7

ORDINARY DIFFERENTIAL EQUATIONS OF 1ST ORDER

7.1 INTRODUCTION

Problem formulation as differential equations arises from elimination, geometry, dynamics, electric-circuits and sound-vibration phenomenon etc. But inspite of its intimate relation to applications, the theory of equations has an independent existence of its own, consisting of a self coherent body of knowledge. Thus, we may define a differential equation as a relation between independent variable x, a dependent variable y (= $f(x)$) and one or more of the derivatives y', y'', ... of y with respect to x.

In this treatise, all the derivatives are total and the corresponding differential equations are some time said to be ordinary to distinguish from partial differential equations which involves partial derivatives.

The **order of a differential equation** is that of the highest order derivative it involves and **its degree** is the degree of the highest order derivative occurring in it.

e.g.

(*i*) $\dfrac{dy}{dx} = x + 5,$ (*ii*) $\dfrac{dy}{dx} = \sin x$

(*iii*) $\dfrac{d^2y}{dx^2} + px = 0$ (*iv*) $y = x\dfrac{dy}{dx} + \sqrt[k]{1 + \left(\dfrac{dy}{dx}\right)^2}$

(*v*) $y\left(\dfrac{d^2y}{dx^2}\right)^3 + (2y-3)\left(\dfrac{dy}{dx}\right)^2 + 7y = 8x^2 - 3$ (*vi*) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial x} = \lambda z$

Equations (*i*) to (*vi*), except (*vi*), all are ordinary differential equations. Equations (*i*) & (*ii*) are of 1st order 1st degree; equation (*iii*) is of 2nd order 1st degree; equation (*iv*) is of 1st order 2nd degree and equation (*v*) is of 2nd order 3rd degree.

A general differential equation between the independent variable x and the dependent variable y has the form

$$a_0 \frac{d^n y}{d^n x} + a_1 \frac{d^{n-1}y}{d^n x} + \ldots\ldots\ldots + a_{n-1}\frac{dy}{dx} + a_n y = f(x) \qquad \ldots(1)$$

where $a_0, a_1, \ldots, a_n$ are either functions of x or merely constant. As in this equation $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}$ occur only to power unity and their product is notably absent, the equation is linear in nature. Clearly, in the set of equations (*i*) to (*v*), equation (*i*) to (*iv*) are linear whereas equation (*v*) is non-linear.

When $f(x) = 0$, the equation (1) is said to be **homogeneous** otherwise, non-homogeneous

The differential equations are either initial value problem or boundary value problem. **An initial value problem** is that differential equation which has certain given initial conditions that must be satisfied by its solution and differential coefficients involved there in.

e.g. (*i*) $\frac{dy}{dx} = 2x + 3,\ y(0) = 1$ i.e. $x = 0$ is the initial point

(*ii*) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = e^x,\ y(1) = 1,\ \frac{dy}{dx} = 4$ i.e. $x = 1$ is the initial point

In boundary value problem, the differential equation together with boundary conditions, must be satisfied by the solution or differential coefficients involved at no less than two separate points.

e.g. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = \cos x,\ y(0) = 1,\ y'(1) = 2$.

7.2 SOLUTIONS OF DIFFERENTIAL EQUATIONS

With regard to solution of a differential equation, we simply mean a relation between variables involved such that this relation and differential coefficients obtained there from satisfy the differential equation. Sometimes it is also termed as **integral** (premitive) of the differential equation and it contains as many as arbitrary constants as the order of the differential equation. If these arbitrary constants are given some particular

Some important points regarding differential equations

1. **Differential Equation:** An equation containing an independent variable, a dependent variable and the derivatives of the dependent variable, is called as differential equation.
2. **Order of a Differential Equation:** The order of the highest order derivative occurring in a differential equation, is called the order of the differential equation.
3. **Degree of a Differential Equation:** The power of the highest order derivative occurring in a differential equation, after it is made free from radicals and fractions, is called the degree of the differential equation.
4. **Solution of a Differential Equation:** A relation between the dependent and independent variables, which when substituted in the differential equation reduces it to an identity, is called its solution.
5. **Formation of a Differential Equation:** Differentiate the given equation as many times as the number of arbitrary constants in the given equation. Now eliminate the constants from these equations. The eliminant will be the required differential equation.
6. **Linear Differential Equation:** A differential equation is said to be linear if the unknown function, together with all of its derivatives appear in the differential equation with power not greater than 1, and not as products either.
7. **First order Differential Equation:** Differential equation in which only the first order derivative appear and there is no higher derivative of the unknown function is called first order differential equation.

Important Mathematical Identities

1. The simplest type of differential equation of first order first degree

$$\frac{dy}{dx} = f(x, y) \text{ or } M(x, y)\,dx + N(x, y)\,dy = 0$$

2. Homogeneous equation

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \text{ or } g\left(\frac{x}{y}\right)$$

3. The most general form of a linear differential equation of the 1st order

$\frac{dy(x)}{dx} + Py(x) = Q$, here P and Q are functions of x only.

4. Leibnitz's Linear equation

$$\frac{1}{1-n}\frac{du}{dx} + Pu = Q \quad \text{or} \quad \frac{du}{dx} + P(1-n)\,u = Q(1-n)$$

5. Equations of the first order and higher degree

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \ldots\ldots\ldots + P_{n-1}\frac{dy}{dx} + P_n = 0$$

values, the solution is termed as **particular solution**. Yet there is another type of solution called **singular solution** i.e. a relation between the variables having no arbitrary constant in it and still satisfies the differential equation. Meaning there by, this solution can not be obtained from the general solution.

e.g. Initial value problem, $\frac{dy}{dx} = xy^{\frac{1}{3}}$, $y(0) = 0$ has the general solution $y = \pm\left(\frac{x^2}{3} + c\right)^{\frac{3}{2}}$ obtained by one of the standard method (viz. variable separable). Here two particular solutions are $y = \pm\frac{x^3}{3\sqrt{3}}$ (with $c = 0$). However, $y = 0$ is also a solution which cannot be obtained from the general solution whatever be the value of c. Thus, $y = 0$ is called the singular solution.

Note: In many cases solutions are in implicit form, although often unavoidable since these are not as satisfactory forms as the explicit one. In certain ways it is incomplete and poses problems than those which are faced in solutions.

7.3 GEOMETRICAL MEANING OF DIFFERENTIAL EQUATION

(*a*) **Geometrical Meaning of Differential Equation $\frac{dy}{dx} = f(x, y)$**: The given differential equation $\frac{dy}{dx} = f(x, y)$ is of 1st order 1st degree, wherein $\frac{dy}{dx}(= m)$ stands for the slope of the tangent to the curve at any general point $P(x, y)$. Let the value of m at the point $A_0(x_0, y_0)$ be m_0 and thus, for a neighbouring point $A_1(x_1, y_1)$, m_0 will be the slope of A_0A_1. Likewise, let the corresponding value of m at A_1 be m_1, then for the neighbouring point $A_2(x_2, y_2)$, m_1 will become slop of A_1A_2 and so on. Thus, if the successive points A_0, A_1, A_2, A_3, ... are chosen very close to one-another, the broken curve $A_0\ A_1\ A_2\ A_3$, ... will lead to an approximately smooth curve $\Gamma(y = f(x))$ passing through $A_0(x_0, y_0)$ and will be a solution to $\frac{dy}{dx} = f(x,\ y)$. Hence represents a family of curves such that through every point, there passes one curve of the family.

(b) Geometrical Meaning of Complete Solution of $\frac{dy}{dx} = f(x,\ y)$:

Let $\phi(x, y, c) = 0$ be the solution of the given differential equation, wherein c is an arbitrary constant. Clearly, this curve $\phi(x, y, c) = 0$ represents a one parameter family of curve such that through each point there passes one curve of the family.

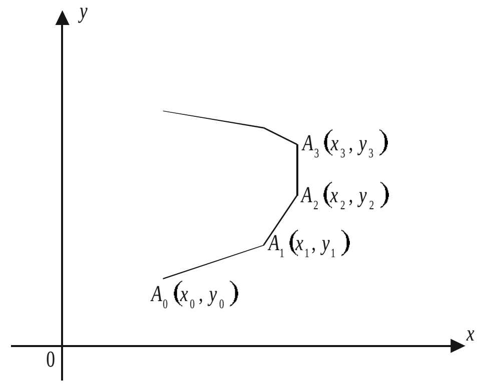

Fig. 7.1

7.4 FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constants from a relation in the variables and constants. Some of the problems are discussed subsequently.

Example 1: Show that $y = A\cos x + B\sin x$ is a solution of $\frac{d^2y}{dx^2} + y = 0$

Solution: The given relation i.e. $y = A\cos x + B\sin x$ ...(1)

Implying $$\frac{dy}{dx} = -A\sin x + B\cos x \qquad ...(2)$$

and $$\frac{d^2y}{dx^2} = -A\cos x - B\sin x \qquad ...(3)$$

Now from (1) and (3), we see that

$$\frac{d^2y}{dx^2} = -y \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0.$$

Example 2: Eliminate the arbitrary constants A and B from the differential equation

$$y = Ax + Bx^2$$

Solution: The given relation $y = Ax + Bx^2$ ...(1)

Implying $$\frac{dy}{dx} = A + 2Bx \qquad ...(2)$$

and $$\frac{d^2y}{dx^2} = 2B \qquad ...(3)$$

On eliminating value of B by using (2) and (3), we get

$$\frac{d^2y}{dx^2} = \frac{1}{x}\left(\frac{dy}{dx} - A\right) \text{ or } A = \frac{dy}{dx} - x\frac{d^2y}{dx^2} \qquad \dots(4)$$

On putting the values of A from (4), and that of B from (3) into equation (1), we get

$$y = \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)x + \frac{1}{2}\frac{d^2y}{dx^2}x^2 \text{ or } x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

Example 3: Find the differential equation of the family of curves $y = Ae^{3x} + Be^{5x}$

Solution: From the given, $\dfrac{dy}{dx} = 3Ae^{3x} + 5Be^{5x}$ $\qquad \dots(1)$

and $\qquad \dfrac{d^2y}{dx^2} = 9Ae^{3x} + 25Be^{5x}$ $\qquad \dots(2)$

On elimination of Ae^{3x} and Be^{5x}, we get

$$\begin{vmatrix} y & 1 & 1 \\ \dfrac{dy}{dx} & 3 & 5 \\ \dfrac{d^2y}{dx^2} & 9 & 25 \end{vmatrix} = 0 \text{ i.e. } \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 5y = 0$$

Example 4: Find the differential equation of all circles $(x-a)^2 + (y-b)^2 = r^2$ where a, b and r are all arbitrary constants.

Solution: The given equation of circles

$$(x-a)^2 + (y-b)^2 = r^2 \qquad \dots(1)$$

Implying $(x-a) + (y-b)\dfrac{dy}{dx} = 0$ $\qquad \dots(2)$

$$1 + (y-b)\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} = 0 \text{ or } 1 + \left(\frac{dy}{dx}\right)^2 + (y-b)\frac{d^2y}{dx^2} = 0$$

$$(y-b) = -\frac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}} \qquad \dots(3)$$

Putting the value of $(y - b)$ into (2), we get

$$(x-a) = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}{\dfrac{d^2y}{dx^2}}\frac{dy}{dx} \qquad \dots(4)$$

Substituting the values of $(x - a)$ and $(y - b)$ in the given equation of circles, results in

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}\left(\frac{dy}{dx}\right)^2+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}=r^2$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=r^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=r^2\left(\frac{d^2y}{dx^2}\right)^2 \text{ is the desired differential equation.}$$

Example 5: Find the differential equation of all conics whose axes coincides with the axes of coordinates.

Solution: Let the general equation of the conic, whose axes are the axis of the ordinate, be $Ax^2 + By^2 = 1$.

On differentiation with respect to x, we get

$$2Ax+2By\frac{dy}{dx}=0 \quad \text{or} \quad \frac{A}{B}=-\frac{y}{x}\frac{dy}{dx} \qquad \ldots(1)$$

Again differentiating it with respect to x, we get

$$A+B\left(\frac{dy}{dx}\right)^2+By\frac{d^2y}{dx^2}=0 \quad \text{implying} \quad \frac{A}{B}+\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=0 \qquad \ldots(2)$$

On substituting value of $\frac{A}{B}$ in above eqn. we get

$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0 \qquad \ldots(3)$$

which is the desired differential equation.

***Note*:** Equation for some standard curves which can be proceeded as above for formation of differential equation.

(*i*) Equation of all lines in the half-plane is $y = mx + c$.

(*ii*) Family of concentric circles is $x^2 + y^2 = a^2$.

(*iii*) Equation of the system of circles touching the Y-axis at the origin is the equation of circles $x^2 + y^2 + 2gx = 0$. touching a given straight line(viz. Y-axis) at a given point (taking centre as the point).

(*iv*) Equation of all circles in the xy-plane is $x^2 + y^2 + 2gx + 2fy + c = 0$; g, f, c are arbitrary constants.

Example 6: Form the differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axis are parallel to the X-axis.

Solution: Let the general equation of the parabola be

$$(y+k)^2 = 4a(x+h) \qquad \ldots(1)$$

Differentiating with respect to x, $(y+k)\dfrac{dy}{dx} = 2a \qquad \ldots(2)$

Further, $(y+k)\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$

$$\frac{2a}{\left(\dfrac{dy}{dx}\right)} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0, \text{ using (2)} \qquad \ldots(3)$$

implies $\quad 2a.\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 = 0$

which is the desired differential equation.

ASSIGNMENT 1

1. Show that

   (*i*) $4y + x^2 = 0$ is a solution of $\left(\dfrac{dy}{dx}\right)^2 + x\dfrac{dy}{dx} - y = 0$

   (*ii*) $y = cx + \dfrac{a}{c}$ is the solution of the equation $y = x + \left(\dfrac{dy}{dx}\right) + \dfrac{a}{\left(\dfrac{dy}{dx}\right)}$.

   Is it the general solution?

   (*iii*) $y = \dfrac{a}{x} + b$ is a solution of $\dfrac{d^2y}{dx^2} + \dfrac{2}{y}\dfrac{dy}{dx} = 0$

2. Eliminate the arbitrary constants involved from the following equations:

   (*i*) $y = Ae^{2x} + Be^{-2x}$ (*ii*) $y = A\cos 3x + B\sin 2x$

3. Form the differential equations from the following

   (*i*) $x = a\sin(wt + b)$, (*ii*) $y = ae^{2x} + be^{-3x} + ce^{x}$.

4. Eliminate

   (*i*) m from $\sqrt{1-x^2} + \sqrt{1-y^2} = m(x-y)$

   (*ii*) c from $y = cx + c - c^3$ (*iii*) k from $ay^2 = (x-k)^3$

5. Form the differential equation of

   (*i*) All the straight line in a plane.

   (*ii*) A family of circles of radius 5 unit with their centre on the Y-axis.

(*iii*) That must be satisfied by the family of concentric circles $x^2 + y^2 = a^2$
(*iv*) All the ellipse centered at the origin.

6. Form the differential equations from all the parabolas with
(*i*) X-axis as the axis and $(a, 0)$ as focus.
(*ii*) Y-axis as the axis and $(0, a)$ as the focus.

$$\begin{pmatrix}\text{Equation of parabola with } X\text{-axis as axis, } y^2 = 4ax \\ \text{Equation of parabola with } Y\text{-axis as axis, } x^2 = 4ay\end{pmatrix}$$

7.5 STANDARD METHODS FOR SOLUTION OF DIFFERENTIAL EQUATIONS OF 1ST ORDER 1ST DEGREE

The simplest type of differential equation is the equation of first order first degree and is generally expressible as

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad M(x, y)\,dx + N(x, y)\,dy = 0$$

As it is not always possible to solve all the differential equations of 1st order, 1st degree in the closed form, only those which belong to the following types can be solved by standard methods:

1. Variable Separable or Reducible to Variable-Separable
2. Homogeneous Equations or Reducible to Homogeneous.
3. Linear Equations or Reducible to Linear Equations.
4. Exact Differential Equation or Reducible to Exact through Integrating Factor.
5. Method of Substitution for reducing to one of the above form.

1. (*a*) Variable Separable Form: If in any differential equation, it is possible to express all functions of x with dx on one side and all the functions of y with dy on the other side, then the variables are said to be separable.

Thus, $$f(x)\,dx + \phi(y)\,dy = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{f(x)}{\phi(y)} \qquad \ldots(1)$$

is a general expression of the above type and integration on both sides, results in

$$\int \phi(y)\,dy = -\int f(x)\,dx + c \qquad \ldots(2)$$

as the general solution.

Observations: The equation $\frac{dy}{dx} = -\frac{f(x)}{\phi(y)}$ has meaning only in parts of xy-plane in which $\phi(y) \neq 0$, as connected parts are considered. Therefore, in such cases, it is assumed either $\phi(y) > 0$ throughout or $\phi(y) < 0$ throughout. Equation (2) gives implicitly the desired solution of (1) as $\phi(y) > 0$ or $\phi(y) < 0$ throughout, means $\int \phi(y)\,dy$ is strictly monotone i.e. increasing or decreasing, respectively. Differentiation of (2) shows that any differentiable function $y(x)$ satisfying (2) for some value of c is a solution of (1).

(*b*) Reducible to Variable-separable: Sometimes, differential equation is not variable-separable type in its given form but after certain substitution, it reduces to the variable-separable form.

Example 7: Solve the following equations:

(*i*) $y\sqrt{1-x^2}\,dy + x\sqrt{1-y^2}\,dx = 0$ (*ii*) $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

(*iii*) $\frac{dy}{dx} = xe^{y-x^2}$, if $y = 0$ when $x = 0$ (*iv*) $x\frac{dy}{dx} + \cot y = 0$: if $y = \frac{\pi}{4}$ when $x = \sqrt{2}$

(*v*) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$ (*vi*) $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

Solution:

(*i*) $y\sqrt{1-x^2}\,dy + x\sqrt{1-y^2}\,dx = 0$

$$\Rightarrow \quad \frac{y}{\sqrt{1-y^2}}\,dy + \frac{x}{\sqrt{1-x^2}}\,dx = 0$$

or $\quad d(1-y^2)^{\frac{1}{2}} + d(1-x^2)^{\frac{1}{2}} = 0$

On integrating both sides $\sqrt{1-y^2} + \sqrt{1-x^2} = C$

(*ii*) $3\,e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

$$\Rightarrow \quad 3\left(\frac{e^x}{1-e^x}\right)dx + \frac{\sec^2 y}{\tan y}\,dy = 0 \quad \text{i.e.} \quad -3\left(\frac{f'(x)}{f(x)}\right)dx + \frac{f'(y)}{f(y)}\,dy = 0$$

$$\Rightarrow \quad -3\log(1-e^x) + \log\tan y = \log C$$

$$\Rightarrow \quad \log\tan y = \log C + 3\log(1-e^x) \qquad \Rightarrow \tan y = C(1-e^x)^3$$

(*iii*) $\frac{dy}{dx} = xe^{y-x^2}$ if $y = 0$ when $x = 0$ ...(1)

i.e. $\quad \frac{dy}{dx} = xe^y e^{-x^2}$ or $\frac{dy}{e^y} = xe^{-x^2}dx$

$$d\left(e^{-y}\right) = \frac{1}{2}d\left(e^{-x^2}\right) \quad \text{i.e.} \quad 2e^{-y} = e^{-x^2} + C \qquad \ldots(2)$$

when $\quad x = 0,\ y = 0;$ we get $2e^0 = e^0 + C$ i.e. $C = 1$

$\therefore \quad 2e^{-y} = e^{-x^2} + 1$

(*iv*) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$ i.e. $\left(y - ay^2\right) = (x+a)\frac{dy}{dx}$

i.e. $\quad \frac{dx}{(x+a)} = \frac{dy}{\left(y-ay^2\right)}$ or $\frac{dx}{x+a} = \left(\frac{1}{y} + \frac{a}{1-ay}\right)dy$, (by partial fractions)

On integration, $\log(x+a) = \log y - \log(1-ay) + \log c$ i.e. $(x+a)(1-ay) = cy$

(*v*) $$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$$

$$(\sin y + y\cos y)dy = x(2\log x + 1)dx \;\Rightarrow\; d(y\sin y) = d(x^2 \log x)$$

$$\therefore \quad y\sin y = x^2 \log x + c$$

Example 8: **Solve the equation**

(*i*) $(x + y + 1)^2 \dfrac{dy}{dx} = 1$ (Jammu Univ. 2002) (*ii*) $\dfrac{dy}{dx} - x\tan(y - x) = 1$

(*iii*) $x^4 \dfrac{dy}{dx} + x^3 y + \operatorname{cosec}(xy) = 0$

Solution:

(*i*) The equation in the given form is not a case of variable-separable but by the substitution given below; this reduces to variable-separable.

Let $(x + y + 1) = z$ so that $1 + \dfrac{dy}{dx} = \dfrac{dz}{dx}$.

With the above substitution, the given equation reduces to

$$z^2\left(\frac{dz}{dx} - 1\right) = 1 \text{ i.e. } \frac{dz}{dx} = \left(\frac{z^2 + 1}{z^2}\right)$$

Now rewrite it as $$\left(\frac{z^2}{z^2 + 1}\right)dz = dx$$

On integration, $$\int\left[\frac{(z^2 + 1) - 1}{(z^2 + 1)}\right]dz = \int dx + c \quad \text{i.e.} \quad z - \tan^{-1} z = x + c$$

$$(x + y + 1) - \tan^{-1}(x + y + 1) = (x + c)$$

(*ii*) Let $y - x = t$ so that $\left(\dfrac{dy}{dx} - 1\right) = \dfrac{dt}{dx}$

with this, the given equation reduces to

$$\frac{dt}{dx} = x\tan t \quad \text{or} \quad \frac{dt}{\tan t} = xdx$$

Integrating both sides, $\log(\sin t) = \dfrac{x^2}{2} + c.$

(*iii*) Let $xy = t$ so that $x\dfrac{dy}{dx} + y = \dfrac{dt}{dx}$

i.e. $$x\frac{dy}{dx} = \left(\frac{dt}{dx} - \frac{t}{x}\right)$$

On substituting the value of xy and $\frac{dy}{dx}$, the given equation reduces to

$$x^3\left[\frac{dt}{dx}-\frac{t}{x}\right]+x^2.t+\operatorname{cosec} t=0$$

or $\quad x^3\frac{dt}{dx}+\operatorname{cosec} t=0 \quad$ implying $\quad \frac{dt}{\operatorname{cosec} t}=-\frac{dx}{x^3}$

Integrating both sides,

$$\int \sin t\, dt=-\int x^{-3}dx+c \quad \text{or} \quad \cos t+\frac{1}{2x^2}=c \quad \text{where } t=xy$$

Note: This problem may be discussed under method of inspection.

ASSIGNMENT 2

1. Solve the following differential equations

(*i*) $\frac{dy}{dx}=e^{2x-3y}+4x^2e^{-3y}$ (*ii*) $(x+1)\frac{dy}{dx}+1=2e^{-y}$

(*iii*) $x\frac{dy}{dx}+\cot y=0$ if $y=\frac{\pi}{4}$ when $x=\sqrt{2}$. (*iv*) $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$

(*v*) $\sin^{-1}\left(\frac{dy}{dx}\right)=x+y.$ (*vi*) $(x-y)^2\frac{dy}{dx}=a^2$

(*vii*) $\frac{dy}{dx}=\cos(x+y+1)^2$ [NIT Jalandhar, 2005]

2. Find the equation of the curve passing through the pt. (1,1) whose differential equation is $(y-yx)\,dx+(x+xy)\,dy=0$.

3. Find the equation of the curve whose slope at any pt. is equal to $\frac{x^2y}{(1+x^3)}$ and which pass through the point (1, 2).

2 (*a*) Homogeneous Equations: Any 1st order 1st degree equation which can be written in the form $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$ or $g\left(\frac{x}{y}\right)$ or more precisely $\frac{dy}{dx}=\frac{f(x,\ y)}{g(x,\ y)}$ is known as homogeneous equation.

To solve a homogeneous equation

(*i*) Put $y=ux$, then $\frac{dy}{dx}=u+x\frac{dy}{dx}$,

(*ii*) in the transformed equation, separate u and x and integrate.

Example 9: Solve

(*i*) $\left(x\tan\left(\frac{y}{x}\right) - y\sec^2\left(\frac{y}{x}\right)\right)dx + x\sec^2\left(\frac{y}{x}\right)dy = 0$

(*ii*) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$, given $y = 1$ at $x = 1$. (*iii*) $y\,dx - x\,dy = \sqrt{x^2 + y^2}\,dx$

(*iv*) $\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx} = 0$

Solution:

(*i*) Rewrite the given equation

$$\frac{dy}{dx} = \frac{y\sec^2\frac{y}{x} - x\tan\frac{y}{x}}{x\sec^2\frac{y}{x}} \qquad \ldots(1)$$

Let $\frac{y}{x} = v$ so that $\frac{dy}{dx} = \left(v + x\frac{dv}{dx}\right)$ ...(2)

The given equation (1) reduces to $\left(v + x\frac{dv}{dx}\right) = \frac{vx\sec^2 v - x\tan v}{x\sec^2 v}$

or $x\frac{dv}{dx} = -\frac{\tan v}{\sec^2 v}$ i.e. $\frac{\sec^2 v\,dv}{\tan v} = -\frac{dx}{x}$ ...(3)

Integrating both sides, $\log\tan v = -\log x + \log c$

or $\tan v = \frac{c}{x}$, where $v = \frac{y}{x}$. ...(4)

(*ii*) Rewrite the given equation as $\frac{dy}{dx} = \frac{xy - y^2}{x^2}$ ...(5)

Taking $y = vx$, we get $\left(v + x\frac{dv}{dx}\right) = \frac{xvx - v^2x^2}{x^2}$

or $x\frac{dv}{dx} = -v^2$ implying $\frac{dv}{-v^2} = \frac{dx}{x}$ ...(6)

Integrating both sides, $\frac{1}{v} = \log x + \log c$ i.e. $\frac{x}{y} = \log cx$

Now $y = 1$ at $x = 1$ implying $1 = \log c$ or $c = e$.

$\therefore$ The particular integral is $x = y\log ex$.

(*iii*) Rewrite the given equation as $\frac{dy}{dx} = \frac{y - \sqrt{x^2 + y^2}}{x}$

Put $y = vx$ so that $\left(v + x\dfrac{dv}{dx}\right) = \dfrac{vx - \sqrt{x^2 + v^2x^2}}{x}$ or $x\dfrac{dv}{dx} = -\sqrt{1+v^2}$

Now by variable separable, $\dfrac{dv}{\sqrt{1+v^2}} = -\dfrac{dx}{x}$

On integration, $\sinh v = \log\dfrac{c}{x}$ or $\log\left[v + \sqrt{v^2+1}\right] = \log\dfrac{c}{x}$

$y + \sqrt{x^2+y^2} = c$

(*iv*) Put $y = vx$ so that $\dfrac{dy}{dx} = \left(v + x\dfrac{dv}{dx}\right) = \dfrac{(x\cos v + vx\sin v)vx}{(vx\sin v - x\cos v)x}$

i.e. $x\dfrac{dv}{dx} = \dfrac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v = \dfrac{2v\cos v}{v\sin v - \cos v}$ (on simplification)

implying $\dfrac{v\sin v - \cos v}{v\cos v}dv = 2\dfrac{dx}{x}$ (case of variable-separable)

Integrating both sides, $\displaystyle\int\left(\tan v - \frac{1}{v}\right)dv = 2\int\frac{1}{x}dx + \log c$

i.e. $-\log\cos v - \log v = 2\log x + \log c$ or $-\log(v\cos v) = \log cx^2$

Implying $\dfrac{1}{v\cos v} = cx^2$ where $v = \dfrac{y}{x}$.

2 (*b*) Equations Reducible to Homogeneous Form:

$$\frac{dy}{dx} = \frac{ax + by + c}{a^1x + b^1y + c} \quad \ldots(1)$$

Case I: When $\dfrac{a}{a^1} \neq \dfrac{b}{b^1}$

Put $x = X + h,\ y = Y + k;$ (h and k are constants)

so that $dx = dX$ and $dy = dY$

On using above suppositions, (1) becomes

$$\frac{dY}{dX} = \frac{(aX + bY) + (ah + bk + c)}{(a^1X + b^1Y) + (a^1h + b^1k + c^1)} \quad \ldots(2)$$

Choose h and k in (2) such that $\left.\begin{aligned} ah + bk + c &= 0 \\ a^1h + b^1k + c^1 &= 0 \end{aligned}\right\}$ …(3)

And on solving, $\dfrac{h}{bc^1 - b^1c} = \dfrac{k}{ca^1 - c^1a} = \dfrac{1}{ab^1 - ba^1}$

or more precisely,

$$h = \frac{bc^1 - b^1c}{ab^1 - ba^1},\quad k = \frac{ca^1 - c^1a}{ab^1 - ba^1};\ \left(\text{provided } ab^1 - ba^1 \neq 0\right) \qquad \ldots(4)$$

By condition (3), equation in X and Y can be solved by putting $Y = VX$ (as usual).

Case II: When $\frac{a}{a^1} = \frac{b}{b^1}$ i.e. $ab^1 - ba^1 = 0$, the above method fails as h and k becomes infinite or indeterminate.

Let $$\frac{a}{a^1} = \frac{b}{b^1} = \frac{1}{m}(\text{say}) \qquad \therefore\ a^1 = am \ \text{ and } \ b^1 = bm \qquad \ldots(5)$$

On putting these values in (1), we get

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c^1} \qquad \ldots(6)$$

Put $$ax + by = t \text{ so that } \left(a + b\frac{dy}{dx}\right) = \frac{dt}{dx} \text{ or } \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right) \qquad \ldots(7)$$

With this, equation (6) reduces to

$$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t + c}{mt + c^1} \text{ or } \frac{dt}{dx} = a + \frac{bt + bc}{mt + c^1} = \frac{(am + b)t + (ac^1 + bc)}{mt + c^1}$$

or $$\frac{(mt + c^1)}{(am + b)t + (ac^1 + bc)}dt = dx$$ (which is a case of variable-separable, where $t = ax + by$)

Example 10: Find the solution of the differential equation

$$(y - x + 1)\,dy - (y + x + 2)\,dx = 0$$ **[Jammu Univ. 2002]**

Solution: Rewrite the given equation as $$\frac{dy}{dx} = \frac{(y + x + 2)}{(y - x + 1)} \qquad \ldots(1)$$

Clearly this equation is comparable to **case** I of homogeneous equation

$$\frac{dy}{dx} = \frac{ax + by + c}{a^1x + b^1y + c^1},\quad \frac{a}{a^1} \neq \frac{b}{b^1}\quad \left(\text{as } a = 1,\ b = 1,\ a^1 = -1,\ b^1 = 1\right)$$

Taking $x = X + h$, $y = Y + k$ (h and k being constants) so that $dx = dX$, $dy = dY$.
With this, equation (1) becomes

$$\frac{dY}{dX} = \frac{(Y + X) + (k + h + 2)}{(Y - X) + (k - h + 1)} \qquad \ldots(2)$$

Choose h, k so that $$\left.\begin{aligned} k + h + 2 &= 0 \\ k - h + 1 &= 0 \end{aligned}\right\} \text{ implying } \left.\begin{aligned} h &= -\frac{1}{2} \\ k &= -\frac{3}{2} \end{aligned}\right\}$$

With above choice of h and k, equation (2) reduces to $\frac{dy}{dx} = \frac{Y+X}{Y-X}$ which is a homogeneous form

Take $\qquad Y = VX \quad$ so that $\frac{dY}{dX} = \left(V + X\frac{dV}{dX}\right)$ $\qquad \ldots(3)$

With above substitution, (2) becomes

$$\left(V + X\frac{dV}{dX}\right) = \left(\frac{V+1}{V-1}\right) \quad \text{or} \quad X\frac{dV}{dX} = \frac{1+2V-V^2}{V-1}$$

which is of variable-separable form i.e.

$$-\left(\frac{V-1}{V^2-2V-1}\right)dV = \frac{dX}{X}. \qquad \ldots(4)$$

or $\qquad -\frac{1}{2}\log\left(V^2-2V-1\right) = \log CX$ i.e. $\frac{1}{\sqrt{V^2-2V-1}} = CX$

which implies $C^2X^2\left(V^2-2V-1\right) = 1$ or $C^2\left(Y^2-2YX-X^2\right) = 1.$ $\qquad \ldots(5)$

Now replace $\quad X = x - h$ and $Y = y - k \quad$ i.e. $\quad X = \left(x + \frac{1}{2}\right)$ and $\quad Y = \left(y + \frac{3}{2}\right)$

We get $\qquad C^2\left[\left(y+\frac{3}{2}\right)^2 - 2\left(x+\frac{1}{2}\right)\left(y+\frac{3}{2}\right) - \left(x+\frac{1}{2}\right)^2\right] = 1$ as the desired solution.

Example 11: Solve the differential equation $(2x + y - 3)\,dy = (x + 2y - 3)\,dx$.

[NIT Kurukshetra, 2007; Chennai, 2000; VTU, 2000]

Solution: Rewrite the given equation as $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ $\qquad \ldots(1)$

and compare with $\frac{dy}{dx} = \frac{ax+by+c}{a^1x+b^1y+c^1}$; $\left(\frac{a}{a^1} \neq \frac{b}{b^1}\right)$, case 1 of non-homogeneous equation,

(as here $a = 1,\ b = 2 : a^1 = 2,\ b^1 = 1$)

Put $\qquad x = X + h,\ y = Y + k$ so that $dx = dX$ and $dy = dY$

The given equation becomes,

$$\frac{dY}{dX} = \frac{(X+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3} = \frac{(X+2Y)+(h+2k-3)}{(2X+Y)+(2h+k-3)} \qquad \ldots(2)$$

Here h and k are so chosen that $\left.\begin{matrix} h+2k-3=0 \\ 2h+k-3=0 \end{matrix}\right\}$ implying $\left.\begin{matrix} h=1 \\ k=1 \end{matrix}\right\}$ $\qquad \ldots(3)$

With above values of h and k, equation (2) reduces to $\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$ which is a homogeneous equation.

Put $\quad Y = uX \quad$ so that $\dfrac{dY}{dX} = \left(u + X\dfrac{du}{dX}\right)$ ...(4)

Thus, equation (4) becomes

$$\left(u + X\frac{du}{dX}\right) = \frac{1+2u}{2+u} \quad \text{or} \quad X\frac{du}{dX} = \left(\frac{1+2u}{2+u} - u\right)$$

or $\quad \left(\dfrac{2+u}{1-u^2}\right)du = \dfrac{dX}{X}$ case of variable-seperable

Integrating both sides,

$$\int\left(\frac{3}{2}.\frac{1}{1-u} + \frac{1}{2}\frac{1}{1+u}\right)du = \log X + C \quad \text{(by partial fractions)}$$

$$\frac{-3}{2}\log(1-u) + \frac{1}{2}\log(1+u) = \log X + C$$

$\Rightarrow \quad 2\log X + 3\log(1-u) - \log(1+u) = -2C$

$\Rightarrow \quad \log\dfrac{X^2(1-u)^3}{(1+u)} = \log k, \quad (-2C = \log k)$

or $\quad \dfrac{X^2\left(1-\dfrac{Y}{X}\right)^3}{\left(1+\dfrac{Y}{X}\right)} = k$ implying $(X-Y)^3 = k(X+Y)$

or $\quad (x-y)^3 = k(x+y-2)$ as $X = x-h = x-1, \quad Y = y-k = y-1$.

Example 12: Solve the differential equation $(x+2y)(dx-dy) = (dx+dy)$.

Solution: The given is rewritten as $(x+2y+1)\,dy = (x+2y-1)\,dx$ ...(1)

On comparing this equation with $\dfrac{dy}{dx} = \dfrac{ax+by+c}{a^1x+b^1y+c^1}$, we see that it is the **caseII** of equations reducible to homogeneous, where $\dfrac{a}{a^1} = \dfrac{b}{b^1}$.

Take $(x+2y) = t$ so that $1 + 2\dfrac{dy}{dx} = \dfrac{dt}{dx}$ or $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{dt}{dx} - 1\right)$

Thus, the given differential becomes

$$\frac{1}{2}\left(\frac{dt}{dx} - 1\right) = \frac{t-1}{t+1} \quad \text{or} \quad \frac{dt}{dx} = \frac{2(t-1)}{t+1} + 1 = \frac{3t-1}{t+1}$$

i.e. $\left(\dfrac{t+1}{3t-1}\right)dt = dx$ (a case of variable-separable) ...(2)

$$\frac{1}{3}\left\{\frac{\left(t-\frac{1}{3}\right)+\frac{1}{3}+1}{\left(t-\frac{1}{3}\right)}\right\}dt = dx$$

On integration, $\dfrac{1}{3}\left\{t+\dfrac{4}{3}\log\left(t-\dfrac{1}{3}\right)\right\} = x+c$

$$(x+2y)+\frac{4}{3}\log\left(x+2y-\frac{1}{3}\right) = 3x+C,$$ Replacing t by $(x + 2y)$,

i.e. $\log\left(x+2y-\dfrac{1}{3}\right)+\dfrac{3}{2}(y-x) = C.$

ASSIGNMENT 3

1. (*i*) $y e^{\frac{x}{y}}\, dx = \left(x e^{\frac{x}{y}} + y^2\right) dy$ (*ii*) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\dfrac{y}{x}$

(*iii*) $xy\log\left(\dfrac{x}{y}\right)dx + \left(y^2 - x^2\log\left(\dfrac{x}{y}\right)\right)dy = 0$ (*iv*) $x^3dx - y^3\,dx = 3xy(ydx - xdy)$

2. (*i*) $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y+3}$ (*ii*) $(2x + 5y + 1)dx - (5x + 2y - 1)dy = 0$

(*iii*) $(4x - 6y - 1)dx + (3y - 2x - 2)dy = 0$ (*iv*) $(x + 2y)\ (dx - dy) = (dx + dy)$

3 (*a*) Linear Differential Equation: A differential equation is said to be linear if the dependent variable and the differential coefficients occur only in first degree and not multiplied together.

The most general form of a linear differential equation of the 1st order is

$$\frac{dy}{dx} + Py = Q \qquad ...(1)$$

where P and Q are functions of x.

This standard form of the equation is known as Leibnitz's Linear Differential Equation. In general, this type of equation will not be exact. However, we can find integrating factor (I.F.), $e^{\int P dx}$ such that on multiplying equation (1) throughout by it, we get.

$$\frac{dy}{dx}\cdot e^{\int Pdx} + Py\cdot e^{\int Pdx} = Q\cdot e^{\int Pdx} \qquad ...(2)$$

We see that both the terms on left hand side, collectively may be as the differential of a single function i.e. $\dfrac{d}{dx}\left(y\cdot e^{\int Pdx}\right)$. Thus, precisely the equation (2) becomes

$$d\left(y\cdot e^{\int Pdx}\right) = Q\cdot e^{\int Pdx} \qquad ...(3)$$

and on integration, we get the general solution $y \cdot e^{\int P dx} = \int \left(Q \cdot e^{\int P dx} \right) dx + c$ ...(4)
where c is the arbitrary constant, called constant of integration.

Note: If the differential equation is $\dfrac{dx(y)}{dy} + Px(y) = Q$, where P and Q are the functions of y, then the integrating factor becomes $e^{\int P dy}$ and the solution will be $x \cdot e^{\int P dy} = \int \left(Q \cdot e^{\int P dy} \right) dy + c.$

Example 13: Solve $\cosh x \dfrac{dy}{dx} + y \sinh x = 2\cosh^2 x \sinh x$

Solution: Rewrite the given equation as: $\dfrac{dy}{dx} + \left(\dfrac{\sinh x}{\cosh x} \right) y = 2\cosh x \sinh x$

This equation is comparable to Leibnitz's Linear equation

$$\frac{dy}{dx} + P(x) y = Q(x) \text{ with } P(x) = \frac{\sinh x}{\cosh x} \text{ and } Q = 2\cosh x \sinh x.$$

$$\therefore \qquad \text{I.F.} = e^{\int P dx} = e^{\int \frac{\sinh x}{\cosh x} dx} = e^{\log \cosh x} = \cosh x \quad \left(\text{using} \int \frac{f'(x)}{f(x)} dx = \log f(x) \right)$$

Thus, solution is

$$y \cdot \cosh x = \int 2\cosh x \sinh x (\cosh x) dx + c$$

$$= \frac{2}{3} \int d\left(\cosh^3 x \right) dx + c = \frac{2}{3} \cosh^3 x + c.$$

Example 14: Solve

(*i*) $\dfrac{dy}{dx} = -\dfrac{x + y\cos x}{1 + \sin x}$ **(*ii*)** $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

(*iii*) $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}} \right) \dfrac{dx}{dy} = 1.$

Solution: (*i*) $\dfrac{dy}{dx} = -\dfrac{x + y\cos x}{1 + \sin x}$

implies $\dfrac{dy}{dx} + \dfrac{\cos x}{1 + \sin x} y = -\dfrac{x}{1 + \sin x}$ ...(1)

comparable with $\dfrac{dy}{dx} + Py = Q$ where $P = \dfrac{\cos x}{1 + \sin x}$

$$\therefore \qquad I.F. = e^{\int \frac{\cos x}{1 + \sin x} dx} = e^{\int \frac{f'(x)}{f(x)} dx} = e^{\log(1 + \sin x)} = (1 + \sin x) \qquad \text{...(2)}$$

Whence the general solution is

$$y(1+\sin x) = -\int\left(\frac{x}{1+\sin x}\right)(1+\sin x)\,dx + c$$

i.e. $y(1+\sin x) = -\dfrac{x^2}{2} + c$ ...(3)

(*ii*) Rewrite the given equation as $\dfrac{dy}{dx} + (\sec^2 x)\,y = \tan x \sec^2 x$

Here $P = \sec^2 x$, $Q = \tan x \sec^2 x$

$\therefore$ $I.F. = e^{\int \sec^2 x dx} = e^{\tan x}$

Whence the general solution, $y \cdot e^{\tan x} = \int \tan x \cdot \sec^2 x \cdot e^{\tan x}\,dx + c$

Let $\tan x = z$ so that $\sec^2 x\,dx = dz$

$$\therefore \quad ye^{\tan x} = \int z.e^z dz + c = \left[ze^z - \int e^z dz\right] + c = \left(ze^z - e^z\right) + c$$

$$= e^{\tan x}(\tan x - 1) + c$$

(*iii*) Rewrite the given equation as

$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \quad \text{or} \quad \frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

which is comparable with $\dfrac{dy}{dx} + Py = Q$, where $P = \dfrac{1}{\sqrt{x}}$, $Q = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$\therefore$ $I.F. = e^{\int \frac{1}{\sqrt{x}} dx} = e^{\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)} = e^{2\sqrt{x}}$ and thus the solution become

$$y.e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} .e^{2\sqrt{x}} dx + c$$

or $y \cdot e^{2\sqrt{x}} = \int x^{-1/2} dx + c$ or $y \cdot e^{2\sqrt{x}} = 2x^{1/2} + c$

Example 15: Solve

(*i*) $x\dfrac{dy}{dx} + \cos^2 y = \tan y \dfrac{dy}{dx}$ **(*ii*)** $\left(1+y^2\right)dx = \left(\tan^{-1}y - x\right)dy$.

(*iii*) $\left(1+y^2\right)dx + \left(x - e^{-\tan^{-1}y}\right)dy = 0$ **[KUK, 2002]**

Solution:

(*i*) Rewrite the given equation as

$$(\tan y - x)\frac{dy}{dx} = \cos^2 y \quad \text{or} \quad \frac{\tan y}{\cos^2 y} - \frac{1}{\cos^2 y}x = \frac{dx}{dy}$$

or $\dfrac{dx}{dy} + \dfrac{1}{\cos^2 y}x = \tan y \sec^2 y$ ...(1)

which is comparable with $\dfrac{dx}{dy}+Px=Q,$ where $P=\sec^2 y$ and $Q=\tan y\sec^2 y.$

Now $I.F.=e^{\int\sec^2 y\,dy}=e^{\tan y}$ and therefore the general solution is

$$x\cdot e^{\tan y}=\int\tan y\sec^2 y\cdot e^{\tan y}dy+c \qquad \ldots(2)$$

where c is an arbitrary constant.

Put $\tan y=z$ so that $\sec^2 y\,dy=dz$

$$\therefore \quad x\cdot e^{\tan y}=\int ze^z dz=e^z(z-1)+c=e^{\tan y}(\tan y-1)+c$$

$$x=\tan y-1+ce^{-\tan y}$$

(*ii*) Rewrite the given equation as $\dfrac{dx}{dy}+\dfrac{1}{1+y^2}x=\dfrac{\tan^{-1}y}{1+y^2}$

which is comparable with previous example, $P=\dfrac{1}{1+y^2},\quad Q=\dfrac{\tan^{-1}y}{1+y^2}$

$$\therefore \quad I.F.=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

and the general solution becomes $x\cdot e^{\tan^{-1}y}=\displaystyle\int\frac{\tan^{-1}y}{1+y^2}\cdot e^{\tan^{-1}y}dy+c$

Take $\tan^{-1}y=t$ so that $y=\tan t$ and $1+y^2=\sec^2 t,\ dy=\sec^2 t\,dt$

Thus $x\cdot e^{\tan^{-1}y}=\displaystyle\int\frac{t\cdot e^t}{\sec^2 t}\sec^2 t\,dt+c$ or $x\cdot e^{\tan^{-1}y}=\displaystyle\int te^t dt+c$

or $\quad x\cdot e^{\tan^{-1}y}=e^t(t-1)+c=e^{\tan^{-1}y}(\tan^{-1}y-1)+c$

$$x=(\tan^{-1}y-1)+ce^{-\tan^{-1}y}.$$

(*iii*) Rewrite the given equation as $\dfrac{dx}{dy}+\dfrac{1}{1+y^2}x=\dfrac{e^{-\tan^{-1}y}}{1+y^2}$

where $\quad P=\dfrac{1}{1+y^2}$ and $Q=\dfrac{e^{-\tan^{-1}y}}{1+y^2}$

$$\therefore \quad x\cdot e^{\tan^{-1}y}=\int\frac{e^{-\tan^{-1}y}}{1+y^2}\cdot e^{\tan^{-1}y}dy+c$$

$$xe^{\tan^{-1}y}=\int\frac{1}{1+y^2}dy+c \text{ i.e. } xe^{\tan^{-1}y}=\tan^{-1}y+c.$$

Example 16: Solve the equations [NIT Kurukshetra, 2006, 2010]

(*i*) $r\sin\theta\,d\theta+(r^3-2r^2\cos\theta+\cos\theta)\,dr=0$ (*ii*) $\dfrac{dy}{dx}-\dfrac{\tan y}{(1+x)}=(1+x)e^x\sec y.$

Solution:

(*i*) The given equation $r\sin\theta\, d\theta + \left(r^3 - 2r^2\cos\theta + \cos\theta\right)dr = 0$ ...(1)

may be written as $\sin\theta \dfrac{d\theta}{dr} + \left(\dfrac{1}{r} - 2r\right)\cos\theta = -r^2$ ...(2)

Put $\cos\theta = t$ so that $-\sin\theta \dfrac{d\theta}{dr} = \dfrac{dt}{dr}$ and hence equation (2) reduces to

$$\frac{dt}{dr} + \left(2r - \frac{1}{r}\right)t = r^2 \qquad \text{...(3)}$$

which is Leibnitz's Linear Equation with $P = \left(2r - \dfrac{1}{r}\right)$ and $Q = r^2$

$\therefore$ Integrating Factor $= e^{\int\left(2r - \frac{1}{r}\right)dr} = e^{\left(r^2 - \log r\right)} = \dfrac{e^{r^2}}{r}$

Whence $t \cdot \dfrac{e^{r^2}}{r} = \displaystyle\int r^2 \cdot \left(\dfrac{e^{r^2}}{r}\right)dr + c$ or $\cos\theta \cdot \dfrac{e^{r^2}}{r} = \displaystyle\int \dfrac{1}{2} d\left(e^{r^2}\right)dr + c$

$\cos\theta \cdot \dfrac{e^{r^2}}{r} = \dfrac{e^{r^2}}{2} + c$ implying $2\cos\theta = r\left(1 + 2ce^{-r^2}\right)$

(*ii*) The given equation $\dfrac{dy}{dx} - \dfrac{\tan y}{(1+x)} = e^x(1+x)\sec y$ ...(1)

may be written as $\cos y \dfrac{dy}{dx} - \dfrac{\sin y}{(1+x)} = e^x(1+x)$ ...(2)

Take $\sin y = t$ so that $\cos y \dfrac{dy}{dx} = \dfrac{dt}{dx}$

With this, equation (4) becomes $\dfrac{dt}{dx} - \dfrac{1}{(1+x)}t = e^x(1+x)$

which is Leibnitz's Linear differential equation with $P = -\dfrac{1}{1+x}$, $Q = e^x(1+x)$

and integrating factor, $I.F. = e^{\int -\frac{1}{1+x}dx} = e^{-\log(1+x)} = \dfrac{1}{1+x}$.

Thus, $t \cdot \dfrac{1}{1+x} = \displaystyle\int e^x(1+x)\dfrac{1}{1+x}dx + c$

or $\sin y \dfrac{1}{1+x} = e^x + c$ implying $\sin y = \left(e^x + c\right)(1+x)$

Example 17: Solve (*i*) $\sec^2 y \dfrac{dy}{dx} + \tan y = x^3$ [NIT Jalandhar, 2005]

(*ii*) $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^2 x$ [KUK, 2002-03]

Solution: (*i*) The given equation is $\sec^2 y \dfrac{dy}{dx} + \tan y = x^3.$

Take $\tan y = t$ so that $\sec^2 y\, dy = dt$ and the equation reduces to $\dfrac{dt}{dx} + t = x^3$

which is comparable to $\dfrac{dt}{dx} + Pt = Q,$ where $P = 1,\quad Q = x^3$

Thus, $I.F. = e^{\int dx} = e^x$ and the general solution becomes

$$t \cdot e^x = \int x^3 e^x dx + c \qquad \text{(by parts, } x^3 \text{ as Ist part)}$$

or $\quad t e^x = \left[x^3 e^x - 3x^2 e^x + 6xe^x - 6\right] + c \quad$ or $\quad \tan y = (x^3 - 3x^2 + 6x) + Ce^{-x}$

(*ii*) Rewrite the given equation as

$$\frac{\tan y}{\cos y}\frac{dy}{dx} + \frac{\tan x}{\cos y} = \cos^2 x \quad \text{or} \quad \sec y \tan y \frac{dy}{dx} + \tan x \sec\, y = \cos^2 x$$

Put $\quad \sec y = t$ so that $\sec y \tan y \dfrac{dy}{dx} = \dfrac{dt}{dx}$

The given equation reduces to

$$\frac{dt}{dx} + (\tan x)\, t = \cos^2 x, \quad \text{where } P = \tan x \text{ and } Q = \cos^2 x$$

$\therefore \qquad IF = e^{\int \tan x\, dx} = e^{-\log \cos x} = \sec x$ and thus the general solution becomes

$$t \cdot \sec x = \int \cos^2 x \cdot \sec x\, dx + c \quad \text{or} \quad t \cdot \sec x = \int \cos x\, dx + c$$

implying $\quad \sec y \cdot \sec x = (\sin x + c),\;$ as $\; t = \sec y.$

3 (*b*) Equations Reducible to Leibnitz's Linear (*Bernoulli's Equation*): The nonlinear equation of first order, $\dfrac{dy}{dx} + P(x)\, y = Q(x)\, y^n,$ is usually known as Bernoulli's equation.

For non-trivial solution divide both sides by y^n, so that

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Now take $y^{1-n} = u$ in $\dfrac{dt}{dx} + (\tan x)\, t = \cos^2 x,$ implying $(1-n)y^{-n}\dfrac{dy}{dx} = \dfrac{du}{dx}$ and thus the above equation reduces to

$$\frac{1}{(1-n)}\frac{du}{dx} + Pu = Q \quad \text{or} \quad \frac{du}{dx} + P(1-n)u = Q(1-n)$$

which is Leibnitz's Linear equation, provided $n \neq 1$.

Bernoulli Equation is due to Jacob Bernoulli (1654-1704), Swiss Mathematician born in **Basel**. These equations usually occur in various applications of mathematics that involves some form of nonlinearity like problem of wave fronts, acceleration waves, shock waves etc.

Note: In Bernoulli's equation, n is any real number i.e. it is not necessarily an integer. Further if, $n = 0$ or 1, then the equation is linear. In case, $n = 0$, the equation is said to be linear homogenous with a property that $y = 0$ is a particular solution, the so called trivial solution. For $n = 1$, the equation becomes a case of variable-separable.

Example 18: Solve the following differential equations.

(*i*) $\dfrac{dy}{dx} + y\tan x = y^3 \sec x$ **(*ii*)** $r\sin\theta - \cos\theta\dfrac{dr}{d\theta} = r^2$.

Solution:

(*i*) On rewriting the given equation as $y^{-3}\dfrac{dy}{dx} + \tan x \cdot y^{-2} = \sec x$ which is Bernoulli's equation reducible to Leibnitz Linear.

Putting $y^{-2} = t$ so that $-2y^{-3}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

With above substitution, the given equation reduces to

$$\frac{dt}{dx} - 2\tan x \cdot t = -2\sec x; \quad \left.\begin{array}{l} P = -2\tan x \\ Q = -2\sec x \end{array}\right\}$$

$$\therefore \qquad I.F. = e^{\int P dx} = e^{\int -2\tan x\, dx} = e^{-2\log\sec x} = \frac{1}{(\sec x)^2} = \cos^2 x$$

Thus, the solution becomes. $t \cdot \cos^2 x = \int \left(e^{\int p dx}\right) Q dx + c$

$$t(\cos x)^2 = \int \cos^2 x \cdot -2\sec x\, dx + c$$

$$y^{-2}(\cos x)^2 = -2\int \cos x\, dx + c \quad \text{or} \quad \cos^2 x = (-2\sin x + c)y^2$$

(*ii*) On rewriting the given equation as $\cos\theta\dfrac{dr}{d\theta} = r\sin\theta - r^2$

$$\text{or} \quad r^{-2}\frac{dr}{d\theta} - r^{-1}\tan\theta = -\sec\theta \qquad \ldots(1)$$

$$\text{Putting} \quad r^{-1} = t \text{ so that } -r^{-2}\frac{dr}{d\theta} = \frac{dt}{d\theta} \qquad \ldots(2)$$

With above substitution, equation (1) becomes $\dfrac{dt}{d\theta} + \tan\theta\, t = \sec\theta$

$$\therefore \qquad I.F. = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta.$$

Whence the solution becomes

$$t \cdot \sec\theta = \int \sec\theta \cdot \sec\theta\, d\theta + c$$

$$\frac{1}{r}\frac{1}{\cos\theta} = \tan\theta + c \quad \text{or} \quad \frac{1}{r} = (\sin\theta + c\cos\theta)$$

Example 19: **Solve the differential equation** $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{xy}}$. **[NIT Kurukshetra, 2010]**

Solution: On rewriting the given equation as

$$\frac{dx}{dy} = \frac{x + \sqrt{xy}}{y} \quad \text{i.e.} \quad \frac{dx}{dy} - xy^{-1} = x^{\frac{1}{2}} y^{-\frac{1}{2}} \qquad \ldots(1)$$

Which is comparable to Bernoulli's equation

$$\frac{dx}{dy} + P \cdot x = Q \cdot x^n, \text{ where } P \text{ and } Q \text{ are functions of } y.$$

$\therefore$ Equation (1) is written as, $x^{-1/2} \dfrac{dx}{dy} - \dfrac{1}{y} x^{1/2} = y^{-1/2}$ $\qquad \ldots(2)$

Taking $x^{1/2} = t$ so that $\dfrac{1}{2} x^{-1/2} \dfrac{dx}{dy} = \dfrac{dt}{dy}$

With above substitution, (2) becomes

$$2\frac{dt}{dy} - \frac{1}{y} t = y^{-1/2} \quad \text{or} \quad \frac{dt}{dy} - \frac{1}{2y} t = \frac{1}{2} y^{-1/2} \qquad \ldots(3)$$

which is a Leibnitz's Linear equation with $P = -\dfrac{1}{2y}$ and $Q = \dfrac{y^{-1/2}}{2}$

$$\therefore \qquad I.F. = e^{\int P dy} = e^{\int -\frac{1}{2y} dy} = e^{-\frac{1}{2}\log y} = y^{-\frac{1}{2}} = \frac{1}{\sqrt{y}}$$

Hence the solution of (3) is

$$t \cdot \frac{1}{\sqrt{y}} = \int \frac{1}{\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} dy + c$$

or $\quad \dfrac{t}{\sqrt{y}} = \dfrac{1}{2} \displaystyle\int \frac{1}{y} dy + c = \dfrac{1}{2} \log y + c = \log \sqrt{y} + c$

or $\quad \sqrt{x} = \sqrt{y}\left(\log \sqrt{y} + c\right)$

Observation: This problem can be dealt under homogeneous category also.

ASSIGNMENT 4

1. (*i*) $\sin x \dfrac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$ (*ii*) $\left(1 + y^2\right) + \left(x - e^{\tan^{-1} y}\right) \dfrac{dy}{dx} = 0$

(*iii*) $y\log y\, dx + (x - \log y)\, dy = 0$ (*iv*) $y e^y dx = \left(y^3 + 2x e^y\right) dy$

(*v*) $x\log x \dfrac{dy}{dx} + y = \log x^2$ (*vi*) $\dfrac{dy}{dx} - \dfrac{2y}{x+1} = (x+1)^3$

(*vii*) $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$ [J & K 2001] (*viii*) $\left(x^3y^2 + xy\right) dx = dy$

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4 (*a*) Exact Differential Equation of First Order: A differential equation of the form $M(x, y)\, dx + N(x, y)\, dy = 0$ is said to be exact if $M(x, y)\, dx + N(x, y)\, dy = 0$ is the exact differential of some function $u(x, y)$ defined over $(x, y) \in S \in IR^2$ or precisely means $du = M\, dx + N dy = 0$ and its solution becomes $u(x, y) = c$, where c is any arbitrary constant.
e.g. $x dy + y dx = 0$ i.e., $d(xy) = 0$ is exact as it is obtainable from $xy = c$ directly by differentiation.

Note: Here $u(x, y)$ is a function of two variables and possesses continuous partial derivatives of 1st order.

Theorem: The necessary and sufficient condition for $M(x, y)dx + N(x, y)dy = 0$ to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Condition is Necessary

Let the equation $M\, dx + N\, dy = 0$ be exact.

Exact means, $Mdx + Ndy = du$, where u is a function of x and y.

Implying $$Mdx + Ndy = \frac{\partial u(x, y)}{\partial x} dx + \frac{\partial u(x, y)}{\partial y} dy \quad \ldots(1)$$

Equating the coefficient of dx and dy on both sides of (1), we get

$$M = \frac{\partial u}{\partial x} \text{ and } N = \frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} \quad \ldots(2)$$

Since $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$, therefore from (2), $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$
which is the required necessary condition.

Condition is Sufficient

If given $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then to prove $Mdx + Ndy = 0$ is exact.

Let $\int M dx = u(x, y)$, where integration is performed treating y as constant.

Then $$\frac{\partial}{\partial x}\left(\int M\, dx\right) = \frac{\partial u}{\partial x} \text{ or } M = \frac{\partial u}{\partial x} \quad \ldots(3)$$

Implying $\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ ...(4)

On using the given condition, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation (4) becomes $\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$ ...(5)

Integrating both sides of (5) with respect to x, keeping y as constant,

$$N = \frac{\partial u}{\partial y} + f(y), \quad \text{(say)} \qquad \text{...(6)}$$

$$\therefore \qquad M\,dx + N dy = \frac{\partial u}{\partial x}dx + \left(\frac{\partial u}{\partial y} + f(y)\right)dy \quad \text{by (3) and (6)}$$

$$= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + f(y)\,dy$$

$$= du + d\int f(y)\,dy$$

$$= d\left[u + \int f(y)\,dy\right] \qquad \text{...(7)}$$

Clearly it is an exact differential. Hence $Mdx + Ndy = 0$ is an exact differential.

Solution: Now $M(x,\ y)\,dx + N(x,\ y)\,dy = d\left[u + \int f(y)\,dy\right] = 0$

On integration gives $u(x, y) + \int f(y)\,dy = 0$ as solution wherein $u(x, y) = \int M(x, y)\,dx$ and $f(y)$ = terms of N not containing x.

Working Rule: The equation $M(x,\ y)\,dx + N(x,\ y)\ dy = 0$ is exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and its solution is

$$\int_{\text{keeping } y \text{ constant}} M(x, y)\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy = c.$$

Example 20: Check the equation $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$ for exactness. If exact, find the solution. **[KUK, 2003-04]**

Solution: Here $M = y^2 e^{xy^2} + 4x^3, \quad N = 2xy\,e^{xy^2} - 3y^2$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2y \cdot e^{xy^2} + y^2 e^{xy^2} 2xy = 2y(1 + xy^2)\,e^{xy^2}$$

$$\frac{\partial N}{\partial x} = 2y \cdot e^{xy^2} + 2xy\,e^{xy^2} y^2 = 2y(1 + xy^2)\,e^{xy^2}$$

As here $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, means the given equation is an exact one and therefore its solution is

$$\int_{y\ const} (y^2 e^{xy^2} + 4x^3)\, dx + \int (-3y^2)\, dy = c$$

implying $\quad y^2 \dfrac{e^{xy^2}}{y^2} + 4 \cdot \dfrac{x^4}{4} - 3\dfrac{y^3}{3} = c \quad$ or $\quad e^{xy^2} + x^4 - y^3 = c'$

Example 21: Solve the differential equation $\left(e^y + 1\right)\cos x\, dx + e^y \sin x\, dy = 0$

Solution: Here $M = \left(e^y + 1\right)\cos x$ and $N = e^y \sin x$

$$\therefore \qquad \frac{\partial M}{\partial y} = e^y \cos x, \quad \frac{\partial N}{\partial x} = e^y \cos x$$

Here $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, therefore the given equation is an exact one and its solution is

$$\int_{y\ const.} \left(e^y + 1\right)\cos x\, dx = c \quad \text{i.e.} \quad \left(e^y + 1\right)\sin x = c.$$

Example 22: Solve $\dfrac{dy}{dx} = -\dfrac{ax + hy + g}{hx + by + f}$.

Show that the equation $(ax + hy + g)dx + (hx + by + f)dy = 0$ is exact and hence solve it. Further, show that this differential equation repersents a family of conics.

Solution: The given equation, $\dfrac{dy}{dx} = -\dfrac{ax + hy + g}{hx + by + f}$ in its simplified from is written as

$$(ax + hy + g)\, dx + (hx + by + f)\, dy = 0 \qquad \ldots(1)$$

Here $\qquad M = ax + hy + g \quad$ and $\quad N = hx + by + f$

$\therefore \qquad \dfrac{\partial M}{\partial y} = h$ and $\dfrac{\partial N}{\partial x} = h$, whence the given equation is an exact one.

Its solution is

$$\int_{y_{const.}} \left(ax + hy + g\right)dx + \int \left(by + f\right)dy = c$$

$$\Rightarrow \qquad \frac{ax^2}{2} + \left(hy + g\right)x + \frac{by^2}{2} + fy = c$$

$$\Rightarrow \qquad ax^2 + 2hxy + by^2 + 2gx + 2fy = C', \text{ where } 2c = C'$$

Which is a second degree equation in x and y, clearly represents a family of conics.

Example 23: Solve the initial value problem $e^x(\cos y\, dx - \sin y\, dy) = 0$.

Solution: Here $M(x, y) = e^x \cos y$ and $N(x, y) = -e^x \sin y$

$$\therefore \qquad \frac{\partial M}{\partial y} = -e^x \sin y \text{ and } \frac{\partial N}{\partial x} = -e^x \sin y$$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is an exact one.

Implying, $\frac{\partial u}{\partial x} = M = e^x \cos y$ ...(1)

and $\frac{\partial u}{\partial y} = N = -e^x \sin y$ ...(2)

Integration of equation (1) gives, $u(x, y) = e^x \cdot \cos y + f(y)$ ...(3)

From above equation (3), we get $\frac{\partial u}{\partial y} = -e^x \sin y + \frac{d}{dy} f(y)$ ...(4)

Comparing (2) and (4) , we see that, $\frac{d}{dy} f(y) = 0$ implying $f(y) = c_1$ ...(5)

Hence the solution (3), becomes, $u(x, y) = e^x \cdot \cos y + c_1$ ...(6)

Further, exactness of the given equation implies,

$$M dx + N dy = du = 0 \quad \text{or} \quad u(x, y) = c_2$$

Therfore from (6), $c_2 = e^x \cdot \cos y + c_1$ or $e^x \cdot \cos y = (c_1 + c_2) = c$

Applying the initial conditions $y(0) = 0$, we obtain $c = 1$.

Hence, the solution of the initial value problem is $e^x \cos y = 1$

Example 24: Solve $(\sec x \tan x \tan y - e^x) dx + (\sec x \cdot \sec^2 y) \cdot dy = 0.$

Solution: Here $M = \sec x \tan x \tan y - e^x$ and $N = \sec x \cdot \sec^2 y$

$$\therefore \qquad \frac{\partial M}{\partial y} = \sec x \cdot \tan x \cdot \sec^2 y, \quad \frac{\partial N}{\partial x} = \sec x \cdot \tan x \cdot \sec^2 y$$

Clearly $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is an exact one.

Whence its solution is

$$\left(\int_{y_{const.}} \sec x . \tan x \tan y - e^x \right) dx = c$$

$$\int_{y_{const.}} \left(\tan y \frac{d}{dx} (\sec x) - e^x \right) dx = c \quad \text{implying} \quad \tan y \cdot \sec x - e^x = c.$$

ASSIGNMENT 5

1. Solve the following equations

(*i*) $\left(x^2 - ay\right)dx = \left(ax - y^2\right)dy$ [KUK, 2003-04]

(*ii*) $ye^{xy}dx + \left(xe^{xy}2y\right)dy = 0$

(*iii*) $\left(x^4 - 2xy^2 + y^4\right)dx - \left(2x^2y - 4xy^3 + \sin y\right)dy = 0$

(*iv*) $\dfrac{2x}{y^3}dx + \dfrac{y^2 - 3x^2}{y^4}dy = 0$ [J&K, 2001]

(*v*) $\left(2xy + y - \tan y\right)dx + \left(x^2 - x\tan^2 y + \sec^2 y\right)dy = 0$

(*vi*) $\left[y\left(1 + \dfrac{1}{x}\right) + \cos y\right]dx + \left[x + \log x - x\sin y\right]dy = 0$

4 (*b*) Equations Reducible to Exact Equations by Use of Integrating Factor: Sometimes the equation is not exact in the given form but it can be made exact by multiplying it by an integrating factor. Here we discuss certain rules for finding integrating factors (I.F.) of the equation, $M(x, y)\,dx + N(x, y)\,dy = 0$ for making it an exact.

I. I.F. Found By Inspection

Many times we come across terms which can be grouped or rearranged in such a way that these are exact differential of a single term and their integrating factors are easily obtainable as explained below:

$$xdy + ydx = d(xy) \qquad \frac{xdy - ydx}{xy} = d\left(\log\left(\frac{y}{x}\right)\right)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right) \qquad \frac{xdy - ydx}{y^2} = -d\left(\frac{x}{y}\right)$$

$$\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\tfrac{y}{x}\right) \qquad \frac{xdy - ydx}{x^2 - y^2} = d\left(\tfrac{1}{2}\log\frac{x + y}{x - y}\right)$$

$$\frac{xdx + ydy}{x^2 + y^2} = d\left(\tfrac{1}{2}\log\left(x^2 + y^2\right)\right). \qquad \left(\frac{dx}{x} + \frac{dy}{y}\right) = d(\log(xy))$$

Example 25: Solve the differential equation $ydx - xdy + e^{\frac{1}{x}}dx = 0$, finding integrating factor by inspection.

Solution: As the given differential equation contains the group of terms $(ydx - xdy)$ and $e^{\frac{1}{x}}$ for which $\dfrac{1}{x^2}$ may be tried as an integrating factor. Therefore, multiplying throughout the given equation by $\dfrac{1}{x^2}$, we see

$$\frac{ydx - xdy}{x^2} + \frac{e^{\frac{1}{x}}}{x^2}dx = 0 \quad \text{or} \quad -d\left(\frac{y}{x}\right) - d\left(e^{\frac{1}{x}}\right) = 0$$

On integration $\quad y + xe^{\frac{1}{x}} = cx$

Example 26: Solve the equations

*(i) $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$ **(ii) $x^4\dfrac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0$

[*KUK 2006; **NIT Kurukshetra, 2009]

Solution:

(i) The given equation $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$ contains group of terms ($ydx - xdy$) and $3x^2e^{x^3}\cdot y^2$. Therefore, $\dfrac{1}{y^2}$ may be tried as integrating factor.

On multiplying the equation by $\dfrac{1}{y^2}$ thoughout, we get

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}dx = 0 \quad \text{or} \quad d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0 \qquad \left(\because\ d\,e^{x^3} = e^{x^3}d(x^3) = e^{x^3}3x^2\right)$$

On integration we get, $\dfrac{x}{y} + e^{x^3} = c$ or $x + ye^{x^3} = cy$

(ii) Rewrite the given equation $x^4\dfrac{dy}{dx} + x^3y + \cos ec(xy) = 0$

as $\quad x^3(xdy + ydx) + \cos ec(xy)dx = 0$

$$\Rightarrow \qquad \frac{(x\,dy + y\,dx)}{\operatorname{cosec}(xy)} + x^{-3}dx = 0$$

$$\Rightarrow \qquad (\sin xy)(x\,dy + y\,dx) + x^{-3}dx = 0$$

Implying further, $d\{\sin(xy)\} - \dfrac{1}{2}d(x^{-2}) = 0$

On integration, $\sin xy - \dfrac{x^{-2}}{2} = c$ or $(2x^2\sin xy - 1) = Cx^2$.

II. Integrating Factor of Homogeneous Equation

Theorem: If the terms $M(x, y)$ and $N(x, y)$ in the equation $M(x, y)\,dx + N(x, y)\,dy = 0$ are homogeneous functions of degree n in x and y, then $\dfrac{1}{x\cdot M + y\cdot N}$ is an integrating factor, provided $x\cdot M + y\cdot N \neq 0$.

Proof: If $\dfrac{1}{x\cdot M+y\cdot N}$ is the integrating factor of the equation $M(x,\ y)\,dx+\ N(x,\ y)\,dy=\ 0.$ Means, multiplication of this equation by integrating factor makes it an exact one i.e.

$$\frac{M}{x\cdot M+y\cdot N}dx+\frac{N}{x\cdot M+y\cdot N}dy=0=du \qquad \ldots(1)$$

Which precisely means, new m and n i.e.

$$m(x,\ y)=\frac{M}{x\cdot M+y\cdot N}\ \text{ and }\ n(x,\ y)=\frac{N}{x\cdot M+y\cdot N} \qquad \ldots(2)$$

must satisfy $\dfrac{\partial}{\partial y}m(x,\ y)=\dfrac{\partial}{\partial x}n(x,\ y)$ $\qquad \ldots(3)$

Implying $\dfrac{\partial}{\partial y}\left[\dfrac{M}{x.M+y.N}\right]=\dfrac{\partial}{\partial x}\left[\dfrac{N}{x.M+y.N}\right]$

$$\frac{(x\cdot M+y\cdot N)\dfrac{\partial M}{\partial y}-M\left[x\dfrac{\partial M}{\partial y}+N+y\dfrac{\partial N}{\partial y}\right]}{(x\cdot M+y\cdot N)^2}=\frac{(x\cdot M+y\cdot N)\dfrac{\partial N}{\partial x}-N\left[x\dfrac{\partial M}{\partial x}+M+y\dfrac{\partial N}{\partial x}\right]}{(x\cdot M+y\cdot N)^2}$$

or

$$N\cdot y\frac{\partial M}{\partial y}-M\cdot y\frac{\partial N}{\partial y}=M\cdot x\frac{\partial N}{\partial x}-N\cdot x\frac{\partial M}{\partial x}$$

$$M\left(x\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y}\right)-N\left(x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}\right)=0 \qquad \ldots(4)$$

Since $M(x,\ y)$ and $N(x,\ y)$ are homogeneous functions of degree n in $(x,\ y)$. Therefore, by Euler's Theorem, equation (4) is identically satisfied i.e. $M(nN)-N(nM)=0$. Hence the result.

Observation: If $M\cdot x+N\cdot y=0$ then $\dfrac{1}{xy}$ *or* $\dfrac{1}{x^2}$ or $\dfrac{1}{y^2}$ may be tried as I.F.

Example 27: Solve the differential equation $\dfrac{dy}{dx}=\dfrac{x^3+y^3}{xy^2}$. [KUK, 2009]

Solution: The given equation is a homogeneous one which can be written in the form

$$\left(x^3+y^3\right)dx-xy^2dy=0 \qquad \ldots(1)$$

$$I.F.=\frac{1}{x\cdot M+y\cdot N}=\frac{1}{x\left(x^3+y^3\right)+y\left(-xy^2\right)}=\frac{1}{x^4} \qquad \ldots(2)$$

On multiplying the given equation by the I.F., we get

$$\frac{1}{x^4}\left(x^3+y^3\right)dx+\frac{1}{x^4}\left(-xy^2\right)\ dy=0\quad \text{i.e.}\quad \left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx-\frac{y^2}{x^3}dy=0 \qquad \ldots(3)$$

Now this equation, with new $M=\left(\frac{1}{x}+\frac{y^3}{x^4}\right)$ and $N=-\frac{y^2}{x^3}$ is an exact one as $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

$\therefore$ Its solution becomes, $\int\limits_{y_{const}}\left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx=c$ or $\log x-\frac{1}{3}\left(\frac{y}{x}\right)^3=c$

Alternately it can be discussed under inspection method also.

Equation (3) implies

$$\frac{dx}{x}-\frac{y^2}{x^2}\left(\frac{xdy-ydx}{x^2}\right)=0 \quad \text{or} \quad \frac{dx}{x}-\frac{y^2}{x^2}d\left(\frac{y}{x}\right)=0$$

or $$\frac{dx}{x}-\frac{1}{3}d\left(\frac{y}{x}\right)^3=0 \quad \text{since} \quad d\left(\frac{y}{x}\right)^3=3\left(\frac{y}{x}\right)^2 d\left(\frac{y}{x}\right)$$

On integration, we get $\log x-\frac{1}{3}\left(\frac{y}{x}\right)^3=c$

Example 28: **Solve $(3xy-2ay^2)dx+(x^2-2\,axy)dy=0$.** **[KUK 2010]**

Solution: The given equation is a homogeneous one with

$$M(x,y)=(3xy-2ay^2) \quad \text{and} \quad N=(x^2-2axy)$$

$$I.F.=\frac{1}{x\cdot M+y\cdot N}=\frac{1}{x(3xy-2ay^2)+y(x^2-2axy)}=\frac{1}{4xy(x-ay)}.$$

Multiplying the given equation throughout by the I.F., we get

$$\frac{(3xy-2ay^2)}{4xy(x-ay)}dx+\frac{(x^2-2axy)}{4xy(x-ay)}dy=0$$

or $$\left[\frac{2y(x-ay)+xy}{4xy(x-ay)}\right]dx+\left[\frac{x(x-ay)-axy}{4xy(x-ay)}\right]dy=0$$

or $$\left[\frac{1}{2x}+\frac{1}{4(x-ay)}\right]dx+\left(\frac{1}{4y}-\frac{a}{4(x-ay)}\right)dy=0$$

Clearly here new $M(x, y)$ and $N(x, y)$ satisfies the condition $\frac{\partial}{\partial y}M(x,y)=\frac{\partial}{\partial x}N(x,y)$ and therefore, its solution is given by

$$\int\limits_{y_{const.}}\left[\frac{1}{2x}+\frac{1}{4}\frac{1}{(x-ay)}\right]dx+\int\frac{1}{4y}dy=c$$

$$\frac{1}{2}\log x + \frac{1}{4}\log(x - ay) + \frac{1}{4}\log y = c$$

$$\log x^{\frac{1}{2}}(x - ay)^{\frac{1}{4}} y^{\frac{1}{4}} = c$$

$$x^{\frac{1}{2}}(x - ay)^{\frac{1}{4}} y^{\frac{1}{4}} = e^{C} \quad \text{or} \quad x^2(x - ay)y = e^{4c} = C$$

Note: This problem has also been discussed under the next category, example no.29. Alternately, it may also be discussed as a case of Homogeneous Equations by substituting $y = vx$.

III.(*i*) If $\dfrac{\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)}{N} = f(x)$, a function of x alone, then $e^{\int f(x)dx}$ is an integrating factor of $M(x, y)dx + N(x, y)dy = 0$.

(*ii*) If $\dfrac{\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)}{M} = -g(y)$, a function y alone, then $e^{\int g(y)dy}$ is an integrating factor of $M(x, y)\,dx + N(x, y)\,dy = 0$. (Proofs are excluded)

Example 29: Solve $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$ **[KUK 2010]**

Solution: For $M = (3xy - 2ay^2)$ and $N = (x^2 - 2axy)$; $\dfrac{\partial M}{\partial Y} \neq \dfrac{\partial N}{\partial x}$.

Hence the equation is not an exact in the given form. But we see,

$$\frac{\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)}{N} = \frac{(3x - 4ay) - (2x - 2ay)}{x(x - 2ay)} = \frac{1}{x} = f(x)$$

$$\therefore \qquad I.F. = e^{\int \frac{1}{x} dx} = x$$

Multiply the given equation throughout by x, we get

$$(3x^2y - 2axy^2)\,dx + (x^3 - 2ax^2y)\,dy = 0$$

Clearly new $M(x, y) = 3x^2y - 2axy^2$ and $N(x, y) = (x^3 - 2ax^2y)$ satisfy the condition $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and thus, in the new form it is an exact.

Therefore its solution becomes, $\displaystyle\int_{y_{const.}} (3x^2y - 2axy^2)\,dx = c$

or $\quad 3\dfrac{x^3}{3}y - 2a\dfrac{x^2}{2}y^2 = c \quad$ or $\quad x^2(x - ay)y = c$.

Example 30: Solve the differential equation $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$

Solution: Here $M = y^4 + 2y,\ N = xy^3 + 2y^4 - 4x$

Implying $M_y = 4y^3 + 2,\ N_x = y^3 - 4$

Clearly $M_y \neq N_x$, whence the given equation is not exact in its present form.

But here, $\dfrac{M_y - N_x}{M} = \dfrac{\left(4y^3 + 2\right) - \left(y^3 - 4\right)}{\left(y^4 + 2y\right)} = \dfrac{3}{y} = -g(y)$, a function of y alone.

So that, $I.F.\ e^{\int g(y)\,dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = \dfrac{1}{y^3}$

On multiplying the given equation throughout by $\dfrac{1}{y^3}$, we get

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0 \text{ which is now an exact one.}$$

Hence its solution is given by

$$\int\left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = c \quad \text{implying} \quad \left(y + \frac{2}{y^2}\right)x + y^2 = c.$$

Example 31: Solve $\left(3x^2 - y^2\right)dy - 2xy\,dx = 0.$

Solution: Here $M = -\,2xy$ and $N = 3x^2 - y^2$

$$\frac{\partial M}{\partial y} = -2x,\ \frac{\partial N}{\partial x} = 6x;\ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -8x.$$

So that $\dfrac{M_y - N_x}{M} = \dfrac{-8x}{-2xy} = \dfrac{4}{y} = -g(y)$

and integrating factor becomes, $e^{\int g(y)\,dy} = e^{\int \frac{-4}{y}dy} = e^{-\frac{1}{4}\log y} = \dfrac{1}{y^4}$

Multiplying the given differential equation throughout by the integrating factor $\dfrac{1}{y^4}$, we get, $\left(3x^2y^{-4} - y^{-2}\right)dy - 2xy^{-3}dx = 0$ which is an exact one.

Whence its solution becomes,

$$\int_{y_{const.}} \left(-2xy^{-3}\right)dx + \int -y^{-2}dy = c$$

or $$\frac{-x^2}{y^3} + \frac{1}{y} = c \quad \text{or} \quad cy^3 - y^2 + x^2 = 0$$

IV. Integrating Factor of the Form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

Theorem: Integrating Factor of the equation $M(x, y)\,dx + N(x, y)\,dy = 0$, if comparable with $f_1(xy)ydx + f_2(xy)xdy = 0$ is $\dfrac{1}{x\cdot M - y\cdot N}$ provided $(x \cdot M - y \cdot N) \neq 0$.

Proof: The given equation $M(x, y)dx + N(x, y)dy = 0$ has $M(x, y) = f_1(xy)y$, $N(x, y) = f_2(xy)x$. (i.e. M, N are functions of product xy.)

Now rewrite $$Mdx + Ndy = \frac{1}{2}\left[(x\cdot M + y\cdot N)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (x\cdot M - y\cdot N)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$\Rightarrow \qquad Mdx + Ndy = \frac{1}{2}\left[(x\cdot M + y\cdot N)\cdot d(\log xy) + (x\cdot M - y\cdot N)\,d\left(\log\frac{x}{y}\right)\right]$$

Dividing throughout by $(x \cdot M - y \cdot N \neq 0)$, we get

$$\frac{Mdx + Ndy}{(x\cdot M - y\cdot N)} = \frac{1}{2}\left[\left(\frac{x\cdot M + y\cdot N}{x\cdot M - y\cdot N}\right)d(\log xy) + d\left(\log\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{f_1(xy)xy + f_2(xy)xy}{f_1(xy)xy - f_2(xy)xy}\,d(\log xy) + d\left(\log\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[f(xy).d[\log(xy)] + d\left(\log\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[F(\log(xy))\cdot d\log(xy) + d\left(\log\frac{x}{y}\right)\right]$$

$$\left(\text{Since } xy = e^{\log xy} \Rightarrow f(xy) = f\left(e^{\log xy}\right) = F(\log xy)\right)$$

$$\therefore \qquad \frac{Mdx + Ndy}{xM - yN} = \frac{1}{2}\left[F(\log xy)\cdot d(\log xy) + d\left(\log\frac{x}{y}\right)\right]$$

which is an exact differential equation.

Implying $\dfrac{1}{x\cdot M - y\cdot N}M\,dx + \dfrac{1}{x\cdot M - y\cdot N}N\,dy = 0$ is an exact.

Whence $\dfrac{1}{M\cdot x - N\cdot y}$ is an integrating factor of $Mdx + Ndy = 0$

Example 32: Solve the differential equation $\left(x^2y^2 + xy + 1\right)\cdot ydx + \left(x^2y^2 - xy + 1\right)xdy = 0$

Solution: The given equation is comparable with

$f_1(xy)ydx + f_2(xy)xdy = 0$ for which integrating factor is $\dfrac{1}{x\cdot M - y\cdot N}$

$$\therefore \qquad I.F. = \frac{1}{(x^2y^2 + xy + 1)yx - (x^2y^2 - xy + 1)yx} = \frac{1}{2x^2y^2}$$

Multiplying the equation throughout by the integrating factor, $\frac{1}{2x^2y^2}$, we get

$$\frac{(x^2y^2 + xy + 1)y}{2x^2y^2}dx + \frac{(x^2y^2 - xy + 1)x}{2x^2y^2}dy = 0$$

Which is now an exact and hence its solution becomes

$$\frac{1}{2}\int_{y\ const.}\left(y + \frac{1}{x} + \frac{1}{y}x^{-2}\right)dx - \frac{1}{2}\int\frac{1}{y}dy = c$$

or $$\frac{1}{2}\left[yx + \log x - \frac{1}{xy} - \frac{1}{2}\log y\right] = c$$

or $$\frac{xy}{2} + \frac{1}{2}\log x - \frac{1}{2xy} - \frac{1}{4}\log y = c$$

or $$\frac{2x^2y^2 + 2xy\log x - 2 - xy\log y}{4xy} = c$$

$$2x^2y^2 + xy\log\left(\frac{x^2}{y}\right) - 2 = 4xyc$$

V. Integrating Factor for the Equation of the Form

$x^ay^b(mydx + nxdy) + x^{a'}y^{b'}(m'ydx + n'xdy) = 0$ is $x^{\alpha}y^{\beta}$

where $\dfrac{a+\alpha+1}{m} = \dfrac{b+\beta+1}{n}$, $\dfrac{a'+\alpha+1}{m'} = \dfrac{b'+\beta+1}{n'}$.

Example 33: Solve the Equation $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$. [KUK, 2004-05]

Solution: Re-write the given equation as:

$$(2x^2ydx + 2x^3dy) + (y^2dx - xydy) = 0 \qquad \ldots(1)$$

or $$x^2y^0(2ydx + 2xdy) + x^0y^1(ydx - xdy) = 0$$

which is comparable to the form $x^ay^b(mydx + nxdy) + x^{a^1}y^{b^1}(m^1ydx + n^1xdy) = 0$

Here $$\left.\begin{matrix} a = 2 \\ b = 0 \end{matrix}\right\} \left.\begin{matrix} m = 2 \\ n = 2 \end{matrix}\right\}; \quad \left.\begin{matrix} a^1 = 0 \\ b^1 = 1 \end{matrix}\right\} \left.\begin{matrix} m^1 = 1 \\ n^1 = -1 \end{matrix}\right\}; \qquad \ldots(2)$$

With the above values, we get

$$\frac{2+\alpha+1}{2}=\frac{0+\beta+1}{2} \quad \text{or} \quad \alpha-\beta=-2 \qquad \ldots(3)$$

and
$$\frac{0+\alpha+1}{1}=\frac{1+\beta+1}{-1} \quad \text{or} \quad \alpha+\beta=-3 \qquad \ldots(4)$$

On solving (3) and (4), we get $\alpha=\frac{-5}{2}$ and $\beta=\frac{-1}{2}$

$$\therefore \quad I.F.=x^{\alpha}y^{\beta}=x^{\frac{-5}{2}}y^{\frac{-1}{2}} \qquad \ldots(5)$$

Multiplying equation (1) by $x^{-\frac{5}{2}}y^{-\frac{1}{2}}$ throughout, we have

$$x^{\frac{-5}{2}}y^{\frac{-1}{2}}\left\{\left(y^2+2x^2y\right)dx+\left(2x^3-xy\right)dy\right\}=0 \qquad \ldots(6)$$

i.e.
$$\left(x^{\frac{-5}{2}}y^{\frac{3}{2}}+2x^{\frac{-1}{2}}y^{\frac{1}{2}}\right)dx+\left(2x^{\frac{1}{2}}y^{\frac{-1}{2}}-x^{\frac{-3}{2}}y^{\frac{1}{2}}\right)dy=0$$

New M and N satisfy the condition $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$\therefore$ The solution is $\int\left(x^{\frac{-5}{2}}y^{\frac{3}{2}}+2x^{\frac{-1}{2}}y^{\frac{1}{2}}\right)dx=c$

$$\frac{x^{\frac{-3}{2}}y^{\frac{3}{2}}}{\frac{-3}{2}}+\frac{2x^{\frac{1}{2}}y^{\frac{1}{2}}}{\frac{1}{2}}=c \quad \text{or} \quad 4(xy)^{\frac{1}{2}}-\frac{2}{3}\left(\frac{y}{x}\right)^{\frac{3}{2}}=c.$$

ASSIGNMENT 6

1. Solve the following equations

(*i*) $xdy-ydx+a\left(x^2+y^2\right)dx=0$ (*ii*) $xdx+ydy=a^2\left(\frac{xdy-ydx}{x^2+y^2}\right)$ **[UP Tech, 2008]**

(*iii*) $ydx-xdy+\log x\, dx=0$ [In *i*, *ii*, *iii*, apply inspection method]

(*iv*) $x^2y\,dx-\left(x^3+y^3\right)dy=0$ $\left[I.F.=\frac{1}{M\cdot x+N\cdot y}=\frac{1}{y^4}\right]$

(*v*) $(1+xy)\,ydx+(1-dy)xdy=0$ $\left[I.F.=x^{\alpha}y^{\beta}=x^{-\frac{11}{7}}\cdot y^{\frac{-10}{7}}\right]$

(*vi*) $\left(2x^2y^2+y\right)dx-\left(x^3y-3x\right)dy=0$ **[NIT Kurukshetra, 08]** $\left[I.F.=x^{\alpha}y^{\beta}=x^{-\frac{11}{7}}\cdot y^{\frac{-10}{7}}\right]$

(*vii*) $\left(xy^2 + 2x^2y^3\right)dx + \left(x^2y - x^3y^2\right)dy = 0$ $\left[I.F. = \dfrac{1}{M\cdot x - N\cdot y} = \dfrac{1}{3x^3y^3}\right]$

(*viii*) $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2y\,dy = 0$ $\left[\dfrac{M_y - N_x}{N} = \dfrac{4}{x},\ I.F. = \dfrac{1}{x^4}\right]$

(*ix*) $\left(x^3y^2 + x\right)dy + \left(x^2y^3 - y\right)dx = 0$ $\left[I.F. = x^{\alpha}y^{\beta} = \dfrac{1}{xy}\right]$

(*x*) $\left(2xy + x^2\right)y' = 3y^2 + 2xy$ $\left[I.F. = \dfrac{1}{M\cdot x + N\cdot y} = \dfrac{1}{xy(x+y)}\right]$

2. Under what conditions the following differential equations are exact?
   (*i*) $(ax + y)dx + (bx + gy)dy = 0$
   (*ii*) $(a \sinh x \cos y + b \cosh x \sin y)\, dx + (c \sinh x \cos y + d \cosh x \sin y)\, dy = 0$
   (*iii*) $[u(x) + v(y)]\, dx + [p(x) + q(y)]dy = 0$ (*vi*) $xy^3dx + \alpha\, x^2y^2dy = 0$
3. Solve the following initial value problems:
   (*i*) $\left(4x^3y^3 + \dfrac{1}{x}\right)dx + \left(3x^4y^2 - \dfrac{1}{y}\right)dy = 0,\quad y(1) = 1$ **[NIT Kurukshetra, 2007]**
   (*ii*) $xy\,dx - \left(x^2 + y^2\right)dy = 0,\ y(0) = 1$
   (*iii*) $\left(x - y\cos x\right)dx - \sin x\,dy = 0,\quad y\left(\dfrac{\pi}{2}\right) = 1$ (*vi*) $(\cos x + y\sin x)\, dx = \cos x\, dy,\ y(\pi) = 0.$
4. Show that $F(x, y)$ is an integrating factor of $M(x, y)\, dx + N(x, y)\, dy = 0$ if and only if
   $$\left(M\frac{\partial F}{\partial y} - N\frac{\partial F}{\partial x}\right) + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)F = 0$$

7.6 EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

Here we deal with differential equations which are of first order but not of first degree. The most general form of this type is

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \ \ldots\ + P_{n-1}\frac{dy}{dx} + P_n = 0$$

i.e. $$p^n + P_1\, p^{n-1} + P_2\, p^{n-2} + \ \ldots\ + P_{n-1}\, p + P_n = 0 \qquad \ldots(1)$$

where $P_1,\ P_2,\ \ldots,\ P_n$ are function of x and y.

These equations are generally divided into 3 types.

Type I. Equations Solvable for *p*

Let the *n*th degree equation $f(x, y, b) = 0$ be expressed as a polynomial of degree '*n*' in the form of equation (1)

Then we have,

$$(p-p_1)(p-p_2)\dots(p-p_n)=0 \qquad \dots(2)$$

i.e. $$[p-F_1(x,\ y)][p-F_2(x,\ y)]\ \dots\ [p-F_n(x,\ y)]=0$$

$$p-F_1(x,\ y)=0,\quad p-F_2(x,\ y)=0,\ \dots,\ p-F_n(x,y)=0 \qquad \dots(3)$$

On solving each of these equations of 1st order and 1st degree, we get solutions

$$f_1(x,\ y,\ c_1)=0,\ \ f_2(x,\ y,\ c_2)=0,\ \dots,\ f_n(x,\ y,\ c_n)=0 \qquad \dots(4)$$

Since the given equation is of 1st order, therefore, it cannot have more than one arbitrary constant. Meaning thereby, $c_1=c_2=\ \dots\ =c_n=c(\text{say})$.

Whence the above solutions reduces to

$$f_1(x,\ y,\ c)=0,\ \ f_2(x,\ y,\ c)=0,\ \dots,\ f_n(x,\ y,\ c)=0 \qquad \dots(5)$$

Therefore, the general solution of (1) becomes

$$f_1(x,\ y,\ c)\cdot f_2(x,\ y,\ c)\cdot\dots\cdot f_n(x,\ y,\ c)=0 \qquad \dots(6)$$

Working Rule

(*i*) Resolve the given equation into linear factors of 'p'.
(*ii*) Equate each factor to zero so that each factor gives independent first order differential equation.
(*iii*) Combine all solutions to get the required general sol.

Example 34: **Solve the differential equation $p^2-5p+6=0$.**

Solution: The given equation is $p^2-5p+6=0$ where $p=\dfrac{dy}{dx}$.

$\Rightarrow$ $(p-2)\ (p-3)=0$ or $p=2,\ 3$

or $$\frac{dy}{dx}=2 \ \text{ and }\ \frac{dy}{dx}=3$$

On integration, we get

$$y=2x+c \ \text{ and } y=3x+c$$

where c is any arbitrary constant.

Hence the required the general solution becomes, $(2x-y+c)(3x-y+c)=0$

Example 35: **Find the complete primitive of $x^2\left(\dfrac{dy}{dx}\right)^2-2xy\left(\dfrac{dy}{dx}\right)+2y^2-x^2=0$.**

Solution: Taking $\dfrac{dy}{dx}=p$, rewrites the given equation,

$$x^2p^2-2xyp+(2y^2-x^2)=0$$

implying $$p = \frac{2xy \pm \sqrt{4x^2y^2 - 8x^2y^2 + 4x^4}}{2x^2} = \frac{2xy \pm 2x\sqrt{x^2 - y^2}}{2x^2} \quad \ldots(1)$$

or $$\frac{dy}{dx} = \frac{y \pm \sqrt{x^2 - y^2}}{x}$$ which is a homogeneous equation in x and y ...(2)

Let $$y = ux, \quad \frac{dy}{dx} = u + x\frac{du}{dx} \quad \ldots(3)$$

From (2) and (3), $$u + x\frac{du}{dx} = \frac{ux \pm \sqrt{x^2 - u^2x^2}}{x}$$

or $$\frac{du}{\sqrt{1-u^2}} = \pm\frac{dx}{x}$$ (a case of variable separable) ...(4)

Integrating both sides

$$\sin^{-1} u = \pm \log x + \log c,$$ where c is an arbitrary constant.

or $$\sin^{-1}\frac{y}{x} = \pm \log cx, \text{ for } u = \frac{y}{x}$$

Example 36: Solve the differential equation $4p^2x - (3x - a)^2 = 0$.

Solution: The given equation is $4p^2x - (3x-a)^2 = 0,\ p = \frac{dx}{dy}$

Implying $$p^2 = \frac{(3x-a)^2}{4x} \quad \text{or} \quad p = \frac{3x-a}{2\sqrt{x}} \quad \text{or} \quad \frac{dy}{dx} = \frac{3x-a}{2\sqrt{x}}$$

$\Rightarrow$ $$dy = \frac{3x-a}{2\sqrt{x}}dx,$$ (a case of variable separable)

On integration,

$$y = \frac{3}{2}\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{a}{2}\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c,$$ where c is an arbitrary constant.

$$y - c = x^{\frac{1}{2}}(x-a) \Rightarrow (y-c)^2 = x(x-a)^2,$$ the complete solution.

Type II. Equations Solvable for y

Let the equation be $f(x, y, p) = 0,\ p = \frac{dy}{dx}$ ...(1)

As this differential equation is solvable for y, therefore, it can be put in the form

$$y = F(x, p) \quad \ldots(2)$$

Differentiating this with respect to x, we get

$$p = \phi\left(x,\ p,\ \frac{dp}{dx}\right) \quad \ldots(3)$$

Equation (3) involves two variables, x and p and therefore, its solution will be of the form

$$\eta(x, p, c) = 0 \quad \ldots(4)$$

where c is any arbitrary constant.

Elimination of 'p' from (2) and (4) gives solution of the given differential equation.

Note: In case, when elimination of 'p' is not possible , then we may solve (2) and (4) for x and y in terms of 'p' i.e. $x = F_1(p, c)$, $y = F_2(p, c)$ as the required solution, where p is the parameter.

Example 37: Solve the differential equation $y + px = x^4 p^2$.

Solution: The given equation is $y + px = x^4 p^2$ ...(1)

On differentiation it with respect to x, results in

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} + x^4\left(2p\frac{dp}{dx}\right) + p^2 \cdot 4x^3$$

$$\Rightarrow \qquad p = -p - x\frac{dp}{dx} + 2px^4\frac{dp}{dx} + 4p^2x^3$$

$$\Rightarrow \qquad \left(2p + x\frac{dp}{dx}\right) - 2px^3\left(x\frac{dp}{dx} + 2p\right) = 0$$

$$\Rightarrow \qquad \left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0 \qquad ...(2)$$

$$\Rightarrow \qquad \text{either} \quad 2p + x\frac{dp}{dx} = 0 \quad ...(i) \qquad \text{or} \quad 1 - 2px^3 = 0 \quad ...(ii) \qquad ...(3)$$

Integrating both sides of 3(i), we get

$$\log x^2 p = \log c \quad i.e. \quad p = \frac{c}{x^2} \qquad ...(4)$$

where c is an arbitrary const.

Using (4), (1) reduces to $y = -x\left(\frac{c}{x^2}\right) + x^4\left(\frac{c^2}{x^4}\right)$

$$\Rightarrow \qquad y = -\frac{c}{x} + c^2 \quad \text{or} \quad xy + c - c^2 x = 0$$

Observation: The factor, $1 - 2px^3 = 0$ has been left in our calculation, being interested only in general solution. However, this concerns to singular solution of equation (1).

Example 38: Solve $y - 2px = f(xp^2)$.

Solution: Differentiating the given equation with respect to x, we get

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + f'(xp^2)\left[p^2 + x.2p\frac{dp}{dx}\right]$$

$$\Rightarrow \qquad p = 2p + 2x\frac{dp}{dx} + f'(xp^2)p\left[p + 2x\frac{dp}{dx}\right]$$

$$\Rightarrow \qquad \left(p + 2x\frac{dp}{dx}\right) + p.f'(xp^2)\left[p + 2x\frac{dp}{dx}\right] = 0$$

$$\Rightarrow \qquad \left(p + 2x\frac{dp}{dx}\right)\left(1 + p.f'(xp^2)\right) = 0 \qquad \ldots(1)$$

Now either $\left(p + 2x\dfrac{dp}{dx}\right) = 0$ or $\left(1 + p.f'\left(xp^2\right)\right) = 0$ ...(2)

$$\Rightarrow \qquad \frac{dx}{x} + 2\cdot\frac{dp}{p} = 0 \qquad \text{(Variable-Separable)}$$

Integrating both sides,

or $\log p^2 x = \log c^2 \Rightarrow p^2 x = c^2$ or $p = \dfrac{c}{\sqrt{x}}$ ...(3)

Using (3), the given equation reduces to $y = 2\dfrac{c}{\sqrt{x}}x + f\left(x.\dfrac{c^2}{x}\right)$

or $y = 2c\sqrt{x} + f\left(c^2\right)$ is the desired solution.

Caution: Some time one is unreasonable to write (3) as $\dfrac{dy}{dx} = c\sqrt{x}$ and integrate it to say that the required solution is $y = \dfrac{2}{3}cx^{\frac{3}{2}} + c'$, which is not correct.

Type III. Solvable for '*x*'

Let the equation be $f(x, y, p) = 0$ ...(1)

Since it is solvable for 'x', therefore, it can be put in the form

$$x = F(y, p) = 0 \qquad \ldots(2)$$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \phi\left(y,\ p,\ \frac{dp}{dy}\right) \qquad \ldots(3)$$

Equation (3) involves two variables y and p, therefore, its solution will be of the form

$$\eta(y, p, c) = 0 \qquad \ldots(4)$$

where c is any arbitrary constant.

Elimination of 'p' from (2) and (4) gives the solution of equation (1).

Example 39: Solve the differential equation $y = 2px + y^2p^3$.

Solution: Rewrite the given equation as

$$x = \frac{y}{2p} - y^2\frac{p^2}{2} \qquad \ldots(1)$$

Differentiating with respect to y, we get

$$\frac{dx}{dy}=\left(\frac{1}{2p}-\frac{y}{2p^2}\frac{dp}{dy}\right)-\left(\frac{2y\quad p^2}{2}+y^2\cdot\frac{2p}{2}\frac{dp}{dy}\right)$$

$$\Rightarrow \qquad \frac{1}{p}=\frac{1}{2p}-\frac{y}{2p^2}\frac{dp}{dy}-yp^2\quad -py^2\frac{dp}{dy}$$

$$\Rightarrow \qquad \left(\frac{1}{2p}+y\quad p^2\right)+\frac{y}{p}\frac{dp}{dy}\left(\frac{1}{2p}+y\quad p^2\right)=0$$

$$\Rightarrow \qquad \left(1+\frac{y}{p}\cdot\frac{dp}{dy}\right)\left(\frac{1}{2p}+yp^2\right)=0 \qquad \ldots(2)$$

Implying, either $\left(1+\frac{y}{p}\cdot\frac{dp}{dy}\right)=0$ or $\frac{1}{2p}+yp^2=0$ $\qquad\ldots(3)$

1st equation imply, $1=-\frac{y}{p}\cdot\frac{dp}{dy}$ or $\frac{dy}{y}=-\frac{dp}{p}$ (variable separable)

Integrating both sides,

$$\log y = -\log p + \log c,$$

$$\Rightarrow \qquad p=\frac{c}{y} \text{ where } c \text{ is an arbitary constant.} \qquad \ldots(4)$$

using (4) , (1) reduces to

$$x=\frac{y^2}{2c}-\frac{y^2}{2}\left(\frac{c^2}{y^2}\right)$$

$$\Rightarrow \qquad x=\frac{y^2}{2c}-\frac{c^2}{2} \quad \text{or} \quad y^2=2cx+c^3 \quad \text{is the desired solution.}$$

ASSIGNMENT 7

1. Solve the following equations

(i) $p^3-p\left(x^2+xy+y^2\right)+xy\left(x+y\right)=0,$

(ii) $p^2+2xp-3x^2=0$

(iii) $p^2-2p\cosh x+1=0$

(iv) $p^2+2py\cot x=y^2$

(v) $xy\left(\frac{dy}{dx}\right)^2-\left(x^2+y^2\right)\frac{dy}{dx}+xy=0$

(vi) $y=x\left[p+\sqrt{1+p^2}\right]$

(vii) $p^3+2xp^2-y^2p^2-2xy^2p=0$

(viii) $p(p+y)=x(x+y)$.

2. Find the complete primitive of the following

(*i*) $y = xp^2 + p$ (*ii*) $xp^2 + x = 2yp$

(*iii*) $y = x + a\tan^{-1} p$ (*vi*) $y = \sin p + \cos p$

(*v*) $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$ (*vi*) $x - yp = ap^2$

3. Solve the differential equations

(*i*) $(y + ap)\sqrt{p^2 - 1} + a\cos h^{-1} p = c,$ (*ii*) $y = c(x - c)^2$

(*iii*) $y + (1 + p^2)^{-1} = c,$ (*vi*) $\log y = c^2 + cx.$

Note: On 2(i) is that in general, the equation of the form $y = xf(p) + \phi(p)$, are solvable for y and lead to Leibnitz's equation in $\dfrac{dx}{dp}$.

7.7 CLAIRAUT'S EQUATION [PTU, 2007]

A non-linear differential equation of the form $y = xp + f(p)$ ...(1)

is called the Clairaut's equation. This equation has a special character of always having a singular solution.

Differentiating (1) with respect to 'x', we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad \text{or} \quad \{x + f'(p)\}\frac{dp}{dx} = 0$$

Therefore, either $dp = 0$ implying $p = c$ ...(2)

or $x + f'(p) = 0$ ...(3)

Here elimination of 'p' from (1) and (2) gives the complete primitive (general solution)

$$y = cx + f(c) \qquad \text{...(4)}$$

viz. one parameter family of straight lines.

Elimination of 'p' from (1) and (3) results in an equation having no constant. This equation is called singular solution of (1) which gives the envelop of the family of straight lines (3).

Observation: Equation obtained from differentiation of (4) with respect to 'c' viz. $o = x + f'(c)$ differs from (3) only in having 'c' instead 'p'. Thus, both the discriminants must represent the envelop.

Example 40: Solve the differential equation $y = px + p^3$ and find the singular solution.

Solution: The given equation $y = px + p^3$ ...(1)

is comparable with $y = px + f(p)$, whence it is a clairaut's equation.

Differentiating (1) with respect to 'x', we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + 3p^2\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + 3p^2\frac{dp}{dx} \quad \text{i.e.} \quad \frac{dp}{dx}(x + 3p^2) = 0$$

$\therefore$ either $\dfrac{dp}{dx}=0$ ...(i) or $x+3p^2=0$ ...(ii) ...(2)

(i) when $\dfrac{dp}{dx}=0 \Rightarrow p=c$, where c is any arbitrary constant

Hence the general solution of (1) becomes $y = cx + x^3$.

(ii) when $x+3p^2=0$, we had $p^2=\dfrac{-x}{3}$ ...(3)

From (1) and (3), we get

$$y=px+p\left(-\frac{x}{3}\right) \text{ or } y=p\left(x-\frac{x}{3}\right) \text{ or } 3y=2px$$

Squaring both sides,

$$9y^2=4p^2x^2 \quad\Rightarrow\quad 9y^2=4\left(-\frac{x}{3}\right)x^2 \text{ or } 4x^3+27y^2=0$$

Which is the required singular solution.

Example 41: Solve the differential equation $\sin px\cos y=\cos px\sin y+p$ and obtain the singular solution.

Solution: The given equation $\sin px\cos y=\cos px\sin y+p$ ...(1)
may be rewritten as

$$\sin(px-y)=p \quad\text{or}\quad y=px-\sin^{-1}p \qquad ...(2)$$

Which is comparable to Clairaut's form $y = px + f(p)$.
Differentiating (2) with respect to x, we get

$$\frac{dy}{dx}=p+x\frac{dp}{dx}-\frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

$$\Rightarrow \qquad p=p+\frac{dp}{dx}\left(x-\frac{1}{1-p^2}\right) \text{ or } \frac{dp}{dx}\left(x-\frac{1}{1-p^2}\right)=0$$

$\therefore$ Either $\dfrac{dp}{dx}=0$ ...(i) or $x-\dfrac{1}{1-p^2}=0$ ...(ii) ...(3)

(i) when $\dfrac{dp}{dx}=0 \Rightarrow p=c$, where c is any arbitrary constant.

Hence the general solution of (1) becomes, $y=cx-\sin^{-1}c$.

(ii) when $x-\dfrac{1}{\sqrt{1-p^2}}=0$ or $1-p^2=\dfrac{1}{x^2}$

$\Rightarrow$ $p^2=1-\dfrac{1}{x^2}$ or $p=\dfrac{\sqrt{x^2-1}}{x}$ ...(4)

Using (4), (2) reduces to $y=\sqrt{x^2-1}-\sin^{-1}\dfrac{\sqrt{x^2-1}}{x}$

which is required singular solution.

ASSIGNMENT 8

1. Solve the following equations and find their singular solutions

(*i*) $y = px + \cos p$ (*ii*) $y = px - e^p$

(*iii*) $y = px + \sqrt{a^2p^2 + b^2}$ (*iv*) $y = px - \sqrt{1 + p^2}$

(*v*) $y = xp + \dfrac{a}{p}$ (*vi*) $p = \log(px - y)$

ANSWERS

ASSIGNMENT 1

1. (*ii*) yes.

2. (*i*) $\dfrac{d^2y}{dx^2} = 4y,$ (*ii*) $\dfrac{d^2y}{dx^2} = -9y.$

3. (*i*) $\dfrac{d^2y}{dx^2} + \omega^2 x = 0$ (*ii*) $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 0$

4. (*i*) $\dfrac{dx}{\sqrt{1-x^2}} = \dfrac{dy}{\sqrt{1-y^2}}.$ (*ii*) $y = x\dfrac{dy}{dx} + \dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2$

5. (*i*) $\dfrac{d^2y}{dx^2} = 0$ (*ii*) $(x^2 - 25)\left(\dfrac{dy}{dx}\right)^2 + x^2 = 0$

(*iii*) $x + y\dfrac{dy}{dx} = 0$ (*iv*) $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 + y\dfrac{dy}{dx} = 0$

6. (*i*) $y\dfrac{dy}{dx} = 2a$ (*ii*) $x\dfrac{dy}{dx} - 2y = 0.$

ASSIGNMENT 2

1. (*i*) $3e^{2x} - 2e^{3y} + 8x^3 = c$ (*ii*) $(x+1)(2 - e^y) = c$

(*iii*) $x = 2\cos y$ (*iv*) $\log\left[1 + \tan\left(\dfrac{x+y}{2}\right)\right] = x + c$

(*v*) $\tan(x + y) - \sec(x + y) = x + c$ (*vi*) $a\log\left(\dfrac{x-y-a}{x-y+a}\right) = 2y + c$

(*vii*) $x = \operatorname{cosec}(x + y + 1) - \cot(x + y + 1) + c$

2. $xy = \pm\ e^{x-y}$ 3. $y^3 = 4(1 + x^3)$

ASSIGNMENT 3

1. (*i*) $e^{\frac{x}{y}} = y + c$ (*ii*) $y = 2x\tan^{-1}(cx)$

(*iii*) $\log y - \dfrac{x^2}{4y^2}\left(x\log\dfrac{y}{x} + 1\right) = c$ (*iv*) $\left(x^2 + y^2\right)^2 = c\left(y^2 - x^2\right)$

2. (*i*) $3(2y - x) + \log(3x + 3y + 4) = c$ (*ii*) $(x+y)^7 = c\left(x - y - \dfrac{2}{3}\right)^3$

(*iii*) $x - y + \dfrac{3}{4}\log\left(8x - 12y - 5\right) = c$ (*iv*) $\log\left(x + y - \dfrac{1}{3}\right) + \dfrac{3}{2}(y - x) = c$.

ASSIGNMENT 4

1. (*i*) $y\sin x = \dfrac{2}{3}\sin^3 x + c$ (*i*) $xe^{\tan^{-1} y} = \dfrac{1}{2}e^{2\tan^{-1} y} + c$

(*iii*) $x = \dfrac{1}{2}\log y + c(\log y)^{-1}$ (*v*) $xe^y = c + \tan y$

(*v*) $y = \log x + \dfrac{c}{\log x}$ (*vi*) $y = \dfrac{1}{2}(x+1)^4 + c(x+1)^2$

(*vii*) $e^{x+y} = \dfrac{1}{2}e^{2x} + c$ (*viii*) $\dfrac{1}{y} = x^2 - 2 + ce^{-\frac{x^2}{2}}$.

ASSIGNMENT 5

1. (*i*) $x^3 + y^3 - 3axy = c$ (*ii*) $e^{xy} + y^2 = c$

(*iii*) $\dfrac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$ (*iv*) $x^2 - y^2 = cy^3$

(*v*) $x^2y + xy - x\tan y + \tan y = c$ (*vi*) $y(x + \log x) + x\cos y = \text{c}$.

ASSIGNMENT 6

1. (*i*) $ax + \tan^{-1}\dfrac{y}{x} = c$ (*ii*) $y + cx + \log x + 1 = 0$

(*iii*) $x^2 + y^2 - 2a^2\tan^{-1}\dfrac{y}{x} = c$ (*iv*) $\dfrac{-x^3}{3y^3} + \log y = c$

(*v*) $-\dfrac{1}{xy} + \log\dfrac{x}{y} = c$ (*vi*) $\dfrac{7}{5}\left(x^{\frac{10}{7}} \quad y^{\frac{-5}{7}}\right) - \dfrac{7}{4}\left(x^{\frac{-4}{7}}y^{\frac{-12}{7}}\right) = c$

(*vii*) $-\dfrac{1}{xy} + 2\log x - \log y = c$ (*viii*) $-\dfrac{y^2}{2x^2} + \dfrac{1}{3}e^{\frac{1}{x^3}} = c$

(*ix*) $\log\left(\frac{y}{x}\right) + \frac{1}{2}x^2y^2 = c$ (*x*) $x^3 = cy(x+y)$

2. (*i*) $b = 1$ (*ii*) $a = -b,\ b = c$

(*iii*) $v'(y) = p'(x)$ (*iv*) $\alpha = \frac{3}{2}$

3. (*i*) Exact, $x^4y^3 + \log\frac{x}{y} = 1$ (*ii*) $I.F. = \frac{1}{y^3},\ y^2 = e^{\frac{x^2}{y^2}}$

(*iii*) Exact, $x^2 - 2y\sin y = \frac{\pi^2}{4} - 2$ (*iv*) Exact, $y\cos x = \sin x$

ASSIGNMENT 7

1. (*i*) $(2y - x^2 - c)(y - ce^x)(y + x - 1 - ce^{-x}) = 0$ (*ii*) $(2y - x^2 - c)(2y + 3x^2 - c) = 0$

(*iii*) $(y - e^x - c)(y + e^{-x} - c) = 0$ (*iv*) $[y(1+\cos x) - c]\ [y(1-\cos x) - c] = 0$

(*v*) $(y - cx)(y^2 - x^2 - c) = 0$ (*vi*) $x^2 + y^2 = cx$

(*vii*) $(y - c)(y + x^2 - c)(xy + cy + 1) = 0$ (*viii*) $(y - c)(y + x^2 - c)(xy + cy + 1) = 0$

2. (*i*) $(p-1)^2 x = (\log\ \ p - p + c),\ (p-1)^2 y = p^2(\log\ \ p - 2 + c) + p$

(*ii*) $2cy = c^2x^2 + 1$, with the given relation. (*iii*) $x + c = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{1+p^2}} - \tan^{-1}p\right]$

(*iv*) $x = \sin p + c$, with the given relation (*v*) $y = 2\sqrt{xc} + c^2$

(*vi*) $x = \frac{p}{\sqrt{1-p^2}}(c + a\ \ \sin^{-1}p),\ y = \frac{1}{\sqrt{1-p^2}}(c + a\ \ \sin^{-1}p) - ap$

3. (*i*) $x - yp = ap^2$ (*ii*) $p^3 - 4xyp + 8y^2 = 0$

(*iii*) $p = \tan\left(x - \frac{p}{1+p^2}\right)$ (*iv*) $y^2\log y = xyp + p^2$

ASSIGNMENT 8

1. (*i*) $y = cx + \cos c,\ (y - x\sin^{-1}x)^2 = 1 - x^2$ (*ii*) $y = cx - e^c,\ y = x(\log x - 1)$

(*iii*) $y = cx + \sqrt{a^2c^2 + b^2},\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (*iv*) $y = cx - \sqrt{1+c^2},\ y + \sqrt{1-x^2} = 0$

(*v*) $y = cx + \frac{a}{c},\ y^2 = 4ax$ (*vi*) $c = \log(cx - y),\ y = x(\log x - 1).$