Partial Differentiation

Chapter 4

4.1 INTRODUCTION

We often come across functions which depend on two or more variables. For example, area of a triangle depends on its base and height, hence we can say that area is the function of two variables, i.e., its base and height. u is called a function of two variables x and y, if u has a definite value for every pair of x and y. It is written as $u = f(x, y)$. The variables x and y are independent variables while u is dependent variable. The set of all the pairs (x, y) for which u is defined is called the domain of the function. Similarly, we can define function of more than two variables.

4.2 PARTIAL DERIVATIVE

A partial derivative of a function of several variables is the ordinary derivative w.r.t. one of the variables, when all the remaining variables are kept constant. Consider a function $u = f(x, y)$, here, u is the dependent variable and x and y are independent variables. The partial derivative of $u = f(x, y)$ w.r.t. x is the ordinary derivative of u w.r.t. x, keeping y constant. It is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x or f_x and is known as first order partial derivative of u w.r.t. x.

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right]$$

Similarly, the partial derivative of $u = f(x, y)$ w.r.t. y is the ordinary derivative of u w.r.t. y treating x as constant. It is denoted by $\frac{\partial u}{\partial y}$ or $\frac{\partial f}{\partial y}$ or u_y or f_y and is known as first order partial derivative of u w.r.t. y.

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right]$$

4.2.1 Geometrical Interpretation

The function $u = f(x, y)$ represents a surface. The point $P[x_1, y_1, f(x_1, y_1)]$ on the surface corresponds to the values x_1, y_1 of the independent variables x, y. The intersection of the plane $y = y_1$ (parallel to the zox–plane) and the surface $u = f(x, y)$ is the curve shown by the dotted line in the Figure. On this curve, x and u vary according to the relation $u = f(x, y_1)$. The ordinary derivative of $f(x, y_1)$ w.r.t. x at x_1 is $\left(\frac{\partial u}{\partial x}\right)_{(x, y_1)}$. Hence, $\left(\frac{\partial u}{\partial x}\right)_{(x_1, y_1)}$ is the slope of the tangent to the curve of the intersection of the surface $u = f(x, y)$ with the plane $y = y_1$ at the point $P[x_1, y_1, f(x_1, y_1)]$.

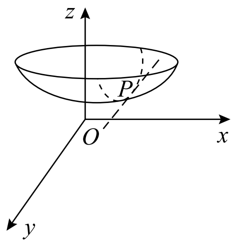


Fig. 4.1

Similarly, $\left(\frac{\partial u}{\partial y}\right)_{(x_1, y_1)}$ is the slope of the tangent to the curve of the intersection of the surface $u = f(x, y)$ with the plane $x = x_1$ at the point $P[x_1, y_1, f(x_1, y_1)]$.

4.3 HIGHER ORDER PARTIAL DERIVATIVES

Partial derivatives of higher order, of a function $u = f(x, y)$, are obtained by partial differentiation of first order partial derivative. Thus, if $u = f(x, y)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

are called second order partial derivatives. Similarly, other higher order derivatives can also be obtained.

Note:

1. If $u = f(x, y)$ possesses continuous second order partial derivatives $\frac{\partial^2 u}{\partial x\, \partial y}$ and $\frac{\partial^2 u}{\partial y\, \partial x}$, then $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$. This is called commutative property.

2. Standard rules for differentiation of sum, difference, product and quotient are also applicable for partial differentiation.

Example 1: If $u = (1 - 2xy + y^2)^{-\frac{1}{2}}$, then show that $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = u^3 y^2$.

Solution: $$u = (1 - 2xy + y^2)^{-\frac{1}{2}}$$

Differentiating u partially w.r.t. x and y,

$$\frac{\partial u}{\partial x} = \frac{-1}{2}(1 - 2xy + y^2)^{-\frac{3}{2}}(-2y)$$

$$\frac{\partial u}{\partial y} = \frac{-1}{2}(1 - 2xy + y^2)^{-\frac{3}{2}}(-2x + 2y)$$

Hence, $$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = (1 - 2xy + y^2)^{-\frac{3}{2}}(xy - xy + y^2)$$

$$= \left[(1 - 2xy + y^2)^{-\frac{1}{2}}\right]^3 y^2$$

$$= u^3 y^2.$$

Example 2: If $u = \log(\tan x + \tan y + \tan z)$, then show that

$\sin 2x\dfrac{\partial u}{\partial x} + \sin 2y\dfrac{\partial u}{\partial y} + \sin 2z\dfrac{\partial u}{\partial z} = 2.$

Solution: $$u = \log(\tan x + \tan y + \tan z)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}\cdot \sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z}\cdot \sec^2 y$$

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z}\cdot \sec^2 z$$

Hence,

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z}$$

$$= \frac{2\sin x\cos x\sec^2 x + 2\sin y\cos y\sec^2 y + 2\sin z\cos z\sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} = 2.$$

Example 3: If $\boldsymbol{u = \dfrac{e^{x+y+z}}{e^x + e^y + e^z}}$**, show that** $\boldsymbol{\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 2u.}$

Solution:
$$u = \frac{e^{x+y+z}}{e^x + e^y + e^z}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{e^{x+y+z}}{e^x + e^y + e^z} - \frac{e^{x+y+z}}{(e^x + e^y + e^z)^2} \cdot e^x$$

$$= \frac{e^{x+y+z}}{e^x + e^y + e^z}\left(1 - \frac{e^x}{e^x + e^y + e^z}\right) \qquad \text{... (1)}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{e^{x+y+z}}{e^x + e^y + e^z}\left(1 - \frac{e^y}{e^x + e^y + e^z}\right) \qquad \text{... (2)}$$

$$\frac{\partial u}{\partial z} = \frac{e^{x+y+z}}{e^x + e^y + e^z}\left(1 - \frac{e^z}{e^x + e^y + e^z}\right) \qquad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{e^{x+y+z}}{e^x + e^y + e^z}\left(3 - \frac{e^x + e^y + e^z}{e^x + e^y + e^z}\right)$$

$$= \frac{e^{x+y+z}}{e^x + e^y + e^z}(3-1)$$

$$= 2u$$

Example 4: If $\boldsymbol{u(x, y) = \dfrac{x^2 + y^2}{x + y}}$**, then show that** $\boldsymbol{\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right).}$

Solution:
$$u(x, y) = \frac{x^2 + y^2}{x + y}$$

$$u(x + y) = x^2 + y^2 \qquad \text{... (1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$u + (x + y)\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{2x - u}{x + y}$$

Differentiating Eq. (1) partially w.r.t. y,

$$u + (x + y)\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = \frac{2y - u}{x + y}$$

$$\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2=\left(\frac{2x-u}{x+y}-\frac{2y-u}{x+y}\right)^2$$

$$=\left[\frac{2(x-y)}{(x+y)}\right]^2 \qquad \text{... (2)}$$

Again,
$$4\left(1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)=4\left(1-\frac{2x-u}{x+y}-\frac{2y-u}{x+y}\right)$$

$$=4\left(1-\frac{2x-u+2y-u}{x+y}\right)=4\left[1-\frac{2(x+y)}{(x+y)}+\frac{2u}{(x+y)}\right]$$

$$=4\left[1-2+2\left\{\frac{x^2+y^2}{(x+y)^2}\right\}\right]=4\left[\frac{-(x+y)^2+2x^2+2y^2}{(x+y)^2}\right]$$

$$=\frac{4(x^2+y^2-2xy)}{(x+y)^2}=\left[\frac{2(x-y)}{(x+y)}\right]^2 \qquad \text{... (3)}$$

From Eqs (1) and (2), we get

$$\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2=4\left(1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right).$$

Example 5: If $z=ct^{-\frac{1}{2}}e^{\frac{-x^2}{4a^2t}}$, prove that $\frac{\partial z}{\partial t}=a^2\frac{\partial^2 z}{\partial x^2}$.

Solution:
$$z=ct^{-\frac{1}{2}}e^{\frac{-x^2}{4a^2t}}$$

Differentiating z partially w.r.t. t,

$$\frac{\partial z}{\partial t}=-\frac{1}{2}ct^{-\frac{3}{2}}e^{\frac{-x^2}{4a^2t}}+ct^{-\frac{1}{2}}e^{\frac{-x^2}{4a^2t}}\left(\frac{x^2}{4a^2t^2}\right)$$

$$=\frac{ce^{\frac{-x^2}{4a^2t}}}{2}t^{-\frac{5}{2}}\left(-t+\frac{x^2}{2a^2}\right) \qquad \text{... (1)}$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x}=ct^{-\frac{1}{2}}e^{\frac{-x^2}{4a^2t}}\left(\frac{-2x}{4a^2t}\right)$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2}=\frac{-2ct^{-\frac{1}{2}}}{4a^2t}\left[e^{\frac{-x^2}{4a^2t}}+xe^{\frac{-x^2}{4a^2t}}\left(\frac{-2x}{4a^2t}\right)\right]$$

$$= \frac{ct^{-\frac{1}{2}}}{2a^2t^2} e^{\frac{-x^2}{4a^2t}} \left(-t + \frac{x^2}{2a^2}\right)$$

$$a^2 \frac{\partial^2 z}{\partial x^2} = \frac{ce^{\frac{-x^2}{4a^2t}}}{2} \cdot t^{-\frac{5}{2}} \left(-t + \frac{x^2}{2a^2}\right) \qquad \ldots (2)$$

From Eqs (1) and (2), we get

$$\frac{\partial z}{\partial t} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Example 6: If $u(x, t) = ae^{-gx} \sin(nt - gx)$, where a, g, n are constants, satisfying the equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$, prove that $g = \frac{1}{a}\sqrt{\frac{n}{2}}$.

Solution: $u(x, t) = ae^{-gx} \sin(nt - gx)$

Differentiating u partially w.r.t. t,

$$\frac{\partial u}{\partial t} = ae^{-gx}[\cos(nt - gx)]n$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = -age^{-gx} \sin(nt - gx) + [ae^{-gx} \cos(nt - gx)](-g)$$

$$= -age^{-gx}[\sin(nt - gx) + \cos(nt - gx)]$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = ag^2e^{-gx}[\sin(nt - gx) + \cos(nt - gx)] - age^{-gx}[-g\cos(nt - gx) + g\sin(nt - gx)]$$

$$= ag^2e^{-gx}[2\cos(nt - gx)]$$

Substituting in $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$,

$$ae^{-gx}[\cos(nt - gx)]n = a^3g^2e^{-gx}[2\cos(nt - gx)]$$

$$g^2 = \frac{n}{2a^2}$$

$$g = \frac{1}{a}\sqrt{\frac{n}{2}}.$$

Example 7: If $u = e^{x^y}$, find $\frac{\partial^2 u}{\partial y\, \partial x}$.

Solution: $u = e^{x^y}$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = e^{x^y} \frac{\partial}{\partial x}(x^y) = e^{x^y} \cdot yx^{y-1}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = e^{x^y} \frac{\partial}{\partial y}(x^y) \cdot yx^{y-1} + e^{x^y} x^{y-1} + e^{x^y} y \frac{\partial}{\partial y}(x^{y-1})$$

$$\frac{\partial^2 u}{\partial y \partial x} = e^{x^y} x^y \log x \cdot yx^{y-1} + e^{x^y} x^{y-1} + e^{x^y} yx^{y-1} \log x$$

$$= e^{x^y} x^{y-1}(yx^y \log x + 1 + y \log x).$$

Example 8: If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2)e^{xyz}$.

Solution: $u = e^{xyz}$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial y \partial z} = xe^{xyz} + x^2 yze^{xyz}$$

Differentiating $\frac{\partial^2 u}{\partial y \partial z}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} + xyze^{xyz} + 2xyze^{xyz} + x^2 y^2 z^2 e^{xyz}$$

$$= (1 + 3xyz + x^2 y^2 z^2)e^{xyz}.$$

Example 9: If $u = x^3 y + e^{xy^2}$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Solution: $u = x^3 y + e^{xy^2}$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 3x^2 y + e^{xy^2} \cdot y^2$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 3x^2 + 2ye^{xy^2} + y^2 e^{xy^2} \cdot 2xy$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3x^2 + 2ye^{xy^2}(1 + xy^2) \qquad \text{... (1)}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = x^3 + e^{xy^2} \cdot 2xy$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 3x^2 + 2ye^{xy^2} + 2xye^{xy^2} \,.\, y^2$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3x^2 + 2ye^{xy^2}(1 + xy^2) \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

Example 10: If $z = x^y + y^x$, prove that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.

Solution:

$$z = x^y + y^x$$

$$z = e^{\log x^y} + e^{\log y^x} = e^{y\log x} + e^{x\log y}$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = e^{y\log x} \cdot \frac{y}{x} + e^{x\log y} \cdot \log y$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{1}{x}(e^{y\log x} + e^{y\log x}\, y\log x) + e^{x\log y} \cdot \frac{x}{y}\log y + e^{x\log y} \cdot \frac{1}{y}$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{e^{y\log x}}{x}(1 + y\log x) + \frac{e^{x\log y}}{y}(x\log y + 1) \qquad \text{... (1)}$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = e^{y\log x} \cdot \log x + e^{x\log y} \cdot \frac{x}{y}$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^{y\log x}\cdot\frac{y}{x}\log x + e^{y\log x}\cdot\frac{1}{x} + e^{x\log y}\cdot\frac{1}{y} + e^{x\log y}\log y\cdot\frac{x}{y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{e^{y\log x}}{x}(y\log x + 1) + \frac{e^{x\log y}}{y}(1 + x\log y) \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}.$$

Example 11: If $u = (3xy - y^3) - (y^2 - 2x)^{\frac{3}{2}}$, show that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

Solution: $$u = (3xy - y^3) - (y^2 - 2x)^{\frac{3}{2}}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 3y - \frac{3}{2}(y^2 - 2x)^{\frac{1}{2}}(-2) = 3y + 3(y^2 - 2x)^{\frac{1}{2}}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 3 + \frac{3}{2}(y^2 - 2x)^{-\frac{1}{2}}(2y)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3 + \frac{3y}{\sqrt{y^2 - 2x}} \qquad \text{... (1)}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = 3x - 3y^2 - \frac{3}{2}(y^2 - 2x)^{\frac{1}{2}}(2y)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 3 - 3y\frac{1}{2\sqrt{y^2 - 2x}}(-2)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3 + \frac{3y}{\sqrt{y^2 - 2x}} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

Example 12: If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$,

prove that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{x^2 - y^2}{x^2 + y^2}$.

Solution:
$$z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) - \frac{y^2}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= 2x \tan^{-1}\frac{y}{x} - \frac{x^2 y}{x^2 + y^2} - \frac{y^3}{x^2 + y^2}$$

$$= 2x \tan^{-1}\frac{y}{x} - y$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x} = 2x \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 1$$

$$= \frac{2x^2}{x^2 + y^2} - 1 = \frac{2x^2 - x^2 - y^2}{x^2 + y^2}$$

$$= \frac{x^2 - y^2}{x^2 + y^2} \qquad \text{... (1)}$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - y^2 \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) - 2y \tan^{-1}\frac{x}{y}$$

$$= \frac{x^3}{x^2 + y^2} + \frac{xy^2}{y^2 + x^2} - 2y \tan^{-1}\frac{x}{y}$$

$$= x - 2y \tan^{-1}\frac{x}{y}$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\,\partial y}=1-2y\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$=1-\frac{2y^2}{y^2+x^2}=\frac{y^2+x^2-2y^2}{y^2+x^2}$$

$$=\frac{x^2-y^2}{x^2+y^2} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}=\frac{x^2-y^2}{x^2+y^2}.$$

Example 13: If $u=\log(x^2+y^2+z^2)$, prove that $x\frac{\partial^2 u}{\partial y\,\partial z}=y\frac{\partial^2 u}{\partial z\,\partial x}=z\frac{\partial^2 u}{\partial x\,\partial y}$.

Solution: $u=\log(x^2+y^2+z^2)$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x}=\frac{1}{x^2+y^2+z^2}\cdot 2x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y}=-\frac{2x}{(x^2+y^2+z^2)^2}\cdot 2y$$

$$z\frac{\partial^2 u}{\partial x\,\partial y}=-\frac{4xyz}{(x^2+y^2+z^2)^2} \qquad \text{... (1)}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y}=\frac{1}{x^2+y^2+z^2}\cdot 2y$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. z,

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial z}\left(\frac{2y}{x^2+y^2+z^2}\right)$$

$$=-\frac{2y}{(x^2+y^2+z^2)^2}\cdot 2z$$

$$x\frac{\partial^2 u}{\partial y\,\partial z}=-\frac{4xyz}{(x^2+y^2+z^2)^2} \qquad \text{... (2)}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. z,

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = -\frac{2x}{(x^2+y^2+z^2)^2}\cdot 2z$$

$$y\frac{\partial^2 u}{\partial z\,\partial x} = -\frac{4xyz}{(x^2+y^2+z^2)^2} \qquad \text{... (3)}$$

From Eqs (1), (2) and (3), we get

$$x\frac{\partial^2 u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y}.$$

Example 14: If $a^2x^2 + b^2y^2 = c^2z^2$, evaluate $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$.

Solution: $\qquad a^2x^2 + b^2y^2 = c^2z^2$

Differentiating partially w.r.t. x,

$$2a^2x = 2c^2z\cdot\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{a^2x}{c^2z}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2}\left(\frac{1}{z} - \frac{x}{z^2}\cdot\frac{\partial z}{\partial x}\right) = \frac{a^2}{c^2z}\left(1 - \frac{x}{z}\cdot\frac{a^2x}{c^2z}\right)$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2z}\left(1 - \frac{a^2x^2}{c^2z^2}\right)$$

Similarly,

$$\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(1 - \frac{b^2y^2}{c^2z^2}\right)$$

Hence,

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(2 - \frac{a^2x^2+b^2y^2}{c^2z^2}\right)$$

$$= \frac{1}{c^2z}\left(2 - \frac{c^2z^2}{c^2z^2}\right) = \frac{1}{c^2z}(2-1)$$

$$= \frac{1}{c^2z}.$$

Example 15: If $u = \log(x^3 + y^3 - x^2y - xy^2)$,

prove that $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$.

Solution:
$$\begin{aligned}u &= \log (x^3 + y^3 - x^2y - xy^2)\\ &= \log[(x+y)(x^2-xy+y^2)-xy(x+y)]\\ &= \log(x+y)(x^2-xy+y^2-xy)\\ &= \log(x+y)(x-y)^2\\ &= \log(x+y)+2\log(x-y)\end{aligned}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2}{x-y}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{1}{(x+y)^2} + \frac{2}{(x-y)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

Example 16: If $u = \log (x^3 + y^3 + z^3 - 3xyz)$,

prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Solution:
$$\begin{aligned}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)\\ &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)v\end{aligned}$$

where,
$$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$u = \log (x^3 + y^3 + z^3 - 3xyz)$$

Differentiating u partially w.r.t. x, y, and z simultaneously,

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2 + y^2 + z^2) - 3(xy + yz + zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} \cdot \frac{(x + y + z)}{(x + y + z)}$$

$$= \frac{3(x^3 + y^3 + z^3 - 3xyz)}{(x^3 + y^3 + z^3 - 3xyz)(x + y + z)}$$

$$= \frac{3}{x + y + z}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right)$$

$$= -\frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2}$$

$$= -\frac{9}{(x + y + z)^2}.$$

Example 17: If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ and $a^2 + b^2 + c^2 = 1$, then show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

Solution:

$$u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 6(ax + by + cz)a - 2x$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = 6a \cdot a - 2 = 6a^2 - 2$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = 6(ax + by + cz)b - 2y$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = 6b \cdot b - 2 = 6b^2 - 2$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = 6(ax + by + cz)c - 2z$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. z,

$$\frac{\partial^2 u}{\partial z^2} = 6c \cdot c - 2 = 6c^2 - 2$$

Hence,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= 6(a^2 + b^2 + c^2) - 6 \\ &= 6(1) - 6 \qquad [\because a^2 + b^2 + c^2 = 1] \\ &= 0\end{aligned}$$

Example 18: If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Solution:

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = -\frac{1}{2(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot 2x = -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= -\left[\frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \frac{3x \cdot 2x}{2(x^2 + y^2 + z^2)^{\frac{5}{2}}}\right] \\ &= -\frac{1}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}(x^2 + y^2 + z^2 - 3x^2)\end{aligned}$$

$$= \frac{-(-2x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{-(x^2 - 2y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

and $$\frac{\partial^2 u}{\partial z^2} = \frac{-(x^2 + y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

Hence, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-(-2x^2 + 2y^2 + 2z^2 + 2x^2 - 2y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0.$$

Example 19: If $u = z\tan^{-1}\left(\frac{x}{y}\right)$, find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Solution: $$u = z\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = z\frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{zy}{y^2 + x^2}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{yz \cdot 2x}{(x^2 + y^2)^2} = -\frac{2xyz}{(x^2 + y^2)^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{z}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) = \frac{-xz}{y^2 + x^2}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{xz \cdot 2y}{(x^2 + y^2)^2} = \frac{2xyz}{(x^2 + y^2)^2}$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = \tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. z,

$$\frac{\partial^2 u}{\partial z^2} = 0$$

Hence, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{2xyz}{(x^2+y^2)^2} + \frac{2xyz}{(x^2+y^2)^2} = 0.$$

Example 20: If $v = (1-2xy+y^2)^{-\frac{1}{2}}$, find the value of

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right).$$

Solution: $$v = (1-2xy+y^2)^{-\frac{1}{2}}$$

Differentiating v partially w.r.t. x,

$$\frac{\partial v}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2y)$$

$$(1-x^2)\frac{\partial v}{\partial x} = y(1-x^2)(1-2xy+y^2)^{-\frac{3}{2}}$$

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] = y\frac{\partial}{\partial x}\left[(1-x^2)(1-2xy+y^2)^{-\frac{3}{2}}\right]$$

$$= y\left[(-2x)(1-2xy+y^2)^{-\frac{3}{2}} - \frac{3}{2}(1-x^2)(1-2xy+y^2)^{-\frac{5}{2}}(-2y)\right]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}[-2x(1-2xy+y^2)+3y(1-x^2)]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(-2x+4x^2y-2xy^2+3y-3x^2y)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(-2x+x^2y-2xy^2+3y) \qquad \text{... (1)}$$

Differentiating v partially w.r.t. y,

$$\frac{\partial v}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2x+2y)$$

$$y^2\frac{\partial v}{\partial y} = -y^2(-x+y)(1-2xy+y^2)^{-\frac{3}{2}}$$

$$\frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right) = -2y(-x+y)(1-2xy+y^2)^{-\frac{3}{2}} - y^2(1-2xy+y^2)^{-\frac{3}{2}}$$

$$+\frac{3y^2}{2}(-x+y)(1-2xy+y^2)^{-\frac{5}{2}}(-2x+2y)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}[2(x-y)(1-2xy+y^2) - y(1-2xy+y^2) + 3y(-x+y)^2]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(2x - 4x^2y + 2xy^2 - 2y + 4xy^2 - 2y^3 - y$$
$$+ 2xy^2 - y^3 + 3yx^2 + 3y^3 - 6xy^2)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(2x - x^2y + 2xy^2 - 3y) \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right) = 0.$$

Example 21: If $u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$,
show that $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0.$

Solution: $u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$

Differentiating u partially w.r.t. r,

$$\frac{\partial u}{\partial r} = (nar^{n-1} - bnr^{-n-1})(\cos n\theta + \sin n\theta)$$

Differentiating $\dfrac{\partial u}{\partial r}$ partially w.r.t. r,

$$\frac{\partial^2 u}{\partial r^2} = n[a(n-1)r^{n-2} + b(n+1)r^{-n-2}](\cos n\theta + \sin n\theta)$$

Differentiating u partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta} = (ar^n + br^{-n})(-n\sin n\theta + n\cos n\theta)$$

Differentiating $\dfrac{\partial u}{\partial \theta}$ partially w.r.t. θ,

$$\frac{\partial^2 u}{\partial \theta^2} = (ar^n + br^{-n})(-n^2\cos n\theta - n^2\sin n\theta)$$
$$= -n^2(ar^n + br^{-n})(\cos n\theta + \sin n\theta)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = n[a(n-1)r^{n-2} + b(n+1)r^{-n-2}](\cos n\theta + \sin n\theta)$$
$$+ n(ar^{n-2} - br^{-n-2})(\cos n\theta + \sin n\theta) - \frac{n^2}{r^2}(ar^n + br^{-n})(\cos n\theta + \sin n\theta)$$
$$= (\cos n\theta + \sin n\theta)r^{n-2}(an^2 - an + bn^2 + bn + an - bn - an^2 - bn^2)$$
$$= 0$$

Example 22: **Show that** $\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial \theta}$ **and** $\frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial \theta}$ **and hence, show that** $\frac{\partial^2 x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2}=0$ **if** $x=e^{r\cos\theta}\cos(r\sin\theta)$ **and** $y=e^{r\cos\theta}\sin(r\sin\theta)$.

Solution: $x=e^{r\cos\theta}\cos(r\sin\theta)$

Differentiating x partially w.r.t. r,

$$\begin{aligned}\frac{\partial x}{\partial r}&=e^{r\cos\theta}\cdot\cos\theta\cos(r\sin\theta)+e^{r\cos\theta}[-\sin(r\sin\theta)]\sin\theta\\&=e^{r\cos\theta}[\cos\theta\cos(r\sin\theta)-\sin\theta\sin(r\sin\theta)]\\&=e^{r\cos\theta}\cos(\theta+r\sin\theta)\qquad\qquad ...(1)\end{aligned}$$

$$y=e^{r\cos\theta}\sin(r\sin\theta)$$

Differentiating y partially w.r.t. r,

$$\begin{aligned}\frac{\partial y}{\partial r}&=e^{r\cos\theta}\cos\theta\sin(r\sin\theta)+e^{r\cos\theta}\cos(r\sin\theta)\sin\theta\\&=e^{r\cos\theta}\sin(r\sin\theta+\theta)\qquad\qquad ...(2)\end{aligned}$$

Differentiating x partially w.r.t. θ,

$$\begin{aligned}\frac{\partial x}{\partial \theta}&=e^{r\cos\theta}(-r\sin\theta)\cos(r\sin\theta)+e^{r\cos\theta}[-\sin(r\sin\theta)\cdot r\cos\theta)]\\&=-re^{r\cos\theta}\sin(\theta+r\sin\theta)\qquad\qquad ...(3)\end{aligned}$$

Differentiating y partially w.r.t. θ,

$$\begin{aligned}\frac{\partial y}{\partial \theta}&=e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta)+e^{r\cos\theta}\cos(r\sin\theta)\cdot r\cos\theta\\&=re^{r\cos\theta}\cos(\theta+r\sin\theta)\qquad\qquad ...(4)\end{aligned}$$

From Eqs (1) and (4), we get

$$\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial \theta}$$

From Eqs (2) and (3), we get

$$\frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial \theta},\qquad \frac{\partial x}{\partial \theta}=-r\frac{\partial y}{\partial r}$$

Differentiating $\frac{\partial x}{\partial r}$ partially w.r.t. r,

$$\frac{\partial^2 x}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial y}{\partial \theta}\right)=\frac{-1}{r^2}\frac{\partial y}{\partial \theta}+\frac{1}{r}\frac{\partial^2 y}{\partial r\,\partial \theta}$$

Differentiating $\frac{\partial x}{\partial \theta}$ partially w.r.t. θ,

$$\frac{\partial^2 x}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-r\frac{\partial y}{\partial r}\right) = -r\frac{\partial^2 y}{\partial \theta\,\partial r} = -r\frac{\partial^2 y}{\partial r\,\partial \theta}$$

Hence, $$\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = \frac{-1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r\,\partial \theta} + \frac{1}{r^2}\frac{\partial y}{\partial \theta} - \frac{1}{r}\frac{\partial^2 y}{\partial r\,\partial \theta} = 0.$$

Example 23: If $\theta = t^n e^{\frac{-r^2}{4t}}$, then find n so that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$.

Solution: $$\theta = t^n e^{\frac{-r^2}{4t}}$$

Differentiating θ partially w.r.t. t,

$$\frac{\partial \theta}{\partial t} = nt^{n-1}e^{\frac{-r^2}{4t}} + t^n e^{-\frac{r^2}{4t}}\left(\frac{r^2}{4t^2}\right) = e^{\frac{-r^2}{4t}}\left(nt^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)$$

Differentiating θ partially w.r.t. r,

$$\frac{\partial \theta}{\partial r} = t^n e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{t^{n-1}}{2}r^3 e^{\frac{-r^2}{4t}}\right)$$

$$= -\frac{t^{n-1}}{2}\left[3r^2 e^{\frac{-r^2}{4t}} + r^3 e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = e^{\frac{-r^2}{4t}}\left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)$$

Substituting in $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$,

$$e^{\frac{-r^2}{4t}}\left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right) = e^{\frac{-r^2}{4t}}\left(nt^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)$$

$$-\frac{3}{2}t^{n-1} = nt^{n-1}$$

$$n = -\frac{3}{2}.$$

Example 24: Find the value of n so that $v = r^n\,(3\cos^2\theta - 1)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\,\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial v}{\partial \theta}\right) = 0.$$

Solution: $$v = r^n (3\cos^2\theta - 1)$$

Differentiating v partially w.r.t. r,

$$\frac{\partial v}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = \frac{\partial}{\partial r}[nr^{n+1}(3\cos^2\theta - 1)]$$

$$= n(n+1)r^n(3\cos^2\theta - 1) \qquad \text{... (1)}$$

Differentiating v partially w.r.t. θ,

$$\frac{\partial v}{\partial \theta} = r^n \cdot 6\cos\theta(-\sin\theta)$$

$$= -3r^n \sin 2\theta$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = \frac{\partial}{\partial \theta}(-3r^n \sin\theta \cdot \sin 2\theta)$$

$$= -3r^n(\cos\theta\sin 2\theta + 2\sin\theta\cos 2\theta)$$

$$= -3r^n[\cos\theta \cdot 2\sin\theta\cos\theta + 2\sin\theta(2\cos^2\theta - 1)]$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = -3r^n(2\cos^2\theta + 4\cos^2\theta - 2) = -6r^n(3\cos^2\theta - 1)$$

Substituting in $\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v}{\partial \theta}\right) = 0$,

$$n(n+1)r^n(3\cos^2\theta - 1) - 6r^n(3\cos^2\theta - 1) = 0$$

$$n(n+1) - 6 = 0$$

$$n^2 + n - 6 = 0$$

$$(n+3)(n-2) = 0$$

$$n = -3,\ 2.$$

Example 25: If $x^x y^y z^z = c$, show that at $x = y = z$,

(a) $\dfrac{\partial^2 z}{\partial x\,\partial y} = -(x\log ex)^{-1}$ **(b)** $\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{2(x^2-2)}{x(1+\log x)}$.

Solution: (a) $x^x y^y z^z = c$

Taking logarithm on both the sides,

$$\log x^x + \log y^y + \log z^z = \log c$$

$$x\log x + y\log y + z\log z = \log c \qquad \text{... (1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$x \cdot \frac{1}{x} + \log x + z \cdot \frac{1}{z}\frac{\partial z}{\partial x} + \log z\frac{\partial z}{\partial x} = 0 \qquad [\because z = f(x, y)]$$

$$\frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -(1+\log x)\left[-\frac{1}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x}\right]$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{(1+\log x)}{z(1+\log z)^2}\left(-\frac{1+\log x}{1+\log z}\right)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{(1+\log x)^2}{z(1+\log z)^3}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)}$$

$$= -[x(\log e + \log x)]^{-1} \qquad [\because \log e = 1]$$

$$= -(x\log ex)^{-1}.$$

(b) Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{1+\log x}{1+\log z}\right)$$

$$= \frac{(1+\log x)}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x} - \frac{1}{x(1+\log z)}$$

$$= -\frac{(1+\log x)}{z(1+\log z)^2}\cdot\frac{(1+\log x)}{(1+\log z)} - \frac{1}{x(1+\log z)}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{x(1+\log x)}$$

Similarly,
$$\frac{\partial^2 z}{\partial y^2} = \frac{-(1+\log y)^2}{z(1+\log z)^3} - \frac{1}{y(1+\log z)}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial y^2} = -\frac{2}{x(1+\log x)}$$

Hence,
$$\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = \frac{-2}{x(1+\log x)} - 2xy\left[\frac{-1}{x(1+\log x)}\right] + \left[\frac{-2}{x(1+\log x)}\right]$$

$$= \frac{2(xy-2)}{x(1+\log x)} = \frac{2(x^2-2)}{x(1+\log x)} \qquad [\because x = y = z]$$

Example 26: If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, **prove that**

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Solution: $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$

Differentiating given equation partially w.r.t. x,

$$\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x}-\frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}-\frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x}=0$$

$$\frac{\partial u}{\partial x}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=\frac{2x}{a^2+u}$$

$$\frac{\partial u}{\partial x}\cdot p=\frac{2x}{(a^2+u)}$$

where,
$$p=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2x}{(a^2+u)p}$$

Similarly,
$$\frac{\partial u}{\partial y}=\frac{2y}{(b^2+u)p}$$

$$\frac{\partial u}{\partial z}=\frac{2z}{(c^2+u)p}$$

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{p^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$=\frac{4}{p^2}(p)=\frac{4}{p} \qquad \text{... (1)}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{2}{p}\left(\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right)$$

$$=\frac{2}{p}(1)=\frac{2}{p} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Example 27: If $u = \phi(x + ky) + \psi(x - ky)$, show that $\frac{\partial^2 u}{\partial y^2} = k^2 \frac{\partial^2 u}{\partial x^2}$.

Solution: $u = \phi(x + ky) + \psi(x - ky)$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \phi'(x+ky)\cdot 1 + \psi'(x-ky)\cdot 1$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \phi''(x+ky) + \psi''(x-ky) \quad \text{... (1)}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \phi'(x+ky)\cdot k + \psi'(x-ky)\cdot(-k)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \phi''(x+ky)\cdot k^2 + \psi''(x-ky)(-k)^2$$

$$= k^2[\phi''(x+ky) + \psi''(x-ky)] \quad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{\partial^2 u}{\partial y^2} = k^2 \frac{\partial^2 u}{\partial x^2}.$$

Example 28: If $u = xf(x+y) + y\phi(x+y)$, then show that $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution: $u = xf(x + y) + y\phi(x + y)$,

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = f(x+y) + xf'(x+y) + y\phi'(x+y)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = f'(x+y) + f'(x+y) + xf''(x+y) + y\phi''(x+y)$$

$$= 2f'(x+y) + xf''(x+y) + y\phi''(x+y)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = f'(x+y) + xf''(x+y) + y\phi''(x+y) + \phi'(x+y)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = xf'(x+y) + \phi(x+y) + y\phi'(x+y)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = xf''(x+y) + \phi'(x+y) + \phi'(x+y) + y\phi''(x+y)$$

$$= xf''(x+y) + 2\phi'(x+y) + y\phi''(x+y)$$

Hence, $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2}$

$$= 2f'(x+y) + xf''(x+y) + y\phi''(x+y) - 2f'(x+y) - 2xf''(x+y)$$
$$-2y\phi''(x+y) - 2\phi'(x+y) + xf''(x+y) + 2\phi'(x+y) + y\phi''(x+y) = 0$$

Example 29: If $u = f\left(\frac{x^2}{y}\right)$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x\,\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0$.

Solution: $u = f\left(\frac{x^2}{y}\right)$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = f'\left(\frac{x^2}{y}\right)\frac{\partial}{\partial x}\left(\frac{x^2}{y}\right) = f'\left(\frac{x^2}{y}\right)\left(\frac{2x}{y}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{2}{y}f'\left(\frac{x^2}{y}\right) + f''\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial x}\left(\frac{x^2}{y}\right)\cdot\left(\frac{2x}{y}\right)$$

$$= \frac{2}{y}f'\left(\frac{x^2}{y}\right) + f''\left(\frac{x^2}{y}\right)\cdot\left(\frac{2x}{y}\right)^2$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = f'\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x^2}{y}\right) = f'\left(\frac{x^2}{y}\right)\cdot\left(\frac{-x^2}{y^2}\right)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2}{y^3}f'\left(\frac{x^2}{y}\right) + \left(\frac{-x^2}{y^2}\right)f''\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x^2}{y}\right)$$

$$= \frac{2x^2}{y^3}f'\left(\frac{x^2}{y}\right) + \left(\frac{x^2}{y^2}\right)^2 f''\left(\frac{x^2}{y}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{2x}{y^2} f'\left(\frac{x^2}{y}\right) + \frac{2x}{y} f''\left(\frac{x^2}{y}\right)\cdot\left(\frac{-x^2}{y^2}\right)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x\,\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = \frac{2x^2}{y} f'\left(\frac{x^2}{y}\right) + \frac{4x^4}{y^2} f''\left(\frac{x^2}{y}\right) - \frac{6x^2}{y} f'\left(\frac{x^2}{y}\right)$$
$$-\frac{6x^4}{y^2} f''\left(\frac{x^2}{y}\right) + \frac{4x^2}{y} f'\left(\frac{x^2}{y}\right) + \frac{2x^4}{y^2} f''\left(\frac{x^2}{y}\right) = 0.$$

Example 30: If $u = e^{xyz} f\left(\frac{xy}{z}\right)$, prove that $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2\,xyzu$ and $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\,xyzu$ and hence, show that $x\frac{\partial^2 u}{\partial z\,\partial x} = y\frac{\partial^2 u}{\partial z\,\partial y}$.

Solution: $$u = e^{xyz} f\left(\frac{xy}{z}\right)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = e^{xyz} yz\cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)$$

$$\frac{\partial u}{\partial y} = e^{xyz} xz\cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)$$

$$\frac{\partial u}{\partial z} = e^{xyz} xy\cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right)$$

(i) $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z}$

$$= e^{xyz} xyzf\left(\frac{xy}{z}\right) + \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) + e^{xyz} xyz\cdot f\left(\frac{xy}{z}\right) - \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) = 2xyzu.$$

(ii) $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$= e^{xyz} xyz\cdot f\left(\frac{xy}{z}\right) + \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) + e^{xyz} xyz\cdot f\left(\frac{xy}{z}\right) - \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) = 2xyzu.$$

(iii) Differentiating $\frac{\partial u}{\partial z}$ w.r.t. x,

$$\frac{\partial^2 u}{\partial z\,\partial x} = e^{xyz} yz\cdot xy\,f\left(\frac{xy}{z}\right) + e^{xyz} y\cdot f\left(\frac{xy}{z}\right) + e^{xyz} xy\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)$$
$$+ e^{xyz} yz\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right) + e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)\left(\frac{-xy}{z^2}\right)$$
$$+ e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{y}{z^2}\right)$$

$$x\frac{\partial^2 u}{\partial z\,\partial x} = e^{xyz}\left[x^2y^2z\cdot f\left(\frac{xy}{z}\right)+xy\cdot f\left(\frac{xy}{z}\right)-\frac{x^2y^2}{z^3}f''\left(\frac{xy}{z}\right)-\frac{xy}{z^2}f'\left(\frac{xy}{z}\right)\right] \quad \text{... (1)}$$

Differentiating $\frac{\partial u}{\partial z}$ w.r.t. y,

$$\frac{\partial^2 u}{\partial z\,\partial y} = e^{xyz}xz\cdot xy\cdot f\left(\frac{xy}{z}\right)+e^{xyz}x\cdot f\left(\frac{xy}{z}\right)+e^{xyz}xy\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)$$

$$+\,e^{xyz}xz\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right)+e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)\left(\frac{-xy}{z^2}\right)$$

$$+\,e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{x}{z^2}\right)+e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)\left(\frac{-xy}{z^2}\right)+e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{x}{z^2}\right)$$

$$y\frac{\partial^2 u}{\partial z\,\partial y} = e^{xyz}\left[x^2y^2z\cdot f\left(\frac{xy}{z}\right)+xy\cdot f\left(\frac{xy}{z}\right)-\frac{x^2y^2}{z^3}f''\left(\frac{xy}{z}\right)-\frac{xy}{z^2}f'\left(\frac{xy}{z}\right)\right] \quad \text{... (2)}$$

From Eqs (1) and (2), we get

$$x\frac{\partial^2 u}{\partial z\,\partial x} = y\frac{\partial^2 u}{\partial z\,\partial y}.$$

Example 31: If $u = r^m$, $r = \sqrt{x^2+y^2+z^2}$, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$.

Solution:

$$u = r^m$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = mr^{m-1}\frac{\partial r}{\partial x}$$

But

$$r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = mr^{m-1}\frac{x}{r} = mr^{m-2}x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = m\left[r^{m-2} + (m-2)r^{m-3}\frac{\partial r}{\partial x}x\right]$$

$$= m\left[r^{m-2} + (m-2)r^{m-3}\frac{x}{r}x\right]$$

$$= m[r^{m-2} + (m-2)r^{m-4}x^2] \qquad \text{... (1)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = m[r^{m-2} + (m-2)r^{m-4}y^2] \qquad \text{... (2)}$$

$$\frac{\partial^2 u}{\partial z^2} = m[r^{m-2} + (m-2)r^{m-4}z^2] \qquad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3mr^{m-2} + m(m-2)r^{m-4}(x^2 + y^2 + z^2)$$

$$= 3mr^{m-2} + m(m-2)r^{m-4} \cdot r^2$$

$$= r^{m-2}(3m + m^2 - 2m)$$

$$= r^{m-2}(m + m^2)$$

$$= m(m+1)r^{m-2}.$$

Example 32: If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r}f'(r)$.

Solution: $u = f(r)$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}f(r) = \frac{\partial}{\partial r}f(r)\cdot\frac{\partial r}{\partial x} = f'(r)\cdot\frac{\partial r}{\partial x}$$

But $r^2 = x^2 + y^2 + z^2$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = f'(r)\cdot\frac{x}{r}$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right]$$

$$= f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r} + \frac{f'(r)}{r} + xf'(r)\left(\frac{-1}{r^2}\right)\cdot\frac{\partial r}{\partial x}$$

$$= f''(r)\frac{x}{r}\frac{x}{r} + \frac{f'(r)}{r} - \frac{x}{r^2}f'(r)\cdot\frac{x}{r}$$

$$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3}f'(r) \qquad \text{... (1)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3}f'(r) \qquad \text{... (2)}$$

and

$$\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3}f'(r) \qquad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}(x^2+y^2+z^2) + \frac{3f'(r)}{r} - \frac{(x^2+y^2+z^2)}{r^3}f'(r)$$

$$= \frac{f''(r)}{r^2}\cdot r^2 + \frac{3f'(r)}{r} - \frac{r^2}{r^3}f'(r)$$

$$= f''(r) + \frac{2f'(r)}{r}.$$

Example 33: If $u = f(r^2)$ where $r^2 = x^2 + y^2 + z^2$,

prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 4r^2 f''(r^2) + 6f'(r^2)$.

Solution: $u = f(r^2)$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}f(r^2) = \frac{\partial}{\partial x}f(l), \text{ where } r^2 = l$$

$$= \frac{\partial}{\partial l}f(l)\cdot\frac{\partial l}{\partial x} = f'(l)\frac{\partial l}{\partial x} = f'(r^2)\frac{\partial r^2}{\partial x}$$

$$= f'(r^2)2r\frac{\partial r}{\partial x}$$

But $r^2 = x^2 + y^2 + z^2$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = f'(r^2)\cdot 2r\frac{x}{r}$$
$$= 2xf'(r^2)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = 2f'(r^2) + 2x\frac{\partial f'(r^2)}{\partial x}$$
$$= 2f'(r^2) + 2xf''(r^2)\cdot 2r\frac{\partial r}{\partial x}$$
$$= 2f'(r^2) + 2xf''(r^2)\cdot 2r\frac{x}{r}$$
$$= 2f'(r^2) + 4x^2 f''(r^2) \qquad \text{... (1)}$$

Similarly,
$$\frac{\partial^2 u}{\partial y^2} = 2f'(r^2) + 4y^2 f''(r^2) \qquad \text{... (2)}$$

and
$$\frac{\partial^2 u}{\partial z^2} = 2f'(r^2) + 4z^2 f''(r^2) \qquad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6f'(r^2) + 4(x^2+y^2+z^2)f''(r^2) = 6f'(r^2) + 4r^2 f''(r^2)$$

Example 34: If $f(r) = r^{-\frac{1}{2}}(a+\log r)$, $r^2 = x^2+y^2+z^2$, prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -\frac{f(r)}{4r^2}$.

Solution:
$$f(r) = r^{-\frac{1}{2}}(a+\log r)$$

Differentiating f partially w.r.t. x,

$$\frac{\partial f}{\partial x} = -\frac{1}{2}r^{-\frac{3}{2}}\frac{\partial r}{\partial x}(a+\log r) + r^{-\frac{1}{2}}\cdot\frac{1}{r}\frac{\partial r}{\partial x}$$
$$= -\frac{1}{2}r^{-\frac{3}{2}}\cdot\frac{x}{r}(a+\log r) + r^{-\frac{3}{2}}\cdot\frac{x}{r} \qquad \left[\text{As proved earlier } \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$
$$= -\frac{xr^{-\frac{5}{2}}}{2}(a+\log r - 2)$$

Differentiating $\frac{\partial f}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 f}{\partial x^2} = -\frac{r^{-\frac{5}{2}}}{2}(a+\log r - 2) + \frac{x}{2}\cdot\frac{5}{2}r^{-\frac{7}{2}}\frac{\partial r}{\partial x}(a+\log r - 2) - \frac{xr^{-\frac{5}{2}}}{2}\cdot\frac{1}{r}\frac{\partial r}{\partial x}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)+\frac{5x}{4}r^{-\frac{7}{2}}\cdot\frac{x}{r}(a+\log r-2)-\frac{x}{2r}r^{-\frac{5}{2}}\frac{x}{r}$$

$$\left[\text{As proved earlier } \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$= -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)\left(1-\frac{5x^2}{2r^2}\right)-\frac{x^2}{2r^2}r^{-\frac{5}{2}}$$

Similarly, $$\frac{\partial^2 f}{\partial y^2} = -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)\left(1-\frac{5y^2}{2r^2}\right)-\frac{y^2}{2r^2}r^{-\frac{5}{2}}$$

and $$\frac{\partial^2 f}{\partial z^2} = -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)\left(1-\frac{5z^2}{2r^2}\right)-\frac{z^2}{2r^2}r^{-\frac{5}{2}}$$

Hence, $$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}(a+\log r-2)\left[3-\frac{5}{2r^2}(x^2+y^2+z^2)\right]-\frac{(x^2+y^2+z^2)}{2r^2}r^{-\frac{5}{2}}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}\left[r^{\frac{1}{2}}f(r)-2\right]\left(3-\frac{5\cdot r^2}{2r^2}\right)-\frac{r^2}{2r^2}r^{-\frac{5}{2}}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}\left[r^{\frac{1}{2}}f(r)-2\right]\left(3-\frac{5}{2}\right)-\frac{r^{-\frac{5}{2}}}{2}$$

$$= -\frac{r^{-\frac{5}{2}}}{2}\left[\frac{r^{\frac{1}{2}}f(r)}{2}-1+1\right]$$

$$= -\frac{f(r)}{4r^2}.$$

Example 35: If $v = x\log(x+r) - r$ where $r^2 = x^2+y^2$, prove that $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=\dfrac{1}{x+r}$.

Solution: $$v = x\log(x+r) - r$$

Differentiating v partially w.r.t. x,

$$\frac{\partial v}{\partial x} = \log(x+r)+\frac{x}{x+r}\left(1+\frac{\partial r}{\partial x}\right)-\frac{\partial r}{\partial x}$$

But $$r^2 = x^2+y^2$$

Differentiating r^2 partially w.r.t. x,

$$\frac{\partial r}{\partial x}=\frac{x}{r}$$

Differentiating r^2 partially w.r.t. y,

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial v}{\partial x} = \log(x+r) + \frac{x}{x+r}\left(1+\frac{x}{r}\right) - \frac{x}{r}$$

$$= \log(x+r) + \frac{x}{(x+r)} \cdot \frac{(r+x)}{r} - \frac{x}{r}$$

$$= \log(x+r) + \frac{x}{r} - \frac{x}{r}$$

$$= \log(x+r)$$

Differentiating $\frac{\partial v}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{x+r}\left(1+\frac{\partial r}{\partial x}\right) = \frac{1}{x+r}\left(1+\frac{x}{r}\right) = \frac{1}{r}$$

Differentiating v partially w.r.t. y,

$$\frac{\partial v}{\partial y} = \frac{x}{x+r} \cdot \frac{\partial r}{\partial y} - \frac{\partial r}{\partial y} = \frac{x}{x+r} \cdot \frac{y}{r} - \frac{y}{r}$$

$$= \frac{y}{r}\left(\frac{x-x-r}{x+r}\right)$$

$$= -\frac{y}{x+r}$$

Differentiating $\frac{\partial v}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 v}{\partial y^2} = -\frac{1}{x+r} + \frac{y}{(x+r)^2} \cdot \frac{\partial r}{\partial y} = -\frac{1}{x+r}\left(1 - \frac{y}{x+r} \cdot \frac{y}{r}\right)$$

$$= -\frac{1}{x+r}\left[\frac{rx+r^2-y^2}{r(x+r)}\right] = -\frac{1}{x+r}\left[\frac{rx+x^2}{r(x+r)}\right]$$

$$= -\frac{x}{r(x+r)}$$

Hence, $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{r}\left(1-\frac{x}{x+r}\right)$$

$$= \frac{1}{r}\left(\frac{x+r-x}{x+r}\right) = \frac{1}{x+r}.$$

4.4 VARIABLES TO BE TREATED AS CONSTANTS

In some problems, it is difficult to identify which variable is to be treated as constant. In such cases, the variable to be treated as constant is written as the suffix of the bracket.

Thus $\left(\frac{\partial r}{\partial x}\right)_y$ means that r is first to be expressed as a function of x and y and then differentiated w.r.t. x keeping y constant. Similarly, $\left(\frac{\partial x}{\partial r}\right)_\theta$ means that x is first to be expressed as a function of r and θ and then differentiated w.r.t. r keeping θ constant.

Example 1: If $x^2 = au + bv,\ y^2 = au - bv,$ prove that
$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$, where a, b are constants.

Solution:

$$x^2 = au + bv$$

$$2x\left(\frac{\partial x}{\partial u}\right)_v = a, \quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$$

$$y^2 = au - bv$$

$$2y\left(\frac{\partial y}{\partial v}\right)_u = -b, \quad \left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y}$$

Now,
$$x^2 = au + bv, \quad y^2 = au - bv$$

$$x^2 + y^2 = 2au,\ u = \frac{x^2 + y^2}{2a}, \quad \left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}$$

and
$$x^2 - y^2 = 2bv, v = \frac{x^2 - y^2}{2b}, \left(\frac{\partial v}{\partial y}\right)_x = -\frac{y}{b}$$

Hence,
$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a}\cdot\frac{a}{2x} = \frac{1}{2}$$

and
$$\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \left(-\frac{y}{b}\right)\left(-\frac{b}{2y}\right) = \frac{1}{2}.$$

Example 2: If $x = r\cos\theta,\ y = r\sin\theta,$ prove that

(a) $\left(\frac{\partial r}{\partial x}\right)_y = \left(\frac{\partial x}{\partial r}\right)_\theta$ **(b)** $r\left(\frac{\partial \theta}{\partial x}\right)_y = \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r$

(c) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ **(d)** $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$

Solution: (a) $x = r\cos\theta$, $y = r\sin\theta$,

$$x^2 + y^2 = r^2$$

Differentiating r^2 partially w.r.t. x keeping y constant,

$$2x = 2r\left(\frac{\partial r}{\partial x}\right)_y$$

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} \qquad \text{... (1)}$$

Again, $\qquad x = r\cos\theta$

Differentiating x partially w.r.t. r keeping θ constant,

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta = \frac{x}{r} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\left(\frac{\partial r}{\partial x}\right)_y = \left(\frac{\partial x}{\partial r}\right)_\theta$$

(b) $x = r\cos\theta$, $y = r\sin\theta$

Differentiating x partially w.r.t. θ keeping r constant,

$$\left(\frac{\partial x}{\partial \theta}\right)_r = -r\sin\theta$$

$$\frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r = -\sin\theta \qquad \text{... (3)}$$

Now, $\qquad \tan\theta = \frac{y}{x}$

Differentiating $\tan\theta$ partially w.r.t. x keeping y constant,

$$\sec^2\theta\left(\frac{\partial \theta}{\partial x}\right)_y = -\frac{y}{x^2}$$

$$\frac{r^2}{x^2}\left(\frac{\partial \theta}{\partial x}\right)_y = -\frac{r\sin\theta}{x^2}$$

$$r\left(\frac{\partial \theta}{\partial x}\right)_y = -\sin\theta = \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r \qquad \text{[From Eq. (3)]}$$

(c) $$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiating $\frac{\partial r}{\partial x}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} \qquad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

Similarly, $$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3} \qquad \left[\because \frac{\partial r}{\partial y} = \frac{y}{r}\right]$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{r^3} + \frac{x^2}{r^3} = \frac{1}{r}\left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right)$$

$$= \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right].$$

(d) $$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

Differentiating θ partially w.r.t. x,

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

Differentiating $\frac{\partial \theta}{\partial x}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = \frac{y}{(x^2+y^2)^2}\cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}$$

Differentiating θ partially w.r.t. y,

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial y}\right) = -\frac{x\cdot 2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 \theta}{\partial y^2} = -\frac{2xy}{(x^2+y^2)^2}$$

Hence, $$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$$

Example 3: If $ux + vy = 0$ and $\frac{u}{x} + \frac{v}{y} = 1$, then prove that

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2+y^2}{y^2-x^2}.$$

Solution:
$$ux + vy = 0 \qquad \ldots (1)$$
$$\frac{u}{x} + \frac{v}{y} = 1 \qquad \ldots (2)$$

From Eq. (1),
$$u = -\frac{-vy}{x}$$

Substituting in Eq. (2),
$$\frac{-vy}{x^2} + \frac{v}{y} = 1$$
$$v(-y^2 + x^2) = x^2 y$$
$$v = \frac{x^2 y}{x^2 - y^2}$$

Differentiating v partially w.r.t. y keeping x constant,
$$\left(\frac{\partial v}{\partial y}\right) = x^2\left[\frac{1}{x^2 - y^2} - \frac{y}{(x^2 - y^2)^2}(-2y)\right]$$
$$= x^2\left[\frac{x^2 - y^2 + 2y^2}{(x^2 - y^2)^2}\right] = \frac{x^2(x^2 + y^2)}{(x^2 - y^2)^2}$$
$$= \frac{x^2(x^2 + y^2)}{(x^2 - y^2)^2}$$

From Eq. (1),
$$v = -\frac{ux}{y}$$

Substituting in Eq. (2),
$$\frac{u}{x} - \frac{ux}{y^2} = 1$$
$$u(y^2 - x^2) = xy^2$$
$$u = \frac{xy^2}{y^2 - x^2}$$

Differentiating u w.r.t. x keeping y constant,
$$\left(\frac{\partial u}{\partial x}\right)_y = y^2\left[\frac{1}{y^2 - x^2} - \frac{x}{(y^2 - x^2)^2}(-2x)\right]$$
$$= \left[\frac{y^2(x^2 + y^2)}{(y^2 - x^2)^2}\right]$$

Hence,
$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{(x^2 + y^2)(y^2 - x^2)}{(y^2 - x^2)^2}$$
$$= \frac{x^2 + y^2}{y^2 - x^2}.$$

Exercise 4.1

1. If $u = \cos\left(\sqrt{x}+\sqrt{y}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\sin\left(\sqrt{x}+\sqrt{y}\right)=0.$$

2. If $z^3 - xz - y = 0$, prove that

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{3z^2+x}{(3z^2-x)^3}.$$

3. If $z = \tan(y+ax)+(y-ax)^{\frac{3}{2}}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}$.

4. If $u = 2(ax+by)^2 - k(x^2+y^2)$ and $a^2+b^2=k$, find the value of $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$.

[**Ans. :** 0]

5. If $e^u = \tan x + \tan y$, show that

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}=2.$$

6. If $z^3 - 3yz - 3x = 0$, show that

(i) $z\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y}$

(ii) $z\left[\dfrac{\partial^2 z}{\partial x\,\partial y}+\left(\dfrac{\partial z}{\partial x}\right)^2\right] = \dfrac{\partial^2 z}{\partial y^2}$.

7. If $z(z^2+3x)+3y=0$, prove that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{2z(x-1)}{(z^2+x)^3}.$$

8. If $u = \log(x^2+y^2)+\tan^{-1}\left(\dfrac{y}{x}\right)$, show that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$.

9. If $u(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$, find the value of $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$.

$$\left[\textbf{Ans.:}\ \frac{2}{(x^2+y^2+z^2)^2}\right]$$

10. If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that

$$\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial \theta},\quad \frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial \theta}$$

Hence, deduce that

$$\frac{\partial^2 x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2}=0.$$

11. If $v = (x^2-y^2)f(x,y)$, prove that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=(x^4-y^4)f''(x,y).$$

12. If $u = f(ax^2+2hxy+by^2)$ and $v = \phi(ax^2+2hxy+by^2)$, show that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

13. If $x = \dfrac{r}{2}(e^{\theta}+e^{-\theta})$, $y = \dfrac{r}{2}(e^{\theta}-e^{-\theta})$, prove that $\left(\dfrac{\partial x}{\partial r}\right)_{\theta} = \left(\dfrac{\partial r}{\partial x}\right)_{y}$.

$$\left[\textbf{Hint:}\ x = r\cosh\theta,\ y = r\sinh\theta,\ x^2-y^2=r^2\right]$$

14. If $\log_e\theta = r - x$, $r^2 = x^2+y^2$, show that $\dfrac{\partial^2\theta}{\partial y^2} = \dfrac{\theta(x^2+ry^2)}{r^3}$.

$$\left[\textbf{Hint:}\ \theta = e^{r-x},\ \frac{\partial r}{\partial y}=\frac{y}{r}\right]$$

15. If $u = e^{ax}\sin by$, prove that

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}.$$

16. If $u = \tan^{-1}\left(\dfrac{xy}{\sqrt{1+x^2+y^2}}\right)$, prove that

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}.$$

17. If $u = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}}$, prove that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

18. If $u = \tan(y+ax) - (y-ax)^{\frac{3}{2}}$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

19. If $u = \frac{xy}{2x+z}$, prove that

$$\frac{\partial^3 u}{\partial y \partial z^2} = \frac{\partial^3 u}{\partial z^2 \partial y}.$$

20. If $u = x^m y^n$, prove that

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial^3 u}{\partial y\,\partial x^2}.$$

21. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following functions:

(i) $\sqrt{x+y-1}$

(ii) $\sqrt{1-x^2-y^2}$

(iii) y^x

(iv) $\log_{10}(ax+by)$

(v) $(y-ax)^{\frac{3}{2}}$.

Ans. :

(i) $\frac{1}{\sqrt{x+y-1}}, \frac{1}{\sqrt{x+y-1}}$

(ii) $\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}$

(iii) $y^x \log y,\ xy^{x-1}$

(iv) $\frac{a}{(\log_e 10)(ax+by)}$, $\frac{b}{(\log_e 10)(ax+by)}$

(v) $-\frac{3a}{2}(y-ax)^{\frac{1}{2}},\ \frac{3}{2}(y-ax)^{\frac{1}{2}}$

22. If $x^4 - xy^2 + yz^2 - z^4 = 6$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\left[\textbf{Ans.:}\ \frac{y^2-4x^3}{2yz-4z^3}, \frac{2xy-z^2}{2yz-4z^3}\right]$$

23. If $z^3 + xy - y^2z = 6$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at (0, 1, 2).

$$\left[\textbf{Ans.:}\ -\frac{1}{11}, \frac{4}{11}\right]$$

24. Find the value of n for which $u = kt^{-\frac{1}{2}} e^{-\frac{x^2}{na^2 t}}$ satisfies the partial differential equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$.

[**Ans. :** $n = 4$]

25. Find the value of n for which $u = t^n e^{-\frac{r^2}{4kt}}$ satisfies the partial differential equation $\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$.

$$\left[\textbf{Ans.:}\ n = -\frac{3}{2}\right]$$

26. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}$ in terms of r, θ, ϕ.

Hint: $r^2 = x^2 + y^2 + z^2$,

$\phi = \tan^{-1}\frac{y}{x}$,

$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$

Ans.: $\sin\theta\cos\phi$, $\frac{\cos\theta\cos\phi}{r}$, $\frac{-\sin\phi}{r\sin\theta}$

27. If $u = x^2(y - z) + y^2(z - x) + z^2(x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

28. If $u = e^x (x \cos y - y \sin y)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

29. Prove that $f(x, y, z) = z \tan^{-1} \frac{y}{x}$ is a harmonic function.

[**Hint :** Prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$]

30. If $z(x + y) = x^2 + y^2$, prove that

$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\, \partial y} = 2\frac{\partial z}{\partial x}.$$

31. If $\frac{x^2}{2+u} + \frac{y^2}{4+u} + \frac{z^2}{6+u} = 1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

32. If $u = (x^2 - y^2) f(r)$, where $r = xy$, show that $\frac{\partial^2 u}{\partial x\, \partial y} = (x^2 - y^2)[3f'(r) + rf''(r)]$.

33. If $z = f(x^2, y)$, prove that $x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$.

34. If $z = e^{ax+by} f(ax - by)$, where a, b are constants, prove that

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

35. Prove that $z = \frac{1}{r}[f(ct + r) + \phi(ct - r)]$ satisfies the partial differential equation $\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$ where c is constant.

36. If $u + iv = f(x + iy)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

[**Hint :** $u + iv = f(x + iy)$,

$u - iv = f(x - iy)$

$u = \frac{1}{2}[f(x + iy) + f(x - iy)]$,

$v = \frac{1}{2i}[f(x + iy) - f(x - iy)]$]

37. If u, v, w are function of x, y, z given as $x = u + v + w$,

$y = u^2 + v^2 + w^2$,

$z = u^3 + v^3 + w^3$,

prove that

$$\frac{\partial u}{\partial x} = \frac{vw(w - v)}{(u - v)(v - w)(w - u)}.$$

[**Hint :** Differentiate x, y, z w.r.t. x and solve the equations using Cramer's rule]

38. If $u = (x^2 + y^2 + z^2)^{\frac{n}{2}}$, find the value of n which satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

[Ans.: 0, –1]

39. If $u = \log(e^x + e^y)$, show that

$$\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2 = 0.$$

40. If $z = y f(x^2 - y^2)$, show that

$$y\left(\frac{\partial z}{\partial x}\right) + x\left(\frac{\partial z}{\partial y}\right) = \frac{xz}{y}.$$

4.5 COMPOSITE FUNCTION

4.5.1 Chain Rule

If $z = f(u)$, where u is again a function of two variables x and y, i.e., $u = \phi(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \text{ or } \frac{df}{du} \cdot \frac{\partial u}{\partial x} \text{ or } f'(u)\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} \text{ or } \frac{df}{du} \cdot \frac{\partial u}{\partial y} \text{ or } f'(u)\frac{\partial u}{\partial y}.$$

4.5.2 Composite Function of One Variable or Total Differential Coefficient

If $u = f(x, y)$, where $x = \phi(t)$, $y = \psi(t)$, then z is a function of t and is called composite function of a single variable t and

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

is called total differential of u.

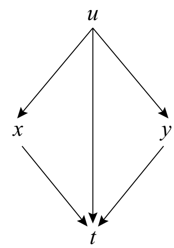


Fig. 4.2

If $u = f(x, y, z)$ and $x = \phi(t)$, $y = \psi(t)$, $z = \xi(t)$, then total differential of u is given as

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}.$$

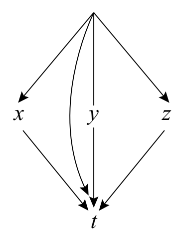


Fig. 4.3

4.5.3 Composite Function of Two Variables

If $z = f(x, y)$, where $x = \phi(u, v)$, $y = \psi(u, v)$, then z is a function of u, v and is called composite function of two variables u and v.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

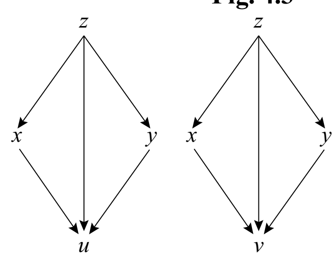


Fig. 4.4

Example 1: If $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$, find $\frac{dz}{dt}$.

Solution: $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$

We know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (y^2 + 2xy)\,2at + (2xy + x^2)\,2a$$

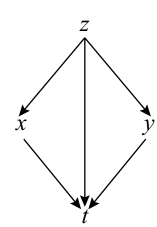


Fig. 4.5

Substituting x, y and z,

$$\begin{aligned}\frac{dz}{dt} &= (4a^2t^2 + 2at^2 \cdot 2at)\,2at + (2at^2 \cdot 2at + a^2t^4)2a \\ &= 4a^2t^2(1+t)\,2at + a^2t^3(4+t)2a \\ &= 8a^3t^3(1+t) + 2a^3t^3(4+t) \\ &= 2a^3t^3(8+5t).\end{aligned}$$

Example 2: If $z = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, prove that $\dfrac{dz}{dt} = \dfrac{3}{\sqrt{1-t^2}}$.

Solution: $z = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$

We know that

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \\ &= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 3 + \frac{1(-1)}{\sqrt{1-(x-y)^2}}\cdot 12t^2 \\ &= \frac{3-12t^2}{\sqrt{1-x^2-y^2+2xy}}\end{aligned}$$

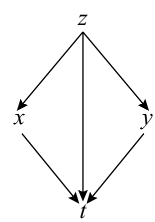


Fig. 4.6

Substituting x and y,

$$\begin{aligned}\frac{dz}{dt} &= \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}} = \frac{3(1-4t^2)}{\sqrt{1-8t^2+16t^4-t^2-16t^6+8t^4}} \\ &= \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2 - t^2(1+16t^4-8t^2)}} = \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2 - t^2(1-4t^2)^2}} \\ &= \frac{3(1-4t^2)}{(1-4t^2)\sqrt{1-t^2}} = \frac{3}{\sqrt{1-t^2}}.\end{aligned}$$

Example 3: If $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$, find $\dfrac{du}{dt}$.

Solution: $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$

We know that

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \\ &= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)(e^t + e^{-t}) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)(e^t - e^{-t}) \\ &= \frac{-y}{x^2+y^2}\cdot y + \frac{x}{x^2+y^2}\cdot x = \frac{x^2-y^2}{x^2+y^2} = \frac{(e^t-e^{-t})^2 - (e^t+e^{-t})^2}{(e^t-e^{-t})^2 + (e^t+e^{-t})^2} \\ &= \frac{-4}{2(e^{2t}+e^{-2t})} = -\frac{2}{e^{2t}+e^{-2t}}.\end{aligned}$$

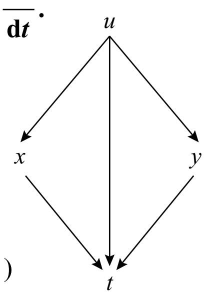


Fig. 4.7

Example 4: **If $u = x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dz}{\sqrt{1-z^2}} = 0$.**

Solution: $u = x^2 + y^2 + z^2 - 2xyz = 1$

We know that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$(2x - 2yz)dx + (2y - 2xz)dy + (2z - 2xy)dz = 0$$

$$(x - yz)dx + (y - xz)dy + (z - xy)dz = 0 \qquad \text{... (1)}$$

We have,

$$x^2 + y^2 + z^2 - 2xyz = 1$$

$$x^2 - 2xyz = 1 - y^2 - z^2$$

$$x^2 - 2xyz + y^2z^2 = 1 - y^2 - z^2 + y^2z^2$$

$$(x - yz)^2 = (1 - y^2)(1 - z^2)$$

$$x - yz = \sqrt{1-y^2}\cdot\sqrt{1-z^2}$$

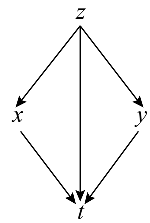


Fig. 4.8

Similarly,

$$y - xz = \sqrt{1-x^2}\cdot\sqrt{1-z^2}$$

and

$$z - xy = \sqrt{1-x^2}\cdot\sqrt{1-y^2}$$

Substituting in Eq. (1),

$$\sqrt{1-y^2}\cdot\sqrt{1-z^2}\,dx + \sqrt{1-x^2}\cdot\sqrt{1-z^2}\,dy + \sqrt{1-x^2}\cdot\sqrt{1-y^2}\,dz = 0$$

$$\sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2}\left(\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}}\right) = 0$$

Hence,

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

Example 5: **If $u = x^2 + y^2 + z^2$, where $x = e^t, y = e^t \sin t, z = e^t \cos t$, find $\dfrac{du}{dt}$.**

Solution: $u = x^2 + y^2 + z^2, x = e^t, y = e^t \sin t, z = e^t \cos t$

We know that

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= 2xe^t + 2y(e^t \sin t + e^t \cos t) + 2z(e^t \cos t - e^t \sin t)$$

$$= 2e^t \cdot e^t + 2e^t \sin t \cdot e^t(\sin t + \cos t) + 2e^t \cos t \cdot e^t(\cos t - \sin t)$$

$$= 2e^{2t}(1 + \sin^2 t + \sin t \cos t + \cos^2 t - \cos t \sin t)$$

$$= 4e^{2t}$$

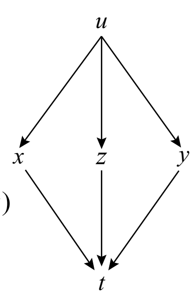


Fig. 4.9

Example 6: **If $z = e^{xy}, x = t \cos t, y = t \sin t$, find $\dfrac{dz}{dt}$ at $t = \dfrac{\pi}{2}$.**

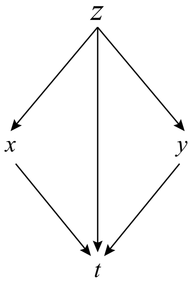


Fig. 4.10

Solution: $$z = e^{xy}, x = t \cos t, y = t \sin t$$

We know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= e^{xy}[y(\cos t - t \sin t) + e^{xy}\, x(\sin t + t \cos t)]$$

At $$t = \frac{\pi}{2},\ x = 0,\ y = \frac{\pi}{2}$$

Hence, $$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = e^0\left[\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right]$$

$$= -\frac{\pi^2}{4}\cdot$$

Example 7: **If $z = f(u, v), u = \log(x^2 + y^2), v = \dfrac{y}{x}$, show that $x\dfrac{\partial z}{\partial y} - y\dfrac{\partial z}{\partial x} = (1+v^2)\dfrac{\partial z}{\partial v}$.**

Solution: $$z = f(u, v), u = \log(x^2 + y^2),\ v = \frac{y}{x},$$

We know that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{1}{x^2+y^2}\cdot 2x + \frac{\partial z}{\partial v}\left(\frac{-y}{x^2}\right)$$

$$y\frac{\partial z}{\partial x} = \frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u} - \frac{y^2}{x^2}\cdot\frac{\partial z}{\partial v}$$

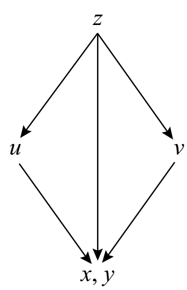


Fig. 4.11

and $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{2y}{x^2+y^2} + \frac{\partial z}{\partial v}\cdot\frac{1}{x}$$

$$x\frac{\partial z}{\partial y} = \frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

Hence, $$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} + \frac{y^2}{x^2}\frac{\partial z}{\partial v} = (1+v^2)\frac{\partial z}{\partial v}.$$

Example 8: **If $w = \phi(u, v), u = x^2 - y^2 - 2xy, v = y$, prove that $\dfrac{\partial w}{\partial v} = 0$ is equivalent to $(x+y)\dfrac{\partial w}{\partial x} + (x-y)\dfrac{\partial w}{\partial y} = 0$.**

Solution: $$w = \phi(u, v), u = x^2 - y^2 - 2xy, v = y$$

We know that

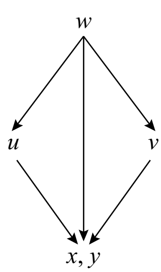


Fig. 4.12

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial w}{\partial u}(2x - 2y) + \frac{\partial w}{\partial v}\cdot 0$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}(2x - 2y)$$

and $$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}(-2y - 2x) + \frac{\partial w}{\partial v}\cdot 1$$

$$\frac{\partial w}{\partial y} = -2(x+y)\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

If $$\frac{\partial w}{\partial v} = 0,$$

then $$\frac{\partial w}{\partial x} = 2(x-y)\frac{\partial w}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial y} = -2(x+y)\frac{\partial w}{\partial u}$$

$$(x+y)\frac{\partial w}{\partial x} + (x-y)\frac{\partial w}{\partial y} = (x+y)\,2(x-y)\frac{\partial w}{\partial u} - (x-y)\,2(x+y)\frac{\partial w}{\partial u} = 0$$

Hence, $\frac{\partial w}{\partial v} = 0$ is equivalent to $(x+y)\frac{\partial w}{\partial x} + (x-y)\frac{\partial w}{\partial y} = 0.$

Example 9: If $z = f(x, y)$ and $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Solution: $z = f(x, y), \quad x = e^u + e^{-v}, \quad y = e^{-u} - e^v$

$z = f(x, y)$

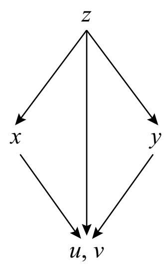


Fig. 4.13

We know that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u})$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$$

Hence, $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v) = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Example 10: If $z = f(x, y)$, $x = u \cosh v$, $y = u \sinh v$,

prove that $$\left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2.$$

Solution: $z = f(x, y), \quad x = u\cosh v, \quad y = u\sinh v$

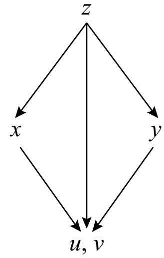


Fig. 4.14

We know that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cosh v + \frac{\partial z}{\partial y}\sinh v$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}u\sinh v + \frac{\partial z}{\partial y}u\cosh v$$

$$\left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cosh^2 v + \left(\frac{\partial z}{\partial y}\right)^2\sinh^2 v + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v$$

$$-\left(\frac{\partial z}{\partial x}\right)^2\sinh^2 v - \left(\frac{\partial z}{\partial y}\right)^2\cosh^2 v - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 (\cosh^2 v - \sinh^2 v) - \left(\frac{\partial z}{\partial y}\right)^2 (\cosh^2 v - \sinh^2 v)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2.$$

Example 11: If $x = r\cos\theta, y = r\sin\theta$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

Solution: Let $z = f(r,\theta)$

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta,$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

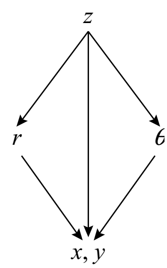


Fig. 4.15

We know that
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial \theta}\left(\frac{-\sin\theta}{r}\right)$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\cdot\frac{\cos\theta}{r}$$

Hence,
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\cos\theta - \frac{\partial z}{\partial \theta}\cdot\frac{\sin\theta}{r}\right)^2 + \left(\frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\cdot\frac{\cos\theta}{r}\right)^2$$

$$= \left(\frac{\partial z}{\partial r}\right)^2 (\cos^2\theta + \sin^2\theta) + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 (\sin^2\theta + \cos^2\theta)$$

$$- \frac{2}{r}\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial \theta}\sin\theta\cos\theta + \frac{2}{r}\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial \theta}\sin\theta\cos\theta$$

$$= \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

Example 12: If $z = f(u, v)$, and $u = x^2 - y^2, v = 2xy$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(u^2+v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Solution: $z = f(u, v)$, and $u = x^2 - y^2$, $v = 2xy$

We know that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot 2x + \frac{\partial z}{\partial v}\cdot 2y = 2\left(x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)$$

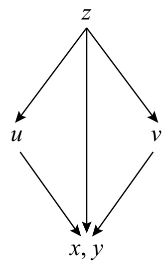


Fig. 4.16

and
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}(2x) = 2\left(-y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)$$

Hence,
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\left(x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)^2 + 4\left(-y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)^2$$

$$= 4\left[x^2\left(\frac{\partial z}{\partial u}\right)^2 + y^2\left(\frac{\partial z}{\partial v}\right)^2 + 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} + y^2\left(\frac{\partial z}{\partial u}\right)^2 \right.$$
$$\left. + x^2\left(\frac{\partial z}{\partial v}\right)^2 - 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right]$$

$$= 4(x^2 + y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2 + y^2)^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2 - y^2)^2 + 4x^2y^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(u^2 + v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Example 13: If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Solution: Let
$$l = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}, \quad m = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial l}{\partial x} = \frac{-1}{x^2}, \quad \frac{\partial l}{\partial y} = \frac{1}{y^2}, \quad \frac{\partial l}{\partial z} = 0$$

and
$$\frac{\partial m}{\partial x} = \frac{-1}{x^2}, \quad \frac{\partial m}{\partial y} = 0, \quad \frac{\partial m}{\partial z} = \frac{1}{z^2}$$

$$u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right) = f(l, m)$$

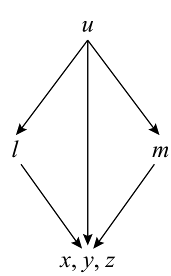


Fig. 4.17

We know that

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x}=\frac{\partial u}{\partial l}\left(\frac{-1}{x^2}\right)+\frac{\partial u}{\partial m}\left(\frac{-1}{x^2}\right)$$

$$x^2\frac{\partial u}{\partial x}=-\left(\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\right) \qquad \text{... (1)}$$

Also,

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}=\frac{\partial u}{\partial l}\left(\frac{1}{y^2}\right)+\frac{\partial u}{\partial m}\cdot 0$$

$$y^2\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l} \qquad \text{... (2)}$$

and

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}=\frac{\partial u}{\partial l}\cdot 0+\frac{\partial u}{\partial m}\left(\frac{1}{z^2}\right)$$

$$z^2\frac{\partial u}{\partial z}=\frac{\partial u}{\partial m} \qquad \text{... (3)}$$

Adding Eqs (1), (2) and (3),

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=-\left(\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\right)+\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}=0.$$

Example 14: If $u=f\left(\frac{x}{y},\ \frac{y}{z},\ \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.

Solution: Let $\frac{x}{y}=l, \qquad \frac{y}{z}=m, \qquad \frac{z}{x}=n$

$$\frac{\partial l}{\partial x}=\frac{1}{y}, \qquad \frac{\partial l}{\partial y}=\frac{-x}{y^2}, \qquad \frac{\partial l}{\partial z}=0$$

$$\frac{\partial m}{\partial x}=0, \qquad \frac{\partial m}{\partial y}=\frac{1}{z}, \qquad \frac{\partial m}{\partial z}=\frac{-y}{z^2}$$

$$\frac{\partial n}{\partial x}=-\frac{z}{x^2}, \qquad \frac{\partial n}{\partial y}=0, \qquad \frac{\partial n}{\partial z}=\frac{1}{x}$$

We know that

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$=\frac{\partial u}{\partial l}\cdot\frac{1}{y}+\frac{\partial u}{\partial m}\cdot 0+\frac{\partial u}{\partial n}\left(\frac{-z}{x^2}\right)$$

$$x\frac{\partial u}{\partial x}=\frac{x}{y}\cdot\frac{\partial u}{\partial l}-\frac{z}{x}\cdot\frac{\partial u}{\partial n}$$

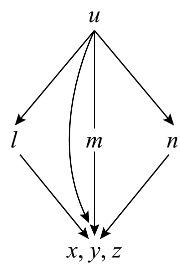


Fig. 4.18

Also,

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial m}\cdot\frac{1}{z} + \frac{\partial u}{\partial n}\cdot 0$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{y}\cdot\frac{\partial u}{\partial l} + \frac{y}{z}\cdot\frac{\partial u}{\partial m}$$

and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\quad\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial n}\cdot\frac{1}{x}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial m} + \frac{z}{x}\cdot\frac{\partial u}{\partial n}$$

Hence,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Example 15: If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.

Solution: Let $x^2 - y^2 = l,\quad y^2 - z^2 = m,\quad z^2 - x^2 = n$

$$\frac{\partial l}{\partial x} = 2x, \qquad \frac{\partial l}{\partial y} = -2y, \qquad \frac{\partial l}{\partial z} = 0$$

$$\frac{\partial m}{\partial x} = 0, \qquad \frac{\partial m}{\partial y} = 2y, \qquad \frac{\partial m}{\partial z} = -2z$$

$$\frac{\partial n}{\partial x} = -2x, \qquad \frac{\partial n}{\partial y} = 0, \qquad \frac{\partial n}{\partial z} = 2z$$

We know that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot 2x + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}(-2x)$$

$$\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial u}{\partial n}$$

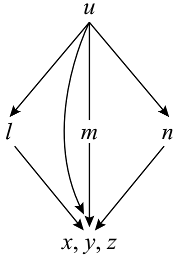


Fig. 4.19

Also,
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}(-2y) + \frac{\partial u}{\partial m}(2y) + \frac{\partial u}{\partial n}(0)$$

$$\frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial l} + 2\frac{\partial u}{\partial m}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$
$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial n}(2z)$$
$$\frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n}$$
$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0.$$

Example 16: If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Solution: Let $e^{y-z} = l,\ e^{z-x} = m,\ e^{x-y} = n$

$$\frac{\partial l}{\partial x} = 0, \qquad \frac{\partial l}{\partial y} = e^{y-z} = l, \qquad \frac{\partial l}{\partial z} = -e^{y-z} = -l$$
$$\frac{\partial m}{\partial x} = -e^{z-x} = -m, \qquad \frac{\partial m}{\partial y} = 0, \qquad \frac{\partial m}{\partial z} = e^{z-x} = m$$
$$\frac{\partial n}{\partial x} = e^{x-y} = n, \qquad \frac{\partial n}{\partial y} = -e^{x-y} = -n, \qquad \frac{\partial n}{\partial z} = 0$$
$$u = f(e^{y-z}, e^{z-x}, e^{x-y}) = f(l, m, n).$$

We know that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$
$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-m) + \frac{\partial u}{\partial n}\cdot n$$
$$= -m\frac{\partial u}{\partial m} + n\frac{\partial u}{\partial n}$$

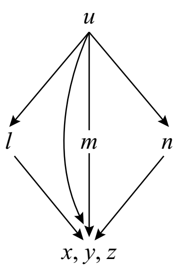


Fig. 4.20

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$
$$= \frac{\partial u}{\partial l}\cdot l + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-n)$$
$$= l\frac{\partial u}{\partial l} - n\frac{\partial u}{\partial n}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$
$$= \frac{\partial u}{\partial l}(-l) + \frac{\partial u}{\partial m}\cdot m + \frac{\partial u}{\partial n}\cdot 0 = -l\frac{\partial u}{\partial l} + m\frac{\partial u}{\partial m}$$

Hence, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Example 17: **If $x=\sqrt{vw},\ y=\sqrt{wu},\ z=\sqrt{uv}$ and ϕ is a function of x, y and z, then prove that $x\dfrac{\partial\phi}{\partial x}+y\dfrac{\partial\phi}{\partial y}+z\dfrac{\partial\phi}{\partial z}=u\dfrac{\partial\phi}{\partial u}+v\dfrac{\partial\phi}{\partial v}+w\dfrac{\partial\phi}{\partial w}$.**

Solution:

$$x=\sqrt{vw}$$

$$\frac{\partial x}{\partial u}=0,\qquad \frac{\partial x}{\partial v}=\frac{1}{2}\sqrt{\frac{w}{v}},\qquad \frac{\partial x}{\partial w}=\frac{1}{2}\sqrt{\frac{v}{w}}$$

$$y=\sqrt{wu}$$

$$\frac{\partial y}{\partial u}=\frac{1}{2}\sqrt{\frac{w}{u}},\qquad \frac{\partial y}{\partial v}=0,\qquad \frac{\partial y}{\partial w}=\frac{1}{2}\sqrt{\frac{u}{w}}$$

$$z=\sqrt{uv}$$

$$\frac{\partial z}{\partial u}=\frac{1}{2}\sqrt{\frac{v}{u}},\quad \frac{\partial z}{\partial v}=\frac{1}{2}\sqrt{\frac{u}{v}},\quad \frac{\partial z}{\partial w}=0$$

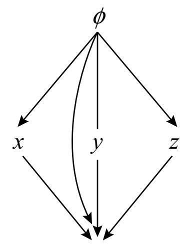


Fig. 4.21

We know that

$$\frac{\partial\phi}{\partial u}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$=\frac{\partial\phi}{\partial x}\cdot 0+\frac{\partial\phi}{\partial y}\cdot\frac{1}{2}\sqrt{\frac{w}{u}}+\frac{\partial\phi}{\partial z}\cdot\frac{1}{2}\sqrt{\frac{v}{u}}$$

$$u\frac{\partial\phi}{\partial u}=\frac{1}{2}\left[\frac{\partial\phi}{\partial y}\sqrt{uw}+\frac{\partial\phi}{\partial z}\sqrt{uv}\right]=\frac{1}{2}\left(y\frac{\partial\phi}{\partial y}+z\frac{\partial\phi}{\partial z}\right)$$

Also,

$$\frac{\partial\phi}{\partial v}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$=\frac{\partial\phi}{\partial x}\cdot\frac{1}{2}\sqrt{\frac{w}{v}}+\frac{\partial\phi}{\partial y}\cdot 0+\frac{\partial\phi}{\partial z}\frac{1}{2}\sqrt{\frac{u}{v}}$$

$$v\frac{\partial\phi}{\partial v}=\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\sqrt{vw}+\frac{\partial\phi}{\partial z}\sqrt{uv}\right)$$

$$=\frac{1}{2}\left(x\frac{\partial\phi}{\partial x}+z\frac{\partial\phi}{\partial z}\right)$$

and

$$\frac{\partial\phi}{\partial w}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial w}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial w}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial w}$$

$$=\frac{\partial\phi}{\partial x}\cdot\frac{1}{2}\sqrt{\frac{v}{w}}+\frac{\partial\phi}{\partial y}\frac{1}{2}\sqrt{\frac{u}{w}}+\frac{\partial\phi}{\partial z}\cdot 0$$

$$w\frac{\partial\phi}{\partial w}=\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\sqrt{vw}+\frac{\partial\phi}{\partial y}\sqrt{uw}\right)$$

$$=\frac{1}{2}\left(x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}\right)$$

Hence, $$u\frac{\partial\phi}{\partial u}+v\frac{\partial\phi}{\partial v}+w\frac{\partial\phi}{\partial w}=x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}+z\frac{\partial\phi}{\partial z}.$$

Example 18: If $f(xy^2, z-2x)=0$, show that $2x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=4x$.

Solution: Let $l=xy^2,\ \ m=z-2x,\ \ f(l,m)=0$

We know that

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial x}+\frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial x}=0 \qquad [\because f(xy^2,\ z-2x)=0]$$

$$\frac{\partial f}{\partial l}(y^2)+\frac{\partial f}{\partial m}\left(\frac{\partial z}{\partial x}-2\right)=0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}}=\frac{2-\frac{\partial z}{\partial x}}{y^2} \qquad \text{... (1)}$$

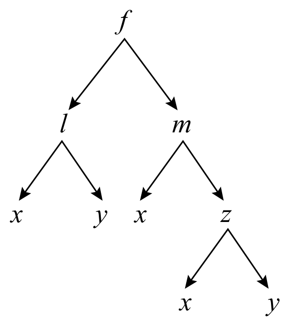


Fig. 4.22

and $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial y}=0$$

$$\frac{\partial f}{\partial l}(2xy)+\frac{\partial f}{\partial m}\left(\frac{\partial z}{\partial y}\right)=0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}}=-\frac{\frac{\partial z}{\partial y}}{2xy} \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\frac{2-\frac{\partial z}{\partial x}}{y^2}=-\frac{\frac{\partial z}{\partial y}}{2xy}$$

$$4x-2x\frac{\partial z}{\partial x}=-y\frac{\partial z}{\partial y}$$

Hence, $$2x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=4x.$$

Example 19: If $f\left(\frac{z}{x^3},\ \frac{y}{x}\right)=0$, prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=3z$.

Solution: Let $l=\left(\frac{z}{x^3}\right),\ m=\frac{y}{x},$ then $f(l,m)=0$

We know that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial x} = 0 \qquad \left[\because f\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0\right]$$

$$\frac{\partial f}{\partial l}\left(\frac{-3z}{x^4} + \frac{1}{x^3}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial m}\left(-\frac{y}{x^2}\right) = 0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}} = \frac{\frac{y}{x^2}}{\frac{1}{x^3}\frac{\partial z}{\partial x} - \frac{3z}{x^4}}$$

$$= \frac{x^2 y}{x\frac{\partial z}{\partial x} - 3z} \qquad \ldots (1)$$

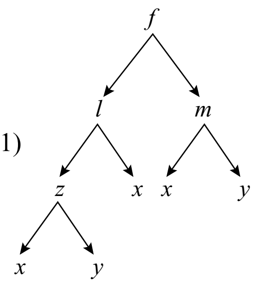


Fig. 4.23

and

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial y} = 0$$

$$\frac{\partial f}{\partial l}\left(\frac{1}{x^3}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial m}\left(\frac{1}{x}\right) = 0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}} = \frac{-\left(\frac{1}{x}\right)}{\frac{1}{x^3}\frac{\partial z}{\partial y}}$$

$$= -\frac{x^2}{\frac{\partial z}{\partial y}} \qquad \ldots (2)$$

From Eqs (1) and (2), we get

$$\frac{x^2 y}{x\frac{\partial z}{\partial x} - 3z} = -\frac{x^2}{\frac{\partial z}{\partial y}}$$

$$y\frac{\partial z}{\partial y} = -x\frac{\partial z}{\partial x} + 3z$$

Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$$

Example 20: If $f(lx + my + nz, x^2 + y^2 + z^2) = 0$,

prove that $(ly - mx) + (ny - mz)\dfrac{\partial z}{\partial x} + (lz - nx)\dfrac{\partial z}{\partial y} = 0$.

Solution: Let $u = lx + my + nz$, $v = x^2 + y^2 + z^2$ then $f(u, v) = 0$

We know that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial f}{\partial u}\left(l + n\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(2x + 2z\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = -\frac{2\left(x + z\frac{\partial z}{\partial x}\right)}{\left(l + n\frac{\partial z}{\partial x}\right)} \quad \ldots (1)$$

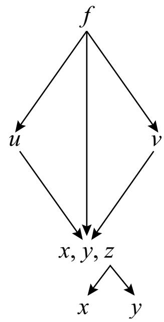


Fig. 4.24

and
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial f}{\partial u}\left(m + n\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(2y + 2z\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = -\frac{2\left(y + z\frac{\partial z}{\partial y}\right)}{\left(m + n\frac{\partial z}{\partial y}\right)} \quad \ldots (2)$$

From Eqs (1) and (2), we get

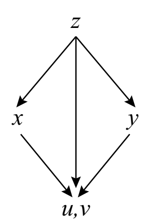
$$\frac{2\left(x + z\frac{\partial z}{\partial x}\right)}{\left(l + n\frac{\partial z}{\partial x}\right)} = \frac{2\left(y + z\frac{\partial z}{\partial y}\right)}{\left(m + n\frac{\partial z}{\partial y}\right)}$$

$$mx + nx\frac{\partial z}{\partial y} + mz\frac{\partial z}{\partial x} + nz\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = ly + lz\frac{\partial z}{\partial y} + ny\frac{\partial z}{\partial x} + nz\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$$

Hence, $(ly - mx) + (ny - mz)\frac{\partial z}{\partial x} + (lz - nx)\frac{\partial z}{\partial y} = 0.$

Example 21: If $z = f(x, y)$ where $x = \log u$, $y = \log v$, show that $\frac{\partial^2 z}{\partial x\, \partial y} = uv\frac{\partial^2 z}{\partial u\, \partial v}$.

Solution: $z = f(x, y)$, $x = \log u$, $y = \log v$



We know that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{1}{u} + \frac{\partial z}{\partial y}\cdot 0 = \frac{1}{u}\frac{\partial z}{\partial x}$$

Differentiating $\frac{\partial z}{\partial u}$ w.r.t. v,

$$\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)=\frac{\partial}{\partial v}\left(\frac{1}{u}\frac{\partial z}{\partial x}\right)$$

$$=\frac{1}{u}\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\left(\frac{\partial x}{\partial v}\right)+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\left(\frac{\partial y}{\partial v}\right)\right]$$

$$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{u}\left(\frac{\partial^2 z}{\partial x^2}\cdot 0+\frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{1}{v}\right)=\frac{1}{uv}\cdot\frac{\partial^2 z}{\partial x\,\partial y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y}=uv\frac{\partial^2 z}{\partial u\,\partial v}.$$

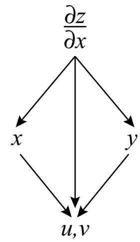


Fig. 4.25

Example 22: If $x=r\cos\theta,\ y=r\sin\theta,$ show that

(i) equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ transforms into $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0.$

(ii) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right].$

Solution: (i) $x=r\cos\theta,\ y=r\sin\theta$

Therefore, $x^2+y^2=r^2$ and $\theta=\tan^{-1}\frac{y}{x}$

$$2x=2r\frac{\partial r}{\partial x},\qquad \frac{\partial r}{\partial x}=\frac{x}{r}=\frac{r\cos\theta}{r}=\cos\theta$$

Similarly,
$$\frac{\partial r}{\partial y}=\frac{y}{r}=\frac{r\sin\theta}{r}=\sin\theta$$

$$\frac{\partial \theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)=\frac{-y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial x}=\frac{-r\sin\theta}{r^2}=\frac{-\sin\theta}{r}$$

Similarly,
$$\frac{\partial \theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}$$

Let $u=f(r,\ \theta)$, given $r^2=x^2+y^2$ and $\theta=\tan^{-1}\frac{y}{x}$

We know that

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial r}\cdot\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\cdot\frac{\partial \theta}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right)$$

$$=\left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{\partial \theta}{\partial x}\right]\frac{\partial r}{\partial x}+\frac{\partial u}{\partial r}\cdot\frac{\partial^2 r}{\partial x^2}$$

$$+\left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\cdot\frac{\partial \theta}{\partial x}\right]\frac{\partial \theta}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial^2 \theta}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2+\frac{\partial^2 u}{\partial \theta\,\partial r}\cdot\frac{\partial \theta}{\partial x}\cdot\frac{\partial r}{\partial x}$$

$$+\frac{\partial u}{\partial r}\cdot\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 u}{\partial r\,\partial \theta}\cdot\frac{\partial r}{\partial x}\cdot\frac{\partial \theta}{\partial x}+\frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2+\frac{\partial u}{\partial \theta}\cdot\frac{\partial^2 \theta}{\partial x^2}$$

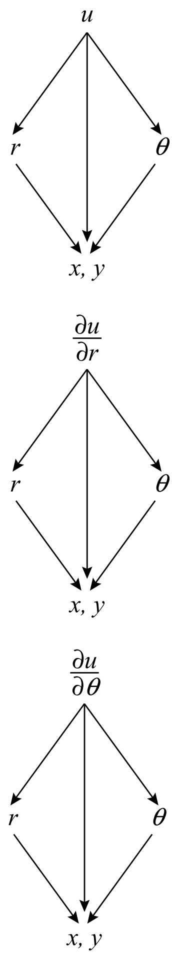


Fig. 4.26

We have,

$$\frac{\partial r}{\partial x}=\cos\theta$$

$$\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right)=-\sin\theta\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 r}{\partial x^2}=-\sin\theta\left(\frac{-\sin\theta}{r}\right)=\frac{\sin^2\theta}{r}$$

Also,

$$\frac{\partial \theta}{\partial x}=-\frac{\sin\theta}{r}$$

$$\frac{\partial^2 \theta}{\partial x^2}=-\left(\frac{\cos\theta}{r}\frac{\partial \theta}{\partial x}-\frac{1}{r^2}\frac{\partial r}{\partial x}\sin\theta\right)$$

$$=-\left[\frac{\cos\theta}{r}\left(\frac{-\sin\theta}{r}\right)-\frac{1}{r^2}\cos\theta\cdot\sin\theta\right]$$

$$=\frac{2\sin\theta\cos\theta}{r^2}=\frac{\sin 2\theta}{r^2}$$

Substituting in $\dfrac{\partial^2 u}{\partial x^2}$,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial r^2}\cos^2\theta+2\frac{\partial^2 u}{\partial r\,\partial \theta}\left(\frac{-\sin\theta}{r}\right)\cos\theta+\frac{\partial u}{\partial r}\cdot\frac{\sin^2\theta}{r}+\frac{\partial^2 u}{\partial \theta^2}\cdot\frac{\sin^2\theta}{r^2}+\frac{\partial u}{\partial \theta}\cdot\frac{\sin 2\theta}{r^2}$$

To get $\dfrac{\partial^2 u}{\partial y^2}$, replace θ by $\dfrac{\pi}{2}+\theta$ in $\dfrac{\partial^2 u}{\partial x^2}$.

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}(-\sin\theta)^2 + \frac{2}{r}\frac{\partial^2 u}{\partial r\,\partial\theta}(-\cos\theta)(-\sin\theta)$$

$$+\frac{1}{r}\cdot\frac{\partial u}{\partial r}\cos^2\theta + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\cos^2\theta + \frac{1}{r^2}\frac{\partial u}{\partial\theta}\sin(\pi+2\theta)$$

$$= \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{2}{r}\frac{\partial^2 u}{\partial r\,\partial\theta}\cos\theta\sin\theta + \frac{1}{r}\frac{\partial u}{\partial r}\cos^2\theta$$

$$+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\cos^2\theta - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\sin 2\theta$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

Hence, equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ transforms into $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0.$

(ii) We have $\quad \frac{\partial^2 r}{\partial x^2} = \frac{\sin^2\theta}{r},\ \frac{\partial r}{\partial y} = \sin\theta,\ \frac{\partial^2 r}{\partial y^2} = \cos\theta\cdot\frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{r}$

Hence, $$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}$$

$$= \frac{1}{r}\left[\left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial x}\right)^2\right].$$

Example 23: If $z = f(x, y)$, $(x + y) = (u + v)^3$ and $x - y = (u - v)^3$, show that $(u^2 - v^2)\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) = 9(x^2 - y^2)\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)$.

Solution:

We have, $$x + y = (u+v)^3 \text{ and } x - y = (u-v)^3$$

$$2x = (u+v)^3 + (u-v)^3,\ x = \frac{1}{2}\left[(u+v)^3 + (u-v)^3\right]$$

$$\frac{\partial x}{\partial u} = \frac{1}{2}\left[3(u+v)^2 + 3(u-v)^2\right] = 3(u^2+v^2),\ \frac{\partial^2 x}{\partial u^2} = 6u$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}\left[3(u+v)^2 + 3(u-v)^2(-1)\right] = 6uv,\ \frac{\partial^2 x}{\partial v^2} = 6u$$

and $$2y = (u+v)^3 - (u-v)^3,\ y = \frac{1}{2}\left[(u+v)^3 - (u-v)^3\right]$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}\left[3(u+v)^2 - 3(u-v)^2\right] = 6uv,\ \frac{\partial^2 y}{\partial u^2} = 6v$$

$$\frac{\partial y}{\partial v} = \frac{1}{2}\left[3(u+v)^2 - 3(u-v)^2(-1)\right] = 3(u^2+v^2),\ \frac{\partial^2 y}{\partial v^2} = 6v$$

We know that

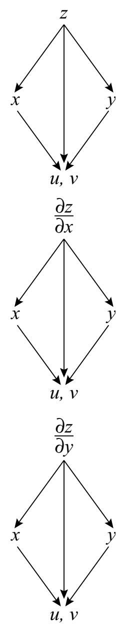


Fig. 4.27

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}\right) + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2} + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial u}\right]\frac{\partial x}{\partial u} + \frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial u}\right]\frac{\partial y}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 + \frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2}$$

$$+ \frac{\partial^2 z}{\partial y\,\partial x}\cdot\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial u} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2 + \frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial u} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2 + \frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial u^2} + \frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial u^2} \quad \text{... (1)}$$

Similarly, $$\frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial v}\right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\cdot\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial v} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial v}\right)^2 + \frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial v^2} + \frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial v^2} \quad \text{... (2)}$$

Substituting derivative values in Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 z}{\partial x^2}9(u^2+v^2)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\cdot 3(u^2+v^2)6uv + \frac{\partial^2 z}{\partial y^2}\cdot 36u^2v^2 + \frac{\partial z}{\partial x}\cdot 6u + \frac{\partial z}{\partial y}\cdot 6v$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}\cdot 36u^2v^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}6uv\cdot 3(u^2+v^2) + \frac{\partial^2 z}{\partial y^2}\cdot 9(u^2+v^2)^2 + \frac{\partial z}{\partial x}\cdot 6u + \frac{\partial z}{\partial y}\cdot 6v$$

$$\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} = 9\frac{\partial^2 z}{\partial x^2}\left[(u^2+v^2)^2 - 4u^2v^2\right] - 9\frac{\partial^2 z}{\partial y^2}\left[(u^2+v^2)^2 - 4u^2v^2\right]$$

$$= 9(u^2-v^2)^2\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)$$

Hence, $$(u^2-v^2)\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) = 9(u^2-v^2)^3\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)$$

$$= 9(u+v)^3(u-v)^3\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)$$

$$= 9(x+y)(x-y)\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)$$

$$= 9(x^2-y^2)\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right).$$

Example 24: If $x+y=2e^{\theta}\cos\phi$ and $x-y=2ie^{\theta}\sin\phi$, prove that

(i) $\dfrac{\partial v}{\partial \theta}+i\dfrac{\partial v}{\partial \phi}=2y\dfrac{\partial v}{\partial y}$ **(ii)** $\dfrac{\partial^2 v}{\partial \theta^2}+\dfrac{\partial^2 v}{\partial \phi^2}=4xy\dfrac{\partial^2 v}{\partial x\,\partial y}$

Solution: $x+y=2e^{\theta}\cos\phi,\ x-y=2ie^{\theta}\sin\phi$

$$2x=2e^{\theta}(\cos\phi+i\sin\phi),\ x=e^{\theta+i\phi}$$

$$\frac{\partial x}{\partial \theta}=e^{\theta+i\phi}=x,\qquad \frac{\partial x}{\partial \phi}=ie^{\theta+i\phi}=ix$$

$$2y=2e^{\theta}(\cos\phi-i\sin\phi),\ y=e^{\theta-i\phi}$$

$$\frac{\partial y}{\partial \theta}=e^{\theta-i\phi}=y,\qquad \frac{\partial y}{\partial \phi}=-ie^{\theta-i\phi}=-iy$$

Let $v=f(x,y)$

We know that

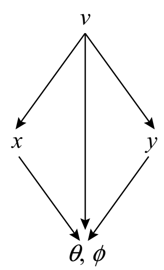


Fig. 4.28

$$\frac{\partial v}{\partial \theta}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta}=\frac{\partial v}{\partial x}x+\frac{\partial v}{\partial y}y$$

$$\frac{\partial^2 v}{\partial \theta^2}=\frac{\partial}{\partial \theta}\left(x\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial \theta}\left(y\frac{\partial v}{\partial y}\right)$$

$$=\frac{\partial x}{\partial \theta}\cdot\frac{\partial v}{\partial x}+x\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial y}{\partial \theta}\cdot\frac{\partial v}{\partial y}+y\cdot\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial y}\right)$$

$$=\frac{\partial x}{\partial \theta}\cdot\frac{\partial v}{\partial x}+x\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{\partial y}{\partial \theta}\right]$$

$$+\frac{\partial y}{\partial \theta}\cdot\frac{\partial v}{\partial y}+y\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)\cdot\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)\cdot\frac{\partial y}{\partial \theta}\right]$$

$$=x\frac{\partial v}{\partial x}+x\left(\frac{\partial^2 v}{\partial x^2}\cdot x+\frac{\partial^2 v}{\partial x\,\partial y}\cdot y\right)+y\frac{\partial v}{\partial y}+y\left(\frac{\partial^2 v}{\partial y\,\partial x}\cdot x+\frac{\partial^2 v}{\partial y^2}\cdot y\right)$$

$$=x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+2xy\frac{\partial^2 v}{\partial y\,\partial x}+x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}$$

and
$$\frac{\partial v}{\partial \phi}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \phi}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \phi}$$

$$=\frac{\partial v}{\partial x}(ix)+\frac{\partial v}{\partial y}(-iy)=i\left(x\frac{\partial v}{\partial x}-y\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial^2 v}{\partial \phi^2}=\frac{\partial}{\partial \phi}\left(ix\frac{\partial v}{\partial x}\right)-\frac{\partial}{\partial \phi}\left(iy\frac{\partial v}{\partial y}\right)$$

$$=i\frac{\partial x}{\partial \phi}\cdot\frac{\partial v}{\partial x}+ix\frac{\partial}{\partial \phi}\left(\frac{\partial v}{\partial x}\right)-i\frac{\partial y}{\partial \phi}\cdot\frac{\partial v}{\partial y}-iy\frac{\partial}{\partial \phi}\left(\frac{\partial v}{\partial y}\right)$$

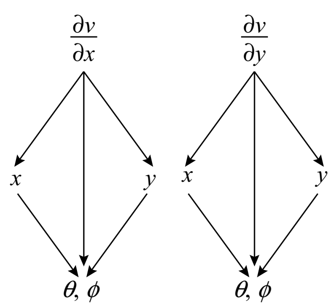


Fig. 4.29

$$= i\left[\frac{\partial x}{\partial\phi}\cdot\frac{\partial v}{\partial x} + x\left\{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{\partial x}{\partial\phi} + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)\cdot\frac{\partial y}{\partial\phi}\right\} - \frac{\partial y}{\partial\phi}\cdot\frac{\partial v}{\partial y}\right.$$

$$\left. - y\left\{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)\cdot\frac{\partial x}{\partial\phi} + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)\cdot\frac{\partial y}{\partial\phi}\right\}\right]$$

$$= i\left[ix\frac{\partial v}{\partial x} + x\left(\frac{\partial^2 v}{\partial x^2}ix - \frac{\partial^2 v}{\partial x\,\partial y}iy\right) + iy\frac{\partial v}{\partial y} - y\left(\frac{\partial^2 v}{\partial y\,\partial x}ix - \frac{\partial^2 v}{\partial y^2}iy\right)\right]$$

$$= i^2 x\frac{\partial v}{\partial x} + i^2x^2\frac{\partial^2 v}{\partial x^2} - 2i^2xy\frac{\partial^2 v}{\partial x\,\partial y} + i^2 y\frac{\partial v}{\partial y} + i^2y^2\frac{\partial^2 v}{\partial y^2}$$

$$= -x\frac{\partial v}{\partial x} - x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} - y\frac{\partial v}{\partial y} - y^2\frac{\partial^2 v}{\partial y^2}$$

Hence, $$\frac{\partial v}{\partial\theta} + i\frac{\partial v}{\partial\phi} = \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) + i^2\left(x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y}\right) = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} - x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$= 2y\frac{\partial v}{\partial y}.$$

and $$\frac{\partial^2 v}{\partial\theta^2} + \frac{\partial^2 v}{\partial\phi^2} = 4xy\frac{\partial^2 v}{\partial x\,\partial y}.$$

Example 25: If $z = f(x, y)$, where $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, where α is a constant, show that

(i) $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$ **(ii)** $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}.$$

Solution: (i) $z = f(x, y)$ and $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$

We know that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-\sin\alpha) + \frac{\partial z}{\partial y}\cos\alpha$$

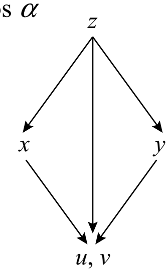


Fig. 4.30

Hence,

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\alpha + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\alpha + 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cos\alpha\sin\alpha$$

$$+ \left(\frac{\partial z}{\partial x}\right)^2\sin^2\alpha + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\alpha - 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cos\alpha\sin\alpha$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

(ii) $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\cos\alpha + \dfrac{\partial z}{\partial y}\sin\alpha$

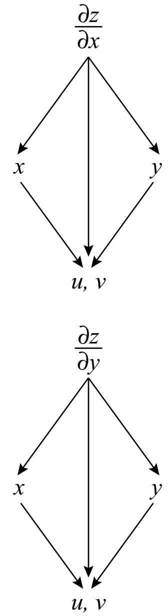


Fig. 4.31

Differentiating $\dfrac{\partial z}{\partial u}$ w.r.t. u,

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\cos\alpha\right) + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\sin\alpha\right)$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial u}\right]\cos\alpha$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial u}\right]\sin\alpha$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2 z}{\partial x\,\partial y}\sin\alpha\cos\alpha + \frac{\partial^2 z}{\partial y^2}\sin^2\alpha$$

Differentiating $\dfrac{\partial z}{\partial v^2}$ w.r.t. v,

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left[\frac{\partial z}{\partial x}(-\sin\alpha)\right] + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\cdot\cos\alpha\right)$$

$$= -\sin\alpha\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial v}\right]$$

$$+ \cos\alpha\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial v}\right]$$

$$= -\sin\alpha\left[\frac{\partial^2 z}{\partial x^2}(-\sin\alpha) + \frac{\partial^2 z}{\partial x\,\partial y}\cos\alpha\right] + \cos\alpha\left[\frac{\partial^2 z}{\partial y\,\partial x}(-\sin\alpha) + \frac{\partial^2 z}{\partial y^2}\cos\alpha\right]$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}\sin^2\alpha - 2\frac{\partial^2 z}{\partial x\,\partial y}\cos\alpha\sin\alpha + \frac{\partial^2 z}{\partial y^2}\cos^2\alpha$$

Hence,
$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}(\sin^2\alpha + \cos^2\alpha) + \frac{\partial^2 z}{\partial y^2}(\sin^2\alpha + \cos^2\alpha)$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.$$

Exercise 4.2

1. If $z = \tan^{-1}\left(\dfrac{x}{y}\right)$, where $x = 2t$, $y = 1 - t^2$, prove that $\dfrac{dz}{dt} = \dfrac{2}{1+t^2}$.

2. If $u = x^3 + y^3$, where $x = a\cos t$, $y = b\sin t$, find $\dfrac{du}{dt}$.

[**Ans.:** $-3a^3\cos^2 t\sin t + 3b^2\sin^2 t\cos t$]

3. If $u = xe^y z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^3 x$, find $\dfrac{du}{dx}$.

$$\left[\textbf{Ans.: } e^y z\left(1 - \frac{x^2}{y} + 3x\cot x\right)\right]$$

4. If $u = e^{\frac{r-x}{l}}$, where $r^2 = x^2 + y^2$ and l is a constant, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2}{l}\cdot\frac{\partial u}{\partial x} = \frac{u}{lr}.$$

5. If $u = \log r$ and $r = \sqrt{(x-a)^2 + (y-b)^2}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ if a, b are constants.

6. If $u^2 (x^2 + y^2 + z^2) = 1$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

$$\left[\textbf{Hint: } \text{Let } x^2 + y^2 + z^2 = r^2,\ u = \frac{1}{r}\right]$$

7. If $u = r^m$, where $r = \sqrt{x^2 + y^2 + z^2}$ find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$.

$$\left[\textbf{Ans.: } m(m+1)r^{m-2}\right]$$

8. If $u = f(r)$, where r is given by the relation $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}.$$

9. If $z = f(u, v)$, where $u = x^2 - y^2$, $v = 2xy$, then show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2 + v^2}\left(\frac{\partial z}{\partial u}\right).$$

10. If $z = f(u, v)$, where $u = x^2 + y^2$, $v = x^2 - y^2$, then show that

(i) $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 4xy\dfrac{\partial z}{\partial u}$.

(ii) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$

$$= 4u\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] + 8v\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}.$$

11. If $w = z\sin^{-1}\left(\dfrac{y}{x}\right)$, where $x = 3u^2 + 2v$, $y = 4u - 2v^3$, $z = 2u^2 - 3v^2$, find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

12. If $w = (x^2 + y - 2)^4 + (x - y + 2)^3$, where $x = u - 2v + 1$ and $y = 2u + v - 2$, find $\dfrac{\partial w}{\partial v}$ at $u = 0$, $v = 0$.

[Ans.: −882]

13. If $w = x + 2y + z^2$, $x = \dfrac{u}{v}$, $y = u^2 + e^v$, $z = 2u$, show that

$$u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v} = 12u^2 + 2ve^v.$$

14. If F is a function of x, y, z, then show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z},$$

where $x = u + v + w$, $y = uv + vw + wu$, $z = uvw$.

15. If $z = f(x, y)$, $x = uv$, $y = \dfrac{u}{v}$, prove that $\dfrac{\partial z}{\partial x} = \dfrac{1}{2v}\dfrac{\partial z}{\partial u} + \dfrac{1}{2u}\dfrac{\partial z}{\partial u}$ and

$$\frac{\partial z}{\partial y} = \frac{v}{2}\frac{\partial z}{\partial u} - \frac{v^2}{2u}\frac{\partial z}{\partial v}.$$

16. If $x = u + v$, $y = uv$ and F is a function of x, y, prove that

$$\frac{\partial^2 F}{\partial u^2} - 2\frac{\partial^2 F}{\partial u\,\partial v} + \frac{\partial^2 F}{\partial v^2} = (x^2 - 4y)\frac{\partial^2 v}{\partial y^2} - 2\frac{\partial v}{\partial y}.$$

$$\left[\textbf{Hint: } \text{L.H.S.} = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right)\right]$$

17. If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$, prove that $\frac{1}{x^{n-1}}\frac{\partial u}{\partial x} + \frac{1}{y^{n-1}}\frac{\partial u}{\partial y} + \frac{1}{z^{n-1}}\frac{\partial u}{\partial z} = 0.$

18. If $z = f(x, y)$, where $x = u - av$, $y = u + av$, prove that

$$a^2\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} = 4a^2\frac{\partial^2 z}{\partial x\, \partial y}.$$

19. If $z = f(u, v)$, where $u = lx + my$, $v = ly - mx$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$

$$= (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

20. If $x = u + av$ and $y = u + bv$ transform the equation $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x\, \partial y} + 3\frac{\partial^2 z}{\partial y^2} = 0$ into the equation $\frac{\partial^2 z}{\partial u \partial v} = 0$, find the values of a and b.

$$\left[\textbf{Ans.: } a = 1,\ b = \frac{2}{3}\right]$$

21. If $z = f(x, y)$, $y = e^x$, $v = e^y$, prove that

$$\frac{\partial^2 z}{\partial x\, \partial y} = uv\frac{\partial^2 z}{\partial u\, \partial v}.$$

22. If $z = f(x, y)$, $x = \frac{\cos u}{v}$, $y = \frac{\sin u}{v}$, prove that

$$v\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} = (y - x)\frac{\partial z}{\partial x} - (y + x)\frac{\partial z}{\partial y}.$$

23. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$

24. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$

25. If $u = f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$ show that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

26. If $x + y = 2e^\theta \cos\phi$ and $x - y = 2ie^\theta \sin\phi$, prove that $\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial \phi} = 2y\frac{\partial v}{\partial y}.$

27. Find the values of the constants a and b such that $u = x + ay$ and $v = x + by$ transform the equation

$$9\frac{\partial^2 f}{\partial x^2} - 9\frac{\partial^2 f}{\partial x\, \partial y} + 2\frac{\partial^2 f}{\partial y^2} = 0$$

into $\frac{\partial^2 f}{\partial u\, \partial v} = 0$, where f is a function of x and y.

$$\left[\textbf{Ans.: } a = \frac{3}{2},\ b = 3\right]$$

28. If $x = r\cosh\theta$, $y = r\sinh\theta$ and $z = f(x, y)$, prove that

(i) $(x - y)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = r\frac{\partial z}{\partial r} - \frac{\partial z}{\partial \theta}$

(ii) $(x^2 - y^2)\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)$

$$= r^2\frac{\partial^2 z}{\partial r^2} + r\frac{\partial z}{\partial r} - \frac{\partial^2 z}{\partial \theta^2}.$$

29. If $x = e^v \sec u$, $y = e^v \tan u$ and $z = f(x, y)$, prove that $\cos u\left(\frac{\partial^2 z}{\partial u\, \partial v} - \frac{\partial z}{\partial u}\right)$

$$= xy\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + (x^2 + y^2)\frac{\partial^2 z}{\partial x\, \partial y}.$$

30. If $f(x^2y^3, z-3x) = 0$, prove that

$$3x\frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial y} = 9x.$$

31. If $f(y+z, x^2+y^2+z^2) = 0$, prove that

$$(y-z)\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = x.$$

32. If $f(cx-az, cy-bz) = 0$, prove that

$$a\frac{\partial z}{\partial x} - b\frac{\partial z}{\partial y} = c.$$

33. If $x^2 = a\sqrt{u} + b\sqrt{v}$ and $y^2 = a\sqrt{u} - b\sqrt{v}$, where a, b are constant, prove that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u.$$

34. If $x = \dfrac{\cos\theta}{u}$, $y = \dfrac{\sin\theta}{u}$, prove that

$$\left(\frac{\partial x}{\partial u}\right)_\theta \left(\frac{\partial u}{\partial x}\right)_y = \cos^2\theta.$$

35. If $x^2 = au + bv$, $y^2 = au - bv$, prove that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u.$$

36. If $u = ax + by$, $v = bx - ay$, prove that

(i) $\left(\dfrac{\partial y}{\partial v}\right)_x \left(\dfrac{\partial v}{\partial y}\right)_u = \dfrac{a^2+b^2}{a^2}$

(ii) $\left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{a^2}{a^2+b^2}.$

37. If $u = ax + by$, $v = bx - ay$, find the value of $\left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial u}\right)_v \left(\dfrac{\partial y}{\partial v}\right)_x \left(\dfrac{\partial v}{\partial y}\right)_u.$

[**Ans. :** 1]

38. If x increases at the rate of 2 cm/sec at the instant, when $x = 3$ cm and $y = 1$ cm, at what rate must y change in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing.

[**Hint :** $\dfrac{dx}{dt} = 2$ at $x = 3, y = 1,$

$u = 2xy - 3x^2y$, find $\dfrac{dy}{dt}$ if $\dfrac{du}{dt} = 0$ at $x = 3, y = 1$]

[**Ans.:** $-\dfrac{32}{21}$ cm/sec (y if decreasing)]

4.6 IMPLICIT FUNCTIONS

Any function of the type $f(x, y) = c$ is called an implicit function, where y is a function of x and c is a constant.

If $$f(x,y) = c \text{ then } \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Proof: If $f(x, y)$ is a function of x and y, where y is a function of x, then total differential coefficient of f w.r.t. x is given by

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

But $f(x, y) = c$

$$\frac{df}{dx} = 0$$

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

Hence,

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Example 1: If $f(x,y)=0$, $\phi(x,z)=0$, show that $\dfrac{\partial\phi}{\partial x}\cdot\dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dz}=\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial\phi}{\partial z}$.

Solution: $f(x,y)=0$ and $\phi(x,z)=0$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}\quad\text{and}\quad\frac{dz}{dx}=-\frac{\partial\phi/\partial x}{\partial\phi/\partial z}$$

$$\frac{dy/dx}{dz/dx}=\frac{-\dfrac{\partial f/\partial x}{\partial f/\partial y}}{-\dfrac{\partial\phi/\partial x}{\partial\phi/\partial z}}$$

$$\frac{dy}{dz}=\frac{\partial f/\partial x}{\partial f/\partial y}\cdot\frac{\partial\phi/\partial z}{\partial\phi/\partial x}$$

Hence,

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz}=\frac{\partial f}{\partial x}\cdot\frac{\partial\phi}{\partial z}.$$

Example 2: If $y\log(\cos x)=x\log(\sin y)$, find $\dfrac{dy}{dx}$.

Solution: Let $f(x,y)=y\log(\cos x)-x\log(\sin y)$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$=-\frac{y\dfrac{1}{\cos x}(-\sin x)-\log(\sin y)}{\log\cos x-\dfrac{x}{\sin y}\cdot\cos y}$$

$$=\frac{y\tan x+\log\sin y}{\log\cos x-x\cot y}.$$

Example 3: If $u=\sin(x^2+y^2)$ and $a^2x^2+b^2y^2=c^2$, find $\dfrac{du}{dx}$.

Solution: $u=\sin(x^2+y^2)$ and $a^2x^2+b^2y^2=c^2$

$$\frac{\partial u}{\partial x}=\cos(x^2+y^2)\cdot 2x$$

$$\frac{\partial u}{\partial y}=\cos(x^2+y^2)\cdot 2y$$

Let $$f(x,y) = a^2x^2 + b^2y^2 - c^2$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2a^2x}{2b^2y} = -\frac{a^2x}{b^2y}$$

We know that

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\left(-\frac{a^2x}{b^2y}\right)$$

$$= 2x\cos(x^2+y^2)\cdot\left(1-\frac{a^2}{b^2}\right)$$

Example 4: If $f(x, y)$ = constant is an implicit function, show that $\dfrac{dy}{dx} = -\dfrac{p}{q}$ and $\dfrac{d^2y}{dx^2} = -\dfrac{1}{q^3}(q^2r - 2pqs + p^2t)$, if $q \neq 0$ where $p = \dfrac{\partial f}{\partial x}$, $q = \dfrac{\partial f}{\partial y}$, $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$.

Solution: $f(x,y)$ = constant.

$$\frac{\partial f}{\partial x} = 0$$

We know that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$\frac{dy}{dx} = -\frac{p}{q}$$

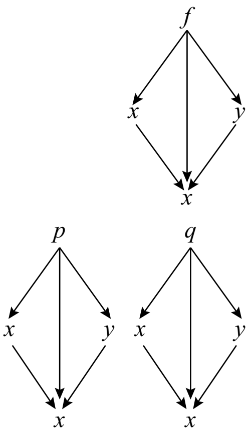


Fig. 4.32

Differentiating $\dfrac{dy}{dx}$ w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{q\cdot\dfrac{dp}{dx} - p\dfrac{dq}{dx}}{q^2}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{\left\{q\left(\dfrac{\partial p}{\partial x}\cdot\dfrac{dx}{dx} + \dfrac{\partial p}{\partial y}\dfrac{dy}{dx}\right) - p\left(\dfrac{\partial q}{\partial x}\cdot\dfrac{dx}{dx} + \dfrac{\partial q}{\partial y}\cdot\dfrac{dy}{dx}\right)\right\}}{q^2}\right]$$

$$= -\left[\frac{q\left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\left(-\frac{p}{q}\right)\right\}-p\left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\left(-\frac{p}{q}\right)\right\}}{q^2}\right]$$

$$= -\left[\frac{q\left\{\frac{\partial^2 f}{\partial x^2}-\frac{p}{q}\frac{\partial^2 f}{\partial x\,\partial y}\right\}-p\left\{\frac{\partial^2 f}{\partial y\,\partial x}-\frac{p}{q}\frac{\partial^2 f}{\partial y^2}\right\}}{q^2}\right]$$

$$= -\left[\frac{q(qr-ps)-p(qs-pt)}{q^3}\right] = -\frac{1}{q^3}(q^2r-2pqs+p^2t).$$

Example 5: If $x^4+y^4+4a^2xy=0$, show that $(y^3+a^2x)^3\dfrac{d^2y}{dx^2}=2a^2xy(x^2y^2+3a^4)$.

Solution: Let $f(x,y)=x^4+y^4+4a^2xy$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{4x^3+4a^2y}{4y^3+4a^2x}$$

$$(4y^3+4a^2x)\frac{dy}{dx} = -(4x^3+4a^2y)$$

Differentiating above equation w.r.t. x,

$$(4y^3+4a^2x)\frac{d^2y}{dx^2}+\left(12y^2\frac{dy}{dx}+4a^2\right)\frac{dy}{dx} = -\left(12x^2+4a^2\frac{dy}{dx}\right)$$

$$(4y^3+4a^2x)\frac{d^2y}{dx^2}+12y^2\left(\frac{4x^3+4a^2y}{4y^3+4a^2x}\right)^2-8a^2\left(\frac{4x^3+4a^2y}{4y^3+4a^2x}\right)+12x^2=0$$

$$(4y^3+4a^2x)\frac{d^2y}{dx^2}$$

$$= \frac{-12y^2(x^6+a^4y^2+2x^3a^2y)+8a^2(x^3+a^2y)\cdot(y^3+a^2x)-12x^2(y^6+a^4x^2+2xy^3a^2)}{(y^3+a^2x)^2}$$

$$= \frac{-12y^2x^6-12y^4a^4-24y^3x^3a^2+8a^2(x^3+a^2y)(y^3+a^2x)-12x^2y^6-12a^4x^4-24x^3y^3a^2}{(y^3+a^2x)^2}$$

$$(y^3+a^2x)^3\frac{d^2y}{dx^2} = -3y^2x^6-3y^4a^4-6y^3a^2x^3+2a^2x^3y^3+2a^4x^4+2a^4y^4$$
$$+2a^6xy-3x^2y^6-3a^4x^4-6x^3y^3a^2$$
$$= -3x^2y^2(x^4+y^4)-a^4(x^4+y^4)-10x^3y^3a^2+2a^6xy$$
$$= -3x^2y^2(-4a^2xy)-a^4(-4a^2xy)-10x^3y^3a^2+2a^6xy$$
$$= 12a^2x^3y^3+4a^6xy-10x^3y^3a^2+2a^6xy$$
$$= 2a^2x^3y^3+6a^6xy = 2a^2xy(x^2y^2+3a^4)$$

Note: It can also be proved by putting values of p, q, r, s, t in the result of Ex. 4.

Exercise 4.3

1. If $x^3 + y^3 - 3axy = 0$, find $\dfrac{dy}{dx}$.

$$\left[\textbf{Ans.:}\ \frac{ay - x^2}{y^2 - ax}\right]$$

2. If $x^3 + 3x^2 + 6xy^2 + y^3 = 1$, find $\dfrac{dy}{dx}$.

$$\left[\textbf{Ans.:}\ -\frac{(x^2 + 2x + 2y^2)}{(4xy + y^2)}\right]$$

3. If $x^y = y^x$, prove that

$$\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}.$$

4. If $f(x, y) = x\sin(x - y) - (x + y) = 0$, find $\dfrac{dy}{dx}$.

$$\left[\textbf{Ans.:}\ \frac{[\sin(x - y)](1 + x) - 1}{x\cos(x - y) + 1}\right]$$

5. If $y^{x^y} = \sin x$, find $\dfrac{dy}{dx}$.

$$\left[\begin{array}{l}\textbf{Hint:}\ f = x^y\log y - \log\sin x,\\ \quad \text{let } x^y = z,\\ \log z = y\log x \text{ find } \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y} \text{ and}\\ \text{then } \dfrac{\partial f}{\partial x} \text{ and } \dfrac{\partial f}{\partial y}\end{array}\right]$$

$$\left[\textbf{Ans.:}\ \frac{-(yx^{y-1}\log y - \cot x)}{x^y\log x\log y + x^y y^{-1}}\right]$$

6. If $x^5 + y^5 = 5a^3x^2$, find $\dfrac{d^2y}{dx^2}$.

$$\left[\textbf{Ans.:}\ \frac{6a^3x^2(a^3 + x^3)}{y^9}\right]$$

7. If $xy^3 - yx^3 = 6$ is the equation of curve, find the slope of the tangent at the point (1, 2).

$$\left[\textbf{Hint:}\ \text{Find } \frac{dy}{dx} \text{ at } (1, 2)\right]$$

$$\left[\textbf{Ans.:}\ -\frac{2}{11}\right]$$

8. Find $\dfrac{d^2y}{dx^2}$, if $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a$.

$$\left[\textbf{Ans.:}\ \frac{a}{(1 - x^2)^{\frac{3}{2}}}\right]$$

9. If $u = x\log xy$ and $x^3 + y^3 + 3xy - 1 = 0$, find $\dfrac{du}{dx}$.

$$\left[\begin{array}{l}\textbf{Hint:}\ \dfrac{du}{dx} = \dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dx} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}\\ \qquad \dfrac{dy}{dx} = -\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}\end{array}\right]$$

$$\left[\textbf{Ans.:}\ 1 + \log xy - \frac{x}{y}\left(\frac{x^2 + ay}{y^2 + ax}\right)\right]$$

10. If $x^m + y^m = b^m$, show that

$$\frac{d^2y}{dx^2} = -(m - 1)b^m \frac{x^{m-2}}{y^{2m-1}}.$$

11. If $u = x^2y$ and $x^2 + xy + y^2 = 1$, find $\dfrac{du}{dx}$.

12. If $x^3 + y^3 = 3ax^2$, find $\dfrac{d^2y}{dx^2}$.

$$\left[\textbf{Ans.:}\ -\frac{2a^2x^2}{y^5}\right]$$

4.7 HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM

A function $f(x, y, z)$ is said to be homogeneous function of degree n, if for any positive number t

$$f(xt, yt, zt) = t^n f(x, y, z)$$

where, n is a real number.

4.7.1 Euler's Theorem for Function of Two Variables

Statement: If u is a homogeneous function of two variables x and y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof: Let $u = f(x, y)$ is a homogeneous function of degree n.

$$u = f(X, Y) = t^n f(x, y)$$

where, $X = xt$ and $Y = yt$

Differentiating $u = f(X, Y)$ w.r.t. t using composite function,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial t} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial t} = x\frac{\partial u}{\partial X} + y\frac{\partial u}{\partial Y}$$

At $t = 1$, $X = x$ and $Y = y$

$$\frac{\partial u}{\partial t} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \qquad \text{... (1)}$$

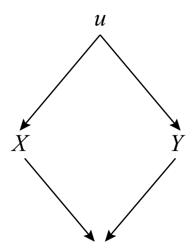


Fig. 4.33

Differentiating $u = t^n f(x, y)$ w.r.t. t,

$$\frac{\partial u}{\partial t} = nt^{n-1} f(x, y)$$

At $t = 1$,

$$\frac{\partial u}{\partial t} = nf(x, y) = nu \qquad \text{... (2)}$$

From Eqs (1) and (2),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

4.7.2 Euler's Theorem for Function of Three Variables

Statement: If u is a homogeneous function of three variables x, y, z of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

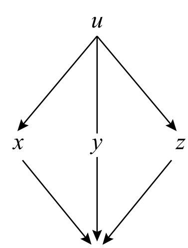


Fig. 4.34

Proof: Let $u = f(x, y, z)$ is a homogeneous function of degree n.

$$u = f(X, Y, Z) = t^n f(x, y, z)$$

where, $X = xt,\ Y = yt,\ Z = zt$.

Differentiating $u = f(X, Y, Z)$ w.r.t t using composite function,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial t} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial t} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial t} = x\frac{\partial u}{\partial X} + y\frac{\partial u}{\partial Y} + z\frac{\partial u}{\partial Z}$$

At $t = 1$, $X = x$, $Y = y$ and $Z = z$

$$\frac{\partial u}{\partial t} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} \qquad \text{... (1)}$$

Differentiating $u = t^n f(x, y, z)$ w.r.t. t,

$$\frac{\partial u}{\partial t} = nt^{n-1} f(x, y, z)$$

At $\quad t = 1$,

$$\frac{\partial u}{\partial t} = nf(x, y, z) = nu \qquad \text{... (2)}$$

From Eqs (1) and (2),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

4.7.3 Deductions from Euler's Theorem

Corollary 1: If u is a homogeneous function of two variables x, y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proof: Let u is a homogeneous function of two variables x and y of degree n.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \text{... (1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x} \qquad \text{... (2)}$$

Differentiating (1) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y} \qquad \text{... (3)}$$

Multiplying Eq. (2) by x and Eq. (3) by y and adding,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = (n-1)nu \qquad \text{[Using Eq. (1)]}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Example 1: Verify Euler's theorem for

(i) $u=x^2yz-4y^2z^2+2xz^3$ **(ii)** $u=x^4y^2\sin^{-1}\dfrac{y}{x}$

(iii) $u=\dfrac{x^2+y^2}{x+y}$ **(iv)** $u=\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$.

Solution: (i) $u=x^2yz-4y^2z^2+2xz^3$

Replacing x by xt, y by yt and z by zt,

$$u=t^3(x^2yz-4y^2z^2+2xz^3)$$

Hence, u is a homogeneous function of degree 3.

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu=3 \quad \ldots (1)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x}=2xyz+2z^3,\ \frac{\partial u}{\partial y}=x^2z-8yz^2,\ \frac{\partial u}{\partial z}=x^2y-8y^2z+6xz^2$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2x^2yz+2xz^3+x^2yz-8y^2z^2+x^2yz-8y^2z^2+6xz^3$$

$$=4x^2yz-16y^2z^2+8xz^3$$

$$=4(x^2yz-4y^2z^2+2xz^3)$$

$$=4u \quad \ldots (2)$$

Hence, from Eqs (1) and (2), theorem is verified.

(ii) $u=x^4y^2\sin^{-1}\dfrac{y}{x}$

Replacing x by xt and y by yt,

$$u=t^6\left(x^4y^2\sin^{-1}\frac{y}{x}\right)$$

Hence, u is a homogeneous function of degree 6.

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=6u \quad \ldots (1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x}=y^2\left[4x^3\sin^{-1}\frac{y}{x}+\frac{x^4}{\sqrt{1-\frac{y^2}{x^2}}}\left(-\frac{y}{x^2}\right)\right]$$

$$=y^2\left(4x^3\sin^{-1}\frac{y}{x}-\frac{yx^3}{\sqrt{x^2-y^2}}\right)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = x^4\left[2y\sin^{-1}\frac{y}{x} + y^2\frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\cdot\frac{1}{x}\right]$$

$$= 2x^4 y\sin^{-1}\frac{y}{x} + \frac{x^4y^2}{\sqrt{x^2-y^2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4x^4y^2\sin^{-1}\frac{y}{x} - \frac{x^4y^3}{\sqrt{x^2-y^2}} + 2x^4y^2\sin^{-1}\frac{y}{x} + \frac{x^4y^3}{\sqrt{x^2-y^2}}$$

$$= 6x^4y^2\sin^{-1}\frac{y}{x} = 6u \qquad \dots (2)$$

Hence, from Eqs (1) and (2), theorem is verified.

(iii) $$u = \frac{x^2+y^2}{x+y}$$

Replacing x by xt and y by yt,

$$u = t\left(\frac{x^2+y^2}{x+y}\right)$$

Hence, u is a homogeneous function of degree 1.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = u \qquad \dots (1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{2x}{x+y} - \frac{(x^2+y^2)}{(x+y)^2} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{2y}{x+y} - \frac{x^2+y^2}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^3-xy^2+2x^2y+y^3-x^2y+2xy^2}{(x+y)^2}$$

$$= \frac{x^3+y^3+xy^2+x^2y}{(x+y)^2}$$

$$= \frac{(x+y)(x^2-xy+y^2)+xy(y+x)}{(x+y)^2}$$

$$= \frac{(x+y)(x^2-xy+y^2+xy)}{(x+y)^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^2+y^2}{x+y} = u \qquad \dots (2)$$

Hence, from Eqs (1) and (2), theorem is verified.

(iv) $$u = \frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}.$$

Replacing x by xt and y by yt,

$$u = t^{\frac{1}{2}}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$$

Hence, u is a homogeneous function of degree $\frac{1}{2}$.

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu = \frac{1}{2}u \qquad \dots (1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{x+y+z}{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}\cdot\frac{1}{2\sqrt{x}}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{x+y+z}{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}\cdot\frac{1}{2\sqrt{y}}$$

and $$\frac{\partial u}{\partial z} = \frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{x+y+z}{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}\cdot\frac{1}{2\sqrt{z}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = \frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{(x+y+z)\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}$$

$$= \frac{1}{2}\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = \frac{1}{2}u \qquad \dots (2)$$

Hence, from Eqs (1) and (2), theorem is verified.

Example 2: If $u = e^{\frac{x}{y}}\sin\left(\frac{x}{y}\right)+e^{\frac{y}{x}}\cos\left(\frac{x}{y}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0$.

Solution: $$u = e^{\frac{x}{y}}\sin\left(\frac{x}{y}\right)+e^{\frac{y}{x}}\cos\left(\frac{x}{y}\right),$$

Replacing x by xt and y by yt,

$$u = t^0\left[e^{\frac{x}{y}}\sin\left(\frac{x}{y}\right)+e^{\frac{y}{x}}\cos\left(\frac{x}{y}\right)\right]$$

Hence, u is a homogeneous function of degree 0.

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0.$$

Example 3: **Find $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$ where $u=(8x^2+y^2)(\log x-\log y)$.**

Solution:

$$u=(8x^2+y^2)(\log x-\log y)$$
$$=(8x^2+y^2)\log\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$u=t^2(8x^2+y^2)\log\left(\frac{x}{y}\right)$$

Hence, u is a homogeneous function of degree 2.
By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u=2(8x^2+y^2)(\log x-\log y).$$

Example 4: **If $u=\dfrac{x^2}{y}f\left(\dfrac{y}{x}\right)+\dfrac{y^2}{x}g\left(\dfrac{x}{y}\right)$, prove that**

$$x^2\left[y\frac{\partial u}{\partial x}-xf\left(\frac{y}{x}\right)\right]+y^2\left[x\frac{\partial u}{\partial y}-yg\left(\frac{x}{y}\right)\right]=0.$$

Solution:

$$u=\frac{x^2}{y}f\left(\frac{y}{x}\right)+\frac{y^2}{x}g\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$u=t\left[\frac{x^2}{y}f\left(\frac{y}{x}\right)+\frac{y^2}{x}g\left(\frac{x}{y}\right)\right]$$

Hence, u is a homogeneous function of degree 1.
By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1\cdot u=\frac{x^2}{y}f\left(\frac{y}{x}\right)+\frac{y^2}{x}g\left(\frac{x}{y}\right)$$

$$x^2y\frac{\partial u}{\partial x}+xy^2\frac{\partial u}{\partial y}=x^3f\left(\frac{y}{x}\right)+y^3g\left(\frac{x}{y}\right)$$

$$x^2\left[y\frac{\partial u}{\partial x}-xf\left(\frac{y}{x}\right)\right]+y^2\left[x\frac{\partial u}{\partial y}-yg\left(\frac{x}{y}\right)\right]=0.$$

Example 5: **If $u(x,y)=\dfrac{1}{x^2}+\dfrac{1}{xy}+\dfrac{\log x-\log y}{x^2}$, prove that**

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2u(x,y)=0.$$

Solution:

$$u(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2}$$
$$=\frac{1}{x^2}+\frac{1}{xy}+\frac{1}{x^2}\log\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$u(x, y) = t^{-2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{x^2}\log\left(\frac{x}{y}\right)\right]$$

Hence, u is a homogeneous function of degree –2.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u(x, y)$$

Hence, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u(x, y) = 0.$$

Example 6: If $z = \log(x^2 + y^2) + \frac{x^2 + y^2}{x + y} - 2\log(x + y)$, find the value of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$.

Solution:

$$\begin{aligned} z &= \log(x^2 + y^2) + \frac{x^2 + y^2}{x + y} - 2\log(x + y) \\ &= \log(x^2 + y^2) + \frac{x^2 + y^2}{x + y} - \log(x + y)^2 \\ &= \log\frac{(x^2 + y^2)}{(x + y)^2} + \frac{x^2 + y^2}{x + y} \\ &= u + v \end{aligned}$$

where, $u = \log\frac{x^2 + y^2}{(x + y)^2}$, $v = \frac{x^2 + y^2}{x + y}$

Replacing x by xt and y by yt in u and v,

$$u = t^0 \log\frac{x^2 + y^2}{(x + y)^2}, \quad v = t\left(\frac{x^2 + y^2}{x + y}\right)$$

Hence, u is a homogeneous function of degree 0 and v is homogeneous function of degree 1.

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \cdot u = 0 \quad \text{... (1)}$$

and $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 \cdot v \quad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = 0 + v$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2 + y^2}{x + y}.$$

Example 7: If $\boldsymbol{u = f\left(\frac{y}{x}\right) + \sqrt{x^2 + y^2}}$**, prove that** $\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2 + y^2}}$**.**

Solution: Let $u = v + w$

where, $$v = f\left(\frac{y}{x}\right), \quad w = \sqrt{x^2 + y^2}$$

Replacing x by xt and y by yt,

$$v = t^0 f\left(\frac{y}{x}\right) \text{ and } w = t\sqrt{x^2 + y^2}$$

Hence, v is a homogeneous function of degree 0 and w is homogeneous function of degree 1.

By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0 \cdot v = 0 \quad \ldots (1)$$

and $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 1 \cdot w = w \quad \ldots (2)$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = w$$

Hence, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2 + y^2}.$$

Example 8: If $\boldsymbol{u = \frac{x^3 y^3 z^3}{x^2 + y^2 + z^2} + \cos\left(\frac{xy + yz + xz}{x^2 + y^2 + z^2}\right)}$**, then show that**

$$\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{7x^3 y^3 z^3}{x^2 + y^2 + z^2}}.$$

Solution: Let $u = v + w$

where, $$v = \frac{x^3 y^3 z^3}{x^2 + y^2 + z^2}, \quad w = \cos\left(\frac{xy + yz + xz}{x^2 + y^2 + z^2}\right)$$

Replacing x by xt, y by yt and z by zt,

$$v = t^7\left(\frac{x^3 y^3 z^3}{x^2 + y^2 + z^2}\right), \quad w = t^0 \cos\left(\frac{xy + yz + xz}{x^2 + y^2 + z^2}\right)$$

Hence, v is a homogeneous function of degree 7 and w is homogeneous function of degree 0.

By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 7v \quad \ldots (1)$$

and $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0 \cdot w = 0 \quad \ldots (2)$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)=7v$$

Hence,
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{7x^3y^3z^3}{x^2+y^2+z^2}.$$

Example 9: If $v=\frac{1}{r}f(\theta)$ where $x=r\cos\theta$, $y=r\sin\theta$, show that $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+v=0$.

Solution: $x=r\cos\theta$, $y=r\sin\theta$

$$r=\sqrt{x^2+y^2} \text{ and } \theta=\tan^{-1}\left(\frac{y}{x}\right)$$

$$v=\frac{1}{r}f(\theta)=\frac{1}{\sqrt{x^2+y^2}}f\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Replacing x by xt and y by yt,

$$v=\frac{t^{-1}}{\sqrt{x^2+y^2}}f\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Hence, v is a homogeneous function of degree -1.
By Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial y}{\partial y}=-1\cdot v$$

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+v=0.$$

Example 10: If $x=e^u\tan v$, $y=e^u\sec v$ and $z=e^{-2u}f(v)$, prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+2z=0$.

Solution:
$$x=e^u\tan v,\ y=e^u\sec v$$

$$y^2-x^2=e^{2u}(\sec^2 v-\tan^2 v)=e^{2u}$$

$$e^{-2u}=\frac{1}{y^2-x^2}$$

$$\frac{x}{y}=\frac{\tan v}{\sec v}=\sin v$$

$$v=\sin^{-1}\left(\frac{x}{y}\right)$$

$$z=e^{-2u}f(v)=\frac{1}{y^2-x^2}f\left(\sin^{-1}\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$z = \frac{1}{t^2(y^2 - x^2)} f\left(\sin^{-1}\frac{x}{y}\right) = t^{-2}\frac{1}{(y^2 - x^2)} f\left(\sin^{-1}\frac{x}{y}\right)$$

Hence, z is a homogeneous function of degree -2.

By Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -2z$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + 2z = 0.$$

Example 11: If $u = f(v)$ where v is a homogeneous function of x, y of degree n, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nvf'(v)$. Hence, deduce that if $u = \log v$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n$.

Solution:

$$u = f(v)$$

$$\frac{\partial u}{\partial x} = f'(v)\frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y} = f'(v)\frac{\partial v}{\partial y}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xf'(v)\frac{\partial v}{\partial x} + yf'(v)\frac{\partial v}{\partial y} = f'(v)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

$$= f'(v)\cdot nv \qquad \text{... (1)}$$

$\left[\because\ v \text{ is a homogeneous function of degree } n\text{, By Euler's theorem } x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv\right]$

If $u = \log v, f(v) = \log v,\ f'(v) = \frac{1}{v}$

Substituting in Eq. (1),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{v}\cdot nv = n.$$

Example 12: If $u = \left(\frac{x}{y}\right)^{\frac{y}{x}}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

Solution:

$$u = \left(\frac{x}{y}\right)^{\frac{y}{x}}$$

Replacing x by xt and y by yt,

$$u = t^0\left(\frac{x}{y}\right)^{\frac{y}{x}}$$

Hence, u is a homogeneous function of degree 0.

By Cor. 1

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0(0-1)u = 0.$$

Example 13: If $u = \log\left(\frac{\sqrt{x^2+y^2}}{x+y}\right)$, find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Solution:
$$u = \log\left(\frac{\sqrt{x^2+y^2}}{x+y}\right)$$

Replacing x by xt and y by yt in u,
$$u = t^0 \log\left(\frac{\sqrt{x^2+y^2}}{x+y}\right)$$

Hence, u is a homogeneous function of degree 0.

By Cor. 1
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0(0-1)u$$
$$= 0.$$

Example 14: If $u = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Solution: Let $u = v + w$,

where,
$$v = xf\left(\frac{y}{x}\right) \text{ and } w = g\left(\frac{y}{x}\right)$$

Replacing x by xt and y by yt,
$$v = txf\left(\frac{y}{x}\right) \text{ and } w = t^0 g\left(\frac{y}{x}\right)$$

Hence, v is a homogeneous function of degree 1 and w is a homogeneous function of degree 0.

By Cor. 1
$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 1(1-1)v = 0 \qquad \text{... (1)}$$
$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 0(0-1)w = 0 \qquad \text{... (2)}$$

Adding Eqs (1) and (2),
$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0$$

Hence,
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Example 15: If $z = \frac{(x^2+y^2)^n}{2n(2n-1)} + xf\left(\frac{y}{x}\right) + \phi\left(\frac{x}{y}\right)$, show that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (x^2+y^2)^n.$$

Solution: Let $z = u + v + w$

where, $$u = \frac{(x^2+y^2)^n}{2n(2n-1)}, \quad v = xf\left(\frac{y}{x}\right), \quad w = \phi\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt in u, v and w

$$u = \frac{t^{2n}(x^2+y^2)^n}{2n(2n-1)}, \quad v = t\,xf\left(\frac{y}{x}\right), \quad w = t^0\phi\left(\frac{x}{y}\right)$$

Hence, u, v and w are homogeneous function of degree $2n$, 1 and 0 respectively.

By Cor. 1,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2n(2n-1)u \qquad \dots (1)$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 1(1-1)\,v = 0 \qquad \dots (2)$$

and $$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 0(0-1)w = 0 \qquad \dots (3)$$

Adding Eqs (1), (2) and (3),

$$x^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\right)$$
$$+ y^2\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) = 2n(2n-1)u$$

Hence, $$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (x^2+y^2)^n.$$

Example 16: If $z = x^n f\left(\frac{y}{x}\right) + y^{-n} f\left(\frac{x}{y}\right)$, prove that

$$\boldsymbol{x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z}.$$

Solution: Let $z = u + v$

where, $$u = x^n f\left(\frac{y}{x}\right), \quad v = y^{-n} f\left(\frac{x}{y}\right)$$

Replacing x by xt and y by yt,

$$u = t^n x^n f\left(\frac{y}{x}\right), \quad v = t^{-n} y^{-n} f\left(\frac{x}{y}\right)$$

Hence, u is a homogeneous function of degree n and v is a homogeneous function of degree $-n$.

By Euler's theorem and Cor. 1

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n(n-1)u + nu$$
$$= n^2 u \qquad \dots (1)$$

and $$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\,\partial y}+y^2\frac{\partial^2 v}{\partial y^2}+x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=-n(-n-1)v-nv$$
$$=n^2 v \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}\right)+2xy\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x\,\partial y}\right)+y^2\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}\right)+x\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)$$
$$+y\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\right)=n^2(u+v)$$

Hence, $$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=n^2 z.$$

Example 17: If $u=\dfrac{x^3+y^3}{y\sqrt{x}}+\dfrac{1}{x^7}\sin^{-1}\left(\dfrac{x^2+y^2}{x^2+2xy}\right)$, find value of

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\ \textbf{at}\ x=1,\ y=2.$$

Solution: Let $u = v + w$

where, $$v=\left(\frac{x^3+y^3}{y\sqrt{x}}\right)\ \text{and}\ w=\left[\frac{1}{x^7}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)\right]$$

Replacing x by xt and y by yt,

$$v=t^{\frac{3}{2}}\left(\frac{x^3+y^3}{y\sqrt{x}}\right)\ \text{and}\ w=t^{-7}\left[\frac{1}{x^7}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)\right]$$

Hence, v is a homogeneous function of degree $\dfrac{3}{2}$ and w is a homogeneous function of degree -7.

By Euler's theorem and Cor. 1,

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\,\partial y}+y^2\frac{\partial^2 v}{\partial y^2}+x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{3}{2}\left(\frac{3}{2}-1\right)v+\frac{3}{2}v=\frac{9}{4}v \qquad \text{... (1)}$$

$$x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\,\partial y}+y^2\frac{\partial^2 w}{\partial y^2}+x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=-7(-7-1)w-7w=49w \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right)+2xy\left(\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial y}\right)+y^2\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right)+x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)$$
$$+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)=\frac{9v}{4}+49w$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{9v}{4}+49w$$

At $x = 1, y = 2$,

$$v = \frac{1+8}{2\sqrt{1}} = \frac{9}{2}.$$

and
$$w = \frac{1}{(1)^7}\sin^{-1}\left(\frac{1+4}{1+4}\right) = \frac{\pi}{2}$$

Hence, $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{81}{8} + \dfrac{49\pi}{2}.$

Example 18: If $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + x^4 \tan^{-1}\left(\frac{y}{x}\right)$, find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ at $x = 1, y = 1$.

Solution: Let $u = v + w$

where, $v = x^3 \sin^{-1}\left(\frac{y}{x}\right)$, $w = x^4 \tan^{-1}\left(\frac{y}{x}\right)$

Replacing x by xt and y by yt,

$$v = t^3\left[x^3 \sin^{-1}\left(\frac{y}{x}\right)\right] \text{ and } w = t^4\left[x^4 \tan^{-1}\left(\frac{y}{x}\right)\right]$$

Hence, v is a homogeneous function of degree 3 and w is a homogeneous function of degree 4.

By Euler's theorem,

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} + x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3(3-1)v + 3v$$

$$= 9v \qquad \text{... (1)}$$

and
$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 4(4-1)w + 4w$$

$$= 16w \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) + x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right)$$

$$= 9v + 16w$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9v + 16w$$

At $x = 1, y = 1$,

$$v = \sin^{-1} 1 = \frac{\pi}{2} \text{ and } w = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence, $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{9\pi}{2} + \dfrac{16\pi}{4} = \dfrac{17\pi}{2}.$

Example 19: If $u = \dfrac{x^4 + y^4}{x^2 y^2} + x^6 \tan^{-1}\left(\dfrac{x^2 + y^2}{x^2 + 2xy}\right)$**, find the value of**

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \text{ at } x = 1, y = 2.$$

Solution: Let $u = v + w$

where, $$v = \frac{x^4 + y^4}{x^2 y^2} \quad \text{and} \quad w = x^6 \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + 2xy}\right)$$

Replacing x by xt and y by yt,

$$v = t^0 \left(\frac{x^4 + y^4}{x^2 y^2}\right) \quad \text{and} \quad w = t^6 \left[x^6 \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + 2xy}\right)\right]$$

Hence, v is a homogeneous function of degree 0 and w is a homogeneous function of degree 6.

By Euler's theorem, and Cor. 1

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x\, \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} + x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 0(0-1)v + 0 \cdot v$$

$$= 0. \qquad \text{... (1)}$$

$$x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x\, \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} + x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = 6(6-1)w + 6w$$

$$= 36w \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy \left(\frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 w}{\partial x\, \partial y}\right) + y^2 \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) + x \left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right)$$

$$+ y \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = 36w$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 36w$$

At $x = 1, y = 2$,

$$w = \tan^{-1}\left(\frac{1+4}{1+4}\right) = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence, $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{36\pi}{4} = 9\pi.$$

Example 20: If $f(x, y, z) = 0$ **where** $f(x, y, z)$ **is a homogeneous function of degree** n**, then show that**

$$x^2 \frac{\partial^2 z}{\partial x^2} = -xy \frac{\partial^2 z}{\partial x\, \partial y} = y^2 \frac{\partial^2 z}{\partial y^2}.$$

Solution: Here z is an implicit function of x and y,

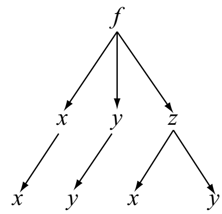


Fig. 4.35

$$f(x, y, z) = 0,$$

$$\frac{\partial f}{\partial x} = 0$$

Using composite function,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \qquad \left[\because \frac{\partial y}{\partial x} = 0\right]$$

$$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} \qquad \text{... (1)}$$

Similarly,

$$\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \qquad \text{... (2)}$$

f is a homogeneous function of degree n.

By Euler's theorem.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf = 0 \qquad [\because f(x, y, z) = 0]$$

Substituting $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ from Eqs (1) and (2),

$$x\left(-\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}\right) + y\left(-\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y}\right) + z\frac{\partial f}{\partial z} = 0$$

$$-x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + z = 0$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \qquad \text{... (3)}$$

Differentiating Eq. (3) w.r.t. x,

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial z}{\partial x}$$

$$x\frac{\partial^2 z}{\partial x^2} = -y\frac{\partial^2 z}{\partial x\,\partial y}$$

$$x^2\frac{\partial^2 z}{\partial x^2} = -xy\frac{\partial^2 z}{\partial x\,\partial y} \qquad \text{... (4)}$$

Again differentiating (3) w.r.t. y,

$$x\frac{\partial^2 z}{\partial y\,\partial x} + y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}$$

$$x\frac{\partial^2 z}{\partial y\,\partial x} = -y\frac{\partial^2 z}{\partial y^2}$$

$$y^2\frac{\partial^2 z}{\partial y^2} = -xy\frac{\partial^2 z}{\partial x\,\partial y} \qquad \text{... (5)}$$

From Eqs (4) and (5), we get

$$x^2\frac{\partial^2 z}{\partial x^2} = -xy\frac{\partial^2 z}{\partial x\,\partial y} = y^2\frac{\partial^2 z}{\partial y^2}.$$

Exercise 4.4

1. Verify Euler's theorem for

(i) $u = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{y}\right)$

(ii) $u = \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

(iii) $u = \log\left(\dfrac{x^2+y^2}{x^2-y^2}\right)$

(iv) $u = 3x^2yz + 5xy^2z + 4z^4$

(v) $u = \dfrac{x^2+y^2+z^2}{x+y+z}$

(vi) $u = \dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}$.

2. If $u = \cos\dfrac{xy+yz+zx}{x^2+y^2+z^2}$, find

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}.$$

[**Ans. :** 0]

3. If $u = \cos\left(\dfrac{xy+yz}{x^2+y^2+z^2}\right)$

$$+\sin\left[\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{(xy)^{\frac{1}{4}}}\right],$$

evaluate $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}$.

[**Ans. :** 0]

4. If $u = \sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$

5. If $z = x^3 e^{-\frac{x}{y}}$, find the value of

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}.$$

[**Ans. :** $6z$]

6. If $u = x^2yz - 4y^2z^2 + 2xz^3$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = -4u.$$

7. If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u} = 1$,

where u is a homogeneous function in x, y, z of degree n, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2nu.$$

8. If $u = \dfrac{x^3y^3}{x^3+y^3}$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3u.$$

9. If $u = \dfrac{x^2+y^2}{\sqrt{x+y}}$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{3}{2}u.$$

10. If $u = \dfrac{xy}{x+y}$, find the value of

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}.$$

[**Ans. :** 0]

11. If $u = \dfrac{x^2y^2z^2}{x^2+y^2+z^2}+\cos\dfrac{xy+yz}{x^2+y^2+z^2}$,

prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = \frac{4x^2y^2z^2}{x^2+y^2+z^2}.$$

12. If $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

13. If $u = 3x^4 \cot^{-1}\left(\frac{y}{x}\right) + 16y^4 \cos^{-1}\left(\frac{y}{x}\right)$,

prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 12u.$$

14. If $u = y^2 e^{\frac{y}{x}} + x^2 \tan^{-1}\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u.$$

15. If $u = x^3 y^2 \sin^{-1}\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 25u.$$

16. If $u = x^2 \log\left(\frac{\sqrt[3]{y} - \sqrt[3]{x}}{\sqrt[3]{y} + \sqrt[3]{x}}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2 \log\left(\frac{\sqrt[3]{y} - \sqrt[3]{x}}{\sqrt[3]{y} + \sqrt[3]{x}}\right).$$

17. If $u = f\left(\frac{x^2 - y^2}{z^2}, \frac{y^2 - z^2}{x^2}, \frac{z^2 - x^2}{y^2}\right)$,

prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

18. If $u = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^n$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

19. If $u = x^2 \sin^{-1}\frac{y}{x} - y^2 \cos^{-1}\frac{x}{y}$, prove

that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u.$

20. If $u = x\sin^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x}$, find the

value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

[**Ans. :** 0]

21. If $y = x\cos u$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

$\left[\textbf{Hint:}\ \cos u = \frac{y}{x},\ u = \cos^{-1}\frac{y}{x}\right]$

22. If $u = x^3\left[\tan^{-1}\left(\frac{y}{x}\right) + \frac{y}{x}e^{-\frac{y}{x}}\right] + y^{-3}\left[\sin^{-1}\left(\frac{x}{y}\right) + \frac{x}{y}\log\frac{x}{y}\right]$,

prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9u.$$

23. If $z = f(x, y)$ and u, v are homogeneous functions of degree n in x, y, then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right).$$

24. If $u = (x^2 + y^2)^{\frac{2}{3}}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{4}{9}u.$$

Corollary 2: If $z = f(u)$ is a homogeneous function of degree n in variables x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}$.

Proof: By Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = nf(u)$$

$$x\frac{\partial}{\partial x}f(u) + y\frac{\partial}{\partial y}f(u) = nf(u)$$

$$xf'(u)\frac{\partial u}{\partial x} + yf'(u)\frac{\partial u}{\partial y} = nf(u)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

Note: If $v = f(u)$ is a homogeneous function of degree n in variables x, y and z, then

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}$.

Corollary 3: If $z = f(u)$ is a homogeneous function of degree n in variables x and y, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where, $g(u) = n\frac{f(u)}{f'(u)}$.

Proof: By Cor. 2,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = g(u) \qquad \text{... (1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = g'(u)\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = [g'(u) - 1]\frac{\partial u}{\partial x} \qquad \text{... (2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = g'(u)\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = [g'(u) - 1]\frac{\partial u}{\partial y} \qquad \text{... (3)}$$

Multiplying Eq. (2) by x and Eq. (3) by y and adding,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=[g'(u)-1]\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=[g'(u)-1]g(u) \qquad \text{[Using Eq. (1)]}$$

where, $g(u)=n\dfrac{f(u)}{f'(u)}$.

Example 1: If $u=\sec^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2\cot u$.

Solution:

$$u=\sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$$

Replacing x by xt and y by yt,

$$u=\sec^{-1}\left[t^2\left(\frac{x^3+y^3}{x+y}\right)\right]$$

u is a non-homogeneous function. But $\sec u=\dfrac{x^3+y^3}{x+y}$ is a homogeneous function of degree 2.

Let $f(u)=\sec u$

By Cor. 2,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=2\frac{\sec u}{\sec u\tan u}=2\cot u$$

Example 2: If $u=\sin^{-1}(xyz)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=3\tan u$.

Solution:

$$u=\sin^{-1}(xyz)$$

Replacing x by xt, y by yt and z by zt,

$$u=\sin^{-1}\left[t^3(xyz)\right]$$

u is a non-homogeneous function. But $\sin u=xyz$ is a homogeneous function of degree 3.

Let $f(u)=\sin u$

By Cor. 2,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{f(u)}{f'(u)}=3\frac{\sin u}{\cos u}=3\tan u.$$

Example 3: If $u = \log x + \log y$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2$.

Solution:
$$u = \log x + \log y = \log xy$$

Replacing x by xt and y by yt,

$$u = \log\left[t^2(xy)\right]$$

u is a non-homogeneous function. But $e^u = xy$ is a homogeneous function of degree 2.

Let $f(u) = e^u$

By Cor. 2,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{e^u}{e^u} = 2.$$

Example 4: If $u = \log(x^2 + y^2 + z^2)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2$.

Solution:
$$u = \log(x^2 + y^2 + z^2)$$

Replacing x by xt, y by yt, and z by zt,

$$u = \log\left[t^2(x^2 + y^2 + z^2)\right]$$

u is a non-homogeneous function. But $e^u = x^2 + y^2 + z^2$ is a homogeneous function of degree 2.

Let $f(u) = e^u$

By Cor. 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)} = 2\frac{e^u}{e^u} = 2$$

Example 5: If $u = e^{x^2 f\left(\frac{y}{x}\right)}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$.

Solution:
$$u = e^{x^2 f\left(\frac{y}{x}\right)}$$

Replacing x by xt and y by yt,

$$u = e^{t^2 x^2 f\left(\frac{y}{x}\right)}$$

u is a non-homogeneous function. But $\log u = x^2 f\left(\dfrac{y}{x}\right)$ is homogeneous function of degree 2.

Let $f(u) = \log u$

By Cor. 2,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{\log u}{1/u} = 2u\log u.$$

Example 6: If $u = \tan\left(\frac{xy+yz}{x^2+y^2+z^2}\right) + \sin\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right).$$

Solution: Let $u = v + w$

where, $v = \tan\left(\frac{xy+yz}{x^2+y^2+z^2}\right)$ and $w = \sin\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$

Replacing x by xt, y by yt, and z by zt,

$$v = t^0 \tan\left(\frac{xy+yz}{x^2+y^2+z^2}\right),$$

$$w = \sin\left[t^{\frac{1}{2}}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\right]$$

v is a homogeneous function of degree 0.

By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 0 \cdot v = 0 \qquad \text{... (1)}$$

w is a non-homogeneous function. But $\sin^{-1} w = \left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$ is a homogeneous function of x, y, z of degree $\frac{1}{2}$.

Let $f(w) = \sin^{-1} w$

By Cor. 2,

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = n\frac{f(w)}{f'(w)} = \frac{1}{2}\frac{\sin^{-1} w}{\frac{1}{\sqrt{1-w^2}}}$$

$$= \frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\sqrt{1-\sin^2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}$$

$$= \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2} \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)=\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2}$$

Hence,
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2}.$$

Example 7: If $x=e^u\tan v, y=e^u\sec v$, prove that $\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=0.$

Solution:
$$x=e^u\tan v,\ y=e^u\sec v$$
$$y^2-x^2=e^{2u}(\sec^2 v-\tan^2 v)=e^{2u}$$

and
$$\frac{x}{y}=\sin v$$
$$v=\sin^{-1}\left(\frac{x}{y}\right)$$

v is homogeneous function of degree 0.
By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0$$

Hence,
$$\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=0.$$

Example 8: If $u=\sin^{-1}\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$, prove that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}(\tan^2 u+13).$

Solution:
$$u=\sin^{-1}\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

Replacing x by xt and y by yt, $u = \sin^{-1}\left[t^{-\frac{1}{12}}\left(\dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right]$

u is a non-homogeneous function. But $\sin u = \left(\dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$ is a homogeneous function with degree $-\dfrac{1}{12}$.

Let $f(u) = \sin u$

By Cor. 2,

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\dfrac{f(u)}{f'(u)} = -\dfrac{1}{12}\dfrac{\sin u}{\cos u} = -\dfrac{1}{12}\tan u.$

By Cor. 3,

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where, $$g(u) = n\frac{f(u)}{f'(u)} = \frac{-1}{12}\tan u$$

$$g'(u) = \frac{-1}{12}\sec^2 u$$

Hence, $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{1}{12}\tan u\left(-\frac{1}{12}\sec^2 u - 1\right)$$

$$= \frac{1}{12}\tan u\left(\frac{1 + \tan^2 u + 12}{12}\right) = \frac{\tan u}{144}(\tan^2 u + 13).$$

Example 9: If $u = \dfrac{1}{3}\log\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right)$, find the value of

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

Solution: $u = \dfrac{1}{3}\log\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right)$

Replacing x by xt and y by yt, $$u = \frac{1}{3}\log\left[t\left(\frac{x^3+y^3}{x^2+y^2}\right)\right]$$

u is a non-homogeneous function. But $e^{3u} = \dfrac{x^3+y^3}{x^2+y^2}$ is a homogeneous function of degree 1.

Let $f(u) = e^{3u}$

By Cor. 2,

(i) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 1\cdot\frac{e^{3u}}{3e^{3u}} = \frac{1}{3}.$$

By Cor. 3,

(ii) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

where, $$g(u) = n\frac{f(u)}{f'(u)} = \frac{1}{3}$$
$$g'(u) = 0.$$

Hence, $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{3}(0-1) = \frac{-1}{3}.$$

Example 10: If $u = \tan^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, prove that

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2\sin^3 u\cos u.}$$

Solution: $$u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

Replacing x by xt and y by yt,

$$u = \tan^{-1}\left[t\left(\frac{x^2+y^2}{x+y}\right)\right]$$

u is a non-homogeneous function. But $\tan u = \dfrac{x^2+y^2}{x+y}$ is a homogeneous function of degree 1.

Let $f(u) = \tan u$

By Cor. 3,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

where,

$$g(u) = n\frac{f(u)}{f'(u)} = 1\cdot\frac{\tan u}{\sec^2 u} = \sin u\cos u = \frac{\sin 2u}{2}$$

$$g'(u) = \cos 2u$$

Hence,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\sin 2u}{2}(\cos 2u - 1)$$

$$= \sin u \cos u (-2\sin^2 u) = -2\sin^3 u \cos u.$$

Example 11: If $u = \sinh^{-1}\left(\frac{x^3 + y^3}{x^2 + y^2}\right)$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\tanh^3 u.$$

Solution:

$$u = \sinh^{-1}\left(\frac{x^3 + y^3}{x^2 + y^2}\right)$$

Replacing x by xt and y by yt,

$$u = \sinh^{-1}\left[t\left(\frac{x^3 + y^3}{x^2 + y^2}\right)\right]$$

u is a non-homogeneous function. But $\sinh u = \frac{x^3 + y^3}{x^2 + y^2}$ is a homogeneous function of degree 1.

Let $f(u) = \sinh u$

By Cor. 3,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

where,

$$g(u) = n\frac{f(u)}{f'(u)} = 1 \cdot \frac{\sinh u}{\cosh u} = \tanh u$$

$$g'(u) = \operatorname{sech}^2 u.$$

Hence, $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \tanh u(\operatorname{sech}^2 u - 1)$$

$$= \tanh u(-\tanh^2 u) = -\tanh^3 u.$$

Example 12: If $u = \log \frac{x + y}{\sqrt{x^2 + y^2}} + \sin^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin w \cos 2w}{4\cos^3 w}, \text{ where } w = \sin^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right).$$

Solution: Let $u = v + w$

where, $$v = \log \frac{x + y}{\sqrt{x^2 + y^2}}, \qquad w = \sin^{-1}\frac{x + y}{\sqrt{x} + \sqrt{y}}$$

Replacing x by xt and y by yt,

$$v = t^0 \log \frac{x+y}{\sqrt{x^2+y^2}}, \qquad w = \sin^{-1}\left[t^{\frac{1}{2}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right]$$

v is a homogeneous function of degree 0.

By Cor. 1,

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 0\cdot v = 0 \qquad \text{... (1)}$$

and w is a non-homogeneous function. But $\sin w = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree $\dfrac{1}{2}$.

Let $f(w) = \sin w$

By Cor. 3,

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = g(w)[g'(w)-1]$$

where,
$$g(w) = n\frac{f(w)}{f'(w)} = \frac{1}{2}\frac{\sin w}{\cos w} = \frac{1}{2}\tan w$$

$$g'(w) = \frac{1}{2}\sec^2 w.$$

Hence,
$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = \frac{1}{2}\tan w\left(\frac{1}{2}\sec^2 w - 1\right)$$

$$= \frac{1}{2}\sin w\frac{(1-2\cos^2 w)}{2\cos^3 w} = -\frac{\sin w\cos 2w}{4\cos^3 w} \qquad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial x\,\partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right) = -\frac{\sin w\cos 2w}{4\cos^3 w}$$

Hence, $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\sin w\cos 2w}{4\cos^3 w}$, where $w = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$.

Example 13: If $u = \log(x^3+y^3+z^3-3xyz)$, then show that

(i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x+y+z}$

(ii) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3$

(iii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial x\,\partial y}+2\frac{\partial^2 u}{\partial y\,\partial z}+2\frac{\partial^2 u}{\partial z\,\partial x}=\frac{-9}{(x+y+z)^2}$$

(iv) $$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+2yz\frac{\partial^2 u}{\partial y\,\partial z}+2zx\frac{\partial^2 u}{\partial z\,\partial x}=-3.$$

Solution: $u = \log\,(x^3 + y^3 + z^3 - 3xyz)$

(i) Differentiating u w.r.t. x, y and z,

$$\frac{\partial u}{\partial x}=\frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial u}{\partial y}=\frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

$$\frac{\partial u}{\partial z}=\frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$=\frac{3}{x+y+z}.$$

(ii) Replacing x by xt, y by yt and z by zt,

$$u=\log t^3(x^3+y^3+z^3-3xyz)$$

u is a non-homogeneous function. But $e^u = x^3 + y^3 + z^3 - 3xyz$ is a homogeneous function of degree 3.

Let $f(u) = e^u$

By Cor. 2,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{f(u)}{f'(u)}=3\frac{e^u}{e^u}=3.$$

(iii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial x\,\partial y}+2\frac{\partial^2 u}{\partial y\,\partial z}+2\frac{\partial^2 u}{\partial z\,\partial x}=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)=\frac{-3}{(x+y+z)}(1+1+1)$$

$$=\frac{-9}{(x+y+z)^2}.$$

(iv) By Cor. 3,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

where,

$$g(u)=n\frac{f(u)}{f'(u)}=3$$

$$g'(u)=0$$

Hence,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=3(0-1)=-3.$$

Example 14: If $u=\log r$ and $r^2=x^2+y^2$, prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+1=0.$

Solution:

$$u=\log r=\log\sqrt{x^2+y^2}$$

Replacing x by xt and y by yt,

$$u=\log\left(t\sqrt{x^2+y^2}\right)$$

u is a non-homogeneous function of x and y. But $e^u=\sqrt{x^2+y^2}$ is a homogeneous function of degree 1.

Let $f(u)=e^u$

By Cor. 3,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\,[g'(u)-1]$$

where,

$$g(u)=n\frac{f(u)}{f'(u)}=\frac{e^u}{e^u}=1$$

$$g'(u)=0.$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=1(0-1)=-1$$

Hence, $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+1=0.$$

Example 15: If $u=\log r$, $r=x^3+y^3-x^2y-xy^2$, show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u=-\frac{4}{(x+y)^2}$ and $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$

Solution: $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)$$

$$u=\log(x^3+y^3-x^2y-xy^2)$$

$$=\log\left[(x+y)(x^2+y^2-xy)-xy(x+y)\right]$$

$$= \log(x+y)(x^2+y^2-2xy) = \log(x+y)+2\log(x-y)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x+y}+\frac{2}{x-y}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y}-\frac{2}{x-y}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = \frac{2}{x+y}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{2}{x+y}\right)$$

$$= -\frac{2}{(x+y)^2}-\frac{2}{(x+y)^2} = -\frac{4}{(x+y)^2}$$

Replacing x by xt and y by yt in u,

$$u = \log t^3(x^3+y^3-x^2y-xy^2)$$

u is a non-homogeneous function. But $e^u = x^3+y^3-x^2y-xy^2$ is a homogeneous function of degree 3.

Let $f(u) = e^u$

By Cor. 2,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 3\frac{e^u}{e^u} = 3.$$

Exercise 4.5

1. If $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

2. If $u = \sin^{-1}\left(\dfrac{x^2y^2}{x+y}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3\tan u.$$

3. If $u = \log(x^2+xy+y^2)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2.$$

4. If $(x-y)\tan u = x^3+y^3$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u.$$

5. If $u = \log(x^3+y^3-x^2y-xy^2)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = 3.$

6. If $u = \sin^{-1}\left(\dfrac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z} = 2\tan u.$

7. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u.$$

8. If $u = \sin^{-1}\left(\dfrac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{6}}+y^{\frac{1}{6}}}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{144}\tan u(\tan^2 u-1).$$

9. If $\left(\sqrt{x}+\sqrt{y}\right)\sin^2 u = x^{\frac{1}{3}}+y^{\frac{1}{3}}$, prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$

$$=\frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right).$$

10. If $u=\cos^{-1}\left(\dfrac{x^5-2y^5+6z^5}{\sqrt{ax^3+by^3+cz^3}}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-\frac{7}{2}\cot u.$

11. If $\left(\sqrt{x}+\sqrt{y}\right)\cot\ u-x-y=0$, prove that $4x\frac{\partial u}{\partial x}+4y\frac{\partial u}{\partial y}+\sin 2u=0.$

$$\left[\textbf{Hint:}\ \cot u=\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

12. If $u=\sin^{-1}\left(\dfrac{ax+by+cz}{\sqrt{x^n+y^n+z^n}}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2\tan u.$

4.8 APPLICATIONS OF PARTIAL DIFFERENTIATION

4.8.1 Jacobians

If u and v are continuous and differentiable functions of two independent variables x and y, i.e., $u=f_1(x, y)$ and $v=f_2(x, y)$, then the determinant $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and is denoted as $J=\dfrac{\partial(u, v)}{\partial(x, y)}$.

Similarly, if u, v and w are continuous and differentiable functions of three independent variables x, y, z, then the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

Jacobian is useful in transformation of variables from cartesian to polar, cylindrical and spherical coordinates in multiple integrals.

Properties of Jacobians

1. If u and v are functions of x and y, then

$$J \cdot J^* = 1 \text{ where } J = \frac{\partial(u, v)}{\partial(x, y)} \text{ and } J^* = \frac{\partial(x, y)}{\partial(u, v)}$$

Proof: Let u and v are two functions of x and y.

$$u = f_1(x, y) \text{ and } v = f_2(x, y) \qquad \text{... (1)}$$

Writing x and y in terms of u and v,

$$x = \phi_1(u, v) \text{ and } y = \phi_2(u, v) \qquad \text{... (2)}$$

Differentiating Eq. (1) partially w.r.t. u and v,

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} \qquad \text{... (3)}$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} \qquad \text{... (4)}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} \qquad \text{... (5)}$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} \qquad \text{... (6)}$$

$$J \,.\, J^* = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad \left[\begin{array}{l}\text{Interchanging rows and columns} \\ \text{of second determinant}\end{array}\right]$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad \text{[Substituting Eqs (3), (4), (5), (6)]}$$

$$= 1$$

Similarly $$\frac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(u, v, w)} = 1$$

2. If u, v are functions of r, s and r, s are functions of x, y, then

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}.$$

Proof:

$$\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix} \qquad \left[\begin{array}{l}\text{Interchanging rows} \\ \text{and columns of} \\ \text{second determinant}\end{array}\right]$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}$$

Similarly, $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{\partial(u, v, w)}{\partial(r, s, t)} \cdot \dfrac{\partial(r, s, t)}{\partial(x, y, z)}.$

3. If functions u, v of two independent variables x, y are dependent, then $\dfrac{\partial(u, v)}{\partial(x, y)} = 0.$

Proof: If u, v are dependent, then there must be a relation $f(u, v) = 0$ ... (1)

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = 0 \qquad \text{... (2)}$$

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \qquad \text{... (3)}$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from Eqs (2) and (3),

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad \left[\begin{array}{l}\text{Interchanging rows and columns} \\ \text{of the second determinant}\end{array}\right]$$

$$\frac{\partial(u, v)}{\partial(x, y)} = 0.$$

Example 1: Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for each of the following functions:

(i) $u = x^2 - y^2$, $v = 2xy$ **(ii) $u = x \sin y$, $v = y \sin x$**

(iii) $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$ **(iv) $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$.**

Solution: (i) $u = x^2 - y^2$ $\qquad v = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4\,(x^2 + y^2)$$

(ii) $u = x \sin y$ $\qquad v = y \sin x$

$$\frac{\partial u}{\partial x} = \sin y \qquad \frac{\partial v}{\partial x} = y \cos x$$

$$\frac{\partial u}{\partial y} = x \cos y \qquad \frac{\partial v}{\partial y} = \sin x$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sin y & x \cos y \\ y \cos x & \sin x \end{vmatrix}$$

$$= \sin x \sin y - xy \cos x \cos y$$

(iii) $u = x + \frac{y^2}{x}$ $\qquad v = \frac{y^2}{x}$

$$\frac{\partial u}{\partial x} = 1 - \frac{y^2}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{-y^2}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x} \qquad \frac{\partial v}{\partial y} = \frac{2y}{x}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1-\frac{y^2}{x^2} & \frac{2y}{x} \\ \frac{-y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = \frac{2y}{x} - \frac{2y^3}{x^3} + \frac{2y^3}{x^3} = \frac{2y}{x}$$

(iv) $u = \dfrac{x+y}{1-xy}$ $\qquad v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$= \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$= \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$$

Example 2: Find the Jacobian for each of the following functions:

(i) $x = r\cos\theta$, $\qquad y = r\sin\theta$

(ii) $x = a\cosh\theta\cos\phi$, $\qquad y = a\sinh\theta\sin\phi$.

Solution:

(i) $x = r\cos\theta$ $\qquad y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

(ii) $x = a\cosh\theta\cos\phi$ $\quad\quad$ $y = a\sinh\theta\sin\phi$

$$\frac{\partial x}{\partial\theta} = a\sinh\theta\cos\phi \qquad \frac{\partial y}{\partial\theta} = a\cosh\theta\sin\phi$$

$$\frac{\partial x}{\partial\phi} = -a\cosh\theta\sin\phi \qquad \frac{\partial y}{\partial\phi} = a\sinh\theta\cos\phi$$

$$J = \frac{\partial(x,y)}{\partial(\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{vmatrix} = \begin{vmatrix} a\sinh\theta\cos\phi & -a\cosh\theta\sin\phi \\ a\cosh\theta\sin\phi & a\sinh\theta\cos\phi \end{vmatrix}$$

$$= a^2(\sinh^2\theta\cos^2\phi + \cosh^2\theta\sin^2\phi)$$

$$= a^2[\sinh^2\theta(1-\sin^2\phi) + (1+\sinh^2\theta)\sin^2\phi]$$

$$= a^2(\sinh^2\theta + \sin^2\phi)$$

$$= \frac{a^2}{2}(\cosh 2\theta - 1 + 1 - \cos 2\phi)$$

$$= \frac{a^2}{2}(\cosh 2\theta - \cos 2\phi)$$

Example 3: Find the Jacobian for each of the following functions:

(i) $u = xyz,\quad v = x^2 + y^2 + z^2,\quad w = x + y + z$

(ii) $x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$

(iii) $x = \dfrac{vw}{u},\quad y = \dfrac{wu}{v},\quad z = \dfrac{uv}{w}.$

Solution:

(i) $u = xyz \qquad v = x^2 + y^2 + z^2 \qquad w = x + y + z$

$$\frac{\partial u}{\partial x} = yz \qquad \frac{\partial v}{\partial x} = 2x \qquad \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = xz \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial w}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = xy \qquad \frac{\partial v}{\partial z} = 2z \qquad \frac{\partial w}{\partial z} = 1$$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y - 2z) - xz\,(2x - 2z) + xy(2x - 2y)$$
$$= 2y^2z - 2yz^2 - 2x^2z + 2xz^2 + 2x^2y - 2xy^2$$
$$= 2[x^2(y - z) - x(y^2 - z^2) + yz(y - z)]$$
$$= 2(y - z)\,[x^2 - x\,(y + z) + yz]$$
$$= 2(y - z)\,[y(z - x) - x(z - x)] = 2(y - z)\,(z - x)\,(y - x)$$

(ii) $x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \qquad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \qquad \frac{\partial z}{\partial \phi} = 0$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\,[\cos\theta(\cos\theta\sin\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi]$$
$$+ \sin\theta(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi)]$$
$$= r^2\,(\sin\theta\cos^2\theta + \sin^3\theta)$$

(iii) $x = \dfrac{vw}{u} \qquad y = \dfrac{wu}{v} \qquad z = \dfrac{uv}{w}$

$$\frac{\partial x}{\partial u} = \frac{-vw}{u^2} \qquad \frac{\partial y}{\partial u} = \frac{w}{v} \qquad \frac{\partial z}{\partial u} = \frac{v}{w}$$

$$\frac{\partial x}{\partial v} = \frac{w}{u} \qquad \frac{\partial y}{\partial v} = \frac{-wu}{v^2} \qquad \frac{\partial z}{\partial v} = \frac{u}{w}$$

$$\frac{\partial x}{\partial w}=\frac{v}{u} \qquad \frac{\partial y}{\partial w}=\frac{u}{v} \qquad \frac{\partial z}{\partial w}=\frac{-uv}{w^2}$$

$$J=\frac{\partial(x,y,z)}{\partial(u,v,z)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}\frac{-vw}{u^2} & \frac{w}{u} & \frac{v}{u}\\ \frac{w}{v} & \frac{-wu}{v^2} & \frac{u}{v}\\ \frac{v}{w} & \frac{u}{w} & \frac{-uv}{w^2}\end{vmatrix}$$

$$=\frac{1}{u^2v^2w^2}\begin{vmatrix}-vw & wu & uv\\ vw & -wu & uv\\ vw & wu & -uv\end{vmatrix}=\frac{u^2v^2w^2}{u^2v^2w^2}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}$$

$$=-1(1-1)-1(-1-1)+1(1+1)=4.$$

Example 4: Verify $J\cdot J^*=1$ for the following functions:

(i) $x=e^u\cos v$, $\quad y=e^u\sin v$

(ii) $x=u$, $\quad y=u\tan v$, $\quad z=w$.

Solution: (i) $x=e^u\cos v \qquad y=e^u\sin v$

$$\frac{\partial x}{\partial u}=e^u\cos v \qquad \frac{\partial y}{\partial u}=e^u\sin v$$

$$\frac{\partial x}{\partial v}=-e^u\sin v \qquad \frac{\partial y}{\partial v}=e^u\cos v$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}e^u\cos v & -e^u\sin v\\ e^u\sin v & e^u\cos v\end{vmatrix}$$

$$=e^{2u}\begin{vmatrix}\cos v & -\sin v\\ \sin v & \cos v\end{vmatrix}$$

$$=e^{2u}(\cos^2 v+\sin^2 v)=e^{2u}$$

Writing u, v in terms of x and y,

$$\frac{y}{x}=\tan v \qquad x^2+y^2=e^{2u}$$

$$v=\tan^{-1}\left(\frac{y}{x}\right) \qquad u=\frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial v}{\partial x}=\frac{-y}{x^2+y^2} \qquad \frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2} \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$J^* = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2 + y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} = \frac{1}{e^{2u}}$$

Hence, $$J.\,J^* = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = e^{2u} \cdot \frac{1}{e^{2u}} = 1$$

(ii) $x = u \qquad y = u \tan v \qquad z = w$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = \tan v \qquad \frac{\partial z}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = 0 \qquad \frac{\partial y}{\partial v} = u \sec^2 v \qquad \frac{\partial z}{\partial v} = 0$$

$$\frac{\partial x}{\partial w} = 0 \qquad \frac{\partial y}{\partial w} = 0 \qquad \frac{\partial z}{\partial w} = 1$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u \sec^2 v$$

Writing u, v, w in terms of x, y and z,

$$u = x \qquad \tan v = \frac{y}{u} = \frac{y}{x} \qquad w = z$$

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2} \qquad \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} \qquad \frac{\partial w}{\partial y} = 0$$

$$\frac{\partial u}{\partial z} = 0 \qquad \frac{\partial v}{\partial z} = 0 \qquad \frac{\partial w}{\partial z} = 1$$

$$J^* = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{x}{x^2+y^2} = \frac{1}{x\left[1+\left(\frac{y}{x}\right)^2\right]} = \frac{1}{u\sec^2 v}$$

Hence, $J.J^* = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} \;= u\sec^2 v . \frac{1}{u\sec^2 v} \;= 1.$

Example 5: If $x = uv$ and $y = \dfrac{u+v}{u-v}$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Solution: $x = uv \qquad\qquad y = \frac{u+v}{u-v}$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial y}{\partial u} = \frac{-2v}{(u-v)^2}$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = \frac{2u}{(u-v)^2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix}$$

$$= \frac{4uv}{(u-v)^2}$$

We know that

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Hence,
$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}.$$

Example 6: If $u = \dfrac{2yz}{x}$, $v = \dfrac{3zx}{y}$, $w = \dfrac{4xy}{z}$, find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

Solution: $u = \dfrac{2yz}{x}$ $\quad v = \dfrac{3zx}{y}$ $\quad w = \dfrac{4xy}{z}$

$$\frac{\partial u}{\partial x} = \frac{-2yz}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{3z}{y} \qquad \frac{\partial w}{\partial x} = \frac{4y}{z}$$

$$\frac{\partial u}{\partial y} = \frac{2z}{x} \qquad \frac{\partial v}{\partial y} = \frac{-3zx}{y^2} \qquad \frac{\partial w}{\partial y} = \frac{4x}{z}$$

$$\frac{\partial u}{\partial z} = \frac{2y}{x} \qquad \frac{\partial v}{\partial z} = \frac{3x}{y} \qquad \frac{\partial w}{\partial z} = \frac{-4xy}{z^2}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & \frac{-3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & \frac{-4xy}{z^2} \end{vmatrix}$$

$$= \frac{-2yz}{x^2}\left(\frac{12x^2yz}{y^2z^2} - \frac{12x^2}{yz}\right) - \frac{2z}{x}\left(\frac{-12xyz}{yz^2} - \frac{12xy}{yz}\right) + \frac{2y}{x}\left(\frac{12xz}{yz} + \frac{12xyz}{zy^2}\right)$$

$$= 96$$

We know that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

Hence, $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{96}.$$

Example 7: If $u = x^2 - y^2$, $v = 2xy$, where $x = r\cos\theta$ and $y = r\sin\theta$, find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$.

Solution: $u = x^2 - y^2$ $\qquad v = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) = 4r^2$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Hence, $$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} . \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^2 .\ r = 4r^3.$$

Example 8: If $u = e^x \cos y$, $v = e^x \sin y$, where, $x = lr + sm$ and $y = mr - ls$, verify chain rule of Jacobians, l, m being constants.

Solution: $u = e^x \cos y \qquad v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$$= e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$$

$x = lr + sm \qquad y = mr - ls$

$$\frac{\partial x}{\partial r} = l \qquad \frac{\partial y}{\partial r} = m$$

$$\frac{\partial x}{\partial s} = m \qquad \frac{\partial y}{\partial s} = -l$$

$$\frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} l & m \\ m & -l \end{vmatrix} = -(l^2 + m^2)$$

$$\frac{\partial(u,v)}{\partial(x,y)} . \frac{\partial(x,y)}{\partial(r,s)} = -e^{2x}(l^2 + m^2) \qquad \ldots (1)$$

Now, $u = e^{lr+ms} \cos(mr - sl) \qquad v = e^{lr+ms} \sin(mr - sl)$

$$\frac{\partial u}{\partial r} = le^{lr+ms}\cos(mr - sl) - me^{lr+ms}\sin(mr - sl) \qquad \frac{\partial v}{\partial r} = le^{lr+ms}\sin(mr - sl) + me^{lr+ms}\cos(mr - sl)$$

$= le^x \cos y - me^x \sin y$

$$\frac{\partial u}{\partial s} = me^{lr+ms} \cos(mr - sl) + le^{lr+ms} \sin(mr - sl)$$

$= me^x \cos y + le^x \sin y$

$= le^x \sin y + me^x \cos y$

$$\frac{\partial v}{\partial s} = me^{lr+ms} \sin(mr - sl) - le^{lr+ms} \cos(mr - ls)$$

$= me^x \sin y - le^x \cos y$

$$\frac{\partial(u,v)}{\partial(r,s)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} = \begin{vmatrix} le^x \cos y - me^x \sin y & me^x \cos y + le^x \sin y \\ le^x \sin y + me^x \cos y & me^x \sin y - le^x \cos y \end{vmatrix}$$

$$= e^{2x} \begin{vmatrix} l\cos y - m\sin y & m\cos y + l\sin y \\ l\sin y + m\cos y & m\sin y - l\cos y \end{vmatrix}$$

$$= e^{2x} [(l \cos y - m \sin y)(m \sin y - l \cos y) - (l \sin y + m \cos y)(m \cos y + l \sin y)]$$

$$= e^{2x} [lm \cos y \sin y - l^2 \cos^2 y - m^2 \sin^2 y + lm \sin y \cos y - lm \sin y \cos y - l^2 \sin^2 y - m^2 \cos^2 y - lm \sin y \cos y]$$

$$= e^{2x} [-l^2(\cos^2 y + \sin^2 y) - m^2(\cos^2 y + \sin^2 y)]$$

$$= -e^{2x}(l^2 + m^2)$$

Hence, $\dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,s)} = \dfrac{\partial(u,v)}{\partial(r,s)}$.

Example 9: If $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$, find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

Solution: $x = \sqrt{vw}$ $\qquad y = \sqrt{uw}$ $\qquad z = \sqrt{uv}$

$$\frac{\partial x}{\partial u} = 0 \qquad \frac{\partial y}{\partial u} = \frac{1}{2}\sqrt{\frac{w}{u}} \qquad \frac{\partial z}{\partial u} = \frac{1}{2}\sqrt{\frac{v}{u}}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}\sqrt{\frac{w}{v}} \qquad \frac{\partial y}{\partial v} = 0 \qquad \frac{\partial z}{\partial v} = \frac{1}{2}\sqrt{\frac{u}{v}}$$

$$\frac{\partial x}{\partial w} = \frac{1}{2}\sqrt{\frac{v}{w}} \qquad \frac{\partial y}{\partial w} = \frac{1}{2}\sqrt{\frac{u}{w}} \qquad \frac{\partial z}{\partial w} = 0$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= -\frac{1}{2}\sqrt{\frac{w}{v}}\left(-\frac{1}{4}\sqrt{\frac{v}{w}}\right) + \frac{1}{2}\sqrt{\frac{v}{w}}\left(\frac{1}{4}\sqrt{\frac{w}{v}}\right) = \frac{1}{4}$$

and
$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(-r\sin\theta\cos\theta\cos\phi)$$
$$-r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi) = r^2\sin\theta$$

Hence,
$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)}\cdot\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \frac{1}{4}r^2\sin\theta.$$

Example 10: Determine whether the following functions are functionally dependent or not. If functionally dependent, find the relation between them.

(i) $u = e^x \sin y, v = e^x \cos y$

(ii) $u = \sin^{-1}x + \sin^{-1}y, \quad v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

(iii) $u = \dfrac{x-y}{x+z} \quad y = \dfrac{x+z}{y+z}$

(iv) $u = x + y - z, \quad v = x - y + z, \quad w = x^2 + y^2 + z^2 - 2yz$

(v) $u = xy + yz + zx, \quad v = x^2 + y^2 + z^2, \quad w = x + y + z$

(vi) $u = x^2e^{-y}\cosh z, \quad v = x^2e^{-y}\sinh z, \quad w = 3x^4e^{-2y}.$

Solution: (i)
$$u = e^x \sin y \qquad\qquad v = e^x\cos y$$
$$\frac{\partial u}{\partial x} = e^x\sin y \qquad\qquad \frac{\partial v}{\partial x} = e^x\cos y$$
$$\frac{\partial u}{\partial y} = e^x\cos y \qquad\qquad \frac{\partial v}{\partial y} = -e^x\sin y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x\sin y & e^x\cos y \\ e^x\cos y & -e^x\sin y \end{vmatrix}$$

$$= e^x(-\sin^2 y - \cos^2 y) = -e^x \neq 0$$

Hence, u and v are functionally independent.

(ii) $u = \sin^{-1}x + \sin^{-1}y \qquad v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\partial v}{\partial x} = \sqrt{1-y^2} + y\left(\frac{-2x}{2\sqrt{1-x^2}}\right) = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2}} \qquad \frac{\partial v}{\partial y} = x\left(\frac{-2y}{2\sqrt{1-y^2}}\right) + \sqrt{1-x^2} = \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= \left(\frac{-xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + 1\right) - \left(1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right) = 0$$

Hence, u and v are functionally dependent.

Relation between u and v

Let $\qquad \sin^{-1}x = \alpha, \qquad x = \sin\alpha$

$\qquad \sin^{-1}y = \beta, \qquad y = \sin\beta$

$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$= \sin(\alpha+\beta) \qquad = \sin(\sin^{-1}x + \sin^{-1}y) = \sin u$$

(iii) $u = \dfrac{x-y}{x+z}, \qquad v = \dfrac{x+z}{y+z}$

Since number of functions are less than the number of variables, for functional dependence, we must have,

$$\frac{\partial(u,v)}{\partial(x,y)} = 0, \qquad \frac{\partial(u,v)}{\partial(y,z)} = 0 \qquad \frac{\partial(u,v)}{\partial(z,x)} = 0$$

$$\frac{\partial u}{\partial x} = \frac{(x+z)-(x-y)}{(x+z)^2} = \frac{y+z}{(x+z)^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{y+z}$$

$$\frac{\partial u}{\partial y} = \frac{-1}{x+z} \qquad \frac{\partial v}{\partial y} = -\frac{(x+z)}{(y+z)^2}$$

$$\frac{\partial u}{\partial z} = -\frac{(x-y)}{(x+z)^2} \qquad \frac{\partial v}{\partial z} = \frac{(y+z)-(x+z)}{(y+z)^2} = \frac{y-x}{(y+z)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{y+z}{(x+z)^2} & -\frac{1}{x+z} \\ \frac{1}{y+z} & \frac{-(x+z)}{(y+z)^2} \end{vmatrix} = 0$$

$$\frac{\partial(u,v)}{\partial(y,z)}=\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\end{vmatrix}=\begin{vmatrix}-\frac{1}{x+z} & \frac{y-x}{(x+z)^2}\\ \frac{-(x+z)}{(y+z)^2} & \frac{y-x}{(y+z)^2}\end{vmatrix}=0$$

$$\frac{\partial(u,v)}{\partial(z,x)}=\begin{vmatrix}\frac{\partial u}{\partial z} & \frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x}\end{vmatrix}=\begin{vmatrix}\frac{y-x}{(x+z)^2} & \frac{y+z}{(x+z)^2}\\ \frac{y-x}{(y+z)^2} & \frac{1}{y+z}\end{vmatrix}=0$$

Hence, u and v are functionally dependent.

Relation between u and v

$$u=\frac{x-y}{x+z},\qquad v=\frac{x+z}{y+z},\qquad \frac{1}{v}=\frac{y+z}{x+z}$$

$$u+\frac{1}{v}=\frac{(x-y)+(y+z)}{x+z}=1$$

(iv) $u=x+y-z$ $\qquad v=x-y+z$ $\qquad w=x^2+y^2+z^2-2yz$

$$\frac{\partial u}{\partial x}=1\qquad \frac{\partial v}{\partial x}=1\qquad \frac{\partial w}{\partial x}=2x$$

$$\frac{\partial u}{\partial y}=1\qquad \frac{\partial v}{\partial y}=-1\qquad \frac{\partial w}{\partial y}=2y-2z$$

$$\frac{\partial u}{\partial z}=-1\qquad \frac{\partial v}{\partial z}=1\qquad \frac{\partial w}{\partial z}=2z-2y$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ 2x & 2y-2z & 2z-2y\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & -1\\ 2x & 2y-2z & 2y-2z\end{vmatrix}\quad [\text{By } (-1)\, c_3]$$

$$=0$$

Hence, u and v are functionally dependent.

Relation among u, v and w

$$u+v=2x\qquad u-v=2y-2z$$

$$x=\frac{u+v}{2}\qquad y-z=\frac{u-v}{2}$$

$$\left(\frac{u+v}{2}\right)^2+\left(\frac{u-v}{2}\right)^2=x^2+(y-z)^2$$

$$\frac{1}{4}(2u^2+2v^2)=x^2+y^2+z^2-2yz$$

$$u^2+v^2=2w$$

(v) $u=xy+yz+zx \qquad v=x^2+y^2+z^2 \qquad w=x+y+z$

$$\frac{\partial u}{\partial x}=y+z \qquad \frac{\partial v}{\partial x}=2x \qquad \frac{\partial w}{\partial x}=1$$

$$\frac{\partial u}{\partial y}=z+x \qquad \frac{\partial v}{\partial y}=2y \qquad \frac{\partial w}{\partial y}=1$$

$$\frac{\partial u}{\partial z}=x+y \qquad \frac{\partial v}{\partial z}=2z \qquad \frac{\partial w}{\partial z}=1$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}y+z & z+x & x+y\\ 2x & 2y & 2z\\ 1 & 1 & 1\end{vmatrix}$$

$$=2(y+z)(y-z)-2(z+x)(x-z)+2(x+y)(x-y)$$

$$=0$$

Hence, u, v and w are functionally dependent.

Relation among u, v and w

$$w^2=(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)=v+2u$$

(vi) $u=x^2e^{-y}\cosh z \qquad v=x^2e^{-y}\sinh z \qquad w=3x^4e^{-2y}$

$$\frac{\partial u}{\partial x}=2xe^{-y}\cosh z \qquad \frac{\partial v}{\partial x}=2xe^{-y}\sinh z \qquad \frac{\partial w}{\partial x}=12x^3e^{-2y}$$

$$\frac{\partial u}{\partial y}=-x^2e^{-y}\cosh z \qquad \frac{\partial v}{\partial y}=-x^2e^{-y}\sinh z \qquad \frac{\partial w}{\partial y}=-6x^4e^{-2y}$$

$$\frac{\partial u}{\partial z}=x^2e^{-y}\sinh z \qquad \frac{\partial v}{\partial z}=x^2e^{-y}\cosh z \qquad \frac{\partial w}{\partial z}=0$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}2xe^{-y}\cosh z & -x^2e^{-y}\cosh z & x^2e^{-y}\sinh z\\ 2xe^{-y}\sinh z & -x^2e^{-y}\sinh z & x^2e^{-y}\cosh z\\ 12x^3e^{-2y} & -6x^4e^{-2y} & 0\end{vmatrix}$$

$$=12\,x^7e^{-4y}(\cosh^2 z-\sinh^2 z)-12x^7e^{-4y}(\cosh^2 z-\sinh^2 z)=0$$

Hence, u, v and w are functionally dependent.

Relation among u, v and w

$$3u^2 - 3v^2 = 3(x^4e^{-2y}\cosh^2 z - x^4e^{-2y}\sinh^2 z) = 3x^4e^{-2y} = w.$$

Exercise 4.6

1. Find the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$ for each of the following functions:

(i) $u = x + y, \quad v = x - y$

(ii) $u = x^2, \quad v = y^2$

(iii) $u = 3x + 5y, \quad v = 4x - 3y$

(iv) $u = \dfrac{y-x}{1+xy}$, $v = \tan^{-1}y - \tan^{-1}x$

(v) $u = x \sin y, \quad v = y \sin x.$

$\left[\textbf{Ans.:}\ \text{(i)}\ \dfrac{-1}{2}\quad \text{(ii)}\ 4xy\quad \text{(iii)}\ -29\quad \text{(iv)}\ 0\quad \text{(v)}\ \sin x \sin y - xy\cos x \cos y\right]$

2. Find the Jacobian for each of the following functions:

(i) $x = e^u \cos v, \quad y = e^u \sin v$

(ii) $x = u(1 - v), \quad y = uv$

(iii) $x = uv, \quad y = \dfrac{u+v}{u-v}.$

$\left[\textbf{Ans.:}\ \text{(i)}\ e^{2u}\quad \text{(ii)}\ u\quad \text{(iii)}\ \dfrac{4uv}{(u-v)^2}\right]$

3. Find the Jacobian for each of the following functions:

(i) $u = \dfrac{yz}{x}, \ v = \dfrac{zx}{y}, \ w = \dfrac{xy}{z}$

(ii) $u = xyz, \quad v = xy + yz + zx, \quad w = x + y + z$

(iii) $u = x^2, \quad v = \sin y, \quad w = e^{-3z}$

(iv) $x = \dfrac{1}{2}(u^2 - v^2), \ y = uv, \ z = w.$

$\left[\textbf{Ans.:}\ \text{(i)}\ 4\quad \text{(ii)}\ (x-y)(y-z)(z-x)\quad \text{(iii)}\ -6e^{-3z}x\cos y\quad \text{(iv)}\ \dfrac{1}{u^2+v^2}\right]$

4. Verify that $J \cdot J^* = 1$ for the following functions:

(i) $u = x + \dfrac{y^2}{x}, \ v = \dfrac{y^2}{x}$

(ii) $x = u(1 - v), \quad y = uv$

(iii) $x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi.$

5. If $u = x + y + z, \quad uv = y + z, uvw = z,$ evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

[**Ans.:** u^2v]

6. If $u^3 + v^3 = x + y, \quad u^2 + v^2 = x^3 + y^3,$ show that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{2}\dfrac{(y^2 - x^2)}{uv(u-v)}$.

7. Calculate $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ if $u = \dfrac{x}{\sqrt{1-r^2}}$, $v = \dfrac{y}{\sqrt{1-r^2}}$, $w = \dfrac{z}{\sqrt{1-r^2}}$ where $r^2 = x^2 + y^2 + z^2$.

$\left[\textbf{Ans.:}\ (1 - r^2)^{-\frac{5}{2}}\right]$

8. If $u = x + y + z, \quad u^2v = y + z,$ $u^3w = z$, show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = u^{-5}.$

9. Show that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$, if

$u = x^2 - 2y^2, v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$.

10. Determine whether the following function are functionally dependent or not. If functionally dependent, find the relation between them.

(i) $u = \dfrac{x-y}{x+y}, \quad v = \dfrac{x+y}{y}$

(ii) $u = \dfrac{x^2 - y^2}{x^2 + y^2}, \quad v = \dfrac{2xy}{x^2 + y^2}$

(iii) $u = \sin x + \sin y$, $v = \sin(x + y)$

(iv) $u = \dfrac{x-y}{x+y}, \quad v = \dfrac{xy}{(x+y)^2}$

(v) $u = x + y + z, \quad v = x^2 + y^2 + z^2$, $w = x^3 + y^3 + z^3 - 3xyz$

(vi) $u = xe^y \sin z, \quad v = xe^y \cos z$, $w = x^2 e^{2y}$

(vii) $u = \dfrac{3x^2}{2(y+z)}, \quad v = \dfrac{2(y+z)}{3(x-y)^2}$, $w = \dfrac{x-y}{x}$.

Ans.:

(i) Dependent, $u = \dfrac{2-v}{v}$

(ii) Dependent, $u^2 + v^2 = 1$

(iii) Indepentent

(iv) Dependent, $4v = 1 - u^2$

(v) Dependent, $2w = u(3v - u^2)$

(vi) Dependent, $u^2 + v^2 = w$

(vii) Dependent, $uvw^2 = 1$

4.8.2 Errors and Approximation

Let $u = f(x, y)$ be a continuous function of x and y. If δx and δy are small increments in x and y respectively and δu is corresponding increment in u, then

$$u + \delta u = f(x + \delta x, y + \delta y)$$

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

[expanding by Taylor's theorem and ignoring higher powers and products of δx and δy.]

or
$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y$$

For a function $u = f(x, y, z)$ of three variables, we have

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

Definition

If δx is the error in x, then

(i) δx is known as **Absolute error** in x.

(ii) $\dfrac{\delta x}{x}$ is known as **Relative error** in x.

(iii) $\dfrac{100}{x}\delta x$ is known as **Percentage error** in x.

Example 1: Find the percentage error in calculating the area of a rectangle when an error of 3% is made in measuring each of its sides.

Solution : Let a and b be the side of the rectangle and A is its area.

$$A = ab$$

$$\log A = \log a + \log b$$

$$\frac{1}{A}\delta A = \frac{1}{a}\delta a + \frac{1}{b}\delta b$$

$$\frac{100}{A}\delta A = \frac{100}{a}\delta a + \frac{100}{b}\delta b$$

Percentage error in measuring each of its sides is 3.

$$\frac{100}{A}\delta A = 3 + 3 = 6$$

Hence, percentage error in calculating the area = 6%.

Example 2: Find the percentage error in the area of an ellipse when an error of 1.5% is made in measuring its major and minor axes.

Solution: Let $2a$ and $2b$ are the major and minor axes of the ellipse and A is its area.

$$A = \pi ab$$

$$\log A = \log \pi + \log a + \log b$$

$$\frac{1}{A}\delta A = 0 + \frac{1}{a}\delta a + \frac{1}{b}\delta b$$

$$\frac{100}{A}\delta A = \frac{100}{a}\delta a + \frac{100}{b}\delta b$$

Percentage error in measuring its major and minor axes is 1.5%.

$$\frac{100\delta A}{A} = 1.5 + 1.5 = 3$$

Hence, percentage error in area of ellipse = 3%.

Example 3: The focal length of mirror is found from the formula $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$. Find the percentage error in f if u and v are both in error by 2% each.

Solution:

$$\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$$

$$\frac{-2}{f^2}\delta f = -\frac{1}{v^2}\delta v + \frac{1}{u^2}\delta u$$

$$\frac{-2}{f}.\frac{100}{f}\delta f = \frac{-1}{v}.\frac{100}{v}\delta v + \frac{1}{u}.\frac{100}{u}\delta u = \frac{-1}{v}(2) + \frac{1}{u}(2)$$

$$= -2\left(\frac{1}{v} - \frac{1}{u}\right) = -2\left(\frac{2}{f}\right)$$

$$\frac{100}{f}\delta f = 2$$

Hence, percentage error in f = 2%.

Example 4: If $D = \frac{a^2}{b} + \frac{c^2}{2}$, find the percentage error in D if error in measuring a is $\frac{1}{2}\%$ and in measuring b and c are 1% each.

Solution:

$$D = \frac{a^2}{b} + \frac{c^2}{2}$$

$$\delta D = \frac{2a}{b}\delta a - \frac{a^2}{b^2}\delta b + \frac{2c}{2}\delta c$$

$$\frac{100}{D}\delta D = \frac{1}{D}\left(\frac{2a^2}{b}\cdot\frac{100}{a}\delta a - \frac{a^2}{b}\cdot\frac{100}{b}\delta b + c^2\frac{100}{c}\delta c\right)$$

But

$$\frac{100}{a}\delta a = \frac{1}{2},\quad \frac{100}{b}\delta b = \frac{100}{c}\delta c = 1$$

$$\frac{100}{D}\delta D = \frac{1}{D}\left(\frac{2a^2}{b}\cdot\frac{1}{2} - \frac{a^2}{b} + c^2\right)$$

$$= \frac{c^2}{D} = \frac{c^2}{\frac{a^2}{b} + \frac{c^2}{2}} = \frac{2bc^2}{2a^2 + bc^2}$$

Hence, percentage error in $D = \frac{2bc^2}{2a^2 + bc^2}$.

Example 5: Find the possible percentage error in computing the parallel resistance R of three resistance R_1, R_2, R_3, if R_1, R_2, R_3, are each in error by 1.2%.

Solution:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$-\frac{1}{R^2}\delta R = -\frac{1}{R_1^2}\delta R_1 - \frac{1}{R_2^2}\delta R_2 - \frac{1}{R_3^2}\delta R_3$$

$$\frac{1}{R}\cdot\frac{100}{R}\delta R = \frac{1}{R_1}\cdot\frac{100}{R_1}\delta R_1 + \frac{1}{R_2}\cdot\frac{100}{R_2}\delta R_2 + \frac{1}{R_3}\cdot\frac{100}{R_3}\delta R_3$$

But

$$\frac{100}{R_1}\delta R_1 = \frac{100}{R_2}\delta R_2 = \frac{100}{R_3}\delta R_3 = 1.2$$

$$\frac{1}{R}\cdot\frac{100}{R}\delta R = \frac{1}{R_1}(1.2) + \frac{1}{R_2}(1.2) + \frac{1}{R_3}(1.2)$$

$$= 1.2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$= 1.2\left(\frac{1}{R}\right)$$

$$\frac{100}{R}\delta R = 1.2$$

Hence, percentage error in $R = 1.2\%$

Example 6: The resonant frequency in a series electrical circuit is given by $f = \dfrac{1}{2\pi\sqrt{LC}}$. If the measurement of L and C are in error by 2% and −1% respectively, find the percentage error in f.

Solution:

$$f = \frac{1}{2\pi\sqrt{LC}}$$

$$\log f = \log \frac{1}{2\pi} - \frac{1}{2}\log L - \frac{1}{2}\log C$$

$$\frac{1}{f}\delta f = 0 - \frac{1}{2}\cdot\frac{1}{L}\delta L - \frac{1}{2}\cdot\frac{1}{C}\delta C$$

$$\frac{100}{f}\delta f = -\frac{1}{2}\cdot\frac{100}{L}\delta L - \frac{1}{2}\cdot\frac{100}{C}\delta C$$

But
$$\frac{100}{L}\delta L = 2,\ \frac{100}{C}\delta C = -1$$

$$\frac{100}{f}\delta f = -\frac{1}{2}(2) - \frac{1}{2}(-1) = -0.5$$

Hence, percentage error in $f = -\,0.5\%$.

Example 7: If $z = 2xy^2 - 3x^2y$ and x increases at the rate of 2 cm/s as it passes through $x = 3$ cm. Show that if y is passing through $y = 1$ cm, y must decrease at the rate of $\dfrac{32}{15}$ cm/s in order that z remains constant.

Solution:

$$z = 2xy^2 - 3x^2y$$

$$\delta z = (2y^2 - 6xy)\,\delta x + (4xy - 3x^2)\delta y$$

But
$$x = 3,\ y = 1,\ \delta x = 2,\ \delta z = 0$$

$$0 = (2 - 18)\,2 + (12 - 27)\,\delta y$$

$$\delta y = -\frac{32}{15}$$

Hence, y must decrease at the rate of $\dfrac{32}{15}$ cm/s.

Example 8: If $e^z = \sec x \cos y$ and errors of magnitude h and $-h$ are made in estimating x and y, where x and y are found to be $\dfrac{\pi}{3}$ and $\dfrac{\pi}{6}$ respectively, find the corresponding error in z.

Solution:
$$e^z = \sec x \cos y$$
$$z \log e = \log \sec x + \log \cos y$$
$$\delta z = \frac{1}{\sec x}\sec x \tan x\, \delta x + \frac{1}{\cos y}(-\sin y)\delta y$$
$$= \tan x\, \delta x - \tan y\, \delta y$$

But
$$x = \frac{\pi}{3},\ y = \frac{\pi}{6},\ \delta x = h,\ \delta y = -h$$
$$\delta z = \tan\frac{\pi}{3}(h) - \tan\frac{\pi}{6}(-h)$$
$$= \sqrt{3}h - \frac{1}{\sqrt{3}}(-h) = \sqrt{3}h + \frac{h}{\sqrt{3}} = \frac{4h}{\sqrt{3}} = \frac{4\sqrt{3}}{3}h$$

Hence, error in $z = \frac{4\sqrt{3}}{3}h$.

Example 9: In calculating the volume of a right circular cone, errors of 2% and 1% are made in height and radius of base respectively. Find the % error in volume.

Solution: Let r be the radius of base, h height and V volume of the right circular cone.
$$V = \frac{1}{3}\pi r^2 h$$
$$\log V = \log\frac{\pi}{3} + 2\log r + \log h$$
$$\frac{1}{V}\delta V = 0 + \frac{2}{r}\delta r + \frac{1}{h}\delta h$$
$$\frac{100}{V}\delta V = 2\left(\frac{100}{r}\right)\delta r + \frac{100}{h}\delta h = 2(1) + 2 = 4$$

Hence, percentage error in volume = 4%.

Example 10: The diameter and the altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the possible error in the values computed for volume and lateral surface.

Solution: Let d and h are diameter and height of the cylinder respectively and V be its volume.

$$V = \pi\left(\frac{d}{2}\right)^2 h = \frac{\pi}{4}d^2 h$$

$$\log V = \log\frac{\pi}{4} + 2\log d + \log h$$

$$\frac{1}{V}\delta V = 0 + \frac{2}{d}\delta d + \frac{1}{h}\delta h$$

$$\frac{1}{V}\delta V = \frac{2}{d}\delta d + \frac{1}{h}\delta h$$

But $d = 4$ cm, $h = 6$ cm, δd = 0.1 cm, $\delta h = 0.1$ cm.

$$V = \frac{\pi}{4}d^2 h = \frac{\pi}{4}\times(4)^2\times 6 = 75.36 \text{ cm}^3$$

$$\frac{1}{V}\delta V = \frac{2}{4}\times 0.1 + \frac{1}{6}\times 0.1$$

$$\delta V = 75.36 \times 0.067$$

$$= 5.05 \text{ cm}^3$$

Hence, error in volume = 5.05 cm³

Lateral surface area,

$$S = 2\pi rh$$

$$= \pi dh$$

$$\log S = \log\pi + \log d + \log h$$

$$\frac{1}{S}\delta S = 0 + \frac{1}{d}\delta d + \frac{1}{h}\delta h$$

$$\frac{1}{S}\delta S = \frac{1}{d}\delta d + \frac{1}{h}\delta h$$

But $d = 4$ cm, $h = 6$ cm, $\delta d = 0.1$ cm, $\delta h = 0.1$ cm.

$$S = \pi \times 4 \times 6 = 75.36 \text{ cm}^2$$

$$\frac{1}{S}\delta S = \left(\frac{1}{4}\right)(0.1) + \left(\frac{1}{6}\right)(0.1)$$

$$\delta S = 75.36 \times 0.0416$$

$$= 3.14 \text{ cm}^2$$

Hence, error in lateral surface area = 3.14 cm².

Example 11: A balloon is in the form of a right circular cylinder of radius 1.5 m and height 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the height by 0.05 m, find the percentage change in the volume of the balloon.

Solution: Radius of the cylinder, $r = 1.5$ m Height of the cylinder, $h = 4$ m

Volume of the cylinder $= \pi r^2 h$ Volume of the hemisphere $= \frac{2}{3}\pi r^3$

Volume of the balloon,

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\delta V = \pi(2rh\ \delta r + r^2 \delta h) + \frac{4}{3}\pi(3r^2 \delta r)$$

But $r = 1.5$ m, $h = 4$ m, $\delta r = 0.01$ m, $\delta h = 0.05$ m

$$\delta V = \pi[2 \times 1.5 \times 4 \times 0.01 + (1.5)^2(0.05)] + 4\pi(1.5)^2(0.01)$$

$$= \pi\,(0.12 + 0.1225 + 0.09) = 3.225\pi$$

$$V = \pi\left[(1.5)^2 \times 4 + \frac{4}{3}(1.5)^3\right]$$

$$= \pi\,(9 + 4.5) = 13.5\pi$$

Percentage change in the volume, $\frac{\delta V}{V} \times 100 = \frac{3.225}{13.5} \times 100$

$= 2.389\%.$

Example 12: At a distance 120 feet from the foot of a tower, the elevation of its top is 60°. If the possible error in measuring the distance and elevation are 1 inch and 1 minute respectively, find the approximate error in the calculated height of the tower.

Solution: Let h, x and θ are height, horizontal distance and angle of elevation of the tower respectively.

$$\tan\theta = \frac{h}{x}$$

$$h = x\tan\theta$$

$$\log h = \log x + \log\tan\theta$$

$$\frac{1}{h}\delta h = \frac{1}{x}\delta x + \frac{1}{\tan\theta}\sec^2\theta\,\delta\theta$$

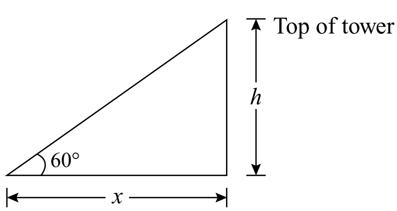


Fig. 4.36

But $x = 120$ ft. and $\theta = 60^\circ$, $h = 120\tan 60^\circ = 120\sqrt{3}$, $\delta x = 1 \text{ inch} = \frac{1}{12}$ ft., $\delta\theta = 1 \text{ minute} = \frac{1}{60}\cdot\frac{\pi}{180}$ radians

$$\frac{1}{120\sqrt{3}}\delta h = \frac{1}{120}\cdot\frac{1}{12} + \frac{1}{\sqrt{3}}\cdot 4\cdot\frac{1}{60}\cdot\frac{\pi}{180}$$

$$\delta h = 0.284 \text{ ft.}$$

Hence, approximate error in height = 0.284 ft

Example 13: In estimating the cost of pile of bricks measured 2 m × 15 m × 1.2 m, the top of the pile is stretched 1% beyond the standard length. If the count is 450 bricks in 1 cubic m and bricks cost Rs. 450 per thousand, find the approximate error in the cost.

Solution: Let l, b and h be the length, breadth and height of the pile and V be its volume.

$$V = lbh$$

$$\log V = \log\ l + \log\ b + \log h$$

$$\frac{1}{V}\delta V = \frac{1}{l}\delta l + \frac{1}{b}\delta b + \frac{1}{h}\delta h$$

$$\frac{100}{V}\delta V = \frac{100}{l}\delta l + \frac{100}{b}\delta b + \frac{100}{h}\delta h$$

Top is stretched 1% beyond the standard length.

Percentage error in height i.e. $\frac{100}{h}\delta h = 1$ and $\delta l = 0$, $\delta b = 0$

$$\frac{100}{V}\delta V = 0 + 0 + 1$$

$$\delta V = \frac{V}{100} = \frac{l \times b \times h}{100} = \frac{2 \times 15 \times 1.2}{100}$$

$$= 0.36 \text{ cubic metre}$$

Hence, error in number of bricks = 0.36 × 450 = 162

Cost of 162 bricks $= 162 \times \frac{450}{1000} = 72.9$

Hence, error in cost = Rs. 72.90

Example 14: Evaluate $\left[(3.82)^2 + 2\,(2.1)^3\right]^{\frac{1}{5}}$ using theory of approximation.

Solution: Let $z = (x^2 + 2y^3)^{\frac{1}{5}}$

$$\delta z = \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}(2x)\delta x + \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}(6y^2)\delta y$$

$$= \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}(2x\delta x + 6y^2\delta y)$$

Consider, $x = 4$, $\delta x = 3.82 - 4 = -0.18$, $y = 2$, $\delta y = 2.1 - 2 = 0.1$

Hence, $$\delta z = \frac{1}{5}(32)^{-\frac{4}{5}}[2(4)(-0.18) + 6(2)^2(0.1) = 0.012$$

Approximate value = $z + \delta z = (32)^{\frac{1}{5}} + 0.012 = 2.012.$

Example 15: Evaluate $(1.99)^2\ (3.01)^3\ (0.98)^{\frac{1}{10}}$ using approximation.

Solution: Let $u = x^2 y^3 z^{\frac{1}{10}}$

$$\log u = 2\log x + 3\log y + \frac{1}{10}\log z$$

$$\frac{1}{u}\delta u = \frac{2}{x}\delta x + \frac{3}{y}\delta y + \frac{1}{10}\frac{1}{z}\delta z$$

Consider, $x = 2,$ $\quad y = 3,$ $\quad z = 1,$

$\delta x = 1.99 - 2 = -0.01,$ $\quad \delta y = 3.01 - 3 = 0.01,$ $\quad \delta z = 0.98 - 1 = -0.02$

Hence, $$u = 2^2 \cdot 3^3 \cdot 1^{\frac{1}{10}} = 108$$

$$\frac{1}{108}\delta u = (-0.01) + 0.01 + \frac{1}{10}(-0.02)$$

$$\delta u = -0.216$$

Approximate value = $u + \delta u = 108 - 0.216$

$= 107.784$.

Example 16: Find the approximate value of $[(0.98)^2 + (2.01)^2 + (1.94)^2]^{\frac{1}{2}}$.

Solution: Let $u = \sqrt{x^2 + y^2 + z^2}$

$$u = \sqrt{x^2 + y^2 + z^2}$$

$$u^2 = x^2 + y^2 + z^2$$

$$2u\ \delta u = 2x\ \delta x + 2y\ \delta y + 2z\ \delta z$$

$$u\ \delta u = x\delta x + y\delta y + z\delta z$$

Consider, $x = 1,$ $\quad y = 2$ and $\quad z = 2$

$\delta x = 0.98 - 1 = -0.02,$ $\quad \delta y = 2.01 - 2 = 0.01,$ $\quad \delta z = 1.94 - 2 = -0.06$

$$u = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$$

Hence, $u\delta u = 1\ (-0.02) + 2\ (0.01) + 2\ (-0.06) = -\ 0.12$

$\delta u = -\ 0.04$

Approximate value of $u = u + \delta u = 3 - 0.04 = 2.96$

Example 17: If the sides and angles of a plane triangle vary in such a way that its circum radius remains constant, prove that $\dfrac{\delta a}{\cos A}+\dfrac{\delta b}{\cos B}+\dfrac{\delta c}{\cos C}=0$, where d*a*, d*b*, d*c* are smaller increments in the sides *a, b, c* respectively.

Solution: From the sine rule,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

We know that, circum radius $R=\dfrac{a}{2\ \sin A}=\dfrac{b}{2\ \sin B}=\dfrac{c}{2\ \sin C}$

Considering,
$$R=\frac{a}{2\ \sin A}$$

$$\delta R=\frac{1}{2\ \sin A}\delta a-\frac{a\ \cos A}{2\ \sin^2 A}\delta A$$

But R is constant,
$$\delta R=0$$

$$0=\frac{\delta a}{2\sin A}-\frac{a\cos A}{2\sin^2 A}\delta A$$

$$\frac{\delta a}{\cos A}=\frac{a}{\sin A}\delta A=2R\delta A$$

Similarly,
$$\frac{\delta b}{\cos B}=\frac{b}{\sin B}\delta B=2R\delta B$$

and
$$\frac{\delta c}{\cos C}=\frac{c}{\sin C}\delta C=2R\delta C$$

$$\frac{\delta a}{\cos A}+\frac{\delta b}{\cos B}+\frac{\delta c}{\cos C}=2R(\delta A+\delta B+\delta C)$$
$$=2R\delta(A+B+C)$$
$$=2R\delta(\pi)=0.$$

Example 18: If Δ be the area of the triangle, prove that the error in Δ resulting from a small error in side *c* is given by $\delta\Delta=\dfrac{\Delta}{4}\left(\dfrac{1}{s}+\dfrac{1}{s-a}+\dfrac{1}{s-b}-\dfrac{1}{s-c}\right)\delta c$, where $\Delta=\sqrt{s\ (s-a)\ (s-b)\ (s-c)}$.

Solution: $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$

$$\log\ \Delta=\frac{1}{2}[\log s+\log(s-a)+\log(s-b)+\log(s-c)]$$

$$\frac{1}{\Delta}\delta\Delta=\frac{1}{2}\left[\frac{1}{s}\delta s+\frac{1}{s-a}\delta(s-a)+\frac{1}{s-b}\delta(s-b)+\frac{1}{s-c}\delta(s-c)\right]$$
$$=\frac{1}{2}\left[\frac{\delta s}{s}+\frac{\delta s-\delta a}{s-a}+\frac{\delta s-\delta b}{s-b}+\frac{\delta s-\delta c}{s-c}\right]$$

But $s = \frac{1}{2}(a+b+c)$, where a and b are constant.

Thus, $\delta a = 0,\ \delta b = 0,\ \delta s = \frac{\delta c}{2}$

Hence, $$\delta\Delta = \frac{\Delta}{2}\left[\frac{\delta c}{2s} + \frac{\delta c}{2(s-a)} + \frac{\delta c}{2(s-b)} + \frac{\frac{\delta c}{2} - \delta c}{s-c}\right]$$

$$= \frac{\Delta}{4}\left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right]\delta c.$$

Exercise 4.7

1. In calculating the volume of right circular cone, errors of 2.75% and 1.25% are made in height and radius of the base. Find the % error in volume.

   [**Ans. :** 5.25%]

2. The height of a cone is $H = 30$ cm, the radius of base $R = 10$ cm. How will the volume of the cone change, if H is increasing by 3 mm while R is decreasing by 1 mm?

   $$\left[\textbf{Hint : } \delta h = 3 \text{ mm} = 0.3 \text{ cm}, \quad \delta r = -1 \text{ mm} = -0.1 \text{ cm}\right]$$

   [**Ans. :** decreased by 10π cm^3]

3. How is the relative change in $V = \pi r^2 h$ related to relative change in r and h? How are percentage changes related?

   $$\left[\textbf{Ans. : } \text{relative change } \frac{\delta V}{V} = \frac{2}{r}\delta r + \frac{1}{h}\delta h \right.$$
   and percentage change in volume = (2% change in radius) + (1% change in height)]

4. In calculating the total surface area of a cylinder, error of 1% each are made in measuring the height and the base radius. Find % error in calculating the total surface area.

   [**Ans. :** 2%]

5. In calculating the volume of a right circular cylinder, errors of 2% and 1% are made in measuring the height and base radius respectively. Find the percentage error in calculating volume of the cylinder.

   [**Ans. :** 4%]

6. Find the percentage error in calculating the area of a rectangle when an error of 2% is made in measuring each of its sides.

   [**Ans. :** 4%]

7. Find the percentage error in calculating the area of a rectangle when an error of 1% is found in measuring its sides.

   [**Ans. :** 2%]

8. If R_1 and R_2 are two resistances in parallel, their resistance R is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If there is an error of 2% in both R_1 and R_2, find percentage error in R.

   [**Ans. :** 2%]

9. One side of a rectangle is $a = 10$ cm and the other side is $b = 24$ cm. How will the diagonal l of the rectangle change if a is increased by 4 mm and b is decreased by 1 mm?

$$\left[\textbf{Ans.:}\ \frac{4}{65}\text{cm}\right]$$

10. The resistance R of a circuit was found by using the formula $I = \dfrac{E}{R}$. If there is an error of 0.1 ampere in reading I and 0.5 volts in reading E, find the corresponding percentage error in R when $I = 15$ ampere and $E = 100$ volts.

[**Ans.:** − 0.167%]

11. The radius and height of a cone are 4 cm and 6 cm respectively. What is the error in its volume if the scale used in taking the measurement is short by 0.01 cm per cm?

$$\left[\begin{aligned}\textbf{Hint:}\ \delta r &= 4 \times 0.01 = 0.04 \text{ cm},\\ \delta h &= 6 \times 0.01 = 0.06 \text{ cm}\end{aligned}\right]$$

[**Ans.:** $0.96\pi\,\text{cm}^3$]

12. In estimating the cost of a pile of bricks measured as $6' \times 50' \times 4'$, the top is stretched 1% beyond its standard length. If the count is 12 bricks per ft³ and bricks cost Rs. 100 per 1000, find the approximate error in the cost.

[**Ans.:** Rs. 43.20]

13. Show that the error in calculating the time period of a pendulum at any place is zero, if an error of μ% is made in measuring its length and gravity at that place.

$$\left[\textbf{Hint:}\ T = 2\pi\sqrt{\frac{l}{g}}\right]$$

14. At distance 20 meters from the foot of a tower, the elevation of its top is 60°. If the possible error in measuring distance and elevation are 1 cm and 1 minute, find the approximate error in calculating height.

[**Ans.:** 0.040]

15. The diameter and the altitude of a right circular cylinder are measured as 24 cm and 30 cm respectively. There is an error of 0.1 cm in each measurement. Find the possible error in the volume of the cylinder.

[**Ans.:** 50.4π cm.]

16. If the measurements of radius, base and height of a right circular cone are changed by −1% and 2%, show that there will be no error in the volume.

17. If $f = x^2 y^3 z^{\frac{1}{10}}$, find the approximate value of f, when $x = 1.99$, $y = 3.01$ and $z = 0.98$.

[**Ans.:** 107.784]

18. If $f = x^3 y^2 z^4$, find the approximate value of f, when $x = 1.99$, $y = 3.01$, $z = 0.99$.

[**Ans.:** 68.5202]

19. If $f = (160 - x^3 - y^3)^{\frac{1}{3}}$, find the approximate value of $f\,(2.1,\ 2.9) - f\,(2, 3)$.

[**Ans.:** 0.016]

20. If $f = e^{xyz}$, find the approximate value of f, when $x = 0.01$, $y = 1.01$, $z = 2.01$.

[**Ans.:** 1.02]

21. Find $[(2.92)^3 + (5.87)^3]^{\frac{1}{5}}$ approximately by using the theory of approximation.

[**Ans.:** 2.96]

4.8.3 Maxima and Minima

Let $u = f(x, y)$ be a continuous function of x and y. Then u will be maximum at $x = a$, $y = b$, if $f(a, b) > f(a + h, b + k)$ and will be minimum at $x = a$, $y = b$, if $f(a, b) < f(a + h, b + k)$ for small positive or negative values of h and k.

The point at which function $f(x, y)$ is either maximum or minimum is known as **stationary point.** The value of the function at stationary point is known as extreme (maximum or minimum) value of the function $f(x, y)$.

Working rule: To determine the maxima and minima (extreme values) of a function $f(x, y)$.

Step I: Solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously for x and y.

Step II: Obtain the values of $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x\, \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$.

Step III: (i) If $rt - s^2 > 0$ and $r < 0$ (or $t < 0$) at (a, b), then $f(x, y)$ is maximum at (a, b) and the maximum value of the function is $f(a, b)$.

(ii) If $rt - s^2 > 0$ and $r > 0$ (or $t > 0$) at (a, b), then $f(x, y)$ is minimum at (a, b) and the minimum value of the function is $f(a, b)$.

(iii) If $rt - s^2 < 0$ at (a, b), then $f(x, y)$ is neither maximum nor minimum at (a, b). Such point is known as **saddle point**.

(iv) If $rt - s^2 = 0$ at (a, b), then no conclusion can be made about the extreme values of $f(x, y)$ and further investigation is required.

Example 1: Show that the minimum value of $f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$ is $3a^2$.

Solution:
$$f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2} = 0 \quad \text{... (1)}$$

$$\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2} = 0 \quad \text{... (2)}$$

Solving Eqs (1) and (2),

$$x^2 y = a^3 \quad \text{... (3)}$$

and

$$xy^2 = a^3 \quad \text{... (4)}$$

Solving Eqs (3) and (4),

$$x = y$$

Substituting in Eq. (3),

$$x^3 = a^3$$
$$x = a$$
$$y = a$$

Stationary point is (a, a).

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$$

At (a, a), $r = 2$, $s = 1$, $t = 2$

Step III: At (a, a),

$$rt - s^2 = (2)\,(2) - (1)^2 = 3 > 0$$

Hence, $f(x, y)$ is minimum at (a, a).

$$f_{\min} = a^2 + a^3\left(\frac{1}{a} + \frac{1}{a}\right) = 3a^2.$$

Example 2: Find the stationary value of $x^3 + y^3 - 3axy$, $a > 0$.

Solution: $f(x, y) = x^3 + y^3 - 3axy$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay = 0 \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0 \qquad \text{... (2)}$$

From Eq. (1),

$$y = \frac{x^2}{a}$$

Substituting in Eq. (2),

$$x^4 - a^3x = 0$$

$$x\,(x - a)\,(x^2 + ax + a^2) = 0$$

$$x = 0,\ x = a$$

Then $y = 0$, $y = a$.

Hence, stationary points are $(0, 0)$ and (a, a).

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = -3a$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Step III: At $(0, 0)$

$$rt - s^2 = (0)\,(0) - (-3a)^2 = -9a^2 < 0$$

Hence, function $f(x, y)$ is neither maximum nor minimum at (0, 0).

At (a, a)

$$rt - s^2 = (6a)(6a) - (-3a)^2 = 27a^2 > 0$$

and $$r = 6a > 0$$

Hence, function $f(x, y)$ is minimum at (a, a).

$$f_{\min} = a^3 + a^3 - 3a^3$$
$$= -a^3.$$

Example 3: Find the extreme values of $u = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$, if any.

Solution: $$u = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$$

Step I: For extreme values,

$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 6x = 0$$

$$x^2 + y^2 - 2x = 0 \qquad \text{... (1)}$$

and $$\frac{\partial u}{\partial y} = 6xy - 6y = 0$$

$$6y(x - 1) = 0$$
$$y = 0, x = 1$$

Substituting $y = 0$ in Eq. (1),

$$x^2 - 2x = 0, x = 0, 2$$

Stationary points are (0, 0), (2, 0)

Substituting $x = 1$ in Eq. (1),

$$1 + y^2 - 2 = 0, y^2 = 1, y = \pm 1$$

Stationary points are (1, 1), (1, –1)

Step II: $$r = \frac{\partial^2 u}{\partial x^2} = 6x - 6 = 6(x - 1)$$

$$s = \frac{\partial^2 u}{\partial x\, \partial y} = 6y$$

$$t = \frac{\partial^2 u}{\partial y^2} = 6x - 6 = 6(x - 1)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
(0, 0)	−6	0	−6	$36 > 0$ and $r < 0$	maximum
(2, 0)	6	0	6	$36 > 0$ and $r > 0$	minimum
(1, 1)	0	6	0	$-36 < 0$	neither maximum nor minimum
(1, −1)	0	−6	0	$-36 < 0$	neither maximum nor minimum

Hence, u is maximum at (0, 0) and minimum at (2, 0).

$$u_{\text{max}} = 0 + 7 = 7$$

and $$u_{\text{min}} = 2^3 + 3(2)(0)^2 - 3(2)^2 - 3(0)^2 + 7 = 3.$$

Example 4: Find the extreme values of $u = x^3 + y^3 - 63(x + y) + 12xy$.

Solution: $$u(x, y) = x^3 + y^3 - 63x - 63y + 12xy$$

Step I: $$\frac{\partial u}{\partial x} = 3x^2 - 63 + 12y$$

$$\frac{\partial u}{\partial y} = 3y^2 - 63 + 12x$$

For extreme values

$$\frac{\partial u}{\partial x} = 0$$

$$3x^2 - 63 + 12y = 0,\ 3x^2 + 12y = 63$$

$$x^2 + 4y = 21 \qquad \dots (1)$$

and $$\frac{\partial u}{\partial y} = 0$$

$$3y^2 - 63 + 12x = 0,\ 12x + 3y^2 = 63$$

$$4x + y^2 = 21 \qquad \dots (2)$$

Solving Eqs (1) and (2),

$$x^2 + 4y = 4x + y^2,\ x^2 - y^2 = 4(x - y)$$

$$(x + y)(x - y) - 4(x - y) = 0$$

$$(x - y)(x + y - 4) = 0$$

$$x + y - 4 = 0,\ x - y = 0$$

$$y = 4 - x,\ y = x$$

Substituting $y = 4 - x$ in Eq. (1),

$$x^2 + 4(4 - x) = 21$$

$$x^2 - 4x - 5 = 0,\ (x + 1)(x - 5) = 0$$

$$x = -1,\ 5$$

$$y = 5,\ -1$$

Stationary points are (–1, 5), (5, –1).

Putting $y = x$ in Eq. (1),

$$x^2 + 4x - 21 = 0,\ (x + 7)(x - 3) = 0$$

$$x = -7,\ 3$$

$$y = -7,\ 3$$

Stationary points are (–7, –7), (3, 3).

Hence, all stationary points are: (–1, 5), (5, –1) (–7, –7), (3, 3).

Step II:
$$r = \frac{\partial^2 u}{\partial x^2} = 6x$$
$$s = \frac{\partial^2 u}{\partial x\, \partial y} = 12$$
$$t = \frac{\partial^2 u}{\partial y^2} = 6y$$

Step III:

(x, y)	**r**	**s**	**t**	**$rt - s^2$**	**Conclusion**
(−1, 5)	−6	12	30	−324 < 0	neither maximum nor minimum
(5, −1)	30	12	−6	−324 < 0	neither maximum nor minimum
(−7, −7)	−42	12	−42	1620 > 0 and r < 0	maximum
(3, 3)	18	12	18	180 > 0 and r > 0	minimum

Hence, at (−7, −7), u is maximum.
$$u_{max} = (-7)^3 + (-7)^3 - 63\,(-7\ -7) + 12(-7)\,(-7) = 2156.$$
and at (3, 3), u is minimum.
$$u_{min} = 3^3 + 3^3 - 63\,(3 + 3) + 12(3)\,(3) = -216.$$

Example 5: Find the stationary value of $xy\,(a - x - y)$.

Solution:
$$f(x, y) = xy\,(a - x - y)$$
$$= axy - x^2y - xy^2$$

Step I: For extreme values,
$$\frac{\partial f}{\partial x} = ay - 2xy - y^2 = 0 \qquad \text{... (1)}$$
$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy = 0 \qquad \text{... (2)}$$

From Eqs (1) and (2), we get
$$y\,(a - 2x - y) = 0$$
$$y = 0,\ a - 2x - y = 0$$
and
$$x\,(a - x - 2y) = 0$$
$$x = 0,\ a - x - 2y = 0$$

Considering four pairs of equations
$$y = 0 \qquad x = 0$$
$$y = 0 \qquad a - x - 2y = 0$$
$$a - 2x - y = 0 \qquad x = 0$$
$$a - 2x - y = 0 \qquad a - x - 2y = 0$$

Solving these equations, following pairs of values of x and y are obtained.

$(0, 0), (0, a), (a, 0), \left(\frac{a}{3}, \frac{a}{3}\right)$

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = -2y$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = a - 2x - 2y$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2x$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
$(0, 0)$	0	a	0	$-a^2 < 0$	neither maximum nor minimum
$(0, a)$	$-2a$	$-a$	0	$-a^2 < 0$	neither maximum nor minimum
$(a, 0)$	0	$-a$	$-2a$	$-a^2 < 0$	neither maximum nor minimum
$\left(\frac{a}{3}, \frac{a}{3}\right)$	$\frac{-2a}{3}$	$\frac{-a}{3}$	$\frac{-2a}{3}$	$\frac{a^2}{3} > 0$	maximum or minimum

Hence, $f(x, y)$ is maximum or minimum at $\left(\frac{a}{3}, \frac{a}{3}\right)$ depending on whether $a > 0$ or $a < 0$.

$$f_{\text{extreme}} = \frac{a}{3} \cdot \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}.$$

Example 6: Examine the function $u = x^3 y^2 (12 - 3x - 4y)$ for extreme values.

Solution:

$$u(x, y) = 12x^3 y^2 - 3x^4 y^2 - 4x^3 y^3$$

Step I:

$$\frac{\partial u}{\partial x} = 36x^2 y^2 - 12x^3 y^2 - 12x^2 y^3$$

$$= 12x^2 y^2 (3 - x - y)$$

$$\frac{\partial u}{\partial y} = 24x^3 y - 6x^4 y - 12x^3 y^2 = 6x^3 y(4 - x - 2y)$$

For extreme values,

$$\frac{\partial u}{\partial x} = 0$$

$$12x^2 y^2 (3 - x - y) = 0$$

$$x = 0, y = 0, x + y = 3 \qquad \text{... (1)}$$

and $$\frac{\partial u}{\partial y} = 0$$

$$6x^3 y(4 - x - 2y) = 0$$

$$x = 0, y = 0, x + 2y = 4 \qquad \ldots (2)$$

Considering six pairs of equations,

$$\begin{array}{rr} x = 0 & y = 0 \\ x = 0 & x + 2y = 4 \\ y = 0 & x + 2y = 4 \\ x = 0 & x + y = 3 \\ y = 0 & x + y = 3 \\ x + y = 3 & x + 2y = 4 \end{array}$$

Solving these equations, following pairs of stationary points are obtained

(0, 0), (0, 2), (4, 0), (0, 3), (3, 0), (2, 1)

Step II: $$r = \frac{\partial^2 u}{\partial x^2} = 72xy^2 - 36x^2y^2 - 24xy^3 = 12\,xy^2(6 - 3x - 2y)$$

$$s = \frac{\partial^2 u}{\partial x\, \partial y} = 72x^2y - 24x^3y - 36x^2y^2 = 12x^2y(6 - 2x - 3y)$$

$$t = \frac{\partial^2 u}{\partial y^2} = 24x^3 - 6x^4 - 24x^3y = 6x^3(4 - x - 4y)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
(0, 0)	0	0	0	0	no conclusion
(0, 2)	0	0	0	0	no conclusion
(4, 0)	0	0	0	0	no conclusion
(0, 3)	0	0	0	0	no conclusion
(3, 0)	0	0	0	0	no conclusion
(2, 1)	−48	−48	−96	2304 > 0 and $r < 0$	maximum

Hence, function is maximum at (2, 1)

$$u_{\max} = (2^3)(1^2)(12 - 6 - 4) = 16.$$

Example 7: Find the extreme values of sin x + sin y + sin(x + y).

Solution: $$f(x, y) = \sin x + \sin y + \sin(x + y)$$

Step I: $$\frac{\partial f}{\partial x} = \cos x + \cos(x + y)$$

$$\frac{\partial f}{\partial y} = \cos y + \cos(x + y)$$

For extreme values,

$$\frac{\partial f}{\partial x} = 0, \quad \cos x + \cos(x+y) = 0 \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial y} = 0, \quad \cos y + \cos(x+y) = 0 \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$\cos x + \cos(x+y) = \cos y + \cos(x+y)$$

$$\cos x = \cos y, \; x = y$$

Substituting $x = y$ in Eq. (1),

$$\cos x + \cos 2x = 0,$$

$$\cos x = -\cos 2x = \cos(\pi - 2x) \text{ or } \cos(\pi + 2x)$$

$$x = \pi - 2x \text{ or } \pi + 2x$$

$$x = \frac{\pi}{3}, -\pi$$

$$y = \frac{\pi}{3}, -\pi$$

Thus, $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $(-\pi, -\pi)$ are stationary points.

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = -\sin(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$ and $r < 0$	maximum
$(-\pi, -\pi)$	0	0	0	0	no conclusion

Hence, function is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$f_{\max} = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

Example 8: Find the extreme values of sin x sin y sin $(x + y)$.

Solution: $f(x, y) = \sin x \sin y \sin(x + y)$

Step I:
$$\frac{\partial f}{\partial x} = \sin y[\cos x \ \sin(x+y) + \sin x \ \cos(x+y)]$$
$$= \sin y \ \sin(2x+y) = \frac{1}{2}[\cos 2x - \cos(2x+2y)]$$

and
$$\frac{\partial f}{\partial y} = \sin x[\cos y \ \sin(x+y) + \sin y \ \cos(x+y)]$$
$$= \sin x \ \sin(x+2y) = \frac{1}{2}[\cos 2y - \cos(2x+2y)]$$

For extreme values,
$$\frac{\partial f}{\partial x} = 0, \ \cos 2x - \cos 2(x+y) = 0 \quad \text{... (1)}$$

and
$$\frac{\partial f}{\partial y} = 0, \ \cos 2y - \cos 2(x+y) = 0 \quad \text{... (2)}$$

From Eqs (1) and (2), we get
$$\cos 2x = \cos 2y, \ x = y$$

Putting $x = y$ in Eq. (1),
$$\cos 2x - \cos 2(x+x) = 0, \ \cos 2x = \cos 4x, \ \cos 2x = 2\cos^2 2x - 1$$
$$2\cos^2 2x - \cos 2x - 1 = 0$$
$$\cos 2x = \frac{1 \pm \sqrt{1+8}}{4}$$
$$= 1, \ -\frac{1}{2}$$
$$\cos 2x = \cos 0, \qquad \cos 2x = \cos\frac{2\pi}{3}$$
$$x = 0, \qquad x = \frac{\pi}{3}$$
$$y = 0, \qquad y = \frac{\pi}{3}$$

Thus, $(0, 0)$, $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ are stationary points.

Step II:
$$r = \frac{\partial^2 f}{\partial x^2} = -\sin 2x + \sin 2(x+y) = 2\sin y \ \cos(2x+y)$$
$$s = \frac{\partial^2 f}{\partial x \, \partial y} = \sin 2(x+y)$$
$$t = \frac{\partial^2 f}{\partial y^2} = -\sin 2y + \sin 2(x+y) = 2\sin x \ \cos(x+2y)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
$(0, 0)$	0	0	0	0	no conclusion
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$ and $r < 0$	maximum

Hence, function is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$f_{\max} = \sin\frac{\pi}{3}\sin\frac{\pi}{3}\sin\frac{2\pi}{3}$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}.$$

Example 9: Find the points on the surface $z^2 = xy + 1$ nearest to the origin. Also find that distance.

Solution: Let $P(x, y, z)$ be any point on the surface $z^2 = xy + 1$.

Its distance from the origin is given by

$$D = \sqrt{(x^2 + y^2 + z^2)}$$
$$D^2 = x^2 + y^2 + z^2$$

Since P lies on the surface $z^2 = xy + 1$

$$D^2 = x^2 + y^2 + xy + 1$$

Let $$f(x, y) = x^2 + y^2 + xy + 1$$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 2x + y = 0 \qquad \text{... (1)}$$

$$\frac{\partial f}{\partial y} = 2y + x = 0 \qquad \text{... (2)}$$

Solving Eqs (1) and (2),

$$x = 0 \text{ and } y = 0$$

Stationary point is (0, 0).

Step II:
$$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

Step III: At (0, 0)

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0$$

and $$r = 2 > 0$$

Thus, $f(x, y)$ is minimum at (0, 0) and hence D is minimum at (0, 0).

At $$x = 0,\ y = 0$$

$$z^2 = xy + 1 = 1$$

$$z = \pm 1$$

Hence, D is minimum at (0, 0, 1) and $(0,\ 0,\ -1)$.

Thus, the points (0, 0, 1) and (0, 0, –1) on the surface $z^2 = xy + 1$ are nearest to the origin.

Minimum distance $= \sqrt{0+0+1} = 1$.

Example 10: A rectangular box open at the top is to have a volume 108 cubic meters. Find the dimensions of the box if its total surface area is minimum.

Solution: Let x, y and z be the dimensions of the box. Let V and S be its volume and surface area respectively.

$$V = xyz$$

$$S = xy + 2xz + 2yz$$

Substituting $z = \dfrac{V}{xy}$,

$$S = xy + 2x \cdot \frac{V}{xy} + 2y \cdot \frac{V}{xy} = xy + \frac{2V}{y} + \frac{2V}{x}$$

Step I:

$$\frac{\partial S}{\partial x} = y - \frac{2V}{x^2}$$

$$\frac{\partial S}{\partial y} = x - \frac{2V}{y^2}$$

For extreme values,

$$\frac{\partial S}{\partial x} = 0$$

$$y - \frac{2V}{x^2} = 0 \qquad \text{... (1)}$$

and $$\frac{\partial S}{\partial y} = 0$$

$$x - \frac{2V}{y^2} = 0 \qquad \text{... (2)}$$

Solving Eqs (1) and (2),

$$y = \frac{2V}{x^2}$$

$$x = 2V\left(\frac{x^4}{4V^2}\right) = 0$$

$$x\left(1 - \frac{x^3}{2V}\right) = 0$$

$$x = (2V)^{\frac{1}{3}}$$

$$y = \frac{2V}{x^2} = \frac{2V}{(2V)^{\frac{2}{3}}} = (2V)^{\frac{1}{3}} \quad [\text{since } x \neq 0 \text{ being the side of the box}]$$

Hence, stationary point is $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$

Step II:

$$r = \frac{\partial^2 S}{\partial x^2} = \frac{4V}{x^3}$$

$$s = \frac{\partial^2 S}{\partial x\, \partial y} = 1$$

$$t = \frac{\partial^2 S}{\partial y^2} = \frac{4V}{y^3}$$

At $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$, $r = \dfrac{4V}{2V} = 2 > 0,\ s = 1,\ t = \dfrac{4V}{2V} = 2$

Step III: At $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$,

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0 \text{ and } r = 2 > 0$$

Hence, S is minimum at $x = y = (2V)^{\frac{1}{3}}$

But $V = 108 \text{ m}^3$

$$x = y = (2 \times 108)^{\frac{1}{3}} = 6$$

and $z = \dfrac{V}{xy} = \dfrac{108}{6 \times 6} = 3$

Hence, dimensions of the box which make its total surface area S minimum are $x = 6,\ y = 6,\ z = 3$.

Example 11: Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Solution: Let x, y, z be the length, breadth and height of the rectangular solid and V be its volume.

$$V = xyz \qquad \text{... (1)}$$

Let given sphere is

$$x^2 + y^2 + z^2 = a^2$$

$$z^2 = a^2 - x^2 - y^2$$

Substituting in Eq. (1),

$$V = xy\sqrt{a^2 - x^2 - y^2}$$

$$V^2 = x^2y^2(a^2 - x^2 - y^2)$$

Let $$f(x, y) = V^2 = x^2y^2(a^2 - x^2 - y^2) \qquad \text{... (2)}$$

Step I: $$\frac{\partial f}{\partial x} = y^2[2x(a^2 - x^2 - y^2) + x^2(-2x)]$$

$$= 2xy^2(a^2 - 2x^2 - y^2)$$

and $$\frac{\partial f}{\partial y} = x^2[2y(a^2 - x^2 - y^2) + y^2(-2y)]$$

$$= 2x^2y(a^2 - x^2 - 2y^2)$$

For extreme values,

$$\frac{\partial f}{\partial y} = 0,\ 2xy^2(a^2 - 2x^2 - y^2) = 0$$

$$x = 0,\ y = 0,\ 2x^2 + y^2 = a^2 \qquad \text{... (3)}$$

and $$\frac{\partial f}{\partial y} = 0,\ 2x^2y(a^2 - x^2 - 2y^2)$$

$$x = 0,\ y = 0,\ x^2 + 2y^2 = a^2 \qquad \text{... (4)}$$

But x and y are the sides of the rectangular solid, therefore cannot be zero.

Solving $2x^2 + y^2 = a^2$ and $x^2 + 2y^2 = a^2$

$$x = \frac{a}{\sqrt{3}},\ y = \frac{a}{\sqrt{3}}$$

$$z = \sqrt{a^2 - \frac{a^2}{3} - \frac{a^2}{3}} = \frac{a}{\sqrt{3}}$$

Thus, stationary points are $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = 2a^2y^2 - 12x^2y^2 - 2y^4$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 4a^2xy - 8x^3y - 8xy^3$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2a^2x^2 - 2x^4 - 12x^2y^2$$

Step III: At $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$,

$$r = \frac{2a^4}{3} - \frac{4a^4}{3} - \frac{2a^4}{9} = -\frac{8a^4}{9}$$

$$s = \frac{4a^4}{3} - \frac{8a^4}{9} - \frac{8a^4}{9} = -\frac{4a^4}{9}$$

$$t = \frac{2a^4}{3} - \frac{2a^4}{9} - \frac{12a^4}{9} = -\frac{8a^4}{9}$$

$$rt - s^2 = \frac{64a^4}{81} - \frac{16a^4}{81} = \frac{48a^4}{81} > 0$$

$rt - s^2 > 0$ and $r < 0$

Therefore, $f(x, y)$ i.e. v^2 is maximum at $x = y = z$ and hence, v is maximum when $x = y = z$, i.e. rectangular solid is a cube.

Exercise 4.8

1. Examine maxima and minima of the following functions and find their extreme values:

(i) $2 + 2x + 2y - x^2 - y^2$

(ii) $x^2y^2 - 5x^2 - 8xy - 5y^2$

(iii) $x^2 + y^2 + xy + x - 4y + 5$

(iv) $x^2 + y^2 + 6x = 12$

(v) $x^3y^2(1 - x - y)$

(vi) $xy(3a - x - y)$

(vii) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

(viii) $x^4 + y^4 - 2(x - y)^2$

(ix) $x^4 + x^2y + y^2$

(x) $x^4 + y^4 - 4a^2xy$

(xi) $y^4 - x^4 + 2(x^2 - y^2)$

(xii) $x^3 + 3x^2 + y^2 + 4xy$

(xiii) $x^2y - 3x^2 - 2y^2 - 4y + 3$

(xiv) $x^4 - y^4 - x^2 - y^2 + 1$.

Ans.: (i) max. at $(1, 1)$; 4

(ii) max. at $(0, 0)$; 0

(iii) min. at $(-2, 3)$; -2

(iv) min at $(-3, 0)$; 3

(v) max. at $\left(\frac{1}{2}, \frac{1}{3}\right)$; $\frac{1}{432}$

(vi) max. at (a, a); a^3

(vii) max. at $(0, 0)$; 4

(viii) min. at $\left(\sqrt{2}, -\sqrt{2}\right)$ and $\left(-\sqrt{2}, \sqrt{2}\right)$; -8

(ix) min. at $(0, 0)$; 0

(x) min. at (a, a) and $(-a, a)$; a^4

(xi) No extreme values

(xii) No extreme values

(xiii) max. at $(0, -1)$; 5

(xiv) max. at $(0, 0)$; 1, min at $\left(\pm\frac{1}{\sqrt{2}}, \pm\sqrt{\frac{1}{\sqrt{2}}}\right)$; $\frac{1}{2}$

2. A rectangular box, open at the top, is to have a volume of 32 cc. Find the dimensions of the box requiring least materials for its construction.

   [**Ans. :** 4, 4, 2]

3. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

   [**Hint :** $f = xy + yz + zx$ where $x + y + z = 120$]

   [**Ans. :** 40, 40, 40]

4. The sum of three positive numbers is 'a'. Determine the maximum value of their product.

   $$\left[\textbf{Ans.:} \frac{a^3}{27} \text{ at} \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)\right]$$

5. Find the volume of the largest rectangular parallelopiped that can be inscribed in an ellipsoid

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

   [**Hint:** Let $2x$, $2y$, $2z$ be the sides of the parallelopiped, then its volume

   $$v = 8xyz = 8xy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$]

   $$\left[\textbf{Ans.:} \frac{8abc}{3\sqrt{3}}\right]$$

6. Prove that area of a triangle with constant perimeter is maximum when the triangle is equilateral.

   [**Hint :**

   Area $= \sqrt{s(s-a)(s-b)(s-c)}$

   where $2s = a + b + c$, $c = 2s - a - b$, s is constant]

7. Find the shortest distance from origin to the surface $xyz^2 = 2$.

   [**Ans. :** 2]

8. Find the shortest distance from the origin to the plane $x - 2y - 2z = 3$.

   [**Ans. :** 1]

9. Find the shortest distance between the lines

   $$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \text{ and}$$

   $$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}.$$

   $$\left[\textbf{Ans.:} 2\sqrt{29}\right]$$

10. Find the maximum value of $\cos A \cos B \cos C$, where A, B, C are angles of a triangle.

    $$\left[\textbf{Ans.:} \text{max. at} \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right); \frac{1}{8}\right]$$

4.8.4 Lagrange's Method of Undetermined Multipliers

Let $f(x, y, z)$ be a function of three variables x, y, z, and the variables be connected by the relation

$$\phi(x, y, z) = 0 \qquad \text{... (1)}$$

Suppose we wish to find the values of x, y, z, for which $f(x, y, z)$ is stationary (maximum and minimum)

For this purpose, we construct an auxiliary equation

$$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z) \qquad \text{... (2)}$$

Differentiating partially w.r.t. x, y, z and equating to zero,

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \qquad \text{... (3)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \qquad \text{... (4)}$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \qquad \text{... (5)}$$

Solving Eqs (1), (3), (4) and (5), we can find the values of x, y, z and λ for which $f(x, y, z)$ has stationary value. This method of obtaining stationary values of $f(x, y, z)$ is called the Lagrange's method of undetermined multipliers and Eqs (3), (4) and (5) are called Lagrange's equations. The term λ is called undetermined multiplier.

Example 1: Find the point on the plane $ax + by + cz = p$ at which the function $f = x^2 + y^2 + z^2$ has a minimum value and find this minimum f.

Solution:

$$f = x^2 + y^2 + z^2 \qquad \text{... (1)}$$

$$ax + by + cz = p \qquad \text{... (2)}$$

$$\phi(x, y, z) = ax + by + cz - p = 0$$

Lagrange's equations

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$2x + \lambda a = 0$$

$$x = \frac{-\lambda a}{2}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$2y + \lambda b = 0$$

$$y = \frac{-\lambda b}{2}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$$2z + \lambda c = 0$$

$$z = \frac{-\lambda c}{2}$$

Substituting x, y, z in Eq. (2),

$$a\left(\frac{-\lambda a}{2}\right) + b\left(\frac{-\lambda b}{2}\right) + c\left(\frac{-\lambda c}{2}\right) = p$$

$$\lambda a^2 + \lambda b^2 + \lambda c^2 = -2p$$

$$\lambda = \frac{-2p}{a^2 + b^2 + c^2}$$

Thus, $$x = \frac{ap}{a^2+b^2+c^2},\ y = \frac{bp}{a^2+b^2+c^2},\ z = \frac{cp}{a^2+b^2+c^2}$$

The minimum value of $$f = \frac{a^2p^2}{(a^2+b^2+c^2)^2} + \frac{b^2p^2}{(a^2+b^2+c^2)^2} + \frac{c^2p^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)^2} = \frac{p^2}{a^2+b^2+c^2}.$$

Example 2: Find the maximum value of $f = x^2y^3z^4$ subject to the condition $x + y + z = 5$.

Solution:
$$f = x^2y^3z^4 \quad \text{... (1)}$$
$$x + y + z = 5 \quad \text{... (2)}$$
$$\phi(x, y, z) = x + y + z - 5 = 0$$

Lagrange's equations

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$

$$2xy^3z^4 + \lambda = 0$$
$$2xy^3z^4 = -\lambda \quad \text{... (3)}$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

$$3x^2y^2z^4 + \lambda = 0$$
$$3x^2y^2z^4 = -\lambda \quad \text{... (4)}$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

$$4x^2y^3z^3 + \lambda = 0$$
$$4x^2y^3z^3 = -\lambda \quad \text{... (5)}$$

From Eqs (3) and (4),

$$2xy^3z^4 = 3x^2y^2z^4$$
$$2y = 3x$$
$$y = \frac{3}{2}x$$

From Eqs (3) and (5),

$$2xy^3z^4 = 4x^2y^3z^3$$
$$z = 2x$$

Substituting y and z in Eq. (2),

$$x + \frac{3}{2}x + 2x = 5$$

$$9x = 10$$

$$x = \frac{10}{9}$$

$$y = \frac{3}{2}x = \frac{3}{2}\left(\frac{10}{9}\right) = \frac{5}{3}$$

$$z = 2x = 2\left(\frac{10}{9}\right) = \frac{20}{9}$$

Maximum value of $f = \left(\frac{10}{9}\right)^2 \left(\frac{5}{3}\right)^3 \left(\frac{20}{9}\right)^4 = \frac{(2^{10})\,(5^9)}{3^{15}}.$

Example 3: Show that the rectangular solid of maximum value that can be inscribed in a sphere is a cube.

Solution: Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid. Let r be the radius of the sphere.

Volume of solid, $V = 8xyz$ … (1)

Equation of the sphere, $x^2 + y^2 + z^2 = r^2$ … (2)

$$\phi\,(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$$

Lagrange's equation

$$\frac{\partial V}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$8yz + \lambda.2x = 0$$

$$2\lambda x = -8yz$$

$$2\lambda x^2 = -8xyz \qquad \dots (3)$$

$$\frac{\partial V}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$8xz + \lambda.2y = 0$$

$$2\lambda y = -8xz$$

$$2\lambda y^2 = -8xyz \qquad \dots (4)$$

$$\frac{\partial V}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$$8xy + \lambda.2z = 0$$

$$2\lambda z = -8xy$$

$$2\lambda z^2 = -8xyz \qquad \dots (5)$$

From Eqs (3), (4) and (5),

$$2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$$

$$x^2 = y^2 = z^2$$

$$x = y = z$$

Hence, rectangular solid is a cube.

Example 4: A rectangular box open at the top is to have volume of 32 cubic units. Find the dimensions of the box requiring least material for its construction.

Solution: Let x, y, z be the dimensions of the box.

Volume $$V = xyz = 32 \qquad \dots (1)$$

The box is open at the top. Therefore, its surface area

$$S = xy + 2xz + 2yz \qquad \dots (2)$$

$$\phi(x, y, z) = xyz - 32 \qquad \dots (3)$$

Lagrange's equation

$$\frac{\partial S}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$y + 2z + \lambda yz = 0 \qquad \dots (4)$$

$$\frac{\partial S}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$x + 2z + \lambda xz = 0 \qquad \dots (5)$$

$$\frac{\partial S}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$$2x + 2y + \lambda xy = 0 \qquad \dots (6)$$

Multiplying Eq. (4) by x,

$$xy + 2xz + \lambda xyz = 0$$

$$xy + 2xz + 32\lambda = 0$$

$$xy + 2xz = -32\lambda \qquad \dots (7)$$

Multiplying Eq. (5) by y,

$$xy + 2yz + \lambda xyz = 0$$

$$xy + 2yz + 32\lambda = 0$$

$$xy + 2yz = -32\lambda \qquad \dots (8)$$

Multiplying Eq. (6) by z,

$$2xz + 2yz + \lambda xyz = 0$$
$$2xz + 2yz + 32\lambda = 0$$
$$2xz + 2yz = -32\lambda \qquad \ldots (9)$$

From Eqs (7) and (8),
$$xy + 2xz = xy + 2yz$$
$$2xz = 2yz$$
$$x = y$$

From Eqs (8) and (9),

$$xy + 2yz = 2xz + 2yz$$
$$xy = 2xz$$
$$y = 2z, \; z = \frac{y}{2}$$

Substituting x, y, z in Eq. (1),

$$y \cdot y \cdot \frac{y}{2} = 32$$
$$y^3 = 64$$
$$y = 4$$
$$x = y = 4$$
$$z = \frac{y}{2} = 2$$

Hence, dimensions of the box requiring least material for its construction are 4, 4, 2.

Example 5: Find the maximum and minimum distances from the origin to the curve $3x^2 + 4xy + 6y^2 = 140$.

Solution: The distance d from the origin (0, 0) to any point (x, y) is given by

$$d = \sqrt{x^2 + y^2}, \; d^2 = x^2 + y^2$$

Let $f(x, y) = x^2 + y^2$

and $\phi(x, y) = 3x^2 + 4xy + 6y^2 - 140$

Lagrange's equations

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$
$$2x + \lambda (6x + 4y) = 0 \qquad \ldots (1)$$
$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$
$$2y + \lambda (4x + 12y) = 0 \qquad \ldots (2)$$

Solving Eqs (1) and (2),

$$\lambda = -\frac{x}{3x+2y} = -\frac{y}{2x+6y}$$

$$-\lambda = \frac{x^2}{3x^2+2xy} = \frac{y^2}{2xy+6y^2} = \frac{x^2+y^2}{3x^2+6y^2+4xy} = \frac{f(x, y)}{140}$$

Substituting λ in Eqs (1) and (2),

$$2x - \frac{f}{140}(6x+4y) = 0, \; 2y - \frac{f}{140}(4x+12y) = 0$$

$$(140-3f)x - 2fy = 0 \quad \dots (3)$$

and $$-2fx + (140-6f)\,y = 0 \quad \dots (4)$$

Substituting $x = \dfrac{2fy}{140-3f}$ from Eq. (3) in Eq. (4),

$$-4f^2 + (140-3f)\,(140-6f) = 0$$

$$14f^2 - 1260f + (140^2) = 0$$

$$f^2 - 90f - 1400 = 0$$

$$f = 70, \; 20$$

Thus, maximum and minimum distances are $\sqrt{70}, \sqrt{20}$.

Example 6: A wire of length *b* is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found.

Solution: Let x and y be two parts of the wire.

$$x + y = b \quad \dots (1)$$

Let the piece of length x is bent in the form of a square so that each side is $\dfrac{x}{4}$.

Thus, the area of the square, $A_1 = \dfrac{x}{4} \cdot \dfrac{x}{4} = \dfrac{x^2}{16}$.

Suppose piece of length y is bent in the form of a circle of radius r so perimeter of the circle is y.

$$2\pi r = y, \; r = \frac{y}{2\pi}$$

Thus, the area of the circle, $$A_2 = \pi\left(\frac{y}{2\pi}\right)^2 = \frac{y^2}{4\pi}.$$

Let sum of the areas is given as

$$f(x, y) = \frac{x^2}{16} + \frac{y^2}{4\pi}$$

and $\phi(x, y) = x + y - b$

Lagrange's equations:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{2x}{16} + \lambda = 0, \ x = -8\lambda$$

and $$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{2y}{4\pi} + \lambda = 0, \ y = -2\pi\lambda$$

Substituting x and y in Eq. (1),

$$(-8\lambda) + (-2\pi\lambda) = b$$

$$\lambda = \frac{-b}{8 + 2\pi}$$

Thus, $$x = \frac{8b}{8 + 2\pi} = \frac{4b}{4 + \pi}$$

$$y = \frac{2\pi b}{8 + 2\pi} = \frac{\pi b}{4 + \pi}$$

Substituting in $f(x, y)$,

$$f(x, y) = \frac{1}{16}\left(\frac{4b}{4 + \pi}\right)^2 + \frac{1}{4\pi}\left(\frac{\pi b}{4 + \pi}\right)^2$$

$$= \frac{b^2}{(4 + \pi)^2}\left(1 + \frac{\pi^2}{4\pi}\right) = \frac{b^2 \pi (4 + \pi)}{4\pi (4 + \pi)^2}$$

$$= \frac{b^2}{4(\pi + 4)}$$

Hence, the least value of the sum of the areas is $\dfrac{b^2}{4(\pi + 4)}$.

Example 7: A closed rectangular box has length twice its breadth and has constant volume *V*. Determine the dimensions of the box requiring least surface area.

Solution: Let x be the breadth and y be the height of the rectangular box so length of the box will be $2x$.

Volume of the box $V = x \cdot 2x \cdot y = 2x^2 y$

Volume of the box is constant

$$2x^2 y = V = \text{constant} \quad \ldots (1)$$

Surface area of the box is given by

$$S = 2\,(2x \cdot x + x \cdot y + y \cdot 2x) = 4x^2 + 6xy \quad \ldots (2)$$

Let $$\phi(x, y) = 2x^2y - V \quad \dots (3)$$

Lagrange's equations:

$$\frac{\partial S}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$8x + 6y + \lambda(4xy) = 0$$

$$2x + 3y + \lambda(2xy) = 0 \quad \dots (4)$$

and $$\frac{\partial S}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$6x + \lambda(2x^2) = 0$$

$$3x + \lambda x^2 = 0, \ x = -\frac{3}{\lambda}$$

Substituting $x = -\frac{3}{\lambda}$ in Eq. (4),

$$2\left(-\frac{3}{\lambda}\right) + 3y + \lambda \ 2y\left(-\frac{3}{\lambda}\right) = 0$$

$$-\frac{6}{\lambda} = 3y, \ y = -\frac{2}{\lambda}$$

Substituting x and y in Eq. (1),

$$2\left(-\frac{3}{\lambda}\right)^2\left(-\frac{2}{\lambda}\right) = V$$

$$\lambda^3 = \frac{-36}{V}, \ \lambda = -\left(\frac{36}{V}\right)^{\frac{1}{3}}$$

$$x = -\frac{3}{\lambda} = 3\left(\frac{V}{36}\right)^{\frac{1}{3}} = \left(\frac{27V}{36}\right)^{\frac{1}{3}} = \left(\frac{3V}{4}\right)^{\frac{1}{3}}$$

$$y = -\frac{2}{\lambda} = 2\left(\frac{V}{36}\right)^{\frac{1}{3}} = \left(\frac{8V}{36}\right)^{\frac{1}{3}} = \left(\frac{2V}{9}\right)^{\frac{1}{3}}$$

Hence, the dimensions of the box requiring least surface area are $2\left(\frac{3V}{4}\right)^{\frac{1}{3}}$, $\left(\frac{3V}{4}\right)^{\frac{1}{3}}$, $\left(\frac{2V}{9}\right)^{\frac{1}{3}}$.

Example 8: Using the Lagrange's method find the minimum and maximum distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$.

Solution: Given sphere is $x^2 + y^2 + z^2 = 36$ … (1)
Let the coordinates of any point on the sphere be (x, y, z), then its distance D from the point (1, 2, 2) is

$$D = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

Let $D^2 = f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$

and $$\phi(x, y, z) = x^2 + y^2 + z^2 - 36$$

Lagrange's equations:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$2(x-1) + \lambda(2x) = 0$$

$$(x-1) + \lambda x = 0 \quad \dots (2)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$2(y-2) + \lambda(2y) = 0$$

$$(y-2) + \lambda y = 0 \quad \dots (3)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

and $$2(z-2) + \lambda(2z) = 0$$

$$(z-2) + \lambda z = 0 \quad \dots (4)$$

Multiplying Eq. (2) by x, Eq. (3) by y and Eq. (4) by z and adding,

$$(x^2 + y^2 + z^2) - (x + 2y + 2z) + \lambda(x^2 + y^2 + z^2) = 0$$

$$36(1 + \lambda) - (x + 2y + 2z) = 0 \quad \text{[Using Eq. (1)]} \quad \dots (5)$$

From Eq. (2),

$$x = \frac{1}{1+\lambda} \quad \dots (6)$$

From Eq. (3),

$$y = \frac{2}{1+\lambda} \quad \dots (7)$$

From Eq. (4),

$$z = \frac{2}{1+\lambda} \quad \dots (8)$$

Substituting x, y, z in Eq. (5),

$$36(1+\lambda) - \left(\frac{1+4+4}{1+\lambda}\right) = 0$$

$$36(1+\lambda)^2 = 9, \quad (1+\lambda)^2 = \frac{1}{4},$$

$$1 + \lambda = \pm\frac{1}{2},$$

Substituting in Eqs (6), (7) and (8),

$$x = \pm 2, y = \pm 4, z = \pm 4$$

Minimum distance $= \sqrt{(2-1)^2 + (4-2)^2 + (4-2)^2} = \sqrt{1+4+4} = 3$

Maximum distance $= \sqrt{(-2-1)^2 + (-4-2)^2 + (-4-2)^2} = \sqrt{9+36+36} = 9.$

Example 9: Use the method of the Lagrange's multipliers to find volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots (1)$$

Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular parallelopiped inscribed in the ellipsoid.

Volume of the parallelopiped, $V = (2x)\,(2y)\,(2z) = 8xyz$.

Let
$$\phi(x, y, z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$

Lagrange's equations:

$$\frac{\partial V}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0$$

$$8yz+\lambda\frac{2x}{a^2}=0,\ 4yz+\lambda\frac{x}{a^2}=0 \quad \ldots (2)$$

$$\frac{\partial V}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0$$

$$8xz+\lambda\frac{2y}{b^2}=0,\ 4xz+\lambda\frac{y}{b^2}=0 \quad \ldots (3)$$

$$\frac{\partial V}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

$$8xy+\lambda\frac{2z}{c^2}=0,\ 4xy+\lambda\frac{z}{c^2}=0 \quad \ldots (4)$$

Multiplying Eq. (2) by x, Eq. (3) by y and Eq. (4) by z and adding,

$$12xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) = 0$$

$$12xyz+\lambda=0 \quad \text{[Using Eq. (1)]}$$

$$\lambda=-12xyz$$

Substituting in Eq. (2),

$$4yz - 12xyz\left(\frac{x}{a^2}\right) = 0$$

$$1 - \frac{3x^2}{a^2} = 0,\ x = \frac{a}{\sqrt{3}}$$

Similarly substituting λ in Eqs (3) and (4),

$$y = \frac{b}{\sqrt{3}},\ z = \frac{c}{\sqrt{3}}$$

Volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid

$$V = 8xyz = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}.$$

Exercise 4.9

1. Find stationary values of the function $f(x, y, z) = x^2 + y^2 + z^2$, given that $z^2 = xy + 1$.

   [Ans. : $(0, 0, -1), (0, 0, 1)$**]**

2. Find the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ subject to the fulfillment of the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, given a, b, c are not zero.

   $$\left[\textbf{Ans.:}\ x = \frac{1}{a}(a+b+c),\ y = \frac{1}{b}(a+b+c),\ z = \frac{1}{c}(a+b+c)\right]$$

3. Find the largest product of the numbers x, y and z when $x + y + z^2 = 16$.

   $$\left[\textbf{Ans.:}\ \frac{4096}{25\sqrt{5}}\right]$$

4. Find the largest product of the numbers x, y and z when $x^2 + y^2 + z^2 = 9$.

   $$\left[\textbf{Ans.:}\ 3\sqrt{3}\right]$$

5. Find a point in the plane $x + 2y + 3z = 13$ nearest to the point $(1, 1, 1)$.

   $$\left[\textbf{Ans.:}\ \left(\frac{3}{2}, 2, \frac{5}{2}\right)\right]$$

6. Find the shortest distance from the point $(1, 2, 2)$ to the sphere $x^2 + y^2 + z^2 = 36$.

   [Ans. : 3**]**

7. Find the maximum distance from the origin $(0, 0)$ to the curve $3x^2 + 3y^2 + 4xy - 2 = 0$.

   $$\left[\textbf{Ans.:}\ \sqrt{2}\right]$$

8. Decompose a positive number a into three parts so that their product is maximum.

   $$\left[\textbf{Ans.:}\ \frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right]$$

9. Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

   $$\left[\textbf{Ans.:}\ \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}\right]$$

10. Find the dimensions of a rectangular box of maximum capacity whose surface area is given when

(i) box is open at the top

(ii) box is closed.

$$\left[\textbf{Ans.:} \text{ (i) } \sqrt{\frac{s}{3}}, \sqrt{\frac{s}{3}}, \frac{1}{2}\sqrt{\frac{s}{3}} \quad \text{(ii) } \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}\right]$$

11. Determine the perpendicular distance of the point (a, b, c) from the plane $lx + my + nz = 0$.

$$\left[\textbf{Ans.:} \text{ minimum distance } \frac{la+mb+nc}{\sqrt{l^2+m^2+n^2}}\right]$$

12. Find the length and breadth of a rectangle of maximum area that can be inscribed in the ellipse $4x^2 + y^2 = 36$.

$$\left[\textbf{Ans.:} \frac{3\sqrt{2}}{2}, \sqrt{2}, \textit{Area} = 12\right]$$

13. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid of revolution $4x^2 + 4y^2 + 9z^2 = 36$.

$$\left[\textbf{Ans.:} 16\sqrt{3}\right]$$

14. Find the extreme volume of $x^2 + y^2 + z^2 + xy + xz + yz$ subject to the conditions $x + y + z = 1$ and $x + 2y + 3z = 3$.

$$\left[\textbf{Ans.:} \frac{1}{6}, \frac{1}{3}, \frac{5}{6}\right]$$

FORMULAE

Chain Rule

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y}$$

where $z = f(u)$ and $u = \phi(x, y)$

Total Differential Coefficient

(i) $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

where $u = f(x, y)$ and $x = \phi(t)$, $y = \psi(t)$

(ii) $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

where $u = f(x, y, z)$ and $x = \phi(t)$, $y = \psi(t)$, $z = \xi(t)$,

Composite Function of Two Variables

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

where $z = f(x, y)$ and $x = \phi(u, v)$, $y = \psi(u, v)$

Implicit Functions

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

where $f(x, y) = c$ and y is a function of x.

Euler's Theorem and deductions

(i) $$x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} = nu$$

(ii) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

(iii) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

(iv) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

(v) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

where $g(u) = n\dfrac{f(u)}{f'(u)}$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. If $z = f(x + ay) + \phi(x - ay)$, then
   (a) $z_{xx} = z_{yy}$ (b) $z_{xx} = a^2 z_{yy}$
   (c) $z_{yy} = a^2 z_{xx}$ (d) none of these
2. If $x = \log(x \tan^{-1} y)$, then f_{xy} is equal to
   (a) $-\frac{1}{x^2}$ (b) 0
   (c) $\frac{1}{x^2}$ (d) none of these
3. If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, then
   $u_x + u_y + u_z$ is equal to
   (a) 0 (b) xyz
   (c) $x + y + z$ (d) none of these
4. If $z = \cos\left(\frac{x}{y}\right) + \sin\left(\frac{x}{y}\right)$, then
   $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ is equal to
   (a) z (b) $2z$
   (c) 0 (d) none of these
5. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then $xu_x + yu_y + zu_z$ is equal to
   (a) $3u$ (b) $2u$
   (c) 3 (d) none of these
6. If $u = x^2 + y^2 + z^2$ be such that $xu_x + yu_y + zu_z = \lambda u$ then, λ is equal to
   (a) 1 (b) 2
   (c) 3 (d) none of these
7. If $f(x, y, z) = 0$, then the value of
   $\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x}$ is
   (a) 1 (b) -1
   (c) 0 (d) none of these
8. If $u(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right)$
   $- y^2 \tan^{-1}\left(\frac{x}{y}\right)$, $x > 0, y > 0$, then
   $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ is equal to
   (a) 0 (b) $2u$
   (c) u (d) $3u$
9. If $f(x, y) = e^{xy^2}$, the total differential of the function at the point (1, 2) is
   (a) $e(dx + dy)$ (b) $e^4(dx + dy)$
   (c) $e^4(4dx + dy)$ (d) $4e^4(dx + dy)$
10. If $f(x, y) = 0$, then $\frac{dy}{dx}$ is equal to
    (a) $\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ (b) $\dfrac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$
    (c) $-\dfrac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$ (d) $-\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$
11. The function $f(x, y) = 2x^2 + 2xy - y^3$ has
    (a) only one stationary point at (0, 0)
    (b) two stationary points at (0, 0) and $\left(\frac{1}{6}, \frac{1}{3}\right)$
    (c) two stationary points at (0, 0) and (1, –1)
    (d) no stationary points
12. If $z = f(x, y)$, dz is equal to
    (a) $\left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy$
    (b) $\left(\frac{\partial f}{\partial y}\right)dx + \left(\frac{\partial f}{\partial x}\right)dy$
    (c) $\left(\frac{\partial f}{\partial x}\right)dx - \left(\frac{\partial f}{\partial y}\right)dy$
    (d) $\left(\frac{\partial f}{\partial y}\right)dx - \left(\frac{\partial f}{\partial x}\right)dy$
13. The function $z = 5xy - 4x^2 + y^2 - 2x - y + 5$ has at $x = \frac{1}{41}, y = \frac{18}{41}$
    (a) maxima (b) saddle point

(c) minima (d) none of these

14. If $f(x, y)$ is such that $f_x = e^x \cos y$ and $f_y = e^x \sin y$, then which of the following is true
(a) $f(x, y) = e^{x+y} \sin(x + y)$
(b) $f(x, y) = e^x \sin(x + y)$
(c) $f(x, y)$ does not exist
(d) none of these

15. The percentage error in the area of a rectangle when an error of 1% is made in measuring its length and breadth is equal to
(a) 1% (b) 2%
(c) 0 (d) 3%

16. The function $f(x) = 10 + x^6$
(a) is a decreasing function of x
(b) has a minimum at $x = 0$
(c) has neither a maximum nor a minimum at $x = 0$
(d) none of these

17. If $u = f(y + ax) + \phi(y - ax)$,
then $\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial^2 u}{\partial y^2}$ is
(a) 0 (b) a^2
(c) $a^2(f'' - \phi'')$ (d) $a^2(f'' + \phi'')$

18. With usual notations, the properties of maxima and minima under various conditions are,

I	II
(P) Maxima	(i) $rt - s^2 = 0$
(Q) Minima	(ii) $rt - s^2 < 0$
(R) Saddle point	(iii) $rt - s^2 > 0$, $r > 0$
(S) Failure case	(iv) $rt - s^2 > 0$ $r < 0$

(a) P-i, Q-iii, R-iv, S-ii
(b) P-ii, Q-i, R-iii, S-iv
(c) P-iii, Q-iv, R-ii, S-i
(d) P-iv, Q-ii, R-i, S-iii

19. The Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ for the function $u = e^x \sin y$, $v = (x + \log \sin y)$ is
(a) 1
(b) $\sin x \sin y - xy \cos x \cos y$
(c) 0
(d) $\frac{e^x}{x}$

20. If the function u, v, w of three independent variables x, y, z are not independent, then the Jacobian of u, v, w w.r.t to x, y, z is always equal to
(a) 1
(b) 0
(c) ∞
(d) Jacobian of x, y, z w.r.t u, v, w

21. The approximate value of $f(0.999)$ where $f(x) = 2x^4 + 7x^3 - 8x^2 + 3x + 1$ is
(a) 4.984 (b) 3.984
(c) 2.984 (d) 1.984

Answers

1. (c)	2. (b)	3. (a)	4. (c)	5. (c)	6. (b)	7. (b)
8. (b)	9. (d)	10. (d)	11. (b)	12. (a)	13. (b)	14. (c)
15. (b)	16. (c)	17. (a)	18. (c)	19. (c)	20. (b)	21. (a)