CHAPTER 8

Partial Derivatives

Chapter Outline

8.1 INTRODUCTION

We often come across functions which depend on two or more variables. For example, area of a triangle depends on its base and height, hence we can say that area is the function of two variables, i.e., its base and height. u is called a function of two variables x and y, if u has a definite value for every pair of x and y. It is written as $u = f(x, y)$. The variables x and y are independent variables while u is dependent variable. The set of all the pairs (x, y) for which u is defined is called the domain of the function. Similarly, we can define function of more than two variables.

8.2 FUNCTIONS OF TWO OR MORE VARIABLES

The function $f(x, y)$ is called a real-valued function of two or more variables if there are two or more independent variables, e.g., total surface area of a rectangular parallelepiped is $2(xy + yz + zx)$ which is a function of three variables.

8.3 LIMIT AND CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

8.3.1 Limits

If $f(x, y)$ is a function of two variables x, y then

$$\lim_{(x, y)\to(a, b)} f(x, y) = l$$

if and only if for any chosen number $\varepsilon > 0$ however small, there exists a number $\delta > 0$ such that

$$|f(x, y) - l| < \varepsilon$$

for all values of (x, y) for which, $|x - a| < \delta$ and $|y - b| < \delta$

8.3.2 Test for Non-existence of a Limit

1. Evaluate limits

(i) $\lim_{x\to a}\left\{\lim_{y\to b} f(x, y)\right\}$ and (ii) $\lim_{y\to b}\left\{\lim_{x\to a} f(x, y)\right\}$

If both the limit values are equal, then $\lim_{(x, y)\to(a, b)} f(x, y)$ exists.

2. If $a = 0$, $b = 0$, evaluate limit along different paths say $y = mx$ or $y = mx^n$, etc.
If all limit values are equal, then $\lim_{(x, y)\to(a, b)} f(x, y)$ exists.

8.3.3 Theorems on Limit

If $\lim_{(x, y)\to(a, b)} f(x, y) = l$ and $\lim_{(x, y)\to(a, b)} g(x, y) = m$,

(i) $$\lim_{(x, y)\to(a, b)} [f(x, y) + g(x, y)] = \lim_{(x, y)\to(a, b)} f(x, y) + \lim_{(x, y)\to(a, b)} g(x, y) = l + m$$

(ii) $$\lim_{(x, y)\to(a, b)} [f(x, y) - g(x, y)] = \lim_{(x, y)\to(a, b)} f(x, y) - \lim_{(x, y)\to(a, b)} g(x, y) = l - m$$

(iii) $$\lim_{(x, y)\to(a, b)} [f(x, y) \cdot g(x, y)] = \lim_{(x, y)\to(a, b)} f(x, y) \cdot \lim_{(x, y)\to(a, b)} g(x, y) = lm$$

(iv) $$\lim_{(x, y)\to(a, b)} \left[\frac{f(x, y)}{g(x, y)}\right] = \frac{\lim_{(x, y)\to(a, b)} f(x, y)}{\lim_{(x, y)\to(a, b)} g(x, y)} = \frac{l}{m}, \text{ provided } m \neq 0$$

8.3.4 Continuity

Let $f(x, y)$ be a function of x and y defined at (a, b) as well as in the neighbourhood of it. The function $f(x, y)$ is continuous at (a, b) if the following three conditions are satisfied:

(i) $f(a, b)$ exists, i.e., $f(x, y)$ is defined at (a, b).

(ii) $\lim\limits_{(x, y)\to(a, b)} f(x, y)$ exists.

(iii) $\lim\limits_{(x, y)\to(a, b)} f(x, y) = f(a, b)$.

A function $f(x, y)$ is continuous in a domain if it is continuous at each point of that domain.

Note: If $f(x, y)$ and $g(x, y)$ are continuous at (a, b), then $f+g, f-g, fg, \dfrac{f}{g}$ (provided $g \neq 0$) are continuous at (a, b).

Example 1

Find $\lim\limits_{(x, y)\to(1, 2)} \dfrac{x^2+y}{3x+y^2}$.

Solution

$$\lim_{(x, y)\to(1, 2)} \frac{x^2+y}{3x+y^2} = \frac{1^2+2}{3(1)+2^2} = \frac{3}{7}$$

Example 2

Find $\lim\limits_{(x, y)\to(0, 0)} \dfrac{x-y}{x+y}$.

Solution

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x-y}{x+y}\right) = \lim_{x\to 0} 1 = 1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x-y}{x+y}\right) = \lim_{y\to 0}(-1) = -1$$

Since both the limits are different, the limit does not exist.

Example 3

By considering different paths of approach, show that the function $f(x, y) = \dfrac{x^4-y^2}{x^4+y^2}$ *has no limit as* $(x, y)\to(0, 0)$. **[Winter 2015]**

Solution

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^4-y^2}{x^4+y^2}\right)=\lim_{x\to 0}\frac{x^4}{x^4}=\lim_{x\to 0}1=1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^4-y^2}{x^4+y^2}\right)=\lim_{y\to 0}\left(\frac{-y^2}{y^2}\right)=\lim_{y\to 0}(-1)=-1$$

Since both the limits are different, $f(x, y)$ has no limit as $(x, y) \to (0, 0)$.

Example 4

Find $\displaystyle\lim_{(x,y)\to(0,0)}\frac{xy}{y^2-x^2}$.

Solution

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{y^2-x^2}\right)=\lim_{x\to 0}0=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{y^2-x^2}\right)=\lim_{y\to 0}0=0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0}\frac{x(mx)}{(mx)^2-x^2}=\lim_{x\to 0}\frac{m}{m^2-1}=\frac{m}{m^2-1}$$

Since the last limit depends on m and m is not fixed, the limit does not exist.

Example 5

Find $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$.

Solution

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2y}{x^4+y^2}\right)=\lim_{x\to 0}0=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2y}{x^4+y^2}\right)=\lim_{y\to 0}0=0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0}\frac{x^2(mx)}{x^4+(mx)^2}=\lim_{x\to 0}\frac{mx}{x^2+m^2}=0$$

Putting $y = mx^2$ and taking limit $x \to 0$,

$$\lim_{x\to 0}\frac{x^2(mx^2)}{x^4+(mx^2)^2}=\lim_{x\to 0}\frac{m}{1+m^2}=\frac{m}{1+m^2}$$

Since the last limit depends on m and m is not fixed, the limit does not exist.

Example 6

Show that $f(x, y) = x^2 + 2y$ *is continuous at* (1, 2).

Solution

$$\lim_{(x,y)\to(1,2)} f(x, y) = \lim_{(x,y)\to(1,2)} (x^2 + 2y) = 1^2 + 2(2) = 5$$

$$f(1, 2) = 1^2 + 2(2) = 5$$

$$\lim_{(x,y)\to(1,2)} f(x, y) = f(1, 2)$$

Hence, $f(x, y)$ is continuous at (1, 2).

Example 7

Show that $f(x, y) = 2x^2 + y, \quad (x, y) \neq (1, 2)$

$$= 0, \qquad (x, y) = (1, 2)$$

is discontinuous at (1, 2).

Solution

$$\lim_{(x,y)\to(1,2)} f(x, y) = \lim_{(x,y)\to(1,2)} (2x^2 + y) = 2(1^2) + 2 = 4$$

$$f(1, 2) = 0$$

$$\lim_{(x,y)\to(1,2)} f(x, y) \neq f(1, 2)$$

Hence, $f(x, y)$ is discontinuous at (1, 2).

Example 8

Discuss the continuity of $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$

$$= 0, \qquad (x, y) = (0, 0)$$

at (0, 0).

Solution

$$\lim_{x\to0}\left[\lim_{y\to0} f(x, y)\right] = \lim_{x\to0}\left(\lim_{y\to0} \frac{x^2 - y^2}{x^2 + y^2}\right)$$

$$= \lim_{x\to0} \frac{x^2}{x^2} = \lim_{x\to0} 1 = 1$$

$$\lim_{y\to0}\left[\lim_{x\to0} f(x, y)\right] = \lim_{y\to0}\left(\lim_{x\to0} \frac{x^2 - y^2}{x^2 + y^2}\right)$$

$$= \lim_{y\to0}\left(-\frac{y^2}{y^2}\right) = \lim_{y\to0}(-1) = -1$$

Since both the limits are not equal, $f(x, y)$ is discontinuous at (0, 0).

Example 9

Determine the set of points at which the given function is continuous:

$$f(x, y) = \begin{cases} \dfrac{3x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

[Winter 2013]

Solution

For $(x, y) \neq (0, 0)$, the function $f(x, y) = \dfrac{3x^2 y}{x^2 + y^2}$ is a rational function and hence, it is continuous.

For $(x, y) = (0, 0)$, $f(x, y) = 0$

$$\lim_{x \to 0}\left[\lim_{y \to 0} f(x, y)\right] = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{3x^2 y}{x^2 + y^2}\right)$$
$$= \lim_{x \to 0} 0 = 0$$
$$\lim_{y \to 0}\left[\lim_{x \to 0} f(x, y)\right] = \lim_{y \to 0}\left(\lim_{x \to 0} \frac{3x^2 y}{x^2 + y^2}\right)$$
$$= \lim_{y \to 0} 0 = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x \to 0} \frac{3x^2 (mx)}{x^2 + m^2 x^2} = \lim_{x \to 0} \frac{3x^3 m}{x^2 (1 + m^2)} = \lim_{x \to 0} \frac{3mx}{1 + m^2} = 0$$

Hence, the limit exists at (0, 0).

$$\lim_{(x, y) \to (0,0)} f(x, y) = f(0, 0)$$

Hence, $f(x, y)$ is continuous at (0, 0).

Example 10

Show that $f(x, y) = \dfrac{2xy}{x^2 + y^2}, \ (x, y) \neq (0, 0)$

$$= 0, \qquad (x, y) = (0, 0)$$

is continuous at every point except at the origin. **[Summer 2017]**

Solution

$$\lim_{x \to 0}\left[\lim_{y \to 0} f(x, y)\right] = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{2xy}{x^2 + y^2}\right)$$
$$= \lim_{x \to 0} 0 = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right] = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{2xy}{x^2+y^2}\right)$$
$$= \lim_{y\to 0} 0 = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0}\frac{2x(mx)}{x^2+(mx)^2} = \lim_{x\to 0}\frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

Since the last limit depends on m and m is not fixed, the limit does not exist.
Hence, $f(x)$ is discontinuous at the origin, i.e., (0, 0).
Let $(x, y) = (a, b) \neq (0, 0)$ be an arbitrary point in xy-plane, where a and b are real numbers.

$$\lim_{(x, y)\to(a, b)} f(x, y) = \lim_{(x, y)\to(a, b)}\frac{2xy}{x^2+y^2}$$
$$= \frac{2ab}{a^2+b^2}$$
$$f(a, b) = \frac{2ab}{a^2+b^2}$$

which is finite for real values of a and b.

$$\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$$

This shows that $f(x, y)$ is continuous at (a, b).
Hence, $f(x, y)$ is continuous at every point except at the origin.

Example 11

Show that $f(x, y) = \begin{cases} \dfrac{2x^2y}{x^3+y^3} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$

is not continuous at the origin. **[Winter 2016]**

Solution

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right] = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{2x^2y}{x^3+y^3}\right)$$
$$= \lim_{x\to 0}(0) = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right] = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{2x^2y}{x^3+y^3}\right)$$
$$= \lim_{y\to 0}(0) = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0} \frac{2x^2(mx)}{x^3 + (mx)^3} = \lim_{x\to 0} \frac{2mx^3}{x^3(1+m^3)}$$
$$= \frac{2m}{1+m^3}$$

Since the last limit depends on m and m is not fixed, the limit does not exist. Hence, $f(x)$ is discontinuous at the origin.

Example 12

Determine the continuity of the function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at origin. **[Summer 2016]**

Solution

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{x\to 0}\left[\lim_{y\to 0} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)\right]$$
$$= \lim_{x\to 0}\left[x^2 \sin\left(\frac{1}{x^2}\right)\right]$$
$$= \lim_{x\to 0}\left[\frac{\sin\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}}\right]$$
$$= 1$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = \lim_{y\to 0}\left[\lim_{x\to 0} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)\right]$$
$$= \lim_{y\to 0}\left[y^2 \sin\left(\frac{1}{y^2}\right)\right]$$
$$= \lim_{y\to 0}\left[\frac{\sin\left(\frac{1}{y^2}\right)}{\frac{1}{y^2}}\right]$$
$$= 1$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0} (x^2 + m^2x^2)\sin\left(\frac{1}{x^2+m^2x^2}\right) = (1+m^2)\sin\left(\frac{1}{1+m^2}\right)$$

Since the last limit depends on m and m is not fixed, the limit does not exist. Hence, $f(x)$ is discontinuous at origin (0, 0).

Let $(x, y) = (a, b)$ be an arbitrary point in xy-plane, where a and b are real numbers.

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(a,b)} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)$$

$$= (a^2+b^2)\sin\left(\frac{1}{a^2+b^2}\right)$$

$$f(a, b) = (a^2+b^2)\sin\left(\frac{1}{a^2+b^2}\right)$$

which is finite for real values of a and b.

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

This shows that $f(x, y)$ is continuous at (a, b).

Hence, $f(x, y)$ is continuous at every point except at the origin.

Example 13

Show that $f(x, y) = \dfrac{xy}{\sqrt{x^2+y^2}}, \quad (x, y) \neq (0, 0)$

$= 0, \quad (x, y) = (0, 0)$

is continuous at the origin. **[Summer 2014]**

Solution

For $(x, y) \neq (0, 0)$, the function $f(x, y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ is a rational function and hence, it is continuous.

For $(x, y) = (0, 0)$, $f(x, y) = 0$

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right] = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{\sqrt{x^2+y^2}}\right)$$

$$= \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right] = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{\sqrt{x^2+y^2}}\right)$$

$$= \lim_{y\to 0} 0 = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0}\frac{x(mx)}{\sqrt{x^2+m^2x^2}} = \lim_{x\to 0}\frac{mx^2}{x\sqrt{1+m^2}} = \lim_{x\to 0}\left(\frac{m}{\sqrt{1+m^2}}\right)x = 0$$

The limit exists at the origin.
Hence, $f(x, y)$ is continuous at the origin.

EXERCISE 8.1

1. Evaluate the following limits:

(i) $\lim\limits_{(x,y)\to(1,2)} \dfrac{3x^2y}{x^2+y^2+5}$ (ii) $\lim\limits_{(x,y)\to(\infty,2)} \dfrac{xy+4}{x^2+2y^2}$ (iii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x+2y}$

(iv) $\lim\limits_{(x,y)\to(0,1)} e^{-\frac{1}{x^2(y-1)^2}}$ (v) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x-y}{x^2+y^2}$

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } \dfrac{3}{5} \quad \text{(ii) } 0 \quad \text{(iii) does not exist} \\ \qquad \text{(iv) } 0 \quad \text{(v) does not exist}\end{array}\right]$$

2. Show that for $f(x, y) = \dfrac{2x-y}{2x+y}$, $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x, y)\right] \ne \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x, y)\right]$

3. Check the continuity of the following functions:

(i) $f(x,y) = \dfrac{x}{3x+5y}$ at $(0, 0)$

(ii) $f(x, y) = \dfrac{xy}{x^2+y^2}$, $(x, y) \ne (0, 0)$

$\quad = 0$, $(x, y) = (0, 0)$

at origin.

(iii) $f(x, y) = \dfrac{x^2y^2}{x^4+y^4}$ at $(0, 0)$.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) Discontinuous} \quad \text{(ii) Discontinuous} \\ \qquad \text{(iii) Discontinuous}\end{array}\right]$$

8.4 PARTIAL DERIVATIVES

A partial derivative of a function of several variables is the ordinary derivative w.r.t. one of the variables, when all the remaining variables are kept constant. Consider a function $u = f(x, y)$. Here, u is the dependent variable and x and y are independent variables. The partial derivative of $u = f(x, y)$ w.r.t. x is the ordinary derivative of u w.r.t. x, keeping y constant. It is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x or f_x and is known as first-order partial derivative of u w.r.t. x.

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right]$$

Similarly, the partial derivative of $u = f(x, y)$ w.r.t. y is the ordinary derivative of u w.r.t. y treating x as constant. It is denoted by $\frac{\partial u}{\partial y}$ or $\frac{\partial f}{\partial y}$ or u_y or f_y and is known as *first-order partial derivative* of u w.r.t. y.

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right]$$

8.5 HIGHER-ORDER PARTIAL DERIVATIVES

Partial derivatives of higher order, of a function $u = f(x, y)$, are obtained by partial differentiation of first-order partial derivative. Thus, if $u = f(x, y)$ then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

are called second-order partial derivatives. Similarly, other higher-order derivatives can also be obtained.

Mixed Derivative Theorem

If $u = f(x, y)$ possesses continuous second-order partial derivatives $\frac{\partial^2 u}{\partial x\, \partial y}$ and $\frac{\partial^2 u}{\partial y\, \partial x}$ then $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$. This is also called *commutative property*.

Note: Standard rules for derivatives of sum, difference, product and quotient are also applicable for partial derivatives.

Example 1

If $f(x, y) = x^2y + xy^2$ *then find* $f_x(1, 2)$ *and* $f_y(1, 2)$ *by definition.*

[Summer 2017]

Solution

$$f(x, y) = x^2y + xy^2$$

$$f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 y + (x+\Delta x)y^2] - x^2y - xy^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2y + 2x\,\Delta xy + (\Delta x)^2 y + xy^2 + \Delta xy^2 - x^2y - xy^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\,\Delta xy + (\Delta x)^2 y + \Delta xy^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2xy + y^2) + \Delta xy$$

$$f_x = \frac{\partial f}{\partial x} = 2xy + y^2$$

$$f_x(1, 2) = 2(1)(2) + (2)^2 = 4 + 4 = 8$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right]$$

$$= \lim_{\Delta y \to 0} \frac{[x^2(y+\Delta y) + x(y+\Delta y)^2] - x^2y - xy^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2y + x^2\Delta y + xy^2 + 2xy\Delta y + x(\Delta y)^2 - x^2y - xy^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2xy(\Delta y) + x^2(\Delta y) + x(\Delta y)^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} (2xy + x^2) + x(\Delta y)$$

$$f_y = \frac{\partial f}{\partial y} = 2xy + x^2$$

$$f_y(1, 2) = 2(1)(2) + (1)^2 = 4 + 1 = 5$$

Example 2

If $u=(1-2xy+y^2)^{-\frac{1}{2}}$ *then show that* $x\dfrac{\partial u}{\partial x}-y\dfrac{\partial u}{\partial y}=u^3y^2$.

Solution

$$u=(1-2xy+y^2)^{-\frac{1}{2}}$$

Differentiating u partially w.r.t. x and y,

$$\frac{\partial u}{\partial x}=-\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2y)$$

$$\frac{\partial u}{\partial y}=-\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2x+2y)$$

Hence, $$x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=(1-2xy+y^2)^{-\frac{3}{2}}(xy-xy+y^2)$$

$$=\left[(1-2xy+y^2)^{-\frac{1}{2}}\right]^3 y^2$$

$$=u^3y^2$$

Example 3

If $u = \log(\tan x + \tan y + \tan z)$ *then show that*

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2.$$

Solution

$$u = \log(\tan x + \tan y + \tan z)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x}=\frac{1}{\tan x+\tan y+\tan z}\cdot\sec^2 x$$

$$\frac{\partial u}{\partial y}=\frac{1}{\tan x+\tan y+\tan z}\cdot\sec^2 y$$

$$\frac{\partial u}{\partial z}=\frac{1}{\tan x+\tan y+\tan z}\cdot\sec^2 z$$

Hence,

$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}$$

$$=\frac{2\sin x\cos x\sec^2 x+2\sin y\cos y\sec^2 y+2\sin z\cos z\sec^2 z}{\tan x+\tan y+\tan z}$$

$$=\frac{2(\tan x+\tan y+\tan z)}{\tan x+\tan y+\tan z}$$

$$=2$$

Example 4

If $u=\dfrac{e^{x+y+z}}{e^x+e^y+e^z}$, *show that* $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=2u.$ **[Summer 2017]**

Solution

$$u=\frac{e^{x+y+z}}{e^x+e^y+e^z}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x}=\frac{e^{x+y+z}}{e^x+e^y+e^z}-\frac{e^{x+y+z}}{(e^x+e^y+e^z)^2}\cdot e^x$$

$$=\frac{e^{x+y+z}}{e^x+e^y+e^z}\left(1-\frac{e^x}{e^x+e^y+e^z}\right) \quad \ldots(1)$$

Similarly,

$$\frac{\partial u}{\partial y}=\frac{e^{x+y+z}}{e^x+e^y+e^z}\left(1-\frac{e^y}{e^x+e^y+e^z}\right) \quad \ldots(2)$$

$$\frac{\partial u}{\partial z}=\frac{e^{x+y+z}}{e^x+e^y+e^z}\left(1-\frac{e^z}{e^x+e^y+e^z}\right) \quad \ldots(3)$$

Adding Eqs (1), (2) and (3),

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{e^{x+y+z}}{e^x+e^y+e^z}\left(3-\frac{e^x+e^y+e^z}{e^x+e^y+e^z}\right)$$

$$=\frac{e^{x+y+z}}{e^x+e^y+e^z}(3-1)$$

$$=2u$$

Example 5

If $u = \dfrac{x^2+y^2}{x+y}$, *show that* $\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)$.

Solution

$$u = \frac{x^2+y^2}{x+y}$$

$$u(x+y) = x^2 + y^2 \qquad \ldots(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$u + (x+y)\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{2x-u}{x+y}$$

Differentiating Eq. (1) partially w.r.t. y,

$$u + (x+y)\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = \frac{2y-u}{x+y}$$

$$\text{LHS} = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = \left(\frac{2x-u}{x+y} - \frac{2y-u}{x+y}\right)^2$$

$$= \left[\frac{2(x-y)}{(x+y)}\right]^2 \qquad \ldots(2)$$

$$\text{RHS} = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 4\left(1 - \frac{2x-u}{x+y} - \frac{2y-u}{x+y}\right)$$

$$= 4\left(1 - \frac{2x-u+2y-u}{x+y}\right)$$

$$= 4\left[1 - \frac{2(x+y)}{(x+y)} + \frac{2u}{(x+y)}\right]$$

$$= 4\left[1 - 2 + 2\left\{\frac{x^2+y^2}{(x+y)^2}\right\}\right]$$

$$= 4\left[\frac{-(x+y)^2 + 2x^2 + 2y^2}{(x+y)^2}\right]$$

$$= \frac{4(x^2+y^2-2xy)}{(x+y)^2}$$

$$= \left[\frac{2(x-y)}{(x+y)}\right]^2 \qquad \ldots(3)$$

From Eqs (1) and (2),

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Example 6

Find the value of n for which $u = kt^{-\frac{1}{2}} e^{-\left(\frac{x^2}{na^2t}\right)}$ *satisfies the equation* $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$.

Solution

$$u = kt^{-\frac{1}{2}} e^{-\left(\frac{x^2}{na^2t}\right)}$$

Differentiating u partially w.r.t. t,

$$\frac{\partial u}{\partial t} = -\frac{1}{2} kt^{-\frac{3}{2}} e^{-\left(\frac{x^2}{na^2t}\right)} + kt^{-\frac{1}{2}} e^{-\left(\frac{x^2}{na^2t}\right)} \left(\frac{x^2}{na^2t^2}\right)$$

$$= -\frac{1}{2} t^{-1} u + u \frac{x^2}{na^2t^2}$$

$$= u\left(\frac{x^2}{na^2t^2} - \frac{1}{2t}\right) \qquad \ldots(1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = kt^{-\frac{1}{2}} e^{-\left(\frac{x^2}{na^2t}\right)} \left(-\frac{2x}{na^2t}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{kt^{-\frac{1}{2}}}{na^2t}\left[2x \cdot e^{-\left(\frac{x^2}{na^2t}\right)} \left(-\frac{2x}{na^2t}\right) + e^{-\left(\frac{x^2}{na^2t}\right)}(2)\right]$$

$$= -\frac{2}{na^2t}\left(-\frac{2x^2}{na^2t} u + u\right)$$

$$= u\left(\frac{4x^2}{n^2a^4t^2} - \frac{2}{na^2t}\right)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = u\left(\frac{4x^2}{n^2 a^2 t^2} - \frac{2}{nt}\right) \quad \ldots(2)$$

From Eqs (1) and (2),

$$n = 4$$

Example 7

Find the value of n for which $v = Ae^{-gx}\sin(nt - gx)$ *satisfies the partial differential equation* $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ *where g, A are constants.*

Solution

$$v = Ae^{-gx}\sin(nt - gx)$$

Differentiating v partially w.r.t. t,

$$\frac{\partial v}{\partial t} = Ae^{-gx}[\cos(nt - gx)]\cdot n$$

Differentiating v partially w.r.t. x,

$$\begin{aligned}\frac{\partial v}{\partial x} &= -Age^{-gx}\sin(nt - gx) + [Ae^{-gx}\cos(nt - gx)](-g)\\ &= -Age^{-gx}[\sin(nt - gx) + \cos(nt - gx)]\end{aligned}$$

Differentiating $\frac{\partial v}{\partial x}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= Ag^2 e^{-gx}[\sin(nt - gx) + \cos(nt - gx)]\\ &\quad - Age^{-gx}[-g\cos(nt - gx) + g\sin(nt - gx)]\\ &= Ag^2 e^{-gx}\cdot 2\cos(nt - gx)\end{aligned}$$

Also,

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

$$n\cdot A\cdot e^{-gx}\cos(nt - gx) = 2Ag^2 e^{-gx}\cos(nt - gx)$$

$$\therefore n = 2g^2$$

Example 8

If $u = e^{x^y}$, *find* $\dfrac{\partial^2 u}{\partial y\,\partial x}$.

Solution

$$u = e^{x^y}$$

Differentiating u partially w.r.t. x,

$$\begin{aligned}\frac{\partial u}{\partial x} &= e^{x^y}\,\frac{\partial}{\partial x}(x^y)\\ &= e^{x^y}\cdot yx^{y-1}\end{aligned}$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. y,

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) &= e^{x^y}\,\frac{\partial}{\partial y}(x^y)\cdot yx^{y-1} + e^{x^y}\,x^{y-1} + e^{x^y}\,y\frac{\partial}{\partial y}(x^{y-1})\\ \frac{\partial^2 u}{\partial y\,\partial x} &= e^{x^y}\,x^y\log x\cdot yx^{y-1} + e^{x^y}\,x^{y-1} + e^{x^y}\,yx^{y-1}\log x\\ &= e^{x^y}\,x^{y-1}(yx^y\log x + 1 + y\log x)\end{aligned}$$

Example 9

If $z^3 - zx - y = 0$, *find* $\dfrac{\partial^2 z}{\partial x\,\partial y}$.

Solution

$$z^3 - zx - y = 0 \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. y,

$$\begin{aligned}3z^2\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} - 1 &= 0\\ \frac{\partial z}{\partial y} &= \frac{1}{3z^2 - x}\end{aligned}$$

Differentiating Eq. (1) partially w.r.t. x,

$$\begin{aligned}3z^2\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial x} - z &= 0\\ \frac{\partial z}{\partial x} &= \frac{z}{3z^2 - x}\end{aligned}$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{(3z^2 - x)^2}\left(6z\frac{\partial z}{\partial x} - 1\right)$$

$$= -\frac{1}{(3z^2 - x)^2}\left(\frac{6z^2}{3z^2 - x} - 1\right)$$

$$= -\frac{3z^2 + x}{(3z^2 - x)^3}$$

Example 10

If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1 + x^2 + y^2}}\right)$, *show that* $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1 + x^2 + y^2)^{\frac{3}{2}}}$.

Solution

$$u = \tan^{-1}\left(\frac{xy}{\sqrt{1 + x^2 + y^2}}\right)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \frac{x^2 y^2}{1 + x^2 + y^2}} \frac{\partial}{\partial x}\left(\frac{xy}{\sqrt{1 + x^2 + y^2}}\right)$$

$$= \frac{1 + x^2 + y^2}{1 + x^2 + y^2 + x^2 y^2}\left(\frac{\sqrt{1 + x^2 + y^2}\, y - xy \cdot \frac{1}{2\sqrt{1 + x^2 + y^2}} 2x}{1 + x^2 + y^2}\right)$$

$$= \frac{1 + x^2 + y^2}{1 + x^2 + y^2(1 + x^2)}\left[\frac{(1 + x^2 + y^2 - x^2)y}{(1 + x^2 + y^2)\sqrt{1 + x^2 + y^2}}\right]$$

$$= \frac{(1 + y^2)y}{(1 + x^2)(1 + y^2)\sqrt{1 + x^2 + y^2}}$$

$$= \frac{y}{\left(1 + x^2\right)\sqrt{1 + x^2 + y^2}}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left[\frac{y}{(1+x^2)\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{1}{1+x^2}\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{1+x^2+y^2}}\right)$$

$$= \frac{1}{1+x^2}\left[\frac{\sqrt{1+x^2+y^2}\,(1) - y\dfrac{2y}{2\sqrt{1+x^2+y^2}}}{1+x^2+y^2}\right]$$

$$= \frac{1}{1+x^2}\left[\frac{1+x^2+y^2-y^2}{(1+x^2+y^2)\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}$$

Example 11

If $u = e^{xyz}$, *show that* $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$.

Solution

$$u = e^{xyz}$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = e^{xyz}\cdot xy$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial y\,\partial z} = xe^{xyz} + x^2yze^{xyz}$$

Differentiating $\frac{\partial^2 u}{\partial y\,\partial z}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\,\partial z}\right) = e^{xyz} + xyze^{xyz} + 2xyze^{xyz} + x^2y^2z^2e^{xyz}$$

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$$

Example 12

If $u = e^{3xyz}$, *show that* $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (3 + 27xyz + 27x^2y^2z^2)e^{3xyz}$.

[Summer 2014]

Solution

$$u = e^{3xyz}$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = 3xy\, e^{3xyz}$$

Differentiating $\dfrac{\partial u}{\partial z}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y \partial z} = 3x\left(e^{3xyz} + ye^{3xy} \cdot 3xz\right)$$

$$= e^{3xyz}(3x + 9x^2yz)$$

Differentiating $\dfrac{\partial^2 u}{\partial y \partial z}$ partially w.r.t. x,

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{3xyz}(3 + 18xyz) + (3x + 9x^2yz)e^{3xyz} \cdot 3yz$$

$$= e^{3xyz}(3 + 18xyz + 9xyz + 27x^2y^2z^2)$$

$$= e^{3xyz}(3 + 27xyz + 27x^2y^2z^2)$$

Example 13

If $u = \log(x^2 + y^2)$, *verify that* $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$.

Solution

$$u = \log(x^2 + y^2)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}(2x) = \frac{2x}{x^2 + y^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2}(2y) = \frac{2y}{x^2 + y^2}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y\,\partial x} = 2x\left[-\frac{1}{(x^2+y^2)^2}\right]2y = -\frac{4xy}{(x^2+y^2)^2} \qquad \ldots(1)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x\,\partial y} = 2y\left[-\frac{1}{(x^2+y^2)^2}\right]2x = -\frac{4xy}{(x^2+y^2)^2} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

Example 14

If $u = x^3 y + e^{xy^2}$, *prove that* $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

Solution

$$u = x^3 y + e^{xy^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = x^3 + e^{xy^2}\cdot 2xy$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 3x^2 + 2ye^{xy^2} + 2xye^{xy^2}\cdot y^2$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3x^2 + 2ye^{xy^2}(1+xy^2) \qquad \ldots(1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 3x^2 y + e^{xy^2}\cdot y^2$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 3x^2 + 2ye^{xy^2} + y^2 e^{xy^2}\cdot 2xy$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3x^2 + 2ye^{xy^2}(1+xy^2) \quad ...(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

Example 15

If $z = x + y^x$, *prove that* $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$. **[Winter 2013]**

Solution

$$z = x + y^x$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = 1 + y^x \log y$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = xy^{x-1}$$

Differentiating $\dfrac{\partial z}{\partial x}$ partially w.r.t. y,

$$\begin{aligned}\frac{\partial^2 z}{\partial y\,\partial x} &= y^x \cdot \frac{1}{y} + \log y \cdot xy^{x-1} \\ &= y^{x-1} + x\log y(y^{x-1}) \\ &= y^{x-1}(1 + x\log y) \quad ...(1)\end{aligned}$$

Differentiating $\dfrac{\partial z}{\partial y}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 z}{\partial x\,\partial y} &= y^{x-1} \cdot 1 + xy^{x-1}\log y \\ &= y^{x-1}(1 + x\log y) \quad ...(2)\end{aligned}$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}$$

Example 16

If $z = x^y + y^x$, *prove that* $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$.

Solution

$$z = x^y + y^x$$

$$z = e^{\log x^y} + e^{\log y^x}$$
$$= e^{y\log x} + e^{x\log y}$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = e^{y\log x}\cdot \log x + e^{x\log y}\cdot \frac{x}{y}$$

Differentiating $\dfrac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^{y\log x}\cdot \frac{y}{x}\log x + e^{y\log x}\cdot \frac{1}{x} + e^{x\log y}\cdot \frac{1}{y} + e^{x\log y}\log y\cdot \frac{x}{y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{e^{y\log x}}{x}(y\log x + 1) + \frac{e^{x\log y}}{y}(1 + x\log y) \qquad \ldots(1)$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = e^{y\log x}\cdot \frac{y}{x} + e^{x\log y}\cdot \log y$$

Differentiating $\dfrac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{1}{x}(e^{y\log x} + e^{y\log x}\, y\log x) + e^{x\log y}\cdot \frac{x}{y}\log y + e^{x\log y}\cdot \frac{1}{y}$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{e^{y\log x}}{x}(1 + y\log x) + \frac{e^{x\log y}}{y}(x\log y + 1) \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}$$

Example 17

If $u = (3xy - y^3) - (y^2 - 2x)^{\frac{3}{2}}$, *show that* $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$.

Solution

$$u = (3xy - y^3) - (y^2 - 2x)^{\frac{3}{2}}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = 3x - 3y^2 - \frac{3}{2}(y^2 - 2x)^{\frac{1}{2}}(2y)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 3 - \frac{3y}{2}(y^2 - 2x)^{-\frac{1}{2}}(-2)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 3 + \frac{3y}{\sqrt{y^2 - 2x}} \qquad \ldots(1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 3y - \frac{3}{2}(y^2 - 2x)^{\frac{1}{2}}(-2)$$

$$= 3y + 3(y^2 - 2x)^{\frac{1}{2}}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 3 + \frac{3}{2}(y^2 - 2x)^{-\frac{1}{2}}(2y)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 3 + \frac{3y}{\sqrt{y^2 - 2x}} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

Example 18

If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, *prove that* $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{x^2 - y^2}{x^2 + y^2}$.

Solution

$$z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - y^2 \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{y^2+x^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = 1 - \frac{2y^2}{y^2+x^2} = \frac{y^2+x^2-2y^2}{y^2+x^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} \qquad \ldots(1)$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial y} = 2x\tan^{-1}\left(\frac{y}{x}\right) + x^2\cdot\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) - \frac{y^2}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2x\cdot\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 1$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{2x^2}{x^2+y^2} - 1$$

$$= \frac{2x^2-x^2-y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{x^2 - y^2}{x^2 + y^2}$$

Example 19

If $u = \log(x^2 + y^2 + z^2)$, *prove that* $x\frac{\partial^2 u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y}$.

Solution

$$u = \log(x^2 + y^2 + z^2)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2 + z^2}\cdot 2y$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. z,

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right) = -\frac{2y}{(x^2 + y^2 + z^2)^2}\cdot 2z$$

$$x\frac{\partial^2 u}{\partial z\,\partial y} = -\frac{4xyz}{(x^2 + y^2 + z^2)^2}$$

or
$$x\frac{\partial^2 u}{\partial y\,\partial z} = -\frac{4xyz}{(x^2 + y^2 + z^2)^2} \qquad \ldots(1)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2 + z^2}\cdot 2x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. z,

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = -\frac{2x}{(x^2 + y^2 + z^2)^2}\cdot 2z$$

$$y\frac{\partial^2 u}{\partial z\,\partial x} = -\frac{4xyz}{(x^2 + y^2 + z^2)^2} \qquad \ldots(2)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{2x}{(x^2 + y^2 + z^2)^2}\cdot 2y$$

$$z\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{4xyz}{(x^2 + y^2 + z^2)^2} \qquad \ldots(3)$$

From Eqs (1), (2) and (3),

$$x\frac{\partial^2 u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y}$$

Example 20

If $z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$, *find the value of* $\dfrac{\partial^2 z}{\partial x^2} - a^2 \dfrac{\partial^2 z}{\partial y^2}$.

Solution

$$z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = a\sec^2(y + ax) - \frac{3}{2}a(y - ax)^{\frac{1}{2}}$$

Differentiating $\dfrac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = 2a^2 \sec^2(y + ax)\tan(y + ax) + \frac{3}{4}a^2 (y - ax)^{-\frac{1}{2}} \qquad \ldots(1)$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = \sec^2(y + ax) + \frac{3}{2}(y - ax)^{\frac{1}{2}}$$

Differentiating $\dfrac{\partial z}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y + ax)\tan(y + ax) + \frac{3}{4}(y - ax)^{-\frac{1}{2}} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$$

Example 21

If $a^2 x^2 + b^2 y^2 = c^2 z^2$, *evaluate* $\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}$.

Solution

$$a^2 x^2 + b^2 y^2 = c^2 z^2$$

Differentiating partially w.r.t. x,

$$2a^2 x = 2c^2 z \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{a^2 x}{c^2 z}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2}\left(\frac{1}{z} - \frac{x}{z^2}\cdot\frac{\partial z}{\partial x}\right)$$

$$= \frac{a^2}{c^2 z}\left(1 - \frac{x}{z}\cdot\frac{a^2 x}{c^2 z}\right)$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2 z}\left(1 - \frac{a^2 x^2}{c^2 z^2}\right)$$

Similarly,

$$\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z}\left(1 - \frac{b^2 y^2}{c^2 z^2}\right)$$

Hence,

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2 z}\left(2 - \frac{a^2 x^2 + b^2 y^2}{c^2 z^2}\right)$$

$$= \frac{1}{c^2 z}\left(2 - \frac{c^2 z^2}{c^2 z^2}\right)$$

$$= \frac{1}{c^2 z}(2-1)$$

$$= \frac{1}{c^2 z}$$

Example 22

If $u = \log(x^3 + y^3 - x^2 y - xy^2)$, *prove that* $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$.

Solution

$$u = \log(x^3 + y^3 - x^2 y - xy^2)$$

$$= \log[(x+y)(x^2 - xy + y^2) - xy(x+y)]$$

$$= \log[(x+y)(x^2 - xy + y^2 - xy)]$$

$$= \log[(x+y)(x-y)^2]$$

$$= \log(x+y) + 2\log(x-y)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2}{x-y}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x\, \partial y} = -\frac{1}{(x+y)^2} + \frac{2}{(x-y)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\, \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$$

Example 23

If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, *prove that* $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Solution

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) v$$

where $$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

Differentiating u partially w.r.t. x, y, and z,

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2 + y^2 + z^2) - 3(xy + yz + zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} \cdot \frac{(x+y+z)}{(x+y+z)}$$

$$= \frac{3(x^3 + y^3 + z^3 - 3xyz)}{(x^3 + y^3 + z^3 - 3xyz)(x+y+z)}$$

$$= \frac{3}{x+y+z}$$

Hence,
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= -\frac{9}{(x+y+z)^2}$$

Example 24

If $u = 3\,(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ *and* $a^2 + b^2 + c^2 = 1$, *show that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

Solution

$$u = 3\,(ax + by + cz)^2 - (x^2 + y^2 + z^2)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = 6(ax + by + cz)a - 2x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = 6a \cdot a - 2$$

$$= 6a^2 - 2$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = 6(ax+by+cz)b - 2y$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = 6b \cdot b - 2$$
$$= 6b^2 - 2$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = 6(ax+by+cz)c - 2z$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. z,

$$\frac{\partial^2 u}{\partial z^2} = 6c \cdot c - 2$$
$$= 6c^2 - 2$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(a^2+b^2+c^2) - 6$$

$$= 6(1) - 6 \qquad [\because a^2+b^2+c^2 = 1]$$
$$= 0$$

Example 25

If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, *find the value of* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Solution

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = -\frac{1}{2(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot 2x$$

$$= -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\left[\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3x \cdot 2x}{2(x^2+y^2+z^2)^{\frac{5}{2}}}\right]$$

$$= -\frac{1}{(x^2+y^2+z^2)^{\frac{5}{2}}}(x^2+y^2+z^2-3x^2)$$

$$= -\frac{(-2x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = -\frac{(x^2-2y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

and $$\frac{\partial^2 u}{\partial z^2} = -\frac{(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Hence, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{(-2x^2+2y^2+2z^2+2x^2-2y^2-2z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$= 0$$

Example 26

If $u = z\tan^{-1}\left(\frac{x}{y}\right)$, *find the value of* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Solution

$$u = z\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = z\frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= \frac{zy}{y^2+x^2}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{yz \cdot 2x}{(x^2+y^2)^2}$$

$$= -\frac{2xyz}{(x^2+y^2)^2}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{z}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)$$

$$= -\frac{xz}{y^2+x^2}$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{xz\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{2xyz}{(x^2+y^2)^2}$$

Differentiating u partially w.r.t. z,

$$\frac{\partial u}{\partial z} = \tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. z,

$$\frac{\partial^2 u}{\partial z^2} = 0$$

Hence, $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = -\frac{2xyz}{(x^2+y^2)^2}+\frac{2xyz}{(x^2+y^2)^2}$$

$$= 0$$

Example 27

If $v = (1-2xy+y^2)^{-\frac{1}{2}}$, *find the value of* $\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right]+\frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right)$.

Solution

$$v = (1-2xy+y^2)^{-\frac{1}{2}}$$

Differentiating v partially w.r.t. x,

$$\frac{\partial v}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2y)$$

$$(1-x^2)\frac{\partial v}{\partial x} = y(1-x^2)(1-2xy+y^2)^{-\frac{3}{2}}$$

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] = y\frac{\partial}{\partial x}\left[(1-x^2)(1-2xy+y^2)^{-\frac{3}{2}}\right]$$

$$= y\left[(-2x)(1-2xy+y^2)^{-\frac{3}{2}}-\frac{3}{2}(1-x^2)(1-2xy+y^2)^{-\frac{5}{2}}(-2y)\right]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}[-2x(1-2xy+y^2)+3y(1-x^2)]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(-2x+4x^2y-2xy^2+3y-3x^2y)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(-2x+x^2y-2xy^2+3y) \qquad \ldots(1)$$

Differentiating v partially w.r.t. y,

$$\frac{\partial v}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-\frac{3}{2}}(-2x+2y)$$

$$y^2\frac{\partial v}{\partial y} = -y^2(-x+y)(1-2xy+y^2)^{-\frac{3}{2}}$$

$$\frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right) = -2y(-x+y)(1-2xy+y^2)^{-\frac{3}{2}} - y^2(1-2xy+y^2)^{-\frac{3}{2}}$$
$$+\frac{3y^2}{2}(-x+y)(1-2xy+y^2)^{-\frac{5}{2}}(-2x+2y)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}[2(x-y)(1-2xy+y^2)-y(1-2xy+y^2)+3y(-x+y)^2]$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(2x-4x^2y+2xy^2-2y+4xy^2-2y^3-y$$
$$+2xy^2-y^3+3yx^2+3y^3-6xy^2)$$

$$= y(1-2xy+y^2)^{-\frac{5}{2}}(2x-x^2y+2xy^2-3y) \qquad \ldots(2)$$

Adding Eqs (1) and (2),

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left(y^2\frac{\partial v}{\partial y}\right) = 0.$$

Example 28

If $u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$, *show that* $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0.$

Solution

$$u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$$

Differentiating u partially w.r.t. r,

$$\frac{\partial u}{\partial r} = (nar^{n-1} - bnr^{-n-1})(\cos n\theta + \sin n\theta)$$

Differentiating $\dfrac{\partial u}{\partial r}$ partially w.r.t. r,

$$\frac{\partial^2 u}{\partial r^2} = n[a(n-1)r^{n-2} + b(n+1)r^{-n-2}](\cos n\theta + \sin n\theta)$$

Differentiating u partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta} = (ar^n + br^{-n})(-n\sin n\theta + n\cos n\theta)$$

Differentiating $\frac{\partial u}{\partial \theta}$ partially w.r.t. θ,

$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} &= (ar^n + br^{-n})(-n^2 \cos n\theta - n^2 \sin n\theta)\\ &= -n^2(ar^n + br^{-n})(\cos n\theta + \sin n\theta)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= n[a(n-1)r^{n-2} + b(n+1)r^{-n-2}](\cos n\theta + \sin n\theta)\\ &\quad + n(ar^{n-2} - br^{-n-2})(\cos n\theta + \sin n\theta) - \frac{n^2}{r^2}(ar^n + br^{-n})(\cos n\theta + \sin n\theta)\\ &= (\cos n\theta + \sin n\theta)r^{n-2}(an^2 - an + bn^2 + bn + an - bn - an^2 - bn^2)\\ &= 0\end{aligned}$$

Example 29

If $x = r\cos\theta,\ y = r\sin\theta,$ *show that* $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1.$

Solution

$$x = r\cos\theta,\ y = r\sin\theta$$

$$x^2 + y^2 = r^2 \qquad \ldots(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$2x = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiating Eq. (1) partially w.r.t. y,

$$2y = 2r\frac{\partial r}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

Hence, $$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Example 30

If $x = \cos\theta - r\sin\theta,\ y = \sin\theta + r\cos\theta,$ *show that*

(i) $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ *(ii)* $\dfrac{\partial \theta}{\partial x} = -\dfrac{\cos\theta}{r}.$

Solution

(i)

$$x = \cos\theta - r\sin\theta \quad \ldots(1)$$

$$y = \sin\theta + r\cos\theta \quad \ldots(2)$$

$$x^2 = \cos^2\theta - 2r\cos\theta\sin\theta + r^2\sin^2\theta$$

$$y^2 = \sin^2\theta + 2r\sin\theta\cos\theta + r^2\cos^2\theta$$

$$x^2 + y^2 = 1 + r^2 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$2x = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

(ii) Multiplying Eq. (1) by $\cos\theta$ and Eq. (2) by $\sin\theta$,

$$x\cos\theta = \cos^2\theta - r\sin\theta\cos\theta \quad \ldots(4)$$

$$y\sin\theta = \sin^2\theta + r\cos\theta\sin\theta \quad \ldots(5)$$

Adding Eqs (4) and (5),

$$x\cos\theta + y\sin\theta = 1$$

$$x\cot\theta + y = \operatorname{cosec}\theta \quad \ldots(6)$$

Differentiating Eq. (6) partially w.r.t. x,

$$\cot\theta - x\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x} = -\cot\theta\operatorname{cosec}\theta\frac{\partial\theta}{\partial x}$$

$$\cot\theta + \cot\theta\operatorname{cosec}\theta\frac{\partial\theta}{\partial x} = x\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x}$$

$$= (\cos\theta - r\sin\theta)\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x}$$

$$= \cos\theta\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x} - r\sin\theta\operatorname{cosec}^2\theta\frac{\partial\theta}{\partial x}$$

$$= \cot\theta\operatorname{cosec}\theta\frac{\partial\theta}{\partial x} - r\operatorname{cosec}\theta\frac{\partial\theta}{\partial x}$$

$$\cot\theta = -r\operatorname{cosec}\theta\frac{\partial\theta}{\partial x}$$

$$\frac{\partial \theta}{\partial x} = -\frac{\cot \theta}{r \operatorname{cosec} \theta} = -\frac{\cos \theta}{r}$$

Example 31

Show that $\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}$ *and* $\frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$ *and hence, show that* $\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = 0$ *if* $x = e^{r\cos\theta}\cos(r\sin\theta)$ *and* $y = e^{r\cos\theta}\sin(r\sin\theta)$.

Solution

$$x = e^{r\cos\theta}\cos(r\sin\theta)$$

Differentiating x partially w.r.t. r,

$$\begin{aligned}\frac{\partial x}{\partial r} &= e^{r\cos\theta}\cdot\cos\theta\cos(r\sin\theta) + e^{r\cos\theta}[-\sin(r\sin\theta)]\sin\theta \\ &= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)] \\ &= e^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad \ldots(1)\end{aligned}$$

Now, $\quad y = e^{r\cos\theta}\sin(r\sin\theta)$

Differentiating y partially w.r.t. r,

$$\begin{aligned}\frac{\partial y}{\partial r} &= e^{r\cos\theta}\cos\theta\sin(r\sin\theta) + e^{r\cos\theta}\cos(r\sin\theta)\sin\theta \\ &= e^{r\cos\theta}\sin(r\sin\theta + \theta) \qquad \ldots(2)\end{aligned}$$

Differentiating x partially w.r.t. θ,

$$\begin{aligned}\frac{\partial x}{\partial \theta} &= e^{r\cos\theta}(-r\sin\theta)\cos(r\sin\theta) + e^{r\cos\theta}[-\sin(r\sin\theta)]\cdot(r\cos\theta) \\ &= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \qquad \ldots(3)\end{aligned}$$

Differentiating y partially w.r.t. θ,

$$\begin{aligned}\frac{\partial y}{\partial \theta} &= e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta) + e^{r\cos\theta}\cos(r\sin\theta)\cdot r\cos\theta \\ &= re^{r\cos\theta}\cos(\theta + r\sin\theta) \qquad \ldots(4)\end{aligned}$$

From Eqs (1) and (4),

$$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta} \qquad \ldots(5)$$

From Eqs (2) and (3),

$$\frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$$

$$\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r} \qquad \ldots(6)$$

Differentiating Eq. (5) partially w.r.t. r,

$$\frac{\partial^2 x}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial y}{r\partial \theta}\right)$$

$$= \frac{-1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta}$$

Differentiating Eq. (6) partially w.r.t. θ,

$$\frac{\partial^2 x}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-r\frac{\partial y}{\partial r}\right)$$

$$= -r\frac{\partial^2 y}{\partial \theta \partial r}$$

$$= -r\frac{\partial^2 y}{\partial r \partial \theta}$$

Hence, $\displaystyle \frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = \frac{-1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta} + \frac{1}{r^2}\frac{\partial y}{\partial \theta} - \frac{1}{r}\frac{\partial^2 y}{\partial r\partial \theta}$

$$= 0$$

Example 32

If $\theta = t^n e^{\frac{-r^2}{4t}}$ *then find n so that* $\displaystyle \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$.

[Winter 2016; Summer 2016]

Solution

$$\theta = t^n e^{\frac{-r^2}{4t}}$$

Differentiating θ partially w.r.t. t,

$$\frac{\partial \theta}{\partial t} = nt^{n-1}e^{\frac{-r^2}{4t}} + t^n e^{\frac{-r^2}{4t}}\left(\frac{r^2}{4t^2}\right)$$

$$= e^{\frac{-r^2}{4t}}\left(nt^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)$$

Differentiating θ partially w.r.t. r,

$$\frac{\partial \theta}{\partial r} = t^n e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{t^{n-1}}{2}r^3 e^{\frac{-r^2}{4t}}\right)$$

$$= -\frac{t^{n-1}}{2}\left[3r^2 e^{\frac{-r^2}{4t}} + r^3 e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=e^{\frac{-r^2}{4t}}\left(-\frac{3}{2}t^{n-1}+\frac{r^2}{4}t^{n-2}\right)$$

Given $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t},$$

$$e^{\frac{-r^2}{4t}}\left(-\frac{3}{2}t^{n-1}+\frac{r^2}{4}t^{n-2}\right)=e^{\frac{-r^2}{4t}}\left(nt^{n-1}+\frac{1}{4}r^2t^{n-2}\right)$$

$$-\frac{3}{2}t^{n-1}=nt^{n-1}$$

$$n=-\frac{3}{2}$$

Example 33

Find the value of n so that $v = r^n\,(3\cos^2\theta - 1)$ *satisfies the equation* $\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=0.$

Solution

$$v = r^n\,(3\cos^2\theta - 1)$$

Differentiating v partially w.r.t. r,

$$\frac{\partial v}{\partial r}=nr^{n-1}(3\cos^2\theta-1)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)=\frac{\partial}{\partial r}[nr^{n+1}(3\cos^2\theta-1)]$$

$$=n(n+1)r^n(3\cos^2\theta-1) \qquad \ldots(1)$$

Differentiating v partially w.r.t. θ,

$$\frac{\partial v}{\partial\theta}=r^n\cdot 6\cos\theta(-\sin\theta)$$

$$=-3r^n\sin 2\theta$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=\frac{\partial}{\partial\theta}(-3r^n\sin\theta\cdot\sin 2\theta)$$

$$=-3r^n(\cos\theta\sin 2\theta+2\sin\theta\cos 2\theta)$$

$$=-3r^n[\cos\theta\cdot 2\sin\theta\cos\theta+2\sin\theta(2\cos^2\theta-1)]$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=-3r^n(2\cos^2\theta+4\cos^2\theta-2)$$

$$=-6r^n(3\cos^2\theta-1)$$

Given $$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=0,$$

$$n(n+1)\,r^n\,(3\cos^2\theta-1)-6r^n\,(3\cos^2\theta-1)=0$$
$$n(n+1)-6=0$$
$$n^2+n-6=0$$
$$(n+3)(n-2)=0$$
$$n=-3,\ 2$$

Example 34

If $x^x\,y^y\,z^z=c$, show that at $x=y=z$,

(*i*) $\dfrac{\partial^2 z}{\partial x\,\partial y}=-(x\log ex)^{-1}$ (*ii*) $\dfrac{\partial^2 z}{\partial x^2}-2xy\dfrac{\partial^2 z}{\partial x\,\partial y}+\dfrac{\partial^2 z}{\partial y^2}=\dfrac{2(x^2-2)}{x(1+\log x)}$.

Solution

(i) $$x^x\,y^y\,z^z=c$$

Taking logarithm on both the sides,

$$\log x^x+\log y^y+\log z^z=\log c$$
$$x\log x+y\log y+z\log z=\log c \qquad \ldots(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$x\cdot\frac{1}{x}+\log x+\frac{\partial z}{\partial x}\cdot\log z+z\cdot\frac{1}{z}\frac{\partial z}{\partial x}=0 \qquad [\because z=f(x,y)]$$
$$\frac{\partial z}{\partial x}=-\frac{1+\log x}{1+\log z}$$

Differentiating $\dfrac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=-(1+\log x)\left[-\frac{1}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x}\right]$$
$$\frac{\partial^2 z}{\partial y\,\partial x}=\frac{(1+\log x)}{z(1+\log z)^2}\left(-\frac{1+\log x}{1+\log z}\right)$$
$$\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{(1+\log x)^2}{z(1+\log z)^3}$$

At $x=y=z$,

$$\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{(1+\log x)^2}{x(1+\log x)^3}$$

$$= -\frac{1}{x(1+\log x)}$$
$$= -[x(\log e + \log x)]^{-1} \qquad [\because \log e = 1]$$
$$= -(x \log ex)^{-1}.$$

(ii) Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{1+\log x}{1+\log z}\right)$$
$$= \frac{(1+\log x)}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x} - \frac{1}{x(1+\log z)}$$
$$= -\frac{(1+\log x)}{z(1+\log z)^2}\cdot\frac{(1+\log x)}{(1+\log z)} - \frac{1}{x(1+\log z)}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{x(1+\log x)}$$

Similarly,

$$\frac{\partial^2 z}{\partial y^2} = \frac{-(1+\log y)^2}{z(1+\log z)^3} - \frac{1}{y(1+\log z)}$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial y^2} = -\frac{2}{x(1+\log x)}$$

Hence, $\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = -\frac{2}{x(1+\log x)} - 2xy\left[-\frac{1}{x(1+\log x)}\right] + \left[-\frac{2}{x(1+\log x)}\right]$

$$= \frac{2(xy-2)}{x(1+\log x)}$$
$$= \frac{2(x^2-2)}{x(1+\log x)} \qquad [\because x = y = z]$$

Example 35

If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Solution

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \qquad ...(1)$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x}-\frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}-\frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x}=0$$

$$\frac{\partial u}{\partial x}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=\frac{2x}{a^2+u}$$

$$\frac{\partial u}{\partial x}\cdot p=\frac{2x}{(a^2+u)}$$

where $$p=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

$$\frac{\partial u}{\partial x}=\frac{2x}{(a^2+u)p}$$

Similarly, $$\frac{\partial u}{\partial y}=\frac{2y}{(b^2+u)p}$$

$$\frac{\partial u}{\partial z}=\frac{2z}{(c^2+u)p}$$

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{p^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$=\frac{4}{p^2}(p)$$

$$=\frac{4}{p} \qquad \ldots(2)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{2}{p}\left(\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right)$$

$$=\frac{2}{p}(1)$$

$$=\frac{2}{p} \qquad \ldots(3)$$

From Eqs (2) and (3),

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$

Example 36

If $z=e^{ax+by}f(ax-by)$, *prove that* $b\dfrac{\partial z}{\partial x}+a\dfrac{\partial z}{\partial y}=2abz.$

Solution

$$z = e^{ax+by} f(ax - by)$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = ae^{ax+by} f(ax-by) + ae^{ax+by} f'(ax-by)$$
$$= az + ae^{ax+by} f'(ax-by)$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = be^{ax+by} f(ax-by) - be^{ax+by} f'(ax-by)$$
$$= bz - be^{ax+by} f'(ax-by)$$

Hence, $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = abz + abe^{ax+by} f'(ax-by) + abz - abe^{ax+by} f'(ax-by)$

$$= 2abz$$

Example 37

If $u = \phi(x + ky) + \psi(x - ky)$, *show that* $\dfrac{\partial^2 u}{\partial y^2} = k^2 \dfrac{\partial^2 u}{\partial x^2}$.

Solution

$$u = \phi(x + ky) + \psi(x - ky)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \phi'(x+ky)\cdot 1 + \psi'(x-ky)\cdot 1$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \phi''(x+ky) + \psi''(x-ky) \qquad \ldots(1)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \phi'(x+ky)\cdot k + \psi'(x-ky)\cdot(-k)$$

Differentiating $\dfrac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \phi''(x+ky)\cdot k^2 + \psi''(x-ky)(-k)^2$$
$$= k^2[\phi''(x+ky) + \psi''(x-ky)] \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 u}{\partial y^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

Example 38

If $u = xf(x + y) + y\phi(x + y)$, *show that* $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

Solution

$$u = xf(x + y) + y\,\phi(x + y),$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = f(x+y) + xf'(x+y) + y\phi'(x+y)$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= f'(x+y) + f'(x+y) + xf''(x+y) + y\phi''(x+y)\\ &= 2f'(x+y) + xf''(x+y) + y\phi''(x+y)\end{aligned}$$

Differentiating $\dfrac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = f'(x+y) + xf''(x+y) + y\phi''(x+y) + \phi'(x+y)$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = xf'(x+y) + \phi(x+y) + y\phi'(x+y)$$

Differentiating $\dfrac{\partial u}{\partial y}$ partially w.r.t. y,

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= xf''(x+y) + \phi'(x+y) + \phi'(x+y) + y\phi''(x+y)\\ &= xf''(x+y) + 2\phi'(x+y) + y\phi''(x+y)\end{aligned}$$

Hence, $\quad \dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2}$

$$\begin{aligned}&= 2f'(x+y) + xf''(x+y) + y\phi''(x+y) - 2f'(x+y) - 2xf''(x+y)\\ &\quad - 2y\phi''(x+y) - 2\phi'(x+y) + xf''(x+y) + 2\phi'(x+y) + y\phi''(x+y)\\ &= 0\end{aligned}$$

Example 39

If $u = f\left(\sqrt{x^2+y^2}\right)$, *prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\sqrt{x^2+y^2}} f'\left(\sqrt{x^2+y^2}\right) + f''\left(\sqrt{x^2+y^2}\right).$$

Solution

Let $\qquad \sqrt{x^2+y^2} = r$

$$u = f(r)$$

Differentiating u partially w.r.t. x,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} f(r) \\
&= \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} \\
&= f'(r) \cdot \frac{\partial}{\partial x} \sqrt{x^2+y^2} \\
&= f'(r) \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x
\end{aligned}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= f''(r) \frac{\partial r}{\partial x} \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{f'(r)}{\sqrt{x^2+y^2}} - x f'(r) \frac{1}{2(x^2+y^2)^{\frac{3}{2}}} \cdot 2x \\
&= f''(r) \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{f'(r)}{\sqrt{x^2+y^2}} - \frac{x^2 f'(r)}{(x^2+y^2)^{\frac{3}{2}}} \\
&= f''(r) \frac{x^2}{x^2+y^2} + \frac{f'(r)}{\sqrt{x^2+y^2}} - \frac{x^2 f'(r)}{(x^2+y^2)^{\frac{3}{2}}}
\end{aligned}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = f''(r) \frac{y^2}{x^2+y^2} + \frac{f'(r)}{\sqrt{x^2+y^2}} - \frac{y^2 f'(r)}{(x^2+y^2)^{\frac{3}{2}}}$$

Hence,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= f''(r) \frac{(x^2+y^2)}{x^2+y^2} + \frac{2f'(r)}{\sqrt{x^2+y^2}} - \frac{(x^2+y^2) f'(r)}{(x^2+y^2)^{\frac{3}{2}}} \\
&= f''(r) + \frac{2f'(r)}{\sqrt{x^2+y^2}} - \frac{f'(r)}{\sqrt{x^2+y^2}}
\end{aligned}$$

$$= f''(r) + \frac{f'(r)}{\sqrt{x^2+y^2}}$$

$$= f''\left(\sqrt{x^2+y^2}\right) + \frac{f'\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$$

Example 40

If $u = f\left(\frac{x^2}{y}\right)$, *show that* $x^2 \frac{\partial^2 u}{\partial x^2} + 3xy \frac{\partial^2 u}{\partial x\, \partial y} + 2y^2 \frac{\partial^2 u}{\partial y^2} = 0.$

Solution

$$u = f\left(\frac{x^2}{y}\right)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = f'\left(\frac{x^2}{y}\right)\frac{\partial}{\partial x}\left(\frac{x^2}{y}\right)$$

$$= f'\left(\frac{x^2}{y}\right)\left(\frac{2x}{y}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{2}{y} f'\left(\frac{x^2}{y}\right) + f''\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial x}\left(\frac{x^2}{y}\right)\cdot\left(\frac{2x}{y}\right)$$

$$= \frac{2}{y} f'\left(\frac{x^2}{y}\right) + f''\left(\frac{x^2}{y}\right)\cdot\left(\frac{2x}{y}\right)^2$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = f'\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x^2}{y}\right)$$

$$= f'\left(\frac{x^2}{y}\right)\cdot\left(-\frac{x^2}{y^2}\right)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2}{y^3} f'\left(\frac{x^2}{y}\right) + \left(-\frac{x^2}{y^2}\right) f''\left(\frac{x^2}{y}\right)\cdot\frac{\partial}{\partial y}\left(\frac{x^2}{y}\right)$$

$$= \frac{2x^2}{y^3} f'\left(\frac{x^2}{y}\right) + \left(\frac{x^2}{y^2}\right)^2 f''\left(\frac{x^2}{y}\right)$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{2x}{y^2} f'\left(\frac{x^2}{y}\right) + \frac{2x}{y} f''\left(\frac{x^2}{y}\right)\cdot\left(-\frac{x^2}{y^2}\right)$$

$$\begin{aligned} x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x\,\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} &= \frac{2x^2}{y} f'\left(\frac{x^2}{y}\right) + \frac{4x^4}{y^2} f''\left(\frac{x^2}{y}\right) - \frac{6x^2}{y} f'\left(\frac{x^2}{y}\right) \\ &\quad - \frac{6x^4}{y^2} f''\left(\frac{x^2}{y}\right) + \frac{4x^2}{y} f'\left(\frac{x^2}{y}\right) + \frac{2x^4}{y^2} f''\left(\frac{x^2}{y}\right) \\ &= 0 \end{aligned}$$

Example 41

If $u = e^{xyz} f\left(\frac{xy}{z}\right)$, *prove that* $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2xyzu$ *and* $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyzu$ *and hence, show that* $x\frac{\partial^2 u}{\partial z\,\partial x} = y\frac{\partial^2 u}{\partial z\,\partial y}$.

Solution

$$u = e^{xyz} f\left(\frac{xy}{z}\right)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = e^{xyz} yz \cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)$$

$$\frac{\partial u}{\partial y} = e^{xyz} xz \cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)$$

$$\frac{\partial u}{\partial z} = e^{xyz} xy \cdot f\left(\frac{xy}{z}\right) + e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right)$$

(i) $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z}$

$$= e^{xyz} xyz \cdot f\left(\frac{xy}{z}\right) + \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) + e^{xyz} xyz \cdot f\left(\frac{xy}{z}\right) - \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right)$$

$$= 2xyze^{xyz} \cdot f\left(\frac{xy}{z}\right)$$

$$= 2xyzu.$$

(ii) $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$= e^{xyz} xyz \cdot f\left(\frac{xy}{z}\right) + \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right) + e^{xyz} xyz \cdot f\left(\frac{xy}{z}\right) - \frac{xy}{z} e^{xyz} f'\left(\frac{xy}{z}\right)$$

$$= 2xyze^{xyz} f\left(\frac{xy}{z}\right)$$
$$= 2xyzu$$

(iii) Differentiating $\frac{\partial u}{\partial z}$ w.r.t. x,

$$\frac{\partial^2 u}{\partial z\,\partial x} = e^{xyz} yz \cdot xy \cdot f\left(\frac{xy}{z}\right) + e^{xyz} y \cdot f\left(\frac{xy}{z}\right) + e^{xyz} xy\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)$$
$$+ e^{xyz} yz\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right) + e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{y}{z}\right)\left(\frac{-xy}{z^2}\right)$$
$$+ e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{y}{z^2}\right)$$

$$x\frac{\partial^2 u}{\partial z \partial x} = e^{xyz}\left[x^2y^2z \cdot f\left(\frac{xy}{z}\right) + xy \cdot f\left(\frac{xy}{z}\right) - \frac{x^2y^2}{z^3} f''\left(\frac{xy}{z}\right) - \frac{xy}{z^2} f'\left(\frac{xy}{z}\right)\right] \quad \ldots(1)$$

Differentiating $\frac{\partial u}{\partial z}$ w.r.t. y,

$$\frac{\partial^2 u}{\partial z\,\partial y} = e^{xyz} xz \cdot xy \cdot f\left(\frac{xy}{z}\right) + e^{xyz} x \cdot f\left(\frac{xy}{z}\right) + e^{xyz} xy\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)$$
$$+ e^{xyz} xz\left[f'\left(\frac{xy}{z}\right)\right]\left(\frac{-xy}{z^2}\right) + e^{xyz}\left[f''\left(\frac{xy}{z}\right)\right]\left(\frac{x}{z}\right)\left(\frac{-xy}{z^2}\right)$$
$$+ e^{xyz}\left[f'\left(\frac{xy}{z}\right)\right]\left(-\frac{x}{z^2}\right)$$

$$y\frac{\partial^2 u}{\partial z\,\partial y} = e^{xyz}\left[x^2y^2z \cdot f\left(\frac{xy}{z}\right) + xy \cdot f\left(\frac{xy}{z}\right) - \frac{x^2y^2}{z^3} f''\left(\frac{xy}{z}\right) - \frac{xy}{z^2} f'\left(\frac{xy}{z}\right)\right] \quad \ldots(2)$$

From Eqs (1) and (2),

$$x\frac{\partial^2 u}{\partial z\,\partial x} = y\frac{\partial^2 u}{\partial z\,\partial y}$$

Example 42

If $u = r^m, r = \sqrt{x^2 + y^2 + z^2}$, *show that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$.

Solution

$$u = r^m$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = mr^{m-1}\frac{\partial r}{\partial x} \quad \ldots(1)$$

But
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$r^2 = x^2 + y^2 + z^2$$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$
$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Substituting in Eq. (1),
$$\frac{\partial u}{\partial x} = mr^{m-1}\frac{x}{r}$$
$$= mr^{m-2}x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = m\left[r^{m-2} + (m-2)r^{m-3}\frac{\partial r}{\partial x}x\right]$$
$$= m\left[r^{m-2} + (m-2)r^{m-3}\frac{x}{r}x\right]$$
$$= m[r^{m-2} + (m-2)r^{m-4}x^2] \qquad ...(2)$$

Similarly,
$$\frac{\partial^2 u}{\partial y^2} = m[r^{m-2} + (m-2)r^{m-4}y^2] \qquad ...(3)$$
$$\frac{\partial^2 u}{\partial z^2} = m[r^{m-2} + (m-2)r^{m-4}z^2] \qquad ...(4)$$

Adding Eqs (2), (3) and (4),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3mr^{m-2} + m(m-2)r^{m-4}(x^2 + y^2 + z^2)$$
$$= 3mr^{m-2} + m(m-2)r^{m-4}\cdot r^2$$
$$= r^{m-2}(3m + m^2 - 2m)$$
$$= r^{m-2}(m + m^2)$$
$$= m(m+1)r^{m-2}$$

Example 43

If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r).$$ **[Summer 2015]**

Solution

$$u = f(r)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r)$$

$$= \frac{\partial}{\partial r} f(r) \cdot \frac{\partial r}{\partial x}$$

$$= f'(r) \cdot \frac{\partial r}{\partial x} \qquad ...(1)$$

But $\qquad r^2 = x^2 + y^2 + z^2$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Subsituting in Eq. (1),

$$\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right]$$

$$= f''(r)\frac{\partial r}{\partial x} \cdot \frac{x}{r} + \frac{f'(r)}{r} + xf'(r)\left(-\frac{1}{r^2}\right) \cdot \frac{\partial r}{\partial x}$$

$$= f''(r)\frac{x}{r}\frac{x}{r} + \frac{f'(r)}{r} - \frac{x}{r^2} f'(r) \cdot \frac{x}{r}$$

$$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3} f'(r) \qquad \ldots(2)$$

Similarly, $\qquad \frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - \frac{y^2}{r^3} f'(r) \qquad \ldots(3)$

and $\qquad \frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - \frac{z^2}{r^3} f'(r) \qquad \ldots(4)$

Adding Eqs (2), (3) and (4),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}(x^2 + y^2 + z^2) + \frac{3f'(r)}{r} - \frac{(x^2 + y^2 + z^2)}{r^3} f'(r)$$

$$= \frac{f''(r)}{r^2} \cdot r^2 + \frac{3f'(r)}{r} - \frac{r^2}{r^3} f'(r)$$

$$= f''(r) + \frac{2f'(r)}{r}$$

Example 44

If $u = f(r^2)$ where $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=4r^2 f''(r^2)+6f'(r^2).$$

Solution

$$u = f(r^2)$$

Differentiating u partially w.r.t. x,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} f(r^2)\\ &= \frac{\partial}{\partial x} f(l), \quad \text{where } r^2 = l\\ &= \frac{\partial}{\partial l} f(l)\cdot\frac{\partial l}{\partial x}\\ &= f'(l)\frac{\partial l}{\partial x}\\ &= f'(r^2)\frac{\partial r^2}{\partial x}\end{aligned}$$

$$\therefore \frac{\partial u}{\partial x} = f'(r^2)\cdot 2r\frac{\partial r}{\partial x} \qquad \text{...(1)}$$

But $\qquad r^2 = x^2 + y^2 + z^2$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Substituting in Eq. (1),

$$\begin{aligned}\frac{\partial u}{\partial x} &= f'(r^2)\cdot 2r\frac{x}{r}\\ &= 2x\,f'(r^2)\end{aligned}$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= 2f'(r^2)+2x\frac{\partial f'(r^2)}{\partial x}\\ &= 2f'(r^2)+2x\,f''(r^2)\cdot 2r\frac{\partial r}{\partial x}\\ &= 2f'(r^2)+2x\,f''(r^2)\cdot 2r\frac{x}{r}\\ &= 2f'(r^2)+4x^2 f''(r^2)\end{aligned} \qquad \text{...(2)}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = 2f'(r^2) + 4y^2 f''(r^2) \qquad \ldots(3)$$

and $$\frac{\partial^2 u}{\partial z^2} = 2f'(r^2) + 4z^2 f''(r^2) \qquad \ldots(4)$$

Adding Eqs (2), (3) and (4),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6f'(r^2) + 4(x^2 + y^2 + z^2) f''(r^2)$$

$$= 6f'(r^2) + 4r^2 f''(r^2)$$

Example 45

If $v = x \log(x + r) - r$ *where* $r^2 = x^2 + y^2$, *prove that* $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{1}{x + r}$.

Solution

$$v = x \log(x + r) - r$$

Differentiating v partially w.r.t. x,

$$\frac{\partial v}{\partial x} = \log(x + r) + \frac{x}{x + r}\left(1 + \frac{\partial r}{\partial x}\right) - \frac{\partial r}{\partial x}$$

But $$r^2 = x^2 + y^2$$

Differentiating r^2 partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\therefore \quad \frac{\partial v}{\partial x} = \log(x + r) + \frac{x}{x + r}\left(1 + \frac{x}{r}\right) - \frac{x}{r}$$

$$= \log(x + r) + \frac{x}{(x + r)} \cdot \frac{(r + x)}{r} - \frac{x}{r}$$

$$= \log(x + r) + \frac{x}{r} - \frac{x}{r}$$

$$= \log(x + r)$$

Differentiating $\frac{\partial v}{\partial x}$ partially w.r.t. x,

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= \frac{1}{x+r}\left(1+\frac{\partial r}{\partial x}\right)\\ &= \frac{1}{x+r}\left(1+\frac{x}{r}\right)\\ &= \frac{1}{r}\end{aligned}$$

Differentiating v partially w.r.t. y,

$$\begin{aligned}\frac{\partial v}{\partial y} &= \frac{x}{x+r}\cdot\frac{\partial r}{\partial y}-\frac{\partial r}{\partial y}\\ &= \frac{x}{x+r}\cdot\frac{y}{r}-\frac{y}{r}\\ &= \frac{y}{r}\left(\frac{x-x-r}{x+r}\right)\\ &= -\frac{y}{x+r}\end{aligned}$$

Differentiating $\frac{\partial v}{\partial y}$ partially w.r.t. y,

$$\begin{aligned}\frac{\partial^2 v}{\partial y^2} &= -\frac{1}{x+r}+\frac{y}{(x+r)^2}\cdot\frac{\partial r}{\partial y}\\ &= -\frac{1}{x+r}\left(1-\frac{y}{x+r}\cdot\frac{y}{r}\right)\\ &= -\frac{1}{x+r}\left[\frac{rx+r^2-y^2}{r(x+r)}\right]\\ &= -\frac{1}{x+r}\left[\frac{rx+x^2}{r(x+r)}\right]\\ &= -\frac{x}{r(x+r)}\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} &= \frac{1}{r}\left(1-\frac{x}{x+r}\right)\\ &= \frac{1}{r}\left(\frac{x+r-x}{x+r}\right)\\ &= \frac{1}{x+r}\end{aligned}$$

EXERCISE 8.2

1. If $u = \cos\left(\sqrt{x} + \sqrt{y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\sin\left(\sqrt{x} + \sqrt{y}\right) = 0$.

2. If $u = 2(ax + by)^2 - k(x^2 + y^2)$ and $a^2 + b^2 = k$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

[Ans. : 0]

3. If $e^u = \tan x + \tan y$, show that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} = 2$.

4. If $z^3 - 3yz - 3x = 0$, show that

(i) $z\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y}$ (ii) $z\left[\dfrac{\partial^2 z}{\partial x \partial y} + \left(\dfrac{\partial z}{\partial x}\right)^2\right] = \dfrac{\partial^2 z}{\partial y^2}$

5. If $z(z^2 + 3x) + 3y = 0$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{2z(x-1)}{(z^2 + x)^3}$.

6. If $u = \log(x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

7. If $u(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$.

$$\left[\textbf{Ans.: } \frac{2}{(x^2 + y^2 + z^2)^2}\right]$$

8. If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that

$$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}, \quad \frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$$

Hence, deduce that $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial x}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 x}{\partial \theta^2} = 0$.

9. If $v = (x^2 - y^2)f(x, y)$, prove that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = (x^4 - y^4)f''(x, y)$.

10. If $u = f(ax^2 + 2hxy + by^2)$ and $v = \phi(ax^2 + 2hxy + by^2)$, show that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

11. If $x = \frac{r}{2}(e^{\theta} + e^{-\theta}),\ y = \frac{r}{2}(e^{\theta} - e^{-\theta})$, prove that $\left(\frac{\partial x}{\partial r}\right)_{\theta} = \left(\frac{\partial r}{\partial x}\right)_{y}$.

$\left[\textbf{Hint:}\ x = r\cos h\theta, y = r\sin h\theta,\ x^2 - y^2 = r^2\right]$

12. If $\log_e \theta = r - x,\ r^2 = x^2 + y^2$, show that $\frac{\partial^2\theta}{\partial y^2} = \frac{\theta(x^2 + ry^2)}{r^3}$.

$\left[\textbf{Hint:}\ \theta = e^{r-x},\ \frac{\partial r}{\partial y} = \frac{y}{r}\right]$

13. If $u = e^{ax}$ sin by, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

14. If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}$.

15. If $u = \frac{1}{\sqrt{y}}e^{-\frac{(x-a)^2}{4y}}$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

16. If $u = \tan(y + ax) - (y - ax)^{\frac{3}{2}}$, prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

17. If $u = \frac{xy}{2x+y}$, prove that $\frac{\partial^3 u}{\partial y\,\partial z^2} = \frac{\partial^3 u}{\partial z^2\,\partial y}$.

18. If $u = x^m y^n$, prove that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial^3 u}{\partial y\,\partial x^2}$.

19. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following functions:

(i) $\sqrt{x+y-1}$ (ii) $\sqrt{1-x^2-y^2}$ (iii) y^x (iv) $\log_{10}(ax+by)$ (v) $(y-ax)^{\frac{3}{2}}$

Ans.: (i) $\frac{1}{\sqrt{x+y-1}}, \frac{1}{\sqrt{x+y-1}}$ (ii) $\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}$

(iii) $y^x \log y,\ xy^{x-1}$ (iv) $\frac{a}{(\log_e 10)(ax+by)}, \frac{b}{(\log_e 10)(ax+by)}$

(v) $-\frac{3a}{2}(y-ax)^{\frac{1}{2}}, \frac{3}{2}(y-ax)^{\frac{1}{2}}$

20. If $x^4 - xy^2 + yz^2 - z^4 = 6$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\left[\textbf{Ans.}: \frac{y^2 - 4x^3}{2yz - 4z^3}, \frac{2xy - z^2}{2yz - 4z^3}\right]$$

21. If $z^3 + xy - y^2 z = 6$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at (0, 1, 2).

$$\left[\textbf{Ans.}: -\frac{1}{11}, \frac{4}{11}\right]$$

22. Find the value of for which $u = t^n e^{-\frac{r^2}{4kt}}$ satisfies the partial differential equation $\dfrac{\partial u}{\partial t} = k\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r}\right)$.

$$\left[\textbf{Ans.}: n = -\frac{3}{2}\right]$$

23. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\dfrac{\partial r}{\partial x}, \dfrac{\partial \theta}{\partial x}$ in terms of r, θ, ϕ.

$$\left[\textbf{Hint}: r^2 = x^2 + y^2 + z^2, \phi = \tan^{-1}\frac{y}{x}, \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}\right]$$

$$\left[\textbf{Ans.}: \sin\theta\cos\phi, \frac{\cos\theta\cos\phi}{r}, \frac{-\sin\phi}{r\sin\theta}\right]$$

24. If $u = x^2(y - z) + y^2(z - x) + z^2(x - y)$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

25. If $u = e^x(x\cos y - y\sin y)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

26. For the function $f(x, y, z) = z\tan^{-1}\dfrac{y}{x}$, prove that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$.

27. If $z(x + y) = x^2 + y^2$, prove that $x\dfrac{\partial^2 z}{\partial x^2} + y\dfrac{\partial^2 z}{\partial x\,\partial y} = 2\dfrac{\partial z}{\partial x}$.

28. If $\dfrac{x^2}{a + u} + \dfrac{y^2}{b + u} = 1$, prove that $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = 2\left(x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right)$

29. If $u = x^y$, prove that $\dfrac{\partial^3 u}{\partial x^2\partial y} = \dfrac{\partial^3 u}{\partial x\,\partial y\,\partial x}$

30. If $\dfrac{x^2}{2+u}+\dfrac{y^2}{4+u}+\dfrac{z^2}{6+u}=1,$ prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

31. If $u = (x^2 - y^2)\, f(r)$, where $r = xy$, show that

$$\frac{\partial^2 u}{\partial x\,\partial y}=(x^2-y^2)[3f'(r)+rf''(r)].$$

32. If $z = f(x^2, y)$, prove that $x\dfrac{\partial z}{\partial x}=2y\dfrac{\partial z}{\partial y}.$

33. Prove that $z=\dfrac{1}{r}[f(ct+r)+\phi(ct-r)]$ satisfies the partial differential equation $\dfrac{\partial^2 u}{\partial t^2}=\dfrac{c^2}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)$ where is constant.

34. If $u+iv=f(x+iy)$, prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0,\ \dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=0.$

$$\left[\begin{array}{l}\textbf{Hint: } u+iv=f(x+iy),\ u-iv=f(x-iy)\\ \qquad u=\dfrac{1}{2}[f(x+iy)+f(x-iy)],\ v=\dfrac{1}{2i}[f(x+iy)-f(x-iy)]\end{array}\right]$$

35. If u, v, w are function of x, y, z given as $x = u + v + w$, $y = u^2 + v^2 + w^2$, $z=u^3+v^3+w^3$, prove that $\dfrac{\partial u}{\partial x}=\dfrac{vw(w-v)}{(u-v)(v-w)(w-u)}.$

[**Hint**: Differentiate x, y, z w.r.t. x and solve the equations using Cramer's rule.]

36. If $u=(x^2+y^2+z^2)^{\frac{n}{2}}$, find the value of which satisfies the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0.$

[**Ans.:** 0, –1]

37. If $u=\log(e^x+e^y)$, show that $\left(\dfrac{\partial^2 u}{\partial x^2}\right)\left(\dfrac{\partial^2 u}{\partial y^2}\right)-\left(\dfrac{\partial^2 u}{\partial x\partial y}\right)^2=0.$

38. If $z=yf(x^2-y^2)$, show that $y\left(\dfrac{\partial z}{\partial x}\right)+x\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{xy}{y}.$

8.6 TOTAL DERIVATIVES

8.6.1 Chain Rule

If $z = f(u)$, where u is again a function of two variables x and y, i.e., $u = \phi(x, y)$ then

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}z}{\mathrm{d}u} \cdot \frac{\partial u}{\partial x} \quad \text{or} \quad \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\partial u}{\partial x} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\mathrm{d}z}{\mathrm{d}u} \cdot \frac{\partial u}{\partial y} \quad \text{or} \quad \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\partial u}{\partial y} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial y}$$

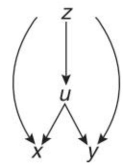

Fig. 8.1

8.6.2 Composite Function of One Variable

If $u = f(x, y)$, where $x = \phi(t)$, $y = \psi(t)$ then u is a function of t and is called the *composite function of a single variable t* and

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t}$$

is called the *total derivative* of u.

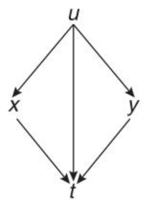

Fig. 8.2

If $u = f(x, y, z)$ and $x = \phi(t)$, $y = \psi(t)$, $z = \xi(t)$ then total derivative of u is given as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t}.$$

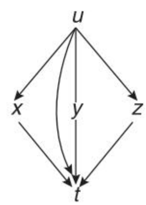

Fig. 8.3

Example 1

If $u = y^2 - 4ax$, $x = at^2$, $y = 2at$, *find* $\dfrac{\mathrm{d}u}{\mathrm{d}t}$.

Solution

$$u = y^2 - 4ax,\ x = at^2,\ y = 2at$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t}$$

$$= (-4a)2at + (2y)2a$$

Substituting y,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -8a^2t + 2(2at)(2a)$$

$$= -8a^2t + 8a^2t$$

$$= 0$$

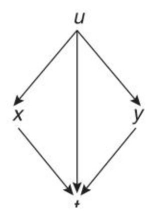

Fig. 8.4

Example 2

If $u = \sin\left(\dfrac{x}{y}\right)$ *where* $x = e^t$, $y = t^2$, *find* $\dfrac{\mathrm{d}u}{\mathrm{d}t}$.

Solution

$$u = \sin\left(\frac{x}{y}\right), x = e^t, y = t^2$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{y}\cos\left(\frac{x}{y}\right)\cdot e^t + \left(-\frac{x}{y^2}\right)\cos\left(\frac{x}{y}\right)\cdot 2t$$

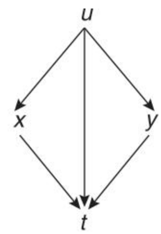

Fig. 8.5

Substituting x and y,

$$\frac{du}{dt} = \frac{1}{t^2}\cos\left(\frac{e^t}{t^2}\right)\cdot e^t - \frac{e^t}{(t^2)^2}\cos\left(\frac{e^t}{t^2}\right)\cdot 2t$$

$$= \frac{1}{t^2}e^t\cos\left(\frac{e^t}{t^2}\right)\left(1-\frac{2}{t}\right)$$

Example 3

If $u = x^2y^3$, $x = \log t$, $y = e^t$, *find* $\dfrac{du}{dt}$.

Solution

$$u = x^2y^3, x = \log t, y = e^t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2xy^3)\frac{1}{t} + (3x^2y^2)e^t$$

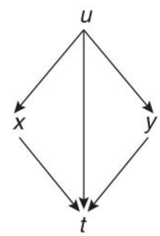

Fig. 8.6

Substituting x and y,

$$\frac{du}{dt} = 2(\log t)e^{3t}\cdot\frac{1}{t} + 3(\log t)^2 e^{2t}\cdot e^t$$

$$= \frac{2}{t}\log t\, e^{3t} + 3(\log t)^2 e^{3t}$$

Example 4

If $u = xy + yz + zx$ *where* $x = \dfrac{1}{t}$, $y = e^t$, $z = e^{-t}$, *find* $\dfrac{du}{dt}$.

Solution

$$u = xy + yz + zx,\ x = \frac{1}{t}, y = e^t, z = e^{-t}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$$

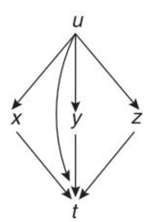

Fig. 8.7

Substituting x, y and z,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{t^2}(e^t + e^{-t}) + \left(\frac{1}{t} + e^{-t}\right)e^t - \left(e^t + \frac{1}{t}\right)e^{-t}$$

$$= -\frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$$

Example 5

If $z = xy^2 + x^2y, x = at^2, y = 2at,$ *find* $\frac{\mathrm{d}z}{\mathrm{d}t}$.

Solution

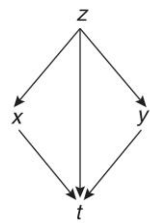

Fig. 8.8

$$z = xy^2 + x^2y, x = at^2, y = 2at$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$= (y^2 + 2xy)2at + (2xy + x^2)2a$$

Substituting x, y and z,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (4a^2t^2 + 2at^2\cdot 2at)2at + (2at^2\cdot 2at + a^2t^4)2a$$

$$= 4a^2t^2(1+t)2at + a^2t^3(4+t)2a$$

$$= 8a^3t^3(1+t) + 2a^3t^3(4+t)$$

$$= 2a^3t^3(8+5t)$$

Example 6

If $z = \sin^{-1}(x-y), x = 3t, y = 4t^3,$ *prove that* $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{3}{\sqrt{1-t^2}}$.

[Summer 2015]

Solution

$$z = \sin^{-1}(x-y), x = 3t, y = 4t^3$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 3 + \frac{1(-1)}{\sqrt{1-(x-y)^2}}\cdot 12t^2$$

$$= \frac{3-12t^2}{\sqrt{1-x^2-y^2+2xy}}$$

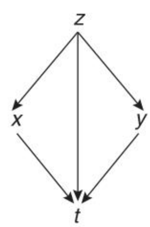

Fig. 8.9

Substituting x and y,

$$\begin{aligned}\frac{dz}{dt} &= \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}\\ &= \frac{3(1-4t^2)}{\sqrt{1-8t^2-t^2-16t^6+16t^4+8t^4}}\\ &= \frac{3(1-4t^2)}{\sqrt{1-8t^2+16t^4-t^2-16t^6+8t^4}}\\ &= \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2-t^2(1+16t^4-8t^2)}}\\ &= \frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2-t^2(1-4t^2)^2}}\\ &= \frac{3(1-4t^2)}{(1-4t^2)\sqrt{1-t^2}}\\ &= \frac{3}{\sqrt{1-t^2}}\end{aligned}$$

Example 7

If $u=\tan^{-1}\left(\frac{y}{x}\right), x=e^t-e^{-t}, y=e^t+e^{-t}$, *find* $\frac{du}{dt}$.

Solution

$$u=\tan^{-1}\left(\frac{y}{x}\right), x=e^t-e^{-t}, y=e^t+e^{-t}$$

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}\\ &= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)(e^t+e^{-t})+\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)(e^t-e^{-t})\\ &= -\frac{y}{x^2+y^2}\cdot y+\frac{x}{x^2+y^2}\cdot x\\ &= \frac{x^2-y^2}{x^2+y^2}\\ &= \frac{(e^t-e^{-t})^2-(e^t+e^{-t})^2}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}\end{aligned}$$

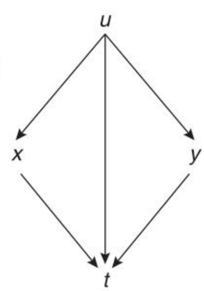

Fig. 8.10

$$= -\frac{4}{2(e^{2t}+e^{-2t})}$$

$$= -\frac{2}{e^{2t}+e^{-2t}}$$

Example 8

For $z = \tan^{-1}\left(\frac{x}{y}\right)$, *x = u cos v, y = u sin v, evaluate* $\frac{\partial z}{\partial u}$ *and* $\frac{\partial z}{\partial v}$ *at the point* $\left(1.3, \frac{\pi}{6}\right)$. **[Winter 2015]**

Solution

$$z = \tan^{-1}\left(\frac{x}{y}\right), \quad x = u\cos v,\ y = u\sin v$$

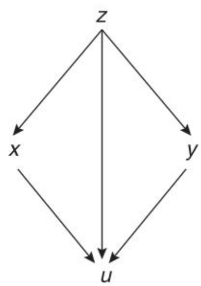

Fig. 8.11

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)\cdot\cos v + \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)\cdot\sin v$$

$$= \frac{y\cos v}{y^2+x^2} - \frac{x\sin v}{y^2+x^2}$$

$$= \frac{u\sin v\cos v - u\cos v\sin v}{x^2+y^2}$$

$$= 0$$

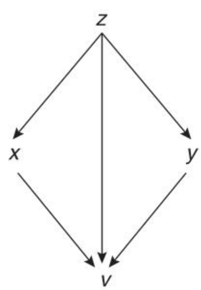

Fig. 8.12

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)(-u\sin v) + \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)(u\cos v)$$

$$= -\frac{yu\sin v}{y^2+x^2} - \frac{xu\cos v}{y^2+x^2}$$

$$= -\frac{(y^2+x^2)}{y^2+x^2}$$

$$= -1$$

Example 9

If $u = x^2 + y^2 + z^2$, *where* $x = e^t, y = e^t \sin t, z = e^t \cos t$, *find* $\dfrac{\mathrm{d}u}{\mathrm{d}t}$.

Solution

$$u = x^2 + y^2 + z^2,\ x = e^t,\ y = e^t \sin t,\ z = e^t \cos t$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}t}$$

$$= 2xe^t + 2y(e^t \sin t + e^t \cos t) + 2z(e^t \cos t - e^t \sin t)$$

$$= 2e^t \cdot e^t + 2e^t \sin t \cdot e^t(\sin t + \cos t) + 2e^t \cos t \cdot e^t(\cos t - \sin t)$$

$$= 2e^{2t}(1 + \sin^2 t + \sin t \cos t + \cos^2 t - \cos t \sin t)$$

$$= 4e^{2t}$$

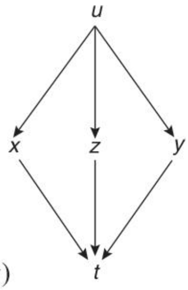

Fig. 8.13

Example 10

If $z = e^{xy}, x = t\cos t, y = t \sin t$, *find* $\dfrac{\mathrm{d}z}{\mathrm{d}t}$ *at* $t = \dfrac{\pi}{2}$.

Solution

$$z = e^{xy}, x = t\cos t, y = t\sin t$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$= e^{xy} y(\cos t - t\sin t) + e^{xy} x(\sin t + t\cos t)$$

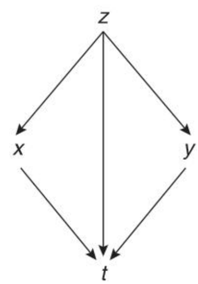

Fig. 8.14

At $t = \dfrac{\pi}{2}, x = 0, y = \dfrac{\pi}{2}$

Hence, $$\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=\frac{\pi}{2}} = e^0\left[\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right]$$

$$= -\frac{\pi^2}{4}$$

Example 11

If $z = x^2y + 3xy^4$ *where* $x = \sin 2t$ *and* $y = \cos t$, *find* $\dfrac{\mathrm{d}z}{\mathrm{d}t}$ *when* $t = 0$.

[Winter 2013]

Solution

$$z = x^2y + 3xy^4,\ x = \sin 2t,\ y = \cos t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

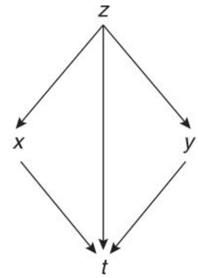

Fig. 8.15

At $t = 0,\ x = 0,\ y = 1$

Hence, $$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)[2(1)] + 0 = 6$$

Example 12

If $u = x^2 + y^2 + z^2 - 2xyz = 1$, *show that* $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dz}{\sqrt{1-z^2}} = 0.$

Solution

$$u = x^2 + y^2 + z^2 - 2xyz = 1$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$(2x - 2yz)dx + (2y - 2xz)dy + (2z - 2xy)dz = 0$$

$$(x - yz)dx + (y - xz)dy + (z - xy)dz = 0 \qquad \ldots(1)$$

We have, $$x^2 + y^2 + z^2 - 2xyz = 1$$

$$x^2 - 2xyz = 1 - y^2 - z^2$$

$$x^2 - 2xyz + y^2z^2 = 1 - y^2 - z^2 + y^2z^2$$

$$(x - yz)^2 = (1 - y^2)(1 - z^2)$$

$$x - yz = \sqrt{1-y^2}\cdot\sqrt{1-z^2}$$

Similarly, $$y - xz = \sqrt{1-x^2}\cdot\sqrt{1-z^2}$$

and $$z - xy = \sqrt{1-x^2}\cdot\sqrt{1-y^2}$$

Substituting in Eq. (1),

$$\sqrt{1-y^2}\cdot\sqrt{1-z^2}\,dx + \sqrt{1-x^2}\cdot\sqrt{1-z^2}\,dy + \sqrt{1-x^2}\cdot\sqrt{1-y^2}\,dz = 0$$

$$\sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2}\left(\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}}\right) = 0$$

Hence, $$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}+\frac{dz}{\sqrt{1-z^2}}=0$$

8.6.3 Composite Function of Two Variables

If $z = f(x, y)$, where $x = \phi(u, v)$, $y = \psi(u, v)$ then z is a function of u, v and is called the composite function of two variables u and v.

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

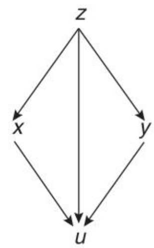

Fig. 8.16

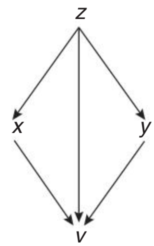

Fig. 8.17

Example 1

If $z = f(u, v), u = \log(x^2 + y^2), v = \dfrac{y}{x}$, *show that*

$$x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x}=(1+v^2)\frac{\partial z}{\partial v}.$$

Solution

$$z = f(u, v), u = \log(x^2 + y^2), v = \frac{y}{x},$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial z}{\partial u}\cdot\frac{1}{x^2+y^2}\cdot 2x+\frac{\partial z}{\partial v}\left(-\frac{y}{x^2}\right)$$

$$y\frac{\partial z}{\partial x}=\frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u}-\frac{y^2}{x^2}\cdot\frac{\partial z}{\partial v} \qquad ...(1)$$

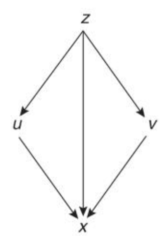

Fig. 8.18

and $$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=\frac{\partial z}{\partial u}\cdot\frac{2y}{x^2+y^2}+\frac{\partial z}{\partial v}\cdot\frac{1}{x}$$

$$x\frac{\partial z}{\partial y}=\frac{2xy}{x^2+y^2}\cdot\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v} \qquad ...(2)$$

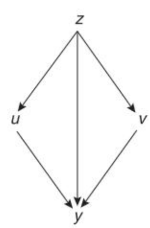

Fig. 8.19

Subtracting Eq. (1) from Eq. (2),

Hence, $x\dfrac{\partial z}{\partial y}-y\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial v}+\dfrac{y^2}{x^2}\dfrac{\partial z}{\partial v}$

$$=(1+v^2)\frac{\partial z}{\partial v}$$

Example 2

If $w=\phi(u,v), u=x^2-y^2-2xy, v=y,$ *prove that* $\dfrac{\partial w}{\partial v}=0$ *is equivalent to* $(x+y)\dfrac{\partial w}{\partial x}+(x-y)\dfrac{\partial w}{\partial y}=0.$

Solution

$$w=\phi(u,v), u=x^2-y^2-2xy, v=y$$

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial w}{\partial u}(2x-2y)+\frac{\partial w}{\partial v}\cdot 0$$

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}(2x-2y)$$

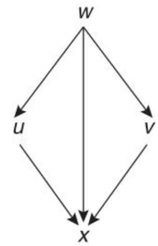

Fig. 8.20

and

$$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=\frac{\partial w}{\partial u}(-2y-2x)+\frac{\partial w}{\partial v}\cdot 1$$

$$\frac{\partial w}{\partial y}=-2(x+y)\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}$$

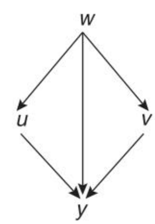

Fig. 8.21

$$(x+y)\frac{\partial w}{\partial x}+(x-y)\frac{\partial w}{\partial y}=(x+y)2(x-y)\frac{\partial w}{\partial u}-(x-y)2(x+y)\frac{\partial w}{\partial u}+(x-y)\frac{\partial w}{\partial v}$$

$$=(x-y)\frac{\partial w}{\partial v}$$

If $\dfrac{\partial w}{\partial v}=0$ then $(x+y)\dfrac{\partial w}{\partial x}+(x-y)\dfrac{\partial w}{\partial y}=0$.

Hence, $\dfrac{\partial w}{\partial v}=0$ is equivalent to $(x+y)\dfrac{\partial w}{\partial x}+(x-y)\dfrac{\partial w}{\partial y}=0.$

Example 3

If $z = f(x, y)$ *and* $x = e^{u} + e^{-v}$ *and* $y = e^{-u} - e^{v}$, *show that*
$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$. **[Summer 2014]**

Solution

$$z = f(x, y),\ \ x = e^{u} + e^{-v},\ y = e^{-u} - e^{v}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}e^{u} + \frac{\partial z}{\partial y}(-e^{-u}) \qquad ...(1)$$

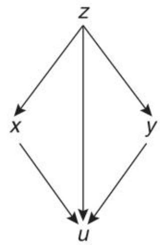

Fig. 8.22

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) \qquad ...(2)$$

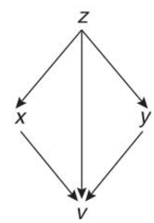

Fig. 8.23

Subtracting Eq. (2) from Eq. (1),

Hence,
$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u} + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^{v})$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Example 4

If $z = f(u, v)$ *and* $u = x\cos\theta - y\sin\theta,\ v = x\sin\theta + y\cos\theta$, *show that*
$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$.

Solution

$$z = f(u, v),\ \ u = x\cos\theta - y\sin\theta,\ \ v = x\sin\theta + y\cos\theta$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cos\theta + \frac{\partial z}{\partial v}\sin\theta$$

$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}x\cos\theta + \frac{\partial z}{\partial v}x\sin\theta \qquad ...(1)$$

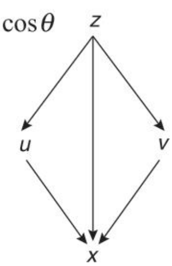

Fig. 8.24

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-\sin\theta) + \frac{\partial z}{\partial v}\cos\theta$$

$$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(-y\sin\theta) + \frac{\partial z}{\partial v}(y\cos\theta) \qquad ...(2)$$

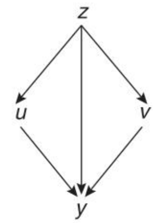

Fig. 8.25

Adding Eqs (1) and (2),

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(x\cos\theta - y\sin\theta) + \frac{\partial z}{\partial v}(x\sin\theta + y\cos\theta)$$

$$= u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$$

Example 5

If $z = f(x, y)$, $x = uv$, $y = \dfrac{u+v}{u-v}$, *prove that* $u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v} = 2x\dfrac{\partial z}{\partial x}$.

Solution

$$z = f(x, y),\ x = uv,\ y = \frac{u+v}{u-v},$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}v + \frac{\partial z}{\partial y}\left\{\frac{(u-v)-(u+v)}{(u-v)^2}\right\}$$

$$= \frac{\partial z}{\partial x}v - \frac{2v}{(u-v)^2}\frac{\partial z}{\partial y}$$

$$u\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}uv - \frac{2uv}{(u-v)^2}\frac{\partial z}{\partial y} \qquad ...(1)$$

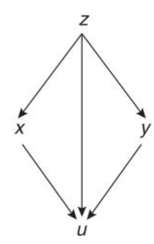

Fig. 8.26

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}u + \frac{\partial z}{\partial y}\left\{\frac{(u-v)-(u+v)(-1)}{(u-v)^2}\right\}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}u + \frac{2u}{(u-v)^2}\cdot\frac{\partial z}{\partial y}$$

$$v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}uv + \frac{2uv}{(u-v)^2}\cdot\frac{\partial z}{\partial y} \qquad ...(2)$$

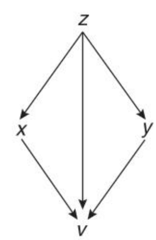

Fig. 8.27

Adding Eqs (1) and (2),

$$u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}=2uv\frac{\partial z}{\partial x}$$
$$=2x\frac{\partial z}{\partial x}\cdot$$

Example 6

If $z=f(x,y)$, $x=u\cosh v$, $y=u\sinh v$, *prove that*

$$\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2.$$

Solution

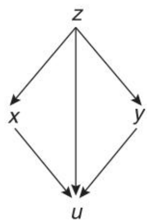

Fig. 8.28

$$z=f(x,y),\quad x=u\cosh v,\quad y=u\sinh v$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$=\frac{\partial z}{\partial x}\cosh v+\frac{\partial z}{\partial y}\sinh v \qquad ...(1)$$

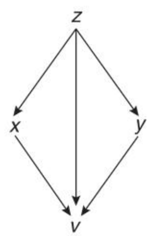

Fig. 8.29

and
$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$
$$=\frac{\partial z}{\partial x}u\sinh v+\frac{\partial z}{\partial y}u\cosh v$$
$$\frac{1}{u}\cdot\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\sinh v+\frac{\partial z}{\partial y}\cosh v \qquad ...(2)$$

Squaring and subtracting Eq. (2) from Eq. (1),

$$\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cosh^2 v+\left(\frac{\partial z}{\partial y}\right)^2\sinh^2 v+2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v$$
$$-\left(\frac{\partial z}{\partial x}\right)^2\sinh^2 v-\left(\frac{\partial z}{\partial y}\right)^2\cosh^2 v-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v$$
$$=\left(\frac{\partial z}{\partial x}\right)^2(\cosh^2 v-\sinh^2 v)-\left(\frac{\partial z}{\partial y}\right)^2(\cosh^2 v-\sinh^2 v)$$
$$=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2$$

Example 7

If $x = r\cosh\theta,\ y = r\sinh\theta,$ *show that* $(x-y)(z_x - z_y) = r\,z_r - z_\theta.$

Solution

$$z = f(x, y), \quad x = r\cosh\theta, \quad y = r\sinh\theta$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x}\cdot\cosh\theta + \frac{\partial z}{\partial y}\cdot\sinh\theta$$

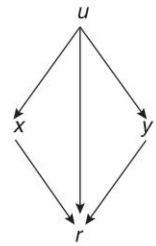

Fig. 8.30

and
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}\cdot r\sinh\theta + \frac{\partial z}{\partial y}\cdot r\cosh\theta$$

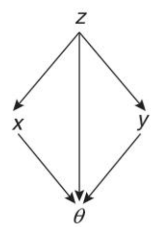

Fig. 8.31

$$r\frac{\partial z}{\partial r} - \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot r\cosh\theta + \frac{\partial z}{\partial y}\cdot r\sinh\theta - \frac{\partial z}{\partial x}\cdot r\sinh\theta - \frac{\partial z}{\partial y}\cdot r\cosh\theta$$

$$= \frac{\partial z}{\partial x}(r\cosh\theta - r\sinh\theta) + \frac{\partial z}{\partial y}(r\sinh\theta - r\cosh\theta)$$

$$= \frac{\partial z}{\partial x}(x-y) - \frac{\partial z}{\partial y}(x-y)$$

$$= (x-y)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Hence, $(x-y)(z_x - z_y) = r\,z_r - z_\theta$

Example 8

If $z = f(x, y)$ *where* $x^2 = au + bv,\ y^2 = au - bv$ *then prove that*

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = \frac{1}{2}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

Solution

$$z = f(x, y), \quad x^2 = au + bv, \quad y^2 = au - bv$$

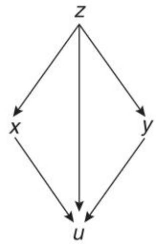

Fig. 8.32

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{a}{2x} + \frac{\partial z}{\partial y}\cdot\frac{a}{2y}$$

$$u\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{au}{2x} + \frac{\partial z}{\partial y}\cdot\frac{au}{2y} \qquad \text{...(1)}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$=\frac{\partial z}{\partial x}\cdot\frac{b}{2x}+\frac{\partial z}{\partial y}\left(-\frac{b}{2y}\right)$$

$$v\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{bv}{2x}-\frac{\partial z}{\partial y}\cdot\frac{bv}{2y} \qquad ...(2)$$

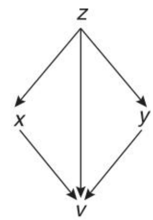

Fig. 8.33

Adding Eqs (1) and (2),

$$u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{au}{2x}+\frac{\partial z}{\partial y}\frac{au}{2y}+\frac{\partial z}{\partial x}\frac{bv}{2x}-\frac{\partial z}{\partial y}\frac{bv}{2y}$$

$$=\frac{\partial z}{\partial x}\left(\frac{au+bv}{2x}\right)+\frac{\partial z}{\partial y}\left(\frac{au-bv}{2y}\right)$$

$$=\frac{\partial z}{\partial x}\left(\frac{x^2}{2x}\right)+\frac{\partial z}{\partial y}\left(\frac{y^2}{2y}\right)$$

$$=\frac{1}{2}\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

Example 9

If $u=f(x,y)$ where $x=r\cos\theta$ and $y=r\sin\theta$, prove that

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2.$$

Solution

$$u=f(x,y),\ x=r\cos\theta,\ y=r\sin\theta$$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta \qquad \ldots(1)$$

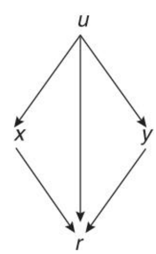

Fig. 8.34

and

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}r\cos\theta$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta}=-\sin\theta\frac{\partial u}{\partial x}+\cos\theta\frac{\partial u}{\partial y} \qquad \ldots(2)$$

Squaring and adding Eqs (1) and (2),

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \cos^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2 + 2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$+\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + \cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2 - 2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

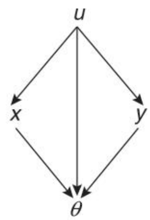

Fig. 8.35

Example 10

If $z = f(u, v)$ where $u = x^2 + y^2$, $v = 2xy$, *show that*

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2 - v^2}\left(\frac{\partial z}{\partial u}\right).$$

Solution

$$z = f(u, v), \quad u = x^2 + y^2, \quad v = 2xy$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cdot 2x + \frac{\partial z}{\partial v}\cdot 2y$$

$$x\frac{\partial z}{\partial x} = 2x^2\frac{\partial z}{\partial u} + 2xy\frac{\partial z}{\partial v} \qquad \text{...(1)}$$

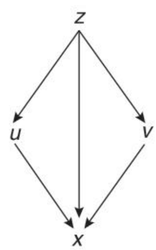

Fig. 8.36

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}\cdot 2y + \frac{\partial z}{\partial v}\cdot 2x$$

$$y\frac{\partial z}{\partial y} = 2y^2\frac{\partial z}{\partial u} + 2xy\frac{\partial z}{\partial v} \qquad \text{...(2)}$$

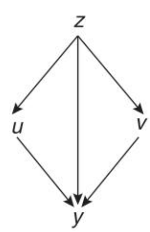

Fig. 8.37

Substracting Eq. (2) from Eq. (1),

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2 - y^2)\frac{\partial z}{\partial u}$$

$$= 2\sqrt{(x^2 + y^2)^2 - 4x^2y^2}\left(\frac{\partial z}{\partial u}\right)$$

$$= 2\sqrt{(u^2 - v^2)}\left(\frac{\partial z}{\partial u}\right)$$

Example 11

If $z = f(u, v)$, *and* $u = x^2 - y^2, v = 2xy$, *show that*

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(u^2 + v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Solution

$$z = f(u, v), \text{ and } u = x^2 - y^2, v = 2xy$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cdot 2x + \frac{\partial z}{\partial v}\cdot 2y$$

$$= 2\left(x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right) \quad \text{...(1)}$$

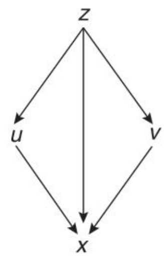

Fig. 8.38

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}(2x)$$

$$= 2\left(-y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right) \quad \text{...(2)}$$

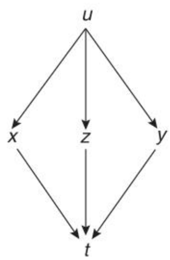

Fig. 8.39

Squaring and adding Eq. (1) and Eq. (2),

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\left(x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)^2 + 4\left(-y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)^2$$

$$= 4\left[x^2\left(\frac{\partial z}{\partial u}\right)^2 + y^2\left(\frac{\partial z}{\partial v}\right)^2 + 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right.$$

$$\left. + y^2\left(\frac{\partial z}{\partial u}\right)^2 + x^2\left(\frac{\partial z}{\partial v}\right)^2 - 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right]$$

$$= 4(x^2 + y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2 + y^2)^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2 - y^2)^2 + 4x^2y^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(u^2 + v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

Example 12

If $x = e^u \operatorname{cosec} v,\ y = e^u \cot v$ *then show that*

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right].$$

Solution

$$z = f(x, y), \quad x = e^u \operatorname{cosec} v, \quad y = e^u \cot v$$

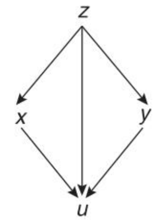

Fig. 8.40

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} e^u \operatorname{cosec} v + \frac{\partial z}{\partial y} e^u \cot v$$

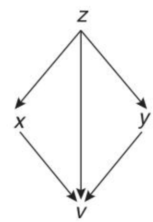

Fig. 8.41

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^u \operatorname{cosec} v \cot v) + \frac{\partial z}{\partial y}(-e^u \operatorname{cosec}^2 v)$$

$$\begin{aligned}
e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right] &= e^{-2u}\Bigg[\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\operatorname{cosec}^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cot^2 v \\
&\quad + 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} e^{2u}\operatorname{cosec} v \cot v \\
&\quad + (-\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2 (e^{2u}\operatorname{cosec}^2 v \cot^2 v) \\
&\quad + (-\sin^2 v)\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\operatorname{cosec}^4 v \\
&\quad + (-\sin^2 v) 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} e^{2u}\operatorname{cosec}^3 v \cot v\Bigg] \\
&= \left(\frac{\partial z}{\partial x}\right)^2 (\operatorname{cosec}^2 v - \cot^2 v) + \left(\frac{\partial z}{\partial y}\right)^2 (\cot^2 v - \operatorname{cosec}^2 v) \\
&= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2
\end{aligned}$$

Example 13

If $z = f(x, y), x = e^u \cos v, y = e^u \sin v,$ *prove that*

(i) $x\dfrac{\partial z}{\partial v} + y\dfrac{\partial z}{\partial u} = e^{2u}\dfrac{\partial z}{\partial y}$ (ii) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\dfrac{\partial z}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2\right]$

Solution

$$z = f(x, y),\ x = e^u \cos v,\ y = e^u \sin v$$

(i)
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$= \frac{\partial z}{\partial x}\cdot e^u \cos v + \frac{\partial z}{\partial y} e^u \sin v$$
$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

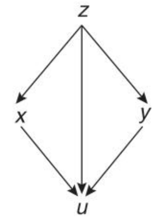

Fig. 8.42

$$y\frac{\partial z}{\partial u} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \quad ...(1)$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$
$$= \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y} e^u \cos v$$
$$= -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

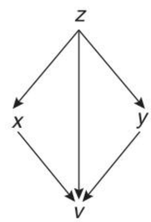

Fig. 8.43

$$x\frac{\partial z}{\partial v} = -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \quad ...(2)$$

Adding Eqs (1) and (2),

$$x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = (y^2 + x^2)\frac{\partial z}{\partial y}$$
$$= e^{2u}\frac{\partial z}{\partial y}$$

(ii)
$$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = e^{-2u}\left[x^2\left(\frac{\partial z}{\partial x}\right)^2 + y^2\left(\frac{\partial z}{\partial y}\right)^2 + 2xy\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right]$$
$$+\left[y^2\left(\frac{\partial z}{\partial x}\right)^2 + x^2\left(\frac{\partial z}{\partial y}\right)^2 - 2xy\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right]$$
$$= e^{-2u}\left[(x^2 + y^2)\left(\frac{\partial z}{\partial x}\right)^2 + (x^2 + y^2)\left(\frac{\partial z}{\partial y}\right)^2\right]$$
$$= e^{-2u}e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Example 14

If $u = f\left(\dfrac{y-x}{xy}, \dfrac{z-x}{xz}\right)$, *show that* $x^2 \dfrac{\partial u}{\partial x} + y^2 \dfrac{\partial u}{\partial y} + z^2 \dfrac{\partial u}{\partial z} = 0.$

[Summer 2016]

Solution

Let
$$l = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}, \; m = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial l}{\partial x} = -\frac{1}{x^2}, \qquad \frac{\partial m}{\partial x} = -\frac{1}{x^2},$$

$$\frac{\partial l}{\partial y} = \frac{1}{y^2}, \qquad \frac{\partial m}{\partial y} = 0,$$

$$\frac{\partial l}{\partial z} = 0, \qquad \frac{\partial m}{\partial z} = \frac{1}{z^2}$$

$$u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right) = f(l, m)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial x}$$

$$= \frac{\partial u}{\partial l}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial m}\left(-\frac{1}{x^2}\right)$$

$$x^2 \frac{\partial u}{\partial x} = -\left(\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m}\right) \qquad \ldots(1)$$

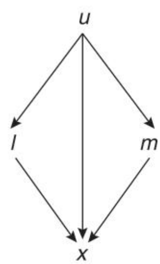

Fig. 8.44

Also,
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial y}$$

$$= \frac{\partial u}{\partial l}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial m} \cdot 0$$

$$y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \qquad \ldots(2)$$

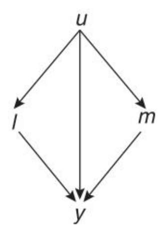

Fig. 8.45

and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial z}$$

$$= \frac{\partial u}{\partial l} \cdot 0 + \frac{\partial u}{\partial m}\left(\frac{1}{z^2}\right)$$

$$z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial m} \qquad \ldots(3)$$

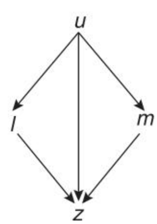

Fig. 8.46

Adding Eqs (1), (2) and (3),

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=-\left(\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\right)+\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}=0$$

Example 15

If $u=f(x-y,\ y-z,\ z-x)$ *then show that* $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0.$

Solution

Let $\quad x-y=l,\qquad y-z=m,\qquad z-x=n$

$$\frac{\partial l}{\partial x}=1,\qquad \frac{\partial m}{\partial x}=0,\qquad \frac{\partial n}{\partial x}=-1,$$

$$\frac{\partial l}{\partial y}=-1,\qquad \frac{\partial m}{\partial y}=1,\qquad \frac{\partial n}{\partial y}=0,$$

$$\frac{\partial l}{\partial z}=0\qquad \frac{\partial m}{\partial z}=-1\qquad \frac{\partial n}{\partial z}=1$$

$$u=f(x-y,\ y-z,\ z-x)=f(l,m,n)$$

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial u}{\partial l}\frac{\partial l}{\partial x}+\frac{\partial u}{\partial m}\frac{\partial m}{\partial x}+\frac{\partial u}{\partial n}\frac{\partial n}{\partial x}\\&=\frac{\partial u}{\partial l}\cdot 1+\frac{\partial u}{\partial m}\cdot 0+\frac{\partial u}{\partial n}(-1)\\&=\frac{\partial u}{\partial l}-\frac{\partial u}{\partial n}\qquad\qquad \text{... (1)}\end{aligned}$$

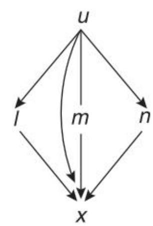

Fig. 8.47

also,

$$\begin{aligned}\frac{\partial u}{\partial y}&=\frac{\partial u}{\partial l}\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}\frac{\partial m}{\partial y}+\frac{\partial u}{\partial n}\frac{\partial n}{\partial y}\\&=\frac{\partial u}{\partial l}(-1)+\frac{\partial u}{\partial m}(1)+\frac{\partial u}{\partial n}\cdot 0\\&=-\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\qquad\qquad \text{... (2)}\end{aligned}$$

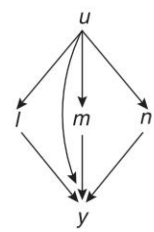

Fig. 8.48

and

$$\begin{aligned}\frac{\partial u}{\partial z}&=\frac{\partial u}{\partial l}\frac{\partial l}{\partial z}+\frac{\partial u}{\partial m}\frac{\partial m}{\partial z}+\frac{\partial u}{\partial n}\frac{\partial n}{\partial z}\\&=\frac{\partial u}{\partial l}\cdot 0+\frac{\partial u}{\partial m}(-1)+\frac{\partial u}{\partial n}(1)\\&=-\frac{\partial u}{\partial m}+\frac{\partial u}{\partial n}\qquad\qquad \text{... (3)}\end{aligned}$$

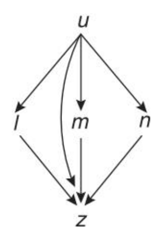

Fig. 8.49

Adding Eqs (1), (2) and (3),

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Example 16

Find $\frac{\partial w}{\partial r}, \frac{\partial w}{\partial s}$ *in terms of r and s if* $w = x + 2y + z^2$, *where* $x=\frac{r}{s}$, $y = r^2 + \log s$, $z = 2r$. **[Winter 2014]**

Solution

$$x=\frac{r}{s}, \qquad y=r^2+\log s, \qquad z=2r$$

$$\frac{\partial x}{\partial r}=\frac{1}{s}, \qquad \frac{\partial y}{\partial r}=2r, \qquad \frac{\partial z}{\partial r}=2$$

$$\frac{\partial x}{\partial s}=-\frac{r}{s^2}, \qquad \frac{\partial y}{\partial s}=\frac{1}{s}, \qquad \frac{\partial z}{\partial s}=0$$

$$w = x + 2y + z^2$$

$$\begin{aligned}\frac{\partial w}{\partial r}&=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}\\&=(1)\left(\frac{1}{s}\right)+(2)(2r)+(2z)(2)\\&=\frac{1}{s}+4r+4z\\&=\frac{1}{s}+4r+4(2r)\\&=\frac{1}{s}+12s\end{aligned}$$

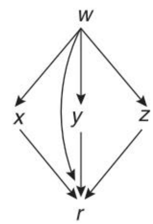

Fig. 8.50

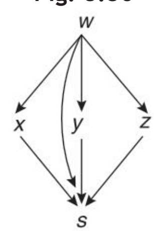

Fig. 8.51

$$\begin{aligned}\frac{\partial w}{\partial s}&=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}\\&=(1)\left(-\frac{r}{s^2}\right)+(2)\left(\frac{1}{s}\right)+(2z)(0)\\&=-\frac{r}{s^2}+\frac{2}{s}\end{aligned}$$

Example 17

If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$ **[Winter 2013]**

Solution

Let $\frac{x}{y} = l, \quad \frac{y}{z} = m, \quad \frac{z}{x} = n$

$$\frac{\partial l}{\partial x} = \frac{1}{y}, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -\frac{z}{x^2},$$

$$\frac{\partial l}{\partial y} = \frac{-x}{y^2}, \quad \frac{\partial m}{\partial y} = \frac{1}{z} \quad \frac{\partial n}{\partial y} = 0,$$

$$\frac{\partial l}{\partial z} = 0 \quad \frac{\partial m}{\partial z} = \frac{-y}{z^2}, \quad \frac{\partial n}{\partial z} = \frac{1}{x}$$

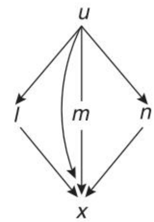

Fig. 8.52

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot\frac{1}{y} + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\left(\frac{-z}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\cdot\frac{\partial u}{\partial l} - \frac{z}{x}\cdot\frac{\partial u}{\partial n} \qquad \text{...(1)}$$

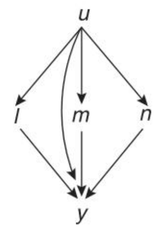

Fig. 8.53

Also, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial m}\cdot\frac{1}{z} + \frac{\partial u}{\partial n}\cdot 0$$

$$y\frac{\partial u}{\partial y} = -\frac{x}{y}\cdot\frac{\partial u}{\partial l} + \frac{y}{z}\cdot\frac{\partial u}{\partial m} \qquad \text{...(2)}$$

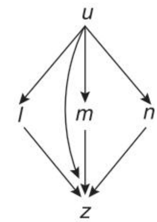

Fig. 8.54

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial n}\cdot\frac{1}{x}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial m} + \frac{z}{x}\cdot\frac{\partial u}{\partial n} \qquad \text{...(3)}$$

Adding Eqs (1), (2) and (3),

Hence, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Example 18

If $u = f\left(x^2 - y^2, y^2 - z^2, z^2 - x^2\right)$, *prove that* $\dfrac{1}{x}\dfrac{\partial u}{\partial x} + \dfrac{1}{y}\dfrac{\partial u}{\partial y} + \dfrac{1}{z}\dfrac{\partial u}{\partial z} = 0.$

Solution

Let
$$l = x^2 - y^2,\ m = y^2 - z^2,\ n = z^2 - x^2$$

$$\frac{\partial l}{\partial x} = 2x, \qquad \frac{\partial m}{\partial x} = 0, \qquad \frac{\partial n}{\partial x} = -2x$$

$$\frac{\partial l}{\partial y} = -2y, \qquad \frac{\partial m}{\partial y} = 2y, \qquad \frac{\partial n}{\partial y} = 0$$

$$\frac{\partial l}{\partial z} = 0, \qquad \frac{\partial m}{\partial z} = -2z, \qquad \frac{\partial n}{\partial z} = 2z$$

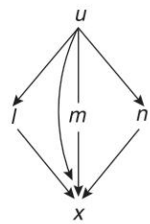

Fig. 8.55

$$u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2) = f(l, m, n)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot 2x + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-2x)$$

$$\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial u}{\partial n} \qquad \text{...(1)}$$

Also,
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}(-2y) + \frac{\partial u}{\partial m}(2y) + \frac{\partial u}{\partial n}(0)$$

$$\frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial l} + 2\frac{\partial u}{\partial m} \qquad \text{...(2)}$$

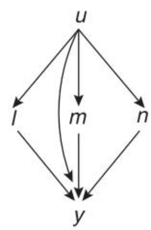

Fig. 8.56

and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial n}(2z)$$

$$\frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n} \qquad \text{...(3)}$$

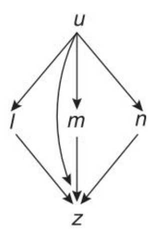

Fig. 8.57

Adding Eqs (1), (2) and (3),

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

Example 19

If $u = f(x^2 + 2yz, y^2 + 2zx)$, *prove that*

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$

Solution

Let $\quad x^2 + 2yz = l, \quad y^2 + 2zx = m$

$$\frac{\partial l}{\partial x} = 2x, \quad \frac{\partial l}{\partial y} = 2z, \quad \frac{\partial l}{\partial z} = 2y$$

$$\frac{\partial m}{\partial x} = 2z, \quad \frac{\partial m}{\partial y} = 2y, \quad \frac{\partial m}{\partial z} = 2x$$

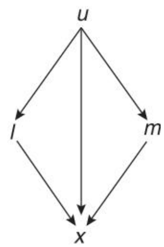

Fig. 8.58

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot 2x + \frac{\partial u}{\partial m}\cdot 2z$$

$$(y^2 - zx)\frac{\partial u}{\partial x} = (2xy^2 - 2x^2z)\frac{\partial u}{\partial l} + (2y^2z - 2z^2x)\frac{\partial u}{\partial m} \quad \text{... (1)}$$

Also,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}$$

$$= \frac{\partial u}{\partial l}\cdot 2z + \frac{\partial u}{\partial m}\cdot 2y$$

$$(x^2 - yz)\frac{\partial u}{\partial y} = (2x^2z - 2yz^2)\frac{\partial u}{\partial l} + (2x^2y - 2y^2z)\frac{\partial u}{\partial m} \quad \text{... (2)}$$

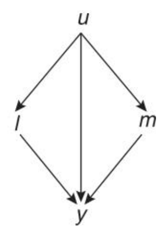

Fig. 8.59

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 2y + \frac{\partial u}{\partial m}\cdot 2x$$

$$(z^2 - xy)\frac{\partial u}{\partial z} = (2yz^2 - 2xy^2)\frac{\partial u}{\partial l} + (2z^2x - 2x^2y)\frac{\partial u}{\partial m} \quad \text{... (3)}$$

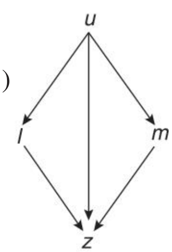

Fig. 8.60

Adding Eqs (1), (2) and (3),

Hence, $\quad (y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$

Example 20

If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, *show that* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$

Solution

Let $\quad l = e^{y-z}, \qquad m = e^{z-x}, \qquad n = e^{x-y}$

$$\frac{\partial l}{\partial x} = 0, \qquad \frac{\partial m}{\partial x} = -e^{z-x} = -m, \qquad \frac{\partial n}{\partial x} = e^{x-y} = n$$

$$\frac{\partial l}{\partial y} = e^{y-z} = l, \qquad \frac{\partial m}{\partial y} = 0, \qquad \frac{\partial n}{\partial y} = -e^{x-y} = -n$$

$$\frac{\partial l}{\partial z} = -e^{y-z} = -l, \qquad \frac{\partial m}{\partial z} = e^{z-x} = m, \qquad \frac{\partial n}{\partial z} = 0$$

$$u = f(e^{y-z}, e^{z-x}, e^{x-y}) = f(l, m, n).$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}\\ &= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}\cdot(-m) + \frac{\partial u}{\partial n}\cdot n\\ &= -m\frac{\partial u}{\partial m} + n\frac{\partial u}{\partial n} \qquad \text{...(1)}\end{aligned}$$

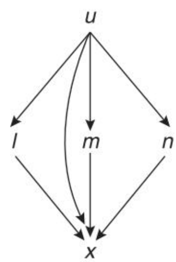

Fig. 8.61

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}\\ &= \frac{\partial u}{\partial l}\cdot l + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-n)\\ &= l\frac{\partial u}{\partial l} - n\frac{\partial u}{\partial n} \qquad \text{...(2)}\end{aligned}$$

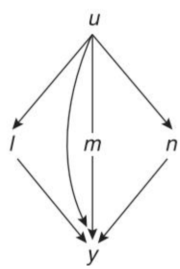

Fig. 8.62

and
$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$
$$=\frac{\partial u}{\partial l}(-l)+\frac{\partial u}{\partial m}\cdot m+\frac{\partial u}{\partial n}\cdot 0$$
$$=-l\frac{\partial u}{\partial l}+m\frac{\partial u}{\partial m} \qquad ...(3)$$

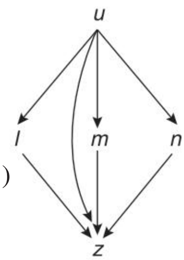

Fig. 8.63

Adding Eqs (1), (2) and (3),
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Example 21

If $x=\sqrt{vw}$, $y=\sqrt{wu}$, $z=\sqrt{uv}$ *and* ϕ *is a function of x, y and z, prove that* $x\dfrac{\partial\phi}{\partial x}+y\dfrac{\partial\phi}{\partial y}+z\dfrac{\partial\phi}{\partial z}=u\dfrac{\partial\phi}{\partial u}+v\dfrac{\partial\phi}{\partial v}+w\dfrac{\partial\phi}{\partial w}$.

Solution

$$x=\sqrt{vw}, \qquad y=\sqrt{wu}, \qquad z=\sqrt{uv}$$
$$\frac{\partial x}{\partial u}=0, \qquad \frac{\partial y}{\partial u}=\frac{1}{2}\sqrt{\frac{w}{u}}, \qquad \frac{\partial z}{\partial u}=\frac{1}{2}\sqrt{\frac{v}{u}}$$
$$\frac{\partial x}{\partial v}=\frac{1}{2}\sqrt{\frac{w}{v}}, \qquad \frac{\partial y}{\partial v}=0, \qquad \frac{\partial z}{\partial v}=\frac{1}{2}\sqrt{\frac{u}{v}}$$
$$\frac{\partial x}{\partial w}=\frac{1}{2}\sqrt{\frac{v}{w}}, \qquad \frac{\partial y}{\partial w}=\frac{1}{2}\sqrt{\frac{u}{w}}, \qquad \frac{\partial z}{\partial w}=0$$

$$\frac{\partial\phi}{\partial u}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial u}$$
$$=\frac{\partial\phi}{\partial x}\cdot 0+\frac{\partial\phi}{\partial y}\cdot\frac{1}{2}\sqrt{\frac{w}{u}}+\frac{\partial\phi}{\partial z}\cdot\frac{1}{2}\sqrt{\frac{v}{u}}$$

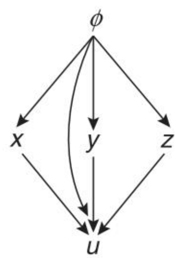

Fig. 8.64

$$u\frac{\partial\phi}{\partial u}=\frac{1}{2}\left[\frac{\partial\phi}{\partial y}\sqrt{uw}+\frac{\partial\phi}{\partial z}\sqrt{uv}\right]$$
$$=\frac{1}{2}\left(y\frac{\partial\phi}{\partial y}+z\frac{\partial\phi}{\partial z}\right) \qquad ...(1)$$

Also,
$$\frac{\partial\phi}{\partial v}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{\partial\phi}{\partial x}\cdot\frac{1}{2}\sqrt{\frac{w}{v}} + \frac{\partial\phi}{\partial y}\cdot 0 + \frac{\partial\phi}{\partial z}\frac{1}{2}\sqrt{\frac{u}{v}}$$

$$v\frac{\partial\phi}{\partial v} = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\sqrt{vw} + \frac{\partial\phi}{\partial z}\sqrt{uv}\right)$$

$$= \frac{1}{2}\left(x\frac{\partial\phi}{\partial x} + z\frac{\partial\phi}{\partial z}\right) \qquad ...(2)$$

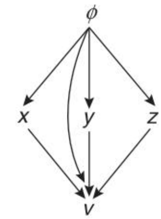

Fig. 8.65

and
$$\frac{\partial\phi}{\partial w} = \frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial w} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial w}$$

$$= \frac{\partial\phi}{\partial x}\cdot\frac{1}{2}\sqrt{\frac{v}{w}} + \frac{\partial\phi}{\partial y}\frac{1}{2}\sqrt{\frac{u}{w}} + \frac{\partial\phi}{\partial z}\cdot 0$$

$$w\frac{\partial\phi}{\partial w} = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\sqrt{vw} + \frac{\partial\phi}{\partial y}\sqrt{uw}\right)$$

$$= \frac{1}{2}\left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\right) \qquad ...(3)$$

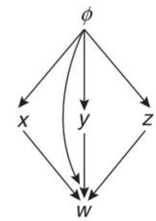

Fig. 8.66

Adding Eqs (1), (2) and (3),

$$u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v} + w\frac{\partial\phi}{\partial w} = x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + z\frac{\partial\phi}{\partial z}$$

Example 22

If $f(xy^2, z-2x) = 0$, *show that* $2x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 4x.$

Solution

Let
$$l = xy^2,\ m = z - 2x,$$
$$f(l, m) = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial x}$$

$$0 = \frac{\partial f}{\partial l}(y^2) + \frac{\partial f}{\partial m}\left(\frac{\partial z}{\partial x} - 2\right) \qquad [\because f(xy^2, z-2x) = 0]$$

$$\frac{\dfrac{\partial f}{\partial l}}{\dfrac{\partial f}{\partial m}} = \frac{2 - \dfrac{\partial z}{\partial x}}{y^2} \qquad \ldots(1)$$

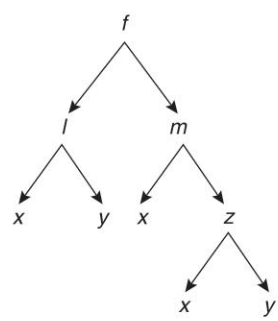

Fig. 8.67

and $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial y} = 0$$

$$\frac{\partial f}{\partial l}(2xy) + \frac{\partial f}{\partial m}\left(\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial m}} = -\frac{\frac{\partial z}{\partial y}}{2xy} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{2-\frac{\partial z}{\partial x}}{y^2} = -\frac{\frac{\partial z}{\partial y}}{2xy}$$

$$4x - 2x\frac{\partial z}{\partial x} = -y\frac{\partial z}{\partial y}$$

Hence, $$2x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 4x$$

Example 23

If $f\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0,$ *prove that* $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z.$

Solution

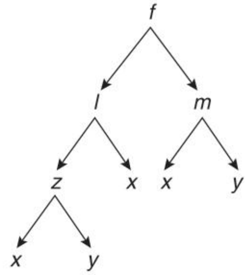

Fig. 8.68

Let $$l = \left(\frac{z}{x^3}\right),\ m = \frac{y}{x}$$

$$f(l, m) = 0.$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial f}{\partial m}\cdot\frac{\partial m}{\partial x}$$

$$0 = \frac{\partial f}{\partial l}\left(\frac{-3z}{x^4} + \frac{1}{x^3}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial m}\left(-\frac{y}{x^2}\right) \qquad \left[\because \quad f\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0\right]$$

$$\frac{\partial f}{\partial l}\left(-\frac{3z}{x^4} + \frac{1}{x^3}\frac{\partial z}{\partial x}\right) = \frac{\partial f}{\partial m}\left(\frac{y}{x^2}\right)$$

$$\frac{\partial f}{\partial l}\left(-3z + x\frac{\partial z}{\partial x}\right) = \frac{\partial f}{\partial m}(x^2 y) \qquad \ldots(1)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial f}{\partial m} \cdot \frac{\partial m}{\partial y}$$

and $$0 = \frac{\partial f}{\partial l}\left(\frac{1}{x^3}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial m}\left(\frac{1}{x}\right)$$

$$\frac{\partial f}{\partial l}\left(\frac{1}{x^3}\frac{\partial z}{\partial y}\right) = -\frac{\partial f}{\partial m}\left(\frac{1}{x}\right)$$

$$\frac{\partial f}{\partial l}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial f}{\partial m}(x^2) \qquad \text{...(2)}$$

Dividing Eq. (1) by Eq. (2),

$$\frac{-3z + x\dfrac{\partial z}{\partial x}}{\dfrac{\partial z}{\partial y}} = \frac{x^2 y}{-x^2}$$

$$-3z + x\frac{\partial z}{\partial x} = -y\frac{\partial z}{\partial y}$$

Hence, $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$$

Example 24

If $f(lx + my + nz, x^2 + y^2 + z^2) = 0$, *prove that*

$$(ly - mx) + (ny - mz)\frac{\partial z}{\partial x} + (lz - nx)\frac{\partial z}{\partial y} = 0.$$

Solution

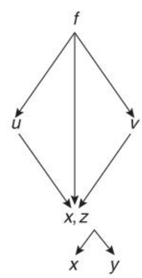

Fig. 8.69

Let $$u = lx + my + nz, \quad v = x^2 + y^2 + z^2$$

$$f(u, v) = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$0 = \frac{\partial f}{\partial u}\left(l + n\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(2x + 2z\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial f}{\partial u}\left(l + n\frac{\partial z}{\partial x}\right) = -\frac{\partial f}{\partial v}\left(2x + 2z\frac{\partial z}{\partial x}\right) \qquad \text{...(1)}$$

and $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$0 = \frac{\partial f}{\partial u}\left(m + n\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(2y + 2z\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial f}{\partial u}\left(m+n\frac{\partial z}{\partial y}\right)=-\frac{\partial f}{\partial v}\left(2y+2z\frac{\partial z}{\partial y}\right) \qquad \ldots(2)$$

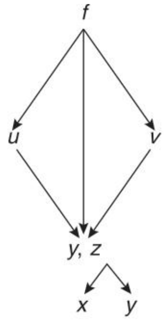

Fig. 8.70

Dividing Eq. (1) by Eq. (2),

$$\frac{l+n\dfrac{\partial z}{\partial x}}{m+n\dfrac{\partial z}{\partial y}}=\frac{x+z\dfrac{\partial z}{\partial x}}{y+z\dfrac{\partial z}{\partial y}}$$

$$ly+lz\frac{\partial z}{\partial y}+ny\frac{\partial z}{\partial x}+nz\frac{\partial z}{\partial x}.\frac{\partial z}{\partial y}=mx+nx\frac{\partial z}{\partial y}+mz\frac{\partial z}{\partial x}+nz\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$$

Hence, $(ly-mx)+(ny-mz)\dfrac{\partial z}{\partial x}+(lz-nx)\dfrac{\partial z}{\partial y}=0$

Example 25

If u is a function of x and y and x and y are functions of r and θ given by $x = e^r \cos\theta$, $y = e^r \sin\theta$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right]$$

Solution

$$u=f(x, y),\ x=e^r\cos\theta,\ y=e^r\sin\theta$$

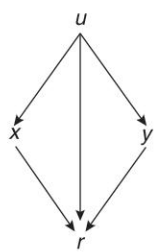

Fig. 8.71

$$\begin{aligned}\frac{\partial u}{\partial r}&=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}\\&=\frac{\partial u}{\partial x}e^r\cos\theta+\frac{\partial u}{\partial y}e^r\sin\theta\\&=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} \qquad \ldots(1)\end{aligned}$$

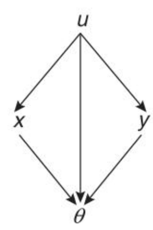

Fig. 8.72

Again,

$$\begin{aligned}\frac{\partial u}{\partial \theta}&=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}\\&=\frac{\partial u}{\partial x}(-e^r\sin\theta)+\frac{\partial u}{\partial y}e^r\cos\theta\\&=-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y} \qquad \ldots(2)\end{aligned}$$

Squaring and adding Eqs (1) and (2),

$$\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2 = x^2\left(\frac{\partial u}{\partial x}\right)^2+2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+y^2\left(\frac{\partial u}{\partial y}\right)^2+y^2\left(\frac{\partial u}{\partial x}\right)^2$$

$$-2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+x^2\left(\frac{\partial u}{\partial y}\right)^2$$

$$=(x^2+y^2)\left[\left(\frac{\partial u}{\partial x}\right)^2\right]+(x^2+y^2)\left[\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$=(x^2+y^2)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$=e^{2r}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right]$$

Example 26

If $x+y=2e^{\theta}\cos\phi$ *and* $x-y=2ie^{\theta}\sin\phi$, *prove that*

$$\frac{\partial v}{\partial \theta}+i\frac{\partial v}{\partial \phi}=2y\frac{\partial v}{\partial y}$$

Solution

$$x+y=2e^{\theta}\cos\phi,\quad x-y=2ie^{\theta}\sin\phi$$

$$2x=2e^{\theta}(\cos\phi+i\sin\phi)$$

$$x=e^{\theta}e^{i\phi}=e^{\theta+i\phi}$$

$$\frac{\partial x}{\partial \theta}=e^{\theta+i\phi}=x$$

$$\frac{\partial x}{\partial \phi}=ie^{\theta+i\phi}=ix$$

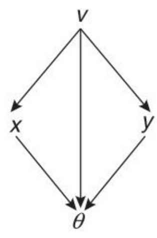

Fig. 8.73

And

$$2y=2e^{\theta}\left(\cos\theta-i\sin\phi\right)$$

$$y=e^{\theta}e^{-i\phi}=e^{\theta-i\phi}$$

$$\frac{\partial y}{\partial \theta}=e^{\theta-i\phi}=y$$

$$\frac{\partial y}{\partial \phi}=-ie^{\theta-i\phi}=-iy$$

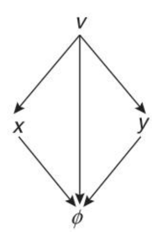

Fig. 8.74

Let $v = f(x, y)$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y \qquad \ldots(1)$$

$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \phi}$$

$$= \frac{\partial v}{\partial x}(ix) + \frac{\partial v}{\partial y}(-iy)$$

$$= i\left(x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y}\right) \qquad \ldots(2)$$

$$\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial \phi} = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + i^2\left(x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y}\right)$$

$$= x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} - x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$= 2y\frac{\partial v}{\partial y}$$

Example 27

If $z = f(x, y)$ *where* $x = \log u$, $y = \log v$, *show that* $\dfrac{\partial^2 z}{\partial x\,\partial y} = uv\dfrac{\partial^2 z}{\partial u\,\partial v}$.

Solution

$$z = f(x, y), \quad x = \log u, \quad y = \log v$$

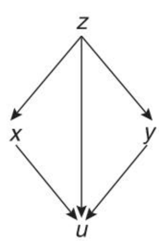

Fig. 8.75

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot\frac{1}{u} + \frac{\partial z}{\partial y}\cdot 0$$

$$= \frac{1}{u}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}\cdot 0 + \frac{\partial z}{\partial y}\frac{1}{v}$$

$$= \frac{1}{v}\frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial v} \equiv \frac{1}{v}\frac{\partial}{\partial y}$$

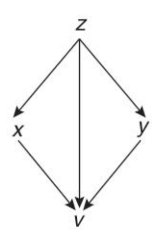

Fig. 8.76

$$\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)=\left(\frac{1}{v}\frac{\partial}{\partial y}\right)\left(\frac{1}{u}\frac{\partial z}{\partial x}\right)$$

Now,

$$\frac{\partial^2 z}{\partial v\partial u}=\frac{1}{uv}\frac{\partial^2 z}{\partial y\partial x}$$

$$\frac{\partial^2 z}{\partial x\partial y}=uv\frac{\partial^2 z}{\partial u\partial v}$$

EXERCISE 8.3

1. If $z=\tan^{-1}\left(\frac{x}{y}\right)$, where $x = 2t$, $y = 1 - t^2$, prove that $\frac{dz}{dt}=\frac{2}{1+t^2}$.

2. If $u = x^3 + y^3$, where $x = a \cos t$, $y = b \sin t$, find $\frac{du}{dt}$.

[**Ans.:** $-3a^3\cos^2 t\sin t+3b^2\sin^2 t\cos t$]

3. If $u = xe^yz$, where $y=\sqrt{a^2-x^2}$, $z = \sin^3 x$, find $\frac{du}{dx}$.

$$\left[\textbf{Ans.: } e^y z\left(1-\frac{x^2}{y}+3x\cot x\right)\right]$$

4. If $u=e^{\frac{r-x}{l}}$, where $r^2 = x^2 + y^2$ and l is a constant, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2}{l}\cdot\frac{\partial u}{\partial x}=\frac{u}{lr}.$$

5. If $u = \log r$ and $r=\sqrt{(x-a)^2+(y-b)^2}$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ if a, b are constants.

6. If $u^2 (x^2 + y^2 + z^2) = 1$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

$$\left[\textbf{Hint: } \text{Let } x^2+y^2+z^2=r^2, u=\frac{1}{r}\right]$$

7. If $u = r^m$, where $r=\sqrt{x^2+y^2+z^2}$, find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$.

[**Ans.:** $m\ (m + 1)\ r^{m-2}$]

8. If $u = f(r)$, where is given by the relation $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}$.

9. If $z = f(u,v)$, where $u = x^2 + y^2$, $y = 2$ then show that

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=2\sqrt{u^2+v^2}\left(\frac{\partial z}{\partial u}\right).$$

10. If $z = f(u,v)$, where $u = x^2 + y^2$, $v = x^2 - y^2$ then show that

(i) $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 4xy\dfrac{\partial z}{\partial u}$.

(ii) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = 4u\left[\left(\dfrac{\partial z}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2\right] + 8v\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial z}{\partial v}$

11. If $w = z\sin^{-1}\left(\dfrac{y}{x}\right)$, where $x = 3u^2 + 2v$, $y = 4u - 2v^3$, $z = 2u^2 - 3v^2$, find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

12. If $w = (x^2 + y - 2)^4 + (x - y + 2)^3$, where $x = u - 2v + 1$ and $y = 2u + v - 2$, find $\dfrac{\partial w}{\partial v}$ at $u = 0$, $v = 0$.

[**Ans.:** -882]

13. If $w = x + 2y + z^2$, $x = \dfrac{u}{v}$, $y = u^2 + e^v$, $z = 2u$, show that

$$u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v} = 12u^2 + 2ve^v.$$

14. If F is a function of x, y, z then show that
$u\dfrac{\partial F}{\partial u} + v\dfrac{\partial F}{\partial v} + w\dfrac{\partial F}{dw} = x\dfrac{\partial F}{\partial x} + 2y\dfrac{\partial F}{\partial y} + 3z\dfrac{\partial F}{dz}$, where $x = u + v + w$, $y = uv + vw + wu$, $z = uvw$.

15. If $z = f(x,y)$, $x = uv$, $y = \dfrac{u}{v}$, prove that

$$\frac{\partial z}{\partial x} = \frac{1}{2v}\frac{\partial z}{\partial u} + \frac{1}{2u}\frac{\partial z}{\partial u} \text{ and } \frac{\partial z}{\partial y} = \frac{v}{2}\frac{\partial z}{\partial u} - \frac{v^2}{2u}\frac{\partial z}{\partial v}.$$

16. If $x = u + v$, $y = uv$ and F is a function of x, y, prove that

$$\frac{\partial^2 F}{\partial u^2} - 2\frac{\partial^2 F}{\partial u\,\partial v} + \frac{\partial^2 F}{\partial v^2} = (x^2 - 4y)\frac{\partial^2 y}{\partial y^2} - 2\frac{\partial v}{\partial y}.$$

$$\left[\textbf{Hint:}\ \text{LHS} = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right)\right]$$

17. If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$, prove that

$$\frac{1}{x^{n-1}}\frac{\partial u}{\partial x} + \frac{1}{y^{n-1}}\frac{\partial u}{\partial y} + \frac{1}{z^{n-1}}\frac{\partial u}{\partial z} = 0.$$

18. If $z = f(x,y)$, where $x = u - av$, $y = u + av$, prove that

$$a^2\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} = 4a^2\frac{\partial^2 z}{\partial x\,\partial y}.$$

19. If $z = f(u, v)$, where $u = lx + my, v = ly - mx$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

20. If $x = u + av$ and $y = u + bv$, transform the equation $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x\, \partial y} + 3\frac{\partial^2 z}{\partial y^2} = 0$ into the equation $\frac{\partial^2 z}{\partial u\, \partial v} = 0$, find the values of a and b.

$\left[\textbf{Ans. :}\ a = 1, b = \frac{2}{3}\right]$

21. If $z = f(x,y), y = e^x, v = e^y$, prove that $\frac{\partial^2 z}{\partial x\, \partial y} = uv\frac{\partial^2 z}{\partial u\, \partial v}$.

22. If $z = f(x,y), x = \frac{\cos u}{v}, y = \frac{\sin u}{v}$, prove that

$$v\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} = (y - x)\frac{\partial z}{\partial x} - (y + x)\frac{\partial z}{\partial y}.$$

23. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0.$

24. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$

25. If $u = f(ax^2 + 2hxy + by^2)$ $v = \phi(ax^2 + 2hxy + by^2)$ show that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

26. If $x + y = 2e^{\theta}\cos\phi$ and $x - y = 2ie^{\theta}\sin\phi$, prove that $\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial \phi} = 2y\frac{\partial v}{\partial y}$.

27. Find the values of the constants and such that $u = x + y$ and $v = x + by$ transform the equation $9\frac{\partial^2 f}{\partial x^2} - 9\frac{\partial^2 f}{\partial x\, \partial y} + 2\frac{\partial^2 f}{\partial y^2} = 0$ into $\frac{\partial^2 f}{\partial u\, \partial v} = 0$, where f is a function of x and y.

$\left[\textbf{Ans. :}\ a = \frac{3}{2}, b = 3\right]$

28. If $x = y\cosh\theta$, $y = r\sinh\theta$ and $z = f(x,y)$, prove that

(i) $(x - y)\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = r\frac{\partial z}{\partial r} - \frac{\partial z}{\partial \theta}$

(ii) $(x^2 - y^2)\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = r^2\frac{\partial^2 z}{\partial r^2} + r\frac{\partial z}{\partial r} - \frac{\partial^2 z}{\partial \theta^2}$

29. If $x = e^v \sec u,\ y = e^v \tan u$ and $z = f(x, y)$, prove that

$$\cos u\left(\frac{\partial^2 z}{\partial u\,\partial v} - \frac{\partial z}{\partial u}\right) = xy\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + (x^2 + y^2)\frac{\partial^2 z}{\partial x\,\partial y}.$$

30. If $f(x^2y^3, z - 3x) = 0$, prove that $3x\dfrac{\partial z}{\partial x} - 2y\dfrac{\partial z}{\partial y} = 9x$.

31. If $f(y + z, x^2 + y^2 + z^2) = 0$, prove that $(y - z)\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = x$.

32. If $f(cx - az, cy - bz) = 0$, prove that $a\dfrac{\partial z}{\partial x} - b\dfrac{\partial z}{\partial y} = c$.

33. If $x = \dfrac{\cos\theta}{u},\ y = \dfrac{\sin\theta}{u}$, prove that $\left(\dfrac{\partial x}{\partial u}\right)_\theta \left(\dfrac{\partial u}{\partial x}\right)_y = \cos^2\theta$.

34. If $x^2 = au + bv,\ y^2 = au - bv$, prove that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u.$$

35. If $u = ax + by,\ v = bx - ay$, prove that

(i) $\left(\dfrac{\partial y}{\partial v}\right)_x \left(\dfrac{\partial v}{\partial y}\right)_u = \dfrac{a^2 + b^2}{a^2}$ (ii) $\left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial u}\right)_v = \dfrac{a^2}{a^2 + b^2}$

8.7 IMPLICIT DIFFERENTIATION

Any function of the type $f(x, y) = c$ is called an implicit function, where y is a function of x and c is a constant.

If $\quad f(x, y) = c \quad$ then $\quad \dfrac{dy}{dx} = -\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}$

Proof If $f(x, y)$ is a function of x and y, where y is a function of x then total differential coefficient of f w.r.t. x is given by

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

But $\quad f(x, y) = c$

$$\frac{df}{dx} = 0$$

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$$

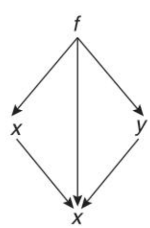

Fig. 8.77

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Example 1

If $y\log(\cos x) = x\log(\sin y)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Let $$f(x, y) = y\log(\cos x) - x\log(\sin y)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{y\frac{1}{\cos x}(-\sin x) - \log(\sin y)}{\log\cos x - \frac{x}{\sin y}\cdot\cos y}$$

$$= \frac{y\tan x + \log\sin y}{\log\cos x - x\cot y}$$

Example 2

If $x^3 + y^3 = 3axy$, *find* $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Let $$f(x, y) = x^3 + y^3 - 3axy$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{3x^2 - 3ay}{3y^2 - 3ax}$$

$$= -\frac{x^2 - ay}{y^2 - ax}$$

$$= \frac{ay - x^2}{y^2 - ax}$$

Example 3

If $y \sin x = x \cos y$, *find* $\frac{dy}{dx}$.

Solution

Let $$f(x, y) = y \sin x - x \cos y$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{y \cos x - \cos y}{\sin x + x \sin y}$$

$$= \frac{\cos y - y \cos x}{\sin x + x \sin y}$$

Example 4

If $x^y + y^x = c$, *find* $\frac{dy}{dx}$.

Solution

Let $$f(x, y) = x^y + y^x - c$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

Example 5

If $(\cos x)^y = (\sin y)^x$, *find* $\frac{dy}{dx}$. **[Winter 2014]**

Solution

$$(\cos x)^y = (\sin y)^x$$

Taking logarithm of both the sides,

$$y \log \cos x = x \log \sin y$$

Let $$f(x, y) = y \log \cos x - x \log \sin y$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{\frac{y}{\cos x}(-\sin x) - \log \sin y}{\log \cos x - \frac{x}{\sin y}(\cos y)}$$

$$= \frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}$$

Example 6

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, *find* $\dfrac{dy}{dx}$.

Solution

Let $$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{2(ax + hy + g)}{2(hx + by + f)}$$

$$= -\frac{ax + hy + g}{hx + by + f}$$

Example 7

If $u = \sin(x^2 + y^2)$ *and* $a^2x^2 + b^2y^2 = c^2$, *find* $\dfrac{du}{dx}$.

Solution

$$u = \sin(x^2 + y^2) \text{ and } a^2x^2 + b^2y^2 = c^2$$

$$\frac{\partial u}{\partial x} = \cos(x^2 + y^2) \cdot 2x$$

$$\frac{\partial u}{\partial y} = \cos(x^2 + y^2) \cdot 2y$$

Let $$f(x, y) = a^2x^2 + b^2y^2 - c^2$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{2a^2x}{2b^2y}$$

$$= -\frac{a^2x}{b^2y}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\left(-\frac{a^2x}{b^2y}\right)$$

$$= 2x\cos(x^2+y^2)\cdot\left(1-\frac{a^2}{b^2}\right)$$

Example 8

If $u = x\log(xy)$ *where* $x^3+y^3+3xy=1$, *find* $\frac{du}{dx}$.

Solution

$$u = x\log(xy)$$

$$= x(\log x + \log y)$$

$$\frac{\partial u}{\partial x} = x\cdot\frac{1}{x} + (\log x + \log y)$$

$$= 1 + \log x + \log y$$

$$\frac{\partial u}{\partial y} = x\frac{1}{y} = \frac{x}{y}$$

Let

$$f(x, y) = x^3 + y^3 + 3xy - 1$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{3x^2+3y}{3y^2+3x}$$

$$= -\frac{x^2+y}{y^2+x}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 1 + \log x + \log y + \frac{x}{y}\left(-\frac{x^2+y}{y^2+x}\right)$$

$$= 1 + \log(xy) - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

Example 9

If $u = \tan^{-1}\left(\frac{x}{y}\right)$ *where* $x^2 + y^2 = a^2$, *find* $\frac{\mathrm{d}u}{\mathrm{d}x}$.

Solution

$$u = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2}$$

Let $\qquad f(x, y) = x^2 + y^2 - a^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{2x}{2y} = -\frac{x}{y}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial x} . \frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial u}{\partial y} . \frac{dy}{dx}$$

$$= \frac{y}{x^2 + y^2} - \frac{x}{x^2 + y^2}\left(-\frac{x}{y}\right)$$

$$= \frac{y}{x^2 + y^2} + \frac{x^2}{y(x^2 + y^2)}$$

$$= \frac{y^2 + x^2}{y(x^2 + y^2)}$$

$$= \frac{1}{y}$$

Example 10

If $u = \phi(x, y)$ *and* $f(x, y) = 0$, *prove that* $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\phi_x f_y - \phi_y f_x}{f_y}$.

Solution

$$f(x, y) = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

$$u = \phi(x, y)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= \phi_x + \phi_y\frac{dy}{dx}$$

$$= \phi_x + \phi_y\left(-\frac{f_x}{f_y}\right)$$

$$= \frac{\phi_x f_y - \phi_y f_x}{f_y}$$

Example 11

If $f(x, y) = 0$, $\phi(x, z) = 0$, *show that* $\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial z}$.

Solution

$$f(x, y) = 0 \text{ and } \phi(x, z) = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \text{ and } \frac{dz}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}$$

$$\frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\left(-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right)}{\left(-\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}\right)}$$

$$\frac{dy}{dz} = \frac{\left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right)}{\left(\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}\right)}$$

Hence, $$\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz}=\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial z}$$

Example 12

If $\phi(x, y, z)=0$, *prove that* $\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial y}{\partial z}\right)_z=-1.$

Solution

$$\phi(x, y, z)=0$$

$$\left(\frac{dz}{dy}\right)_x=\left(\frac{\partial z}{\partial y}\right)_x=-\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}} \qquad \text{...(1)}$$

$$\left(\frac{dx}{dz}\right)_y=\left(\frac{\partial x}{\partial z}\right)_y=-\frac{\frac{\partial \phi}{\partial z}}{\frac{\partial \phi}{\partial x}} \qquad \text{...(2)}$$

$$\left(\frac{dy}{dx}\right)_z=\left(\frac{\partial y}{\partial x}\right)_z=-\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} \qquad \text{...(3)}$$

From Eqs (1), (2) and (3),

$$\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial y}{\partial x}\right)_z=\left(-\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}\right)\left(-\frac{\frac{\partial \phi}{\partial z}}{\frac{\partial \phi}{\partial x}}\right)\left(-\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}\right)$$
$$=-1$$

EXERCISE 8.4

1. If $x^3+y^3-3axy=0$, find $\frac{dy}{dx}$.

$$\left[\textbf{Ans.:}\ \frac{ay-x^2}{y^2-ax}\right]$$

2. If $x^3+3x^2+6xy^2+y^3=1$, find $\frac{dy}{dx}$.

$$\left[\textbf{Ans.:}\ -\frac{(x^2+2x+2y^2)}{(4xy+y^2)}\right]$$

3. If $x^y=y^x$, prove that $\frac{dy}{dx}=\frac{y(y-x\log y)}{x(x-y\log x)}$.

4. If $f(x, y) = x \sin(x - y) - (x + y) = 0$, find $\dfrac{dy}{dx}$.

$$\left[\textbf{Ans.: } \frac{[\sin(x-y)](1+x)-1}{x\cos(x-y)+1}\right]$$

5. If $y^{x^y} = \sin x$, find $\dfrac{dy}{dx}$.

$$\left[\textbf{Hint: } f = x^y \log y - \log\sin x, \text{ let } x^y = z, \log z = y\log x \text{ find } \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \text{ and then } \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}\right]$$

$$\left[\textbf{Ans.: } \frac{-(yx^{y-1}\log y - \cot x)}{x^y \log x \log y + x^y y^{-1}}\right]$$

6. If $x^5 + y^5 = 5a^3x^2$, find $\dfrac{d^2y}{dx^2}$.

$$\left[\textbf{Ans. : } \frac{6a^3x^2(a^3+x^3)}{y^9}\right]$$

7. If $xy^3 - yx^3 = 6$ is the equation of curve, find the slope of the tangent at the point (1, 2).

$$\left[\textbf{Hint : } \text{Find } \frac{dy}{dx} \text{ at } (1, 2)\right]$$

$$\left[\textbf{Ans. : } -\frac{2}{11}\right]$$

8. Find $\dfrac{d^2y}{dx^2}$, if $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$.

$$\left[\textbf{Ans. : } \frac{a}{(1-x^2)^{\frac{3}{2}}}\right]$$

9. If $u = x \log xy$ and $x^3 + y^3 + 3xy - 1 = 0$, find $\dfrac{du}{dx}$.

$$\left[\textbf{Hint : } \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}\ \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right]$$

$$\left[\textbf{Ans. : } 1 + \log xy - \frac{x}{y}\left(\frac{x^2+ay}{y^2+ax}\right)\right]$$

10. If $x^m + y^m = b^m$, show that $\dfrac{d^2y}{dx^2} = -(m-1)b^m \dfrac{x^{m-2}}{y^{2m-1}}$.

11. If $u = x^2y$ and $x^2 + xy + y^2 = 1$, find $\dfrac{du}{dx}$.

12. If $x^3 + y^3 = 3ax^2$, find $\dfrac{d^2y}{dx^2}$.

$$\left[\textbf{Ans.: } -\frac{2a^2x^2}{y^5}\right]$$

8.8 GRADIENT AND DIRECTIONAL DERIVATIVE

8.8.1 Gradient

The gradient of a function $z = f(x, y)$ is written as grad f and is defined as

$$\text{grad } f = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

grad f is a vector function.

If $f(x, y, z)$ is a function of three independent variables, its total differential df is given as

$$\begin{aligned} \mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z \\ &= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\cdot(\hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z) \\ &= \nabla f \cdot \mathrm{d}\bar{r} \qquad \begin{bmatrix} \because \ \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \\ \mathrm{d}\bar{r} = \hat{i}\mathrm{d}x + \hat{j}\mathrm{d}y + \hat{k}\mathrm{d}z \end{bmatrix} \\ &= |\nabla f||\mathrm{d}\bar{r}|\cos\theta \end{aligned}$$

where θ is the angle between the vectors ∇f and $\mathrm{d}\bar{r}$. If $\mathrm{d}\bar{r}$ and $\nabla\phi$ are in the same direction, then $\theta = 0$.

$$\mathrm{d}f = |\nabla f||\mathrm{d}\bar{r}|$$

$\cos\theta = 1$ is the maximum value of $\cos\theta$. Hence, df is maximum at $\theta = 0$.

8.8.2 Directional Derivative

The rate of change of a function of several variables in the direction of the coordinate axes is called as directional derivative.

For a function $z = f(x, y, z)$, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are the directional derivative of f in the direction of coordinate axes.

The directional derivative of a function $f(x, y, z)$ in the direction of vector $\bar{a}$ is the component of ∇f in the direction of $\bar{a}$. If $\hat{a}$ is the unit vector in the direction of $\bar{a}$, the directional derivative of f in the direction of $\bar{a}$ is given by

$$D_a f = \nabla f \cdot \hat{a} = \nabla f \cdot \frac{a}{|\bar{a}|}$$

Example 1

Find ∇f *at* $(1, -2, 1)$ *if* $f = 3x^2y - y^3z^2$.

Solution

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}(6xy - 0) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(0 - 2y^3z)$$

At $x = 1$, $y = -2$, $z = 1$,

$$\nabla f = \hat{i}(-12) + \hat{j}(3 - 12) + \hat{k}(16)$$

$$\nabla f \text{ at } (1, -2, 1) = -12\hat{i} - 9\hat{j} + 16\hat{k}$$

Example 2

Find the directional derivatives of $f = xy^2 + yz^2$ *at the point* $(2, -1, 1)$ *in the direction of the vector* $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}y^2 + \hat{j}(2xy + z^2) + \hat{k}(2yz)$$

At the point $(2, -1, 1)$,

$$\nabla f = \hat{i} + \hat{j}(-4 + 1) + \hat{k}(-2)$$

$$= \hat{i} - 3\hat{j} - 2\hat{k}$$

Directional derivative in the direction of the vector $\bar{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ is

$$D_a f = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= (\hat{i} - 3\hat{j} - 2\hat{k}) \cdot \frac{(\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{1 + 4 + 4}}$$

$$= \frac{(1 - 6 - 4)}{3}$$

$$= -3$$

Example 3

Find the directional derivatives of $f = \dfrac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$ *at the point* $P(1, -1, 1)$ *in the direction of* $\bar{a} = \hat{i} + \hat{j} + \hat{k}$.

Solution

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\left[-\frac{2x}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \hat{j}\left[-\frac{2y}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \hat{k}\left[-\frac{2z}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$$

$$= -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

At the point (1, –1, 1),

$$\Delta f = -\frac{(\hat{i}-\hat{j}+\hat{k})}{(3)^{\frac{3}{2}}}$$

Directional derivative in the direction of $\overline{a} = \hat{i}+\hat{j}+\hat{k}$ is

$$D_a f = \nabla f \cdot \frac{\overline{a}}{|\overline{a}|}$$

$$= -\frac{(\hat{i}-\hat{j}-\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})}{(3)^{\frac{3}{2}}\sqrt{1+1+1}}$$

$$= \frac{-1+1+1}{3^2}$$

$$= -\frac{1}{9}$$

Example 4

Evaluate ∇e^{r^2}, *where* $r^2 = x^2 + y^2 + z^2$.

Solution

$$r^2 = x^2 + y^2 + z^2$$

Differentiating partially w.r.t. x, y, and z respectively,

$$2r\frac{\partial r}{\partial x} = 2x, \qquad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z, \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla e^{r^2} = \hat{i}\frac{\partial e^{r^2}}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial z}$$

$$= \hat{i}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \hat{i}\left(e^{r^2}\cdot 2r\right)\frac{x}{r} + \hat{j}\left(e^{r^2}\cdot 2r\right)\frac{y}{r} + \hat{k}\left(e^{r^2}\cdot 2r\right)\frac{z}{r}$$

$$= 2e^{r^2}(x\hat{i} + y\hat{j} + z\hat{k})$$

Example 5

Prove that $\nabla r^n = nr^{n-2}\overline{r}, \overline{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\overline{r}|$.

Solution

$$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned}
\nabla r^n &= \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}\\
&= \hat{i}\frac{\partial r^n}{\partial r}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r^n}{\partial r}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r^n}{\partial r}\frac{\partial r}{\partial z}\\
&= \hat{i}nr^{n-1}\frac{x}{r} + \hat{j}nr^{n-1}\frac{y}{r} + \hat{k}nr^{n-1}\frac{z}{r}\\
&= nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})\\
&= nr^{n-2}\overline{r}
\end{aligned}$$

EXERCISE 8.5

1. Find ∇f if

(i) $f = \log(x^2 + y^2 + z^2)$

(ii) $f = (x^2 + y^2 + z^2)e^{-\sqrt{x^2+y^2+z^2}}$

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{2\overline{r}}{r^2} \qquad \text{(ii)}\ (2-r)e^{-r}\overline{r} \quad \text{where } r = x\hat{i} + y\hat{i} + z\hat{k}, r = |\overline{r}|\right]$$

2. Find ∇f and $|\nabla f|$ if

(i) $f = 2xz^4 - x^2y$ at $(2, -2, -1)$

(ii) $f = 2xz^2 - 3xy - 4x$ at $(1, -1, 2)$

$$\left[\begin{array}{ll}\textbf{Ans.:}\ \text{(i)}\ 10\hat{i} - 4\hat{j} - 16\hat{k}, & 2\sqrt{93}\\ \qquad\ \ \text{(ii)}\ 7\hat{i} - 3\hat{j} + 8\hat{k}, & 2\sqrt{29}\end{array}\right]$$

3. Find the directional derivative of $f = xy + yz + zx$ at (1, 2, 0) in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

$$\left[\textbf{Ans.:}\ \frac{10}{3}\right]$$

4. Find the directional derivative of $f = x^2yz + 4xz^2$ at (1, –2, –1) in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

$$\left[\textbf{Ans.}: \frac{37}{\sqrt{3}}\right]$$

5. Find the directional derivative of $f = x^2yz^2$ along the curve $x = e^{-t}$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t = 0$.

$$\left[\textbf{Ans.}: -\frac{1}{\sqrt{6}}\right]$$

8.9 TANGENT PLANE AND NORMAL LINE

Let P be any point on the surface $f(x, y, z) = 0$ or $z = f(x, y)$ and let Q be any other point on it. Any curve is taken on the surface joining Q to P. As Q tends to P along this curve, the chord PQ tends, in general, to a definite straight line. This straight line is called a *tangent line* to the surface at P. Since different curves can be obtained by joining Q to P, different tangent lines are obtained at P. All these tangent lines lie in a plane called the *tangent plane* to the surface at P.

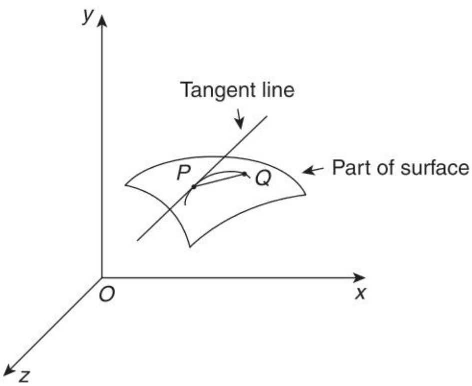

Fig. 8.78

The equation of the tangent plane at $P(x_0, y_0, z_0)$ to the surface $f(x, y, z) = 0$ is

$$(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$$

where

$$f_x(x_0, y_0, z_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0, z_0)}$$

$$f_y(x_0, y_0, z_0) = \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0, z_0)}$$

$$f_z(x_0, y_0, z_0) = \left.\frac{\partial f}{\partial z}\right|_{(x_0, y_0, z_0)}$$

The line through P, perpendicular to the tangent plane is called the *normal* to the surface at P. The equations of the normal line to the surface at $P(x_0, y_0, z_0)$ are

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

Example 1

Find the equations of the tangent plane and normal line to the surface $xyz = 6$ *at* $(1, 2, 3)$. **[Winter 2014]**

Solution

Let $$f(xyz) = xyz - 6$$

$$f_x(x, y, z) = yz, \qquad f_x(1, 2, 3) = 6$$

$$f_y(x, y, z) = xz, \qquad f_y(1, 2, 3) = 3$$

$$f_z(x, y, z) = xy, \qquad f_z(1, 2, 3) = 2$$

Hence, the equation of the tangent plane at (1, 2, 3) is

$$6(x-1)+3(y-2)+2(z-3)=0$$

$$6x-6+3y-6+2z-6=0$$

$$6x+3y+2z=18$$

The set of equations of the normal line is

$$\frac{x-1}{6}=\frac{y-2}{3}=\frac{z-3}{2}$$

Example 2

Find the equations of the tangent plane and normal line to the surface $x^2 + y^2 + z - 9 = 0$ *at the point* $(1, 2, 4)$. **[Winter 2016]**

Solution

Let $$f(x, y, z) = x^2 + y^2 + z - 9$$

$$f_x(x, y, z) = 2x, \qquad f_x(1, 2, 4) = 2$$

$$f_y(x, y, z) = 2y, \qquad f_y(1, 2, 4) = 4$$

$$f_z(x, y, z) = 1, \qquad f_z(1, 2, 4) = 1$$

Hence, the equation of the tangent plane at (1, 2, 4) is

$$2(x-1) + 4(y-2) + 1(z-4) = 0$$

$$2x - 2 + 4y - 8 + z - 4 = 0$$

$$2x + 4y + z - 14 = 0$$

$$2x + 4y + z = 14$$

The set of equations of the normal line is

$$\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-4}{1}$$

Example 3

Find the equations of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 3$ *at the point* (1, 1, 1).

Solution

Let $$f(x, y, z) = x^2 + y^2 + z^2 - 3$$

$$f_x(x, y, z) = 2x, \qquad f_x(1, 1, 1) = 2$$
$$f_y(x, y, z) = 2y, \qquad f_y(1, 1, 1) = 2$$
$$f_z(x, y, z) = 2z, \qquad f_z(1, 1, 1) = 2$$

Hence, the equation of the tangent plane at (1, 1, 1) is

$$2(x-1)+2(y-1)+2(z-1)=0$$
$$2x-2+2y-2+2z-2=0$$
$$2x+2y+2z=6$$
$$x+y+z=3$$

The set of equations of the normal line is

$$\frac{x-1}{2}=\frac{y-1}{2}=\frac{z-1}{2}$$
$$x-1=y-1=z-1$$

Example 4

Find the equations of the tangent plane and normal line to the surface $2x^2 + y + 2z = 3$ *at* (2, 1, – 3). **[Summer 2017]**

Solution

Let $$f(x, y, z) = 2x^2 + y^2 + 2z - 3$$

$$f_x(x, y, z) = 4x, \qquad f_x(2, 1, -3) = 8$$
$$f_y(x, y, z) = 2y, \qquad f_y(2, 1, -3) = 2$$
$$f_z(x, y, z) = 2, \qquad f_z(2, 1, -3) = 2$$

Hence, the equation of the tangent plane at (2, 1, –3) is

$$8(x-2) + 2(y-1) + 2(z+3) = 0$$
$$8x - 16 + 2y - 2 + 2z + 6 = 0$$
$$8x + 2y + 2z - 12 = 0$$
$$8x + 2y + 2z = 12$$
$$4x + y + z = 6$$

The set of equations of the normal line is

$$\frac{x-2}{8}=\frac{y-1}{2}=\frac{z+3}{2}$$
$$\frac{x-2}{4}=y-1=z+3$$

Example 5

Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ *to the ellipsoid* $\dfrac{x^2}{4}+y^2+\dfrac{z^2}{9}=3$.

[Winter 2013; Summer 2016]

Solution

Let $$f(x, y, z)=\frac{x^2}{4}+y^2+\frac{z^2}{9}-3$$

$$f_x(x, y, z)=\frac{x}{2}, \qquad f_x(-2, 1, -3)=-1$$

$$f_y(x, y, z)=2y, \qquad f_y(-2, 1, -3)=2$$

$$f_z(x, y, z)=\frac{2z}{9}, \qquad f_z(-2, 1, -3)=-\frac{2}{3}$$

Hence, the equation of the tangent plane at $(-2, 1, -3)$ is

$$-1(x+2)+2(y-1)-\frac{2}{3}(z+3)=0$$

$$3(x+2)-6(y-1)+2(z+3)=0$$

$$3x+6-6y+6+2z+6=0$$

$$3x-6y+2z=-18$$

The set of equations of the normal line is

$$\frac{x+2}{-1}=\frac{y-1}{2}=\frac{z+3}{\frac{-2}{3}}$$

Example 6

Find the equations of the tangent plane and normal line to the surface $z=2x^2+y^2$ *at the point* $(1, 1, 3)$.

Solution

Let $$f(x, y, z)=2x^2+y^2-z$$

$$f_x(x, y, z)=4x, \qquad f_x(1, 1, 3)=4$$

$$f_y(x, y, z)=2y, \qquad f_y(1, 1, 3)=2$$

$$f_z(x, y, z)=-1, \qquad f_z(1, 1, 3)=-1$$

Hence, the equation of the tangent plane at (1, 1, 3) is

$$\begin{aligned} 4(x-1)+2(y-1)-1(z-3)&=0 \\ 4x-4+2y-2-z+3&=0 \\ 4x+2y-z&=3 \end{aligned}$$

The set of equations of the normal line is

$$\frac{x-1}{4}=\frac{y-1}{2}=\frac{z-3}{-1}$$

Example 7

Find the equations of the tangent plane and normal line to the surface $x^2yz+3y^2=2xz^2-8z$ *at the point* (1, 2, –1).

Solution

Let $$f(x,y,z)=x^2yz+3y^2-2xz^2+8z$$

$$\begin{aligned} f_x(x,y,z)&=2xyz-2z^2, & f_x(1,2,-1)&=2(1)(2)(-1)-2(-1)^2=-6 \\ f_y(x,y,z)&=x^2z+6y, & f_y(1,2,-1)&=(1)^2(-1)+6(2)=11 \\ f_z(x,y,z)&=x^2y-4xz+8, & f_z(1,2,-1)&=(1)^2(2)-4(1)(-1)+8=14 \end{aligned}$$

Hence, the equation of the tangent plane at (1, 2, –1) is

$$\begin{aligned} -6(x-1)+11(y-2)+14(z+1)&=0 \\ -6x+6+11y-22+14z+14&=0 \\ 6x-11y-14z+2&=0 \end{aligned}$$

The set of equations of the normal line is

$$\frac{x-1}{-6}=\frac{y-2}{11}=\frac{z+1}{14}$$

Example 8

Find the equations of the tangent plane and normal line to the surface $2xz^2-3xy-4x=7$ *at the point* (1, –1, 2). **[Summer 2015]**

Solution

Let $$f(x,y,z)=2xz^2-3xy-4x=7$$

$$\begin{aligned} f_x(x,y,z)&=2z^2-3y-4, & f_x(1,-1,2)&=(2)^2-3(-1)-4=7 \\ f_y(x,y,z)&=-3x, & f_y(1,-1,2)&=-3(1)=-3 \\ f_z(x,y,z)&=4xz, & f_z(1,-1,2)&=4(1)(2)=8 \end{aligned}$$

Hence, the equation of the tangent plane at (1, –1, 2) is

$$7(x-1)+(-3)(y+1)+8(z-2)=0$$
$$7x-7-3y-3+8z-16=0$$
$$7x-3y+8z=26$$

The set of equations of the normal line is

$$\frac{x-1}{7}=\frac{y+1}{-3}=\frac{z-2}{8}$$

Example 9

Find the equation of the tangent plane and normal line to the surface $z+8=xe^{y}\cos z$ *at the point* (8, 0, 0).

Solution

Let $f(x, y, z)=xe^{y}\cos z-z-8$

$$f_x(x, y, z)=e^{y}\cos z \qquad f_x(8, 0, 0)=1$$
$$f_y(x, y, z)=xe^{y}\cos z \qquad f_y(8, 0, 0)=8$$
$$f_z(x, y, z)=\sin z-1 \qquad f_z(8, 0, 0)=-1$$

Hence, the equation of the tangent plane at (8,0,0) is

$$1(x-8)+8(y-0)-1(z-0)=0$$
$$x-8+8y-z=0$$
$$x+8y-z-8=0$$

The set of equations of the normal line is

$$\frac{x-8}{1}=\frac{y-0}{8}=\frac{z-0}{-1}$$
$$x-8=\frac{y}{8}=-z$$

Example 10

Find the equations of the tangent plane and normal line to the surface $\cos \pi x - x^2y + e^{xz} + yz = 4$ *at the point* $P(0, 1, 2)$. **[Winter 2015]**

Solution

Let $f(x, y, z) = \cos \pi x - x^2y + e^{xz} + yz = 4$

$$f_x(x, y, z)=-\pi\sin\pi x-2xy+ze^{xz}, \qquad f_x(0, 1, 2)=2$$
$$f_y(x, y, z)=-x^2+z, \qquad f_y(0, 1, 2)=2$$
$$f_z(x, y, z)=xe^{xz}+y, \qquad f_z(0, 1, 2)=1$$

Hence, the equation of the tangent plane at $P(0, 1, 2)$ is

$$2(x-0)+2(y-1)+1(z-2)=0$$
$$2x+2y-2+z-2=0$$
$$2x+2y+z=-4$$

The set of equations of the normal line is

$$\frac{x}{2}=\frac{y-1}{2}=\frac{z-2}{1}$$

Example 11

Show that the plane tangent to the surface $z = x^2 + y^2$ at the point (a, b, c) intersects the z-axis at a point where $z = -c$.

Solution

Let $$f(x, y, z) = x^2 + y^2 - z$$

$$f_x(x, y, z) = 2x, \qquad f_x(a, b, c) = 2a$$
$$f_y(x, y, z) = 2y, \qquad f_y(a, b, c) = 2b$$
$$f_z(x, y, z) = -1, \qquad f_z(a, b, c) = -1$$

Hence, the equation of the tangent plane at (a, b, c) is

$$2a(x-a)+2b(y-b)-1(z-c)=0$$
$$2ax-2a^2+2by-2b^2-z+c=0$$
$$2ax+2by-z-2a^2-2b^2+c=0 \qquad \ldots(1)$$

The point of intersection of tangent plane with the z-axis is obtained by putting $x = 0$, $y = 0$ in Eq. (1),

$$-z-2a^2-2b^2+c=0$$
$$z=c-2a^2-2b^2$$
$$=c-2(a^2+b^2) \qquad \ldots(2)$$

Since the point (a, b, c) lies on the surface $x^2 + y^2 = z$, $\qquad \ldots(3)$

$$a^2+b^2=c$$

Substituting Eq. (3) in Eq. (2),

$$z=c-2c=-c$$

Example 12

Show that the plane $4x-6y-z+14=0$ *touches the surface* $x^2+3y^2+2z=0$ *and find the point of contact.*

Solution

Let
$$f(x, y, z) = x^2 + 3y^2 + 2z$$
$$f_x(x, y, z) = 2x$$
$$f_y(x, y, z) = 6y$$
$$f_z(x, y, z) = 2$$

Let $P(x_0, y_0, z_0)$ be a point on the surface

$$f_x(x_0, y_0, z_0) = 2x_0$$
$$f_y(x_0, y_0, z_0) = 6y_0$$
$$f_z(x_0, y_0, z_0) = 2$$

Hence, the equation of the tangent plane at (x_0, y_0, z_0) is

$$2x_0(x-x_0)+6y_0(y-y_0)+2(z-z_0)=0$$
$$2xx_0-2x_0^2+6yy_0-6y_0^2+2z-2z_0=0$$
$$xx_0+3yy_0+z+z_0=0$$

Comparing with the plane equation $4x - 6y - z + 14 = 0$

$$\frac{x_0}{4}=\frac{3y_0}{-6}=\frac{1}{-1}=\frac{z_0}{14}$$
$$\therefore\ x_0 = -4,\ y_0 = 2,\ z_0 = -14$$

The point $(-4, 2, -14)$ satisfies the equation of the surface and the tangent plane. Hence, the point of contact is $(-4, 2, -14)$.

Example 13

Show that the surfaces $z = xy - 2$ *and* $x^2+y^2+z^2=3$ *have the same tangent plane at* $(1, 1, -1)$. **[Summer 2014]**

Solution

For the surface $f(x, y, z) = xy - z - 2$,

$$f_x(x, y, z) = y, \qquad f_x(1, 1, -1) = 1$$
$$f_y(x, y, z) = x, \qquad f_y(1, 1, -1) = 1$$
$$f_z(x, y, z) = -1, \qquad f_z(1, 1, -1) = -1$$

Hence, the equation of the tangent plane is

$$1(x-1)+1(y-1)-1(z+1)=0$$
$$x-1+y-1-z-1=0$$
$$x+y-z-3=0$$

For the surface $f(x, y, z)=x^2+y^2+z^2-3$,

$$f_x(x, y, z)=2x \qquad f_x(1, 1, -1)=2$$
$$f_y(x, y, z)=2y \qquad f_y(1, 1, -1)=2$$
$$f_z(x, y, z)=2z \qquad f_z(1, 1, -1)=-2$$

Hence, the equation of the tangent plane is

$$2(x-1)+2(y-1)-2(z+1)=0$$
$$2x-2+2y-2-2z-2=0$$
$$x+y-z-3=0$$

Hence, the surfaces $z=xy-2$ and $x^2+y^2+z^2=3$ have the same tangent plane at (1, 1, −1).

Example 14

Find equations of the normal line of the sphere $x^2+y^2+z^2=6$ at the point (a, b, c). Show that the normal line passes through the origin.

Solution

Let $f(x, y, z)=x^2+y^2+z^2-6$

$$f_x(x, y, z)=2x, \qquad f_x(a, b, c)=2a$$
$$f_y(x, y, z)=2y, \qquad f_y(a, b, c)=2b$$
$$f_z(x, y, z)=2z, \qquad f_z(a, b, c)=2c$$

Hence, the set of equations of the normal line at (a, b, c) is

$$\frac{x-a}{2a}=\frac{y-b}{2b}=\frac{z-c}{2c}$$

At the origin, $x=0$, $y=0$, $z=0$.

$$\frac{0-a}{2a}=\frac{0-b}{2b}=\frac{0-c}{2c}=-\frac{1}{2}$$

The point (0,0,0) satisfies the equation of the normal line.
Hence, the normal line passes through the origin (0, 0, 0).

EXERCISE 8.6

1. Find the equations of the tangent plane and the normal line to the following surfaces at indicated points.

(a) $z = \sqrt{4 - x^2 - 2y^2}$, $(1, -1, 1)$

(b) $z = \sqrt{1 - x^2 - y^2}$, $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

(c) $z = 1 - \frac{1}{10}\left(x^2 + 4y^2\right)$, $\left(1, 1, \frac{1}{2}\right)$

(d) $x^2y^2 + xz - 2y^3 = 10$, $(2, 1, 4)$

$$\left[\begin{array}{lll} \textbf{Ans.:} \text{ (a) } x - 2y + z - 4 = 0 & , & \frac{x-1}{1} = \frac{y+1}{-2} = \frac{z-1}{1} \\ \text{(b) } 2x + 2y + z - 3 = 0 & , & \frac{3x-2}{2} = \frac{3y-2}{2} = \frac{3z-1}{1} \\ \text{(c) } 2x + 8y + 10z - 15 = 0 & , & \frac{x-1}{2} = \frac{y-1}{8} = \frac{z-\frac{1}{2}}{10} \\ \text{(d) } 4x + y + z = 13 = 0 & , & \frac{x-2}{4} = \frac{y-1}{1} = \frac{z-4}{1} \end{array}\right]$$

2. Show that the tangent plane to the surface $x^2 = y(x + z)$ at any point passes through the origin.

3. Show that the plane $3x + 12y - 6z - 17 = 0$ touches the surface $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find also the point of contact.

$$\left[\textbf{Ans.:} \left(-1, 2, \frac{2}{3}\right)\right]$$

4. Show that the plane $ax + by + cz + d = 0$ touches the surface

$$px^2 + 9y^2 + 2z = 0, \text{ if } \frac{a^2}{p} + \frac{b^2}{q} + 2cd = 0$$

8.10 LOCAL EXTREME VALUES (MAXIMUM AND MINIMUM VALUES)

Let $u = f(x, y)$ be a continuous function of x and y. u will be maximum at $x = a$, $y = b$, if $f(a, b) > f(a + h, b + k)$ and will be minimum at $x = a$, $y = b$, if $f(a, b) < f(a + h, b + k)$ for small positive or negative values of h and k.

The point at which function $f(x, y)$ is either maximum or minimum is known as *stationary point.* The value of the function at stationary point is known as extreme (maximum or minimum) value of the function $f(x, y)$.

Working Rule to determine extreme values of a function f(x, y)

Step I Solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously for x and y.

Step II Obtain the values of $r = \dfrac{\partial^2 f}{\partial x^2}, s = \dfrac{\partial^2 f}{\partial x\,\partial y}, t = \dfrac{\partial^2 f}{\partial y^2}$.

Step III

(i) If $rt - s^2 > 0$ and $r < 0$ (or $t < 0$) at (a, b) then $f(x, y)$ is maximum at (a, b) and the maximum value of the function is $f(a, b)$.

(ii) If $rt - s^2 > 0$ and $r > 0$ (or $t > 0$) at (a, b) then $f(x, y)$ is minimum at (a, b) and the minimum value of the function is $f(a, b)$.

(iii) If $rt - s^2 < 0$ at (a, b) then $f(x, y)$ is neither maximum nor minimum at (a, b). Such a point is known as *saddle point*.

(iv) If $rt - s^2 = 0$ at (a, b) then no conclusion can be made about the extreme values of $f(x, y)$ and further investigation is required.

Example 1

Discuss the maxima and minima of the function $x^2 + y^2 + 6x + 12$.

Solution

Let $\quad f(x, y) = x^2 + y^2 + 6x + 12$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$
$$2x + 6 = 0$$
$$2(x + 3) = 0$$
$$x + 3 = 0$$
$$x = -3$$

and

$$\frac{\partial f}{\partial y} = 0$$
$$2y = 0$$
$$y = 0$$

Stationary point is (– 3, 0).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

Step III At (–3, 0)

$$rt - s^2 = 2(2) - 0 = 4 > 0 \quad \text{and } r > 0$$

Hence, $f(x, y)$ is minimum at (–3, 0).

$$f_{\min} = (-3)^2 + 0 + 6(-3) + 12 = 3$$

Example 2

Show that the minimum value of $f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$ *is* $3a^2$.

Solution

$$f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$y - \frac{a^3}{x^2} = 0$$

$$x^2 y = a^3 \quad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$x - \frac{a^3}{y^2} = 0$$

$$xy^2 = a^3 \quad \ldots(2)$$

Solving Eqs (1) and (2),

$$x = y$$

Substituting in Eq. (1),

$$x^3 = a^3$$

$$x = a$$

$$\therefore \quad y = a$$

Stationary point is (a, a).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$$

Step III At (a, a), $r = 2$, $s = 1$, $t = 2$

$$rt - s^2 = (2)\,(2) - (1)^2 = 3 > 0$$

Also, $$r = 2 > 0$$

Hence, $f(x, y)$ is minimum at (a, a).

$$f_{\min} = a^2 + a^3\left(\frac{1}{a} + \frac{1}{a}\right) = 3a^2.$$

Example 3

Discuss the maxima and minima of the function $3x^2 - y^2 + x^3$.

Solution

Let $$f(x, y) = 3x^2 - y^2 + x^3$$

Step I For extreme values, $$\frac{\partial f}{\partial x} = 0$$

$$6x + 3x^2 = 0$$

$$3x(x+2) = 0$$

$$x = 0, -2$$

and $$\frac{\partial f}{\partial y} = 0$$

$$-2y = 0$$

$$y = 0$$

Stationary points are $(0, 0)$, $(-2, 0)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6 + 6x$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2$$

Step III

(x,y)	*r*	*s*	*t*	$rt - s^2$	*Conclusion*
(0, 0)	6	0	–2	$-12 < 0$	neither maximum nor minimum
(–2, 0)	–6	0	–2	$12 > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at (–2, 0)

$$f_{\max} = 3(-2)^2 - 0 + (-2)^3 = 4$$

Example 4

Find the stationary value of $x^3 + y^3 - 3axy,\ a > 0.$

Solution

Let
$$f(x, y) = x^3 + y^3 - 3axy$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 - 3ay = 0$$

$$x^2 - ay = 0 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$3y^2 - 3ax = 0$$

$$y^2 - ax = 0 \qquad \ldots(2)$$

From Eq. (1),

$$y = \frac{x^2}{a}$$

Substituting in Eq. (2),

$$x^4 - a^3x = 0$$

$$x(x-a)(x^2 + ax + a^2) = 0$$

$$x = 0,\ x = a$$

$$\therefore \quad y = 0,\ y = a.$$

Hence, stationary points are (0, 0) and (a, a).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = -3a$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Step III At (0, 0), $r = 0, \quad s = -3a, \quad t = 0$

$$rt - s^2 = (0)\,(0) - (-3a)^2 = -9a^2 < 0$$

Hence, $f(x, y)$ is neither maximum nor minimum at (0, 0).

At (a, a), $r = 6a, \quad s = -3a, \quad t = 6a$

$$rt - s^2 = (6a)\,(6a) - (-3a)^2 = 27a^2 > 0$$

Also,

$$r = 6a > 0$$

Hence, $f(x, y)$ is minimum at (a, a).

$$f_{\min} = a^3 + a^3 - 3a^3 = -\,a^3.$$

Example 5

Find the extreme values of the function $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$*, if any.*

[Summer 2017]

Solution

Let

$$f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 + 3y^2 - 6x = 0$$

$$x^2 + y^2 - 2x = 0 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$6xy - 6y = 0$$

$$6y(x-1) = 0$$

$$y = 0,\ x = 1$$

Putting $y = 0$ in Eq. (1),

$$x^2 - 2x = 0,$$

$$x = 0, 2$$

Stationary points are (0, 0), (2, 0).

Putting $x = 1$ in Eq. (1),

$$1 + y^2 - 2 = 0,$$

$$y^2 = 1,\ y = \pm 1$$

Stationary points are (1, 1), (1, –1).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 6 = 6(x-1)$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 6 = 6(x-1)$$

Step III

(x, y)	r	s	t	$rt - s^2$	*Conclusion*
(0, 0)	$-6 < 0$	0	-6	$36 > 0$ and $r < 0$	maximum
(2, 0)	$6 > 0$	0	6	$36 > 0$ and $r > 0$	minimum
(1, 1)	0	6	0	$-36 < 0$	neither maximum nor minimum
(1, –1)	0	-6	0	$-36 < 0$	neither maximum nor minimum

Hence, $f(x, y)$ is maximum at (0, 0) and minimum at (2, 0).

$$f_{\max} = 0 + 4 = 4$$

and

$$\begin{aligned} f_{\min} &= 2^3 + 3(2)(0)^2 - 3(2)^2 - 3(0)^2 + 4 \\ &= 8 + 0 - 12 + 4 \\ &= 0 \end{aligned}$$

Example 6

Find the extreme values of the function $x^3 + y^3 - 63\,(x + y) + 12xy$.

Solution

Let

$$f(x, y) = x^3 + y^3 - 63x - 63y + 12xy$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 - 63 + 12y = 0,$$

$$3x^2 + 12y = 63$$

$$x^2 + 4y = 21 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$3y^2 - 63 + 12x = 0,$$

$$12x + 3y^2 = 63$$

$$4x + y^2 = 21 \qquad \ldots(2)$$

Equating Eqs (1) and (2),

$$x^2 + 4y = 4x + y^2$$

$$x^2 - y^2 - 4(x - y) = 0$$

$$(x + y)(x - y) - 4(x - y) = 0$$

$$(x - y)(x + y - 4) = 0$$

$$x - y = 0,\ x + y - 4 = 0$$

$$y = x,\ y = 4 - x$$

Putting $y = x$ in Eq. (1),

$$\begin{aligned} x^2 + 4x - 21 &= 0, \\ (x+7)(x-3) &= 0 \\ x &= -7, 3 \\ \therefore\ y &= -7, 3 \end{aligned}$$

Stationary points are (–7, –7), (3, 3).
Putting $y = 4 - x$ in Eq. (1),

$$\begin{aligned} x^2 + 4(4-x) &= 21 \\ x^2 - 4x - 5 &= 0, \\ (x+1)(x-5) &= 0 \\ x &= -1, 5 \\ \therefore\ y &= 5, -1 \end{aligned}$$

Stationary points are (–1, 5), (5, –1).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 12$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Step III

(x, y)	***r***	***s***	***t***	***rt – s²***	***Conclusion***
(–7, –7)	–42	12	–42	$1620 > 0$ and $r < 0$	maximum
(3, 3)	18	12	18	$180 > 0$ and $r > 0$	minimum
(–1, 5)	–6	12	30	$-324 < 0$	neither maximum nor minimum
(5, –1)	30	12	–6	$-324 < 0$	neither maximum nor minimum

Hence, $f(x, y)$ is maximum at (–7, –7).

$$f_{\max} = (-7)^3 + (-7)^3 - 63(-7-7) + 12(-7)(-7) = 784.$$

and $f(x, y)$ is minimum at (3, 3).

$$f_{\min} = 3^3 + 3^3 - 63(3+3) + 12(3)(3) = -216.$$

Example 7

Find the extreme value of xy $(a - x - y)$.

Solution

Let
$$f(x, y) = xy(a - x - y)$$
$$= axy - x^2y - xy^2$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$
$$ay - 2xy - y^2 = 0 \qquad \ldots(1)$$
$$y(a - 2x - y) = 0$$
$$y = 0,\ a - 2x - y = 0$$

and
$$\frac{\partial f}{\partial y} = 0$$
$$ax - x^2 - 2xy = 0 \qquad \ldots(2)$$
$$x(a - x - 2y) = 0$$
$$x = 0,\ a - x - 2y = 0$$

Considering four pairs of equations of Eqs (1) and (2),

$$y = 0 \qquad x = 0$$
$$y = 0 \qquad a - x - 2y = 0$$
$$a - 2x - y = 0 \qquad x = 0$$
$$a - 2x - y = 0 \qquad a - x - 2y = 0$$

Solving these equations, the following pairs of values of stationary points are obtained: $(0, 0), (a, 0), (0, a), \left(\frac{a}{3}, \frac{a}{3}\right)$

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = -2y$$
$$s = \frac{\partial^2 f}{\partial x\,\partial y} = a - 2x - 2y$$
$$t = \frac{\partial^2 f}{\partial y^2} = -2x$$

Step III

(x, y)	*r*	*s*	*t*	*rt – s²*	*Conclusion*
$(0, 0)$	0	a	0	$-a^2 < 0$	neither maximum nor minimum
$(a, 0)$	0	$-a$	$-2a$	$-a^2 < 0$	neither maximum nor minimum
$(0, a)$	$-2a$	$-a$	0	$-a^2 < 0$	neither maximum nor minimum
$\left(\frac{a}{3}, \frac{a}{3}\right)$	$\frac{-2a}{3}$	$\frac{-a}{3}$	$\frac{-2a}{3}$	$\frac{a^2}{3} > 0$	maximum or minimum

Hence, $f(x, y)$ is maximum or minimum at $\left(\frac{a}{3}, \frac{a}{3}\right)$ depending on whether $a > 0$ or $a < 0$.

$$f_{\text{extreme}} = \frac{a}{3} \cdot \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}.$$

Example 8

Examine the function $x^3\, y^2(12 - 3x - 4y)$ for extreme values.

Solution

Let $$f(x, y) = 12x^3y^2 - 3x^4y^2 - 4x^3y^3$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$36x^2y^2 - 12x^3y^2 - 12x^2y^3 = 0$$

$$12x^2y^2(3 - x - y) = 0$$

$$x = 0,\ y = 0,\ x + y = 3 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$24x^3y - 6x^4y - 12x^3y^2 = 0$$

$$6x^3y(4 - x - 2y) = 0$$

$$x = 0,\ y = 0,\ x + 2y = 4 \qquad \ldots(2)$$

Considering six pairs of equations of Eqs (1) and (2),

$$\begin{array}{rr} x = 0 & y = 0 \\ x = 0 & x + 2y = 4 \\ y = 0 & x + 2y = 4 \\ x + y = 3 & x = 0 \\ x + y = 3 & y = 0 \\ x + y = 3 & x + 2y = 4 \end{array}$$

Solving these equations, the following pairs of stationary points are obtained:

(0, 0), (0, 2), (4, 0), (0, 3), (3, 0), (2, 1)

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 72xy^2 - 36x^2y^2 - 24xy^3 = 12xy^2(6 - 3x - 2y)$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 72x^2y - 24x^3y - 36x^2y^2 = 12x^2y(6 - 2x - 3y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 24x^3 - 6x^4 - 24x^3y = 6x^3(4 - x - 4y)$$

Step III

(x, y)	*r*	*s*	*t*	$rt - s^2$	*Conclusion*
(0, 0)	0	0	0	0	no conclusion
(0, 2)	0	0	0	0	no conclusion
(4, 0)	0	0	0	0	no conclusion
(0, 3)	0	0	0	0	no conclusion
(3, 0)	0	0	162	0	no conclusion
(2, 1)	–48	–48	–96	$2304 > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at (2, 1).

$$f_{\max} = (2^3)(1^2)(12-6-4) = 16.$$

Example 9

Find the maxima and minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Solution

Let $$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$4x^3 - 4x + 4y = 0$$

$$4(x^3 - x + y) = 0$$

$$x^3 - x + y = 0 \qquad \ldots(1)$$

and
$$\frac{\partial f}{\partial y} = 0$$

$$4y^3 + 4x - 4y = 0$$

$$4(y^3 + x - y) = 0$$

$$y^3 + x - y = 0 \qquad \ldots(2)$$

Adding Eqs (1) and (2),

$$x^3 + y^3 = 0$$

$$y = -x$$

Substituting $y = -x$ in Eq. (2),

$$-x^3 + x + x = 0$$

$$x^3 - 2x = 0$$

$$x(x^2 - 2) = 0$$

$$x = 0, \quad \pm\sqrt{2}$$

$$y = 0, \quad \mp\sqrt{2}$$

Stationary points are $(0, 0), \left(\sqrt{2}, -\sqrt{2}\right), \left(-\sqrt{2}, \sqrt{2}\right)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 4$$

$$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

Step III

(x, y)	*r*	*s*	*t*	*rt – s²*	*Conclusion*
(0, 0)	–4	4	–4	0	no conclusion
$(\sqrt{2}, -\sqrt{2})$	20	4	20	$384 > 0$ and $r > 0$	minimum
$(-\sqrt{2}, \sqrt{2})$	20	4	20	$384 > 0$ and $r > 0$	minimum

Hence, $f(x, y)$ is minimum at the point $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

At $(\sqrt{2}, -\sqrt{2})$, $f_{\min} = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2 = -8$

Example 10

Find the extreme values of the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$$ **[Summer 2014]**

Solution

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

and
$$\frac{\partial f}{\partial y} = 0$$
$$3y^2 - 12 = 0$$
$$3(y^2 - 4) = 0$$
$$y^2 - 4 = 0$$
$$y^2 = 4$$
$$y = \pm 2$$

Stationary points are (1, 2), (1, –2), (–1, 2), (–1, –2).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$
$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 0$$
$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Step III

(x, y)	*r*	*s*	*t*	$rt - s^2$	*Conclusion*
(1, 2)	6	0	12	$72 > 0$ and $r > 0$	minimum
(1, –2)	6	0	–12	$-72 < 0$	neither maximum nor minimum
(–1, 2)	–6	0	12	$-72 < 0$	neither maximum nor minimum
(–1, –2)	–6	0	–12	$72 > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at (–1, –2) and minimum at (1, 2).

$$f_{\text{max}} = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20 = 38$$
$$f_{\text{min}} = (1)^3 + (2)^3 - 3(1) - 12(2) + 20 = 2$$

Example 11

Find the extreme values of the function $x^3 + xy^2 + 21x - 12x^2 - 2y^2$.

Solution

Let
$$f(x, y) = x^3 + xy^2 + 21x - 12x^2 - 2y^2$$

Step I For extreme values,
$$\frac{\partial f}{\partial x} = 0,$$

$$3x^2 + y^2 - 24x + 21 = 0 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$2xy - 4y = 0$$

$$2y(x-2) = 0$$

$$y = 0,\ x = 2$$

Putting $y = 0$ in Eq. (1),

$$3x^2 - 24x + 21 = 0,\ x^2 - 8x + 7 = 0$$

$$(x-1)(x-7) = 0$$

$$x = 1,\ 7$$

Stationary points are (1, 0), (7, 0).

Putting $x = 2$ in Eq. (1),

$$12 + y^2 - 48 + 21 = 0$$

$$y^2 = 15,\ y = \pm\sqrt{15}$$

Stationary points are $\left(2, \sqrt{15}\right), \left(2, -\sqrt{15}\right)$.

Hence, all stationary points are (1, 0), (7, 0), $\left(2, \sqrt{15}\right), \left(2, -\sqrt{15}\right)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 24 = 6(x-4)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 2y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x - 4 = 2(x-2)$$

Step III

(x, y)	*r*	*s*	*t*	$rt - s^2$	*Conclusion*
(1, 0)	–18	0	–2	$36 > 0$ and $r < 0$	maximum
(7, 0)	18	0	10	$180 > 0$ and $r > 0$	minimum
$\left(2, \sqrt{15}\right)$	–12	$2\sqrt{15}$	0	$-60 < 0$	neither maximum nor minimum
$\left(2, -\sqrt{15}\right)$	–12	$-2\sqrt{15}$	0	$-60 < 0$	neither maximum nor minimum

Hence, $f(x, y)$ is maximum at (1, 0) and minimum at (7, 0).

$$f_{\max} = 1^3 + (1 \times 0^2) + 21 - (12 \times 1^2) - (2 \times 0^2) = 10,$$

$$f_{\min} = 7^3 + (7 \times 0^2) + (21 \times 7) - (12 \times 7^2) - (2 \times 0^2) = -98$$

Example 12

Find all the stationary points of the function

$x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

after examining whether the function is maximum or minimum at those points. **[Summer 2016]**

Solution

Let $$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 + 3y^2 - 30x + 72 = 0 \qquad \ldots(1)$$

$$\frac{\partial f}{\partial y} = 0$$

$$6xy - 30y = 0 \qquad \ldots(2)$$

$$6y(x - 5) = 0$$

$$y = 0, x = 5$$

Putting $y = 0$ in Eq. (1),

$$3x^2 - 30x + 72 = 0$$

$$x = 4, x = 6$$

Stationary points are (4, 0), (6, 0).

Putting $x = 5$ in Eq. (1),

$$3y^2 - 3 = 0$$

$$y = \pm 1$$

Stationary points are (5, 1) and (5, –1).

Hence, all stationary points are (5, 1) (5, –1), (4, 0) and (6, 0).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

Step III

(x, y)	*r*	*s*	*t*	$rt - s^2$	*Conclusion*
(4, 0)	–6	0	–6	$36 > 0$ and $r < 0$	maximum
(6, 0)	6	0	6	$36 > 0$ and $r > 0$	minimum
(5, 1)	0	6	0	$-36 < 0$	neither maximum nor minimum
(5, –1)	0	–6	0	$-36 < 0$	neither maximum nor minimum

Hence, $f(x, y)$ is maximum at (4, 0) and $f(x, y)$ is minimum at (6, 0).

$$f_{max} = (4)^3 + 0 - 15\,(4)^2 - 0 + 72\,(4) = 112$$
$$f_{min} = (6)^3 + 0 - 15\,(6)^2 - 0 + 72\,(6) = 108$$

Example 13

Find the extreme values of $\sin x + \sin y + \sin(x + y)$.

Solution

Let $$f(x, y) = \sin x + \sin y + \sin(x + y)$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$\cos x + \cos(x + y) = 0 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$\cos y + \cos(x + y) = 0 \qquad \ldots(2)$$

Equating Eqs (1) and (2),

$$\cos x + \cos(x + y) = \cos y + \cos(x + y)$$
$$\cos x = \cos y$$
$$x = y$$

Substituting $y = x$ in Eq. (1),

$$\cos x + \cos 2x = 0,$$
$$\cos x = -\cos 2x$$
$$= \cos(\pi - 2x) \text{ or } \cos(\pi + 2x)$$
$$x = \pi - 2x \text{ or } \pi + 2x$$
$$x = \frac{\pi}{3}, -\pi$$
$$y = \frac{\pi}{3}, -\pi$$

Stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right), (-\pi, -\pi)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

Step III

(x, y)	*r*	*s*	*t*	*rt – s²*	*Conclusion*
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$ and $r < 0$	maximum
$(-\pi, -\pi)$	0	0	0	0	no conclusion

Hence, $f(x, y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$f_{\max} = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

Example 14

Find the extreme values of $\sin x \sin y \sin (x + y)$.

Solution

Let $\quad f(x, y) = \sin x \sin y \sin (x + y)$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$\sin y\,[\cos x \sin(x+y) + \sin x \cos(x+y)] = 0$$

$$\sin y \sin(2x+y) = 0$$

$$\frac{1}{2}[\cos 2x - \cos(2x+2y)] = 0$$

$$\cos 2x - \cos(2x+2y) = 0 \qquad \text{...(1)}$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$\sin x\,[\cos y \sin(x+y) + \sin y \cos(x+y)] = 0$$

$$\sin x \sin(x+2y) = 0$$
$$\frac{1}{2}[\cos 2y - \cos(2x+2y)] = 0$$
$$\cos 2y - \cos(2x+2y) = 0 \qquad \text{...(2)}$$

Equating Eqs (1) and (2),

$$\cos 2x = \cos 2y$$
$$x = y$$

Substituting $x = y$ in Eq. (1),

$$\cos 2x - \cos(2x+2x) = 0$$
$$\cos 2x = \cos 4x = 2\cos^2 2x - 1$$
$$2\cos^2 2x - \cos 2x - 1 = 0$$
$$\cos 2x = \frac{1 \pm \sqrt{1+8}}{4}$$
$$= 1, -\frac{1}{2}$$

$$\cos 2x = 1 = \cos 0, \qquad \cos 2x = -\frac{1}{2} = \cos\frac{2\pi}{3}$$
$$x = 0, \qquad x = \frac{\pi}{3}$$
$$y = 0, \qquad y = \frac{\pi}{3}$$

Stationary points are $(0,0), \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin 2x + \sin 2(x+y) = 2\sin y \cos(2x+y)$$
$$s = \frac{\partial^2 f}{\partial x\, \partial y} = \sin 2(x+y)$$
$$t = \frac{\partial^2 f}{\partial y^2} = -\sin 2y + \sin 2(x+y) = 2\sin x \cos(x+2y)$$

Step III

(x, y)	*r*	*s*	*t*	*rt – s*^{2}	*Conclusion*
(0, 0)	0	0	0	0	no conclusion
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$f_{\max} = \sin\frac{\pi}{3}\sin\frac{\pi}{3}\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

8.11 EXTREME VALUES WITH CONSTRAINED VARIABLES

Sometimes we have to find the extreme values of a function of three (or more) variables, say $f(x, y, z)$, which are not independent but are connected by some given relation $\phi(x, y, z) = 0$. The extreme values of $f(x, y, z)$ in such a situation are called *constrained extreme values*.

In such situations, we use $\phi(x, y, z) = 0$ to eliminate one of the variables, say z, from the given function, thus converting the function as a function of only two variables and then find the extreme values of the function.

Example 1

Find the minimum value of $x^2 + y^2 + z^2$ with the constraint $x + y + z = 3a$.

Solution

Let

$$f = x^2 + y^2 + z^2$$

$$x + y + z = 3a$$

$$z = 3a - x - y \quad \text{...(1)}$$

Substituting the value of z in Eq. (1),

$$f = x^2 + y^2 + (3a - x - y)^2$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$2x - 2(3a - x - y) = 0$$

$$4x - 6a + 2y = 0$$

$$2x + y = 3a \quad \text{...(2)}$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$2y - 2(3a - x - y) = 0$$

$$2y - 6a + 2x + 2y = 0$$

$$x + 2y = 3a \quad \text{...(3)}$$

Solving Eqs (2) and (3),

$$x = y = a$$

The stationary point is (a, a).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 2$$

$$t = \frac{\partial^2 f}{\partial y^2} = 4$$

Step III At (a, a), $r = 4$, $s = 2$, $t = 4$

$$rt - s^2 = (4)(4) - (2)^2 = 12 > 0$$

Also, $r = 4 > 0$
Hence, $f(x, y)$ is minimum at (a, a)

$$f_{\min} = a^2 + a^2 + (3a - a - a)^2$$
$$= 3a^2$$

Example 2

Find the minimum value of x^3y^2z subject to the condition $x + y + z = 1$.

Solution

Let
$$f = x^3y^2z$$
$$x + y + z = 1$$
$$z = 1 - x - y \qquad ...(1)$$

Substituting the value of z in Eq. (1),

$$f = x^3y^2(1 - x - y)$$
$$= x^3y^2 - x^4y^2 - x^3y^3$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(3 - 4x - 3y) = 0$$

$$x = 0,\ y = 0,\ 4x + 3y = 3 \qquad \ldots(2)$$

and
$$\frac{\partial f}{\partial y} = 0$$

$$2x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^3y(2 - 2x - 3y) = 0$$

$$x = 0,\ y = 0,\ 2x + 3y = 2 \qquad \ldots(3)$$

Considering six pairs of Eqs (2) and (3),

$$\begin{array}{rr} x=0 & y=0 \\ x=0 & 2x+3y=2 \\ y=0 & 2x+3y=2 \\ 4x+3y=3 & x=0 \\ 4x+3y=3 & y=0 \\ 4x+3y=3 & 2x+3y=2 \end{array}$$

Solving these equations, the following pairs of stationary points are formed:

$$(0, 0),\ \left(0, \frac{2}{3}\right),\ (1, 0),\ (0, 1),\ \left(\frac{3}{4}, 0\right),\ \left(\frac{1}{2}, \frac{1}{3}\right)$$

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1-2x-y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6-8x-9y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y = 2x^3(1-x-3y)$$

Step III

(x, y)	*r*	*s*	*t*	*rt – s*^{2}	*Conclusion*
(0, 0)	0	0	0	0	no conclusion
$\left(0, \frac{2}{3}\right)$	0	0	0	0	no conclusion
(1, 0)	0	0	0	0	no conclusion
(0, 1)	0	0	0	0	no conclusion
$\left(\frac{3}{4}, 0\right)$	0	0	$\frac{27}{128}$	0	no conclusion
$\left(\frac{1}{2}, \frac{1}{3}\right)$	$-\frac{1}{9}$	$-\frac{1}{12}$	$-\frac{1}{8}$	$\frac{1}{144} > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$.

$$f_{\max} = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{432}$$

Example 3

Divide 120 *into three parts so that the sum of their products taken two at a time shall be maximum.*

Solution

Step I Let x, y, z be three numbers.

Let
$$x + y + z = 120$$
$$\begin{aligned} f &= xy + yz + zx \\ &= xy + y(120 - x - y) + (120 - x - y)x \\ &= xy + 120y - xy - y^2 + 120x - x^2 - xy \\ &= 120x + 120y - xy - x^2 - y^2 \end{aligned}$$

For extreme values,
$$\frac{\partial f}{\partial x} = 0$$
$$120 - y - 2x = 0 \qquad \text{...(1)}$$

and
$$\frac{\partial f}{\partial y} = 0$$
$$120 - x - 2y = 0 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),
$$x = 40$$
$$y = 40$$

Stationary point is (40, 40).

Step II
$$r = \frac{\partial^2 f}{\partial x^2} = -2$$
$$s = \frac{\partial^2 f}{\partial x \partial y} = -1$$
$$t = \frac{\partial^2 f}{\partial y^2} = -2$$

Step III At (40, 40).
$$rt - s^2 = (-2)(-2) - (-1)^2 = 3 > 0$$

and
$$r = -2 < 0$$

$f(x, y)$ is maximum at (40, 40).
Hence, three parts are 40, 40 and 40.

Example 4

Find a point on the plane $2x + 3y - z = 5$ *which is nearest to the origin.*

[Summer 2017]

Solution

Let $P(x, y, z)$ be any point on the plane $2x + 3y - z = 5$.
Its distance from the origin is given by

$$d = \sqrt{x^2 + y^2 + z^2}$$

$$d^2 = x^2 + y^2 + z^2$$

Since P lies on the plane $z = 2x + 3y - 5$,

$$d^2 = x^2 + y^2 + (2x + 3y - 5)^2$$

Let

$$f(x, y) = x^2 + y^2 + 4x^2 + 9y^2 + 25 + 12xy - 30y - 20x$$
$$= 5x^2 + 10y^2 + 12xy - 30y - 20x + 25$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$10x + 12y = 20$$

$$5x + 6y = 10 \quad \text{...(1)}$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$20y + 12x = 30$$

$$10y + 6x = 15 \quad \text{...(2)}$$

Solving Eqs (1) and (2),

$$x = \frac{5}{7}, y = \frac{15}{14}$$

Stationary point is $\left(\frac{5}{7}, \frac{15}{14}\right)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 10$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6$$

$$t = \frac{\partial^2 f}{\partial y^2} = 10$$

Step III At $\left(\frac{5}{7}, \frac{15}{14}\right)$, $r = 10$, $s = 6$, $t = 10$

$$rt - s^2 = 100 - 36 = 64 > 0$$

Also, $r = 10 > 0$

$f(x, y)$ i.e. d^2 is minimum at $\left(\frac{5}{7}, \frac{15}{14}\right)$ and hence d is minimum at $\left(\frac{5}{7}, \frac{15}{14}\right)$.

At $\left(\frac{5}{7}, \frac{15}{14}\right)$,

$$z = 2x + 3y - 5 = 2\left(\frac{5}{7}\right) + 3\left(\frac{15}{14}\right) - 5 = -\frac{5}{14}$$

Hence, d is minimum at $\left(\frac{5}{7}, \frac{15}{14}, -\frac{5}{14}\right)$.

The point $\left(\frac{5}{7}, \frac{15}{14}, -\frac{5}{14}\right)$ on the plane $2x + 3y - z = 5$ is nearest to the origin.

Example 5

Find the points on the surface $z^2 = xy + 1$ nearest to the origin. Also find that distance.

Solution

Let $P(x, y, z)$ be any point on the surface $z^2 = xy + 1$.

Its distance from the origin is given by

$$d = \sqrt{(x^2 + y^2 + z^2)}$$

$$d^2 = x^2 + y^2 + z^2$$

Since P lies on the surface $z^2 = xy + 1$,

$$d^2 = x^2 + y^2 + xy + 1$$

Let $$f(x, y) = x^2 + y^2 + xy + 1$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$2x + y = 0 \qquad \ldots(1)$$

and $$\frac{\partial f}{\partial y} = 0$$

$$2y + x = 0 \qquad \ldots(2)$$

Solving Eqs (1) and (2),

$$x = 0, y = 0$$

Stationary point is (0, 0).

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

Step III At (0, 0), $r = 2$, $t = 2$, $s = 1$

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0$$

Also, $$r = 2 > 0$$

$f(x, y)$, i.e., d^2 is minimum at (0, 0) and hence d is minimum at (0, 0).

At (0, 0),

$$z^2 = xy + 1 = 1$$
$$z = \pm 1$$

Hence, d is minimum at (0, 0, 1) and (0, 0, –1).

The points (0, 0, 1) and (0, 0, –1) on the surface $z^2 = xy + 1$ are nearest to the origin.

Minimum distance $= \sqrt{0+0+1} = 1$.

Example 6

A rectangular box open at the top is to have a volume of 108 *cubic metres. Find the dimensions of the box if its total surface area is minimum.*

Solution

Let x, y and z be the dimensions of the box. Let V and S be its volume and surface area respectively.

$$V = xyz$$
$$S = xy + 2xz + 2yz$$

Substituting $z = \dfrac{V}{xy}$,

$$S = xy + 2x \cdot \frac{V}{xy} + 2y \cdot \frac{V}{xy}$$
$$= xy + \frac{2V}{y} + \frac{2V}{x}$$

Step I For extreme values,

$$\frac{\partial S}{\partial x} = 0$$
$$y - \frac{2V}{x^2} = 0 \qquad \dots(1)$$

and

$$\frac{\partial S}{\partial y} = 0$$
$$x - \frac{2V}{y^2} = 0 \qquad \dots(2)$$

Substituting $y = \dfrac{2V}{x^2}$ from Eq. (1) in Eq. (2),

$$x - 2V\left(\frac{x^4}{4V^2}\right) = 0$$
$$x\left(1 - \frac{x^3}{2V}\right) = 0$$

$$x = (2V)^{\frac{1}{3}}$$

$$\therefore \quad y = \frac{2V}{x^2} = \frac{2V}{(2V)^{\frac{2}{3}}} = (2V)^{\frac{1}{3}} \quad \text{[Since } x \neq 0 \text{ being the side of the box]}$$

Stationary point is $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$.

Step II

$$r = \frac{\partial^2 S}{\partial x^2} = \frac{4V}{x^3}$$

$$s = \frac{\partial^2 S}{\partial x \, \partial y} = 1$$

$$t = \frac{\partial^2 S}{\partial y^2} = \frac{4V}{y^3}$$

Step III At $\left[(2V)^{\frac{1}{3}}, (2V)^{\frac{1}{3}}\right]$, $r = \frac{4V}{2V} = 2 > 0$, $s = 1$, $t = \frac{4V}{2V} = 2$

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0 \text{ and } r = 2 > 0$$

Hence, S is minimum at $x = y = (2V)^{\frac{1}{3}}$.

Putting $\quad V = 108 \text{ m}^3$,

$$x = y = (2 \times 108)^{\frac{1}{3}} = 6$$

and $\quad z = \frac{V}{xy} = \frac{108}{6 \times 6} = 3$

Hence, dimensions of the box which make its total surface area S minimum are $x = 6$, $y = 6$, $z = 3$.

Example 7

Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Solution

Let x, y, z be the length, breadth and height of the rectangular solid and V be its volume.

$$V = xyz \qquad \ldots(1)$$

Let the given sphere be

$$x^2 + y^2 + z^2 = a^2$$

$$z^2 = a^2 - x^2 - y^2$$

Substituting in Eq. (1),

$$V = xy\sqrt{a^2 - x^2 - y^2}$$

$$V^2 = x^2y^2(a^2 - x^2 - y^2)$$

Let $$f(x, y) = V^2 = x^2y^2(a^2 - x^2 - y^2) \qquad \ldots(2)$$

Step I For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$y^2[2x(a^2 - x^2 - y^2) + x^2(-2x)] = 0$$

$$2xy^2(a^2 - 2x^2 - y^2) = 0$$

$$x = 0,\ y = 0,\ 2x^2 + y^2 = a^2 \qquad \ldots(3)$$

and $$\frac{\partial f}{\partial y} = 0$$

$$x^2[2y(a^2 - x^2 - y^2) + y^2(-2y)] = 0$$

$$2x^2y(a^2 - x^2 - 2y^2) = 0$$

$$x = 0,\ y = 0,\ x^2 + 2y^2 = a^2 \qquad \ldots(4)$$

But x and y are the sides of the rectangular solid, and therefore cannot be zero.
Solving $2x^2 + y^2 = a^2$ and $x^2 + 2y^2 = a^2$,

$$x^2 = \frac{a^2}{3},\ y^2 = \frac{a^2}{3}$$

$$x = \frac{a}{\sqrt{3}},\ y = \frac{a}{\sqrt{3}} \qquad [\because \text{side cannot be negative}]$$

$$z = \sqrt{a^2 - \frac{a^2}{3} - \frac{a^2}{3}} = \frac{a}{\sqrt{3}}$$

Stationary points are $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Step II

$$r = \frac{\partial^2 f}{\partial x^2} = 2a^2y^2 - 12x^2y^2 - 2y^4$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 4a^2xy - 8x^3y - 8xy^3$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2a^2x^2 - 2x^4 - 12x^2y^2$$

Step III At $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$, $r = \frac{2a^4}{3} - \frac{4a^4}{3} - \frac{2a^4}{9} = -\frac{8a^4}{9}$

$$s = \frac{4a^4}{3} - \frac{8a^4}{9} - \frac{8a^4}{9} = -\frac{4a^4}{9}$$

$$t = \frac{2a^4}{3} - \frac{2a^4}{9} - \frac{12a^4}{9} = -\frac{8a^4}{9}$$

$$rt - s^2 = \frac{64a^8}{81} - \frac{16a^8}{81} = \frac{48a^8}{81} > 0$$

$rt - s^2 > 0$ and $r < 0$

$f(x, y)$, i.e. V^2 is maximum at $x = y = z$ and hence, V is maximum when $x = y = z$, i.e. the rectangular solid is a cube.

EXERCISE 8.7

1. Examine maxima and minima of the following functions and find their extreme values:

(i) $2 + 2x + 2y - x^2 - y^2$ (ii) $x^2y^2 - 5x^2 - 8xy - 5y^2$

(iii) $x^2 + y^2 + xy + x - 4y + 5$ (iv) $x^2 + y^2 + 6x = 12$

(v) $x^3y^2(1 - x - y)$ (vi) $xy(3a - x - y)$

(vii) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ (viii) $x^4 + y^4 - 2(x - y)^2$

(ix) $x^4 + x^2y + y^2$ (x) $x^4 + y^4 - 4a^2xy$

(xi) $y^4 - x^4 + 2(x^2 - y^2)$ (xii) $x^3 + 3x^2 + y^2 + 4xy$

(xiii) $x^2y - 3x^2 - 2y^2 - 4y + 3$ (xiv) $x^4 - y^4 - x^2 - y^2 + 1$.

[**Ans.:** (i) Max. at $(1, 1); 4$ (ii) Max. at $(0, 0); 0$
(iii) Min. at $(-2, 3); -2$ (iv) Min. at $(-3, 0); 3$
(v) Max. at $\left(\frac{1}{2}, \frac{1}{3}\right); \frac{1}{432}$ (vi) Max. at $(a, a); a^3$
(vii) Max. at $(0, 0); 4$ (viii) Min. at $\left(\sqrt{2}, -\sqrt{2}\right)$ and $\left(-\sqrt{2}, \sqrt{2}\right); -8$
(ix) Min. at $(0, 0); 0$ (x) Min. at (a, a) and $(-a, a); a^4$
(xi) No extreme values (xii) No extreme values
(xiii) Max. at $(0, -1); 5$ (xiv) Max. at $(0, 0); 1$, min at $\left(\pm\frac{1}{\sqrt{2}}, \pm\sqrt{\frac{1}{\sqrt{2}}}\right); \frac{1}{2}$]

2. A rectangular box, open at the top, is to have a volume of 32 cc. Find the dimensions of the box requiring least materials for its construction.

[**Ans.:** 4, 4, 2]

3. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

[**Hint:** $f = xy + yz + zx$ where $x + y + z = 120$]

[**Ans.:** 40, 40, 40]

4. The sum of three positive numbers is 'a'. Determine the maximum value of their product.

$$\left[\textbf{Ans.: } \frac{a^3}{27} \text{ at } \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)\right]$$

5. Find the volume of the largest rectangular parallelepiped that can be inscribed in an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$\left[\begin{array}{l}\textbf{Hint: } \text{Let } 2x, 2y, 2z \text{ be the sides of the parallelepiped; then its volume} \\ \quad v = 8xyz = 8xy\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}\end{array}\right]$$

$$\left[\textbf{Ans.: } \frac{8abc}{3\sqrt{3}}\right]$$

6. Prove that area of a triangle with constant perimeter is maximum when the triangle is equilateral.

$$\left[\begin{array}{l}\textbf{Hint: } \text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \\ \qquad \text{where } 2s = a + b + c, c = 2s - a - b, s \text{ is constant}\end{array}\right]$$

7. Find the shortest distance from the origin to the surface $xyz^2 = 2$.

[**Ans.:** 2]

8. Find the shortest distance from the origin to the plane $x - 2y - 2z = 3$.

[**Ans.:** 1]

9. Find the shortest distance between the lines $\dfrac{x-3}{1} = \dfrac{y-5}{-2} = \dfrac{z-7}{1}$ and $\dfrac{x+1}{7} = \dfrac{y+1}{-6} = \dfrac{z+1}{1}$.

$$\left[\textbf{Ans.: } 2\sqrt{29}\right]$$

10. Find the maximum value of $\cos A \cos B \cos C$, where A, B, C are angles of a triangle.

$$\left[\textbf{Ans.: } \text{max. at } \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right); \frac{1}{8}\right]$$

8.12 METHOD OF LAGRANGE MULTIPLIERS

Let $f(x, y, z)$ be a function of three variables x, y, z, and the variables be connected by the relation

$$\phi (x, y, z) = 0 \qquad \ldots(1)$$

Suppose we wish to find the values of x, y, z, for which $f(x, y, z)$ is stationary (maximum and minimum).

For this purpose, we construct an auxiliary equation

$$f (x, y, z) + \lambda\phi(x, y, z) = 0 \qquad \ldots(2)$$

Differentiating Eq. (2) partially w.r.t. x, y, z,

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \qquad \ldots(3)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \qquad \ldots(4)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \qquad \ldots(5)$$

Eliminating λ from Eqs (3), (4) and (5), the values of x, y, and z are obtained for which $f(x, y, z)$ has stationary value. This method of obtaining stationary values of $f(x, y, z)$ is called Lagrange's method of undetermined multipliers, and equations (3), (4) and (5) are called *Lagrange's equations*. The term λ is called *undetermined multiplier.*

Example 1

Find the minimum value of $x^2 + y^2$, subject to the condition $ax + by = c$.

Solution

Let $\quad f (x, y) = x^2 + y^2 \qquad \ldots(1)$

$\quad ax + by = c \qquad \ldots(2)$

Let $\quad \phi (x, y) = ax + by - c = 0$

Let the auxiliary equation be

$$(x^2 + y^2) + \lambda(ax + by - c) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2x + \lambda a = 0$$

$$\lambda = -\frac{2x}{a} \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$2y + \lambda b = 0$$

$$\lambda = -\frac{2y}{b} \qquad \ldots(5)$$

From Eqs (4) and (5),

$$\frac{2x}{a} = \frac{2y}{b}$$

$$y = \frac{b}{a}x$$

Substituting y in Eq. (2),

$$ax + b\left(\frac{b}{a}x\right) = c$$

$$ax + \frac{b^2}{a}x = c$$

$$(a^2 + b^2)\,x = ac$$

$$x = \frac{ac}{a^2 + b^2}$$

$$\therefore\ y = \frac{b}{a}\left(\frac{ac}{a^2 + b^2}\right) = \frac{bc}{a^2 + b^2}$$

Minimum value of $x^2 + y^2 = \dfrac{a^2c^2}{(a^2 + b^2)^2} + \dfrac{b^2c^2}{(a^2 + b^2)^2}$

Example 2

Find the minimum values of x^2yz^3, subject to the condition $2x + y + 3z = a$.

[Summer 2014]

Solution

Let $$f(x, y, z) = x^2yz^3 \qquad \ldots(1)$$

$$2x + y + 3z = a \qquad \ldots(2)$$

Let $$\phi(x, y, z) = 2x + y + 3z - a = 0$$

Let the auxiliary equation be

$$x^2yz^3 + \lambda(2x + y + 3z - a) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2xyz^3 + 2\lambda = 0$$

$$\lambda = -xyz^3 \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$x^2z^3 + \lambda = 0$$

$$\lambda = -x^2z^3 \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$3x^2yz^2 + 3\lambda = 0$$

$$\lambda = -x^2yz^2 \qquad \ldots(6)$$

From Eqs (4), (5), and (6),

$$xyz^3 = x^2z^3 = x^2yz^2$$

$$yz = xz = xy$$

$$\therefore \; y = x \text{ and } z = y$$

Substituting $y = z = x$ in Eq. (2),

$$2x + x + 3x = a$$

$$6x = a$$

$$x = \frac{a}{6}$$

$$\therefore \; y = \frac{a}{6}, z = \frac{a}{6}$$

Minimum value of $x^2yz^3 = \left(\frac{a}{6}\right)^2\left(\frac{a}{6}\right)\left(\frac{a}{6}\right)^3 = \left(\frac{a}{6}\right)^6$

Example 3

Find the maximum value of $f = x^2y^3z^4$, *subject to the condition* $x + y + z = 5$.

Solution

Let $\qquad f(x, y, z) = x^2y^3z^4 \qquad \ldots(1)$

$\qquad x + y + z = 5 \qquad \ldots(2)$

Let $\qquad \phi(x, y, z) = x + y + z - 5 = 0$

Let the auxiliary equation be

$$x^2y^3z^4 + \lambda(x + y + z - 5) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2xy^3z^4 + \lambda = 0$$

$$\lambda = -2xy^3z^4 \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$3x^2y^2z^4 + \lambda = 0$$

$$\lambda = -3x^2y^2z^4 \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$4x^2y^3z^3 + \lambda = 0$$

$$\lambda = -4x^2y^3z^3 \qquad \ldots(6)$$

From Eqs (4), (5), and (6),

$$2xy^3z^4 = 3x^2y^2z^4 = 4x^2y^3z^3$$
$$2yz = 3xz = 4xy$$
$$\therefore\ y = \frac{3}{2}x \quad \text{and} \quad z = 2x$$

Substituting y and z in Eq. (2),

$$x + \frac{3}{2}x + 2x = 5$$
$$9x = 10$$
$$x = \frac{10}{9}$$
$$\therefore\ y = \frac{3}{2}x = \frac{3}{2}\left(\frac{10}{9}\right) = \frac{5}{3}$$

and
$$z = 2x = 2\left(\frac{10}{9}\right) = \frac{20}{9}$$

Maximum value of $x^2y^3z^4 = \left(\frac{10}{9}\right)^2\left(\frac{5}{3}\right)^3\left(\frac{20}{9}\right)^4 = \frac{(2^{10})(5^9)}{3^{15}}$

Example 4

Find the maximum value of $x^my^nz^p$ when $x + y + z = a$.

Solution

Let
$$f(x, y, z) = x^m y^n z^p \qquad \ldots(1)$$
$$x + y + z = a \qquad \ldots(2)$$

Let
$$\phi(x, y, z) = x + y + z - a = 0$$

Let the auxiliary equation be

$$x^m y^n z^p + \lambda(x + y + z - a) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$mx^{m-1}y^nz^p + \lambda = 0$$
$$\lambda = -mx^{m-1}y^nz^p \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$ny^{n-1}x^mz^p + \lambda = 0$$
$$\lambda = -ny^{n-1}x^mz^p \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$px^my^nz^{p-1} + \lambda = 0$$

$$\lambda = -px^m y^n z^{p-1} \quad \ldots(6)$$

From Eqs (4), (5) and (6),

$$mx^{m-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z}$$

$$\therefore \quad y = \frac{n}{m}x \quad \text{and} \quad z = \frac{p}{m}x$$

Substituting y and z in Eq. (2),

$$x + \frac{n}{m}x + \frac{p}{m}x = a$$

$$x = \frac{am}{m+n+p}$$

$$\therefore \quad y = \frac{n}{m}x = \frac{an}{m+n+p}$$

and

$$z = \frac{p}{m}x = \frac{ap}{m+n+p}$$

Maximum value of $x^m y^n z^p = \dfrac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}$

Example 5

Find the minimum value of $x^2 + y^2 + z^2$ with the constraint $xy + yz + zx = 3a^2$.

Solution

Let

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \ldots(1)$$

$$xy + yz + zx = 3a^2 \quad \ldots(2)$$

Let

$$\phi(x, y, z) = xy + yz + zx - 3a^2$$

Let the auxiliary equation be

$$(x^2 + y^2 + z^2) + \lambda(xy + yz + zx - 3a^2) = 0 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2x + \lambda(y+z) = 0$$

$$\lambda = -\frac{2x}{y+z} \quad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$2y + \lambda(x+z) = 0$$

$$\lambda = -\frac{2y}{z+x} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$2z + \lambda(y+x) = 0$$

$$\lambda = -\frac{2z}{x+y} \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{2x}{y+z} = \frac{2y}{z+x} = \frac{2z}{x+y} = \frac{2x+2y+2z}{y+z+z+x+x+y} = 1$$

$$2x - y - z = 0$$
$$-x + 2y - z = 0$$
$$-x - y + 2z = 0$$

Solving these equations,

$$x = y = z$$

Substituting $y = z = x$ in Eq. (2),

$$3x^2 = 3a^2$$
$$x = \pm a$$
$$x = y = z = \pm a$$

Minimum value of $x^2 + y^2 + z^2 = 3a^2$

Example 6

Using Lagrange's method of multipliers, show that the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ occur at $x = \frac{a+b+c}{a}, y = \frac{a+b+c}{b}, z = \frac{a+b+c}{c}$.

Solution

Let $$f(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2 \qquad \ldots(1)$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \qquad \ldots(2)$$

Let $$\phi(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$$

Let the auxiliary equation be

$$\left(a^3x^2 + b^3y^2 + c^3z^2\right) + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$2a^3x - \frac{\lambda}{x^2} = 0$$

$$2a^3x^3 - \lambda = 0$$

$$\lambda = 2a^3x^3 \quad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$2b^3y - \frac{\lambda}{y^2} = 0$$

$$2b^3y^3 - \lambda = 0$$

$$\lambda = 2b^3y^3 \quad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$2c^3z - \frac{\lambda}{z^2} = 0$$

$$2c^3z^3 - \lambda = 0$$

$$\lambda = 2c^3z^3 \quad \text{...(6)}$$

From Eqs (3), (4) and (5),

$$2a^3x^3 = 2b^3y^3 = 2c^3z^3$$

$$ax = by = cz$$

$$\therefore y = \frac{ax}{b} \quad \text{and} \quad z = \frac{ax}{c}$$

Substituting y and z in Eq. (2),

$$\frac{1}{x} + \frac{b}{ax} + \frac{c}{ax} = 1$$

$$\frac{a+b+c}{ax} = 1$$

$$x = \frac{a+b+c}{a}$$

$$\therefore y = \frac{ax}{b} = \frac{a+b+c}{b}$$

and

$$z = \frac{ax}{c} = \frac{a+b+c}{c}$$

Hence, the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ occurs at

$$x = \frac{a+b+c}{a}, y = \frac{a+b+c}{b}, z = \frac{a+b+c}{c}$$

Example 7

Find the point on the plane $ax + by + cz = p$ at which the function $f = x^2 + y^2 + z^2$ has a minimum value and find this minimum value of f.

[Summer 2015]

Solution

Let $$f(x, y, z) = x^2 + y^2 + z^2 \quad \dots(1)$$

$$ax + by + cz = p \quad \dots(2)$$

Let $$\phi(x, y, z) = ax + by + cz - p = 0$$

Let the auxiliary equation be

$$(x^2 + y^2 + z^2) + \lambda(ax + by + cz - p) = 0 \quad \dots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2x + \lambda a = 0$$

$$\lambda = -\frac{2x}{a} \quad \dots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$2y + \lambda b = 0$$

$$\lambda = -\frac{2y}{b} \quad \dots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$2z + \lambda c = 0$$

$$\lambda = -\frac{2z}{c} \quad \dots(6)$$

From Eqs (4), (5) and (6),

$$\frac{2x}{a} = \frac{2y}{b} = \frac{2z}{c}$$

$$y = \frac{bx}{a} \text{ and } z = \frac{cx}{a}$$

Substituting y and z in Eq. (2),

$$ax + \frac{b^2x}{a} + \frac{c^2x}{a} = p$$

$$x = \frac{ap}{a^2 + b^2 + c^2}$$

$$\therefore y = \frac{bp}{a^2 + b^2 + c^2}$$

and $$z = \frac{cp}{a^2 + b^2 + c^2}$$

Thus, $(x^2 + y^2 + z^2)$ is minimum at $\left(\frac{ap}{a^2 + b^2 + c^2}, \frac{bp}{a^2 + b^2 + c^2}, \frac{cp}{a^2 + b^2 + c^2}\right)$

Minimum value of $x^2 + y^2 + z^2 = \frac{a^2p^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2p^2}{(a^2 + b^2 + c^2)^2} + \frac{c^2p^2}{(a^2 + b^2 + c^2)^2}$

$$= \frac{p^2(a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}$$

Example 8

Find the length of the shortest line from the point $\left(0, 0, \dfrac{25}{9}\right)$ *to the surface* $z = xy$.

Solution

Let (x, y, z) be a point on the surface $z = xy$.

The distance d between (x, y, z) and $\left(0, 0, \dfrac{25}{9}\right)$ is

$$d = \sqrt{x^2 + y^2 + \left(z - \frac{25}{9}\right)^2}$$

$$d^2 = x^2 + y^2 + \left(z - \frac{25}{9}\right)^2$$

Let $$f(x, y, z) = d^2 = x^2 + y^2 + \left(z - \frac{25}{9}\right)^2 \qquad \ldots(1)$$

$$z = xy \qquad \ldots(2)$$

Let $$\phi(x, y, z) = z - xy = 0$$

Let the auxiliary equation be

$$\left[x^2 + y^2 + \left(z - \frac{25}{9}\right)^2\right] + \lambda(z - xy) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$2x + \lambda(-y) = 0$$

$$\lambda = \frac{2x}{y} \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$2y + \lambda(-x) = 0$$

$$\lambda = \frac{2y}{x} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$2\left(z - \frac{25}{9}\right) + \lambda = 0$$

$$\lambda = -2\left(z - \frac{25}{9}\right) \qquad \ldots(6)$$

From Eqs (4) and (5),

$$\frac{2x}{y} = \frac{2y}{x}$$
$$x^2 = y^2$$
$$x = \pm y \qquad \text{...(6)}$$

Substituting $x = y$, in Eq. (4),

$$\lambda = 2$$

Substituting $\lambda = 2$ in Eq. (6),

$$2 = -2\left(z - \frac{25}{9}\right)$$
$$-1 = z - \frac{25}{9}$$
$$z = -1 + \frac{25}{9} = \frac{16}{9}$$

Substituting $y = x$ and $z = \frac{16}{9}$ in Eq. (2),

$$\frac{16}{9} = x^2$$
$$x = \pm\frac{4}{3}$$
$$\therefore y = \pm\frac{4}{3}$$

Similarly, when $x = -y$, $z = \frac{34}{9}$. But this gives a complex value of x and y.

Thus $f(x, y, z)$, i.e., d^2 is minimum when $x = \pm\frac{4}{3}, y = \pm\frac{4}{3}, z = \frac{16}{9}$.

Minimum distance $$d = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{16}{9} - \frac{25}{9}\right)^2} = \frac{\sqrt{41}}{3}$$

Hence, the length of the shortest line from $\left(0, 0, \frac{25}{9}\right)$ to the surface $z = xy$ is $\frac{\sqrt{41}}{3}$.

Example 9

Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube. **[Summer 2016]**

Solution

Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid.

Let r be the radius of the sphere.

Volume of solid, $$V = 8xyz \qquad \text{...(1)}$$

Equation of the sphere, $$x^2 + y^2 + z^2 = r^2 \qquad \text{...(2)}$$

Let $$\phi(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$$

Let the auxiliary equation be

$$8xyz + \lambda(x^2 + y^2 + z^2 - r^2) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$8yz + \lambda \cdot 2x = 0$$

$$\lambda = -\frac{4yz}{x} \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t y,

$$8xz + \lambda \cdot 2y = 0$$

$$\lambda = -\frac{4xz}{y} \qquad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t z,

$$8xy + \lambda \cdot 2z = 0$$

$$\lambda = -\frac{4xy}{z} \qquad \text{...(6)}$$

From Eqs (4), (5) and (6),

$$\frac{4yz}{x} = \frac{4xz}{y} = \frac{4xy}{z}$$

$$y^2 = x^2 \quad \text{and} \quad z^2 = y^2$$

$$x^2 = y^2 = z^2$$

$$x = y = z$$

Hence, the rectangular solid is a cube.

Example 10

Find the minimum and maximum distance from the point (1, 2, 2) *to the sphere* $x^2 + y^2 + z^2 = 36$.

Solution

Let (x, y, z) be any point on the sphere. Its distance D from the point (1, 2, 2) is

$$D = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

Let $$D^2 = f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2 \qquad \text{...(1)}$$

$$x^2 + y^2 + z^2 = 36 \qquad \text{...(2)}$$

Let $$\phi(x, y, z) = x^2 + y^2 + z^2 - 36$$

Let the auxiliary equation be

$$\left[(x-1)^2+(y-2)^2+(z-2)^2\right]+\lambda(x^2+y^2+z^2-36)=0 \qquad ...(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2(x-1)+\lambda(2x)=0$$

$$\lambda=-\frac{x-1}{x}=-1+\frac{1}{x} \qquad ...(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$2(y-2)+\lambda(2y)=0$$

$$\lambda=-\frac{y-2}{y}=-1+\frac{2}{y} \qquad ...(5)$$

Differentiating Eq. (3) partially, w.r.t. z,

$$2(z-2)+\lambda(2z)=0$$

$$\lambda=-\frac{z-2}{z}=-1+\frac{2}{z} \qquad ...(6)$$

From Eqs (4), (5) and (6),

$$-1+\frac{1}{x}=-1+\frac{2}{y}=-1+\frac{2}{z}$$

$$\frac{1}{x}=\frac{2}{y}=\frac{2}{z}$$

$$y=2x \quad \text{and} \quad z=2x$$

Substituting y and z in Eq. (2),

$$x^2+4x^2+4x^2=36$$

$$9x^2=36$$

$$x^2=4$$

$$x=\pm 2$$

$$\therefore y=\pm 4$$

and

$$z=\pm 4$$

Minimum distance $=\sqrt{(2-1)^2+(4-2)^2+(4-2)^2}=\sqrt{1+4+4}=3$

Maximum distance $=\sqrt{(-2-1)^2+(-4-2)^2+(-4-2)^2}=\sqrt{9+36+36}=9$

Example 11

A rectangular box open at the top is to have a volume of 32 cubic units. Find the dimensions of the box requiring least material for its construction.

[Winter 2016, 2014]

Solution

Let x, y, z be the dimensions of the box.
The box is open at the top.

Surface area $\quad S = xy + 2xz + 2yz \quad \text{...(1)}$

Volume $\quad V = xyz = 32 \quad \text{...(2)}$

Let $\quad \phi(x, y, z) = xyz - 32$

Let the auxiliary equation be

$$(xy + 2xz + 2yz) + \lambda(xyz - 32) = 0 \quad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$y + 2z + \lambda yz = 0$$

$$\lambda = -\frac{y+2z}{yz} = -\frac{1}{z} - \frac{2}{y} \quad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$x + 2z + \lambda xz = 0$$

$$\lambda = -\frac{x+2z}{xz} = -\frac{1}{z} - \frac{2}{x} \quad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$2x + 2y + \lambda xy = 0$$

$$\lambda = -\frac{2x+2y}{xy} = -\frac{2}{y} - \frac{2}{x} \quad \text{...(6)}$$

From Eqs (4), (5) and (6),

$$-\frac{1}{z} - \frac{2}{y} = -\frac{1}{z} - \frac{2}{x} = -\frac{2}{y} - \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x} \quad \text{and} \quad \frac{1}{z} = \frac{2}{x}$$

$$y = x \quad \text{and} \quad z = \frac{x}{2}$$

Substituting y and z in Eq. (2),

$$x(x)\left(\frac{x}{2}\right) = 32$$

$$x^3 = 64$$

$$x = 4$$

$$\therefore y = 4, \quad z = 2$$

Hence, dimensions of the box requiring least material for its construction are 4, 4, 2.

Example 12

A rectangular box without a lid is to be made from 12 m^2 *of cardboard. Find the maximum volume of such a box.* **[Winter 2013, 2015]**

Solution

Let x, y, z be the dimensions of the box.
The box is open at the top.

Volume $\qquad V = xyz \qquad$ ...(1)

Surface area $\qquad S = xy + 2xz + 2yz = 12 \qquad$ ...(2)

Let $\qquad \phi(x, y, z) = xy + 2xz + 2yz - 12$

Let the auxiliary equation be

$$xyz + \lambda(xy + 2xz + 2yz - 12) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$yz + \lambda(y + 2z) = 0$$

$$\lambda = -\frac{yz}{y+2z}$$

$$\frac{1}{\lambda} = -\frac{y+2z}{yz} = -\frac{1}{z} - \frac{2}{y} \qquad \text{...(4)}$$

Differentiating Eq. (4) partially w.r.t. y,

$$xz + \lambda(x + 2z) = 0$$

$$\lambda = -\frac{xz}{x+2z}$$

$$\frac{1}{\lambda} = -\frac{x+2z}{xz} = -\frac{1}{z} - \frac{2}{x} \qquad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$xy + \lambda(2x + 2y) = 0$$

$$\lambda = -\frac{xy}{2(x+y)}$$

$$\frac{1}{\lambda} = -\frac{2x+2y}{xy} = -\frac{2}{y} - \frac{2}{x} \qquad \text{...(6)}$$

From Eqs (4), (5) and (6),

$$-\frac{1}{z} - \frac{2}{y} = -\frac{1}{z} - \frac{2}{x} = -\frac{2}{y} - \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x} \quad \text{and} \quad \frac{1}{z} = \frac{2}{x}$$

$$y = x \quad \text{and} \quad z = \frac{x}{2}$$

Substituting y and z in Eq. (2),

$$x(x) + 2x\left(\frac{x}{2}\right) + 2x\left(\frac{x}{2}\right) = 12$$

$$3x^2 = 12$$

$$x^2 = 4$$

$$x = 2$$

$$\therefore \; y = 2, \; z = 1$$

Hence, maximum volume = $xyz = 2(2)(1) = 4$ m^3

Example 13

A wire of length b is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found.

Solution

Let the piece of length x be bent in the form of a square so that each side is $\frac{x}{4}$.

The area of the square, $A_1 = \frac{x}{4} \cdot \frac{x}{4} = \frac{x^2}{16}$.

Suppose a piece of length y is bent in the form of a circle of radius r; so perimeter of the circle is y.

$$2\pi r = y$$

$$r = \frac{y}{2\pi}$$

The area of the circle, $A_2 = \pi\left(\frac{y}{2\pi}\right)^2 = \frac{y^2}{4\pi}$.

Let sum of the areas be given as

$$f(x, y) = \frac{x^2}{16} + \frac{y^2}{4\pi} \qquad \text{...(1)}$$

Also, $\quad x + y = b \qquad \text{...(2)}$

Let $\quad \phi(x, y) = x + y - b$

Let the auxiliary equation be

$$\left(\frac{x^2}{16} + \frac{y^2}{4\pi}\right) + \lambda(x + y - b) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$\frac{2x}{16} + \lambda = 0$$

$$\lambda = -\frac{x}{8} \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$\frac{2y}{4\pi} + \lambda = 0$$

$$\lambda = -\frac{y}{2\pi} \qquad \text{...(5)}$$

From Eqs (4) and (5),

$$\frac{x}{8} = \frac{y}{2\pi}$$

$$y = \frac{\pi}{4}x$$

Substituting y in Eq. (2),

$$x + \frac{\pi}{4}x = b$$

$$x = \frac{4b}{4+\pi}$$

$$\therefore y = \frac{\pi b}{4+\pi}$$

Hence, the least value of the sum of the areas is

$$\begin{aligned}\frac{x^2}{16} + \frac{y^2}{4\pi} &= \frac{1}{16}\left(\frac{4b}{4+\pi}\right)^2 + \frac{1}{4\pi}\left(\frac{\pi b}{4+\pi}\right)^2 \\ &= \frac{b^2}{(4+\pi)^2}\left(1 + \frac{\pi^2}{4\pi}\right) \\ &= \frac{b^2\pi(4+\pi)}{4\pi(4+\pi)^2} \\ &= \frac{b^2}{4(\pi+4)}\end{aligned}$$

Example 14

A closed rectangular box has length twice its breadth and has constant volume V. Determine the dimensions of the box requiring least surface area.

Solution

Let x be the breadth and y be the height of the rectangular box so length of the box will be $2x$.

Surface area of the box

$$S = 2\,(2x \cdot x + x \cdot y + y \cdot 2x) = 4x^2 + 6xy$$

Let $\quad f(x, y) = 4x^2 + 6xy \quad$ ...(1)

Volume of the box $\quad V = x \cdot 2x \cdot y = 2x^2y \quad$ ...(2)

Let $\quad \phi(x, y) = 2x^2y - V$

Let the auxiliary equation be

$$\left(4x^2 + 6xy\right) + \lambda(2x^2y - V) = 0 \qquad ...(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$8x + 6y + \lambda(4xy) = 0$$

$$\lambda = -\frac{4x+3y}{2xy} \qquad ...(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$6x + \lambda(2x^2) = 0$$

$$\lambda = -\frac{3}{x} \qquad ...(5)$$

From Eqs (4) and (5),

$$\frac{4x+3y}{2xy} = \frac{3}{x}$$

$$4x + 3y = 6y$$

$$y = \frac{4x}{3}$$

Substituting y in Eq. (2),

$$2x^2 \cdot \frac{4x}{3} = V$$

$$x^3 = \frac{3V}{8}$$

$$x = \left(\frac{3V}{8}\right)^{\frac{1}{3}}$$

$$\therefore y = \frac{4}{3}\left(\frac{3V}{8}\right)^{\frac{1}{3}} = \left(\frac{8V}{9}\right)^{\frac{1}{3}}$$

Hence, the dimensions of the box requiring least surface area are $2\left(\frac{3V}{8}\right)^{\frac{1}{3}}, \left(\frac{3V}{8}\right)^{\frac{1}{3}}, \left(\frac{8V}{9}\right)^{\frac{1}{3}}$.

Example 15

Show that if the perimeter of a triangle is a constant, the triangle has maximum area when it is equilateral.

Solution

Let x, y and z be the sides of the triangle.

Perimeter of the triangle $\quad s = \dfrac{x+y+z}{2}$

Area of the triangle $\quad A = \sqrt{s(s-x)(s-y)(s-z)}$

Let $\quad f(x, y, z) = A^2 = s(s-x)(s-y)(s-z) \qquad \ldots(1)$

Also, $\quad x + y + z = 2s \qquad \ldots(2)$

Let $\quad \phi(x, y, z) = x + y + z - 2s$

Let the auxiliary equation be

$$\left[s(s-x)(s-y)(s-z)\right] + \lambda(x+y+z-2s) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$-s(s-y)(s-z) + \lambda = 0$$

$$\lambda = s(s-y)(s-z) \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t y,

$$-s(s-x)(s-z) + \lambda = 0$$

$$\lambda = s(s-x)(s-z) \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t z,

$$-s(s-x)(s-y) + \lambda = 0$$

$$\lambda = s(s-x)(s-y) \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$s(s-y)(s-z) = s(s-x)(s-z) = s(s-x)(s-y)$$

$$s-y = s-x \quad \text{and} \quad s-z = s-y$$

$$y = x \quad \text{and} \quad z = y$$

$$\therefore x = y = z$$

Hence, the triangle is equilateral.

Example 16

The temperature $u(x, y, z)$ at any point in space is $u = 400\, xyz^2$. Find the highest temperature on surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution

Let $$f(x, y, z) = u = 400\,xyz^2 \qquad \text{...(1)}$$

$$x^2 + y^2 + z^2 = 1 \qquad \text{...(2)}$$

Let $$\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

Let the auxiliary equation be

$$400\,xyz^2 + \lambda(x^2 + y^2 + z^2 - 1) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$\lambda = -\frac{200yz^2}{x} \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$400xz^2 + \lambda(2y) = 0$$

$$\lambda = -\frac{200xz^2}{y} \qquad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$800xyz + \lambda(2z) = 0$$

$$\lambda = -400xy$$

$$\lambda = -400xy \qquad \text{...(6)}$$

From Eqs (4), (5) and (6),

$$\frac{200yz^2}{x} = \frac{200xz^2}{y} = 400xy$$

$$200y^2z^2 = 200x^2z^2 = 400x^2y^2$$

$$\frac{1}{x^2} = \frac{1}{y^2} = \frac{2}{z^2}$$

$$x^2 = y^2 = \frac{z^2}{2}$$

Substituting y^2 and z^2 in Eq. (2),

$$x^2 + x^2 + 2x^2 = 1$$

$$4x^2 = 1$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}$$

$$\therefore\ y = \pm\frac{1}{2}, \quad z = \pm\frac{1}{\sqrt{2}}$$

Considering positive sign,

$$x = \frac{1}{2}, \quad y = \frac{1}{2}, \quad z = \frac{1}{\sqrt{2}}$$

Highest temperature on the surface of the sphere

$$u = 400xyz^2 = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 50.$$

Example 17

Use the method of the Lagrange's multipliers to find volume of largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Solution

Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular parallelepiped inscribed in the ellipsoid.

Volume of the parallelepiped, $V = (2x)\,(2y)\,(2z) = 8xyz$.

Let
$$f(x, y, z) = 8\,xyz \qquad \text{...(1)}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{...(2)}$$

Let
$$\phi(x, y, z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$

Let the auxiliary equation be

$$8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$8yz + \lambda\frac{2x}{a^2} = 0$$

$$\lambda = -\frac{4yza^2}{x} \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$8xz + \lambda \frac{2y}{b^2} = 0$$

$$\lambda = -\frac{4xzb^2}{y} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$8xy + \lambda \frac{2z}{c^2} = 0$$

$$\lambda = -\frac{4xyc^2}{z} \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{4yza^2}{x} = \frac{4xzb^2}{y} = \frac{4xyc^2}{z}$$

$$\frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}$$

$$\therefore y^2 = \frac{b^2}{a^2}x^2 \quad \text{and} \quad z^2 = \frac{c^2}{a^2}x^2$$

Substituting y^2, z^2 in Eq. (2),

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\frac{3x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{3}$$

$$x = \pm \frac{a}{\sqrt{3}}$$

$$\therefore y = \pm \frac{b}{\sqrt{3}}, \quad z = \pm \frac{c}{\sqrt{3}}$$

Since sides cannot be negative,

$$x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}$$

Volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid

$$V = 8xyz = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$$

Example 18

Find the minimum distance from origin to the plane $3x + 2y + z = 12$.

Solution

Let (x, y, z) be a point on the plane $3x + 2y + z = 12$.
The distance d between (x, y, z) and origin $(0, 0, 0)$ is

$$d = \sqrt{x^2 + y^2 + z^2}$$
$$d^2 = x^2 + y^2 + z^2$$

Let $$f(x, y, z) = d^2 = x^2 + y^2 + z^2 \qquad \text{...(1)}$$

$$3x + 2y + z = 12 \qquad \text{...(2)}$$

Let $$\phi(x, y, z) = 3x + 2y + z - 12 = 0$$

Let the auxiliary equation be

$$(x^2 + y^2 + z^2) + \lambda(3x + 2y + z - 12) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$2x + \lambda(3) = 0$$
$$\lambda = -\frac{2x}{3} \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$2y + \lambda(2) = 0$$
$$\lambda = -\frac{2y}{2} = -y \qquad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$2z + \lambda(1) = 0$$
$$\lambda = -2z \qquad \text{...(6)}$$

From Eqs (3), (4) and (5),

$$\frac{2x}{3} = y = 2z$$
$$y = \frac{2x}{3}$$
$$z = \frac{x}{3}$$

Substituting y, z in Eq. (2),

$$3x + \frac{4x}{3} + \frac{x}{3} = 12$$

$$\frac{14x}{3} = 12$$

$$\therefore \quad x = \frac{18}{7}, y = \frac{12}{7}, z = \frac{6}{7}$$

The minimum distance is

$$d = \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{\sqrt{504}}{7}$$

Example 19

Divide 24 into three parts such that the continued product of the first, square of the second and cube of the third may be maximum.

Solution

Let x, y and z be three parts of 24.

Let $$f(x, y, z) = xy^2 z^3 \qquad \text{...(1)}$$

Let $$x + y + z = 24 \qquad \text{...(2)}$$

$$\phi(x, y, z) = x + y + z - 24 = 0$$

Let the auxiliary equation be

$$xy^2 z^3 + \lambda\,(x + y + z - 24) = 0 \qquad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$y^2z^3 + \lambda = 0$$

$$\lambda = -\,y^2z^3 \qquad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$2xyz^3 + \lambda = 0$$

$$\lambda = -\,2xyz^3 \qquad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$3xy^2z^2 + \lambda = 0$$

$$\lambda = -\,3xy^2z^2 \qquad \text{...(6)}$$

From Eqs (4), (5), (6),

$$y^2z^3 = 2xyz^3 = 3xy^2z^2$$

Dividing by xy^2z^3,

$$\frac{1}{x} = \frac{2}{y} = \frac{3}{z}$$

$$y = 2x, z = 3x$$

Substituting y, z in Eq. (2)

$$x + 2x + 3x = 24$$

$$6x = 24$$

$$x = 4$$

$$\therefore \qquad y = 8, z = 12$$

Hence, 4, 8 and 12 are three parts of 24.

Example 20

A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ *enters the earth's atmosphere and its surface begins to heat. After* 1 *hour, the temperature at the point* (x, y, z) *on the surface of the probe* $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. *Find the hottest points on the probe's surface.*

Solution

Let $$f(x, y, z) = 8x^2 + 4yz - 16z + 600 \qquad \ldots(1)$$

$$4x^2 + y^2 + 4z^2 = 16 \qquad \ldots(2)$$

Let $$\phi(x, y, z) = 4x^2 + y^2 + 4z^2 - 16 = 0$$

Let the auxiliary equation be

$$(8x^2 + 4yz - 16z + 600) + \lambda(4x^2 + y^2 + 4z^2 - 16) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$16x + \lambda(8x) = 0$$

$$\lambda = -2 \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$4z + \lambda(2y) = 0$$

$$\lambda = -\frac{2z}{y} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$4y - 16 + \lambda(8z) = 0$$

$$\lambda = \frac{16 - 4y}{8z} = \frac{4 - y}{2z} \qquad \ldots(6)$$

From Eqs (4) and (5),

$$-2 = -\frac{2z}{y}$$

$$y = z \qquad \ldots(7)$$

From Eqs (4) and (6),

$$-2 = \frac{4-y}{2y}$$

$$-4y = 4 - y$$

$$-3y = 4$$

$$y = -\frac{4}{3}$$

$$z = -\frac{4}{3}$$

Substituting in Eq. (2),

$$4x^2 + \frac{16}{9} + \frac{64}{9} = 16$$

$$4x^2 = \frac{64}{9}$$

$$x^2 = \frac{16}{9}$$

$$x = \pm\frac{4}{3}$$

The hottest points on the probe's surface are $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

Example 21

Find the points on the surface $z^2 = xy + 1$ *nearest to the origin.*

Solution

Let (x, y, z) be any point on the surface $z^2 = xy + 1$.

The distance d between (x, y, z) and origin $(0, 0, 0)$ is

$$d = \sqrt{x^2 + y^2 + z^2}$$

$$d^2 = x^2 + y^2 + z^2$$

Let $$f(x, y, z) = d^2 = x^2 + y^2 + z^2 \quad \ldots(1)$$

$$z^2 = xy + 1 \quad \ldots(2)$$

Let $$\phi(x, y, z) = z^2 - xy - 1 = 0$$

Let the auxiliary equation be

$$(x^2 + y^2 + z^2) + \lambda(z^2 - xy - 1) = 0 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$2x - \lambda y = 0$$

$$\lambda = \frac{y}{2x} \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$2y - \lambda x = 0$$

$$\lambda = \frac{x}{2y} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$2z + \lambda(2z) = 0$$

$$\lambda = -1 \qquad \ldots(6)$$

From Eqs (4) and (6),

$$\frac{y}{2x} = -1$$

$$y = -2x$$

From Eqs (5) and (6),

$$\frac{x}{2y} = -1$$

$$x = -2y = -2(-2x) = 4x$$

$$x = 0$$

$$\therefore \quad y = 0$$

Substituting in Eq. (2)

$$z^2 = 1$$

$$z = \pm 1$$

The nearest points on the surface from the origin are (0, 0, ±1).

Example 22

If $u = \frac{x^2}{a^3} + \frac{y^2}{b^3} + \frac{z^2}{c^3}$ *where* $x + y + z = 1$ *then prove that stationary value of u is given by* $x = \frac{a^3}{a^3 + b^3 + c^3}$, $y = \frac{b^3}{a^3 + b^3 + c^3}$, $z = \frac{c^3}{a^3 + b^3 + c^3}$.

Solution

Let

$$f(x, y, z) = u = \frac{x^2}{a^3} + \frac{y^2}{b^3} + \frac{z^2}{c^3} \qquad \ldots(1)$$

$$x + y + z = 1 \qquad \ldots(2)$$

Let

$$\phi(x, y, z) = x + y + z - 1 = 0$$

Let the auxiliary equation be

$$\left(\frac{x^2}{a^3}+\frac{y^2}{b^3}+\frac{z^2}{c^3}\right)+\lambda(x+y+z-1)=0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$\frac{2x}{a^3}+\lambda=0$$

$$\lambda=-\frac{2x}{a^3} \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$\frac{2y}{b^3}+\lambda=0$$

$$\lambda=-\frac{2y}{b^3} \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$\frac{2z}{c^3}+\lambda=0$$

$$\lambda=-\frac{2z}{c^3} \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{2x}{a^3}=\frac{2y}{b^3}=\frac{2z}{c^3}$$

$$\frac{x}{a^3}=\frac{y}{b^3}=\frac{z}{c^3}$$

$$y=\frac{b^3}{a^3}x, \quad z=\frac{c^3}{a^3}x$$

Substituting y, z in Eq. (2),

$$x+\frac{b^3}{a^3}x+\frac{c^3}{a^3}x=1$$

$$\frac{(a^3+b^3+c^3)x}{a^3}=1$$

$$x=\frac{a^3}{a^3+b^3+c^3}$$

$$\therefore \qquad y=\frac{b^3}{a^3+b^3+c^3}, \quad z=\frac{c^3}{a^3+b^3+c^3}$$

Example 23

If $u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}$ *where* $x + y + z = 1$ *then find the stationary values.*

Solution

Let $$f(x, y, z) = u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} \quad \ldots(1)$$

$$x + y + z = 1 \quad \ldots(2)$$

Let $$\phi(x, y, z) = x + y + z - 1 = 0$$

Let the auxiliary equation be

$$\left(\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}\right) + \lambda(x + y + z - 1) = 0 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$-\frac{2a^3}{x^3} + \lambda = 0$$

$$\lambda = \frac{2a^3}{x^3} \quad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$-\frac{2b^3}{y^3} + \lambda = 0$$

$$\lambda = \frac{2b^3}{y^3} \quad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$-\frac{2c^3}{z^3} + \lambda = 0$$

$$\lambda = \frac{2c^3}{z^3} \quad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\frac{2a^3}{x^3} = \frac{2b^3}{y^3} = \frac{2c^3}{z^3}$$

$$\frac{a^3}{x^3} = \frac{b^3}{y^3} = \frac{c^3}{z^3}$$

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$$

$$y = \frac{b}{a}x$$
$$z = \frac{c}{a}x$$

Substituting y, z in Eq. (2),

$$x + \frac{b}{a}x + \frac{c}{a}x = 1$$
$$\frac{(a+b+c)}{a}x = 1$$
$$x = \frac{a}{a+b+c}$$
$$\therefore \qquad y = \frac{b}{a+b+c}, \; z = \frac{c}{a+b+c}$$

Example 24

Prove that the stationary values of $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ *where* $lx + my + nz = 0$ *and* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *are given by the roots of the equation* $\frac{l^2a^4}{1-a^2u} + \frac{m^2b^4}{1-b^2u} + \frac{n^2c^4}{1-c^2u} = 0$.

Solution

Let
$$f(x, y, z) = u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \qquad \ldots(1)$$
$$lx + my + nz = 0 \qquad \ldots(2)$$
$$\phi(x, y, z) = lx + my + nz = 0$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \ldots(3)$$
$$\psi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

Let the auxiliary equation be

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) + \lambda_1(lx + my + nz) + \lambda_2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0 \qquad \ldots(4)$$

Differentiating Eq. (4) partially w.r.t. x,

$$\frac{2x}{a^4} + \lambda_1 l + \lambda_2\left(\frac{2x}{a^2}\right) = 0 \qquad \ldots(5)$$

Differentiating Eq. (4) partially w.r.t. y,

$$\frac{2y}{b^4} + \lambda_1 m + \lambda_2\left(\frac{2y}{b^2}\right) = 0 \qquad \ldots(6)$$

Differentiating Eq. (4) partially w.r.t. z,

$$\frac{2z}{c^4} + \lambda_1 n + \lambda_2\left(\frac{2z}{c^2}\right) = 0 \qquad \ldots(7)$$

Multiplying Eq. (5) by x, Eq. (6) by y, Eq. (7) by z and then adding,

$$2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) + \lambda_1(lx + my + nz) + 2\lambda_2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$2u + \lambda_1(0) + 2\lambda_2(1) = 0$$

$$\lambda_2 = -u$$

Substituting $\lambda_2 = -u$ in Eq. (5),

$$\frac{2x}{a^4} + \lambda_1 l - \frac{2xu}{a^2} = 0$$

$$2x\left(\frac{1 - a^2 u}{a^4}\right) + \lambda_1 l = 0$$

$$x = -\frac{a^4 \lambda_1 l}{2(1 - a^2 u)}$$

Similarly

$$y = -\frac{b^4 \lambda_1 m}{2(1 - b^2 u)}$$

$$z = -\frac{c^4 \lambda_1 n}{2(1 - c^2 u)}$$

Substituting x, y, z in Eq. (2),

$$-\frac{l^2 a^4 \lambda_1}{2(1 - a^2 u)} - \frac{m^2 b^4 \lambda_1}{2(1 - b^2 u)} - \frac{n^2 c^4 \lambda_1}{2(1 - c^2 u)} = 0$$

$$\left(\frac{l^2 a^4}{1 - a^2 u} + \frac{m^2 b^4}{1 - b^2 u} + \frac{n^2 c^4}{1 - c^2 u}\right)\lambda_1 = 0$$

But $\lambda_1 \neq 0$

$$\therefore \qquad \frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0$$

Example 25

In a plane triangle ABC, find the extreme values of cos A cos B cos C.

[Summer 2015]

Solution

Let $$f(A, B, C) = \cos A \cos B \cos C \qquad \ldots(1)$$

In a triangle ABC,

$$A + B + C = 180° \qquad \ldots(2)$$

Let $$\phi(A, B, C) = A + B + C - 180°$$

Let the auxiliary equation be

$$\cos A \cos B \cos C + \lambda\ (A + B + C - 180°) = 0 \qquad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. A,

$$-\sin A \cos B \cos C + \lambda = 0$$

$$\lambda = \sin A \cos B \cos C \qquad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. B,

$$-\cos A \sin B \cos C + \lambda = 0$$

$$\lambda = \cos A \sin B \cos C \qquad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. C,

$$-\cos A \cos B \sin C + \lambda = 0$$

$$\lambda = \cos A \cos B \sin C \qquad \ldots(6)$$

From Eqs (4), (5) and (6),

$$\sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C$$

Dividing by cos A cos B cos C,

$$\tan A = \tan B = \tan C$$

$$A = B = C = \frac{\pi}{3}$$

Hence, $$f_{\max} = \cos A \cos B \cos C = \cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{\pi}{3}$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

EXERCISE 8.8

1. Find stationary values of the function $f(x, y, z) = x^2 + y^2 + z^2$, given that $z^2 = xy + 1$.

[Ans.: (0, 0, –1), (0, 0, 1)**]**

2. Find the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ subject to the fulfillment of the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1,$ given a, b, c are not zero.

$$\left[\textbf{Ans.:}\ x=\frac{1}{a}(a+b+c),\ y=\frac{1}{b}(a+b+c),\ z=\frac{1}{c}(a+b+c)\right]$$

3. Find the largest product of the numbers x, y and z when $x + y + z^2 = 16$.

$$\left[\textbf{Ans.:}\ \frac{4096}{25\sqrt{5}}\right]$$

4. Find the largest product of the numbers x, y and z when $x^2 + y^2 + z^2 = 9$.

$$\left[\textbf{Ans.:}\ 3\sqrt{3}\right]$$

5. Find a point in the plane $x + 2y + 3z = 13$ nearest to the point (1, 1, 1).

$$\left[\textbf{Ans.:}\ \left(\frac{3}{2}, 2, \frac{5}{2}\right)\right]$$

6. Find the shortest distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$.

[Ans.: 3**]**

7. Find the maximum distance from the origin (0, 0) to the curve $3x^2 + 3y^2 + 4xy - 2 = 0$.

$$\left[\textbf{Ans.:}\ \sqrt{2}\right]$$

8. Decompose a positive number a into three parts so that their product is maximum.

$$\left[\textbf{Ans.:}\ \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)\right]$$

9. Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

$$\left[\textbf{Ans.:}\ \frac{a^{m+n+p}m^m n^n p^p}{(m+n+p)^{m+n+p}}\right]$$

10. Find the dimensions of a rectangular box of maximum capacity whose surface area is given when

(i) box is open at the top (ii) box is closed

$$\left[\textbf{Ans.:}\ \text{(i)}\ \sqrt{\frac{s}{3}}, \sqrt{\frac{s}{3}}, \frac{1}{2}\sqrt{\frac{s}{3}} \quad \text{(ii)}\ \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}\right]$$

11. Determine the perpendicular distance of the point (a, b, c) from the plane $lx + my + nz = 0$.

$$\left[\textbf{Ans.}: \text{minimum distance} \frac{la+mb+nc}{\sqrt{l^2+m^2+n^2}}\right]$$

12. Find the length and breadth of a rectangle of maximum area that can be inscribed in the ellipse $4x^2 + y^2 = 36$.

$$\left[\textbf{Ans.}: \frac{3\sqrt{2}}{2}, \sqrt{2}, \textit{Area} = 12\right]$$

13. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid of revolution $4x^2 + 4y^2 + 9z^2 = 36$.

$$\left[\textbf{Ans.}: 16\sqrt{3}\right]$$

14. Find the extreme volume of $x^2 + y^2 + z^2 + xy + xz + yz$ subject to the conditions $x + y + z = 1$ and $x + 2y + 3z = 3$.

$$\left[\textbf{Ans.}: \frac{1}{6}, \frac{1}{3}, \frac{5}{6}\right]$$

Points to Remember

Chain Rule

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{df}{du}\frac{\partial u}{\partial x} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} \quad \text{or} \quad \frac{df}{du}\frac{\partial u}{\partial y} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial y}$$

Composite Function of One Variable

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Composite Function of Two Variables

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Implicit Differentiation

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

Gradient and Directional Derivative

Gradient

$$\text{grad } f = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$df = |\nabla f||d\bar{r}|\cos\theta$$

Directional Derivative

$$D_a f = \nabla f \cdot \hat{a} = \nabla f \cdot \frac{a}{|\bar{a}|}$$

Tangent Plane and Normal to a Surface

The equation of the tangent plane at $P(x_0, y_0, z_0)$ to the surface $f(x, y, z) = 0$ is

$$(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$$

The equations of the normal line to the surface at $P(x_0, y_0, z_0)$ are

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

Local Extreme Values (Maximum and Minimum Values)

To determine the maxima and minima (extreme values) of a function $f(x, y)$

Step 1: Solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously for x and y.

Step 2: Obtain the values of $r = \dfrac{\partial^2 f}{\partial x^2}, s = \dfrac{\partial^2 f}{\partial x\, \partial y}, t = \dfrac{\partial^2 f}{\partial y^2}$.

Step 3: (i) If $rt - s^2 > 0$ and $r < 0$ (or $t < 0$) at (a, b) then $f(x, y)$ is maximum at (a, b) and the maximum value of the function is $f(a, b)$.

(ii) If $rt - s^2 > 0$ and $r > 0$ (or $t > 0$) at (a, b) then $f(x, y)$ is minimum at (a, b) and the minimum value of the function is $f(a, b)$.

(iii) If $rt - s^2 < 0$ at (a, b) then $f(x, y)$ is neither maximum nor minimum at (a, b).

(iv) If $rt - s^2 = 0$ at (a, b) then no conclusion can be made about the extreme values of $f(x, y)$.

Method of Lagrange Multipliers

Let $f(x, y, z)$ be a function of three variables x, y, z, and the variables be connected by the relation

$$\phi(x, y, z) = 0$$

Let $f(x, y, z) + \lambda\phi(x, y, z) = 0$ be an auxiliary equation.
Differentiating this equation partially w.r.t x, y and z

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

Eliminating λ from these equations, the values of x, y and z are obtained for which $f(x, y, z)$ has a stationary (maximum and minimum) value.

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x+y}$ is

(a) limit does not exist (b) 0
(c) 1 (d) –1

2. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+\sqrt{y}}{\sqrt{x^2+y}}$, $x \neq 0$, $y \neq 0$ is

(a) limit does not exist (b) 0
(c) 1 (d) –1

3. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$ is

(a) 0 (b) $\dfrac{1}{2}$ (c) 1 (d) does not exist

4. The value of $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{y^2-x^2}$ is

(a) 0 (b) 1 (c) –1 (d) does not exist

5. The value of $\lim\limits_{\substack{x \to 1 \\ y \to 1}} \dfrac{8x^2 y}{x^2 + y^2 + 5}$ is

(a) $\dfrac{3}{7}$ (b) $\dfrac{8}{5}$ (c) $\dfrac{8}{7}$ (d) $\dfrac{3}{5}$

6. The value of $\lim\limits_{\substack{x \to 1 \\ y \to 2}} \dfrac{4xy}{6x^2 + y^2}$ is

(a) $\dfrac{4}{5}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{3}{10}$ (d) $\dfrac{2}{5}$

7. The value of $\lim\limits_{\substack{x \to 1 \\ y \to 0}} \dfrac{2x^2 + y}{4x - y}$ is

(a) $\dfrac{3}{2}$ (b) $\dfrac{1}{2}$ (c) 1 (d) $\dfrac{5}{2}$

8. The value of $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{2x^2 + y}{4x^2 - y}$ is

(a) –1 (b) $\dfrac{1}{2}$ (c) 1 (d) does not exist

9. The value of $\lim\limits_{(x,y) \to (0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ is **[Winter 2015]**

(a) 1 (b) 0 (c) –1 (d) does not exist

10. The value of $\lim\limits_{(x,y) \to (0,0)} \dfrac{x^2 - yx}{x + y}$ is **[Winter 2014]**

(a) 2 (b) 1 (c) 0 (d) –1

11. If $x = r\cos\theta$, $y = r\sin\theta$ then $\dfrac{\partial r}{\partial x} = $ — and $\dfrac{\partial r}{\partial y} = $ —

(a) $\dfrac{x}{r}, \tan\theta$ (b) $\dfrac{x}{r}, \dfrac{y}{r}$ (c) $\tan\theta, \sin\theta$ (d) $\dfrac{x}{r}, \sin\theta$

12. If $u = \sin(ax + by + cz)$ then $\dfrac{\partial u}{\partial x} =$

(a) $a\cos(ax + by + cz)$ (b) $a\sin(ax + by + cz)$
(c) $b\cos(ax + by + cz)$ (d) $b\sin(ax + by + cz)$

13. If $u = x^2 y + y^2 z + z^2 x$ then $\dfrac{\partial u}{\partial x} + \dfrac{\partial x}{\partial y} + \dfrac{\partial u}{\partial z} =$

(a) $x + y + z$ (b) $(x + y + z)^2$ (c) $\dfrac{1}{x+y+z}$ (d) $\dfrac{1}{(x+y+z)^2}$

14. If $f = x^2 + y^2$ then $\dfrac{\partial^2 f}{\partial x \partial y} =$

(a) 1 (b) 0 (c) –1 (d) 2

15. If $u = \log (x^2 + y^2)$ then $\dfrac{\partial u}{\partial x} =$

(a) $\dfrac{2y}{x^2 + y^2}$ (b) $\dfrac{2}{x^2 + y^2}$ (c) $\dfrac{2x}{x^2 + y^2}$ (b) $\dfrac{y}{x^2 + y^2}$

16. If $u = \sin (x + y)$ then $\dfrac{\partial u}{\partial y} =$

(a) $\sin x$ (b) $\cos(x + y)$ (c) $\tan(x + y)$ (d) $\cos x$

17. If $u = e^x (x \cos y - y \sin y)$ then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} =$

(a) 1 (b) –1 (c) 0 (d) 2

18. If $u = x^y$ then $\dfrac{\partial u}{\partial x}$ is

(a) 0 (b) yx^{y-1} (c) $x^y \log x$ (d) yx^y

19. If $u = y^x$ then $\dfrac{\partial u}{\partial x} =$ ——— and $\dfrac{\partial u}{\partial y} =$ ———

(a) $\log y, y^{x-1}$ (b) $y^x \log x, x$ (c) y^x, xy^{x-1} (d) $y^x \log y, xy^{x-1}$

20. If $u = x^3 + y^3$ then $\dfrac{\partial^2 u}{\partial x \partial y} =$

(a) –3 (b) 3 (c) 0 (d) 1

21. If $u = f(x + ay) + g(x - ay)$ then $\dfrac{\partial^2 u}{\partial y^2}$ is

(a) $\dfrac{\partial^2 u}{\partial x^2}$ (b) $a \dfrac{\partial^2 u}{\partial x^2}$ (c) $a^2 \dfrac{\partial^2 u}{\partial x^2}$ (d) $\dfrac{\partial^2 u}{\partial x \partial y}$

22. If $u = (x - y)^4 + (y - z)^4 + (z - x)^4$ then $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}$ is

(a) 1 (b) u (c) $4u$ (d) 0

23. If $u = \tan^{-1}(x + y)$ then $u_x - u_y$ is

(a) 0 (b) 1 (c) –1 (d) $\sin x \cos y$

24. If $f = \log (x \tan^{-1} y)$ then f_{xy} is equal to

(a) $-\dfrac{1}{x^2}$ (b) 0 (c) $\dfrac{1}{x^2}$ (d) $\dfrac{1}{x}$

25. If $z = x^2 - y^2$ then $\dfrac{\partial z}{\partial x} =$ **[Winter 2014]**

(a) $2y$ (b) 0 (c) $2z$ (d) $2x$

26. If $f(x, y, z, w) = \dfrac{3\cos(xw)\sin y^5}{e^y + \dfrac{(1+y^2)}{xyw}} + 5xzw$ then $\dfrac{\partial f}{\partial z}$ at $(1,2,3,4)$ is **[Winter 2015]**

(a) 20 (b) 200 (c) 0 (d) 1

27. If $z = f(x,y)$, $\mathrm{d}z$ is equal to

(a) $\left(\dfrac{\partial f}{\partial x}\right)\mathrm{d}x + \left(\dfrac{\partial f}{\partial y}\right)\mathrm{d}y$ (b) $\left(\dfrac{\partial f}{\partial y}\right)\mathrm{d}x + \left(\dfrac{\partial f}{\partial x}\right)\mathrm{d}y$

(c) $\left(\dfrac{\partial f}{\partial x}\right)\mathrm{d}x - \left(\dfrac{\partial f}{\partial y}\right)\mathrm{d}y$ (d) $\left(\dfrac{\partial f}{\partial y}\right)\mathrm{d}x - \left(\dfrac{\partial f}{\partial x}\right)\mathrm{d}y$

28. For an implicit function $f(x,y) = c$, the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is **[Summer 2017, 2015]**

(a) $\dfrac{f_x}{f_y}$ (b) $\dfrac{f_y}{f_x}$ (c) $-\dfrac{f_x}{f_y}$ (d) $-\dfrac{f_y}{f_x}$

29. If $x^3 + y^3 + 3xy = 0$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} =$

(a) $\dfrac{x^2 - y}{y^2 - x}$ (b) $-\dfrac{x^2 + y}{y^2 + x}$ (c) $\dfrac{x^2 + y}{x^2 - y}$ (d) $\dfrac{x^2 + y}{x - y}$

30. If $u = \sin(xy^2)$, $x = \log t$, $y = e^t$ then $\dfrac{\mathrm{d}u}{\mathrm{d}t} =$

(a) $y^2\left(\dfrac{1}{t} + 2x\right)\cos xy^2$ (b) 0

(c) 1 (d) -1

31. If f is a function of u, v, w and u, v, w are functions of x, y, z then $\dfrac{\partial f}{\partial y}$ is

(a) $\dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial z} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial z} + \dfrac{\partial f}{\partial w}\cdot\dfrac{\partial w}{\partial z}$ (b) $\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial u} + \dfrac{\partial f}{\partial z}\cdot\dfrac{\partial z}{\partial u}$

(c) $\dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial y} + \dfrac{\partial f}{\partial w}\cdot\dfrac{\partial w}{\partial y}$ (d) $\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial v} + \dfrac{\partial f}{\partial z}\cdot\dfrac{\partial z}{\partial v}$

32. If $u = x^2 - y^2$, $v = xy$ then $\dfrac{\partial x}{\partial u}$ is

(a) $\dfrac{x}{2(x^2 + y^2)}$ (b) $\dfrac{y}{2(x^2 + y^2)}$ (c) $\dfrac{y}{x^2 + y^2}$ (d) $\dfrac{x}{x^2 + y^2}$

33. If $f(x,y) = e^{xy^2}$, the total differential of the function at the point (1,2) is

(a) $e(dx + dy)$ (b) $e^4(dx + dy)$ (c) $e^4(4dx + dy)$ (d) $4e^4(dx + dy)$

34. If $f(x, y) = x^2 + y^2 + 3$, the minimum value of $f(x, y)$ is

(a) 3 (b) ∞ (c) 0 (d) 1

35. The stationary points of $x^3 + y^3 - 3axy$ are

(a) $(0, 0), (a, a)$ (b) $(0, 0), (a, 0)$ (c) $(a, 0), (0, -a)$ (d) $(0, a), (a, 0)$

36. If $f(x,y) = xy + (x - y)$, the stationary points are

(a) (0, 0) (b) (1, –1) (c) (1, 2) (d) (1, –2)

37. The stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ are

(a) $(\sqrt{2}, \sqrt{2})$ (b) $(-\sqrt{2}, -\sqrt{2})$ (c) $(\sqrt{2}, -\sqrt{2})$ (d) $(0, \sqrt{2})$

38. In a plane triangle ABC, the maximum value of $\cos A \cos B \cos C$ is

(a) 0 (b) $\frac{1}{8}$ (c) $\frac{\sqrt{3}}{8}$ (d) $\frac{3}{8}$

39. For the function $f(x,y) = x^2 + y^2 + 6x + 12$, minima occurs at

(a) (0, 3) (b) (–3, 0) (c) (3, 0) (d) (0, –3)

40. The function $f(x, y) = 2x^2 + 2xy - y^3$ has

(a) only one stationary point at (0, 0)

(b) two stationary points at (0, 0) and $\left(\frac{1}{6}, -\frac{1}{3}\right)$

(c) two stationary points at (0, 0) and (1, –1)

(d) no stationary points

41. The function $z = 5xy - 4x^2 + y^2 - 2x - y + 5$ has at $x = \frac{1}{41}, y = \frac{18}{41}$

(a) maxima (b) saddle point (c) minima (d) no conclusion

42. With usual notations, the properties of maxima and minima under various conditions are

I		II	
(P)	Maxima	(i)	$rt - s^2 = 0$
(Q)	Minima	(ii)	$rt - s^2 < 0$
(R)	Saddle Point	(iii)	$rt - s^2 > 0, r > 0$
(S)	Failure Case	(iv)	$rt - s^2 > 0, r < 0$

(a) P – i, Q – iii, R – iv, S – ii (b) P – ii Q – i, R – iii, S – iv
(c) P–iv, Q – iii, R–ii, S – i (d) P – iv, Q – ii, R – i, S – iii

43. The minimum value of $f(x,y) = x^2y^2$ is **[Winter 2015]**

(a) 1 (b) 2 (c) 4 (d) no conclusion

44. The sum of the squares of two positive numbers is 200, their minimum product is

(a) 200 (b) $25\sqrt{7}$ (c) 28 (d) 0

45. The minimum value of $x^2 + y^2 + z^2$ given that $xy + yz + zx = 3a^2$ is

(a) $3a$ (b) $4a^2$ (c) $\frac{1}{3}a^2$ (d) $3a^2$

46. For the auxiliary equation $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z) = 0$, the Lagrange's equations are **[Winter 2014]**

(a) $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ (b) $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$

(c) $\frac{\partial \phi}{\partial x} = 0, \frac{\partial \phi}{\partial y} = 0, \frac{\partial \phi}{\partial z} = 0$ (d) $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial \phi} = 0, \frac{\partial f}{\partial z} = 0$

47. The equation of the tangent plane of $z = x$ at (2, 0, 2) is

(a) $z = x$ (b) $x + y + z = 2$ (c) $z + x = 0$ (d) $x + y = 2$

48. A point (a, b) is said to be saddle point if at (a, b) **[Summer 2016]**

(a) $rt - s^2 > 0$ (b) $rt - s^2 = 0$ (c) $rt - s^2 < 0$ (d) $rt - s^2 \geq 0$

49. The minimum value of $f(x, y) = x^2 + y^2$ is **[Winter 2016]**

(a) 1 (b) 2 (c) 4 (d) 0

Answers

1. (a)	2. (c)	3. (d)	4. (d)	5. (c)	6. (a)	7. (b)	8. (d)	9. (d)
10. (c)	11. (b)	12. (a)	13. (b)	14. (b)	15. (c)	16. (b)	17. (c)	18. (b)
19. (d)	20. (c)	21. (c)	22. (d)	23. (a)	24. (b)	25. (d)	26. (a)	27. (a)
28. (c)	29. (b)	30. (a)	31. (c)	32. (a)	33. (d)	34. (a)	35. (a)	36. (b)
37. (c)	38. (b)	39. (b)	40. (b)	41. (b)	42. (c)	43. (d)	44. (d)	45. (d)
46. (a)	47. (a)	48. (b)	49. (d)					