Laplace Transform

Chapter 12

12.1 INTRODUCTION

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing and probability theory.

12.2 LAPLACE TRANSFORM

If $f(t)$ is a function of t defined for all $t \geq 0$, then $\int_0^\infty e^{-st} f(t)\, dt$ is defined as Laplace transform of $f(t)$, provided the integral exists and is denoted by $L\{f(t)\}$.

$$\mathrm{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

The integral is a function of the parameter s and is denoted by $F(s)$, $\bar{f}(s)$ or $\phi(s)$.

12.2.1 Sufficient Conditions for the Existence of Laplace Transform

The Laplace transform of function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous, i.e, $f(t)$ is continuous in every subinterval and $f(t)$ has finite limits at the end points of each subinterval.

(ii) $f(t)$ is of exponential order of α, i.e, there exists M, α such that $|f(t)| \leq Me^{\alpha t}$, for all $t \geq 0$. In other words,

$$\lim_{t\to\infty}\left\{e^{-\alpha t} f(t)\right\} = \text{finite quantity}$$

12.3 LAPLACE TRANSFORM OF SOME STANDARD FUNCTIONS

(i) $\boldsymbol{f(t) = k}$ **where $\boldsymbol{k}$ is a constant**

Proof: $L\{k\} = \int_0^\infty e^{-st}\, k\, \mathrm{d}t = k\left|\dfrac{e^{-st}}{-s}\right|_0^\infty = \dfrac{k}{s}$

(ii) $\boldsymbol{f(t) = t^n}$

Proof: $L\{t^n\} = \int_0^\infty e^{-st}\, t^n \mathrm{d}t$

Putting $st = x$, $\mathrm{d}t = \dfrac{\mathrm{d}x}{s}$

$$L\{t^n\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{\mathrm{d}x}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}\; x^n\; \mathrm{d}x = \frac{\overline{|n+1}}{s^{n+1}} \qquad s > 0,\; n+1 > 0$$

If n is a positive integer, $\overline{|n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

(iii) $\boldsymbol{f(t) = e^{at}}$

Proof: $L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, \mathrm{d}t = \int_0^\infty e^{-(s-a)t}\, \mathrm{d}t = \left|\dfrac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \dfrac{1}{s-a}$

(iv) $\boldsymbol{f(t) = \sin at}$

Proof: $L\{\sin at\} = L\left\{\dfrac{e^{iat} - e^{-iat}}{2i}\right\} = \dfrac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$

$$= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) = \frac{1}{2i}\left(\frac{s+ia-s+ia}{s^2+a^2}\right)$$

$$= \frac{1}{2i}\;\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

(v) $\boldsymbol{f(t) = \cos at}$

Proof: $L\{\cos at\} = L\left\{\dfrac{e^{iat} + e^{-iat}}{2}\right\} = \dfrac{1}{2}\left[L\{e^{iat}\} + L\{e^{-iat}\}\right]$

$$= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) = \frac{1}{2}\left(\frac{s+ia+s-ia}{s^2+a^2}\right)$$

$$= \frac{s}{s^2+a^2}$$

(vi) $\boldsymbol{f(t) = \sinh at}$

Proof: $L\{\sinh at\} = L\left\{\dfrac{e^{at} - e^{-at}}{2}\right\} = \dfrac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{s+a-s+a}{s^2-a^2}\right)$$

$$= \frac{a}{s^2-a^2}$$

(vii) $\boldsymbol{f(t) = \cosh at}$

Proof: $L\{\cosh at\} = L\left\{\dfrac{e^{at} + e^{-at}}{2}\right\} = \dfrac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{s+a+s-a}{s^2-a^2}\right)$$

$$= \frac{s}{s^2-a^2}$$

Example 1: Find the Laplace transforms by definition:

(i) $f(t) = 3 \quad 0 < t < 5$
$\quad = 0 \quad t > 5$

(ii) $f(t) = t \quad 0 < t < a$
$\quad = b \quad t > a$

(iii) $f(t) = (t-2)^2 \quad t > 2$
$\quad = 0 \quad 0 < t < 2$

(iv) $f(t) = 1 \quad 0 < t < 1$
$\quad = e^t \quad 1 < t < 4$
$\quad = 0 \quad t > 4$

(v) $f(t) = \cos t \quad 0 < t < \pi$
$\quad = \sin t \quad t > \pi$

(vi) $f(t) = \cos(t-a) \quad t > a$
$\quad = 0 \quad t < a$

(vii) $f(t) = t \quad 0 < t < \dfrac{1}{2}$
$\quad = t - 1 \quad \dfrac{1}{2} < t < 1$
$\quad = 0 \quad t > 1$

(viii) $f(t) = 0 \quad 0 < t < \pi$
$\quad = \sin^2(t-\pi) \quad t > \pi.$

Solution:

(i) $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^5 e^{-st}\cdot 3\,\mathrm{d}t + \int_5^\infty e^{-st}\cdot 0\,\mathrm{d}t = 3\left|\frac{e^{-st}}{-s}\right|_0^5 + 0$$

$$= 3\left|\frac{e^{-5s}}{-s} - \frac{1}{-s}\right| = \frac{3}{s}(1 - e^{-5s})$$

(ii) $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^a e^{-st}\cdot t\,\mathrm{d}t + \int_a^\infty e^{-st}\cdot b\,\mathrm{d}t$$

$$= \left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\cdot 1\right|_0^a + b\left|\frac{e^{-st}}{-s}\right|_a^\infty = e^{-as}\left(-\frac{a}{s} - \frac{1}{s^2}\right) - e^0\left(0 - \frac{1}{s^2}\right) - \frac{b}{s}(0 - e^{-as})$$

$$= \frac{1}{s^2} + \left[\frac{(b-a)}{s} - \frac{1}{s^2}\right]e^{-as}$$

(iii) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^2 e^{-st} \cdot 0\, dt + \int_2^\infty e^{-st} (t-2)^2\, dt$

$$= 0 + \left| \frac{e^{-st}}{-s}(t-2)^2 - \frac{e^{-st}}{s^2} 2(t-2) + \frac{e^{-st}}{-s^3} 2 \right|_2^\infty$$

$$= 0 - \frac{e^{-2s}}{-s^3} \cdot 2 = \frac{2}{s^3} e^{-2s}$$

(iv) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st} \cdot 1\, dt + \int_1^4 e^{-st} e^t\, dt + \int_4^\infty e^{-st} \cdot 0\, dt$

$$= \left| \frac{e^{-st}}{-s} \right|_0^1 + \left| \frac{e^{t(1-s)}}{1-s} \right|_1^4 = \frac{e^{-s} - 1}{-s} + \frac{e^{4(1-s)} - e^{(1-s)}}{1-s}$$

$$= \frac{1 - e^{-s}}{s} + \frac{e^{(1-s)} - e^{4(1-s)}}{s-1}$$

(v) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\pi e^{-st} \cos t\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$

$$= \left| \frac{e^{-st}}{s^2+1}(-s\cos t + \sin t) \right|_0^\pi + \left| \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t) \right|_\pi^\infty$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s}(-s\cos\pi) - (-s\cos 0) + 0 - e^{-\pi s}(-\cos\pi) \right]$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s}(s-1) - s \right]$$

(vi) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cos(t-a)\, dt$

Putting $t - a = x$

$dt = dx$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{f(t)\} = \int_0^\infty e^{-s(x+a)} \cos x\, dx = e^{-as} \int_0^\infty e^{-xs} \cos x\, dx$$

$$= e^{-as} \left| \frac{e^{-xs}}{s^2+1}(-s\cos x + \sin x) \right|_0^\infty$$

$$= \frac{e^{-as}}{s^2+1}(0+s) = \frac{se^{-as}}{s^2+1}$$

(vii) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^{\frac{1}{2}} e^{-st} t\,\mathrm{d}t + \int_{\frac{1}{2}}^{1} e^{-st}(t-1)\,\mathrm{d}t + \int_1^\infty e^{-st}\cdot 0\,\mathrm{d}t$

$$= \left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\cdot 1\right|_0^{\frac{1}{2}} + \left|\frac{e^{-st}}{-s}(t-1) - \frac{e^{-st}}{s^2}\cdot 1\right|_{\frac{1}{2}}^{1}$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{2s} - \frac{1}{s^2}\right) - e^0\left(0 - \frac{1}{s^2}\right) - \frac{e^{-s}}{s^2} - e^{-\frac{s}{2}}\left(\frac{1}{2s} - \frac{1}{s^2}\right)$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{s}\right) + \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-\frac{s}{2}}}{s}$$

(viii) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\pi e^{-st}\cdot 0\,\mathrm{d}t + \int_\pi^\infty e^{-st}\sin^2(t-\pi)\,\mathrm{d}t$

$$= \int_\pi^\infty e^{-st}\left[\frac{1-\cos 2(t-\pi)}{2}\right]\mathrm{d}t$$

$$= \frac{1}{2}\int_\pi^\infty e^{-st}\left[1-\cos(2\pi - 2t)\right]\mathrm{d}t = \frac{1}{2}\int_\pi^\infty e^{-st}(1-\cos 2t)\mathrm{d}t$$

$$= \frac{1}{2}\left[\int_\pi^\infty e^{-st}\mathrm{d}t - \int_\pi^\infty e^{-st}\cos 2t\,\mathrm{d}t\right]$$

$$= \frac{1}{2}\left[\left|\frac{e^{-st}}{-s}\right|_\pi^\infty - \left|\frac{e^{-st}}{s^2+4}(-s\cos 2t + 2\sin 2t)\right|_\pi^\infty\right]$$

$$= \frac{1}{2}\left[\left(0 + \frac{e^{-\pi s}}{s}\right) - \left\{0 - \frac{e^{-\pi s}}{s^2+4}(-s\cos 2\pi + 2\sin 2\pi)\right\}\right]$$

$$= \frac{e^{-\pi s}}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Exercise 12.1

Find the Laplace transforms of the following functions:

1. $f(t) = t, \quad 0 < t < 3$
$\quad = 6, \quad t > 3.$

$\left[\textbf{Ans. :} \frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-3s}\right]$

2. $f(t) = t^2, \quad 0 < t < 1$
$\quad = 1, \quad t > 1.$

$\left[\textbf{Ans. :} \frac{1}{s}(1-e^{-s}) - \frac{2e^{-s}}{s^2} + \frac{2}{s^3}(1-e^{-s})\right]$

3. $f(t) = (t-a)^3, \quad t > a$
$\quad = 0, \quad t < a.$
$$\left[\textbf{Ans. :}\ \frac{6}{s^4}e^{-as}\right]$$

4. $f(t) = 0, \quad 0 \le t \le 1$
$\quad = t, \quad 1 < t < 2$
$\quad = 0, \quad t > 2.$
$$\left[\textbf{Ans. :}\ \left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s} - \left(\frac{1}{s^2}+\frac{2}{s}\right)e^{-2s}\right]$$

5. $f(t) = t^2, \quad 0 < t < 2$
$\quad = t-1, \quad 2 < t < 3$
$\quad = 7, \quad t > 3.$
$$\left[\begin{aligned}\textbf{Ans. :}\ &\frac{2}{s^3} - \frac{e^{-2s}}{s^3}(2+3s+3s^2) \\ &+ \frac{e^{-3s}}{s^2}(5s-1)\end{aligned}\right]$$

6. $f(t) = e^t, \quad 0 < t < 1$
$\quad = 0, \quad t > 1.$
$$\left[\textbf{Ans. :}\ \frac{1}{1-s}(e^{1-s}-1)\right]$$

7. $f(t) = \cos\left(t - \frac{2\pi}{3}\right), \quad t > \frac{2\pi}{3}$
$\quad = 0, \quad t < \frac{2\pi}{3}.$
$$\left[\textbf{Ans. :}\ e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2+1}\right]$$

8. $f(t) = \sin 2t, \quad 0 < t < \pi$
$\quad = 0, \quad t > \pi.$
$$\left[\textbf{Ans. :}\ \frac{2(1-e^{-ps})}{s^2+4}\right]$$

12.4 PROPERTIES OF LAPLACE TRANSFORM

12.4.1 Linearity

If $\quad L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$, then
$$L\{a\,f_1(t) + b\,f_2(t)\} = a\,F_1(s) + b\,F_2(s)$$
where a and b are constants.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$\begin{aligned}L\{af_1(t) + bf_2(t)\} &= \int_0^\infty e^{-st}\{af_1(t) + bf_2(t)\}\,\mathrm{d}t \\ &= a\int_0^\infty e^{-st} f_1(t)\,\mathrm{d}t + b\int_0^\infty e^{-st} f_2(t)\,\mathrm{d}t \\ &= a\,F_1(s) + b\,F_2(s)\end{aligned}$$

Example 1: Find the Laplace transforms of the following functions:

(i) $4t^2 + \sin 3t + e^{2t}$ **(ii)** $t^2 - e^{-2t} + \cosh^2 3t$ **(iii)** $\cosh^5 t$
(iv) $(\sin 2t - \cos 2t)^2$ **(v)** $\cos(\omega t + b)$ **(vi)** $\cos t \cos 2t \cos 3t$
(vii) $\sin^5 t$ **(viii)** $\sin\sqrt{t}$ **(ix)** $\frac{\cos\sqrt{t}}{\sqrt{t}}$
(x) $\left(\sqrt{t}-1\right)^2$ **(xi)** $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3.$

Solution:

(i) $L\left\{4t^2+\sin 3t+e^{2t}\right\}=4L\left\{t^2\right\}+L\{\sin 3t\}+L\left\{e^{2t}\right\}$

$$=4\cdot\frac{2}{s^3}+\frac{3}{s^2+9}+\frac{1}{s-2}$$

$$=\frac{8}{s^3}+\frac{3}{s^2+9}+\frac{1}{s-2}$$

(ii) $L\left\{t^2-e^{-2t}+\cosh^2 3t\right\}=L\left\{t^2\right\}-L\left\{e^{-2t}\right\}+L\left\{\cosh^2 3t\right\}$

$$=L\left\{t^2\right\}-L\left\{e^{-2t}\right\}+\frac{1}{2}L\{1+\cosh 6t\}$$

$$=\frac{2}{s^3}-\frac{1}{s+2}+\frac{1}{2s}+\frac{s}{2(s^2-36)}$$

(iii) $L\left\{\cosh^5 t\right\}=L\left\{\left(\frac{e^t+e^{-t}}{2}\right)^5\right\}$

$$=L\left\{\frac{1}{2^5}(e^{5t}+5e^{4t}e^{-t}+10e^{3t}e^{-2t}+10\,e^{2t}e^{-3t}+5e^{t}e^{-4t}+e^{-5t})\right\}$$

$$=\frac{1}{32}L\left\{(e^{5t}+e^{-5t})+5(e^{3t}+e^{-3t})+10(e^{t}+e^{-t})\right\}$$

$$=\frac{1}{16}L\{\cosh 5t+5\cosh 3t+10\cosh t\}$$

$$=\frac{1}{16}\left[L\{\cosh 5t\}+5L\{\cosh 3t\}+10L\{\cosh t\}\right]$$

$$=\frac{1}{16}\left[\frac{s}{s^2-25}+\frac{5s}{s^2-9}+\frac{10s}{s^2-1}\right]$$

(iv) $L\left\{(\sin 2t-\cos 2t)^2\right\}=L\left\{\sin^2 2t+\cos^2 2t-2\cos 2t\sin 2t\right\}$

$$=L\{1-\sin 4t\}=L\{1\}-L\{\sin 4t\}$$

$$=\frac{1}{s}-\frac{4}{s^2+16}$$

(v) $L\{\cos(\omega t+b)\}=L\{\cos\omega t\cos b-\sin\omega t\sin b\}$

$$=\cos b\,L\{\cos\omega t\}-\sin b\,L\{\sin\omega t\}$$

$$=\cos b\cdot\frac{s}{s^2+\omega^2}-\sin b\cdot\frac{\omega}{s^2+\omega^2}$$

(vi) $L\{\cos t\cos 2t\cos 3t\} = L\left\{\frac{1}{2}(\cos 3t+\cos t)\cos 3t\right\} = \frac{1}{2}L\{\cos^2 3t+\cos t\cos 3t\}$

$$= \frac{1}{2}L\left\{\frac{1+\cos 6t}{2}+\frac{\cos 4t+\cos 2t}{2}\right\}$$

$$= L\left\{\frac{1}{4}+\frac{1}{4}\cos 6t+\frac{1}{4}\cos 4t+\frac{1}{4}\cos 2t\right\}$$

$$= L\left\{\frac{1}{4}\right\}+\frac{1}{4}L\{\cos 6t\}+\frac{1}{4}L\{\cos 4t\}+\frac{1}{4}L\{\cos 2t\}$$

$$= \frac{1}{4s}+\frac{s}{4(s^2+36)}+\frac{s}{4(s^2+36)}+\frac{s}{4(s^2+4)}$$

(vii) $L\{\sin^5 t\} = L\left\{\left(\frac{e^{it}-e^{-it}}{2i}\right)^5\right\}$

$$= L\left\{\frac{1}{(2i)^5}(e^{i5t}-5e^{i4t}e^{-it}+10e^{i3t}e^{-i2t}-10e^{i2t}e^{-i3t}+5e^{it}e^{-i4t}-e^{-i5t})\right\}$$

$$= \frac{1}{32i}L\{(e^{i5t}-e^{-i5t})-5(e^{i3t}-e^{-i3t})+10(e^{it}-e^{-it})\}$$

$$= \frac{1}{16}L\{\sin 5t-5\sin 3t+10\sin t\}$$

$$= \frac{1}{16}\left[L\{\sin 5t\}-5L\{\sin 3t\}+10L\{\sin t\}\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2+25}-\frac{15}{s^2+9}+\frac{10}{s^2+1}\right]$$

$$= \frac{5}{16(s^2+25)}-\frac{15}{16(s^2+9)}+\frac{5}{8(s^2+1)}$$

(viii) We know that

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

$$\sin\sqrt{t} = t^{\frac{1}{2}}-\frac{t^{\frac{3}{2}}}{3!}+\frac{t^{\frac{5}{2}}}{5!}-\cdots$$

$$L\{\sin\sqrt{t}\} = L\left\{t^{\frac{1}{2}}\right\}-\frac{1}{3!}L\left\{t^{\frac{3}{2}}\right\}+\frac{1}{5!}L\left\{t^{\frac{5}{2}}\right\}-\cdots = \frac{\left\lceil\frac{3}{2}\right.}{s^{\frac{3}{2}}}-\frac{1}{3!}\frac{\left\lceil\frac{5}{2}\right.}{s^{\frac{5}{2}}}+\frac{1}{5!}\frac{\left\lceil\frac{7}{2}\right.}{s^{\frac{7}{2}}}-\cdots$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{7}{2}}} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{2s^{\frac{3}{2}}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \cdots\right] = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{(4s)}}$$

(ix) We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-\frac{1}{2}} - \frac{t^{\frac{1}{2}}}{2!} + \frac{t^{\frac{3}{2}}}{4!} - \cdots$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\left\{t^{-\frac{1}{2}}\right\} - \frac{1}{2!}L\left\{t^{\frac{1}{2}}\right\} + \frac{1}{4!}L\left\{t^{\frac{3}{2}}\right\} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} - \cdots = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} - \cdots$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \cdots\right] = \sqrt{\frac{\pi}{s}}e^{-\frac{1}{(4s)}}$$

(x) $$L\left\{\left(\sqrt{t}-1\right)^2\right\} = L\left\{t - 2\sqrt{t} + 1\right\} = L\{t\} - 2L\left\{\sqrt{t}\right\} + L\{1\}$$

$$= \frac{1}{s^2} - \frac{2\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{s} = \frac{1}{s^2} - \frac{\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{1}{s}$$

(xi) $$L\left\{\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{t^{\frac{3}{2}} - 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}} - t^{-\frac{3}{2}}\right\}$$

$$= L\left\{t^{\frac{3}{2}}\right\} - 3L\left\{t^{\frac{1}{2}}\right\} + 3L\left\{t^{-\frac{1}{2}}\right\} - L\left\{t^{-\frac{3}{2}}\right\}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} - \frac{3\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{3\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{\Gamma\left(\frac{-1}{2}\right)}{s^{\frac{-1}{2}}}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\frac{1}{2}}}{s^{\frac{5}{2}}} - \frac{3\cdot\frac{1}{2}\cdot\sqrt{\frac{1}{2}}}{s^{\frac{3}{2}}} + \frac{3\sqrt{\frac{1}{2}}}{s^{\frac{1}{2}}} - \frac{\sqrt{\frac{1}{2}}}{-\frac{1}{2}s^{-\frac{1}{2}}} \qquad \left[\because \quad \Gamma(n+1) = n\Gamma(n), \quad \Gamma(n) = \frac{\Gamma(n+1)}{n}\right]$$

$$= \sqrt{\frac{\pi}{s}}\left(\frac{3}{4s^2} - \frac{3}{2s} + 3 + 2s\right)$$

Example 2: If $J_o(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$, find $L\{J_o(t)\}$.

Solution:

$$J_o(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$$

$$= 1 - \left(\frac{t}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{t}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{t}{2}\right)^6 + \cdots$$

$$= 1 - \frac{t^2}{4} + \frac{t^4}{64} - \frac{t^6}{2304} + \cdots$$

$$L\{J_o(t)\} = L\{1\} - \frac{1}{4}L\{t^2\} + \frac{1}{64}L\{t^4\} - \frac{1}{2304}L\{t^6\} + \cdots$$

$$= \frac{1}{s} - \frac{1}{4}\cdot\frac{2!}{s^3} + \frac{1}{64}\cdot\frac{4!}{s^5} - \frac{1}{2304}\cdot\frac{6!}{s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{3}{8}\left(\frac{1}{s^2}\right)^2 - \frac{15}{48}\left(\frac{1}{s^2}\right)^3 + \cdots\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+s^2}}$$

Exercise 12.2

Find the Laplace transforms of the following functions:

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

$$\left[\textbf{Ans.: } \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^4+9}\right]$$

2. $e^{2t} + 4t^3 - \sin 2t \cos 3t$

$$\left[\textbf{Ans.: } \frac{1}{s-2} + \frac{24}{s^4} - \frac{5}{2}\cdot\frac{1}{s^2+25} + \frac{1}{2(s^2+1)}\right]$$

3. $3t^2 + e^{-t} + \sin^3 2t$

$$\left[\textbf{Ans.: } \frac{6}{s^3} + \frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s^2+4} - \frac{3}{2}\cdot\frac{1}{s^2+36}\right]$$

4. $(t^2 + a)^2$

$$\left[\textbf{Ans.: } \frac{a^2s^4 + 4as^2 + 24}{s^5}\right]$$

5. $\sin(\omega t+\alpha)$

$$\left[\textbf{Ans. :} \cos\alpha\cdot\frac{\omega}{s^2+\omega^2}+\sin\alpha\cdot\frac{s}{s^2+\omega^2}\right]$$

6. $\sin 2t\cos 3t$

$$\left[\textbf{Ans. :} \frac{2(s^2-5)}{(s^2+1)(s^2+25)}\right]$$

7. $\cos^3 2t$

$$\left[\textbf{Ans. :} \frac{s(s^2+28)}{(s^2+36)(s^2+4)}\right]$$

8. $\sinh^3 3t$

$$\left[\textbf{Ans. :} \frac{162}{(s^2-81)(s^2-8)}\right]$$

9. $\dfrac{1+2t}{\sqrt{t}}$

$$\left[\textbf{Ans. :} \sqrt{\frac{\pi}{s}}\left(1+\frac{1}{s}\right)\right]$$

10. $\sin(t+\alpha)\cos(t-\alpha)$

$$\left[\textbf{Ans. :} \frac{1}{s^2+4}+\frac{\sin 2\alpha}{s}\right]$$

12.4.2 Change of Scale

If $L\{f(t)\}=F(s)$, then $L\{f(at)\}=\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

Proof:

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)\mathrm{d}t$$

$$L\{f(at)\}=\int_0^\infty e^{-st}f(at)\mathrm{d}t$$

Putting $at=x$, $\mathrm{d}t=\dfrac{\mathrm{d}x}{a}$

$$L\{f(at)\}=\int_0^\infty e^{-s\left(\frac{x}{a}\right)}f(x)\frac{\mathrm{d}x}{a}=\frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x}f(x)\mathrm{d}x$$

$$=\frac{1}{a}F\left(\frac{s}{a}\right)$$

Example 1: If $L\{f(t)\}=\log\left(\dfrac{s+3}{s+1}\right)$, find $L\{f(2t)\}$.

Solution: $L\{f(t)\}=\log\left(\dfrac{s+3}{s+1}\right)$

By change of scale property,

$$L\{f(2t)\}=\frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right)=\frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Example 2: If $L\left\{\sin\sqrt{t}\right\} = \dfrac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{(4s)}}$, **find** $L\left\{\sin 2\sqrt{t}\right\}$.

Solution: $L\left\{\sin\sqrt{t}\right\} = \dfrac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{(4s)}}$

By change of scale property,

$$L\left\{\sin 2\sqrt{t}\right\} = L\left\{\sin\sqrt{4t}\right\} = \frac{1}{4}\frac{\sqrt{\pi}}{2\cdot\frac{s}{4}\sqrt{\frac{s}{4}}} e^{-\frac{1}{4\left(\frac{s}{4}\right)}} = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{s}}$$

Example 3: If $L\left\{\operatorname{erf}\sqrt{t}\right\} = \dfrac{1}{s\sqrt{s+1}}$, **find** $L\left\{\operatorname{erf} 2\sqrt{t}\right\}$.

Solution: $L\left\{\operatorname{erf}\sqrt{t}\right\} = \dfrac{1}{s\sqrt{s+1}}$

By change of scale property,

$$L\left\{\operatorname{erf} 2\sqrt{t}\right\} = L\left\{\operatorname{erf}\sqrt{4t}\right\} = \frac{1}{4}\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} = \frac{2}{s\sqrt{s+4}}$$

Example 4: If $L\{J_o(t)\} = \dfrac{1}{\sqrt{s^2+1}}$, **find** $L\{J_o(3t)\}$.

Solution: $L\{J_o(t)\} = \dfrac{1}{\sqrt{s^2+1}}$

By change of scale property,

$$L\{J_o(3t)\} = \frac{1}{3}\frac{1}{\sqrt{\left(\frac{s}{3}\right)^2+1}} = \frac{1}{\sqrt{s^2+9}}$$

Exercise 12.3

1. If $L\{f(t)\} = \dfrac{8(s-3)}{(s^2-6s+25)^2}$, find $L\{f(2t)\}$.

$\left[\textbf{Ans. :}\ \dfrac{1}{4}\dfrac{(s-6)}{(s^2-12s+100)^2}\right]$

2. If $L\{f(t)\} = \dfrac{2}{s^3}e^{-s}$, find $L\{f(3t)\}$.

$\left[\textbf{Ans. :}\ \dfrac{18}{s^3}e^{-\frac{s}{3}}\right]$

3. If $L\{f(t)\} = \dfrac{s^2-s-1}{\cdots s+ \quad ^2\ s-}$, find $L\{f(2t)\}$.

$\left[\textbf{Ans. :}\ \dfrac{s^2-2s+4}{4(s+1)^2(s-2)}\right]$

12.4.3 First Shifting Theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at} f(t)\} = F(s+a)$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-st} e^{-at} f(t)\mathrm{d}t = \int_0^\infty e^{-(s+a)t} f(t)\mathrm{d}t = F(s+a)$$

Example 1: Find the Laplace transforms of the following functions:

(i) $e^{-3t} t^4$

(ii) $(t+1)^2 e^t$

(iii) $e^t \left(1+\sqrt{t}\right)^4$

(iv) $e^{4t} \sin^3 t$

(v) $\cosh at \cos at$

(vi) $\sin\frac{h}{2} \sin\frac{\sqrt{3}}{2} t$

(vii) $e^{-3t} \cosh 4t \sin 3t$

(viii) $\sin 2t \cos t \cosh 2t$

(ix) $\dfrac{\cos 2t \sin t}{e^t}$

(x) $e^{-4t} \sinh t \sin t$.

Solution:

(i) $$L\{t^4\} = \frac{4!}{s^5}$$

By first shifting theorem,

$$L\{e^{-3t} t^4\} = \frac{4!}{(s+3)^5}$$

(ii) $$L\{(t+1)^2\} = L\{t^2 + 2t + 1\} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

By first shifting theorem,

$$L\{(t+1)^2 e^t\} = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$$

(iii) $$L\left\{\left(1+\sqrt{t}\right)^4\right\} = L\left\{1 + 4\sqrt{t} + 6\left(\sqrt{t}\right)^2 + 4\left(\sqrt{t}\right)^3 + \left(\sqrt{t}\right)^4\right\}$$

$$= L\left\{1 + 4t^{\frac{1}{2}} + 6t + 4t^{\frac{3}{2}} + t^2\right\} = \frac{1}{s} + \frac{4\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{6\Gamma(2)}{s^2} + \frac{4\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} + \frac{\Gamma(3)}{s^3}$$

$$= \frac{1}{s} + \frac{4\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{6}{s^2} + \frac{4\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} + \frac{2}{s^3} = \frac{1}{s} + \frac{2\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{6}{s^2} + \frac{3\sqrt{\pi}}{s^{\frac{5}{2}}} + \frac{2}{s^3}$$

By first shifting theorem,

$$L\left\{e^{t}\left(1+\sqrt{t}\right)^{4}\right\}=\frac{1}{s-1}+\frac{2\sqrt{\pi}}{(s-1)^{\frac{3}{2}}}+\frac{6}{(s-1)^{2}}+\frac{3\sqrt{\pi}}{(s-1)^{\frac{5}{2}}}+\frac{2}{(s-1)^{3}}$$

(iv) $$L\left\{\sin^{3} t\right\}=\frac{1}{4}L\left\{3\sin t-\sin 3t\right\}=\frac{3}{4(s^{2}+1)}-\frac{3}{4(s^{2}+9)}$$

By first shifting theorem,

$$L\left\{e^{4t}\sin^{3} t\right\}=\frac{3}{4\left[(s-4)^{2}+1\right]}-\frac{3}{4\left[(s-4)^{2}+9\right]}$$

$$=\frac{3}{4(s^{2}-8s+17)}-\frac{3}{4(s^{2}-8s+25)}=\frac{6}{(s^{2}-8s+7)(s^{2}-8s+25)}$$

(v) $$\cosh at\cos at=\left(\frac{e^{at}+e^{-at}}{2}\right)\cos at=\frac{1}{2}(e^{at}\cos at+e^{-at}\cos at)$$

$$L\left\{\cos at\right\}=\frac{s}{s^{2}+a^{2}}$$

$$L\left\{\cosh at\cos at\right\}=\frac{1}{2}L\ \{e^{at}\cos at+e^{-at}\cos at\}$$

By first shifting theorem,

$$L\left\{\cosh at\cos at\right\}=\frac{1}{2}\left[\frac{s-a}{(s-a)^{2}+a^{2}}+\frac{s+a}{(s+a)^{2}+a^{2}}\right]$$

$$=\frac{1}{2}\left[\frac{s-a}{s^{2}+2a^{2}-2as}+\frac{s+a}{s^{2}+2a^{2}+2as}\right]$$

$$=\frac{1}{2}\left[\frac{(s-a)(s^{2}+2a^{2}+2as)+(s+a)(s^{2}+2a^{2}-2as)}{(s^{2}+2a^{2})^{2}-4a^{2}s^{2}}\right]$$

$$=\frac{s^{3}}{s^{4}+4a^{4}}$$

(vi) $$\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t=\left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2}\right)\sin\frac{\sqrt{3}}{2}t=\frac{1}{2}\left(e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t-e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right)$$

$$L\left\{\sin\frac{\sqrt{3}}{2}t\right\}=\frac{\frac{\sqrt{3}}{2}}{s^{2}+\frac{3}{4}}$$

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\}=\frac{1}{2}L\left\{e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t-e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right\}$$

By first shifting theorem,

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\}=\frac{1}{2}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$=\frac{\sqrt{3}}{4}\left[\frac{1}{s^2+1-s}-\frac{1}{s^2+1+s}\right]=\frac{\sqrt{3}}{2}\frac{s}{s^4+s^2+1}$$

(vii) $$e^{-3t}\cosh 4t\sin 3t=e^{-3t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\sin 3t=\frac{1}{2}(e^{t}\sin 3t+e^{-7t}\sin 3t)$$

$$L\{\sin 3t\}=\frac{3}{s^2+9}$$

$$L\left\{e^{-3t}\cosh 4t\sin 3t\right\}=\frac{1}{2}L\left\{e^{t}\sin 3t+e^{-7t}\sin 3t\right\}$$

By first shifting theorem,

$$L\left\{e^{-3t}\cosh 4t\,\sin 3t\right\}=\frac{1}{2}\left[\frac{3}{(s-1)^2+9}+\frac{3}{(s+7)^2+9}\right]$$

$$=\frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$$

(viii) $$\sin 2t\cos t\cosh 2t=\left(\frac{\sin 3t+\sin t}{2}\right)\left(\frac{e^{2t}+e^{-2t}}{2}\right)$$

$$=\frac{1}{4}(e^{2t}\sin 3t+e^{2t}\sin t+e^{-2t}\sin 3t+e^{-2t}\sin t)$$

$$L\{\sin t\}=\frac{1}{s^2+1}$$

$$L\{\sin 3t\}=\frac{3}{s^2+9}$$

$$L\{\sin 2t\cos t\cosh 2t\}=\frac{1}{4}L\left\{e^{2t}\sin 3t+e^{2t}\sin t+e^{-2t}\sin 3t+e^{-2t}\sin t\right\}$$

By first shifting theorem,

$$L\{\sin 2t\cos t\cosh 2t\}=\frac{1}{4}\left[\frac{3}{(s-2)^2+9}+\frac{1}{(s-2)^2+1}+\frac{3}{(s+2)^2+9}+\frac{1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{(s^2-4s+13)(s^2+4s+13)} + \frac{s^2+5}{(s^2-4s+5)(s^2+4s+5)}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{s^4+10s^2+169} + \frac{s^2+5}{s^4-6s^2+25}\right]$$

(ix) $$\frac{\cos 2t \sin t}{e^t} = e^{-t}\left(\frac{\sin 3t - \sin t}{2}\right) = \frac{1}{2}(e^{-t}\sin 3t - e^{-t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2} L\,\{e^{-t}\sin 3t - e^{-t}\sin t\}$$

By first shifting theorem,

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right] = \frac{1}{2}\frac{2s^2+4s-4}{(s^2+2s+10)(s^2+2s+2)}$$

$$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

(x) $$e^{-4t}\sinh t \sin t = e^{-4t}\left(\frac{e^t - e^{-t}}{2}\right)\sin t = \frac{1}{2}(e^{-3t}\sin t - e^{-5t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{e^{-4t}\sinh t \sin t\right\} = \frac{1}{2} L\left\{e^{-3t}\sin t - e^{-5t}\sin t\right\}$$

By first shifting theorem,

$$L\left\{e^{-4t}\sinh t \sin t\right\} = \frac{1}{2}\left[\frac{1}{(s+3)^2+1} - \frac{1}{(s+5)^2+1}\right]$$

$$= \frac{1}{2}\frac{4s+16}{(s^2+6s+10)(s^2+10s+26)}$$

$$= \frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}$$

Exercise 12.4

Find the Laplace transforms of the following functions:

1. $t^3 e^{-3t}$

$$\left[\textbf{Ans.}: \frac{6}{(s+3)^4}\right]$$

2. $e^{-t}\cos 2t$

$$\left[\textbf{Ans.}: \frac{s+1}{s^2+2s+5}\right]$$

3. $2e^{3t}\sin 4t$

$$\left[\textbf{Ans.}: \frac{8}{s^2-6s+25}\right]$$

4. $(t+2)^2 e^t$

$$\left[\textbf{Ans.}: \frac{4s^2-4s+2}{(s-1)^3}\right]$$

5. $e^{2t}(3\sin 4t-4\cos 4t)$

$$\left[\textbf{Ans.}: \frac{20-4s}{s^2-4s+20}\right]$$

6. $e^{-4t}\cosh 2t$

$$\left[\textbf{Ans.}: \frac{s+4}{s^2+8s+12}\right]$$

7. $(1+te^{-t})^3$

$$\left[\textbf{Ans.}: \frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}\right]$$

8. $e^{-t}(3\sinh 2t-5\cosh 2t)$

$$\left[\textbf{Ans.}: \frac{1-5s}{s^2+2s-3}\right]$$

9. $e^t\sin 2t\sin 3t$

$$\left[\textbf{Ans.}: \frac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}\right]$$

10. $e^{-3t}\cosh 5t\sin 4t$

$$\left[\textbf{Ans.}: \frac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+20)}\right]$$

11. $e^{-4t}\cosh t\sin t$

$$\left[\textbf{Ans.}: \frac{s^2+8s+18}{(s^2+6s+10)(s^2+10s+26)}\right]$$

12. $e^{2t}\sin^4 t$

$$\left[\textbf{Ans.}: \frac{3}{8(s-2)}-\frac{s-2}{2(s^2-4s+8)}+\frac{s-4}{8(s^2-8s+32)}\right]$$

12.4.4 Second Shifting Theorem

If $\quad L\{f(t)\}=F(s)$

and $\quad g(t)=f(t-a) \quad t>a$

$\qquad\qquad =0 \qquad\quad t<a$

then $\quad L\{g(t)\}=e^{-as}F(s)$

Proof: $L\{f(t)\}=\int_0^\infty e^{-st}f(t)\mathrm{d}t$

$$L\{g(t)\}=\int_0^\infty e^{-st}g(t)\mathrm{d}t=\int_0^a e^{-st}\cdot 0\,\mathrm{d}t+\int_a^\infty e^{-st}f(t-a)\mathrm{d}t$$

Putting $t - a = x$

$dt = dx$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{g(t)\} = \int_0^\infty e^{-s(a+x)} f(x)\mathrm{d}x = e^{-as}\int_0^\infty e^{-sx} f(x)\mathrm{d}x$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\mathrm{d}t = e^{-as}F(s)$$

Example 1: Find the Laplace transforms of the following functions:

(i) $g(t) = \cos(t-a) \quad t > a$

$= 0 \quad t < a$

(ii) $g(t) = e^{t-a} \quad t > a$

$= 0 \quad t < a$

(iii) $g(t) = \sin\left(t - \frac{\pi}{4}\right) \quad t > \frac{\pi}{4}$

$= 0 \quad t < \frac{\pi}{4}$

(iv) $g(t) = (t-1)^3 \quad t > 1$

$= 0 \quad t < 1.$

Solution:

(i) Let $f(t) = \cos t$

$$L\{f(t)\} = F(s) = \frac{s}{s^2+1}$$

By second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{s}{s^2+1}$$

(ii) Let $f(t) = e^t$

$$L\{f(t)\} = F(s) = \frac{1}{s-1}$$

By second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{1}{s-1}$$

(iii) Let $f(t) = \sin t$

$$L\{f(t)\} = F(s) = \frac{1}{s^2+1}$$

By second shifting theorem,

$$L\{g(t)\} = e^{-\frac{\pi s}{4}}\frac{1}{s^2+1}$$

(iv) Let $f(t) = t^3$

$$L\{f(t)\} = F(s) = \frac{3!}{s^4}$$

By second shifting theorem,

$$L\{g(t)\} = e^{-s} \cdot \frac{3!}{s^4}$$

Exercise 12.5

Find the Laplace transforms of the following functions:

1. $f(t) = \cos\left(t - \frac{2\pi}{3}\right) \qquad t > \frac{2\pi}{3}$

$\quad = 0 \qquad t < \frac{2\pi}{3}$

$\left[\textbf{Ans. :}\ e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}\right]$

2. $f(t) = (t-2)^2 \qquad t > 2$

$\quad = 0 \qquad t < 2$

$\left[\textbf{Ans. :}\ e^{-2s} \frac{2}{s^3}\right]$

3. $f(t) = 5\sin 3\left(t - \frac{\pi}{4}\right) \qquad t > \frac{\pi}{4}$

$\quad = 0 \qquad t < \frac{\pi}{4}$

$\left[\textbf{Ans. :}\ e^{-\frac{\pi s}{4}} \frac{1}{s^2+9}\right]$

12.4.5 Multiplication by *t*

If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$

Proof: $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

Differentiating both the sides w.r.t. s using DUIS,

$$\frac{\mathrm{d}}{\mathrm{d}s} F(s) = \frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^\infty (-t e^{-st}) f(t)\mathrm{d}t = \int_0^\infty e^{-st} \{-t f(t)\}\mathrm{d}t = -L\{t f(t)\}$$

$$L\{t f(t)\} = (-1)\frac{\mathrm{d}}{\mathrm{d}s} F(s)$$

Similarly, $$L\{t^2 f(t)\} = (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s)$$

In general,

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Example 1: Find the Laplace transforms of the following functions:

(i) $t \sin at$	**(ii)** $t\cos^2 t$
(iii) $t\sin^3 t$	**(iv)** $t\sin 2t \cosh t$
(v) $t\sqrt{1+\sin t}$	**(vi)** $t\, e^{3t} \sin t$
(vii) $t\left(\frac{\sin t}{e^t}\right)^2$	**(viii)** $t^2 \cos at$
(ix) $t^2 e^t \sin 4t$	**(x)** $(1+t\,e^{-t})^3$.

Solution:

(i) $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

$$L\{t\sin at\} = -\frac{d}{ds} L\{\sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

(ii) $$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2} L\{1+\cos 2t\} = \frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$L\{t\cos^2 t\} = -\frac{d}{ds} L\{\cos^2 t\} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}+\frac{(s^2+4)\cdot 1 - s\cdot 2s}{(s^2+4)^2}\right] = \frac{1}{2s^2}+\frac{s^2-4}{2(s^2+4)^2}$$

(iii) $$L\{\sin^3 t\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\} = \frac{1}{4}\left(\frac{3}{s^2+1}-\frac{3}{s^2+9}\right) = \frac{3}{4}\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right)$$

$$L\{t\sin^3 t\} = -\frac{d}{ds} L\{\sin^3 t\} = -\frac{3}{4}\frac{d}{ds}\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right)$$

$$= -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2}+\frac{2s}{(s^2+9)^2}\right] = \frac{3s}{2}\left[\frac{(s^2+9)^2-(s^2+1)^2}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{s^4+18s^2+81-s^4-2s^2-1}{(s^2+1)^2(s^2+9)^2}\right] = \frac{3s}{2}\cdot\frac{16(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

$$= \frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

(iv) $$L\{\sin 2t\cosh t\} = L\left\{\sin 2t\left(\frac{e^t+e^{-t}}{2}\right)\right\} = \frac{1}{2}L\left\{e^t\sin 2t + e^{-t}\sin 2t\right\}$$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^2+4} + \frac{2}{(s+1)^2+4}\right]$$

$$= \frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}$$

$$L\{t\sin 2t\cosh t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 2t\cosh t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}\right)$$

$$= \frac{2s-2}{\left(s^2-2s+5\right)^2} + \frac{2s+2}{\left(s^2+2s+5\right)^2}$$

(v) $$L\left\{\sqrt{1+\sin t}\right\} = L\left\{\sin\frac{t}{2} + \cos\frac{t}{2}\right\} = \frac{\frac{1}{2}}{s^2+\frac{1}{4}} + \frac{s}{s^2+\frac{1}{4}}$$

$$= \frac{1}{2}\cdot\frac{4}{4s^2+1} + \frac{4s}{4s^2+1} = \frac{4s+2}{4s^2+1}$$

$$L\left\{t\sqrt{1+\sin t}\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\left\{\sqrt{1+\sin t}\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{4s+2}{4s^2+1}\right)$$

$$= -\left[\frac{(4s^2+1)4-(4s+2)8s}{(4s^2+1)^2}\right] = \frac{-16s^2-4+32s^2+16s}{(4s^2+1)^2}$$

$$= \frac{16s^2+16s-4}{(4s^2+1)^2} = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

(vi) $$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{t\sin t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

By first shifting theorem,

$$L\left\{e^{3t}\,t\sin t\right\} = \frac{2(s-3)}{\left[(s-3)^2+1\right]^2} = \frac{2(s-3)}{(s^2-6s+10)^2}$$

(vii) $$f(t) = t\left(\frac{\sin t}{e^t}\right)^2 = t\,e^{-2t}\sin^2 t = t\,e^{-2t}\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}\,t\,e^{-2t}\left(1-\cos 2t\right)$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{t(1-\cos 2t)\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}L(1-\cos 2t) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$= -\left[-\frac{1}{s^2}-\frac{(s^2+4)\cdot 1 - s\cdot 2s}{(s^2+4)^2}\right] = \frac{1}{s^2}+\frac{4-s^2}{(s^2+4)^2}$$

By first shifting theorem,

$$L\left\{\frac{1}{2}t\cdot e^{-2t}(1-\cos 2t)\right\} = \frac{1}{2}\left[\frac{1}{(s+2)^2}+\frac{4-(s+2)^2}{\left\{(s+2)^2+4\right\}^2}\right]$$

(viii) $L\left\{\cos at\right\} = \dfrac{s}{s^2+a^2}$

$$L\left\{t^2\cos at\right\} = (-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}L\left\{\cos at\right\}$$

$$= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{s}{s^2+a^2}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{(s^2+a^2)\cdot 1 - s\cdot 2s}{(s^2+a^2)^2}\right] = \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2(-2s)-(a^2-s^2)\cdot 2(s^2+a^2)(2s)}{(s^2+a^2)^4}$$

$$= \frac{-2s^3-2a^2s-4a^2s+4s^3}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

(ix) $L\left\{\sin 4t\right\} = \dfrac{4}{s^2+16}$

$$L\left\{t^2\sin 4t\right\} = (-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}L\left\{\sin 4t\right\}$$

$$= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{4}{s^2+16}\right) = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{4(2s)}{(s^2+16)^2}\right] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{8s}{(s^2+16)^2}\right]$$

$$= -\left[\frac{(s^2+16)^2\cdot 8 - 8s\cdot 2(s^2+16)(2s)}{(s^2+16)^4}\right]$$

$$= \frac{-8s^2-128+32s^2}{(s^2+16)^3} = \frac{24s^2-128}{(s^2+16)^3} = \frac{8(3s^2-16)}{(s^2+16)^3}$$

By first shifting theorem,

$$L\left\{t^2e^t\sin 4t\right\} = \frac{8\left[3(s-1)^2-16\right]}{\left[(s-1)^2+16\right]^3} = \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

(x) $L\left\{\left(1+te^{-t}\right)^3\right\} = L\left\{1+3t\,e^{-t}+3t^2\,e^{-2t}+t^3e^{-3t}\right\}$

$$= L\left\{1\right\}+3L\left\{t\,e^{-t}\right\}+3L\left\{t^2\,e^{-2t}\right\}+L\left\{t^3\,e^{-3t}\right\}$$

$$= L\{1\} + 3(-1)\frac{d}{ds}L\{e^{-t}\} + 3(-1)^2\frac{d^2}{ds^2}L\{e^{-2t}\} + (-1)^3\frac{d^3}{ds^3}L\{e^{-3t}\}$$

$$= \frac{1}{s} - 3\frac{d}{ds}\left(\frac{1}{s+1}\right) + 3\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) - \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$$

Example 2: If $L\{\text{erf}\sqrt{t}\} = \dfrac{1}{s\sqrt{s^2+1}}$, find

(i) $L\{t\,\text{erf}\,2\sqrt{t}\}$ (ii) $L\{t\,e^{3t}\,\text{erf}\sqrt{t}\}$.

Solution: $L\{\text{erf}\sqrt{t}\} = \dfrac{1}{s\sqrt{s^2+1}}$

(i) By change of scale property,

$$L\{\text{erf}\,2\sqrt{t}\} = \frac{1}{4}\frac{1}{\left(\frac{s}{4}\right)\sqrt{\left(\frac{s}{4}\right)+1}} = \frac{2}{s\sqrt{s+4}}$$

$$L\{t\,\text{erf}\,2\sqrt{t}\} = -\frac{d}{ds}L\{\text{erf}\,2\sqrt{t}\} = -\frac{d}{ds}\left(\frac{2}{s\sqrt{s+4}}\right)$$

$$= -\left[\frac{-\left(2\sqrt{s+4} + 2\cdot s\cdot\frac{1}{2\sqrt{s+4}}\right)}{s^2(s+4)}\right] = \frac{2(s+4)+s}{s^2(s+4)\sqrt{s+4}} = \frac{3s+8}{s^2(s+4)^{\frac{3}{2}}}$$

(ii) $L\{e^{3t}\text{ erf }\sqrt{t}\} = \dfrac{1}{(s-3)\sqrt{s-3+1}} = \dfrac{1}{(s-3)\sqrt{s-2}}$

$$L\{t\,e^{3t}\text{ erf }\sqrt{t}\} = -\frac{d}{ds}L\{e^{3t}\text{ erf }\sqrt{t}\}$$

$$= -\frac{d}{ds}\left[\frac{1}{(s-3)\sqrt{s-2}}\right] = \frac{\sqrt{s-2} + (s-3)\frac{1}{2\sqrt{s-2}}}{(s-3)^2(s-2)}$$

$$= \frac{2(s-2)+(s-3)}{2(s-3)^2(s-2)^{\frac{3}{2}}} = \frac{3s-7}{2(s-3)^2(s-2)^{\frac{3}{2}}}$$

Exercise 12.6

Find the Laplace transforms of the following functions:

1. $t\cos at$

$$\left[\textbf{Ans.: } \frac{s^2-a^2}{(s^2+a^2)^2}\right]$$

2. $t\cos^3 t$

$$\left[\textbf{Ans.: } \frac{1}{4}\left[\frac{-s^2+9}{(s^2+9)^2}+\frac{s^2+3}{(s^2+1)^2}\right]\right]$$

3. $t\cos(\omega t-\alpha)$

$$\left[\textbf{Ans.: } \frac{(s^2-\omega^2)\cos\alpha+2\omega s\sin\alpha}{(s^2+\omega^2)^2}\right]$$

4. $t\sqrt{1-\sin t}$

$$\left[\textbf{Ans.: } \frac{4(4s^2-4s-1)}{(4s^2+1)^2}\right]$$

5. $t\cosh 3t$

$$\left[\textbf{Ans.: } \frac{s^2+9}{(s^2-9)^2}\right]$$

6. $t\sinh 2t\sin 3t$

$$\left[\textbf{Ans.: } 3\left[\frac{s-2}{(s^2-4s+13)^2}-\frac{s-2}{(s^2+4s+13)^2}\right]\right]$$

7. $t(3\sin 2t-2\cos 2t)$

$$\left[\textbf{Ans.: } \frac{8+12s-2s^2}{(s^2+4)^2}\right]$$

8. $t\, e^{3t}\sin 2t$

$$\left[\textbf{Ans.: } \frac{4(s-3)}{(s^2-6s+13)^2}\right]$$

9. $(t^2-3t+2)\sin 3t$

$$\left[\textbf{Ans.: } \frac{6s^4-18s^3+126s^2-162s+432}{(s^2+9)^3}\right]$$

10. $(t+\sin 2t)^2$

$$\left[\textbf{Ans.: } \frac{2}{s^3}+\frac{s}{(s^2+1)^2}+\frac{1}{2s}-\frac{s}{2(s^2+4)}\right]$$

11. $(t\sinh 2t)^2$

$$\left[\textbf{Ans.: } \frac{1}{2}\left[\frac{1}{(s-4)^3}+\frac{1}{(s+4)^3}\right]\right]$$

12. $t^2e^{-3t}\cosh 2t$

$$\left[\textbf{Ans.: } \frac{1}{(s+1)^3}+\frac{1}{(s+5)^3}\right]$$

13. $t^2e^{-2t}\sin 3t$

$$\left[\textbf{Ans.: } \frac{18(s^2+4s+1)}{(s^2+4s+13)^2}\right]$$

14. $t\sqrt{1+\sin 2t}$

$$\left[\textbf{Ans.: } \frac{s^2+2s-1}{(s^2+1)^2}\right]$$

15. $t\, e^{2t}(\cos t-\sin t)$

$$\left[\textbf{Ans.: } \frac{s^2-6s+7}{(s^2-4s+5)^2}\right]$$

16. $(t\cos 2t)^2$

$$\left[\textbf{Ans.: } \frac{1}{s^3}-\frac{s(48-s^2)}{(s^2+16)^3}\right]$$

17. $t^2\sin t\cos 2t$

$$\left[\textbf{Ans.: } \frac{9(s^2-3)}{(s^2+9)^3}+\frac{1-3s^2}{(s^2+1)^3}\right]$$

18. $t^3\cos t$

$$\left[\textbf{Ans.: } \frac{6s^4-36s^2+6}{(s^2+9)^3}\right]$$

12.4.6 Division by *t*

If $$L\{f(t)\} = F(s), \text{ then } L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\mathrm{d}s$$

Proof: $$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t$$

Integrating both the sides w.r.t s from s to ∞,

$$\int_s^{\infty} F(s)\mathrm{d}s = \int_s^{\infty}\int_0^{\infty} e^{-st} f(t)\mathrm{d}t\,\mathrm{d}s$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^{\infty} F(s)\mathrm{d}s = \int_0^{\infty}\left[\int_s^{\infty} e^{-st} f(t)\,\mathrm{d}s\right]\mathrm{d}t = \int_0^{\infty}\left[\frac{e^{-st}}{-t} f(t)\right]_s^{\infty}\mathrm{d}t$$

$$= \int_0^{\infty} e^{-st}\frac{f(t)}{t}\mathrm{d}t = L\left\{\frac{f(t)}{t}\right\}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\mathrm{d}s$$

Example 1: Find the Laplace transforms of the following functions:

(i) $\dfrac{1-e^{-t}}{t}$ (ii) $\dfrac{e^{-at}-e^{-bt}}{t}$ (iii) $\dfrac{\sinh t}{t}$

(iv) $\dfrac{\cosh 2t\sin 2t}{t}$ (v) $\dfrac{1-\cos t}{t}$ (vi) $\dfrac{\cos at-\cos bt}{t}$

(vii) $\dfrac{e^{-t}\sin t}{t}$ (viii) $\dfrac{e^{-2t}\sin 2t\cosh t}{t}$ (ix) $\dfrac{\sin^2 t}{t^2}$.

Solution:

(i) $$L\{1-e^{-t}\} = \frac{1}{s}-\frac{1}{s+1}$$

$$L\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^{\infty} L\{1-e^{-t}\}\mathrm{d}s = \int_s^{\infty}\left(\frac{1}{s}-\frac{1}{s+1}\right)\mathrm{d}s = \left|\log s-\log(s+1)\right|_s^{\infty}$$

$$= \left|\log\frac{s}{s+1}\right|_s^{\infty} = \log\left|\frac{1}{1+\frac{1}{s}}\right|_s^{\infty} = \log 1-\log\left(\frac{1}{1+\frac{1}{s}}\right)$$

$$= -\log\frac{s}{s+1} = \log\frac{s+1}{s}$$

(ii) $$L\{e^{-at}-e^{-bt}\} = \frac{1}{s+a}-\frac{1}{s+b}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^{\infty} L\{e^{-at}-e^{-bt}\}\mathrm{d}s = \int_s^{\infty}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)\mathrm{d}s$$

$$= \left|\log(s+a) - \log(s+b)\right|_s^\infty = \left|\log\frac{s+a}{s+b}\right|_s^\infty = \left|\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right|_s^\infty$$

$$= \log 1 - \log\frac{1+\frac{a}{s}}{1+\frac{b}{s}} = -\log\frac{s+a}{s+b} = \log\frac{s+b}{s+a}$$

(iii) $L\{\sinh t\} = L\left\{\frac{e^t - e^{-t}}{2}\right\} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$

$$L\left\{\frac{\sinh t}{t}\right\} = \int_s^\infty L\{\sinh t\}\,ds = \frac{1}{2}\int_s^\infty\left(\frac{1}{s-1} - \frac{1}{s+1}\right)ds$$

$$= \frac{1}{2}\left|\log(s-1) - \log(s+1)\right|_s^\infty = \frac{1}{2}\left|\log\frac{s-1}{s+1}\right|_s^\infty$$

$$= \frac{1}{2}\left|\log\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right|_s^\infty = \frac{1}{2}\left(\log 1 - \log\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right)$$

$$= -\frac{1}{2}\log\frac{s-1}{s+1} = \frac{1}{2}\log\frac{s+1}{s-1}$$

(iv) $L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = L\left\{\left(\frac{e^{2t} + e^{-2t}}{2t}\right)\sin 2t\right\} = \frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right]$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,ds = \int_s^\infty \frac{2}{s^2+4}\,ds$$

$$= \left|\tan^{-1}\frac{s}{2}\right|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{2} = \cot^{-1}\frac{s}{2}$$

By first shifting theorem,

$$L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = \frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right] = \frac{1}{2}\left[\cot^{-1}\left(\frac{s-2}{2}\right) + \cot^{-1}\left(\frac{s+2}{2}\right)\right]$$

(v) $L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$

$$L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty L\{1-\cos t\}\,ds = \int_s^\infty\left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds$$

$$= \left|\log s - \frac{1}{2}\log\left(s^2+1\right)\right|_s^\infty = -\frac{1}{2}\left|\log(s^2+1) - \log s^2\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log\frac{s^2+1}{s^2}\right|_s^\infty = -\frac{1}{2}\left|\log\left(1+\frac{1}{s^2}\right)\right|_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}\log\left(1+\frac{1}{s^2}\right) = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

(vi) $L\{\cos at - \cos bt\} = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty L\{\cos at - \cos bt\}\,\mathrm{d}s = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)\mathrm{d}s$$

$$= \left|\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log\frac{s^2+a^2}{s^2+b^2}\right|_s^\infty = \frac{1}{2}\left|\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right|_s^\infty$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}} = -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

(vii) $L\{\sin t\} = \dfrac{1}{s^2+1}$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty L\{e^{-t}\sin t\}\,\mathrm{d}s = \int_s^\infty \frac{1}{(s+1)^2+1}\,\mathrm{d}s$$

$$= \left|\tan^{-1}(s+1)\right|_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

(viii) $L\left\{\dfrac{e^{-2t}\sin 2t\cosh t}{t}\right\} = L\left\{\dfrac{e^{-2t}\sin 2t}{t}\,\dfrac{(e^t+e^{-t})}{2}\right\}$

$$= \frac{1}{2}\left[L\left\{\frac{e^{-t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-3t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,\mathrm{d}s = \int_s^\infty \frac{2}{s^2+4}\,\mathrm{d}s$$

$$= 2 \cdot \frac{1}{2} \left| \tan^{-1} \frac{s}{2} \right|_s^\infty = \frac{\pi}{2} - \tan^{-1} \frac{s}{2} = \cot^{-1} \frac{s}{2}$$

$$L\left\{ \frac{e^{-2t} \sin 2t \cosh t}{t} \right\} = \frac{1}{2} \left[L\left\{ \frac{e^{-t} \sin 2t}{t} \right\} + L\left\{ \frac{e^{-3t} \sin 2t}{t} \right\} \right]$$

$$= \frac{1}{2} \left[\cot^{-1} \left(\frac{s+1}{2} \right) + \cot^{-1} \left(\frac{s+3}{2} \right) \right]$$

(ix) $$L\left\{ \frac{\sin^2 t}{t^2} \right\} = L\left\{ \frac{1 - \cos 2t}{2t^2} \right\} = \frac{1}{2} L\left\{ \frac{1 - \cos 2t}{t^2} \right\}$$

$$L\{1 - \cos 2t\} = \frac{1}{s} - \frac{s}{s^2 + 4}$$

$$L\left\{ \frac{1 - \cos 2t}{t} \right\} = \int_s^\infty L\{1 - \cos 2t\} \, \mathrm{d}s$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 4} \right) \mathrm{d}s = \left| \log s - \frac{1}{2} \log(s^2 + 4) \right|_s^\infty$$

$$= \left| \log \frac{s}{\sqrt{s^2 + 4}} \right|_s^\infty = \left| \log \frac{1}{\sqrt{1 + \frac{4}{s^2}}} \right|_s^\infty = \log 1 - \log \frac{1}{\sqrt{1 + \frac{4}{s^2}}}$$

$$= -\log \frac{s}{\sqrt{s^2 + 4}} = \frac{1}{2} \log \frac{s^2 + 4}{s^2}$$

$$L\left\{ \frac{1 - \cos 2t}{t^2} \right\} = \int_s^\infty L\left\{ \frac{1 - \cos 2t}{t} \right\} \mathrm{d}s = \frac{1}{2} \int_s^\infty \log \left\{ \frac{s^2 + 4}{s^2} \right\} \mathrm{d}s$$

$$= \frac{1}{2} \left[\left| s \cdot \log \frac{s^2 + 4}{s^2} \right|_s^\infty - \int_s^\infty s \cdot \frac{s^2}{s^2 + 4} \left\{ \frac{2s \cdot s^2 - 2s\left(s^2 + 4\right)}{s^4} \right\} \mathrm{d}s \right]$$

$$= \frac{1}{2} \left[-s \log \frac{s^2 + 4}{s^2} - \int_s^\infty - \frac{8}{s^2 + 4} \mathrm{d}s \right] = \frac{1}{2} \left[-s \log \frac{s^2 + 4}{s^2} + 8 \cdot \frac{1}{2} \left| \tan^{-1} \frac{s}{2} \right|_s^\infty \right]$$

$$= \frac{1}{2} \left[-s \log \frac{s^2 + 4}{s^2} + 4\left(\frac{\pi}{2} - \tan^{-1} \frac{s}{2} \right) \right] = \frac{1}{2} \left[-s \log \frac{s^2 + 4}{s^2} + 4 \cot^{-1} \frac{s}{2} \right]$$

$$L\left\{ \frac{\sin^2 t}{t^2} \right\} = \frac{1}{2} L\left\{ \frac{1 - \cos 2t}{t^2} \right\} = \frac{1}{4} \left[-s \log \frac{s^2 + 4}{s^2} + 4 \cot^{-1} \frac{s}{2} \right]$$

Exercise 12.7

Find the Laplace transforms of the following functions:

1. $\dfrac{\sin t}{t}$

$\left[\textbf{Ans.:} \cot^{-1} s\right]$

2. $\dfrac{\sin^2 t}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{4}\log\left(\dfrac{s^2+4}{s^2}\right)\right]$

3. $\left(\dfrac{\sin 2t}{\sqrt{t}}\right)^2$

$\left[\textbf{Ans.:} \dfrac{1}{4}\log\left(\dfrac{s^2+16}{s^2}\right)\right]$

4. $\dfrac{\sin^3 t}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{4}\left(3\cot^{-1} s-\cot^{-1}\dfrac{s}{3}\right)\right]$

5. $\dfrac{1-\cos at}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{2}\log\left(\dfrac{s^2+a^2}{s^2}\right)\right]$

6. $\dfrac{\sin t\sin 5t}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{2}\log\left(\dfrac{s^2+36}{s^2+16}\right)\right]$

7. $\dfrac{2\sin t\sin 2t}{t}$

$\left[\textbf{Ans.:} \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+1}\right)\right]$

8. $\dfrac{e^{2t}\sin t}{t}$

$\left[\textbf{Ans.:} \cot^{-1}(s-2)\right]$

9. $\dfrac{e^{2t}\sin^3 t}{t}$

$\left[\begin{aligned}&\textbf{Ans.:} \dfrac{3}{4}\cot^{-1}(s-2)\\ &\qquad-\dfrac{1}{4}\cot^{-1}\left(\dfrac{s-2}{3}\right)\end{aligned}\right]$

10. $\dfrac{1-\cos t}{t^2}$

$\left[\textbf{Ans.:} s\log\dfrac{s}{\sqrt{s^2+1}}+\cot^{-1} s\right]$

12.4.7 Laplace Transforms of Derivatives

If $L\{f(t)\}=F(s)$, then

$$L\{f'(t)\}=sF(s)-f(0)$$

$$L\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$$

In general

$$L\{f^n(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)\ldots-f^{(n-1)}(0)$$

Proof: $L\{f'(t)\}=\int_0^\infty e^{-st}f'(t)\mathrm{d}t$

Integrating by parts,

$$L\{f'(t)\} = \left[e^{-st} f(t)\right]_0^{\infty} - \int_0^{\infty} (-se^{-st}) f(t)\mathrm{d}t \; = -f(0) + s\int_0^{\infty} e^{-st} f(t)\mathrm{d}t$$

$$= -f(0) + s\,L\{f(t)\}$$

Similarly, $$L\{f''\dot{t}\dot{}\} = -f' \quad + s\; L\{f' \; t\;\}$$

$$= -f'(0) + s\left[-f(0) + sL\{f(t)\}\right]$$

$$= -f'(0) - s\,f(0) + s^2 L\{f(t)\}$$

In general, $$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{n-1}(0)$$

Example 1: Find $L\{f(t)\}$ and $L\{f'(t)\}$ of the following functions:

(i) $f(t) = \dfrac{\sin t}{t}$

(ii) $f(t) = 3 \quad 0 \le t < 5$
$\quad = 0 \quad t > 5$

(iii) $f(t) = e^{-5t} \sin t$

(iv) $f(t) = t \quad 0 \le t < 3$
$\quad = 6 \quad t > 3.$

Solution:

(i) $$L\{f(t)\} = F(s) = L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} L\{\sin t\}\mathrm{d}s = \int_s^{\infty} \frac{1}{s^2+1}\mathrm{d}s$$

$$= \left|\tan^{-1} s\right|_s^{\infty} = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cot^{-1} s - \lim_{t\to 0} \frac{\sin t}{t} = s\cot^{-1} s - 1$$

(ii) $$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t = \int_0^{5} e^{-st} \cdot 3\mathrm{d}t + \int_5^{\infty} 0 \cdot \mathrm{d}t$$

$$= 3\left|\frac{e^{-st}}{-s}\right|_0^5 + 0 = \frac{-3}{s}(e^{-5s} - 1) = \frac{3}{s}(1 - e^{-5s})$$

$$L\{f'(t)\} = sF(s) - f(0) = s \cdot \frac{3}{s}(1 - e^{-5s}) - 3 = -3e^{-5s}$$

(iii) $$L\{f(t)\} = F(s) = L\{e^{-5t} \sin t\} = \frac{1}{(s+5)^2 + 1}$$

$$L\{f'(t)\} = sF(s) - f(0) = s \cdot \frac{1}{s^2 + 10s + 26} - e^0 \sin 0 = \frac{s}{s^2 + 10s + 26}$$

(iv) $$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^{3} e^{-st} t\ \mathrm{d}t + \int_3^{\infty} e^{-st} \cdot 6\,\mathrm{d}t$$

$$= \left|\frac{e^{-st}}{-s}\cdot t\right|_0^3 - \left|\frac{e^{-st}}{s^2}\right|_0^3 + 6\left|\frac{e^{-st}}{-s}\right|_3^\infty = -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6}{s}e^{-3s}$$

$$= \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

$$L\{f'(t)\} = s\,F(s) - f(0) = \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right)$$

Exercise 12.8

Find $L\{f'(t)\}$ of the following functions:

1. $f(t) = \left(\dfrac{1-\cos 2t}{t}\right)$

$$\left[\textbf{Ans.: } s\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$$

2. $f(t) = t+1 \qquad 0 \le t \le 2$

$\qquad = 3 \qquad t > 2$

$$\left[\textbf{Ans.: } \frac{1}{s}(1-e^{-2s})\right]$$

12.4.8 Laplace Transforms of Integrals

If $\qquad L\{f(t)\} = F(s)$, then $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \dfrac{F(s)}{s}$

Proof: $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \int_0^\infty e^{-st}\left\{\int_0^t f(t)\mathrm{d}t\right\}\mathrm{d}t$

Integrating by parts

$$L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \left[\int_0^t f(t)\mathrm{d}t\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty\left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t)\mathrm{d}t\right)\right]\mathrm{d}t$$

$$= \int_0^\infty \frac{1}{s}e^{-st}f(t)\mathrm{d}t = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

Example 1: Find Laplace transforms of the following functions:

(i) $\int_0^t e^{-2t}t^3\,\mathrm{d}t$ (ii) $\int_0^t t\cosh t\,\mathrm{d}t$ (iii) $\int_0^t t\,e^{-4t}\sin 3t\,\mathrm{d}t$

(iv) $e^{-4t}\int_0^t t\sin 3t\,\mathrm{d}t$ (v) $t\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t$ (vi) $\int_0^t t\,e^{-3t}\sin^2 t\,\mathrm{d}t$

(vii) $\cosh t\int_0^t e^t\cosh t\,\mathrm{d}t$ (viii) $e^{-t}\int_0^t \dfrac{\sin t}{t}\mathrm{d}t$ (ix) $\int_0^t\int_0^t\int_0^t t\sin t\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$.

Solution:

(i) $$L\left\{e^{-2t}t^3\right\} = \frac{3!}{(s+2)^4} = \frac{6}{(s+2)^4}$$

$$L\left\{\int_0^t e^{-2t}t^3 \mathrm{d}t\right\} = \frac{1}{s}L\left\{e^{-2t}t^3\right\} = \frac{6}{s(s+2)^4}$$

(ii) $$L\{t\cosh t\} = L\left\{t\left(\frac{e^t + e^{-t}}{2}\right)\right\} = \frac{1}{2}L\left\{t\,e^t + t\,e^{-t}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right] = \frac{1}{2}\cdot\frac{2(s^2+1)}{(s^2-1)^2} = \frac{s^2+1}{(s^2-1)^2}$$

$$L\left\{\int_0^t t\cosh t\,\mathrm{d}t\right\} = \frac{1}{s}L\{t\cosh t\} = \frac{s^2+1}{s(s^2-1)^2}$$

(iii) $$L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\left\{t\,e^{-4t}\sin 3t\right\} = \frac{6(s+4)}{\left[(s+4)^2+9\right]^2} = \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$L\left\{\int_0^t t\,e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{t\,e^{-4t}\sin 3t\right\} = \frac{6(s+4)}{s(s^2+8s+25)^2}$$

(iv) $$L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{1}{s}L\{t\sin 3t\} = \frac{6}{(s^2+9)^2}$$

$$L\left\{e^{-4t}\int_0^t t\sin 3t\right\} = \frac{6}{\left[(s+4)^2+9\right]^2} = \frac{6}{(s^2+8s+25)^2}$$

(v) $$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\left\{e^{-4t}\sin 3t\right\} = \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+25}$$

$$L\left\{\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{e^{-4t}\sin 3t\right\} = \frac{3}{s^3+8s^2+25s}$$

$$L\left\{t\int_0^t e^{-4t}\sin 3t\,dt\right\} = -\frac{d}{ds}L\left\{\int_0^t e^{-4t}\sin 3t\,dt\right\} = -\frac{d}{ds}\left(\frac{3}{s^3+8s^2+25s}\right)$$

$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

(vi) $$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$L\{t\sin^2 t\} = -\frac{d}{ds}L\{\sin^2 t\} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}-\left\{\frac{s^2+4-s\cdot 2s}{(s^2+4)^2}\right\}\right] = \frac{1}{2}\left[\frac{1}{s^2}-\frac{s^2-4}{(s^2+4)^2}\right]$$

$$L\{t\,e^{-3t}\sin^2 t\} = \frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{(s+3)^2-4}{\{(s+3)^2+4\}^2}\right] = \frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

$$L\left\{\int_0^t t\,e^{-3t}\sin^2 t\,dt\right\} = \frac{1}{s}\,L\{t\,e^{-3t}\sin^2 t\} = \frac{1}{2s}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

(vii) $L\{\cosh t\} = \dfrac{s}{s^2-1}$

$$L\{e^t\cosh t\} = \frac{s-1}{(s-1)^2-1} = \frac{s-1}{s^2-2s+1-1} = \frac{s-1}{s(s-2)}$$

$$L\left\{\int_0^t e^t\cosh t\,dt\right\} = \frac{1}{s}L\{e^t\cosh t\} = \frac{s-1}{s^2(s-2)}$$

$$L\left\{\cosh t\int_0^t e^t\cosh t\,dt\right\} = L\left\{\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t e^t\cosh t\,dt\right\}$$

$$= \frac{1}{2}\left[L\left\{e^t\int_0^t e^t\cosh t\,dt\right\}+L\left\{e^{-t}\int_0^t e^t\cosh t\,dt\right\}\right]$$

$$= \frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)}+\frac{(s+1)-1}{(s+1)^2(s+1-2)}\right] = \frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)}+\frac{s}{(s+1)^2(s-1)}\right]$$

(viii) $$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds = \left|\tan^{-1}s\right|_s^\infty = \frac{\pi}{2}-\tan^{-1}s = \cot^{-1}s$$

$$L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}L\left\{\frac{\sin t}{t}\right\} = \frac{1}{s}\cot^{-1}s$$

$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$$

(ix) $$L\{t\sin t\} = -\frac{d}{ds}L\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$L\left\{\int_0^t t\sin t\,dt\right\} = \frac{1}{s}L\{t\sin t\}$$

$$L\left\{\int_0^t\int_0^t t\sin t\,dt\right\} = \frac{1}{s}L\left\{\int_0^t t\sin t\,dt\right\} = \frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\}$$

$$L\left\{\int_0^t\int_0^t\int_0^t t\sin t\,dt\right\} = \frac{1}{s}L\left\{\int_0^t\int_0^t t\sin t\,dt\right\} = \frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\}$$

$$= \frac{1}{s^3}\cdot\frac{2s}{(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}$$

Exercise 12.9

Find the Laplace transforms of the following functions:

1. $\int_0^t e^{-t}t^4\,dt$

$$\left[\textbf{Ans.: } \frac{4!}{s(s+1)^5}\right]$$

2. $\int_0^t \frac{1+e^{-t}}{t}\,dt$

$$\left[\textbf{Ans.: } \frac{1}{s}\log[s(s+1)]\right]$$

3. $\int_0^t \frac{e^t\sin t}{t}\,dt$

$$\left[\textbf{Ans.: } \frac{1}{s}\cot^{-1}(s-1)\right]$$

4. $\int_0^t t\,e^{-2t}\sin 3t\,dt$

$$\left[\textbf{Ans.: } \frac{1}{s}\cdot\frac{3(2s+4)}{(s^2+4s+13)^2}\right]$$

5. $e^{-3t}\int_0^t t\sin 3t\,dt$

$$\left[\textbf{Ans.: } -\frac{6}{(s^2+6s+18)^2}\right]$$

6. $\int_0^t t^2\sin t\,dt$

$$\left[\textbf{Ans.: } -\frac{2\,(1-3s^2)}{s\,(s^2+1)^3}\right]$$

7. $\int_0^t t\cos^2 t\,dt$

$$\left[\textbf{Ans.: } \frac{1}{2s^3}+\frac{1}{2}\cdot\frac{s^2-4}{s(s^2+4)^2}\right]$$

8. $\int_0^t t\,e^{-3t}\cos^2 2t\,dt$

$$\left[\textbf{Ans.: } \frac{1}{2s(s+3)^2}+\frac{1}{2}\cdot\frac{s^2+6s-7}{s(s^2+6s+25)^2}\right]$$

12.4.9 Initial Value Theorem

If $L\{f(t)\} = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof: We know that,

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0)$$

$$= \int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$= \int_0^\infty \lim_{s\to\infty} \left[e^{-st} f'(t)\right]\mathrm{d}t + f(0)$$

$$= 0 + f(0) = f(0)$$

$$= \lim_{t\to 0} f(t)$$

12.4.10 Final Value Theorem

If $L\{f(t)\} = F(s)$, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof: We know that

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0)$$

$$= \int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\mathrm{d}t + f(0)$$

$$= \int_0^\infty \lim_{s\to 0} \left[e^{-st} f'(t)\right]\mathrm{d}t + f(0)$$

$$= \int_0^\infty f'(t)\,\mathrm{d}t + f(0)$$

$$= \left|f(t)\right|_0^\infty + f(0)$$

$$= \lim_{t\to\infty} f(t) - f(0) + f(0)$$

$$= \lim_{t\to\infty} f(t)$$

Example1 : Verify the initial and final value theorems for the following functions:

(i) $e^{-t}(t+1)^2$ **(ii) $e^{-t}(t^2 + \cos 3t)$.**

Solution:

(i) $f(t) = e^{-t}(t+1)^2 = e^{-t}(t^2 + 2t + 1)$

$$F(s) = \frac{2}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{1}{s+1}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{2s}{(s+1)^2} + \frac{s}{s+1}$$

$$\lim_{t \to 0} f(t) = 1$$

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \left[\frac{\frac{2}{s^2}}{\left(1 + \frac{1}{s}\right)^3} + \frac{\frac{2}{s}}{\left(1 + \frac{1}{s}\right)^2} + \frac{1}{1 + \frac{1}{s}} \right] = 1$$

Hence, initial value theorem is verified.

$$\lim_{t \to \infty} f(t) = 0$$

$$\lim_{s \to 0} sF(s) = 0$$

Hence, final value theorem is verified.

(ii) $f(t) = e^{-t}(t^2 + \cos 3t)$

$$F(s) = \frac{2}{(s+1)^3} + \frac{s+1}{(s+1)^2 + 9}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2 + 9}$$

$$\lim_{t \to 0} f(t) = 1$$

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \left[\frac{\frac{2}{s^2}}{\left(1 + \frac{1}{s}\right)^3} + \frac{\left(1 + \frac{1}{s}\right)}{\left(1 + \frac{1}{s}\right)^2 + \frac{9}{s^2}} \right] = 1$$

Hence, initial value theorem is verified.

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} (t^2 + \cos 3t)e^{-t} = 0$$

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} \left[\frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2 + 9} \right] = 0$$

Hence, final value theorem is verified.

Exercise 12.10

1. Verify the initial value theorem for the functions
   (i) $3 - 2\cos t$ **(ii)** $(2t + 3)^2$
   (iii) $t + \sin 3t$.

2. Verify the final value theorem for the functions
   (i) $1 + e^{-t}(\sin t + \cos t)$
   (ii) $t^3 e^{-2t}$.

12.5 EVALUATION OF AN INTEGRAL USING LAPLACE TRANSFORM

Example 1: Evaluate $\int_0^\infty e^{-2t} \sin^3 t\, dt$.

Solution:

$$\int_0^\infty e^{-st} \sin^3 t\, dt = L\{\sin^3 t\}$$

$$= L\left\{\frac{3\sin t - \sin 3t}{4}\right\} = \frac{3}{4}\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$$

$$= \frac{3}{4}\left[\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{6}{(s^2+1)(s^2+9)} \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t} \sin^3 t\, dt = \frac{6}{(4+1)(4+9)} = \frac{6}{65}$$

Example 2: Evaluate $\int_0^\infty e^{-4t} \cosh^3 t\, dt$.

Solution: $$\int_0^\infty e^{-st} \cosh^3 t\, dt = L\{\cosh^3 t\} = L\left\{\frac{\cosh 3t + 3\cosh t}{4}\right\}$$

$$= \frac{1}{4}\frac{s}{s^2-9} + \frac{3}{4}\frac{s}{s^2-1} = \frac{1}{4}\left[\frac{s^3 - s + 3s^3 - 27s}{(s^2-9)(s^2-1)}\right]$$

$$= \frac{1}{4}\left[\frac{4s^3 - 28s}{(s^2-9)(s^2-1)}\right] = \frac{s(s^2-7)}{(s^2-9)(s^2-1)} \qquad \text{... (1)}$$

Putting $s = 4$ in Eq. (1),

$$\int_0^\infty e^{-4t} \cosh^3 t\, dt = \frac{4(16-7)}{(16-9)(16-1)} = \frac{12}{35}$$

Example 3: Evaluate $\int_0^\infty e^{-3t} t^5\, dt$.

Solution:

$$\int_0^\infty e^{-st} t^5\, dt = L\{t^5\} = \frac{5!}{s^6} = \frac{120}{s^6}$$

Putting $s = 3$ in Eq. (1),

$$\int_0^\infty e^{-3t} t^5\, dt = \frac{120}{3^6} = \frac{40}{243}$$

Example 4: If $\int_0^{\infty} e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\mathrm{d}t = \frac{3}{8}$, find α.

Solution: $\int_0^{\infty} e^{-st}\sin(t+\alpha)\cos(t-\alpha)\mathrm{d}t = \frac{1}{2}\int_0^{\infty} e^{-st}(\sin 2t + \sin 2\alpha)\mathrm{d}t$

$$= \frac{1}{2}L\{\sin 2t + \sin 2\alpha\}$$

$$= \frac{1}{2}\left(\frac{2}{s^2+4} + \sin 2\alpha \cdot \frac{1}{s}\right) \quad \dots (1)$$

Putting $s = 2$ in Eq. (1),

$$\int_0^{\infty} e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\,\mathrm{d}t = \frac{1}{2}\left(\frac{2}{4+4} + \frac{1}{2}\sin 2\alpha\right)$$

$$= \frac{1}{8} + \frac{1}{4}\sin 2\alpha$$

But $\int_0^{\infty} e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\mathrm{d}t = \frac{3}{8}$

$$\frac{1}{8} + \frac{1}{4}\sin 2\alpha = \frac{3}{8}$$

$$\frac{1}{4}\sin 2\alpha = \frac{1}{4}$$

$$\sin 2\alpha = 1$$

$$2\alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4}$$

Example 5: Show that $\int_0^{\infty} e^{-3t}\, t\sin t\,\mathrm{d}t = \frac{3}{50}$.

Solution: $\int_0^{\infty} e^{-st}\, t\sin t\,\mathrm{d}t = L\{t\sin t\}$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} \quad \dots (1)$$

Putting $s = 3$ in Eq. (1),

$$\int_0^{\infty} e^{-3t}\, t\sin t\,\mathrm{d}t = \frac{6}{(9+1)^2} = \frac{3}{50}$$

Example 6: Show that $\int_0^{\infty} e^{-2t}t^2 \sin 3t\,\mathrm{d}t = \frac{18}{2197}$.

Solution: $\int_0^{\infty} e^{-st}t^2 \sin 3t\,\mathrm{d}t = L\{t^2 \sin 3t\}$

$$= (-1)^2 \frac{d^2}{ds^2} L\{\sin 3t\} = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) = \frac{d}{ds}\left[-\frac{3\cdot 2s}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{(s^2+9)^2\cdot 1 - s\cdot 2(s^2+9)2s}{(s^2+9)^4}\right] = -6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right]$$

$$= \frac{-6(-3s^2+9)}{(s^2+9)^3} = \frac{18(s^2-3)}{(s^2+9)^3} \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t} t^2 \sin 3t \, dt = \frac{18(4-3)}{(4+9)^3} = \frac{18}{2197}$$

Example 7: Show that $\int_0^\infty e^{-t} t^3 \sin t \, dt = 0$.

Solution: $\int_0^\infty e^{-st} t^3 \sin t \, dt = L\{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3} L\{\sin t\} = -\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$

If we differentiate $\frac{1}{s^2+1}$ three times, problem becomes tedious. Hence, we will solve this problem by different method.

$$\int_0^\infty e^{-st} t^3 \sin t \, dt = \int_0^\infty e^{-st}\left[\text{Imaginary part of } e^{it}\right] t^3 \, dt$$

$$= \text{Img}\cdot\text{part} \int_0^\infty e^{-st}\cdot e^{it} t^3 \, dt = \text{Img}\cdot\text{part } L\{e^{it} t^3\}$$

$$= \text{Img}\cdot\text{part} \frac{3!}{(s-i)^4} \qquad \text{... (1)}$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t} t^3 \sin t \, dt = \text{Img}\cdot\text{part}\frac{6}{(1-i)^4} = \text{Img}\cdot\text{part}\frac{6}{\left(\sqrt{2}\, e^{\frac{-i\pi}{4}}\right)^4} = \text{Img}\cdot\text{part}\frac{6}{4}e^{i\pi}$$

$$= \text{Img}\cdot\text{part}\left[\frac{3}{2}(\cos\pi + i\sin\pi)\right] = \text{Img}\cdot\text{part}\left[\frac{3}{2}(-1+i\cdot 0)\right] = 0$$

Example 8: If $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$, prove that $\int_0^\infty e^{-3t} t J_0(4t) dt = \frac{3}{125}$.

Solution: $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$

By change of scale property,

$$L\{J_0(4t)\} = \frac{1}{4} \cdot \frac{1}{\sqrt{\left(\frac{s}{4}\right)^2 + 1}} = \frac{1}{\sqrt{s^2 + 16}}$$

$$\int_0^\infty e^{-st} t\, J_0(4t)\mathrm{d}t = L\{t \cdot J_0(4t)\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{J_0(4t)\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\sqrt{s^2+16}}\right) = \frac{1}{2} \cdot \frac{2s}{(s^2+16)^{\frac{3}{2}}}$$

$$= \frac{s}{(s^2+16)^{\frac{3}{2}}} \qquad \text{... (1)}$$

Putting $s = 3$ in Eq. (1),

$$\int_0^\infty e^{-3t} t\, J_0(4t)\,\mathrm{d}t = \frac{3}{(9+16)^{\frac{3}{2}}} = \frac{3}{125}$$

Example 9: Show that $\displaystyle\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{t e^t}\right) \mathrm{d}t = \frac{3\pi}{4}$.

Solution: $\displaystyle\int_0^\infty e^{-st}\left(\frac{\sin 2t + \sin 3t}{t}\right)\mathrm{d}t = L\left\{\frac{\sin 2t + \sin 3t}{t}\right\}$

$$= \int_s^\infty L\{\sin 2t + \sin 3t\}\,\mathrm{d}s = \int_s^\infty \left(\frac{2}{s^2+4} + \frac{3}{s^2+9}\right)\mathrm{d}s$$

$$= 2 \cdot \frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty + 3 \cdot \frac{1}{3}\left|\tan^{-1}\frac{s}{3}\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{2} + \frac{\pi}{2} - \tan^{-1}\frac{s}{3}$$

$$= \pi - \tan^{-1}\frac{s}{2} - \tan^{-1}\frac{s}{3} \qquad \text{... (1)}$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}\left(\frac{\sin 2t + \sin 3t}{t}\right)\mathrm{d}t = \pi - \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3} = \pi - \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$= \pi - \tan^{-1} 1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Example 10: Show that $\displaystyle\int_0^\infty e^{-t}\,\frac{\sin^2 t}{t}\,\mathrm{d}t = \frac{1}{4}\log 5$.

Solution: $\displaystyle\int_0^\infty e^{-st}\,\frac{\sin^2 t}{t}\mathrm{d}t = L\left\{\frac{\sin^2 t}{t}\right\} = L\left\{\frac{1-\cos 2t}{2t}\right\}$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds = \frac{1}{2}\left|\log s - \frac{1}{2}\log(s^2+4)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log \frac{s}{\sqrt{s^2+4}}\right|_s^\infty = \frac{1}{2}\left[\lim_{s\to\infty} \log \frac{s}{\sqrt{s^2+4}} - \log \frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\left(\lim_{s\to\infty} \log \frac{1}{\sqrt{1+\frac{4}{s^2}}} - \log \frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\left(\log 1 - \log \frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\log \sqrt{\frac{s^2+4}{s}} \qquad \text{... (1)}$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt = \frac{1}{2}\log\frac{\sqrt{5}}{1} = \frac{1}{4}\log 5$$

Example 11: Show that $\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t \sinh t}{t}\,dt = \frac{\pi}{8}$.

Solution: $\int_0^\infty e^{-st}\frac{\sin t \sinh t}{t}\,dt = L\left\{\frac{\sin t \sinh t}{t}\right\} = L\left\{\left(\frac{e^t - e^{-t}}{2}\right)\frac{\sin t}{t}\right\}$

We will first find $L\left\{\frac{\sin t}{t}\right\}$ and then apply shifting theorem.

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds = \left|\tan^{-1} s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st}\frac{\sin t \sinh t}{t}\,dt = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right]. \qquad \text{... (1)}$$

Putting $s = \sqrt{2}$ in Eq. (1),

$$\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t \sinh t}{t}\,dt = \frac{1}{2}\left[\tan^{-1}\left(\sqrt{2}+1\right) - \tan^{-1}\left(\sqrt{2}-1\right)\right]$$

$$= \frac{1}{2}\tan^{-1}\frac{\sqrt{2}+1-\sqrt{2}+1}{1+\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)} = \frac{1}{2}\tan^{-1}\left(\frac{2}{1+2-1}\right)$$

$$= \frac{1}{2}\tan^{-1} 1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

Example 12: Show that $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \log\frac{2}{3}$.

Solution: $\displaystyle\int_0^\infty e^{-st}\left(\frac{\cos 6t-\cos 4t}{t}\right)dt = L\left\{\frac{\cos 6t-\cos 4t}{t}\right\}$

$$= \int_s^\infty L\{\cos 6t - \cos 4t\}\,ds = \int_s^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right)ds$$

$$= \left|\frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)\right|_s^\infty = \frac{1}{2}\left[\lim_{s\to\infty}\log\left(\frac{s^2+36}{s^2+16}\right) - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\left[\lim_{s\to\infty}\log\frac{\left(1+\frac{36}{s^2}\right)}{\left(1+\frac{16}{s^2}\right)} - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+36}{s^2+16}\right)\right] = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right) \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \frac{1}{2}\log\frac{16}{36}$$

$$= \frac{1}{2}\log\left(\frac{4}{6}\right)^2 = \log\frac{4}{6} = \log\frac{2}{3}$$

Example 13: Evaluate $\displaystyle\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u\,du\right)dt$.

Solution: $L\{\sinh u\cosh u\} = L\left\{\frac{1}{2}\sinh 2u\right\} = \frac{1}{2}\cdot\frac{2}{s^2-4} = \frac{1}{s^2-4}$

$$L\{u^2\sinh u\cosh u\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s^2-4}\right) = \frac{d}{ds}\left[\frac{-2s}{(s^2-4)^2}\right]$$

$$= -2\left[\frac{(s^2-4)^2 - s\cdot 2(s^2-4)\cdot 2s}{(s^2-4)^4}\right]$$

$$= -2\left[\frac{s^2-4-4s^2}{(s^2-4)^3}\right]$$

$$= \frac{2(3s^2+4)}{(s^2-4)^3}$$

$$L\left\{\int_0^t u^2\sinh u\cosh u\,du\right\} = \frac{1}{s}L\{u^2\sinh u\cosh u\} = \frac{2(3s^2-4)}{s(s^2+4)^3}$$

Now, $\int_0^\infty e^{-st}\left\{\int_0^t u^2 \sinh u \cosh u \, du\right\} dt = \dfrac{2(3s^2-4)}{s(s^2+4)^3}$ ... (1)

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}\left(\int_0^t u \sinh u \cosh u \, du\right) dt = \frac{2(3\cdot 1-4)}{1(1+4)^3} = -\frac{2}{125}$$

Example 14: Evaluate $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u} du \, dt$.

Solution: $L\{\sin u\} = \dfrac{1}{s^2+1}$

$$L\left\{\frac{\sin u}{u}\right\} = \int_s^\infty L\{\sin u\} ds = \int_s^\infty \frac{1}{s^2+1} ds = \left|\tan^{-1} s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{1}{s} L\left\{\frac{\sin u}{u}\right\} = \frac{1}{s}\cot^{-1} s$$

Now, $\int_0^\infty e^{-st}\int_0^t \frac{\sin u}{u} du \, dt = \dfrac{1}{s}\cot^{-1} s$

Putting $s = 1$,

$$\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u} du \, dt = \cot^{-1} 1 = \frac{\pi}{4}.$$

Exercise 12.11

Evaluate the following integrals using the Laplace transform:

1. $\int_0^\infty e^{-3t} \cos^2 t \, dt$ $\left[\textbf{Ans.: } \dfrac{11}{39}\right]$

2. $\int_0^\infty e^{-5t} \sinh^3 t \, dt$ $\left[\textbf{Ans.: } \dfrac{1}{64}\right]$

3. $\int_0^\infty e^{-3t} \cos^3 t \, dt$ $\left[\textbf{Ans.: } \dfrac{4}{15}\right]$

4. $\int_0^\infty e^{-2t} t^3 \sin t \, dt$ $\left[\textbf{Ans.: } -\dfrac{576}{25}\right]$

5. $\int_0^\infty e^{-3t} t^2 \sinh 2t \, dt$ $\left[\textbf{Ans.: } \dfrac{124}{125}\right]$

6. $\int_0^\infty e^{-2t} t \sin^2 t \, dt$ $\left[\textbf{Ans.: } \dfrac{1}{8}\right]$

7. $\int_0^\infty \dfrac{e^{-t} - e^{-3t}}{t} dt$ **[Ans.:** log 3**]**

8. $\int_0^\infty e^{-t} \dfrac{(1-\cos 2t)}{2t} dt$ $\left[\textbf{Ans.: } \dfrac{1}{4}\log 5\right]$

9. $\int_0^\infty e^{-t} \frac{(\cos 3t - \cos 2t)}{t} \mathrm{d}t$

$\left[\textbf{Ans.:} \frac{1}{2} \log \frac{1}{2}\right]$

10. $\int_0^\infty e^{-t} \frac{\sin \sqrt{3}t}{t} \mathrm{d}t$

$\left[\textbf{Ans.:} \frac{\pi}{3}\right]$

11. $\int_0^\infty e^{-2t} \frac{\sinh t}{t} \mathrm{d}t$

$\left[\textbf{Ans.:} \frac{1}{2} \log 3\right]$

12. $\int_0^\infty e^{-t} \int_0^t t \cos^2 t \, \mathrm{d}t \, \mathrm{d}t$

$\left[\textbf{Ans.:} \frac{12}{50}\right]$

13. $\int_0^\infty e^{-t} \left(t \int_0^t e^{-4u} \cos u \, \mathrm{d}u\right) \mathrm{d}t$

$\left[\textbf{Ans.:} \frac{9}{64}\right]$

14. $\int_0^\infty e^{-t} \left(\frac{1}{t} \int_0^t e^{-u} \sin u \, \mathrm{d}u\right) \mathrm{d}t$

$\left[\textbf{Ans.:} \frac{1}{4} \log 5 - \frac{1}{2} \cot^{-1} 2\right]$

12.6 HEAVISIDE'S UNIT STEP FUNCTION

It is defined as

$$\begin{aligned} u(t) &= 0 \qquad t < 0 \\ &= 1 \qquad t > 0 \end{aligned}$$

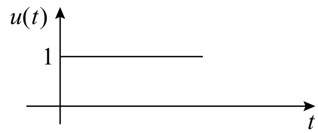


Fig. 12.1

The displaced (delayed) unit step function $u(t-a)$ represents the function $u(t)$ which is displaced by a distance 'a' to the right.

$$\begin{aligned} u(t-a) &= 0 \qquad t < a \\ &= 1 \qquad t > a \end{aligned}$$

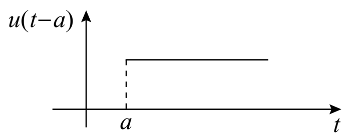


Fig. 12.2

Heaviside's unit step functions $u(t-a)$ and $u(t)$ are used to represent a portion of the curve of the function $f(t)$.

Case I: When any function $f(t)$ is multiplied by the unit step function $u(t)$, the resultant function $f(t)\,u(t)$ represents the part of the function $f(t)$ to the right of the origin.

$$\begin{aligned} f(t)\,u(t) &= 0 \qquad t < 0 \\ &= f(t) \quad t > 0 \end{aligned}$$

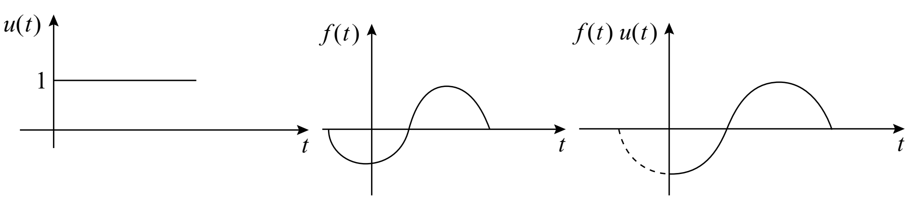


Fig. 12.3

Case II: When any function $f(t)$ is multiplied by displaced unit step function $u(t-a)$, the resultant function $f(t)\,u(t-a)$ represents the part of the function $f(t)$ to the right of $t = a$.

$$\begin{aligned} f(t)\,u(t-a) &= 0 \qquad t < a \\ &= f(t) \quad t > a \end{aligned}$$

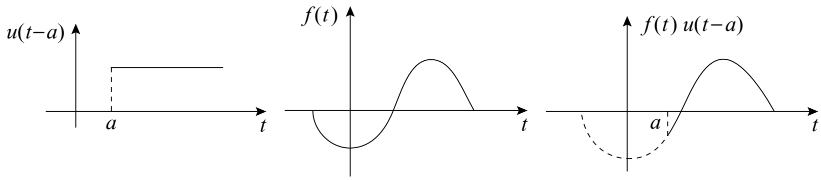


Fig. 12.4

Case III: When the displaced unit step function $f(t-a)$ is multiplied by $u(t-b)$, the resultant function $f(t-a)\,u(t-b)$ represents the part of the function $f(t-a)$ to the right of $t = b$.

$$\begin{aligned} f(t-a)\,u(t-b) &= 0 \qquad & t < b \\ &= f(t-a) \qquad & t > b \end{aligned}$$

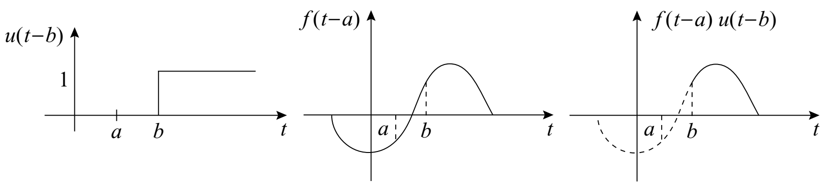


Fig. 12.5

Case IV: When any function $f(t)$ is multiplied by the function $[u(t-a) - u(t-b)]$ lying between $a < t < b$, the resultant function $f(t)\,[u(t-a) - u(t-b)]$ represents the part of the function $f(t)$ lying between $a < t < b$.

$$\begin{aligned} f(t)\,[u(t-a) - u(t-b)] &= 0 \qquad & t < a \\ &= f(t) \qquad & a < t < b \\ &= 0 \qquad & t > b \end{aligned}$$

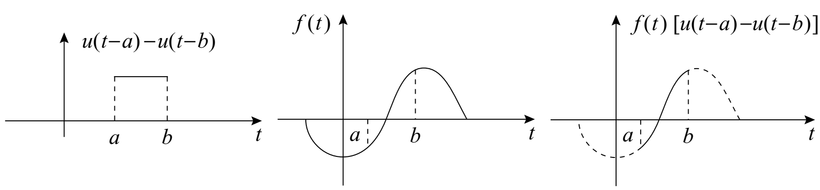


Fig. 12.6

12.6.1 Laplace Transform of Heaviside's Unit Step Functions

(i) *Laplace transform of unit step function u(t)*

$$\begin{aligned} u(t) &= 0 \qquad t < 0 \\ &= 1 \qquad t > 0 \end{aligned}$$

$$L\{u(t)\} = \int_0^\infty e^{-st} u(t)\,dt$$

$$= \int_0^\infty e^{-st}\,dt = \left| \frac{e^{-st}}{-s} \right|_0^\infty = \frac{1}{s}$$

(ii) *Laplace transform of the displaced unit step function u(t – a)*

$$u(t-a) = 0 \quad t < a$$
$$= 1 \quad t > a$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) \mathrm{d}t$$

$$= \int_a^\infty e^{-st} \mathrm{d}t = \left| \frac{e^{-st}}{-s} \right|_a^\infty$$

$$= \frac{1}{s} e^{-as}$$

(iii) *Laplace transform of the function f(t – a) u(t – a)*

$$f(t-a)\, u(t-a) = 0 \qquad t < a$$
$$= f(t-a) \qquad t > a$$

$$L\{f(t-a)\, u(t-a)\} = \int_0^\infty e^{-st} f(t-a) u(t-a) \mathrm{d}t = \int_a^\infty e^{-st} f(t-a) \mathrm{d}t$$

Putting $t - a = x$, $\mathrm{d}t = \mathrm{d}x$

When $t = a$, $x = 0$

$t \to \infty$, $x \to \infty$

$$L\{f(t-a)\, u(t-a)\} = \int_0^\infty e^{-s(a+x)} f(x) \mathrm{d}x = e^{-as} \int_0^\infty e^{-sx} f(x) \mathrm{d}x$$

$$= e^{-as} L\{f(x)\}$$

$$= e^{-as} F(s)$$

If $a = 0$,

$L\{f(t)\, u(t)\} = F(s)$

(iv) *Laplace transform of the function f(t) u(t – a)*

$$L\{f(t)\, u(t-a)\} = \int_0^\infty e^{-st} f(t) u(t-a) \mathrm{d}t = \int_a^\infty e^{-st} f(t) \mathrm{d}t$$

Putting $t - a = x$, $\mathrm{d}t = \mathrm{d}x$

When $t = a$, $x = 0$

$t \to \infty$, $x \to \infty$

$$L\{f(t)\, u(t-a)\} = \int_0^\infty e^{-s(x+a)} f(x+a) \mathrm{d}x = e^{-as} \int_0^\infty e^{-sx} f(x+a) \mathrm{d}x$$

$$= e^{-as} \int_0^\infty e^{-st} f(t+a) \mathrm{d}t = e^{-as} L\{f(t+a)\}$$

Example 1: Find the Laplace transform of $t^2\, u(t-2)$.

Solution:

$$L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$$

$$L\{t^2\, u(t-2)\} = e^{-2s} L\{(t+2)^2\}$$
$$= e^{-2s} L\{t^2 + 4t + 4\}$$
$$= e^{-2s} \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right)$$

Example 2: Find the Laplace transform of $(1 + 2t - 3t^2 + 4t^3)\, u(t - 2)$ and hence, evaluate $\int_0^\infty e^{-t}\,(1 + 2t - 3t^2 + 4t^3)\, u(t - 2)dt$.

Solution: $L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$

$$\begin{aligned}
L\{(1 + 2t - 3t^2 + 4t^3)\, u(t - 2)\} &= e^{-2s} L[1 + 2(t + 2) - 3(t + 2)^2 + 4(t + 2)^3] \\
&= e^{-2s} L\{1 + 2(t + 2) - 3(t^2 + 4t + 4) \\
&\quad + 4(t^3 + 6t^2 + 12t + 8)\} \\
&= e^{-2s} L\{25 + 38t + 21t^2 + 4t^3\} \\
&= e^{-2s}\left(\frac{25}{s} + 38\cdot\frac{1}{s^2} + 21\cdot\frac{2!}{s^3} + 4\cdot\frac{3!}{s^4}\right) \\
&= e^{-2s}\left(\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right)
\end{aligned}$$

Now, $$\int_0^\infty e^{-st}\,(1 + 2t - 3t^2 + 4t^3)\, u(t - 2)\,dt = e^{-2s}\left(\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right) \qquad \ldots (1)$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}\,(1 + 2t - 3t^2 + 4t^3)\, u(t - 2)dt = e^{-2}\left(\frac{25}{1} + \frac{38}{1^2} + \frac{42}{1^3} + \frac{24}{1^4}\right) = \frac{129}{e^2}$$

Example 3: Find the Laplace transform of $\sin t \left[u\left(t - \frac{\pi}{2}\right) - u\left(t - \frac{3\pi}{2}\right)\right]$.

Solution: $L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$

$$\begin{aligned}
L\left\{\sin t\, u\left(t - \frac{\pi}{2}\right) - u\left(t - \frac{3\pi}{2}\right)\right\} &= L\left\{\sin t\, u\left(t - \frac{\pi}{2}\right)\right\} - L\left\{u\left(t - \frac{3\pi}{2}\right)\right\} \\
&= e^{-\frac{\pi s}{2}} L\left\{\sin\left(t + \frac{\pi}{2}\right)\right\} - \frac{e^{-\frac{3\pi s}{2}}}{s} \\
&= e^{\frac{-\pi s}{2}} L\{\cos t\} - \frac{e^{-\frac{3\pi s}{2}}}{s} \\
&= e^{\frac{-\pi s}{2}} \frac{s}{s^2 + 1} - e^{-\frac{3\pi s}{2}} \cdot \frac{1}{s}
\end{aligned}$$

Example 4: Find the Laplace transform of $e^{-t} \sin t\, u(t - \pi)$.

Solution: $L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$

$$\begin{aligned}
L\{e^{-t} \sin t\, u(t - \pi)\} &= e^{-\pi s} L\{e^{-(t+\pi)} \sin(t + \pi)\} = -e^{-\pi s} e^{-\pi} L\{e^{-t} \sin t\} \\
&= -e^{-\pi(s+1)} \frac{1}{(s+1)^2 + 1} = -e^{-\pi(s+1)} \frac{1}{s^2 + 2s + 2}
\end{aligned}$$

Example 5: Find the Laplace transforms of the following functions:

(i) $f(t) = t^2 \quad 0 < t < 1$
$\quad\;\; = 4t \quad t > 1$

(ii) $f(t) = \sin 2t \quad 2\pi < t < 4\pi$
$\quad\;\; = 0 \quad$ otherwise

(iii) $f(t) = \cos t \quad 0 < t < \pi$
$\quad\;\; = \sin t \quad t > \pi$

(iv) $f(t) = \cos t \quad 0 < t < \pi$
$\quad\;\; = \cos 2t \quad \pi < t < 2\pi$
$\quad\;\; = \cos 3t \quad t > 2\pi$

Solution:

(i) Expressing $f(t)$ in terms of unit step function,

$$f(t) = t^2\, u(t) - t^2\, u(t-1) + 4t\, u(t-1)$$

$$\begin{aligned}
L\{f(t)\} &= L\{t^2\, u(t) - t^2\, u(t-1) + 4t\, u(t-1)\}\\
&= L\{t^2\, u(t)\} - L\{t^2\, u(t-1) + 4\, L\{t\, u(t-1)\}\\
&= \frac{2}{s^3} - e^{-s} L\{(t+1)^2\} + 4e^{-s} L\{(t+1)\}\\
&= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + 4e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)\\
&= \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right)
\end{aligned}$$

(ii) Expressing $f(t)$ in terms of unit step function,

$$f(t) = \sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)$$

$$\begin{aligned}
L\{f(t)\} &= L\{\sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)\}\\
&= L\{\sin 2t\, u(t-2\pi)\} - L\{\sin 2t\, u(t-4\pi)\}\\
&= e^{-2\pi s} L\{\sin 2(t+2\pi)\} - e^{-4\pi s} L\{\sin 2(t+4\pi)\}\\
&= e^{-2\pi s} L\{\sin 2t\} - e^{-4\pi s} L\{\sin 2t\}\\
&= e^{-2\pi s} \frac{2}{s^2+4} - e^{-4\pi s} \frac{2}{s^2+4} = \frac{2}{s^2+4}\left(e^{-2\pi s} - e^{-4\pi s}\right)
\end{aligned}$$

(iii) Expressing $f(t)$ in terms of unit step function,

$$f(t) = \cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)$$

$$\begin{aligned}
L\{f(t)\} &= L\{\cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)\}\\
&= L\{\cos t\, u(t)\} - L\{\cos t\, u(t-\pi)\} + L\{\sin t\, u(t-\pi)\}\\
&= \frac{s}{s^2+1} - e^{-\pi s} L\{\cos(t+\pi)\} + e^{-\pi s} L\{\sin(t+\pi)\}\\
&= \frac{s}{s^2+1} - e^{-\pi s} L\{-\cos t\} + e^{-\pi s} L\{-\sin t\}\\
&= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos t\} - e^{-\pi s} L\{\sin t\}\\
&= \frac{s}{s^2+1} + e^{-\pi s}\cdot\frac{s}{s^2+1} - e^{-\pi s}\cdot\frac{1}{s^2+1}\\
&= \frac{1}{s^2+1}\left[s + e^{-\pi s}(s-1)\right]
\end{aligned}$$

(iv) Expressing $f(t)$ in terms of unit step function

$$f(t) = [\cos t\, u(t) - \cos t\, u(t-\pi)] + [\cos 2t\, u(t-\pi) - \cos 2t\, u(t-2\pi)] + \cos 3t\, u(t-2\pi)$$

$$= \cos t\, u(t) + (\cos 2t - \cos t)\, u(t-\pi) + (\cos 3t - \cos 2t)\, u(t-2\pi)$$

$$L\{f(t)\} = L\{\cos t\, u(t)\} + L\{(\cos 2t - \cos t)\, u(t-\pi)\} + L\{(\cos 3t - \cos 2t)\, u(t-2\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\} + e^{-2\pi s} L\{\cos 3(t+2\pi) - \cos 2(t+2\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2t + \cos t\} + e^{-2\pi s} L\{\cos 3t - \cos 2t\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Exercise 12.12

(I) Find the Laplace transforms of the following functions:

1. $t^4\, u(t-2)$

$$\left[\textbf{Ans.: } e^{-4s}\left(\frac{16}{s} + \frac{32}{s^2} + \frac{48}{s^3} + \frac{48}{s^4} + \frac{24}{s^5}\right)\right]$$

2. $(1 + 3t - 4t^2 + 2t^3)\, u(t-3)$

$$\left[\textbf{Ans.: } e^{-3s}\left(\frac{28}{s} + \frac{33}{s^2} + \frac{28}{s^3} + \frac{12}{s^4}\right)\right]$$

3. $t\, e^{-2t}\, u(t-1)$

$$\left[\textbf{Ans.: } e^{-(s+2)}\frac{s+3}{(s+2)^2}\right]$$

4. $\cos t\, u(t-1)$

$$\left[\textbf{Ans.: } e^{-s}\left(\frac{s\cos 1 - \sin 1}{s^2+1}\right)\right]$$

(II) Express the following functions in terms of Heaviside's unit step function and hence, find the Laplace transform.

1.
$$\begin{aligned} f(t) &= t & 0<t<2 \\ &= t^2 & t>2 \end{aligned}$$

$$\left[\textbf{Ans.: } \frac{1}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)\right]$$

2.
$$\begin{aligned} f(t) &= e^t \cos t & 0<t<\pi \\ &= e^t \sin t & t>\pi \end{aligned}$$

$$\left[\textbf{Ans.: } \frac{s-1}{s^2-2s+2} + e^{-\pi(s-1)}\cdot\frac{s-2}{s^2-2s+2}\right]$$

3.
$$\begin{aligned} f(t) &= \sin t & 0<t<\pi \\ &= \sin 2t & \pi<t<2\pi \\ &= \sin 3t & t>2\pi \end{aligned}$$

$$\left[\textbf{Ans.: } \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right) - e^{-2\pi s}\left(\frac{3}{s^2+9} + \frac{2}{s^2+4}\right)\right]$$

4.
$$\begin{aligned} f(t) &= t-1 & 1<t<2 \\ &= 3-t & 2<t<3 \\ &= 0 & t>3 \end{aligned}$$

$$\left[\textbf{Ans.: } \frac{(1-e^{-s})^2}{s^2}\right]$$

5.
$$\begin{aligned} f(t) &= \sin t & 0<t<\pi \\ &= t & t>\pi \end{aligned}$$

$$\left[\textbf{Ans.: } \frac{1+e^{-\pi s}}{s^2+1} + e^{-\pi s}\left(\frac{\pi s+1}{s^2}\right)\right]$$

12.7 DIRAC DELTA OR UNIT IMPULSE FUNCTION

Consider the function $f(t)$ as shown in Fig. 12.7.

$$f(t) = \frac{1}{T} \qquad -\frac{T}{2} < t < \frac{T}{2}$$
$$= 0 \qquad \text{otherwise}$$

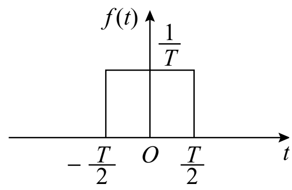


Fig. 12.7

The width of this function is T and its amplitude is $\frac{1}{T}$. Hence, the area of this function is one unit. As $T \to 0$, the function becomes a delta function or unit impulse function.

$$\lim_{T\to 0} f(t) = \delta(t)$$

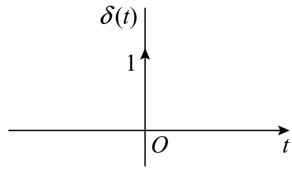


Fig. 12.8

Dirac delta function has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude of the function is infinitely large such that the area under its curve is equal to one unit. Hence, it is defined as,

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \qquad t = 0$$

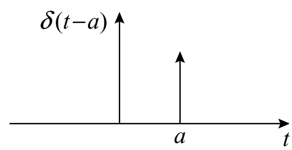


Fig. 12.9

The displaced (delayed) delta or unit impulse function $\delta(t - a)$ represents the function $\delta(t)$ which is displaced by a distance 'a' to the right.

$$\delta(t - a) = 0 \qquad t \neq a$$

and $$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1 \qquad t = a$$

Some properties of Dirac delta function:

(i) $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,\mathrm{d}t = f(0)$

(ii) $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,\mathrm{d}t = f(a)$

12.7.1 Laplace Transform of Dirac Delta Functions

(i) *Laplace transform of $\delta(t)$*

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \qquad t = 0$$

$$L\{\delta(t)\} = \int_0^{\infty} e^{-st}\,\delta(t)\,\mathrm{d}t$$
$$= [e^{-st}]_{t=0}$$
$$= 1$$

(ii) *Laplace transform of δ(t – a)*

$$\delta(t-a)=0 \qquad t\neq a$$

and $$\int_{-\infty}^{\infty}\delta(t-a)\,\mathrm{d}t=1 \qquad t=a$$

$$L\{\delta(t-a)\}=\int_0^{\infty}e^{-st}\delta(t-a)\,\mathrm{d}t$$
$$=[e^{-st}]_{t=a} \qquad \text{[From property (ii)]}$$
$$=e^{-as}$$

(iii) *Laplace transform of f(t) δ(t – a)*

$$f(t)\,\delta(t-a)=0 \qquad t\neq a$$

and $$\int_{-\infty}^{\infty}f(t)\delta(t-a)\,\mathrm{d}t=f(a) \qquad t=a$$

$$L\{f(t)\,\delta(t-a)\}=\int_0^{\infty}e^{-st}f(t)\delta(t-a)\,\mathrm{d}t$$
$$=[e^{-st}f(t)]_{t=a} \qquad \text{[From property (ii)]}$$
$$=e^{-as}f(a)$$

Example 1: Find the Laplace transforms of the following functions:

(i) $\sin 2t\;\delta\left(t-\dfrac{\pi}{4}\right)-t^2\,\delta(t-2)$ **(ii)** $t\,u(t-4)+t^2\,\delta(t-4)$

(iii) $t^2\,u(t-2)-\cosh t\;\delta(t-2)$**.**

Solution:

(i) $$L\{f(t)\,\delta(t-a)\}=e^{-as}f(a)$$

$$L\left\{\sin 2t\,\delta(t-\frac{\pi}{4})-t^2\delta(t-2)\right\}=e^{\frac{-\pi s}{4}}\sin 2\left(\frac{\pi}{4}\right)-e^{-2s}(2)^2=e^{\frac{-\pi s}{4}}\sin\frac{\pi}{2}-4e^{-2s}$$
$$=e^{\frac{-\pi s}{4}}-4e^{-2s}$$

(ii) $$L\{f(t)\,\delta(t-a)\}=e^{-as}f(a)$$

$$L\{t\,u(t-4)+t^2\,\delta(t-2)\}=e^{-4s}L\{f(t+4)\}+L\{t^2\,\delta(t-4)\}$$
$$=e^{-4s}L\{t+4\}+e^{-4s}(4)^2$$
$$=e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}\right)+16\,e^{-4s}=e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}+16\right)$$

(iii) $$L\{f(t)\,\delta(t-a)\}=e^{-as}f(a)$$

and $$L\{f(t)\,u(t-a)\}=e^{-as}L\{f(t+a)\}$$

$$L\{t^2\,u(t-2)-\cos\mathrm{h}t\;\delta(t-2)\}=L\{t^2\,u(t-2)\}-L\{\cos\mathrm{h}t\;\delta(t-2)\}$$
$$=e^{-2s}L\{(t+2)^2\}-e^{-2s}\cosh 2$$
$$=e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)-e^{-2s}\cosh 2$$
$$=e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}-\cosh 2\right)$$

Example 2: Evaluate the following integrals

(i) $\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)dt.$ **(ii)** $\int_0^\infty t^2 e^{-t}\sin t\,\delta(t-2)dt.$

(iii) $\int_0^\infty t^m (\log t)^n\,\delta(t-3)\,dt.$

Solution:

(i) $$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

$$\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)dt = \cos\frac{2\pi}{4} = 0$$

(ii) $$\int_0^\infty f(t)\,\delta(t-a)dt = f(a)$$

$$\int_0^\infty t^2 e^{-t}\sin t\,\delta(t-2)\,dt = (2)^2 e^{-2}\sin 2 = 4e^{-2}\sin 2$$

(iii) $$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

$$\int_0^\infty t^m (\log t)^n\,\delta(t-3)\,dt = 3^m (\log 3)^n$$

Exercise 12.13

(I) Find the Laplace transforms of the following functions:

1. $t\,u(t-4) - t^2\,\delta(t-2)$

$$\left[\textbf{Ans.: } e^{-4s}\frac{1}{s^2}(1+4s) - 4e^{-2s}\right]$$

2. $\sin 2t\,\delta(t-2)$

[**Ans.:** $e^{-2s}\sin 4$]

3. $t^2\,u(t-2) - \cosh t\,\delta(t-4)$

$$\left[\textbf{Ans.: } \frac{2e^{-2s}}{s^3}(2s^2+2s+1) - e^{-4s}\cosh 4\right]$$

4. $t\,e^{-2t}\,\delta(t-2)$

[**Ans.:** $2e^{-(4+2s)}$]

5. $\dfrac{e^{-t}\sin t}{t}\delta(t-3)$

$$\left[\textbf{Ans.: } \frac{1}{3}e^{-(s+3)}\sin 3\right]$$

6. $(e^{-4t} + \log t)\,\delta(t-2)$

[**Ans.:** $(e^{-8} + \log 2)\,e^{-2s}$]

(II) Evaluate the following integrals:

1. $\int_0^\infty \sin 4t\,\delta\left(t-\frac{\pi}{8}\right)dt$

$$\left[\textbf{Ans.: } e^{\frac{-\pi s}{8}}\right]$$

2. $\int_0^\infty e^{-t}\sin t\,\delta(t-a)\,dt$

[**Ans.:** $e^{-a}(\sin a - \cos a)$]

12.8 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

A function $f(t)$ is said to be periodic if there exists a constant $T(T > 0)$ such that $f(t+T) = f(t)$, for all values of t.

$$f(t+2T) = f(t+T+T) = f(t+T) = f(t)$$

In general, $f(t+nT) = f(t)$ for all t, where n is an integer (positive or negative) and T is the period of the function.

If $f(t)$ is a piecewise continuous periodic function with period T, then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\mathrm{d}t$$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^T e^{-st} f(t)\mathrm{d}t + \int_T^\infty e^{-st} f(t)\mathrm{d}t$$

In the second integral, putting $t = x + T$, $\mathrm{d}t = \mathrm{d}x$

When $\quad t = T, \quad x = 0$

$\qquad t \to \infty, \quad x \to \infty$

$$\begin{aligned} L\{f(t)\} &= \int_0^T e^{-st} f(t)\mathrm{d}t + \int_0^\infty e^{-s(x+T)} f(x+T)\mathrm{d}x \\ &= \int_0^T e^{-st} f(t)\mathrm{d}t + e^{-Ts}\int_0^\infty e^{-sx} f(x)\mathrm{d}x \\ &= \int_0^T e^{-st} f(t)\mathrm{d}t + e^{-Ts}\int_0^\infty e^{-st} f(t)\mathrm{d}t \\ &= \int_0^T e^{-st} f(t)\mathrm{d}t + e^{-Ts} L\{f(t)\} \\ (1-e^{-Ts})\, L\{f(t)\} &= \int_0^T e^{-st} f(t)\mathrm{d}t \\ L\{f(t)\} &= \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\mathrm{d}t \end{aligned}$$

Example 1: Find the Laplace transform of $f(t) = k\dfrac{t}{T} \quad 0 < t < T$ if $f(t) = f(t+T)$.

Solution: The function $f(t)$ is known as a sawtooth function.

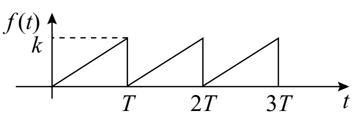


Fig. 12.10

The function $f(t)$ is a periodic function with period T.

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\mathrm{d}t = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}\frac{kt}{T}\mathrm{d}t$$

$$= \frac{1}{1-e^{-Ts}}\frac{k}{T}\int_0^T e^{-st}\, t\, \mathrm{d}t$$

$$= \frac{k}{T(1-e^{-Ts})}\left| t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right|_0^T = \frac{k}{T(1-e^{-Ts})}\left(-T\frac{e^{-Ts}}{s} - \frac{e^{-Ts}}{s^2} + \frac{1}{s^2} \right)$$

$$= \frac{k}{T(1-e^{-Ts})}\left[-\frac{Te^{-Ts}}{s} + \frac{1}{s^2}(1-e^{-Ts}) \right] = \frac{k}{Ts^2} - \frac{ke^{-Ts}}{s(1-e^{-Ts})}$$

Example 2: Find the Laplace transform of

$$\begin{aligned} f(t) &= t \qquad 0 < t < 1 \\ &= 0 \qquad 1 < t < 2 \end{aligned}$$

if $\quad f(t) = f(t + T)$.

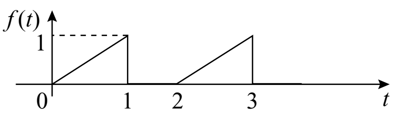


Fig. 12.11

Solution: The function $f(t)$ is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,\mathrm{d}t = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\, t\,\mathrm{d}t + \int_1^2 e^{-st}\cdot 0\,\mathrm{d}t\right]$$

$$= \frac{1}{1-e^{-2s}}\left| \frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2} \right|_0^1 = \frac{1}{1-e^{-2s}}\left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right)$$

$$= \frac{1}{s^2(1-e^{-2s})}(1 - e^{-s} - se^{-s})$$

Example 3: Find the Laplace transform of

$$\begin{aligned} f(t) &= \frac{t}{a} \qquad\qquad 0 < t < a \\ &= \frac{1}{a}(2a - t) \quad a < t < 2a \end{aligned}$$

if $\quad f(t) = f(t + 2a)$.

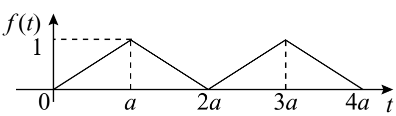


Fig. 12.12

Solution: The function $f(t)$ is a periodic function with period $2a$.

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\frac{t}{a}\,\mathrm{d}t + \int_a^{2a} e^{-st}\frac{1}{a}(2a-t)\,\mathrm{d}t\right]$$

$$= \frac{1}{a(1-e^{-2as})}\left[\left| \frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2} \right|_0^a + \left| \frac{e^{-st}}{-s}(2a-t) + \frac{e^{-st}}{s^2} \right|_a^{2a} \right]$$

$$= \frac{1}{a(1-e^{-2as})}\left(-\frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2} \right)$$

$$= \frac{-2e^{-as} + 1 + e^{-2as}}{as^2(1-e^{-2as})} = \frac{(1-e^{-as})^2}{as^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1-e^{-as}}{as^2(1+e^{-as})} = \frac{e^{\frac{as}{2}} - e^{\frac{-as}{2}}}{as^2\left(e^{\frac{as}{2}} + e^{\frac{-as}{2}}\right)} = \frac{\tanh\left(\frac{as}{2}\right)}{as^2}$$

Example 4: Find the Laplace transform of

$$f(t) = 1 \qquad 0 \le t < a$$
$$= -1 \qquad a < t < 2a$$

and $f(t)$ is periodic with period with period $2a$.

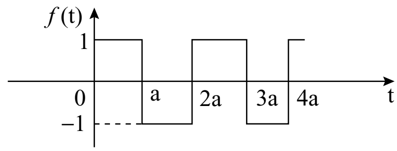


Fig. 12.13

Solution: The function $f(t)$ is known as a square function.

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\,dt + \int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left|\frac{e^{-st}}{-s}\right|_0^a + \left|\frac{e^{-st}}{s}\right|_a^{2a}\right] = \frac{1}{1-e^{-2as}}\left(-\frac{e^{-as}}{s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right)$$

$$= \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})} = \frac{1}{s}\cdot\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}}\right)}$$

$$= \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

Example 5: Find the Laplace transform of

$$f(t) = a\sin\omega t \qquad 0 < t < \frac{\pi}{\omega}$$
$$= 0 \qquad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$$

if $$f(t) = f\left(t + \frac{2\pi}{\omega}\right).$$

Solution: The function $f(t)$ is known as a half-sine wave rectifier function with period $\frac{2\pi}{\omega}$.

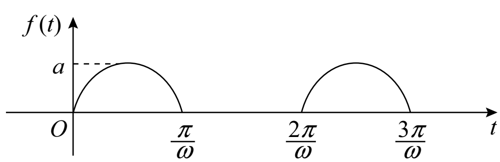


Fig. 12.14

The function $f(t)$ is a periodic function.

$$L\{f(t)\} = \frac{1}{1-e^{-\left(\frac{2\pi}{\omega}\right)s}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left(\int_0^{\frac{\pi}{\omega}} e^{-st}\, a\sin\omega t\,dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}\cdot 0\,dt\right)$$

$$= \frac{a}{1-e^{-\frac{2\pi s}{\omega}}}\left|\frac{1}{s^2+\omega^2}\cdot e^{-st}(-s\sin\omega t - \omega\cos\omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{a}{1-e^{-\frac{2\pi s}{\omega}}} \cdot \frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega)+\omega\right]$$

$$= \frac{a\omega\left(1+e^{-\frac{\pi s}{\omega}}\right)}{\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)} \cdot \frac{1}{s^2+\omega^2} = \frac{a\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)} \cdot \frac{1}{s^2+\omega^2}$$

Example 6: Find the Laplace transform of

$$\boldsymbol{f(t) = |\sin \omega t| \qquad t \geq 0.}$$

Solution: $f\left(t+\dfrac{\pi}{\omega}\right) = \left|\sin \omega\left(t+\dfrac{\pi}{\omega}\right)\right|$

$$= |\sin(\omega t+\pi)|$$

$$= |-\sin \omega t| = |\sin \omega t|$$

Hence, the function $f(t)$ is periodic with period $\dfrac{\pi}{\omega}$.

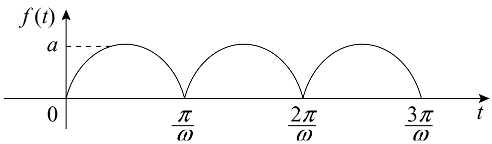


Fig. 12.15

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} |\sin \omega t|\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\,\mathrm{d}t \qquad \left[\because |\sin \omega t| = \sin \omega t \quad 0 < t < \frac{\pi}{\omega}\right]$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left|\frac{e^{-st}}{s^2+\omega^2}(-s\sin \omega t - \omega\cos \omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega)-(-\omega)\right]$$

$$= \frac{1}{s^2+\omega^2} \cdot \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\,\omega\left(1+e^{-\frac{\pi s}{\omega}}\right) = \frac{\omega}{s^2+\omega^2}\left(\frac{e^{\frac{\pi s}{2\omega}}+e^{-\frac{\pi s}{2\omega}}}{e^{\frac{\pi s}{2\omega}}-e^{-\frac{\pi s}{2\omega}}}\right)$$

$$= \frac{\omega}{s^2+\omega^2} \cdot \coth\left(\frac{\pi s}{2\omega}\right)$$

Example 7: Find the Laplace transform of

$$f(t) = t^2 \qquad 0 < t < 2$$

if $$f(t) = f(t+2).$$

Solution: The function $f(t)$ is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,dt = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}\cdot t^2\,dt$$

$$= \frac{1}{1-e^{-2s}}\left| t^2\cdot\left(\frac{e^{-st}}{-s}\right) - 2t\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right|_0^2$$

$$= \frac{1}{1-e^{-2s}}\left(-4\frac{e^{-2s}}{s} - 4\frac{e^{-2s}}{s^2} - 2\frac{e^{-2s}}{s^3} + \frac{2}{s^3}\right)$$

$$= \frac{1}{\left(1-e^{-2s}\right)s^3}\left(2 - 2e^{-2s} - 4s\,e^{-2s} - 4s^2 e^{-2s}\right)$$

Example 8: Find the Laplace transform of

$$f(t) = e^t \qquad 0 < t < 2\pi$$

if $$f(t) = f(t+2\pi).$$

Solution: The function $f(t)$ is a periodic function with period 2π.

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}e^t\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{(1-s)t}\,dt = \frac{1}{1-e^{-2\pi s}}\left|\frac{e^{(1-s)t}}{1-s}\right|_0^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)2\pi}}{1-s} - \frac{1}{1-s}\right] = \frac{e^{(1-s)2\pi} - 1}{(1-e^{-2\pi s})\,(1-s)}$$

Example 9: Find the Laplace transform of the function shown in Fig. 12.16.

Solution: The function $f(t)$ can be represented in terms of Heaviside unit step function.

$$f(t) = [u(t-T) - u(t-2T)] + 2[u(t-2T) - u(t-3T)] + 3[u(t-3T) - u(t-4t)] + \ldots\infty$$

$$= u(t-T) + u(t-2T) + u(t-3T) + \ldots\infty$$

$$L\{f(t)\} = L\{u(t-T) + u(t-2T) + u(t-3T) + \ldots.\}$$

$$= \frac{1}{s}e^{-Ts} + \frac{1}{s}e^{-2Ts} + \frac{1}{s}e^{-3Ts} + \ldots$$

$$= \frac{1}{s}\left[e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \ldots\right] = \frac{e^{-Ts}}{s(1-e^{-Ts})}$$

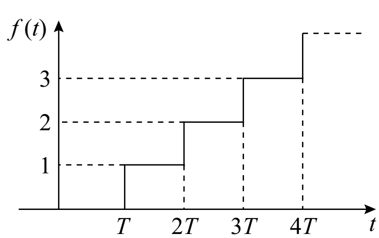


Fig. 12.16

Exercise 12.14

Find the Laplace transforms of the following periodic functions:

1. $f(t) = 1 \quad 0 < t < 1$
$= 0 \quad 1 < t < 2$
$= -1 \quad 2 < t < 3$
$f(t) = f(t+3)$

$$\left[\textbf{Ans.: } \frac{1}{s}\left(\frac{3}{1-e^{-3s}} - \frac{1}{1-e^{-s}} - 1\right)\right]$$

2. $f(t) = t \quad 0 < t < a$
$= 2a - t \quad a < t < 2a$
$f(t) = f(t+2a)$

$$\left[\textbf{Ans.: } \frac{1}{s^2}\tanh\frac{as}{2}\right]$$

3. $f(t) = t \quad 0 < t < \pi$
$= \pi - t \quad \pi < t < 2\pi$
$f(t) = f(t+2\pi)$

$$\left[\textbf{Ans.: } \frac{1-(1+\pi s)e^{-\pi s}}{(1+e^{-\pi s})s^2}\right]$$

4. $f(t) = |\cos \omega t| \quad t > 0$

$$\left[\textbf{Ans.: } \frac{1}{s^2+\omega^2}\left(s + \omega \operatorname{cosech}\frac{\pi s}{2\omega}\right)\right]$$

5. $f(t) = \cos \omega t \quad 0 < t < \dfrac{\pi}{\omega}$
$= 0 \quad \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega}$

$$\left[\textbf{Ans.: } \frac{s}{\left(1-e^{-\frac{\pi s}{\omega}}\right)(s^2+\omega^2)}\right]$$

6. $f(t) = E \quad 0 < t < \dfrac{P}{2}$
$= -E \quad \dfrac{P}{2} < t < P$
$f(t) = f(t+P)$

$$\left[\textbf{Ans.: } \frac{E}{s}\tanh\left(\frac{Ps}{4}\right)\right]$$

7. $f(t) = \left(\dfrac{\pi - t}{2}\right)^2 \quad 0 < t < 2\pi$
$f(t) = f(t+2\pi)$

$$\left[\textbf{Ans.: } \frac{1}{s^3}(2\pi s \coth \pi s - \pi^2 s^2 - 2)\right]$$

12.9 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$, then $f(t)$ is called inverse Laplace transform of $F(s)$ and symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform can be found by the following methods:

(i) Standard results
(ii) Second shifting theorem
(iii) Differentiation of $F(s)$
(iv) Partial fraction expansion
(v) Convolution theorem

12.9.1 Standard Results

Inverse Laplace transforms of some simple functions can be found by standard results and properties of Laplace transform.

Example 1: Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s^2-3s+4}{s^3}$ **(ii)** $\dfrac{3s+4}{s^2+9}$ **(iii)** $\dfrac{4s+15}{16s^2-25}$ **(iv)** $\dfrac{2s+2}{s^2+2s+10}$

(v) $\dfrac{2s+3}{s^2+2s+2}$ **(vi)** $\dfrac{3s+7}{s^2-2s-3}$ **(vii)** $\dfrac{s}{(2s+1)^2}$ **(viii)** $\dfrac{1}{\sqrt{2s+3}}$ **(ix)** $\dfrac{3s+1}{(s+1)^4}$.

Solution: (i) $F(s)=\dfrac{s^2-3s+4}{s^3}=\dfrac{1}{s}-\dfrac{3}{s^2}+\dfrac{4}{s^3}$

$$L^{-1}\{F(s)\} = 1-3t+2t^2$$

(ii) $F(s)=\dfrac{3s+4}{s^2+9}=\dfrac{3s}{s^2+9}+\dfrac{4}{s^2+9}$

$$L^{-1}\{F(s)\} = 3\cos 3t+\frac{4}{3}\sin 3t$$

(iii) $F(s)=\dfrac{4s+15}{16s^2-25}=\dfrac{4s+15}{16\left(s^2-\dfrac{25}{16}\right)}=\dfrac{1}{4}\dfrac{s}{s^2-\dfrac{25}{16}}+\dfrac{15}{16}\dfrac{1}{s^2-\dfrac{25}{16}}$

$$L^{-1}\{F(s)\} = \frac{1}{4}\cosh\frac{5}{4}t+\frac{3}{4}\sinh\frac{5}{4}t$$

(iv) $F(s)=\dfrac{2s+2}{s^2+2s+10}=\dfrac{2(s+1)}{(s+1)^2+9}$

$$L^{-1}\{F(s)\} = 2e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\}=2e^{-t}\cos 3t$$

(v) $F(s)=\dfrac{2s+3}{s^2+2s+2}=\dfrac{2s+2+1}{(s+1)^2+1}=\dfrac{2(s+1)+1}{(s+1)^2+1}=2\dfrac{(s+1)}{(s+1)^2+1}+\dfrac{1}{(s+1)^2+1}$

$$L^{-1}\{F(s)\} = 2e^{-t}L^{-1}\left\{\frac{s}{s^2+1}\right\}+e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\}=2e^{-t}\cos t+e^{-t}\sin t$$

(vi) $F(s)=\dfrac{3s+7}{s^2-2s-3}=\dfrac{3(s-1)+10}{(s-1)^2-4}=3\dfrac{(s-1)}{(s-1)^2-4}+10\dfrac{1}{(s-1)^2-4}$

$$L^{-1}\{F(s)\} = 3e^{t}L^{-1}\left\{\frac{s}{s^2-4}\right\}+10e^{t}L^{-1}\left\{\frac{1}{s^2-4}\right\}=3e^{t}\cosh 2t+5e^{t}\sinh 2t$$

(vii) $F(s)=\dfrac{s}{(2s+1)^2}=\dfrac{1}{4}\dfrac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2}=\dfrac{1}{4}\left[\dfrac{1}{s+\frac{1}{2}}-\dfrac{1}{2}\cdot\dfrac{1}{\left(s+\frac{1}{2}\right)^2}\right]$

$$L^{-1}\{F(s)\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}-\frac{1}{8}e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2}\right\}=\frac{1}{4}e^{-\frac{t}{2}}-\frac{1}{8}e^{-\frac{t}{2}}t$$

(viii) $F(s)=\dfrac{1}{\sqrt{2s+3}}=\dfrac{1}{\sqrt{2}}\dfrac{1}{\left(s+\dfrac{3}{2}\right)^{\frac{1}{2}}}$

$$L^{-1}\{F(s)\}=\frac{1}{\sqrt{2}}e^{-\frac{3t}{2}}L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}=\frac{1}{\sqrt{2}}e^{-\frac{3t}{2}}\frac{t^{-\frac{1}{2}}}{\boxed{\frac{1}{2}}}=\frac{1}{\sqrt{2\pi}}t^{-\frac{1}{2}}e^{-\frac{3t}{2}}$$

(ix) $F(s)=\dfrac{3s+1}{(s+1)^4}=\dfrac{3(s-1)-2}{(s+1)^4}=\dfrac{3}{(s+1)^3}-\dfrac{2}{(s+1)^4}$

$$L^{-1}\{F(s)\}=3e^{-t}L\left\{\frac{1}{s^3}\right\}-2e^{-t}\left\{\frac{1}{s^4}\right\}=3e^{-t}\frac{t^2}{2!}-2e^{-t}\frac{t^3}{3!}=\frac{3}{2}e^{-t}t^2-\frac{1}{3}e^{-t}t^3$$

Exercise 12.15

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{3s-12}{s^2+8}$

$\left[\textbf{Ans.: } 3\cos 2\sqrt{2}\,t-3\sqrt{2}\sin 2\sqrt{2}\,t\right]$

2. $\dfrac{s+1}{s^{\frac{4}{3}}}$

$\left[\textbf{Ans.: } \dfrac{t^{-\frac{2}{3}}+3t^{\frac{1}{3}}}{\boxed{\frac{1}{3}}}\right]$

3. $\left(\dfrac{\sqrt{s}-1}{s}\right)^2$

$\left[\textbf{Ans.: } 1+t-\dfrac{4t^{\frac{1}{2}}}{\sqrt{\pi}}\right]$

4. $\dfrac{5}{(s+2)^5}$

$\left[\textbf{Ans.: } \dfrac{5}{24}t^4e^{-2t}\right]$

5. $\dfrac{4s+12}{s^2+8s+16}$

$\left[\textbf{Ans.: } 4e^{-4t}(1-t)\right]$

6. $\dfrac{1}{(s^2+2s+5)^2}$

$\left[\textbf{Ans.: } \dfrac{e^{-t}}{16}(\sin 2t-2t\cos 2t)\right]$

7. $\dfrac{(s^2-1)^2}{s^5}$

$\left[\textbf{Ans.: } 1-t^2+\dfrac{t^4}{24}\right]$

8. $\dfrac{s}{(s-2)^6}$

$\left[\textbf{Ans.: } e^{2t}\left(\dfrac{t^4}{24}+\dfrac{t^5}{60}\right)\right]$

9. $\dfrac{s}{s^2+2s+2}$

$\left[\textbf{Ans.: } e^{-t}(\cos t-\sin t)\right]$

10. $\dfrac{1}{(s+2)^4}$

$\left[\textbf{Ans.: } \dfrac{1}{6}e^{-2t}t^3\right]$

12.9.2 Partial Fraction Expansion

Any function $F(s)$ can be written as $\dfrac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials in s. For performing partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that of $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I: Factors are linear and distinct,

$$F(s) = \frac{P(s)}{(s+a)(s+b)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case II: Factors are linear and repeated,

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \ldots + \frac{B_n}{(s+b)^n}$$

Case III: Factors are quadratic and distinct,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case IV: Factors are quadratic and repeated,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{C_1 s+D_1}{s^2+cs+d} + \frac{C_2 s+D_2}{(s^2+cs+d)^2} + \ldots + \frac{C_n s+D_n}{(s^2+cs+d)^n}$$

Example 1: Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{s+2}{s(s+1)(s+3)}$ **(ii)** $\dfrac{s+2}{s^2(s+3)}$ **(iii)** $\dfrac{s^2-15s-11}{(s+1)(s-2)^2}$

(iv) $\dfrac{s+2}{(s+3)(s+1)^3}$ **(v)** $\dfrac{s^3+6s^2+14s}{(s+2)^4}$ **(vi)** $\dfrac{3s+1}{(s+1)(s^2+2)}$

(vii) $\dfrac{s+4}{s(s-1)(s^2+4)}$ **(viii)** $\dfrac{s}{(s^2+1)(s^2+4)}$ **(ix)** $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

(x) $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$ **(xi)** $\dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)}$

(xii) $\dfrac{2s}{s^4+4}$ **(xiii)** $\dfrac{s}{s^4+s^2+1}$ **(xiv)** $\dfrac{1}{s^3+1}$

(xv) $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$.

Solution:

(i) $F(s) = \dfrac{s+2}{s(s+1)(s+3)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+3}$$

$$s + 2 = A(s + 1)(s + 3) + Bs(s + 3) + Cs(s + 1) \quad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3A$$

$$A = \frac{2}{3}$$

Putting $s = -1$ in Eq. (1),

$$1 = B(-1)(2)$$

$$B = -\frac{1}{2}$$

Putting $s = -3$ in Eq. (1),

$$-1 = C(-3)(-2)$$

$$C = -\frac{1}{6}$$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s}-\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{2}{3}-\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}$$

(ii) $F(s) = \dfrac{s+2}{s^2(s+3)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$s + 2 = As(s + 3) + B(s + 3) + Cs^2 \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3B$$

$$B = \frac{2}{3}$$

Putting $s = -3$ in Eq. (1),

$$-1 = 9C$$

$$C = -\frac{1}{9}$$

Equating the coefficients of s^2,

$$0 = A + C$$

$$A = \frac{1}{9}$$

$$F(s) = \frac{1}{9}\cdot\frac{1}{s} + \frac{2}{3}\cdot\frac{1}{s^2} - \frac{1}{9}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

(iii) $F(s) = \dfrac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$5s^2 - 15s - 11 = A(s - 2)^2 + B(s + 1)(s - 2) + C(s + 1) \qquad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$9 = 9A$$

$$A = 1$$

Putting $s = 2$ in Eq. (1),

$$-21 = 3C$$

$$C = -7$$

Equating the coefficients of s^2,

$$5 = A + B$$

$$B = 4$$

$$F(s) = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} - 7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= e^{-t} + 4e^{2t} - 7\,te^{2t}$$

(iv) $F(s) = \dfrac{s+2}{(s+3)(s+1)^3}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$$s + 2 = A(s+1)^3 + B(s+3)(s+1)^2 + C(s+3)(s+1) + D(s+3) \quad \ldots (1)$$

Putting $s = -3$ in Eq. (1),

$$-1 = -8A$$

$$A = \frac{1}{8}$$

Putting $s = -1$ in Eq. (1),

$$1 = 2D$$

$$D = \frac{1}{2}$$

Equating the coefficients of s^3,

$$0 = A + B$$

$$B = -\frac{1}{8}$$

Equating the coefficients of s^2,

$$0 = 3A + 5B + C$$

$$C = -\frac{3}{8} + \frac{5}{8} = \frac{1}{4}$$

$$F(s) = \frac{1}{8}\cdot\frac{1}{s+3} - \frac{1}{8}\cdot\frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

$$L^{-1}\{F(s)\} = \frac{1}{8}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{8}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= \frac{1}{8}e^{-3t} - \frac{1}{8}e^{-t} + \frac{1}{4}t\,e^{-t} + \frac{1}{2}\cdot\frac{t^2}{2}\cdot e^{-t}$$

$$= \frac{1}{8}[e^{-3t} + (2t^2 + 2t - 1)e^{-t}]$$

(v) $F(s) = \dfrac{s^3 + 6s^2 + 14s}{(s+2)^4}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}$$

$$s^3 + 6s^2 + 14s = A(s+2)^3 + B(s+2)^2 + C(s+2) + D$$

$$= As^3 + (6A + B)s^2 + (12A + 4B + C)s + (8A + 4B + 2C + D) \quad \ldots (1)$$

Equating the coefficients of s^3,

$$A = 1$$

Equating the coefficients of s^2,

$$6 = 6A + B$$

$$B = 0$$

Equating the coefficients of s,

$$14 = 12A + 4B + C$$

$$C = 14 - 12 - 0 = 2$$

Equating the coefficients of s^0,

$$0 = 8A + 4B + 2C + D$$

$$D = -8 - 0 - 4 = -12$$

$$F(s) = \frac{1}{s+2} + \frac{2}{(s+2)^3} - \frac{12}{(s+2)^4}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} + 2L^{-1}\left\{\frac{1}{(s+2)^3}\right\} - 12L^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

$$= e^{-2t} + 2\cdot\frac{t^2}{2}\cdot e^{-2t} - 12\cdot\frac{t^3}{6}\cdot e^{-2t} = e^{-2t}(1 + t^2 - 2t^3)$$

(vi) $F(s) = \dfrac{3s+1}{(s+1)(s^2+2)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$$

$$3s + 1 = A(s^2 + 2) + (Bs + C)(s + 1) \quad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$-2 = 3A$$

$$A = -\frac{2}{3}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B = \frac{2}{3}$$

Equating the coefficients of $s°$,

$$1 = 2A + C$$

$$C = 1 + \frac{4}{3} = \frac{7}{3}$$

$$F(s) = -\frac{2}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{s}{s^2+2} + \frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\}$$

$$= -\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t$$

(vii) $F(s) = \dfrac{s+4}{s(s-1)(s^2+4)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$s + 4 = A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)\,s(s-1) \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$4 = -4A$$

$$A = -1$$

Putting $s = 1$ in Eq. (1),

$$5 = 5B$$

$$B = 1$$

Equating the coefficients of s^3,

$$0 = A + B + C$$

$$C = 1 - 1 = 0$$

Equating the coefficients of s,

$$1 = 4A + 4B - D$$

$$D = -4 + 4 - 1 = -1$$

$$F(s) = -\frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}$$

$$L^{-1}\{F(s)\} = -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s^2+4}\right\} = -1 + e^t - \frac{1}{2}\sin 2t$$

(viii) $$F(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right] = \frac{1}{3}[\cos t - \cos 2t]$$

(ix) $F(s) = \dfrac{s^2}{(s^2 + a^2)(s^2 + b^2)}$

Let $s^2 = x$

$$G(x) = \frac{x}{(x + a^2)(x + b^2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x + a^2} + \frac{B}{x + b^2}$$

$$x = A(x + b^2) + B(x + a^2) \quad \ldots (1)$$

Putting $x = -a^2$ in Eq. (1),

$$-a^2 = A(-a^2 + b^2)$$

$$A = \frac{a^2}{a^2 - b^2}$$

Putting $x = -b^2$ in Eq. (1),

$$-b^2 = B(-b^2 + a^2)$$

$$B = -\frac{b^2}{a^2 - b^2}$$

$$G(x) = \frac{a^2}{a^2 - b^2} \cdot \frac{1}{x + a^2} - \frac{b^2}{a^2 - b^2} \cdot \frac{1}{x + b^2}$$

$$F(s) = \frac{a^2}{a^2 - b^2} \cdot \frac{1}{s^2 + a^2} - \frac{b^2}{a^2 - b^2} \cdot \frac{1}{s^2 + b^2}$$

$$L^{-1}\{F(s)\} = \frac{a^2}{a^2 - b^2} L^{-1}\left\{\frac{1}{s^2 + a^2}\right\} - \frac{b^2}{a^2 - b^2} L^{-1}\left\{\frac{1}{s^2 + b^2}\right\}$$

$$= \frac{a^2}{a^2 - b^2} \frac{1}{a} \sin at - \frac{b^2}{a^2 - b^2} \frac{1}{b} \sin bt$$

$$= \frac{1}{a^2 - b^2} (a \sin at - b \sin bt)$$

(x) $F(s) = \dfrac{s^2 + 2s + 3}{(s^2 + 2s + 5)(s^2 + 2s + 2)}$

Let $s^2 + 2s = x$

$$G(x) = \frac{x + 3}{(x + 5)(x + 2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x + 5} + \frac{B}{x + 2}$$

$$x + 3 = A(x + 2) + B(x + 5) \quad \ldots (1)$$

Putting $x = -5$ in Eq. (1),

$$-2 = -3A$$

$$A = \frac{2}{3}$$

Putting $x = -2$ in Eq. (1),

$$1 = 3B$$

$$B = \frac{1}{3}$$

$$G(x) = \frac{2}{3} \cdot \frac{1}{x+5} + \frac{1}{3} \cdot \frac{1}{x+2}$$

$$F(s) = \frac{2}{3} \cdot \frac{1}{(s^2+2s+5)} + \frac{1}{3} \cdot \frac{1}{(s^2+2s+2)} = \frac{2}{3} \cdot \frac{1}{(s+1)^2+4} + \frac{1}{3} \cdot \frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\} = \frac{2}{3} L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{2}{3} e^{-t} \cdot \frac{1}{2} \sin 2t + \frac{1}{3} e^{-t} \sin t = \frac{1}{3} e^{-t} (\sin 2t + \sin t)$$

(xi) $$F(s) = \frac{s+2}{(s^2+4s+8)(s^2+4s+13)} = \frac{s+2}{5}\left[\frac{s^2+4s+13-s^2-4s-8}{(s^2+4s+8)(s^2+4s+13)}\right]$$

$$= \frac{1}{5}\left[\frac{s+2}{s^2+4s+8} - \frac{s+2}{s^2+4s+13}\right] = \frac{1}{5}\left[\frac{s+2}{(s+2)^2+4} - \frac{s+2}{(s+2)^2+9}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{5}\left[L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - L^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\}\right]$$

$$= \frac{1}{5}\left[e^{-2t} L^{-1}\left\{\frac{s}{s^2+4}\right\} - e^{-2t} L^{-1}\left\{\frac{s}{s^2+9}\right\}\right]$$

$$= \frac{1}{5}\left(e^{-2t} \cos 2t - e^{-2t} \cos 3t\right) = \frac{e^{-2t}}{5} (\cos 2t - \cos 3t)$$

(xii) $$F(s) = \frac{2s}{s^4+4} = \frac{2s}{(s^4+4s^2+4)-4s^2} = \frac{2s}{(s^2+2)^2-(2s)^2}$$

$$= \frac{2s}{(s^2+2+2s)(s^2+2-2s)} = \frac{1}{2}\left[\frac{s^2+2+2s-s^2-2+2s}{(s^2+2+2s)(s^2+2-2s)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+2-2s} - \frac{1}{s^2+2+2s}\right] = \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}\right]$$

$$= \frac{1}{2}\left[e^{t} L^{-1}\left\{\frac{1}{s^2+1}\right\} - e^{-t} L^{-1}\left\{\frac{1}{s^2+1}\right\}\right] = \frac{1}{2}\left[e^{t} \sin t - e^{-t} \sin t\right]$$

$$= \sin t \sinh t$$

(xiii) $$F(s)=\frac{s}{s^4+s^2+1}=\frac{s}{s^4+2s^2+1-s^2}=\frac{s}{(s^2+1)^2-s^2}$$

$$=\frac{s}{(s^2+1+s)\,(s^2+1-s)}=\frac{1}{2}\left[\frac{s^2+1+s-s^2-1+s}{(s^2+1+s)\,(s^2+1-s)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{s^2+1-s}-\frac{1}{s^2+1+s}\right]=\frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$L^{-1}\{F(s)\}=\frac{1}{2}\left[L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}-L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}\right]$$

$$=\frac{1}{2}\left[e^{\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}-e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}\right]$$

$$=\frac{1}{2}\left[e^{\frac{t}{2}}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t-e^{-\frac{t}{2}}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$

$$=\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2}\right)=\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\,\sinh\frac{t}{2}$$

(xiv) $F(s)=\dfrac{1}{s^3+1}=\dfrac{1}{(s+1)\,(s^2-s+1)}$

By partial fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2-s+1}$$

$$1=A(s^2-s+1)+(Bs+C)\,(s+1) \qquad \dots (1)$$

Putting $s=-1$ in Eq. (1),

$$1=3A$$

$$A=\frac{1}{3}$$

Equating coefficients of s^2,

$$0=A+B$$

$$B=-\frac{1}{3}$$

Equating coefficients of s,

$$0=-A+B+C$$

$$C=\frac{2}{3}$$

$$F(s)=\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\frac{s}{s^2-s+1}+\frac{2}{3}\frac{1}{s^2-s+1}=\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\left(\frac{s-2}{s^2-s+1}\right)$$

$$=\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\left[\frac{s-\frac{1}{2}-\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right]=\frac{1}{3}\cdot\frac{1}{s+1}-\frac{1}{3}\cdot\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{1}{3}\cdot\frac{\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$L^{-1}\left\{F(s)\right\}=\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{3}L^{-1}\left\{\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$=\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{3}e^{\frac{t}{2}}L^{-1}\left\{\frac{s}{s^2+\frac{3}{4}}\right\}+\frac{1}{2}e^{\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}$$

$$=\frac{1}{3}e^{-t}-\frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t+\frac{1}{2}e^{\frac{t}{2}}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$=\frac{1}{3}e^{-t}-\frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t$$

(xv) $F(s)=\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$

By partial fraction expansion,

$$F(s)=\frac{As+B}{(s^2-2s+2)}+\frac{Cs+D}{(s^2-2s+2)^2}$$

$$s^3-3s^2+6s-4=(As+B)(s^2-2s+2)+Cs+D$$

$$=As^3+s^2(B-2A)+s(2A-2B+C)+2B+D$$

Equating coefficients of s^3,

$$A=1$$

Equating coefficients of s^2,

$$-3=B-2A$$
$$B=-3+2=-1$$

Equating coefficients of s,

$$6=2A-2B+C$$
$$C=6-2-2=2$$

Equating coefficients of s^0,

$$-4=2B+D$$
$$D=-4+2=-2$$

$$F(s) = \frac{s-1}{(s^2-2s+2)} + \frac{2s-2}{(s^2-2s+2)^2} = \frac{s-1}{(s-1)^2+1} + \frac{2(s-1)}{\left[(s-1)^2+1\right]^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + 2\,L^{-1}\left\{\frac{s-1}{\left[(s-1)^2+1\right]^2}\right\}$$

$$= e^t\,L^{-1}\left\{\frac{s}{s^2+1}\right\} + 2e^t\,L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = e^t\cos t + 2e^t\,\frac{t}{2}\sin t$$

$$= e^t(\cos t + t\sin t)$$

Exercise 12.16

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}$

$\left[\textbf{Ans. :}\ -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}\right]$

2. $\dfrac{s+2}{s^2(s+3)}$

$\left[\textbf{Ans. :}\ \frac{1}{9}(1+6t-e^{-3t})\right]$

3. $\dfrac{1}{s(s+1)^2}$

$\left[\textbf{Ans. :}\ 1-e^{-t}-te^{-t}\right]$

4. $\dfrac{1}{s^2(s+3)^2}$

$\left[\textbf{Ans. :}\ \frac{1}{27}(-2+3t+2e^{-3t}+3t^2e^{-3t})\right]$

5. $\dfrac{s^2}{(s+4)^3}$

$\left[\textbf{Ans. :}\ e^{-4t}(1-8t+8t^2)\right]$

6. $\dfrac{1}{(s-2)^4(s+3)}$

$\left[\textbf{Ans. :}\ \frac{e^{2t}}{6}\left[\frac{t^3}{5} - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625}\right] + \frac{1}{625}e^{-3t}\right]$

7. $\dfrac{5s^2-7s+17}{(s-1)(s^2+4)}$

$\left[\textbf{Ans. :}\ 3e^t + 2\cos 2t - \frac{5}{2}\sin 2t\right]$

8. $\dfrac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}$

$\left[\textbf{Ans. :}\ \frac{1}{2}\sin t + \frac{1}{2}t\cos t - te^{-t}\right]$

9. $\dfrac{1}{s^3(s-1)}$

$\left[\textbf{Ans. :}\ 1-t+\frac{t^2}{2}-e^{-t}\right]$

10. $\dfrac{s}{(s+1)^2(s^2+1)}$

$\left[\textbf{Ans. :}\ \frac{1}{2}(\sin t - te^{-t})\right]$

11. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

$$\left[\textbf{Ans.: } e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t\right]$$

12. $\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}$

$$\left[\textbf{Ans.: } \frac{1}{2}\sin t \sinh t\right]$$

13. $\dfrac{10}{s(s^2-2s+5)}$

$$\left[\textbf{Ans.: } 2 - e^t(2\cos 2t - \sin 2t)\right]$$

14. $\dfrac{s^2+8s+27}{(s+1)(s^2+4s+13)}$

$$\left[\textbf{Ans.: } 2e^{-t} + e^{-2t}(\sin 3t - \cos 3t)\right]$$

15. $\dfrac{2s-1}{s^4+s^2+1}$

$$\left[\begin{aligned}\textbf{Ans.: } & \frac{1}{2}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t \\ & -\frac{1}{2}e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t - \frac{5}{2\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\end{aligned}\right]$$

16. $\dfrac{s}{s^4+4a^4}$

$$\left[\textbf{Ans.: } \frac{1}{2a^2}\sin at \sinh at\right]$$

17. $\dfrac{s^2}{s^4+4a^4}$

$$\left[\begin{aligned}\textbf{Ans.: } & \frac{1}{2a}\sinh at \cos at \\ & + \frac{1}{2a}\cosh at \sin at\end{aligned}\right]$$

12.9.3 Convolution Theorem

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$, then

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

where $\int_0^t f_1(u) f_2(t-u)\mathrm{d}u = f_1(t) * f_2(t)$

Proof: $F_1(s)\cdot F_2(s) = L\{f_1(t)\}\cdot L\{f_2(t)\} = \int_0^\infty e^{-su} f_1(u)\mathrm{d}u \cdot \int_0^\infty e^{-sv} f_2(v)\mathrm{d}v$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f_1(u) f_2(v)\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^\infty f_1(u)\left[\int_0^\infty e^{-s(u+v)} f_2(v)\mathrm{d}v\right]\mathrm{d}u$$

Putting $u+v=t$, $\mathrm{d}v = \mathrm{d}t$

When $v = 0$, $t = u$

$v \to \infty$, $t \to \infty$

$$F_1(s)\cdot F_2(s) = \int_0^\infty f_1(u)\left[\int_u^\infty e^{-st} f_2(t-u)\mathrm{d}t\right]\mathrm{d}u = \int_0^\infty \int_u^\infty e^{-st} f_1(u) f_2(t-u)\mathrm{d}t\,\mathrm{d}u$$

The region of integration is bounded by the lines $u = 0$ and $u = t$. To change the order of integration, draw a vertical strip which starts from the line $u = 0$ and terminates on the line $u = t$. Therefore, u varies from 0 to t and t varies from 0 to ∞.

$$F_1(s)\cdot F_2(s)=\int_0^\infty e^{-st}\int_0^t f_1(u)f_2(t-u)\mathrm{d}u\,\mathrm{d}t$$
$$=L\left\{\int_0^t f_1(u)f_2(t-u)\mathrm{d}u\right\}$$

Hence, $L^{-1}\{F_1(s)\cdot F_2(s)\}=\int_0^t f_1(u)f_2(t-u)\mathrm{d}u$

Note: Convolution operation is commutative i.e.

$$L\left\{\int_0^t f_1(u)f_2(t-u)\mathrm{d}u\right\}=L\left\{\int_0^t f_1(t-u)f_2(u)\mathrm{d}u\right\}$$

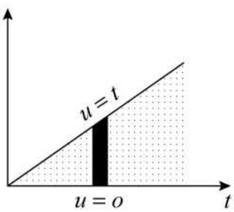


Fig. 12.17

Example1: Find the inverse Laplace transforms of the following functions:

(i) $\dfrac{1}{(s+2)\,(s-1)}$ **(ii)** $\dfrac{1}{s^2(s+1)^2}$ **(iii)** $\dfrac{1}{s\sqrt{s+4}}$

(iv) $\dfrac{1}{(s-2)\,(s+2)^2}$ **(v)** $\dfrac{1}{(s-2)^4(s+3)}$ **(vi)** $\dfrac{1}{s(s^2+a^2)}$

(vii) $\dfrac{1}{s^2(s^2+1)}$ **(viii)** $\dfrac{s^2}{(s^2+a^2)^2}$ **(ix)** $\dfrac{s}{(s^2+a^2)^2}$

(x) $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$ **(xi)** $\dfrac{1}{(s+1)(s^2+1)}$ **(xii)** $\dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$

(xiii) $\dfrac{1}{(s^2+4s+13)^2}$ **(xiv)** $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ **(xv)** $\dfrac{1}{(s+3)(s^2+2s+2)}$

(xvi) $\dfrac{1}{(s^2+4)(s+1)^2}$.

Solution:

(i) $F(s)=\dfrac{1}{(s+2)\,(s-1)}$

Let $F_1(s)=\dfrac{1}{s+2}$ $F_2(s)=\dfrac{1}{s-1}$

$f_1(t)=e^{-2t}$ $f_2(t)=e^t$

By convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-2u}\,e^{t-u}\mathrm{d}u=e^t\int_0^t e^{-3u}\mathrm{d}u=e^t\left|\frac{e^{-3u}}{-3}\right|_0^t=\frac{e^t}{3}(1-e^{-3t})$$

(ii) $F(s)=\dfrac{1}{s^2(s+1)^2}$

Let $F_1(s)=\dfrac{1}{(s+1)^2}$ $F_2(s)=\dfrac{1}{s^2}$

$f_1(t)=te^{-t}$ $f_2(t)=t$

By convolution theorem,

$$L^{-1}\left\{F(s)\right\} = \int_0^t ue^{-u}(t-u)\,\mathrm{d}u = \int_0^t (ut-u^2)e^{-u}\mathrm{d}u$$

$$= \left|(ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\right|_0^t$$

$$= te^{-t} + 2e^{-t} + t - 2$$

(iii) $F(s) = \dfrac{1}{s\sqrt{s+4}}$

Let $F_1(s) = \dfrac{1}{\sqrt{s+4}}$ $\qquad F_2(s) = \dfrac{1}{s}$

$$f_1(t) = e^{-4t}\frac{t^{-\frac{1}{2}}}{\left|\frac{1}{2}\right.} \qquad f_2(t) = 1$$

$$= e^{-4t}\sqrt{\frac{1}{\pi t}}$$

By convolution theorem,

$$L^{-1}\left\{F(s)\right\} = \int_0^t e^{-4u}\sqrt{\frac{1}{\pi u}}\,\mathrm{d}u = \frac{1}{\sqrt{\pi}}\int_0^t e^{-4u}u^{-\frac{1}{2}}\mathrm{d}u$$

Putting $4u = x^2$, $\mathrm{d}u = \dfrac{x}{2}\mathrm{d}x$

When $\quad u = 0, \quad x = 0$

$\qquad u = t, \quad x = 2\sqrt{t}$

$$L^{-1}\left\{F(s)\right\} = \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2}\cdot\frac{2}{x}\cdot\frac{x}{2}\,\mathrm{d}x = \frac{1}{2}\,\mathrm{erf}\,2\sqrt{t}$$

(iv) $F(s) = \dfrac{1}{(s-2)(s+2)^2}$

Let $F_1(s) = \dfrac{1}{(s+2)^2}$ $\qquad F_2(s) = \dfrac{1}{s-2}$

$$f_1(t) = te^{-2t} \qquad f_2(t) = e^{2t}$$

By convolution theorem,

$$L^{-1}\left\{F(s)\right\} = \int_0^t ue^{-2u}\,e^{2(t-u)}\mathrm{d}u = e^{2t}\int_0^t ue^{-4u}\mathrm{d}u = e^{2t}\left|\frac{ue^{-4u}}{-4} - \frac{e^{-4u}}{16}\right|_0^t$$

$$= e^{2t}\left(\frac{-te^{-4t}}{4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right) = \frac{e^{2t}}{16} - \frac{te^{-2t}}{4} - \frac{e^{-2t}}{16}$$

$$= \frac{1}{16}(e^{2t} - e^{-2t} - 4t\,e^{-2t})$$

(v) $F(s) = \dfrac{1}{(s-2)^4(s+3)}$

Let $F_1(s) = \dfrac{1}{(s-2)^4}$ $\qquad F_2(s) = \dfrac{1}{s+3}$

$f_1(t) = e^{2t}\dfrac{t^3}{6}$ $\qquad f_2(t) = e^{-3t}$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{2u}\frac{u^3}{6}\, e^{-3(t-u)}\mathrm{d}u = \frac{e^{-3t}}{6}\int_0^t u^3 e^{5u}\mathrm{d}u$$

$$= \frac{e^{-3t}}{6}\left| u^3\frac{e^{5u}}{5} - 3u^2\frac{e^{5u}}{25} + 6u\frac{e^{5u}}{125} - 6\frac{e^{5u}}{625}\right|_0^t$$

$$= \frac{e^{-3t}}{6}\left(t^3\frac{e^{5t}}{5} - 3t^2\frac{e^{5t}}{25} + 6t\frac{e^{5t}}{125} - 6\frac{e^{5t}}{625} + \frac{6}{625}\right)$$

$$= \frac{e^{-3t}}{625} + \frac{e^{2t}}{6}\left(\frac{t^3}{5} - \frac{3t^2}{25} + \frac{6t}{125} - \frac{6}{625}\right)$$

(vi) $F(s) = \dfrac{1}{s(s^2+a^2)}$

Let $F_1(s) = \dfrac{1}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{1}{s}$

$f_1(t) = \dfrac{1}{a}\sin at$ $\qquad f_2(t) = 1$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \frac{1}{a}\sin au\,\mathrm{d}u = \frac{1}{a}\left| -\frac{\cos au}{a}\right|_0^t = \frac{1}{a^2}(1-\cos at)$$

(vii) $F(s) = \dfrac{1}{s^2(s^2+1)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\qquad F_2(s) = \dfrac{1}{s^2}$

$f_1(t) = \sin t$ $\qquad f_2(t) = t$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u\,(t-u)\mathrm{d}u$$

$$= \left|(t-u)(-\cos u) - \sin u\right|_0^t = t - \sin t$$

(viii) $F(s) = \dfrac{s^2}{(s^2+a^2)^2}$

Let $F_1(s) = \dfrac{s}{s^2+a^2}$ $\quad F_2(s) = \dfrac{s}{s^2+a^2}$

$f_1(t) = \cos at$ $\quad f_2(t) = \cos at$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cos a(t-u)\mathrm{d}u = \frac{1}{2}\int_0^t [\cos at + \cos(2au - at)]\mathrm{d}u$$

$$= \frac{1}{2}\left| u\cos at + \frac{1}{2a}\sin(2au - at)\right|_0^t = \frac{1}{2}\left(t\cos at + \frac{1}{a}\sin at\right)$$

$$= \frac{1}{2a}(\sin at + at\cos at)$$

(ix) $F(s) = \dfrac{s}{(s^2+a^2)^2}$

Let $F_1(s) = \dfrac{s}{s^2+a^2}$ $\quad F_2(s) = \dfrac{1}{s^2+a^2}$

$f_1(t) = \cos at$ $\quad f_2(t) = \dfrac{1}{a}\sin at$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cdot \frac{1}{a}\sin a(t-u)\mathrm{d}u$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin a(t-2u)]\mathrm{d}u$$

$$= \frac{1}{2a}\left| u\sin at + \frac{1}{2a}\cos a(t-2u)\right|_0^t = \frac{1}{2a}t\sin at$$

(x) $F(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)}$

Let $F_1(s) = \dfrac{1}{s^2+a^2}$ $\quad F_2(s) = \dfrac{1}{s^2+b^2}$

$f_1(t) = \dfrac{1}{a}\sin at$ $\quad f_2(t) = \dfrac{1}{b}\sin bt$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \frac{1}{a}\sin au \cdot \frac{1}{b}\sin b(t-u)\mathrm{d}u = \frac{1}{ab}\int_0^t \sin au \sin b(t-u)\mathrm{d}u$$

$$= -\frac{1}{2ab}\int_0^t \left[\cos\{(a-b)u + bt\} - \cos\{(a+b)u - bt\}\right]\mathrm{d}u$$

$$= -\frac{1}{2ab}\left|\frac{\sin\{(a-b)u+bt\}}{a-b} - \frac{\sin\{(a+b)u-bt\}}{a+b}\right|_0^t$$

$$= -\frac{1}{2ab}\left(\frac{\sin at}{a-b} - \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)$$

$$= -\frac{1}{2ab}\left(2b\,\frac{\sin at}{a^2-b^2} - 2a\,\frac{\sin bt}{a^2-b^2}\right)$$

$$= \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}$$

(xi) $F(s) = \dfrac{1}{(s+1)(s^2+1)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\qquad F_2(s) = \dfrac{1}{s+1}$

$f_1(t) = \sin t$ $\qquad f_2(t) = e^{-t}$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u\, e^{-(t-u)}\mathrm{d}u = \int_0^t e^{u-t}\sin u\,\mathrm{d}u = e^{-t}\left|\frac{e^u}{2}(\sin u - \cos u)\right|_0^t$$

$$= \frac{e^{-t}}{2}\left[e^t(\sin t - \cos t) + 1\right] = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t}$$

(xii) $F(s) = \dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$

Let $F_1(s) = \dfrac{s+1}{s^2+2s+2}$ $\qquad F_2(s) = \dfrac{s}{s^2+1}$

$= \dfrac{s+1}{(s+1)^2+1}$ $\qquad f_2(t) = \cos t$

$f_1(t) = e^{-t}\cos t$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-u}\cos u\cos(t-u)\,\mathrm{d}u = \frac{1}{2}\int_0^t e^{-u}\left[\cos t + \cos(2u-t)\right]\mathrm{d}u$$

$$= \frac{1}{2}\left|-e^{-u}\cos t + \frac{e^{-u}}{5}\{-\cos(2u-t) + 2\sin(2u-t)\}\right|_0^t$$

$$= \frac{1}{2}\left[-e^{-t}\cos t + \frac{e^{-t}}{5}(-\cos t + 2\sin t) + \cos t - \frac{1}{5}(-\cos t - 2\sin t)\right]$$

$$= \frac{1}{10}\left[e^{-t}(2\sin t - 6\cos t) + (2\sin t + 6\cos t)\right]$$

(xiii) $F(s) = \dfrac{1}{(s^2+4s+13)^2}$

Let $F_1(s) = \dfrac{1}{s^2+4s+13}$ $\qquad F_2(s) = \dfrac{1}{s^2+4s+13}$

$= \dfrac{1}{(s+2)^2+9}$ $\qquad f_2(t) = \dfrac{e^{-2t}}{3}\sin 3t$

$f_1(t) = \dfrac{e^{-2t}}{3}\sin 3t$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \frac{e^{-2u}}{3}\sin 3u \cdot \frac{e^{-2(t-u)}}{3}\sin 3(t-u)\,\mathrm{d}u$$

$$= \frac{e^{-2t}}{9}\int_0^t \sin 3u \sin 3(t-u)\,\mathrm{d}u = -\frac{e^{-2t}}{18}\int_0^t [\cos 3t - \cos(6u-3t)]\,\mathrm{d}u$$

$$= -\frac{e^{-2t}}{18}\left| u\cos 3t - \frac{\sin(6u-3t)}{6}\right|_0^t = -\frac{e^{-2t}}{18}\left(t\cos 3t - \frac{\sin 3t}{6} - \frac{\sin 3t}{6}\right)$$

$$= \frac{e^{-2t}}{18}\left(\frac{\sin 3t}{3} - t\cos 3t\right)$$

(xiv) $F(s) = \dfrac{(s+2)^2}{(s^2+4s+8)^2}$

Let $F_1(s) = \dfrac{s+2}{s^2+4s+8}$ $\qquad F_2(s) = \dfrac{s+2}{s^2+4s+8}$

$= \dfrac{s+2}{(s+2)^2+4}$ $\qquad f_2(t) = e^{-2t}\cos 2t$

$f_1(t) = e^{-2t}\cos 2t$

By convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u}\cos 2u\, e^{-2(t-u)}\cos 2(t-u)\,\mathrm{d}u$$

$$= e^{-2t}\int_0^t \cos 2u\cos 2(t-u)\,\mathrm{d}u$$

$$= \frac{e^{-2t}}{2}\int_0^t [\cos 2t + \cos(4u-2t)]\,\mathrm{d}u$$

$$= \frac{e^{-2t}}{2}\left| u\cos 2t + \frac{\sin(4u-2t)}{4}\right|_0^t$$

$$= \frac{e^{-2t}}{2}\left(t\cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4}\right)$$

$$= \frac{e^{-2t}}{4}(\sin 2t + 2t\cos 2t)$$

(xv) $F(s)=\dfrac{1}{(s+3)\,(s^2+2s+2)}$

Let $F_1(s)=\dfrac{1}{s^2+2s+2}$ $\qquad F_2(s)=\dfrac{1}{s+3}$

$=\dfrac{1}{(s+1)^2+1}$ $\qquad f_2(t)=e^{-3t}$

$f_1(t)=e^{-t}\,\sin t$

By convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-u}\,\sin u\, e^{-3(t-u)}\mathrm{d}u=e^{-3t}\int_0^t e^{2u}\,\sin u\;\mathrm{d}u=e^{-3t}\left|\frac{e^{2u}}{5}(2\sin u-\cos u)\right|_0^t$$

$$=\frac{e^{-3t}}{5}\left[e^{2t}(2\sin t-\cos t)+1\right]=\frac{1}{5}\left[e^{-t}(2\sin t-\cos t)+e^{-3t}\right]$$

(xvi) $F(s)=\dfrac{1}{(s^2+4)\,(s+1)^2}$

Considering $F(s)$ as a product of three functions,

$$F(s)=\frac{1}{(s^2+4)}\cdot\frac{1}{s+1}\cdot\frac{1}{s+1}$$

Let $F_1(s)=\dfrac{1}{s^2+4}$ $\qquad F_2(s)=\dfrac{1}{s+1}$ $\qquad F_3(s)=\dfrac{1}{s+1}$

$f_1(t)=\dfrac{1}{2}\sin 2t$ $\qquad f_2(t)=e^{-t}$ $\qquad f_3(t)=e^{-t}$

By convolution theorem,

$$L^{-1}\{F_1(s)\cdot F_2(s)\}=\int_0^t\frac{1}{2}\sin 2u\; e^{-(t-u)}\mathrm{d}u=\frac{e^{-t}}{2}\left|\frac{e^{u}}{5}(\sin 2u-2\cos 2u)\right|_0^t$$

$$=\frac{e^{-t}}{10}\left[e^{t}(\sin 2t-2\cos 2t)+2\right]=\frac{\sin 2t-2\cos 2t}{10}+\frac{e^{-t}}{5}$$

$$L^{-1}\{F_1(s)F_2(s)F_3(s)\}=\int_0^t\left(\frac{\sin 2u-2\cos 2u}{10}+\frac{e^{-u}}{5}\right)e^{-(t-u)}\mathrm{d}u$$

$$=\frac{e^{-t}}{10}\int_0^t\left[e^{u}(\sin 2u-2\cos 2u)+2\right]\mathrm{d}u$$

$$=\frac{e^{-t}}{10}\left|\frac{e^{u}}{5}\left\{(\sin 2u-2\cos 2u)-2(\cos 2u+2\sin 2u)\right\}+2u\right|_0^t$$

$$=\frac{e^{-t}}{10}\left[\frac{e^{t}}{5}(-3\sin 2t-4\cos 2t)+2t+\frac{4}{5}\right]$$

$$=\frac{2}{25}e^{-t}+\frac{te^{-t}}{5}-\frac{1}{50}(3\sin 2t+4\cos 2t)$$

Exercise 12.17

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{1}{(s+3)(s-1)}$

$\left[\textbf{Ans. :} \dfrac{e^{t}}{4}(1-e^{-4t})\right]$

2. $\dfrac{1}{s(s^2+4)}$

$\left[\textbf{Ans. :} \dfrac{1}{4}(1-\cos 2t)\right]$

3. $\dfrac{1}{(s-3)(s+3)^2}$

$\left[\textbf{Ans. :} \dfrac{1}{36}(e^{3t}-e^{-3t}-6te^{-3t})\right]$

4. $\dfrac{s}{(s^2+4)^2}$

$\left[\textbf{Ans. :} \dfrac{1}{4}t\sin 2t\right]$

5. $\dfrac{s^2}{(s^2-a^2)^2}$

$\left[\textbf{Ans. :} \dfrac{1}{2}(\sinh at + at\cosh at)\right]$

6. $\dfrac{1}{s(s^2-a^2)}$

$\left[\textbf{Ans. :} \dfrac{1}{a^2}(\cosh at - 1)\right]$

7. $\dfrac{1}{s^3(s^2+1)}$

$\left[\textbf{Ans. :} \dfrac{t^2}{2}+\cos t - 1\right]$

8. $\dfrac{s^2}{(s^2+4)^2}$

$\left[\textbf{Ans. :} \dfrac{1}{4}(\sin 2t + 2t\cos 2t)\right]$

9. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

$\left[\textbf{Ans. :} \dfrac{1}{3}(2\sin 2t - \sin t)\right]$

10. $\dfrac{s}{(s^2-a^2)^2}$

$\left[\textbf{Ans. :} \dfrac{1}{2a}(at\cosh at + \sinh at)\right]$

11. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

$\left[\textbf{Ans. :} \dfrac{1}{b^2-a^2}(\sin at - \sin bt)\right]$

12. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

$\left[\textbf{Ans. :} \dfrac{1}{a^2-b^2}(a\sin at - b\sin bt)\right]$

13. $\dfrac{s}{(s^2+a^2)^3}$

$\left[\textbf{Ans. :} \dfrac{t}{8a^3}(\sin at - at\cos at)\right]$

14. $\dfrac{s+3}{(s^2+6s+13)^2}$

$\left[\textbf{Ans. :} \dfrac{1}{4}e^{-3t}\,t\sin 2t\right]$

15. $\dfrac{s}{s^4+8s^2+16}$

$\left[\textbf{Ans. :} \dfrac{1}{4}t\sin 2t\right]$

16. $\dfrac{(s+3)^2}{(s^2+6s+5)^2}$

$\left[\textbf{Ans. :} \dfrac{1}{4}(2t\cosh 2t + \sinh 2t)\right]$

17. $\dfrac{1}{s(s+1)(s+2)}$

$\left[\textbf{Ans. :} \dfrac{1}{2}+\dfrac{1}{2}e^{-2t}-e^{-t}\right]$

12.9.4 Differentiation of $F(s)$

We know that, If $L\{f(t)\} = F(s)$, then

$$L\{t\, f(t)\} = -F'(s)$$

i.e., $L^{-1}\{F'(s)\} = -t\, f(t)$

Hence, $L^{-1}\{F(s)\} = f(t) = -\dfrac{1}{t} L^{-1}\{F'(s)\}$

Example 1: Find the inverse Laplace transforms of the following functions:

(i) $\log\dfrac{s+a}{s+b}$ (ii) $\log\dfrac{s^2+b^2}{s^2+a^2}$ (iii) $\log\dfrac{s^2+a^2}{(s+b)^2}$

(iv) $\log\sqrt{\dfrac{s^2-a^2}{s^2}}$ (v) $\log\sqrt{\dfrac{s-1}{s+1}}$ (vi) $\log\sqrt{\dfrac{s^2+1}{s(s+1)}}$

(vii) $\tan^{-1}\dfrac{2}{s^2}$ (viii) $\tan^{-1}\dfrac{2}{s}$ (ix) $\tan^{-1}\left(\dfrac{s+a}{b}\right)$

(x) $\cot^{-1} s$ (xi) $\cot^{-1}(s+1)$ (xii) $2\tanh^{-1} s$.

Solution:

(i) $F(s) = \log\dfrac{s+a}{s+b} = \log(s+a) - \log(s+b)$

$$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = -\frac{1}{t}(e^{-at} - e^{-bt})$$

(ii) $F(s) = \log\dfrac{s^2+b^2}{s^2+a^2} = \log(s^2+b^2) - \log(s^2+a^2)$

$$F'(s) = \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right\} = -\frac{1}{t}(2\cos bt - 2\cos at)$$

$$= \frac{2}{t}(\cos at - \cos bt)$$

(iii) $F(s) = \log\dfrac{s^2+a^2}{(s+b)^2} = \log(s^2+a^2) - \log(s+b)^2$

$$= \log(s^2+a^2) - 2\log(s+b)$$

$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2}{s+b}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2}{s+b}\right\}$$

$$= -\frac{1}{t}(2\cos at - 2e^{-bt}) = \frac{2}{t}(e^{-bt} - \cos at)$$

(iv) $F(s) = \log\sqrt{\dfrac{s^2-a^2}{s^2}} = \log\sqrt{s^2-a^2} - \log\sqrt{s^2} = \dfrac{1}{2}\log(s^2-a^2) - \log s$

$$F'(s) = \frac{1}{2}\frac{2s}{s^2-a^2} - \frac{1}{s}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{s}{s^2-a^2} - \frac{1}{s}\right\} = -\frac{1}{t}(\cosh at - 1) = \frac{1}{t}(1 - \cosh at)$$

(v) $F(s) = \log\sqrt{\dfrac{s-1}{s+1}} = \log\sqrt{s-1} - \log\sqrt{s+1} = \dfrac{1}{2}\log(s-1) - \dfrac{1}{2}\log(s+1)$

$$F'(s) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right\} = -\frac{1}{t}\left(\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right) = -\frac{1}{t}\sinh t$$

(vi) $F(s) = \log\dfrac{s^2+1}{s(s+1)} = \log(s^2+1) - \log s - \log(s+1)$

$$F'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}\{F'(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\} = -\frac{1}{t}(2\cos t - 1 - e^{-t})$$

(vii) $F(s) = \tan^{-1}\dfrac{2}{s^2}$

$$F'(s) = \frac{1}{1+\frac{4}{s^4}}\left(-\frac{4}{s^3}\right) = -\frac{4s}{s^4+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{-\frac{4s}{s^4+4}\right\} = \frac{4}{t} L^{-1}\left\{\frac{s}{s^4+4}\right\}$$

$$= \frac{4}{t} L^{-1}\left\{\frac{s}{(s^2+2)^2 - (2s)^2}\right\} = \frac{4}{t}\cdot\frac{1}{4} L^{-1}\left\{\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right\}$$

$$= \frac{1}{t} L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\} = \frac{1}{t}(e^{t}\sin t - e^{-t}\sin t)$$

$$= \frac{\sin t}{t}(e^{t} - e^{-t}) = \frac{2}{t}\sin t \sinh t$$

(viii) $F(s) = \tan^{-1}\dfrac{2}{s}$

$$F'(s) = \frac{1}{1+\dfrac{4}{s^2}}\left(-\frac{2}{s^2}\right) = -\frac{2}{s^2+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{-\frac{2}{s^2+4}\right\} = \frac{2}{t}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{2}{t}\cdot\frac{1}{2}\sin 2t = \frac{1}{t}\sin 2t$$

(ix) $F(s) = \tan^{-1}\left(\dfrac{s+a}{b}\right)$

$$F'(s) = \frac{1}{1+\left(\dfrac{s+a}{b}\right)^2}\cdot\frac{1}{b} = \frac{b}{(s+a)^2+b^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\} = -\frac{1}{t}e^{-at}\sin bt$$

(x) $F(s) = \cot^{-1} s$

$$F'(s) = -\frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{-\frac{1}{s^2+1}\right\} = \frac{1}{t}\sin t$$

(xi) $F(s) = \cot^{-1}(s+1)$

$$F'(s) = -\frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F'(s)\} = -\frac{1}{t}L^{-1}\left\{-\frac{1}{(s+1)^2+1}\right\} = \frac{1}{t}e^{-t}\sin t$$

(xii) $F(s) = 2\tanh^{-1} s = 2\cdot\dfrac{1}{2}\log\dfrac{1+s}{1-s} = \log(1+s) - \log(1-s)$

$$F'(s) = \frac{1}{1+s} + \frac{1}{1-s}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{1+s}+\frac{1}{1-s}\right\} = -\frac{1}{t}(e^{-t}-e^{t}) = \frac{2}{t}\sinh t$$

Exercise 12.18

Find the inverse Laplace transforms of the following functions:

1. $\log\left(1+\dfrac{a^2}{s^2}\right)$

$$\left[\textbf{Ans. :}\ \frac{2}{t}(1-\cos at)\right]$$

2. $\log\left(1-\dfrac{1}{s^2}\right)$

$$\left[\textbf{Ans. :}\ \frac{2}{t}(1-\cosh t)\right]$$

3. $\log\dfrac{s^2-4}{(s-3)^2}$

$$\left[\textbf{Ans. :}\ \frac{2}{t}(e^{3t}-\cosh 2t)\right]$$

4. $\log\sqrt{\dfrac{s^2+1}{s^2}}$

$$\left[\textbf{Ans. :}\ \frac{1}{t}(1-\cos t)\right]$$

5. $\log\dfrac{(s-2)^2}{s^2+1}$

$$\left[\textbf{Ans. :}\ \frac{2}{t}(\cos t-e^{2t})\right]$$

6. $\log\left(\dfrac{s^2-4}{s^2}\right)^{\frac{1}{3}}$

$$\left[\textbf{Ans. :}\ \frac{2}{3t}(1-\cosh 2t)\right]$$

7. $\log\dfrac{1}{s}\left(1+\dfrac{1}{s^2}\right)$

$$\left[\textbf{Ans. :}\ \int_0^t \frac{2(1-\cos t)}{t}\,\mathrm{d}t\right]$$

8. $\dfrac{1}{s}\log\dfrac{s+1}{s+2}$

$$\left[\textbf{Ans. :}\ \int_0^t \frac{e^{-2t}-e^{-t}}{t}\,\mathrm{d}t\right]$$

9. $\tan^{-1}(s+1)$

$$\left[\textbf{Ans. :}\ -\frac{1}{t}e^{-t}\sin t\right]$$

10. $\tan^{-1}\dfrac{s}{2}$

$$\left[\textbf{Ans. :}\ -\frac{1}{t}\sin 2t\right]$$

11. $\cot^{-1} as$

$$\left[\textbf{Ans. :}\ \frac{1}{t}\sin\frac{t}{a}\right]$$

12. $\cot^{-1}\left(\dfrac{2}{s^2}\right)$

$$\left[\textbf{Ans. :}\ -\frac{2}{t}\sin t\sinh t\right]$$

12.9.5 Second Shifting Theorem (Heaviside's Unit Step Function)

We know that if $L\{f(t)\}=F(s)$, then by second shifting theorem of Laplace transform,

$$L\{f(t-a)u(t-a)\}=e^{-as}F(s)$$

Hence, $L^{-1}\{e^{-as}F(s)\}=f(t-a)\,u(t-a)$

Example 1: Find the inverse Laplace transforms of following functions:

(i) $\dfrac{e^{-2s}}{(s+4)^3}$ (ii) $\dfrac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$ (iii) $\dfrac{e^{-3s}}{s^2+4}$ (iv) $\dfrac{e^{-2s}}{s^2+8s+25}$

(v) $\dfrac{e^{-\pi s}}{s^2-2s+2}$ (vi) $\dfrac{(s+1)e^{-2s}}{s^2+2s+2}$ (vii) $\dfrac{se^{-2s}}{s^2+2s+2}$ (viii) $e^{-s}\left(\dfrac{1+\sqrt{s}}{s^3}\right)$.

Solution:

(i) Let $F(s)=\dfrac{1}{(s+4)^3}$

$$L^{-1}\{F(s)\}=e^{-4t}\,L^{-1}\left\{\frac{1}{s^3}\right\}=e^{-4t}\cdot\frac{t^2}{2}$$

$$L^{-1}\{e^{-2s}F(s)\}=e^{-4(t-2)}\,\frac{(t-2)^2}{2}\,u(t-2)$$

(ii) Let $F(s)=\dfrac{1}{(s+4)^{\frac{5}{2}}}$

$$L^{-1}\{F(s)\}=e^{-4t}\,L^{-1}\left\{\frac{1}{s^{\frac{5}{2}}}\right\}=e^{-4t}\,\frac{t^{\frac{3}{2}}}{\left\lceil\frac{5}{2}\right.}=\frac{e^{-4t}\,t^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.}=\frac{4e^{-4t}\,t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\{e^{4-3s}F(s)\}=\frac{e^4\cdot 4}{3\sqrt{\pi}}\,e^{-4(t-3)\cdot(t-3)^{\frac{3}{2}}\,u(t-3)}$$

$$=\frac{4}{3\sqrt{\pi}}\,e^{-4(t-4)}\,(t-3)^{\frac{3}{2}}u(t-3)$$

(iii) Let $F(s)=\dfrac{1}{s^2+4}$

$$L^{-1}\{F(s)\}=\frac{1}{2}\sin 2t$$

$$L^{-1}\{e^{-3s}F(s)\}=\frac{1}{2}\sin 2(t-3)\,u(t-3)$$

(iv) Let $F(s)=\dfrac{1}{s^2+8s+25}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{(s+4)^2+9}\right\}=e^{-4t}L^{-1}\left\{\frac{1}{s^2+9}\right\}=\frac{e^{-4t}}{3}\sin 3t$$

$$L^{-1}\{e^{-2s}F(s)\}=\frac{e^{-4(t-2)}}{3}\sin 3(t-2)\,u(t-2)$$

(v) Let $F(s)=\dfrac{1}{s^2-2s+2}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$=e^t\ L^{-1}\left\{\frac{1}{s^2+1}\right\}=e^t\ \sin t$$

$$L^{-1}\left\{e^{-\pi s}F(s)\right\}=e^{(t-\pi)}\sin(t-\pi)u(t-\pi)$$

(vi) Let $F(s)=\dfrac{s+1}{s^2+2s+2}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{(s+1)}{(s+1)^2+1}\right\}$$

$$=e^{-t}\ L^{-1}\left\{\frac{s}{s^2+1}\right\}=e^{-t}\ \cos t$$

$$L^{-1}\left\{e^{-2s}F(s)\right\}=e^{-(t-2)}\cos(t-2)u(t-2)$$

(vii) Let $F(s)=\dfrac{s}{s^2+2s+2}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{s+1-1}{(s+1)^2+1}\right\}=L^{-1}\left\{\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$=e^{-t}\ L^{-1}\left\{\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}=e^{-t}(\cos t-\sin t)$$

$$L^{-1}\left\{e^{-2s}F(s)\right\}=e^{-(t-2)}\left[\cos(t-2)-\sin(t-2)\right]u(t-2)$$

(viii) Let $F(s)=\left(\dfrac{1+\sqrt{s}}{s^3}\right)$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{s^3}+\frac{1}{s^{\frac{5}{2}}}\right\}=\frac{t^2}{2!}+\frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}$$

$$=\frac{t^2}{2}+\frac{t^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}=\frac{t^2}{2}+\frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\left\{e^{-s}F(s)\right\}=\left[\frac{(t-1)^2}{2}+\frac{4(t-1)^{\frac{3}{2}}}{3\sqrt{\pi}}\right]u(t-1)$$

Exercise 12.19

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{e^{-as}}{(s+b)^{\frac{5}{2}}}$

$\left[\textbf{Ans. :}\ \dfrac{4}{3\sqrt{\pi}}\, e^{-b(t-a)}(t-a)^{\frac{3}{2}}u(t-a)\right]$

2. $\dfrac{e^{-\pi s}}{s^2+9}$

$\left[\textbf{Ans. :}\ \dfrac{1}{3}\sin 3(t-\pi)u(t-\pi)\right]$

3. $\dfrac{e^{-\pi s}}{s^2(s^2+1)}$

$\left[\textbf{Ans. :}\ [(t-\pi)+\sin(t-\pi)]u(t-\pi)\right]$

4. $\dfrac{e^{-4s}}{\sqrt{2s+7}}$

$\left[\textbf{Ans. :}\ \dfrac{e^{\frac{-7(t-4)}{2}}}{\sqrt{2\pi(t-4)}}u(t-4)\right]$

5. $\dfrac{(s+1)e^{-s}}{s^2+s+1}$

$\left[\textbf{Ans. :}\ e^{\frac{-(t-1)}{2}}\left[\cos\left(\sqrt{3}\,\dfrac{(t-1)}{2}\right) + \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{\sqrt{3}(t-1)}{2}\right)\right]u(t-1)\right]$

6. $\dfrac{se^{-3s}}{s^2-1}$

$\left[\textbf{Ans. :}\ \cosh(t-3)u(t-3)\right]$

7. $\dfrac{se^{-as}}{s^2+b^2}$

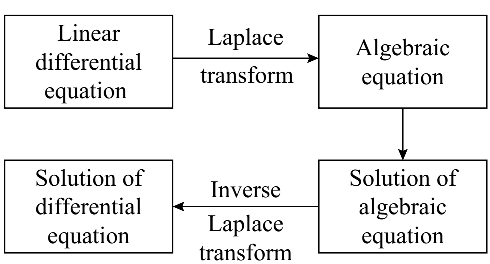
$\left[\textbf{Ans. :}\ \cos b(t-a)u(t-a)\right]$

8. $e^{-s}\left(\dfrac{1-\sqrt{s}}{s^2}\right)^2$

$\left[\textbf{Ans. :}\left[\dfrac{(t-1)^3}{6} - \dfrac{16}{15\sqrt{\pi}}(t-1)^{\frac{5}{2}} + \dfrac{(t-1)^2}{2}\right]u(t-1)\right]$

12.10 APPLICATION OF LAPLACE TRANSFORM TO DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform is useful in solving linear differential equations with given initial conditions by using algebraic methods. Initial conditions are included from the very beginning of the solution.



Example 1: Solve $\frac{dy}{dt}+2y=e^{-3t}$, $y(0)=1$.

Solution: Taking Laplace transform of both the sides,

$$sY(s)-y(0)+2Y(s)=\frac{1}{s+3}$$

$$sY(s)-1+2Y(s)=\frac{1}{s+3} \qquad [\because y(0)=1]$$

$$(s+2)\,Y(s)=\frac{1}{s+3}+1=\frac{s+4}{s+3}$$

$$Y(s)=\frac{s+4}{(s+2)(s+3)}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s+2}+\frac{B}{s+3}$$

$$s+4=A(s+3)+B(s+2) \qquad \dots (1)$$

Putting $s=-2$ in Eq. (1),

$A=2$

Putting $s=-3$ in Eq. (1),

$B=-1$

$$Y(s)=\frac{2}{s+2}-\frac{1}{s+3}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=2e^{-2t}-e^{-3t}$$

Example 2: Solve $\frac{dy}{dt}+y=\cos 2t$, $y(0)=1$.

Solution: Taking Laplace transform of both the sides,

$$sY(s)-y(0)+Y(s)=\frac{s}{s^2+4}$$

$$sY(s)-1+Y(s)=\frac{s}{s^2+4} \qquad [\because y(0)=1]$$

$$(s+1)\,Y(s)=\frac{s}{s^2+4}+1=\frac{s^2+s+4}{(s^2+4)}$$

$$Y(s)=\frac{s^2+s+4}{(s+1)(s^2+4)}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

$$s^2+s+4=A(s^2+4)+Bs+C)\,(s+1) \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$4 = 5A$$

$$A = \frac{4}{5}$$

Equating coefficients of s^2,

$$1 = A + B$$

$$B = 1 - \frac{4}{5} = \frac{1}{5}$$

Equating coefficients of s^0,

$$4 = 4A + C$$

$$C = 4\left(1 - \frac{4}{5}\right) = \frac{4}{5}$$

$$Y(s) = \frac{4}{5} \cdot \frac{1}{s+1} + \frac{1}{5} \cdot \frac{s}{s^2+4} + \frac{4}{5} \cdot \frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{4}{5} e^{-t} + \frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t$$

Example 3: Solve $y'' + 4y' + 8y = 1$, $y(0) = 0$, $y'(0) = 1$.

Solution: Taking Laplace transform of both the sides,

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 8Y(s) = \frac{1}{s}$$

$$\left[s^2 Y(s) - 1\right] + 4sY(s) + 8Y(s) = \frac{1}{s} \qquad [\because y^{\cdots\cdot} = \quad y' \quad = \ ]$$

$$(s^2 + 4s + 8)Y(s) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$Y(s) = \frac{s+1}{s(s^2 + 4s + 8)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}$$

$$s + 1 = A(s^2 + 4s + 8) + (Bs + C)s \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$1 = 8A$$

$$A = \frac{1}{8}$$

Equating coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{1}{8}$$

Equating coefficients of s,

$$1 = 4A + C$$

$$C = 1 - \frac{1}{2} = \frac{1}{2}$$

$$Y(s) = \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s}{s^2+4s+8} + \frac{1}{2}\cdot\frac{1}{s^2+4s+8} = \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{(s+2)-2}{(s+2)^2+4} + \frac{1}{2}\cdot\frac{1}{(s+2)^2+4}$$

$$= \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s+2}{(s+2)^2+4} + \frac{3}{4}\cdot\frac{1}{(s+2)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t$$

Example 4: Solve $y'' + y = t$, $y(0) = 1$, $y'(0) = 0$.

Solution: Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - s + Y(s) = \frac{1}{s^2} \qquad \left[\because y(0) = 1,\ y'(0) = 0\right]$$

$$(s^2+1)\,Y(s) = \frac{1}{s^2} + s = \frac{s^3+1}{s^2}$$

$$Y(s) = \frac{s^3+1}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{1}{s^2(s^2+1)} = \frac{s}{s^2+1} + \frac{s^2+1-s^2}{s^2(s^2+1)}$$

$$= \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \cos t + t - \sin t$$

Example 5: Solve $y'' + y = t^2 + 2t$, $y(0) = 4$, $y'(0) = -2$.

Solution: Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2Y(s) - 4s + 2 + sY(s) - 4 = \frac{2}{s^3} + \frac{2}{s^2}$$

$$(s^2+s)\,Y(s) = \frac{2}{s^3}+\frac{2}{s^2}+4s+2 = \frac{2(1+s)}{s^3}+4s+2$$

$$Y(s) = \frac{2(1+s)}{s^3(s^2+s)}+\frac{4s}{s^2+s}+\frac{2}{s^2+s} = \frac{2}{s^4}+\frac{4}{s+1}+\frac{2}{s}-\frac{2}{s+1}$$

$$= \frac{2}{s^4}+\frac{2}{s}-\frac{2}{s+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{t^3}{3}+2+2e^{-t}$$

Example 6: Solve $(D^2+9)y = 18t,\ \ y(0) = 0,\ \ y\left(\frac{\pi}{2}\right) = 1$.

Solution: Taking Laplace transform of both the sides,

$$\left[s^2\,Y(s) - sy(0) - y'(0)\right] + 9Y(s) = \frac{18}{s^2}$$

Let $y'(0) = A$

$$s^2\,Y(s) - A + 9Y(s) = \frac{18}{s^2} \qquad [\because y(0) = 0]$$

$$(s^2+9)\,Y(s) = \frac{18}{s^2}+A$$

$$Y(s) = \frac{18}{s^2(s^2+9)}+\frac{A}{s^2+9} = \frac{18}{9}\left(\frac{1}{s^2}-\frac{1}{s^2+9}\right)+\frac{A}{s^2+9} = \frac{2}{s^2}+\frac{A-2}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2t + \frac{A-2}{3}\sin 3t$$

Putting $t = \frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right) = 1$,

$$1 = 2\cdot\frac{\pi}{2}+\frac{A-2}{3}\sin\frac{3\pi}{2} = \pi - \frac{A-2}{3}$$

$$3 = 3\pi - A + 2$$

$$A = 3\pi - 1$$

Hence, $y(t) = 2t + \dfrac{3\pi-1-2}{3}\sin 3t \ = 2t + (\pi - 1)\sin 3t$

Example 7: Solve $(D^2-2D+1)y = e^t$, $y = 2$ and $Dy = -1$ at $t = 0$.

Solution: Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 2\left[sY(s) - y(0)\right] + Y(s) = \frac{1}{s-1}$$

$$\left[s^2Y(s)-2s+1\right]-2\left[sY(s)-2\right]+Y(s)=\frac{1}{s-1}$$

$$(s^2-2s+1)\,Y(s)=\frac{1}{s-1}+2s-5$$

$$(s-1)^2\,Y(s)=\frac{1+2s(s-1)-5(s-1)}{s-1}$$

$$Y(s)=\frac{2s^2-7s+6}{(s-1)^3}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}$$

$$2s^2-7s+6=A(s-1)^2+B(s-1)+C \qquad \ldots (1)$$

Putting $s = 1$ in Eq. (1),

$C = 1$

Equating coefficients of s^2,

$A = 2$

Equating coefficients of s,

$-7 = -2A + B,$

$B = -7 + 4 = -3$

$$Y(s)=\frac{2}{s-1}-\frac{3}{(s-1)^2}+\frac{1}{(s-1)^3}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=2e^t-3t\,e^{-t}+\frac{t^2}{2}e^t$$

Example 8: Solve $y''-6y'+9y=t^2e^{3t}$, $y(0)=2, y'(0)=6$.

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s)-s\,y(0)-y'(0)]-6\,[sY(s)-y(0)]+9Y(s)=\frac{2}{(s-3)^3}$$

$$[s^2Y(s)-2s-6]-6\,[sY(s)-2]+9Y(s)=\frac{2}{(s-3)^3}$$

$$(s^2-6s+9)\,Y(s)=\frac{2}{(s-3)^3}+2s-6$$

$$(s-3)^2\,Y(s)=\frac{2}{(s-3)^3}+2(s-3)$$

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2}{(s-3)}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^{3t}\frac{t^4}{4!} + 2e^{3t}$$

$$= \frac{1}{12}t^4 e^{3t} + 2e^{3t}$$

Example 9: Solve $(D^2 + 2D + 5)y = e^{-t}\sin t, y(0) = 0, y'(0) = 1.$

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = \frac{1}{(s+1)^2 + 1}$$

$$s^2Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{1}{s^2 + 2s + 2} \qquad [\because y(0) = 0, y'(0) = 1]$$

$$(s^2 + 2s + 5)\ Y(s) = \frac{1}{s^2 + 2s + 2} + 1$$

$$= \frac{s^2 + 2s + 3}{s^2 + 2s + 2}$$

$$Y(s) = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

By partial fraction expansion,

$$Y(s) = \frac{As + B}{s^2 + 2s + 2} + \frac{Cs + D}{(s^2 + 2s + 5)}$$

$$s^2 + 2s + 3 = (As + B)(s^2 + 2s + 5) + (Cs + D)(s^2 + 2s + 2)$$

$$= (A + C)\ s^3 + (2A + B + 2C + D)s^2$$

$$+(5A + 2B + 2C + 2D)s + (5B + 2D)$$

Equating the coefficients of s^3, s^2, s and s^0,

$$A + C = 0$$

$$2A + B + 2C + D = 1$$

$$5A + 2B + 2C + 2D = 2$$

$$5B + 2D = 3$$

Solving these equations,

$$A = 0, B = \frac{1}{3}, C = 0, D = \frac{2}{3}$$

$$Y(s) = \frac{1}{3}\cdot\frac{1}{s^2+2s+2}+\frac{2}{3}\cdot\frac{1}{s^2+2s+5} = \frac{1}{3}\cdot\frac{1}{(s+1)^2+1}+\frac{2}{3}\cdot\frac{1}{(s+1)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{3}e^{-t}\sin t+\frac{1}{3}e^{-t}\sin 2t = \frac{e^{-t}}{3}(\sin t+\sin 2t)$$

Example 10: Solve $y''+9y=\cos 2t,\ y(0)=1, y\left(\frac{\pi}{2}\right)=-1.$

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s)-s\,y(0)-y'(0)]+9Y(s)=\frac{s}{s^2+4}$$

Let $y'(0)=A$

$$s^2Y(s)-s-A+9Y(s)=\frac{s}{s^2+4} \qquad [\because y(0)=1]$$

$$(s^2+9)\ Y(s)=\frac{s}{s^2+4}+s+A$$

$$Y(s) = \frac{s}{(s^2+4)(s^2+9)}+\frac{s}{s^2+9}+\frac{A}{s^2+9} = \frac{s}{5}\left[\frac{(s^2+9)-(s^2+4)}{(s^2+4)(s^2+9)}\right]+\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

$$= \frac{1}{5}\cdot\frac{s}{s^2+4}-\frac{1}{5}\cdot\frac{s}{s^2+9}+\frac{s}{s^2+9}+\frac{A}{s^2+9} = \frac{1}{5}\cdot\frac{s}{s^2+4}+\frac{4}{5}\cdot\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{A}{3}\sin 3t$$

Putting $t=\frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right)=-1$,

$$-1 = -\frac{1}{5}-\frac{A}{3}$$

$$A = \frac{12}{5}$$

Hence, $y\ (t) = \frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t$

Example 11: Solve $y'''-2y''+5y'=0,\ y(0)=0, y'(0)=0, y''(0)=1.$

Solution: Taking Laplace transform of both the sides,

$$[s^3Y(s)-s^2y(0)-sy'(0)-y''(0)]-2[s^2Y(s)-sy(0)-y'(0)]+5[sY(s)-y(0)]=0$$

$$s^3Y(s)-1-2s^2Y(s)+5s\ Y(s)=0 \qquad [\because y(0)=0, y'(0)=0, y''(0)=1]$$

$$(s^3-2s^2+5s)\ Y(s)=1$$

$$Y(s) = \frac{1}{s^3 - 2s^2 + 5s} = \frac{1}{s(s^2 - 2s + 5)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{s^2 - 2s + 5}$$

$$1 = A\,(s^2 - 2s + 5) + (Bs + C)s \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$1 = 5A$$

$$A = \frac{1}{5}$$

Equating coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{1}{5}$$

Equating coefficients of s,

$$0 = -2A + C$$

$$C = \frac{2}{5}$$

$$Y(s) = \frac{1}{5}\cdot\frac{1}{s} - \frac{1}{5}\cdot\frac{s}{s^2 - 2s + 5} + \frac{2}{5}\cdot\frac{1}{s^2 - 2s + 5} = \frac{1}{5}\cdot\frac{1}{s} - \frac{1}{5}\cdot\frac{s}{(s-1)^2 + 4} + \frac{2}{5}\cdot\frac{1}{(s-1)^2 + 4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{5} - \frac{1}{5}e^t \cos 2t + \frac{1}{5}e^t \sin 2t$$

Example 12: Solve $y''' - 3y'' + 3y' - y = t^2 e^t,\ y(0) = 1, y'(0) = 0, y''(0) = -2.$

Solution: Taking Laplace transform of both the sides,

$$[s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2 Y(s) - sy(0) - y'(0)]$$

$$+ 3[s\ Y(s) - y(0)] - Y(s) = \frac{2}{(s-1)^3}$$

$$[s^3 Y(s) - s^2 + 2] - 3[s^2 Y(s) - s] + 3[sY(s) - 1] - Y(s) = \frac{2}{(s-1)^3}$$

$$[\because y(0) = 1, y'(0) = 0, y''(0) = -2]$$

$$(s^3 - 3s^2 + 3s - 1)Y(s) = \frac{2}{(s-1)^3} + (s^2 - 3s + 1)$$

$$(s-1)^3\ Y(s) = \frac{2}{(s-1)^3} + (s^2 - 3s + 1)$$

$$Y(s) = \frac{2}{(s-1)^6} + \frac{s^2-3s+1}{(s-1)^3} = \frac{2}{(s-1)^6} + \frac{(s-1)^2-(s-1)-1}{(s-1)^3}$$

$$= \frac{2}{(s-1)^6} + \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{60}e^t t^5 + e^t - e^t t - \frac{1}{2}e^t t^2$$

Example 13: Solve $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t,\ y(0) = 1.$

Solution: Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1}$$

$$sY(s) - 1 + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1} \qquad [\because y(0) = 1]$$

$$\left(s + 2 + \frac{1}{s}\right)Y(s) = \frac{1}{s^2+1} + 1 = \frac{s^2+2}{s^2+1}$$

$$\frac{s^2+2s+1}{s}Y(s) = \frac{s^2+2}{s^2+1}$$

$$Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+2s+1)} = \frac{s(s^2+2)}{(s^2+1)(s+1)^2}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$s(s^2+2) = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$-3 = 2B$$

$$B = -\frac{3}{2} \qquad \dots (2)$$

Equating coefficients of s^0

$$0 = A + B + D \qquad \dots (3)$$

Equating the coefficients of s^3,

$$1 = A + C \qquad \dots (4)$$

Equating the coefficients of s^2,

$$0 = A + B + 2C + D \qquad \dots (5)$$

Solving Eqs. (2), (3), (4) and (5),

$$A = 1, C = 0, D = \frac{1}{2}$$

$$Y(s) = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = e^{-t} - \frac{3}{2}e^{-t}t + \frac{1}{2}\sin t$$

Example 14: Solve $y'' + 4y = \delta(t)$, $y(0) = 0, y'(0) = 0$.

Solution: Taking Laplace transform of both the sides,

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 1$$

$$s^2\,Y(s) + 4\,Y(s) = 1 \qquad [\because y(0) = 0, y'(0) = 0]$$

$$(s^2 + 4)\,Y(s) = 1$$

$$Y(s) = \frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{2}\sin 2t$$

Example 15: Solve $y'' + 3y' + 2y = t\delta(t-1)$, $y(0) = 0, y'(0) = 0$.

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s) - s\,y(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = e^{-s}$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = e^{-s} \qquad [\because y(0) = 0, y'(0) = 0]$$

$$(s^2 + 3s + 2)\,Y(s) = e^{-s}$$

$$Y(s) = \frac{e^{-s}}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} = e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$$

Taking inverse Laplace transform of both the sides,

$$y(t) = e^{-(t-1)}\,u(t-1) - e^{-2(t-1)}\,u(t-1)$$

Example 16: Solve $y'' + 4y = u(t-2)$, $y(0) = 0, y'(0) = 1$.

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \frac{e^{-2s}}{s}$$

$$s^2Y(s) - 1 + 4Y(s) = \frac{e^{-2s}}{s} \qquad [\because y(0) = 0, y'(0) = 1]$$

$$(s^2+4)Y(s) = \frac{e^{-2s}}{s} + 1$$

$$Y(s) = \frac{e^{-2s}}{s(s^2+4)} + \frac{1}{s^2+4} = \frac{e^{-2s}}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) + \frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{4}u(t-2) - \frac{1}{4}\cos 2(t-2)\,u(t-2) + \frac{1}{2}\sin 2t$$

Example 17: Solve $\frac{d^2y}{dt^2} + 4y = f(t)$ with conditions $y(0) = 0$ and $y'(0) = 1$

and $f(t) = 1 \qquad 0 < t < 1$
$\quad\quad = 0 \qquad t > 1.$

Solution: $f(t) = u(t) - u(t-1)$

Taking Laplace transform of both the sides,

$$[s^2Y(s) - s\,y(0) - y'(0)] + 4Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$s^2Y(s) - 1 + 4Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} \qquad [\because y(0) = 0, y'(0) = 1]$$

$$(s^2+4)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} + 1$$

$$Y(s) = \frac{1}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)} + \frac{1}{s^2+4}$$

$$= \frac{s}{4}\left[\frac{s^2+4-s^2}{s^2(s^2+4)}\right] - e^{-s}\frac{s}{4}\left[\frac{s^2+4-s^2}{s^2(s^2+4)}\right] + \frac{1}{s^2+4}$$

$$= \frac{s}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right) - e^{-s}\frac{s}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right) + \frac{1}{s^2+4}$$

$$= \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{s}{s^2+4} - e^{-s}\frac{1}{4}\cdot\frac{1}{s} + e^{-s}\frac{1}{4}\cdot\frac{s}{s^2+4} + \frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{4}u(t-1) + \frac{1}{4}\cos 2(t-1)\,u(t-1) + \frac{1}{2}\sin 2t$$

Example 18: Solve $y'' + 4y = f(t)$ where $y(0) = 0$ and $y'(0) = 0$

and $f(t) = 0 \qquad 0 < t < 3$
$\quad\quad = t \qquad t \geq 3.$

Solution: $f(t) = t\,u(t-3)$

Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = e^{-3s}\,L\{t+3\}$$

$$s^2Y(s) + 4\,Y(s) = e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right) \qquad [\because y(0)=0, y'(0)=0]$$

$$(s^2+4)\,Y(s) = e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$$

$$Y(s) = e^{-3s}\left[\frac{1}{s^2(s^2+4)}+\frac{3}{s(s^2+4)}\right]$$

$$= \frac{e^{-3s}}{4}\left[\left(\frac{1}{s^2}-\frac{1}{s^2+4}\right)+\frac{3}{4}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\right]$$

$$= e^{-3s}\left(\frac{1}{4}\cdot\frac{1}{s^2}-\frac{1}{4}\cdot\frac{1}{s^2+4}+\frac{3}{4}\cdot\frac{1}{s}-\frac{3}{4}\cdot\frac{s}{s^2+4}\right)$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \left[\frac{1}{4}(t-3)-\frac{1}{8}\sin 2(t-3)+\frac{3}{4}-\frac{3}{4}\cos 2(t-3)\right]u(t-3)$$

$$= \frac{1}{8}[2t-\sin 2(t-3)-6\cos 2(t-3)]\,u(t-3)$$

Exercise 12.20

Using Laplace transform, solve the following differential equations:

1. $y'+4y=1;\ y(0)=-3.$

$$\left[\textbf{Ans. :}\ y(t)=\frac{1}{4}-\frac{13}{4}e^{-4t}\right]$$

2. $y'+6y=e^{4t};\ y(0)=2.$

$$\left[\textbf{Ans. :}\ y(t)=\frac{1}{10}e^{4t}+\frac{19}{10}e^{-6t}\right]$$

3. $y'+4y=\cos t;\ y(0)=0.$

$$\left[\textbf{Ans. :}\ y(t)=-\frac{4}{17}e^{-4t}+\frac{4}{17}\cos t+\frac{1}{17}\sin t\right]$$

4. $y'+3y=10\sin t;\ y(0)=0.$

$$\left[\textbf{Ans. :}\ y(t)=e^{-3t}-\cos t+3\sin t\right]$$

5. $y'+0.2y=0.01t;\ y(0)=-0.25.$

$$[\textbf{Ans. :}\ y(t)=0.05\,t-0.25]$$

6. $y'-2y=1-t;\ y(0)=1$

$$\left[\textbf{Ans. :}\ y(t)=-\frac{1}{4}+\frac{1}{2}t+\frac{5}{4}e^{2t}\right]$$

7. $y''+5y'+4y=0,$
$y(0)=1, y'(0)=-1.$

$$\left[\textbf{Ans. :}\ y(t)=\frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t}\right]$$

8. $y''+2y'-3y=6e^{-2t},$
$y(0)=2, y'(0)=-14.$

$$\left[\textbf{Ans. :}\ y(t)=-2e^{-2t}+\frac{11}{2}e^{-3t}-\frac{3}{2}e^{t}\right]$$

9. $y''-4y'+4y=1,$
$y(0)=1, y'(0)=4.$

$$\left[\textbf{Ans. :}\ y(t)=\frac{1}{4}+\frac{3}{4}e^{2t}+\frac{5}{2}te^{2t}\right]$$

10. $y'' - 4y' + 3y = 6t - 8,$
$y(0) = 0,\ y'(0) = 0.$

$$\left[\textbf{Ans.:}\ y(t) = 2t + e^{t} - e^{3t}\right]$$

11. $y'' - 3y' + 2y = 4t + e^{3t},$
$y(0) = 1,\ y'(0) = -1.$

$$\left[\begin{aligned}\textbf{Ans.:}\ y(t) &= -\frac{1}{2}e^{t} - 2e^{2t}\\ &+\frac{1}{2}e^{3t} + 2t + 3\end{aligned}\right]$$

12. $y'' + 2y' + y = 3te^{-t},$
$y(0) = 4,\ y'(0) = 2.$

$$\left[\textbf{Ans.:}\ y(t) = 4e^{-t} + 6t\,e^{-t} + \frac{t^3}{2}e^{-t}\right]$$

13. $y'' + y = \sin t \cdot \sin 2t,$
$y(0) = 1,\ y'(0) = 0.$

$$\left[\begin{aligned}\textbf{Ans.:}\ y(t) &= \frac{15}{16}\cos t + \frac{t}{4}\sin t\\ &+\frac{1}{16}\cos 3t\end{aligned}\right]$$

14. $y'' + y = e^{-2t}\sin t,$
$y(0) = 0, y'(0) = 0.$

$$\left[\begin{aligned}\textbf{Ans.:}\ y(t) &= \frac{1}{8}\sin t - \frac{1}{8}\cos t\\ &+\frac{1}{8}e^{-2t}\sin t + \frac{1}{8}e^{-2t}\cos t\end{aligned}\right]$$

15. $y'' + y = t\cos 2t,$
$y(0) = 0, y'(0) = 0.$

$$\left[\begin{aligned}\textbf{Ans.:}\ y(t) &= \frac{4}{9}\sin 2t - \frac{5}{9}\sin t\\ &-\frac{1}{3}t\cos 2t\end{aligned}\right]$$

16. $y''' + 4y'' + 5y' + 2y = 10\cos t,$
$y(0) = 0,\ y'(0) = 0,\ y''(0) = 3.$

$$\left[\begin{aligned}\textbf{Ans.:}\ y(t) &= -e^{-2t} + 2e^{-t}\\ &-2te^{-t} - \cos t + 2\sin t\end{aligned}\right]$$

17. $y' + y - 2\int_0^t y\,dt = \frac{t^2}{2},$

$y(0) = 1,\ \ y'(0) = -2.$

$$\left[\begin{aligned}\textbf{Ans.:}\ y(t) &= \frac{1}{3}e^{t} + \frac{11}{12}e^{-2t}\\ &-\frac{1}{2}t - \frac{1}{4}\end{aligned}\right]$$

12.11 APPLICATION OF LAPLACE TRANSFORM TO A SYSTEM OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

The Laplace transform can also be used to solve two or more simultaneous differential equations. The Laplace transform method transforms the differential equations into algebraic equations.

Example 1: Solve $\dfrac{dx}{dt} + y = \sin t$

$$\frac{dy}{dt} + x = \cos t$$

where $x(0) = 0$ and $y(0) = 2$.

Solution: Taking Laplace transform of both the equations,

$$sX(s)-x(0)+Y(s)=\frac{1}{s^2+1}$$

$$s\,X(s)+Y(s)=\frac{1}{s^2+1} \qquad \ldots (1)$$

and

$$s\,Y(s)-y(0)+X(s)=\frac{s}{s^2+1}$$

$$s\,Y(s)+X(s)=\frac{s}{s^2+1}+2$$

$$s\,Y(s)+X(s)=\frac{2s^2+s+2}{s^2+1} \qquad \ldots (2)$$

Multiplying Eq. (1) by s,

$$s^2\,X(s)+s\;Y(s)=\frac{s}{s^2+1} \qquad \ldots (3)$$

Subtracting Eq. (3) from Eq. (2),

$$(s^2-1)\,X(s)=-2$$

$$X(s)=-\frac{2}{s^2-1} \qquad \ldots (4)$$

Substituting $X(s)$ in Eq. (1),

$$Y(s)=\frac{1}{s^2+1}+2\frac{s}{s^2-1} \qquad \ldots (5)$$

Taking inverse Laplace transform of Eqs. (4) and (5),

$$x(t)=-2\sinh t$$

and

$$y(t)=\sin t+2\cosh t$$

Example 2: Solve $\boldsymbol{\frac{dx}{dt}-y=e^t}$

$\boldsymbol{\frac{dy}{dt}+x=\sin t}$

where $\boldsymbol{x(0)=1}$ and $\boldsymbol{y(0)=0}$.

Solution: Taking Laplace transform of both the equations,

$$s\,X(s)-x(0)-Y(s)=\frac{1}{s-1}$$

$$s\,X(s)-Y(s)=\frac{1}{s-1}+1=\frac{s}{s-1} \qquad \ldots (1)$$

and

$$s\;Y(s)-y(0)+X(s)=\frac{1}{s^2+1}$$

$$s\,Y(s)+X(s)=\frac{1}{s^2+1} \qquad \ldots (2)$$

Multiplying Eq. (1) by s,

$$s^2 X(s) - s\, Y(s) = \frac{s^2}{s-1} \qquad \dots (3)$$

Adding Eqs. (2) and (3),

$$(s^2+1)\, X(s) = \frac{1}{s^2+1} + \frac{s^2}{s-1}$$

$$X(s) = \frac{1}{(s^2+1)^2} + \frac{s^2}{(s-1)(s^2+1)}$$

$$= \frac{1}{(s^2+1)^2} + \frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) \qquad \dots (4)$$

Substituting $X(s)$ in Eq. (1),

$$Y(s) = s\,X(s) - \frac{s}{s-1} = \frac{s}{(s^2+1)} - \frac{s^3}{(s-1)(s^2+1)} - \frac{s}{s-1}$$

$$Y(s) = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)\,(s^2+1)}$$

$$= \frac{s}{(s^2+1)^2} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) \qquad \dots (5)$$

Taking the inverse Laplace transform of Eqs. (4) and (5),

$$x(t) = \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}(e^t + \cos t + \sin t) = \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$$

and $$y(t) = \frac{1}{2}t\sin t - \frac{1}{2}(e^t - \cos t + \sin t) = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t)$$

Example 3: Solve $\dfrac{dx}{dt} + 5x - 2y = t$

$$\frac{dy}{dt} + 2x + y = 0$$

where $x(0) = 0$ and $y(0) = 0$.

Solution: Taking Laplace transform of both the equations,

$$s\,X(s) - x(0) + 5X(s) - 2Y(s) = \frac{1}{s^2}$$

$$(s+5)\,X(s) - 2Y(s) = \frac{1}{s^2} \qquad \dots (1)$$

and $$s\,Y(s) - y(0) + 2X(s) + Y(s) = 0$$

$$2X(s) + (s+1)\,Y(s) = 0 \qquad \dots (2)$$

Multiplying Eq. (1) by $\frac{1}{2}(s+1)$,

$$\frac{1}{2}(s+5)(s+1)X(s) - (s+1)Y(s) = \frac{s+1}{2s^2} \qquad \dots (3)$$

Adding Eqs. (2) and (3),

$$X(s) = \frac{s+1}{s^2(s+3)^2} \qquad \dots (4)$$

Substituting $X(s)$ in Eq. (2),

$$Y(s) = -\frac{2}{s^2(s+3)^2} \qquad \dots (5)$$

Now,

$$X(s) = \frac{s+1}{s^2(s+3)^2}$$

By partial fraction expansion,

$$X(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$s + 1 = A\, s(s+3)^2 + B(s+3)^2 + C(s+3)s^2 + D\, s^2 \qquad \dots (6)$$

Putting $s = 0$ in Eq. (6),

$$1 = 9\,B$$

$$B = \frac{1}{9}$$

Putting $s = -3$ in Eq. (6),

$$-2 = 9\,D$$

$$D = -\frac{2}{9}$$

Equating the coefficients of s^3,

$$A + C = 0$$

$$A = -C$$

Equating the coefficients of s^2,

$$6A + B + 3C + D = 0$$

$$-3C = \frac{1}{9}$$

$$C = -\frac{1}{27}$$

$$A = \frac{1}{27}$$

$$X(s) = \frac{1}{27}\cdot\frac{1}{s} + \frac{1}{9}\cdot\frac{1}{s^2} - \frac{1}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,

$$x(t) = \frac{1}{27} + \frac{1}{9}t - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

Similarly,
$$Y(s) = \frac{-2}{s^2(s+3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$
$$= \frac{4}{27}\cdot\frac{1}{s} - \frac{2}{9}\cdot\frac{1}{s^2} - \frac{4}{27}\cdot\frac{1}{s+3} - \frac{2}{9}\cdot\frac{1}{(s+3)^2}$$

Taking inverse Laplace transform of both the sides,
$$y(t) = \frac{4}{27} - \frac{2}{9}t - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

Example 4: Solve $\frac{\mathbf{d}x}{\mathbf{d}t} + \frac{\mathbf{d}y}{\mathbf{d}t} + x - y = e^{-t}$

$$\frac{\mathbf{d}x}{\mathbf{d}t} + \frac{\mathbf{d}y}{\mathbf{d}t} + 2x + y = e^{t}$$

where $x(0) = 1$ and $y(0) = 0$.

Solution: Taking Laplace transform of both the equations,
$$s\,X(s) - x(0) + sY(s) - y(0) + X(s) - Y(s) = \frac{1}{s+1}$$
$$(s+1)\,X(s) + (s-1)Y(s) = \frac{1}{s+1} + 1 = \frac{s+2}{s+1} \quad \dots (1)$$

and
$$s\,X(s) - x(0) + sY(s) - y(0) + 2X(s) + Y(s) = \frac{1}{s-1}$$
$$(s+2)\,X(s) + (s+1)Y(s) = \frac{1}{s-1} + 1 = \frac{s}{s-1} \quad \dots (2)$$

Multiplying Eq. (1) by $(s + 1)$ and Eq. (2) by $(s - 1)$,
$$(s+1)^2\,X(s) + (s-1)(s+1)Y(s) = s+2 \quad \dots (3)$$
$$(s+2)(s-1)\,X(s) + (s-1)(s+1)Y(s) = s \quad \dots (4)$$

Subtracting Eq. (4) from Eq. (3),
$$(s + 3)\,X(s) = 2$$
$$X(s) = \frac{2}{s+3} \quad \dots (5)$$

Substituting $X(s)$ in Eq. (1),
$$\frac{2(s+1)}{s+3} + (s-1)Y(s) = \frac{s+2}{s+1}$$
$$Y(s) = \frac{s+2}{(s+1)(s-1)} - \frac{2(s+1)}{(s+3)(s-1)}$$
$$= \frac{(s+2)(s+3) - 2(s+1)^2}{(s-1)(s+1)(s+3)}$$

$$= \frac{-s^2 + s + 4}{(s-1)(s+1)(s+3)}$$

$$= \frac{-s^2 + s + 4}{(s^2-1)(s+3)}$$

By partial fraction expansion,

$$Y(s) = \frac{As + B}{s^2 - 1} + \frac{C}{s+3}$$

$$-s^2 + s + 4 = (As + B)(s+3) + C(s^2 - 1) \quad \dots (6)$$

Putting $s = -3$ in Eq. (6),

$$-8 = 8C$$
$$C = -1$$

Equating the coefficient of s^2,

$$-1 = A + C$$
$$A = 0$$

Equating the coefficient of s°,

$$4 = 3B - C$$
$$B = 1$$

$$Y(s) = \frac{1}{s^2 - 1} - \frac{1}{s+3} \quad \dots (7)$$

Taking inverse Laplace transform of Eqs. (5) and (7),

$$x(t) = 2e^{-3t}$$

and $$y(t) = \sinh t - e^{-3t}$$

Example 5: Solve $\dfrac{d^2x}{dt^2} - \dfrac{dy}{dt} = te^{-t} - 2e^{-t} - 3$

$$\frac{dx}{dt} - 2y - x = -2te^{-t} + e^{-t} - 6t$$

where $x(0) = 0$, $x'(0) = 1$ and $y(0) = 0$.

Solution: Taking Laplace transform of both the equations,

$$[s^2 X(s) - sx(0) - x'(0)] - [sY(s) - y(0)] = \frac{1}{(s+1)^2} - \frac{2}{s+1} - \frac{3}{s}$$

$$s^2 X(s) - s\,Y(s) = 1 + \frac{1}{(s+1)^2} - \frac{2}{s+1} - \frac{3}{s}$$

$$s^2 X(s) - s\,Y(s) = \frac{s^2}{(s+1)^2} - \frac{3}{s} \quad \dots (1)$$

and $$s\,X(s) - x(0) - 2\,Y(s) - X(s) = -\frac{2}{(s+1)^2} + \frac{1}{s+1} - \frac{6}{s^2}$$

$$(s-1)\,X(s) - 2\,Y(s) = \frac{s-1}{(s+1)^2} - \frac{6}{s^2} \quad \dots (2)$$

Multiplying Eq. (2) by $\frac{s}{2}$,

$$\frac{s(s-1)}{2}X(s)-sY(s)=\frac{s(s-1)}{2(s+1)^2}-\frac{3}{s} \qquad \ldots (3)$$

Subtracting Eq. (3) from Eq. (1),

$$\frac{(s^2+s)}{2}X(s)=\frac{s^2+s}{2(s+1)^2}$$

$$X(s)=\frac{1}{(s+1)^2} \qquad \ldots (4)$$

Substituting $X(s)$ in Eq. (1),

$$\frac{s^2}{(s+1)^2}-sY(s)=\frac{s^2}{(s+1)^2}-\frac{3}{s}$$

$$Y(s)=\frac{3}{s^2} \qquad \ldots (5)$$

Taking inverse Laplace transform of Eqs. (4) and (5),

$$x(t)=t\,e^{-t}$$

and $$y(t)=3t$$

Example 6: Solve $\frac{d^2x}{dt^2}-x-3y=0$

$$\frac{d^2y}{dt^2}-4x=-4e^t$$

where $x(0)=2, x'(0)=3, y(0)=1, y'(0)=2$.

Solution: Taking Laplace transform of both the equations,

$$[s^2X(s)-sx(0)-x'(0)]-X(s)-3Y(s)=0$$

$$s^2X(s)-2s-3-X(s)-3Y(s)=0$$

$$(s^2-1)X(s)-3Y(s)=2s+3 \qquad \ldots (1)$$

and $$[s^2Y(s)-sy(0)-y'(0)]-4X(s)=-\frac{4}{s-1}$$

$$s^2Y(s)-s-2-4X(s)=-\frac{4}{s-1}$$

$$s^2Y(s)-4X(s)=-\frac{4}{s-1}+s+2 \qquad \ldots (2)$$

Multiplying Eq. (1) by $\frac{s^2}{3}$,

$$\frac{s^2(s^2-1)}{3}X(s)-s^2Y(s)=\frac{s^2(2s+3)}{3} \qquad \ldots (3)$$

Adding Eqs. (2) and (3),

$$\left[\frac{s^2(s^2-1)}{3}-4\right]X(s)=\frac{s^2(2s+3)}{3}+\left[-\frac{4}{s-1}+s+2\right]$$

$$(s^4-s^2-12)\,X(s)=s^2(2s+3)+\frac{3(s+3)(s-2)}{s-1}$$

$$X(s)=\frac{s^2(2s+3)(s-1)+3(s+3)(s-2)}{(s-1)(s^2+3)(s^2-4)}$$

$$=\frac{2s^4+s^3+3s-18}{(s-1)(s^2+3)(s^2-4)}=\frac{(s+2)(2s-3)(s^2+3)}{(s-1)(s^2+3)(s^2-4)}$$

$$=\frac{2s-3}{(s-1)(s-2)}=\frac{1}{s-1}+\frac{1}{s-2} \qquad \ldots (4)$$

Substituting $X(s)$ in Eq. (1),

$$(s^2-1)\frac{(2s-3)}{(s-1)(s-2)}-3Y(s)=2s+3$$

$$Y(s)=\frac{1}{3}\left[\frac{(s+1)(2s-3)-(2s+3)(s-2)}{s-2}\right]=\frac{1}{3}\left(\frac{3}{s-2}\right)=\frac{1}{s-2} \qquad \ldots (5)$$

Taking inverse Laplace transform of Eqs. (5) and (6),

$$x(t)=e^t+e^{2t}$$

and $\quad y(t)=e^{2t}$

Exercise 12.21

Solve the following simultaneous equations:

1. $\dfrac{dx}{dt}+\dfrac{dy}{dt}+x=e^{-t}$

$\dfrac{dx}{dt}+2\dfrac{dy}{dt}+2x+2y=0$

where $x(0)=-1, y(0)=1$.

$$\left[\textbf{Ans.: } \begin{aligned} x(t)&=-e^{-t}(\cos t+\sin t),\\ y(t)&=e^{-t}(1+\sin t)\end{aligned}\right]$$

2. $\dfrac{dx}{dt}=2x-3y$

$\dfrac{dy}{dt}=y-2x$

where $x(0)=8, y(0)=3$.

$$\left[\textbf{Ans.: } \begin{aligned} x(t)&=5e^{-t}+8e^{4t},\\ y(t)&=5e^{-t}-2e^{4t}\end{aligned}\right]$$

3. $\dfrac{dx}{dt}-\dfrac{dy}{dt}+2y=\cos 2t$

$\dfrac{dx}{dt}+\dfrac{dy}{dt}-2x=\sin 2t$

where $x(0)=0, y(0)=-1$.

$$\left[\begin{aligned} &\textbf{Ans.: }\\ &x(t)=\frac{1}{2}e^t(\cos t+\sin t)-\frac{1}{2}\cos 2t,\\ &y(t)=-e^t(\cos t-\sin t)-\sin 2t\end{aligned}\right]$$

4. $2\frac{dx}{dt}+\frac{dy}{dt}-x-y=e^{-t}$

$\frac{dx}{dt}+\frac{dy}{dt}+2x+y=e^{-t}$

where $x(0)=2,\ y(0)=1.$

$$\left[\textbf{Ans.: } \begin{aligned} x(t) &= 2\cos t+8\sin t,\\ y(t) &= \cos t-13\sin t+\sinh t\end{aligned}\right]$$

5. $\frac{d^2x}{dt^2}+y=-5\cos 2t$

$\frac{d^2y}{dt^2}+x=5\cos 2t$

where $x(0)=1, x'(0)=1,$
$y'(0)=1, y(0)=-1.$

$$\left[\textbf{Ans.: } \begin{aligned} x(t) &= \sin t+\cos 2t,\\ y(t) &= \sin t-\cos 2t\end{aligned}\right]$$

6. $2\frac{d^2x}{dt^2}+3\frac{dy}{dt}=4$

$2\frac{d^2y}{dt^2}-3\frac{dx}{dt}=0$

where $x(0)=x'(0)=y(0)$
$=y'(0)=0.$

$$\left[\textbf{Ans.: } \begin{aligned} x(t) &= \frac{8}{9}\left(1-\cos\frac{3}{2}t\right),\\ y(t) &= \frac{8}{9}\left(\frac{3}{2}t-\sin\frac{3}{2}t\right)\end{aligned}\right]$$

FORMULAE

Laplace Transform

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$$

Properties of Laplace Transform

(i) Linearity

$L\{af_1(t)+bf_2(t)\} = aF_1(s)+bF_2(s)$

(ii) Change of scale

$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

(iii) First shifting theorem

$L\{e^{-at}f(t)\} = F(s+a)$

(iv) Second shifting theorem

$L\{u(t-a)\} = e^{-as}F(s)$

(v) Multiplication by t

$L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}f(s)$

(vi) Division by t

$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

(vii) Laplace transform of derivatives

$L\{f'(t)\} = sF(s)-f(0)$

$L\{f''(t)\} = s^2F(s)-sf(0)-f'(0)$

$L\{f^n(t)\} = s^nF(s)-s^{n-1}f(0)$
$-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-f^{(n-1)}(0)$

(viii) Laplace transform of integrals

$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$

(ix) Initial value theorem

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

(x) Final value theorem

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

(xi) Convolution theorem

$L\{f_1(t)*f_2(t)\}$

$= L\left\{\int_0^t f_1(u)f_2(t-u)\,du\right\}$

$= F_1(s)\cdot F_2(s)$

(xii) Laplace transform of periodic functions

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\,dt$$

Table of Laplace Transformation

Sr. No.	*f(t)*	*F(s)*
1	k	$\frac{k}{s}$
2	t	$\frac{1}{s^2}$
3	t^n	$\frac{\overline{\left\| n+1\right.}}{s^{n+1}}$
4	e^{at}	$\frac{1}{s-a}$
5	$\sin at$	$\frac{a}{s^2+a^2}$
6	$\cos at$	$\frac{s}{s^2+a^2}$
7	$\sinh at$	$\frac{a}{s^2-a^2}$
8	$\cosh at$	$\frac{s}{s^2-a^2}$
9	$e^{-bt}\sin at$	$\frac{a}{(s+b)^2+a^2}$
10	$e^{-bt}\cos at$	$\frac{s+b}{(s+b)^2+a^2}$
11	$e^{-bt}\sinh at$	$\frac{a}{(s+b)^2-a^2}$
12	$e^{-bt}\cosh at$	$\frac{s+b}{(s+b)^2-a^2}$
13	$u(t)$	$\frac{1}{s}$
14	$u(t-a)$	$\frac{e^{-as}}{s}$
15	$\delta(t)$	1
16	$\delta(t-a)$	e^{-as}

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. Given that $F(s)$ is the Laplace transform of $f(t)$, the Laplace transform of $\int_0^t f(\tau)\,d\tau$ is
   (a) $sF(s) - f(0)$
   (b) $\frac{1}{s}F(s)$
   (c) $\int_0^s f(\tau)\,d\tau$
   (d) $\frac{1}{s}[F(s) - f(0)]$
2. If the Laplace transform of a signal $y(t)$ is $Y(s) = \frac{1}{s(s-1)}$, then its final value is
   (a) –1 (b) 0
   (c) 1 (d) unbounded
3. A solution for the differential equation $\dot{x}(t) + 2x(t) = \delta(t)$ with initial condition $x(0) = 0$ is
   (a) $e^{-2t}u(t)$ (b) $e^{2t}u(t)$
   (c) $e^{-t}u(t)$ (d) $e^{t}u(t)$
4. The Dirac delta function $\delta(t)$ is defined as
   (a) $\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$
   (b) $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$
   (c) $\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$
   (d) $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$
5. Consider the function $f(t)$ having Laplace transform

$F(s) = \dfrac{\omega_0}{s^2 + \omega_0^2}$, Re$(s) > 0$

The final value of $f(t)$ would be

(a) 0
(b) 1
(c) $-1 \le f(\infty) \le 1$
(d) ∞

6. The Laplace transform of $i(t)$ is given by $I(s) = \dfrac{2}{s(1+s)}$. As $t \to \infty$, the value of $i(t)$ tends to

(a) 0 (b) 1
(c) 2 (d) ∞

7. Consider the function $F(s) = \dfrac{5}{s(s^2 + 3s + 2)}$, where $F(s)$ is the Laplace transform of the function $f(t)$. The initial value of $f(t)$ is equal to

(a) 5 (b) $\dfrac{5}{2}$
(c) $\dfrac{5}{3}$ (d) 0

8. The Laplace transform of the function $\sin^2 2t$ is

(a) $\dfrac{\left(\dfrac{1}{2s}\right) - s}{[2(s^2+16)]}$ (b) $\dfrac{8}{s(s^2+16)}$
(c) $\dfrac{\left(\dfrac{1}{s}\right) - s}{s^2+4}$ (d) $\dfrac{s}{s^2+4}$

9. The Laplace transform of $(t^2 - 2t)\, u(t-1)$ is

(a) $\dfrac{2}{s^3}e^{-s} - \dfrac{2}{s^2}e^{-s}$
(b) $\dfrac{2}{s^3}e^{-2s} - \dfrac{2}{s^2}e^{-s}$
(c) $\dfrac{2}{s^3}e^{-s} - \dfrac{2}{s}e^{-s}$
(d) None of these

10. The Laplace transform of the function $f(t) = t$, starting at $t = a$, is

(a) $\dfrac{1}{(s+a)^2}$ (b) $\dfrac{e^{-as}}{(s+a)^2}$
(c) $\dfrac{e^{-as}}{s^2}$ (d) $\dfrac{a}{s^2}$

11. If $L[f(t)] = \dfrac{2(s+1)}{s^2 + 2s + 5}$, then $f(0)$ and $f(\infty)$ are given by

(a) 0, 2 respectively
(b) 2, 0 respectively
(c) 0, 1 respectively
(d) $\dfrac{2}{5}$, 0 respectively

12. The inverse Laplace transform of the function $\dfrac{s+5}{(s+1)(s+3)}$ is

(a) $2e^{-t} - e^{-3t}$
(b) $2e^{-t} + e^{-3t}$
(c) $e^{-t} - 2e^{-3t}$
(d) $e^{-t} + e^{-3t}$

13. Given that $L[f(t)] = \dfrac{s+2}{s^2+1}$,

$L[g(t)] = \dfrac{s^2+1}{(s+3)(s+2)}$,

$h(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$.

Then $L[h(t)]$ is

(a) $\dfrac{s^2+1}{s+3}$
(b) $\dfrac{1}{s+3}$
(c) $\dfrac{s^2+1}{(s+3)(s+2)} + \dfrac{s+2}{s^2+1}$
(d) None of these

14. For the equation $\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 5$, the solution $x(t)$ approaches the following values at $t \to \infty$, with all initial conditions as zero

(a) 0 (b) $\dfrac{5}{2}$
(c) 5 (d) 10

15. The delayed unit step function is defined as $u(t-a) = 0 \quad t < a$.
$ = 1 \quad t > a$
Its Laplace transform is
(a) ae^{-as} (b) $\dfrac{e^{-as}}{s}$
(c) $\dfrac{e^{as}}{s}$ (d) $\dfrac{e^{as}}{a}$

16. $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt$ is equal to
(a) π (b) $\dfrac{\pi}{2}$
(c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{3}$

17. If $f(t) = 2e^{\log t}$, then $F(s)$ is
(a) $\dfrac{2}{s^2}$ (b) $\dfrac{1}{s^2}$
(c) $\dfrac{2}{s}$ (d) $\dfrac{2}{s^3}$

18. Inverse Laplace transform of $\dfrac{e^{-3s}}{(s-2)^4}$ is
(a) $1 \quad t < 3$
$\dfrac{1}{5}\dfrac{(t-3)^3}{4} \quad t > 3$
(b) $0 \quad t < 3$
$\dfrac{1}{6}\dfrac{t^3}{6}e^2 \quad t > 3$
(c) $0 \quad t < 3$
$0 \quad t > 3$
(d) $0 \quad t < 3$
$\dfrac{1}{6}\dfrac{(t-3)^3}{6}e^{2(t-3)} \quad t > 3$

19. Match List I (functions) with List II (Laplace transforms) and select the correct answer.

List I	*List II*
(A) $e^{-t}u(t)$	1. $\dfrac{1}{s^2}$
(B) $tu(t)$	2. $\dfrac{1}{(s+1)^2}$
(C) $u(t)$	3. $\dfrac{1}{s}$
(D) $te^{-t}u(t)$	4. $\dfrac{1}{s+1}$

$u(t)$ denotes the unit step function

	A	B	C	D
(a)	4	1	3	2
(b)	2	3	1	4
(c)	4	3	1	2
(d)	2	1	3	4

20. The expression for the waveform in terms of the unit step function is given by

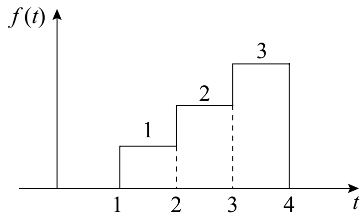


Fig. 12.18

(a) $f(t) = u(t-1) - u(t-2) + u(t-3)$
(b) $f(t) = u(t-1) + u(t-2) + u(t-3)$
(c) $f(t) = u(t-1) + u(t-2) - u(t-3)$
(d) $f(t) = u(t-1) + u(t-2) + u(t-3) - 3u(t-4)$

21. The Laplace transform of the function shown in Fig. 12.19 is

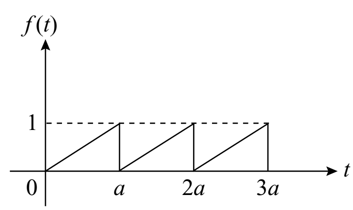


Fig. 12.19

(a) $F(s) = \dfrac{1}{1-e^{-as}}$
(b) $F(s) = \dfrac{1-e^{-as}}{2s^2} - \dfrac{e^{-as}}{s}$

(c) $F(s) = \dfrac{1 - e^{-as}}{2}$

(d) $F(s) = \dfrac{1}{as^2} - \dfrac{e^{-as}}{s(1 - e^{-as})}$

22. Given, $L\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st}\,\mathrm{d}t$ which of the following expressions are correct?

(1) $L\{f(t-a)\,u(t-a)\} = F(s)\,e^{-as}$

(2) $L\{tf(t)\} = -\dfrac{\mathrm{d}}{\mathrm{d}s} F(s)$

(3) $L\{(t-a)f(t)\} = asF(s)$

(4) $L\left\{\dfrac{\mathrm{d}}{\mathrm{d}t} f(t)\right\} = sF(s) - f(0)$

Select the correct answer using the codes given below:

(a) 1, 2 and 3 (b) 1, 2 and 4

(c) 2, 3 and 4 (d) 1, 3 and 4

23. If $h(t) = 10e^{-10t}$ and $e(t) = \sin 10t$, the Laplace transform of the function

$f(t) = \int_0^t h(t-\tau)\,e(\tau)$ is given by,

(a) $\dfrac{10}{(s+10)(s^2+100)}$

(b) $\dfrac{10(s+10)}{s^2+100}$

(c) $\dfrac{100}{(s+10)(s^2+100)}$

(d) $\dfrac{1}{(s+10)(s^2+100)}$

24. The Laplace transform of $\sin 2t\,\delta\left(t - \dfrac{\pi}{4}\right)$ is

(a) $e^{\frac{-\pi s}{4}}$ (b) $e^{\frac{\pi s}{4}}$

(c) $e^{\frac{-\pi s}{2}}$ (d) $e^{\frac{\pi s}{2}}$

25. Which one of the following is the correct Laplace transform of the function in Fig. 12.20?

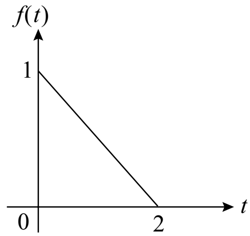


Fig. 12.20

(a) $\dfrac{1}{Ts^2}[1 - e^{-Ts}(1 + Ts)]$

(b) $\dfrac{1}{Ts^2}[e^{-Ts} - 1 + Ts]$

(c) $\dfrac{1}{Ts^2}[e^{-Ts} + 1 - Ts]$

(d) $\dfrac{1}{Ts^2}[1 - e^{-Ts} + Ts]$

Answers

1. (b)	2. (a)	3. (a)	4. (d)	5. (c)	6. (c)	7. (d)
8. (b)	9. (a)	10. (c)	11. (b)	12. (a)	13. (b)	14. (b)
15. (b)	16. (b)	17. (a)	18. (d)	19. (a)	20. (d)	21. (d)
22. (b)	23. (c)	24. (a)	25. (b)			