phi(y + m_1 x) + \psi(y + \mathrm{m}_2 x)$$

Case II: When the roots are equal (repeated):

Take $m_1 = m_2 = m$(say), so equation (2) becomes

$$(D - mD')(D - mD')z = 0 \qquad \ldots(5)$$

Let $(D - mD')z = u$, then the above equation reduces to $(D - mD')u = 0$ which is again a Lagrange's linear equation and has solution

$$u = \phi(y + mx)$$

Now with this value of u, equation $(D - mD')z = u$

becomes, $p - mq = \phi(y + mx)$

Its auxiliary equation is $\underset{\text{I}}{\dfrac{dx}{1}} = \underset{\text{II}}{\dfrac{dy}{-m}} = \underset{\text{III}}{\dfrac{dz}{\phi(y + mx)}}$

from which on considering I and II, we get $(y + mx) = a$

and on considering I and II, we get $dz = \phi(a)dx$

i.e., $z = \phi(a)x + b$

or $z = x\,\phi(y + mx) + \phi_1(y + mx)$ which is the complete solution.

Example 49: Solve the following homogenous linear partial differential equations

(*i*) $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0.$ **(*ii*)** $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^4 z}{\partial y^4} = 0.$

(*iii*) $25r - 40s + 16t = 0.$ **(*iv*)** $r = a^2 t.$

Solution: (*i*) Its auxiliary equation is

$$m^4 - 2m^3 + 2m - 1 = 0, \text{ where } D/D' = m$$

$\Rightarrow$ $(m + 1)(m - 1)^3 = 0$ or $m = -1, 1, 1, 1$

Hence the solution (the complementary function) is

$$z = \phi_1(y - x) + \phi_2(y + x) + x\,\phi_3(y + x) + x^2\phi_4(y + x)$$

(*ii*) Here the given equation in its symbolic form is written as

$(D^4 - D'^4)z = 0$

Its auxiliary equation is

$$(m^4 - 1) = 0 \quad \text{or} \quad (m-1)(m+1)(m^2+1) = 0$$

i.e., $m = -1, 1, i, -i$

$\therefore$ The solution

$$z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix).$$

(*iii*) The given equation is symbolic form is as follows:

$$25D^2z - 40DD'z + 16D'^2 z = 0$$

Its auxiliary equation is

$$25(D/D')^2 - 40(D/D') + 16 = 0$$

i.e., $25m^2 - 40m + 16 = 0$ where $D/D' = m$

or $(5m-4)^2 = 0 \quad \Rightarrow \quad m = \dfrac{4}{5}, \dfrac{4}{5}$

which is a case of repeated roots

$\therefore$ The solution is

$$z = \phi_1(5y + 4x) + x\,\phi_2(5y + 4x).$$

(*iv*) The given equation $r = a^2t$ in its symbolic form is

$D^2z - a^2D'^2 z = 0$

Its auxiliary equation is $m^2 - a^2 = 0$

or $m = \pm a$, which is a case of distinct roots.

$\therefore$ $z = \phi_1(y + ax) + \phi(y - ax).$

To Find Particular Integral

Consider the symbolic form of the equation as

$$f(D, D')z = F(x, y) \qquad \ldots(1)$$

For this, Particular Integral $(\text{P.I.}) = \dfrac{1}{f(D, D')}F(x, y)$ …(2)

Case I: When $F(x, y) = e^{ax+by}$

$$\text{P.I.} = \frac{1}{f(D, D')}e^{ax+by} = \frac{1}{f(a, b)}e^{ax+by}, \quad f(a, b) \neq 0. \qquad \ldots(3)$$

(*i.e.*, replace D by a, D' by b)

Case II: When $F(x, y) = \sin(ax + by)$ or $\cos(ax + by)$,

$$\text{P.I.} = \frac{1}{f(D, D')}\sin(ax + by)$$

$$= \frac{1}{f(-a^2, -ab, -b^2)}\sin(ax + by), \text{ Provided } f(-a^2, -ab, -b^2) \neq 0. \qquad \ldots(4)$$

(*i.e.*, replace $D^2 = -a^2$, $D'^2 - b^2$, $DD' = -ab$)

Case III: When $F(x, y) = x^m y^n$, where m, n are positive integer

$$\text{P.I.} = \frac{1}{f(D, D')}x^m y^n = [f(D, D')]^{-1} x^m y^n \qquad \ldots(5)$$

(*i.e.*, expand $[f(D, D')]^{-1}$ in ascending powers of D/D' and operate on $x^m y^n$ term by term.)

Case IV: When $F(x, y)$ is any function of x and y.

$$\text{P.I.} = \frac{1}{f(D, D')} F(x, y) = \frac{1}{(D - m_1 D')(D - m_2 D') \ldots} F(x, y)$$

and $\dfrac{1}{\mathrm{D} - mD'} F(x, y) = \int F(x, c - mx) dx$, where $c = y + mx$.

Case V: When $F(x, y) = e^{ax + by} U(x, y)$,

$$\text{P.I.} = \frac{1}{f(\mathrm{D}, D')} e^{ax+by} \cdot U(x, y) = e^{ax+by} \frac{1}{f(D + a, D' + b)} U(x, y)$$

(*i.e.*, replace D by $D + a$ and D' by $D' + b$ in $f(D, D')$)

Example 50: Solve $(D^3 + D^2 D' - DD'^2 - D'^3)z = e^x \cos 2y$. **[KUK, 2000]**

Solution: The given equation

$$(D^3 + D^2 D' - DD'^2 - D'^3)z = e^x \cos 2y$$

has auxiliary equation as

$$(m^3 + m^2 - m - 1) = 0$$

or $(m - 1)(m + 1)^2 = 0$ or $m = 1, -1, -1$

whence complementary function

$$z = \phi_1(y + x) + \phi_2(y - x) + x\phi_3(y - x)$$

Now
$$\text{P.I.} = \frac{1}{D^3 + D^2 D' - DD'^2 - D'^3} e^x \cos 2y$$

$$= \frac{1}{D^3 + D^2 D' - DD'^2 - D'^3} e^x \left(\frac{e^{2iy} + e^{-2i0ta}}{2} \right)$$

$$= \frac{1}{2} \frac{1}{D^3 + D^2 D' - DD'^2 - D'^3} \{ \underset{\text{I}}{e^{x+2iy}} + \underset{\text{II}}{e^{x-2iy}} \}$$

$$\left(\text{Discuss under,} \quad \text{P.I.} = \frac{1}{f(D, D')} e^{ax+by} = \frac{e^{ax+by}}{f(a, b)}, \ f(a, b) \neq 0 \right)$$

$$= \frac{1}{2} \frac{1}{(1 + 2i + 4 + 8i)} e^{x+2iy} + \frac{1}{2} \frac{1}{(1 - 2i + 4 - 8i)} e^{x-2iy}$$

(as in Ist, $a = 1$, $b = 2i$; in II, $a = 1$, $b = -2i$)

$$= \frac{1}{2} \cdot \frac{1}{5(1 + 2i)} e^{x+2iy} + \frac{1}{2} \cdot \frac{1}{5(1 - 2i)} e^{x-2iy}$$

$$= \frac{1}{2}\cdot\frac{1}{5}\frac{(1-2i)\,e^{x+2iy}}{5(1+2i)(1-2i)} + \frac{1}{2}\cdot\frac{(1+2i)}{5(1-2i)(1+2i)}e^{x-2iy}$$

$$= \frac{1}{25}\left[\frac{(1-2i)\,e^{x+2iy}}{2} + \frac{(1+2i)\,e^{x-2iy}}{2}\right]$$

$$= \frac{e^x}{25}\left[\frac{e^{2iy}+e^{-2iy}}{2} - 2i\,\frac{e^{2iy}-e^{-2iy}}{2}\right]$$

$$\text{P.I.} = \frac{e^x}{25}[\cos 2y + 2\sin 2y]$$

Therefore complete solution

$$z = \phi_1(y+x) + \phi_2(y-x) + x\,\phi_3(y-x) + \frac{e^x}{25}[\cos 2y + 2\sin 2y]$$

Example 51: $\dfrac{\partial^2 y}{\partial t^2} - a^2\dfrac{\partial^2 y}{\partial x^2} = E\sin pt.$

Solution: The given equation in its symbolic form is written as

$$(D^2 - a^2D'^2)y = E\sin pt \qquad \ldots(1)$$

Corresponding A.E. is

$$m^2 - a^2 = 0 \quad i.e., \quad m = \pm a$$

$\therefore$ $$y_{\text{C.F.}} = \phi(t+ax) + \phi_2(t-ax) \qquad \ldots(2)$$

Now $$\text{P.I.} = \frac{1}{D^2 - a^2D'^2}E\sin pt$$

$$\left[\begin{array}{l}\text{using, P.I.} = \dfrac{1}{f(D^2, DD', D'^2)}\sin(ax+by), \text{ replacing } D^2 \text{ by } -a^2, DD\text{' by provided} \\ f(-a^2, -ab, -b^2) \neq 0)\end{array}\right]$$

i.e., $$\text{P.I.} = \frac{E\sin pt}{-p^2} \qquad \ldots(3)$$

$\therefore$ Complete solution $y = \phi_1(t+ax) + \phi_2(t-ax) + \dfrac{E\sin pt}{(-p^2)}$

$$y = \phi_1(t+ax) + \phi_2(t-ax) - E\sin pt/p_2$$

Example 52: Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x\partial y} = E\cos x\cdot\cos 2y.$ **[NIT Kurukshetra, 2003]**

Solution: The given equation can be expressed as

$$(D^2 - DD')z = \cos x\cdot\cos 2y$$

Corresponding A.E. is $m^2 - m = 0$, when $D/D' = m \Rightarrow m = 0,\ m = 1$

whence, C.F. $= \phi_1(y) + \phi_2(y + x)$

Now
$$\text{P.I.} = \frac{1}{(D^2 - DD')}\cos x \cos 2y$$

$$= \frac{1}{2}\frac{1}{D^2 - DD'}[\underset{\text{I}}{\cos(x + 2y)} + \underset{\text{II}}{\cos(x - 2y)}]$$

$$= \frac{1}{2}\left[\frac{1}{-1+2}\cos(x+2y) + \frac{1}{-1-2}\cos(x-2y)\right]$$

$\left(\text{When } \cos(ax + by) \text{ replace } D^2 \text{ by } -a^2,\ D'^2 \text{ by } -b^2,\ DD' = -ab\right)$

$\Rightarrow$
$$\text{P.I.} = \frac{1}{2}\left[\cos(x+2y) - \frac{1}{3}\cos(x-2y)\right]$$

Therefore, complete solution,

$$z = \phi_1(y) + \phi_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Example 53: Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \cos mx \cos ny.$

Find complete solution if $F(x, y) = \cos x \cos 2y$.

Solution: The given equation in its symbolic form is

$$(D^2 + D'^2)z = \cos mx \cos ny = \frac{1}{2}[\cos(mx+ny) + \cos(mx-ny)] \qquad \ldots(1)$$

Corresponding auxiliary equation is

$m^2 + 1 = 0$ *i.e.,* $m = \pm i$

whence C.F. $= \phi_1(y + ix) + \phi_2(y - ix)$ …(2)

Now
$$\text{P.I.} = \frac{1}{2}\frac{1}{(D^2 + D'^2)}[\cos(mx+ny) + \cos(mx-ny)]$$

Replace D^2 by $-m^2$, DD' by $-mn$, D'^2 by $-n^2$;

i.e.,
$$\text{P.I.} = \frac{1}{-2(m^2+n^2)}\cos(mx+ny) + \frac{1}{-2(m^2+n^2)}\cos^2(mx-ny)$$

$$= \frac{1}{-2(m^2+n^2)}[\cos(mx+ny) + \cos(mx-ny)]$$

$\Rightarrow$
$$\text{P.I.} = \frac{-1}{(m^2+n^2)}\cos mx \cos ny$$

Therefore complete solution is

$$z = \phi_1(y + ix) + \phi_2(y - ix) + \frac{-1}{(m^2 + n^2)} \cos mx \cos ny$$

Further if $m = 1$, $n = 2$ as $F(x, y) = \cos x \cos 2y$, we have

$$\text{P.I.} = -\frac{1}{(1^2 + 2^2)} \cos x \cos 2y = -\frac{1}{5} \cos x \cos 2y$$

Hence complete solution in this case is

$$z = \phi_1(y + ix) + \phi_2(y - ix) - \frac{1}{5} \cos x \cos 2y$$

Example 54: Solve $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 4\sin(2x + y)$**.**

Solution: The symbolic form of the above equation is

$$(D^3 - 4D^2D' + 4DD'^2)z = 4\sin(2x + y)$$

Corresponding A.E. is $m^3 - 4m^2 + 4m = 0$

or $m(m - 2)^2 = 0$ *i.e.*, $m = 0, 2, 2$

whence C.F. $= \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$

Now $$\text{P.I.} = \frac{1}{D(D^2 - 4DD' + 4D'^2)} 4\sin(2x + y)a = \frac{4}{D} \cdot \frac{1}{(-4 + 8 - 4)} \sin(2x + y)$$

$$= \frac{4}{D} \cdot \frac{1}{0} \sin(2x + y)$$

Hence method fails.

$\therefore$ $$\text{PI} = \frac{1}{(D + 2D')^2} \cdot \frac{1}{D} 4\sin(2x + y)$$

(Here, differentiate $f(D, D')$ twice with respect to D and multiply the numerator twice by x)

i.e., $$\text{P.I.} = \frac{x^2}{2}\{-2\cos(2x + y)\} = -x^2 \cos(2x + y)$$

$\therefore$ The complete solution is

$$z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) - x^2\cos(2x + y).$$

Example 55: Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(2x + y)$.

Solution: The given equation is a homogenous linear partial differential equation of 2nd order and its symbolic form is

$$(D^2 + DD' - 6D'^2)z = \cos(2x + y) \quad \ldots(1)$$

Auxiliary equation is

$$(m^2 + m - 6) = 0 \quad \text{or} \quad m = 2, -3. \qquad \ldots(2)$$

whence complementary function (C.F.) = $\phi_1(y + 2x) + \phi_2(y - 3x)$ …(3)

$$\text{P.I.} = \frac{1}{(D^2 + DD' - 6D'^2)} \cos(2x + y), \quad \left.\begin{aligned} D^2 &= -4 \\ DD' &= -2 \\ D'^2 &= -1 \end{aligned}\right\}$$

(*i.e.*, on replacing D^2 by a^2, DD' by $-ab$, $D'^2 = -b^2$ provided $\phi(-a^2, -ab, -b^2) \neq 0$)

Clearly it is a case of failure as $\phi(-a^2, -ab, -b^2) = 0$.

Therefore, we discuss it under category IV *i.e.*, factorise $\phi(D, D')$ and apply these factors turn by turn

$$\therefore \quad \text{P.I.} = \frac{1}{(D - 2D')(D + 3D')} \cos(2x + y)$$

$$= \frac{1}{D + 3D'} \int_{D-2D'} \cos(2x + c_1 - 2x)\, dx \quad \text{as here } y + 2x = c_1$$

$$= \frac{1}{D + 3D'} \int (\cos c_1)\, dx$$

$$= \frac{1}{D + 3D'} (x \cos c_1) = \frac{1}{(D + 3D')} x \cos(y + 2x)$$

$$= \int_{D+3D'} x \cos(y + 2x)\, dx = \int x \cos(c_2 + 3x + 2x) dx \quad \because\ y - 3x = c_2$$

$$= \int x \cos(5x + c_2)\, dx = \frac{x \sin(5x + c_2)}{5} + \frac{1}{5} \frac{\cos(5x + c_2)}{5}$$

$$= \frac{x}{5} \sin(2x + y) + \frac{1}{25} \cos(2x + y) \qquad \ldots(4)$$

$\therefore$ Complete Integral z = C.I. + P.I.

$$= \phi_1(y + 2x) + \phi_2(y - 3x) + \frac{x}{5} \sin(2x + y) + \frac{1}{25} \cos(2x + y).$$

Alternately

$$\text{P.I.} = \frac{1}{f(D, D')} F(x, y) = \frac{1}{(D - 2D'(D + 3D')} \cos(2x + y) \qquad \ldots(5)$$

$$= \frac{1}{D'\left(\frac{D}{D'} - 2\right) D'\left(\frac{D}{D'} + 3\right)} \cos(2x + y) \qquad \ldots(6)$$

Now $$\frac{1}{\left(\frac{D}{D'} - 2\right)\left(\frac{D}{D'} + 3\right)} = \frac{1}{(m - 2)(m + 3)'} \quad \text{where } \frac{D}{D'} = m$$

$$= \frac{1}{5}\left[\frac{1}{(m-2)} - \frac{1}{(m+3)}\right] \quad \text{(By Partial Fraction)}$$

$$= \frac{1}{5}\left[\frac{1}{\left(\frac{D}{D'}-2\right)} - \frac{1}{\left(\frac{D}{D'}+3\right)}\right]$$

$$= \frac{D'}{5}\left[\frac{1}{(D-2D')} - \frac{1}{(D+3D')}\right] \qquad \ldots(7)$$

Therefore on substituting (7) in (6), we get

$$\text{P.I.} = \frac{1}{5D'}\left[\frac{1}{(D-2D')} - \frac{1}{(D+3D')}\right]\cos(2x+y)$$

$$= \frac{1}{5D'}\int_{(D-2D')}\cos(2x+y)\,dx - \frac{1}{5}\frac{1}{D'}\int_{D+3D'}\cos(2x+y)dx$$

$$= \frac{1}{5D'}\int\cos(2x+c_1-2x)\,dx - \frac{1}{5D'}\int\cos(2x+c_2+3x)\,dx$$

$$\Rightarrow \qquad = \frac{1}{5D'}x\cos c_1 - \frac{1}{5D'}\frac{\sin(5x+c_2)}{5}$$

$$= \frac{x}{5}\frac{1}{D'}\cos(2x+y) - \frac{1}{25D'}\sin(2x+y) \qquad \text{replacing} \quad \begin{array}{l} c_1 = y+2x \\ c_2 = y-3x \end{array}$$

$$\text{P.I.} = \frac{x}{5}\sin(2x+y) + \frac{1}{25}\cos(2x+y)$$

Example 56: Solve $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 3\dfrac{\partial^2 z}{\partial y^2} = \sqrt{x+3y}$.

Solution: The given equation which is an homogenous linear partial differential equation of 2nd order can be written in its symbolic form as

$$(D^2 - 4DD' + 3D'^2)z = (x+3y)^{1/2} \qquad \ldots(1)$$

Its auxiliary equation is as

$$(m^2 - 4m + 3) = 0 \quad \text{or} \quad m = 1, 3$$

whence $\quad$ C.F. $= \phi_1(y+x) + \phi_2(y+3x)$ $\qquad \ldots(2)$

For Particular Integral,

$$\text{P.I.} = \frac{1}{f(D,D')}F(x,y) = \frac{1}{(D-D')(D-3D')}(x+3y)^{1/2}$$

$$= \frac{1}{(D-3D')} \int_{D-D'} (x+3y)^{1/2} dx$$

$$= \frac{1}{(D-3D')} \int \left[x+3(c_1-x)\right]^{\frac{1}{2}} dx \qquad \text{since } y+x=c_1 \text{ for } m=1$$

$$= \frac{1}{(D-3D')} \int (3c_1-2x)^{\frac{1}{2}} dx$$

$$= \frac{1}{(D-3D')} \left[\frac{(3c_1-2x)^{3/2}}{-2\times\frac{3}{2}}\right]$$

$$= \frac{1}{(D-3D')} \left[-\frac{1}{3}(x+3y)^{3/2}\right], \quad \text{On replacing } c_1 = y+x$$

$$= -\frac{1}{3} \int_{D-3D'} (x+3y)^{3/2} dx$$

$$= -\frac{1}{3} \int (x+3c_2-9x)^{3/2} dx, \quad \text{as } (y+3x)=c_2 \text{ for } m=3$$

$$= -\frac{1}{3} \frac{(3c_2-8x)^{5/2}}{-8\times\frac{5}{2}}$$

$$= \frac{(3c_2-8x)^{5/2}}{60} = \frac{(x+3y)^{5/2}}{60}, \quad \text{replacing } c_2 = y+3x \qquad \ldots(3)$$

Here, complete solution

$$z = \phi_1(y+x) + \phi_2(y+3x) + \frac{(x+3y)^{5/2}}{60}$$

Alternately: $\text{P.I.} = \dfrac{1}{(D-D')(D-3D')}(x+3y)^{1/2}$

$$= \frac{1}{D'^2} \frac{1}{\left(\frac{D}{D'}-1\right)\left(\frac{D}{D'}-3\right)}(x+3y)^{1/2} \qquad \ldots(4)$$

Considering $\dfrac{1}{\left(\frac{D}{D'}-1\right)\left(\frac{D}{D'}-3\right)} = \dfrac{1}{(m-1)(m-3)}$, when $\dfrac{D}{D'} = m$

$$= \frac{1}{2}\left[\frac{1}{(m-3)} - \frac{1}{(m-1)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\left(\frac{D}{D'}-3\right)} - \frac{1}{\left(\frac{D}{D'}-1\right)}\right]$$

$$= \frac{D'}{2}\left[\frac{1}{D-3D'} - \frac{1}{D-D'}\right] \qquad \ldots(5)$$

Therefore using (5), we have

$$\text{P.I.} = \frac{1}{2D'}\left[\frac{1}{D-3D'} - \frac{1}{D-D'}\right](x+3y)^{1/2}$$

$$= \frac{1}{2D'}\int_{D-3D'}(x+3y)^{1/2}dx - \frac{1}{2D'}\int_{D-D'}(x+3y)^{1/2}dx$$

$$= \frac{1}{2D'}\int(x+3c_2-9x)^{1/2}dx - \frac{1}{2D'}\int(x+3c_1-3x)^{1/2}dx$$

$$= \frac{1}{2D'}\int(3c_2-8x)^{1/2}dx - \frac{1}{2D'}\int(3c_1-2x)^{1/2}dx$$

$$= \frac{1}{2D'}\frac{(3c_2-8x)^{3/2}}{-8\times\frac{3}{2}} - \frac{1}{2D'}\frac{(3c_1-2x)^{3/2}}{-2\times\frac{3}{2}}$$

$$= -\frac{1}{24}\frac{1}{D'}(3c_2-8x)^{3/2} + \frac{1}{6}\frac{1}{D'}(3c_1-2x)^{3/2}$$

$$\text{P.I.} = -\frac{1}{24}\frac{1}{D'}(x+3y)^{3/2} + \frac{1}{6}\frac{1}{D'}(x+3y)^{3/2}, \qquad (\text{as } c_2 = y+3x,\ c_1 = y+x)$$

$$= \frac{3}{24}\frac{1}{D'}(x+3y)^{3/2}$$

$$= \frac{3}{24}\int(x+3y)^{3/2}dy = \frac{3}{24}\frac{(x+3y)^{5/2}}{3\times\frac{5}{2}}$$

$$\Rightarrow \qquad \text{P.I.} = \frac{(x+3y)^{5/2}}{60}$$

Hence the complete solution

$$z = \phi_1(y+x) + \phi_2(y+3x) + \frac{(x+3y)^{5/2}}{60}.$$

Example 57: Solve the equation $4r + 12s + 9t = e^{3x-2y}$.

Solution: For $z(x, y)$, we know that $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$, $t = \dfrac{\partial^2 z}{\partial y^2}$ for z as a function of x and y.

So the given equation becomes

$$4\frac{\partial^2 z}{\partial x^2} + 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = e^{(3x-2y)}$$

or $(4D^2 + 12DD' + 9D'^2)z = e^{(3x-2y)}$ ...(1)

where $D = \dfrac{\partial}{\partial x}$ and $D' = \dfrac{\partial}{\partial y}$

Now for equation (1) A.E. is as follows:

$4m^2 + 12m + 9 = 0$, where $D/D' = m$ ...(2)

or $(2m + 3)^2 = 0$ *i.e.*, $m = -\dfrac{3}{2}, -\dfrac{3}{2}$

whence C.F. $= \phi_1(2y - 3x) + x\phi_2(2y - 3x)$ $[\because \; y + mx = c]$

Now for obtaining particular integral, we have

$$\text{P.I.}(z) = \frac{1}{f(D, D')} e^{ax+by} = \frac{e^{ax+by}}{f(a, b)}, \text{ provided } f(a, b) \neq 0$$

Here $z = \dfrac{1}{4D^2 + 12DD' + 9D'^2} e^{(3x-2y)}$ is clearly a case of failure as $D = a = 3$ and $D' = b = -2$, $f(a, b) = 0$ for

Therefore,

$$\text{P.I.} = \frac{1}{4\left(D + \frac{3}{2}D'\right)^2} e^{3x-2y}$$

$$= \frac{1}{4\left(D + \frac{3}{2}D'\right)} \int_{\left(D+\frac{3}{2}D'\right)} e^{3x-2y} dx$$

$$= \frac{1}{4\left(D + \frac{3}{2}D'\right)} \int e^{2c} dx \quad \text{as} \quad y = c - \frac{3}{2}x$$

$$= \frac{1}{4\left(D + \frac{3}{2}D'\right)} x e^{2c} = \frac{1}{4\left(D + \frac{3}{2}D'\right)} x e^{3x-2y}$$

$$= \frac{1}{4} \int_{\left(D+\frac{3}{2}D'\right)} x e^{3x-2y} dx$$

$$= \frac{1}{4} \int x\, e^{2c}\, dx = \frac{1}{4} \cdot \frac{x^2}{2} \cdot e^{2c}$$

$\Rightarrow$ $$\text{P.I.} = \frac{x^2}{8} e^{3x-2y}; \text{ replace, } c = \left(y + \frac{3}{2}x\right)$$

Therefore the complete solution

$$z = \phi_1(2y - 3x) + x\,\phi_2(2y - 3x) + \frac{x^2}{8} e^{3x-2y}$$

Example 58: Solve $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 5\dfrac{\partial^3 z}{\partial x \partial y^2} - 2\dfrac{\partial^3 z}{\partial y^3} = e^{2x+y}$.

Solution: The symbolic form of the above equation is

$$(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{2x+y}$$

The corresponding A.E. is $m^3 - 4m^2 + 5m - 2 = 0$

i.e., $(m - 2)(m^2 - 2m + 1) = 0$

or $(m - 2)(m - 1)(m - 1) = 0$ *i.e.*, $m = 1, 1, 2$

$\therefore$ $$\text{C.F.} = \phi_1(y + x) + x\,\phi_2(y + x) + \phi_3(y + 2x)$$

Now $$\text{P.I.} = \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} e^{2x+y}$$

It is a case of failure as $f(a, b) = 0$ for $D = a = 2$ and $D' = b = 1$

Therefore,

$$\text{P.I.} = \frac{1}{(D - 2D')(D - D')^2} e^{2x+y}$$

$$\text{P.I.} = \frac{1}{D - 2D'} \cdot \frac{1}{(2-1)^2} e^{2x+y}$$

$$= \frac{1}{D - 2D'} e^{2x+y}$$

$$= \int e^{2x+c-2x}\, dx, \text{ as } y + 2x = c$$

$$= \int e^c\, dx = x\, e^c$$

$$= x\, e^{y+2x}$$

$\Rightarrow$ $$\text{P.I.} = x\, e^{2x+y}$$

Therefore the complete solution,

$$z = \phi_1(y + x) + x\,\phi_2(y + x) + \phi_3(y + 2x) + x\, e^{2x+y}.$$

Example 59: Solve $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$.

Solution: Corresponding A.E. is

$$(m^2 - m - 2) = 0 \quad i.e., \quad (m - 2)(m + 1) = 0 \qquad \ldots(1)$$

or $\qquad m = 2, -1$

whence $\qquad \text{C.F.} = \phi_1(y + 2x) + \phi_2(y - x) \qquad \ldots(2)$

Now

$$\text{P.I.} = \frac{1}{D^2 - DD' - 2D'^2}(y-1)e^x$$

$$= \frac{1}{(D - 2D')(D + D')}(y-1)e^x$$

$$= \frac{1}{(D - 2D')}\int_{D+D'}(y-1)e^x\,dx$$

Corresponding to the factor $D + D'$, $y = c_1 + x$

$$\Rightarrow \qquad \text{P.I.} = \frac{1}{(D - 2D')}\int (c_1 + x - 1)e^x\,dx$$

$$= \frac{1}{(D - 2D')}\int \left((c_1 - 1)e^x + x\,e^x\right)dx$$

$$= \frac{1}{D - 2D'}\left[(c_1 - 1)e^x + (x - 1)\,e^x\right]$$

$$= \frac{1}{D - 2D'}\left[(y - 2)\,e^x\right]; \quad \text{replacing, } c_1 = (y - x)$$

$$\text{P.I.} = \int_{D-2D'}(y-2)\,e^x\,dx$$

Expressing $(y - 2)$ in terms of x as $y + 2x = c_2$ corresponding to the factor $(D - 2D')$

$$\Rightarrow \qquad \text{P.I.} = \int \left(c_2 - 2x - 2\right)e^x dx = \int \left((c_2 - 2) - 2x\right)e^x dx$$

$$= (c_2 - 2)\,e^x - 2\,(x - 1)\,e^x$$

$$\text{P.I.} = ye^x, \text{ on replacing } c_2 \text{ by } (y + 2x)$$

Therefore complete solution $z = \phi_1(y + 2x) + \phi_2(y - x) + ye^x$.

Example 60: Solve $(r + s - 6t) = y\cos x$.

Solution: The given equation can be written as

$$(D^2 + DD' - 6D'^2)z = y\cos x \qquad \ldots(1)$$

Corresponding A.E. is $(m^2 + m - 6) = 0$

$\Rightarrow \quad (m + 3)(m - 2) = 0 \quad i.e., \quad m = -3, +2$

$\therefore \qquad \text{C.F.} = \phi_1(y - 3x) + \phi_2(y + 2x)$

Now $$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\, y\cos x$$

$$= \frac{1}{(D+3D')(D-2D')}\, y\cos x$$

Now apply these operators *viz.*, $(D + 3D')$ and $(D - 2D')$ turn by turn on $y \cos x$ as a function of x only

$$= \frac{1}{(D+3D')} \int_{D-2D'} y\cos x\, dx$$

$$= \frac{1}{(D+3D')} \int (c_1 - 2x)\cos x\, dx, \qquad \text{Here } y + 2x = c_1$$

$$= \frac{1}{(D+3D')}\left[(c_1 - 2x)\sin x - 2\cos x\right] \quad \text{replace } c_1 = y + 2x$$

$$= \frac{1}{(D+3D')}(y\sin x - 2\cos x)$$

$$= \int_{D+3D'} y\sin x\, dx - \int_{D+3D'} 2\cos x\, dx$$

$$= \int_{D+3D'} (c_2 + 3x)\sin x\, dx - 2\int_{D+3D'} \cos x\, dx \quad y - 3x = c_2$$

$$= -(c_2 + 3x)\cos x - \int 3(-\cos x)\, dx - 2\int \cos x\, dx$$

$$\text{P.I.} = -y\cos x + \int \cos x\, dx = -y\cos x + \sin x, \quad \text{replacing } (c_2 + 3x) \text{ by } y$$

$\therefore$ $z = \phi_1(y - 3x) + \phi_2(y + 2x) - y\cos x + \sin x$ as complete solution.

Example 61: Solve $(D^2 + 3DD' + 2D'^2)z = 24xy$.

Solution: Here Auxiliary Equation (A.E.) is

$(m^2 + 3m + 2) = 0$ where $m = D/D'$ …(1)

$\Rightarrow$ $m^2 + 2m + m + 2 = 0$

$\Rightarrow$ $(m + 2)(m + 1) = 0$ *i.e.*, $m = -1, -2$ …(2)

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y - 2x)$ …(3)

Now $$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}\, 24xy$$

$$= \frac{1}{D^2\left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]}\, 24xy$$

$$= \frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+2\frac{D'^2}{D^2}\right)\right]^{-1} 24xy$$

$$= \frac{1}{D^2}\left[1+1-3\frac{D'}{D}\right]24xy$$

(On leaving higher order terms of $\frac{D'}{D}$ since in the term xy, y is in power one only)

$$= \frac{1}{D^2}24xy - 3\cdot\frac{D'}{D^3}24xy$$

$\Rightarrow$ P.I. $= 4x^3y - 3x^4$

Therefore the complete solution is

$$z = \phi_1(y-x) + \phi_2(y-2x) + 4x^3y - 3x^4.$$

Example 62: Solve $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3y^3$. **[NIT Kurukshetra, 2007]**

Solution: The given equation can be written as

$$(D^3 - D'^3)z = x^3y^3 \qquad \ldots(1)$$

Corresponding auxiliary equation is

$$m^3 - 1 = 0 \quad i.e., \quad (m-1)(m^2+m+1) = 0$$

or $(m-1)(m-w)(m-w^2) = 0$, for w to be the cube root of unity …(2)

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+wx) + \phi_3(y+w^2x)$ …(3)

Now

$$\text{P.I.} = \frac{1}{D^3 - D'^3}x^3y^3 = \frac{1}{D^3\left[1-\left(\frac{D'}{D}\right)^3\right]}x^3y^3$$

$$= \frac{1}{D^3}\left[1-\left(\frac{D'}{D}\right)^3\right]^{-1}x^3y^3 = \frac{1}{D^3}\left[1+\frac{D'^3}{D^3}+\ldots\right]x^3y^3$$

$$= \frac{1}{D^3}x^3y^3 + \frac{D'^3}{D^6}x^3y^3$$

(On taking $\frac{D'}{D}$ to the powers to which y is appearing in $F(x, y) = x^ny^m$ i.e. neglecting terms containing powers more than 3 in D')

$$= \frac{x^6y^3}{4\cdot5\cdot6} + \frac{6x^9}{4\cdot5\cdot6\cdot7\cdot8\cdot9}$$

$$\text{P.I.} = \frac{x^6y^3}{120} + \frac{x^9}{10080}$$

Hence complete solution

$$z = \phi_1(y + x) + \phi_2(y + wx) + \phi^3(y + w^2x) + \frac{x^6y^3}{120} + \frac{x^9}{10080}$$

Example 63: Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = x^2 \sin(x + y)$.

Solution: The given equation in its symbolic form is written as

$$(D^2 + DD' - 6D'^2)z = x^2 \sin(x + y) \qquad \ldots(1)$$

$\sin(x + y)$ = imaginary part of $e^{i(x+y)}$; since we know that $e^{i\theta} = \underbrace{\cos\theta}_{\text{Real Part}} + \underbrace{i\sin\theta}_{\text{Imaginary Part}}$

$\therefore \qquad f(D, D')z = I_m.\ e^{i(x+y)} \cdot x^2$

Its auxiliary equation is $m^2 + m - 6 = 0$ *i.e.*, $m = 2, -3$

$$\therefore \qquad \text{C.F.} = \phi_1(y + 2x) + \phi_2(y - 3x) \qquad \ldots(2)$$

Now

$$\text{P.I.} = \text{Im}\cdot \frac{1}{D^2 + DD' - 6D'^2} e^{i(x+y)} \cdot x^2$$

$$= \text{Im}\cdot e^{i(x+y)} \cdot x^2 \frac{1}{(D+i)^2 + (D+i)(D'+i) - 6(D'+i)^2} x^2$$

$$= \text{Im}\cdot e^{i(x+y)}\ x^2 \frac{1}{D^2 + 3iD + DD' - 11iD' - 6D'^2 + 4} x^2$$

$$\text{P.I.} = \text{Im}\cdot e^{i(x+y)} \frac{1}{4}\left[1 + \left(\frac{D^2}{4} + \frac{3iD}{4}\frac{DD'}{4} - \frac{11}{4}iD' - \frac{6}{4}D'^2\right)\right]^{-1} x^2$$

$$= \text{Im}\cdot e^{i(x+y)} \frac{1}{4}\left[1 - \left(\frac{D^2}{4} + \frac{3i}{4}D + \frac{DD'}{4} - \frac{11}{4}iD' - \frac{6}{4}D'^2 + \frac{9}{16}i^2D^2\right)\right] x^2$$

$$= \text{Im}\cdot e^{i(x+y)} \frac{1}{4}\left[x^2 - \frac{1}{2} - \frac{3i}{4}2x - \frac{9}{8}\right]$$

$$= \text{Im}\cdot \frac{1}{4}\left[\left(x^2 - \frac{13}{8}\right) - \frac{3i}{2}x\right][\cos(x + y) + i\sin(x + y)]$$

$$\text{P.I.} = \left[\frac{1}{4}\left(x^2 - \frac{13}{8}\right)\sin(x + y) - \frac{3}{8}x\cos(x + y)\right] \qquad \ldots(3)$$

Hence the complete solution

$$z = \phi_1(y + 2x) + \phi_2(y - 3x) + \frac{1}{4}\left(x^2 - \frac{13}{8}\right)\sin(x + y) - \left(\frac{3}{8}\right)x\ \cos(x + y)$$

MISCELLANEOUS PROBLEMS

Example 64: Solve $4r - 4s + t = 16 \log(x + 2y)$.

Solution: The symbolic form of the above equation is

$$(4D^2 - 4DD' + D'^2)z = 16\log(x + 2y) \qquad \ldots(1)$$

Corresponding A.E. is $4m^2 - 4m + 1 = 0$

i.e., $(2m - 1)^2 = 0$ or $m = \frac{1}{2}, \frac{1}{2}$

$\therefore$ C.F. $= \phi_1(2y + x) + x\phi_2(2y + x)$

$$\text{P.I.} = \frac{1}{4\left(D - \frac{1}{2}D'\right)^2} \cdot 16 \log(x + 2y)$$

$$= 4 \cdot \frac{1}{D - \frac{1}{2}D'} \int \log 2c \, dx \quad \text{as } (2y + x) = c$$

$$= 4 \cdot \frac{1}{D - \frac{1}{2}D'} \cdot x \log(x + 2y)$$

$$= 4 \cdot \int x \log 2c \, dx$$

$$= 4 \cdot \frac{x^2}{2} \cdot \log 2c$$

$$= 2x^2 \log 2c(x + 2y)$$

Hence the complete solution,

$$z = \phi_1(y + 2x) + x\phi_2(y + 2x) + 2x^2 \log(x + 2y).$$

Example 65: Find a real function V of x and y reducing to zero when $y = 0$ and satisfying $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + 4\pi(x^2 + y^2) = 0$.

Solution: Here the given function $V(x, y)$ will be obtainable from the Particular Integral only, since C.F. $= \phi_1(y + ix) + \phi_2(y - ix)$ is simply an imaginary function, if $y = 0$.

Now
$$\text{P.I.} = \frac{1}{D^2 + D'^2}\{-4\pi(x^2 + y^2)\}$$

$$= \frac{1}{D^2\left(1 + \frac{D'^2}{D^2}\right)}\{-4\pi(x^2 + y^2)\}$$

$$= \frac{-4\pi}{D^2}\left[\left(1 + \frac{D'^2}{D^2}\right)^{-1}\right](x^2 + y^2)$$

$$= \frac{-4\pi}{D^2}\left[1 - \frac{D'^2}{D^2}\right](x^2 + y^2), \quad \text{On neglecting higher powers of } \frac{D'^2}{D^2}$$

$$= -4\pi\left\{\frac{1}{D^2}(x^2 + y^2) - \frac{D'^2}{D^4}(x^2 + y^2)\right\}$$

$$= -\pi\left[4\left(\frac{x^4}{3\cdot 4} + \frac{y^2x^2}{1\cdot 2}\right) - 4\times 2\times \frac{x^4}{1\cdot 2\cdot 3\cdot 4}\right]$$

$$= -\frac{\pi x^4}{3} - 2\pi x^2y^2 + \pi\frac{x^4}{3}$$

$$\text{P.I.} = -2\pi x^2y^2$$

$$\therefore \quad V(x, y) = -2\pi x^2y^2.$$

Example 66: Solve $r - t = \tan^3 x \tan y - \tan x \tan^3 y$.

Solution: The given equation can be written in symbolic form as

$$(D^2 - D'^2)z = \tan^3 x \tan y - \tan x \tan^3 y \qquad \ldots(1)$$

Corresponding auxiliary equation is

$$(m^2 - 1) = 0 \quad i.e., \quad m = 1, -1$$

$$\therefore \quad \text{C.F.} = \phi_1(y + x) + \phi_2(y - x) \qquad \ldots(2)$$

Now $\quad \text{P.I.} = \dfrac{1}{D^2 - D'^2}(\tan^3 x\tan y - \tan x\tan^3 y)$

$$= \frac{1}{(D + D')(D - D')}(\tan^3 x\tan y - \tan x\tan^3 y)$$

$$= \frac{1}{(D + D')}\int_{D-D'}(\tan^3 x\tan(c_1 - x) - \tan x\tan^3(c_1 - x))\,dx \quad \text{here } (y + x) = c_1$$

$$= \frac{1}{(D + D')}\int\left[(\sec^2 x - 1)\tan x\tan(c_1 - x) - \tan x\tan(c - x)\left\{\sec^2(c_1 - x) - 1\right\}\right]dx$$

$$= \frac{1}{(D + D')}\int\left[\tan(c_1 - x)\tan x\sec^2 x - \tan x\tan(c_1 - x)\sec^2(c_1 - x)\right]dx$$

$$= \frac{1}{D + D'}\int\underset{\text{I}}{\tan(c_1 - x)}\,\underset{\text{II}}{\tan x\sec^2 x}\,dx - \frac{1}{D + D'}\int\underset{\text{I}}{\tan x}\,\underset{\text{II}}{\tan(c_1 - x)\sec^2(c_1 - x)}\,dx$$

[Integration by parts *viz*., Ist function Int. II $-\int$(Diff. Ist) (Int. IInd) dx]

$$= \frac{1}{D+D'}\left[\tan(c_1 - x)\,\frac{\tan^2 x}{2} + \frac{1}{2}\int \sec^2(c_1 - x)\,\tan^2 x\,dx\right]$$

$$+\frac{1}{D+D'}\left[\tan x\,\frac{\tan^2(c_1 - x)}{2} - \frac{1}{2}\int \sec^2 x\,\tan^2(c_1 - x)dx\right]$$

$$\left[\text{since } d\left(\frac{\tan^2 x}{2}\right) = \tan x \sec^2 x \text{ and } d\left(\frac{\tan^2(c_1 - x)}{2}\right) = \tan(c_1 - x)\sec^2(c_1 - x)(-1)\right]$$

$$\Rightarrow \qquad \text{P.I.} = \frac{1}{2(D+D')}\Big[\tan^2 x \tan(c_1 - x) + \tan x \tan^2(c_1 - x)$$

$$+\int\left\{(\sec^2 x - 1)\sec^2(c_1 - x) - \sec^2 x\left(\sec^2(c_1 - x) - 1\right)dx\right\}\Big]$$

$$= \frac{1}{2(D+D')}\left[\tan^2 x\cdot\tan(c_1 - x) + \tan x\cdot\tan^2(c_1 - x) + \int\left\{\sec^2 x - \sec^2(c_1 - x)\right\}dx\right],$$

$$= \frac{1}{2(D+D')}\left[\tan^2 x\cdot\tan y + \tan x\cdot\tan^2 y + (\tan x + \tan y)\right],$$

(on replacing $(c_1 - x)$ by y in all terms)

$$= \frac{1}{2(D+D')}\left[\tan^2 x\cdot\tan y + \tan y + \tan x + \tan x\cdot\tan^2 y\right]$$

$$= \frac{1}{2(D+D')}\left[\tan y\sec^2 x + \tan x\sec^2 y\right]$$

$$= \frac{1}{2}\int_{D+D'}\left\{\tan(c_2 + x)\sec^2 x + \tan x\sec^2(c_2 + x)\right\}dx$$

(As here $y - x = c_2$ for the factor $D + D'$ in $f(D, D')$)

$$= \frac{1}{2}\left[\int \tan(c_2 + x)\,\sec^2 x\,dx + \int \underset{\text{I}}{\tan x}\,\underset{\text{II}}{\sec^2(c_2 + x)}dx\right],$$

(i.e. taking up 2nd integral only)

$$= \frac{1}{2}\left[\int \tan(c_2 + x)\,\sec^2 x\,dx + \tan x\,\tan(c_2 + x) - \int \sec^2 x\,\tan(c_2 + x)dx\right]$$

$$= \frac{1}{2}\tan x\,\tan(c_2 + x)$$

$$\text{P.I.} = \frac{1}{2}\tan x\cdot\tan y \qquad \text{(replacing } c_2 + x\text{, by } y\text{)}$$

Therefore complete solution,

$$z = \phi_1(y+x) + \phi_2(y-x) + \frac{1}{2}\tan x \cdot \tan y.$$

Example 67: A surface is drawn satisfying $r + t = 0$ and touching $x^2 + z^2 = 1$ along its section by $y = 0$. Obtain its equation in the form $z^2(x^2 + z^2 - 1) = y^2(x^2 + z^2)$.

Solution: The symbolic form of the given equation is

$$(D^2 + D'^2)z = 0 \quad \text{or} \quad (D + iD')(D - iD')z = 0 \qquad \ldots(1)$$

$$\therefore \qquad \text{C.F. } (z) = \phi_1(y + ix) + \phi_2(y - ix) \qquad \ldots(2)$$

Now we know that

$$p = \frac{\partial z}{\partial x} = i\phi_1'(y+ix) - i\phi_2'(y-ix) \qquad \ldots(3)$$

and $$q = \frac{\partial z}{\partial y} = \phi_1'(y+ix) + \phi_2'(y-ix) \qquad \ldots(4)$$

Also from the given $x^2 + z^2 = 1$ or $z = \sqrt{1-x^2}$ …(5)

Thus, $$p = \frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1-x^2}}$$

and $$q = \frac{\partial z}{\partial y} = 0 \qquad \ldots(6)$$

Now under the given condition that at $y = 0$, the surface touches $x_2 + z_2 = 1$

(*i.e.*, at $y = 0$, the tangents $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$ must be equal)

Precisely,

$$\left.\begin{aligned} [i\phi_1'(y+ix) - i\phi_2'(y-ix)]_{y=0} &= -\frac{x}{\sqrt{1-x^2}} \\ \text{and} \quad [\phi_1'(y+ix) + \phi_2'(y-ix)]_{y=0} &= 0 \end{aligned}\right\} \qquad \ldots(7)$$

or $$2i\phi_1'(y+ix) = \frac{-x}{\sqrt{1-x^2}} \quad \text{or} \quad \phi_1'(y+ix) = \frac{x}{2\sqrt{1+(ix)^2}}, \; y = 0$$

or $$\phi_1'(y+ix) = \frac{y+ix}{2\sqrt{1+(y+ix)^2}} \qquad \text{(no matter, since } y = 0) \qquad \ldots(8)$$

Integrating both sides,

$$\Rightarrow \qquad \phi_1(y+ix) = \frac{1}{2}\sqrt{1+(y+ix)^2} + a \qquad \ldots(9)$$

Now from (7), we get

$$\phi_2'(y - ix) = -\phi_1'(y + ix) \quad \text{at} \quad y = 0$$

$$= -\frac{1}{2}\frac{(y+ix)}{\sqrt{1+(y+ix)^2}} = \frac{1}{2}\left(-\frac{ix}{\sqrt{1-x^2}}\right) \quad \text{at } y = 0$$

$$\phi_2'(y - ix) = \frac{1}{2}\frac{(y - ix)}{\sqrt{1 + (y - ix)^2}} \quad \text{at } y = 0$$

On integrating both sides, we get

$$\phi_2(y - ix) = \frac{1}{2}\sqrt{1 + (y - ix)^2} + b$$

whence $$z = \frac{1}{2}\left\{\sqrt{1 + (y + ix)^2} + \sqrt{1 + (y - ix)^2}\right\} + c \qquad \ldots(10)$$

where $c = (a + b)$

Now on equating the two values of z as in (5) and (10), we get

$$\sqrt{1 - x^2} = \left[\frac{1}{2}\left\{2\sqrt{1 - x^2}\right\} \text{at } y = 0\right]_{y=0} + c \quad \Rightarrow \quad c = 0$$

Therefore $$z = \frac{1}{2}\left\{\sqrt{1 + (y + ix)^2} + \sqrt{1 + (y - ix)^2}\right\}$$

$$\Rightarrow \quad 2z - \sqrt{1 + (y + ix)^2} = \sqrt{1 + (y - ix)^2}$$

Squaring on both sides, we get

$$4z^2 + 1 + (y + ix)^2 - 4z\sqrt{1 + (y + ix)^2} = 1 + (y - ix)^2$$

or $$(z^2 + ixy) = z\sqrt{1 + (y + ix)^2}$$

Again squaring on both sides, we get

$$z^4 + 2ixyz^2 + i^2x^2y^2 = z^2\{1 + (y + ix)^2\}$$

$$\Rightarrow \quad z^4 + 2ixyz^2 - x^2y^2 = z^2 + z^2y^2 - z^2x^2 + 2z^2xyi$$

$$\Rightarrow \quad z^2(z^2 + x^2 - 1) = y^2(z^2 + x^2)$$

Example 68: Find a surface passing through the two lines $z = x = 0$, $z - 1 = x - y = 0$ satisfying $r - 4s + t = 0$.

Solution: The given equation may be written as

$$(D^2 - 4DD' + 4D'^2)z = 0 \qquad \ldots(1)$$

Its auxiliary equation is $m^2 - 4m + 4 = 0$

or $$(m - 2)^2 = 0 \quad i.e., \quad m = 2, 2 \qquad \ldots(2)$$

$$\therefore \quad \text{C.F. } (z) = \phi_1(y + 2x) + x\,\phi_2(y + 2x) \qquad \ldots(3)$$

Since the above surface (2) passes through the lines $z = x = 0$ and $z - 1 = x - y = 0$

$$\therefore \quad z = 0 = \phi_1(y + 2x) + 0 \cdot \phi_2(y + 2x)$$

i.e., $$\phi_1(y + 2x) = 0 \qquad \ldots(4)$$

and $$z - 1 = 0 = x - y \quad i.e., \quad z = 1 \quad \text{and} \quad x = y$$

$$\Rightarrow \quad 1 = \phi_1(y + 2x) + x\phi_2(y + 2x)$$

On using (4), we get

$$\phi_2(y + 2x) = \frac{1}{x} \quad \text{or} \quad \phi_2(y + 2x) = \frac{1}{x} = \frac{3}{3x} = \frac{3}{2x + x} = \frac{3}{2x + y}$$

$$\Rightarrow \qquad \phi_2(y+2x) = \frac{3}{2x+y} \qquad \ldots(5)$$

Hence the required solution is

$$z = x\frac{3}{2x+y} \quad \text{or} \quad z(2x+y) = 3x$$

11.9 NON-HOMOGENOUS LINEAR EQUATIONS

If the differential coefficients involving in the partial differential equation $f(D, D') = F(x, y)$ are not of the same order than it is called 'non-homogenous linear partial differential equation'.

Alike homogenous linear p.d.e. its solution also consists of two parts *viz*. complementary function and particular integral.

To Find Complementary Function

For Complementary Function, factorise $f(D, D')$ into factors of the form $(D - mD' - \alpha)$, say, so that $(D - mD' - \alpha)z = 0$ or $p - mq = az$ is solved for z.

The subsidiary equations for above equation are

$$\underset{\text{I}}{\frac{dx}{1}} = \underset{\text{II}}{\frac{dy}{-m}} = \underset{\text{III}}{\frac{dz}{\alpha z}}$$

On considering I & II, we get

$$\frac{dx}{1} = \frac{dy}{-m} \quad \text{i.e.,} \quad (y + mx) = c_1$$

From I and III, we get

$$\frac{dx}{1} = \frac{dz}{\alpha z} \quad \text{or} \quad \alpha\, dx = \frac{dz}{z} \quad \text{i.e.,} \quad z = c_2 e^{\alpha x}$$

or $\qquad z = e^{\alpha x}\phi_0(c_1) = e^{\alpha x}\phi(y + mx)$

Likewise, find solutions for other such factors and add them to get complementary function.

Example 69: Solve $(D - D' - 1)(D - D' - 2)z = e^{2x-y} + x$.

Solution: Here $f(D, D') = (D - D' - 1)(D - D' - 2)$

On comparing it with $f(D, D') = (D - m_1D' - \alpha_1)(D - m_2D' - \alpha_2)$

we get $\qquad \left.\begin{matrix} m_1 = 1 \\ \alpha_1 = 1 \end{matrix}\right\}, \quad \left.\begin{matrix} m_2 = 1 \\ \alpha_2 = 2 \end{matrix}\right\}$

$$\therefore \qquad C.F. = e^{\alpha_1 x}\phi_1(y + m_1x) + e^{\alpha_2 x}\phi_2(y + m_2x)$$

Now $\qquad \text{P.I.}_1 = \dfrac{1}{f(D, D')}F(x, y) = \dfrac{1}{(D - D' - 1)(D - D' - 2)}e^{2x-y}$

$$= \frac{1 \cdot e^{2x-y}}{(2+1-1)(2+1-2)}$$

(Replace D by $a = 2$ and D' by $b = -1$, provided $f(a, b) \neq 0$)

$$= \frac{1}{2} e^{2x-y}$$

$$\text{P.I.}_2 = \frac{1}{(D - D' - 1)(D - D' - 2)} x$$

$$= \frac{1}{-\left[1 - (D - D')\right] \times -2\left(1 - \frac{1}{2}(D - D')\right)} x$$

$$= \frac{1}{2}[1 - (D' - D)]^{-1} \left[1 - \frac{1}{2}(D - D')\right]^{-1} x$$

$$= \frac{1}{2}\left[1 + (D' - D) + \ldots\right]\left[1 + \frac{1}{2}(D - D') + \ldots\right] x$$

$$= \frac{1}{2}\left(1 + D + \frac{D}{2} + \ldots\right) x = \frac{1}{2} x + \frac{1}{2}\frac{3}{2} = \frac{x}{2} - \frac{3}{4}$$

Therefore complete solution,

$$z = e^x \phi_1(y + x) + e^{2x} \phi_2(y + x) + \frac{e^{2x-y}}{2} + \frac{x}{2} + \frac{3}{4}$$

Example 70: Solve the following equation

$$(D + D' - 1)(D + 2D' - 3)z = (4 + 3x + 6y)$$

Solution: Here $f(D, D') = (D + D' - 1)(D + 2D' - 3)$ is non linear and comparable to

$$(D - m_1 D' - c_1)(D - m_2 D' - c_2)$$

where $\left.\begin{matrix} m_1 = -1 \\ c_1 = 1 \end{matrix}\right\}$ and $\left.\begin{matrix} m_2 = -2 \\ c_2 = 3 \end{matrix}\right\}$

From $(D - m_1 D' - c_1)z = 0$, we have

$$z = \phi(y + m_1 x) e^{c_1 x} = \phi((y - x) e^x$$

and from $(D - m_2 D' - c_2)z = 0$, we get

$$z = \phi(y + m_2 x) e^{c_2 x} = \phi(y - 2x) e^{3x}$$

$\therefore$ Complementary Function (C.F.) = $\phi(y - x)e^x + \phi(y - 2x)e^{3x}$

Now for Particular Integral

$$\text{P.I.}(z) = \frac{1}{f(D, D')} F(x, y) = \frac{1}{(D^2 + 2D'^2 + 3DD' - 4D - 5D' + 3)}(4 + 3x + 6y)$$

or
$$\text{P.I.}(z) = \frac{1}{3\left[1 + \frac{1}{3}(D^2 + 2D'^2 + 3DD' - 4D - 5D')\right]}(4 + 3x + 6y)$$

$$= \frac{1}{3}\left[1 + \frac{1}{3}(D^2 + 2D'^2 + 3DD' - 4D - 5D')\right]^{-1}(4 + 3x + 6y)$$

$$= \frac{1}{3}\left[1 - \frac{1}{3}(-4D - 5D') + \dots\right](4 + 3x + 6y)$$

(Taking higher order terms as zero

$$= \frac{1}{3}\left[(4 + 3x + 6y) + \frac{4}{3}D(3x) + \frac{5}{3}D'(6y)\right]$$

$$= \frac{1}{3}\left[4 + 3x + 6y + 4 + 10\right]$$

$$= \frac{18 + 3x + 6y}{3} = (x + 2y + 6)$$

$\therefore$ Complete solution = $\phi(y - x)e^{x} + \phi(y - 2x)e^{-2x} + (x + 2y + 6)$.

Example 71: Solve $(2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)$.

Solution: Here $f(D, D') = (2DD' + D'^2 - 3D') = D'(2D + D' - 3)$

$\therefore$ Complementary function = $e^{0y}\,\phi_1(x) + e^{3y}\,\phi_2(2y - x)$

Now $$\text{P.I.} = \frac{1}{2DD' + D'^2 - 3D'}3\cos(3x - 2y)$$

$$= \frac{(-1)}{(3D' - 8)}3\cos(3x - 2y) \qquad [\because \quad f(D^2, DD', D'^2) = f(-a^2, -ab, -b^2) \neq 0]$$

On replacing $D^2 = -a^2 = 9$, $DD' = -ab = 6$, $D'^2 = -b^2 = -4$

$$= -3\frac{(3D' + 8)}{9D'^2 - 64}\cos(3x - 2y)$$

$$= \frac{3}{100}(3D' + 8)\cos(3x - 2y)$$

$$= \frac{3}{50}[3\sin(3x - 2y) + 4\cos(3x - 2y)]$$

Therefore complete solution,

$$z = \phi_1(x) + e^{3y}\,\phi_2(2y - x) + \frac{3}{50}[4\cos(3x - 2y) + 3\sin(3x - 2y)]$$

Example 72: Solve $r - s + p = 1$.

Solution: The given equation can be expressed in symbolic form as

$$(D^2 - DD' + D)z = 1$$

For complementary function, we write it as

$$D(D - D' + 1)z = 0$$

$\therefore$ C.F. $= e^{0x}\,\phi_1(y) + e^{-x}\,\phi_2(y + x)$

Now
$$\text{P.I.} = \frac{1}{D(D - D' + 1)} \cdot 1$$

$$= \frac{1}{D}(1 + D - D')^{-1} \cdot 1 = \frac{1}{D}[1 - D + \ldots]1 = \left[\frac{1}{D} - 1 + \ldots\right] \cdot 1$$

$$= x - 1$$

$\therefore$ Complete solution is $z = \phi_1(y) + e^{-x}\phi_2(y + x) + x - 1$.

Example 73: Solve $(D^2 - D'^2 - 3D + 3D')z = e^{x + 2y} + xy$.

Solution: Rewriting the given equation as

$(D - D')(D + D' - 3)z = xy + e^{x + 2y}$ comparable to

$$(D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2)z = F(x, y)$$

$$\text{C.F.} = e^{\alpha_1 x}\phi_1(y + m_1 x) + e^{\alpha_2 x}\phi_2(y + m_2 x)$$

or C.F. $= e^{0x}\,\phi_1(y + x) + e^{3x}\,\phi_2(y - x)$ $\quad\because\ m_1 = 1;\quad m_2 = -1$

$c_1 = 0;\quad c_2 = 3$

Now Particular Integrals corresponding to xy and $e^{x + 2y}$ are as follows:

$$\text{P.I.}_1 = \frac{1}{(D - D')(D + D' - 3)}xy = \frac{1}{-3D\left(1 - \frac{D'}{D}\right)\left(1 - \frac{D}{3} - \frac{D'}{3}\right)}xy$$

$$= \frac{-1}{3D}\left(1 - \frac{D'}{D}\right)^{-1} \cdot \left(1 - \frac{D}{3} - \frac{D'}{3}\right)^{-1} xy$$

$$= -\frac{1}{3}\left[\frac{1}{D} + \frac{1}{3} + \frac{2}{3}\frac{D'}{D} + \frac{1}{9}D + \frac{1}{3}D' + \frac{1}{9}DD' + \frac{D'}{D^2} + \ldots\right]xy$$

(On taking terms of D, D', DD' to the power 1 since in $F(x, y)$, x and y are in power 1 only)

$$\text{P.I.}_1 = -\frac{1}{3}\left[\frac{1}{2}x^2y + \frac{1}{3}xy + \frac{1}{3}x^2 + \frac{1}{9}y + \frac{1}{3}x + \frac{1}{9} + \frac{1}{6}x^3\right]$$

$$\text{P.I.}_2 = \frac{1}{(D - D')(D + D' - 3)}e^{x + 2y},$$ Replacing D by 1 and D' by 2

$$= \frac{1}{(1 - 2)(D + D' - 3)}e^{x + 2y}$$

$$= \frac{1}{D + D' - 3}e^{x + 2y}$$

$$= -e^{x+2y}\frac{1}{[D + 1 + D' + 2 - 3]} \cdot 1$$

$$= -e^{x+2y} \cdot \frac{1}{D+D'} \cdot 1$$

$$= -e^{x+2y} \cdot \frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1} \cdot 1$$

$$= -x\, e^{x+2y}.$$

Therefore complete solution $z = \phi_1(y + x) + e^{3x}\phi_2(y - x) - x\, e^{x + 2y}$.

Example 74: Solve $(D - 3D' - 2)^2 z = 2e^{2x} \tan(y + 3x)$.

Solution: Here symbolic form of the equation

$(D - 3D' - 2)z = 2e^{2x} \tan(y + 3x)$

is comparable to $f(D, D') = (D - 3D' - 2)^2 z = 2e^{2x} \tan(y + 3x)$

a non-homogenous linear equation with repeated factors $(D - mD' - a)$.

whence C.F. $= e^{2x}\phi_1(y + 3x) + xe^{2x}\phi_2(y + 3x)$

Now
$$\text{P.I.} = \frac{1}{(D-3D'-2)} 2e^{2x} \tan(y+3x),$$

$$= 2e^{2x} \frac{1}{(D+2-3D'-2)} \tan(y+3x), \quad \text{(Replace } D \text{ by } (D+2))$$

$$= 2e^{2x} \frac{1}{(D-3D')^2} \tan(y+3x)$$

$$= 2e^{2x} \cdot \frac{1}{D-3D'} \cdot \int \tan c\, dx, \quad c = (y+3x)$$

$$= 2e^{2x} \cdot \frac{1}{D-3D'} \cdot x \tan c = 2e^{2x} \int_{D-3D'} x \tan c\, dx$$

$$= 2e^{2x} \cdot \frac{x^2}{2} \tan(y+3x)$$

P.I. $= x^2 e^{2x} \tan(y + 3x)$

Therefore the complete solution

$$z = e^{2x}\phi_1(y + 3x) + xe^{2x}\phi_2(y + 3x) + x^2e^{2x} \tan(y + 3x).$$

Example 75: Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y)$.

Solution: The symbolic form of the given equation is

$(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y)$

Thus $f(D, D') = (D + D')(D + D' - 2)$ is comparable to

$(D - m_1D' - \alpha)(D - m_2D' - \alpha_2)$

so that

$$\text{C.F.} = e^{\alpha_1 x}\phi_1(y + m_1x) + e^{\alpha_2 x}\phi_2(y + m_2x)$$

$= e^{0x}\,\phi_1(y-x) + e^{2x}\,\phi_2(y-x)$ $\quad \because \quad m_1 = 1, \quad \alpha_1 = 0$

$m_2 = 1, \quad \alpha_2 = 2$

Now $$\text{P.I.} = \frac{1}{(D^2 + 2DD' + D^2 - 2D - 2D')}\sin(x+2y)$$

$$= \frac{1}{(-1 + 2(-2) + (-4) - 2D - 2D')}\sin(x+2y)$$

(Replacing $D^2 = -1$, $DD' = -2$, $D'^2 = -4$)

$$= \frac{-1}{2(D+D')+9}\sin(x+2y)$$

$$= \frac{-1[2(D+D')-9]}{[2(D+D')+9][2(D+D')-9]}\sin(x+2y)$$

$$= \frac{-1[2(D-D')-9]}{4[-1+2(-2)-4]-81}\sin(x+2y)$$

$$= \frac{1}{117}[2D\sin(x+2y) - 2D'\sin(x+2y) - 9\sin(x+2y)]$$

$$\text{P.I.} = \frac{-1}{117}[2\cos(x+2y) - 9\sin(x+2y)]$$

Hence the complete solution

$$z = \phi_1(y-x) + e^{2x}\phi_2(y-x) - \frac{1}{117}[2\cos(x+2y) - 9\sin(x+2y)]$$

Example 76: Solve $\displaystyle\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = x^2 + y^2.$

Solution: The symbolic form of the above equation is

$(D^2 - DD' + D)z = (x^2 + y^2)$

Here $D, D') = D(D - D' + 1)$ comparable to $= (D - m_1D' - \alpha_1)(D - m_2D' - \alpha_2)$,

with $m_1 = 0$, $\alpha_1 = 0$; $m_2 = 1$, $\alpha_2 = -1$

So that C.F. $= \phi_1(y) + e^{-x}\,\phi_2(y + x)$

Now $$\text{P.I.} = \frac{1}{D(D-D'+1)}(x^2+y^2)$$

$$= \frac{1}{D[1+(D-D')]}(x^2+y^2)$$

$$= \frac{1}{D}\left[1+(D-D')\right]^{-1}(x^2+y^2)$$

$$= \frac{1}{D}\left[1-(D-D')+(D^2+D'^2-2DD')+(D^3-3D^2D'+3DD'^2-D'^3)\right](x^2+y^2)$$

$$= \frac{1}{D}\left[\frac{1}{D}-1+\frac{D'}{D}+D+\frac{D'^2}{D}-2D'+D^2+3D'^2\right](x^2+y^2)$$

$$= \left[\frac{x^3}{3}+xy^2-x^2-y^2+2yx+2x-2y-2-6\right]$$

$$= \frac{x^3}{3}+xy^2-x^2-y^2+2xy+4x-2y+8$$

Therefore complete solution

$$z = \phi(y)+e^{-x}\phi_2(y+x)+\frac{x^3}{3}+xy^2-x^2+2xy+4x-y^2-2y+8$$

Example 77: Solve $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y$.

Solution: Here $f(D, D) = (D^2 - DD' + D' - 1) = (D - 1)(D - D' + 1)$

On comparing it with $(D - m_1D' - \alpha_1)(D - m_2D' - \alpha_2)$

we see $m_1 = 0, \alpha_1 = 0 \quad m_2 = 1, \alpha_2 = -1;$

whence C.F. $= e^x \phi_1(y) + e^{-x} \phi_2(y + x)$.

Now P.I. = P.I.$_1$ + P.I.$_2$, where P.I.$_1$ Particular Integral corresponding to $\cos(x + 2y)$ and P.I.$_2$, Particular Integral corresponding to e^y

$$\therefore \qquad \text{P.I.}_1 = \frac{1}{(D^2-DD'+D'-1)}\cos(x+2y)$$

$$= \frac{1}{-1-(-2)+D'-1}\cos(x+2y), \qquad (\text{Replace } D^2 = -1,\ DD'\ -2,\ D'^2 = -4)$$

$$= \frac{1}{D'}\cos(x+2y)$$

$$= \frac{1}{2}\sin(x+2y)$$

and

$$\text{P.I.}_2 = \frac{1}{D^2-DD'+D'-1}e^y$$

It is a case of failure as $f(D, D') = f(a, b) = 0$ for $a = 0$ and $b = 1$ in e^y.

Therefore differentiate $f(D, D')$ with respect to D and multiply $F(x, y)$ by x.

$$\therefore \qquad \text{P.I.}_2 = x \cdot \frac{1}{2D-D'}e^y$$

Now replace D by a (= 0) and D' by b (= 1)

$\Rightarrow \qquad \text{P.I.}_2 = \dfrac{1}{0-1} x\,e^y = -xe^y$

Therefore, $\quad z = \phi_1(y) + e^{-x}\phi_2(y+x) + \dfrac{1}{2}\sin(x+2y) - xe^y$

Example 78: Solve $(D^3 - 3DD' + D' + 4)z = e^{2x+y}$

Solution: Here $f(D, D')z$ = (D3 – 3DD' + D' + 4)z which can not be factorised in terms of D and D'.

Let its solution be $z = e^{ax+by}$

Implies $(D^3 - 3DD' + D' + 4)z = \phi(a, b)e^{ax+by} = (a^3 - 3ab + b + 4)e^{ax+by}$

then $\quad (D^3 - 3DD' + D' + 4)z = 0$, if $(a^3 - 3ab + b + 4) = 0$

Whence $\quad$ C.F. $(z) = \sum \lambda\, e^{ax+by}$, where $(a^3 - 3ab + b + 4) = 0$

Now

$$\text{P.I.} = \frac{1}{D^3 - 3DD' + D' + 4} e^{2x+y}$$

$$= \frac{e^{2x+y}}{2^3 - 3(2) + 1 + 4} \quad \text{as here } a = 2, b = 1$$

$$= \frac{1}{7} e^{2x+y}$$

$\therefore$ Complete solution $(z) = \sum \lambda\, e^{ax+by} + \dfrac{1}{7}e^{2x+y}$, where $(a^3 - 3ab + b + 4) = 0$

ASSIGNMENT 7

Solve the following equations:

1. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} - z = e^{-x}$
2. $DD'(D + 2D' + 1)z = 0.$
3. $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y.$
4. $(D^2 - D')z = 2y - x^2.$

ANSWERS

Assignment 1

1. (*i*) $px + qy = 0$, (*ii*) $p + q = mz$, (*iii*) $q = px + p^2$

2. (*i*) $x(y - z)p + y(z - x)q = z(x - y)$ (*ii*) $z\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y} = 0$

3. (*i*) $z = px + qy + p^2 + q^2$ (*ii*) $px - qy = 0$

(*iii*) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial t}$ (*iv*) $z(px + qy) = z^2 - 1$

Assignment 3

1. $z = \dfrac{x^2}{2}\log y + axy + \phi(x) + \psi(y)$

2. $z = -\dfrac{1}{x^2}\sin xy + y\,f(x) + \phi(x)$

3. $z = \sin x + e^y \cos x$

4. $4 = e^{-t}\sin x + \phi(x) + \phi(y)$

5. $z = \log x \cdot \log y\ \phi(x) + \phi(y)$

6. $z = e^{x+y} + \phi(y) + \phi(x)$

Assignment 4

1. $f(xy, x^2 + y^2 + z^2) = 0$

2. $f\left(\dfrac{y}{x^2} + \dfrac{x}{z^2}, xyz^{1/3}\right) = 0$

3. $f(x^3 - y^3, x^2 - z^2) = 0$

4. $f(lx + my + nz, x^2 + y^2 + z^2) = 0$

5. $f\left(\dfrac{y}{x}, \dfrac{z-k}{x}\right) = 0$

6. $f\left(\dfrac{x-y}{y-z}, \dfrac{z-u}{y-z}, (z-u)(x+y+z+u)^{1/3}\right) = 0$

Assignment 5

1. $z^2 = a^2 + (x + ay + b)^2$ or $z = b$

2. $z = ax + by + a^2b^2$

3. $z = ax + \sin x + \dfrac{1}{a}\sin y + c$

4. $z = ax + by + \dfrac{ab}{ab - a - b}\Big]$

Assignment 6

1. $z^2 = (a + bx)^2 + a^2y^2$

2. $z = ay + b(x^2 - a)$

3. $z = \dfrac{ax}{y^2} - \dfrac{a^2}{4y^3} + \dfrac{b}{y}.$

4. $4z = (a\log x + 6\log y + c)^2$ wehre $a^2 + b^2 = 1$

5. $z^2 = \left(b + x\sqrt{x^2 + a^2} + a\log\left\{x + \sqrt{x^2} + a^2\right\} + y\sqrt{y^2 - a^2} + a\log y\left\{y + \sqrt{y^2 - a^2}\right\}\right)$

Assignment 7

1. $z = e^{-x}\phi_1(y) + e^x\phi_2(y - x) - \dfrac{xe^{-x}}{2}$

2. $z = \phi_1(y) + \phi_2(-x) + e^{-x}\,\phi_3(y - 2x)$

3. $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) - \dfrac{1}{2}\sin(x + 2y) - xe^y$

4. $z = \Sigma\lambda e^{\,ax+by} - y^2 - \dfrac{x^4}{12}$