CHAPTER 5

Sequences and Series

Chapter Outline

5.1 INTRODUCTION

In this chapter, we will learn about the convergence and divergence of sequence and series. There are various methods to test the convergence and divergence of an infinite series. We will study Comparison Test, D'Alembert's ratio test, Cauchy's root test and Cauchy's integral test. We will also study alternating series, absolute and uniform convergence of the series and power series.

5.2 SEQUENCE

An ordered set of real numbers as $u_1, u_2, u_3, \ldots\ldots u_n, \ldots\ldots$ is called a sequence and is denoted by $\{u_n\}$. If the number of terms in a sequence is infinite, it is said to be an infinite sequence, otherwise it is a finite sequence and u_n is called the n^{th} term of the sequence.

5.2.1 Limit of a Sequence

A sequence $\{u_n\}$ tends to a finite number l as $n \to \infty$ if for every $\in\, > 0$ there exists an integer m such that, $|u_n - l| < \in$ for all $n > m$, i.e., $\lim\limits_{n\to\infty} u_n = l$.

5.2.2 Continuous Function Theorem for Sequences

Let $\{u_n\}$ be a sequence of real numbers. If $u_n \to l$ and if f is a function that is continuous at l and defined at all u_n, then $f(u_n) \to f(l)$.

5.2.3 Convergence, Divergence and Oscillation of Finite Series

(i) If the sequence $\{u_n\}$ has a finite limit, i.e., $\lim\limits_{n\to\infty} u_n$ is finite, the sequence is said to be convergent.

e.g.

$$\{u_n\} = \left\{\frac{1}{1+\frac{1}{n}}\right\}$$

$$\lim_{n\to\infty} u_n = 1$$

Since limit is finite, the sequence is convergent.

(ii) If the sequence $\{u_n\}$ has infinite limit, i.e., $\lim\limits_{n\to\infty} u_n$ is infinite, the sequence is said to be divergent.

e.g.

$$\{u_n\} = \{2n+1\}$$

$$\lim_{n\to\infty} u_n = \infty$$

Since limit is infinite, the sequence is divergent.

(iii) If the limit of the sequence $\{u_n\}$ is not unique, the sequence is said to be oscillatory.

e.g.

$$\{u_n\} = (-1)^n + \frac{1}{2^n}$$

$$\lim_{n\to\infty} u_n = 1, \text{ if } n \text{ is even}$$

$$= -1, \text{ if } n \text{ is odd}$$

Since limit is not unique, the sequence is oscillatory.

5.2.4 Monotonic Sequence

A sequence is said to be monotonically increasing if $u_{n+1} \geq u_n$ for each value of n and is monotonically decreasing if $u_{n+1} \leq u_n$ for each value of n. The sequence is called alternating sequence if the terms are alternate positive and negative.

For example, (i) 1, 2, 3, 4, … is a monotonically increasing sequence.

(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a monotonically decreasing sequence.

(iii) 1, –2, 3, –4, … is an alternating sequence.

5.2.5 Bounded Sequence

A sequence $\{u_n\}$ is said to be a bounded sequence if there exists numbers m and M such that $m < u_n < M$ for all n.

***Note* 1:** Every convergent sequence is bounded but the converse is not true.

***Note* 2:** A monotonic increasing sequence converges if it is bounded above and diverges to $+\infty$ if it is not bounded above.

***Note* 3:** A monotonic decreasing sequence converges if it is bounded below and diverges to $-\infty$ if it is not bounded below.

***Note* 4:** If sequence $\{u_n\}$ and $\{v_n\}$ converges to l_1 and l_2 respectively then

(i) Sequence $\{u_n + v_n\}$ converges to $l_1 + l_2$

(ii) Sequence $\{u_n \,.\, v_n\}$ converges to $l_1 \cdot l_2$

(iii) Sequence $\left\{\frac{u_n}{v_n}\right\}$ converges to $\frac{l_1}{l_2}$ provided $l_2 \neq 0$

Example 1

Test the convergence of the sequence $\left\{\frac{n^2+n}{2n^2-n}\right\}$.

Solution

Let

$$u_n = \frac{n^2+n}{2n^2-n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2+n}{2n^2-n}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}}$$

$$= \frac{1}{2}$$

Hence, $\{u_n\}$ is convergent.

Example 2

Test the convergence of the sequence $\{\tanh n\}$.

Solution

Let $$u_n = \tanh n$$

$$\begin{aligned}\lim_{n\to\infty} u_n &= \lim_{n\to\infty} \tanh n \\ &= \lim_{n\to\infty} \frac{\sinh n}{\cosh n} \\ &= \lim_{n\to\infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} \\ &= \lim_{n\to\infty} \frac{e^{2n} - 1}{e^{2n} + 1} \\ &= \lim_{n\to\infty} \frac{1 - \frac{1}{e^{2n}}}{1 + \frac{1}{e^{2n}}} \\ &= 1\end{aligned}$$

Hence, $\{u_n\}$ is convergent.

Example 3

Test the convergence of the sequence $\{2^n\}$.

Solution

Let $u_n = 2^n$

$$\begin{aligned}\lim_{n\to\infty} u_n &= \lim_{n\to\infty} 2^n \\ &= \infty\end{aligned}$$

Hence, $\{u_n\}$ is divergent.

Example 4

Test the convergence of the sequence $\left\{2 - (-1)^n\right\}$.

Solution

Let $$u_n = 2 - (-1)^n$$

$$\begin{aligned}\lim_{n\to\infty} u_n &= \lim_{n\to\infty} 2 - (-1)^n \\ &= 2 - 1 = 1 \quad , \qquad \text{if } n \text{ is even} \\ &= 2 - (-1) = 3 \quad , \qquad \text{if } n \text{ is odd}\end{aligned}$$

Since limit is not unique, the sequence $\{u_n\}$ is oscillatory.

Example 5

Show that the sequence $\{u_n\}$ whose n^{th} term is $u_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$, is monotonic increasing and bounded. Is it convergent?

Solution

$$u_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$$

$$u_{n+1} = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} + \frac{1}{3^{n+1}}$$

$$u_{n+1} - u_n = \frac{1}{3^{n+1}} > 0$$

Hence, $\{u_n\}$ is monotonic increasing sequence.

Also,
$$u_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$$

$$= \frac{1\left(1 - \frac{1}{3^{n+1}}\right)}{1 - \frac{1}{3}}$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right) < \frac{3}{2}$$

$\{u_n\}$ is bounded above by $\frac{3}{2}$.

Since $\{u_n\}$ is monotonic increasing and bounded above, it is convergent.

Example 6

Show that the sequence $\{u_n\}$ whose n^{th} term is $u_n = \frac{1}{1!} + \frac{2}{2!} + \ldots + \frac{1}{n!}$, $n \in N$, is monotonic increasing and bounded. Is it convergent?

Solution

$$u_n = \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$$

$$u_{n+1} = \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$u_{n+1} - u_n = \frac{1}{(n+1)!} > 0$$

$$u_{n+1} > u_n$$

Hence, $\{u_n\}$ is a monotonic increasing sequence.

Also,
$$
\begin{aligned}
u_n &= \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\
&= 1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\
&< 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n} \\
&< \frac{1\left(1-\frac{1}{2^{n+1}}\right)}{1-\frac{1}{2}} \qquad \text{[Using sum of G.P]} \\
&< 2\left(1-\frac{1}{2^{n+1}}\right) \\
&< 2
\end{aligned}
$$

$\{u_n\}$ is bounded above by 2.
Since $\{u_n\}$ is monotonic increasing and bounded above, it is convergent.

Example 7

Show that the sequence $\left\{\frac{n}{n^2+1}\right\}$ *is monotonic decreasing and bounded. Is it convergent?*

Solution

Let
$$
\begin{aligned}
u_n &= \frac{n}{n^2+1} \\
u_{n+1} &= \frac{n+1}{(n+1)^2+1} \\
u_{n+1}-u_n &= \frac{n+1}{(n+1)^2+1}-\frac{n}{n^2+1} \\
&= \frac{(n+1)(n^2+1)-n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)} \\
&= \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0
\end{aligned}
$$

Hence, $\{u_n\}$ is a monotonic decreasing sequence.

Also,
$$u_n = \frac{n}{n^2+1} > 0$$

$\{u_n\}$ is bounded below by 0.
Since $\{u_n\}$ is monotonic decreasing and bounded below, it is convergent.

5.2.6 Sandwich Theorem for Sequences

Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be three sequences such that $u_n \le v_n \le w_n$ for all n. If $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} w_n = l$, then $\lim\limits_{n\to\infty} v_n = l$

Example 1

Show that the sequence $\{u_n\}$, where $u_n = \dfrac{\sin n}{n}$ converges to zero.

Solution

We know that

$$-1 \le \sin n \le 1$$

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

By sandwich theorem,

$$\lim_{n\to\infty}\frac{\sin n}{n} = 0$$

Hence, $\{u_n\}$ converges to zero.

Example 2

If $x \in R$ with $|x| < 1$ then prove that $x^n \to 0$ as $n \to \infty$.

Solution

For $\quad x = 0, \quad x^n = 0$

For $\quad x \ne 0, \quad$ let $|x| = \dfrac{1}{1+y}$

$$|x|^n = \frac{1}{(1+y)^n}$$

$$= \frac{1}{1 + ny + \dfrac{n(n-1)y^2}{2!} + \cdots}$$

$$< \frac{1}{ny} \qquad \left[\because \left\{1 + ny + \frac{n(n-1)y^2}{2!} + \cdots\right\} > ny\right]$$

$$0 < |x|^n < \frac{1}{ny}$$

$$\lim_{n\to\infty} \frac{1}{ny} = 0$$

By sandwich theorem,

$$\lim_{n\to\infty} |x|^n = 0$$

Hence, $$|x|^n \to 0 \quad \text{as } n \to \infty$$

EXERCISE 5.1

1. Test the convergence of the following sequences:

(i) $\frac{2n+1}{1-3n}$ (ii) $2 + (0.1)^n$ (iii) $1 + (-1)^n$

(iv) e^n (v) $1 + (-1)^n$ (vi) $\frac{n^2}{2n-1}\sin\left(\frac{1}{n}\right)$

(vii) $\tan^{-1} n$

[**Ans.:** (i) convergent (ii) convergent
(iii) divergent (iv) divergent
(v) oscillatory (vi) convergent
(vii) convergent]

2. Determine whether the following sequences are monotonically increasing/decreasing, bounded or convergent/divergent.

(i) $1+\frac{1}{n}$ (ii) $\frac{2n-7}{3n+2}$

[**Ans.:** **(i)** decreasing, bounded, convergent
(ii) increasing, bounded, convergent]

3. Show that the sequence $\{u_n\}$, where $u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!}$; $n \ge 2$, is convergent.

4. Does the sequence $\{u_n\}$ convergent where $u_n = \left(\frac{n+1}{n-1}\right)^n$?

[**Ans.: yes**]

5.3 INFINITE SERIES

If $u_1, u_2, u_3, \ldots u_n, \ldots$ is an infinite sequence of real numbers, then the sum of the terms of the sequence, $u_1 + u_2 + u_3 + \ldots + u_n + \ldots\infty$ is called an infinite series.

The infinite series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$ is usually denoted by $\sum_{n=1}^{\infty} u_n$ or Σu_n.
The sum of its first n terms is denoted by S_n and is also known as n^{th} partial sum of Σu_n.

5.3.1 Convergence, Divergence and Oscillation of Finite Series

Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots u_n + \ldots \infty$ and let the sum of the first n terms be $S_n = u_1 + u_2 + u_3 + \ldots + u_n$. As $n \to \infty$, three possibilities arise for S_n:

(i) If S_n tends to a finite limit as $n \to \infty$, the series Σu_n is said to be convergent.

(ii) If S_n tends to $\pm\infty$ as $n \to \infty$, the series Σu_n is said to be divergent.

(iii) If S_n does not tend to a unique limit as $n \to \infty$, i.e., limit does not exist, the series Σu_n is said to be oscillatory.

5.3.2 Properties of Infinite Series

1. The convergence or divergence of an infinite series remains unaffected:
   (i) by addition or removal of a finite number of terms
   (ii) by multiplication of each term with a finite number

2. If two series Σu_n and Σv_n are convergent, then $\Sigma(u_n + v_n)$ is also convergent.

3. If two series Σu_n and Σv_n are divergent, then $\Sigma(u_n + v_n)$ may be convergent.

4. If each term of a series Σu_n of positive terms does not exceed the corresponding term of a convergent series Σv_n of positive terms, then Σu_n is convergent.

5. If each term of a series Σu_n of positive terms exceeds the corresponding term of a divergent series Σv_n of positive terms, then Σu_n is divergent.

5.4 THE n^{th} TERM TEST FOR DIVERGENCE

If a positive term series Σu_n is convergent, then $\lim_{n\to\infty} u_n = 0.$

The converse of this result is not true, i.e., if $\lim_{n\to\infty} u_n = 0,$ it is not necessary that the series will be convergent.

For example,
$$\sum u_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \infty$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Now,
$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 1 + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}}$$
$$S_n > \frac{n}{\sqrt{n}}$$
$$S_n > \sqrt{n}$$
and
$$\lim_{n\to\infty} \sqrt{n} = \infty$$

Thus, the series is divergent.

Hence, $\lim\limits_{n\to\infty} u_n = 0$ is a necessary but not sufficient condition for convergence of Σu_n.

If $\lim\limits_{n\to\infty} u_n \neq 0$ or $\lim\limits_{n\to\infty} u_n$ does not exist, then Σu_n is divergent.

5.5 GEOMETRIC SERIES

Consider the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ ... (1)

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$= \frac{a(1-r^n)}{1-r}, \qquad \text{if } r < 1$$
$$= \frac{a(r^n - 1)}{r-1}, \qquad \text{if } r > 1$$

(i) When $|r| < 1$,
$$\lim_{n\to\infty} r^n = 0$$
$$\lim_{n\to\infty} S_n = \frac{a}{1-r} \text{ is finite.}$$

Hence, the series is convergent.

(ii) When $r > 1$,
$$\lim_{n\to\infty} r^n \to \infty$$
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n - 1)}{r-1} \to \infty$$

Hence, the series is divergent.

(iii) When $r = 1$,
$$S_n = a + a + a + \cdots = na$$
$$\lim_{n\to\infty} S_n \to \infty$$

Hence, the series is divergent.

(iv) When $r = -1$,

$$S_n = a - a + a - \cdots (-1)^{n-1} a$$
$$= 0, \text{ if } n \text{ is even}$$
$$= a, \text{ if } n \text{ is odd}$$

Hence, the series is oscillatory.

(v) When $r < -1$, let $r = -k$ where $k > 0$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a[1-(-k)^n]}{1+k}$$

$$= \lim_{n\to\infty} \frac{a[1-(-1)^n k^n]}{1+k}$$
$$= -\infty, \text{ if } n \text{ is even}$$
$$= +\infty, \text{ if } n \text{ is odd}$$

Hence, the series is oscillatory.

From all the above cases, we conclude that the geometric series (1) is

(i) convergent if $|r| < 1$

(ii) divergent if $r \geq 1$

(iii) oscillatory if $r \leq -1$

Example 1

Prove that $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$ *converges and find its sum.*

[Winter 2014]

Solution

The given series is geometric series with $a = 1$ and $r = \frac{2}{3}$.

$$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1}$$

$$|r| = \frac{2}{3} < 1$$

Hence, the series is convergent.

$$S_n = \frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = 3$$

Example 2

Test the convergence of the series $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\cdots$

[Summer 2017]

Solution

The given series is geometric series with $a = 5$ and $r = -\frac{2}{3}$.

$$5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\cdots=\sum_{n=1}^{\infty} ar^{n-1}$$

$$=\sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1}$$

$$|r|=\left|-\frac{2}{3}\right|=\frac{2}{3}<1$$

Hence, the series is convergent.

$$S_n=\frac{a}{1-r}$$

$$=\frac{5}{1-\left(-\frac{2}{3}\right)}$$

$$=\frac{5}{1+\frac{2}{3}}$$

$$=\frac{5}{\frac{5}{3}}$$

$$=3$$

Example 3

Let $S=\sum_{n=1}^{\infty} n\alpha^n$ *where* $|\alpha|<1$. *Find the value of* α *in* $(0, 1)$ *such that* $S = 2\alpha$.

[Winter 2016]

Solution

$$S=\alpha+2\alpha^2+3\alpha^3+\ldots \qquad \ldots (1)$$

$$\alpha S=\alpha^2+2\alpha^3+3\alpha^4+\ldots \qquad \ldots (2)$$

Subtracting Eq. (2) from Eq. (1),

$$S(1-\alpha) = \alpha + \alpha^2 + \alpha^3 + \ldots$$

$$S(1-\alpha) = \frac{\alpha}{1-\alpha}$$

$$S = \frac{\alpha}{(1-\alpha)^2}$$

If $S = 2\,\alpha$,

$$2\alpha = \frac{\alpha}{(1-\alpha)^2}$$

$$(1-\alpha)^2 = \frac{1}{2}$$

$$1-\alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = 0.2929$$

Example 4

A ball is dropped from 'a' meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh where $0 < r < 1$. Find the total distance the ball travels up and down, when $a = 6$ m and $r = \frac{2}{3}$ m. **[Winter 2016]**

Solution

$$\text{Total distance } (h) = a + 2ar + 2ar^2 + 2ar^3 + \ldots$$

$$= a + \frac{2ar}{1-r}$$

$$= \frac{a(1+r)}{1-r}$$

Here, $a = 6 \text{ m} \quad r = \frac{2}{3} \text{ m}$

$$h = \frac{6\left(1+\frac{2}{3}\right)}{\left(1-\frac{2}{3}\right)} = 6 \cdot \frac{5}{2} \cdot \frac{3}{1} = 30 \text{ m}$$

Example 5

The figure 5.1 shows the first seven of a sequence of squares. The outermost square has an area of $4m^2$. Each of the other squares is obtained by joining the midpoints of the sides of the squares in the infinite sequence. Find sum of the areas of all the squares in the infinite sequence. **[Winter 2015]**

Solution

Since each square is obtained by joining the midpoints of the square before its, area of each square is half the area of previous square.

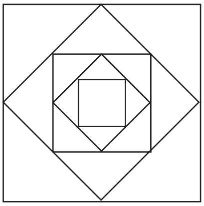

Fig. 5.1

$$\text{Area of } 2^{\text{nd}} \text{ square } = \frac{1}{2}(\text{area of outermost square})$$
$$= \frac{1}{2}(4) = 2$$
$$\text{Area of } 3^{\text{rd}} \text{ square } = \frac{1}{2}(\text{area of } 2^{\text{nd}} \text{ square})$$
$$= \frac{1}{2}(2) = 1$$
$$\text{Area of } 4^{\text{th}} \text{ square } = \frac{1}{2}(\text{area of } 3^{\text{rd}} \text{ square})$$
$$= \frac{1}{2}(1) = \frac{1}{2}$$

and so on.

Sum(s) of areas of all squares in the infinite sequence is

$$S = 4 + 2 + 1 + \frac{1}{2} + \cdots$$
$$= 4\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)$$
$$= 4\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right)$$

which is an infinite geometric series with $a = 1$, $r = \frac{1}{2}$.

$$S = 4\left(\frac{a}{1-r}\right)$$

$$= 4\left(\frac{1}{1-\frac{1}{2}}\right)$$

$$= 8$$

5.6 TELESCOPING SERIES

A telescoping series is a series in which the n^{th} partial sum S_n (sum of first n terms) can be represented in such a manner that almost each term cancels with a preceeding or following term except fixed number of terms.

Example 1

Test the convergence of the series $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots.$

[Summer 2014]

Solution

$$u_n = \frac{1}{n(n+1)}$$

$$= \frac{n+1-n}{n(n+1)}$$

$$= \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1-\frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)$$

$$= 1-0$$

$$= 1 \quad \text{[finite]}$$

Hence, the series Σu_n is convergent.

Example 2

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$.

Solution

$$
\begin{aligned}
u_n &= \frac{1}{n^2+3n+2} \\
&= \frac{1}{(n+1)(n+2)} \\
&= \frac{(n+2)-(n+1)}{(n+1)(n+2)} \\
&= \frac{1}{n+1} - \frac{1}{n+2}
\end{aligned}
$$

$$
\begin{aligned}
S_n &= u_1 + u_2 + u_3 + \cdots + u_n \\
&= \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{n+1}-\frac{1}{n+2}\right) \\
&= \frac{1}{2} - \frac{1}{n+2}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right) \\
&= \frac{1}{2} \quad \text{[finite]}
\end{aligned}
$$

Hence, the series Σu_n is convergent.

Example 3

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$.

Solution

$$
\begin{aligned}
u_n &= \frac{1}{n^2+4n+3} \\
&= \frac{1}{(n+1)(n+3)} \\
&= \frac{(n+3)-(n+1)}{2(n+1)(n+3)}
\end{aligned}
$$

$$= \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+3}\right]$$

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)\right.$$

$$\left. + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \left(\frac{1}{n+1} - \frac{1}{n+3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{5}{12} \quad \text{[finite]}$$

Hence, the series Σu_n is convergent.

Example 4

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2n+1}{n^2 + (n+1)^2}$.

Solution

$$u_n = \frac{2n+1}{n^2 (n+1)^2}$$

$$= \frac{(n+1)^2 - n^2}{n^2 (n+1)^2}$$

$$= \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

$$= \left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \cdots + \left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]$$

$$= 1 - \frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1-\frac{1}{(n+1)^2}\right]$$
$$=1 \text{ [finite]}$$

Hence, the series Σu_n is convergent.

5.7 COMBINING SERIES

If two series Σu_n and Σv_n are convergent then the basic mathematical operations between the series do not change the behaviour (convergence) of these series, i.e. combine series $\Sigma(u_n + v_n)$, $\Sigma(u_n - v_n)$, $\Sigma k u_n$ are also convergent, where k is a constant.

Example 1

Find the sum of the series $\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}}$. **[Summer 2015]**

Solution

$$\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left[\left(\frac{1}{2}\right)^{n-1}-\left(\frac{1}{6}\right)^{n-1}\right]$$
$$=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}-\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1}$$
$$=\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)-\left(1+\frac{1}{6}+\frac{1}{36}+\cdots\right)$$

Both the series are geometric series with $a = 1$ and $r=\frac{1}{2}$ and $r=\frac{1}{6}$ respectively.

$$S_n = \frac{a_1}{1-r_1}-\frac{a_2}{1-r_2}$$
$$=\frac{1}{1-\frac{1}{2}}-\frac{1}{1-\frac{1}{6}}$$
$$=2-\frac{6}{5}$$
$$=\frac{4}{5}$$

Example 2

Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}$.

[Summer 2015]

Solution

$$\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=1}^{\infty} \left[\left(\frac{2}{3}\right)^n + 5\left(\frac{1}{3^n}\right)^n \right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \left(\frac{5}{3}\right)\left(\frac{1}{3}\right)^{n-1}$$

Both the series are geometric series with $r_1 = \frac{2}{3}$ and $r_2 = \frac{1}{3}$ respectively.

$$|r_1| < 1 \quad \text{and} \quad |r_2| < 1$$

Hence, both the series are convergent.

$$S_1 = \frac{a_1}{1 - r_1} = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

$$S_2 = \frac{a_2}{1 - r_2} = \frac{\frac{5}{3}}{1 - \frac{1}{3}} = \frac{\frac{5}{3}}{\frac{2}{3}} = \frac{5}{2}$$

Hence, $$\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} = 2 + \frac{5}{2} = \frac{9}{2}.$$

5.8 HARMONIC SERIES

The harmonic series is expressed as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

This series is divergent in nature.

5.9 *p*-SERIES

The generalisation of harmonic series is known as p-series. It is represented as

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$$

This series is

(i) convergent if $p > 1$

(ii) divergent if $p \le 1$

5.10 COMPARISON TEST

If Σu_n and Σv_n are series of positive terms such that $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l$ (finite and non-zero) then both series converge or diverge together.

Proof

$$\lim_{n\to\infty}\frac{u_n}{v_n}=l$$

By definition of limit, for a positive number $\in$, however small, there exists an integer m such that

$$\left|\frac{u_n}{v_n}-l\right|<\in \qquad \text{for all } n>m$$

$$-\in<\frac{u_n}{v_n}-l<\in \qquad \text{for all } n>m$$

$$l-\in<\frac{u_n}{v_n}<l+\in \qquad \text{for all } n>m$$

Neglecting the first m terms of Σu_n and Σv_n,

$$l-\in<\frac{u_n}{v_n}<l+\in \qquad \text{for all } n \text{ ... (1)}$$

Case I If Σv_n is convergent then $\lim\limits_{n\to\infty}(v_1+v_2+v_3+\cdots+v_n)=\text{finite}=k,$ say

From Eq. (1),

$$\frac{u_n}{v_n}<l+\in$$

$$u_n<(l+\in)v_n \qquad \text{for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n)<(l+\in)\lim_{n\to\infty}(v_1+v_2+v_3+\cdots+v_n)$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n)<(l+\in)k \text{ (finite)}$$

Hence, Σu_n is also convergent.

Case II If Σv_n is divergent then

$$\lim_{n\to\infty}(v_1+v_2+v_3+...+v_n)\to\infty \quad \text{... (2)}$$

From Eq. (1),

$$l-\in<\frac{u_n}{v_n}$$

$$u_n>(l-\in)v_n \quad \text{for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n)>(l-\in)\lim_{n\to\infty}(v_1+v_2+v_3+\cdots+v_n)$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n)\to\infty \quad \text{[From Eq. (2)]}$$

Hence, Σu_n is also divergent.

Note

The following standard limits can be used to solve the problems:

(i) $\lim_{n\to\infty}\frac{\log n}{n}=0$

(ii) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

(iii) $\lim_{n\to\infty}(n)^{\frac{1}{n}}=1$

(iv) $\lim_{n\to\infty}(n!)^{\frac{1}{n}}=\infty$

(v) $\lim_{n\to\infty}\left(\frac{n!}{n}\right)^{\frac{1}{n}}=\frac{1}{e}$

(vi) $\lim_{n\to\infty}x^n=0$ if $x<1$

(vii) $\lim_{n\to\infty}x^n=\infty$ if $x>1$

(viii) $\lim_{n\to\infty}\frac{x^n}{n!}=0$ for all x

(ix) $\lim_{n\to 0}\left(\frac{a^n-1}{n}\right)=\log a$

(x) $\lim_{n\to\infty}\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}=\log a$

Example 1

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$.

Solution

Let $$u_n = \frac{\sqrt{n}}{n^2+1} = \frac{1}{n^{\frac{3}{2}}\left(1+\frac{1}{n^2}\right)}$$

Let $$v_n = \frac{1}{n^{\frac{3}{2}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}}$$

$$= 1 \qquad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{3}{2}}}$ is convergent as $p = \frac{3}{2} > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 2

Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{2n-1}{n(n+1)(n+2)}$.

Solution

Let $$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{\left(2-\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= 2 \text{ [finite and non-zero]}$$

and the series $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 3

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{1+2^2+3^2+\cdots+n^2}$.

[Winter 2015]

Solution

Let
$$u_n = \frac{1}{1+2^2+3^2+\cdots+n^2}$$
$$= \frac{6}{n(n+1)(2n+1)}$$
$$= \frac{6}{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

Let
$$v_n = \frac{1}{n^3}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$
$$= 6 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^3}$ is convergent as $p = 3 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 4

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)\sqrt{n}}$.

Solution

Let
$$u_n = \frac{n+2}{(n+1)\sqrt{n}}$$

$$= \frac{1+\frac{2}{n}}{n^{\frac{1}{2}}\left(1+\frac{1}{n}\right)}$$

Let $$v_n = \frac{1}{n^{\frac{1}{2}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent as $p = \frac{1}{2} < 1.$

Hence, by comparison test, Σu_n is also divergent.

Example 5

Is the series $\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}$ *convergent or divergent?* **[Summer 2015]**

Solution

Let $$u_n = \frac{2n+1}{(n+1)^2}$$

$$= \frac{n\left(2+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}$$

$$= \frac{1}{n}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2}$$

$$=\frac{2}{1}=2 \qquad \text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 6

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$. **[Winter 2013]**

Solution

Let $$u_n=\frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$$

$$=\frac{n^{\frac{2}{3}}\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{3}{4}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$

$$=\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{1}{12}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$

Let $$v_n=\frac{1}{n^{\frac{1}{12}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$

$$=\frac{(2)^{\frac{1}{3}}}{(3)^{\frac{1}{4}}} \qquad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{1}{12}}}$ is divergent as $p = \frac{1}{12} < 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 7

Test the convergence of the series $\frac{1}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{3}{5\cdot 7} + \frac{4}{7\cdot 9} + \cdots$.

Solution

Let
$$u_n = \frac{n}{(2n-1)(2n+1)}$$
$$= \frac{1}{n\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$
$$= \frac{1}{4} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 8

Test the convergence of the series $\frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \cdots$.

Solution

Let
$$u_n = \frac{1}{(2n+1)^p}$$
$$= \frac{1}{n^p\left(2+\frac{1}{n}\right)^p}$$

Let
$$v_n = \frac{1}{n^p}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(2+\frac{1}{n}\right)^p}$$

$$=\frac{1}{2^p} \qquad \text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p\le 1$.

Hence, by comparison test, Σu_n is also convergent if $p>1$ and divergent if $p\le 1$.

Example 9

Test the convergence of the series $\frac{2}{1}+\frac{3}{8}+\frac{4}{27}+\frac{5}{64}+\cdots+\frac{n+1}{n^3}+\cdots$.

Solution

Let
$$u_n=\frac{n+1}{n^3}$$
$$=\frac{1}{n^2}\left(1+\frac{1}{n}\right)$$

Let
$$v_n=\frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$=1 \qquad \text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n^2}$ is convergent as $p=2>1$.

Hence, by comparison test, Σu_n is also convergent.

Example 10

Test the convergence of the series $\frac{1}{1+\sqrt{2}}+\frac{2}{1+2\sqrt{3}}+\frac{3}{1+3\sqrt{4}}+\cdots$.

Solution

Let
$$u_n=\frac{n}{1+n\sqrt{n+1}}$$
$$=\frac{n}{n^{\frac{3}{2}}\left(\frac{1}{n^{\frac{3}{2}}}+\sqrt{1+\frac{1}{n}}\right)}$$

$$= \frac{1}{n^{\frac{1}{2}}\left(\frac{1}{n^{\frac{3}{2}}} + \sqrt{1+\frac{1}{n}}\right)}$$

Let $$v_n = \frac{1}{n^{\frac{1}{2}}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n^{\frac{3}{2}}} + \sqrt{1+\frac{1}{n}}\right)}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \frac{1}{n^{\frac{1}{2}}}$ is divergent as $p = \frac{1}{2} < 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 11

Test the convergence of the series $\sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + \cdots$.

Solution

Let $$u_n = \sqrt{\frac{n}{(n+1)^3}}$$

$$= \frac{n^{\frac{1}{2}}}{(n+1)^{\frac{3}{2}}}$$

$$= \frac{n^{\frac{1}{2}}}{n^{\frac{3}{2}}\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}$$

$$= \frac{1}{n\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 12

Test the convergence of the series $\frac{14}{1^3}+\frac{24}{2^3}+\frac{34}{3^3}+\cdots.$

Solution

n^{th} term of the numerator $= a + (n-1)d = 14 + (n-1)10 = 10n + 4$
n^{th} term of the denominator $= n^3$

$$u_n=\frac{10n+4}{n^3}$$

$$=\frac{1}{n^2}\left(10+\frac{4}{n}\right)$$

Let $$v_n=\frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(10+\frac{4}{n}\right)$$

$$= 10 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 13

Test the convergence of the series $\frac{1}{a\cdot 1^2+b}+\frac{2}{a\cdot 2^2+b}+\frac{3}{a\cdot 3^2+b}+\cdots.$

Solution

Let $$u_n=\frac{n}{a\cdot n^2+b}$$

$$=\frac{1}{n\left(a+\frac{b}{n^2}\right)}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(a + \frac{b}{n^2}\right)}$$

$$= \frac{1}{a} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 14

Test the convergence of the series $\frac{1}{1^2+m} + \frac{2}{2^2+m} + \frac{3}{3^2+m} + \cdots$.

Solution

Let $$u_n = \frac{n}{n^2+m}$$

$$= \frac{1}{n\left(1 + \frac{m}{n^2}\right)}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1 + \frac{m}{n^2}\right)}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 15

Test the convergence of the series $\frac{2\cdot 1^3+5}{4\cdot 1^5+1} + \frac{2\cdot 2^3+5}{4\cdot 2^5+1} + \cdots + \frac{2\cdot n^3+5}{4\cdot n^5+1} + \cdots$.

Solution

Let $$u_n = \frac{2n^3+5}{4n^5+1}$$

$$= \frac{\left(2+\frac{5}{n^3}\right)}{n^2\left(4+\frac{1}{n^5}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\left(2+\frac{5}{n^3}\right)}{\left(4+\frac{1}{n^5}\right)}$$

$$= \frac{2}{4} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 16

Test the convergence of the series $\frac{1\cdot 2}{3^2\cdot 4^2}+\frac{3\cdot 4}{5^2\cdot 6^2}+\frac{5\cdot 6}{7^2\cdot 8^2}+\cdots.$

[Winter 2013]

Solution

Let $$u_n = \frac{(2n-1)\cdot 2n}{(2n+1)^2(2n+2)^2} \quad \text{[Using A.P.]}$$

$$= \frac{\left(2-\frac{1}{n}\right)\cdot 2}{n^2\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}$$

$$= \frac{1}{4} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 17

Test the convergence of the series $\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\frac{7}{4\cdot 5\cdot 6}+\cdots.$

[Winter 2014]

Solution

Let
$$u_n = \frac{(2n+1)}{n(n+1)(n+2)} \qquad \text{[Using A.P.]}$$
$$= \frac{\left(2+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let
$$v_n = \frac{1}{n^2}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$
$$= 2 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 18

Test the convergence of the series $\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right).$

[Summer 2016]

Solution

Let
$$u_n = \sqrt{n^4+1}-\sqrt{n^4-1}$$
$$= \frac{\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)}{\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)}\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)$$

$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{1}{n^2} \cdot \frac{2}{\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2}{\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)}$$

$$= 2 \qquad \text{[finite and non-zero]}$$

and $\sum v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 19

Check for convergence of the series $\sum_{n=1}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$.

[Winter 2016]

Solution

$$u_n = \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$$

$$= \frac{n^3\left(5-\frac{3}{n^2}\right)}{n^5\left(1-\frac{2}{n}\right)\left(1+\frac{5}{n^2}\right)}$$

$$= \frac{\left(5-\frac{3}{n^2}\right)}{n^2\left(1-\frac{2}{n}\right)\left(1+\frac{5}{n^2}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(5 - \frac{3}{n^2}\right)}{\left(1 - \frac{2}{n}\right)\left(1 + \frac{5}{n^2}\right)}$$

$$= 5 \qquad \text{[finite and non zero]}$$

and $\sum v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison text, $\sum u_n$ is also convergent.

Example 20

Test the convergence of the series $\sum_{n=1}^{\infty} \left[\frac{1}{n} - \log\left(\frac{n+1}{n}\right)\right]$.

Solution

Let $$u_n = \frac{1}{n} - \log\left(\frac{n+1}{n}\right)$$

$$= \frac{1}{n} - \log\left(1 + \frac{1}{n}\right)$$

$$= \frac{1}{n} - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right)$$

$$= \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \cdots$$

$$= \frac{1}{n^2}\left(\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots\right)$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots\right)$$

$$= \frac{1}{2} \qquad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 21

Test the convergence of the series $\sum_{n=1}^{\infty} \sin\frac{1}{n}$.

Solution

Let
$$u_n = \sin\frac{1}{n}$$
$$= \frac{1}{n} - \frac{1}{3!n^3} + \frac{1}{5!n^5} - \cdots$$
$$= \frac{1}{n}\left(1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots\right)$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1 - \frac{1}{3!n^2} + \frac{1}{5!n^4} - \cdots\right)$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 22

Test the convergence of the series $\sum_{n=1}^{\infty}\left[\left(n^3+1\right)^{\frac{1}{3}} - n\right]$. **[Summer 2017]**

Solution

Let
$$u_n = \left[(n^3+1)^{\frac{1}{3}} - n\right]$$
$$= n\left(1 + \frac{1}{n^3}\right)^{\frac{1}{3}} - n$$
$$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\left(\frac{1}{n^3}\right)^3 + \cdots\right] - n$$
$$= \frac{1}{3}\cdot\frac{1}{n^2} - \frac{1}{3^2}\cdot\frac{1}{n^5} + \frac{5}{3^4}\cdot\frac{1}{n^8} - \cdots$$

$$= \frac{1}{n^2}\left(\frac{1}{3} - \frac{1}{3^2}\cdot\frac{1}{n^3} + \frac{5}{3^4}\cdot\frac{1}{n^6} - \cdots\right)$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{3^2}\cdot\frac{1}{n^3} + \frac{5}{3^4}\cdot\frac{1}{n^6} - \cdots\right)$$

$$= \frac{1}{3} \quad \text{[finite and non-zero]}$$

and the series $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

Example 23

Test the convergence of the series $\sum_{n=1}^{\infty}\left[(n+1)^{\frac{1}{3}} - n^{\frac{1}{3}}\right]$.

Solution

Let $$u_n = (n+1)^{\frac{1}{3}} - n^{\frac{1}{3}}$$

$$= n^{\frac{1}{3}}\left[\left(1+\frac{1}{n}\right)^{\frac{1}{3}} - 1\right]$$

$$= n^{\frac{1}{3}}\left[\left\{1 + \frac{1}{3n} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^2} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\cdot\frac{1}{n^3} + \cdots\right\} - 1\right]$$

$$= \frac{1}{3n^{\frac{2}{3}}} - \frac{1}{9n^{\frac{5}{3}}} + \frac{5}{81n^{\frac{8}{3}}} - \cdots$$

$$= \frac{1}{n^{\frac{2}{3}}}\left(\frac{1}{3} - \frac{1}{9n} + \frac{5}{81n^2} - \cdots\right)$$

Let $$v_n = \frac{1}{n^{\frac{2}{3}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{9n}+\frac{5}{81n^2}-\cdots\right)$$

$$=\frac{1}{3}\quad\text{[finite and non-zero]}$$

and $\Sigma v_n=\sum\frac{1}{n^{\frac{2}{3}}}$ is divergent as $p=\frac{2}{3}<1$.

Hence, by comparison test, Σu_n is also divergent.

Example 24

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n}$.

Solution

Let
$$u_n=\frac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n}$$

$$=\left[1+\left(\frac{1}{n}+\frac{1}{n^2}\right)\right]^{\frac{1}{2}}-\left[1+\left(-\frac{1}{n}+\frac{1}{n^2}\right)\right]^{\frac{1}{2}}$$

Expanding using binomial expansion,

$$u_n=\left[1+\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n^2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{1}{n}+\frac{1}{n^2}\right)^2+\cdots\right]$$

$$-\left[1+\frac{1}{2}\left(-\frac{1}{n}+\frac{1}{n^2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{n}+\frac{1}{n^2}\right)^2+\cdots\right]$$

$$=\left[1+\frac{1}{2n}+\frac{1}{2n^2}-\frac{1}{8}\left(\frac{1}{n^2}+\frac{2}{n^3}+\frac{1}{n^4}\right)+\cdots\right]$$

$$-\left[1-\frac{1}{2n}+\frac{1}{2n^2}-\frac{1}{8}\left(\frac{1}{n^2}-\frac{2}{n^3}+\frac{1}{n^4}\right)+\cdots\right]$$

$$= \frac{1}{n} - \frac{1}{2n^3} + \cdots$$
$$= \frac{1}{n}\left(1 - \frac{1}{2n^2} + \cdots\right)$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1 - \frac{1}{2n^2} + \cdots\right)$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 25

Test the convergence of the series $\sum\left(\dfrac{\sqrt{n^2+1}-n}{n^p}\right)$.

Solution

Let $$u_n = \frac{\sqrt{n^2+1}-n}{n^p}$$
$$= \frac{n\left[\left(1+\frac{1}{n^2}\right)^{\frac{1}{2}} - 1\right]}{n^p}$$
$$= \frac{n}{n^p}\left[\left\{1 + \frac{1}{2n^2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\cdot\frac{1}{n^4} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\cdot\frac{1}{n^6} + \cdots\right\} - 1\right]$$
$$= \frac{n}{n^p}\left(\frac{1}{2n^2} - \frac{1}{8n^4} + \frac{1}{16n^6} - \cdots\right)$$
$$= \frac{1}{n^{p+1}}\left(\frac{1}{2} - \frac{1}{8n^2} + \frac{1}{16n^4} - \cdots\right)$$

Let $$v_n = \frac{1}{n^{p+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{8n^2} + \frac{1}{16n^4} - \cdots \right)$$

$$= \frac{1}{2} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{p+1}}$ is convergent if $p + 1 > 1$, i.e., $p > 0$ and divergent if $p + 1 \le 1$, i.e., $p \le 0$.

Hence, by comparison test, Σu_n is also convergent if $p > 0$ and divergent if $p \le 0$.

Example 26

Test the convergence of the series $\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} + \cdots$, *where x is a positive fraction.*

Solution

Since it is an infinite series, by ignoring the first term, the series can be rewritten as

$$\sum u_n = \left(\frac{1}{x-1} + \frac{1}{x+1} \right) + \left(\frac{1}{x-2} + \frac{1}{x+2} \right) + \cdots$$

$$= \frac{2x}{x^2 - 1^2} + \frac{2x}{x^2 - 2^2} + \cdots$$

$$= \sum \frac{2x}{x^2 - n^2}$$

$$u_n = \frac{2x}{x^2 - n^2}$$

$$= \frac{2x}{n^2 \left(\frac{x^2}{n^2} - 1 \right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2x}{\left(\frac{x^2}{n^2} - 1 \right)}$$

$$= -2x \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \dfrac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

EXERCISE 5.2

1. Test the convergence of the following series:

(i) $\sum \dfrac{1}{n^2+1}$ (ii) $\sum\left(\sqrt{n+1}-\sqrt{n}\right)$

(iii) $\sum\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$ (iv) $\sum\left(\dfrac{n^p}{\sqrt{n+1}+\sqrt{n}}\right)$

(v) $\sum \dfrac{n^p}{(n+1)^q}$ (vi) $\sum \dfrac{1}{\sqrt{n}}\tan\left(\dfrac{1}{n}\right)$

(vii) $\sum \tan^{-1}\left(\dfrac{1}{n}\right)$ (viii) $\sum \dfrac{1}{n^{\left(a+\frac{b}{n}\right)}}$

[**Ans.:**
(i) Convergent (ii) Divergent (iii) Convergent
(iv) Convergent if $p < -\dfrac{1}{2}$ Divergent if $p \geq -\dfrac{1}{2}$
(v) Convergent if $p - q + 1 < 0$, Divergent if $p - q + 1 \geq 0$
(vi) Convergent (vii) Divergent
(viii) Convergent if $a > 1$, Divergent if $a \leq 1$]

2. Test the convergence of the series

$$\frac{(1+a)(1+b)}{1\cdot 2\cdot 3}+\frac{(2+a)(2+b)}{2\cdot 3\cdot 4}+\frac{(3+a)(3+b)}{3\cdot 4\cdot 5}+\dots.$$

[**Ans.:** Divergent]

5.11 D'ALEMBERT'S RATIO TEST

If Σu_n is a positive-term series and $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = l$ then

(i) Σu_n is convergent if $l < 1$

(ii) Σu_n is divergent if $l > 1$

Proof

Case I If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l < 1.$

Consider a number $l < r < 1$ such that $\frac{u_{n+1}}{u_n} < r$ for all $n > m$ … (1)

Neglecting the first m terms,

$$\begin{aligned}
\sum_{n=m+1}^{\infty} u_n &= u_{m+1} + u_{m+2} + u_{m+3} + \ldots \infty \\
&= u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+1}} + \ldots\right) \\
&= u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+3}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \ldots\right) \\
&< u_{m+1}(1 + r + r \cdot r + r \cdot r \cdot r + \ldots) \qquad \text{[Using Eq. (1)]} \\
&= u_{m+1}(1 + r + r^2 + r^3 + \ldots) \\
&= u_{m+1} \cdot \frac{1}{1-r} \qquad (r < 1)
\end{aligned}$$

$$\therefore \qquad \sum_{n=m+1}^{\infty} u_n < \frac{u_{m+1}}{1-r} \quad \text{(finite)}$$

Thus, the series $\sum_{n=m+1}^{\infty} u_n$ is convergent.

The nature of a series remains unchanged if we neglect a finite number of terms in the beginning. Hence, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Case II If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l > 1,$

$$\frac{u_{n+1}}{u_n} > 1 \text{ for all } n > m \qquad \text{... (2)}$$

Neglecting the first m terms,

$$\sum_{n=m+1}^{\infty} u_n = u_{m+1} + u_{m+2} + u_{m+3} + u_{m+4} + \ldots \infty$$

$$= u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+1}} + \ldots\right)$$

$$= u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+3}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \ldots\right)$$

$$> u_{m+1}(1+1+1+1+\ldots)$$

$$\therefore \quad (u_{m+1} + u_{m+2} + \ldots \text{to } n \text{ terms}) > u_{m+1}(1+1+1\ldots \text{ to } n \text{ terms})$$

$$S_n > u_{m+1}.n$$

$$\lim_{n\to\infty} S_n > \lim_{n\to\infty} n u_{m+1} \to \infty \qquad [\because u_{m+1} \text{ is positive}]$$

Thus, the series $\sum_{n=m+1}^{\infty} u_n$ is divergent.

The nature of a series remains unchanged if we neglect a finite number of terms in the beginning. Hence, the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Note 1: If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ the ratio test fails, i.e. no conclusion can be drawn about the convergence or divergence of the series.

Note 2: It is convenient to use D'Alembert's ratio test in the following form:

If Σu_n is a positive term series and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$, then

(i) Σu_n is convergent if $l > 1$

(ii) Σu_n is divergent if $l < 1$

(iii) The ratio test fails if $l = 1$

Example 1

Test the convergence of the series $\sum_{n\le 0}^{\infty} \frac{3^{2n}}{2^{3n}}$. **[Summer 2014]**

Solution

Let $$u_n = \frac{3^{2n}}{2^{3n}}$$

$$u_{n+1} = \frac{3^{2(n+1)}}{2^{3(n+1)}} = \frac{3^{2n+2}}{2^{3n+3}}$$

$$\frac{u_n}{u_{n+1}} = \frac{3^{2n}}{2^{3n}} \cdot \frac{2^{3n+3}}{3^{2n+2}}$$

$$= \frac{2^3}{3^2} = \frac{8}{9}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{8}{9} = \frac{8}{9} < 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 2

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{5^{n-1}}{n!}$.

Solution

Let
$$u_n = \frac{5^{n-1}}{n!}$$

$$u_{n+1} = \frac{5^n}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{5^{n-1}}{n!} \cdot \frac{(n+1)!}{5^n}$$

$$= \frac{n+1}{5}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n+1}{5} \to \infty > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 3

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n}{n^3+1}$. **[Winter 2016]**

Solution

Let
$$u_n = \frac{2^n}{n^3+1}$$

$$u_{n+1} = \frac{2^{n+1}}{(n+1)^3+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^n}{n^3+1} \cdot \frac{(n+1)^3+1}{2^{n+1}}$$

$$= \frac{\left(1+\frac{1}{n}\right)^3 + \frac{1}{n^3}}{1+\frac{1}{n^3}} \cdot \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^3 + \frac{1}{n^3}}{1+\frac{1}{n^3}} \cdot \frac{1}{2}$$

$$= \frac{1}{2} < 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 4

Test the convergence of the series $\sum \frac{n!}{n^n}$.

Solution

Let
$$u_n = \frac{n!}{n^n}$$

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{\frac{n!}{n^n}}{\frac{(n+1)!}{(n+1)^{n+1}}}$$

$$= \frac{(n+1)(n+1)^n}{(n+1)n^n}$$

$$= \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= e > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 5

Test the convergence of the series $\sum \frac{n!(2)^n}{n^n}$.

Solution

Let
$$u_n = \frac{n!(2)^n}{n^n}$$

$$u_{n+1} = \frac{(n+1)!(2)^{n+1}}{(n+1)^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{n!2^n}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!2^{n+1}}$$

$$= \frac{1}{2}\left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{2} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= \frac{e}{2} > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 6

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^3}{(n-1)!}$.

Solution

Let
$$u_n = \frac{n^3}{(n-1)!}$$

$$u_{n+1} = \frac{(n+1)^3}{n!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^3}{(n-1)!} \cdot \frac{n!}{(n+1)^3}$$

$$= \frac{n^3}{(n-1)!} \cdot \frac{n(n-1)!}{(n+1)^3}$$

$$= \frac{n}{\left(1+\frac{1}{n}\right)^3}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{n}{\left(1+\frac{1}{n}\right)^3}\to\infty>1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 7

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{(n+1)^n}{n!}$.

Solution

Let

$$u_n=\frac{(n+1)^n}{n!}$$

$$u_{n+1}=\frac{(n+2)^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}}=\frac{(n+1)^n}{n!}\cdot\frac{(n+1)!}{(n+2)^{n+1}}$$

$$=\frac{(n+1)^n}{n!}\cdot\frac{(n+1)(n!)}{[(n+1)+1]^{n+1}}$$

$$=\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}$$

$$=\frac{1}{e}<1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 8

Test the convergence of the series $\sum_{n=1}^{\infty}\frac{2^n+1}{3^n+1}$.

Solution

Let

$$u_n=\frac{2^n+1}{3^n+1}$$

$$u_{n+1}=\frac{2^{n+1}+1}{3^{n+1}+1}$$

$$\frac{u_n}{u_{n+1}}=\left(\frac{2^n+1}{3^n+1}\right)\left(\frac{3^{n+1}+1}{2^{n+1}+1}\right)$$

$$= \frac{\left(1+\frac{1}{2^n}\right)\left(3+\frac{1}{3^n}\right)}{\left(1+\frac{1}{3^n}\right)\left(2+\frac{1}{2^n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{2^n}\right)\left(3+\frac{1}{3^n}\right)}{\left(1+\frac{1}{3^n}\right)\left(2+\frac{1}{2^n}\right)}$$

$$= \frac{3}{2} > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 9

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{5^n + a}{3^n + b}, a > 0, b > 0.$

Solution

Let
$$u_n = \frac{5^n + a}{3^n + b}$$

$$u_{n+1} = \frac{5^{n+1} + a}{3^{n+1} + b}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{5^n + a}{3^n + b} \cdot \frac{3^{n+1} + b}{5^{n+1} + a}$$

$$= \lim_{n\to\infty} \frac{1+\frac{a}{5^n}}{1+\frac{b}{3^n}} \cdot \frac{3+\frac{b}{3^n}}{5+\frac{a}{5^n}}$$

$$= \frac{3}{5} < 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 10

Test the convergence of the series $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \cdots.$

Solution

Let
$$u_n = \frac{(n+1)!}{3^n}$$

$$u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)!}{3^n} \cdot \frac{3^{n+1}}{(n+2)!}$$

$$= \lim_{n\to\infty} \left(\frac{3}{n+2} \right)$$

$$= 0 < 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

Example 11

Test the convergence of the series $1 + \frac{4}{2!} + \frac{4^2}{3!} + \frac{4^3}{4!} + \frac{4^4}{5!} + \cdots$.

Solution

Let
$$u_n = \frac{4^{n-1}}{n!}$$

$$u_{n+1} = \frac{4^n}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{4^{n-1}}{n!} \cdot \frac{(n+1)!}{4^n}$$

$$= \frac{(n+1)}{4}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)}{4} \to \infty > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 12

Test the convergence of the series $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots$.

Solution

Let
$$u_n = \frac{n^2}{n!}$$

$$u_{n+1} = \frac{(n+1)^2}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2}{n!} \cdot \frac{(n+1)!}{(n+1)^2}$$

$$= \frac{n^2(n+1)\cdot n!}{n!(n+1)^2}$$

$$= \frac{n^2}{n+1}$$

$$= \frac{n}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n}{1+\frac{1}{n}} \to \infty > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 13

Test the convergence of $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots,(p>0)$.

Solution

Let
$$u_n = \frac{n^p}{n!}$$

$$u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{n^p}{n!}}{\frac{(n+1)^p}{(n+1)!}}$$

$$= \lim_{n\to\infty} \frac{n^p}{(n+1)^p}\cdot\frac{(n+1)!}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{\left(1+\frac{1}{n}\right)^p} \to \infty > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 14

Test the convergence of the series $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\frac{4}{1+2^4}+\cdots$.

Solution

Let
$$u_n = \frac{n}{1+2^n}$$
$$u_{n+1} = \frac{n+1}{1+2^{n+1}}$$
$$\frac{u_n}{u_{n+1}} = \frac{n}{1+2^n} \cdot \frac{1+2^{n+1}}{n+1}$$
$$= \frac{\frac{1}{2^n}+2}{\left(\frac{1}{2^n}+1\right)\left(1+\frac{1}{n}\right)}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{2^n}+2}{\left(\frac{1}{2^n}+1\right)\left(1+\frac{1}{n}\right)}$$
$$= 2 > 1 \qquad \left[\because \lim_{n\to\infty} 2^n \to \infty\right]$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 15

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\ldots 2n}{5\cdot 8\cdot 11\ldots(3n+2)}$.

Solution

Let
$$u_n = \frac{2\cdot 4\cdot 6\ldots 2n}{5\cdot 8\cdot 11\ldots(3n+2)} \qquad \text{[Using A.P.]}$$
$$u_{n+1} = \frac{2\cdot 4\cdot 6\ldots 2n(2n+2)}{5\cdot 8\cdot 11\ldots(3n+2)(3n+5)}$$
$$\frac{u_n}{u_{n+1}} = \frac{2\cdot 4\cdot 6\ldots 2n}{5\cdot 8\cdot 11\ldots(3n+2)} \cdot \frac{5\cdot 8\cdot 11\ldots(3n+2)(3n+5)}{2\cdot 4\cdot 6\ldots 2n(2n+2)}$$
$$= \frac{3n+5}{2n+2}$$
$$= \frac{3+\frac{5}{n}}{2+\frac{2}{n}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{3+\dfrac{5}{n}}{2+\dfrac{2}{n}}$$

$$=\frac{3}{2}>1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 16

Test the convergence of the series $\dfrac{2}{1}+\dfrac{2\cdot5}{1\cdot5}+\dfrac{2\cdot5\cdot8}{1\cdot5\cdot9}+\dfrac{2\cdot5\cdot8\cdot11}{1\cdot5\cdot9\cdot13}+\cdots.$

Solution

Let $u_n=\dfrac{2\cdot5\cdot8\cdot11\ldots(3n-1)}{1\cdot5\cdot9\cdot13\ldots(4n-3)}$ [Using A.P.]

$$u_{n+1}=\frac{2\cdot5\cdot8\cdot11\ldots(3n-1)(3n+2)}{1\cdot5\cdot9\cdot13\ldots(4n-3)(4n+1)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\dfrac{2\cdot5\cdot8\cdot11\ldots(3n-1)}{1\cdot5\cdot9\cdot13\ldots(4n-3)}}{\dfrac{2\cdot5\cdot8\cdot11\ldots(3n-1)(3n+2)}{1\cdot5\cdot9\cdot13\ldots(4n-3)(4n+1)}}$$

$$=\lim_{n\to\infty}\frac{4n+1}{3n+2}$$

$$=\lim_{n\to\infty}\frac{4+\dfrac{1}{n}}{3+\dfrac{2}{n}}$$

$$=\frac{4}{3}>1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 17

Test the convergence of the series $\left(\dfrac{1}{3}\right)^2+\left(\dfrac{1\cdot2}{3\cdot5}\right)^2+\left(\dfrac{1\cdot2\cdot3}{3\cdot5\cdot7}\right)^2+\ldots\infty.$

Solution

Let $u_n=\left[\dfrac{1\cdot2\cdot3\ldots n}{3\cdot5\cdot7\ldots(2n+1)}\right]^2$ [Using A.P.]

$$u_{n+1}=\left[\frac{1\cdot2\cdot3\,\ldots\,n(n+1)}{3\cdot5\cdot7\,\ldots\,(2n+1)(2n+3)}\right]^2$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left[\dfrac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right]^2}{\left[\dfrac{1\cdot 2\cdot 3\ldots n(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)(2n+3)}\right]^2}$$

$$= \lim_{n\to\infty}\left[\frac{2n+3}{n+1}\right]^2$$

$$= \lim_{n\to\infty}\left[\frac{2+\dfrac{3}{n}}{1+\dfrac{1}{n}}\right]^2$$

$$= 4 > 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

Example 18

Test the convergence of the series $2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+\cdots$.

[Summer 2014]

Solution

Let

$$u_n = \frac{n+1}{n}x^{n-1}$$

$$u_{n+1} = \frac{n+2}{n+1}x^n$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)x^{n-1}}{n}\cdot\frac{n+1}{(n+2)x^n}$$

$$= \frac{1+\dfrac{1}{n}}{1}\cdot\frac{1+\dfrac{1}{n}}{1+\dfrac{2}{n}}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1+\dfrac{1}{n}}{1}\cdot\frac{1+\dfrac{1}{n}}{1+\dfrac{2}{n}}\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$, i.e., $x = 1$.

Then
$$u_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$= 1 \quad \text{[finite and non-zero]}$$

Σu_n is divergent for $x = 1$.

Hence, the series is convergent for $x < 1$ and is divergent for $x \geq 1$.

Example 19

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(n+1)}{n^2} x^n$.

Solution

Let
$$u_n = \frac{n+1}{n^2} x^n$$
$$u_{n+1} = \frac{n+2}{(n+1)^2} x^{n+1}$$
$$\frac{u_n}{u_{n+1}} = \frac{(n+1)x^n}{n^2} \cdot \frac{(n+1)^2}{(n+2)x^{n+1}}$$
$$= \frac{(n+1)^3}{n^2(n+2)x}$$
$$= \frac{\left(1+\frac{1}{n}\right)^3}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^3}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$
$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$, i.e., $x = 1$.

Then
$$u_n = \frac{n+1}{n^2}$$
$$= \frac{1}{n}\left(1+\frac{1}{n}\right)$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, Σu_n is also divergent for $x = 1$.
Hence, the series is convergent for $x < 1$ and is divergent for $x \geq 1$.

Example 20

Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{x^n}{n^2+1}\right)$, *for* $x > 0$.

Solution

Let
$$u_n = \frac{x^n}{n^2+1}$$
$$u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$
$$\frac{u_n}{u_{n+1}} = \frac{x^n}{n^2+1}\cdot\frac{(n+1)^2+1}{x^{n+1}}$$
$$= \frac{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}{1+\frac{1}{n^2}}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}{1+\frac{1}{n^2}}\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $x = 1$.

Then $$u_n = \frac{1}{n^2+1}$$

$$= \frac{1}{n^2\left(1+\frac{1}{n^2}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n^2}\right)}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum\frac{1}{n^2}$ is convergent as $p = 2 > 1$.

By comparison test, Σu_n is also convergent if $x = 1$.

Hence, the series is convergent for $x \le 1$ and is divergent for $x > 1$.

Example 21

Test the convergence of the series $\sum\frac{2^n}{n^4+1}x^n$, $x > 0$.

Solution

Let $$u_n = \frac{2^n}{n^4+1}x^n$$

$$u_{n+1} = \frac{2^{n+1}}{(n+1)^4+1} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^n x^n}{n^4+1} \cdot \frac{(n+1)^4+1}{2^{n+1} x^{n+1}}$$

$$= \frac{\left(1+\frac{1}{n}\right)^4 + \frac{1}{n^4}}{\left(1+\frac{1}{n^4}\right) 2x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^4 + \frac{1}{n^4}}{\left(1+\frac{1}{n^4}\right) 2x}$$

$$= \frac{1}{2x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{2x} > 1$ or $x < \frac{1}{2}$

(ii) divergent if $\frac{1}{2x} < 1$ or $x > \frac{1}{2}$

The test fails if $\frac{1}{2x} = 1$ or $x = \frac{1}{2}$.

Then
$$u_n = \frac{2^n}{n^4+1} \cdot \frac{1}{2^n}$$

$$= \frac{1}{n^4+1}$$

$$= \frac{1}{n^4\left(1+\frac{1}{n^4}\right)}$$

Let
$$v_n = \frac{1}{n^4}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n^4}\right)}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^4}$ is convergent as $p = 4 > 1$.

By comparison test, Σu_n is also convergent if $x = \frac{1}{2}$.

Hence, the series is convergent for $x \le \frac{1}{2}$ and is divergent for $x > \frac{1}{2}$.

Example 22

Test the convergence of the series $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$.

Solution

Let
$$u_n = \sqrt{\frac{n}{n^2+1}} \cdot x^n$$

$$u_{n+1} = \sqrt{\frac{(n+1)}{(n+1)^2+1}} \cdot x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n^2+1}} \cdot x^n \sqrt{\frac{(n+1)^2+1}{n+1}} \cdot \frac{1}{x^{n+1}}$$

$$= \sqrt{\frac{n}{(n+1)} \cdot \frac{(n^2+2n+2)}{(n^2+1)}} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \sqrt{\frac{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right)}} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$, or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$, or $x > 1$

The test fails for $x = 1$.

Then
$$u_n = \sqrt{\frac{n}{n^2+1}}$$

$$= \frac{n^{\frac{1}{2}}}{n\sqrt{1+\frac{1}{n^2}}}$$

$$= \frac{1}{n^{\frac{1}{2}}\sqrt{1+\frac{1}{n^2}}}$$

Let
$$v_n = \frac{1}{n^{\frac{1}{2}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent for $p = \frac{1}{2} < 1$

By comparison test, Σu_n is also divergent for $x = 1$.
Hence, the series is convergent for $x < 1$ and is divergent for $x \geq 1$.

Example 23

Test the convergence of the series $\frac{x}{x+1}+\frac{x^2}{x+2}+\frac{x^3}{x+3}+\frac{x^4}{x+4}+\cdots$.

Solution

Let
$$u_n = \frac{x^n}{x+n}$$

$$u_{n+1} = \frac{x^{n+1}}{x+n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{x+n}\cdot\frac{x+n+1}{x^{n+1}}$$

$$= \frac{\frac{x}{n}+1+\frac{1}{n}}{\frac{x}{n}+1}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\frac{x}{n}+1+\frac{1}{n}}{\frac{x}{n}+1}\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$ or $x = 1$.

Then
$$u_n = \frac{1}{1+n}$$
$$= \frac{1}{n\left(\frac{1}{n}+1\right)}$$

Let
$$v_n = \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n}+1\right)}$$
$$= 1 \text{ [finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, Σu_n is also divergent for $x = 1$.
Hence, the series is convergent for $x < 1$ and is divergent for $x \geq 1$.

Example 24

Test the convergence of the series $\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \frac{x^4}{7\cdot 8} + \cdots$.

Solution

Let
$$u_n = \frac{x^n}{(2n-1)2n}$$
$$u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$$
$$\frac{u_n}{u_{n+1}} = \frac{x^n}{(2n-1)2n} \cdot \frac{(2n+1)(2n+2)}{x^{n+1}}$$
$$= \frac{\left(2+\frac{1}{n}\right)}{\left(2-\frac{1}{n}\right)}\left(1+\frac{1}{n}\right)\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{\left(2-\frac{1}{n}\right)}\left(1+\frac{1}{n}\right)\cdot\frac{1}{x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$ or $x = 1$.

Then
$$u_n = \frac{1}{(2n-1)2n}$$

$$= \frac{1}{2n^2\left(2-\frac{1}{n}\right)}$$

Let
$$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{2\left(2-\frac{1}{n}\right)}$$

$$= \frac{1}{4} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

By comparison test, Σu_n is also convergent.

Hence, the series is convergent for $x \le 1$ and is divergent for $x > 1$.

Example 25

Test the convergence of the series $\frac{1}{1\cdot2\cdot3} + \frac{x}{4\cdot5\cdot6} + \frac{x^2}{7\cdot8\cdot9} + \cdots$.

[Summer 2017]

Solution

Let $$u_n = \frac{x^{n-1}}{(3n-2)(3n-1)\,3n}$$

$$u_{n+1} = \frac{x^n}{(3n+1)\,(3n+2)(3n+3)}$$

$$\frac{u_n}{u_n+1} = \frac{x^{n-1}}{(3n-2)\,(3n-1)\,(3n)} \cdot \frac{(3n+1)\,(3n+2)\,(3n+3)}{x^n}$$

$$= \frac{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)3} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)3} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

By D′ Alembert's ratio text, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$ or $x = 1$.

Then $$u_n = \frac{1}{(3n-2)(3n-1)(3n)}$$

$$= \frac{1}{n^3\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)3}$$

Let $$v_n = \frac{1}{n^3}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)3}$$

$$= \frac{1}{27} \qquad \text{[finite and non-zero]}$$

and $\sum v_n = \sum \frac{1}{n^3}$ is convergent as $p = 3 > 1$.

By comparison test, $\sum u_n$ is also convergent for $x = 1$.

Hence, the series is convergent for $x \le 1$ and is divergent for $x > 1$.

Example 26

Test the convergence of the series $2x + \frac{3}{8}x^2 + \frac{4}{27}x^3 + \cdots + \frac{(n+1)}{n^3}x^n + \cdots$.

Solution

Let
$$u_n = \frac{n+1}{n^3} \cdot x^n$$
$$u_{n+1} = \frac{n+2}{(n+1)^3} \cdot x^{n+1}$$
$$\frac{u_n}{u_{n+1}} = \frac{(n+1)x^n}{n^3} \cdot \frac{(n+1)^3}{(n+2)x^{n+1}}$$
$$= \frac{\left(1+\frac{1}{n}\right)^4}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^4}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$
$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $\frac{1}{x} = 1$ or $x = 1$.

Then
$$u_n = \frac{n+1}{n^3}$$
$$= \frac{1}{n^2}\left(1+\frac{1}{n}\right)$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

By comparison test, Σu_n is also convergent for $x = 1$.

Hence, the series is convergent for $x \le 1$ and is divergent for $x > 1$.

Example 27

Test the convergence of the series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$.

[Winter 2013]

Solution

Let $$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$= \frac{(n+2)}{(n+1)}\sqrt{\frac{n+1}{n}} \cdot \frac{1}{x^2}$$

$$= \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x^2} > 1$ or $x^2 < 1$

(ii) divergent if $\frac{1}{x^2} < 1$ or $x^2 > 1$

The test fails if $\frac{1}{x^2} = 1$ or $x^2 = 1$.

Then
$$u_n = \frac{1}{(n+1)\sqrt{n}}$$
$$= \frac{1}{n^{\frac{3}{2}}\left(1+\frac{1}{n}\right)}$$

Let
$$v_n = \frac{1}{n^{\frac{3}{2}}}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)}$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \frac{1}{n^{\frac{3}{2}}}$ is convergent as $p = \frac{3}{2} > 1$.

By comparison test, Σu_n is also convergent for $x^2 = 1$.

Hence, the series is convergent for $x^2 \le 1$ and is divergent for $x^2 > 1$.

Example 28

Test the convergence of the series $1+\frac{3}{2}x+\frac{5}{9}x^2+\frac{7}{28}x^3+\frac{9}{65}x^4+\cdots$.

Solution

Let
$$u_n = \frac{2n+1}{n^3+1}x^n \quad \text{[Neglecting the first term]}$$
$$u_{n+1} = \frac{2n+3}{(n+1)^3+1}x^{n+1}$$
$$\frac{u_n}{u_{n+1}} = \frac{(2n+1)x^n}{n^3+1}\cdot\frac{[(n+1)^3+1]}{(2n+3)x^{n+1}}$$
$$= \frac{\left(2+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}{\left(2+\frac{3}{n}\right)\left(1+\frac{1}{n^3}\right)x}$$
$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}{\left(2+\frac{3}{n}\right)\left(1+\frac{1}{n^3}\right)x}$$

$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $x = 1$.

Then
$$u_n = \frac{2n+1}{n^3+1}$$
$$= \frac{2+\frac{1}{n}}{n^2\left(1+\frac{1}{n^3}\right)}$$

Let
$$v_n = \frac{1}{n^2}$$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\left(1+\frac{1}{n^3}\right)}$$
$$= 2 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

By comparison test, Σu_n is also convergent if $x = 1$.

Hence, the series is convergent for $x \le 1$ and is divergent for $x > 1$.

EXERCISE 5.3

Test the convergence of the following series:

1. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots \infty$

[**Ans.:** Convergent]

2. $\sum_{n=1}^{\infty} \frac{4 \cdot 7 \cdot 10 \ldots (3n+1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots n}$

[**Ans.:** Convergent]

3. $\frac{1}{1+5} + \frac{2}{1+5^2} + \frac{3}{1+5^3} + \ldots \infty$

[**Ans.:** Convergent]

4. $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \ldots \infty$ [**Ans.:** Convergent]

5. $\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\cdots$ [**Ans.:** Convergent]

6. $1+\frac{3}{2!}+\frac{3^2}{3!}+\frac{3^3}{4!}+\frac{3^4}{5!}+\cdots$ [**Ans.:** Convergent]

7. $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$ [**Ans.:** Convergent]

8. $\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n+1}$ [**Ans.:** Convergent]

9. $\sum_{n=1}^{\infty}\frac{1}{n!}$ [**Ans.:** Convergent]

10. $\sum_{n=1}^{\infty}\frac{n^2(n+1)^2}{n!}$ [**Ans.:** Convergent]

11. $\sum_{n=1}^{\infty}\frac{3^n+4^n}{4^n+5^n}$ [**Ans.:** Divergent]

12. $\sum_{n=1}^{\infty}\frac{x^n}{3^n\cdot n^2},\ x>0$

[**Ans.:** Convergent for $x < 3$, divergent for $x > 3$]

13. $\sum_{n=1}^{\infty}\frac{3^n-2}{3^n+1}\cdot x^{n-1},\ x>0$

[**Ans.:** Convergent for $x < 1$, divergent for $x > 3$]

14. $\sum_{n=1}^{\infty}\frac{x^n}{(2^n)!}$ [**Ans.:** Convergent]

15. $\sum_{n=1}^{\infty}\sqrt{\frac{n+1}{n^3+1}}\cdot x^n,\ x>0$ [**Ans.:** Convergent for $x < 1$, divergent for $x > 1$]

16. $x+2x^2+3x^3+4x^4+\ldots\infty$

[**Ans.:** Convergent for $x < 1$, divergent for $x > 1$]

17. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}+\ldots\infty$

[**Ans.:** Convergent for $x < 1$, divergent for $x > 1$]

18. $\frac{x}{1\cdot 3}+\frac{x^2}{3\cdot 5}+\frac{x^3}{5\cdot 7}+\ldots\infty$

[**Ans.:** Convergent for $x < 1$, divergent for $x > 1$]

19. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\ldots+\frac{n^2-1}{n^2+1}x^n+\ldots\infty$

[**Ans.:** Convergent for $x < 1$, divergent for $x > 1$]

20. $\frac{x}{2\sqrt{3}}+\frac{x^2}{3\sqrt{4}}+\frac{x^3}{4\sqrt{5}}+\ldots\infty$

[**Ans.:** Convergent for $x < 1$, divergent for $x > 1$]

5.12 RAABE'S TEST

If Σu_n is a positive term series and $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=l,$ then

(i) Σu_n is convergent if $l > 1$

(ii) Σu_n is divergent if $l < 1$

(iii) Test fails if $l = 1$

Proof:

(i) Consider a number p such that $p > 1$. The series $\Sigma u_n = \Sigma\frac{1}{n^p}$ is convergent if $p > 1$. By comparison test, Σu_n will be convergent if from and after some term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$$

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p = 1+\frac{p}{n}+\frac{p(p-1)}{2!n^2}+\ldots$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) > n\left[\frac{p}{n}+\frac{p(p-1)}{2n^2}+\ldots\right]$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) > \lim_{n\to\infty}\left[p+\frac{p(p-1)}{2n}+\ldots\right]$$

$$l > p > 1$$

Hence, Σu_n is convergent if $l > 1$.

(ii) Consider a number p such that $p < 1$. The series $\Sigma v_n = \Sigma\frac{1}{n^p}$ is divergent if $p < 1$.

By comparison test, Σu_n will be divergent if from and after some term

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

Proceeding as in case (i), we get

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) < \lim_{n\to\infty}\left[p+\frac{p(p-1)}{2n}+\dots\right]$$

$$l < p < 1$$

Hence, Σu_n is divergent if $l < 1$.

(iii) Raabe's test fails if $l = 1$ and other tests are required to check the nature of the series.

Note: When Raabe's test fails, logarithmic test can be applied.

Example 1

Test the convergence of the series $\dfrac{2}{7}+\dfrac{2\cdot 5}{7\cdot 10}+\dfrac{2\cdot 5\cdot 8}{7\cdot 10\cdot 13}+\dots$.

Solution

$$u_n = \frac{2\cdot 5\cdot 8\ldots\ldots(3n-1)}{7\cdot 10\cdot 13\ldots\ldots(3n+4)}$$

$$u_{n+1} = \frac{2\cdot 5\cdot 8\ldots(3n-1)(3n+2)}{7\cdot 10\cdot 13\ldots(3n+4)(3n+7)}$$

$$\frac{u_n}{u_{n+1}} = \frac{2\cdot 5\cdot 8\ldots(3n-1)}{7\cdot 10\cdot 13\ldots(3n+4)}\cdot\frac{7\cdot 10\cdot 13\ldots(3n+4)(3n+7)}{2\cdot 5\cdot 8\ldots(3n-1)(3n+2)}$$

$$= \frac{3n+7}{3n+2}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+2}-1\right)$$

$$= \lim_{n\to\infty}\frac{5n}{3n+2}$$

$$= \lim_{n\to\infty}\frac{5}{3+\frac{2}{n}} = \frac{5}{3} > 1$$

Hence, by Raabe's test, the series is convergent.

Example 2

Test the convergence of the series $\sum\dfrac{4\cdot 7\ldots\ldots(3n+1)x^n}{n!}$.

Solution

$$u_n = \frac{4\cdot 7\ldots(3n+1)x^n}{n!}$$

$$u_{n+1} = \frac{4\cdot 7\ldots(3n+1)(3n+4)x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{4\cdot 7\ldots(3n+1)x^n}{n!}\cdot\frac{(n+1)!}{4\cdot 7\ldots(3n+1)(3n+4)x^{n+1}}$$

$$= \frac{n+1}{(3n+4)x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{\left(3+\frac{4}{n}\right)x}$$

$$= \frac{1}{3x}$$

By ratio test, the series is

(i) Convergent if $\frac{1}{3x} > 1$ or $x < \frac{1}{3}$

(ii) Divergent if $\frac{1}{3x} < 1$ or $x > \frac{1}{3}$

(iii) Test fails if $x = \frac{1}{3}$

Then
$$\frac{u_n}{u_{n+1}} = \frac{n+1}{(3n+4)\ \frac{1}{3}}$$

$$= \frac{3n+3}{3n+4}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{3n+3}{3n+4}-1\right)$$

$$= \lim_{n\to\infty}\left(\frac{-n}{3n+4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-1}{3+\frac{4}{n}}\right)$$

$$= -\frac{1}{3} < 1$$

By Raabe's test, the series is divergent if $x = \frac{1}{3}$.

Hence, the series is convergent if $x < \frac{1}{3}$ and divergent if $x \geq \frac{1}{3}$.

Example 3

Test the convergence of the series $\sum \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)x^{2n+1}}{2\cdot 4\cdot 6\ldots\ldots 2n(2n+1)}$.

Solution

$$u_n = \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)x^{2n+1}}{2\cdot 4\cdot 6\ldots\ldots(2n+1)}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)\,x^{2n+3}}{2\cdot 4\cdot 6\ldots 2n\,(2n+2)(2n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)x^{2n+1}}{2\cdot 4\cdot 6\ldots\ldots 2n(2n+1)}\cdot\frac{2\cdot 4\cdot 6\ldots\ldots 2n(2n+2)(2n+3)}{1\cdot 3\cdot 5\ldots\ldots(2n-1)(2n+1)x^{2n+3}}$$

$$= \frac{(2n+2)(2n+3)}{(2n+1)^2\,x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\cdot\left(2+\frac{3}{n}\right)}{\left(2+\frac{1}{n}\right)^2 x^2}$$

$$= \frac{1}{x^2}$$

By ratio test, the series is

(i) Convergent if $\frac{1}{x^2} > 1$ or $x^2 < 1$

(ii) Divergent if $\frac{1}{x^2} < 1$ or $x^2 > 1$

(iii) Test fails if $x^2 = 1$

Then
$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+2)(2n+3)}{(2n+1)^2}-1\right]$$

$$= \lim_{n\to\infty} \frac{n(6n+5)}{(2n+1)^2}$$

$$= \lim_{n\to\infty} \frac{\left(6+\frac{5}{n}\right)}{\left(2+\frac{1}{n}\right)^2}$$

$$= \frac{3}{2} > 1$$

By Raabe's test, the series is convergent if $x^2 = 1$.

Hence, the series is convergent if $x^2 \le 1$ and is divergent if $x^2 > 1$.

Example 4

Test the convergence of the series

$$\sum \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)\cdot b(b+1)(b+2)\ldots\ldots(b+n-1)x^n}{1\cdot 2\cdot 3\ldots\ldots n\cdot c(c+1)(c+2)\ldots\ldots(c+n-1)}.$$

Solution

$$u_n = \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)\cdot b(b+1)(b+2)\ldots\ldots(b+n-1)x^n}{1\cdot 2\cdot 3\ldots\ldots n\cdot c(c+1)(c+2)\ldots\ldots(c+n-1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{a(a+1)\ldots\ldots(a+n-1)\cdot b(b+1)\ldots\ldots(b+n-1)x^n}{1\cdot 2\cdot 3\ldots\ldots n\cdot c(c+1)\ldots\ldots(c+n-1)}\cdot \frac{1\cdot 2\ldots\ldots n(n+1)\cdot c(c+1)\ldots(c+n-1)(c+n)}{a(a+1)\ldots\ldots(a+n-1)(a+n)\cdot b(b+1)\ldots\ldots(b+n-1)(b+n)x^{n+1}}$$

$$= \frac{(n+1)(c+n)}{(a+n)(b+n)x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(\frac{c}{n}+1\right)}{\left(\frac{a}{n}+1\right)\left(\frac{b}{n}+1\right)x}$$

$$= \frac{1}{x}$$

By ratio test, the series is

(i) Convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) Divergent if $\frac{1}{x} < 1$ or $x > 1$

(iii) Test fails if $x = 1$

Then
$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(c+n)}{(a+n)(b+n)}$$

Applying Raabe's test,

$$\begin{aligned}\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) &= \lim_{n\to\infty} n\left[\frac{(n+1)(c+n)}{(a+n)(b+n)} - 1\right]\\ &= \lim_{n\to\infty} n\left[\frac{(c-ab)+n(1+c-a-b)}{(a+n)(b+n)}\right]\\ &= \lim_{n\to\infty}\left[\frac{\frac{(c-ab)}{n} + (1+c-a-b)}{\left(\frac{a}{n}+1\right)\left(\frac{b}{n}+1\right)}\right]\\ &= 1+c-a-b\end{aligned}$$

By Raabe's test, the series is (i) convergent if $1+c-a-b > 1$ or $c > a+b$, and (ii) divergent if $1+c-a-b < 1$ or $c < a+b$.

Hence, the series is convergent if $x < 1$ and divergent if $x > 1$.

For $x = 1$, the series is convergent if $c > a + b$ and divergent if $c < a + b$.

EXERCISE 5.4

Test the convergence of the following series:

1. $1 + \frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1}{2\cdot 4\cdot 6} + \ldots.$

[Ans.: Divergent]

2. $\sum \frac{1^2 \cdot 5^2 \cdot 9^2 \ldots\ldots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \ldots\ldots (4n)^2}.$

[Ans.: Convergent]

3. (i) $1 + \frac{2^2}{3\cdot 4} + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6} + \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} + \ldots.$

[Ans.: Convergent]

(ii) $1+\dfrac{(1!)^2}{2!}x+\dfrac{(2!)^2}{4!}x^2+\dfrac{(3!)^2}{6!}x^3+\ldots.$

[**Ans.:** Convergent for $x < 4$ and divergent for $x \geq 4$]

(iii) $1+\dfrac{3}{7}x+\dfrac{3\cdot 6}{7\cdot 10}x^2+\dfrac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\ldots.$

[**Ans.:** Convergent for $x \leq 1$ and divergent for $x > 1$]

4. $1+\dfrac{2^2}{3^2}+\dfrac{2^2\cdot 4^2}{3^2\cdot 5^2}+\dfrac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\ldots.$

[**Ans. :** Divergent]

5. $\dfrac{a(a+1)}{2!}+\dfrac{(a+1)(a+2)}{3!}+\dfrac{(a+2)(a+3)}{4!}+\ldots.$

[**Ans.:** Convergent for $a \leq 0$]

6. $\sum \dfrac{(n!)^2}{(2n)!}x^{2n}.$

[**Ans.:** Convergent for $x < 4$ and divergent for $x^2 \geq 4$]

5.13 CAUCHY'S ROOT TEST

If Σu_n is a positive term series and if $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = l$ then

(i) Σu_n is convergent if $l < 1$.

(ii) Σu_n is divergent if $l > 1$.

Proof

Case I If $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = l < 1.$

Consider a number $l < r < 1$ such that $(u_n)^{\frac{1}{n}} < r$ for all $n > m$

$$u_n < r^n \text{ for all } n > m \qquad \ldots (1)$$

The geometric series, $\Sigma r^n = r + r^2 + r^3 + \ldots\infty$

$$
\begin{aligned}
S_n &= r + r^2 + r^3 + \ldots + r^n \\
&= \frac{r(1-r^n)}{1-r} \\
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{r(1-r^n)}{1-r} \\
&= \frac{r}{1-r}, \text{ which is finite}
\end{aligned}
\qquad
\left[\begin{array}{l} \because r < 1 \\ \therefore \lim\limits_{x\to\infty} r^n = 0 \end{array}\right]
$$

Hence, the series Σr^n is convergent.

From Eq. (1), $u_n < r^n$ for all $n > m$

$$\Sigma u_n < \Sigma r^n$$

Since Σr^n is convergent, Σu_n is also convergent.

Case II: If $\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}} = l > 1$.

$$(u_n)^{\frac{1}{n}} > 1 \text{ for all } n > m \qquad \ldots (2)$$

Neglecting the first m terms,

$$
\begin{aligned}
\Sigma(u_n)^{\frac{1}{n}} &= (u_{m+1})^{\frac{1}{m+1}} + (u_{m+2})^{\frac{1}{m+2}} + (u_{m+3})^{\frac{1}{m+3}} + \ldots\infty \\
&> 1 + 1 + 1 \ldots\infty \qquad \text{[Using Eq. (2)]}
\end{aligned}
$$

$$
\begin{aligned}
S_n &= (u_{m+1})^{\frac{1}{m+1}} + (u_{m+2})^{\frac{1}{m+2}} + (u_{m+3})^{\frac{1}{m+3}} + \ldots + (u_{m+n})^{\frac{1}{m+n}} \\
&> 1 + 1 + 1 \ldots n \text{ terms} = n
\end{aligned}
$$

$$\lim_{n\to\infty} S_n > \lim_{n\to\infty} n \to \infty$$

The series $\sum\limits_{n=m+1}^{\infty} u_n$ is divergent. The nature of a series remains unchanged if we neglect a finite number of terms in the beginning. Hence, the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

***Note*:** If $\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}} = 1$, the root test fails, i.e., no conclusion can be drawn about the convergence or divergence of the series.

Example 1

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

Solution

Let
$$u_n = \frac{1}{(\log n)^n}$$
$$(u_n)^{\frac{1}{n}} = \frac{1}{\log n}$$
$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\log n}$$
$$= 0 < 1 \qquad [\because \log\infty \to \infty]$$

Hence, by Cauchy's root test, the series is convergent.

Example 2

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{a^{n+1}}{n^n}$.

Solution

Let
$$u_n = \frac{a^{n+1}}{n^n}$$
$$(u_n)^{\frac{1}{n}} = \frac{(a)^{1+\frac{1}{n}}}{n}$$
$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a\, a^{\frac{1}{n}}}{n}$$
$$= 0 < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 3

Test the convergence of the series $\sum \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$.

Solution

Let
$$u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$$

$$(u_n)^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$=\frac{1}{e}<1$$

Hence, by Cauchy's root test, the series is convergent.

Example 4

Test the convergence of the series $\sum \dfrac{(n-\log n)^n}{2^n \cdot n^n}$.

Solution

Let
$$u_n = \frac{(n-\log n)^n}{2^n \cdot n^n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{(n-\log n)}{2n} \qquad \left[\frac{\infty}{\infty}\text{form}\right]$$

$$=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{\log n}{2n}\right)$$

$$=\frac{1}{2}-\lim_{n\to\infty}\frac{\frac{1}{n}}{2} \qquad \text{[Using L'Hospital's rule]}$$

$$=\frac{1}{2}<1$$

Hence, by Cauchy's root test, the series is convergent.

Example 5

Test the convergence of the series

$$\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\cdots+\left(\frac{n}{2n+1}\right)^n+\cdots.$$

Solution

Let $$u_n = \left(\frac{n}{2n+1}\right)^n$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{2n+1}$$

$$= \lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}$$

$$= \frac{1}{2} < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 6

Test the convergence of the series $\frac{1^3}{3}+\frac{2^3}{3^2}+\frac{3^3}{3^3}+\frac{4^3}{3^4}+\cdots$.

Solution

Let $$u_n = \frac{n^3}{3^n}$$

$$(u_n)^{\frac{1}{n}} = \left(\frac{n^3}{3^n}\right)^{\frac{1}{n}}$$

$$= \frac{n^{\frac{3}{n}}}{3}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^{\frac{3}{n}}}{3}$$

$$= \frac{1}{3}\lim_{n\to\infty}\left(n^{\frac{1}{n}}\right)^3 \qquad \ldots(1)$$

Let $$l = \lim_{n\to\infty}(n)^{\frac{1}{n}}$$

$$\log l = \lim_{n\to\infty}\frac{1}{n}\log n$$

$$= \lim_{n\to\infty}\frac{\log n}{n} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$\log l = 0$$

$$l = e^0$$

$$= 1$$

$$\therefore \lim_{n\to\infty} (n)^{\frac{1}{n}} = 1$$

Substituting in Eq. (1),

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \frac{1}{3} < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 7

Test the convergence of the series

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots.$$

Solution

Let
$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$$

$$(u_n)^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$$

$$= \left[\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= (e-1)^{-1}$$

$$= \frac{1}{e-1} < 1$$

Hence, by Cauchy's root test, the series is convergent.

Example 8

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n x^n}{(n+1)^n}, x > 0.$

Solution

Let
$$u_n = \frac{n^n x^n}{(n+1)^n}$$

$$(u_n)^{\frac{1}{n}} = \left[\frac{n^n x^n}{(n+1)^n}\right]^{\frac{1}{n}}$$

$$= \frac{nx}{n+1}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{x}{1+\frac{1}{n}}$$

$$= x$$

Hence, by Cauchy's root test, the series is

(i) convergent if $x < 1$

(ii) divergent if $x > 1$

The test fails if $x = 1$.

Then
$$u_n = \frac{n^n}{(n+1)^n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} \neq 0$$

The series is divergent for $x = 1$.

Hence, the series is convergent if $x < 1$ and is divergent if $x \geq 1$.

Example 9

Test the convergence of the series $\sum \frac{(n+1)^n x^n}{n^{n+1}}$. **[Summer 2014]**

Solution

Let
$$u_n = \frac{(n+1)^n x^n}{n^{n+1}}$$

$$(u_n)^{\frac{1}{n}} = \frac{(n+1)x}{n^{\frac{n+1}{n}}}$$

$$= \frac{(n+1)x}{n.n^{\frac{1}{n}}}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{x}{n^{\frac{1}{n}}}$$

$$= x \qquad \left[\because \lim_{x\to\infty} n^{\frac{1}{n}} = 1 \text{ as solved in Ex 6}\right]$$

Hence, by Cauchy's root test, the series is convergent, if $x < 1$ and divergent if $x > 1$.

The test fails for $x = 1$.

Then
$$u_n = \frac{(n+1)^n}{n^{n+1}}$$

$$= \frac{(n+1)^n}{n \cdot n^n}$$

$$= \frac{1}{n}\left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{n}\left(1+\frac{1}{n}\right)^n$$

Let
$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e \qquad \text{[finite and non-zero]}$$

$\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, Σu_n is also divergent for $x = 1$.

Hence, the series is convergent for $x < 1$ and divergent for $x \geq 1$.

Example 10

Test the convergence of the series $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots\infty$.

Solution

Let
$$u_n = \left(\frac{n}{n+1}\right)^{n-1} x^{n-1}$$

$$(u_n)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^{\frac{n-1}{n}} x^{\frac{n-1}{n}}$$

$$= \left(\frac{n}{n+1}\right)^{1-\frac{1}{n}} (x)^{1-\frac{1}{n}}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{1-\frac{1}{n}} (x)^{1-\frac{1}{n}}$$

$$= x$$

Hence, by Cauchy's root test, the series is convergent if $x < 1$ and divergent if $x > 1$. Root test fails for $x = 1$.

EXERCISE 5.5

Test the convergence of the following series:

1. $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots\infty$ **[Ans.:** Convergent**]**

2. $\sum\left(\frac{n+1}{3n}\right)^n$ **[Ans.:** Convergent**]**

3. $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

[Ans.: Convergent**]**

4. $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\ldots(x>0)$

[Ans.: Convergent**]**

5. $\sum\left(1+\frac{1}{n}\right)^{n^2}$

[Ans.: Divergent**]**

6. $\sum\frac{(1+nx)^n}{n^n}$

[Ans.: Convergent if $x < 1$ and divergent if $x > 1$**]**

5.14 CAUCHY'S INTEGRAL TEST

If $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} f(n)$ is a positive term series where $f(n)$ decreases as n increases and let $\int_1^{\infty} f(x)\mathrm{d}x = I$ then

(i) Σu_n is convergent if I is finite

(ii) Σu_n is divergent if I is infinite

Proof Consider the area under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$ represented as $\int_1^{n+1} f(x)\mathrm{d}x$. Plot the terms $f(1), f(2), f(3), \ldots, f(n), f(n + 1)$.

The area $\int_1^{n+1} f(x)\mathrm{d}x$ lies between the sum of the areas of smaller rectangles and sum of the areas of larger rectangles

$$f(2)+f(3)+\cdots+f(n+1) \le \int_1^{n+1} f(x)\mathrm{d}x \le f(1)+f(2)+f(3)+\cdots+f(n)$$

$$S_{n+1} - f(1) \le \int_1^{n+1} f(x)\mathrm{d}x \le S_n$$

As $n \to \infty$ first inequality reduces to

$$\lim_{n\to\infty} S_{n+1} \le \int_1^{\infty} f(x)\mathrm{d}x + f(1)$$

This shows that if $\int_1^{\infty} f(x)\mathrm{d}x$ is finite, $\Sigma f(n) = \Sigma u_n$ is convergent.

As $n \to \infty$ second inequality reduces to

$$\int_1^{\infty} f(x)\mathrm{d}x \le \lim_{n\to\infty} S_n$$

or $$\lim_{n\to\infty} S_n \ge \int_1^{\infty} f(x)\mathrm{d}x$$

This shows that if $\int_1^{\infty} f(x)\mathrm{d}x$ is infinite,

$\Sigma f(n) = \Sigma u_n$ is divergent.

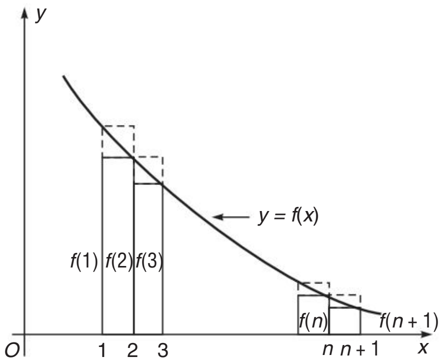

Fig. 5.2

Example 1

Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

Solution

Let
$$u_n = \frac{1}{n\log n} = f(n)$$
$$f(x) = \frac{1}{x\log x}$$
$$\int_2^\infty f(x)\mathrm{d}x = \int_2^\infty \frac{1}{x\log x}\mathrm{d}x$$
$$= \lim_{m\to\infty}\int_2^m \frac{1}{x\log x}\mathrm{d}x$$
$$= \lim_{m\to\infty}\left|\log\log x\right|_2^m$$
$$= \lim_{m\to\infty}(\log\log m - \log\log 2) \to \infty \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log f(x)\right]$$

Hence, by Cauchy's integral test, the series is divergent.

Example 2

Test the convergence of the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$.

Solution

Let
$$u_n = n^2 e^{-n^3} = f(n)$$
$$f(x) = x^2 e^{-x^3}$$
$$\int_1^\infty f(x)\mathrm{d}x = \int_1^\infty x^2 e^{-x^3}\mathrm{d}x$$
$$= \lim_{m\to\infty}\left[-\frac{1}{3}\int_1^m e^{-x^3}(-3x^2)\mathrm{d}x\right]$$
$$= \lim_{m\to\infty}\left[-\frac{1}{3}\left|e^{-x^3}\right|_1^m\right] \qquad \left[\because e^{f(x)}f'(x)\mathrm{d}x = e^{f(x)}\right]$$
$$= \lim_{m\to\infty}\left[-\frac{1}{3}\left(e^{-m^3} - e^{-1}\right)\right]$$
$$= -\frac{1}{3}(e^{-\infty} - e^{-1})$$
$$= -\frac{1}{3}\left(0 - \frac{1}{e}\right)$$
$$= \frac{1}{3e} \quad \text{[finite]}$$

Hence, by Cauchy's integral test, the series is convergent.

Example 3

Test the convergence of the series $\sum_{n=3}^{\infty} \frac{1}{n \log n\sqrt{\log^2 n-1}}$.

[Winter 2015]

Solution

Let
$$u_n = \frac{1}{n\log n\sqrt{\log^2 n-1}}$$
$$f(x) = \frac{1}{x\log x\sqrt{\log^2 x-1}}$$
$$\int_3^{\infty} f(x)\,\mathrm{d}x = \int_3^{\infty} \frac{1}{x\log x\sqrt{\log^2 x-1}}\,\mathrm{d}x$$

Putting $\log x = t$, $\quad \frac{1}{x}\mathrm{d}x = \mathrm{d}t$

When $x = 3$, $\quad t = \log 3$

When $x \to \infty$, $\quad t = \log \infty = \infty$

$$\begin{aligned}
\int_3^{\infty} f(x)\mathrm{d}x &= \int_{\log 3}^{\infty} \frac{\mathrm{d}t}{t\sqrt{t^2-1}}\\
&= \lim_{m\to\infty} \int_{\log 3}^{m} \frac{\mathrm{d}t}{t\sqrt{t^2-1}}\\
&= \lim_{m\to\infty} \left|\sec^{-1} t\right|_{\log 3}^{m}\\
&= \lim_{m\to\infty} \left[\sec^{-1} m - \sec^{-1}(\log 3)\right]\\
&= \sec^{-1}\infty - \sec^{-1}(\log 3)\\
&= \frac{\pi}{2} - \sec^{-1}(\log 3)\\
&= \operatorname{cosec}^{-1}(\log 3) \qquad [\text{finite}]
\end{aligned}$$

Hence, by Cauchy's integral test, the series is convergent.

Example 4

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2\tan^{-1} n}{1+n^2}$. **[Winter 2014]**

Solution

Let
$$u_n = \frac{2\tan^{-1} n}{1+n^2} = f(n)$$
$$f(x) = \frac{2\tan^{-1} x}{1+x^2}$$
$$\int_1^{\infty} f(x)\,\mathrm{d}x = \int_1^{\infty} \frac{2\tan^{-1} x}{1+x^2}\,\mathrm{d}x$$
$$= \lim_{m\to\infty} \int_1^{m} \frac{2\tan^{-1} x}{1+x^2}\,\mathrm{d}x$$

Putting $\tan^{-1}x = t$, $\quad \dfrac{\mathrm{d}x}{1+x^2} = \mathrm{d}t$

When $x = 1$, $\quad t = \tan^{-1}(1) = \dfrac{\pi}{4}$

When $x = m$, $\quad t = \tan^{-1} m$

$$\int_1^{\infty} f(x)\mathrm{d}x = \lim_{m\to\infty} \int_{\frac{\pi}{4}}^{\tan^{-1} m} 2t\,\mathrm{d}t$$
$$= \lim_{m\to\infty} \left|\frac{2t^2}{2}\right|_{\frac{\pi}{4}}^{\tan^{-1} m}$$
$$= \lim_{m\to\infty} \left[\left(\tan^{-1} m\right)^2 - \frac{\pi^2}{16}\right]$$
$$= \left(\tan^{-1} \infty\right)^2 - \frac{\pi^2}{16}$$
$$= \frac{\pi^2}{4} - \frac{\pi^2}{16}$$
$$= \frac{3\pi^2}{16} \qquad \text{[finite]}$$

Hence, by Cauchy's integral test, the series is convergent.

Example 5

Show that the harmonic series of order p,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\infty$$ *is convergent if* $p > 1$ *and is divergent if* $p \le 1$.

[Summer 2015]

Solution

Let

$$u_n = \frac{1}{n^p} = f(n)$$

$$f(x) = \frac{1}{x^p}$$

$$\int_1^\infty f(x)\mathrm{d}x = \int_1^\infty \frac{1}{x^p}\mathrm{d}x$$

$$= \lim_{m\to\infty} \left| \frac{x^{-p+1}}{-p+1} \right|_1^m$$

$$= \lim_{m\to\infty} \left(\frac{m^{1-p}}{1-p} - \frac{1}{1-p} \right)$$

$$= -\frac{1}{1-p}, \qquad p > 1$$

$$= \infty, \qquad p < 1$$

If $p = 1$,

$$\int_1^\infty f(x)\mathrm{d}x = \int_1^\infty \frac{1}{x}\mathrm{d}x$$

$$= \lim_{m\to\infty} \int_1^m \frac{1}{x}\mathrm{d}x$$

$$= \lim_{m\to\infty} \left|\log x\right|_1^m$$

$$= \lim_{m\to\infty} (\log m - \log 1)$$

$$= \log\infty \to \infty$$

The integral $\int_1^\infty f(x)\mathrm{d}x$ is finite if $p > 1$ and is infinite if $p \le 1$.

Hence, by Cauchy's integral test, the series is convergent if $p > 1$ and is divergent if $p \le 1$.

EXERCISE 5.5

Test the convergence of the following series:

1. $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ **[Ans.:** Divergent]

2. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$ **[Ans.:** Convergent]

3. $\displaystyle\sum_{n=1}^\infty ne^{-n^2}$ **[Ans.:** Convergent]

4. $\displaystyle\sum_{n=1}^\infty \frac{1}{n(\log n)^2}$ **[Ans.:** Convergent]

5.15 ALTERNATING SERIES

An infinite series with alternate positive and negative terms is called an alternating series.

Leibnitz's Test for Alternating Series

An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}v_n = \sum_{n=1}^{\infty}u_n$ is convergent if

(i) each term is numerically less than its preceding term, i.e. $|u_{n+1}| < |u_n|$ or $|u_n| > |u_{n+1}|$

(ii) $\lim_{n\to\infty}|u_n| = 0$

Example 1

Test the convergence of the series $1-\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}-\dfrac{1}{\sqrt{4}}+\dots$.

Solution

Let $\quad u_n = (-1)^{n-1}\dfrac{1}{\sqrt{n}}$

$$|u_n| = \frac{1}{\sqrt{n}}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} > 0 \qquad \text{for all } n \in N$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\left|\frac{1}{\sqrt{n}}\right|$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 2

Test the convergence of the series $1-\dfrac{1}{2\sqrt{2}}+\dfrac{1}{3\sqrt{3}}-\dfrac{1}{4\sqrt{4}}+\cdots$.

Solution

Let $$u_n = \frac{(-1)^{n-1}}{n\sqrt{n}}$$

$$|u_n| = \frac{1}{n\sqrt{n}}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{n\sqrt{n}} - \frac{1}{(n+1)\sqrt{n+1}}$$

$$= \frac{(n+1)\sqrt{n+1} - n\sqrt{n}}{\left(n\sqrt{n}\right)\left[(n+1)\sqrt{(n+1)}\right]} > 0 \quad \text{for all } n \in N$$

$$\therefore |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n\sqrt{n}}$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 3

Test the convergence of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$

Solution

Let
$$u_n = (-1)^{n-1} \cdot \frac{1}{n^2}$$

$$|u_n| = \frac{1}{n^2}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$= \frac{2n+1}{n^2(n+1)^2} > 0 \quad \text{for all } n \in N$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n^2}$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 4

Test the convergence of the series $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots.$

Solution

Let $$u_n = (-1)^n \frac{1}{n^p}$$

$$|u_n| = \frac{1}{n^p}$$

The given series is an alternating series.

Case I: If $p > 0$,

(i) $$|u_n| - |u_{n+1}| = \frac{1}{n^p} - \frac{1}{(n+1)^p}$$

$$= \frac{(n+1)^p - n^p}{n^p (n+1)^p} > 0 \qquad [\because p > 0]$$

$$\therefore |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n^p}$$

$$= 0 \qquad [\because p > 0]$$

Hence, by Leibnitz's test, the series is convergent if $p > 0$.

Case II: If $p < 0$

In this case the conditions (i) and (ii) of the Leibnitz's test are not satisfied.
Hence, the given series is not convergent if $p < 0$.

Example 5

Test the convergence of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$.

Solution

Let $$u_n = (-1)^{n-1} \frac{1}{2^{n-1}}$$

$$|u_n| = \frac{1}{2^{n-1}}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{2^{n-1}} - \frac{1}{2^n}$$

$$= \frac{1}{2^{n-1}}\left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2^{n-1}} \cdot \frac{1}{2}$$

$$= \frac{1}{2^n} > 0 \quad \text{for all } n \in N$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{2^{n-1}}$$
$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 6

Test the convergence of the series $\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \cdots$.

Solution

Let $$u_n = (-1)^{n-1} \frac{n}{n^2+1}$$

$$|u_n| = \frac{n}{n^2+1}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1}$$
$$= \frac{n(n^2+2n+2)-(n+1)(n^2+1)}{(n^2+1)(n^2+2n+2)}$$
$$= \frac{n^2+n-1}{(n^2+1)(n^2+2n+2)} > 0 \quad \text{for all } n \in N$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{n}{n^2+1}$$
$$= \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}}$$
$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 7

Test the convergence of the series $\sum_{n\le 1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$. **[Summer 2014]**

Solution

Let $$u_n = \frac{(-1)^{n+1}}{\log(n+1)}$$

$$|u_n| = \frac{1}{\log(n+1)}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{\log(n+1)} - \frac{1}{\log(n+2)}$$

$$= \frac{\log(n+2) - \log(n+1)}{\log(n+1)\cdot\log(n+2)} > 0$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{\log(n+1)}$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 8

Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2(n+1)}$.

Solution

Let $$u_n = \frac{(-1)^{n-1}}{n^2(n+1)}$$

$$|u_n| = \frac{1}{n^2(n+1)}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{n^2(n+1)} - \frac{1}{(n+1)^2(n+2)}$$

$$= \frac{(n+1)(n+2) - n^2}{n^2(n+1)^2(n+2)}$$

$$= \frac{3n+2}{n^2(n+1)^2(n+2)} > 0 \quad \text{for all } n \in N$$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n^2(n+1)}$$

$$= 0$$

Hence, by Leibnitz's test, the series is convergent.

Example 9

Test the convergence of the series $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots(if\ x<1)$.

Solution

Let
$$u_n=(-1)^{n-1}\frac{x^n}{n}$$
$$|u_n|=\frac{x^n}{n}$$

The given series is an alternating series.

(i) $$|u_n|-|u_{n+1}|=\frac{x^n}{n}-\frac{x^{n+1}}{n+1}$$
$$=\frac{x^n[(n+1)-nx]}{n(n+1)}$$
$$=\frac{x^n[1+(1-x)n]}{n(n+1)}>0 \qquad [\because n\ge 1 \text{ and } 0<x<1]$$
$$\therefore \quad |u_n|>|u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n|=\lim_{n\to\infty}\frac{x^n}{n} \qquad \left[\because \lim_{n\to\infty}x^n=0 \text{ if } x<1\right]$$
$$=0$$

Hence, by Leibnitz's test, the series is convergent.

Example 10

Test the convergence of the series

$$\log\left(\frac{1}{2}\right)-\log\left(\frac{2}{3}\right)+\log\left(\frac{3}{4}\right)-\log\left(\frac{4}{5}\right)+\cdots$$

Solution

$$\log\left(\frac{1}{2}\right)-\log\left(\frac{2}{3}\right)+\log\left(\frac{3}{4}\right)-\log\left(\frac{4}{5}\right)+\cdots$$
$$=-\log\left(\frac{2}{1}\right)+\log\left(\frac{3}{2}\right)-\log\left(\frac{4}{3}\right)+\log\left(\frac{5}{4}\right)-\cdots$$

Let
$$u_n=(-1)^n\log\left(\frac{n+1}{n}\right)$$
$$|u_n|=\log\left(\frac{n+1}{n}\right)$$

The given series is an alternating series.

(i) $|u_n| - |u_{n+1}| = \log\dfrac{n+1}{n} - \log\dfrac{n+2}{n+1} > 0 \quad \left[\because \dfrac{n+1}{n} > \dfrac{n+2}{n+1} \text{ for all } n \in N\right]$

$$\therefore \quad |u_n| > |u_{n+1}|$$

(ii)
$$\begin{aligned}\lim_{n\to\infty}|u_n| &= \lim_{n\to\infty}\log\left(\frac{n+1}{n}\right)\\ &= \lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)\\ &= \log 1\\ &= 0\end{aligned}$$

Hence, by Leibnitz's test, the series is convergent.

Example 11

Test the convergence of the series $\dfrac{1}{2} - \dfrac{2}{3} + \dfrac{3}{4} - \dfrac{4}{5} + \cdots$.

Solution

Let
$$u_n = (-1)^{n-1}\cdot\frac{n}{n+1}$$

$$|u_n| = \frac{n}{n+1}$$

The given series is an alternating series.

(i)
$$\begin{aligned}|u_n| - |u_{n+1}| &= \frac{n}{n+1} - \frac{n+1}{n+2}\\ &= \frac{n^2+2n-n^2-2n-1}{(n+1)(n+2)}\\ &= -\frac{1}{(n+1)(n+2)} < 0\end{aligned}$$

Since each term of the series is not numerically less than the preceding term, Leibnitz's test cannot be applied.

The series can be written as

$$\begin{aligned}\sum_{n=1}^{\infty} u_n &= \left(1-\frac{1}{2}\right) - \left(1-\frac{1}{3}\right) + \left(1-\frac{1}{4}\right) - \left(1-\frac{1}{5}\right) + \cdots\\ &= (1-1+1-1+\cdots) + \left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right)\\ &= \sum_{n=1}^{\infty}(-1)^{n-1} + (\log 2 - 1)\end{aligned}$$

As $n \to \infty$, the sum of this series tends to $(-1 + \log 2 - 1)$ or $(1 + \log 2 - 1)$ according as n is even or odd.
Hence, the given series is an oscillatory series.

EXERCISE 5.6

Test the convergence of the following series:

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots\ldots$

[Ans.: Convergent]

2. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2n-1}$

[Ans.: Oscillatory]

3. $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots$

[Ans.: Convergent]

4. $1 - 2x + 3x^2 - 4x^3 + \ldots (x < 1)$

[Ans.: Convergent]

5. $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots (0 < x < 1)$

[Ans.: Convergent]

5.16 ABSOLUTE AND CONDITIONAL CONVERGENT OF A SERIES

Absolute Convergence of a Series The series $\sum_{n=1}^{\infty} u_n$ with both positive and negative terms (not necessarily alternative) is called absolutely convergent if the corresponding series $\sum_{n=1}^{\infty} |u_n|$ with all positive terms is convergent.

Conditional Convergence of a Series If the series $\sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} |u_n|$ is divergent, then the series $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent.

Note 1: Every absolutely convergent series is a convergent series but converse is not true.
Note 2: Any convergent series of positive terms is also absolutely convergent.

Example 1

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2^{3n}}{3^{2n}}$.

Solution

Let
$$u_n = \frac{(-1)^n 2^{3n}}{3^{2n}}$$
$$|u_n| = \frac{2^{3n}}{3^{2n}}$$
$$|u_{n+1}| = \frac{2^{3(n+1)}}{3^{2(n+1)}}$$
$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{2^{3n}}{3^{2n}} \cdot \frac{3^{2n+2}}{2^{3n+3}}$$
$$= \lim_{n\to\infty} \frac{9}{8}$$
$$= \frac{9}{8} > 1$$

By D'Alembert's ratio test, $\sum_{n=1}^{\infty} |u_n|$ is convergent. Thus, the series is absolutely convergent and hence convergent.

Example 2

Test the series for absolute or conditional convergence
$$1 - \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \cdots .$$

Solution

Let
$$u_n = (-1)^{n-1} \cdot \frac{n}{3^{n-1}}$$
$$\sum_{n=1}^{\infty} |u_n| = 1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \ldots$$
$$|u_n| = \frac{n}{3^{n-1}}$$
$$|u_{n+1}| = \frac{n+1}{3^n}$$
$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{n}{3^{n-1}} \cdot \frac{3^n}{n+1}$$
$$= \lim_{n\to\infty} \frac{3}{1+\frac{1}{n}}$$
$$= 3 > 1$$

By D'Alembert's ratio test, $\sum_{n=1}^{\infty}|u_n|$ is convergent and hence, the series is absolutely convergent.

Example 3

Test the series for absolute or conditional convergence $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$.

[Winter 2016]

Solution

Let $$u_n=\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$$

$$|u_n|=\frac{1}{\sqrt{n}+\sqrt{1+n}}$$

The given series is an alternating series.

(i) $$|u_n|-|u_{n+1}|=\frac{1}{\sqrt{n}+\sqrt{1+n}}-\frac{1}{\sqrt{n+1}+\sqrt{n+2}}$$

$$=\frac{\sqrt{n+1}+\sqrt{n+2}-\sqrt{n}-\sqrt{n+1}}{(\sqrt{n}+\sqrt{1+n})(\sqrt{1+n}+\sqrt{n+2})}$$

$$=\frac{\sqrt{n+2}-\sqrt{n}}{(\sqrt{n}+\sqrt{1+n})(\sqrt{1+n}+\sqrt{n+2})}>0 \qquad \text{for all } n\in N$$

$$\therefore \quad |u_n|>|u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n|=\lim_{n\to\infty}\frac{1}{\sqrt{n}+\sqrt{1+n}}=0$$

By Leibnitz's test, $\sum u_n$ is convergent.

$$|u_n|=\frac{1}{\sqrt{n}+\sqrt{1+n}}$$

$$=\frac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$$

Let $$v_n=\frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{|u_n|}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}$$

$$= 6 \qquad \text{[finite and non-zero]}$$

and $\sum v_n = \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent as $p = \frac{1}{2}$.

By comparison test, $\sum |u_n|$ is also divergent.

The series $\sum u_n$ is convergent and $\sum |u_n|$ is divergent.
Hence, the series is conditionally convergent.

Example 4

Determine absolute or conditional convergence of the series

$$\sum_{n=1}^{\infty}(-1)^n \cdot \frac{n^2}{n^3+1}$$ **[Winter 2013; Summer 2017]**

Solution

Let $$u_n = (-1)^n \cdot \frac{n^2}{n^3+1}$$

$$|u_n| = \frac{n^2}{n^3+1}$$

$$= \frac{1}{n\left(1+\frac{1}{n^3}\right)}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{|u_n|}{v_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^3}}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, $\sum |u_n|$ is also divergent.

Hence, Σu_n is not absolutely convergent.
To check the conditional convergence, applying Leibnitz's test,

(i) $|u_n| - |u_{n+1}| = \frac{n^2}{n^3+1} - \frac{(n+1)^2}{(n+1)^3+1}$

$$= \frac{n^2(n^3+3n^2+3n+2)-(n^3+1)(n^2+2n+1)}{(n^3+1)[(n+1)^3+1]}$$

$$= \frac{n^4+2n^3+n^2-2n-1}{(n^3+1)[(n+1)^3+1]}$$

$$= \frac{n^4+n^2(2n+1)-1(2n+1)}{(n^3+1)[(n+1)^3+1]}$$

$$= \frac{n^4+(2n+1)(n^2-1)}{(n^3+1)[(n+1)^3+1]} > 0 \quad \text{for all } n \in N$$

$$|u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{n^2}{n^3+1}$$

$$= \lim_{n\to\infty} \frac{1}{n\left(1+\frac{1}{n^3}\right)}$$

$$= 0$$

By Leibnitz's test, $\sum u_n$ is convergent.

The series Σu_n is convergent and $\sum |u_n|$ is divergent.

Hence, the series is conditionally convergent.

Example 5

Test the series for absolute or conditional convergence

$$\frac{2}{3} - \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} - \frac{5}{6}\cdot\frac{1}{4} + \dots$$

Solution

Let $$u_n = (-1)^{n-1}\left(\frac{n+1}{n+2}\cdot\frac{1}{n}\right)$$

$$\sum_{n=1}^{\infty} |u_n| = \frac{2}{3} + \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} + \frac{5}{6}\cdot\frac{1}{4} + \cdots.$$

$$|u_n| = \frac{n+1}{n+2}\cdot\frac{1}{n}$$

Let $$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{|u_n|}{v_n} = \lim_{n\to\infty} \frac{n+1}{n+2}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}}$$

$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, $\Sigma |u_n|$ is also divergent.

Hence, the series is not absolutely convergent.

To check the conditional convergence, applying Leibnitz's test,

(i) $$|u_n| - |u_{n+1}| = \frac{n+1}{n(n+2)} - \frac{n+2}{(n+1)(n+3)}$$

$$= \frac{n^2+3n+3}{n(n+1)(n+2)(n+3)} > 0 \qquad \text{for all } n \in N$$

$$|u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{n+1}{n(n+2)}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{n\left(1+\frac{2}{n}\right)}$$

$$= 0$$

By Leibnitz's test, Σu_n is convergent.
The series Σu_n is convergent and $\Sigma |u_n|$ is divergent.
Hence, the series is conditionally convergent.

Example 6

Test the convergence of the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, x > 0.$

[Summer 2016]

Solution

Let $$u_n = (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$$

$$|u_n| = \frac{x^{2n-1}}{2n-1}$$

$$|u_{n+1}| = \frac{x^{2n+1}}{2n+1}$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{x^{2n-1}}{2n-1} \cdot \frac{2n+1}{x^{2n+1}}$$

$$= \frac{2+\frac{1}{n}}{2-\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \left(\frac{2+\frac{1}{n}}{2-\frac{1}{n}} \right) \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

By D'Alembert's ratio test, $\sum |u_n|$ is convergent if $\frac{1}{x^2} > 1$ or $x^2 < 1$ or $x < 1 \quad [\because x > 0]$

Thus, the given series is absolutely convergent and hence, is convergent for $x < 1$.

If $x^2 = 1$ or $x = 1 \quad [\because x > 0]$

$$u_n = \frac{(-1)^{n-1}}{2n-1}$$

$$|u_n| = \frac{1}{2n-1}$$

The given series is an alternating series.

(i) $|u_n| - |u_{n+1}| = \frac{1}{2n-1} - \frac{1}{2n+1}$

$$= \frac{2}{4n^2-1} > 0 \quad \text{for all } n \in N$$

(ii) $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{2n-1}$

$$= 0$$

By Leibnitz's test, the series is convergent for $x = 1$

Hence, the series is convergent for $x \le 1$.

EXERCISE 5.7

Test the following series for absolute or conditional convergence:

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

[**Ans.:** Conditionally convergent]

2. $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \ldots$

[**Ans.:** Absolutely convergent]

3. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$

[Ans.: Conditionally convergent]

4. $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\ldots$

[Ans.: Absolutely convergent]

5.17 POWER SERIES

A power series is an infinite series of the form $\sum_{n=1}^{\infty} a_n x^n = a_0 + a_1(x-c)+..$ $a_2(x-c)^2 + a_3(x-c)^3 + \cdots$, where a_n represents the coefficient of the n^{th} term, c is a constant and x varies around c. When $c = 0$, the series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

5.17.1 Interval and Radius of Convergence

A power series will converge only for certain values of x. An interval $(-R, R)$ in which a power series converges is called the interval of convergence. The number R is called the radius of convergence, e.g., if a power series converges for all the values of x, then interval of convergence will be $(-\infty, \infty)$ and the radius of convergence will be ∞.

5.17.2 Test for Convergence

Since a power series may be positive, alternating or mixed series, the concept of absolute convergence is used to test the convergence of a power series. Applying D'Alembert's ratio test,

$$u_n = a_n x^n$$

$$u_{n+1} = a_{n+1} x^{n+1}$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{a_n x^n}{a_{n+1} x^{n+1}}\right|$$

$$= \left|\frac{1}{x}\right| \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

If $$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = l,$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{1}{x}\right| \cdot l = \left|\frac{l}{x}\right|$$

By D' Alembert's ratio test, the series is absolutely convergent and hence is convergent

If $\left|\frac{l}{x}\right| > 1$, i.e., $|x| < l, -l < x < l$.

Here, interval of convergence of the series is $(-l, l)$ and the radius of convergence is l.

Example 1

Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{2^n}, x > 0.$

Solution

Let
$$u_n = \frac{x^n}{2^n}$$
$$u_{n+1} = \frac{x^{n+1}}{2^{n+1}}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n}{2^n} \cdot \frac{2^{n+1}}{x^{n+1}}$$
$$= \frac{2}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{2}{x} > 1$ or $x < 2$

(ii) divergent if $\frac{2}{x} < 1$ or $x > 2$

The test fails if $\frac{2}{x} = 1$, or $x = 2$.

Then
$$u_n = \frac{2^n}{2^n} = 1$$
$$\sum_{n=1}^{\infty} u_n = 1 + 1 + 1 + \cdots \infty$$

which is a divergent series.

Hence, the series is convergent for $0 < x < 2$ and the range of convergence is $0 < x < 2$.

Example 2

Determine the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$ *and also, their behaviour at each end point.*

Solution

Let
$$u_n = \frac{2^n x^n}{n!}$$
$$u_{n+1} = \frac{2^{n+1} x^{n+1}}{(n+1)!}$$
$$\frac{u_n}{u_{n+1}} = \frac{2^n x^n}{n!} \cdot \frac{(n+1)!}{2^{n+1} x^{n+1}}$$
$$= \frac{n+1}{2x}$$
$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2x}\right|$$
$$= \infty > 1$$

Hence, By D'Alembert's ratio test, the series is convergent for all values of x i.e. $-\infty < x < \infty$ and interval of convergence is $(-\infty, \infty)$.

Example 3

Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{a+\sqrt{n}}, x > 0, a > 0.$

Solution

Let
$$u_n = \frac{x^n}{a+\sqrt{n}}$$
$$u_{n+1} = \frac{x^{n+1}}{a+\sqrt{n+1}}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n}{a+\sqrt{n}} \cdot \frac{a+\sqrt{n+1}}{x^{n+1}}$$
$$= \lim_{n\to\infty} \frac{\frac{a}{\sqrt{n}} + \sqrt{1+\frac{1}{n}}}{\frac{a}{\sqrt{n}}+1} \cdot \frac{1}{x}$$
$$= \frac{1}{x}$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{1}{x} > 1$ or $x < 1$

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

The test fails if $x = 1$.

Then
$$u_n = \frac{1}{a+\sqrt{n}}$$
$$= \frac{1}{\sqrt{n}}\left(\frac{1}{\frac{a}{\sqrt{n}}+1}\right)$$

Let
$$v_n = \frac{1}{\sqrt{n}}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\frac{a}{\sqrt{n}}+1}$$
$$= 1 \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent as $p = \frac{1}{2} < 1$.

By comparison test, Σu_n is also divergent for $x = 1$.
Hence, the series is convergent for $0 < x < 1$ and the range of convergence is $0 < x < 1$.

Example 4

Obtain the range of convergence of $\sum_{n=1}^{\infty}\frac{(x+1)^n}{3^n \cdot n}$.

Solution

Let
$$u_n = \frac{(x+1)^n}{3^n \cdot n}$$
$$u_{n+1} = \frac{(x+1)^{n+1}}{3^{n+1}(n+1)}$$
$$\frac{u_n}{u_{n+1}} = \frac{(x+1)^n}{3^n \cdot n}\cdot\frac{3^{n+1}(n+1)}{(x+1)^{n+1}}$$
$$= \frac{3(n+1)}{(x+1)n}$$
$$= \frac{3\left(1+\frac{1}{n}\right)}{x+1}$$
$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{3\left(1+\frac{1}{n}\right)}{x+1}\right|$$
$$= \left|\frac{3}{x+1}\right|$$

The series is convergent if

$$\left|\frac{3}{x+1}\right| > 1$$

$$3 > |x+1|$$

$$|x+1| < 3$$

$$-3 < (x+1) < 3$$

$$-4 < x < 2$$

At $x = 2$,

$$u_n = \frac{1}{n}$$

$\therefore \quad \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent as $p = 1$.

At $x = -4$,

$$u_n = \frac{(-1)^n}{n}$$

$$|u_n| = \frac{1}{n}$$

The given series is an alternating series.

(i) $|u_n| - |u_{n+1}| = \frac{1}{n} - \frac{1}{n+1}$

$$= \frac{1}{n(n+1)} > 0 \qquad \text{for all } n \in N$$

$\therefore \quad |u_n| > |u_{n+1}|$

(ii) $\lim\limits_{n\to\infty} |u_n| = \lim\limits_{n\to\infty} \frac{1}{n}$

$= 0$

By Leibnitz's test, the series is convergent at $x = -4$.
Hence, the series is convergent for $-4 \le x < 2$ and the range of convergence is $-4 \le x < 2$.

Example 5

Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n+\sqrt{1+n^2}}$.

Solution

Let

$$u_n = \frac{x^n}{n+\sqrt{1+n^2}}$$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)+\sqrt{1+(n+1)^2}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{n+\sqrt{1+n^2}} \cdot \frac{(n+1)+\sqrt{1+(n+1)^2}}{x^{n+1}}$$

$$= \frac{\left(1+\frac{1}{n}\right)+\sqrt{\frac{1}{n^2}+\left(1+\frac{1}{n}\right)^2}}{\left(1+\sqrt{\frac{1}{n^2}+1}\right)x}$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)+\sqrt{\frac{1}{n^2}+\left(1+\frac{1}{n}\right)^2}}{\left(1+\sqrt{\frac{1}{n^2}+1}\right)x}\right|$$

$$= \frac{1}{|x|}$$

The series is convergent if

$$\frac{1}{|x|} > 1$$

$$|x| < 1$$

$$-1 < x < 1$$

At $x = 1$,

$$u_n = \frac{1}{n+\sqrt{1+n^2}}$$

$$= \frac{1}{n\left(1+\sqrt{\frac{1}{n^2}+1}\right)}$$

Let

$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(1+\sqrt{\frac{1}{n^2}+1}\right)}$$

$$= \frac{1}{2} \quad \text{[finite and non-zero]}$$

and $\Sigma v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

Thus, by comparison test, Σu_n is also divergent if $x = 1$.
At $x = -1$,

$$u_n = \frac{(-1)^n}{n+\sqrt{1+n^2}}$$

$$|u_n| = \frac{1}{n+\sqrt{1+n^2}}$$

The given series is an alternating series.

(i) $$|u_n|-|u_{n+1}|=\frac{1}{n+\sqrt{1+n^2}}-\frac{1}{(n+1)+\sqrt{1+(n+1)^2}}$$

$$=\frac{(n+1)+\sqrt{1+(n+1)^2}-n-\sqrt{1+n^2}}{\left(n+\sqrt{1+n^2}\right)\left[(n+1)+\sqrt{1+(n+1)^2}\right]}$$

$$=\frac{1+\sqrt{1+(n+1)^2}-\sqrt{1+n^2}}{\left(n+\sqrt{1+n^2}\right)\left[(n+1)+\sqrt{1+(n+1)^2}\right]}>0 \qquad \text{for all } n\in N$$

$$\therefore \quad |u_n|>|u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n|=\lim_{n\to\infty}\frac{1}{n+\sqrt{1+n^2}}$$

$$=0$$

Thus, by Leibnitz's test, the series is convergent if $x=-1$.

Hence, the series is convergent for $-1\le x<1$ and the range of convergence is $-1\le x<1$.

Example 6

For the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$, *find the radius and interval of convergence.* **[Winter 2016]**

Solution

Let $$u_n=\frac{(-1)^{n-1}x^{2n-1}}{2n-1}$$

$$u_{n+1}=\frac{(-1)^n x^{2n+2-1}}{2n+2-1}$$

$$=\frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\frac{u_n}{u_{n+1}}=\frac{(-1)^{n-1}x^{2n-1}}{2n-1}\cdot\frac{2n+1}{(-1)^n x^{2n+1}}$$

$$=-\frac{(2n+1)}{(2n-1)}\cdot\frac{1}{x^2}$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)}{(2n-1)}\cdot\frac{1}{x^2}\right|$$

$$= \left|\frac{1}{x^2}\right|$$

By D'Alembert's ratio test, the series is convergent if $\left|\frac{1}{x^2}\right| > 1$ or $1 > |x^2|$ or $|x^2| < 1$ i.e., $-1 < x < 1$ and divergent for $x > 1$.

At $x = 1$,

$$u_n = \frac{(-1)^{n-1}\,(1)^{2n-1}}{2n-1}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$= \frac{(-1)^{n-1}}{2n-1}$$

$$|u_n| = \frac{1}{2n-1}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{1}{2n-1} - \frac{1}{2n+1}$$

$$= \frac{2n+1-2n+1}{(2n-1)\,(2n+1)}$$

$$= \frac{2}{(2n-1)\,(2n+1)} > 0 \quad \text{for all } n \in N$$

$$|u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{2n-1} = 0$$

By Leibnitz's test, $\sum |u_n|$ is convergent.

At $x = -1$,

$$u_n = \frac{(-1)^{n-1}\,(-1)^{2n-1}}{2n-1}$$

$$= \frac{(-1)^{3n-2}}{2n-1}$$

$$|u_n| = \frac{1}{2n-1}$$

The given series is an alternating series.

Hence, by Leibnitz's test, $\sum |u_n|$ is convergent.

Thus, for the interval $-1 \le x \le 1$, given series is convergent.

Since condition for convergence is $|x^2| < 1$, radius of convergence = 1.

Since $\sum_{n=1}^{\infty} |u_n|$ is convergent, the series is absolutely convergent for $-1 \le x \le 1$.

Example 7

Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

[Winter 2013; Summer 2016]

Solution

$$u_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$$u_{n+1} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(-3)^n x^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n+2}}{(-3)^{n+1} x^{n+1}}$$

$$= \frac{\sqrt{n+2}}{(-3)x\sqrt{n+1}}$$

$$= \frac{\sqrt{1+\frac{2}{n}}}{(-3)x\sqrt{1+\frac{1}{n}}}$$

$$\lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{1+\frac{2}{n}}}{(-3)x\sqrt{1+\frac{1}{n}}}\right|$$

$$= \left|\frac{1}{-3x}\right|$$

$$= \left|\frac{1}{3x}\right|$$

By D'Alembert's ratio test, the series is convergent if $\left|\frac{1}{3x}\right| > 1$ or $|3x| < 1$ or $|x| < \frac{1}{3}$ or $-\frac{1}{3} < x < \frac{1}{3}$.

The interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

At $x = -\frac{1}{3}$,

$$\begin{aligned} u_n &= \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} \\ &= \frac{1}{\sqrt{n+1}} \\ &= \frac{1}{\sqrt{n}\sqrt{1+\frac{1}{n}}} \end{aligned}$$

Let $v_n = \frac{1}{\sqrt{n}}$

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{u_n}{v_n}\right| &= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}} \\ &= 1 \quad \text{[finite and non-zero]} \end{aligned}$$

and $\Sigma v_n = \sum \frac{1}{\sqrt{n}}$ is divergent as $p = \frac{1}{2}$.

Thus, by comparison test, Σu_n is also divergent if $x = -\frac{1}{3}$.

At $x = \frac{1}{3}$,

$$\begin{aligned} u_n &= \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} \\ &= \frac{(-1)^n}{\sqrt{n+1}} \\ |u_n| &= \frac{1}{\sqrt{n+1}} \\ \lim_{n\to\infty}|u_n| &= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0 \end{aligned}$$

By Leibnitz's test, the series is convergent at $x = \frac{1}{3}$.

Hence, the series is convergent at each end point.

Example 8

Determine the interval of convergence for the series $\sum_{n=1}^{\infty} (-1)^n \frac{n(x+1)^n}{2^n}$ *and also its behaviour at each end point.*

Solution

Let
$$u_n = (-1)^n \frac{n(x+1)^n}{2^n}$$
$$u_{n+1} = (-1)^{n+1} \frac{(n+1)(x+1)^{n+1}}{2^{n+1}}$$
$$\frac{u_n}{u_{n+1}} = \frac{(-1)^n n\cdot(x+1)^n}{2^n} \cdot \frac{2^{n+1}}{(-1)^{(n+1)}(n+1)(x+1)^{n+1}}$$
$$= \frac{n}{n+1} \cdot \frac{2}{x+1}$$
$$= \frac{1}{1+\frac{1}{n}} \cdot \frac{2}{x+1}$$
$$\lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \left| \left(\frac{1}{1+\frac{1}{n}}\right)\left(\frac{2}{x+1}\right) \right|$$
$$= \left|\frac{2}{x+1}\right|$$

The series is convergent if
$$\left|\frac{2}{x+1}\right| > 1$$
$$2 > |x+1|$$
$$|x+1| < 2$$
$$-2 < (x+1) < 2$$
$$-3 < x < 1$$

The series is convergent in the interval (– 3, 1).

At $x = -3$,

$$u_n = (-1)^n \frac{n(-3+1)^n}{2^n}$$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} n$$

which is a divergent series.
At $x = 1$,

$$\begin{aligned} u_n &= (-1)^n \frac{n(1+1)^n}{2^n} \\ &= (-1)^n n \\ |u_n| &= n \\ \lim_{n\to\infty} |u_n| &\neq 0 \end{aligned}$$

By Leibnitz's test, the series is not convergent at $x = 1$.
Hence, the series is not convergent at each end point and the interval of convergence is $(-3, 1)$.

Example 9

For the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$, *find the radius and interval of convergence. For what values of x does the series converge absolutely, conditionally?* **[Winter 2015]**

Solution

Let

$$\begin{aligned} u_n &= \frac{(-1)^n (x+2)^n}{n} \\ u_{n+1} &= \frac{(-1)^{n+1} (x+2)^{n+1}}{n+1} \\ \frac{u_n}{u_{n+1}} &= \frac{(-1)^n (x+2)^n}{n} \cdot \frac{(n+1)}{(-1)^{n+1} (x+2)^{n+1}} \\ &= -\left(1+\frac{1}{n}\right) \cdot \frac{1}{(x+2)} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| &= \lim_{n\to\infty} \left| \left(1+\frac{1}{n}\right) \cdot \frac{1}{(x+2)} \right| \\ &= \left| \frac{1}{x+2} \right| \end{aligned}$$

The series is convergent if

$$\left|\frac{1}{x+2}\right| > 1$$
$$1 > |x+2|$$
$$|x+2| < 1$$
$$-1 < (x+2) < 1$$
$$-3 < x < -1$$

At $x = -3$,

$$u_n = \frac{(-1)^n(-3+2)^n}{n} = \frac{1}{n}$$
$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a divergent series.
At $x = -1$,

$$u_n = \frac{(-1)^n(-1+2)^n}{n}$$
$$= \frac{(-1)^n}{n}$$
$$|u_n| = \frac{1}{n}$$
$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n} = 0$$

By Leibnitz's test, the series is convergent at $x = -1$.

Hence, interval of convergence is $(-3, -1]$, i.e. $-3 < x \le -1$.

Since condition for convergence is $|x+2| < 1$, radius of convergence = 1.

Since $\sum_{n=1}^{\infty} |u_n|$ is convergent, the series is absolutely convergent for $-3 < x \le -1$.

Example 10

Obtain the range of convergence of $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$.

Solution

Let

$$u_n = \frac{1}{x^n + x^{-n}}$$

$$
\begin{aligned}
&= \frac{x^n}{x^{2n}+1} \\
u_{n+1} &= \frac{x^{n+1}}{x^{2n+2}+1} \\
\frac{u_n}{u_{n+1}} &= \frac{x^n}{x^{2n}+1} \cdot \frac{x^{2n+2}+1}{x^{n+1}} \\
&= \frac{x^{2n+2}+1}{x(x^{2n}+1)} \\
\left|\frac{u_n}{u_{n+1}}\right| &= \left|\frac{x^{2n+2}+1}{x(x^{2n}+1)}\right|
\end{aligned}
$$

If $|x| > 1$,

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| &= \lim_{n\to\infty}\left|\frac{x^2+\frac{1}{x^{2n}}}{x\left(1+\frac{1}{x^{2n}}\right)}\right| \\
&= |x| > 1 \quad \left[\because \lim_{n\to\infty} x^{2n} \to \infty\right]
\end{aligned}
$$

Thus, the series is convergent for $|x| > 1,$ i.e. $x > 1$ and $x < -1$

At $x = 1$,

$$
\begin{aligned}
u_n &= \frac{1}{2} \\
\sum_{n=1}^{\infty} u_n &= \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\infty
\end{aligned}
$$

which is a divergent series.

At $x = -1$,

$$
\begin{aligned}
u_n &= \frac{(-1)^n}{2} \\
|u_n| &= \frac{1}{2} \\
\lim_{n\to\infty}|u_n| &= \frac{1}{2} \neq 0
\end{aligned}
$$

Thus, by Leibnitz's test, the series is not convergent at $x = -1$.

Hence, the series is convergent for $|x| > 1$ and range of convergence is $|x| > 1$.

EXERCISE 5.9

Obtain the range of convergence of the following series:

1. $1 + x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$ **[Ans.: $-1 < x < 1$]**

2. $\frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \cdots + \frac{x^n}{n+2} + \cdots$ **[Ans.: $-1 < x < 1$]**

3. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n+1)}$ **[Ans.: $|x| \leq 1$]**

4. $\sum_{n=0}^{\infty} \frac{(x+2)}{\sqrt{n+1}}$ **[Ans.: $-3 \leq x \leq -1$]**

5. $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{\log(n+1)}$ **[Ans.: $|x| < 1$]**

6. $\sum_{n=0}^{\infty} (-2)^n (n+1)(x-1)^n$ $\left[\textbf{Ans.: } \frac{1}{2} < x < \frac{3}{2}\right]$

7. $\sum_{n=1}^{\infty} n!(x-1)^n$ **[Ans.: $x = 1$]**

8. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ **[Ans.: $|x| < 4$]**

9. $\sum_{n=1}^{\infty} \frac{(-2)^n (2x+1)^n}{n^2}$ $\left[\textbf{Ans.: } -\frac{3}{4} \leq x \leq -\frac{1}{4}\right]$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)^{\frac{3}{2}}}$ **[Ans.: $-1 \leq x \leq 1$]**

Points to Remember

Sequence

A sequence $\{u_n\}$ is said to be convergent, divergent or oscillatory according as $\lim_{n\to\infty} u_n$ is finite, infinite or not unique respectively.

The infinite series Σu_n is said to be convergent, divergent or oscillatory according as $\lim_{n\to\infty} S_n$ is finite, infinite or not unique respectively.

If a positive term series Σu_n is convergent then $\lim_{n\to\infty} u_n = 0$ but converse is not true, i.e., if $\lim_{n\to\infty} u_n = 0$, the series may converge or diverge. If $\lim_{n\to\infty} u_n \neq 0$, the series is not convergent.

Comparison Test

If Σu_n and Σv_n are series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite and non-zero) then both series converge or diverge together.

D'Alembert's Ratio Test

If Σu_n is a positive term series and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ then

(i) Σu_n is convergent if $l > 1$.

(ii) Σu_n is divergent if $l < 1$.

(iii) The test fails if $l = 1$.

Raabe's Test

If Σu_n is a positive term series and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then

(i) Σu_n is convergent if $l > 1$

(ii) Σu_n is divergent if $l < 1$

(iii) Test fails if $l = 1$

Cauchy's Root Test

If Σu_n is a positive term series and if $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$ then

(i) Σu_n is convergent if $l < 1$.

(ii) Σu_n is divergent if $l > 1$.

This test is preferred when u_n contains n^{th} powers of itself.

Cauchy's Integral Test

If $\Sigma u_n = \Sigma f(n)$ is a positive term series where $f(n)$ decreases as n increases and let $\int_1^\infty f(x)\,dx = I$ then

(i) Σu_n is convergent if I is finite.

(ii) Σu_n is divergent if I is infinite.

This test is preferred when evaluation of the integral of $f(x)$ is easy.

Leibnitz's Test

An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent if

(i) each term is numerically less than its preceding term, i.e, $u_{n+1} < u_n$ or $u_n > u_{n+1}$

(ii) $\lim_{n\to\infty} u_n = 0$

Absolute Convergence

The series $\sum_{n=1}^{\infty} u_n$ with both positive and negative terms (not necessarily alternative) is called absolutely convergent if the corresponding series $\sum_{n=1}^{\infty} |u_n|$ with all positive terms is convergent.

Conditional Convergence

If the series $\sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} |u_n|$ is divergent then the series $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The value of $\lim_{x \to 0} x \sin \frac{1}{x}$ is **[Summer 2014]**

(a) 1 (b) π (c) 0 (d) ∞

2. The sum of the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is **[Winter 2015]**

(a) 0 (b) $\frac{3}{4}$ (c) 1 (d) 2

3. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is **[Winter 2015]**

(a) divergent (b) absolutely convergent
(c) conditionally convergent (d) nothing can be said

4. The series $\sum n(5^{\frac{1}{n}} - 1)$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

5. The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)}$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

6. The series $\sum \frac{1}{n^p}$ is divergent if

(a) $p > 1$ (b) $p \le 1$ (c) $p = 1$ (d) $p = 0$

7. If $\lim\limits_{x \to 0} \frac{u_{n+1}}{u_n} > 1$, then $\sum\limits_{n=1}^{\infty} u_n$ is

(a) convergent (b) divergent
(c) may or may not be convergent (d) oscillatory

8. The series $a + ar + ar^2 + ar^3 + \cdots$ oscillates finitely if

(a) $|r| < 1$ (b) $r > 1$ (c) $r = 1$ (d) $r \le -1$

9. The series $\frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \frac{81}{32} + \cdots$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

10. The series $\sum\limits_{n=1}^{\infty} \frac{1}{n5^n}$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

11. The series $a + ar + ar^2 + ar^3 + \cdots$ diverges if

(a) $|r| < 1$ (b) $r \ge 1$ (c) $r \le -1$ (d) $r = 1$

12. The series $\sum \frac{1}{n^p}$ converges if

(a) $p > 1$ (b) $p < 1$ (c) $p = 0$ (d) $p = 1$

13. The series $\sum \frac{1}{n^{\frac{1}{4}}}$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

14. The series $\sum\limits_{n=1}^{\infty} \frac{3^{2n}}{4n}$ is

(a) convergent (b) divergent
(c) oscillates finitely (d) oscillates infinitely

15. The series $\sum\limits_{n=1}^{\infty} \sin \frac{1}{n}$ is

(a) convergent (b) divergent
(c) oscillatory (d) may or may not be convergent

16. $\lim\limits_{n \to a} \frac{u_n}{v_n} = 1$ and $\sum v_n$ diverges then $\sum u_n$ is

(a) divergent (b) convergent
(c) oscillatory (d) oscillates infinitely

17. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\cdots\infty$ converges if

(a) $p<2$ (b) $p>2$ (c) $p>1$ (d) $p\geq 1$

18. If Σu_n is a series of positive terms such that $\lim_{n\to\infty} u_n = 0$ then Σu_n is

(a) convergent (b) divergent
(c) may or may not be convergent (d) oscillatory

19. $\sum \frac{1}{n(\log n)^p}$ is convergent

(a) for $p > 1$ (b) for $p < 1$
(c) for all real values of p (d) for no value of p

20. $\sum_{n=0}^{\infty}(2x)^n$ is divergent if

(a) $-1\leq x\leq 1$ (b) $-\frac{1}{2}<x<\frac{1}{2}$ (c) $-2\leq x\leq 2$ (d) $-\frac{1}{2}\geq x\geq\frac{1}{2}$

21. The geometric series $\sum_{n=0}^{\infty} ar^n$, when $r = -1 \times 2$ is

(a) convergent (b) divergent (c) oscillatory (d) none of these

22. If $u_n = \frac{n!}{n^n}$ then $\lim_{x\to\infty}\frac{u_n}{u_{n+1}} =$

(a) e^2 (b) e (c) e^{-1} (d) 1

23. The series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ is

(a) convergent (b) divergent (c) oscillatory (d) none of these

24. The series $\sum\frac{2^{3n}}{3^{2n}}$ is

(a) convergent (b) divergent (c) oscillatory (d) none of these

25. The power series $\sum_{n=1}^{\infty}(3x)^n$ is convergent if

(a) $x=\frac{1}{3}$ (b) $x>\frac{1}{3}$ (c) $-\frac{1}{3}<x<\frac{1}{3}$ (d) $\frac{1}{3}<x<1$

26. The series $\sum_{n=1}^{\infty}\frac{n^2}{n!}$ is

(a) oscillatory (b) divergent (c) convergent (d) none of these

27. The series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ is

(a) oscillatory (b) convergent (c) divergent (d) none of these

28. The series $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\cdots+\frac{x^n}{(2n-1)2n}+\cdots$ is

(a) p series (b) geometric series
(c) alternating series (d) power series

29. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ converges only if

(a) $-1 < x < 1$ (b) $-1 \le x \le 1$ (c) $-1 < x \le 1$ (d) $-1 \le x < 1$

30. Which of the following series is divergent?

(a) $\sum\left(1+\frac{1}{n}\right)$ (b) $\sum\frac{1}{n^2}$ (c) $\sum\frac{1}{n^\pi}$ (d) $\sum\frac{1}{n^e}$

31. Which of the following series is convergent?

(a) $\sum_{n=1}^{\infty}\frac{1}{n^{0.9}}$ (b) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (c) $\sum\frac{1}{n}$ (d) $\sum\frac{1}{n^{1.001}}$

32. The series $\sum\frac{\cos n\pi}{1+n^2}$ is

(a) absolutely convergent (b) conditionally convergent
(c) convergent (d) divergent

33. The series $\sum(-n)$ is

(a) divergent (b) convergent (c) oscillatory (d) none of these

34. The sum of the series $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\cdots$ is **[Summer 2016]**

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) none of these

35. The series $\sum_{n=1}^{\infty}\frac{1}{(\log n)^n}$ is **[Summer 2016]**

(a) oscillatory (b) divergent (c) convergent (d) none of these

36. The sequence $\sin\left(\frac{\pi}{6}+\frac{1}{n}\right)$ converges to **[Winter 2016]**

(a) 0 (b) 1 (c) -1 (d) 0.5

37. The sum of the series $\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n$ is **[Winter 2016]**

(a) $\frac{\pi}{\pi - e}$ (b) $\frac{e}{\pi - e}$ (c) $\frac{\pi}{e - \pi}$ (d) $\frac{e}{\pi}$

38. $\sum_{n=1}^{\infty}\frac{2^n}{3n-1}$ is **[Winter 2016]**

(a) convergent and sum is 0 (b) convergent and sum is 1
(c) divergent (d) oscillating

39. Infinite series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$ is **[Summer 2017]**

(a) divergent (b) convergent (c) oscillatory (d) none of these

Answers

1. (c)	2. (c)	3. (c)	4. (b)	5. (b)	6. (b)	7. (b)	8. (d)	9. (b)
10. (a)	11. (b)	12. (a)	13. (b)	14. (b)	15. (b)	16. (a)	17. (b)	18. (c)
19. (a)	20. (b)	21. (c)	22. (b)	23. (a)	24. (a)	25. (c)	26. (c)	27. (c)
28. (d)	29. (c)	30. (a)	31. (d)	32. (a)	33. (a)	34. (a)	35. (c)	36. (d)
37. (b)	38. (c)	39. (b)						