CHAPTER 3

Ordinary Differential Equations and Applications

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3.1 INTRODUCTION

Differential equations are very important in engineering mathematics. A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders. It provides the medium for the interaction between mathematics and various branches of science and engineering. Most common differential equations are radioactive decay, chemical reactions, Newton's law of cooling, series *RL*, *RC*, and *RLC* circuits, simple harmonic motions, etc.

3.2 DIFFERENTIAL EQUATIONS

A differential equation is an equation which involves variables (dependent and independent) and their derivatives, e.g.,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \qquad \ldots(3.1)$$

$$\left(\frac{d^2y}{dx^2}\right)^2 - \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^3 = 0 \qquad \ldots(3.2)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \qquad \ldots(3.3)$$

Equations (3.1) and (3.2) involve ordinary derivatives and, hence, are called *ordinary differential equations* whereas Eq. (3.3) involves partial derivatives and, hence, is called a *partial differential equation*.

3.2.1 Order

The order of a differential equation is the order of the highest derivative present in the equation, e.g., the order of Eqs (3.1) and (3.2) is 2.

3.2.2 Degree

The degree of a differential equation is the power of the highest order derivative after clearing the radical sign and fraction, e.g., the degree of Eq. (3.1) is 1 and the degree of Eq. (3.2) is 2.

3.2.3 Solution or Primitive

The solution of a differential equation is a relation between the dependent and independent variables (excluding derivatives), which satisfies the equation.

The solution of a differential equation is not always unique. It may have more than one solution or sometimes no solution.

The general solution of a differential equation of order n contains n arbitrary constants.

The particular solution of a differential equation is obtained from the general solution by giving particular values to the arbitrary constants.

3.2.4 Formation of Differential Equations

Ordinary differential equations are formed by elimination of arbitrary constants c_1, $c_2, \ldots, c_n$ from a relation like $f(x, y, c_1, c_2, \ldots, c_n) = 0$

Consider $\qquad f(x, y, c_1, c_2, \ldots, c_n) = 0 \qquad \ldots(3.4)$

Differentiating Eq. (3.4) successively w.r.t. x, n times and eliminating n arbitrary constants $c_1, c_2, ..., c_n$ from the above $(n + 1)$ equations a differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}. \cdots\right) = 0$$

is obtained. Its general solution is given by Eq. (3.4) itself.

Example 1

Form the differential equation by eliminating arbitrary constants from $\log\left(\frac{y}{x}\right) = cx$.

Solution

$$\log\left(\frac{y}{x}\right) = cx \qquad ...(1)$$

Differentiating Eq.(1) w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} - \frac{1}{x} = c$$

Eliminating c from Eq. (1),

$$\log\left(\frac{y}{x}\right) = x\left(\frac{1}{y}\frac{dy}{dx} - \frac{1}{x}\right)$$

$$= \frac{x}{y}\frac{dy}{dx} - 1$$

which is the differential equation of first order.

Example 2

Find the differential equation of the family of circles of radius r whose centre lies on the x-axis. **[Winter 2014]**

Solution

Let $(a, 0)$ be the centre and r be the radius of the family of circles. The equation of the family of circles is

$$(x - a)^2 + (y - 0)^2 = r^2$$

$$(x - a)^2 + y^2 = r^2 \qquad ...(1)$$

where a is an arbitrary constant.

Differentiating Eq. (1) w.r.t. x,

$$2x-2a+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$x-a=-y\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{...(2)}$$

Eliminating a from Eqs (1) and (2),

$$\left(-y\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+y^2=r^2$$

$$y^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+y^2=r^2$$

$$\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]y^2=r^2$$

which is the equation of the family of circles.

Example 3

Form the differential equation by eliminating arbitrary constants from $y = Ae^{-3x} + Be^{2x}$.

Solution

$$y=Ae^{-3x}+Be^{2x} \qquad \text{...(1)}$$

Differentiating Eq. (1) twice w.r.t. x,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-3Ae^{-3x}+2Be^{2x} \qquad \text{...(2)}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=9Ae^{-3x}+4Be^{2x} \qquad \text{...(3)}$$

Eliminating A and B from Eqs (1), (2), and (3),

$$\begin{vmatrix} e^{-3x} & e^{2x} & y \\ -3e^{-3x} & 2e^{2x} & -\frac{\mathrm{d}y}{\mathrm{d}x} \\ 9e^{-3x} & 4e^{2x} & -\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \end{vmatrix}=0$$

$$(-1)e^{-3x}e^{2x}\begin{vmatrix} 1 & 1 & y \\ -3 & 2 & \frac{\mathrm{d}y}{\mathrm{d}x} \\ 9 & 4 & -\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \end{vmatrix}=0$$

$$\begin{vmatrix} 1 & 1 & y \\ -3 & 2 & \dfrac{dy}{dx} \\ 9 & 4 & \dfrac{d^2y}{dx^2} \end{vmatrix} = 0$$

$$1\left(2\frac{d^2y}{dx^2} - 4\frac{dy}{dx}\right) - 1\left(-3\frac{d^2y}{dx^2} - 9\frac{dy}{dx}\right) + y(-12-18) = 0$$

$$5\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 30y = 0$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

which is the differential equation of order two.

3.3 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation which contains first-order and first-degree derivatives of y (dependent variable) and known functions of x (independent variable) and y is known as an ordinary differential equation of first order and first degree. The general form of this equation can be written as

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

or in explicit form as

$$\frac{dy}{dx} = f(x, y) \text{ or } M(x, y)dx + N(x, y)dy = 0$$

Solution of the differential equation can be obtained by classifying them as follows:

(i) Variable separable
(ii) Homogeneous differential equations
(iii) Nonhomogeneous differential equations
(iv) Exact differential equations
(v) Non-exact differential equations reducible to exact form
(vi) Linear differential equations
(vii) Nonlinear differential equations reducible to linear form

3.3.1 Variable Separable

A differential equation of the form

$$M(x)dx + N(y)dy = 0 \qquad \text{...(3.5)}$$

where M (x) is the function of x only and N (y) is the function of y only, is called a differential equation with variables separable as in Eq. (3.5), the function of x and the function of y can be separated easily.

Integrating Eq. (3.5), we get the solution as

$$\int M(x)\,dx + \int N(y)\,dy = c$$

or

$$\int g(y)\,dy = \int f(x)\,dx + c$$

where c is the arbitrary constant.

Example 1

Solve $y(1+x^2)^{\frac{1}{2}}\,dy + x\sqrt{1+y^2}\,dx = 0$.

Solution

$$y(1+x^2)^{\frac{1}{2}}\,dy = -x\sqrt{1+y^2}\,dx$$

$$\frac{y}{\sqrt{1+y^2}}\,dy = -\frac{x}{\sqrt{1+x^2}}\,dx$$

Integrating both the sides,

$$\int \frac{y}{\sqrt{1+y^2}}\,dy = -\int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\frac{1}{2}\int (1+y^2)^{-\frac{1}{2}}(2y)\,dy = -\frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}(2x)\,dx$$

$$\frac{1}{2}\cdot\frac{(1+y^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}\cdot\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

Example 2

Solve $9yy' + 4x = 0$. **[Summer 2016]**

Solution

$$9yy' = -4x$$

$$9y\frac{dy}{dx} = -4x$$

$$9y\,dy = -4x\,dy$$

Integrating both the sides,

$$9\frac{y^2}{2} = -\frac{4x^2}{2} + c$$

$$9y^2 + 4x^2 = 2c = c' \qquad \text{where } c' = 2c$$

Example 3

Solve $3e^x \tan y\, \mathrm{d}x + (1 + e^x)\sec^2 y\, \mathrm{d}y = 0.$ **[Winter 2017]**

Solution

$$3e^x \tan y\, \mathrm{d}x = -(1 + e^x)\sec^2 y\, \mathrm{d}y$$

$$\frac{3e^x}{1+e^x}\mathrm{d}x = -\frac{\sec^2 y}{\tan y}\mathrm{d}y$$

Integrating both the sides,

$$\int \frac{3e^x}{1+e^x}\mathrm{d}x = -\int \frac{\sec^2 y}{\tan y}\mathrm{d}y$$

$$3\log(1+e^x) = -\log\tan y + \log c \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log\left|f(x)\right|\right]$$

$$\log(1+e^x)^3 = \log\frac{c}{\tan y}$$

$$(1+e^x)^3 = \frac{c}{\tan y}$$

$$(1+e^x)^3 \tan y = c$$

Example 4

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{x-y} + x^2 e^{-y}.$ **[Summer 2018]**

Solution

$$e^y \frac{\mathrm{d}y}{\mathrm{d}x} = e^x + x^2$$

$$e^y \mathrm{d}y = (e^x + x^2)\,\mathrm{d}x$$

Integrating both the sides,

$$\int e^y \mathrm{d}y = \int (e^x + x^2)\mathrm{d}x$$

$$e^y = e^x + \frac{x^3}{3} + c$$

3.3.2 Homogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \qquad \ldots(3.6)$$

is called a homogeneous equation if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree, i.e., degree of the RHS of Eq. (3.6) is zero.

Equation (3.6) can be reduced to variable-separable form by putting $y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Equation (3.6) reduces to

$$v + x\frac{dv}{dx} = \frac{M(x,vx)}{N(x,vx)} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

$$\frac{dv}{g(v)-v} = \frac{dx}{x}$$

This equation is in variable-separable form and can be solved by integrating

$$\int \frac{dv}{g(v)-v} = \int \frac{dx}{x} + c$$

After integrating and replacing v by $\frac{y}{x}$, we get the solution of Eq. (3.6).

Note: *Homogeneous functions*: A function $f(x, y, z)$ is said to be a homogeneous function of degree n, if for any positive number t,

$$f(xt, yt, zt) = t^n f(x, y, z),$$

where n is a real number.

Example 1

Solve $x(x-y)dy + y^2 dx = 0.$

Solution

$$\frac{dy}{dx} = \frac{-y^2}{x^2 - xy} = \frac{M(x,y)}{N(x,y)} \qquad \ldots(1)$$

The equation is homogeneous since M and N are of the same degree 2.

Let $\qquad y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{-v^2x^2}{x^2(1-v)}=\frac{-v^2}{1-v}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{-v^2}{1-v}-v=\frac{-v}{1-v}$$

$$\left(\frac{v-1}{v}\right)\mathrm{d}v=\frac{\mathrm{d}x}{x}$$

$$\left(1-\frac{1}{v}\right)\mathrm{d}v=\frac{\mathrm{d}x}{x}$$

Integrating both the sides,

$$\int\left(1-\frac{1}{v}\right)\mathrm{d}v=\int\frac{\mathrm{d}x}{x}$$

$$v-\log v=\log x+\log c$$

$$v=\log v+\log cx=\log c\,xv$$

$$\frac{y}{x}=\log cy$$

$$y=x\log cy$$

Example 2

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y+\sqrt{x^2+y^2}}{x}$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y+\sqrt{x^2+y^2}}{x}=\frac{\mathrm{M}(x,y)}{\mathrm{N}(x,y)} \qquad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 1.

Let $y=vx$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$$

Substituting in Eq. (1),

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{vx+\sqrt{x^2+v^2x^2}}{x}=v+\sqrt{1+v^2}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x}=\sqrt{1+v^2}$$

$$\frac{\mathrm{d}v}{\sqrt{1+v^2}}=\frac{\mathrm{d}x}{x}$$

Integrating both the sides,

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\log\left(v+\sqrt{v^2+1}\right) = \log x + \log c = \log cx$$

$$v+\sqrt{v^2+1} = cx$$

$$\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1} = cx$$

$$y+\sqrt{y^2+x^2} = cx^2$$

Example 3

Solve $x^2y\,dx - (x^3 + xy^2)\,dy = 0.$ **[Winter 2012]**

Solution

$$\frac{dy}{dx} = \frac{x^2y}{x^3+xy^2} = \frac{xy}{x^2+xy} = \frac{M(x, y)}{N(x, y)} \qquad ...(1)$$

The equation is homogeneous since M and N are of the same degree 2.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{x(vx)}{x^2+x(vx)} = \frac{v}{1+v}$$

$$x\frac{dv}{dx} = \frac{v}{1+v} - v = \frac{v-v-v^2}{1+v}$$

$$\frac{1+v}{v^2}dv = -\frac{dx}{x}$$

$$\left(\frac{1}{v^2}+\frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrating both the sides,

$$-\frac{1}{v} + \log v = -\log x + \log c$$

$$\log v + \log x = \frac{1}{v} + \log c$$

$$= \frac{1}{v}\log e + \log c$$

$$= \log e^{\frac{1}{v}} + \log c$$

$$\log vx = \log c\, e^{\frac{1}{v}}$$

$$vx = ce^{\frac{1}{v}}$$

$$y = ce^{\frac{x}{y}}$$

3.3.3 Nonhomogeneous Differential Equations

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad \ldots(3.7)$$

is called a nonhomogeneous equation where $a_1, b_1, c_1, a_2, b_2, c_2$ are all constants. These equations are classified into two parts and can be solved by the following methods:

Case I If

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$$

$$a_1 = a_2m,\ b_1 = b_2m,$$

then Eq. (3.7) reduces to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{m(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2} \qquad \ldots(3.8)$$

Putting $a_2x + b_2y = t$, $a_2 + b_2\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}t}{\mathrm{d}x}$, Eq. (3.8) reduces to variable-separable form and can be solved using the method of variable-separable equation.

Case II If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, then substituting

$x = X + h$, $y = Y + k$ in Eq. (3.7),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X}$$

$$= \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$$= \frac{(a_1X + b_1Y) + (a_1h + b_1k + c_1)}{(a_2X + b_2Y) + (a_2h + b_2k + c_2)} \qquad \ldots(3.9)$$

Choosing h, k such that

$$a_1h + b_1k + c_1 = 0,\ a_2h + b_2k + c_2 = 0,$$

then Eq. (3.9) reduces to

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is a homogeneous equation and can be solved using the method of homogeneous equation. Finally, substituting $X = x - h$, $Y = y - k$, we get the solution of Eq. (3.7).

Problems Based on Case I: $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$

Example 1

Solve $(x + y - 1)\,\mathrm{d}x + (2x + 2y - 3)\,\mathrm{d}y = 0.$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x + y - 1}{2x + 2y - 3} = \frac{-x - y + 1}{2x + 2y - 3} \qquad \ldots(1)$$

The equation is nonhomogeneous and $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = -\dfrac{1}{2}$

Let $x + y = t$

$$1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x} - 1$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}t}{\mathrm{d}x} - 1 = \frac{-t + 1}{2t - 3}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-t + 1}{2t - 3} + 1$$

$$= \frac{-t + 1 + 2t - 3}{2t - 3}$$

$$= \frac{t - 2}{2t - 3}$$

$$\left(\frac{2t - 3}{t - 2}\right)\mathrm{d}t = \mathrm{d}x$$

$$\left(2 + \frac{1}{t - 2}\right)\mathrm{d}t = \mathrm{d}x$$

Integrating both the sides,

$$\int\left(2+\frac{1}{t-2}\right)\mathrm{d}t=\int\mathrm{d}x$$

$$2t+\log(t-2)=x+c$$

$$2(x+y)+\log(x+y-2)=x+c$$

$$x+2y+\log(x+y-2)=c$$

Example 2

Solve $(x+y)\,\mathrm{d}x+(3x+3y-4)\,\mathrm{d}y=0,\ y(1)=0.$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-x-y}{3x+3y-4} \qquad \ldots(1)$$

The equation is nonhomogeneous and $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{-1}{3}$

Let $x+y=t$

$$1+\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}t}{\mathrm{d}x}-1$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}t}{\mathrm{d}x}-1=\frac{-t}{3t-4}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{-t}{3t-4}+1=\frac{-t+3t-4}{3t-4}=\frac{2t-4}{3t-4}$$

$$\left(\frac{3t-4}{2t-4}\right)\mathrm{d}t=\mathrm{d}x$$

$$\frac{1}{2}\left(3+\frac{2}{t-2}\right)\mathrm{d}t=\mathrm{d}x$$

Integrating both the sides,

$$\frac{1}{2}\int\left(3+\frac{2}{t-2}\right)\mathrm{d}t=\int\mathrm{d}x$$

$$\frac{1}{2}\left[3t+2\log|(t-2)|\right]=x+c$$

$$3(x+y)+2\log|(x+y-2)|=2x+2c$$

$$x+3y+2\log|(x+y-2)|=k,\ \text{where } 2c=k$$

Given $y\,(1) = 0$
Putting $x = 1$, $y = 0$ in the above equation,

$$1 + 2\log|-1| = k$$
$$1 + 2\log 1 = k$$
$$k = 1$$

Hence, the solution is

$$x + 3y + 2\log|x + y - 2| = 1$$

Problems Based on Case II: $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

Example 1

Solve $(x+2y)\,\mathrm{d}x + (y-1)\,\mathrm{d}y = 0.$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x-2y}{y-1} \qquad \ldots (1)$$

The equation is nonhomogeneous and $\dfrac{-1}{0} \neq \dfrac{-2}{1}$

Let $x = X + h$, $\quad y = Y + k$

$\mathrm{d}x = \mathrm{d}X$, $\quad \mathrm{d}y = \mathrm{d}Y$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X}$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{-(X+h) - 2(Y+k)}{(Y+k) - 1}$$
$$= \frac{(-X-2Y) + (-h-2k)}{Y + (k-1)} \qquad \ldots (2)$$

Choosing h, k such that

$$-h - 2k = 0,\ k - 1 = 0 \qquad \ldots (3)$$

Solving these equations,

$$k = 1,\ h = -2$$

Substituting Eq. (3) in Eq. (2),

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{-X-2Y}{Y} \qquad \ldots (4)$$

which is a homogeneous equation.
Let $\quad Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

Substituting in Eq. (4),

$$v + X\frac{dv}{dX} = \frac{-X - 2vX}{vX}$$

$$= \frac{-1-2v}{v}$$

$$X\frac{dv}{dX} = \frac{-1-2v}{v} - v$$

$$= \frac{-1-2v-v^2}{v}$$

$$= \frac{-(v+1)^2}{v}$$

$$\frac{v}{(v+1)^2}dv = -\frac{dX}{X}$$

$$\left[\frac{1}{v+1} - \frac{1}{(v+1)^2}\right]dv = -\frac{dX}{X}$$

Integrating both the sides,

$$\int\frac{1}{v+1}dv - \int\frac{1}{(v+1)^2}dv = -\int\frac{dX}{X}$$

$$\log(v+1) + \frac{1}{v+1} = -\log X + c$$

$$\log\left(\frac{Y}{X}+1\right) + \frac{1}{\frac{Y}{X}+1} = -\log X + c$$

$$\log\left(\frac{Y+X}{X}\right) + \frac{X}{Y+X} = -\log X + c$$

$$\log(Y+X) - \log X + \frac{X}{Y+X} = -\log X + c$$

$$\log(Y+X) + \frac{X}{Y+X} = c$$

Now, $X = x - h = x + 2$

$Y = y - k = y - 1$

Hence, the general solution is

$$\log(x+y+1) + \left(\frac{x+2}{x+y+1}\right) = c$$

Example 2

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x-5y+3}{2x+4y-6}$.

Solution

The equation is nonhomogeneous and $\dfrac{2}{2}\neq\dfrac{-5}{4}$

Let $x = X + h, \quad y = Y + k$

$$\mathrm{d}x = \mathrm{d}X, \quad \mathrm{d}y = \mathrm{d}Y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}Y}{\mathrm{d}X}$$

Substituting in the given equation,

$$\frac{\mathrm{d}Y}{\mathrm{d}X}=\frac{2(X+h)-5(Y+k)+3}{2(X+h)+4(Y+k)-6}$$

$$=\frac{(2X-5Y)+(2h-5k+3)}{(2X+4Y)+(2h+4k-6)} \qquad \ldots(1)$$

Choosing h, k such that

$$2h - 5k + 3 = 0, \qquad 2h + 4k - 6 = 0 \qquad \ldots(2)$$

Solving the equations,

$$h = k = 1$$

Substituting Eq. (2) in Eq. (1),

$$\frac{\mathrm{d}Y}{\mathrm{d}X}=\frac{2X-5Y}{2X+4Y} \qquad \ldots(3)$$

which is a homogeneous equation.

Let $Y = vX$

$$\frac{\mathrm{d}Y}{\mathrm{d}X}=v+X\frac{\mathrm{d}v}{\mathrm{d}X}$$

Substituting in Eq. (3),

$$v+X\frac{\mathrm{d}v}{\mathrm{d}X}=\frac{2X-5vX}{2X+4vX}=\frac{2-5v}{2+4v}$$

$$X\frac{\mathrm{d}v}{\mathrm{d}X}=\frac{2-5v}{2+4v}-v$$

$$=\frac{2-5v-2v-4v^2}{2+4v}$$

$$=\frac{-4v^2-7v+2}{2+4v}$$

$$\frac{2+4v}{4v^2+7v-2}\mathrm{d}v = -\frac{\mathrm{d}X}{X}$$

$$\frac{2+4v}{(4v-1)(v+2)}\mathrm{d}v = -\frac{\mathrm{d}X}{X} \qquad \ldots(4)$$

Now,
$$\frac{2+4v}{(4v-1)(v+2)} = \frac{A}{4v-1}+\frac{B}{v+2}$$

$$2+4v = A(v+2)+B(4v-1)$$
$$= (A+4B)v+(2A-B)$$

Comparing coefficients on both the sides,

$$A+4B=4, \qquad 2A-B=2$$

$$A=\frac{4}{3}, \qquad B=\frac{2}{3}$$

$$\frac{2+4v}{(4v-1)(v+2)} = \frac{4}{3(4v-1)}+\frac{2}{3(v+2)}$$

Substituting in Eq. (4),

$$\left[\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right]\mathrm{d}v = -\frac{\mathrm{d}X}{X}$$

Integrating both the sides,

$$\int\left[\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right]\mathrm{d}v = -\int\frac{\mathrm{d}X}{X}$$

$$\frac{4}{3}\frac{\log(4v-1)}{4}+\frac{2}{3}\log(v+2) = -\log X+\log c$$

$$\frac{1}{3}\log(4v-1)(v+2)^2 = \log\frac{c}{X}$$

$$\log(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}} = \log\frac{c}{X}$$

$$(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}} = \frac{c}{X}$$

$$\left(\frac{4Y}{X}-1\right)^{\frac{1}{3}}\left(\frac{Y}{X}+2\right)^{\frac{2}{3}} = \frac{c}{X}$$

$$(4Y-X)^{\frac{1}{3}}(Y+2X)^{\frac{2}{3}} = c$$

$$(4Y-X)(Y+2X)^2 = c^3 = k$$

Now,
$$X = x - h = x - 1$$
$$Y = y - k = y - 1$$

Hence, the general solution is
$$(4y - x - 3)(y + 2x - 3)^2 = k$$

3.3.4 Exact Differential Equations

Any first-order differential equation which is obtained by differentiation of its general solution without any elimination or reduction of terms is known as exact differential equation.

If $f(x, y) = c$ is the general solution then
$$\mathrm{d}f = 0$$
$$\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = 0$$
$$\mathrm{M}(x, y)\mathrm{d}x + \mathrm{N}(x, y)\mathrm{d}y = 0 \qquad \ldots(3.10)$$

represents an exact differential equation,

where $\quad \mathrm{M}(x, y) = \dfrac{\partial f}{\partial x}, \qquad \mathrm{N}(x, y) = \dfrac{\partial f}{\partial y}$
$$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial \mathrm{N}}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

But
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Therefore,
$$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

Thus, the necessary condition for a differential equation to be an exact differential equation is
$$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

The solution of Eq. (3.10) can be written as
$$\int\limits_{y\text{ constant}} \mathrm{M}(x, y)\mathrm{d}x + \int (\text{terms of N not containing } x)\mathrm{d}y = c$$

Sometimes, integration of M w.r.t. x is tedious whereas N can be integrated easily w.r.t. y. In this case, the solution can be written as
$$\int (\text{terms of M not containing } y)\mathrm{d}x + \int\limits_{x\text{ constant}} \mathrm{N}(x, y)\mathrm{d}y = c$$

Example 1

Check whether the given differential equation is exact or not

$$(x^4 - 2xy^2 + y^4)\,\mathrm{d}x - (2x^2y - 4xy^3 + \sin y)\,\mathrm{d}y = 0$$

Hence, find the general solution. **[Winter 2017]**

Solution

$$\mathrm{M} = x^4 - 2xy^2 + y^4, \qquad \mathrm{N} = -2x^2y + 4xy^3 - \sin y$$

$$\frac{\partial \mathrm{M}}{\partial y} = -4xy + 4y^3, \qquad \frac{\partial \mathrm{N}}{\partial x} = -4xy + 4y^3$$

Since $\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\ \text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int (x^4 - 2xy^2 + y^4)\,\mathrm{d}x + \int (-\sin y)\,\mathrm{d}y = c$$

$$\frac{x^5}{5} - 2\frac{x^2}{2}y^2 + xy^4 + \cos y = c$$

$$\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$$

Example 2

Solve $(y^2 - x^2)\mathrm{d}x + 2xy\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = y^2 - x^2, \qquad \mathrm{N} = 2xy$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2y, \qquad \frac{\partial \mathrm{N}}{\partial x} = 2y$$

Since $\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\ \text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int (y^2 - x^2)\,\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$xy^2 - \frac{x^3}{3} = c$$

Example 3

Solve $(x^3 + 3xy^2)\,\mathrm{d}x + (3x^2y + y^3)\,\mathrm{d}y = 0.$ **[Winter 2014]**

Solution

$$\mathrm{M} = x^3 + 3xy^2, \qquad \mathrm{N} = 3x^2y + y^3$$

$$\frac{\partial \mathrm{M}}{\partial y} = 3x(2y), \qquad \frac{\partial \mathrm{N}}{\partial x} = 3y(2x)$$

$$= 6xy, \qquad = 6xy$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int (x^3 + 3xy^2)\,\mathrm{d}x + \int y^3\,\mathrm{d}y = c$$

$$\frac{x^4}{4} + 3y^2\frac{x^2}{2} + \frac{y^4}{4} = c$$

Example 4

Solve $(2xy\cos x^2 - 2xy + 1)\mathrm{d}x + (\sin x^2 - x^2)\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = 2xy\cos x^2 - 2xy + 1, \qquad \mathrm{N} = \sin x^2 - x^2$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2x\cos x^2 - 2x, \qquad \frac{\partial \mathrm{N}}{\partial x} = (\cos x^2)(2x) - 2x$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{constant}} \mathrm{M}\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int(2xy\cos x^2 - 2xy + 1)\,dx + \int 0\,dy = c$$

$$y\sin x^2 - x^2 y + x = c \qquad \left[\because \int\{\cos f(x)\}f'(x)\,dx = \sin f(x)\right]$$

Example 5

Solve $ye^x dx + (2y + e^x)\,dy = 0.$ **[Summer 2015]**

Solution

$$M = ye^x, \qquad N = 2y + e^x$$

$$\frac{\partial M}{\partial y} = e^x, \qquad \frac{\partial N}{\partial x} = e^x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M\,dx + \int(\text{terms of N not containg } x)\,dy = c$$

$$\int ye^x\,dx + \int 2y\,dy = c$$

$$ye^x + 2\frac{y^2}{2} = c$$

$$ye^x + y^2 = c$$

Example 6

Solve $\frac{dy}{dx} = \frac{x^2 - x - y^2}{2xy}.$ **[Winter 2015]**

Solution

$$(x^2 - x - y^2)dx = 2xy\,dy$$

$$(x^2 - x - y^2)dx - 2xy\,dy = 0$$

$$M = x^2 - x - y^2, \qquad N = -2xy$$

$$\frac{\partial M}{\partial y} = -2y, \qquad \frac{\partial N}{\partial x} = -2y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} M\,dx + \int(\text{terms of N not containing } x)dy = c$$

$$\int (x^2 - x - y^2)\,\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$\frac{x^3}{3} - \frac{x^2}{2}y - xy^2 = c$$

Example 7

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$ **[Winter 2016; Summer 2013]**

Solution

$$(y\cos x + \sin y + y)\mathrm{d}x + (\sin x + x\cos y + x)\mathrm{d}y = 0$$

$$\mathrm{M} = y\cos x + \sin y + y, \qquad \mathrm{N} = \sin x + x\cos y + x$$

$$\frac{\partial \mathrm{M}}{\partial y} = \cos x + \cos y + 1, \qquad \frac{\partial \mathrm{N}}{\partial x} = \cos x + \cos y + 1$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int (y\cos x + \sin y + y)\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$y\sin x + x(\sin y + y) = c$$

Example 8

Solve $[(x + 1)\,e^x - e^y]\,\mathrm{d}x - xe^y\,\mathrm{d}y = 0,\ y(1) = 0.$ **[Summer 2014]**

Solution

$$\mathrm{M} = (x+1)e^x - e^y, \quad \mathrm{N} = -xe^y$$

$$\frac{\partial \mathrm{M}}{\partial y} = -e^y, \qquad \frac{\partial \mathrm{N}}{\partial x} = -e^y$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int\left[(x+1)e^x - e^y\right]\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$\int (x+1)e^x\mathrm{d}x - e^y \int \mathrm{d}x = c$$

$$(x+1)e^x - e^x - x\,e^y = c$$

$$x\,e^x - x\,e^y = c \qquad \text{...(1)}$$

Given $y(1) = 0$

Substituting $x = 1$, $y = 0$ in Eq. (1),

$$e - 1 = c$$

Hence, the solution is

$$x\,e^x - x\,e^y = e - 1$$

Example 9

Solve $\left(1+e^{\frac{x}{y}}\right)\mathrm{d}x + e^{\frac{x}{y}}\left(1-\dfrac{x}{y}\right)\mathrm{d}y = 0,\ y(0) = 4.$

Solution

$$\mathrm{M} = 1 + e^{\frac{x}{y}}, \qquad \mathrm{N} = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$$

$$\frac{\partial \mathrm{M}}{\partial y} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right), \qquad \frac{\partial \mathrm{N}}{\partial x} = e^{\frac{x}{y}}\left(\frac{1}{y}\right)\left(1-\frac{x}{y}\right) + e^{\frac{x}{y}}\left(-\frac{1}{y}\right)$$

$$= \frac{-x}{y^2}e^{\frac{x}{y}}, \qquad = -\frac{x}{y^2}e^{\frac{x}{y}}$$

Since $\dfrac{\partial \mathrm{M}}{\partial \mathrm{y}} = \dfrac{\partial \mathrm{N}}{\partial \mathrm{x}}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int \left(1+e^{\frac{x}{y}}\right)\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$x + ye^{\frac{x}{y}} = c \qquad \text{...(1)}$$

Given $y(0) = 4$
Substituting in Eq. (1),

$$0 + 4e^0 = c$$
$$4 = c$$

Hence, the solution is

$$x + ye^{\frac{x}{y}} = 4$$

Example 10

Solve $\left[\log(x^2 + y^2) + \dfrac{2x^2}{x^2 + y^2}\right]\mathrm{d}x + \dfrac{2xy}{x^2 + y^2}\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = \left[\log(x^2 + y^2) + \frac{2x^2}{x^2 + y^2}\right], \qquad \mathrm{N} = \frac{2xy}{x^2 + y^2}$$

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{1}{x^2 + y^2}\cdot 2y - \frac{2x^2}{(x^2 + y^2)^2}\cdot 2y, \qquad \frac{\partial \mathrm{N}}{\partial x} = \frac{2y}{x^2 + y^2} - \frac{2xy}{(x^2 + y^2)^2}\cdot 2x$$

$$= \frac{2y}{x^2 + y^2} - \frac{4x^2 y}{(x^2 + y^2)^2} \qquad = \frac{2y}{x^2 + y^2} - \frac{4x^2 y}{(x^2 + y^2)^2}$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int (\text{terms of M not containing } y)\mathrm{d}x + \int_{x\text{ constant}} \mathrm{N}\,\mathrm{d}y = c$$

$$\int 0\,\mathrm{d}x + \int \frac{2xy}{x^2 + y^2}\mathrm{d}y = c$$

$$x\log(x^2 + y^2) = c$$

Example 11

For what values of a and b is the differential equation $(y + x^3)\mathrm{d}x + (ax + by^3)\mathrm{d}y = 0$ *exact? Also, find the solution of the equation.*

Solution

$$\mathrm{M} = y + x^3, \qquad \mathrm{N} = ax + by^3$$

$$\frac{\partial \mathrm{M}}{\partial y} = 1, \qquad \frac{\partial \mathrm{N}}{\partial x} = a$$

The equation will be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = a$$

Hence, the equation is exact for $a = 1$ and for all values of b.
Substituting $a = 1$ in the equation, $(y + x^3)\,dx + (x + by^3)\,dy = 0$, which is exact.
Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (y + x^3)\,dx + \int by^3\,dy = c$$

$$xy + \frac{x^4}{4} + \frac{by^4}{4} = c$$

Example 12

Solve $(\cos x + y\sin x)\,dx = (\cos x)dy,\ y(\pi) = 0.$

Solution

$$(\cos x + y\sin x)\,dx - (\cos x)\,dy = 0$$

$$M = \cos x + y\sin x, \qquad N = -\cos x$$

$$\frac{\partial M}{\partial y} = \sin x, \qquad \frac{\partial N}{\partial x} = \sin x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int (\cos x + y\sin x)\,dx + \int 0\,dy = c$$

$$\sin x - y\cos x = c \qquad \text{...(1)}$$

Given $y(\pi) = 0$

Substituting $x = \pi$, $y = 0$ in Eq. (1),

$$\sin \pi - 0 = c$$

$$0 = c$$

Hence, the solution is

$$\sin x - y\cos x = 0$$

$$y = \tan x$$

Example 13

Solve $(ye^{xy} + 4y^3)\,\mathrm{d}x + (xe^{xy} + 12xy^2 - 2y)\,\mathrm{d}y = 0,\ y(0) = 2.$

Solution

$$\mathrm{M} = ye^{xy} + 4y^3, \qquad \mathrm{N} = xe^{xy} + 12xy^2 - 2y$$

$$\frac{\partial \mathrm{M}}{\partial y} = e^{xy} + ye^{xy} \cdot x + 12y^2, \qquad \frac{\partial \mathrm{N}}{\partial x} = e^{xy} + xe^{xy} \cdot y + 12y^2$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}\,\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int (ye^{xy} + 4y^3)\,\mathrm{d}x + \int -2y\,\mathrm{d}y = c$$

$$y\frac{e^{xy}}{y} + 4y^3x - y^2 = c$$

$$e^{xy} + 4xy^3 - y^2 = c \qquad \text{...(1)}$$

Given $y(0) = 2$
Substituting $x = 0$, $y = 2$ in Eq. (1),

$$e^0 + 0 - 4 = c, \qquad -3 = c$$

Hence, the solution is

$$e^{xy} + 4xy^3 - y^2 = -3$$

EXERCISE 3.1

Solve the following differential equations:

1. $(2x^3 + 3y)\,\mathrm{d}x + (3x + y - 1)\,\mathrm{d}y = 0$

$$\left[\textbf{Ans.:}\ x^4 + 6xy + y^2 - 2y = c\right]$$

2. $(1 + e^x)\,\mathrm{d}x + y\,\mathrm{d}y = 0$

$$\left[\textbf{Ans.:}\ x + e^x + \frac{y^2}{2} = c\right]$$

3. $\sinh x \cos y \, dx - \cosh x \sin y \, dy = 0$

$$\left[\textbf{Ans.:}\ \cosh x \cos y = c\right]$$

4. $xe^{x^2+y^2} dx + y(1+e^{x^2+y^2})dy = 0,\ y(0) = 0$

$$\left[\textbf{Ans.:}\ y^2 + e^{x^2+y^2} = 1\right]$$

5. $\left(4x^3y^3 + \dfrac{1}{x}\right)dx + \left(3x^4y^2 - \dfrac{1}{y}\right)dy = 0,\ y(1) = 1$

$$\left[\textbf{Ans.:}\ x^4y^3 + \log\left(\frac{x}{y}\right) = 1\right]$$

6. $(4x^3y^3 dx + 3x^4y^2 dy) - (2xy\,dx + x^2 dy) = 0$

$$\left[\textbf{Ans.:}\ x^4y^3 - x^2y = c\right]$$

7. $2x(ye^{x^2} - 1)dx + e^{x^2} dy = 0$

$$\left[\textbf{Ans.:}\ ye^{x^2} - x^2 = c\right]$$

8. $(1 + x^2\sqrt{y})y\,dx + (x^2\sqrt{y} + 2)x\,dy = 0$

$$\left[\textbf{Ans.:}\ 2xy + \frac{2}{3}x^3y^{\frac{3}{2}} = c\right]$$

9. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

$$\left[\textbf{Ans.:}\ \sin x(e^y + 1) = c\right]$$

10. $(x^2 + 1)\dfrac{dy}{dx} = x^3 - 2xy + x$

$$\left[\textbf{Ans.:}\ x^4 - 4x^2y + 2x^2 - 4y = c\right]$$

11. $\dfrac{dy}{dx} = \dfrac{x^2 - 2xy}{x^2 - \sin y}$

$$\left[\textbf{Ans.:}\ x^3 - 3(x^2y + \cos y) = c\right]$$

12. $\dfrac{dy}{dx} = \dfrac{y+1}{(y+2)e^y - x}$

$$\left[\textbf{Ans.:}\ (y+1)(x - e^y) = c\right]$$

13. $(x - y\cos x)dx - \sin x\, dy = 0,\ y\left(\frac{\pi}{2}\right) = 1$

$$\left[\textbf{Ans.:}\ x^2 - 2y\sin x = \frac{\pi^2}{4} - 2\right]$$

14. $(2xy + e^y)dx + (x^2 + xe^y)dy = 0,\ y(1) = 1$

$$\left[\textbf{Ans.:}\ x^2y + xe^y = e + 1\right]$$

3.3.5 Non-Exact Differential Equations Reducible to Exact Form

Sometimes, a differential equation is not exact but can be made exact by multiplying with a suitable function. This function is known as Integrating factor (IF). There may exists more than one integrating factor to a differential equation.
Here, we will discuss different methods to find an IF to a non-exact differential equation,

$$\mathrm{M}\,dx + \mathrm{N}\,dy = 0$$

Case I

If $\dfrac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{N} = f(x)$, (function of x alone) then $\mathrm{IF} = e^{\int f(x)dx}$

After multiplication with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

Example 1

Solve $(x^2 + y^2 + 3)\,dx - 2xy\,dy = 0.$ **[Summer 2017]**

Solution

$$\mathrm{M} = x^2 + y^2 + 3, \qquad \mathrm{N} = -2xy$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2y, \qquad \frac{\partial \mathrm{N}}{\partial x} = -2y$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{2y - (-2y)}{-2xy} = -\frac{2}{x}$$

$$\text{IF} = e^{\int -\frac{2}{x}\mathrm{d}x} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2}(x^2+y^2+3)\mathrm{d}x - \frac{1}{x^2}2xy\,\mathrm{d}y = 0$$

$$\left(1+\frac{y^2+3}{x^2}\right)\mathrm{d}x - \frac{2y}{x}\mathrm{d}y = 0$$

$$\mathrm{M}_1 = 1+\frac{y^2+3}{x^2}, \qquad \mathrm{N}_1 = -\frac{2y}{x}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{2y}{x^2}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{2y}{x^2}$$

Since $\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\text{ constant}} \mathrm{M}_1\mathrm{d}x + \int(\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int\left(1+\frac{y^2+3}{x^2}\right)\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$x - \frac{y^2+3}{x} = c$$

$$x^2 - y^2 - 3 = cx$$

Example 2

Solve $\left(xy^2 - e^{\frac{1}{x^3}}\right)\mathrm{d}x - x^2y\,\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = xy^2 - e^{\frac{1}{x^3}}, \qquad \mathrm{N} = -x^2y$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2xy, \qquad \frac{\partial \mathrm{N}}{\partial x} = -2xy$$

Since $\frac{\partial \mathrm{M}}{\partial y} \neq \frac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2xy-(-2xy)}{-x^2 y}=-\frac{4}{x}$$

$$\text{IF}=e^{\int -\frac{4}{x}dx}=e^{-4\log x}=e^{\log x^{-4}}=x^{-4}=\frac{1}{x^4}$$

Multiplying the DE by the IF,

$$\frac{1}{x^4}(xy^2-e^{\frac{1}{x^3}})\mathrm{d}x-\frac{1}{x^4}(x^2 y)\mathrm{d}y=0$$

$$\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)\mathrm{d}x-\frac{y}{x^2}\mathrm{d}y=0$$

$$\mathrm{M}_1=\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}, \qquad \mathrm{N}_1=-\frac{y}{x^2}$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=\frac{2y}{x^3}, \qquad \frac{\partial \mathrm{N}_1}{\partial x}=\frac{2y}{x^3}$$

Since $\frac{\partial \mathrm{M}_1}{\partial y}=\frac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}_1\mathrm{d}x+\int(\text{terms of } \mathrm{N}_1 \text{ not containing } x)\mathrm{d}y=c$$

$$\int\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)\mathrm{d}x+\int 0\,\mathrm{d}y=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}\int e^{\frac{1}{x^3}}\left(-\frac{3}{x^4}\right)\mathrm{d}x=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}}=c \qquad \left[\because \int e^{f(x)}f'(x)\mathrm{d}x=e^{f(x)}+c\right]$$

Example 3

Solve $(2x\log x-xy)\mathrm{d}y+2y\,\mathrm{d}x=0.$

Solution

$$2y\,\mathrm{d}x+(2x\log x-xy)\mathrm{d}y=0$$

$$\mathrm{M} = 2y, \qquad \mathrm{N} = 2x\log x - xy$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2, \qquad \frac{\partial \mathrm{N}}{\partial x} = 2\log x + 2 - y$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{2-(2\log x + 2 - y)}{2x\log x - xy}$$

$$= \frac{-(2\log x - y)}{x(2\log x - y)} = -\frac{1}{x}$$

$$\mathrm{IF} = e^{\int -\frac{1}{x}\mathrm{d}x} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying the DE by the IF,

$$\frac{1}{x}(2y)\,\mathrm{d}x + \frac{1}{x}(2x\log x - xy)\,\mathrm{d}y = 0$$

$$\frac{2y}{x}\mathrm{d}x + (2\log x - y)\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{2y}{x}, \qquad \mathrm{N}_1 = 2\log x - y$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{2}{x}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{2}{x}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}_1\mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \frac{2y}{x}\,\mathrm{d}x + \int (-y)\,\mathrm{d}y = c$$

$$2y\log x - \frac{y^2}{2} = c$$

Example 4

Solve $x\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y(x\cos x - \sin x) = 2.$

Solution

$$x\sin x\,\mathrm{d}y + (xy\cos x - y\sin x - 2)\,\mathrm{d}x = 0$$

$$(xy\cos x - y\sin x - 2)\,\mathrm{d}x + x\sin x\,\mathrm{d}y = 0$$

$$\mathrm{M} = xy\cos x - y\sin x - 2 \qquad \mathrm{N} = x\sin x$$

$$\frac{\partial \mathrm{M}}{\partial y} = x\cos x - \sin x \qquad \frac{\partial \mathrm{N}}{\partial x} = \sin x + x\cos x$$

Since $\frac{\partial \mathrm{M}}{\partial y} \neq \frac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial \mathrm{M}}{\partial y} - \frac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{(x\cos x - \sin x) - (\sin x + x\cos x)}{x\sin x}$$

$$= -\frac{2\sin x}{x\sin x} = -\frac{2}{x}$$

$$\mathrm{IF} = e^{\int -\frac{2}{x}\mathrm{d}x} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2}(xy\cos x - y\sin x - 2)\,\mathrm{d}x + \frac{1}{x^2}(x\sin x)\,\mathrm{d}y = 0$$

$$\left(\frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}\right)\mathrm{d}x + \frac{1}{x}\sin x\,\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}, \qquad \mathrm{N}_1 = \frac{\sin x}{x}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\cos x}{x} - \frac{\sin x}{x^2}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

Since $\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int (\text{terms of } \mathrm{M}_1 \text{ not containing } y)\,\mathrm{d}x + \int \mathrm{N}_1\,\mathrm{d}y = c$$

$$\int -\frac{2}{x^2}\,\mathrm{d}x + \int \frac{\sin x}{x}\,\mathrm{d}y = c$$

$$\frac{2}{x} + \left(\frac{\sin x}{x}\right)y = c$$

$$\frac{2}{x} + \frac{y\sin x}{x} = c$$

Case II

If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M} = f(y)$, (function of y alone), then $\text{IF} = e^{\int f(y)\,dy}$

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equation.

Example 1

Solve $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy=0.$

Solution

$$M = y^4+2y, \qquad N = xy^3+2y^4-4x$$

$$\frac{\partial M}{\partial y} = 4y^3+2, \qquad \frac{\partial N}{\partial x} = y^3-4$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M} = \frac{y^3-4-(4y^3+2)}{y^4+2y} = \frac{-3(y^3+2)}{y(y^3+2)} = -\frac{3}{y}$$

$$\text{IF} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

Multiplying the DE by the IF,

$$\frac{1}{y^3}(y^4+2y)dx+\frac{1}{y^3}(xy^3+2y^4-4x)\,dy=0$$

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0$$

$$M_1 = y+\frac{2}{y^2}, \qquad N_1 = x+2y-\frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y} = 1-\frac{4}{y^3}, \qquad \frac{\partial N_1}{\partial x} = 1-\frac{4}{y^3}$$

Since, $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$$

$$\int\left(y+\frac{2}{y^2}\right)\mathrm{d}x+\int 2y\,\mathrm{d}y=c$$

$$\left(y+\frac{2}{y^2}\right)x+y^2=c$$

Example 2

Solve $(2xy^4e^y+2xy^3+y)\,\mathrm{d}x+(x^2y^4e^y-x^2y^2-3x)\,\mathrm{d}y=0.$

Solution

$$\mathrm{M}=2xy^4e^y+2xy^3+y, \qquad \mathrm{N}=x^2y^4e^y-x^2y^2-3x$$

$$\frac{\partial \mathrm{M}}{\partial y}=2x(y^4e^y+4y^3e^y+3y^2)+1, \qquad \frac{\partial \mathrm{N}}{\partial x}=2xy^4e^y-2xy^2-3$$

Since $\dfrac{\partial \mathrm{M}}{\partial y}\neq\dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial \mathrm{N}}{\partial x}-\dfrac{\partial \mathrm{M}}{\partial y}}{\mathrm{M}}=\frac{(2xy^4e^y-2xy^2-3)-(2xy^4e^y+8xy^3e^y+6xy^2+1)}{2xy^4e^y+2xy^3+y}$$

$$=\frac{-4(2xy^3e^y+2xy^2+1)}{y(2xy^3e^y+2xy^2+1)}=-\frac{4}{y}$$

$$\mathrm{IF}=e^{\int-\frac{4}{y}\mathrm{d}y}=e^{-4\log y}=e^{\log y^{-4}}=y^{-4}=\frac{1}{y^4}$$

Multiplying the DE by the IF,

$$\frac{1}{y^4}(2xy^4e^y+2xy^3+y)\mathrm{d}x+\frac{1}{y^4}(x^2y^4e^y-x^2y^2-3x)\mathrm{d}y=0$$

$$\left(2xe^y+\frac{2x}{y}+\frac{1}{y^3}\right)\mathrm{d}x+\left(x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}\right)\mathrm{d}y=0$$

$$\mathrm{M}_1=2xe^y+\frac{2x}{y}+\frac{1}{y^3}, \qquad \mathrm{N}_1=x^2e^y-\frac{x^2}{y^2}-\frac{3x}{y^4}$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=2xe^y-\frac{2x}{y^2}-\frac{3}{y^4}, \qquad \frac{\partial \mathrm{N}_1}{\partial x}=2xe^y-\frac{2x}{y^2}-\frac{3}{y^4}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y}=\dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}_1 \mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right) \mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

Example 3

Solve $xe^x(\mathrm{d}x - \mathrm{d}y) + e^x \mathrm{d}x + ye^y \mathrm{d}y = 0.$

Solution

$$(xe^x + e^x)\mathrm{d}x + (ye^y - xe^x)\mathrm{d}y = 0$$

$$\mathrm{M} = xe^x + e^x, \qquad \mathrm{N} = ye^y - xe^x$$

$$\frac{\partial \mathrm{M}}{\partial y} = 0, \qquad \frac{\partial \mathrm{N}}{\partial x} = -e^x - xe^x$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial \mathrm{N}}{\partial x} - \dfrac{\partial \mathrm{M}}{\partial y}}{\mathrm{M}} = \frac{-e^x(x+1) - 0}{e^x(x+1)} = -1$$

$$\mathrm{IF} = e^{\int -\mathrm{d}y} = e^{-y}$$

Multiplying the DE by the IF,

$$e^{-y}(xe^x + e^x)\mathrm{d}x + e^{-y}(ye^y - xe^x) = 0$$

$$\mathrm{M}_1 = e^{-y}(xe^x + e^x), \qquad \mathrm{N}_1 = y - xe^{x-y}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = -e^{-y}(xe^x + e^x), \qquad \frac{\partial \mathrm{N}_1}{\partial x} = -e^{-y}(xe^x + e^x)$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}_1 \mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int e^{-y}(xe^x + e^x)\mathrm{d}x + \int y\,\mathrm{d}y = c$$

$$e^{-y}(xe^x - e^x + e^x) + \frac{y^2}{2} = c$$

$$xe^{x-y} + \frac{y^2}{2} = c$$

Example 4

Solve $\left(\frac{y}{x}\sec y - \tan y\right)dx + (\sec y \log x - x)dy = 0.$

Solution

$$M = \frac{y}{x}\sec y - \tan y, \qquad N = \sec y \log x - x$$

$$\frac{\partial M}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \tan y - \sec^2 y, \qquad \frac{\partial N}{\partial x} = \frac{\sec y}{x} - 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\frac{\sec y}{x} - 1 - \frac{\sec y}{x} - \frac{y}{x}\sec y \tan y + \sec^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{-\frac{y}{x}\sec y \tan y + \tan^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= -\tan y$$

$$\text{IF} = e^{\int -\tan y\, dy} = e^{-\log \sec y} = e^{\log(\sec y)^{-1}} = (\sec y)^{-1} = \cos y$$

Multiplying the DE by the IF,

$$\cos y\left(\frac{y}{x}\sec y - \tan y\right)dx + \cos y(\sec y \log x - x)\,dy = 0$$

$$\left(\frac{y}{x} - \sin y\right)dx + (\log x - x\cos y)\,dy = 0$$

$$M_1 = \frac{y}{x} - \sin y, \qquad N_1 = \log x - x\cos y$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{x} - \cos y, \qquad \frac{\partial N_1}{\partial x} = \frac{1}{x} - \cos y$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 \mathrm{d}x + \int (\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \left(\frac{y}{x} - \sin y \right) \mathrm{d}x + \int 0 \,\mathrm{d}y = c$$

$$y \log x - x \sin y = c$$

Case III

If the differential equation is of the form $f_1(xy)y\,\mathrm{d}x + f_2(xy)x\,\mathrm{d}y = 0$ then $\text{IF} = \dfrac{1}{Mx - Ny}$, where $M = f_1(xy)y$, $N = f_2(xy)x$ provided $Mx - Ny \neq 0$.

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

Example 1

Solve $(x^2y^2 + 2)\, y\,\mathrm{d}x + (2 - x^2y^2)\, x\,\mathrm{d}y = 0.$ **[Winter 2014]**

Solution

$$M = x^2y^3 + 2y,\ N = 2x - x^3y^2$$

The equation is of the form

$$f_1(xy)\, y\, \mathrm{d}x + f_2(xy)\, x\, \mathrm{d}y = 0$$

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{x^3y^3 + 2xy - 2yx + x^3y^3} = \frac{1}{2x^3y^3}$$

Multiplying the DE by the IF,

$$(x^2y^2 + 2)y\left(\frac{1}{2x^3y^3}\right)\mathrm{d}x + (2 - x^2y^2)x\left(\frac{1}{2x^3y^3}\right)\mathrm{d}y = 0$$

$$\frac{1}{2}\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)\mathrm{d}x + \frac{1}{2}\left(\frac{2}{x^2y^3} - \frac{1}{y}\right)\mathrm{d}y = 0$$

$$M_1 = \frac{1}{2}\left(\frac{1}{x} + \frac{2}{x^3y^2}\right), \qquad N_1 = \frac{1}{2}\left(\frac{2}{x^2y^3} - \frac{1}{y}\right)$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2}\left[2\left(\frac{-2}{x^3y^3}\right)\right], \qquad \frac{\partial N_1}{\partial x} = \frac{1}{2}\left[2\left(\frac{-2}{x^3y^3}\right)\right]$$

$$= -\frac{2}{x^3y^3}, \qquad = -\frac{2}{x^3y^3}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 \mathrm{d}x + \int (\text{terms of } N_1 \text{ not containing } x)\mathrm{d}y = c$$

$$\int \frac{1}{2}\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)\mathrm{d}x + \int \frac{1}{2}\left(-\frac{1}{y}\right)\mathrm{d}y = c$$

$$\frac{1}{2}\left(\log x - \frac{1}{x^2y^2}\right) - \frac{1}{2}\log y = c$$

$$\log x - \log y - \frac{1}{x^2y^2} = c$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{x^2y^2} = c$$

Example 2

Solve $y(1+xy+x^2y^2)\,\mathrm{d}x + x(1-xy+x^2y^2)\,\mathrm{d}y = 0.$

Solution

The equation is of the form

$$f_1(xy)y\,\mathrm{d}x + f_2(xy)x\,\mathrm{d}y = 0$$

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{(xy + x^2y^2 + x^3y^3) - (xy - x^2y^2 + x^3y^3)} = \frac{1}{2x^2y^2}$$

Multiplying the DE by the IF,

$$\frac{y}{2x^2y^2}(1+xy+x^2y^2)\,\mathrm{d}x + \frac{x}{2x^2y^2}(1-xy+x^2y^2)\,\mathrm{d}y = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}\right)\mathrm{d}x + \left(\frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}\right)\mathrm{d}y = 0$$

$$M_1 = \frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}, \qquad N_1 = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}$$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{2x^2y^2} + \frac{1}{2}, \qquad \frac{\partial N_1}{\partial x} = -\frac{1}{2x^2y^2} + \frac{1}{2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M_1 \mathrm{d}x + \int (\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2} \right) \mathrm{d}x + \int -\frac{1}{2y}\,\mathrm{d}y = c$$

$$-\frac{1}{2xy} + \frac{1}{2}\log x + \frac{xy}{2} - \frac{1}{2}\log y = c$$

$$-\frac{1}{2xy} + \frac{xy}{2} + \frac{1}{2}\log\frac{x}{y} = c$$

Example 3

Solve $(xy \sin xy + \cos xy)y\,\mathrm{d}x + (xy \sin xy - \cos xy)x\,\mathrm{d}y = 0.$

Solution

$M = xy^2 \sin xy + y \cos xy, \qquad N = x^2 y \sin xy - x \cos xy$

The equation is in the form

$$f_1(xy)y\,\mathrm{d}x + f_2(xy)x\,\mathrm{d}y = 0$$

$$\text{IF} = \frac{1}{Mx - Ny} = \frac{1}{x^2y^2 \sin xy + xy \cos xy - x^2y^2 \sin xy + xy \cos xy}$$

$$= \frac{1}{2xy \cos xy}$$

Multiplying the DE by the IF,

$$\frac{1}{2xy\cos xy}(xy \sin xy + \cos xy)y\,\mathrm{d}x + \frac{1}{2xy\cos xy}(xy \sin xy - \cos xy)x\,\mathrm{d}y = 0$$

$$\left(\frac{y \tan xy}{2} + \frac{1}{2x} \right)\mathrm{d}x + \left(\frac{x \tan xy}{2} - \frac{1}{2y} \right)\mathrm{d}y = 0$$

$$M_1 = \frac{y\tan xy}{2} + \frac{1}{2x}, \qquad N_1 = \frac{x\tan xy}{2} - \frac{1}{2y}$$

$$\frac{\partial M_1}{\partial y} = \frac{\tan xy + xy\sec^2 xy}{2}, \qquad \frac{\partial N_1}{\partial x} = \frac{\tan xy + xy\sec^2 xy}{2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} M_1 \mathrm{d}x \quad + \int (\text{terms of } N_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\frac{1}{2}\int\left(y\tan xy + \frac{1}{x}\right)\mathrm{d}x + \int -\frac{1}{2y}\,\mathrm{d}y = c$$

$$\frac{1}{2}\left(\frac{y}{y}\log\sec xy + \log x\right) - \frac{1}{2}\log y = c$$

$$\log(x\sec xy) - \log y = 2c$$

$$\log\left(\frac{x}{y}\sec xy\right) = 2c$$

$$\frac{x}{y}\sec xy = e^{2c} = k, \qquad \frac{x}{y}\sec xy = k$$

Case IV

If the differential equation $M\mathrm{d}x + N\mathrm{d}y = 0$ is a homogeneous equation in x and y (degree of each term is same) then $\text{IF} = \dfrac{1}{Mx + Ny}$ provided $Mx + Ny \neq 0$.

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

Example 1

Solve $(x^4 + y^4)\mathrm{d}x - xy^3\,\mathrm{d}y = 0$. **[Summer 2018]**

Solution

$$M = x^4 + y^4, \qquad N = -xy^3$$

The differential equation is homogeneous as each term is of degree 4.

$$\text{IF} = \frac{1}{Mx + Ny} = \frac{1}{x^5 + xy^4 - xy^4} = \frac{1}{x^5}$$

Multiplying the DE by the IF,

$$\frac{1}{x^5}(x^4+y^4)\,\mathrm{d}x-\frac{1}{x^5}(xy^3)\,\mathrm{d}y=0$$

$$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)\mathrm{d}x-\frac{y^3}{x^4}\,\mathrm{d}y=0$$

$$\mathrm{M}_1=\frac{1}{x}+\frac{y^4}{x^5},\qquad \mathrm{N}_1=-\frac{y^3}{x^4}$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=\frac{4y^3}{x^5},\qquad \frac{\partial \mathrm{N}_1}{\partial x}=\frac{4y^3}{x^5}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y}=\dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\text{ constant}} \mathrm{M}_1\mathrm{d}x+\int(\text{terms of }\mathrm{N}_1\text{ not containing }x)\,\mathrm{d}y=c$$

$$\int\left(\frac{1}{x}+\frac{y^4}{x^5}\right)\mathrm{d}x+\int 0\,\mathrm{d}y=c$$

$$\log x-\frac{y^4}{4x^4}=c$$

Example 2

Solve $x^2y\,\mathrm{d}x-(x^3+xy)^2\,\mathrm{d}y=0.$ **[Winter 2014]**

Solution

$$\mathrm{M}=x^2y,\qquad \mathrm{N}=-x^3-xy^2$$

The differential equation is homogeneous as each term is of degree 3.

$$\mathrm{IF}=\frac{1}{\mathrm{M}x+\mathrm{N}y}=\frac{1}{x^3y-x^3y-xy^3}=-\frac{1}{xy^3}$$

Multiplying the DE by the IF,

$$-\frac{1}{xy^3}(x^2y)\,\mathrm{d}x-\left(-\frac{1}{xy^3}\right)(x^3+xy^2)\,\mathrm{d}y=0$$

$$-\frac{x}{y^2}\,\mathrm{d}x+\left(\frac{x^2}{y^3}+\frac{1}{y}\right)\mathrm{d}y=0$$

$$M_1 = -\frac{x}{y^2}, \qquad N_1 = \frac{x^2}{y^3} + \frac{1}{y}$$

$$\frac{\partial M_1}{\partial y} = \frac{2x}{y^3}, \qquad \frac{\partial N_1}{\partial x} = \frac{2x}{y^3}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\text{ constant}} M_1 \mathrm{d}x + \int (\text{terms of } N_1 \text{ not containing } x)\, \mathrm{d}y = c$$

$$\int -\frac{x}{y^2}\mathrm{d}x + \int \frac{1}{y}\mathrm{d}y = c$$

$$-\frac{x^2}{2y^2} + \log y = c$$

Example 3

Solve $(xy - 2y^2)\,\mathrm{d}x - (x^2 - 3xy)\,\mathrm{d}y = 0.$ **[Winter 2013]**

Solution

$$M = xy - 2y^2, \qquad N = -x^2 + 3xy$$

The differential equation is homogeneous as each term is of degree 2.

$$\text{IF} = \frac{1}{Mx + Ny} = \frac{1}{x^2y - 2xy^2 - x^2y + 3xy^2} = \frac{1}{xy^2}$$

Multiplying the DE by the IF,

$$\frac{1}{xy^2}(xy - 2y^2)\mathrm{d}x - \frac{1}{xy^2}(x^2 - 3xy)\mathrm{d}y = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)\mathrm{d}x + \left(-\frac{x}{y^2} + \frac{3}{y}\right)\mathrm{d}y = 0$$

$$M_1 = \frac{1}{y} - \frac{2}{x}, \qquad N_1 = -\frac{x}{y^2} + \frac{3}{y}$$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2}, \qquad \frac{\partial N_1}{\partial x} = -\frac{1}{y^2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{ constant}} \mathrm{M}_1 \mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right)\mathrm{d}x + \int \frac{3}{y}\mathrm{d}y = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

Example 4

Solve $(x^2 y - 2xy^2)\,\mathrm{d}x - (x^3 - 3x^2 y)\,\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = x^2 y - 2xy^2, \qquad \mathrm{N} = -x^3 + 3x^2 y$$

The differential equation is homogeneous as each term is of degree 3.

$$\mathrm{IF} = \frac{1}{\mathrm{M}x + \mathrm{N}y} = \frac{1}{x^3 y - 2x^2 y^2 - x^3 y + 3x^2 y^2} = \frac{1}{x^2 y^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2 y^2}(x^2 y - 2xy^2)\mathrm{d}x - \frac{1}{x^2 y^2}(x^3 - 3x^2 y)\mathrm{d}y = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)\mathrm{d}x - \left(\frac{x}{y^2} - \frac{3}{y}\right)\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{1}{y} - \frac{2}{x}, \qquad \mathrm{N}_1 = -\frac{x}{y^2} + \frac{3}{y}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = -\frac{1}{y^2}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = -\frac{1}{y^2}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\text{constant}} \mathrm{M}_1 \mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right)\mathrm{d}x + \int \frac{3}{y}\mathrm{d}y = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 5

Solve $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y^2}{x} = y.$

Solution

$$x^2\mathrm{d}y + y^2\mathrm{d}x = xy\,\mathrm{d}x$$

$$(y^2 - xy)\mathrm{d}x + x^2\mathrm{d}y = 0$$

$$\mathrm{M} = y^2 - xy, \qquad \mathrm{N} = x^2$$

The differential equation is homogeneous as each term is of degree 2.

$$\mathrm{IF} = \frac{1}{\mathrm{M}x + \mathrm{N}y} = \frac{1}{xy^2 - x^2y + x^2y} = \frac{1}{xy^2}$$

Multiplying the DE by the IF,

$$\frac{1}{xy^2}(y^2 - xy)\mathrm{d}x + \frac{x^2}{xy^2}\mathrm{d}y = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right)\mathrm{d}x + \frac{x}{y^2}\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{1}{x} - \frac{1}{y}, \qquad \mathrm{N}_1 = \frac{x}{y^2}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{1}{y^2}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{1}{y^2}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \mathrm{M}_1\mathrm{d}x + \int(\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int\left(\frac{1}{x} - \frac{1}{y}\right)\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$\log x - \frac{x}{y} = c$$

Example 6

Solve $3x^2y^4\mathrm{d}x + 4x^3y^3\mathrm{d}y = 0,\ y(1) = 1.$

Solution

$$\mathrm{M} = 3x^2y^4, \qquad \mathrm{N} = 4x^3y^3$$

The differential equation is homogeneous as each term is of degree 6.

$$\mathrm{IF} = \frac{1}{\mathrm{M}x + \mathrm{N}y} = \frac{1}{3x^3y^4 + 4x^3y^4} = \frac{1}{7x^3y^4}$$

Multiplying the DE by the IF,

$$\frac{1}{7x^3y^4}(3x^2y^4)\mathrm{d}x + \frac{1}{7x^3y^4}(4x^3y^3)\mathrm{d}y = 0$$

$$\frac{3}{7x}\mathrm{d}x + \frac{4}{7y}\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{3}{7x}, \qquad \mathrm{N}_1 = \frac{4}{7y}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = 0, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = 0$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}_1\mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int \frac{3}{7x}\mathrm{d}x + \int \frac{4}{7y}\mathrm{d}y = \log c$$

$$\frac{3}{7}\log x + \frac{4}{7}\log y = \log c$$

$$\log x^{\frac{3}{7}} + \log y^{\frac{4}{7}} = \log c$$

$$\log\left(x^{\frac{3}{7}}y^{\frac{4}{7}}\right) = \log c$$

$$x^{\frac{3}{7}}y^{\frac{4}{7}} = c \qquad \text{...(1)}$$

Given $y(1) = 1$

Substituting $x = 1$, $y = 1$ in Eq. (1),

$$(1)^{\frac{3}{7}} \cdot (1)^{\frac{4}{7}} = c, \quad 1 = c$$

Hence, the solution is

$$x^{\frac{3}{7}} y^{\frac{4}{7}} = 1$$

Case V

If the differential equation is of the type

$$x^{m_1} y^{n_1} (a_1 y\,\mathrm{d}x + b_1 x\,\mathrm{d}y) + x^{m_2} y^{n_2} (a_2 y\,\mathrm{d}x + b_2 x\,\mathrm{d}y) = 0$$

then $\text{IF} = x^h y^k$

where $$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

and $$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

Solving these two equations, we get the values of h and k.

Example 1

Solve $x(4y\,\mathrm{d}x + 2x\,\mathrm{d}y) + y^3(3y\,\mathrm{d}x + 5x\,\mathrm{d}y) = 0.$

Solution

$$xy^0(4y\,\mathrm{d}x + 2x\,\mathrm{d}y) + x^0 y^3(3y\,\mathrm{d}x + 5x\,\mathrm{d}y) = 0$$

$$m_1 = 1,\ n_1 = 0,\ a_1 = 4,\ b_1 = 2,\ m_2 = 0,\ n_2 = 3,\ a_2 = 3,\ b_2 = 5$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{1 + h + 1}{4} = \frac{0 + k + 1}{2}$$

$$2h + 4 = 4k + 4$$

$$h = 2k \qquad \text{...(1)}$$

and $$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

$$\frac{0 + h + 1}{3} = \frac{3 + k + 1}{5}$$

$$5h + 5 = 3k + 12$$

$$5h - 3k = 7 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$h = 2,\ k = 1$$

$$\text{IF} = x^2 y$$

Multiplying the DE by the IF,

$$x^3 y(4y\,\mathrm{d}x + 2x\,\mathrm{d}y) + x^2 y^4 (3y\,\mathrm{d}x + 5x\,\mathrm{d}y) = 0$$

$$(4x^3y^2 + 3x^2y^5)\,\mathrm{d}x + (2x^4y + 5x^3y^4)\,\mathrm{d}y = 0$$

$$\mathrm{M} = 4x^3y^2 + 3x^2y^5, \qquad \mathrm{N} = 2x^4y + 5x^3y^4$$

$$\frac{\partial \mathrm{M}}{\partial y} = 8x^3y + 15x^2y^4, \qquad \frac{\partial \mathrm{N}}{\partial x} = 8x^3y + 15x^2y^4$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}\mathrm{d}x + \int (\text{terms of N not containing } x)\,\mathrm{d}y = c$$

$$\int (4x^3y^2 + 3x^2y^5)\,\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$x^4y^2 + x^3y^5 = c$$

Example 2

Solve $(x^7y^2 + 3y)\,\mathrm{d}x + (3x^8y - x)\,\mathrm{d}y = 0.$

Solution

$$\mathrm{M} = x^7y^2 + 3y, \qquad \mathrm{N} = 3x^8y - x$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2x^7y + 3, \qquad \frac{\partial \mathrm{N}}{\partial x} = 24x^7y - 1$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, the equation is not exact.

Rewriting the equation,

$$x^7y^2\,\mathrm{d}x + 3x^8y\,\mathrm{d}y + 3y\,\mathrm{d}x - x\,\mathrm{d}y = 0$$

$$x^7y(y\,\mathrm{d}x + 3x\,\mathrm{d}y) + (3y\,\mathrm{d}x - x\,\mathrm{d}y) = 0$$

$$m_1 = 7,\ n_1 = 1,\ a_1 = 1,\ b_1 = 3,\ m_2 = 0,\ n_2 = 0,\ a_2 = 3,\ \ b_2 = -1$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{7+h+1}{1} = \frac{1+k+1}{3}$$

$$3h+24 = k+2$$

$$3h-k = -22 \qquad \text{...(1)}$$

and

$$\frac{m_2+h+1}{a_2} = \frac{n_2+k+1}{b_2}$$

$$\frac{0+h+1}{3} = \frac{0+k+1}{-1}$$

$$-h-1 = 3k+3$$

$$h+3k = -4 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$h = -7,\ k = 1$$

$$\text{IF} = x^{-7}y$$

Multiplying the DE by the IF,

$$x^{-7}y(x^7y^2+3y)\mathrm{d}x + x^{-7}y(3x^8y-x)\mathrm{d}y = 0$$

$$(y^3+3x^{-7}y^2)\mathrm{d}x + (3xy^2-x^{-6}y)\mathrm{d}y = 0$$

$$\mathrm{M}_1 = y^3+3x^{-7}y^2, \qquad \mathrm{N}_1 = 3xy^2-x^{-6}y$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = 3y^2+6x^{-7}y, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = 3y^2+6x^{-7}y$$

Since $\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\partial \mathrm{N}_1}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int\limits_{y\,\text{constant}} \mathrm{M}_1\mathrm{d}x + \int (\text{terms of } \mathrm{N}_1 \text{ not containing } x)\,\mathrm{d}y = c$$

$$\int (y^3+3x^{-7}y^2)\,\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$xy^3 + \frac{3x^{-6}y^2}{-6} = c$$

$$xy^3 - \frac{x^{-6}y^2}{2} = c$$

Case VI *Integrating Factors by Inspection* Sometimes, the integrating factor can be identified by regrouping the terms of the differential equation. The following table helps in identifying the IF after regrouping the terms.

Sr. No.	*Group of Terms*	*Integrating Factor*	*Exact Differential Equation*
1.	$\mathrm{d}x \pm \mathrm{d}y$	$\dfrac{1}{x \pm y}$	$\dfrac{\mathrm{d}x \pm \mathrm{d}y}{x \pm y} = \mathrm{d}[\log(x \pm y)]$
2.	$y\,\mathrm{d}x + x\,\mathrm{d}y$	$\dfrac{1}{2xy}$ $\dfrac{1}{xy}$ $\dfrac{1}{(xy)^n}$	$y\,\mathrm{d}x + x\,\mathrm{d}y = \mathrm{d}(xy)$ $2x^2y\,\mathrm{d}y + 2xy^2\mathrm{d}x = \mathrm{d}(x^2y^2)$ $\dfrac{y\,\mathrm{d}x + x\,\mathrm{d}y}{xy} = \mathrm{d}[\log(xy)]$ $\dfrac{y\,\mathrm{d}x + x\,\mathrm{d}y}{(xy)^n} = \mathrm{d}\left[\dfrac{(xy)^{1-n}}{1-n}\right],\ n \neq 1$
3.	$y\,\mathrm{d}x - x\,\mathrm{d}y$	$\dfrac{1}{y^2}$ $\dfrac{1}{x^2 + y^2}$ $\dfrac{1}{x^2}$ $\dfrac{1}{xy}$	$\dfrac{y\,\mathrm{d}x - x\,\mathrm{d}y}{y^2} = \mathrm{d}\left(\dfrac{x}{y}\right)$ $\dfrac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2 + y^2} = \mathrm{d}\left[\tan^{-1}\left(\dfrac{x}{y}\right)\right]$ $\dfrac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2} = \mathrm{d}\left(-\dfrac{y}{x}\right)$ $\dfrac{y\,\mathrm{d}x - x\,\mathrm{d}y}{xy} = \mathrm{d}\left[\log\left(\dfrac{x}{y}\right)\right]$
4.	$x\,\mathrm{d}x \pm y\,\mathrm{d}y$	2 $\dfrac{1}{(x^2 \pm y^2)}$ $\dfrac{1}{(x^2 \pm y^2)^n}$	$2x\,\mathrm{d}x \pm 2y\,\mathrm{d}y = \mathrm{d}(x^2 \pm y^2)$ $\dfrac{2x\,\mathrm{d}x \pm 2y\,\mathrm{d}y}{x^2 \pm y^2} = \mathrm{d}[\log(x^2 \pm y^2)]$ $\dfrac{2x\,\mathrm{d}x \pm 2y\,\mathrm{d}y}{(x^2 \pm y^2)^n} = \mathrm{d}\left[\dfrac{(x^2 \pm y^2)^{1-n}}{2(1-n)}\right]$
5.	$2y\,\mathrm{d}x + x\,\mathrm{d}y$	x	$2xy\,\mathrm{d}x + x^2\,\mathrm{d}y = \mathrm{d}(x^2y)$
6.	$y\,\mathrm{d}x + 2x\,\mathrm{d}y$	y	$y^2\,\mathrm{d}x + 2xy\,\mathrm{d}y = \mathrm{d}(xy^2)$
7.	$2y\,\mathrm{d}x - x\,\mathrm{d}y$	$\dfrac{x}{y^2}$	$\dfrac{2xy\,\mathrm{d}x - x^2\mathrm{d}y}{y^2} = \mathrm{d}\left(\dfrac{x^2}{y}\right)$
8.	$2x\,\mathrm{d}y - y\,\mathrm{d}x$	$\dfrac{y}{x^2}$	$\dfrac{2xy\,\mathrm{d}y - y^2\mathrm{d}x}{x^2} = \mathrm{d}\left(\dfrac{y^2}{x}\right)$

Example 1

Solve $x\,\mathrm{d}y - y\,\mathrm{d}x + 2x^3\mathrm{d}x = 0.$

Solution

Dividing the equation by x^2,

$$\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2} + 2x\,\mathrm{d}x = 0$$

$$\mathrm{d}\left(\frac{y}{x}\right) + \mathrm{d}(x^2) = 0$$

Integrating both the sides,

$$\frac{y}{x} + x^2 = c$$

Example 2

Solve $x\,\mathrm{d}x + y\,\mathrm{d}y + 2(x^2 + y^2)\,\mathrm{d}x = 0.$

Solution

Dividing the equation by $x^2 + y^2$,

$$\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2} + 2\,\mathrm{d}x = 0$$

$$\frac{1}{2}\mathrm{d}[\log(x^2 + y^2)] + 2\,\mathrm{d}x = 0$$

Integrating both the sides,

$$\frac{1}{2}\log(x^2 + y^2) + 2x = c$$

Example 3

Solve $(1 + xy)y\,\mathrm{d}x + (1 - xy)x\,\mathrm{d}y = 0.$

Solution

$$y\,\mathrm{d}x + xy^2\,\mathrm{d}x + x\,\mathrm{d}y - x^2y\,\mathrm{d}y = 0$$

Regrouping the terms,

$$(y\,\mathrm{d}x + x\,\mathrm{d}y) + (xy^2\mathrm{d}x - x^2y\,\mathrm{d}y) = 0$$

Dividing the equation by x^2y^2,

$$\frac{y\,dx + x\,dy}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

$$d\left(-\frac{1}{xy}\right) + \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating both the sides,

$$-\frac{1}{xy} + \log x - \log y = c$$

$$-\frac{1}{xy} + \log\frac{x}{y} = c$$

Example 4

Solve $x\,dy - y\,dx = 3x^2(x^2 + y^2)\,dx.$

Solution

Dividing the equation by $(x^2 + y^2)$,

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = 3x^2dx$$

$$d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = d(x^3)$$

Integrating both the sides,

$$\tan^{-1}\left(\frac{y}{x}\right) = x^3 + c$$

Example 5

Solve $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0.$

Solution

$$xy\,dx - 2y^2dx - x^2dy + 3xy\,dy = 0$$

Regrouping the terms,

$$x(y\,dx - x\,dy) - 2y^2\,dx + 3xy\,dy = 0$$

Dividing the equation by xy^2,

$$\frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{y^2} - \frac{2}{x}\mathrm{d}x + \frac{3}{y}\mathrm{d}y = 0$$

$$\mathrm{d}\left(\frac{x}{y}\right) - \frac{2}{x}\mathrm{d}x + \frac{3}{y}\mathrm{d}y = 0$$

Integrating both the sides,

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} - \log x^2 + \log y^3 = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 6

Solve $y(2xy + e^x)\,\mathrm{d}x = e^x\mathrm{d}y$.

Solution

$$2xy^2\mathrm{d}x + e^x y\,\mathrm{d}x - e^x\mathrm{d}y = 0$$

Dividing the equation by y^2,

$$2x\,\mathrm{d}x + \frac{ye^x\mathrm{d}x - e^x\mathrm{d}y}{y^2} = 0$$

$$2x\,\mathrm{d}x + \mathrm{d}\left(\frac{e^x}{y}\right) = 0$$

Integrating both the sides,

$$x^2 + \frac{e^x}{y} = c$$

Example 7

Solve $y\mathrm{d}x + x(x^2y - 1)\mathrm{d}y = 0$.

Solution

$$y\,\mathrm{d}x + x^3y\,\mathrm{d}y - x\,\mathrm{d}y = 0$$

Regrouping the terms,

$$y\,\mathrm{d}x - x\,\mathrm{d}y + x^3y\,\mathrm{d}y = 0$$

Dividing the equation by $\dfrac{x^3}{y}$,

$$\frac{y^2\mathrm{d}x - xy\,\mathrm{d}y}{x^3} + y^2\mathrm{d}y = 0$$

$$\frac{1}{2}\left(\frac{y^2\cdot 2x\,\mathrm{d}x - x^2\cdot 2y\,\mathrm{d}y}{x^4}\right) + y^2\mathrm{d}y = 0$$

$$\frac{1}{2}\mathrm{d}\left(-\frac{y^2}{x^2}\right) + y^2\mathrm{d}y = 0$$

Integrating both the sides,

$$-\frac{1}{2}\frac{y^2}{x^2} + \frac{y^3}{3} = c$$

$$-\frac{y^2}{2x^2} + \frac{y^3}{3} = c$$

Example 8

Solve $y(x^3e^{xy} - y)\mathrm{d}x + x(y + x^3e^{xy})\mathrm{d}y = 0.$

Solution

$$x^3ye^{xy}\mathrm{d}x - y^2\mathrm{d}x + xy\,\mathrm{d}y + x^4e^{xy}\mathrm{d}y = 0$$

Regrouping the terms,

$$x^3ye^{xy}\mathrm{d}x + x^4e^{xy}\mathrm{d}y - y^2\mathrm{d}x + xy\,\mathrm{d}y = 0$$

Dividing the equation by x^3,

$$ye^{xy}\mathrm{d}x + xe^{xy}\mathrm{d}y - \frac{1}{2}\left(\frac{y^2\cdot 2x\,\mathrm{d}x - x^2\cdot 2y\,\mathrm{d}y}{x^4}\right) = 0$$

$$\mathrm{d}(e^{xy}) + \frac{1}{2}\mathrm{d}\left(\frac{y^2}{x^2}\right) = 0$$

Integrating both the sides,

$$e^{xy} + \frac{1}{2}\frac{y^2}{x^2} = c$$

Example 9

If x^n is an integrating factor of $(y-2x^3)\,dx - x(1-xy)\,dy = 0$ then find n and solve the equation.

Solution

If x^n is an IF then after multiplication with x^n, the equation becomes exact.

$$(x^n y - 2x^{n+3})dx - x^{n+1}(1-xy)dy = 0 \text{ is an exact DE}$$

where $$M = x^n y - 2x^{n+3}, \qquad N = -x^{n+1} + x^{n+2} y$$

and $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$x^n = -(n+1)x^n + (n+2)x^{n+1} y$$

$$(n+2)x^n(1+xy) = 0$$

$$n+2 = 0$$

$$n = -2$$

Putting $n = -2$ in the equation,

$$(x^{-2} y - 2x)dx - x^{-1}(1-xy)dy = 0$$

$$\left(\frac{y}{x^2} - 2x\right)dx - \left(\frac{1}{x} - y\right)dy = 0$$

$$M = \frac{y}{x^2} - 2x, \qquad N = -\frac{1}{x} + y$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2}, \qquad \frac{\partial N}{\partial x} = \frac{1}{x^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} M\,dx + \int (\text{terms of N not containing } x)\,dy = c$$

$$\int\left(\frac{y}{x^2} - 2x\right)dx + \int y\,dy = c$$

$$-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$$

EXERCISE 3.2

Solve the following differential equations:

1. $(x^2 + y^2 + x)\,dx + xy\,dy = 0$

$$\left[\textbf{Ans.:}\ 3x^4 + 4x^3 + 6x^2y^2 = c\right]$$

2. $(y - 2x^3)\,dx - (x - x^2y)\,dy = 0$

$$\left[\textbf{Ans.:}\ xy^2 - 2y - 2x^3 = cx\right]$$

3. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

$$\left[\textbf{Ans.:}\ x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c\right]$$

4. $\left(2x\sinh\frac{y}{x} + 3y\cosh\frac{y}{x}\right)dx - \left(3x\cosh\frac{y}{x}\right)dy = 0$

$$\left[\textbf{Ans.:}\ -3\sinh\frac{y}{x} = cx^{\frac{2}{3}}\right]$$

5. $(e^x x^4 - 2mxy^2)\,dx + 2mx^2y\,dy = 0$

$$\left[\textbf{Ans.:}\ x^2e^x + my^2 = cx^2\right]$$

6. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}(x + xy^2)\,dy = 0$

$$\left[\textbf{Ans.:}\ x^6 + 3x^4y + x^4y^3 = c\right]$$

7. $(x\sec^2 y - x^2\cos y)\,dy = (\tan y - 3x^4)dx$

$$\left[\textbf{Ans.:}\ \frac{\tan y}{x} + x^3 - \sin y = c\right]$$

8. $(4xy + 3y^2 - x)\,dx + x(x + 2y)\,dy = 0$

$$\left[\textbf{Ans.:}\ x^4y + x^3y^2 - \frac{x^4}{4} = c\right]$$

9. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$

$$\left[\textbf{Ans.:}\ e^x(x^2 + y^2) = c\right]$$

10. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

$$\left[\textbf{Ans.:}\ x^3y^3 + x^2 = cy\right]$$

11. $y(xy + e^x)\,dx - e^x\,dy = 0$

$$\left[\textbf{Ans.:}\ \frac{x^2}{2} + \frac{e^x}{y} = c\right]$$

12. $(3x^2\,y^3\,e^y + y^3 + y^2)dx + (x^3\,y^3\,e^y - xy)dy = 0$

$$\left[\textbf{Ans.:}\ x^3e^y + x + \frac{x}{y} = c\right]$$

13. $y(x^2y + e^x)\,dx - e^x\,dy = 0$

$$\left[\textbf{Ans.:}\ \frac{x^3}{3} + \frac{e^x}{y} = c\right]$$

14. $(xy^3 + y)dx + 2(x^2\,y^2 + x + y^4)\,dy = 0$

$$\left[\textbf{Ans.:}\ 3x^2y^4 + 6xy^2 + 2y^6 = c\right]$$

15. $(2x^2y + e^x)y\,dx - (e^x + y^3)\,dy = 0$

$$\left[\textbf{Ans.:}\ 4x^3y - 3y^3 + 6e^x = cy\right]$$

16. $y\log y\,dx + (x - \log y)\,dy = 0$

$$\left[\textbf{Ans.:}\ 2x\log y = c + (\log y)^2\right]$$

17. $(x - y^2)\,dx + 2xy\,dy = 0$

$$\left[\textbf{Ans.:}\ \frac{y^2}{x} + \log x = c\right]$$

18. $2xy\,dx + (y^2 - x^2)\,dy = 0$

$$\left[\textbf{Ans.:}\ x^2 + y^2 = cy\right]$$

19. $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$

$$\left[\textbf{Ans.:}\ \log\left(\frac{x}{y}\right) = c + \frac{1}{xy}\right]$$

20. $(1 + xy + x^2y^2 + x^3y^3)y\,dx + (1 - xy - x^2y^2 + x^3y^3)x\,dy = 0$

$$\left[\textbf{Ans.:}\ xy - \frac{1}{xy} - \log y^2 = c\right]$$

21. $\dfrac{dy}{dx} = -\dfrac{x^2y^3 + 2y}{2x - 2x^3y^2}$

$$\left[\textbf{Ans.:}\ \frac{1}{3}\log\frac{x}{y^2} - \frac{1}{3x^2y^2} = c\right]$$

22. $y(\sin xy + xy\cos xy)\,dx + x(xy\cos xy - \sin xy)\,dy = 0$

$$\left[\textbf{Ans.:}\ \frac{x\sin(xy)}{y} = c\right]$$

23. $y(x+y)\,dx - x(y-x)\,dy = 0$

$$\left[\textbf{Ans.:}\ \log\sqrt{xy} - \frac{y}{2x} = c\right]$$

24. $x^2y\,dx - (x^3 + y^3)\,dy = 0$

$$\left[\textbf{Ans.:}\ y = ce^{\frac{x^3}{3y^3}}\right]$$

25. $3y\,dx + 2x\,dy = 0,\ y(1) = 1$

$$\left[\textbf{Ans.:}\ yx^{\frac{3}{2}} = 1\right]$$

26. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

$$\left[\textbf{Ans.:}\ -\frac{2}{3}x^{-\frac{3}{2}}y^{\frac{3}{2}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} = c\right]$$

27. $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0$

$$\left[\textbf{Ans.:}\ \frac{7}{5}x^{\frac{10}{7}}y^{-\frac{5}{7}} - \frac{7}{4}x^{-\frac{4}{7}}y^{-\frac{12}{7}} = c\right]$$

28. If y^n is an integrating factor of

$$(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$$

find n and solve the equation.

$$\left[\textbf{Ans.:}\ n = -4,\ x^2y^3e^y + x^2y^2 + x = cy^3\right]$$

3.3.6 Linear Differential Equations

If each term in a differential equation including the derivative is linear in terms of dependent variable then the equation is called linear.
A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{...(3.11)}$$

where P and Q are functions of x, is called a linear differential equation and is linear in y. To solve Eq. (3.11), obtain the integrating factor (IF) as

$$\text{IF} = e^{\int \text{P}\,\text{d}x}$$

Multiplying Eq. (3.11) by the IF,

$$e^{\int \text{P}\,\text{d}x} \frac{\text{d}y}{\text{d}x} + \text{P}e^{\int \text{P}\,\text{d}x} y = \text{Q}e^{\int \text{P}\,\text{d}x}$$

$$\frac{\text{d}}{\text{d}x}\left[e^{\int \text{P}\,\text{d}x} y\right] = \text{Q}e^{\int \text{P}\,\text{d}x}$$

Integrating w.r.t x,

$$e^{\int \text{P}\,\text{d}x} y = \int \text{Q}e^{\int \text{P}\,\text{d}x}\,\text{d}x + c$$

or

$$(\text{IF})\,y = \int (\text{IF})\,\text{Q} + c \qquad ...(3.12)$$

Equation (3.12) is the solution of the differential equation (3.12).

Example 1

Solve $\dfrac{\text{d}y}{\text{d}x} + y\sin x = e^{\cos x}$. **[Summer 2018]**

Solution

The equation is linear in y.

$$\text{P} = \sin x, \qquad \text{Q} = e^{\cos x}$$

$$\text{IF} = e^{\int \sin x\,\text{d}x} = e^{-\cos x}$$

Hence, the general solution is

$$\begin{aligned} e^{-\cos x} y &= \int e^{-\cos x} \cdot e^{\cos x}\,\text{d}x + c \\ &= \int e^0\,\text{d}x + c \\ &= \int \text{d}x + c \\ &= x + c \\ y &= (x+c)e^{\cos x} \end{aligned}$$

Example 2

Solve $\dfrac{\text{d}y}{\text{d}x} + \dfrac{3y}{x} = \dfrac{\sin x}{x^3}$.

Solution

The equation is linear in y.

$$P = \frac{3}{x}, \quad Q = \frac{\sin x}{x^3}$$

$$\text{IF} = e^{\int \frac{3}{x} dx} = e^{3\log x} = e^{\log x^3} = x^3$$

Hence, the general solution is

$$\begin{aligned} x^3 y &= \int x^3 \frac{\sin x}{x^3} dx + c \\ &= \int \sin x \, dx + c \\ &= -\cos x + c \\ y &= -\frac{\cos x}{x^3} + \frac{c}{x^3} \end{aligned}$$

Example 3

Solve $\dfrac{dy}{dx} + 2y\tan x = \sin x.$ **[Winter 2014]**

Solution

The equation is linear in y.

$$P = 2\tan x, \qquad Q = \sin x$$

$$\text{IF} = e^{\int 2\tan x \, dx} = e^{2\log \sec x} = e^{\log \sec^2 x} = \sec^2 x$$

Hence, the general solution is

$$\begin{aligned} (\sec^2 x)y &= \int \sec^2 x \sin x \, dx + c \\ y\sec^2 x &= \int \sec x \frac{\sin x}{\cos x} dx + c \\ &= \int \sec x \tan x \, dx + c \\ &= \sec x + c \end{aligned}$$

Example 4

Solve $\dfrac{dy}{dx} + y\cot x = 2\cos x.$ **[Summer 2016]**

Solution

The equation is linear in y.

$$P = \cot x, \qquad Q = 2\cos x$$

$$\text{IF} = e^{\int \cot x \, dx} = e^{\log \sin x} = \sin x$$

Hence, the general solution is

$$(\sin x)y = \int \sin x(2\cos x)\,dx + c$$

$$y \sin x = \int \sin 2x \, dx + c$$

$$= -\frac{1}{2}\cos 2x + c$$

Example 5

Solve $\frac{dy}{dx} + (\tan x)y = \sin 2x \qquad y(0) = 0.$ **[Summer 2017]**

Solution

The equation is linear in y.

$$P = \tan x, \quad Q = \sin x$$

$$\text{IF} = e^{\int \tan dx = e^{\log \sec x}} = \sec x$$

Hence, the general solution is

$$(\sec x)y = \int \sec x(\sin 2x)\,dx + c$$

$$y \sec x = \int \frac{1}{\cos x}(2\sin x \cos x)\,dx + c$$

$$= 2\int \sin x \, dx + c$$

$$= -2\cos x + c$$

$$y \cdot \frac{1}{\cos x} = -2\cos x + c$$

$$y = -2\cos^2 x + c\cos x \qquad \text{...(1)}$$

Putting $y(0) = 0$ in Eq. (1),

$$y(0) = -2 + c$$

$$0 = -2 + c$$

$$c = 2$$

Hence, $\quad y = -2\cos^2 x + 2\cos x = 2\cos(1 - \cos x)$

Example 6

Solve $(x+1)\dfrac{dy}{dx}-y=e^{3x}(x+1)^2.$ **[Winter 2016; Summer 2014]**

Solution

Rewriting the equation,

$$\frac{dy}{dx}-\frac{y}{x+1}=e^{3x}(x+1)$$

The equation is linear in y.

$$P=-\frac{1}{x+1}, \qquad Q=e^{3x}(x+1)$$

$$IF=e^{\int -\frac{1}{x+1}dx}=e^{-\log(x+1)}=e^{\log(x+1)^{-1}}=\frac{1}{x+1}$$

Hence, the general solution is

$$\left(\frac{1}{x+1}\right)y=\int\left(\frac{1}{x+1}\right)e^{3x}(x+1)dx+c$$

$$=\int e^{3x}dx+c$$

$$=\frac{e^{3x}}{3}+c$$

$$y=(x+1)\left(\frac{e^{3x}}{3}+c\right)$$

Example 7

Solve $\dfrac{dy}{dx}+\dfrac{1}{x^2}y=6e^{\frac{1}{x}}.$ **[Winter 2012]**

Solution

The equation is linear in y.

$$P=\frac{1}{x^2}, \qquad Q=6e^{\frac{1}{x}}$$

$$IF=e^{\int \frac{1}{x^2}dx}=e^{-\frac{1}{x}}$$

Hence, the general solution is

$$e^{-\frac{1}{x}}y=\int e^{-\frac{1}{x}}(6e^{\frac{1}{x}})dx+c$$

$$= 6\int dx + c$$
$$= 6x + c$$
$$y = (6x + c)e^{\frac{1}{x}}$$

Example 8

Solve $\dfrac{dy}{dx} + \dfrac{4x}{1+x^2} y = \dfrac{1}{(x^2+1)^3}$. **[Winter 2013]**

Solution

The equation is linear in y.

$$P = \frac{4x}{1+x^2}, \quad Q = \frac{1}{(x^2+1)^3}$$

$$IF = e^{\int \frac{4x}{1+x^2} dx} = e^{2\log(1+x^2)} = e^{\log(1+x^2)^2} = (1+x^2)^2$$

Hence, the general solution is

$$(1+x^2)^2 y = \int (1+x^2)^2 \cdot \frac{1}{(x^2+1)^3} dx + c$$
$$= \int \frac{1}{x^2+1} dx + c$$
$$= \tan^{-1} x + c$$

Example 9

Solve $(1-x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1-x^2}.$

Solution

Rewriting the equation,

$$\frac{dy}{dx} + \left(\frac{2x}{1-x^2}\right) y = \frac{x}{\sqrt{1-x^2}}$$

The equation is linear in y.

$$P = \frac{2x}{1-x^2}, \quad Q = \frac{x}{\sqrt{1-x^2}}$$

$$\text{IF} = e^{\int \frac{2x}{1-x^2}\mathrm{d}x} = e^{-\log(1-x^2)} = e^{\log(1-x^2)^{-1}} = (1-x^2)^{-1} = \frac{1}{1-x^2}$$

Hence, the general solution is

$$\left(\frac{1}{1-x^2}\right)y = \int \left(\frac{1}{1-x^2}\right)\left(\frac{x}{\sqrt{1-x^2}}\right)\mathrm{d}x + c$$

$$= \int \frac{x}{(1-x^2)^{\frac{3}{2}}}\mathrm{d}x + c$$

$$= -\frac{1}{2}\int (1-x^2)^{-\frac{3}{2}}(-2x)\,\mathrm{d}x + c$$

$$= -\frac{1}{2}\cdot\frac{(1-x^2)^{-\frac{1}{2}}}{-\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)\,\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\frac{y}{1-x^2} = (1-x^2)^{-\frac{1}{2}} + c$$

$$y = \sqrt{1-x^2} + c(1-x^2)$$

Example 10

Solve $x\log x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 2\log x.$

Solution

Rewriting the equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{1}{x\log x}\right)y = \frac{2}{x}$$

The equation is linear in y.

$$\text{P} = \frac{1}{x\log x}, \quad \text{Q} = \frac{2}{x}$$

$$\text{IF} = e^{\int \frac{1}{x\log x}\mathrm{d}x} = e^{\log(\log x)} = \log x$$

Hence, the general solution is

$$(\log x)y = \int (\log x)\cdot\frac{2}{x}\,\mathrm{d}x + c$$

$$= 2\frac{(\log x)^2}{2} + c \qquad \left[\because \int f(x)\cdot f'(x)\,\mathrm{d}x = \frac{[f(x)]^2}{2}\right]$$

$$= (\log x)^2 + c$$

$$y\log x = (\log x)^2 + c$$

Example 11

Solve $(1+x+xy^2)\,\mathrm{d}y + (y+y^3)\,\mathrm{d}x = 0.$

Solution

Rewriting the equation,

$$(1+x+xy^2) + (y+y^3)\frac{\mathrm{d}x}{\mathrm{d}y} = 0$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{(1+y^2)x}{y+y^3} + \frac{1}{y+y^3} = 0$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \left(\frac{1}{y}\right)x = -\frac{1}{y(1+y^2)} \qquad \text{...(1)}$$

The equation is linear in x.

$$\mathrm{P} = \frac{1}{y}, \quad \mathrm{Q} = -\frac{1}{y(1+y^2)}$$

$$\mathrm{IF} = e^{\int \frac{1}{y}\mathrm{d}y} = e^{\log y} = y$$

Hence, the general solution is

$$yx = \int y\left[-\frac{1}{y(1+y^2)}\right]\mathrm{d}y + c$$

$$= -\int \frac{1}{1+y^2}\,\mathrm{d}y + c$$

$$= -\tan^{-1} y + c$$

$$xy = c - \tan^{-1} y$$

Example 12

Solve $y \log y \, \mathrm{d}x + (x - \log y)\, \mathrm{d}y = 0.$

Solution

Rewriting the equation,

$$y \log y \frac{\mathrm{d}x}{\mathrm{d}y} + x - \log y = 0$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \left(\frac{1}{y \log y}\right) x = \frac{1}{y}$$

The equation is linear in x.

$$\mathrm{P} = \frac{1}{y \log y}, \quad \mathrm{Q} = \frac{1}{y}$$

$$\mathrm{IF} = e^{\int \frac{1}{y \log y} \mathrm{d}y} = e^{\log(\log y)} \qquad \left[\because \int \frac{f'(y)}{f(y)} \mathrm{d}y = \log f(y) + c\right]$$

$$= \log y$$

Hence, the general solution is

$$(\log y) x = \int (\log y) \frac{1}{y} \mathrm{d}y + c$$

$$x \log y = \frac{(\log y)^2}{2} + c$$

Example 13

Solve $(1 + \sin y)\, \mathrm{d}x = (2y \cos y - x \sec y - x \tan y)\, \mathrm{d}y.$

Solution

Rewriting the equation,

$$(1 + \sin y) \frac{\mathrm{d}x}{\mathrm{d}y} = 2y \cos y - (\sec y + \tan y) x$$

$$(1 + \sin y) \frac{\mathrm{d}x}{\mathrm{d}y} + \left(\frac{1 + \sin y}{\cos y}\right) x = 2y \cos y$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \left(\frac{1}{\cos y}\right) x = \frac{2y \cos y}{1 + \sin y}$$

The equation is linear in x.

$$\mathrm{P} = \frac{1}{\cos y}, \quad \mathrm{Q} = \frac{2y\cos y}{1+\sin y}$$

$$\mathrm{IF} = e^{\int \frac{1}{\cos y}\mathrm{d}y} = e^{\int \sec y\,\mathrm{d}y} = e^{\log(\sec y + \tan y)} = \sec y + \tan y$$

Hence, the general solution is

$$\begin{aligned}
(\sec y + \tan y)x &= \int (\sec y + \tan y)\left(\frac{2y\cos y}{1+\sin y}\right)\mathrm{d}y + c \\
&= 2\int \left(\frac{1+\sin y}{\cos y}\right)\left(\frac{y\cos y}{1+\sin y}\right)\mathrm{d}y + c \\
&= 2\int y\,\mathrm{d}y + c \\
(\sec y + \tan y)x &= y^2 + c
\end{aligned}$$

Example 14

Solve $(1+y^2)\,\mathrm{d}x = (\tan^{-1} y - x)\,\mathrm{d}y.$ **[Summer 2013]**

Solution

Rewriting the equation,

$$(1+y^2)\frac{\mathrm{d}x}{\mathrm{d}y} = \tan^{-1} y - x$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1} y}{1+y^2}$$

The equation is linear in x.

$$\mathrm{P} = \frac{1}{1+y^2}, \quad \mathrm{Q} = \frac{\tan^{-1} y}{1+y^2}$$

$$\mathrm{IF} = e^{\int \frac{1}{1+y^2}\mathrm{d}y} = e^{\tan^{-1} y}$$

Hence, the general solution is

$$(e^{\tan^{-1} y})x = \int e^{\tan^{-1} y}\left(\frac{\tan^{-1} y}{1+y^2}\right)\mathrm{d}y + c$$

Let $\tan^{-1} y = t$

$$\frac{1}{1+y^2}\,\mathrm{d}y = \mathrm{d}t$$

$$(e^{\tan^{-1} y})x = \int e^t t\,\mathrm{d}t + c$$

$$= te^t - e^t + c$$

$$= e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$$

$$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$$

Example 15

Solve $dr + (2r\cot\theta + \sin 2\theta)\,\mathrm{d}\theta = 0.$

Solution

Rewriting the equation,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + (2\cot\theta)r = -\sin 2\theta$$

The equation is linear in r.

$$\mathrm{P} = 2\cot\theta, \quad \mathrm{Q} = -\sin 2\theta$$

$$\mathrm{IF} = e^{\int 2\cot\theta\,\mathrm{d}\theta} = e^{2\log\sin\theta} = e^{\log\sin^2\theta} = \sin^2\theta$$

Hence, the general solution is

$$\sin^2\theta \cdot r = \int \sin^2\theta(-\sin 2\theta)\,\mathrm{d}\theta + c$$

$$= -2\int \sin^3\theta\cos\theta\,\mathrm{d}\theta + c$$

$$= -2\frac{\sin^4\theta}{4} + c \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$r\sin^2\theta = -\frac{\sin^4\theta}{2} + c$$

Example 16

Solve $\cosh x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cosh^2 x\sinh x - y\sinh x.$

Solution

$$\frac{dy}{dx}+(\tanh x)y = 2\cosh x\sinh x$$

The equation is linear in y.

$$P = \tanh x, \quad Q = 2\cosh x \sinh x$$

$$IF = e^{\int \tanh x\, dx} = e^{\int \frac{\sinh x}{\cosh x}dx} = e^{\log\cosh x} = \cosh x$$

Hence, the general solution is

$$\begin{aligned}(\cosh x)y &= \int \cosh x(2\cosh x\sinh x)\,dx + c\\ &= 2\int \cosh^2 x\cdot \sinh x\,dx + c\\ &= 2\frac{\cosh^3 x}{3}+c \quad \left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]\\ y\cosh x &= \frac{2}{3}\cosh^3 x + c\end{aligned}$$

Example 17

Solve $x(x-1)\dfrac{dy}{dx}-(x-2)y = x^3(2x-1).$

Solution

$$\frac{dy}{dx}-\frac{(x-2)}{x(x-1)}y = \frac{x^2(2x-1)}{(x-1)}$$

The equation is linear in y.

$$\begin{aligned}P &= -\frac{x-2}{x(x-1)}, \qquad Q = \frac{x^2(2x-1)}{x-1}\\ &= -\left(\frac{2}{x}-\frac{1}{x-1}\right)\end{aligned}$$

$$IF = e^{\int\left(-\frac{2}{x}+\frac{1}{x-1}\right)dx} = e^{-2\log x+\log(x-1)} = e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

Hence, the general solution is

$$\begin{aligned}\left(\frac{x-1}{x^2}\right)\cdot y &= \int\left(\frac{x-1}{x^2}\right)\cdot x^2\left(\frac{2x-1}{x-1}\right)dx + c\\ &= x^2 - x + c\end{aligned}$$

$$y = \frac{x^3(x-1)}{x-1} + \frac{cx^2}{x-1}$$

$$y = x^3 + \frac{cx^2}{x-1}$$

Example 18

Solve $(x^2-1)\sin x\dfrac{dy}{dx} + [2x\sin x + (x^2-1)\cos x]y = (x^2-1)\cos x.$

Solution

$$\frac{dy}{dx} + \left(\frac{2x}{x^2-1} + \cot x\right)y = \cot x$$

The equation is linear in y.

$$P = \frac{2x}{x^2-1} + \cot x, \quad Q = \cot x$$

$$IF = e^{\int\left(\frac{2x}{x^2-1}+\cot x\right)dx} = e^{\log(x^2-1)+\log\sin x} = e^{\log\left[(x^2-1)\sin x\right]} = (x^2-1)\sin x$$

Hence, the general solution is

$$\begin{aligned}(x^2-1)\sin x\cdot y &= \int (x^2-1)\sin x\cdot\cot x\,dx + c\\ &= \int (x^2-1)\cos x\,dx + c\\ &= (x^2-1)\sin x - 2x(-\cos x) + 2(-\sin x) + c\\ y(x^2-1)\sin x &= (x^2-3)\sin x + 2x\cos x + c\end{aligned}$$

Example 19

If $\dfrac{dy}{dx} + y\tan x = \sin 2x,\ y(0) = 0,$ *show that the maximum value of y is* $\dfrac{1}{2}$.

Solution

The equation is linear in y.

$$P = \tan x, \qquad Q = \sin 2x$$

$$IF = e^{\int \tan x\,dx} = e^{\log\sec x} = \sec x$$

Hence, the general solution is

$$
\begin{aligned}
(\sec x)y &= \int \sec x \cdot \sin 2x \, dx + c \\
&= \int \sec x \cdot 2\sin x \cos x \, dx + c \\
&= 2\int \sin x \, dx + c \\
y \sec x &= -2\cos x + c \\
y &= -2\cos^2 x + c\cos x \qquad \text{...(1)}
\end{aligned}
$$

Given $y(0) = 0$
Putting $x = 0$, $y = 0$ in Eq. (1),

$$
\begin{aligned}
0 &= -2\cos 0 + c\cos 0 = -2 + c \\
c &= 2
\end{aligned}
$$

Hence, the general solution is

$$y = -2\cos^2 x + 2\cos x \qquad \text{...(2)}$$

For maximum or minimum value,

$$\frac{dy}{dx} = 0$$

$$
\begin{aligned}
-4\cos x(-\sin x) - 2\sin x &= 0 \\
2\sin x(2\cos x - 1) &= 0
\end{aligned}
$$

$\sin x = 0$, $x = 0$ and $2\cos x - 1 = 0$, $\cos x = \frac{1}{2}$, $x = \frac{\pi}{3}$

$x = 0$ and $x = \frac{\pi}{3}$ are the points of extreme values.

Now,
$$\frac{dy}{dx} = 2\sin 2x - 2\sin x$$

$$\frac{d^2y}{dx^2} = 4\cos 2x - 2\cos x$$

When $x = 0$, $\frac{d^2y}{dx^2} = 2 > 0$, y is minimum at $x = 0$.

When $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} = 4\cos\frac{2\pi}{3} - 2\cos\frac{\pi}{3} = 4\left(-\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) = -3 < 0$, y is maximum at $x = \frac{\pi}{3}$.

Putting $x = \dfrac{\pi}{3}$ in Eq. (2), we get maximum value of y.

$$y_{\max} = -2\cos^2\frac{\pi}{3} + 2\cos\frac{\pi}{3} = -\frac{1}{2} + 1 = \frac{1}{2}$$

EXERCISE 3.3

Solve the following differential equations:

1. $x^2\dfrac{dy}{dx} = 3x^2 - 2xy + 1$

$$\left[\textbf{Ans.: } y = \frac{c}{x^2} + x + \frac{1}{x}\right]$$

2. $(2y - 3x)\,dx + x\,dy = 0$

$$\left[\textbf{Ans.: } x^2y = x^3 + c\right]$$

3. $(x+1)\dfrac{dy}{dx} - 2y = (x+1)^4$

$$\left[\textbf{Ans.: } y = \left(\frac{x^2}{2} + x + c\right)(x+1)^2\right]$$

4. $\dfrac{dy}{dx} + y\cot x = \cos x$

$$\left[\textbf{Ans.: } y\sin x = \frac{\sin^2 x}{2} + c\right]$$

5. $\dfrac{1}{x}\dfrac{dy}{dx} + 2y = e^{-x^2}$

$$\left[\textbf{Ans.: } ye^{x^2} = \frac{x^2}{2} + c\right]$$

6. $(y+1)\,dx + [x - (y+2)e^y]\,dy = 0$

$$\left[\textbf{Ans.: } (y+1)(x - e^y) = c\right]$$

7. $dx + x\,dy = e^{-y}\sec^2 y\,dy$

$$\left[\textbf{Ans.: } xe^y - \tan y + c\right]$$

8. $(1+x)\dfrac{dy}{dx} - y = e^x(x+1)^2$

$$\left[\textbf{Ans.: } y = (1+x)(e^x + c)\right]$$

9. $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}-\dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy}=1$

$\left[\textbf{Ans.: } ye^{2\sqrt{x}}=2\sqrt{x}+c\right]$

10. $x\cos x\dfrac{dy}{dx}+y(x\sin x+\cos x)=1$

$\left[\textbf{Ans.: } xy=\sin x+c\cos x\right]$

11. $\cos^2 x\dfrac{dy}{dx}+y=\tan x$

$\left[\textbf{Ans.: } y=\tan x-1+ce^{-\tan x}\right]$

12. $(2x+y^4)\dfrac{dy}{dx}=y$

$\left[\textbf{Ans.: } \dfrac{2x}{y^2}=y^2+c\right]$

13. $\sqrt{a^2+x^2}\dfrac{dy}{dx}+y=\sqrt{a^2+x^2}-x$

$\left[\textbf{Ans.: } \left(x+\sqrt{x^2+a^2}\right)y=a^2x+c\right]$

14. $\dfrac{dy}{dx}=\dfrac{1}{x+e^y}$

$\left[\textbf{Ans.: } xe^{-y}=c+y\right]$

15. $\dfrac{dy}{dx}-\left(\dfrac{3}{x}\right)y=x^3, y(1)=4$

$\left[\textbf{Ans.: } y=x^3(x+3)\right]$

16. $(1+x^2)\dfrac{dy}{dx}-2xy=2x(1+x^2),\ \ y(0)=1$

$\left[\textbf{Ans.: } y=(1+x^2)[1+\log(1+x^2)]\right]$

17. $x\dfrac{dy}{dx}-3y=x^4(e^x+\cos x)-2x^2,\ y(\pi)=\pi^3e^\pi+2\pi^2$

$\left[\textbf{Ans.: } y=2x^2+(e^x+\sin x)x^3\right]$

18. If $\dfrac{dy}{dx}+2y\tan x=\sin x,\ y\left(\dfrac{\pi}{3}\right)=0,$ show that maximum value of y is $\dfrac{1}{8}$.

19. $\dfrac{dy}{dx}+\dfrac{y}{x}=\log x,\ y(1)=1$

$$\left[\textbf{Ans.: } y=\frac{x\log x}{2}-\frac{x}{4}+\frac{5}{4x}\right]$$

20. $\dfrac{dy}{dx}+2xy=xe^{-x^2}$

$$\left[\textbf{Ans.: } ye^{x^2}=\frac{x^2}{2}+c\right]$$

3.3.7 Nonlinear Differential Equations Reducible to Linear Form

Type 1 Bernoulli's Equation

The equation of the form

$$\frac{dy}{dx}+Py=Qy^n \qquad ...(3.13)$$

where P and Q are functions of x or constants is a nonlinear equation, known as Bernoulli's equation. This equation can be made linear using the following method:

Dividing Eq. (3.13) by y^n,

$$\frac{1}{y^n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q \qquad ...(3.14)$$

Let $\dfrac{1}{y^{n-1}}=v$

$$\frac{(1-n)}{y^n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx}=\frac{1}{(1-n)}\cdot\frac{dv}{dx}$$

Substituting in Eq. (3.14),

$$\frac{1}{1-n}\cdot\frac{dv}{dx}+Pv=Q$$

$$\frac{dv}{dx}+(1-n)Pv=Q$$

The equation is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v=\dfrac{1}{y^{n-1}}$, we get the solution of Eq. (3.13).

Example 1

Solve $\dfrac{dy}{dx}+\dfrac{2y}{x}=y^2x^2.$

Solution

The equation is in Bernoulli's form.
Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot\frac{2}{x}=x^2 \qquad ...(1)$$

Let $\dfrac{1}{y}=v,$ $\qquad -\dfrac{1}{y^2}\dfrac{dy}{dx}=\dfrac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx}+\left(\frac{2}{x}\right)v=x^2$$

$$\frac{dv}{dx}-\left(\frac{2}{x}\right)v=-x^2 \qquad ...(2)$$

The equation is linear in v.

$$P=-\frac{2}{x},\ Q=-x^2$$

$$IF=e^{\int -\frac{2}{x}dx}=e^{-2\log x}=e^{\log x^{-2}}=x^{-2}=\frac{1}{x^2}$$

The general solution of Eq. (2) is

$$\frac{1}{x^2}\,v=\int\frac{1}{x^2}(-x^2)dx+c$$

$$=\int -dx+c$$

$$=-x+c$$

$$v=-x^3+cx^2$$

Hence, $\qquad \dfrac{1}{y}=-x^3+cx^2$

Example 2

Solve $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{y^2}{x^2}.$ **[Winter 2017]**

Solution

The equation is in Bernoulli's form.
Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot\frac{1}{x}=\frac{1}{x^2} \qquad \text{...(1)}$$

Let $\frac{1}{y}=v$,
$$-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$$

Substituting in Eq. (1),

$$-\frac{dv}{dx}+\left(\frac{1}{x}\right)v=\frac{1}{x^2}$$

$$\frac{dv}{dx}-\left(\frac{1}{x}\right)v=-\frac{1}{x^2} \qquad \text{...(2)}$$

The equation is linear in v.

$$P=-\frac{1}{x},\ Q=-\frac{1}{x^2}$$

$$IF=e^{\int -\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$$

The general solution of Eq. (2) is

$$\frac{1}{x}v=\int\frac{1}{x}\left(-\frac{1}{x^2}\right)dx+c$$

$$=-\int x^{-3}dx+c$$

$$=-\frac{x^{-2}}{-2}+c$$

$$=\frac{1}{2x^2}+c$$

$$v=\frac{1}{2x}+cx$$

Hence,
$$\frac{1}{y}=\frac{1}{2x}+cx$$

Example 3

Solve $\dfrac{dy}{dx}+y=y^2(\cos x-\sin x)$.

Solution

The equation is in Bernoulli's form.

Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}=\cos x-\sin x \quad ...(1)$$

Let $\frac{1}{y}=v,\quad -\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx}+v=\cos x-\sin x$$

$$\frac{dv}{dx}-v=-\cos x+\sin x \quad ...(2)$$

The equation is linear in v.

$$P=-1,\ Q=-\cos x+\sin x$$

$$IF=e^{\int -dx}=e^{-x}$$

The general solution of Eq. (2) is

$$e^{-x}\cdot v=\int e^{-x}(-\cos x+\sin x)\,dx+c$$

$$=-\int e^{-x}\cos x\,dx+\int e^{-x}\sin x\,dx+c$$

$$=-\left[\frac{e^{-x}}{2}(-\cos x+\sin x)\right]+\left[\frac{e^{-x}}{2}(-\sin x-\cos x)\right]+c$$

$$e^{-x}v=-e^{-x}\sin x+c$$

$$v=-\sin x+ce^{x}$$

Hence, $\frac{1}{y}=-\sin x+ce^{x}$

Example 4

Solve $xy(1+xy^2)\frac{dy}{dx}=1.$

Solution

Rewriting the equation, $\frac{dx}{dy}=xy+x^2y^3$

$$\frac{dx}{dy}-xy=x^2y^3$$

The equation is in Bernoulli's form, where x is a dependent variable.

Dividing the equation by x^2,

$$\frac{1}{x^2}\frac{dx}{dy}-\left(\frac{1}{x}\right)y=y^3 \qquad ...(1)$$

Let $\quad -\dfrac{1}{x}=v, \qquad \dfrac{1}{x^2}\dfrac{dx}{dy}=\dfrac{dv}{dy}$

Substituting in Eq. (1),

$$\frac{dv}{dy}+vy=y^3 \qquad ...(2)$$

The equation is linear in v.

$$P=y, \qquad Q=y^3$$

$$IF=e^{\int y\,dy}=e^{\frac{y^2}{2}}$$

The general solution of Eq. (2) is

$$e^{\frac{y^2}{2}}\cdot v=\int e^{\frac{y^2}{2}}y^3dy+c$$

Putting $\dfrac{y^2}{2}=t, \quad y\,dy=dt$

$$e^{\frac{y^2}{2}}\cdot v=\int e^t\cdot 2t\,dt+c$$

$$=2(e^t t-e^t)+c$$

$$=2e^t(t-1)+c$$

$$=2e^{\frac{y^2}{2}}\left(\frac{y^2}{2}-1\right)+c$$

$$v=y^2-2+ce^{-\frac{y^2}{2}}$$

Hence, $$-\frac{1}{x}=y^2-2+ce^{\frac{-y^2}{2}}$$

Example 5

Solve $y^4dx=\left(x^{-\frac{3}{4}}-y^3x\right)dy.$

Solution

Rewriting the equation,

$$\frac{dx}{dy} = \frac{x^{-\frac{3}{4}}}{y^4} - \frac{x}{y}$$

$$\frac{dx}{dy} + \frac{x}{y} = \frac{x^{-\frac{3}{4}}}{y^4}$$

The equation is in Bernoulli's form, where x is a dependent variable.

Dividing the equation by $x^{-\frac{3}{4}}$,

$$x^{\frac{3}{4}} \frac{dx}{dy} + x^{\frac{7}{4}} \left(\frac{1}{y}\right) = \frac{1}{y^4} \qquad ...(1)$$

Let $\qquad x^{\frac{7}{4}} = v, \quad \frac{7}{4} x^{\frac{3}{4}} \frac{dx}{dy} = \frac{dv}{dy}$

Substituting in Eq. (1),

$$\frac{4}{7} \frac{dv}{dy} + \left(\frac{1}{y}\right) v = \frac{1}{y^4}$$

$$\frac{dv}{dy} + \left(\frac{7}{4y}\right) v = \frac{7}{4y^4} \qquad ...(2)$$

The equation is linear in v.

$$P = \frac{7}{4y}, \qquad Q = \frac{7}{4y^4}$$

$$\text{IF} = e^{\int \frac{7}{4y} dy} = e^{\frac{7}{4}\log y} = e^{\log y^{\frac{7}{4}}} = y^{\frac{7}{4}}$$

The general solution of Eq. (2) is

$$y^{\frac{7}{4}} v = \int y^{\frac{7}{4}} \cdot \frac{7}{4y^4} dy + c$$

$$= \frac{7}{4} \int y^{\frac{-9}{4}} dy + c$$

$$= \frac{7}{4} \left(\frac{4y^{-\frac{5}{4}}}{-5} \right) + c$$

$$y^{\frac{7}{4}}v = -\frac{7}{5}y^{-\frac{5}{4}} + c$$

$$y^3 v = -\frac{7}{5} + cy^{\frac{5}{4}}$$

Hence,
$$y^3 x^{\frac{7}{4}} = -\frac{7}{5} + cy^{\frac{5}{4}}$$

Example 6

Solve $\dfrac{dr}{d\theta} = r\tan\theta - \dfrac{r^2}{\cos\theta}$.

Solution

Rewriting the equation, $\dfrac{dr}{d\theta} - r\tan\theta = -\dfrac{r^2}{\cos\theta}$

The equation is in Bernoulli's form, where r is a dependent variable.
Dividing the equation by r^2,

$$\frac{1}{r^2}\frac{dr}{d\theta} - \frac{\tan\theta}{r} = -\frac{1}{\cos\theta} \qquad \text{...(1)}$$

Let $-\dfrac{1}{r} = v$, $\qquad \dfrac{1}{r^2}\dfrac{dr}{d\theta} = \dfrac{dv}{d\theta}$

Substituting in Eq. (1),

$$\frac{dv}{d\theta} + v\tan\theta = -\frac{1}{\cos\theta} \qquad \text{...(2)}$$

The equation is linear in v.

$$P = \tan\theta, \qquad Q = -\frac{1}{\cos\theta}$$

$$IF = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$$

The general solution of Eq. (2) is

$$\sec\theta \cdot v = \int \sec\theta\left(-\frac{1}{\cos\theta}\right)d\theta + c$$

$$= \int -\sec^2\theta\, d\theta + c$$

$$= -\tan\theta + c$$

Hence,
$$\sec\theta\left(-\frac{1}{r}\right) = -\tan\theta + c$$
$$\frac{\sec\theta}{r} = \tan\theta - c$$

Type 2

The equation of the form $f'(y)\dfrac{dy}{dx} + Pf(y) = Q$ ...(3.15)

where P and Q are functions of x or constants can be reduced to the linear form by putting $f(y) = v,\ f'(y)\dfrac{dy}{dx} = \dfrac{dv}{dx}$ in Eq. (3.15)

$$\frac{dv}{dx} + Pv = Q \qquad ...(3.16)$$

Equation (3.16) is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v = f(y)$, we get the solution of Eq. (3.15).

Example 1

Solve $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$. **[Summer 2015]**

Solution

Dividing the equation by e^y,

$$e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2} \qquad ...(1)$$

Let $e^{-y} = v$, $\quad -e^{-y}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2} \qquad ...(2)$$

The equation is linear in v.

$$P = \frac{1}{x}, \qquad Q = \frac{1}{x^2}$$

$$\text{IF} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

The general solution of Eq. (2) is

$$xv = \int x \cdot \frac{1}{x^2} + c$$

$$= \int \frac{1}{x} + c$$

$$= \log x + c$$

$$v = \frac{1}{x}(\log x + c)$$

Hence, $$e^{-y} = \frac{1}{x}(\log x + c)$$

Example 2

Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 y^3.$ **[Winter 2015]**

Solution

Dividing the equation by y^3,

$$y^{-3}\frac{dy}{dx} + \frac{y^{-2}}{x} = x^3 \qquad ...(1)$$

Let $y^{-2} = v$

$$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

Substituting in Eq. (1),

$$-\frac{1}{2}\frac{dv}{dx} + \frac{v}{x} = x^3$$

$$\frac{dv}{dx} - \frac{2v}{x} = -2x^3 \qquad ...(2)$$

The equation is linear in v.

$$P = -\frac{2}{x}, \qquad Q = -2x^3$$

$$\text{IF} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

The general solution of Eq. (2) is

$$\begin{aligned}\frac{1}{x^2}v &= \int \frac{1}{x^2}(-2x^3)\,\mathrm{d}x + c\\ &= -2\int x\,\mathrm{d}x + c\\ &= -2\frac{x^2}{2} + c\\ &= -x^2 + c\end{aligned}$$

Hence,
$$\frac{1}{x^2y^2} = -x^2 + c$$

Example 3

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x\sin 2y = x^3\cos^2 y.$

Solution

Dividing the equation by $\cos^2 y$,

$$\frac{1}{\cos^2 y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2\sin y\cos y}{\cos^2 y}x = x^3$$

$$\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x} + 2\tan y\cdot x = x^3 \qquad \text{...(1)}$$

Let $\tan y = v$, $\qquad \sec^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x} + 2vx = x^3 \qquad \text{...(2)}$$

The equation is linear in *v*.

$$\mathrm{P} = 2x, \quad \mathrm{Q} = x^3$$

$$\mathrm{IF} = e^{\int 2x\,\mathrm{d}x} = e^{x^2}$$

The general solution of Eq. (2) is

$$e^{x^2}v = \int e^{x^2}\cdot x^3\,\mathrm{d}x + c$$

Putting $x^2 = t, \quad 2x\,\mathrm{d}x = \mathrm{d}t, x\,\mathrm{d}x = \dfrac{\mathrm{d}t}{2}$

$$e^{x^2} v = \int e^t t \frac{\mathrm{d}t}{2} + c$$

$$= \frac{1}{2}\left(te^t - e^t\right) + c$$

$$= \frac{1}{2} e^{x^2}(x^2 - 1) + c$$

$$v = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Hence,
$$\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Example 4

Solve $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y\log y = xye^x$.

Solution

Dividing the equation by xy,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\log y}{x} = e^x \qquad \text{...(1)}$$

Let $\log y = v$,
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{v}{x} = e^x \qquad \text{...(2)}$$

The equation is linear in v.

$$\mathrm{P} = \frac{1}{x}, \qquad \mathrm{Q} = e^x$$

$$\mathrm{IF} = e^{\int \frac{1}{x}\mathrm{d}x} = e^{\log x} = x$$

The general solution of Eq. (2) is

$$xv = \int xe^x \mathrm{d}x + c$$

$$= xe^x - e^x + c$$

$$= e^x(x - 1) + c$$

Hence, $$x \log y = e^x (x-1) + c.$$

Example 5

Solve $\dfrac{dy}{dx} + \tan x \tan y = \cos x \sec y.$

Solution

Dividing the equation by sec y,

$$\frac{1}{\sec y}\frac{dy}{dx} + \tan x \sin y = \cos x$$

$$\cos y \frac{dy}{dx} + \tan x \sin y = \cos x \qquad \text{...(1)}$$

Let $\sin y = v, \cos y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx} + \tan x \cdot v = \cos x \qquad \text{...(2)}$$

The equation is linear in v.

$$\text{P} = \tan x, \qquad \text{Q} = \cos x$$

$$\text{IF} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

The general solution of Eq. (2) is

$$\begin{aligned} \sec x \cdot v &= \int \sec x \cdot \cos x\, dx + c \\ &= \int dx + c \\ &= x + c \end{aligned}$$

Hence, $$\sec x \cdot \sin y = x + c$$

Example 6

Solve $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y).$

Solution

Dividing the equation by e^{-y},

$$e^y \frac{dy}{dx} = e^{2x} - e^x e^y$$

$$e^y \frac{\mathrm{d}y}{\mathrm{d}x} + e^x e^y = e^{2x} \qquad \text{...(1)}$$

Let $e^y = v$, $\quad e^y \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x} + e^x v = e^{2x} \qquad \text{...(2)}$$

The equation is linear in v.

$$\mathrm{P} = e^x, \quad \mathrm{Q} = e^{2x}$$

$$\mathrm{IF} = e^{\int e^x \,\mathrm{d}x} = e^{e^x}$$

The general solution of Eq. (2) is

$$e^{e^x} \cdot v = \int e^{e^x} \cdot e^{2x}\,\mathrm{d}x + c$$

Let $e^x = t$, $e^x\,\mathrm{d}x = \mathrm{d}t$

$$e^{e^x} \cdot v = \int e^t t\,\mathrm{d}t + c$$

$$= e^t \cdot t - e^t + c$$

$$= e^t(t-1) + c$$

$$= e^{e^x}(e^x - 1) + c$$

$$v = e^x - 1 + ce^{-e^x}$$

Hence, $\quad e^y = e^x - 1 + ce^{-e^x}$

Example 7

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^3}{e^{2x} + y^2}$.

Solution

Rewriting the equation, $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{e^{2x}}{y^3} + \dfrac{1}{y}$

$$e^{-2x} \frac{\mathrm{d}x}{\mathrm{d}y} - \frac{e^{-2x}}{y} = \frac{1}{y^3} \qquad \text{...(1)}$$

Let $e^{-2x} = v$, $\quad -2e^{-2x} \dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{\mathrm{d}v}{\mathrm{d}y}$, $\quad e^{-2x} \dfrac{\mathrm{d}x}{\mathrm{d}y} = -\dfrac{1}{2} \dfrac{\mathrm{d}v}{\mathrm{d}y}$

Substituting in Eq. (1),

$$-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}y}-\frac{v}{y}=\frac{1}{y^3}$$

$$\frac{\mathrm{d}v}{\mathrm{d}y}+\frac{2}{y}\cdot v=\frac{-2}{y^3} \qquad \text{...(2)}$$

The equation is linear in v.

$$\mathrm{P}=\frac{2}{y}, \qquad \mathrm{Q}=-\frac{2}{y^3}$$

$$\mathrm{IF}=e^{\int\frac{2}{y}\mathrm{d}y}=e^{2\log y}=e^{\log y^2}=y^2$$

The general solution of Eq. (2) is

$$y^2\cdot v=\int y^2\left(-\frac{2}{y^3}\right)\mathrm{d}y+c$$

$$=-2\int\frac{1}{y}\mathrm{d}y+c$$

$$=-2\log y+c$$

Hence, $$y^2e^{-2x}=-2\log y+c$$

Example 8

Solve $\dfrac{\mathrm{d}z}{\mathrm{d}x}+\dfrac{z}{x}\log z=\dfrac{z}{x^2}(\log z)^2.$

Solution

Rewriting the equation,

$$\frac{1}{z(\log z)^2}\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{1}{\log z}\cdot\frac{1}{x}=\frac{1}{x^2} \qquad \text{...(1)}$$

Let $\dfrac{-1}{\log z}=v,\qquad \dfrac{1}{(\log z)^2}\cdot\dfrac{1}{z}\dfrac{\mathrm{d}z}{\mathrm{d}x}=\dfrac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x}-\frac{v}{x}=\frac{1}{x^2} \qquad \text{...(2)}$$

The equation is linear in v.

$$\mathrm{P} = -\frac{1}{x}, \qquad \mathrm{Q} = \frac{1}{x^2}$$

$$\mathrm{IF} = e^{\int -\frac{1}{x}\mathrm{d}x} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

The general solution of Eq. (2) is

$$\begin{aligned}\frac{1}{x}\cdot v &= \int \frac{1}{x}\cdot\frac{1}{x^2}\mathrm{d}x + c\\ &= \int x^{-3}\mathrm{d}x + c\\ &= \frac{x^{-2}}{-2} + c\end{aligned}$$

Hence,
$$\frac{1}{x}\left(-\frac{1}{\log z}\right) = -\frac{1}{2x^2} + c$$

$$\frac{1}{x\log z} = \frac{1}{2x^2} - c$$

Example 9

Solve $(\sec x \tan x \tan y - e^x)\,\mathrm{d}x + \sec x \sec^2 y\,\mathrm{d}y = 0.$

Solution

Rewriting the equation,

$$\sec x \sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x} + \sec x \tan x \tan y - e^x = 0$$

$$\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x} + \tan x \tan y = \frac{e^x}{\sec x} \qquad \text{...(1)}$$

Let $\tan y = v$, $\quad \sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}v}{\mathrm{d}x} + (\tan x)v = e^x \cos x \qquad \text{...(2)}$$

The equation is linear in v.

$$\mathrm{P} = \tan x, \qquad \mathrm{Q} = e^x \cos x$$

$$\mathrm{IF} = e^{\int \tan x\,\mathrm{d}x} = e^{\log\sec x} = \sec x$$

The general solution of Eq. (2) is

$$(\sec x)v = \int \sec x\, e^x \cos x\,\mathrm{d}x + c$$
$$= \int e^x \mathrm{d}x + c$$
$$= e^x + c$$

Hence, $\sec x \tan y = e^x + c$

Example 10

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x(x+y) = x^3(x+y)^3 - 1.$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} + x(x+y) = x^3(x+y)^3 - 1 \quad ...(1)$$

Let $x + y = z$, $\quad 1 + \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}z}{\mathrm{d}x}, \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}z}{\mathrm{d}x} - 1$

Substituting in Eq. (1),

$$\frac{\mathrm{d}z}{\mathrm{d}x} - 1 + xz = x^3z^3 - 1$$
$$\frac{\mathrm{d}z}{\mathrm{d}x} + xz = x^3z^3 \quad ...(2)$$

Dividing the Eq. (2) by z^3,

$$\frac{1}{z^3}\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{x}{z^2} = x^3 \quad ...(3)$$

Let $\dfrac{1}{z^2} = v, \ -\dfrac{2}{z^3}\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{\mathrm{d}v}{\mathrm{d}x}, \ \dfrac{1}{z^3}\dfrac{\mathrm{d}z}{\mathrm{d}x} = -\dfrac{1}{2}\dfrac{\mathrm{d}v}{\mathrm{d}x}$

Substituting in Eq. (3),

$$-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} + xv = x^3$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} - 2xv = -2x^3 \quad ...(4)$$

The equation is linear in v.

$$\mathrm{P} = -2x, \qquad \mathrm{Q} = -2x^3$$
$$\mathrm{IF} = e^{\int -2x\,\mathrm{d}x} = e^{-x^2}$$

The general solution of Eq. (4) is

$$e^{-x^2} \cdot v = \int e^{-x^2}(-2x^3)\mathrm{d}x + c$$

Let $x^2 = t, 2x\,\mathrm{d}x = \mathrm{d}t$

$$\begin{aligned} e^{-x^2} \cdot v &= -\int te^{-t}\mathrm{d}t + c \\ &= te^{-t} + e^{-t} + c \\ &= (x^2+1)e^{-x^2} + c \\ v &= (x^2+1) + ce^{x^2} \end{aligned}$$

Substituting value of v,

$$\frac{1}{z^2} = (x^2+1) + ce^{x^2}$$

Hence,
$$\frac{1}{(x+y)^2} = (x^2+1) + ce^{x^2}$$

EXERCISE 3.4

Solve the following differential equations:

1. $\dfrac{dy}{dx} = x^3y^3 - xy$

$$\left[\textbf{Ans.:}\ \frac{1}{y^2} = x^2 + 1 + ce^{x^2}\right]$$

2. $x^2y - x^3\dfrac{dy}{dx} = y^4 \cos x$

$$\left[\textbf{Ans.:}\ x^3 = y^3(3\sin x - c)\right]$$

3. $x(3x+2y^2)\,dx + 2y(1+x^2)\,dy = 0$

$$\left[\textbf{Ans.:}\ y^2(1+x^2) = -x^3 + c\right]$$

4. $y\,dx + x(1-3x^2y^2)\,dy = 0$

$$\left[\textbf{Ans.:}\ y^6 = ce^{-\frac{1}{x^2y^2}}\right]$$

5. $x\,dy - [y + xy^3(1+\log x)]\,dx = 0$

$$\left[\textbf{Ans.:}\ x^2 = -\frac{2}{3}x^3y^2\left(\frac{2}{3} + \log x\right) + cy^2\right]$$

6. $\dfrac{dy}{dx} + y = y^2 e^x$

$$\left[\textbf{Ans.:} \quad -\frac{e^{-x}}{y} = x + c\right]$$

7. $x\,dy + y\,dx = x^3 y^6\,dx$

$$\left[\textbf{Ans.:} \quad \frac{2}{y^5} = 5x^3 + cx^5\right]$$

8. $x\dfrac{dy}{dx} + y = y^3 x^{n+1}$

$$\left[\textbf{Ans.:} \quad \frac{n-1}{y^2} = cx^2 - 2x^{n+1}\right]$$

9. $xy\,(1 + x^2 y^2)\dfrac{dy}{dx} = 1$

$$\left[\textbf{Ans.:} \quad \frac{1}{x^2} = ce^{-y^2} - y^2 + 1\right]$$

10. $x^2 y^3\,dx + (x^3 y - 2)\,dy = 0$

$$\left[\textbf{Ans.:} \ x^3 = \frac{2}{y} + \frac{2}{3} + ce^{\frac{3}{y}}\right]$$

11. $y\dfrac{dx}{dy} = x - yx^2 \cos y$

$$\left[\textbf{Ans.:} \quad \frac{y}{x} = y\sin y + \cos y + c\right]$$

12. $\dfrac{dy}{dx} = \dfrac{e^y}{x^2} - \dfrac{1}{x}$

$$\left[\textbf{Ans.:} \ 2xe^{-y} = 1 + 2cx^2\right]$$

13. $y\dfrac{dy}{dx} + \dfrac{4}{3}x - \dfrac{y^2}{3x} = 0$

$$\left[\textbf{Ans.:} \ y^2 x^{-\frac{2}{3}} + 2x^{\frac{4}{3}} = c\right]$$

14. $\dfrac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$

$$\left[\textbf{Ans.:}\ 2\tan^{-1}y=(x^2-1)+ce^{-x^2}\right]$$

15. $\tan y\dfrac{dy}{dx}+\tan x=\cos y\cos^2 x$

$$\left[\textbf{Ans.:}\ \sec y\sec x=\sin x+c\right]$$

16. $(y+e^y-e^{-x})dx+(1+e^y)dy=0$

$$\left[\textbf{Ans.:}\ y+e^y=(x+c)e^{-x}\right]$$

17. $x^2\cos y\dfrac{dy}{dx}=2x\sin y-1$

$$\left[\textbf{Ans.:}\ 3x\sin y=cx^3+1\right]$$

18. $4x^2y\dfrac{dy}{dx}=3x(3y^2+2)+2(3y^2+2)^3$

$$\left[\textbf{Ans.:}\ 4x^9=(3y^2+2)^2(-3x^8+c)\right]$$

19. $\dfrac{dy}{dx}+\dfrac{1}{x}\tan y=\dfrac{1}{x^2}\tan y\sin y$

$$\left[\textbf{Ans.:}\ \operatorname{cosec} y=1+cx\right]$$

20. $x\dfrac{dy}{dx}+3y=x^4e^{\frac{1}{x^2}}y^3$

$$\left[\textbf{Ans.:}\ \frac{1}{y^2}=\left(e^{\frac{1}{x^2}}+c\right)x^6\right]$$

21. $x^2\dfrac{dy}{dx}=\sin^2 y-(\sin y\cos y)x$

$$\left[\textbf{Ans.:}\ \cot y=\frac{1}{2x}+cx\right]$$

22. $\dfrac{dr}{d\theta} = \dfrac{r\sin\theta - r^2}{\cos\theta}$

$$\left[\textbf{Ans.:}\ \frac{1}{r} = c\cos\theta + \sin\theta\right]$$

23. $\cos x\dfrac{dy}{dx} + 4y\sin x = 4\sqrt{y}\sec x$

$$\left[\textbf{Ans.:}\ \sqrt{y}\sec^2 x = 2\left(\tan x + \frac{\tan^3 x}{3}\right) + c\right]$$

24. $\sin y\dfrac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$

$$\left[\textbf{Ans.:}\ 4\cos y = 2\sin^2 x - 2\sin x + 1 - 4ce^{-2\sin x}\right]$$

25. $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$

$$\left[\textbf{Ans.:}\ e^{x+y} = \frac{e^{2x}}{2} + c\right]$$

3.4 APPLICATIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

Orthogonal Trajectories

Two families of curves are called orthogonal trajectories of each other if every curve of one family cuts each curve of another family at right angles.

Working Rules

1. Cartesian curve $f(x, y, c) = 0$

(i) Obtain the differential equation $F\left(x, y, \dfrac{dy}{dx}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(x, y, -\dfrac{dx}{dy}\right) = 0.$

(iii) Solve the differential equation $F\left(x, y, -\frac{\mathrm{d}x}{\mathrm{d}y}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

2. Polar curve $f(r, \theta, c) = 0$

(i) Obtain the differential equation $F\left(r, \theta, \frac{\mathrm{d}r}{\mathrm{d}\theta}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ by $-r^2 \frac{\mathrm{d}\theta}{\mathrm{d}r}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(r, \theta, -r^2 \frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = 0.$

(iii) Solve the differential equation $F\left(r, \theta, -r^2 \frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

Example 1

Find the orthogonal trajectories of the family of semicubical parabolas $ay^2 = x^3$.

Solution

The equation of the family of curves is

$$ay^2 = x^3 \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$a \cdot 2y \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$$

Substituting $a = \frac{x^3}{y^2}$ from Eq. (1),

$$\frac{x^3}{y^2} \cdot 2y \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$$

$$\frac{2x}{y} \frac{\mathrm{d}y}{\mathrm{d}x} = 3 \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (2),

$$\frac{-2x}{y}\frac{dx}{dy} = 3 \qquad \text{...(3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int -2x\,dx = \int 3y\,dy$$

$$-x^2 = \frac{3y^2}{2} + c$$

$$-2x^2 = 3y^2 + 2c$$

$$2x^2 + 3y^2 + 2c = 0$$

which is the equation of the required orthogonal trajectories.

Example 2

Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, *where* λ *is a parameter.*

Solution

The equation of the family of curves is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$$\frac{x}{a^2} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0 \qquad \text{...(2)}$$

$$\frac{y}{b^2+\lambda} = -\frac{x}{a^2\left(\frac{dy}{dx}\right)}$$

$$\frac{y^2}{b^2+\lambda} = -\frac{xy}{a^2\left(\frac{dy}{dx}\right)} \qquad \text{...(3)}$$

Substituting Eq. (3) in Eq. (1),

$$\frac{x^2}{a^2} - \frac{xy}{a^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)} = 1$$

$$(x^2 - a^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy \qquad \text{...(4)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $-\frac{\mathrm{d}x}{\mathrm{d}y}$ in Eq. (4),

$$(a^2 - x^2)\frac{\mathrm{d}x}{\mathrm{d}y} = xy \qquad \text{...(5)}$$

This is the differential equation of the orthogonal trajectories.
Separating the variables and integrating Eq. (5),

$$\int y\,\mathrm{d}y = \int \frac{a^2 - x^2}{x}\mathrm{d}x + c$$

$$\frac{1}{2}y^2 = a^2 \log x - \frac{1}{2}x^2 + c$$

$$x^2 + y^2 = 2a^2 \log x + 2c$$

which is the equation of the required orthogonal trajectories.

Example 3

Find the equation of the family of all orthogonal trajectories of the family of circles which pass through the origin (0, 0) and have centres on the y-axis.

Solution

The equation of the family of circles passing through (0, 0) and having centres on the y-axis is

$$x^2 + y^2 + 2fy = 0 \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2f\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y+f} \qquad \text{...(2)}$$

From Eq. (1),

$$f = -\frac{x^2 + y^2}{2y}$$

$$y + f = y - \frac{x^2 + y^2}{2y}$$

$$= \frac{y^2 - x^2}{2y}$$

Substituting in Eq. (2),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2xy}{y^2 - x^2} \qquad \text{...(3)}$$

This is the differential equation of the given family of circles.

Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $-\frac{\mathrm{d}x}{\mathrm{d}y}$ in Eq. (3),

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2xy}{y^2 - x^2}$$

This is the differential equation of the family of orthogonal trajectories.

$$(y^2 - x^2)\mathrm{d}x - 2xy\,\mathrm{d}y = 0 \qquad \text{...(4)}$$

$$\mathrm{M} = y^2 - x^2, \qquad \mathrm{N} = -2xy$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2y, \qquad \frac{\partial \mathrm{N}}{\partial x} = -2y$$

$$\frac{\partial \mathrm{M}}{\partial y} \neq \frac{\partial \mathrm{N}}{\partial x}$$

The equation is not exact.

$$\frac{\frac{\partial \mathrm{M}}{\partial y} - \frac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{4y}{-2xy} = -\frac{2}{x}$$

$$\mathrm{IF} = e^{\int -\frac{2}{x}\mathrm{d}x} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying Eq. (4) by $\frac{1}{x^2}$,

$$\left(\frac{y^2}{x^2} - 1\right)\mathrm{d}x - \frac{2y}{x}\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{y^2}{x^2} - 1, \qquad \mathrm{N}_1 = -\frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$
$$= \frac{2y}{x^2},$$

The equation is exact.

Hence, the general solution is

$$\int_{y\,\text{constant}} \left(\frac{y^2}{x^2} - 1\right) dx - \int 0\, dy = c$$
$$\frac{-y^2}{x} - x = c$$
$$x^2 + y^2 + cx = 0$$

which is the equation of the required orthogonal trajectories representing the equation of the family of the circles with centre on the x-axis and passing through the origin.

Example 4

Show that the family of confocal conics $\dfrac{x^2}{a} + \dfrac{y^2}{a-b} = 1$ *is self-orthogonal. Here, a is the parameter and b is the constant.*

Solution

The equation of the family of curves is

$$\frac{x^2}{a} + \frac{y^2}{a-b} = 1 \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{2x}{a} + \frac{2y}{a-b}\frac{dy}{dx} = 0$$
$$\frac{yy'}{a-b} = -\frac{x}{a}, \qquad \text{where } y' = \frac{dy}{dx}$$
$$ayy' = -ax + bx$$
$$a(x + yy') = bx$$
$$a = \frac{bx}{x + yy'}$$

Putting the value of a in Eq. (1),

$$\frac{x^2(x + yy')}{bx} + \frac{y^2}{\dfrac{bx}{x + yy'} - b} = 1$$

$$\frac{x(x+yy')}{b}+\frac{y^2(x+yy')}{-byy'}=1$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'} \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing y' by $-\dfrac{1}{y'}$ in Eq. (2),

$$\frac{-\dfrac{x}{y'}-y}{-\dfrac{1}{y'}}=\frac{b}{x+\left(-\dfrac{y}{y'}\right)}$$

$$x+yy'=\frac{by'}{xy'-y}$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'}$$

which is same as Eq. (2). Therefore, the differential equation of the family of orthogonal trajectories is the same as the differential equation of the family of curves. Hence, the given family of curves is self-orthogonal.

Example 5

Find the orthogonal trajectories of the family of curves $r^n \sin n\theta = a^n$.

Solution

The family of curves is given by the equation

$$r^n \sin n\theta = a^n \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$nr^{n-1}\frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\sin n\theta + r^n n\cos n\theta = 0$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=-r\cot n\theta \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ by $-r^2\dfrac{\mathrm{d}\theta}{\mathrm{d}r}$ in Eq. (2),

$$-r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}=-r\cot n\theta$$

$$r\frac{\mathrm{d}\theta}{\mathrm{d}r}=\cot n\theta \qquad \text{...(3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int \tan n\theta \, d\theta = \int \frac{dr}{r}$$

$$\frac{\log \sec n\theta}{n} = \log r + \log c$$

$$\log \sec n\theta = n \log rc$$

$$= \log (rc)^n$$

$$\sec n\theta = c^n r^n$$

$$r^n \cos n\theta = k \quad \text{where} \quad k = \frac{1}{c^n}$$

which is the equation of the required orthogonal trajectories.

Example 6

Find the orthogonal trajectory of the cardioids $r = a\,(1 - \cos\,\theta)$.

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Solution

The family of curves is given by the equation

$$r = a\,(1 - \cos\,\theta) \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{dr}{d\theta} = a \sin\theta$$

Substituting $a = \dfrac{r}{1 - \cos\theta}$ from Eq. (1),

$$\frac{dr}{d\theta} = \frac{r}{1 - \cos\theta} \sin\theta \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing $\dfrac{dr}{d\theta}$ by $-r^2 \dfrac{d\theta}{dr}$ in Eq. (2),

$$-r^2 \frac{d\theta}{dr} = \frac{r \sin\theta}{1 - \cos\theta}$$

$$-r \frac{d\theta}{dr} = \frac{\sin\theta}{1 - \cos\theta} \qquad \text{...(3)}$$

This is the differential equation of the family of orthogonal trajectories.

Separating the variables and integrating Eq. (3),

$$-\int \frac{1-\cos\theta}{\sin\theta}\,d\theta = \int \frac{dr}{r}$$

$$-\int (\operatorname{cosec}\theta - \cot\theta)\,d\theta = \int \frac{dr}{r}$$

$$\log(\operatorname{cosec}\theta + \cot\theta) + \log\sin\theta = \log r - \log c$$

$$\log\left[(\operatorname{cosec}\theta + \cot\theta)\sin\theta\right] = \log\frac{r}{c}$$

$$\left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\sin\theta = \frac{r}{c}$$

$$c(1+\cos\theta) = r$$

$$r = c\,(1+\cos\theta)$$

which is the equation of the family of orthogonal trajectories.

Example 7

Find the orthogonal trajectories of the family of curves $r = 4a\sec\theta\tan\theta$.

Solution

The equation of the family of curves is

$$r = 4a\sec\theta\tan\theta \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{dr}{d\theta} = 4a(\sec\theta\tan\theta\tan\theta + \sec\theta\sec^2\theta)$$

Substituting $4a = \dfrac{r}{\sec\theta\tan\theta}$ from Eq. (1),

$$\frac{dr}{d\theta} = r(\tan\theta + \cot\theta\sec^2\theta) \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in Eq. (2),

$$-r^2\frac{d\theta}{dr} = r(\tan\theta + \cot\theta\sec^2\theta)$$

$$-r\frac{d\theta}{dr} = \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta\sin\theta}$$

$$-r\frac{d\theta}{dr} = \frac{\sin^2\theta + 1}{\cos\theta\sin\theta} \qquad \text{...(3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$
\begin{aligned}
-\frac{1}{2}\int \frac{2\cos\theta\sin\theta}{\sin^2\theta+1}\,\mathrm{d}\theta &= \int \frac{\mathrm{d}r}{r} \\
-\frac{1}{2}\log(1+\sin^2\theta) &= \log r - \log c \\
-\log(1+\sin^2\theta) &= 2\log r - 2\log c \\
&= \log r^2 - \log c^2 \\
\log r^2(1+\sin^2\theta) &= \log c^2 \\
r^2(1+\sin^2\theta) &= c^2
\end{aligned}
$$

which is the equation of the family of orthogonal trajectories.

Example 8

Find the orthogonal trajectories of the family of curves $r = a\,(1+\sin^2\theta)$.

Solution

The equation of the family of curves is

$$r = a\,(1+\sin^2\theta) \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = a\cdot 2\sin\theta\cos\theta$$

Substituting $a = \dfrac{r}{1+\sin^2\theta}$ from Eq. (1),

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r}{1+\sin^2\theta}\cdot 2\sin\theta\cos\theta \qquad \text{...(2)}$$

This is the differential equation of the given family of curves.

Replacing $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ by $-r^2\dfrac{\mathrm{d}\theta}{\mathrm{d}r}$ in Eq. (2),

$$
\begin{aligned}
-r^2\frac{\mathrm{d}\theta}{\mathrm{d}r} &= \frac{r}{1+\sin^2\theta}\cdot 2\sin\theta\cos\theta \\
-r\frac{\mathrm{d}\theta}{\mathrm{d}r} &= \frac{2\sin\theta\cos\theta}{1+\sin^2\theta} \qquad \text{...(3)}
\end{aligned}
$$

This is the differential equation of the family of orthogonal trajectories.

Separating the variables and integrating Eq. (3),

$$\int\left(\frac{1+\sin^2\theta}{2\sin\theta\cos\theta}\right)\mathrm{d}\theta=-\int\frac{\mathrm{d}r}{r}$$

$$\int\left(\operatorname{cosec}\,2\theta+\frac{\tan\theta}{2}\right)\mathrm{d}\theta=-\int\frac{\mathrm{d}r}{r}$$

$$\frac{\log\left(\operatorname{cosec}\,2\theta-\cot 2\theta\right)}{2}+\frac{\log\sec\theta}{2}=-\log r+\log c$$

$$\log\left[\sec\theta\left(\frac{1-\cos 2\theta}{\sin 2\theta}\right)\right]=-2\log r+2\log c$$

$$=-\log r^2+\log c^2$$

$$\log\left[\sec\theta\cdot\frac{2\sin^2\theta}{2\sin\theta\cos\theta}\right]=\log\frac{c^2}{r^2}$$

$$\sec\theta\tan\theta=\frac{c^2}{r^2}$$

$$r^2=c^2\cos\theta\cot\theta$$

which is the equation of the family of orthogonal trajectories.

EXERCISE 3.5

1. Find the orthogonal trajectories of the families of the following curves:

(i) $y^2 = 4\,ax$

(ii) $x^2 - y^2 = ax$

(iii) $y^2 = \dfrac{x^3}{a-x}$

(iv) $x^2 + y^2 + 2ay + b = 2$

(v) $(a+x)y^2 = x^2(3a-x)$

Ans.: (i) $2x^2+y^2=c$

(ii) $y(y^2+3x^2)=c$

(iii) $(x^2+y^2)^2=c(2x^2+y^2)$

(iv) $x^2+y^2+2cx-b=0$

(v) $(x^2+y^2)^5=cy^3(5x^2+y^2)$

2. Show that the family of confocal conics $\dfrac{x^2}{a^2+c}+\dfrac{y^2}{b^2+c}=1$ is self-orthogonal. Here, a and b are constants and c is the parameter.

3. Find the value of the constant d such that the parabolas $y = c_1x^2 + d$ are the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 - y = c_2$.

$$\left[\textbf{Ans.: } d = \frac{1}{4}\right]$$

4. Find the orthogonal trajectories of the families of the following curves:

(i) $r = a(1 + \cos\theta)$

(ii) $r = \dfrac{2a}{1+\cos\theta}$

(iii) $r^2 = a\sin^2\theta$

(iv) $r^n = a^n \cos n\theta$

(v) $r = a(\sec\theta + \tan\theta)$

(vi) $r = ae^{\theta}$

$$\left[\begin{array}{ll} \textbf{Ans.: } \text{(i)} & r = c(1-\cos\theta) \\ \text{(ii)} & r = \dfrac{c}{1-\cos\theta} \\ \text{(iii)} & r^2 = c^2\cos 2\theta \\ \text{(iv)} & r^n = c^n \sin n\theta \\ \text{(v)} & \log r = -\sin\theta + c \\ \text{(vi)} & r = ce^{-\theta} \end{array}\right]$$

3.5 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + a_2\frac{\mathrm{d}^{n-2} y}{\mathrm{d}x^{n-2}} + \ldots + a_n y = 0 \qquad \text{...(3.17)}$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, is known as a homogeneous linear differential equation of order n with constant coefficients. This equation is known as linear since the degree of the dependent variable y and all its differential coefficients is one.

Equation (3.17) can also be written as

$$(a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + a_2\mathrm{D}^{n-2} + \ldots + a_n)y = 0$$

$$f(\mathrm{D})y = 0$$

where $f(\mathrm{D}) = a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + a_2\mathrm{D}^{n-2} + \ldots + a_n$.

Here, $\mathrm{D} \equiv \dfrac{d}{dx}$ is known as the *differential operator*.

The operator D obeys the laws of algebra.

General Solution of a Homogeneous Linear Differential Equation

The homogeneous equation

$$f(\mathrm{D})y = 0 \qquad \text{...(3.18)}$$

can be solved by replacing D by m in $f(\mathrm{D})$ and solving the auxiliary equation (AE)

$$f(m) = 0 \qquad \text{...(3.19)}$$

The general solution of Eq. (3.18) depends upon the nature of the roots of the auxiliary Eq. (3.19).

If $m_1, m_2, m_3, \ldots, m_n$ are n roots of the auxiliary equation, the following cases arise:

Case I *Real and distinct roots*: If roots $m_1, m_2, m_3, \ldots, m_n$ are real and distinct then the solution of Eq. (3.17) is given as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

Case II *Real and repeated roots*: If two roots m_1, m_2 are real and equal, and the remaining (n – 2) roots $m_3, m_4, \ldots, m_n$ are all real and distinct then the solution of Eq. (3.17) is given as

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Note: If, however, r roots $m_1, m_2, m_3, \ldots, m_r$ are equal and remaining $(n - r)$ roots m_{r+1}, $m_{r+2}, \ldots, m_n$ are all real and distinct then the solution of Eq. (3.17) is given as

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_r x^{r-1})e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \ldots + c_n e^{m_n x}$$

Case III *Complex roots*: If two roots m_1, m_2 are complex say, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ (conjugate pair) and remaining (n – 2) roots $m_3, m_4, \ldots, m_n$ are real and distinct then the solution of Eq. (3.17) is given as

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Here, α is the real part and β is the imaginary part of the conjugate pair of complex roots.

Note: If, however, two pairs of complex roots m_1, m_2 and m_3, m_4 are equal, say, $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ and remaining (n – 4) roots $m_5, m_6, \ldots, m_n$ are real and distinct then the solution of Eq. (3.17) is given as

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$$

Remark

(i) In all the above cases, $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(ii) In the general solution of a homogeneous equation, the number of arbitrary constants is always equal to the order of that homogeneous equation.

Example 1

Solve $(D^2 + 2D - 1)y = 0.$

Solution

The auxiliary equation is

$$m^2 + 2m - 1 = 0$$

$$m = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2} \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1 e^{(-1+\sqrt{2})x} + c_2 e^{(-1-\sqrt{2})x}$$

Example 2

Solve $2D^2y + Dy - 6y = 0.$

Solution

The equation can be written as

$$(2D^2 + D - 6)y = 0$$

The auxiliary equation is

$$2m^2 + m - 6 = 0$$

$$(2m - 3)(m + 2) = 0$$

$$m = -2, \ \frac{3}{2} \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}$$

Example 3

Solve $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0.$

Solution

$$(D^2 + 6D + 9)x = 0$$

The auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$m = -3, -3 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$x = (c_1 + c_2 t)e^{-3x}$$

Example 4

Solve $y'' + 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$. **[Summer 2016]**

Solution

$$(D^2 + 4D + 4)y = 0$$

The auxiliary equation is

$$m^2 + 4m + 4 = 0$$
$$(m + 2)^2 = 0$$
$$m = -2, -2 \text{ (real and repeated)}$$

Hence, the general solution is

$$y = (c_1 + c_2x)\, e^{-2x} \quad \text{...(1)}$$

Differentiating Eq. (1),

$$y' = -2(c_1 + c_2x)\, e^{-2x} + e^{-2x}c_2 \quad \text{...(2)}$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1$$
$$c_1 = 1 \quad \text{...(3)}$$
$$y'(0) = -2c_1 + c_2$$
$$1 = -2c_1 + c_2$$
$$1 = -2 + c_2$$
$$c_2 = 3 \quad \text{...(4)}$$

Hence, the particular solution is

$$y = (1 + 3x)e^{-2x}$$

Example 5

Solve the initial-value problem $y'' - 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 1$.

[Winter 2014]

Solution

$$(D^2 - 4D + 4)y = 0$$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$
$$(m - 2)^2 = 0$$
$$m = 2, 2 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{2x} \quad \text{...(1)}$$

Differentiating Eq. (1),

$$y' = 2c_1e^{2x} + 2c_2e^{2x}x + c_2e^{2x} \quad \text{...(2)}$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1$$
$$3 = c_1$$
$$y'(0) = 2c_1 + c_2$$
$$1 = 2c_1 + c_2$$
$$1 = 2(3) + c_2$$
$$c_2 = -5$$

Hence, the particular solution is

$$y = (3 - 5x)e^{2x}$$

Example 6

Solve the initial-value problem $y'' - 9y = 0$, $y(0) = 2$, $y'(0) = -1$.

[Winter 2014]

Solution

$$(D^2 - 9)y = 0$$

The auxiliary equation is

$$m^2 - 9 = 0$$
$$m = \pm 3 \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x} \qquad \text{...(1)}$$

Differentiating Eq. (1),

$$y' = 3c_1 e^{3x} - 3c_2 e^{-3x} \qquad \text{...(2)}$$

Putting $x = 0$ in Eqs (1) and (2),

$$y(0) = c_1 + c_2$$
$$2 = c_1 + c_2 \qquad \text{...(3)}$$
$$y'(0) = 3c_1 - 3c_2$$
$$-1 = 3c_1 - 3c_2 \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$c_1 = \frac{5}{6}, \qquad c_2 = \frac{7}{6}$$

Hence, the particular solution is

$$y = \frac{5}{6}e^{3x} + \frac{7}{6}e^{-3x}$$

Example 7

Solve $y''' - 6y'' + 11y' - 6y = 0.$ **[Summer 2018]**

Solution

$$(\mathrm{D}^3 - 6\mathrm{D}^2 + 11\mathrm{D} - 6)y = 0$$

The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$(m - 1)(m^2 - 5m + 6) = 0$$

$$(m - 1)(m - 2)(m - 3) = 0$$

$$m = 1, 2, 3 \text{ (real and distinct)}$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}$$

Example 8

Solve $(\mathrm{D}^3 - 3\mathrm{D}^2 - \mathrm{D} + 3)y = 0.$

Solution

The auxiliary equation is

$$m^3 - 3m^2 - m + 3 = 0$$

$$(m - 3)(m^2 - 1) = 0$$

$$m = 3, -1, 1 \qquad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1e^{3x} + c_2e^{-x} + c_3e^x$$

Example 9

Solve $(\mathrm{D}^3 - 5\mathrm{D}^2 + 8\mathrm{D} - 4)y = 0.$

Solution

The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$

$$(m - 1)(m^2 - 4m + 4) = 0$$

$$(m - 1)(m - 2)^2 = 0$$

$$m = 1 \text{ (real and distinct)}, \quad m = 2, 2 \text{ (real and repeated)}$$

Hence, the general solution is

$$y = c_1e^x + (c_2 + c_3x)e^{2x}$$

Example 10

Solve $(\mathrm{D}^3 + 1)\, y = 0.$

Solution

The auxiliary equation is

$$m^3 + 1 = 0$$

$$(m+1)(m^2 - m + 1) = 0$$

$$m + 1 = 0,\ m^2 - m + 1 = 0$$

$$m = -1 \quad \text{(real and distinct)}, \qquad m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \quad \text{(complex)}$$

Hence, the general solution is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

Example 11

Solve $(\mathrm{D}^4 - 2\mathrm{D}^3 + \mathrm{D}^2)y = 0.$

Solution

The auxiliary equation is

$$m^4 - 2m^3 + m^2 = 0$$

$$m^2(m^2 - 2m + 1) = 0$$

$$m^2(m-1)^2 = 0$$

$$m = 0, 0, 1, 1 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^x$$

$$= c_1 + c_2 x + (c_3 + c_4 x)e^x$$

Example 12

Solve $(\mathrm{D}^4 - 6\mathrm{D}^3 + 12\mathrm{D}^2 - 8\mathrm{D})y = 0.$

Solution

The auxiliary equation is

$$m^4 - 6m^3 + 12m^2 - 8m = 0$$
$$m(m^3 - 6m^2 + 12m - 8) = 0$$
$$m(m-2)(m^2 - 4m + 4) = 0$$
$$m(m-2)(m-2)^2 = 0$$

$m = 0$ (real and distinct), $m = 2,2,2$ (real and repeated)

Hence, the general solution is

$$y = c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2) e^{2x}$$
$$= c_1 + (c_2 + c_3 x + c_4 x^2) e^{2x}$$

Example 13

Solve $(\mathrm{D}^4 - 1)y = 0.$

Solution

The auxiliary equation is

$$m^4 - 1 = 0$$
$$m^4 = 1$$
$$m^2 = 1, \quad m^2 = -1$$

$m = \pm 1$ ((real and distinct), $m = \pm i$ (complex)

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$$
$$= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

Example 14

Solve $(\mathrm{D}^4 + 4\mathrm{D}^2)y = 0.$

Solution

The auxiliary equation is

$$m^4 + 4m^2 = 0$$
$$m^2(m^2 + 4) = 0$$

$m = 0,\ 0$ (real and distinct), $m = \pm 2i$ (complex)

Hence, the general solution is

$$\begin{aligned} y &= (c_1 + c_2 x)e^{0x} + c_3 \cos 2x + c_4 \sin 2x \\ &= c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x \end{aligned}$$

Example 15

Solve $(\mathrm{D}^4 + 4)y = 0.$

Solution

The auxiliary equation is

$$\begin{aligned} m^4 + 4 &= 0 \\ m^4 + 4 + 4m^2 - 4m^2 &= 0 \\ (m^2 + 2)^2 - (2m)^2 &= 0 \end{aligned}$$

$$(m^2 + 2 + 2m)(m^2 + 2 - 2m) = 0$$

$$(m^2 + 2m + 2)(m^2 - 2m + 2) = 0$$

$$m = -1 \pm i \text{ and } m = 1 \pm i \quad \text{(complex)}$$

Hence, the general solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$$

Example 16

Solve $(\mathrm{D}^4 + 8\mathrm{D}^2 + 16)y = 0.$

Solution

The auxiliary equation is

$$\begin{aligned} m^4 + 8m^2 + 16 &= 0 \\ (m^2 + 4)^2 &= 0 \\ m &= \pm 2i, \ \pm 2i \quad \text{(complex)} \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= e^{0x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x] \\ &= (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x \end{aligned}$$

3.6 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS: METHOD OF REDUCTION OF ORDER

This method is used to obtain the second solution of a homogeneous linear ordinary differential equation of second order if one solution is known. Since a second linearly independent solution is obtained by solving a first-order ordinary differential equation, it is known as the *method of reduction of order*.

Consider the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad \text{...(3.20)}$$

Let y_1 be the known solution of Eq. (3.20).

Putting $y = y_1$ in Eq. (1),

$$y_1'' + P(x)\, y_1' + Q(x)y_1 = 0 \qquad \text{...(3.21)}$$

Let $y = y_2 = uy_1$ be the second solution of Eq. (3.20).

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1''$$

Substituting in Eq. (3.20),

$$(u''y_1 + 2u'y_1' + uy_1'') + P(u'y_1 + uy_1') + Qy_1 = 0$$

$$u''y_1 + 2u'y_1' + Pu'y_1 + u(y_1'' + Py_1' + Qy_1) = 0$$

$$u''y_1 + u'(2y_1' + Py_1) = 0 \qquad \text{[Using Eq. (3.21)]}$$

$$u'' + u'\left(\frac{2y_1'}{y_1} + P\right) = 0 \qquad \text{...(3.22)}$$

Putting $u' = U$ in Eq. (3.22),

$$U' + \left(\frac{2y_1'}{y_1} + P\right)U = 0$$

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -\left(\frac{2y_1'}{y_1} + P\right)U$$

$$\frac{\mathrm{d}U}{U} = -\left(\frac{2y_1'}{y_1} + P\right)\mathrm{d}x$$

Integrating both the sides,

$$\int \frac{\mathrm{d}U}{U} = -2\int \frac{y_1'}{y_1}\,\mathrm{d}x - \int P\mathrm{d}x$$

$$\ln U = -2\ln y_1 - \int P\mathrm{d}x$$

$$= -\ln y_1^2 - \int P\mathrm{d}x$$

$$\ln U + \ln y_1^2 = -\int P\mathrm{d}x$$

$$\ln Uy_1^2 = -\int P\mathrm{d}x$$

$$Uy_1^2 = e^{-\int P\mathrm{d}x}$$

$$U = \frac{1}{y_1^2} e^{-\int P\mathrm{d}x}$$

$$u' = \frac{1}{y_1^2} e^{-\int P\mathrm{d}x}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{y_1^2} e^{-\int P\mathrm{d}x}$$

Integrating both the sides,

$$u = \int \frac{1}{y_1^2} e^{-\int P\mathrm{d}x} \mathrm{d}x$$

Hence,
$$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int P\mathrm{d}x} \mathrm{d}x$$

Since $U > 0$,

$$u = \int U \, \mathrm{d}x \text{ cannot be constant.}$$

$$\frac{y_2}{y_1} = u \neq \text{constant}$$

Hence, y_1 and y_2 are linearly independent solutions.

Example 1

If $y_1 = x$ is one solution of $x^2y'' + xy' - y = 0$, find the second solution.

Solution

Rewriting the equation,

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0 \qquad \text{...(1)}$$

Comparing Eq. (1) with the standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P = \frac{1}{x}$$

Let $y_2 = uy_1$ be the second solution of Eq. (1).

where
$$u = \int \frac{1}{y_1^2} e^{-\int P\mathrm{d}x} \mathrm{d}x, \; y_1 = x$$

$$e^{-\int P\mathrm{d}x} = e^{-\int \frac{1}{x}\mathrm{d}x} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$\begin{aligned}
u &= \int \frac{1}{x^2} \cdot \frac{1}{x} \mathrm{d}x \\
&= \int x^{-3} \mathrm{d}x \\
&= \frac{x^{-2}}{-2} \\
&= -\frac{1}{2x^2} \\
y_2 &= \left(-\frac{1}{2x^2}\right) x \\
&= -\frac{1}{2x}
\end{aligned}$$

Example 2

If $y_1 = x^2$ is one solution of $x^2 y'' - 4xy' + 6y = 0$, $x > 0$ find the second solution. Also, determine the general solution.

Solution

Rewriting the equation,

$$y'' - \frac{4}{x} y' + \frac{6}{x^2} y = 0 \qquad \text{...(1)}$$

Comparing Eq. (1) with standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P = -\frac{4}{x}$$

Let $y_2 = uy_1$ be the second solution of Eq. (1).

where $u = \int \frac{1}{y_1^2} e^{-\int P \mathrm{d}x} \mathrm{d}x, \quad y_1 = x^2$

$$\begin{aligned}
e^{-\int P \mathrm{d}x} &= e^{-\int -\frac{4}{x} \mathrm{d}x} \\
&= e^{4 \ln x} \\
&= e^{\ln x^4} \\
&= x^4
\end{aligned}$$

$$\begin{aligned}
u &= \int \frac{1}{x^4} \cdot x^4 \mathrm{d}x \\
&= \int \mathrm{d}x \\
&= x
\end{aligned}$$

$$y_2 = x \cdot x^2 = x^3$$

Hence, the general solution is

$$y = c_1x^2 + c_2x^3$$

Example 3

If $y_1 = \dfrac{\sin x}{x}$ *is one solution of* $xy'' + 2y' + xy = 0$*, find the second solution. Also, determine the general solution.*

Solution

Rewriting the equation,

$$y'' + \frac{2}{x}y' + y = 0 \qquad ...(1)$$

Comparing Eq. (1) with the standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

$$P = \frac{2}{x}$$

Let $y_2 = uy_1$ be the second solution of Eq. (1).

where $$u = \int \frac{1}{y_1^2} e^{-\int P\mathrm{d}x}\mathrm{d}x, \qquad y_1 = \frac{\sin x}{x}$$

$$\begin{aligned} e^{-\int P\mathrm{d}x} &= e^{-\int \frac{2}{x}\mathrm{d}x} \\ &= e^{-2\log x} \\ &= e^{\ln x^{-2}} \\ &= x^{-2} \end{aligned}$$

$$\begin{aligned} u &= \int \frac{x^2}{\sin^2 x} \cdot \frac{1}{x^2}\mathrm{d}x \\ &= \int \operatorname{cosec}^2 x\,\mathrm{d}x \\ &= -\cot x \end{aligned}$$

$$\begin{aligned} y_2 &= (-\cot x)\frac{\sin x}{x} \\ &= -\frac{\cos x}{x} \end{aligned}$$

Hence, the general solution is

$$
\begin{aligned}
y &= c_1 \frac{\sin x}{x} - c_2 \frac{\cos x}{x} \\
&= c_1 \frac{\sin x}{x} + c_2' \frac{\cos x}{x}, \qquad c_2' = -c_2
\end{aligned}
$$

EXERCISE 3.6

Solve the following differential equations:

1. $(D^2 + D - 2)y = 0$

$$\left[\textbf{Ans.}: y = c_1 e^{-2x} + c_2 e^{x}\right]$$

2. $(4D^2 + 8D - 5y) = 0$

$$\left[\textbf{Ans.}: y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{5x}{2}}\right]$$

3. $(D^2 - 4D - 12)y = 0$

$$\left[\textbf{Ans.}: y = c_1 e^{6x} + c_2 e^{-2x}\right]$$

4. $(D^2 + 2D - 8)y = 0$

$$\left[\textbf{Ans.}: y = c_1 e^{2x} + c_2 e^{-4x}\right]$$

5. $(D^2 + 4D + 1)y = 0$

$$\left[\textbf{Ans.}: y = c_1 e^{(-2+\sqrt{3})x} + c_2 e^{(-2-\sqrt{3})x}\right]$$

6. $(4D^2 + 4D + 1)y = 0$

$$\left[\textbf{Ans.}: y = (c_1 + c_2 x) e^{-\frac{x}{2}}\right]$$

7. $(D^2 + 2\pi D + \pi^2)y = 0$

$$\left[\textbf{Ans.}: y = (c_1 + c_2 x) e^{-\pi x}\right]$$

8. $(9D^2 - 12D + 4)y = 0$

$$\left[\textbf{Ans.}: y = (c_1 + c_2 x) e^{\frac{2x}{3}}\right]$$

9. $(25D^2 - 20D + 4)y = 0$

$$\left[\textbf{Ans.}: y = (c_1 + c_2 x) e^{\frac{2x}{5}}\right]$$

10. $(9D^2 - 30D + 25)y = 0$

$$\left[\textbf{Ans.}: y = (c_1 + c_2 x) e^{\frac{5x}{3}}\right]$$

11. $(D^2 - 6D + 25)y = 0$

$$\left[\textbf{Ans.: } y = e^{3x}\left(c_1 \cos 4x + c_2 \sin 4x\right)\right]$$

12. $(D^2 + 6D + 11)y = 0$

$$\left[\textbf{Ans.: } y = e^{-3x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)\right]$$

13. $[D^2 - 2aD + (a^2 + b^2)y] = 0$

$$[\textbf{Ans.: } y = e^{ax}(c_1 \cos bx + c_2 \sin bx)]$$

14. $(D^3 - 9D)y = 0$

$$\left[\textbf{Ans.: } y = c_1 + c_2 e^{3x} + c_3 e^{-3x}\right]$$

15. $(D^3 - 3D^2 - D + 3)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}\right]$$

16. $(D^3 - 6D^2 + 11D - 6)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}\right]$$

17. $(D^3 - 6D^2 + 12D - 8)y = 0$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x + c_3 x^2)e^{2x}\right]$$

18. $(D^3 + D)y = 0$

$$[\textbf{Ans.: } y = c_1 + c_2 \cos x + c_3 \sin x]$$

19. $(D^3 + 5D^2 + 8D + 6)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{-3x} + e^{-x}\left(c_2 \cos x + c_3 \sin x\right)\right]$$

20. $(8D^4 - 6D^3 - 7D^2 + 6D - 1)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{\frac{x}{4}} + c_2 e^{\frac{x}{2}} + c_3 e^{x} + c_4 e^{-x}\right]$$

21. $(D^4 - 2D^3 + D^2)y = 0$

$$\left[\textbf{Ans.: } y = c_1 + c_2 x + (c_3 + c_4 x)e^{x}\right]$$

22. $(D^4 - 3D^3 + 3D^2 - D)y = 0$

$$\left[\textbf{Ans.: } y = c_1 + (c_2 + c_3 x + c_4 x^2)e^{x}\right]$$

23. $(D^4 + 8D^2 - 9)y = 0$

$$\left[\textbf{Ans.: } y = c_1 e^{-3x} + c_2 e^{-x} c_3 \cos 3x + c_4 \sin 3x\right]$$

24. $(D^4 + D^3 + 14D^2 + 16D - 32)y = 0$

$$\left[\textbf{Ans.:}\ y = c_1e^{x} + c_2e^{-2x}c_3\cos 4x + c_4\sin 4x\right]$$

25. $(D^4 + 2D^3 - 9D^2 - 10D + 50)y = 0$

$$\left[\textbf{Ans.:}\ y = e^{2x}\left(c_1\cos x + c_2\sin x\right) + e^{-3x}\left(c_3\cos x + c_4\sin x\right)\right]$$

26. $(D^4 + 18D^3 + 81)y = 0$

$$\left[\textbf{Ans.:}\ y = \left(c_1 + c_2x\right)\cos 3x + \left(c_3 + c_4x\right)\sin 3x\right]$$

27. $(D^4 - 4D^3 + 14D^2 - 20D + 25)y = 0$

$$\left[\textbf{Ans.:}\ y = e^{x}\left[\left(c_1 + c_2x\right)\cos 2x + \left(c_3 + c_4x\right)\sin 2x\right]\right]$$

3.7 NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n y = Q(x) \qquad \text{...(3.23)}$$

where a_0, a_1, a_2, ..., a_n are constants and Q is a function of x, is known as a *nonhomogeneous linear differential equation with constant coefficients.*
Equation (3.23) can also be written as

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n)y = Q(x) \qquad \text{...(3.24)}$$

$$f(D)y = Q(x)$$

where $f(D) = a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n$

3.7.1 General Solution of a Nonhomogeneous Linear Differential Equation

A general solution of Eq. (3.23) is obtained in two parts as

General solution = Complementary function + Particular integral

$$y = \text{CF} + \text{PI}$$

The complementary function (CF) is the general solution of the homogeneous equation obtained by putting $Q(x) = 0$ in Eq. (3.23).

The particular integral (PI) is any particular solution of the nonhomogeneous Eq. (3.23) and contains no arbitrary constants.

Inverse Operator and Particular Integral

$f(\mathrm{D})$ is known as the *differential operator* and $\frac{1}{f(\mathrm{D})}$ is known as the *inverse differential operator.*

$$f(\mathrm{D})\left[\frac{1}{f(\mathrm{D})}\mathrm{Q}(x)\right] = Q(x)$$

This shows that $\frac{1}{f(\mathrm{D})}\mathrm{Q}(x)$ satisfies the equation $f(\mathrm{D})y = \mathrm{Q}(x)$ and since $\frac{1}{f(\mathrm{D})}\mathrm{Q}(x)$ does not contain any arbitrary constants, it gives the PI of the equation $f(\mathrm{D})y = \mathrm{Q}(x)$.

Hence,
$$\mathrm{PI} = \frac{1}{f(\mathrm{D})}\mathrm{Q}(x)$$

(i) If $f(\mathrm{D}) = \mathrm{D}$ then

$$\mathrm{PI} = \frac{1}{\mathrm{D}}\mathrm{Q}(x) = \int \mathrm{Q}(x)\,\mathrm{d}x$$

(ii) If $f(\mathrm{D}) = \mathrm{D} - a$ then the equation $f(\mathrm{D})y = \mathrm{Q}(x)$ becomes

$$(\mathrm{D} - a)y = \mathrm{Q}(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - ay = \mathrm{Q}(x)$$

is a first-order linear differential equation.

$$\mathrm{IF} = e^{\int -a\mathrm{d}x} = e^{-ax}$$

The solution is

$$ye^{-ax} = \int e^{-ax}\mathrm{Q}(x)\,\mathrm{d}x + c$$

$$y = e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x + ce^{ax}$$

Here, ce^{ax} is the complementary function since it contains the arbitrary constant c and $e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x$ is the particular integral.

Hence,
$$\mathrm{PI} = \frac{1}{\mathrm{D} - a}\mathrm{Q}(x) = e^{ax}\int \mathrm{Q}(x)\,e^{-ax}\mathrm{d}x$$

3.7.2 Direct (Short-cut) Method of Obtaining Particular Integrals

This method depends on the nature of Q(x) in Eq. (3.23). The particular integral by this method can be obtained when Q(x) has the following forms:

(i) $Q(x) = e^{ax+b}$

(ii) $Q(x) = \sin(ax+b)$ or $\cos(ax+b)$

(iii) $Q(x) = x^m$ or polynomial in x

(iv) $Q(x) = e^{ax}v(x)$

(v) $Q(x) = xv(x)$

Case I $Q(x) = e^{ax+b}$

$$f(\mathrm{D})y = e^{ax+b}$$

Now, $\mathrm{D}(e^{ax+b}) = ae^{ax+b}$, $\mathrm{D}^2(e^{ax+b}) = a^2e^{ax+b}, \ldots, \mathrm{D}^n e^{ax+b} = a^n e^{ax+b}$

Consider

$$\begin{aligned} f(\mathrm{D})(e^{ax+b}) &= (a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \ldots + a_n)e^{ax+b} \\ &= (a_0a^n + a_1a^{n-1} + \ldots + a_n)e^{ax+b} \\ &= f(a)e^{ax} \end{aligned}$$

Operating both the sides with $\dfrac{1}{f(\mathrm{D})}$,

$$\begin{aligned} \frac{1}{f(\mathrm{D})}\left[f(\mathrm{D})\left(e^{ax+b}\right)\right] &= \frac{1}{f(\mathrm{D})}\left[f(a)e^{ax+b}\right] \\ e^{ax+b} &= f(a)\frac{1}{f(\mathrm{D})}e^{ax+b} \\ \frac{1}{f(a)}e^{ax+b} &= \frac{1}{f(\mathrm{D})}e^{ax+b}, \quad f(a) \neq 0 \\ \frac{1}{f(\mathrm{D})}e^{ax+b} &= \frac{1}{f(a)}e^{ax+b}, \quad f(a) \neq 0 \end{aligned}$$

Hence, $\mathrm{PI} = \dfrac{1}{f(a)}e^{ax+b}$ if $f(a) \neq 0$

Note: If $f(a) = 0$ then (D – a) is a factor of $f(\mathrm{D})$ and, hence, the above rule fails.

Let $f(\mathrm{D}) = (\mathrm{D} - a)\,\phi(\mathrm{D})$, where $\phi(a) \neq 0$

$$\begin{aligned}
\text{PI} &= \frac{1}{f(\text{D})} e^{ax+b} \\
&= \frac{1}{(\text{D}-a)\phi(\text{D})} e^{ax+b} \\
&= \frac{1}{\phi(a)} \cdot \frac{1}{(\text{D}-a)} e^{ax+b} \\
&= \frac{1}{\phi(a)} \cdot e^{ax} \int e^{-ax} e^{ax+b} \text{d}x \\
&= \frac{1}{\phi(a)} \cdot e^{ax} \cdot x e^{b} \\
&= x \frac{1}{\phi(a)} e^{ax+b} \qquad \text{...(3.25)}
\end{aligned}$$

Since $\qquad f\,(\text{D}) = (\text{D}-a)\phi(\text{D})$

$$f'\,(\text{D}) = (\text{D}-a)\phi'(\text{D}) + \phi(\text{D})$$

$$f'\,(a) = \phi(a)$$

Substituting in Eq. (3.25),

$$\frac{1}{f(\text{D})} e^{ax+b} = x \cdot \frac{1}{f'(a)} e^{ax+b} \quad \text{where } f'(a) \neq 0$$

If $f'(a) = 0$ then repeating the above process,

$$\begin{aligned}
\frac{1}{f(\text{D})} e^{ax+b} &= x\left[x \cdot \frac{1}{f''(a)} e^{ax+b}\right] \\
&= x^2 \frac{1}{f''(a)} e^{ax+b} \qquad \text{where } f''(a) \neq 0
\end{aligned}$$

In general, if $(\text{D} - a)^r$ is a factor of $f(\text{D})$ then

$$\frac{1}{f(\text{D})} e^{ax} = x^r \frac{1}{f^{(r)}(a)} e^{ax+b}$$

Hence, $\qquad \text{PI} = x^r \dfrac{1}{f^{(r)}(a)} e^{ax+b}$

Example 1

Solve $(4\text{D}^2 - 4\text{D} + 1)y = 4.$

Solution

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$
$$(2m-1)^2 = 0$$
$$m = \frac{1}{2}, \frac{1}{2} \quad \text{(real and repeated)}$$
$$\text{CF} = (c_1 + c_2 x)e^{\frac{x}{2}}$$
$$\text{PI} = \frac{1}{4\text{D}^2 - 4\text{D} + 1} 4e^{0x}$$
$$= 4 \cdot \frac{1}{4(0) - 4(0) + 1} e^{0x}$$
$$= 4$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{\frac{x}{2}} + 4$$

Example 2

Solve $(\text{D}^2 + 5\text{D} + 6)y = e^x$. **[Summer 2014]**

Solution

The auxiliary equation is

$$m^2 + 5m + 6 = 0$$
$$(m+3)(m+2) = 0$$
$$m = -2, -3 \quad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^{-2x} + c_2 e^{-3x}$$
$$\text{PI} = \frac{1}{\text{D}^2 + 5\text{D} + 6} e^x$$
$$= \frac{1}{1^2 + 5(1) + 6} e^x$$
$$= \frac{1}{12} e^x$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{12} e^x$$

Example 3

Solve $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{4x}$. **[Winter 2013]**

Solution

$$(D^2 - 5D + 6)y = e^{4x}$$

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$
$$(m-2)(m-3) = 0$$
$$m = 2, 3 \quad \text{(real and distinct)}$$
$$\text{CF} = c_1e^{2x} + c_2e^{3x}$$
$$\text{PI} = \frac{1}{D^2 - 5D + 6}e^{4x}$$
$$= \frac{1}{4^2 - 5(4) + 6}e^{4x}$$
$$= \frac{1}{2}e^{4x}$$

Hence, the general solution is

$$y = c_1e^{2x} + c_2e^{3x} + \frac{1}{2}e^{4x}$$

Example 4

Solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 12y = e^{6x}$. **[Winter 2012]**

Solution

$$(D^2 + D - 12)y = e^{6x}$$

The auxiliary equation is

$$m^2 + m - 12 = 0$$
$$(m-3)(m+4) = 0$$
$$m = 3, -4 \quad \text{(real and distinct)}$$
$$\text{CF} = c_1e^{3x} + c_2e^{-4x}$$
$$\text{PI} = \frac{1}{D^2 + D - 12}e^{6x}$$
$$= \frac{1}{6^2 + 6 - 12}e^{6x}$$
$$= \frac{1}{30}e^{6x}$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-4x} + \frac{1}{30} e^{6x}$$

Example 5

Solve $y'' - 3y' + 2y = e^{3x}$. **[Summer 2018]**

Solution

$$(D^2 - 3D + 2)y = e^{3x}$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{x} + c_2 e^{2x}$$

$$\text{PI} = \frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= \frac{1}{3^2 - 3(3) + 2} e^{3x}$$

$$= \frac{1}{2} e^{3x}$$

Hence, the general solution is

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{3x}$$

Example 6

Solve $(D^2 + 1)y = e^{-x}$.

Solution

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m^2 = -1$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$\text{PI} = \frac{1}{D^2 + 1} e^{-x}$$

$$= \frac{1}{(-1)^2 + 1} e^{-x}$$

$$= \frac{1}{2} e^{-x}$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^{-x}$$

Example 7

Solve $(\text{D}^2 + 2\text{D} + 1)\, y = e^{-x}$.

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1 \qquad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)e^{-x}$$

$$\text{PI} = \frac{1}{\text{D}^2 + 2\text{D} + 1} e^{-x}$$

$$= x \frac{1}{2\text{D} + 2} e^{-x}$$

$$= x^2 \frac{1}{2} e^{-x}$$

$$= \frac{1}{2} x^2 e^{-x}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + \frac{1}{2} x^2 e^{-x}$$

Example 8

Solve $(\text{D}^2 - 2\text{D} + 1)y = 10\, e^x$. **[Summer 2015]**

Solution

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m - 1)^2 = 0$$

$$m = 1, 1 \qquad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)\, e^x$$

$$\text{PI} = \frac{1}{D^2 - 2D + 1} 10e^x$$

$$= \frac{1}{(D-1)^2} 10\, e^x$$

$$= \frac{x}{2(D-1)} 10e^x$$

$$= \frac{x^2}{2}(10e^x)$$

$$= 5x^2\, e^x$$

Hence, the general solution is

$$y = (c_1 + c_2 x)\, e^x + 5x^2 e^x$$

Example 9

Solve $(4D^2 - 4D + 1)y = e^{\frac{x}{2}}$.

Solution

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2} \qquad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)e^{\frac{x}{2}}$$

$$\text{PI} = \frac{1}{4D^2 - 4D + 1} e^{\frac{x}{2}}$$

$$= x \cdot \frac{1}{8D - 4} e^{\frac{x}{2}}$$

$$= x^2 \cdot \frac{1}{8} e^{\frac{x}{2}}$$

$$= \frac{x^2}{8} e^{\frac{x}{2}}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{\frac{x}{2}} + \frac{x^2}{8} e^{\frac{x}{2}}$$

Example 10

Solve $(D^2 - 4)y = e^{2x} + e^{-4x}$.

Solution

The auxiliary equation is

$$m^2 - 4 = 0$$

$$(m-2)(m+2) = 0$$

$$m = 2, -2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{PI} = \frac{1}{D^2 - 4}(e^{2x} + e^{-4x})$$

$$= \frac{1}{D^2 - 4}e^{2x} + \frac{1}{D^2 - 4}e^{-4x}$$

$$= x \cdot \frac{1}{2D}e^{2x} + \frac{1}{(-4)^2 - 4}e^{-4x}$$

$$= x \cdot \frac{1}{2(2)}e^{2x} + \frac{1}{12}e^{-4x}$$

$$= \frac{x}{4}e^{2x} + \frac{1}{12}e^{-4x}$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4}e^{2x} + \frac{1}{12}e^{-4x}$$

Example 11

Solve $(D^2 + 4D + 5)y = -2 \cosh x$.

Solution

The auxiliary equation is

$$m^2 + 4m + 5 = 0$$

$$m = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i \quad \text{(complex)}$$

$$\text{CF} = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$$
\begin{aligned}
\text{PI} &= \frac{1}{D^2+4D+5}(-2\cosh x) \\
&= -2\frac{1}{D^2+4D+5}\left(\frac{e^x+e^{-x}}{2}\right) \\
&= -\frac{1}{D^2+4D+5}e^x - \frac{1}{D^2+4D+5}e^{-x} \\
&= -\frac{1}{(1)^2+4(1)+5}e^x - \frac{1}{(-1)^2+4(-1)+5}e^{-x} \\
&= -\frac{1}{10}e^x - \frac{1}{2}e^{-x}
\end{aligned}
$$

Hence, the general solution is

$$y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{1}{10}e^x - \frac{1}{2}e^{-x}$$

Example 12

Solve $(D^2+6D+9)y = 5^x - \log 2.$

Solution

The auxiliary equation is

$$
\begin{aligned}
m^2+6m+9 &= 0 \\
(m+3)^2 &= 0 \\
m &= -3, -3 \quad \text{(real and repeated)}
\end{aligned}
$$

$$
\begin{aligned}
\text{CF} &= (c_1+c_2x)e^{-3x} \\
\text{PI} &= \frac{1}{D^2+6D+9}(5^x - \log 2) \\
&= \frac{1}{(D+3)^2}(e^{x\log 5}) - \frac{1}{(D+3)^2}(\log 2)e^{0\cdot x} \\
&= \frac{1}{(\log 5+3)^2}e^{x\log 5} - \log 2\cdot\frac{1}{(0+3)^2}e^{0\cdot x} \\
&= \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}
\end{aligned}
$$

Hence, the general solution is

$$y = (c_1+c_2x)e^{-3x} + \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}$$

Example 13

Solve $y''' - 3y'' + 3y' - y = 4e^t$. **[Winter 2014]**

Solution

$$(D^3 - 3D^2 + 3D - 1)y = 4e^t$$

The auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0$$

$$m = 1, 1, 1 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 t + c_3 t^2)e^t$$

$$\text{PI} = \frac{1}{D^3 - 3D^2 + 3D - 1} 4e^t$$

$$= \frac{1}{(D-1)^3} 4e^t$$

$$= 4t \frac{1}{3(D-1)^2} e^t$$

$$= \frac{4t^2}{3} \frac{1}{2(D-1)} e^t$$

$$= \frac{4}{3} t^3 \frac{1}{2} e^t$$

$$= \frac{2}{3} t^3 e^t$$

Hence, the general solution is

$$y = (c_1 + c_2 t + c_3 t^2)e^t + \frac{2}{3} t^3 e^t$$

Example 14

Solve $(D^3 - D^2 + 4D - 4)y = e^x$.

Solution

The auxiliary equation is

$$m^3 - m^2 + 4m - 4 = 0$$

$$(m-1)(m^2 + 4) = 0$$

$$m - 1 = 0, \qquad m^2 + 4 = 0$$

$$m = 1 \text{ (real and distinct)}, \qquad m = \pm 2i \text{ (complex)}$$

$$\begin{aligned}
\text{CF} &= c_1e^x + (c_2\cos 2x + c_3\sin 2x)e^{0x} \\
&= c_1e^x + c_2\cos 2x + c_3\sin 2x \\
\text{PI} &= \frac{1}{D^3 - D^2 + 4D - 4}e^x \\
&= x\cdot\frac{1}{3D^2 - 2D + 4}e^x \\
&= x\frac{1}{3 - 2 + 4}e^x \\
&= \frac{x}{5}e^x
\end{aligned}$$

Hence, the general solution is

$$y = c_1e^x + c_2\cos 2x + c_3\sin 2x + \frac{x}{5}e^x$$

Example 15

Solve $\dfrac{d^3y}{dx^3} + 8y = \cosh 2x.$ **[Winter 2015]**

Solution

$$(D^3 + 8)y = \left(\frac{e^{2x} + e^{-2x}}{2}\right)$$

The auxiliary equation is

$$\begin{aligned}
m^3 + 8 &= 0 \\
(m + 2)(m^2 - 2m + 4) &= 0
\end{aligned}$$

$$m = -2 \quad \text{(real and distinct)}, \quad m = \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{2 \pm \sqrt{12}\,i}{2} = \frac{2 \pm 2\sqrt{3}\,i}{2}$$

$$= 1 \pm i\sqrt{3} \qquad \text{(complex)}$$

$$\begin{aligned}
\text{CF} &= c_1e^{-2x} + e^x(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x) \\
\text{PI} &= \frac{1}{D^3 + 8}\left(\frac{e^{2x} + e^{-2x}}{2}\right) \\
&= \frac{1}{2}\left[\frac{1}{D^3 + 8}e^{2x} + \frac{1}{D^3 + 8}e^{-2x}\right]
\end{aligned}$$

$$= \frac{1}{2}\left[\frac{1}{(2)^3+8}e^{2x} + \frac{x}{3D^2}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{16}e^{2x} + \frac{x}{3(-2)^2}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{16}e^{2x} + \frac{x}{12}e^{-2x}\right]$$

$$= \frac{1}{32}e^{2x} + \frac{x}{24}e^{-2x}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x) + \frac{1}{32}e^{2x} + \frac{x}{24}e^{-2x}$$

Example 16

Solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x + 3e^{-x} + 2.$

Solution

The auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$

$$(m-1)(m^2 - 4m + 4) = 0$$

$$(m-1)(m-2)^2 = 0$$

$$m = 1 \text{ (real and distinct)}, \quad m = 2, 2 \text{ (real and repeated)}$$

$$\text{CF} = c_1 e^x + (c_2 + c_3 x)e^{2x}$$

$$\text{PI} = \frac{1}{D^3 - 5D^2 + 8D - 4}(e^{2x} + 2e^x + 3e^{-x} + 2e^{0x})$$

$$= \frac{1}{D^3 - 5D^2 + 8D - 4}e^{2x} + \frac{1}{D^3 - 5D^2 + 8D - 4}2e^x + \frac{1}{D^3 - 5D^2 + 8D - 4}3e^{-x} + \frac{1}{D^3 - 5D^2 + 8D - 4}2e^{0x}$$

$$= x \cdot \frac{1}{3D^2 - 10D + 8}e^{2x} + x \cdot \frac{1}{3D^2 - 10D + 8}2e^x + \frac{1}{-1-5-8-4}3e^{-x} + \frac{1}{-4}2e^{0x}$$

$$= x^2 \cdot \frac{1}{6D - 10}e^{2x} + x\frac{1}{3-10+8}2e^x - \frac{1}{18}\cdot 3e^{-x} - \frac{1}{2}$$

$$= x^2\frac{1}{12-10}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

$$= \frac{x^2}{2}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

Hence, the general solution is

$$y = c_1e^x + (c_2 + c_3x)e^{2x} + \frac{x^2}{2}e^{2x} + 2xe^x - \frac{1}{6}e^{-x} - \frac{1}{2}$$

Example 17

Solve $(D^3 - 12D + 16)y = (e^x + e^{-2x})^2$.

Solution

$$\begin{aligned}(D^3 - 12D + 16)y &= (e^x + e^{-2x})^2 \\ &= e^{2x} + 2e^{-x} + e^{-4x}\end{aligned}$$

The auxiliary equation is

$m^3 - 12m + 16 = 0$

$$m = -4 \text{ (real and distinct)}, m = 2, 2 \text{(real and repeated)}$$

$$\text{CF} = c_1e^{-4x} + (c_2 + c_3x)e^{2x}$$

$$\begin{aligned}\text{PI} &= \frac{1}{D^3 - 12D + 16}(e^{2x} + 2e^{-x} + e^{-4x}) \\ &= \frac{1}{D^3 - 12D + 16}e^{2x} + 2\frac{1}{D^3 - 12D + 16}e^{-x} + \frac{1}{D^3 - 12D + 16}e^{-4x} \\ &= \frac{1}{(D+4)(D-2)^2}e^{2x} + 2\frac{1}{(-1)^3 - 12(-1) + 16}e^{-x} + \frac{1}{(D+4)(D-2)^2}e^{-4x} \\ &= \frac{1}{(D-2)^2}\left[\frac{1}{D+4}e^{2x}\right] + 2\frac{1}{-1+12+16}e^{-x} + \frac{1}{(D+4)}\left[\frac{1}{(D-2)^2}e^{-4x}\right] \\ &= \frac{1}{(D-2)^2}\frac{e^{2x}}{6} + \frac{2}{27}e^{-x} + \frac{1}{(D+4)}\cdot\frac{1}{(-4-2)^2}e^{-4x} \\ &= \frac{1}{6}x\frac{1}{2(D-2)}e^{2x} + \frac{2}{27}e^{-x} + \frac{1}{36}\frac{1}{D+4}e^{-4x} \\ &= \frac{x}{12}x\cdot\frac{1}{1}e^{2x} + \frac{2}{27}e^{-x} + \frac{1}{36}x\cdot\frac{1}{1}e^{-4x} \\ &= \frac{x^2}{12}e^{2x} + \frac{2}{27}e^{-x} + \frac{x}{36}e^{-4x}\end{aligned}$$

Hence, the general solution is

$$y = c_1e^{-4x} + (c_2 + c_3x)e^{2x} + \frac{x^2}{12}e^{2x} + \frac{2}{27}e^{-x} + \frac{x}{36}e^{-4x}$$

Example 18

Solve $(\mathrm{D}^6 - 64)y = e^x \cosh 2x.$

Solution

The auxiliary equation is

$$m^6 - 64 = 0$$
$$(m^3)^2 - (8)^2 = 0$$
$$(m^3 + 8)(m^3 - 8) = 0$$
$$(m+2)(m^2 - 2m + 4)(m-2)(m^2 + 2m + 4) = 0$$
$$m + 2 = 0,\ m^2 - 2m + 4 = 0$$
$$m - 2 = 0,\ m^2 + 2m + 4 = 0$$
$$m = -2,\quad m = 1 \pm i\sqrt{3},\qquad m = 2,\quad m = -1 \pm i\sqrt{3}$$

Two roots are real and the two pairs of the roots are complex.

$$\mathrm{CF} = c_1 e^{-2x} + c_2 e^{2x} + e^x (c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x) \ + e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x)$$

$$\begin{aligned}
\mathrm{PI} &= \frac{1}{\mathrm{D}^6 - 64} e^x \cosh 2x \\
&= \frac{1}{\mathrm{D}^6 - 64}\left[e^x\left(\frac{e^{2x} + e^{-2x}}{2}\right)\right] \\
&= \frac{1}{\mathrm{D}^6 - 64}\left[\frac{1}{2}(e^{3x} + e^{-x})\right] \\
&= \frac{1}{2}\left[\frac{1}{3^6 - 64} e^{3x} + \frac{1}{(-1)^6 - 64} e^{-x}\right] \\
&= \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)
\end{aligned}$$

Hence, the general solution is

$$\begin{aligned}
y = c_1 e^{-2x} &+ c_2 e^{2x} + e^x (c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x) \\
&+ e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x) + \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)
\end{aligned}$$

Case II **Q(x) = sin ($ax + b$) or cos ($ax + b$)**

(i) If $\mathrm{Q}(x) = \sin(ax + b)$ then Eq. (3.24) reduces to

$$f(\mathrm{D})y = \sin(ax + b)$$

Now $$D[\sin(ax+b)] = a\cos(ax+b)$$
$$D^2[\sin(ax+b)] = (-a^2)\sin(ax+b)$$
$$D^3[\sin(ax+b)] = -a^3\cos(ax+b)$$
$$D^4[\sin(ax+b)] = a^4\sin(ax+b)$$
$$(D^2)^2[\sin(ax+b)] = (-a^2)^2\sin(ax+b)$$

In general, $$(D^2)^r[\sin(ax+b)] = (-a^2)^r\sin(ax+b)$$

This shows that

$$\phi(D^2)\sin(ax+b) = \phi(-a^2)\sin(ax+b)$$

Operating both the sides with $\dfrac{1}{\phi(D^2)}$,

$$\frac{1}{\phi(D^2)}\left[\phi(D^2)\sin(ax+b)\right] = \frac{1}{\phi(D^2)}\left[\phi(-a^2)\sin(ax+b)\right]$$
$$\sin(ax+b) = \phi(-a^2)\frac{1}{\phi(D^2)}\sin(ax+b)$$
$$\frac{1}{\phi(-a^2)}\sin(ax+b) = \frac{1}{\phi(D^2)}\sin(ax+b)$$
$$\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b)$$

If $f(D) = \phi(D^2)$ then

$$\begin{aligned}\text{PI} &= \frac{1}{f(D)}\sin(ax+b)\\ &= \frac{1}{\phi(D^2)}\sin(ax+b)\\ &= \frac{1}{\phi(-a^2)}\sin(ax+b), \text{ if } \phi(-a^2)\neq 0\end{aligned}$$

If $\phi(-a^2) = 0$ then $(D^2 + a^2)$ is a factor of $\phi(D^2)$ and, hence, the above rule fails.

$$\begin{aligned}\text{PI} &= \frac{1}{\phi(D^2)}\sin(ax+b)\\ &= \frac{1}{\phi(D^2)}\left[\text{I.P. of } e^{i(ax+b)}\right]\\ &= \text{IP of } \frac{1}{\phi(D^2)}e^{i(ax+b)}\end{aligned}$$

$$= \text{IP of } x \cdot \frac{1}{\phi'(D^2)} e^{i(ax+b)} \qquad \left[\because \phi(i^2a^2) = \phi(-a^2) = 0\right]$$

$$= \text{IP of } x \cdot \frac{1}{\phi'(i^2a^2)} e^{i(ax+b)}$$

$$= \text{IP of } x \cdot \frac{1}{\phi'(-a^2)} e^{i(ax+b)}$$

$$= x \cdot \frac{1}{\phi'(-a^2)} \sin(ax+b)$$

If $\phi'(-a^2) = 0$ then

$$\frac{1}{\phi(D^2)} \sin(ax+b) = x^2 \cdot \frac{1}{\phi''(-a^2)} \sin(ax+b), \qquad \text{where } \phi''(-a^2) \neq 0$$

In general, if $\phi^{(r)}(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)} \sin(ax+b)$$

$$= x^{(r+1)} \frac{1}{\phi^{(r+1)}(-a^2)} \sin(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2) \neq 0$$

(ii) Similarly, if $Q(x) = \cos(ax + b)$

$$\text{PI} = \frac{1}{\phi(D^2)} \cos(ax+b)$$

$$= \frac{1}{\phi(-a^2)} \cos(ax+b), \quad \phi(-a^2) \neq 0$$

If $\phi(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)} \cos(ax+b)$$

$$= x \cdot \frac{1}{\phi'(-a^2)} \cos(ax+b)$$

If $\phi'(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)} \cos(ax+b)$$

$$= x^2 \cdot \frac{1}{\phi''(-a^2)} \cos(ax+b), \qquad \text{where } \phi''(-a^2) \neq 0$$

In general, if $\phi^{(r)}(-a^2)=0$ then

$$\begin{aligned}\text{PI} &= \frac{1}{\phi(\text{D}^2)}\cos(ax+b)\\ &= x^{r+1}\frac{1}{\phi^{(r+1)}(-a^2)}\cos(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2)\neq 0\end{aligned}$$

Note: If after replacing D^2 by $-a^2$, $f(\text{D})$ contains terms of D then the denominator is rationalized to obtain the even powers of D.

Example 1

Solve $(\text{D}^2 + 9)y = \cos 4x.$ **[Summer 2018]**

Solution

The auxiliary equation is

$$\begin{aligned}m^2+9&=0\\ m&=\pm 3i \quad \text{(complex)}\\ \text{CF}&=c_1\cos 3x+c_2\sin 3x\\ \text{PI}&=\frac{1}{\text{D}^2+9}\sin 4x\\ &=\frac{1}{-4^2+9}\cos 4x\\ &=\frac{1}{-7}\cos 4x\end{aligned}$$

Hence, the general solution is

$$y=c_1\cos 3x+c_2\sin 3x-\frac{1}{7}\cos 4x$$

Example 2

Solve $(\text{D}^2 + 1)y = \sin^2 x.$

Solution

The auxiliary equation is

$$\begin{aligned}m^2+1&=0\\ m&=\pm i \quad \text{(complex)}\\ \text{CF}&=c_1\cos x+c_2\sin x\\ \text{PI}&=\frac{1}{\text{D}^2+1}\sin^2 x\end{aligned}$$

$$= \frac{1}{D^2+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{D^2+1}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{D^2+1}\cos 2x$$

$$= \frac{1}{2}\cdot\frac{1}{D^2+1}e^{0x} - \frac{1}{2}\cdot\frac{1}{-2^2+1}\cos 2x$$

$$= \frac{1}{2}\cdot\frac{1}{0+1}e^{0x} - \frac{1}{2}\cdot\frac{1}{-3}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{6}\cos 2x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{1}{6}\cos 2x$$

Example 3

Solve $(D^2 + 3D + 2)y = \sin 2x.$

Solution

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{-x} + c_2 e^{-2x}$$

$$\text{PI} = \frac{1}{D^2+3D+2}\sin 2x$$

$$= \frac{1}{-4+3D+2}\sin 2x$$

$$= \frac{1}{3D-2}\sin 2x$$

$$= \frac{1}{(3D-2)}\cdot\frac{(3D+2)}{(3D+2)}\sin 2x$$

$$= \frac{(3D+2)}{9D^2-4}\sin 2x$$

$$= \frac{3D+2}{9(-2^2)-4}\sin 2x$$

$$= \frac{3D+2}{-40}\sin 2x$$

$$= -\frac{3}{40}(D\sin 2x) - \frac{1}{20}\sin 2x$$
$$= -\frac{3}{40}\cdot 2\cos 2x - \frac{1}{20}\sin 2x$$
$$= -\frac{1}{20}(3\cos 2x + \sin 2x)$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{20}(3\cos 2x + \sin 2x)$$

Example 4

Solve $(D^2 + 9)y = 2 \sin 3x + \cos 3x.$ **[Summer 2016]**

Solution

The auxiliary equation is

$$m^2 + 9 = 0$$
$$m = \pm 3i \qquad \text{(complex)}$$
$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$
$$\text{PI} = \frac{1}{D^2 + 9}(2 \sin 3x + \cos 3x)$$
$$= 2\frac{1}{D^2 + 9}\sin 3x + \frac{1}{D^2 + 9}\cos 3x$$
$$= 2\frac{x}{2D}\sin 3x + \frac{x}{2D}\cos 3x$$
$$= x\int \sin 3x\,dx + \frac{x}{2}\int \cos 3x\,dx$$
$$= -\frac{x}{3}\cos 3x + \frac{x}{2}\left(\frac{\sin 3x}{3}\right)$$
$$= -\frac{x}{3}\cos 3x + \frac{x}{6}\sin 3x$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{3}\cos 3x + \frac{x}{6}\sin 3x$$

Example 5

Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x.$ **[Winter 2013]**

Solution

The auxiliary equation is

$$m^2 - 4m + 3 = 0$$
$$(m-1)(m-3) = 0$$
$$m = 1, 3 \quad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^x + c_2 e^{3x}$$
$$\text{PI} = \frac{1}{D^2 - 4D + 3}(\sin 3x \cos 2x)$$
$$= \frac{1}{D^2 - 4D + 3}\frac{1}{2}(\sin 5x + \sin x)$$
$$= \frac{1}{2}\cdot\frac{1}{D^2 - 4D + 3}\sin 5x + \frac{1}{2}\cdot\frac{1}{D^2 - 4D + 3}\sin x$$
$$= \frac{1}{2}\cdot\frac{1}{-5^2 - 4D + 3}\sin 5x + \frac{1}{2}\cdot\frac{1}{-1^2 - 4D + 3}\sin x$$
$$= \frac{1}{2}\cdot\frac{1}{-4D - 22}\sin 5x + \frac{1}{2}\cdot\frac{1}{2 - 4D}\sin x$$
$$= -\frac{1}{4}\cdot\frac{1}{2D + 11}\cdot\frac{2D - 11}{2D - 11}\sin 5x + \frac{1}{4}\cdot\frac{1}{1 - 2D}\cdot\frac{1 + 2D}{1 + 2D}\sin x$$
$$= -\frac{1}{4}\cdot\frac{2D - 11}{4D^2 - 121}\sin 5x + \frac{1}{4}\cdot\frac{1 + 2D}{1 - 4D^2}\sin x$$
$$= -\frac{1}{4}\cdot\frac{2D - 11}{4\left(-5^2\right) - 121}\sin 5x + \frac{1}{4}\cdot\frac{1 + 2D}{1 - 4\left(-1^2\right)}\sin x$$
$$= \frac{2}{884}\left(D\sin 5x\right) - \frac{11}{884}\sin 5x + \frac{1}{20}\sin x + \frac{2}{20}\left(D\sin x\right)$$
$$= \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x + \frac{1}{20}\sin x + \frac{1}{10}\cos x$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{3x} + \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x + \frac{1}{20}\sin x + \frac{1}{10}\cos x$$

Example 6

Solve $(D^2 + 6D + 8)\, y = \cos^2 x.$

Solution

The auxiliary equation is

$$m^2 + 6m + 8 = 0$$
$$(m+4)(m+2) = 0$$

$$m = -4, -2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{-2x} + c_2 e^{-4x}$$

$$\begin{aligned}
\text{PI} &= \frac{1}{D^2 + 6D + 8}(\cos^2 x) \\
&= \frac{1}{D^2 + 6D + 8}\left(\frac{1 + \cos 2x}{2}\right) \\
&= \frac{1}{D^2 + 6D + 8}\frac{1}{2} + \frac{1}{D^2 + 6D + 8}\frac{\cos 2x}{2} \\
&= \frac{1}{2}\frac{1}{D^2 + 6D + 8}e^{0x} + \frac{1}{2}\frac{1}{-2^2 + 6D + 8}\cos 2x \\
&= \frac{1}{2}\frac{1}{0 + 0 + 8}e^{0x} + \frac{1}{2}\frac{1}{6D + 4}\cos 2x \\
&= \frac{1}{16}e^{0x} + \frac{1}{4}\frac{1}{3D + 2}\cos 2x \\
&= \frac{1}{16} + \frac{1}{4}\frac{3D - 2}{9D^2 - 4}\cos 2x \\
&= \frac{1}{16} + \frac{1}{4}\frac{3D(\cos 2x) - 2\cos 2x}{9(-2)^2 - 4} \\
&= \frac{1}{16} - \frac{1}{160}(-6\sin 2x - 2\cos 2x) \\
&= \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)
\end{aligned}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-4x} + \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

Example 7

Solve $\quad \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \cos 2x \sin x.$ **[Winter 2015]**

Solution

$$(D^2 - 6D + 9)y = \frac{1}{2}(2\cos 2x \sin x)$$

$$(D^2 - 6D + 9)y = \frac{1}{2}(\sin 3x - \sin x)$$

The auxiliary equation is

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$
$$m = 3, 3 \qquad \text{(real and repeated)}$$
$$\text{CF} = (c_1 + c_2x)\, e^{3x}$$
$$\text{PI} = \frac{1}{\text{D}^2 - 6\text{D} + 9} \cdot \frac{1}{2}(\sin 3x - \sin x)$$
$$= \frac{1}{2}\frac{1}{\text{D}^2 - 6\text{D} + 9}\sin 3x - \frac{1}{2}\frac{1}{\text{D}^2 - 6\text{D} + 9}\sin x$$
$$= \frac{1}{2}\frac{1}{-9 - 6\text{D} + 9}\sin 3x - \frac{1}{2}\frac{1}{-1 - 6\text{D} + 9}\sin x$$
$$= -\frac{1}{12}\int \sin 3x - \frac{1}{2}\frac{1}{8 - 6\text{D}}\sin x$$
$$= \frac{1}{12}\left(\frac{-\cos 3x}{3}\right) - \frac{1}{2}\frac{8 + 6\text{D}}{64 - 36\text{D}^2}\sin x$$
$$= \frac{1}{36}\cos 3x - \frac{4 + 3\text{D}}{64 + 36}\sin x$$
$$= \frac{1}{36}\cos 3x - \frac{1}{100}(4\sin x + 3\cos x)$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{36}\cos 3x - \frac{2}{50}\sin x - \frac{3}{100}\cos x$$

Example 8

Solve $(\text{D}^2 - 4\text{D} + 4) = e^{2x} + \cos 2x.$

Solution

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$
$$(m-2)^2 = 0$$
$$m = 2, 2 \quad \text{(real and repeated)}$$
$$\text{CF} = (c_1 + c_2 x)e^{2x}$$
$$\text{PI} = \frac{1}{\text{D}^2 - 4\text{D} + 4}(e^{2x} + \cos 2x)$$
$$= \frac{1}{\text{D}^2 - 4\text{D} + 4}e^{2x} + \frac{1}{\text{D}^2 - 4\text{D} + 4}\cos 2x$$
$$= x\frac{1}{2\text{D} - 4}e^{2x} + \frac{1}{-2^2 - 4\text{D} + 4}\cos 2x$$

$$= x^2 \frac{1}{2} e^{2x} + \frac{1}{-4D} \cos 2x$$

$$= \frac{x^2}{2} e^{2x} - \frac{1}{4} \int \cos 2x \, dx$$

$$= \frac{x^2}{2} e^{2x} - \frac{1}{4} \frac{\sin 2x}{2}$$

$$= \frac{x^2}{2} e^{2x} - \frac{1}{8} \sin 2x$$

Hence, the general solution is

$$y = (c_1 + c_2 x) e^{2x} + \frac{x^2}{2} e^{2x} - \frac{1}{8} \sin 2x$$

Example 9

Solve $(D^2 - 3D + 2)y = 2\cos(2x + 3) + 2e^x$.

Solution

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 2, 1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^x + c_1 e^{2x}$$

$$\text{PI} = \frac{1}{D^2 - 3D + 2} \left[2\cos(2x+3) + 2e^x \right]$$

$$= 2 \frac{1}{D^2 - 3D + 2} \cos(2x+3) + 2 \frac{1}{D^2 - 3D + 2} e^x$$

$$= 2 \frac{1}{-2^2 - 3D + 2} \cos(2x+3) + 2 \frac{1}{(D-1)(D-2)} e^x$$

$$= 2 \frac{1}{-2 - 3D} \cos(2x+3) + 2 \frac{1}{D-1} \frac{1}{(1-2)} e^x$$

$$= -2 \frac{3D - 2}{9D^2 - 4} \cos(2x+3) - 2x \frac{1}{1} e^x$$

$$= \frac{-2\left[3D\cos(2x+3) - 2\cos(2x+3) \right]}{9\left(-2^2\right) - 4} - 2xe^x$$

$$= \frac{-2\left[-3\sin(2x+3)(2) - 2\cos(2x+3) \right]}{-36 - 4} - 2xe^x$$

$$= \frac{1}{20} \left[-6\sin(2x+3) - 2\cos(2x+3) \right] - 2xe^x$$

$$= -\frac{1}{10}\left[3\sin\left(2x+3\right)+\cos\left(2x+3\right)\right]-2xe^{x}$$

Hence, the general solution is

$$y = c_1 e^{x} + c_2 e^{2x} - \frac{1}{10}\left[3\sin(2x+3)+\cos(2x+3)\right]-2xe^{x}$$

Example 10

Solve $(D^3 - 3D^2 + 9D - 27)y = \cos 3x.$ **[Winter 2016]**

Solution

The auxiliary equation is

$$m^3 - 3m^2 + 9m - 27 = 0$$

$$m^2(m-3) + 9(m-3) = 0$$

$$(m-3)\,(m^2+9) = 0$$

$$m = 3 \text{ (real)}, \quad m = \pm 3i \quad \text{(complex)}$$

$$\text{CF} = c_1\, e^{3x} + c_2 \cos 3x + c_3 \sin 3x$$

$$\text{PI} = \frac{1}{D^3 - 3D^2 + 9D - 27}\cos 3x$$

$$= \frac{x}{3D^2 - 6D + 9}\cos 3x$$

$$= \frac{x}{-27 - 6D + 9}\cos 3x$$

$$= -\frac{x}{6D+18}\cos 3x$$

$$= -\frac{x}{6}\frac{1}{(D+3)}\cos 3x$$

$$= -\frac{x}{6}\frac{D-3}{D^2-9}\cos 3x$$

$$= -\frac{x}{6}\frac{D-3}{-18}\cos 3x$$

$$= \frac{x}{6\cdot 18}[D-3]\cos 3x$$

$$= \frac{x}{6\cdot 18}[-3\sin 3x - 3\cos 3x]$$

$$= -\frac{x}{36}(\sin 3x + \cos 3x)$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x - \frac{x}{36}(\sin 3x + \cos 3x)$$

Example 11

Solve $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$.

Solution

The auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$(m - 1)(m^2 - 2m + 2) = 0$$

$$m - 1 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = 1 \text{ (real)}, \qquad m = 1 \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$$

$$\text{PI} = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x)$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4} e^x + \frac{1}{D(-1^2) - 3(-1^2) + 4D - 2} \cos x$$

$$= x \frac{1}{3 - 6 + 4} \cdot e^x + \frac{1}{3D + 1} \cos x$$

$$= xe^x + \frac{1}{(3D + 1)} \cdot \frac{(3D - 1)}{(3D - 1)} \cos x$$

$$= xe^x + \frac{3D - 1}{9D^2 - 1} \cos x$$

$$= xe^x + \frac{3D - 1}{9(-1^2) - 1} \cos x$$

$$= xe^x - \frac{1}{10}(3D \cos x - \cos x)$$

$$= xe^x - \frac{1}{10}(-3 \sin x - \cos x)$$

$$= xe^x + \frac{1}{10}(3 \sin x + \cos x)$$

Hence, the general solution is

$$y = (c_1 + c_2 \cos x + c_3 \sin x)e^x + xe^x + \frac{1}{10}(3 \sin x + \cos x)$$

Example 12

Solve $(D^4 + 2a^2D^2 + a^4)y = 8\cos ax.$

Solution

The auxiliary equation is

$$m^4 + 2a^2m^2 + a^4 = 0$$

$$(m^2 + a^2)^2 = 0$$

$$m = \pm ia, \pm ia \text{ (complex and repeated)}$$

$$\text{CF} = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax$$

$$\begin{aligned}\text{PI} &= \frac{1}{D^4 + 2a^2D^2 + a^4} 8\cos ax \\ &= x \cdot \frac{1}{4D^3 + 4a^2D} 8\cos ax \\ &= x^2 \cdot \frac{1}{12D^2 + 4a^2} 8\cos ax \\ &= x^2 \cdot \frac{1}{12(-a^2) + 4a^2} 8\cos ax \\ &= -\frac{x^2}{a^2}\cos ax\end{aligned}$$

Hence, the general solution is

$$y = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax - \frac{x^2}{a^2}\cos ax$$

Example 13

Solve $(D-1)^2(D^2+1)y = e^x + \sin^2 \dfrac{x}{2}.$

Solution

The auxiliary equation is

$$(m-1)^2(m^2+1) = 0$$

$$(m-1)^2 = 0, \quad m^2 + 1 = 0$$

$$m = 1, 1 \text{ (real and repeated)}, \ m = \pm i \text{ (complex)}$$

$$\text{CF} = (c_1 + c_2x)e^x + c_3\cos x + c_4\sin x$$

$$\text{PI} = \frac{1}{(D-1)^2(D^2+1)}\left(e^x + \sin^2\frac{x}{2}\right)$$

$$= \frac{1}{(D-1)^2(D^2+1)}\left(e^x + \frac{1-\cos x}{2}\right)$$

$$= \frac{1}{(D-1)^2(D^2+1)}\left(e^x + \frac{e^{0x}}{2} - \frac{\cos x}{2}\right)$$

$$= \frac{1}{(D-1)^2}\cdot\frac{1}{(1^2+1)}e^x + \frac{1}{(0-1)^2(0+1)}\cdot\frac{e^{0x}}{2} - \frac{1}{(D^2+1)(D^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$= x\cdot\frac{1}{2(D-1)}\cdot\frac{e^x}{2} + \frac{1}{2} - \frac{1}{(D^2+1)(-1^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$= \frac{x^2}{2}\cdot\frac{e^x}{2} + \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{(D^2+1)}\frac{1}{D}\cos x$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{1}{4}\frac{1}{(D^2+1)}\int \cos x\,dx$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{D^2+1}\sin x$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{1}{4}x\frac{1}{2D}\sin x$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{x}{8}\int \sin\ x\ dx$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{x}{8}(-\cos x)$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^x + c_3\cos x + c_4\sin x + \frac{x^2e^x}{4} + \frac{1}{2} - \frac{x\cos x}{8}$$

Case III $\mathbf{Q}(x) = x^m$

In this case, Eq. (3.24) reduces to $f(D)y = x^m$.

Hence,
$$\text{PI} = \frac{1}{f(D)}x^m$$
$$= [f(D)]^{-1}x^m$$
$$= [1+\phi(D)]^{-1}x^m$$

Expanding in ascending powers of D up to D^m using Binomial Expansion, since $D^nx^m = 0$ when $n > m$,

$$\text{PI} = (a_0 + a_1D + a_2D^2 + \ldots + a_mD^m)x^m$$

Example 1

Solve $(D^2 + 2D + 1)y = x.$

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$
$$(m+1)^2 = 0$$
$$m = -1, -1 \text{ (real and repeated)}$$
$$\text{CF} = (c_1 + c_2 x)e^{-x}$$
$$\begin{aligned}\text{PI} &= \frac{1}{\text{D}^2 + 2\text{D} + 1} x \\ &= \frac{1}{(1+\text{D})^2} x \\ &= (1+\text{D})^{-2} x \\ &= (1 - 2\text{D} + 3\text{D}^2 - \ldots)x \\ &= x - 2\text{D}x + 3\text{D}^2 x - \ldots \\ &= x - 2 + 0 \\ &= x - 2\end{aligned}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + x - 2$$

Example 2

Solve $y'' + 2y' + 3y = 2x^2$. **[Winter 2014]**

Solution

$$(\text{D}^2 + 2\text{D} + 3)y = 2x^2$$

The auxiliary equation is

$$m^2 + 2m + 3 = 0$$
$$m = \frac{-2 \pm \sqrt{4 - 12}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{2}i \quad \text{(complex)}$$
$$\text{CF} = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$$
$$\begin{aligned}\text{PI} &= \frac{1}{\text{D}^2 + 2\text{D} + 3}(2x^2) \\ &= \frac{1}{3\left(1 + \frac{\text{D}^2 + 2\text{D}}{3}\right)}(2x^2) \\ &= \frac{2}{3}\left(1 + \frac{\text{D}^2 + 2\text{D}}{3}\right)^{-1} x^2\end{aligned}$$

$$= \frac{2}{3}\left[1 - \left(\frac{D^2 + 2D}{3}\right) + \left(\frac{D^2 + 2D}{3}\right)^2 - \cdots\right] x^2$$

$$= \frac{2}{3}\left[x^2 - \frac{2}{3}Dx^2 - \frac{D^2}{3}x^2 + \frac{4}{9}D^2x^2 - \cdots\right]$$

$$= \frac{2}{3}\left[x^2 - \frac{2}{3}(2x) - \frac{2}{3} + \frac{4}{9}(2)\right]$$

$$= \frac{2}{3}\left(x^2 - \frac{4}{3}x + \frac{2}{9}\right)$$

Hence, the general solution is

$$y = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{2}{3}\left(x^2 - \frac{4}{3}x + \frac{2}{9}\right)$$

Example 3

Solve $(D^2 + D)y = x^2 + 2x + 4$.

Solution

The auxiliary equation is

$$m^2 + m = 0$$

$$m(m+1) = 0$$

$$m = 0, -1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{0x} + c_2 e^{-x}$$

$$= c_1 + c_2 e^{-x}$$

$$\text{PI} = \frac{1}{D^2 + D}(x^2 + 2x + 4)$$

$$= \frac{1}{D(D+1)}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1 + D)^{-1}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1 - D + D^2 - D^3 + \cdots)(x^2 + 2x + 4)$$

$$= \frac{1}{D}\left[(x^2 + 2x + 4) - D(x^2 + 2x + 4) + D^2(x^2 + 2x + 4) - D^3(x^2 + 2x + 4) + \cdots\right]$$

$$= \frac{1}{D}\left[(x^2 + 2x + 4) - (2x + 2) + 2 - 0\right]$$

$$= \frac{1}{D}(x^2 + 4)$$

$$= \int (x^2 + 4)\,dx$$

$$= \frac{x^3}{3} + 4x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x$$

Example 4

Solve $(D^2 + 16)y = x^4 + e^{3x} + \cos 3x$. **[Winter 2014]**

Solution

The auxiliary equation is

$$m^2 + 16 = 0$$

$$m = \pm 4i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos 4x + c_2 \sin 4x$$

$$\text{PI} = \frac{1}{D^2 + 16}(x^4 + e^{3x} + \cos 3x)$$

$$= \frac{1}{D^2 + 16}x^4 + \frac{1}{D^2 + 16}e^{3x} + \frac{1}{D^2 + 16}\cos 3x$$

$$= \frac{1}{16}\left(1 + \frac{D^2}{16}\right)^{-1} x^4 + \frac{1}{9 + 16}e^{3x} + \frac{\cos 3x}{(-3^2) + 16}$$

$$= \frac{1}{16}\left(1 - \frac{D^2}{16} + \frac{(-1)(-2)}{2!}\frac{D^4}{256} - \cdots\right)x^4 + \frac{1}{25}e^{3x} + \frac{1}{7}\cos 3x$$

$$= \frac{1}{16}\left(x^4 - \frac{3}{4}x^2 + \frac{3}{32}\right) + \frac{1}{25}e^{3x} + \frac{1}{7}\cos 3x$$

Hence, the general solution is

$$y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{16}\left(x^4 - \frac{3}{4}x^2 + \frac{3}{32}\right) + \frac{1}{25}e^{3x} + \frac{1}{7}\cos 3x$$

Example 5

Solve $(D^2 + 2)y = x^3 + x^2 + e^{-2x} + \cos 3x$.

Solution

The auxiliary equation is

$$m^2 + 2 = 0,$$

$$m = \pm i\sqrt{2} \quad \text{(imaginary)}$$

$$\text{CF} = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$$

$$\text{PI} = \frac{1}{D^2+2}(x^3 + x^2 + e^{-2x} + \cos 3x)$$

$$= \frac{1}{2\left(1+\frac{D^2}{2}\right)}(x^3+x^2) + \frac{1}{D^2+2}e^{-2x} + \frac{1}{D^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1+\frac{D^2}{2}\right)^{-1}(x^3+x^2) + \frac{1}{4+2}e^{-2x} + \frac{1}{-3^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1-\frac{D^2}{2}+\frac{D^4}{4}-\ldots\right)(x^3+x^2) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= \left[\frac{1}{2}(x^3+x^2) - \frac{1}{4}D^2(x^3+x^2) + \frac{D^4}{8}(x^3+x^2) - \cdots\right] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= \left[\frac{1}{2}(x^3+x^2) - \frac{1}{4}(6x+2) + 0\right] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Hence, the general solution is

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{1}{2}(x^3 + x^2 - 3x - 1) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Example 6

Solve $(D^3 - D)y = x^3$. **[Winter 2016]**

Solution

The auxiliary equation is

$$m^3 - m = 0$$

$$m(m^2 - 1) = 0$$

$$m = 0, \pm 1 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 + c_2 e^x + c_3 e^{-x}$$

$$\text{PI} = \frac{1}{D^3 - D}x^3$$

$$= -\frac{1}{D}\left[\frac{1}{1-D^2}\right]x^3$$

$$= -\frac{1}{D}[(1-D^2)^{-1}]x^3$$

$$= -\frac{1}{D}[1 + D^2 + \cdots]x^3$$

$$= -\frac{1}{D}[x^3 + D^2(x^3)]$$

$$= -\frac{1}{D}[x^3 + 6x]$$

$$= -\int (x^3 + 6x)\,dx$$

$$= -\frac{x^4}{4} - 3x^2$$

$$= -\frac{1}{4}x^4 - 3x^2$$

Hence, the general solution is

$$y = c_1 + c_2\, e^x + c_3\, e^{-x} - \frac{1}{4}x^4 - 3x^2$$

Example 7

Solve $(D^3 + 8)y = x^4 + 2x + 1.$

Solution

The auxiliary equation is

$$m^3 + 8 = 0$$

$$m = -2 \text{ (real)}, \quad m = 1 \pm i\sqrt{3} \quad \text{(imaginary)}$$

$$\text{CF} = c_1 e^{-2x} + e^x\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right)$$

$$\text{PI} = \frac{1}{D^3 + 8}(x^4 + 2x + 1)$$

$$= \frac{1}{8\left(1 + \dfrac{D^3}{8}\right)}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 + \frac{D^3}{8}\right)^{-1}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 - \frac{D^3}{8} + \frac{D^6}{64} - \cdots\right)(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left[(x^4 + 2x + 1) - \frac{1}{8}D^3(x^4 + 2x + 1) + \frac{1}{64}D^6(x^4 + 2x + 1) - \cdots\right]$$

$$= \frac{1}{8}(x^4 + 2x + 1 - 3x)$$

$$= \frac{1}{8}(x^4 - x + 1)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x (c_2 \cos\sqrt{3x} + c_3 \sin\sqrt{3}x) + \frac{1}{8}(x^4 - x + 1)$$

Example 8

Solve $(\mathrm{D}^3 - \mathrm{D}^2 - 6\mathrm{D})y = 1 + x^2$. **[Summer 2013]**

Solution

The auxiliary equation is

$$m^3 - m^2 - 6m = 0$$

$$m(m^2 - m - 6) = 0$$

$$m(m-3)(m+2) = 0$$

$$m = 0,\ 3,\ -2 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x}$$

$$= c_1 + c_2 e^{3x} + c_3 e^{-2x}$$

$$\text{PI} = \frac{1}{\mathrm{D}^3 - \mathrm{D}^2 - 6\mathrm{D}}(1 + x^2)$$

$$= \frac{1}{-6\mathrm{D}\left[1 - \frac{\mathrm{D}^2 - \mathrm{D}}{6}\right]}(1 + x^2)$$

$$= -\frac{1}{6\mathrm{D}}\left[1 - \left(\frac{\mathrm{D}^2 - \mathrm{D}}{6}\right)\right]^{-1}(1 + x^2)$$

$$= -\frac{1}{6\mathrm{D}}\left[1 + \left(\frac{\mathrm{D}^2 - \mathrm{D}}{6}\right) + \left(\frac{\mathrm{D}^2 - \mathrm{D}}{6}\right)^2 + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6\mathrm{D}}\left[1 + \frac{\mathrm{D}^2 - \mathrm{D}}{6} + \frac{\mathrm{D}^4 - 2\mathrm{D}^3 + \mathrm{D}^2}{36} + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6\mathrm{D}}\left[1 - \frac{\mathrm{D}}{6} + \frac{7\mathrm{D}^2}{36} - \frac{\mathrm{D}^3}{18} + \cdots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[(1+x^2) - \frac{1}{6}D(1+x^2) + \frac{7}{36}D^2(1+x^2) - \frac{1}{18}D^3(1+x^2) + \cdots\right]$$

$$= -\frac{1}{6D}\left[1+x^2 - \frac{1}{6}(2x) + \frac{7}{36}(2) - 0\right]$$

$$= -\frac{1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right]$$

$$= -\frac{1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx$$

$$= -\frac{1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right)$$

Hence, the general solution is

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6}x\right)$$

Example 9

Solve $(D^3 - 2D + 4)y = x^4 + 3x^2 - 5x + 2.$

Solution

The auxiliary equation is

$$m^3 - 2m + 4 = 0$$

$$(m+2)(m^2 - 2m + 2) = 0$$

$$m + 2 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = -2 \text{ (real)}, \qquad m = 1 \pm i \text{ (complex)}$$

$$\text{CF} = c_1 e^{-2x} + e^x(c_2\cos x + c_3\sin x)$$

$$\text{PI} = \frac{1}{(D^3 - 2D + 4)}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left(1 + \frac{D^3 - 2D}{4}\right)^{-1}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 - \left(\frac{D^3 - 2D}{4}\right)^3\right.$$

$$\left. + \left(\frac{D^3 - 2D}{4}\right)^4 - \ldots\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \frac{4D^2}{16} - \frac{4D^4}{16} + \frac{8D^3}{64}\right.$$

$$\left. + \frac{16D^4}{256} + \text{higher powers of D}\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 + \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} - \frac{3D^4}{16} + \text{higher powers of D}\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[(x^4 + 3x^2 - 5x + 2) + \frac{1}{2}D(x^4 + 3x^2 - 5x + 2) + \frac{1}{4}D^2(x^4 + 3x^2 - 5x + 2)\right.$$

$$- \frac{1}{8}D^3(x^4 + 3x^2 - 5x + 2) - \frac{3}{16}D^4(x^4 + 3x^2 - 5x + 2)$$

$$\left. + \text{higher powers of D}(x^4 + 3x^2 - 5x + 2)\right]$$

$$= \frac{1}{4}\left[(x^4 + 3x^2 - 5x + 2) + \frac{1}{2}(4x^3 + 6x - 5) + \frac{1}{4}(12x^2 + 6) - \frac{1}{8}(24x) - \frac{3}{16}(24) + 0\right]$$

$$= \frac{1}{4}\left(x^4 + 2x^3 + 6x^2 - 5x - \frac{7}{2}\right)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) + \frac{1}{4}\left(x^4 + 2x^3 + 6x^2 - 5x - \frac{7}{2}\right)$$

Example 10

Solve $(D^4 - 2D^3 + D^2)y = x^3$.

Solution

The auxiliary equation is

$$m^4 - 2m^3 + m^2 = 0$$

$$m^2(m^2 - 2m + 1) = 0$$

$$m^2(m-1)^2 = 0$$

$$m = 0, 0, 1, 1 \quad \text{(real and repeated)}$$

Both the roots are real and repeated twice.

$$\text{CF} = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^x$$

$$= c_1 + c_2 x + (c_3 + c_4 x)e^x$$

$$\text{PI} = \frac{1}{D^4 - 2D^3 + D^2}x^3$$

$$= \frac{1}{D^2(D^2-2D+1)} x^3$$
$$= \frac{1}{D^2(1-D)^2} \cdot x^3$$
$$= \frac{1}{D^2}(1-D)^{-2} x^3$$
$$= \frac{1}{D^2}(1+2D+3D^2+4D^3+5D^4+\dots)x^3$$
$$= \frac{1}{D^2}(x^3+2Dx^3+3D^2x^3+4D^3x^3+5D^4x^3+\dots)$$
$$= \frac{1}{D^2}(x^3+2\cdot 3x^2+3\cdot 6x+4\cdot 6+0)$$
$$= \frac{1}{D^2}(x^3+6x^2+18x+24)$$
$$= \frac{1}{D}\left[\int (x^3+6x^2+18x+24)\mathrm{d}x\right]$$
$$= \frac{1}{D}\left(\frac{x^4}{4}+6\frac{x^3}{3}+18\frac{x^2}{2}+24x\right)$$
$$= \int\left(\frac{x^4}{4}+2x^3+9x^2+24x\right)\mathrm{d}x$$
$$= \frac{x^5}{20}+2\frac{x^4}{4}+9\frac{x^3}{3}+24\frac{x^2}{2}$$
$$= \frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2$$

Hence, the general solution is

$$y = c_1 + c_2 x + (c_3 + c_4 x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Example 11

Solve $(D^4-16)y = e^{2x} + x^4$ where $D = \frac{\mathrm{d}}{\mathrm{d}x}$. **[Summer 2017]**

Solution

The auxiliary equation is

$$m^4 - 16 = 0$$

$$(m^2 - 4)(m^2 + 4) = 0$$

$$(m - 2)(m + 2)(m^2 + 4) = 0$$

$$m = 2, -2 \text{ (real and distinct)}, \quad m = \pm 2i \text{ (complex)}$$

$$\text{CF} = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

$$\begin{aligned}
\text{PI} &= \frac{1}{\text{D}^4 - 16}(e^{2x} + x^4) \\
&= \frac{1}{\text{D}^4 - 16} e^{2x} + \frac{1}{\text{D}^4 - 16} x^4 \\
&= \frac{x}{4\text{D}^3} e^{2x} + \left(-\frac{1}{16}\right) \frac{1}{1 - \dfrac{\text{D}^4}{16}} x^4 \\
&= \frac{x}{4(2)^3} e^{2x} - \frac{1}{16}\left[1 - \frac{\text{D}^4}{16}\right]^{-1} x^4 \\
&= \frac{x}{32} e^{2x} - \frac{1}{16}\left[1 + \frac{\text{D}^4}{16} + \cdots\right] x^4 \\
&= \frac{x}{32} e^{2x} - \frac{1}{16}\left[x^4 + \frac{1}{16}\text{D}^4(x^4)\right] \\
&= \frac{x}{32} e^{2x} - \frac{1}{16}\left[x^4 + \frac{24}{16}\right] \\
&= \frac{x}{32} e^{2x} - \frac{x^4}{16} - \frac{1}{16}\,\frac{24}{16} \\
&= \frac{x}{32} e^{2x} - \frac{x^4}{16} - \frac{3}{32} \\
&= \frac{1}{32}(xe^{2x} - 3 - 2x^4)
\end{aligned}$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x + \frac{1}{32}(xe^{2x} - 3 - 2x^4)$$

Example 12

Solve $(D^4+2D^3-3D^2)y = x^2+3e^{2x}$.

Solution

The auxiliary equation is

$$m^4+2m^3-3m^2=0$$
$$m^2(m^2+2m-3)=0$$
$$m^2(m-1)(m+3)=0$$
$$m=0,\,0 \text{ (real and repeated)}, \qquad m=1,-3 \text{ (real and distinct)}$$

$$\begin{aligned}
\text{CF} &= (c_1+c_2x)e^{0x}+c_3e^x+c_4e^{-3x}\\
&= c_1+c_2x+c_3e^x+c_4e^{-3x}
\end{aligned}$$

$$\begin{aligned}
\text{PI} &= \frac{1}{D^4+2D^3-3D^2}\left(x^2+3e^{2x}\right)\\
&= \frac{1}{D^4+2D^3-3D^2}x^2+\frac{1}{D^4+2D^3-3D^2}\,3e^{2x}\\
&= \frac{1}{-3D^2\left(1-\dfrac{D^2+2D}{3}\right)}x^2+\frac{1}{16+16-12}\,3e^{2x}\\
&= -\frac{1}{3D^2}\left(1-\frac{D^2+2D}{3}\right)^{-1}x^2+\frac{3e^{2x}}{20}\\
&= -\frac{1}{3D^2}\left[1+\frac{D^2+2D}{3}+\left(\frac{D^2+2D}{3}\right)^2+\ldots\right]x^2+\frac{3e^{2x}}{20}\\
&= -\frac{1}{3D^2}\left(1+\frac{D^2+2D}{3}+\frac{D^4+4D^2+4D^3}{9}+\cdots\right)x^2+\frac{3e^{2x}}{20}\\
&= -\frac{1}{3D^2}\left(x^2+\frac{2}{3}Dx^2+\frac{7}{9}D^2x^2+\frac{4}{9}D^3x^2+\cdots\right)+\frac{3}{20}e^{2x}\\
&= -\frac{1}{3D^2}\left[x^2+\frac{2}{3}(2x)+\frac{7}{9}(2)+0\right]+\frac{3e^{2x}}{20}
\end{aligned}$$

$$= -\frac{1}{3\text{D}}\left[\int\left(x^2+\frac{4}{3}x+\frac{14}{9}\right)\text{d}x\right]+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3\text{D}}\left(\frac{x^3}{3}+\frac{4}{3}\frac{x^2}{2}+\frac{14}{9}x\right)+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\int\left(\frac{x^3}{3}+\frac{2}{3}x^2+\frac{14}{9}x\right)\text{d}x+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\left(\frac{x^4}{12}+\frac{2x^3}{9}+\frac{7x^2}{9}\right)+\frac{3e^{2x}}{20}$$

Hence, the general solution is

$$y = c_1+c_2x+c_3e^x+c_4e^{-3x}-\frac{x^2}{9}\left(\frac{x^2}{4}+\frac{2x}{3}+\frac{7}{3}\right)+\frac{3e^{2x}}{20}$$

Case IV $\mathbf{Q} = e^{ax}V$**, where** V **is a function of** x**.**

In this case, Eq. (3.24) reduces to $f(\text{D})y = e^{ax}\text{V}$.

Let u be a function of x.

$$\text{D}(e^{ax}u) = e^{ax}\text{D}u+ae^{ax}u$$

$$= e^{ax}(\text{D}+a)u$$

$$\text{D}^2(e^{ax}u) = \text{D}\left[e^{ax}(\text{D}+a)u\right]$$

$$= ae^{ax}(\text{D}+a)u+e^{ax}(\text{D}^2+a\text{D})u$$

$$= e^{ax}(\text{D}^2+2a\text{D}+a^2)u$$

$$= e^{ax}(\text{D}+a)^2u$$

In general,

$$\text{D}^r(e^{ax}u) = e^{ax}(\text{D}+a)^r u$$

Let $\text{D}^r = f(\text{D})$, $\qquad (\text{D}+a)^r = f(\text{D}+a)$

$$f(\text{D})(e^{ax}u) = e^{ax}f(\text{D}+a)u$$

Operating both the sides with $\dfrac{1}{f(\text{D})}$,

$$\frac{1}{f(\text{D})}\left[f(\text{D})(e^{ax}u)\right] = \frac{1}{f(\text{D})}\left[e^{ax}f(\text{D}+a)u\right]$$

$$e^{ax}u = \frac{1}{f(\text{D})}\left[e^{ax}f(\text{D}+a)u\right]$$

Putting $f(\mathrm{D}+a)u = \mathrm{V},\ u = \dfrac{1}{f(\mathrm{D}+a)}\mathrm{V}$

$$e^{ax}\cdot\frac{1}{f(\mathrm{D}+a)}\mathrm{V} = \frac{1}{f(\mathrm{D})}(e^{ax}\mathrm{V})$$

Hence,

$$\begin{aligned}\mathrm{PI} &= \frac{1}{f(\mathrm{D})}\cdot e^{ax}\mathrm{V}\\ &= e^{ax}\cdot\frac{1}{f(\mathrm{D}+a)}\mathrm{V}\end{aligned}$$

Example 1

Solve $(\mathrm{D}+2)^2 y = e^{-2x}\sin x$.

Solution

The auxiliary equation is

$$\begin{aligned}(m+2)^2 &= 0\\ m &= -2, -2 \quad \text{(real and repeated)}\\ \mathrm{CF} &= (c_1 + c_2 x)e^{-2x}\\ \mathrm{PI} &= \frac{1}{(\mathrm{D}+2)^2}e^{-2x}\sin x\\ &= e^{-2x}\frac{1}{(\mathrm{D}-2+2)^2}\sin x\\ &= e^{-2x}\frac{1}{\mathrm{D}^2}\sin x\\ &= e^{-2x}\frac{1}{-1^2}\sin x\\ &= -e^{-2x}\sin x\end{aligned}$$

Hence, the general solution is

$$\begin{aligned}y &= (c_1 + c_2 x)e^{-2x} - e^{-2x}\sin x\\ &= (c_1 + c_2 x - \sin x)e^{-2x}\end{aligned}$$

Example 2

Solve $(\mathrm{D}^2 + 2\mathrm{D} + 1)y = \dfrac{e^{-x}}{x^2}$.

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$
\begin{aligned}
(m+1)^2 &= 0 \\
m &= -1, -1 \quad \text{(real and repeated)} \\
\text{CF} &= (c_1 + c_2 x)e^{-x} \\
\text{PI} &= \frac{1}{\text{D}^2 + 2\text{D} + 1}\left(\frac{e^{-x}}{x^2}\right) \\
&= \frac{1}{(\text{D}+1)^2}\left(\frac{e^{-x}}{x^2}\right) \\
&= e^{-x}\frac{1}{(\text{D}-1+1)^2}\left(\frac{1}{x^2}\right) \\
&= e^{-x}\frac{1}{\text{D}^2}x^{-2} \\
&= e^{-x}\frac{1}{\text{D}}\int x^{-2}\,\text{d}x \\
&= e^{-x}\frac{1}{\text{D}}\left(\frac{x^{-2+1}}{-2+1}\right) \\
&= e^{-x}\frac{1}{\text{D}}x^{-1} \\
&= -e^{-x}\int\frac{\text{d}x}{x} \\
&= e^{-x}\log x
\end{aligned}
$$

Hence, the general solution is

$$
\begin{aligned}
y &= (c_1 + c_2 x)e^{-x} - e^{-x}\log x \\
&= e^{-x}(c_1 + c_2 x - \log x)
\end{aligned}
$$

Example 3

Solve $(\text{D}^2 - 2\text{D} - 1)y = e^x \cos x.$

Solution

The auxiliary equation is

$$
\begin{aligned}
m^2 - 2m - 1 &= 0 \\
m &= 1 \pm \sqrt{2} \quad \text{(real and distinct)} \\
\text{CF} &= c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})}x \\
\text{PI} &= \frac{1}{\text{D}^2 - 2\text{D} - 1}e^x \cos x
\end{aligned}
$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) - 1} \cos x$$

$$= e^x \frac{1}{(D^2 - 2)} \cos x$$

$$= e^x \frac{1}{-1^2 - 2} \cos x$$

$$= -\frac{1}{3} e^x \cos x$$

Hence, the general solution is

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x} - \frac{1}{3} e^x \cos x$$

Example 4

Solve $\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} + 4y = e^x \cos x.$ **[Winter 2017]**

Solution

$$(D^3 - 2D + 4)y = e^x \cos x$$

The auxiliary equation is

$$m^3 - 2m + 4 = 0$$

$$m^2(m+2) - 2m(m+2) + 2(m+2) = 0$$

$$(m+2)(m^2 - 2m + 2) = 0$$

$$m = -2 \quad \text{(real)}, \quad m = 1 \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x)$$

$$\text{PI} = \frac{1}{D^3 - 2D + 4} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^3 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^3 + 3D^2 + D + 3} \cos x$$

$$= e^x \left[x \frac{1}{3D^2 + 6D + 1} \cos x \right] \qquad \left[\begin{array}{r} \because \; D^3 + 3D^2 + D + 3 = 0 \\ \text{at } D^2 = -1^2 = -1 \end{array} \right]$$

$$= e^x x \frac{1}{3(-1)^2 + 6D + 1} \cos x$$

$$= e^x x \frac{1}{6D - 2} \cos x$$

$$= e^x x \frac{1}{2(3D-1)} \cdot \frac{(3D+1)}{(3D+1)} \cos x$$

$$= e^x x \frac{3D+1}{2(9D^2-1)} \cos x$$

$$= e^x x \frac{(3D+1)\cos x}{2\left[9(-1^2)-1\right]}$$

$$= -\frac{e^x x}{20}(3D\cos x + \cos x)$$

$$= -\frac{e^x x}{20}(-3\sin x + \cos x)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) + \frac{e^x x}{20}(3\sin x - \cos x)$$

Example 5

Solve $(D^2 - 2D + 5)y = e^{2x} \sin x.$

Solution

The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = 1 \pm 2i \quad \text{(complex)}$$

$$\text{CF} = e^x(c_1 \cos 2x + c_2 \sin 2x)$$

$$\text{PI} = \frac{1}{D^2 - 2D + 5} e^{2x} \sin x$$

$$= e^{2x} \frac{1}{(D+2)^2 - 2(D+2) + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 4 + 4D - 2D - 4 + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 2D + 5} \sin x$$

$$= e^{2x} \frac{1}{-1^2 + 2D + 5} \sin x$$

$$= e^{2x} \frac{1}{2D + 4} \sin x$$

$$= \frac{1}{2} e^{2x} \frac{1}{D+2} \sin x$$

$$= \frac{1}{2} e^{2x} \frac{D-2}{D^2-4} \sin x$$

$$= \frac{1}{2} e^{2x} \frac{D-2}{-1^2-4} \sin x$$

$$= -\frac{1}{10} e^{2x} (D \sin x - 2 \sin x)$$

$$= -\frac{1}{10} e^{2x} (\cos x - 2 \sin x)$$

Hence, the general solution is

$$y = e^x (c_1 \cos 2x + c_2 \sin 2x) - \frac{1}{10} e^{2x} (\cos x - 2 \sin x)$$

Example 5

Solve $(D^2 + 2D + 2)y = e^x \sin x + 7.$

Solution

The auxiliary equation is

$$m^2 + 2m + 2 = 0$$

$$m = -1 \pm \quad \text{(complex)}$$

$$CF = e^{-x} (c_1 \cos x + c_2 \sin x)$$

$$PI = \frac{1}{D^2 + 2D + 2} (e^x \sin x + 7)$$

$$= \frac{1}{D^2 + 2D + 2} e^x \sin x + \frac{1}{D^2 + 2D + 2} 7$$

$$= e^x \frac{1}{(D+1) + 2(D+1) + 2} \sin x + 7 \frac{1}{D^2 + 2D + 2} e^{0x}$$

$$= e^x \frac{1}{D^2 + 2D + 1 + 2D + 2 + 2} \sin x + 7 \frac{1}{0 + 0 + 2} e^{0x}$$

$$= e^x \frac{1}{D^2 + 4D + 5} \sin x + \frac{7}{2} e^{0x}$$

$$= e^x \frac{1}{(-1)^2 + 4D + 5} \sin x + \frac{7}{2}$$

$$= e^x \frac{1}{4D + 4} \sin x + \frac{7}{2}$$

$$= \frac{e^x}{4} \frac{1}{D+1} \sin x + \frac{7}{2}$$

$$= \frac{e^x}{4}\frac{D-1}{D^2-1}\sin x + \frac{7}{2}$$

$$= \frac{e^x}{4}\frac{D-1}{-1^2-1}\sin x + \frac{7}{2}$$

$$= \frac{e^x}{-8}(D\sin x - \sin x) + \frac{7}{2}$$

$$= -\frac{1}{8}e^x(\cos x - \sin x) + \frac{7}{2}$$

Hence, the general solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) - \frac{1}{8}e^x(\cos x - \sin x) + \frac{7}{2}$$

Example 6

Solve $(D^2 - 2D + 2)y = e^x x^2 + 5 + e^{-2x}$.

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i \quad \text{(complex)}$$

$$CF = e^x(c_1 \cos x + c_2 \sin x)$$

$$PI = \frac{1}{D^2 - 2D + 2}(e^x x^2 + 5 + e^{-2x})$$

$$= \frac{1}{D^2 - 2D + 2}e^x x^2 + \frac{1}{D^2 - 2D + 2}5 + \frac{1}{D^2 - 2D + 2}e^{-2x}$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}x^2 + \frac{1}{D^2 - 2D + 2}5e^{0x} + \frac{1}{4 - 2(-2) + 2}e^{-2x}$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 2}x^2 + \frac{1}{0 - 0 + 2}5e^{0x} + \frac{1}{10}e^{-2x}$$

$$= e^x \frac{1}{D^2 + 1}x^2 + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

$$= e^x(1 + D^2)^{-1}x^2 + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

$$= e^x(1 - D^2 + D^4 - \ldots)x^2 + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

$$= e^x\left[x^2 - D^2(x^2) + D^4(x^2) - \cdots\right] + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

$$= e^x(x^2-2)+\frac{5}{2}+\frac{1}{10}e^{-2x}$$

Hence, the general solution is

$$y = e^x(c_1\cos x + c_2\sin x) + e^x(x^2-2)+\frac{5}{2}+\frac{1}{10}e^{-2x}$$

Example 7

Solve $(\mathrm{D}^2 - 4\mathrm{D} - 5)y = xe^{2x} + 3\cos 4x.$

Solution

The auxiliary equation is

$$m^2 - 4m - 5 = 0$$

$$(m-5)(m+1) = 0$$

$$m = 5, -1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1e^{5x} + c_2e^{-x}$$

$$\text{PI} = \frac{1}{\mathrm{D}^2-4\mathrm{D}-5}(xe^{2x}+3\cos 4x)$$

$$= \frac{1}{\mathrm{D}^2-4\mathrm{D}-5}xe^{2x} + \frac{1}{\mathrm{D}^2-4\mathrm{D}-5}3\cos 4x$$

$$= e^{2x}\frac{1}{(\mathrm{D}+2)^2-4(\mathrm{D}+2)-5}x + 3\frac{1}{-4^2-4\mathrm{D}-5}\cos 4x$$

$$= e^{2x}\frac{1}{\mathrm{D}^2+4\mathrm{D}+4-4\mathrm{D}-8-5}x + 3\frac{1}{-4\mathrm{D}-21}\cos 4x$$

$$= e^{2x}\frac{1}{\mathrm{D}^2-9}x - 3\frac{1}{4\mathrm{D}+21}\cos 4x$$

$$= \frac{e^{2x}}{-9}\frac{1}{\left(1-\frac{\mathrm{D}^2}{9}\right)}x - 3\frac{4\mathrm{D}-21}{16\mathrm{D}^2-441}\cos 4x$$

$$= -\frac{e^{2x}}{9}\left(1-\frac{\mathrm{D}^2}{9}\right)^{-1}x - 3\frac{4\mathrm{D}-21}{16(-4^2)-441}\cos 4x$$

$$= -\frac{e^{2x}}{9}\left[1+\frac{\mathrm{D}^2}{9}+\left(\frac{\mathrm{D}^2}{9}\right)^2+\ldots\right]x + 3\frac{1}{697}\left(4\mathrm{D}\cos 4x - 21\cos 4x\right)$$

$$= -\frac{e^{2x}}{9}\left[x+\frac{1}{9}\mathrm{D}^2x+\cdots\right] + \frac{3}{697}\left[4(-\sin 4x)4 - 21\cos 4x\right]$$

$$= -\frac{e^{2x}}{9}(x+0) + \frac{3}{697}\left(-16\sin 4x - 21\cos 4x\right)$$

$$= -\frac{1}{9}xe^{2x} - \frac{3}{697}\left(16\sin 4x + 21\cos 4x\right)$$

Hence, the general solution is

$$y = c_1 e^{5x} + c_2 e^{-x} - \frac{1}{9}xe^{2x} - \frac{3}{697}(16\sin 4x + 21\cos 4x)$$

Example 8

Solve $(\mathrm{D}^2 + 4\mathrm{D} + 3)y = e^{-x}x \sin x + xe^{3x}$.

Solution

The auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$$(m+3)(m+1) = 0$$

$$m = -3, -1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{-3x} + c_2 e^{-x}$$

$$\text{PI} = \frac{1}{\mathrm{D}^2 + 4\mathrm{D} + 3}(e^{-x}\sin x + xe^{3x})$$

$$= \frac{1}{\mathrm{D}^2 + 4\mathrm{D} + 3}e^{-x}\sin x + \frac{1}{\mathrm{D}^2 + 4\mathrm{D} + 3}xe^{3x}$$

$$= e^{-x}\frac{1}{(\mathrm{D}-1)^2 + 4(\mathrm{D}-1) + 3}\sin x + e^{3x}\frac{1}{(\mathrm{D}+3)^2 + 4(\mathrm{D}+3) + 3}x$$

$$= e^{-x}\frac{1}{\mathrm{D}^2 - 2\mathrm{D} + 1 + 4\mathrm{D} - 4 + 3}\sin x + e^{3x}\frac{1}{\mathrm{D}^2 + 6\mathrm{D} + 9 + 4\mathrm{D} + 12 + 3}x$$

$$= e^{-x}\frac{1}{\mathrm{D}^2 + 2\mathrm{D}}\sin x + e^{3x}\frac{1}{\mathrm{D}^2 + 10\mathrm{D} + 24}x$$

$$= e^{-x}\frac{1}{-1^2 + 2\mathrm{D}}\sin x + \frac{e^{3x}}{24}\frac{1}{\left(1 + \frac{10\mathrm{D} + \mathrm{D}^2}{24}\right)}x$$

$$= e^{-x}\frac{2\mathrm{D}+1}{4\mathrm{D}^2 - 1}\sin x + \frac{e^{3x}}{24}\left[1 + \frac{10\mathrm{D} + \mathrm{D}^2}{24}\right]^{-1}x$$

$$= e^{-x}\frac{(2\mathrm{D}+1)\sin x}{4(-1^2) - 1} + \frac{e^{3x}}{24}\left[1 - \frac{10\mathrm{D} + \mathrm{D}^2}{24} + \cdots\right]x$$

$$= -\frac{e^{-x}}{5}\left(2\mathrm{D}\sin x + \sin x\right) + \frac{e^{3x}}{24}\left[x - \frac{5}{12}\mathrm{D}(x) - \frac{1}{24}\mathrm{D}^2(x) + \cdots\right]$$

$$= -\frac{e^{-x}}{5}\left(2\cos x + \sin x\right) + \frac{e^{3x}}{24}\left(x - \frac{5}{12} - 0\right)$$

$$= -\frac{1}{5}e^{-x}\left(2\cos x + \sin x\right) + \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$$

Hence, the general solution is

$$y = c_1 e^{-3x} + c_2 e^{-x} - \frac{1}{5}e^{-x}(2\cos x + \sin x) + \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$$

Example 9

Solve $(\mathrm{D}^3 + 3\mathrm{D}^2 - 4\mathrm{D} - 12)y = 12xe^{-2x}$.

Solution

The auxiliary equation is

$$m^3 + 3m^2 - 4m - 12 = 0$$

$$m^2(m+3) - 4(m+3) = 0$$

$$(m+3)(m^2 - 4) = 0$$

$$m = -3, -2, 2 \text{ (real and distinct)}$$

$$\mathrm{CF} = c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{2x}$$

$$\mathrm{PI} = \frac{1}{(\mathrm{D}+3)(\mathrm{D}+2)(\mathrm{D}-2)} 12xe^{-2x}$$

$$= 12e^{-2x} \frac{1}{(\mathrm{D}-2+3)(\mathrm{D}-2+2)(\mathrm{D}-2-2)} x$$

$$= 12e^{-2x} \frac{1}{(\mathrm{D}+1)\mathrm{D}(\mathrm{D}-4)} x$$

$$= 12e^{-2x} \frac{1}{\mathrm{D}(\mathrm{D}^2 - 3\mathrm{D} - 4)} x$$

$$= 12e^{-2x} \frac{1}{-4\mathrm{D}\left(1 + \frac{3\mathrm{D} - \mathrm{D}^2}{4}\right)} x$$

$$= -3e^{-2x} \frac{1}{\mathrm{D}}\left(1 + \frac{3\mathrm{D} - \mathrm{D}^2}{4}\right)^{-1} x$$

$$= -3e^{-2x}\frac{1}{D}\left(1-\frac{3D-D^2}{4}+\cdots\right)x$$

$$= -3e^{-2x}\frac{1}{D}\left[x-\frac{3}{4}D(x)+\frac{1}{4}D^2(x)+\cdots\right]$$

$$= -3e^{-2x}\frac{1}{D}\left(x-\frac{3}{4}+0\right)$$

$$= -3e^{-2x}\int\left(x-\frac{3}{4}\right)dx$$

$$= -3e^{-2x}\left(\frac{x^2}{2}-\frac{3}{4}x\right)$$

Hence, the general solution is

$$y = c_1e^{-3x}+c_2e^{-2x}+c_3e^{2x}-3e^{-2x}\left(\frac{x^2}{2}-\frac{3}{4}x\right)$$

Example 10

Solve $(D^3+1)y = e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

Solution

The auxiliary equation is

$$m^3+1=0$$

$$(m+1)(m^2-m+1)=0$$

$$m=-1 \quad \text{(real)}, \quad m=\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \quad \text{(complex)}$$

$$\text{CF} = c_1e^{-x}+e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{PI} = \frac{1}{D^3+1}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}\frac{1}{\left[\left(D+\frac{1}{2}\right)^3+1\right]}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}\frac{1}{\left[\left(D^3+\frac{1}{8}+\frac{3}{2}D^2+\frac{3}{4}D\right)+1\right]}\sin\left(\frac{\sqrt{3}}{2}x\right) \qquad \left[\because\ (a+b)=a^3+b^3+3a^2b+3ab^2\right]$$

$$= e^{\frac{x}{2}}\left[x\frac{1}{3D^2+3D+\frac{3}{4}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right] \quad \left[\begin{array}{l}\because \left(D^3+\frac{1}{8}+\frac{3}{2}D^2+\frac{3}{4}D\right)+1=0 \\ \text{at } D^2=-\left(\frac{\sqrt{3}}{2}\right)^2=-\frac{3}{4}\end{array}\right]$$

$$= e^{\frac{x}{2}}x\frac{1}{3\left[-\left(\frac{\sqrt{3}}{2}\right)^2\right]+3D+\frac{3}{4}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{1}{\left(-\frac{9}{4}+3D+\frac{3}{4}\right)}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{1}{3D-\frac{3}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2}{3(2D-1)}\frac{(2D+1)}{(2D+1)}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2(2D+1)}{3(4D^2-1)}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2(2D+1)}{3\left[4\left\{-\left(\frac{\sqrt{3}}{2}\right)^2\right\}-1\right]}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}x\frac{2\left[2D\left\{\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]}{3(-4)}$$

$$= -\frac{1}{6}xe^{\frac{x}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Hence, the general solution is

$$y = c_1e^{-x}+e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)-\frac{1}{6}xe^{\frac{x}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Case V Q = *x*V, where V is a function of *x*.

In this case Eq. (3.24) reduces to $f(\mathrm{D})y = x\mathrm{V}$.

Let u be a function of x.

$$\begin{aligned}
\mathrm{D}(xu) &= x\mathrm{D}u + u \\
\mathrm{D}^2(xu) &= \mathrm{D}(x\mathrm{D}u + u) = x\mathrm{D}^2u + \mathrm{D}u + \mathrm{D}u = x\mathrm{D}^2u + 2\mathrm{D}u \\
\mathrm{D}^3(xu) &= \mathrm{D}(x\mathrm{D}^2u + 2\mathrm{D}u) = x\mathrm{D}^3u + \mathrm{D}^2u + 2\mathrm{D}^2u = x\mathrm{D}^3u + 3\mathrm{D}^2u
\end{aligned}$$

In general,

$$\mathrm{D}^r(xu) = x\mathrm{D}^r u + r\mathrm{D}^{r-1}u = x\mathrm{D}^r u + \left[\frac{\mathrm{d}}{\mathrm{dD}}(\mathrm{D}^r)\right]u$$

Let $\quad \mathrm{D}^r = f(\mathrm{D})$

$$\begin{aligned}
f(\mathrm{D})(xu) &= x\,f(\mathrm{D})u + \left[\frac{\mathrm{d}}{\mathrm{dD}}f(\mathrm{D})\right]u \\
&= xf(\mathrm{D})u + f'(\mathrm{D})u
\end{aligned}$$

Putting $f(\mathrm{D})u = \mathrm{V},\ u = \dfrac{1}{f(\mathrm{D})}\mathrm{V}$ in the above equation,

$$\begin{aligned}
f(\mathrm{D})\left[x\frac{1}{f(\mathrm{D})}\mathrm{V}\right] &= x\mathrm{V} + f'(\mathrm{D})\left[\frac{1}{f(\mathrm{D})}\mathrm{V}\right] \\
x\mathrm{V} &= f(\mathrm{D})\left[x\frac{1}{f(\mathrm{D})}\mathrm{V}\right] - f'(\mathrm{D})\left[\frac{1}{f(\mathrm{D})}\mathrm{V}\right]
\end{aligned}$$

Operating both the sides with $\dfrac{1}{f(\mathrm{D})}$,

$$\begin{aligned}
\frac{1}{f(\mathrm{D})}x\mathrm{V} &= \frac{1}{f(\mathrm{D})}\left[f(\mathrm{D})\left(x\frac{1}{f(\mathrm{D})}\mathrm{V}\right)\right] - \frac{1}{f(\mathrm{D})}\left[f'(\mathrm{D})\left(\frac{1}{f(\mathrm{D})}\right)\mathrm{V}\right] \\
&= x\frac{1}{f(\mathrm{D})}\mathrm{V} - \frac{f'(\mathrm{D})}{[f(\mathrm{D})]^2}\mathrm{V}
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{PI} &= \frac{1}{f(\mathrm{D})}x\mathrm{V} \\
&= x\frac{1}{f(\mathrm{D})}\mathrm{V} - \frac{f'(\mathrm{D})}{[f(\mathrm{D})]^2}\mathrm{V}
\end{aligned}$$

Example 1

Solve $(\mathrm{D}^2 - 5\mathrm{D} + 6)y = x\cos 2x.$

Solution

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$
$$(m-2)(m-3) = 0$$
$$m = 2,\ 3 \text{ (real and distinct)}$$
$$\mathrm{CF} = c_1 e^{2x} + c_2 e^{3x}$$
$$\mathrm{PI} = \frac{1}{\mathrm{D}^2 - 5\mathrm{D} + 6} x\cos 2x$$
$$= x\frac{1}{\mathrm{D}^2 - 5\mathrm{D} + 6}\cos 2x - \frac{2\mathrm{D} - 5}{(\mathrm{D}^2 - 5\mathrm{D} + 6)^2}\cos 2x$$
$$= x\frac{1}{-2^2 - 5\mathrm{D} + 6}\cos 2x - \frac{2\mathrm{D} - 5}{(-2^2 - 5\mathrm{D} + 6)^2}\cos 2x$$
$$= x\frac{1}{(2 - 5\mathrm{D})}\cdot\frac{(2 + 5\mathrm{D})}{(2 + 5\mathrm{D})}\cos 2x - \frac{2\mathrm{D} - 5}{(4 - 20\mathrm{D} + 25\mathrm{D}^2)}\cos 2x$$
$$= x\frac{(2 + 5\mathrm{D})}{4 - 25\mathrm{D}^2}\cos 2x - \frac{2\mathrm{D} - 5}{\left[4 - 20\mathrm{D} + 25(-2^2)\right]}\cos 2x$$
$$= x\frac{(2 + 5\mathrm{D})}{4 + 100}\cos 2x + \frac{2\mathrm{D} - 5}{4(5\mathrm{D} + 24)}\cos 2x$$
$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{2\mathrm{D} - 5}{4(5\mathrm{D} + 24)}\cdot\frac{(5\mathrm{D} - 24)}{(5\mathrm{D} - 24)}\cos 2x$$
$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{(10\mathrm{D}^2 - 73\mathrm{D} + 120)}{4(25\mathrm{D}^2 - 576)}\cos 2x$$
$$= \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{(10\mathrm{D}^2 - 73\mathrm{D} + 120)}{4(-100 - 576)}\cos 2x$$
$$= \frac{1}{52}x(\cos 2x - 5\sin 2x) - \frac{1}{2704}(-40\cos 2x + 146\sin 2x + 120\cos 2x)$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{52}x(\cos 2x - 5\sin 2x) - \frac{1}{1352}(40\cos 2x + 73\sin 2x)$$

Example 2

Solve $(D^2 - 1)y = xe^x$ where $D = \dfrac{d}{dx}$. **[Summer 2017]**

Solution

The auxiliary equation is

$$\begin{aligned}
m^2 - 1 &= 0\\
m^2 &= 1\\
m &= \pm 1 \text{ (real and distinct)}\\
\text{CF} &= c_1 e^x + c_2 e^{-x}\\
\text{PI} &= \frac{1}{D^2 - 1} xe^x\\
&= e^x \frac{1}{(D+1)^2 - 1} x\\
&= e^x \frac{1}{D^2 + 2D + 1 - 1} x\\
&= e^x \frac{1}{D^2 + 2D} x\\
&= \frac{e^x}{2D} \frac{1}{1 + \frac{D}{2}} x\\
&= \frac{e^x}{2D} \left[1 + \frac{D}{2}\right]^{-1} x\\
&= \frac{e^x}{2D} \left[1 - \frac{D}{2}\right] x\\
&= \frac{e^x}{2D} \left[x - \frac{1}{2}\right]\\
&= \frac{e^x}{2} \int \left(x - \frac{1}{2}\right) dx\\
&= \frac{e^x}{2} \left(\frac{x^2}{2} - \frac{x}{2}\right)\\
&= \frac{1}{4} e^x (x^2 - x)
\end{aligned}$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{4} e^x (x^2 - x)$$

Example 3

Solve $(\mathrm{D}^2 + 2\mathrm{D} + 1)\, y = xe^{-x} \cos x.$

Solution

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$
$$(m+1)^2 = 0$$
$$m = -1, -1 \quad \text{(real and repeated)}$$
$$\mathrm{CF} = (c_1 + c_2 x)e^{-x}$$
$$\mathrm{PI} = \frac{1}{\mathrm{D}^2 + 2\mathrm{D} + 1} xe^{-x} \cos x$$
$$= \frac{1}{(\mathrm{D}+1)^2} xe^{-x} \cos x$$
$$= e^{-x} \frac{1}{(\mathrm{D}-1+1)^2} x \cos x$$
$$= e^{-x} \frac{1}{\mathrm{D}^2} x \cos x$$
$$= e^{-x} \left[x \frac{1}{\mathrm{D}^2} \cos x - \frac{2\mathrm{D}}{(\mathrm{D}^2)^2} \cos x \right]$$
$$= e^{-x} \cdot \left(x \frac{1}{-1^2} \cos x - \frac{2\mathrm{D}}{(-1^2)^2} \cos x \right)$$
$$= e^{-x}(-x \cos x - 2\mathrm{D} \cos x)$$
$$= e^{-x}(-x \cos x + 2 \sin x)$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-x} + e^{-x}(-x \cos x + 2 \sin x)$$

Example 4

Solve $(\mathrm{D}^2 + 3\mathrm{D} + 2)y = xe^x \sin x.$

Solution

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$
$$(m+1)(m+2) = 0$$
$$m = -1, \; -2 \text{ (real and distinct)}$$

$$\text{CF} = c_1e^{-x} + c_2e^{-2x}$$

$$\begin{aligned}
\text{PI} &= \frac{1}{(\text{D}+1)(\text{D}+2)} xe^x \sin x \\
&= e^x \frac{1}{(\text{D}+1+1)(\text{D}+1+2)} x\sin x \\
&= e^x \frac{1}{(\text{D}+2)(\text{D}+3)} x\sin x \\
&= e^x \frac{1}{\text{D}^2+5\text{D}+6} x\sin x \\
&= e^x \left[x\frac{1}{\text{D}^2+5\text{D}+6}\sin x - \frac{2\text{D}+5}{(\text{D}^2+5\text{D}+6)^2}\sin x \right] \\
&= e^x \left[x\frac{1}{-1^2+5\text{D}+6}\sin x - \frac{2\text{D}+5}{(-1^2+5\text{D}+6)^2}\sin x \right] \\
&= e^x \left[x\frac{1}{5(\text{D}+1)}\cdot\frac{(\text{D}-1)}{(\text{D}-1)}\sin x - \frac{2\text{D}+5}{25(\text{D}^2+2\text{D}+1)}\sin x \right] \\
&= e^x \left[\frac{x}{5}\cdot\frac{(\text{D}-1)}{(\text{D}^2-1)}\sin x - \frac{2\text{D}+5}{25(-1^2+2\text{D}+1)}\sin x \right] \\
&= e^x \left[\frac{x}{5}\cdot\frac{(\text{D}-1)}{(-1^2-1)}\sin x - \frac{2\text{D}+5}{25(2\text{D})}\sin x \right] \\
&= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(1+\frac{5}{2\text{D}}\right)\sin x \right] \\
&= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(\sin x+\frac{5}{2}\int \sin x\,\text{d}x\right) \right] \\
&= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(\sin x-\frac{5}{2}\cos x\right) \right]
\end{aligned}$$

Hence, the general solution is

$$y = c_1e^{-x} + c_2e^{-2x} - \frac{1}{5}e^x\left[\frac{x}{2}(\cos x - \sin x) + \frac{1}{5}\left(\sin x - \frac{5}{2}\cos x\right)\right]$$

Example 5

Solve $(4\text{D}^2+8\text{D}+3)y = xe^{-\frac{x}{2}}\cos x.$

Solution

The auxiliary equation is

$$4m^2+8m+3=0$$

$$(2m+1)(2m+3)=0$$

$$m=-\frac{1}{2},\ -\frac{3}{2} \qquad \text{(real and distinct)}$$

$$\text{CF}=c_1e^{-\frac{x}{2}}+c_2e^{-\frac{3x}{2}}$$

$$\text{PI}=\frac{1}{4\text{D}^2+8\text{D}+3}xe^{-\frac{x}{2}}\cos x$$

$$=e^{-\frac{x}{2}}\frac{1}{4\left(\text{D}-\frac{1}{2}\right)^2+8\left(\text{D}-\frac{1}{2}\right)+3}x\cos x$$

$$=e^{-\frac{x}{2}}\frac{1}{4\left(\text{D}^2+\frac{1}{4}-\text{D}\right)+8\text{D}-4+3}x\cos x$$

$$=e^{-\frac{x}{2}}\frac{1}{(4\text{D}^2+4\text{D})}x\cos x$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left(\frac{1}{\text{D}+1}x\cos x\right)$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left[x\cdot\frac{1}{\text{D}+1}\cos x-\frac{1}{(\text{D}+1)^2}\cos x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left(x\cdot\frac{\text{D}-1}{\text{D}^2-1}\cos x-\frac{1}{\text{D}^2+2\text{D}+1}\cos x\right)$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left[x\frac{(\text{D}-1)}{(-1^2-1)}\cos x-\frac{1}{-1^2+2\text{D}+1}\cos x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left[-\frac{x}{2}(\text{D}\cos x-\cos x)-\frac{1}{2}\int\cos x\,\text{d}x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{\text{D}}\left[-\frac{x}{2}(-\sin x-\cos x)-\frac{1}{2}\sin x\right]$$

$$=\frac{e^{-\frac{x}{2}}}{8}\left[\int x(\sin x+\cos x)\text{d}x+\int\sin x\ \text{d}x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}\left[x(-\cos x + \sin x) - (-\sin x - \cos x) - \cos x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}[x(\sin x - \cos x) + \sin x]$$

Hence, the general solution is

$$y = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{3x}{2}} + \frac{e^{-\frac{x}{2}}}{8}\left[x(\sin x - \cos x) + \sin x\right]$$

Example 6

Solve $(\mathrm{D}^2 - 1)y = \sin x + e^x + x^2 e^x$.

Solution

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m = 1, -1 \qquad \text{(real and distinct)}$$

$$\mathrm{CF} = c_1 e^x + c_2 e^{-x}$$

$$\mathrm{PI} = \frac{1}{\mathrm{D}^2 - 1}(x\sin x + e^x + x^2 e^x)$$

$$= \frac{1}{\mathrm{D}^2 - 1} x\sin x + \frac{1}{\mathrm{D}^2 - 1} e^x + \frac{1}{\mathrm{D}^2 - 1} x^2 e^x$$

$$= \left[x\frac{1}{\mathrm{D}^2 - 1}\sin x - \frac{2\mathrm{D}}{(\mathrm{D}^2 - 1)^2}\sin x\right] + x\frac{1}{2\mathrm{D}}e^x + e^x\frac{1}{(\mathrm{D}+1)^2 - 1}x^2$$

$$= \left[x\frac{1}{-1^2 - 1}\sin x - \frac{2\mathrm{D}}{(-1^2 - 1)^2}\sin x\right] + \frac{x}{2(1)}e^x + e^x\frac{1}{\mathrm{D}^2 + 2\mathrm{D}}x^2$$

$$= \left[-\frac{x\sin x}{2} - \frac{2\mathrm{D}\sin x}{4}\right] + \frac{xe^x}{2} + e^x\frac{1}{2\mathrm{D}\left(1 + \frac{D}{2}\right)}x^2$$

$$= \left[-\frac{x\sin x}{2} - \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x\frac{1}{2\mathrm{D}}\left(1 + \frac{\mathrm{D}}{2}\right)^{-1} x^2$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right]\frac{xe^x}{2} + e^x\frac{1}{2\mathrm{D}}\left(1 - \frac{\mathrm{D}}{2} + \frac{\mathrm{D}^2}{4} - \frac{\mathrm{D}^3}{8} + \cdots\right)x^2$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x \frac{1}{2\mathrm{D}}\left(x^2 - \frac{2x}{2} + \frac{2}{4} - 0\right)$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x \frac{1}{2}\int\left(x^2 - x + \frac{1}{2}\right)\mathrm{d}x$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right)$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{1}{2}e^x\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{4}\right)$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{-x} - \left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{1}{2}e^x\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{4}\right)$$

EXERCISE 3.7

Solve the following differential equations:

1. $(\mathrm{D}^2 + \mathrm{D} + 2)y = e^{\frac{x}{2}}$

$$\left[\textbf{Ans.:}\ y = e^{-\frac{x}{2}}\left[c_1\cos\left(\frac{\sqrt{7}x}{2}\right) + c_2\sin\left(\frac{\sqrt{7}x}{2}\right)\right] + -\frac{4}{11}e^x + \frac{1}{4}xe^{\frac{x}{2}}\right]$$

2. $(\mathrm{D}^2 - 4)y = (1 + e^x)^2$

$$\left[\textbf{Ans.:}\ y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}\right]$$

3. $(\mathrm{D}^2 + \mathrm{D} + 1)y = e^{3x} + 6e^x - 3e^{-2x} + 5$

$$\left[\textbf{Ans.:}\ y = e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}x}{2} + c_2\sin\frac{\sqrt{3}x}{2}\right) + \frac{e^{3x}}{13} + 2e^x - e^{-2x} + 5\right]$$

4. $(\mathrm{D}^2 + 4\mathrm{D} + 5)y = -2\cosh x + 2^x$

$$\left[\textbf{Ans.:}\ y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2} + \frac{2^x}{(\log 2)^2 + 4(\log 2) + 5}\right]$$

5. $(D^3 + D^2 + D + 1)y = \sin 2x$

$$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{15}(2\cos 2x - \sin 2x)\right]$$

6. $(3D^2 - 7D + 2)y = \sin x + \cos x$

$$\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{\frac{x}{3}} + \frac{1}{25}(3\cos x - 4\sin x)\right]$$

7. $(D^3 - 2D^2 + 4D)y = e^{2x} + \sin 2x$

$$\left[\textbf{Ans.: } y = c_1 + e^{x}\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right) + \frac{1}{8}(e^{2x} + \sin 2x)\right]$$

8. $(D^3 + 2D^2 + D)y = \sin^2 x$

$$\left[\textbf{Ans.: } y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{x}{2} + \frac{1}{100}(3\sin 2x + 4\cos 2x)\right]$$

9. $(D^2 + D - 6)y = e^{2x}$

$$\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{-3x} + \frac{xe^{2x}}{5}\right]$$

10. $(9D^2 + 6D + 1)y = e^{-\frac{x}{3}}$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{-\frac{x}{3}} + \frac{x^2}{18}e^{-\frac{x}{3}}\right]$$

11. $(D^2 + 4)y = e^{x} + \sin 2x$

$$\left[\textbf{Ans.: } y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^{x}}{5} - \frac{x}{4}\cos 2x\right]$$

12. $(D^2 - 4)y = x^2$

$$\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)\right]$$

13. $(D^2 + D)y = x^2 + 2x + 4$

$$\left[\textbf{Ans.: } y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x\right]$$

14. $(D^2+1)y = e^{2x} + \cosh 2x + x^3$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x + \frac{e^{2x}}{5} + \frac{1}{5}\cosh 2x + x^3 - 6x\right]$$

15. $(D-1)^2(D+1)^2 y = \sin^2\dfrac{x}{2} + e^x + x$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x} + \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8}e^x + x\right]$$

16. $(D^2 - 3D + 2)y = 2e^x\cos\dfrac{x}{2}$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{2x} - \frac{8}{5}e^x\left(2\sin\frac{x}{2} + \cos\frac{x}{2}\right)\right]$$

17. $(D^2 - 2D + 10)y = 16e^x\cos 3x + 24e^x\sin 3x$

$$\left[\textbf{Ans.: } y = e^x(c_1\cos 3x + c_2\sin 3x) + \frac{xe^x}{3}(8\sin 3x - 12\cos 3x)\right]$$

18. $(D^3 - 4D^2 + 9D - 10)y = 24e^x\sin 2x$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + e^x(c_2\cos 2x + c_3\sin 2x) - \frac{6xe^x}{5}(2\sin 2x - \cos 2x)\right]$$

19. $(4D^3 - 12D^2 + 13D - 10)y = 16e^{\frac{x}{2}}\cos x$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + e^{\frac{x}{2}}(c_2\cos x + c_3\sin x) - \frac{4xe^{\frac{x}{2}}}{13}(2\cos x + 3\sin x)\right]$$

20. $(D^2 + 4D + 8)y = 12e^{-2x}\sin x\sin 3x$

$$\left[\textbf{Ans.: } y = e^{-2x}(c_1\cos 2x + c_2\sin 2x) + \frac{1}{2}e^{-2x}(3x\sin 2x + \cos 4x)\right]$$

21. $(4D^2 + 9D + 2)y = xe^{-2x}$

$$\left[\textbf{Ans.: } y = c_1e^{-2x} + c_2e^{-\frac{x}{4}} - \frac{1}{98}(7x^2 + 8x)e^{-2x}\right]$$

22. $(D^2+4)y = x\sin x$

$$\left[\textbf{Ans.: } y = c_1\cos 2x + c_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)\right]$$

23. $(D^2+9)y = xe^{2x}\cos x$

$$\left[\textbf{Ans.: } y = c_1\cos 3x + c_2\sin 3x + \frac{e^{2x}}{400}\left[(30x-11)\cos x + (10x-2)\sin x\right]\right]$$

24. $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$

$$\left[\textbf{Ans.: } y = (c_1+c_2x)e^{2x} - e^{2x}[4x\cos 2x + (2x^2-3)\sin 2x]\right]$$

25. $(D^2-1)y = x\sin x + (1+x^2)e^x$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2-3x+9)\right]$$

3.8 METHOD OF VARIATION OF PARAMETERS

This method is used to find the particular integral if the complementary function is known. In this method, the particular integral is obtained by varying the arbitrary constants of the complementary function and, hence, is known as variation-of-parameters method.

Consider a linear nonhomogeneous differential equation of second order with constant coefficients

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = Q(x) \qquad ...(3.26)$$

Let the complementary function be

$$\text{CF} = c_1y_1 + c_2y_2 \qquad ...(3.27)$$

where y_1, y_2 are the solution of

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0 \qquad ...(3.28)$$

Let the particular integral be

$$y = u(x)\,y_1 + v(x)\,y_2 \qquad ...(3.29)$$

where u and v are unknown functions of x.

Differentiating Eq. (3.29) w.r.t. x,

$$y' = uy_1' + vy_2' + u'y_1 + v'y_2$$

Let u, v satisfy the equation

$$u'y_1 + v'y_2 = 0 \qquad ...(3.30)$$

Then $$y' = uy_1' + vy_2'$$

Differentiating y' again w.r.t. x,

$$y'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'$$

Substituting y'', y' and y in Eq. (3.26),

$$uy_1'' + vy_2'' + u'y_1' + v'y_2' + a_1(uy_1' + vy_2') + a_2(uy_1 + vy_2) = Q(x)$$
$$u(y_1'' + a_1y_1' + a_2y_1) + v(y_2'' + a_1y_2' + a_2y_2) + u'y_1' + v'y_2' = Q(x)$$

Since y_1 and y_2 satisfy Eq. (3.28),

$$u'y_1' + v'y_2' = Q \qquad ...(3.31)$$

Solving Eqs (3.30) and (3.31) by using Cramer's rule,

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ Q & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2Q}{y_1y_2' - y_1'y_2}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & Q \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1Q}{y_1y_2' - y_1'y_2}$$

$$u = \int -\frac{y_2Q}{y_1y_2' - y_1'y_2}\,dx = \int -\frac{y_2Q}{W}\,dx \qquad ...(3.32)$$

$$v = \int \frac{y_1Q}{y_1y_2' - y_1'y_2}\,dx = \int \frac{y_1Q}{W}\,dx \qquad ...(3.33)$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is known as the Wronskian of y_1, y_2.

Hence, the required general solution of Eq. (3.26) is

$$y = \text{CF} + \text{PI}$$
$$= c_1y_1 + c_2y_2 + uy_1 + vy_2$$

where u, v are obtained using equations (3.32) and (3.33).

Note: The above method can also be extended to third-order differential equation. Let the complementary function of a third-order differential equation be

$$\text{CF} = c_1y_1 + c_2y_2 + c_3y_3$$

Let PI = $u(x)y_1 + v(x)y_2 + w(x)y_3$

where
$$u(x) = \int \frac{(y_2 y_3' - y_3 y_2')\mathrm{Q}}{W}\,\mathrm{d}x$$
$$v(x) = \int \frac{(y_3 y_1' - y_1 y_3')\mathrm{Q}}{W}\,\mathrm{d}x$$
$$w(x) = \int \frac{(y_1 y_2' - y_2 y_1')\mathrm{Q}}{W}\,\mathrm{d}x$$

Wronskian,
$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Working Rules

1. Find the complementary function as $\mathrm{CF} = c_1 y_1 + c_2 y_2$.
2. Find the Wronskian of y_1, y_2 as $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.
3. Assume the particular integral as $\mathrm{PI} = u(x)y_1 + v(x)y_2$.
4. Find u and v by evaluating the integrals $u = \int \frac{-y_2 Q}{W}\,\mathrm{d}x$, $v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$.
5. Substitute u and v in PI and write the general solution as $y = \mathrm{CF} + \mathrm{PI}$.

Example 1

Find the Wronskian of y_1, y_2 of $y'' - 2y' + y = e^x \log x$.

Solution

$$(\mathrm{D}^2 - 2\mathrm{D} + 1)\, y = e^x \log x$$

The auxiliary equation is
$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1 \quad \text{(real and repeated)}$$
$$\mathrm{CF} = (c_1 + c_2 x)e^x = c_1 e^x + c_2 x e^x$$
$$y_1 = e^x, \qquad y_2 = xe^x$$
$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$
$$= e^x(xe^x + e^x) - xe^{2x}$$
$$= xe^{2x} + e^{2x} - xe^{2x}$$
$$= e^{2x}$$

Example 2

Solve $\dfrac{d^2y}{dx^2}+y=\sin x.$ **[Winter 2017]**

Solution

$$(D^2+1)y=\sin x$$

The auxiliary equation is

$$m^2+1=0$$

$$m=\pm i \quad \text{(complex)}$$

$$\text{CF}=c_1\cos x+c_2\sin x$$

$$y_1=\cos x, \qquad y_2=\sin x$$

Wronskian $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}=\cos^2 x+\sin^2 x=1$

Let $\text{PI}=u\cos x+v\sin x$ ...(1)

where

$$u=\int -\frac{y_2 Q}{W}\,dx$$

$$=\int -\frac{\sin x\sin x}{1}\,dx$$

$$=-\int\frac{(1-\cos 2x)}{2}\,dx$$

$$=-\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)$$

and

$$v=\int\frac{y_1 Q}{W}\,dx$$

$$=\int\frac{\cos x\sin x}{1}\,dx$$

$$=\int\frac{\sin 2x}{2}\,dx$$

$$=\frac{1}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$=-\frac{1}{4}\cos 2x$$

Substituting u and v in Eq. (1),

$$\text{PI}=-\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)\cos x-\frac{1}{4}\cos 2x\sin x$$

$$=-\frac{1}{2}x+\frac{1}{4}(\sin 2x\cos x-\cos 2x\sin x)$$

$$= -\frac{1}{2}x + \frac{1}{4}\sin(2x - x)$$
$$= -\frac{1}{2}x + \frac{1}{4}\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x + \frac{1}{4}\sin x$$

Example 3

Solve $(\mathrm{D}^2 + 1)y = \operatorname{cosec} x.$

Solution

The auxiliary equation is

$$m^2 = -1$$
$$m = \pm i \quad \text{(complex)}$$
$$\mathrm{CF} = c_1 \cos x + c_2 \sin x$$
$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let $\mathrm{PI} = u\cos x + v\sin x$ ...(1)

where
$$u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$$
$$= -\int \frac{\sin x \operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= -\int \mathrm{d}x$$
$$= -x$$

and
$$v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$$
$$= \int \frac{\cos x \operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= \int \cot x\,\mathrm{d}x$$
$$= \log \sin x$$

Substituting u and v in Eq. (1),

$$\mathrm{PI} = -x\cos x + (\log \sin x)\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \cdot \log \sin x$$

Example 4

Solve $y'' + 9y = \sec 3x$. **[Summer 2015]**

Solution

$$(D^2 + 9)y = \sec 3x$$

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \qquad \text{(complex)}$$

$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$

$$y_1 = \cos 3x, \qquad y_2 = \sin 3x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$

Let $\text{PI} = u\cos 3x + v\sin 3x$ ...(1)

where $u = \int -\frac{y_2 Q}{W}\,dx$

$$= -\int \frac{\sin 3x \sec 3x}{3}\,dx$$

$$= -\frac{1}{3}\int \frac{\sin 3x}{\cos 3x}\,dx$$

$$= -\frac{1}{3}\int \tan 3x\,dx$$

$$= -\frac{1}{3}\left(-\frac{1}{3}\log\cos 3x\right)$$

$$= \frac{1}{9}\log\cos 3x$$

and $v = \int \frac{y_1 Q}{W}\,dx$

$$= \int \frac{\cos 3x \sec 3x}{3}\,dx$$

$$= \frac{1}{3}\int dx$$

$$= \frac{x}{3}$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{9}\cos 3x \log\cos 3x + \frac{x}{3}\sin 3x$$

Hence, the general solution is

$$y = c_1\cos 3x + c_2\sin 3x + \frac{1}{9}\cos 3x\log\cos 3x + \frac{x}{3}\sin 3x$$

Example 5

Solve $y'' + a^2 y = \tan ax.$ **[Summer 2016]**

Solution

$$(D^2 + a^2)y = \tan ax$$

The auxiliary equation is

$$m^2 + a^2 = 0$$

$$m = \pm ai \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos ax + c_2 \sin ax$$

$$y_1 = \cos ax, \qquad y_2 = \sin ax$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = a$

Let $\text{PI} = u \cos ax + v \sin x$ ...(1)

where

$$u = \int -\frac{y_2 Q}{W} \mathrm{d}x$$

$$= \int -\frac{\sin ax \tan ax}{a} \mathrm{d}x$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax} \mathrm{d}x$$

$$= -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax} \mathrm{d}x$$

$$= -\frac{1}{a}\int (\sec ax - \cos ax)\, \mathrm{d}x$$

$$= -\frac{1}{a}\cdot\frac{1}{a}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax$$

$$= \frac{1}{a^2}\sin ax - \frac{1}{a^2}\log(\sec ax + \tan ax)$$

and

$$v = \int \frac{y_1 Q}{W} \mathrm{d}x$$

$$= \int \frac{\cos ax \tan ax}{a} \mathrm{d}x$$

$$= \frac{1}{a}\int \sin ax\, \mathrm{d}x$$

$$= \frac{1}{a}\left(-\frac{1}{a}\cos ax\right)$$

$$= -\frac{1}{a}\cos ax$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{a^2}\sin ax \cos ax - \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax) - \frac{1}{a^2}\sin ax \cos ax$$

$$= -\frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

Hence, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log(\sec ax + \tan ax)$$

Example 6

Solve $(D^2 + 4)y = \tan 2x.$ **[Summer 2014]**

Solution

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos 2x + c_2 \sin 2x$$

$$y_1 = \cos 2x, \qquad y_2 = \sin 2x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Let $$\text{PI} = u\cos 2x + v\sin 2x \qquad \text{...(1)}$$

where
$$u = \int -\frac{y_2 Q}{W}\,dx$$

$$= \int -\frac{\sin 2x \tan 2x}{2}\,dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x)\,dx$$

$$= -\frac{1}{2}\cdot\frac{1}{2}\log(\sec 2x + \tan 2x) + \frac{1}{2}\,\frac{\sin 2x}{2}$$

$$= \frac{1}{4}[\sin 2x - \log(\sec 2x + \tan 2x)$$

and
$$v = \int \frac{y_1 Q}{W}\,dx$$

$$= \int \frac{\cos 2x \tan 2x}{2}\,dx$$

$$= \frac{1}{2}\int \sin 2x dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right)$$

$$= -\frac{1}{4}\cos 2x$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{4}\left[\sin 2x - \log(\sec 2x + \tan 2x)\right]\cos 2x - \frac{1}{4}\cos 2x \sin 2x$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Hence, the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Example 7

Solve $\dfrac{d^2y}{dx^2} + 9y = \tan 3x.$ **[Winter 2015]**

Solution

$$(D^2 + 9)y = \tan 3x$$

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \qquad \text{(complex)}$$

$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$

$$y_1 = \cos 3x, \qquad y_2 = \sin 3x$$

Wronskian $W = \begin{vmatrix} y_i & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$

Let $\text{PI} = u \cos 3x + v \sin 3x$ ...(1)

where $u = \displaystyle\int -\frac{y_2 Q}{W}\, dx$

$$= \int -\frac{\sin 3x \tan 3x}{3}\, dx$$

$$= -\frac{1}{3}\int \frac{\sin^2 3x}{\cos 3x}\, dx$$

$$= -\frac{1}{3}\int \left(\frac{1-\cos^2 3x}{\cos 3x}\right) dx$$

$$= -\frac{1}{3}\int (\sec 3x - \cos 3x)\, dx$$

$$= -\frac{1}{3}\left[\log(\sec 3x + \tan 3x)\cdot\frac{1}{3} - \frac{1}{3}\sin 3x\right]$$

$$= -\frac{1}{9}\left[\log(\sec 3x + \tan 3x)\right] + \frac{1}{9}\sin 3x$$

and
$$v = \int \frac{y_1 Q}{W}\,dx$$

$$= \int \frac{\cos 3x \tan 3x}{3}\,dx$$

$$= \frac{1}{3}\int \sin 3x\,dx$$

$$= \frac{1}{3}\left(-\frac{1}{3}\cos 3x\right)$$

$$= -\frac{1}{9}\cos 3x$$

Substituting u and v in Eq. (1)

$$\text{PI} = -\frac{1}{9}\cos 3x\left[(\log(\sec 3x + \tan 3x)\right] + \frac{1}{9}\sin 3x\cos 3x - \frac{1}{9}\cos 3x\sin 3x$$

$$= -\frac{1}{9}\cos 3x[\log(\sec 3x + \tan 3x)]$$

Hence, the general solution is

$$y = c_1\cos 3x + c_2\sin 3x - \frac{1}{9}\cos 3x\ [\log(\sec 3x + \tan 3x)]$$

Example 8

Solve $(D^2 + 4)y = \cot 2x$.

Solution

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i \qquad \text{(complex)}$$

$$\text{CF} = c_1\cos 2x + c_2\sin 2x$$

$$y_1 = \cos 2x, \qquad y_2 = \sin 2x \qquad \text{...(1)}$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Let $\text{PI} = u\cos 2x + v\sin 2x$

where $u = \int -\frac{y_2 Q}{W}\,dx$

$$= \int -\frac{\sin 2x \cot 2x}{2}\,dx$$
$$= -\frac{1}{2}\int \sin 2x\left(\frac{\cos 2x}{\sin 2x}\right)dx$$
$$= -\frac{1}{2}\int \cos 2x\,dx$$
$$= -\frac{1}{2}\frac{\sin 2x}{2}$$
$$= -\frac{1}{4}\sin 2x$$

and
$$v = \int \frac{y_1 Q}{W}\,dx$$
$$= \int \frac{\cos 2x \cot 2x}{2}\,dx$$
$$= \frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}\,dx$$
$$= \frac{1}{2}\int \frac{1-\sin^2 2x}{\sin 2x}\,dx$$
$$= \frac{1}{2}\int (\operatorname{cosec} 2x - \sin 2x)\,dx$$
$$= \frac{1}{2}\left[\frac{\log(\operatorname{cosec} 2x - \cot 2x)}{2} + \frac{\cos 2x}{2}\right]$$
$$= \frac{1}{4}\left[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = -\frac{1}{4}\sin 2x\cos 2x + \frac{1}{4}\left[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x\right]\sin 2x$$

Hence, the general solution is

$$y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}\sin 2x\cos 2x + \frac{1}{4}\left[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x\right]\sin 2x$$

Example 9

Solve $y'' - 3y' + 2y = e^x$. **[Winter 2016]**

Solution

$$(D^2 - 3D + 2)y = e^x$$

The auxiliary equation is

$$m^3 - 3m + 2 = 0$$
$$(m-2)(m-1) = 0$$
$$m = 1, 2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1\, e^x + \text{c}_2\, e^{2x}$$

$$y_1 = e^x, \qquad y_2 = e^{2x}$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^x \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

Let
$$\text{PI} = ue^x + ve^{2x} \qquad \text{...(1)}$$

where
$$u = \int -\frac{y_2\, Q}{W}\, \text{d}x$$

$$= \int -\frac{e^{2x}\, e^x}{e^{3x}}\, \text{d}x$$

$$= -\int \frac{e^{3x}}{e^{3x}}\, \text{d}x$$

$$= -\int \text{d}x$$

$$= -x$$

and
$$v = \int \frac{y_1\, Q}{W}\, \text{d}x$$

$$= \int \frac{e^x\, e^x}{e^{3x}}\, \text{d}x$$

$$= \int \frac{e^{2x}}{e^{3x}}\, \text{d}x$$

$$= \int e^{-x}\, \text{d}x$$

$$= -e^{-x}$$

Substituting u and v in Eq. (1),

$$\text{PI} = -xe^x - e^{2x}\, e^{-x}$$

$$= -xe^x - e^x$$

$$= -(x+1)e^x$$

Hence, the general solution is

$$y = c_1\, e^x + c_2 e^{2x} - (x+1)\, e^x$$

Example 10

Solve $(\text{D}^2 - 3\text{D} + 2)y = \dfrac{e^x}{1+e^x}$. **[Winter 2012]**

Solution

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-1)\,(m-2) = 0$$

$$m = 1, 2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^x + c_2 e^{2x}$$

$$y_1 = e^x, \qquad y_2 = e^{2x}$$

Wronskian $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

Let $$\text{PI} = ue^x + v\,e^{2x} \qquad \text{...(1)}$$

where $$u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$$

$$= -\int e^{2x} \cdot \frac{e^x}{1+e^x} \cdot \frac{1}{e^{3x}}\,\mathrm{d}x$$

$$= -\int \frac{1}{1+e^x}\,\mathrm{d}x$$

$$= -\int \frac{e^{-x}}{1+e^{-x}}\,\mathrm{d}x \qquad \left[\text{Multiplying and dividing by } e^{-x}\right]$$

$$= \log(1+e^{-x}) \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log f(x)\right]$$

and $$v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$$

$$= \int e^x \cdot \frac{e^x}{1+e^x} \cdot \frac{1}{e^{3x}}\,\mathrm{d}x$$

$$= \int \frac{1}{e^x(1+e^x)}\,\mathrm{d}x$$

$$= \int \frac{e^{-x}}{1+e^x}\,\mathrm{d}x$$

$$= \int \frac{e^{-x} \cdot e^{-x}}{e^{-x}+1}\,\mathrm{d}x$$

$$= \int \frac{e^{-x}(e^{-x}+1-1)}{e^{-x}+1}\,\mathrm{d}x$$

$$= \int \left(e^{-x} - \frac{e^{-x}}{1+e^{-x}}\right)\mathrm{d}x$$

$$= \int e^{-x}\mathrm{d}x - \int \frac{e-x}{1+e^{-x}}\,\mathrm{d}x$$

$$= -e^{-x} + \log(1+e^{-x}) \qquad \left[\because \int \frac{f'(x)}{f(x)}\mathrm{d}x = \log f(x)\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = \log(1+e^{-x})e^{x} + \left[-e^{-x} + \log(1+e^{-x})\right]e^{2x}$$

$$= \log(1+e^{-x})e^{x} - e^{x} + e^{2x}\log(1+e^{-x})$$

Hence, the general solution is

$$y = c_1e^{x} + c_2e^{2x} + e^{x}\log(1+e^{-x}) - e^{x} + e^{2x}\log(1+e^{-x})$$

Example 11

Solve $(D^2+1)y = x\sin x$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \qquad \text{(complex)}$$

$$\text{CF} = c_1\cos x + c_2\sin x$$

$$\text{Wronskian} \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Let $\qquad \text{PI} = u\cos x + v\sin x \qquad \text{...(1)}$

where
$$u = \int -\frac{y_2 Q}{W}\,dx$$

$$= -\int \frac{\sin x \cdot x\sin x}{1}\,dx$$

$$= -\int x\sin^2 x\,dx$$

$$= -\int x\frac{(1-\cos 2x)}{2}\,dx$$

$$= -\frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx$$

$$= -\frac{x^2}{4} + \frac{1}{2}\left[x\frac{\sin 2x}{2} - 1\frac{(-\cos 2x)}{4}\right]$$

$$= -\frac{x^2}{4} + \frac{1}{8}(2x\sin 2x + \cos 2x)$$

and
$$v = \int \frac{y_1 Q}{W}\,dx$$

$$= \int \frac{\cos x \cdot x\sin x}{1}\,dx$$

$$= \int x\frac{\sin 2x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]$$

$$= \frac{1}{8}\left(-2x\cos 2x + \sin 2x\right)$$

Substituting u and v in Eq. (1),

$$\text{PI} = \left[-\frac{x^2}{4} + \frac{1}{8}(2x\sin 2x + \cos 2x)\right]\cos x + \left[\frac{1}{8}(-2x\cos 2x + \sin 2x)\right]\sin x$$

$$= -\frac{x^2}{4}\cos x + \frac{1}{8}\left[2x(\sin 2x\cos x - \cos 2x\sin x) + (\cos 2x\cos x + \sin 2x\sin x)\right]$$

$$= -\frac{x^2}{4}\cos x + \frac{1}{8}\left[2x\sin(2x - x) + \cos(2x - x)\right]$$

$$= -\frac{x^2}{4}\cos x + \frac{1}{8}(2x\sin x + \cos x)$$

Hence, the general solution is

$$y = c_1\cos x + c_2\sin x - \frac{x^2}{4}\cos x + \frac{1}{8}(2x\sin x + \cos x)$$

Example 12

Solve $(D^2 + 1) = \text{cosec } x \cot x$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1\cos x + c_2\sin x$$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Let $\quad \text{PI} = u\cos x + v\sin x \qquad \text{...(1)}$

where

$$u = \int -\frac{y_2 Q}{W}\,dx$$

$$= \int -\frac{\sin x\ \text{cosec } x\ \cot x}{1}\,dx$$

$$= -\int \cot x\,dx$$

$$= -\log(\sin x)$$

and
$$v = \int \frac{y_1 Q}{W} \mathrm{d}x$$
$$= \int \frac{\cos x \operatorname{cosec} x \cot x}{1} \mathrm{d}x$$
$$= \int \cot^2 x \, \mathrm{d}x$$
$$= \int \left(\operatorname{cosec}^2 x - 1\right) \mathrm{d}x$$
$$= -\cot x - x$$

Substituting u and v in Eq. (1),
$$\text{PI} = -\log(\sin x)\cos x + (-\cot x - x)\sin x$$
$$= -\cos x \log(\sin x) - (\cot x + x) \sin x$$

Hence, the general solution is
$$y = c_1 \cos x + c_2 \sin x - \cos x \log (\sin x) - (\cot x + x) \sin x$$

Example 13

Solve $(\mathrm{D}^2 - 1)y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$.

Solution

The auxiliary equation is
$$m^2 - 1 = 0$$
$$m = \pm 1 \text{ (real and distinct)}$$
$$\text{CF} = c_1 e^x + c_2 e^{-x}$$
$$y_1 = e^x, \qquad y_2 = e^{-x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^0 - e^0 = -2$

Let
$$\text{PI} = ue^x + ve^{-x} \qquad \ldots(1)$$

where
$$u = \int -\frac{y_2 Q}{W} \mathrm{d}x$$
$$= -\int \frac{e^{-x}\left[e^{-x} \sin(e^{-x}) + \cos(e^{-x})\right]}{-2} \mathrm{d}x$$

Let $e^{-x} = t, \quad -e^{-x}\mathrm{d}x = \mathrm{d}t$,
$$u = -\frac{1}{2} \int (t \sin t + \cos t) \mathrm{d}t$$
$$= -\frac{1}{2}\left[t(-\cos t) - (-\sin t) + \sin t\right]$$

$$= \frac{1}{2} t \cos t - \sin t$$

$$= \frac{1}{2} e^{-x} \cos(e^{-x}) - \sin(e^{-x})$$

and
$$v = \int \frac{y_1 Q}{W}\, dx$$

$$= \int \frac{e^x \left[e^{-x} \sin(e^{-x}) + \cos(e^{-x}) \right]}{-2} dx$$

$$= \int \frac{e^x \left[\cos(e^{-x}) + e^{-x} \sin(e^{-x}) \right]}{-2} dx$$

$$= -\frac{1}{2} e^x \cos(e^{-x}) \quad \left[\begin{array}{c} \because \int e^x \{ f(x) + f'(x) \} dx = e^x f(x) \\ \text{Here } f(x) = \cos e^{-x} \end{array} \right]$$

Substituting u and v in Eq. (1) ,

$$\text{PI} = \frac{1}{2} \cos(e^{-x}) - e^x \sin(e^{-x}) - \frac{1}{2} \cos(e^{-x})$$

$$= -e^x \sin(e^{-x})$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x})$$

Example 14

Solve $(\mathrm{D}^2 + 3\mathrm{D} + 2)y = e^{e^x}$.

Solution

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m + 1)(m + 2) = 0$$

$$m = -1, -2 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_1 = e^{-x}, \qquad y_2 = e^{-2x}$$

Wronskian
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-2x} e^{-x} + e^{-2x} e^{-x} = -e^{-3x}$$

Let
$$\text{PI} = u e^{-x} + v e^{-2x} \qquad \ldots(1)$$

where
$$u = \int -\frac{y_2 Q}{W}\, dx$$

$$= -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}} \mathrm{d}x$$

$$= \int e^{e^x} e^x \, \mathrm{d}x$$

$$= e^{e^x} \qquad \left[\begin{array}{l} \because \int e^{f(x)} f'(x) \mathrm{d}x = e^{f(x)} \\ \text{Here, } f(x) = e^x \end{array}\right]$$

and $$v = \int \frac{y_1 Q}{W} \, \mathrm{d}x$$

$$= \int \frac{e^{-x} e^{e^x}}{-e^{-3x}} \mathrm{d}x$$

$$= -\int e^{2x} e^{e^x} \, \mathrm{d}x$$

$$= -\int e^x e^{e^x} \cdot e^x \, \mathrm{d}x$$

Let $e^x = t$, $\qquad e^x \mathrm{d}x = \mathrm{d}t$

$$v = -\int t e^t \mathrm{d}t$$

$$= -(te^t - e^t)$$

$$= -e^x e^{e^x} + e^{e^x}$$

Substituting u and v in Eq. (1),

$$\mathrm{PI} = e^{e^x} e^{-x} + \left(-e^x e^{e^x} + e^{e^x}\right) e^{-2x}$$

$$= e^{-2x} e^{e^x}$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$

Example 15

Solve $(\mathrm{D}^2 + 5\mathrm{D} + 6)y = e^{-2x} \sec^2 x(1 + 2\tan x)$.

Solution

The auxiliary equation is

$$m^2 + 5m + 6 = 0$$

$$(m + 2)(m + 3) = 0$$

$$m = -2, -3 \text{ (real and distinct)}$$

$$\mathrm{CF} = c_1 e^{-2x} + c_2 e^{-3x}$$

$$y_1 = e^{-2x}, \qquad y_2 = e^{-3x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -3e^{-5x} + 2e^{-5x} = -e^{-5x}$

Let $\qquad \text{PI} = ue^{-2x} + ve^{-3x} \qquad \ldots(1)$

where

$$u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$$

$$= -\int \frac{e^{-3x}e^{-2x}\sec^2 x(1+2\tan x)}{-e^{-5x}}\,\mathrm{d}x$$

$$= \int (1+2\tan x)\frac{2\sec^2 x}{2}\,\mathrm{d}x$$

[Multiplying and dividing by 2]

$$= \frac{1}{2}\cdot\frac{(1+2\tan x)^2}{2} \qquad \left[\begin{array}{l} \because \int f(x)\cdot f'(x)\,\mathrm{d}x = \dfrac{\{f(x)\}^2}{2} \\ \text{Here, } f(x) = (1+2\tan x) \end{array}\right]$$

and

$$v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$$

$$= \int \frac{e^{-2x}e^{-2x}\sec^2 x(1+2\tan x)}{-e^{-5x}}\,\mathrm{d}x$$

$$= -\int e^x \sec^2 x(1+2\tan x)\,\mathrm{d}x$$

$$= -\int e^x(\sec^2 x + 2\sec^2 x\tan x)\,\mathrm{d}x$$

$$= -e^x\sec^2 x \qquad \left[\begin{array}{l} \because \int e^x\{f(x)+f'(x)\}\mathrm{d}x = e^x f(x) \\ \text{Here } f(x) = \sec^2 x \end{array}\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{4}(1+2\tan x)^2 e^{-2x} + \left(-e^x\sec^2 x\right)e^{-3x}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{4}(1+2\tan x)^2 e^{-2x} - e^{-2x}\sec^2 x$$

Example 16

Solve $(\mathrm{D}^2 - 2\mathrm{D} + 2)y = e^x \tan x.$

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i \text{ (complex)}$$

$$\mathrm{CF} = e^x (c_1 \cos x + c_2 \sin x)$$

$$y_1 = e^x \cos x, \qquad y_2 = e^x \sin x$$

Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^x \cos x\left(e^x \sin x + e^x \cos x\right) - e^x \sin x\left(e^x \cos x - e^x \sin x\right)$$

$$= e^{2x} \cos x \sin x + e^{2x} \cos^2 x - e^{2x} \cos x \sin x + e^{2x} \sin^2 x$$

$$= e^{2x}\left(\cos^2 x + \sin^2 x\right)$$

$$= e^{2x}$$

Let

$$\mathrm{PI} = ue^x \cos x + ve^x \sin x \qquad \ldots(1)$$

where

$$u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$$

$$= \int -\frac{e^x \sin x \cdot e^x \tan x}{e^{2x}}\,\mathrm{d}x$$

$$= -\int \frac{\sin^2 x}{\cos x}\,\mathrm{d}x$$

$$= -\int \frac{1 - \cos^2 x}{\cos x}\,\mathrm{d}x$$

$$= -\int \sec x\,\mathrm{d}x + \int \cos x\,\mathrm{d}x$$

$$= -\log\left(\sec x + \tan x\right) + \sin x$$

and

$$v = \int \frac{y_1 Q}{W}\,\mathrm{d}x$$

$$= \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} \mathrm{d}x$$

$$= \int \sin x \, \mathrm{d}x$$

$$= -\cos x$$

Substituting u and v in Eq. (1),

$$\text{PI} = \left[-\log\left(\sec x + \tan x\right) + \sin x\right] \cdot e^x \cos x + (-\cos x) \cdot e^x \sin x$$

$$= -e^x \cos x \cdot \log(\sec x + \tan x)$$

Hence, the general solution is

$$y = e^x \left(c_1 \cos x + c_2 \sin x\right) - e^x \cos x \cdot \log(\sec x + \tan x)$$

Example 17

Solve $(\mathrm{D}^2 + 1)y = \dfrac{1}{1+\sin x}$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let $\qquad \text{PI} = u \cos x + v \sin x \qquad \ldots(1)$

where

$$u = \int -\frac{y_2 Q}{W} \mathrm{d}x$$

$$= \int -\frac{\sin x}{1} \cdot \frac{1}{1+\sin x} \mathrm{d}x$$

$$= -\int \frac{\sin x}{1+\sin x} \cdot \frac{(1-\sin x)}{(1-\sin x)} \mathrm{d}x$$

$$= -\int \frac{\sin x - \sin^2 x}{1-\sin^2 x} \mathrm{d}x$$

$$= -\int \frac{\sin x - \sin^2 x}{\cos^2 x} \mathrm{d}x$$

$$= -\int \left(\tan x \sec x - \tan^2 x\right) \mathrm{d}x$$

$$= -\int \left(\tan x \sec x - \sec^2 x + 1\right) \mathrm{d}x$$

$$= -\left(\sec x - \tan x + x\right)$$

and $$v = \int \frac{y_1 Q}{W}\, \mathrm{d}x$$

$$= \int \frac{\cos x}{1} \cdot \frac{1}{1+\sin x} \mathrm{d}x$$

$$= \int \frac{\cos x}{1+\sin x} \mathrm{d}x$$

$$= \log\left(1+\sin x\right) \qquad \left[\begin{array}{l} \because \int \frac{f'(x)}{f(x)} \mathrm{d}x = \log\left\{f(x)\right\} \\ \text{Here } f(x) = 1+\sin x \end{array}\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = -\left(\sec x - \tan x + x\right)\cos x + \left[\log(1+\sin x)\right]\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - \left(\sec x - \tan x + x\right)\cos x + \left[\log(1+\sin x)\right]\sin x$$

Example 18

Solve $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$. **[Summer 2017]**

Solution

$$(\mathrm{D}^2 - 4\mathrm{D} + 4)y = \frac{e^{2x}}{x}$$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x)e^{2x}$$

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = 2xe^{4x} + e^{4x} - 2xe^{4x} = e^{4x}$$

Let $$\text{PI} = ue^{2x} + vxe^{2x} \qquad \text{...(1)}$$

where
$$u = \int -\frac{y_2 Q}{W}\,\mathrm{d}x$$
$$= \int -\frac{xe^{2x}}{e^{4x}} \cdot \frac{e^{2x}}{x}\,\mathrm{d}x$$
$$= \int -\,\mathrm{d}x$$
$$= -x$$

and
$$v = \int \frac{y_1\, Q}{W}\,\mathrm{d}x$$
$$= \int \frac{e^{2x}}{e^{4x}} \cdot \frac{e^{2x}}{x}\,\mathrm{d}x$$
$$= \int \frac{1}{x}\,\mathrm{d}x$$
$$= \log x$$

Substituting u and v in Eq. (1),
$$\text{PI} = -xe^{2x} + x(\log x)e^{2x}$$
$$= xe^{2x}(\log x - 1)$$

Hence, the general solution is
$$y = (c_1 + c_2 x)\, e^{2x} + xe^{2x}(\log x - 1)$$

Example 19

Solve $(\mathrm{D}^3 + \mathrm{D})y = \operatorname{cosec} x.$

Solution

The auxiliary equation is
$$m^3 + m = 0$$
$$m(m^2 + 1) = 0$$
$$m = 0 \text{ (real)}, \quad m = \pm i \text{ (complex)}$$
$$\text{CF} = c_1 + c_2 \cos x + c_3 \sin x$$
$$y_1 = 1, \quad y_2 = \cos x, \quad y_3 = \sin x$$

Wronskian $$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= 1\left(\sin^2 x + \cos^2 x\right) - \cos x \cdot (0-0) + \sin x \cdot (0-0) = 1$$

Let $$\text{PI} = u \cdot 1 + v\cos x + w\sin x \qquad \ldots(1)$$

where
$$u = \int \frac{(y_2 y_3' - y_3 y_2')Q}{W}\,\mathrm{d}x$$
$$= \int \frac{\left[\cos x\cos x - \sin x(-\sin x)\right]\operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= \int \left(\cos^2 x + \sin^2 x\right)\operatorname{cosec} x\,\mathrm{d}x$$
$$= \int \operatorname{cosec} x\,\mathrm{d}x$$
$$= \log\left(\operatorname{cosec} x - \cot x\right)$$
$$v = \int \frac{(y_3 y_1' - y_1 y_3')Q}{W}\,\mathrm{d}x$$
$$= \int \frac{\left[\sin x \cdot 0 - 1 \cdot \cos x\right]\operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= \int \left(-\cos x\right)\operatorname{cosec} x\,\mathrm{d}x$$
$$= -\int \cot x\,\mathrm{d}x$$
$$= -\log\sin x$$
$$w = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W}\,\mathrm{d}x$$
$$= \int \frac{\left[1 \cdot \left(-\sin x\right) - \cos x \cdot 0\right]\operatorname{cosec} x}{1}\,\mathrm{d}x$$
$$= \int -\mathrm{d}x$$
$$= -x$$

Substituting u, v and w in Eq. (1),

$$\text{PI} = \log(\operatorname{cosec} x - \cot x) \cdot 1 + (-\log\sin x)\cos x + (-x)\sin x$$
$$= \log(\operatorname{cosec} x - \cot x) - \cos x\log\sin x - x\sin x$$

Hence, the general solution is

$$y = c_1 + c_2\cos x + c_3\sin x + \log(\operatorname{cosec} x - \cot x) - \cos x\log\sin x - x\sin x$$

Example 20

Solve $(\mathrm{D}^3 - 6\mathrm{D}^2 + 12\mathrm{D} - 8)y = \dfrac{e^{2x}}{x}$.

Solution

The auxiliary equation is

$$m^3 - 6m^2 + 12m - 8 = 0$$

$$(m-2)^3 = 0$$

$$m = 2,\ 2,\ 2 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 x + c_3 x^2)\, e^{2x} = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$$

$$y_1 = e^{2x},\ y_2 = xe^{2x},\ \ y_3 = x^2 e^{2x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

$$= \begin{vmatrix} e^{2x} & xe^{2x} & x^2 e^{2x} \\ 2e^{2x} & (2x+1)e^{2x} & (2x^2+2x)e^{2x} \\ 4e^{2x} & 4(x+1)e^{2x} & (4x^2+8x+2)e^{2x} \end{vmatrix}$$

$$= e^{2x}\left[(2x+1)e^{2x}\cdot(4x^2+8x+2)e^{2x} - (2x^2+2x)e^{2x}\cdot 4(x+1)e^{2x}\right]$$
$$- xe^{2x}\left[2e^{2x}\cdot(4x^2+8x+2)e^{2x} - 4e^{2x}\cdot(2x^2+2x)e^{2x}\right]$$
$$+ x^2e^{2x}\left[2e^{2x}\cdot 4(x+1)e^{2x} - 4e^{2x}\cdot(2x+1)e^{2x}\right]$$

$$= 2e^{6x}$$

Let $\quad \text{PI} = ue^{2x} + vxe^{2x} + wx^2e^{2x}$ ...(1)

where $\quad u = \int \frac{(y_2 y_3' - y_3 y_2')Q}{W}\,\mathrm{d}x$

$$= \int \frac{\left[xe^{2x}(2x^2+2x)e^{2x} - x^2e^{2x}(2x+1)e^{2x}\right]}{2e^{6x}} \cdot \frac{e^{2x}}{x}\,\mathrm{d}x$$

$$= \int \frac{x}{2}\,\mathrm{d}x$$

$$= \frac{x^2}{4}$$

$$v = \int \frac{(y_3 y_1' - y_1 y_3')Q}{W}\,\mathrm{d}x$$

$$= \int \frac{\left[x^2e^{2x}\cdot 2e^{2x} - e^{2x}\cdot(2x^2+2x)e^{2x}\right]}{2e^{6x}} \cdot \frac{e^{2x}}{x}\,\mathrm{d}x$$

$$= \int -\mathrm{d}x$$
$$= -x$$
$$w = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W} \mathrm{d}x$$
$$= \int \frac{\left[e^{2x} \cdot (2x+1)e^{2x} - xe^{2x} \cdot 2e^{2x}\right]}{2e^{6x}} \cdot \frac{e^{2x}}{x} \mathrm{d}x$$
$$= \int \frac{1}{2x} \mathrm{d}x$$
$$= \frac{1}{2} \log x$$

Substituting u, v and w in Eq. (1),

$$\text{PI} = \left(\frac{x^2}{4}\right)e^{2x} - (x)xe^{2x} + \left(\frac{1}{2}\log x\right)x^2 e^{2x}$$
$$= -\frac{3x^2}{4}e^{2x} + \frac{x^2}{2}e^{2x} \log x$$

Hence, the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{x^2}{2}e^{2x} \log x$$

EXERCISE 3.9

Solve the following differential equations:

1. $(\mathrm{D}^2 + 3\mathrm{D} + 2)y = \sin e^x$

$\left[\textbf{Ans.:}\ y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin e^x\right]$

2. $(\mathrm{D}^2 + 1)y = \operatorname{cosec} x$

$\left[\textbf{Ans.:}\ y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log(\sin x)\right]$

3. $(\mathrm{D}^2 + 4)y = \tan 2x$

$\left[\textbf{Ans.:}\ y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)\right]$

4. $(\mathrm{D}^2 + 1)y = x - \cot x$

$\left[\textbf{Ans.:}\ y = c_1 \cos x + c_2 \sin x - x\cos^2 x + x\sin^2 x - \sin x \log(\operatorname{cosec} x - \cot x)\right]$

5. $(D^2 + D)y = \dfrac{1}{1+e^x}$

$$\left[\textbf{Ans.: } y = c_1 + c_2 e^{-x} - e^{-x}[e^x \log(e^{-x}+1) + \log(e^x+1)]\right]$$

6. $(D^2 - 2D + 2)y = e^x \tan x$

$$\left[\textbf{Ans.: } y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)\right]$$

7. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{2x} - e^{2x}(2x^2 \sin 2x + 4x \cos 2x - 3\sin 2x)\right]$$

8. $(D^2 + 2D + 1)y = e^{-x} \log x$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{-x} + \frac{x^2}{2} e^{-x}\left(\log x - \frac{3}{2}\right)\right]$$

3.9 CAUCHY'S LINEAR EQUATIONS

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x) \qquad \ldots(3.34)$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called Cauchy's linear equation.

To solve Eq. (3.34),

let $x = e^z,\ 1 = e^z \dfrac{dz}{dx},\ \dfrac{dz}{dx} = \dfrac{1}{e^z} = \dfrac{1}{x}$

Now,
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} = \frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx} = \frac{dy}{dz},\ xDy = \mathcal{D}y, \quad \text{where } \mathcal{D} \equiv \frac{d}{dz} \text{ and } D = \frac{d}{dx}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2} \cdot \frac{1}{x}$$

$$x^2 \frac{d^2 y}{dx^2} = \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) \quad \text{or} \quad x^2 D^2 y = \mathcal{D}(\mathcal{D}-1)y$$

Similarly,
$$x^3 D^3 y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

....

....

$$x^n D^n y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots\left[\mathcal{D}-(n-1)\right]y$$

Substituting these derivatives in Eq. (3.34),

$$\left[a_0\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1)+a_1\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2)\right.$$
$$\left.+\ldots+a_{n-1}\mathcal{D}+a_n\right]y=\mathrm{Q}(e^z)$$

which is a linear differential equation with constant coefficients and can be solved by the usual methods described in previous sections.

Example 1

Solve $x^2y'' - 20\,y = 0.$

Solution

$$(x^2\mathrm{D}^2 - 20)\,y = 0$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)-20\right]y=0 \qquad \text{where } \mathcal{D}\equiv\frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2-\mathcal{D}-20)y=0$$

The auxiliary equation is

$$m^2 - m - 20 = 0$$
$$(m - 5)(m + 4) = 0$$
$$m = 5, -4 \quad \text{(real and distinct)}$$

Hence, the general solution is

$$y = c_1e^{5z}+c_2e^{-4z}$$
$$= c_1x^5+c_2x^{-4}$$
$$= c_1x^5+\frac{c_2}{x^4}$$

Example 2

Solve $(x^2\mathrm{D}^2 + x\mathrm{D})y = 0.$

Solution

$$(x^2\mathrm{D}^2+x\mathrm{D})y=0$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}\right]y=0 \qquad \text{where } \mathcal{D}\equiv\frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2-\mathcal{D}+\mathcal{D})y=0$$
$$\mathcal{D}^2y=0$$

The auxiliary equation is

$$m^2 = 0$$

$$m = 0, 0 \qquad \text{(real and repeated)}$$

Hence, the general solution is

$$\begin{aligned} y &= (c_1 + c_1 z)e^{0z} \\ &= c_1 + c_2 z \\ &= c_1 + c_2 \log x \end{aligned}$$

Example 3

Solve $(4x^2\mathrm{D}^2 + 16x\mathrm{D} + 9)y = 0.$

Solution

$$(4x^2\ \mathrm{D}^2 + 16x\mathrm{D} + 9)y = 0$$

Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D} - 1) + 16\mathcal{D} + 9]y = 0 \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(4\mathcal{D}^2 + 12\mathcal{D} + 9)y = 0$$

The auxiliary equation is

$$4m^2 + 12m + 9 = 0$$

$$(2m + 3)^2 = 0$$

$$m = -\frac{3}{2},\ -\frac{3}{2} \quad \text{(real and repeated)}$$

Hence, the general solution is

$$\begin{aligned} y &= (c_1 + c_2 z)e^{-\frac{3}{2}z} \\ &= (c_1 + c_2 \log x)x^{-\frac{3}{2}} \end{aligned}$$

Example 4

Solve $(x^2\mathrm{D}^2 - x\mathrm{D} + 2)y = 6.$

Solution

$$(x^2\mathrm{D}^2 - x\mathrm{D} + 2)y = 6$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D} - 1) - \mathcal{D} + 2]y = 6 \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 2)y = 6$$

The auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i \text{ (complex)}$$

$$\text{CF} = e^z (c_1 \cos z + c_2 \sin z)$$

$$= x[c_1 \cos(\log x) + c_2 \sin(\log x)]$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 2} 6e^{0z}$$

$$= \frac{1}{2} \cdot 6$$

$$= 3$$

Hence, the general solution is

$$y = x\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right] + 3$$

Example 5

Solve $x^2 y'' - xy' + y = x.$

Solution

$$(x^2\text{D}^2 - x\text{D} + 1)y = x$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 1\right]y = e^z \qquad \text{where } \mathcal{D} \equiv \frac{\text{d}}{\text{d}z}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1)y = e^z$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m - 1)^2 = 0$$

$$m = 1, 1 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^z$$

$$= (c_1 + c_2 \log x)x$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 1} e^z$$

$$= \frac{1}{(\mathcal{D}-1)^2} e^z$$

$$= z \frac{1}{2(\mathcal{D}-1)} e^z$$

$$= z^2 \frac{1}{2} e^z$$

$$= \frac{(\log x)^2 x}{2}$$

$$= \frac{x}{2} (\log x)^2$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x + \frac{x}{2}(\log x)^2$$

Example 6

Solve $(x^2D^2 - 7xD + 12)y = x^2$.

Solution

$$(x^2D^2 - 7xD + 12)y = x^2$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) - 7\mathcal{D} + 12\right]y = (e^z)^2 \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 8\mathcal{D} + 12)y = e^{2z}$$

The auxiliary equation is

$$m^2 - 8m + 12 = 0$$

$$(m-6)(m-2) = 0$$

$$m = 2, 6 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{2z} + c_2 e^{6z}$$

$$= c_1 x^2 + c_2 x^6$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 8\mathcal{D} + 12} e^{2z}$$

$$= z\frac{1}{2\mathcal{D} - 8} e^{2z}$$

$$= z\frac{1}{4-8} 2^{2z}$$

$$= -\frac{z}{4} e^{2z}$$

$$= -\frac{\log x}{4} x^2$$

Hence, the general solution is

$$y = c_1 x^2 + c_2 x^6 - \frac{x^2}{4} \log x$$

Example 7

Solve $\left(xD^2 + D - \dfrac{1}{x}\right)y = -ax^2$.

Solution

$$\left(x\mathrm{D}^2+\mathrm{D}-\frac{1}{x}\right)y=-ax^2$$

Multiplying the given equation by x,

$$(x^2\mathrm{D}^2+x\mathrm{D}-1)y=-ax^3$$

Putting $x=e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}-1\right]y=-ae^{3z} \qquad \text{where } \mathcal{D}\equiv\frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2-1)y=-ae^{3z}$$

The auxiliary equation is

$$m^2-1=0$$

$$m=\pm 1 \text{ (real and distinct)}$$

$$\begin{aligned}\mathrm{CF}&=c_1e^z+c_2e^{-z}\\&=c_1x+c_2x^{-1}\\&=c_1x+\frac{c_2}{x}\end{aligned}$$

$$\begin{aligned}\mathrm{PI}&=\frac{1}{\mathcal{D}^2-1}(-ae^{3z})\\&=-a\frac{1}{8}e^{3z}\\&=-\frac{a}{8}x^3\end{aligned}$$

Hence, the general solution is

$$y=c_1x+\frac{c_2}{x}-\frac{a}{8}x^3$$

Example 8

Solve $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y=x^2+\dfrac{1}{x^2}$.

Solution

$$(x^2\mathrm{D}^2+4x\mathrm{D}+2)y=x^2+\frac{1}{x^2}$$

Putting $x=e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)+4\mathcal{D}+2\right]y=e^{2z}+\frac{1}{e^{2z}} \qquad \text{where } \mathcal{D}\equiv\frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2+3\mathcal{D}+2)y=e^{2z}+e^{-2z}$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -2, -1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{-2z} + c_2 e^{-z}$$

$$= \frac{c_1}{x^2} + \frac{c_2}{x}$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 + 3\mathcal{D} + 2}(e^{2z} + e^{-2z})$$

$$= \frac{1}{\mathcal{D}^2 + 3\mathcal{D} + 2} e^{2z} + \frac{1}{\mathcal{D}^2 + 3\mathcal{D} + 2} e^{-2z}$$

$$= \frac{1}{4 + 3(2) + 2} e^{2z} + \frac{1}{(\mathcal{D}+2)(\mathcal{D}+1)} e^{-2z}$$

$$= \frac{e^{2z}}{12} + \frac{1}{(\mathcal{D}+2)}\left[\frac{1}{-2+1}\right] e^{-2z}$$

$$= \frac{e^{2z}}{12} - \frac{1}{(\mathcal{D}+2)} e^{-2z}$$

$$= \frac{e^{2z}}{12} - z \cdot \frac{1}{1} e^{-2z}$$

$$= \frac{x^2}{12} - (\log x)\frac{1}{x^2}$$

Hence, the general solution is

$$y = \frac{c_1}{x^2} + \frac{c_2}{x} + \frac{x^2}{12} - \frac{1}{x^2}\log x$$

Example 9

Solve $x^2 \dfrac{d^2 y}{dx^2} - x\dfrac{dy}{dx} + y = \sin(\log x)$. **[Summer 2017]**

Solution

$$(x^2 D^2 + xD + 1)y = \sin(\log x)$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 1]y = \sin z \qquad \text{where } \mathcal{D} = \frac{d}{dz}$$

$$(\mathcal{D}^2 - \mathcal{D} - \mathcal{D} + 1)\,y = \sin z$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1)y = \sin z$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m - 1)^2 = 0$$

$$m = 1, 1 \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)\, e^z$$

$$= (c_1 + c_2 \log x)x$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 1} \sin z$$

$$= \frac{1}{-1 - 2\mathcal{D} + 1} \sin z$$

$$= -\frac{1}{2\mathcal{D}} \sin z$$

$$= -\frac{1}{2} \int \sin z \, \mathrm{d}z$$

$$= -\frac{1}{2} (-\cos z)$$

$$= \frac{1}{2} \cos z$$

$$= \frac{1}{2} \cos(\log x)$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x + \frac{1}{2} \cos(\log x)$$

Example 10

Solve $(4x^2\mathrm{D}^2 + 1)y = 19\cos(\log x) + 22\sin(\log x)$.

Solution

$$(4x^2\mathrm{D}^2 + 1)y = 19\cos(\log x) + 22\sin(\log x)$$

Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D} - 1) + 1]\, y = 19\cos z + 22\sin z \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(4\mathcal{D}^2 - 4\mathcal{D} + 1)y = 19\cos z + 22\sin z$$

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2} \text{ (real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^{\frac{1}{2}z}$$

$$= (c_1 + c_2 \log x)x^{\frac{1}{2}}$$

$$\text{PI} = \frac{1}{4\mathcal{D}^2 - 4\mathcal{D} + 1}(19\cos z + 22\sin z)$$

$$= \frac{1}{4(-1^2) - 4\mathcal{D} + 1}(19\cos z + 22\sin z)$$

$$= \frac{1}{-(4\mathcal{D}+3)} \cdot \frac{(4\mathcal{D}-3)}{(4\mathcal{D}-3)}(19\cos z + 22\sin z)$$

$$= \frac{4\mathcal{D}-3}{-(16\mathcal{D}^2 - 9)}(19\cos z + 22\sin z)$$

$$= \frac{4\mathcal{D}-3}{-\left[16(-1^2) - 9\right]}(19\cos z + 22\sin z)$$

$$= \frac{1}{25}\left[4(-19\sin z + 22\cos z) - 3(19\cos z + 22\sin z)\right]$$

$$= \frac{1}{25}(31\cos z - 142\sin z)$$

$$= \frac{1}{25}\left[31\cos(\log x) - 142\sin(\log x)\right]$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x^{\frac{1}{2}} + \frac{1}{25}\left[31\cos(\log x) - 142\sin(\log x)\right]$$

Example 11

Solve $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = 12\dfrac{\log x}{x^2}$.

Solution

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=12\frac{\log x}{x^2}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=12\log x$$

$$(x^2D^2+xD)y=12\log x$$

Putting $x=e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}\right]y=12z \qquad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$(\mathcal{D}^2-\mathcal{D}+\mathcal{D})y=12z$$

$$\mathcal{D}^2y=12z$$

The auxiliary equation is

$$m^2=0$$

$$m=0,0 \qquad \text{(real and repeated)}$$

$$\begin{aligned}\text{CF}&=(c_1+c_2z)e^{0z}\\&=c_1+c_2z\\&=c_1+c_2\log x\end{aligned}$$

$$\begin{aligned}\text{PI}&=\frac{1}{\mathcal{D}^2}12z\\&=12\frac{1}{\mathcal{D}^2}z\\&=12\frac{1}{\mathcal{D}}\int z\,dz\\&=12\frac{1}{\mathcal{D}}\left[\frac{z^2}{2}\right]\\&=12\int\frac{z^2}{2}dz\\&=12\frac{z^3}{6}\\&=2z^3\\&=2(\log x)^3\end{aligned}$$

Hence, the general solution is

$$y=c_1+c_2\log x+2(\log x)^3$$

Example 12

Solve $(4x^2D^2+1)y = \log x,\ x>0.$

Solution

$$(4x^2D^2+1)y = \log x,$$

Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D}-1)+1]y = z \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2-4\mathcal{D}+1)y = z$$

The auxiliary equation is

$$4m^2-4m+1=0$$

$$(2m-1)^2=0$$

$$m=\frac{1}{2},\ \frac{1}{2} \ \text{(real and repeated)}$$

$$\text{CF} = (c_1+c_2z)e^{\frac{1}{2}z}$$

$$= (c_1+c_2\log x)x^{\frac{1}{2}}$$

$$\text{PI} = \frac{1}{4\mathcal{D}^2-4\mathcal{D}+1}z$$

$$= \frac{1}{(2\mathcal{D}-1)^2}z$$

$$= \frac{1}{(1-2\mathcal{D})^2}z$$

$$= (1-2\mathcal{D})^{-2}z$$

$$= (1+4\mathcal{D}+12\mathcal{D}^2+\ldots)z$$

$$= z+4\mathcal{D}z+6\mathcal{D}^2z+\ldots$$

$$= z+4+0$$

$$= \log x+4$$

Hence, the general solution is

$$y = (c_1+c_2\log x)\sqrt{x}+\log x+4$$

Example 13

Solve $x^2 \dfrac{dy}{dx} + 4x\dfrac{dy}{dx} + 2y = x^2 \sin(\log x).$ **[Winter 2016]**

Solution

$$(x^2D^2 + 4xD + 2)\,y = x^2 \sin(\log x)$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) + 4\mathcal{D} + 2]\,y = (e^z)^2 \sin z = e^{2z} \sin z \qquad \text{where } \frac{d}{dx} = \mathcal{D}$$

$$(\mathcal{D}^2 + 3\mathcal{D} + 2)\,y = e^{2z}$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, -2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{-z} + c_2 e^{-2z}$$

$$= c_1 x^{-1} + c_2 x^{-2}$$

$$= \frac{c_1}{x} + \frac{c_2}{x^2}$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 + 3\mathcal{D} + 2} e^{2z} \sin z$$

$$= e^{2z} \frac{1}{(\mathcal{D}+2)^2 + 3(\mathcal{D}+2) + 2} \sin z$$

$$= e^{2z} \frac{1}{\mathcal{D}^2 + 4\mathcal{D} + 4 + 3\mathcal{D} + 6 + 2} \sin z$$

$$= e^{2z} \frac{1}{\mathcal{D}^2 + 7\mathcal{D} + 12} \sin z$$

$$= e^{2z} \frac{1}{(-1) + 7\mathcal{D} + 12} \sin z$$

$$= e^{2z} \frac{1}{7\mathcal{D} + 11} \sin z$$

$$= e^{2z} \frac{7\mathcal{D} - 11}{49\mathcal{D}^2 - 121} \sin z$$

$$= e^{2z} \frac{7\mathcal{D} - 11}{-49 - 121} \sin z$$

$$= -\frac{1}{170} e^{2z} [7\mathcal{D} - 11] \sin z$$

$$= -\frac{1}{170} e^{2z} [7\cos z - 11\sin z]$$

$$= -\frac{1}{170} x^2 [7\cos(\log x) - 11\sin(\log x)]$$

Hence, the general solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} - \frac{1}{170} x^2 [7\cos(\log x) - 11\sin(\log x)]$$

Example 14

Solve $(x^2\mathrm{D}^2 + 5x\mathrm{D} + 3)y = \dfrac{\log x}{x^2}$.

Solution

$$(x^2\mathrm{D}^2 + 5x\mathrm{D} + 3)y = \frac{\log x}{x^2}$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) + 5\mathcal{D} + 3\right] y = \frac{z}{e^{2z}} \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 + 4\mathcal{D} + 3)y = e^{-2z} z$$

The auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$$(m+1)(m+3) = 0$$

$$m = -1, -3 \text{ (real and distinct)}$$

$$\text{CF} = c_1 e^{-z} + c_2 e^{-3z}$$

$$= c_1 (x)^{-1} + c_2 (x)^{-3}$$

$$= \frac{c_1}{x} + \frac{c_2}{x^3}$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 + 4\mathcal{D} + 3} e^{-2z} z$$

$$= e^{-2z} \frac{1}{(\mathcal{D}-2)^2 + 4(\mathcal{D}-2) + 3} z$$

$$= e^{-2z} \frac{1}{\mathcal{D}^2 - 1} z$$

$$= -e^{-2z} (1 - \mathcal{D}^2)^{-1} z$$

$$= -e^{-2z} (1 + \mathcal{D}^2 + \mathcal{D}^4 + \ldots) z$$

$$= -e^{-2z}(z + \mathcal{D}^2 z + \mathcal{D}^4 z + \ldots)$$
$$= -e^{-2z}(z + 0)$$
$$= -(x)^{-2}(\log x)$$
$$= -\frac{\log x}{x^2}$$

Hence, the general solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^3} - \frac{\log x}{x^2}$$

Example 15

Solve $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x \log x.$

Solution

$$(x^2D^2 + 4xD + 2)y = x \log x$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) + 4\mathcal{D} + 2\right]y = e^z \cdot z \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 + 3\mathcal{D} + 2)y = e^z$$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$
$$(m+2)(m+1) = 0$$
$$m = -1, -2 \qquad \text{(real and distinct)}$$
$$\text{CF} = c_1 e^{-z} + c_2 e^{-2z}$$
$$= c_1 x^{-1} + c_2 x^{-2}$$
$$= \frac{c_1}{x} + \frac{c_2}{x^2}$$
$$\text{PI} = \frac{1}{\mathcal{D}^2 + 3\mathcal{D} + 2} z e^z$$
$$= e^z \frac{1}{(\mathcal{D}+1)^2 + 3(\mathcal{D}+1) + 2} z$$
$$= e^z \frac{1}{\mathcal{D}^2 + 2\mathcal{D} + 1 + 3\mathcal{D} + 3 + 2} z$$
$$= e^z \frac{1}{\mathcal{D}^2 + 5\mathcal{D} + 6} z$$

$$= \frac{e^z}{6} \frac{1}{\left(1 + \frac{5\mathcal{D} + \mathcal{D}^2}{6}\right)} z$$

$$= \frac{e^z}{6} \left[1 + \frac{5\mathcal{D} + \mathcal{D}^2}{6}\right]^{-1} z$$

$$= \frac{e^z}{6} \left[1 - \frac{5\mathcal{D} + \mathcal{D}^2}{6} + \cdots\right] z$$

$$= \frac{e^z}{6} \left[z - \frac{5}{6}\mathcal{D}z - \frac{1}{6}\mathcal{D}^2 z + \cdots\right]$$

$$= \frac{e^z}{6} \left[z - \frac{5}{6}\right]$$

$$= \frac{1}{6} x \left[\log x - \frac{5}{6}\right]$$

Hence, the general solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{6} x \left[\log x - \frac{5}{6}\right]$$

Example 16

Solve $\quad x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6x \frac{\mathrm{d}y}{\mathrm{d}x} + 6y = x^{-3} \log x.$ **[Winter 2015]**

Solution

$$(x^2 \mathrm{D}^2 - 6x\,\mathrm{D} + 6)\, y = x^{-3} \log x$$

Putting $x = e^z$,

$$(\mathcal{D}^2 - 7\mathcal{D} + 6)y = z e^{-3z} \qquad \text{where} \quad \mathcal{D} = \frac{d}{\mathrm{d}z}$$

The auxiliary equation is

$$m^2 - 7m + 6 = 0$$

$$m^2 - 6m - m + 6 = 0$$

$$m = 1, 6 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^z + c_2 e^{6z}$$

$$= c_1 x + c_2 x^6$$

$$\begin{aligned}
\text{PI} &= \frac{1}{\mathcal{D}^2 - 7\mathcal{D} + 6} z\, e^{-3z} \\
&= \frac{e^{-3z}}{(\mathcal{D}-3)^2 - 7(\mathcal{D}-3) + 6} z \\
&= e^{-3z} \frac{1}{\mathcal{D}^2 - 6\mathcal{D} + 9 - 7\mathcal{D} + 21 + 6} z \\
&= e^{-3z} \frac{1}{\mathcal{D}^2 - 13\mathcal{D} + 36} z \\
&= \frac{e^{-3z}}{36}\left[1 + \left(-\frac{13\mathcal{D} + \mathcal{D}^2}{36}\right)\right]^{-1} z \\
&= \frac{e^{-3z}}{36}\left[z + \frac{13}{36}\right] \\
&= \frac{1}{36} x^{-3}\left(\log x + \frac{13}{36}\right)
\end{aligned}$$

Hence, the general solution is

$$y = c_1 x + c_2 x^6 + \frac{1}{36} x^{-3}\left(\log x + \frac{13}{36}\right)$$

Example 17

Solve $(x^2\mathrm{D}^2 - x\mathrm{D} + 1)y = \left(\frac{\log x}{x}\right)^2$.

Solution

$$(x^2\mathrm{D}^2 - x\mathrm{D} + 1)y = \left(\frac{\log x}{x}\right)^2$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 1]y = \left(\frac{z}{e^z}\right)^2 \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1)y = z^2 e^{-2z}$$

$$(\mathcal{D} - 1)^2 y = z^2 e^{-2z}$$

The auxiliary equation is

$$(m-1)^2 = 0$$

$$m = 1, 1 \qquad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^z$$

$$= (c_1 + c_2 \log x)x$$

$$\text{PI} = \frac{1}{(\mathcal{D}-1)^2}(z^2 e^{-2z})$$

$$= e^{-2z}\frac{1}{(\mathcal{D}-2-1)^2}z^2$$

$$= e^{-2z}\frac{1}{(\mathcal{D}-3)^2}z^2$$

$$= \frac{e^{-2z}}{9}\frac{1}{\left(1-\frac{\mathcal{D}}{3}\right)^2}z^2$$

$$= \frac{e^{-2z}}{9}\left(1-\frac{\mathcal{D}}{3}\right)^{-2}z^2$$

$$= \frac{e^{-2z}}{9}\left[1+\frac{2\mathcal{D}}{3}+3\frac{\mathcal{D}^2}{9}+3\frac{\mathcal{D}^3}{27}+\cdots\right]z^2$$

$$= \frac{e^{-2z}}{9}\left[z^2+\frac{2}{3}\text{D}z^2+\frac{1}{3}\text{D}^2z^2+\frac{1}{9}\text{D}^3z^2+\cdots\right]$$

$$= \frac{e^{-2z}}{9}\left[z^2+\frac{2}{3}(2z)+\frac{1}{3}(2)+0\right]$$

$$= \frac{e^{-2z}}{9}\left[z^2+\frac{4}{3}z+\frac{2}{3}\right]$$

$$= \frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)x + \frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

Example 18

Solve $\quad x^2\frac{\text{d}^2y}{\text{d}x^2}+x\frac{\text{d}y}{\text{d}x}+y=\log x\cdot\sin(\log x).$ **[Winter 2017]**

Solution

$$(x^2 D^2 - x D + 1)\, y = \log x \sin(\log x)$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D} - 1) + \mathcal{D} + 1]y = z \sin z$$
$$(\mathcal{D}^2 + 1)y = z \sin z$$

The auxiliary equation is

$$m^2 + 1 = 0$$
$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos z + c_2 \sin z$$
$$= c_1 \cos(\log x) + c_2 \sin(\log x)$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 + 1} z \sin z$$

$$= z \frac{1}{\mathcal{D}^2 + 1} \sin z - \frac{2\mathcal{D}}{(\mathcal{D}^2 + 1)^2} \sin z$$

$$= z\left(z \frac{1}{2\mathcal{D}} \sin z \right) - \frac{2}{(\mathcal{D}^2 + 1)^2} \mathcal{D} \sin z$$

$$= \frac{z^2}{2} \int \sin z \, dz - \frac{2}{(\mathcal{D}^2 + 1)^2} \cos z$$

$$= \frac{z^2}{2}(-\cos z) + \frac{2}{2(\mathcal{D}^2 + 1)2\mathcal{D}} \cos z$$

$$= -\frac{z^2}{2} \cos z + \frac{1}{2(\mathcal{D}^3 + \mathcal{D})} \cos z$$

$$= -\frac{z^2}{2} \cos z + \frac{z}{2} \frac{1}{3\mathcal{D}^2 + 1} \cos z$$

$$= -\frac{z^2}{2} \cos z + \frac{z}{2} \frac{1}{3(-1)^2 + 1} \cos z$$

$$= -\frac{z^2}{2} \cos z - \frac{z}{4} \cos z$$

$$= -\frac{(\log x)^2}{2} \cos(\log x) - \frac{\log x}{4} \cos(\log x)$$

Hence, the general solution is

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{2} \cos(\log x) - \frac{\log x}{4} \cos(\log x)$$

Example 19

Solve $(x^3D^3 + x^2D^2 - 2)y = x + \dfrac{1}{x^3}$.

Solution

$$(x^3D^3 + x^2D^2 - 2)y = x + \frac{1}{x^3}$$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + \mathcal{D}(\mathcal{D}-1) - 2\right]y = e^z + e^{-3z} \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2)y = e^z + e^{-3z}$$

The auxiliary equation is

$$m^3 - 2m^2 + m - 2 = 0$$

$$(m-2)(m^2+1) = 0$$

$$m = 2 \text{ (real)}, \ m = \pm i \text{ (complex)}$$

$$\text{CF} = c_1 e^{2z} + c_2 \cos z + c_3 \sin z$$

$$= c_1 x^2 + c_2 \cos(\log x) + c_3 \sin(\log x)$$

$$\text{PI} = \frac{1}{\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2}(e^z + e^{-3z})$$

$$= \frac{1}{1-2+1-2}e^z + \frac{1}{(-3)^3 - 2(-3)^2 - 3 - 2}e^{-3z}$$

$$= -\frac{1}{2}e^z - \frac{1}{50}e^{-3z}$$

$$= -\frac{1}{2}x - \frac{1}{50}(x)^{-3}$$

$$= -\frac{1}{2}x - \frac{1}{50x^3}$$

Hence, the general solution is

$$y = c_1 x^2 + c_2 \cos(\log x) + c_3 \sin(\log x) - \frac{1}{2}x - \frac{1}{50x^3}$$

EXERCISE 3.10

Solve the following differential equations:

1. $(x^2D^2 + xD - 1)y = 0$

$$\left[\textbf{Ans.: } y = c_1 x + \frac{c_2}{x}\right]$$

2. $(9x^2D^2 + 3xD + 10)y = 0$

$$\left[\textbf{Ans.: } y = x^{\frac{1}{3}}\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right]\right]$$

3. $(x^3D^3 - 2xD + 4)y = 0$

$$\left[\textbf{Ans.: } y = \frac{c_1}{x} + (c_2 + c_3 \log x)x^2\right]$$

4. $(x^3D^3 + 3x^2D^2 + 14xD + 34)y = 0$

$$\left[\textbf{Ans.: } \frac{c_1}{x^2} + x[c_2 \cos(4\log x) + c_3 \sin(4\log x)]\right]$$

5. $(x^2D^2 - 3xD + 4)y = x^3$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 \log x)x^2 + x^3\right]$$

6. $(x^3D^3 + 6x^2D^2 - 12)y = \dfrac{12}{x^2}$

$$\left[\textbf{Ans.: } y = c_1x^2 + \frac{c_2}{x^2} + \frac{c_3}{x^3} - \frac{3}{x^2}\log x\right]$$

7. $(4x^3D^3 + 12x^2D^2 + xD + 1)y = 50\sin(\log x)$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 \log x)x^{\frac{1}{2}} + \frac{c_3}{x} + \sin(\log x) + 7\cos(\log x)\right]$$

8. $(x^2D^2 - 3xD + 3)y = 2 + 3\log x$

$$\left[\textbf{Ans.: } y = c_1x + c_2x^3 + \log x + 2\right]$$

9. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\sin(\log x) + 1}{x}$

$$\left[\begin{aligned}&\textbf{Ans.: } y = x^2\left[c_1 \cosh\left(\sqrt{3}\log x\right) + c_2 \sinh\left(\sqrt{3}\log x\right)\right] + \frac{1}{6x}\\ &\qquad + \frac{1}{61x}[5\sin(\log x) + 6\cos(\log x)]\end{aligned}\right]$$

10. $(x^2D^2 - 3xD + 5)y = x^2 \sin(\log x)$

$$\left[\textbf{Ans.: } y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] - \frac{x^2}{2}\log x \cos(\log x)\right]$$

11. $(x^2D^3 + 3xD^2 + D)y = x^2 \log x$

$$\left[\textbf{Ans.:}\ c_1 + c_2 \log x + c_3(\log x)^2 + \frac{x^3}{27}(\log x - 1)\right]$$

12. $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x + \frac{1}{x}\right)$

$$\left[\textbf{Ans.:}\ y = \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x}\log x\right]$$

13. $(x^2D^2 - 2xD + 2)y = (\log x)^2 - \log x^2$

$$\left[\textbf{Ans.:}\ y = c_1x + c_2x^2 + \frac{1}{2}[(\log x)^2 + \log x] + \frac{1}{4}\right]$$

14. $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1-x)^2}$

$$\left[\textbf{Ans.:}\ y = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{x}\log\frac{x}{x-1}\right]$$

3.10 LEGENDRE'S LINEAR EQUATIONS

An equation of the form

$$a_0(a+bx)^n \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(a+bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots$$

$$\ldots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = Q(x) \qquad \text{...(3.35)}$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called *Legendre's linear equation.*

Let $(a+bx) = e^z$

$$b = e^z \frac{dz}{dx}, \qquad \frac{dz}{dx} = \frac{b}{e^z} = \frac{b}{a+bx}$$

Now, $\dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx} = \dfrac{dy}{dz}\cdot\dfrac{b}{(a+bx)}$

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz}$$

$$(a+bx)Dy = b\mathcal{D}y \qquad \text{where } D \equiv \frac{d}{dx} \text{ and } \mathcal{D} \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{b}{a+bx} \cdot \frac{\mathrm{d}y}{\mathrm{d}z}\right)$$

$$= -\frac{b}{(a+bx)^2} \cdot b \cdot \frac{\mathrm{d}y}{\mathrm{d}z} + \frac{b}{(a+bx)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)$$

$$= -\frac{b^2}{(a+bx)^2} \cdot \frac{\mathrm{d}y}{\mathrm{d}z} + \frac{b}{(a+bx)} \cdot \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right) \cdot \frac{\mathrm{d}z}{\mathrm{d}x}$$

$$= -\frac{b^2}{(a+bx)^2} \cdot \frac{\mathrm{d}y}{\mathrm{d}z} + \frac{b}{(a+bx)} \cdot \frac{\mathrm{d}^2 y}{\mathrm{d}z^2}\left(\frac{b}{a+bx}\right)$$

$$(a+bx)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = b^2\left(\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\right)$$

$$(a+bx)^2 \mathrm{D}^2 y = b^2(\mathcal{D}^2 - \mathcal{D})y = b^2\mathcal{D}(\mathcal{D}-1)y$$

Similarly, $$(a+bx)^3 \mathrm{D}^3 y = b^3\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

....

....

$$(a+bx)^n \mathrm{D}^n y = b^n\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (3.35),

$$\left[\left\{a_0 b^n \mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1)\right\} + \left\{a_1 b^{n-1}\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2)\right\} + \ldots + a_{n-1}\mathcal{D} + a_n\right]y$$
$$= Q\left(\frac{e^z - a}{b}\right)$$

which is a linear differential equation with constant coefficients and can be solved by the usual methods described in previous sections.

Example 1

Solve $(2x+3)^2 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (2x+3)\dfrac{\mathrm{d}y}{\mathrm{d}x} - 12y = 6x.$

Solution

$$[(2x+3)^2\mathrm{D}^2 - (2x+3)\mathrm{D} - 12]y = 6x$$

Putting $2x + 3 = e^z$,

$$\left[4\mathcal{D}(\mathcal{D}-1) - 2\mathcal{D} - 12\right]y = 6\left(\frac{e^z - 3}{2}\right) \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(4\mathcal{D}^2 - 6\mathcal{D} - 12)y = 3e^z - 9$$

The auxiliary equation is

$$4m^2-6m-12=0$$
$$2m^2-3m-6=0$$
$$m=\frac{3\pm\sqrt{57}}{4} \qquad \text{(real and distinct)}$$
$$\text{CF}=c_1e^{\left(\frac{3+\sqrt{57}}{4}\right)z}+c_2e^{\left(\frac{3-\sqrt{57}}{4}\right)z}$$
$$=c_1(2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)}+c_2(2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)}$$
$$\text{PI}=\frac{1}{4\mathcal{D}^2-6\mathcal{D}-12}(3e^z-9)$$
$$=3\frac{1}{4\mathcal{D}^2-6\mathcal{D}-12}e^z-9\frac{1}{4\mathcal{D}^2-6\mathcal{D}-12}e^{0z}$$
$$=\frac{3}{4(1)-6(1)-12}e^z-\frac{9}{-12}e^{0z}$$
$$=-\frac{3}{14}e^z+\frac{3}{4}$$
$$=-\frac{3}{14}(2x+3)+\frac{3}{4}$$

Hence, the general solution is

$$y=c_1(2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)}+c_2(2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)}-\frac{3}{14}(2x+3)+\frac{3}{4}$$

Example 2

Solve $(x+2)^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-(x+2)\dfrac{\mathrm{d}y}{\mathrm{d}x}+y=3x+4.$

Solution

$$[(x+2)^2D^2-(x+2)\,D+1]y=3x+4$$

Putting $x+2=e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)-\mathcal{D}+1\right]y=3(e^z-2)+4 \qquad \text{where } \mathcal{D}\equiv\frac{\mathrm{d}}{\mathrm{d}z}$$
$$(\mathcal{D}^2-2\mathcal{D}+1)y=3e^z-2$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^z = \left[c_1 + c_2 \log(x+2)\right](x+2)$$

$$\text{PI} = \frac{1}{(\mathcal{D}-1)^2}(3e^z - 2)$$

$$= \frac{1}{(\mathcal{D}-1)^2}3e^z - 2\frac{1}{(\mathcal{D}-1)^2}e^{0z}$$

$$= \frac{1}{(\mathcal{D}-1)^2}3e^z - 2\frac{1}{(0-1)^2}e^{0z}$$

$$= 3z\frac{1}{2(\mathcal{D}-1)}e^z - 2e^{0z}$$

$$= 3z^2\frac{1}{2}e^z - 2$$

$$= \frac{3}{2}\left[\log(x+2)\right]^2(x+2) - 2$$

Hence, the general solution is

$$y = \left[c_1 + c_2 \log(x+2)\right](x+2) + \frac{3}{2}\left[\log(x+2)\right]^2(x+2) - 2$$

Example 3

Solve $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

Solution

$$[(3x+2)^2D^2 + 3(3x+2)D - 36]\,y = 3x^2 + 4x + 1$$

Putting $3x + 2 = e^z$

$$\left[9\mathcal{D}(\mathcal{D}-1) + 3(3\mathcal{D}) - 36\right]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1 \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(9\mathcal{D}^2 - 36)y = \frac{1}{3}(e^{2z} - 4e^z + 4) + \frac{4}{3}e^z - \frac{8}{3} + 1$$

$$9(\mathcal{D}^2 - 4)y = \frac{1}{3}(e^{2z} - 1)$$

$$(\mathcal{D}^2 - 4)y = \frac{1}{27}(e^{2z} - 1)$$

The auxiliary equation is

$$m^2 - 4 = 0$$

$$m = \pm 2 \qquad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{2z} + c_2 e^{-2z}$$

$$= c_1 (3x+2)^2 + c_2 (3x+2)^{-2}$$

$$\text{PI} = \frac{1}{\mathcal{D}^2 - 4}\left[\frac{1}{27}(e^{2z} - 1)\right]$$

$$= \frac{1}{27}\left[\frac{1}{\mathcal{D}^2 - 4} e^{2z} - \frac{1}{\mathcal{D}^2 - 4} e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{(\mathcal{D}-2)(\mathcal{D}+2)} e^{2z} - \frac{1}{0-4} e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{\mathcal{D}-2} \cdot \frac{1}{2+2} e^{2z} + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[z \frac{1}{4} e^{2z} + \frac{1}{4}\right]$$

$$= \frac{1}{108}(ze^{2z} + 1)$$

$$= \frac{1}{108}\left[(3x+2)^2 \log(3x+2) + 1\right]$$

Hence, the general solution is

$$y = c_1 (3x+2)^2 + c_2 (3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2 \log(3x+2) + 1\right]$$

Example 4

Solve $[(x+1)^2 \mathrm{D}^2 + (x+1)\mathrm{D}]\, y = (2x+3)(2x+4).$

Solution

$$[(x+1)^2 \mathrm{D}^2 + (x+1)\mathrm{D}]\, y = (2x+3)(2x+4)$$

Putting $x + 1 = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) + \mathcal{D}]\, y = [2(e^z - 1) + 3][2(e^z - 1) + 4] \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$\mathcal{D}^2 y = 4e^{2z} + 6e^z + 2$$

The auxiliary equation is

$$m^2 = 0$$

$$m = 0, 0 \quad \text{(real and repeated)}$$

$$
\begin{aligned}
\text{CF} &= (c_1 + c_2 z)e^{0z} \\
&= c_1 + c_2 z \\
&= c_1 + c_2 \log(x+1) \\
\text{PI} &= \frac{1}{\mathcal{D}^2}(4e^{2z} + 6e^z + 2) \\
&= 4\frac{1}{\mathcal{D}^2}e^{2z} + 6\frac{1}{\mathcal{D}^2}e^z + 2\frac{1}{\mathcal{D}^2}e^{0\cdot z} \\
&= 4\frac{1}{2^2}e^{2z} + 6\frac{1}{1^2}e^z + 2z\frac{1}{2\mathcal{D}}e^{0\cdot z} \\
&= e^{2z} + 6e^z + 2z^2\frac{1}{2}e^{0\cdot z} \\
&= e^{2z} + 6e^z + z^2 \\
&= (x+1)^2 + 6(x+1) + [\log(x+1)]^2 \\
&= x^2 + 8x + 7 + [\log(x+1)]^2
\end{aligned}
$$

Hence, the general solution is

$$y = c_1 + c_2 \log(x+1) + x^2 + 8x + 7 + [\log(x+1)]^2$$

Aliter: After putting $x + 1 = e^z$,

$$
\begin{aligned}
\mathcal{D}^2 y &= 4e^{2z} + 6e^z + 2 \\
y &= \frac{1}{\mathcal{D}^2}(4e^{2z} + 6e^z + 2) \\
&= \int\left[\int(4e^{2z} + 6e^z + 2)\,dz\right]dz \\
&= \int(2e^{2z} + 6e^z + 2z + A)\,dz \\
&= e^{2z} + 6e^z + z^2 + Az + B \\
&= (x+1)^2 + 6(x+1) + [\log(x+1)]^2 + A\log(x+1) + B \\
&= x^2 + 8x + 7 + [\log(x+1)]^2 + A\log(x+1) + B
\end{aligned}
$$

Example 5

Solve $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin\left[\log(1+x)\right].$

Solution

$$[(1+x)^2D^2 + (1+x)D + 1]y = 2\sin[\log(1+x)]$$

Putting $1 + x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}+1\right]y = 2\sin z \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2+1)y = 2\sin z$$

The auxiliary equation is

$$m^2+1=0$$

$$m = \pm i \qquad \text{(complex)}$$

$$\begin{aligned}
\text{CF} &= c_1\cos z + c_2\sin z \\
&= c_1\cos\left[\log(1+x)\right] + c_2\sin\left[\log(1+x)\right]
\end{aligned}$$

$$\begin{aligned}
\text{PI} &= \frac{1}{\mathcal{D}^2+1}2\sin z \\
&= 2z\cdot\frac{1}{2\mathcal{D}}\sin z \\
&= z\int\sin z\,dz \\
&= z(-\cos z) \\
&= -\log(1+x)\cos\left[\log(1+x)\right]
\end{aligned}$$

Hence, the general solution is

$$y = c_1[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$$

Example 6

Solve $(x-1)^3\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2(x-1)^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4(x-1)\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4y = 4\log(x-1).$

Solution

$$(x-1)^3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2(x-1)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4(x-1)\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 4\log(x-1)$$

Putting $(x-1) = e^z$,

$$[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)+2\mathcal{D}(\mathcal{D}-1)-4\mathcal{D}+4]y = 4z \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^3-\mathcal{D}^2-4\mathcal{D}+4)y = 4z$$

The auxiliary equation is

$$m^3-m^2-4m+4=0$$

$$(m^2-4)(m-1)=0$$

$$m = \pm 2,\ 1 \quad \text{(real and distinct)}$$

$$\begin{aligned}
\text{CF} &= c_1 e^{z} + c_2 e^{2z} + c_3 e^{-2z} \\
&= c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} \\
\text{PI} &= \frac{1}{\mathcal{D}^3 - \mathcal{D}^2 - 4\mathcal{D} + 4} \cdot 4z \\
&= \frac{1}{4\left(1 - \dfrac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)} \cdot 4z \\
&= \left(1 - \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^{-1} z \\
&= \left[1 + \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4} + \left(\frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^2 + \ldots\right] z \\
&= z + \mathcal{D}(z) + (\text{Higher powers of } \mathcal{D})\, z \\
&= z + 1 + 0 \\
&= \log(x-1) + 1
\end{aligned}$$

Hence, the general solution is

$$y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x-1) + 1$$

EXERCISE 3.11

Solve the following differential equations:

1. $[(1+x)^2 D^2 + (1+x) D + 1] y = 2 \sin \log (x+1)$

$$\left[\textbf{Ans.: } y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \log(1+x)\cos\log(1+x)\right]$$

2. $[(x+2)^2 D^2 - (x+2) D + 1] y = 3x + 4$

$$\left[\textbf{Ans.: } y = [c_1 + c_2 \log(x+2)]\,(x+2) + \frac{3}{2}[\log(x+2)]^2\,(x+2) - 2\right]$$

3. $[(x-1)^3 D^3 + 2(x-1)^2 D^2 - 4(x-1) D + 4] y = 4 \log (x-1)$

$$\left[\textbf{Ans.: } y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x-1) + 1\right]$$

4. $[(x+2)^2 D^2 - (x+2) D + 1] y = 3x + 4$

$$\left[\textbf{Ans.: } y = (x+2)\left[c_1 + c_2 \log(x+2) + \frac{3}{2}\left\{\log(x+2)^2\right\}\right] - 2\right]$$

5. $[(2x+1)^2 D^2 - 2(2x+1) D - 12] y = 6x$

$$\left[\textbf{Ans.: } y = c_1(2x+1)^{-1} + c_2(2x+1)^3 - \frac{3}{8}x + \frac{1}{16}\right]$$

6. $[(x+a)^2D^2 - 4D + 6]y = x$

$$\left[\textbf{Ans.: } y = c_1(x+a)^3 + c_2(x+a)^2 + \frac{1}{6}(3x+2a)\right]$$

7. $[3x+1)^2D^2 - 3(3x+1)D - 12]y = 9x$

$$\left[\textbf{Ans.: } y = (3x+1)\left[c_1(3x+1)^{\sqrt{\frac{7}{12}}} + c_2(3x+1)^{-\sqrt{\frac{7}{12}}}\right] - 3\left[\frac{3x+1}{7} + \frac{1}{4}\right]\right]$$

8. $[(2x+5)^2\,D^2 - 6D + 8]y = 6x$

$$\left[\textbf{Ans.: } y = (2x+5)^2\left[c_1(2x+5)^{\sqrt{2}} + c_2(2x+5)^{-\sqrt{2}}\right] - \frac{3}{2}x - \frac{45}{8}\right]$$

9. $[(2+3x)^2D^2 + 5(2+3x)\,D - 3]y = x^2 + x + 1$

$$\left[\textbf{Ans.: } c_1(2+3x)^{\frac{1}{3}} + c_2(2+3x)^{-1} + \frac{1}{27}\left[\frac{1}{15}(2+3x)^2 + \frac{1}{4}(2+3x) - 7\right]\right]$$

10. $[(2x-1)^3D^3 + (2x-1)D - 2]y = 0$

$$\left[\textbf{Ans.: } y = c_1(2x-1) + (2x-1)\left[c_2(2x-1)^{\frac{\sqrt{3}}{2}} + c_3(2x-1)^{-\frac{\sqrt{3}}{2}}\right]\right]$$

3.11 METHOD OF UNDETERMINED COEFFICIENTS

This method can be used to find the particular integral only if linearly independent derivatives of Q(x) are finite in number. This restriction implies that Q(x) can only have the terms such as k, x^n, e^{ax}, sin ax, cos ax, and combinations of such terms where k, a are constants and n is a positive integer. However, when $Q(x) = \frac{1}{x}$ or tan x or sec$x$, etc., this method fails, since each function has an infinite number of linearly independent derivatives.

In this method, a particular integral is assumed as a linear combination of the terms in Q(x) and all its linearly independent derivatives. Some of the choices of the particular integral are given below.

Sr. No.	Q(x)	Particular Integral
1.	ke^{ax}	Ae^{ax}
2.	$k\sin(ax+b)$ or $k\cos(ax+b)$	$A\sin(ax+b) + B\cos(ax+b)$
3.	$ke^{ax}\sin(bx+c)$ or $ke^{ax}\cos(bx+c)$	$Ae^{ax}\sin(bx+c) + Be^{ax}\cos(bx+c)$
4.	kx^n, $n = 0, 1, 2, \ldots$	$A_nx^n + A_{n-1}x^{n-1} + \ldots + A_2x^2 + A_1x + A_0$

Sr. No.	Q(x)	Particular Integral
5.	$kx^n e^{ax}$ $n = 0, 1, 2, ...$	$e^{ax}(A_n x^n + A_{n-1} x^{n-1} + ... + A_2 x^2 + A_1 x + A_0)$
6.	$kx^n \sin(ax+b)$ or $kx^n \cos(ax+b)$	$x^n [A_n \sin(ax+b) + B_n \cos(ax+b)] + x^{n-1}$ $[A_{n-1} \sin(ax+b) + B_{n-1} \cos(ax+b)] + ...$ $+ x[A_1 \sin(ax+b) + B_1 \cos(ax+b)]$ $+ [A_0 \sin(ax+b) + B_0 \cos(ax+b)]$
7.	$kx^n e^{ax} \sin(bx+c)$ or $kx^n e^{ax} \cos(bx+c)$	$e^{ax}[x^n \{A_n \sin(ax+b) + B_n \cos(ax+b)\}$ $+ x^{n-1}\{A_{n-1} \sin(ax+b) + B_{n-1} \cos(ax+b)\}$ $+ ... + x\{A_1 \sin(ax+b) + B_1 \cos(ax+b)$ $+ \{A_0 \sin(ax+b) + B_0 \cos(ax+b)\}]$

In the table, $A_0, A_1, A_2, \ldots, A_n$ are coefficients to be determined. To obtain the values of these coefficients, we use the fact that the particular integral satisfies the given differential equation.

However, before assuming the particular integral, it is necessary to compare the terms of Q(x) with the complementary function. While comparing the terms following different cases arise.

Case I If no terms of Q(x) occur in the complementary function then particular integral is assumed from the table depending on the nature of Q(x).

Case II If a term u of Q(x) is also a term of the complementary function corresponding to an r-fold root then the assumed particular integral corresponding to u should be multiplied by x^r.

Case III If $x^s u$ is a term of Q(x) and only u is a term of the complementary function corresponding to an r-fold root then the assumed particular integral corresponding to $x^s u$ should be multiplied by x^r.

Note: In cases (ii) and (iii), initially similar types of terms appear in the complementary function and in the assumed particular integral. After multiplication by x^r, the terms of the particular integral change. Hence, this method avoids the repetition of similar terms in the complementary function and particular integral.

Example 1

Solve $y'' + 4y = 8x^2$. **[Summer 2016]**

Solution

$$(D^2 + 4)y = 8x^2$$

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm 2i \qquad \text{(complex)}$$

$$\text{CF} = c_1 \sin 2x + c_2 \cos 2x$$

$$Q = 8x^2$$

Let the particular integral be

$$y = A_1x^2 + A_2x + A_3$$
$$Dy = 2A_1x + A_2$$
$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 + 4(A_1x^2 + A_2x + A_3) = 8x^2$$
$$4A_1x^2 + 4A_2x + (2A_1 + 4A_3) = 8x^2$$

Comparing coefficients on both the sides,

$$4A_1 = 8, \qquad A_1 = 2$$
$$4A_2 = 0, \qquad A_2 = 0$$
$$2A_1 + 4A_3 = 0, \qquad A_3 = -1$$
$$\text{PI} = 2x^2 - 1$$

Hence, the general solution is

$$y = c_1 \sin 2x + c_2 \cos 2x + 2x^2 - 1$$

Example 2

Solve $y'' + 9y = 2x^2$. **[Summer 2017]**

Solution

$$(D^2 + 9)y = 2x^2$$

The auxiliary equation is

$$m^2 + 9 = 0$$
$$m = \pm 3i \text{ (complex)}$$
$$\text{CF} = c_1 \cos 3x + c_2 \sin 3x$$
$$Q = 2x^2$$

Let the particular integral be

$$y = A_1x^2 + A_2x + A_3$$
$$Dy = 2A_1x + A_2$$
$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 + 9(A_1x^2 + A_2x + A_3) = 2x^2$$
$$9A_1x^2 + 9A_2x + (2A_1 + 9A_3) = 2x^2$$

Comparing the coefficient on both the sides,

$$9A_1 = 2, \qquad A_1 = \frac{2}{9}$$
$$9A_2 = 0, \qquad A_2 = 0$$

$$2A_1 + 9A_3 = 0$$

$$2 \cdot \frac{2}{9} + 9A_3 = 0, \qquad A_3 = -\frac{4}{81}$$

$$\text{PI} = \frac{2}{9}x^2 - \frac{4}{81}$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{2}{9}x^2 - \frac{4}{81}$$

Example 3

Solve $(D^2 - 2D + 5)y = 25x^2 + 12$.

Solution

The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \quad \text{(complex)}$$

$$\text{CF} = e^x(c_1 \cos 2x + c_2 \sin 2x)$$

$$Q = 25x^2 + 12$$

Let the particular integral be

$$y = A_1x^2 + A_2x + A_3$$

$$Dy = 2A_1x + A_2$$

$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 - 2(2A_1x + A_2) + 5(A_1x^2 + A_2x + A_3) = 25x^2 + 12$$

$$5A_1x^2 + (-4A_1 + 5A_2)x + (2A_1 - 2A_2 + 5A_3) = 25x^2 + 12$$

Comparing coefficients on both the sides,

$$5A_1 = 25, \qquad A_1 = 5$$

$$-4A_1 + 5A_2 = 0, \qquad A_2 = \frac{4}{5}A_1 = 4$$

$$2A_1 - 2A_2 + 5A_3 = 12, \qquad A_3 = \frac{1}{5}(12 - 10 + 8) = 2$$

$$\text{PI} = 5x^2 + 4x + 2$$

Hence, the general solution is

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + 2$$

Example 4

Solve $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+4y=2x^2+3e^{-x}$. **[Winter 2017]**

Solution

$$(D^2+2D+4)y=2x^2+3e^{-x}$$

The auxiliary equation is

$$m^2+2m+4=0$$

$$m=-1\pm i\sqrt{3} \qquad \text{(complex)}$$

$$\text{CF}=e^{-x}\left(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x\right)$$

$$Q=2x^2+3e^{-x}$$

Let the particular integral be

$$y=A_1x^2+A_2x+A_3+A_4e^{-x}$$

$$\frac{dy}{dx}=2A_1x+A_2-A_4e^{-x}$$

$$\frac{d^2y}{dx^2}=2A_1+A_4e^{-x}$$

Substituting these derivatives in the given equation,

$$2A_1+A_4e^{-x}+2(2A_1x+A_2-A_4e^{-x})+4(A_1x^2+A_2x+A_3+A_4e^{-x})=2x^2+3e^{-x}$$

$$(3A_4)e^{-x}+(4A_1)x^2+(4A_1+4A_2)x+(2A_1+2A_2+4A_3)=2x^2+3e^{-x}$$

Comparing coefficients on both the sides,

$$3A_4=3, \qquad A_4=1$$

$$4A_1=2, \qquad A_1=\frac{1}{2}$$

$$4A_1+4A_2=0, \qquad A_2=-A_1=-\frac{1}{2}$$

$$2A_1+2A_2+4A_3=0, \qquad A_3=\frac{1}{2}(A_1+A_2)=0$$

$$\text{PI}=\frac{1}{2}x^2-\frac{1}{2}x+e^{-x}$$

Hence, the general solution is

$$y=e^{-x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)+\frac{1}{2}x^2-\frac{1}{2}x+e^{-x}$$

Example 5

Solve $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x.$ **[Summer 2018]**

Solution

$$(D^2 - 2D + 5)y = 5x^3 - 6x^2 + 6x$$

The auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \quad \text{(complex)}$$

$$\text{CF} = e^x \left(c_1 \cos 2x + c_2 \sin 2x\right)$$

$$Q = 5x^3 - 6x^2 + 6x$$

Let the particular integral be

$$y = A_1x^3 + A_2x^2 + A_3x + A_4$$

$$y' = 3A_1x^2 + 2A_2x + A_3$$

$$y'' = 6A_1x + 2A_2$$

Substituting these derivatives in the given equation,

$$(6A_1x + 2A_2) - 2(3A_1x^2 + 2A_2x + A_3) + 5(A_1x^3 + A_2x^2 + A_3x + A_4) = 5x^3 - 6x^2 + 6x$$

$$(5A_1)x^3 + (-6A_1 + 5A_2)x^2 + (6A_1 - 4A_2 + 5A_3)x + (2A_2 - 2A_3 + 5A_4)$$
$$= 5x^3 - 6x^2 + 6x$$

Comparing the coefficients on both the sides,

$$5A_1 = 5, \qquad A_1 = 1$$

$$-6A_1 + 5A_2 = -6, \qquad A_2 = \frac{1}{5}(-6 + 6A_1) = 0$$

$$6A_1 - 4A_2 + 5A_3 = 6, \qquad A_3 = \frac{1}{5}(6 - 6A_1 + 4A_2) = 0$$

$$2A_2 - 2A_3 + 5A_4 = 0, \qquad A_4 = \frac{1}{5}(-2A_2 - 2A_3) = 0$$

$$\text{PI} = x^3$$

Hence, the general solution is

$$y = e^x \left(c_1 \cos 2x + c_2 \sin 2x\right) + x^3$$

Example 6

Solve $(D^2 - 2D + 3)y = x^3 + \sin x.$

Solution

The auxiliary equation is

$$m^2 - 2m + 3 = 0$$

$$m = 1 \pm i\sqrt{2} \qquad \text{(complex)}$$

$$\text{CF} = e^x \left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right)$$

$$\text{Q} = x^3 + \sin x$$

Let the particular integral be

$$y = \text{A}_1 x^3 + \text{A}_2 x^2 + \text{A}_3 x + \text{A}_4 + \text{A}_5 \sin x + \text{A}_6 \cos x$$

$$\text{D}y = 3\text{A}_1 x^2 + 2\text{A}_2 x + \text{A}_3 + \text{A}_5 \cos x - \text{A}_6 \sin x$$

$$\text{D}^2 y = 6\text{A}_1 x + 2\text{A}_2 - \text{A}_5 \sin x - \text{A}_6 \cos x$$

Substituting these derivatives in the given equation,

$$(6\text{A}_1 x + 2\text{A}_2 - \text{A}_5 \sin x - \text{A}_6 \cos x) - 2(3\text{A}_1 x^2 + 2\text{A}_2 x + \text{A}_3 + \text{A}_5 \cos x - \text{A}_6 \sin x) + 3(\text{A}_1 x^3 + \text{A}_2 x^2 + \text{A}_3 x + \text{A}_4 + \text{A}_5 \sin x + \text{A}_6 \cos x) = x^3 + \sin x$$

$$3\text{A}_1 x^3 + (-6\text{A}_1 + 3\text{A}_2)x^2 + (6\text{A}_1 - 4\text{A}_2 + 3\text{A}_3)x + (2\text{A}_2 - 2\text{A}_3 + 3\text{A}_4) - 2(\text{A}_5 - \text{A}_6)\cos x + 2(\text{A}_5 + \text{A}_6)\sin x = x^3 + \sin x$$

Comparing coefficients on both the sides,

$$3\text{A}_1 = 1, \qquad \text{A}_1 = \frac{1}{3}$$

$$-6\text{A}_1 + 3\text{A}_2 = 0, \qquad \text{A}_2 = 2\text{A}_1 = \frac{2}{3}$$

$$6\text{A}_1 - 4\text{A}_2 + 3\text{A}_3 = 0, \qquad \text{A}_3 = \frac{1}{3}(4\text{A}_2 - 6\text{A}_1) = \frac{2}{9}$$

$$2\text{A}_2 - 2\text{A}_3 + 3\text{A}_4 = 0, \qquad \text{A}_4 = \frac{2}{3}(\text{A}_3 - \text{A}_2) = -\frac{8}{27}$$

$$2(\text{A}_5 - \text{A}_6) = 0, \qquad \text{A}_5 = \text{A}_6$$

$$2(\text{A}_5 + \text{A}_6) = 1, \qquad 2(\text{A}_5 + \text{A}_5) = 1, \qquad \text{A}_5 = \frac{1}{4}, \ \text{A}_6 = \frac{1}{4}$$

$$\text{PI} = \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Hence, the general solution is

$$y = e^x(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Example 7

Solve $(D^2-9)y = x + e^{2x} - \sin 2x$.

Solution

The auxiliary equation is

$$m^2 - 9 = 0$$

$$m = \pm 3 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{3x} + c_2 e^{-3x}$$

$$\text{Q} = x + e^{2x} - \sin 2x$$

Let the particular integral be

$$y = \text{A}_1 x + \text{A}_2 + \text{A}_3 e^{2x} + \text{A}_4 \sin 2x + \text{A}_5 \cos 2x$$

$$\text{D}y = \text{A}_1 + 2\text{A}_3 e^{2x} + 2\text{A}_4 \cos 2x - 2\text{A}_5 \sin 2x$$

$$\text{D}^2 y = 4\text{A}_3 e^{2x} - 4\text{A}_4 \sin 2x - 4\text{A}_5 \cos 2x$$

Substituting these derivatives in the given equation,

$$4\text{A}_3 e^{2x} - 4\text{A}_4 \sin 2x - 4\text{A}_5 \cos 2x - 9(\text{A}_1 x + \text{A}_2 + \text{A}_3 e^{2x} + \text{A}_4 \sin 2x + \text{A}_5 \cos 2x)$$
$$= x + e^{2x} - \sin 2x$$

$$(-5\text{A}_3)e^{2x} - 9\text{A}_1 x - 9\text{A}_2 + \sin 2x(-13\text{A}_4) + \cos 2x(-13\text{A}_5) = x + e^{2x} - \sin 2x$$

Comparing coefficients on both the sides,

$$-5\text{A}_3 = 1, \qquad \text{A}_3 = -\frac{1}{5}$$

$$-9\text{A}_1 = 1, \qquad \text{A}_1 = -\frac{1}{9}$$

$$-9\text{A}_2 = 0, \qquad \text{A}_2 = 0$$

$$-13\text{A}_4 = -1, \qquad \text{A}_4 = \frac{1}{13}$$

$$-13\text{A}_5 = 0, \qquad \text{A}_5 = 0$$

$$\text{PI} = -\frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{x}{9} - \frac{e^{2x}}{5} + \frac{\sin 2x}{13}$$

Example 8

Solve $(D^2 - 2D)y = e^x \sin x$.

Solution

The auxiliary equation is

$$m^2 - 2m = 0$$

$$m = 0,\ -2 \qquad \text{(real and distinct)}$$

$$CF = c_1 + c_2 e^{2x}$$

$$Q = e^x \sin x$$

Let the particular integral be

$$y = A_1 e^x \sin x + A_2 e^x \cos x$$

$$Dy = A_1(e^x \sin x + e^x \cos x) + A_2(e^x \cos x - e^x \sin x)$$

$$= (A_1 - A_2)e^x \sin x + (A_1 + A_2)e^x \cos x$$

$$D^2 y = (A_1 - A_2)(e^x \sin x + e^x \cos x) + (A_1 + A_2)(e^x \cos x - e^x \sin x)$$

$$= -2A_2\, e^x \sin x + 2A_1 e^x \cos x$$

Substituting these derivatives in the given equation,

$$-2A_2 e^x \sin x + 2A_1 e^x \cos x - 2(A_1 - A_2)e^x \sin x - 2(A_1 + A_2)e^x \cos x = e^x \sin x$$

$$-2A_1 e^x \sin x - 2A_2 e^x \cos x = e^x \sin x$$

Comparing coefficients on both the sides,

$$-2A_1 = 1, \qquad A_1 = -\frac{1}{2}$$

$$2A_2 = 0, \qquad A_2 = 0$$

$$PI = -\frac{1}{2} e^x \sin x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$$

Example 9

Solve $(D^3 + 3D^2 + 2D)y = x^2 + 4x + 8$.

Solution

The auxiliary equation is

$$m^3 + 3m^2 + 2m = 0$$

$$m(m+1)(m+2)=0$$

$$m=0,-1,-2 \qquad \text{(real and distinct)}$$

$$\text{CF}=c_1+c_2e^{-x}+c_3e^{-2x}$$

$$\text{Q}=x^2+4x+8$$

Let the particular integral be

$$y=\text{A}_1x^2+\text{A}_2x+\text{A}_3$$

Since the constant occurs in Q(x) and is also a part of CF corresponding to the 1-fold root $m = 0$, multiplying the assumed particular integral by x, we get

$$y=\text{A}_1x^3+\text{A}_2x^2+\text{A}_3x$$

$$\text{D}y=3\text{A}_1x^2+2\text{A}_2x+\text{A}_3$$

$$\text{D}^2y=6\text{A}_1x+2\text{A}_2$$

$$\text{D}^3y=6\text{A}_1$$

Substituting these derivatives in the given equation,

$$6\text{A}_1+3(6\text{A}_1x+2\text{A}_2)+2(3\text{A}_1x^2+2\text{A}_2x+\text{A}_3)=x^2+4x+8$$

$$6\text{A}_1x^2+(18\text{A}_1+4\text{A}_2)x+(6\text{A}_1+6\text{A}_2+2\text{A}_3)=x^2+4x+8$$

Comparing coefficients on both the sides,

$$6\text{A}_1=1, \qquad \text{A}_1=\frac{1}{6}$$

$$18\text{A}_1+4\text{A}_2=4, \qquad \text{A}_2=\frac{1}{4}(4-3)=\frac{1}{4}$$

$$6\text{A}_1+6\text{A}_2+2\text{A}_3=8, \qquad \text{A}_3=\frac{1}{2}(8-6\text{A}_1-6\text{A}_2)=\frac{1}{2}\left(8-1-\frac{3}{2}\right)=\frac{11}{4}$$

$$\text{PI}=\frac{1}{6}x^3+\frac{1}{4}x^2+\frac{11}{4}x$$

Hence, the general solution is

$$y=c_1+c_2e^{-x}+c_3e^{-2x}+\frac{x^3}{6}+\frac{x^2}{4}+\frac{11x}{4}$$

EXERCISE 3.12

Solve the following differential equations using the method of undetermined coefficients:

1. $(\text{D}^2+6\text{D}+8)y=e^{-3x}+e^x$

$$\left[\textbf{Ans.: } y=c_1e^{-2x}+c_2e^{-4x}-e^{-3x}+\frac{e^x}{15}\right]$$

2. $(4D^2 - 1)y = e^x + e^{3x}$

$$\left[\textbf{Ans.: } y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + \frac{1}{105}(35e^x + 3e^{3x})\right]$$

3. $(D^2 + D - 6)y = 39\cos 3x$

$$\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{2}(\sin 3x - 5\cos 3x)\right]$$

4. $(D^2 + 2D + 5)y = 6\sin 2x + 7\cos 2x$

$$\left[\textbf{Ans.: } y = e^{-x}(c_1\cos 2x + c_2\sin 2x) + 2\sin 2x - \cos 2x\right]$$

5. $(D^2 + 4D - 5)y = 34\cos 2x - 2\sin 2x$

$$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{-5x} + 2(\sin 2x - \cos 2x)\right]$$

6. $(D^3 - D^2 + D - 1)y = 6\cos 2x$

$$\left[\textbf{Ans.: } y = c_1 e^x + c_2\cos x + c_3\sin x + \frac{2}{5}(\cos 2x - 2\sin 2x)\right]$$

7. $(2D^2 - D - 3)y = x^3 + x + 1$

$$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{\frac{3x}{2}} - \frac{1}{27}(9x^3 - 9x^2 + 51x - 20)\right]$$

8. $(D^2 + 4)y = 8x^2$

$$\left[\textbf{Ans.: } y = c_1\cos 2x + c_2\sin 2x + 2x^2 - 1\right]$$

9. $(3D^2 + 2D - 1)y = e^{-2x} + x$

$$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{\frac{x}{3}} + \frac{1}{7}(e^{-2x} - 7x - 14)\right]$$

10. $(D^2 - 2D + 3)y = x^2 + \sin x$

$$\left[\textbf{Ans.: } y = e^x\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right) + \frac{1}{27}(9x^2 + 6x - 8) + \frac{1}{4}(\sin x + \cos x)\right]$$

11. $(D^4 - 1)y = x^4 + 1$

$$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{-x} + c_3\cos x + c_4\sin x - x^4 - 25\right]$$

12. $(D^2-1)y = e^{3x}\cos 2x - e^{2x}\sin 3x$

$$\left[\textbf{Ans.: } y = c_1e^{x} + c_2e^{-x} + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x) + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)\right]$$

13. $(D^2+3D+2)y = 12e^{-x}\sin^3 x$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{-2x} + \frac{e^{-x}}{10}[(\cos 3x + 3\sin 3x) - 45(\cos x + \sin x)]\right]$$

14. $(D^2+4D+3)y = 6e^{-x}$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{-3x} + 3xe^{-x}\right]$$

15. $(D^2-D-6)y = 5e^{-2x} + 10e^{3x}$

$$\left[\textbf{Ans.: } y = c_1e^{3x} + c_2e^{-2x} + 2xe^{3x} - xe^{-2x}\right]$$

16. $(D^2+16)y = 16\sin 4x$

$$\left[\textbf{Ans.: } y = c_1\cos 4x + c_2\sin 4x - 2x\cos 4x\right]$$

17. $(D^2+25)y = 50\cos 5x + 30\sin 5x$

$$\left[\textbf{Ans.: } y = c_1\cos 5x + c_2\sin 5x - x(3\cos 5x - 5\sin 5x)\right]$$

18. $(D^3 - 2D^2 + 4D - 8)y = 8(x^2 + \cos 2x)$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2\cos 2x + c_3\sin 2x - (x^2 + x) - \frac{x}{2}(\cos 2x + \sin 2x)\right]$$

19. $(D^2-4D+5)y = 16e^{2x}\cos x$

$$\left[\textbf{Ans.: } y = e^{2x}(c_1\cos x + c_2\sin x) + 8xe^{2x}\sin x\right]$$

20. $(D^2-6D+13)y = 6e^{3x}\sin x\cos x$

$$\left[\textbf{Ans.: } y = e^{3x}(c_1\cos 2x + c_2\sin 2x) - \frac{3x}{4}e^{3x}\cos 2x\right]$$

21. $(D^3+2D^2-D-2)y = e^{x} + x^2$

$$\left[\textbf{Ans.: } y = c_1e^{x} + c_2e^{-x} + c_3e^{-2x} + \frac{1}{6}xe^{x} - \frac{x^2}{2} + \frac{x}{2} - \frac{5}{4}\right]$$

22. $(D^2-4D+4)y = x^3e^{2x} + xe^{2x}$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{2x} + \left(\frac{x^5}{20} + \frac{x^3}{6}\right)e^{2x}\right]$$

23. $(D^2 - 3D + 2)y = xe^{2x} + \sin x$

$$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{2x} + \left(\frac{x^2}{2} - x\right)e^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x\right]$$

24. $(D^2 + 1)y = \sin^3 x$

$$\left[\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x + \frac{1}{32}\sin 3x - \frac{3}{8}x\cos x\right]$$

25. $(D^2 + 2D + 1)y = x^2 e^{-x}$

$$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{-x} + \frac{x^4}{12}e^{-x}\right]$$

26. $(D^3 - D^2 - 4D + 4)y = 2x^2 - 4x$
$-1 + 2x^2 e^{2x} + 5xe^{2x} + e^{2x}$

$$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{2x} + c_3 e^{-2x} + \frac{x^2}{2} + \frac{x^3}{6}e^{2x}\right]$$

27. $(D^2 - 5D - 6)y = e^{3x}$, $y(0) = 2$, $y'(0) = 1$

$$\left[\textbf{Ans.: } y = \frac{10}{21}e^{6x} + \frac{45}{28}e^{-x} - \frac{1}{12}e^{3x}\right]$$

28. $(D^2 - 5D + 6)y = e^x(2x - 3)$, $y(0) = 1$, $y'(0) = 3$

$$\left[\textbf{Ans.: } y = e^{2x} + xe^x\right]$$

29. $(D^3 - D)y = 4e^{-x} + 3e^{2x}$, $y(0) = 0$, $y'(0) = -1$, $y''(0) = 2$

$$\left[\textbf{Ans.: } y = c_1 + c_2 e^x + c_3 e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}\right]$$

30. $(D^3 - 2D^2 + D)y = 2e^x + 2x$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$

$$\left[\textbf{Ans.: } y = x^2 + 4x + 4 + e^x(x^2 - 4)\right]$$

3.12 APPLICATIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

3.12.1 Oscillation of a Spring

Consider a spring suspended vertically from a fixed point support (Fig. 3.1). Let a mass m attached to the lower end A of the spring stretches the spring by a length e called elongation and comes to rest at B. This position is called *static equilibrium*.

Now, the mass is set in motion from the equilibrium position. Let at any time t the mass is at P such that $BP = x$. The mass m experiences the following forces:

(i) Gravitational force mg acting downwards

(ii) Restoring force $k\,(e + x)$ due to displacement of the spring acting upwards

(iii) Damping (frictional or resistance) force $c\dfrac{\mathrm{d}x}{\mathrm{d}t}$ of the medium opposing the motion (acting upward)

(iv) External force $F(t)$ considering the downward direction as positive

By Newton's second law, the differential equation of motion of the mass m is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \mathrm{mg} - k(e+x) - c\frac{\mathrm{d}x}{\mathrm{d}t} + F(t)$$

At the equilibrium position B,

$$\mathrm{mg} = ke$$

Hence,
$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - c\frac{\mathrm{d}x}{\mathrm{d}t} + F(t)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{c}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = F(t)$$

Let $\dfrac{c}{m} = 2\lambda$ and $\dfrac{k}{m} = \omega^2$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = F(t) \qquad \text{...(3.36)}$$

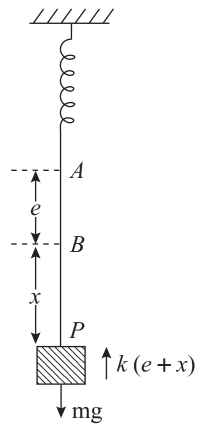

Fig. 3.1 ***Spring suspended vertically***

which represents the equation of motion and its solution gives the displacement x of the mass m at any instant t.

Let us consider the different cases of motion.

Free Oscillation If the external force $F(t)$ is absent and damping force is negligible then Eq. (3.36) reduces to

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0$$

which represents the equation of simple harmonic motion.
Hence, the motion of the mass m is SHM.

$$\text{Time period } = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$\text{Frequency} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Free Damped Oscillations If the external force $F(t)$ is absent and damping is present then Eq. (3.36) reduces to

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

Forced Undamped Oscillation If an external periodic force $F(t)$ = Q cos nt is applied to the support of the spring and damping force is negligible then Eq. (3.36) reduces to

$$\frac{d^2x}{dt^2} + \omega^2 x = Q\cos nt$$

$$(D^2 + \omega^2)x = Q\cos nt \qquad ...(3.37)$$

$$CF = c_1 \cos \omega t + c_2 \sin \omega t$$

$$PI = \frac{1}{D^2 + \omega^2} Q\cos nt$$

Hence, the general solution of Eq. (3.37) is

$$x = CF + PI$$

If the frequency of the external force $\left(\frac{n}{2\pi}\right)$ and the natural frequency $\left(\frac{\omega}{2\pi}\right)$ are same, i.e., $\omega = n$ then resonance occurs.

Forced Damped Oscillation If an external periodic force $F(t)$ = Q cos nt is applied to the support of the spring and damping force is present then Eq. (3.36) reduces to

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = Q\cos nt$$

The auxiliary equation is

$$p^2 + 2\lambda p + \omega^2 = 0 \qquad ...(3.38)$$

The general solution is

$$x = CF + PI = x_c + x_p$$

The x_c is known as the *transient term* and tends to zero as $t \to \infty$. This term represents damped oscillations. The x_p is known as the *steady-state term*. This term represents simple harmonic motion of period $\frac{2\pi}{n}$.

$$p = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

The motion of the mass depends on the nature of the roots of Eq. (3.38), i.e., on the discriminant $\lambda^2 - \omega^2$.

Case I If $\lambda^2 - \omega^2 > 0$ then the roots of Eq. (3.38) are real and distinct.

$$x_c = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right).$$

$$x_c \to 0 \text{ as } t \to \infty.$$

This shows that in this case, damping is so large that no oscillation can occur. Hence, the motion is called *overdamped* or *dead-beat motion*.

Case II If $\lambda^2 - \omega^2 = 0.$ then the roots of Eq. (3.37) are equal and real.

$$x_c = (c_1 + c_2 t)e^{-\lambda t}$$

$$x_c \to 0 \text{ as } t \to \infty.$$

In this case, damping is just enough to prevent oscillation. Hence, the motion is called *critically damped*.

Case III If $\lambda^2 - \omega^2 < 0$ then the roots of Eq. (3.38) are complex.

$$p = -\lambda \pm i\sqrt{\omega^2 - \lambda^2}$$

Hence,

$$x_c = e^{-\lambda t}\left[c_1 \cos\left(\sqrt{\omega^2 - \lambda^2}\right)t + c_2 \sin\left(\sqrt{\omega^2 - \lambda^2}\right)t\right]$$

In this case, the motion is oscillatory due to the presence of the trigonometric factor. Such a motion is called *damped oscillatory motion*.

Free Oscillation

Example 1

A body weighing 20 kg is hung from a spring. A pull of 40 kg weight will stretch the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds, the maximum velocity, and the period of oscillation.

Solution

Since a pull of 40 kg weight stretches the spring to 10 cm, i.e., 0·1 m,

$$40 = k \times 0\cdot 1$$

$$k = 400 \text{ kg/m}$$

Weight of the body, $W = 20$ kg

$$m = \frac{W}{g} = \frac{20}{9.8}$$

The equation of motion is

$$\frac{d^2x}{dt^2}+\omega^2 x=0, \text{ where } \omega^2=\frac{k}{m}=196$$

$$\frac{d^2x}{dt^2}+196x=0$$

$$(D^2+196)x=0 \qquad \text{...(1)}$$

The auxiliary equation is

$$m^2+196=0$$

$$m=\pm 14i \qquad \text{(complex)}$$

Hence, the general solution of Eq. (1) is

$$x=c_1\cos 14t+c_2\sin 14t$$

$$\frac{dx}{dt}=-14c_1\sin 14t+14c_2\cos 14t$$

At $t=0$, $x=20$ cm $=0.2$ m, $v=\dfrac{dx}{dt}=0$,

$0.2=c_1$ and $0=14c_2$, $c_2=0$

(i) Hence, displacement of the body from its equilibrium position at the time t is given by

$$x=0{\cdot}2\cos 14t$$

(ii) Amplitude = 20 cm = 0.2 m

Maximum velocity $=\omega\times\text{Amplitude}=14\times 0{\cdot}2=2{\cdot}8$ m/s

(iii) Period of oscillation $=\dfrac{2\pi}{\omega}=\dfrac{2\pi}{14}=0.45$ s

Free Damped Oscillation

Example 2

A 3 lb weight on a spring stretches it to 6 inches. Suppose a damping force λv is present ($\lambda>0$). Show that the motion is (a) critically damped if $\lambda=1.5$, (b) overdamped if $\lambda>1.5$, and (c) oscillatory if $\lambda<1.5$.

Solution

A 3 lb weight stretches the spring to 6 inches, i.e., $\frac{1}{2}ft$

$$3=k\times\frac{1}{2}$$

$$k=6\,\text{lb/ft}$$

$$\text{Weight} = 3 \text{ lb}$$

$$\text{Mass} = \frac{W}{g} = \frac{3}{32}$$

$$\text{Damping force} = \lambda v = \lambda \frac{dx}{dt} \qquad \text{where } \lambda > 0$$

The equation of motion is

$$m \frac{d^2 x}{dt^2} + kx + \lambda \frac{dx}{dt} = 0$$

$$\frac{3}{32} D^2 x + 6x + \lambda Dx = 0$$

$$\left(D^2 + \frac{32}{3}\lambda D + 64\right)x = 0 \qquad \text{where } D = \frac{d}{dt}$$

The auxiliary equation is

$$m^2 + \frac{32}{3}\lambda m + 64 = 0 \qquad ...(1)$$

$$m = \frac{-\frac{32}{3}\lambda \pm \sqrt{\left(\frac{32}{3}\lambda\right)^2 - 256}}{2}$$

$$= \frac{-32\lambda \pm \sqrt{1024\lambda^2 - 2304}}{6}$$

(a) The motion is critically damped when the roots of Eq. (1) are equal, i.e., $1024\lambda^2 - 2304 = 0$.

$$\lambda = 1{\cdot}5.$$

(b) The motion is overdamped when the roots of Eq. (1) are real and distinct, i.e., $1024\lambda^2 - 2304 > 0 \cdot$

$$\lambda > 1.5.$$

(c) The motion is oscillatory when the roots of Eq. (1) are imaginary, i.e., $1024\lambda^2 - 2304 < 0.$

$$\lambda < 1.5.$$

Forced Undamped Oscillation

Example 3

Determine whether resonance occurs in a system consisting of a 32 lb weight attached to a spring with constant k = 4 lb/ft and an external *force of* 16 sin 2*t and no damping force present. Initially,* $x=\frac{1}{2}$ *and* $\frac{dx}{dt}=-4$.

Solution

The equation of motion is

$$m\frac{d^2x}{dt^2}+kx=16\sin 2t$$

$$\frac{32}{g}\frac{d^2x}{dt^2}+4x=16\sin 2t$$

$$(D^2+4)x=16\sin 2t \qquad \left[\because g=32\text{ ft}/\sec^2\right] \qquad ...(1)$$

The auxiliary equation is

$$m^2+4=0$$

$$m=\pm 2i \qquad \text{(complex)}$$

$$\text{CF}=c_1\cos 2t+c_2\sin 2t$$

$$\text{PI}=\frac{1}{D^2+4}16\sin 2t$$

$$=16t\frac{1}{2D}\sin 2t$$

$$=8t\int\sin 2t\,dt=8t\left(-\frac{\cos 2t}{2}\right)$$

$$=-4t\cos 2t$$

Hence, the general solution of Eq. (1) is

$$x=c_1\cos 2t+c_2\sin 2t-4t\cos 2t$$

$$\frac{dx}{dt}=-2c_1\sin 2t+2c_2\cos 2t-4\cos 2t+8t\sin 2t$$

Initially, at $t=0,\quad x=\frac{1}{2}$ and $\frac{dx}{dt}=-4$

$$\frac{1}{2}=c_1$$

and $-4=2c_2-4$

$$c_2=0$$

Hence, $x=\frac{1}{2}\cos 2t-4t\cos 2t$

$$\omega^2=\frac{k}{m}=\frac{4}{1}$$

$$\omega = 2$$

Also, $n = 2$

Frequency of the external force $= \dfrac{n}{2\pi} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$ cycles / second

Natural frequency $= \dfrac{\omega}{2\pi} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$ cycles/second

Since both the frequencies are same, resonance occurs in the system.

Forced Damped Oscillations

Example 4

Determine the transient and steady-state solutions of mechanical system with 6 *lb weight,* 12 *lb/ft stiffness constant, damping force of* 1.5 *times the instantaneous velocity, external force of* 24 cos 8*t, and initial conditions* $x = \dfrac{1}{3}\,ft,\ \dfrac{dx}{dt} = 0.$

Solution

Weight = 6 lb, k = 12 lb/ft

$$m = \frac{W}{g} = \frac{6}{32} \qquad [\because g = 32 \text{ ft/s}^2]$$

Damping force $= 1.5\dfrac{dx}{dt}$

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx + 1.5\frac{dx}{dt} = 24\cos 8t$$

$$\frac{6}{32}\frac{d^2x}{dt^2} + 12x + 1.5\frac{dx}{dt} = 24\cos 8t$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 64x = 128\cos 8t$$

$$(D^2 + 8D + 64)x = 128\cos 8t \qquad \text{...(1)}$$

The auxiliary equation is

$$m^2 + 8m + 64 = 0$$

$$m = \frac{-8 \pm \sqrt{64 - 256}}{2} = -4 \pm i4\sqrt{3} \quad \text{(complex)}$$

$$\text{CF} = e^{-4t}(c_1 \cos 4\sqrt{3}t + c_2 \sin 4\sqrt{3}t) = x, \text{ say}$$

which gives the transient solution

$$\begin{aligned}
\text{PI} &= \frac{1}{D^2 + 8D + 64} 128 \cos 8t \\
&= 128 \frac{1}{-64 + 8D + 64} \cos 8t \\
&= 16 \int \cos 8t \, dt \\
&= 16 \frac{\sin 8t}{8} \\
&= 2 \sin 8t
\end{aligned}$$

which gives the steady-state solution.
Hence, the general solution of Eq. (1) is

$$x = e^{-4t}(c_1 \cos 4\sqrt{3}t + c_2 \sin 4\sqrt{3}t) + 2 \sin 8t$$

$$\begin{aligned}
\frac{dx}{dt} &= -4e^{-4t}(c_1 \cos 4\sqrt{3}t + c_2 \sin 4\sqrt{3}t) \\
&\quad + e^{-4t}(-4\sqrt{3}c_1 \sin 4\sqrt{3}t + 4\sqrt{3}\, c_2 \cos 4\sqrt{3}t) + 16 \cos 8t
\end{aligned}$$

Initially, at $t = 0$, $x = \frac{1}{3}$ and $\frac{dx}{dt} = 0$

$$\frac{1}{3} = c_1$$

and

$$0 = -4c_1 + 4\sqrt{3}\, c_2 + 16$$

$$c_2 = -\frac{11\sqrt{3}}{9}$$

Hence, transient solution is

$$\begin{aligned}
x_e &= e^{-4t}\left(\frac{1}{3} \cos 4\sqrt{3}t - \frac{11\sqrt{3}}{9} \sin 4\sqrt{3}t\right) \\
&= \frac{e^{-4t}}{9}(3 \cos 4\sqrt{3}t - 11\sqrt{3} \sin 4\sqrt{3}\, t)
\end{aligned}$$

and steady-state solution is

$$x_p = 2 \sin 8t$$

EXERCISE 3.13

1. A body weighing 4·9 kg is hung from a spring. A pull of 10 kg will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds, the maximum velocity and the period of oscillation.

$$\left[\textbf{Ans.:}\, 0.06 \cos 20t, 1.2 \text{ m/s}, 0.314 \text{ s}\right]$$

2. A mass of 200 g is tied at the end of a spring which extends to 4 cm under a force of 196, 000 dynes. The spring is pulled 5 cm and released. Find the displacement t seconds after release, if there be a damping force of 2000 dynes per cm per second. What should be the damping force for the dead-beat motion?

$$\left[\textbf{Ans.:}\, e^{-5t}\left(5\cos\sqrt{220}t + \frac{25}{\sqrt{220}}\sin\sqrt{220}t\right), 6261\right]$$

3. A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of W lb at the other. It is found that resonance occurs when an axial periodic force of 2 cos $2t$ lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by $x = ct \sin 2t$, and find the values of W and c.

$$\left[\textbf{Ans.:}\, W = 6g, c = \frac{1}{12}\right]$$

4. Find the steady-state and transient oscillations of the mechanical system corresponding to the differential equation $\ddot{x} + 2\dot{x} + 2x = \sin 2t - 2\cos 2t$, $x(0) = \dot{x}(0) = 0$.

$$[\textbf{Ans.:}\, -0.5 \sin 2t,\; e^{-t} \sin t]$$

5. If weight $W = 16$ lb, spring constant $k = 10$ lb/ft, damping force $= 2\dfrac{d\dot{x}}{dt}$, external force $F(t)$ is 5 cos $2t$, find the motion of the weight given $x(0) = \dot{x}(0) = 0$. Write the transient and steady-state solutions.

$$\left[\begin{array}{l}\textbf{Ans.:}\, x(t) = e^{-2t}\left(-\frac{3}{8}\sin 4t - \frac{1}{2}\cos 4t\right) + \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t \\ \text{Transient solution}: \frac{5e^{-2t}}{8}\cos(4t - 0.64) \\ \text{Steady-state}: \frac{\sqrt{5}}{4}\cos(2t - 0.46)\end{array}\right]$$

3.12.2 Electrical Circuits

A second-order electrical circuit consists of a resistor, an inductor, and a capacitor in series with an emf $e(t)$ as shown in the Fig. 3.2.
Applying Kirchhoff's voltage law to the circuit,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e(t) \qquad ...(3.39)$$

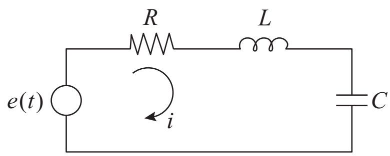

Fig. 3.2 ***Second-order electrical circuit***

But

$$i = \frac{dq}{dt}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t) \qquad ...(3.40)$$

Differentiating Eq. (3.39) w.r.t. t

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{de(t)}{dt} \qquad ...(3.41)$$

Equations (3.40) and (3.41) are both second-order linear nonhomogeneous ordinary differential equations.

Example 1

A circuit consists of an inductance of 2 henrys, a resistance of 4 ohms and capacitance of 0.05 farads. If $q = i = 0$ at $t = 0$, (a) find $q(t)$ and $i(t)$ when there is a constant emf of 100 volts. (b) Find the steady-state solutions.

Solution

(a) The differential equation of the *RLC* circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = e(t)$$

$$2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + \frac{q}{0.05} = 100$$

$$\frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 10q = 50$$

$$(D^2 + 2D + 10)q = 50$$

The auxiliary equation is

$$m^2 + 2m + 10 = 0$$

$$m = -1 \pm 3i \quad \text{(complex)}$$

$$\text{CF} = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$\begin{aligned}\text{PI} &= \frac{1}{\text{D}^2 + 2\text{D} + 10}(50e^{0t}) \\ &= \frac{1}{10} \cdot 50 \\ &= 5\end{aligned}$$

The general solution is

$$q = e^{-t}(c_1 \cos 3t + c_2 \sin 3t) + 5 \qquad \text{...(1)}$$

At $t = 0, \quad q = 0$

$$0 = c_1 + 5$$

$$c_1 = -5$$

Differentiating Eq. (1) w.r.t. t,

$$i = \frac{\text{d}q}{\text{d}t} = e^{-t}(-3c_1 \sin 3t + 3c_2 \cos 3t) - e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

At $t = 0, i = 0$

$$0 = 3c_2 - c_1$$

$$3c_2 = c_1$$

$$c_2 = -\frac{5}{3}$$

Hence, $q(t) = 5 + e^{-t}\left(-5\cos 3t - \frac{5}{3}\sin 3t\right)$

$$= 5 - \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$$

and $i(t) = e^{-t}(15\sin 3t - 5\cos 3t) + e^{-t}\left(5\cos 3t + \frac{5}{3}\sin 3t\right)$

$$= -\frac{5}{3}e^{-t}(3\cos 3t - 9\sin 3t) + \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$$

(b) The steady-state solution is obtained by putting $t = \infty$.

$$q(t) = 5$$

$$i(t) = 0$$

Example 2

(a) Determine q and i in an RLC circuit with $L = 0.5$ H, $R = 6\ \Omega$, $C = 0.02$ F, $e = 24 \sin 10t$ and initial conditions $q = i = 0$ at $t = 0$. (b) Find steady-state and transient solutions.

Solution

The differential equation of the *RLC* circuit is

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2}+R\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{q}{C}=e$$

$$0.5\frac{\mathrm{d}^2q}{\mathrm{d}t^2}+6\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{q}{0.02}=24\sin 10t$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2}+12\frac{\mathrm{d}q}{\mathrm{d}t}+100q=48\sin 10t$$

$$(\mathrm{D}^2+12\mathrm{D}+100)q=48\sin 10t$$

The auxiliary solution is

$$m^2+12m+100=0$$

$$m=-6\pm 8i \quad \text{(complex)}$$

$$\mathrm{CF}=e^{-6t}(c_1\cos 8t+c_2\sin 8t)$$

$$\mathrm{PI}=\frac{1}{\mathrm{D}^2+12\mathrm{D}+100}48\sin 10t$$

$$=48\cdot\frac{1}{-10^2+12\mathrm{D}+100}\sin 10t$$

$$=\frac{48}{12}\int\sin 10t\,\mathrm{d}t$$

$$=4\left(-\frac{\cos 10t}{10}\right)$$

$$=-\frac{2}{5}\cos 10t$$

The general solution is

$$q=e^{-6t}(c_1\cos 8t+c_2\sin 8t)-\frac{2}{5}\cos 10t \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. t

$$i=\frac{\mathrm{d}q}{\mathrm{d}t}=-6e^{-6t}(c_1\cos 8t+c_2\sin 8t)+e^{-6t}(-8c_1\sin 8t+8c_2\cos 8t)+\frac{2}{5}\cdot 10\sin 10t$$

$$=e^{-6t}[(-6c_1+8c_2)\cos 8t-(6c_2+8c_1)\sin 8t]+4\sin 10t$$

At $t=0$, $q=0$, $i=0$

$$0=c_1-\frac{2}{5}$$

$$c_1=\frac{2}{5}$$

and $$0 = -6c_1 + 8c_2$$

$$c_2 = \frac{6c_1}{8} = \frac{3}{10}$$

Hence, $$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right) - \frac{2}{5}\cos 10t$$

and $$i(t) = e^{-6t}(-5\sin 8t) + 4\sin 10t$$

The steady-state solution is obtained by putting $t = \infty$.

$$q(t) = -\frac{2}{5}\cos 10t$$

$$i(t) = 4\sin 10t$$

The transient solution

$$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right)$$

$$i(t) = e^{-6t}(-5\sin 8t)$$

EXERCISE 3.14

1. A circuit consists of a resistance of 5 ohms, an inductance of 0.05 henrys and capacitance of 4×10^{-4} farads. If $q(0) = 0$, $i(0) = 0$, find $q(t)$ and $i(t)$ when an emf of 110 volts is applied.

$$\left[\begin{array}{l}\textbf{Ans.:}\ q(t) = e^{-50t}\left(-\dfrac{11}{250}\cos 50\sqrt{19}t - \dfrac{11\sqrt{19}}{4750}\sin 50\sqrt{19}t\right) + \dfrac{11}{250}, \\ \qquad i(t) = \dfrac{44}{\sqrt{19}}e^{-50t}\sin 50\sqrt{19}t\end{array}\right]$$

2. Determine the charge on the capacitor at any time t in the series circuit having a resistor of 2 Ω, inductor of 0.1 H, capacitor of $\frac{1}{260}$F and $e(t) = 100\sin 60t$. If the initial current and initial charge on the capacitor are both zero, find the steady-state solution.

$$\left[\begin{array}{l}\textbf{Ans.:}\ q(t) = \dfrac{6e^{-10t}}{61}(6\sin 50t + 5\cos 50t) - \dfrac{5}{\sqrt{61}}(5\sin 60t + 6\cos 60t), \\ \text{steady-state solution: } q(t) = -\dfrac{5}{61}(5\sin 60t + 6\cos 60t)\end{array}\right]$$

Points to remember

First-Order Differential Equation

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
1.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Exact, i.e., $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$	–	(i) $\int M(x, y)\,dx + \int (\text{terms of N not containing } x)\,dy = c$ (ii) $\int (\text{terms of M not containing } y)\,dx + \int N(x, y)\,dy = c$
2.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-exact and $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$	$IF = e^{\int f(x)\,dx}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
3.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-exact and $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$	$IF = e^{\int f(y)\,dy}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
4.	$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0,$	Non-exact	$IF = \dfrac{1}{Mx - Ny}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
5.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-exact and homogeneous	$IF = \dfrac{1}{Mx + Ny}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
6.	$x^{m_1} y^{n_1} (a_1 y\,dx + b_1 x\,dy)$ $+ x^{m_2} y^{n_2} (a_2 y\,dx + b_2 x\,dy) = 0$	Non-exact	$IF = x^h y^k$ where $\frac{m_1+h+1}{a_1} = \frac{n_1+k+1}{b_1}$ and $\frac{m_2+h+1}{a_2} = \frac{n_2+k+1}{b_2}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
7.	$\frac{dy}{dx} + Py = Q$, where P and Q are functions of x	Linear in y	$IF = e^{\int P\,dx}$	$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$
8.	$\frac{dy}{dx} + Py = Qy^n$	Nonlinear	$IF = e^{\int P_1\,dx}$ where $P_1 = (1-n)v$ and $v = y^{1-n}$	$ve^{\int P_1\,dx} = \int Q_1 e^{\int P_1\,dx}\,dx + c$ where $Q_1 = (1-n)Q$
9.	$f'(y)\frac{dy}{dx} + Pf(y) = Q$	Nonlinear	$IF = e^{\int P\,dx}$	$ve^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$ where $f(y) = v$

Note: In the cases 1 to 6 after multiplication by IF, differential equation reduces to $M_1(x, y)\,dx + N_1(x, y)\,dy = 0$

Higher Order Differential Equations

Homogeneous Linear Differential Equations with constant coefficients

Sr. No.	Roots	Complementary Function (CF)
1.	Real and distinct roots $(m_1, m_2, \dots m_n)$	$y = c_1e^{m_1x} + c_2e^{m_2x} + c_2e^{m_3x} + \dots + c_ne^{m_nx}$
2.	Real and repeated roots $(m_1 = m_2)$	$y = (c_1 + c_2x)e^{m_1x} + c_3e^{m_3x} + c_4e^{m_4x} + \dots + c_ne^{m_nx}$
3.	Complex roots $(m_1 = \alpha + i\beta, m_2 = \alpha - i\beta)$	$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + c_4e^{m_4x} + \dots + c_ne^{m_nx}$
4.	Complex and repeated roots $(m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta)$	$y = e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x] + c_5e^{m_5x} + c_6e^{m_6x} + \dots + c_ne^{m_nx}$

Nonhomogeneous Linear Differential Equations with constant coefficients

Sr. No.	Q(x)	Particular Integral (PI)
1.	e^{ax+b}	(i) $\dfrac{1}{f(a)}e^{ax+b}$ if $f(a) \neq 0$ (ii) $x^r\dfrac{1}{f^{(r)}(a)}e^{ax+b}$ if $f^{(r-1)}(a) = 0$ and $f^{(r)}(a) \neq 0$
2.	$\sin(ax+b)$ or $\cos(ax+b)$	(i) $\dfrac{1}{\phi(-a^2)}\sin(ax+b)$ or $\dfrac{1}{\phi(-a^2)}\cos(ax+b)$ if $\phi(-a^2) \neq 0$ (ii) $x^r\dfrac{1}{\phi^{(r)}(-a^2)}\cos(ax+b)$, if $\phi^{(r-1)}(-a^2) = 0$ and $\phi^{(r)}(-a^2) \neq 0$
3.	x^m	$[f(D)]^{-1}x^m = [1 + \phi(D)]^{-1}x^m = (a_0 + a_1D + a_2D^2 + \dots + a_mD^m)x^m$
4.	$e^{ax}V$	$e^{ax}\cdot\dfrac{1}{f(D+a)}V$
5.	xV	$x\dfrac{1}{f(D)}V - \dfrac{f'(D)}{[f(D)]^2}V$

If Q(x) is not in any of the above 5 forms then the solution of the differential equation can be obtained by the following methods:

(i) f(D) is factorized as linear factors of D and PI is obtained using the formula

$$\frac{1}{D-a}Q(x) = e^{ax}\int Q(x)e^{-ax}\,dx$$

(ii) Variation of parameters: If CF = $c_1y_1 + c_2y_2$, assume PI = $y = v_1(x)y_1 + v_2(x)y_2$

where $v_1 = \int \frac{-y_2 Q}{W}\,dx$, $v_2 = \int \frac{y_1 Q}{W}\,dx$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. Integrating factor of the differential equation $\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$ is **[Summer 2016]**

(a) y^2 (b) y (c) y^3 (d) $2y^3$

2. The general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \tan 2x$ is

[Summer 2016]

(a) $\sin(yx) = c$ (b) $\sin\left(\frac{y}{x}\right) = c$

(c) $\sin y = c$ (d) $\sin x = c$

3. The orthogonal trajectory of the family of curve $x^2 + y^2 = c^2$ is

[Winter 2016; Summer 2016]

(a) $y = xc$ (b) $y = x + c$ (c) $y = x - c$ (d) $y = \frac{x}{c}$

4. Particular integral of $(D^2 + 4)y = \cos 2x$ is **[Summer 2016]**

(a) $\frac{x\sin 2x}{2}$ (b) $x \sin 2x$ (c) $\frac{x\sin 2x}{4}$ (d) $\frac{x\sin x}{4}$

5. The type, order and degree of the differential equation $\left(\frac{dx}{dy}\right)^2 + 5y^{\frac{1}{3}} = x$ are

[Summer 2016]

(a) Linear, First, Two (b) Nonlinear, First, Two

(c) Linear, Second, First (d) Nonlinear, Second, First

6. The Wronskian of the two functions sin $2x$ and cos $2x$ is **[Winter 2016]**

(a) 1 (b) 2 (c) –1 (d) –2

7. The solution of $(D^2 + 6D + 9)x = 0$ is **[Winter 2016]**

(a) $(c_1 + c_2 t)e^{-3t}$ (b) $c_1 e^{-3t}$

(c) $c_1 c_2 e^t$ (d) $c_1 e^{-t}$

8. The solution of $\dfrac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$ is **[Winter 2016]**

(a) $3e^{2y} = 2(e^{3x} + x^3) + 6c$ (b) $e^{2y} = e^{3x} + x^3 + c$

(c) $3e^{2y} = (e^{3x} + x^3) + 6c$ (d) $e^{2y} = 2(e^{3x} + x^3) + 6c$

9. Integrating factor of the differential equation $\dfrac{dy}{dx} + \dfrac{y}{1+x^2} = x^2$ is

(a) $e^{\frac{1}{1+y^2}}$ (b) $e^{\frac{x^2}{y^2}}$ (c) y^2 (d) $e^{\log y}$

10. The Bernoulli's differential equation $\dfrac{dy}{dx} - y\tan x = y^4 \sec x$ reduces to linear differential equation

(a) $\dfrac{du}{dx} + (3\tan x)u = -3\sec x$ where $y^{-3} = u$

(b) $\dfrac{du}{dx}(\tan x)u = 3\sec x$ where $y^{-3} = u$

(c) $\dfrac{du}{dx} + (\tan x)u = -\sec x$ where $y^{-3} = u$

(d) None of these

11. The value of α so that $e^{\alpha y^2}$ is an integrating factor of the linear differential equation $\dfrac{dx}{dy} + xy = e^{-\frac{y^2}{2}}$ is

(a) –1 (b) $-\dfrac{1}{2}$ (c) 1 (d) $\dfrac{1}{2}$

12. The general solution of $\dfrac{dy}{dx} + (\cot x)y = \sin 2x$ with integrating factor $\sin x$ is

(a) $y\sin x = \dfrac{2}{3}\sin^2 x + c$ (b) $y \sin x = \sin^3 x + c$

(c) $y\sin x = \dfrac{2}{3}\sin^3 x + c$ (d) None of these

13. In solving differential equation $\frac{d^2y}{dx^2} + y = \tan x$ by method of variation of parameters, complementary function = $c_1 \cos x + c_2 \sin x$, particular integral = $u \cos x + v \sin x$, then v is equal to

(a) $-\cos x$ (b) $\log(\sec x + \tan x) - \sin x$

(c) $-\log(\sec xx + \tan x)$ (d) $\cos x$

14. On putting $x = e^z$, the transformed differential equation of $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin(\log x)$ using $D = \frac{d}{dz}$ is

(a) $(D^2 - 4D + 5)y = e^{2z} \sin z$ (b) $(D^2 - 4D + 5)y = x^2 \sin(\log x)$

(c) $(D^2 - 4D - 4)y = e^z \sin z$ (d) $(D^2 - 3D + 5)y = e^{z^2} \sin z$

15. Solution of differential equation $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{1}{x^2}$ is

(a) $(c_1 x + c_2) - \frac{x^2}{4}$ (d) $(c_1 x^2 + c_2) + \frac{x^2}{4}$

(c) $c_1 + c_2\frac{1}{x} + \frac{1}{2x^2}$ (d) $(c_1 \log x + c_2) + \frac{x^2}{4}$

16. Which of the following differential equation is not exact? **[Winter 2015; Summer 2017]**

(a) $(y^2 - x^2)dx + 2xy\,dy = 0$ (b) $(x^3 + 3xy^2)dx + (3x^2y + y^3)\,dy = 0$

(c) $\frac{dy}{dx} = \frac{y}{x}$ (d) $ye^x dx + (2y + e^x)dy = 0$

17. The differential equation of the orthogonal trajectory to the equation $y = cx^2$ is **[Winter 2015]**

(a) $x^2 + 2y^2 + c = 0$ (b) $x^2 + y^2 + c = 0$

(c) $x^2 - 2y^2 + c = 0$ (d) $x^2 - y^2 + c = 0$

18. If $y = c_1y_1 + c_2y_2 = e^x(c_1 \cos x + c_2 \sin x)$ is a complementary function of a second order differential equation, Wronskian $W(y_1, y_2)$ is **[Winter 2015]**

(a) e^x (b) e^{3x} (c) e^{2x} (d) e^{-2x}

19. The general solution of $(D^2 + D + 1)y = 0$ is **[Winter 2015]**

(a) $e^t\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2\frac{\sqrt{3}}{2}t\right)$ (b) $e^{-t}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2\frac{\sqrt{3}}{2}t\right)$

(c) $e^{\frac{1}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2\frac{\sqrt{3}}{2}t\right)$ (d) $e^{-\frac{1}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2\frac{\sqrt{3}}{2}t\right)$

20. The order and degree of the differential equation $y'' + 3y^2 = 3\cos x$ are **[Summer 2017]**

(a) 2, 1 (b) 1, 2 (c) 1, 1 (d) 2, 2

21. The integrating factor of the linear differential equation $y' - \left(\frac{1}{x}\right)y = x^2$ is **[Summer 2017]**

(a) $\frac{1}{x^2}$ (b) x (c) $\frac{1}{x}$ (d) x^2

22. The solution of the differential equation $y'' + 11y' + 10y = 0$ is

(a) $c_1 e^{-x} + c_2 e^{-10x}$ (b) $c_1 e^{x} + c_2 e^{-10x}$

(c) $c_1 e^{-x} + c_2 e^{10x}$ (d)

23. If $y = (c_1 + c_2 x)e^x$ is the complementary function of a second order differential equation, then Wronskian $W(y_1, y_2)$ is **[Summer 2017]**

(a) e^x (b) e^{-x} (c) e^{2x} (d) e^{-2x}

24. The particular integral of $y''' + y' = e^{2x}$ is **[Summer 2017]**

(a) e^{2x} (b) $\frac{1}{10}e^{2x}$ (c) $\frac{1}{10}e^{x}$ (d) e^x

Answers

1. (c)	2. (b)	3. (a)	4. (c)	5. (b)	6. (d)	7. (a)	8. (a)
9. (a)	10. (a)	11. (d)	12. (c)	13. (a)	14. (a)	15. (c)	16. (c)
17. (a)	18. (c)	19. (d)	20. (a)	21. (c)	22. (a)	23. (c)	24. (b)