CHAPTER 6

Taylor's and Maclaurin's Series

Chapter Outline

6.1 INTRODUCTION

In this chapter, we will study Taylor's and Maclaurin's series. A Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. The concept of a Taylor series was discovered by the Scottish mathematician James Gregory and formally introduced by the English mathematician Brook Taylor in 1715. A Maclaurin series is a Taylor series expansion of a function about zero. It is named after the Scottish mathematician Colin Maclaurin, who made extensive use of this special case of Taylor series. It is common practice to approximate a function by using a finite number of terms of its Taylor series and Maclaurin's series covering expansions by definition, by standard expansion, by differentiation and integration and by substitution.

6.2 TAYLOR'S SERIES

Statement If $f(x + h)$ is a given function of h which can be expanded into a convergent series of positive ascending integral powers of h then

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots + \frac{h^n}{n!}f^n(x) + \ldots\ldots$$

Proof Let $f(x + h)$ be a function of h which can be expanded into positive ascending integral powers of h, then

$$f(x+h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 + \ldots\ldots\ldots\ldots \qquad \text{... (1)}$$

Differentiating w.r.t. h successively,

$$f'(x+h) = a_1 + a_2 \cdot 2h + a_3 \cdot 3h^2 + a_4 \cdot 4h^3 + \ldots\ldots \ldots\ldots \quad \ldots (2)$$

$$f''(x+h) = a_2 \cdot 2 + a_3 \cdot 6h + a_4 \cdot 12h^2 + \ldots\ldots \ldots\ldots \quad \ldots (3)$$

$$f'''(x+h) = a_3 \cdot 6 + a_4 \cdot 24h + \ldots\ldots \ldots\ldots \quad \ldots (4)$$

and so on.

Putting $h = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(x)$$

$$a_1 = f'(x)$$

$$a_2 = \frac{1}{2!} f''(x)$$

$$a_3 = \frac{1}{3!} f'''(x) \text{ and so on}$$

Substituting a_0, a_1, a_2 and a_3 in Eq. (1),

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots + \frac{h^n}{n!} f^n(x) + \ldots\ldots$$

This is known as **Taylor's series.**

Putting $x = a$ and $h = x - a$ in above series, we get Taylor's series in powers of $(x - a)$ as

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots + \frac{(x-a)^n}{n!} f^n(a) + \ldots\ldots$$

Note: To express the function in ascending powers of x, express h in terms of x.

Example 1

Prove that $f(mx) = f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + \ldots.$

Solution

$$f(mx) = f(mx - x + x) = f[x + (m-1)x]$$

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots$$

Putting $h = (m-1)x$,

$$f[x+(m-1)x] = f(mx) = f(x) + (m-1)x f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + \ldots\ldots\ldots$$

Example 2

Prove that

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!(1+x)^2} f''(x) - \frac{x^3}{3!(1+x)^3} f'''(x) + \ldots.$$

Solution

$$\frac{x^2}{1+x} = x - \frac{x}{1+x}$$

By Taylor's series,

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots$$

Putting $h = -\dfrac{x}{1+x}$,

$$f\left(x - \frac{x}{1+x}\right) = f\left(\frac{x^2}{1+x}\right)$$

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x}f'(x) + \frac{x^2}{2!\,(1+x)^2}f''(x) - \frac{x^3}{3!\,(1+x)^3}f'''(x) + \ldots$$

Example 3

Express $f(x) = 2x^3 + 3x^2 - 8x + 7$ *in terms of* $(x - 2)$.

Solution

$$f(x) = 2x^3 + 3x^2 - 8x + 7$$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

Putting $a = 2$,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots \qquad \ldots(1)$$

$$\begin{aligned}
f(x) &= 2x^3 + 3x^2 - 8x + 7, & f(2) &= 16 + 12 - 16 + 7 = 19\\
f'(x) &= 6x^2 + 6x - 8, & f'(2) &= 24 + 12 - 8 = 28\\
f''(x) &= 12x + 6, & f''(2) &= 24 + 6 = 30\\
f'''(x) &= 12, & f'''(2) &= 12
\end{aligned}$$

Substituting in Eq. (1),

$$\begin{aligned}
f(x) &= 19 + (x-2)28 + \frac{(x-2)^2}{2!}\cdot 30 + \frac{(x-2)^3}{3!}\cdot 12\\
&= 19 + 28(x-2) + 15(x-2)^2 + 2(x-2)^3
\end{aligned}$$

Example 4

Express $2x^3 + 7x^2 + x - 6$ *in ascending powers of* $(x - 2)$.

Solution

Let $f(x) = 2x^3 + 7x^2 + x - 6$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Putting $a = 2$,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots \qquad ...(1)$$

$$\begin{aligned} f(x) &= 2x^3 + 7x^2 + x - 6, & f(2) &= 40 \\ f'(x) &= 6x^2 + 14x + 1, & f'(2) &= 53 \\ f''(x) &= 12x + 14, & f''(2) &= 38 \\ f'''(x) &= 12, & f'''(2) &= 12 \end{aligned}$$

Substituting in Eq.(1),

$$\begin{aligned} f(x) &= 40 + (x-2)(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12) \\ &= 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3 \end{aligned}$$

Example 5

Expand $x^3 + 7x^2 + x - 6$ *in powers of* $(x-3)$.

Solution

Let $f(x) = x^3 + 7x^2 + x - 6$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Putting $a = 3$,

$$f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \frac{(x-3)^3}{3!}f'''(3) + \cdots \qquad ...(1)$$

$$\begin{aligned} f(x) &= x^3 + 7x^2 + x - 6, & f(3) &= 87 \\ f'(x) &= 3x^2 + 14x + 1, & f'(3) &= 70 \\ f''(x) &= 6x + 14, & f''(3) &= 32 \\ f'''(x) &= 6, & f'''(3) &= 6 \end{aligned}$$

Substituting in Eq. (1),

$$\begin{aligned} f(x) &= 87 + (x-3)(70) + \frac{(x-3)^2}{2!}(32) + \frac{(x-3)^3}{3!}(6) \\ &= 87 + 70(x-3) + 16(x-3)^2 + (x-3)^3 \end{aligned}$$

Example 6

Expand $x^4 - 3x^3 + 2x^2 - x + 1$ *in powers of* $(x - 3)$.

Solution

Let $f(x) = x^4 - 3x^3 + 2x^2 - x + 1$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{IV}(a) + \cdots$$

Putting $a = 3$,

$$f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \frac{(x-3)^3}{3!}f'''(3) + \frac{(x-3)^4}{4!}f^{IV}(3) + \cdots \quad \text{...(1)}$$

$$\begin{aligned}
f(x) &= x^4 - 3x^3 + 2x^2 - x + 1, & f(3) &= 16 \\
f'(x) &= 4x^3 - 9x^2 + 4x - 1, & f'(3) &= 38 \\
f''(x) &= 12x^2 - 18x + 4, & f''(3) &= 58 \\
f'''(x) &= 24x - 18, & f'''(3) &= 54 \\
f^{IV}(x) &= 24, & f^{IV}(x) &= 24
\end{aligned}$$

Substituting in Eq. (1),

$$\begin{aligned}
f(x) &= 16 + (x-3)(38) + \frac{(x-3)^2}{2!}(58) + \frac{(x-3)^3}{3!}(54) + \frac{(x-3)^4}{4!}(24) \\
&= 16 + 38(x-3) + 29(x-3)^2 + 9(x-3)^3 + (x-3)^4
\end{aligned}$$

Example 7

Expand $49 + 69x + 42x^2 + 11x^3 + x^4$ *in powers of* $(x + 2)$.

Solution

Let $f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{IV}(a) + \cdots \quad \text{...(1)}$$

Putting $a = -2$,

$$f(x) = f(-2) + (x+2)f'(-2) + \frac{(x+2)^2}{2!}f''(-2) + \frac{(x+2)^3}{3!}f'''(-2) + \frac{(x+2)^4}{4!}f^{IV}(-2) + \cdots$$

$$\begin{aligned}
f(x) &= 49 + 69x + 42x^2 + 11x^3 + x^4, & f(-2) &= 7 \\
f'(x) &= 69 + 84x + 33x^2 + 4x^3, & f'(-2) &= 1 \\
f''(x) &= 84 + 66x + 12x^2, & f''(-2) &= 0 \\
f'''(x) &= 66 + 24x, & f'''(-2) &= 18 \\
f^{IV}(x) &= 24, & f^{IV}(-2) &= 24
\end{aligned}$$

Substituting in Eq. (1),

$$\begin{aligned}
f(x) &= 7 + (x+2)(1) + \frac{(x+2)^2}{2!}(0) + \frac{(x+2)^3}{3!}(18) + \frac{(x+2)^4}{4!}(24) \\
&= 7 + (x+2) + 3(x+2)^3 + (x+2)^4
\end{aligned}$$

Example 8

Expand $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$ in powers of $(x - 1)$ and find $f(0.99)$.

Solution

$$f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots$$

Putting $a = 1$,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{iv}(1) + \frac{(x-1)^5}{5!}f^{v}(1) + \ldots\ldots. \quad \ldots (1)$$

$$\begin{aligned}
f(x) &= x^5 - x^4 + x^3 - x^2 + x - 1, & f(1) &= 0 \\
f'(x) &= 5x^4 - 4x^3 + 3x^2 - 2x + 1, & f'(1) &= 5 - 4 + 3 - 2 + 1 = 3 \\
f''(x) &= 20x^3 - 12x^2 + 6x - 2, & f''(1) &= 20 - 12 + 6 - 2 = 12 \\
f'''(x) &= 60x^2 - 24x + 6, & f'''(1) &= 60 - 24 + 6 = 42 \\
f^{IV}(x) &= 120x - 24, & f^{IV}(1) &= 120 - 24 = 96 \\
f^{V}(x) &= 120, & f^{V}(1) &= 120
\end{aligned}$$

Substituting in Eq. (1),

$$f(x) = 0 + (x-1)(3) + \frac{(x-1)^2}{2!}(12) + \frac{(x-1)^3}{3!}(42) + \frac{(x-1)^4}{4!}(96) + \frac{(x-1)^5}{5!}(120)$$
$$= 3(x-1) + 6(x-1)^2 + 7(x-1)^3 + 4(x-1)^4 + (x-1)^5$$

Putting $x = 0.99$,

$$f(0.99) = 3\,(0.99-1) + 6\,(0.99-1)^2 + 7\,(0.99-1)^3 + 4\,(0.99-1)^4 + (0.99-1)^5$$
$$= 3\,(-0.01) + 6\,(-0.01)^2 + 7\,(-0.01)^3 + 4\,(-0.01)^4 + (-0.01)^5$$
$$= -0.02939 \text{ approx.}$$

Example 9

Expand $f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$ *in ascending powers of* $(x-1)$.

Solution

$$f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{IV}(a) + \cdots$$

Putting $a = 1$,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{IV}(1) + \cdots \quad ...(1)$$

$$f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8, \qquad f(1) = 299$$
$$f'(x) = 4(x+2)^3 + 15(x+2)^2 + 12(x+2) + 7, \qquad f'(1) = 286$$
$$f''(x) = 12(x+2)^2 + 30(x+2) + 12, \qquad f''(1) = 210$$
$$f'''(x) = 24(x+2) + 30, \qquad f'''(1) = 102$$
$$f^{IV}(x) = 24, \qquad f^{IV}(1) = 24$$

Substituting in Eq. (1),

$$f(x) = 299 + (x-1)(286) + \frac{(x-1)^2}{2!}(210) + \frac{(x-1)^3}{3!}(102) + \frac{(x-1)^4}{4!}(24)$$
$$= 299 + 286(x-1) + 105(x-1)^2 + 17(x-1)^3 + (x-1)^4$$

Example 10

Prove that $\dfrac{1}{1-x} = \dfrac{1}{3} + \dfrac{(x+2)}{3^2} + \dfrac{(x+2)^2}{3^3} + \dfrac{(x+2)^3}{3^4} + \dots$.

Solution

Let $\qquad f(x) = \dfrac{1}{1-x}$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots \dots\dots\dots$$

Putting $a = -2$,

$$f(x) = f(-2) + (x+2)f'(-2) + \frac{(x+2)^2}{2!}f''(-2) + \frac{(x+2)^3}{3!}f'''(-2) + \dots\dots\dots \qquad \dots (1)$$

$$f(x) = \frac{1}{1-x}, \qquad f(-2) = \frac{1}{3}$$

$$f'(x) = \frac{1}{(1-x)^2}, \qquad f'(-2) = \frac{1}{3^2}$$

$$f''(x) = \frac{2}{(1-x)^3}, \qquad f''(-2) = \frac{2!}{3^3}$$

$$f'''(x) = \frac{2\cdot 3}{(1-x)^4}, \qquad f'''(-2) = \frac{3!}{3^4} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x) = \frac{1}{3} + \frac{(x+2)}{3^2} + \frac{(x+2)^2}{3^3} + \frac{(x+2)^3}{3^4} + \dots\dots\dots\dots$$

Example 11

Expand log x *in powers of* $(x - 1)$.

Solution

Let $f(x) = \log x$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

Putting $a = 1$,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f''(1) + \cdots \qquad \dots (1)$$

$$f(x) = \log x, \qquad f(1) = \log 1 = 0$$

$$f'(x) = \frac{1}{x}, \qquad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, \qquad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2 \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x) = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \cdots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

Example 12

Expand log sin x *in powers of* $(x - 2)$. **[Summer 2014]**

Solution

Let $f(x) = \log \sin x$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Putting $a = 2$,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots \qquad \text{...(1)}$$

$$f(x) = \log \sin x, \qquad f(2) = \log \sin (2)$$

$$f'(x) = \frac{\cos x}{\sin x} = \cot x, \qquad f'(2) = \cot (2)$$

$$f''(x) = -\operatorname{cosec}^2 x, \qquad f''(2) = -\operatorname{cosec}^2(2)$$

$$f'''(x) = 2\operatorname{cosec}^2 x \cot x, \qquad f'''(2) = 2\operatorname{cosec}^2(2)\cot(2) \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x) = \log \sin (2) + (x-2)\cot(2) + \frac{(x-2)^2}{2!}\left[-\operatorname{cosec}^2(2)\right] + \frac{(x-2)^3}{3!}\left[2\operatorname{cosec}^2(2)\cot(2)\right] + \cdots$$

$$\log \sin x = \log \sin (2) + (x-2)\cot (2) - \frac{(x-2)^2}{2}\operatorname{cosec}^2(2) + \frac{(x-2)^3}{3}\operatorname{cosec}^2(2)\cot(2) + \cdots$$

Example 13

Expand log (cos *x*) *about* $\frac{\pi}{3}$.

Solution

Let $f(x) = \log(\cos x)$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots$$

Putting $a = \frac{\pi}{3}$,

$$f(x) = f\left(\frac{\pi}{3}\right) + \left(x - \frac{\pi}{3}\right)f'\left(\frac{\pi}{3}\right) + \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 f''\left(\frac{\pi}{3}\right) + \frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 f'''\left(\frac{\pi}{3}\right) + \ldots\ldots \quad (1)$$

$$f(x) = \log(\cos x), \qquad f\left(\frac{\pi}{3}\right) = \log\left(\cos\frac{\pi}{3}\right) = \log\left(\frac{1}{2}\right) = -\log 2$$

$$f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x, \qquad f'\left(\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$$

$$f''(x) = -\sec^2 x, \qquad f''\left(\frac{\pi}{3}\right) = -\sec^2\frac{\pi}{3} = -4$$

$$f'''(x) = -2\sec^2 x \tan x, \qquad f'''\left(\frac{\pi}{3}\right) = -2\sec^2\frac{\pi}{3}\tan\frac{\pi}{3}$$

$$= -2(4)\sqrt{3}$$

$$= -8\sqrt{3} \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x) = -\log 2 + \left(x-\frac{\pi}{3}\right)\left(-\sqrt{3}\right) + \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2(-4) + \frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3\left(-8\sqrt{3}\right) + \ldots$$

$$\log(\cos x) = -\log 2 - \sqrt{3}\left(x-\frac{\pi}{3}\right) - 2\left(x-\frac{\pi}{3}\right)^2 - \frac{4\sqrt{3}}{3}\left(x-\frac{\pi}{3}\right)^3 - \ldots$$

Example 14

Obtain $\tan^{-1} x$ *in powers of* $(x-1)$.

Solution

Let $\quad f(x) = \tan^{-1} x$

By Taylor's series,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots$$

Putting $a = 1$,

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \ldots \qquad \ldots (1)$$

$$f(x) = \tan^{-1} x, \qquad f(1) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$f'(x) = \frac{1}{1+x^2}, \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}, \qquad f''(1) = -\frac{2}{4} = -\frac{1}{2}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}, \qquad f'''(1) = \frac{1}{2} \qquad \text{and so on}$$

Substituting in Eq. (1),

$$f(x) = \frac{\pi}{4} + (x-1)\left(\frac{1}{2}\right) + \frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3!}\left(\frac{1}{2}\right) + \ldots\ldots$$

$$\tan^{-1} x = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + \cdots$$

Example 15

Express $7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$ *in ascending powers of x.*

Solution

Let $\quad f(x+2) = 7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$

$$f(x) = 7 + x + 3x^3 + x^4 - x^5$$

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{IV}(x)\frac{h^5}{5!}f^{V}(x) + \cdots$$

Putting $h = 2$,

$$\begin{aligned} f(x+2) &= (7+x+3x^3+x^4-x^5) + 2(1+9x^2+4x^3-5x^4) \\ &\quad + \frac{2^2}{2!}(18x+12x^2-20x^3) + \frac{2^3}{3!}(18+24x-60x^2) \\ &\quad + \frac{2^4}{4!}(24-120x) + \frac{2^5}{5!}(-120) \\ &= 17-11x-38x^2-29x^3-9x^4-x^5 \end{aligned}$$

Example 16

Express $(x-1)^4 + 2(x-1)^3 + 5(x-1) + 2$ *in ascending powers of x.*

[Summer 2016]

Solution

Let $f(x-1) = (x-1)^4 + 2(x-1)^3 + 5(x-1) + 2$

$$f(x) = 2 + 5x + 2x^3 + x^4$$

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \cdots$$

Putting $h = -1$,

$$\begin{aligned} f(x-1) &= f(x) + (-1)f'(x) + \frac{(-1)^2}{2!}f''(x) + \frac{(-1)^3}{3!}f''' + \frac{(-1)^4}{4!}f^{iv}(x) + \cdots \\ &= (2+5x+2x^3+x^4) + (-1)(5+6x^2+4x^3) \\ &\qquad + \frac{1}{2!}(12x+12x^2) + \frac{(-1)}{3!}(12+24x) + \frac{1}{4!}(24) + 0 \\ &= (2+5x+2x^3+x^4) + (-1)(5+6x^2+4x^3) + (6x+6x^2) + (-1)(2+4x) + 1 \\ &= 2+5x+2x^3+x^4-5-6x^2-4x^3+6x+6x^2-2-4x+1 \\ &= x^4-2x^3+7x-4 \end{aligned}$$

Example 17

Express $5+4(x-1)^2-3(x-1)^3+(x-1)^4$ *in ascending powers of x.*

[Winter 2013]

Solution

Let $$f(x-1)=5+4(x-1)^2-3(x-1)^3+(x-1)^4$$

$$f(x)=5+4x^2-3x^3+x^4$$

By Taylor's series,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{IV}(x)+\cdots$$

Putting $h=-1$,

$$f(x-1)=f(x)+(-1)f'(x)+\frac{(-1)^2}{2!}f''(x)+\frac{(-1)^3}{3!}f'''(x)+\frac{(-1)^4}{4!}f^{IV}(x)+\cdots$$

$$=(5+4x^2-3x^3+x^4)+(-1)(8x-9x^2+4x^3)+\frac{(-1)^2}{2!}(8-18x+12x^2)$$

$$+\frac{(-1)^3}{3!}(-18+24x)+\frac{(-1)^4}{4!}(24)$$

$$=5+4x^2-3x^3+x^4-8x+9x^2-4x^3+4-9x+6x^2+3-4x+1$$

$$=x^4-7x^3+19x^2-21x+13$$

Example 18

Find the expansion of $\tan\left(x+\frac{\pi}{4}\right)$ *in ascending powers of x up to terms in* x^4 *and find the approximate value of* tan (43°).

[Winter 2013; Summer 2016]

Solution

Let $$f\left(x+\frac{\pi}{4}\right)=\tan\left(x+\frac{\pi}{4}\right)$$

$$f(x)=\tan x$$

By Taylor's series,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{IV}(x)+\cdots$$

Putting $\quad x = \dfrac{\pi}{4}, h = x,$

$$f\left(\frac{\pi}{4}+x\right) = f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{x^3}{3!}f'''\left(\frac{\pi}{4}\right) + \frac{x^4}{4!}f^{IV}\left(\frac{\pi}{4}\right) + \cdots \qquad ...(1)$$

$$f(x) = \tan x, \qquad f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

$$f'(x) = \sec^2 x, \qquad f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = 2$$

$$\begin{aligned} f''(x) &= 2\sec x \cdot \sec x \tan x, \qquad f''\left(\frac{\pi}{4}\right) = 2\tan\frac{\pi}{4} + 2\tan^3\frac{\pi}{4} = 4 \\ &= 2\sec^2 x \tan x \\ &= 2(1+\tan^2 x)\tan x \\ &= 2\tan x + 2\tan^3 x \end{aligned}$$

$$\begin{aligned} f'''(x) &= 2\sec^2 x + 6\tan^2 x \sec^2 x, \qquad f'''\left(\frac{\pi}{4}\right) = 2 + 8\tan^2\frac{\pi}{4} + 6\tan^4\frac{\pi}{4} = 16 \\ &= 2(1+\tan^2 x) + 6\tan^2 x\left(1+\tan^2 x\right) \\ &= 2 + 8\tan^2 x + 6\tan^4 x \end{aligned}$$

$$\begin{aligned} f^{IV}(x) = 16\tan x \cdot \sec^2 x + 24\tan^3 x \cdot \sec^2 x, \quad f^{IV}\left(\frac{\pi}{4}\right) &= 16\tan\frac{\pi}{4}\cdot\sec^2\frac{\pi}{4} \\ &\quad + 24\tan^3\frac{\pi}{4}\cdot\sec^2\frac{\pi}{4} \\ &= 80 \quad \text{and so on} \end{aligned}$$

Substituting in Eq. (1),

$$\begin{aligned} f\left(\frac{\pi}{4}+x\right) &= 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(16) + \frac{x^4}{4!}(80) + \cdots \\ \tan\left(\frac{\pi}{4}+x\right) &= 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \cdots \end{aligned} \qquad \ldots(2)$$

Now
$$\begin{aligned} \tan 43^\circ &= \tan(45^\circ - 2^\circ) \\ &= \tan\left(\frac{\pi}{4} - \frac{2\pi}{180}\right) \\ &= \tan\left(\frac{\pi}{4} - 0.0349\right) \end{aligned}$$

$$= 1 + 2(-0.0349) + 2(-0.0349)^2 + \frac{8}{3}(-0.0349)^3 + \frac{10}{3}(-0.0349)^4$$

$$= 0 \cdot 9326 \text{ approx.}$$

Example 19

Prove that

$$\log[\sin(x+h)] = \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \cdots\cdots$$

Solution

Let $f(x+h) = \log[\sin(x+h)]$

$f(x) = \log(\sin x)$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots\ldots \qquad \ldots(1)$$

$$f'(x) = \frac{1}{\sin x}\cos x = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x$$

$$f'''(x) = 2\operatorname{cosec}^2 x \cot x = \frac{2\cos x}{\sin^3 x} \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x+h) = \log\sin x + h\cot x - \frac{h^2}{2!}\operatorname{cosec}^2 x + \frac{h^3}{3!}\frac{2\cos x}{\sin^3 x} + \ldots\ldots\ldots$$

$$\log[\sin(x+h)] = \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \ldots\ldots\ldots$$

Example 20

Expand $\log \cos\left(x + \frac{\pi}{4}\right)$ *using Taylor's theorem in ascending powers of x and hence find the value of* $\log(\cos 48°)$ *correct up to three decimal places.*

Solution

Let $f\left(x+\dfrac{\pi}{4}\right)=\log\cos\left(x+\dfrac{\pi}{4}\right)$

$$f(x)=\log\cos x$$

By Taylor's series,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\cdots$$

Putting $x=\dfrac{\pi}{4}, h=x,$

$$f\left(\frac{\pi}{4}+x\right)=f\left(\frac{\pi}{4}\right)+xf'\left(\frac{\pi}{4}\right)+\frac{x^2}{2!}f''\left(\frac{\pi}{4}\right)+\cdots \qquad ...(1)$$

$$f(x)=\log\cos x, \qquad f\left(\frac{\pi}{4}\right)=\log\cos\frac{\pi}{4}=\log\left(\frac{1}{\sqrt{2}}\right)=-\log\sqrt{2}$$

$$f'(x)=\frac{1}{\cos x}(-\sin x), \quad f'\left(\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$$

$$=-\tan x$$

$$f''(x)=-\sec^2 x, \qquad f''\left(\frac{\pi}{4}\right)=-\sec^2\left(\frac{\pi}{4}\right)=-2 \quad \text{and so on}$$

Substituting in Eq. (1),

$$f\left(\frac{\pi}{4}+x\right)=-\log\sqrt{2}+x(-1)+\frac{x^2}{2!}(-2)+\cdots$$

$$\log\cos\left(\frac{\pi}{4}+x\right)=-\log\sqrt{2}-x-x^2+\cdots \qquad ...(2)$$

Now,

$$\begin{aligned}\log(\cos 48^\circ)&=\log\left[\cos(45^\circ+3^\circ)\right]\\&=\log\left[\cos\left(\frac{\pi}{4}+\frac{3\pi}{180}\right)\right]\\&=\log\left[\cos\left(\frac{\pi}{4}+0.0523\right)\right]\\&=-\log\sqrt{2}-0.0523-(0.0523)^2\\&=-0.402 \quad \text{approx.}\end{aligned}$$

Example 21

Show that $\tan^{-1}(x+h) = \tan^{-1} x + (h\sin\alpha)\dfrac{\sin\alpha}{1} - (h\sin\alpha)^2\dfrac{\sin 2\alpha}{2}$

$$+(h\sin\alpha)^3\frac{\sin 3\alpha}{3} + \cdots$$

where $\alpha = \cot^{-1}x$. **[Summer 2015]**

Solution

Let $f(x+h) = \tan^{-1}(x+h)$

$$f(x) = \tan^{-1} x$$

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \quad \text{... (1)}$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{2x\cdot 4x}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3} \quad \text{and so on}$$

Putting $x = \cot\alpha$,

$$f'(\cot\alpha) = \frac{1}{1+\cot^2\alpha} = \frac{1}{\operatorname{cosec}^2\alpha} = \sin^2\alpha$$

$$f''(\cot\alpha) = -\frac{2\cot\alpha}{(\operatorname{cosec}^2\alpha)^2} = -2\sin^2\alpha\sin^2\alpha\frac{\cos\alpha}{\sin\alpha}$$

$$= -2\sin^2\alpha\sin\alpha\cos\alpha = -\sin^2\alpha\sin 2\alpha$$

$$f'''(\cot\alpha) = \frac{2(3\cot^2\alpha-1)}{(1+\cot^2\alpha)^3} = \frac{2(3\cos^2\alpha-\sin^2\alpha)}{\operatorname{cosec}^6\alpha\cdot\sin^2\alpha}$$

$$= 2(3-4\sin^2\alpha)\sin^4\alpha = 2(3\sin\alpha-4\sin^3\alpha)\sin^3\alpha$$

$$= 2\sin 3\alpha\sin^3\alpha$$

Substituting in Eq. (1),

$$f(x+h) = \tan^{-1} x + h\sin^2\alpha + \frac{h^2}{2!}(-\sin^2\alpha\sin 2\alpha) + \frac{h^3}{3!}(2\sin^3\alpha\sin 3\alpha) + \cdots$$

$$= \tan^{-1} x + h\sin\alpha\left(\frac{\sin\alpha}{1}\right) - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} + \cdots$$

Example 22

Expand $\tan^{-1}(x+h)$ *in powers of h and hence, find the value of* $\tan^{-1}(1.003)$ *up to five places of decimal.*

Solution

Let $\quad f(x+h) = \tan^{-1}(x+h)$

$$f(x) = \tan^{-1} x$$

By Taylor's series,

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots \qquad \ldots (1)$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{2x\cdot 4x}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x+h) = \tan^{-1}(x+h) = \tan^{-1} x + h\cdot\frac{1}{1+x^2} + \frac{h^2}{2!}\left[-\frac{2x}{(1+x^2)^2}\right] + \frac{h^3}{3!}\left[\frac{2(3x^2-1)}{(1+x^2)^3}\right] + \ldots\ldots\ldots$$

Putting $\quad x = 1, h = 0.003,$

$$\tan^{-1}(1+0.003) = \tan^{-1} 1 + \frac{0.003}{2} + \frac{(0.003)^2}{2!}\left(-\frac{2}{4}\right) + \frac{(0.003)^3}{3!}\left(\frac{1}{2}\right) + \ldots\ldots\ldots$$

$$\tan^{-1}(1.003) = \frac{\pi}{4} + 0.00015 - 2.25\times 10^{-8} + 2.25\times 10^{-12} \qquad \text{[Considering first 4 terms]}$$

$$= 0.78540 \text{ approx.}$$

Example 23

Prove that $\sqrt{1+x+2x^2} = 1 + \dfrac{x}{2} + \dfrac{7x^2}{8} - \dfrac{7x^3}{16} + \cdots$.

Solution

Let $\quad f(x) = \sqrt{x}$

$$f(x+h) = \sqrt{x+h}$$

By Taylor's series,

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

Putting $\qquad x = 1,\ h = x + 2x^2,$

$$f(x+h) = \sqrt{x+h} = \sqrt{1+x+2x^2}$$

$$= f(1) + (x+2x^2)\,f'(1) + \frac{(x+2x^2)^2}{2!} f''(1) + \frac{(x+2x^2)^3}{3!} f'''(1) + \ldots\ldots\ldots \quad \ldots (1)$$

$$f(x) = \sqrt{x}, \qquad f(1) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{\frac{3}{2}}}, \qquad f''(1) = -\frac{1}{4}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{x^{\frac{5}{2}}}, \qquad f'''(1) = \frac{3}{8} \quad \text{and so on}$$

Substituting in Eq. (1),

$$\sqrt{1+x+2x^2} = 1 + \frac{1}{2}(x+2x^2) - \frac{1}{4}\frac{(x^2+4x^3+4x^4)}{2} + \frac{3}{8}\frac{(x^3+\ldots)}{6} + \ldots\ldots\ldots$$

$$= 1 + \frac{x}{2} + \frac{7x^2}{8} - \frac{7x^3}{16} + \ldots$$

Example 24

Expand $\sqrt{1+x+2x^2}$ *in powers of* $(x-1)$.

Solution

$$\sqrt{1+x+2x^2} = \sqrt{4+2(x-1)^2+5(x-1)} \qquad \text{[Expressing in terms of } (x-1)\text{]}$$

Let

$$f(x) = \sqrt{x}$$

$$f(x+h) = \sqrt{x+h}$$

By Taylor's series,

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots$$

Putting $x = 4$, $h = 2(x-1)^2 + 5(x-1)$,

$$f(x+h) = \sqrt{x+h} = \sqrt{4+2(x-1)^2+5(x-1)}$$

$$= f(4) + [2(x-1)^2+5(x-1)]\,f'(4) + \frac{[2(x-1)^2+5(x-1)]^2}{2!} f''(4) + \ldots \quad \text{... (1)}$$

$$f(x) = \sqrt{x}, \qquad f(4) = 2$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(4) = \frac{1}{4}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{\frac{3}{2}}}, \qquad f''(4) = -\frac{1}{32} \qquad \text{and so on}$$

Substituting in Eq. (1),

$$\sqrt{4+2(x-1)^2+5(x-1)} = 2 + [2(x-1)^2+5(x-1)]\left(\frac{1}{4}\right) + \frac{[2(x-1)^2+5(x-1)]^2}{2!}\left(-\frac{1}{32}\right) + \ldots\ldots$$

$$\sqrt{1+x+2x^2} = 2 + \frac{5}{4}(x-1) + \frac{7}{64}(x-1)^2 + \ldots\ldots\ldots\ldots$$

Example 25

Show that

$$\frac{1}{2}[f(x) - f(2a-x)] = (x-a)f'(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^5}{5!} f^{V}(a) + \cdots$$

Solution

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{IV}(a) + \frac{(x-a)^5}{5!}f^{V}(a) + \cdots \quad \text{...(1)}$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{IV}(x) + \frac{h^5}{5!}f^{V}(x) + \cdots \quad \text{...(2)}$$

Now, $f(2a-x) = f[a+(a-x)]$

Putting $x = a$, $h = a - x$ in Eq. (2),

$$f[a+(a-x)] = f(a)+(a-x)f'(a)+\frac{(a-x)^2}{2!}f''(a)+\frac{(a-x)^3}{3!}f'''(a) + \frac{(a-x)^4}{4!}f^{IV}(a)+\frac{(a-x)^5}{5!}f^{V}(a)+\cdots$$

$$f(2a-x) = f(a)-(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)-\frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{IV}(a)-\frac{(x-a)^5}{5!}f^{V}(a)+\cdots \quad ...(3)$$

From Eqs (1) and (3),

$$\frac{1}{2}\left[f(x)-f(2a-x)\right] = \frac{1}{2}\left[2(x-a)f'(a)+2\cdot\frac{(x-a)^3}{3!}f'''(a)+2\cdot\frac{(x-a)^5}{5!}f^{V}(a)+\cdots\right]$$

$$= (x-a)f'(a)+\frac{(x-a)^3}{3!}f'''(a)+\frac{(x-a)^5}{5!}f^{V}(a)+\cdots$$

Example 26

Using Taylor's theorem, evaluate up to four places of decimals:

(i) $\sqrt{1.02}$ (ii) $\sqrt{25.15}$ (iii) $\sqrt{9.12}$

(iv) $\sqrt{10}$ (v) $\sqrt{36.12}$ **[Winter 2014]**

Solution

Let $f(x) = \sqrt{x}$

$f(x+h) = \sqrt{x+h}$

By Taylor's series,

$$f(x+h) = f(x)+h\,f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\ldots\ldots\ldots \quad ...\ (1)$$

(i) Putting $x = 1$, $h = 0.02$,

$$f(x+h) = \sqrt{x+h} = \sqrt{1+0.02} = \sqrt{1.02}$$

$$= f(1)+(0.02)f'(1)+\frac{(0.02)^2}{2!}f''(1)+\ldots\ldots\ldots \quad ...\ (2)$$

$f(x) = \sqrt{x}$, $\quad f(1) = 1$

$f'(x) = \dfrac{1}{2\sqrt{x}}$, $\quad f'(1) = \dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4x^{\frac{3}{2}}}$, $\quad f''(1) = -\dfrac{1}{4}$ and so on

Substituting in Eq. (2) and considering only first three terms,

$$\sqrt{1.02} = 1 + (0.02)\left(\frac{1}{2}\right) + \frac{(0.02)^2}{2!}\left(-\frac{1}{4}\right)$$
$$= 1.0099 \text{ approx.}$$

(ii) Putting $x = 25$, $h = 0.15$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{25+0.15}$$
$$= f(25) + (0.15)\,f'(25) + \frac{(0.15)^2}{2!}f''(25) + \ldots \qquad \text{... (3)}$$

$$f(x) = \sqrt{x}, \qquad f(25) = 5$$
$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(25) = \frac{1}{10} = 0.1$$
$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(25) = -\frac{1}{500} = -0.002 \qquad \text{and so on}$$

Substituting in Eq. (3) and considering only first three terms,

$$\sqrt{25.15} = 5 + (0.15)(0.1) + \frac{(0.15)^2}{2}(-0.002)$$
$$= 5.0150 \text{ approx.}$$

(iii) Putting $x = 9$, $h = 0.12$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+0.12}$$
$$= f(9) + (0.12)\,f'(9) + \frac{(0.12)^2}{2!}f''(9) + \ldots \qquad \text{... (4)}$$

$$f(x) = \sqrt{x}, \qquad f(9) = 3$$
$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(9) = \frac{1}{6}$$
$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(9) = -\frac{1}{108} \quad \text{and so on}$$

Substituting in Eq. (4) and considering only first three terms,

$$\sqrt{9.12} = 3 + (0.12)\left(\frac{1}{6}\right) + \frac{(0.12)^2}{2}\left(-\frac{1}{108}\right)$$
$$= 3 + 0.02 - (0.12)(0.06)(0.0093)$$
$$= 3.0199 \text{ approx.}$$

(iv) Putting $x = 9$, $h = 1$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+1} = f(9) + f'(9) + \frac{1}{2!}f''(9) + \ldots \qquad \text{... (5)}$$

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{216} \qquad \text{[Refer (iii)]}$$

$$= 3.1620 \text{ approx.}$$

(v) Putting $x = 36$, $h = 0.12$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{36+0.12} = \sqrt{36 \cdot 12}$$

$$= f(36) + (0.12) f'(36) + \frac{(0.12)^2}{2!} f''(36) + \cdots \qquad \text{... (6)}$$

$$f(x) = \sqrt{x}, \qquad f(36) = \sqrt{36} = 6$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(36) = \frac{1}{2\sqrt{36}} = \frac{1}{12}$$

$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(36) = -\frac{1}{4(36)^{\frac{3}{2}}} = -\frac{1}{864} \qquad \text{and so on}$$

Substituting in Eq. (6) and considering only first three terms,

$$\sqrt{36.12} = 6 + (0.12)\left(\frac{1}{12}\right) + \frac{(0.12)^2}{2!}\left(-\frac{1}{864}\right) + \cdots$$

$$= 6.0099 \text{ approx.}$$

Example 27

Find cosh (1.505), *given* sinh (1.5) = 2.1293 *and* cosh (1.5) = 2.3524.

Solution

Let $f(x) = \cosh x$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots$$

Putting $x = 1.5$, $h = 0.005$,

$f(x + h) = \cosh(x + h) = \cosh(1.5 + 0.005)$

$$= f(1.5) + (0.005) f'(1.5) + \frac{(0.005)^2}{2!} f''(1.5) + \frac{(0.005)^3}{3!} f'''(1.5) + \cdots \qquad \text{... (1)}$$

$$f(x) = \cosh x, \qquad f(1.5) = \cosh(1.5) = 2.3524$$

$$f'(x) = \sinh x, \qquad f'(1.5) = \sinh(1.5) = 2.1293$$

$$f''(x) = \cosh x, \qquad f''(1.5) = \cosh(1.5) = 2.3524 \qquad \text{and so on}$$

Substituting in Eq. (1) and considering only first three terms,

$$\begin{aligned}\cosh(1.505) &= \cosh(1.5) + (0.005)\sinh(1.5) + \frac{(0.005)^2}{2!}\cosh(1.5) + \ldots \\ &= 2.3524 + (0.005)(2.1293) + (12.5)(10^{-6})(2.3524) \\ &= 2.3631 \text{ approx.}\end{aligned}$$

Example 28

Find the approximate value of sin (30°30′).

Solution

Let $\qquad f(x) = \sin x$

$$\begin{aligned}\sin(30°30') = \sin(30° + 30') &= \sin\left(\frac{\pi}{6} + \frac{30}{60}\cdot\frac{\pi}{180}\right) \\ &= \sin\left(\frac{\pi}{6} + 0.0087\right)\end{aligned}$$

By Taylor's series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

$$\sin(x+h) = \sin x + h\cos x + \frac{h^2}{2!}(-\sin x) + \cdots$$

Putting $\quad x = \frac{\pi}{6}, h = 0.0087,$

$$\sin\left(\frac{\pi}{6} + 0.0087\right) = \sin\frac{\pi}{6} + (0.0087)\left(\cos\frac{\pi}{6}\right) + \frac{(0.0087)^2}{2!}\left(-\sin\frac{\pi}{6}\right)$$

[Considering first 3 terms]

$$\sin 30°\,30' = 0.50752 \text{ approx.}$$

EXERCISE 6.1

1. Expand e^x in powers of $(x - 1)$.

$$\left[\textbf{Ans.: } e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right)\right]$$

2. Expand $2x^3 + 7x^2 + x - 1$ in powers of $(x - 2)$.

[**Ans.:** $45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$]

3. Expand $x^5 - 5x^4 + 6x^3 - 7x^2 + 8x - 9$ in powers of $(x - 1)$.

$$\left[\textbf{Ans.: } -6-3(x-1)-9(x-1)^2-4(x-1)^3+(x-1)^5\right]$$

4. Expand $x^4 - 3x^3 + 2x^2 - x + 1$ in powers of $(x - 3)$.

$$\left[\textbf{Ans.: } 16+38(x-3)+29(x-3)^2+9(x-3)^3+(x-3)^4\right]$$

5. Expand $x^3 - 2x^2 + 3x - 5$ in powers of $(x - 2)$.

$$\left[\textbf{Ans.: } 11 + 7(x-2) + 4(x-2)^2 + (x-2)^3\right]$$

6. Expand $2x^3 + 3x^2 - 8x + 7$ in terms of $(x - 2)$.

$$\left[\textbf{Ans.: } 19 + 28(x-2) + 15(x-2)^2 + 2(x-2)^3\right]$$

7. Expand $2x^3 + 5x^2 + 3x - 4$ in powers of $(x + 3)$.

$$\left[\textbf{Ans.: } -22 + 27(x+3) - 13(x+3)^2 + 2(x+3)^3\right]$$

8. Expand $\sqrt{x}$ in powers of $(x - a)$.

$$\left[\textbf{Ans.: } \sqrt{a}+\frac{(x-a)}{2\sqrt{a}}-\frac{(x-a)^3}{8a\sqrt{a}}-\dots\right]$$

9. Expand $\sqrt{1+x+2x^2}$ in powers of $(x - 1)$.

$$\left[\textbf{Ans.: } 2+\frac{5}{4}(x-1)+\frac{7}{32}(x-1)^2+\dots\right]$$

10. Expand $\sin x$ in powers of $(x - a)$.

$$\left[\textbf{Ans.: } \sin a+(x-a)\cos a-\frac{(x-a)^2}{2!}\sin a-\frac{(x-a)^3}{3!}\cos a+\dots\right]$$

11. Expand $\cos x$ in powers of $\left(x-\frac{\pi}{2}\right)$.

$$\left[\textbf{Ans.: } -\left(x-\frac{\pi}{2}\right)+\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3-\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5+\dots\right]$$

12. Expand $\tan x$ in powers of $\left(x-\frac{\pi}{4}\right)$.

$$\left[\textbf{Ans.: } 1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\dots\right]$$

13. Expand $\sin\left(\frac{\pi}{6}+x\right)$ in powers of x up to x^4.

$$\left[\textbf{Ans.: } \frac{1}{2}+\frac{\sqrt{3}}{2}x-\frac{1}{2}\cdot\frac{x^2}{2!}-\frac{\sqrt{3}}{2}\cdot\frac{x^3}{3!}+\frac{1}{2}\cdot\frac{x^4}{4!}+\dots\right]$$

14. Expand $\tan\left(\frac{\pi}{4}+x\right)$ in powers of x up to x^4 and hence, find the value of tan (46° 36').

$$\left[\textbf{Ans.: } \left(1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\ldots\right), 1.0574\right]$$

15. Find the approximate value of cos 64°.

[Ans.: 0.4384**]**

16. Expand log x in powers of $(x - 2)$.

$$\left[\textbf{Ans.: } \log 2+\frac{1}{2}(x-2)-\frac{1}{2!}\cdot\frac{(x-2)^2}{4}+\frac{1}{3!}\cdot\frac{(x-2)^3}{4}+\ldots\right]$$

17. Expand $\log\tan\left(\frac{\pi}{4}+x\right)$ in powers of x.

$$\left[\textbf{Ans.: } 2x+\frac{4}{3}x^3+\frac{4}{3}x^5+\ldots\right]$$

18. Expand $7 + (x + 2) + 3(x + 2)^3 + (x + 2)^4$ in powers of x.

[Ans.: $49 + 69x + 42x^2 + 11x^3 + x^4$**]**

19. Expand $17 + 6(x + 2) + 3(x + 2)^3 + (x + 2)^4 - (x + 2)^5$ in powers of x.

[Ans.: $37 - 6x - 38x^2 - 29x^3 - 9x^4 - x^5$**]**

20. Expand $(x - 2)^4 - 3(x - 2)^3 + 4(x - 2)^2 + 5$ in powers of x.

[Ans.: $61 - 84x + 46x^2 - 11x^3 + x^4$**]**

21. Expand $(x + 2)^4 + 5(x + 2)^3 + 6(x + 2)^2 + 7(x + 2) + 8$ in powers of $(x + 1)$.

$$\left[\begin{array}{l}\textbf{Hint: } f(x)=x^4+5x^3+6x^2+7x+8, f[(x+1)+1]=f(1)\\ \qquad +(x+1)f'(1)+\dfrac{(x+1)^2}{2!}f''(1)+\ldots\end{array}\right]$$

$$\left[\textbf{Ans.: } 27+38(x+1)+27(x+1)^2+9(x+1)^3+(x+1)^4\right]$$

22. Prove that $\sinh(x + a) = \sinh a + x\cosh a + \frac{x^2}{2!}\sinh a + \ldots$ If $\sinh(1.5) = 2.1293$, $\cosh(1.5) = 2.3524$, find the value of sinh (1.505).

[Ans.: 2.1411**]**

6.3 MACLAURIN'S SERIES

Statement If $f(x)$ is a given function of x which can be expanded in positive ascending integral powers of x then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots\dots + \frac{x^n}{n!} f^n(0) + \dots\dots\dots$$

Proof Let $f(x)$ be a function of x which can be expanded into positive ascending integral powers of x.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots\dots\dots \qquad \text{... (1)}$$

Differentiating w.r.t. x successively,

$$f'(x) = a_1 + a_2 \cdot 2x + a_3 \cdot 3x^2 + a_4 \cdot 4x^3 + \dots\dots\dots\dots\dots \qquad \text{... (2)}$$

$$f''(x) = a_2 \cdot 2 + a_3 \cdot 6x + a_4 \cdot 12x^2 + \dots\dots\dots\dots\dots \qquad \text{... (3)}$$

$$f'''(x) = a_3 \cdot 6 + a_4 \cdot 24x + \dots\dots\dots\dots\dots \qquad \text{... (4)}$$

and so on.

Putting $x = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(0)$$

$$a_1 = f'(0)$$

$$a_2 = \frac{1}{2!} f''(0)$$

$$a_3 = \frac{1}{3!} f'''(0) \quad \text{and so on.}$$

Substituting a_0, a_1, a_2 and a_3 in Eq. (1),

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots\dots + \frac{x^n}{n!} f^n(0) + \dots\dots\dots$$

This is known as **Maclaurin's series.**
This series can also be written as,

$$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots\dots + \frac{x^n}{n!} y_n(0) + \dots\dots\dots$$

Standard Expansions

Using Maclaurin's series, expansion of some standard functions can be obtained. These expansions can be directly used while solving the examples.

(i) Expansion of e^x (Exponential series)

Proof Let $y = e^x$, $y(0) = e^0 = 1$

Now, $y_n = \dfrac{d^n}{dx^n}(e^x) = e^x$, $y_n(0) = e^0 = 1$

Substituting in Maclaurin's series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots$$

This series is known as the exponential series.

(a) Replacing x by $-x$ in the above series,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\dots\dots$$

(b) Replacing x by ax in the above series,

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \dots\dots\dots$$

(ii) Expansion of sin x (Sine series)

Proof Let $y = \sin x$, $y(0) = \sin 0 = 0$

Now, $y_n = \dfrac{d^n}{dx^n}(\sin x) = \sin\left(x + \dfrac{n\pi}{2}\right)$, $y_n(0) = \sin\left(\dfrac{n\pi}{2}\right)$

Putting $n = 1, 2, 3, 4, 5, \dots\dots$,

$y_1(0) = 1$, $y_2(0) = 0$, $y_3(0) = -1$, $y_4(0) = 0$, $y_5(0) = 1$, and so on.

Substituting in Maclaurin's series,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots\dots$$

This series is known as the sine series.

(iii) Expansion of cos x (Cosine series)

Proof Let $y = \cos x$, $y(0) = \cos 0 = 1$

Now, $y_n = \dfrac{d^n}{dx^n}(\cos x) = \cos\left(x + \dfrac{n\pi}{2}\right)$, $y_n(0) = \cos\left(\dfrac{n\pi}{2}\right)$,

Putting $n = 1, 2, 3, 4, \dots\dots$,

$y_1(0) = 0$, $y_2(0) = -1$, $y_3(0) = 0$, $y_4(0) = 1$, and so on.

Substituting in Maclaurin's series,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\dots\dots$$

This series is known as the cosine series.

(iv) Expansion of tan x (Tangent series)

Proof Let $y = \tan x,$ $\qquad y(0) = 0$

$$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2, \qquad y_1(0) = 1$$

$$y_2 = 2yy_1, \qquad y_2(0) = 2y(0)y_1(0) = 2(0)(1) = 0$$

$$y_3 = 2y_1^{\,2} + 2yy_2, \qquad y_3(0) = 2(1)^2 + 2(0)(0) = 2$$

$$\begin{aligned} y_4 &= 4y_1y_2 + 2y_1y_2 + 2yy_3 & y_4(0) &= 6(1)(0) + 2(0)(2) \\ &= 6y_1y_2 + 2yy_3, & &= 0 \end{aligned}$$

$$\begin{aligned} y_5 &= 6y_2^{\,2} + 6y_1y_3 + 2y_1y_3 + 2yy_4 & y_5(0) &= 0 + 8(1)(2) + 0 \\ &= 6y_2^{\,2} + 8y_1y_3 + 2yy_4, & &= 16 \end{aligned}$$

Substituting in Maclaurin's series,

$$\begin{aligned} \tan x &= x + \frac{x^3}{3!}(2) + \frac{x^5}{5!}(16) + \ldots\ldots\ldots \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots\ldots \end{aligned}$$

This series is known as the tangent series.

***Note*:** This series can also be obtained by dividing the sine and cosine series since $\tan x = \dfrac{\sin x}{\cos x}$.

(v) Expansion of sinh x

Proof $\sinh\ x = \dfrac{e^x - e^{-x}}{2}$

Substituting exponential series e^x and e^{-x},

$$\begin{aligned} \sinh x &= \frac{\left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\right) - \left(1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \ldots\right)}{2} \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots \end{aligned}$$

(vi) Expansion of cosh x

Proof $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Substituting exponential series e^x and e^{-x},

$$\begin{aligned} \cosh x &= \frac{\left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\right) + \left(1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \ldots\right)}{2} \\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots \end{aligned}$$

(vii) Expansion of tanh x

Proof Expansion of tanh x can be obtained by dividing the series of sinh x and cosh x.

$$\tanh x = \frac{\sinh x}{\cosh x}$$
$$= \frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots}$$
$$= x - \frac{x^3}{3} + \frac{2}{15}x^5 - \ldots\ldots\ldots$$

***Note*:** This series can also be obtained by using Maclaurin's series (refer tangent series).

(viii) Expansion of log (1 + x) (Logarithmic series)

Proof Let $y = \log(1 + x)$, $y(0) = \log 1 = 0$

Now, $$y_n = \frac{d^n}{dx^n}[\log(1+x)] = (-1)^{n-1} \cdot \frac{(n-1)!}{(x+1)^n}$$

$$y_n(0) = (-1)^{n-1} \cdot (n-1)!$$

Putting $n = 1, 2, 3, 4, \ldots..$

$$y_1(0) = 1,\ y_2(0) = -1,\ y_3(0) = 2! \quad \text{and so on}$$

Substituting in Maclaurin's series,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots$$

This series is known as the logarithmic series and is valid for $-1 < x < 1$.

***Note*:** In the above series, replacing x by $-x$, we get expansion of log $(1 - x)$.

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$$

(ix) Expansion of $(1 + x)^m$ (Binomial series)

Proof Let $y = (1+x)^m$, $y(0) = (1+0)^m = 1$

Now, $$y_n = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)(1+x)^{m-n}$$
$$y_n(0) = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)$$

Putting $n = 1, 2, 3, 4, \ldots..$

$$y_1(0) = m,\ y_2(0) = m(m-1),\ y_3(0) = m(m-1)(m-2) \text{ and so on}$$

Substituting in Maclaurin's series,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots\dots\dots$$

This series is known as the binomial series and is valid for $-1 < x < 1$.

By Definition

Example 1

Expand 5^x up to the first three non-zero terms of the series.

Solution

Let $\quad f(x) = 5^x$

By Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots\dots\dots \quad \dots(1)$$

$$f(x) = 5^x, \qquad f(0) = 5^0 = 1$$

$$f'(x) = 5^x \log 5, \qquad f'(0) = 5^0 \log 5 = \log 5$$

$$f''(x) = 5^x (\log 5)^2, \qquad f''(0) = 5^0 (\log 5)^2 = (\log 5)^2$$

Substituting in Eq. (1),

$$5^x = 1 + x\log 5 + \frac{x^2}{2!}(\log 5)^2 + \dots\dots\dots$$

Aliter: $\quad f(x) = 5^x = e^{\log 5^x} = e^{x \log 5}$

$$= 1 + x\log 5 + \frac{(x\log 5)^2}{2!} + \dots\dots\dots \quad \text{[Using exponential series]}$$

Example 2

Obtain the series $\log(1+x)$ *and find the series* $\log\left(\frac{1+x}{1-x}\right)$ *and hence, find the value of* $\log_e\left(\frac{11}{9}\right)$. **[Winter 2016]**

Solution

Let $\quad y = \log(1+x)$

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \dots\dots\dots \quad \dots(1)$$

$$y = \log(1+x), \qquad y(0) = 0$$

$$y_1 = \frac{1}{1+x}, \qquad y_1(0) = 1$$

$$y_2 = -\frac{1}{(1+x)^2}, \qquad y_2(0) = -1$$

$$y_3 = \frac{(2!)}{(1+x)^3}, \qquad y_3(0) = 2!$$

$$y_4 = -\frac{(3!)}{(1+x)^4}, \qquad y_4(0) = -(3!)$$

Substituting in Eq. (1),

$$y = 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!}(2!) - \frac{x^4}{4!}(3!) + \dots\dots\dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots$$

Replacing x by $-x$,

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\dots\dots$$

Now, $\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

Putting $x = \frac{1}{10}$, and considering first three terms,

$$\log_e\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\cdot\frac{1}{(10)^3} + \frac{1}{5}\cdot\frac{1}{(10)^5}\right] = 0.20067$$

Example 3

If $x^3 + y^3 + xy - 1 = 0$, *prove that* $y = 1 - \frac{x}{3} - \frac{26}{81}x^3 - \cdots$.

Solution

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots\cdots \qquad \text{...(1)}$$

$$x^3 + y^3 + xy - 1 = 0 \qquad \text{...(2)}$$

Putting $x = 0, y(0) = 1$

Differentiating Eq. (2) w.r.t. x,

$$3x^2 + 3y^2 y_1 + xy_1 + y = 0 \qquad ...(3)$$

Putting $x = 0$, $y_1(0) = -\frac{1}{3}$

Differentiating Eq. (3) w.r.t. x,

$$6x + 6yy_1^2 + 3y^2 y_2 + 2y_1 + xy_2 = 0 \qquad ...(4)$$

Putting $x = 0$,

$$6\left(-\frac{1}{3}\right)^2 + 3y_2(0) + 2\left(-\frac{1}{3}\right) = 0$$

$$y_2(0) = 0$$

Differentiating Eq. (4) w.r.t. x,

$$6 + 6y_1^3 + 12yy_1y_2 + 3y^2y_3 + 6yy_1y_2 + 3y_2 + xy_3 = 0$$

Putting $x = 0$,

$$6 + 6\left(-\frac{1}{27}\right) + 0 + 3y_3(0) = 0$$

$$y_3(0) = \frac{-52}{27} \text{ and so on.}$$

Substituting in Eq. (1),

$$y = 1 - \frac{x}{3} + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\left(-\frac{52}{27}\right) + \cdots\cdots$$

$$= 1 - \frac{x}{3} - \frac{26}{81}x^3 - \cdots\cdots$$

Example 4

If $x^3 + 2xy^2 - y^3 + x - 1 = 0$, *expand* y *in ascending powers of* x.

Solution

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \cdots\cdots \qquad ...(1)$$

$$x^3 + 2xy^2 - y^3 + x - 1 = 0 \qquad ...(2)$$

Putting $x = 0, y(0) = -1$

Differentiating Eq. (2) w.r.t. x,

$$3x^2 + 2y^2 + 4xyy_1 - 3y^2y_1 + 1 = 0 \qquad ...(3)$$

Putting $x = 0$,

$$2 - 3y_1(0) + 1 = 0$$
$$y_1(0) = 1$$

Differentiating Eq. (3) w.r.t. x,

$$6x + 4yy_1 + 4yy_1 + 4xy_1^2 + 4xyy_2 - 6yy_1^2 - 3y^2y_2 = 0$$

Putting $x = 0$,

$$-8 + 6 - 3y_2(0) = 0$$
$$y_2(0) = -\frac{2}{3} \text{ and so on.}$$

Substituting in Eq. (1),

$$y = -1 + x + \frac{x^2}{2!}\left(-\frac{2}{3}\right) + \cdots\cdots$$
$$= -1 + x - \frac{x^2}{3} + \cdots\cdots$$

Example 5

If $x = y(1 + y^2)$, *prove that* $y = x - x^3 + 3x^5 + \ldots$.

Solution

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \cdots\cdots \quad \text{... (1)}$$

$$x = y(1 + y^2) \quad \text{.... (2)}$$

Putting $x = 0$, $y(0) = 0$

Differentiating Eq. (2) w.r.t. x,

$$1 = y_1 + 3y^2y_1 \quad \text{... (3)}$$

Putting $x = 0$,

$$1 = y_1(0)$$

$$y_1(0) = 1$$

Differentiating Eq. (3) w.r.t. x,

$$0 = y_2 + 6yy_1^2 + 3y^2y_2 \quad \text{... (4)}$$

Putting $x = 0$, $y_2(0) = 0$,

Differentiating Eq. (4) w.r.t. x,

$$0 = y_3 + 12yy_1y_2 + 6y_1^3 + 6yy_1y_2 + 3y^2y_3$$

$$0 = y_3\,(1 + 3y^2) + 18yy_1y_2 + 6y_1^3 \qquad \ldots (5)$$

Putting $x = 0$,

$$0 = y_3\,(0) + 6$$
$$y_3\,(0) = -6$$

Differentiating Eq. (5) w.r.t. x,

$$0 = (1 + 3y^2)\,y_4 + 6yy_1y_3 + 18y_1^2\,y_2 + 18yy_2^2 + 18yy_1y_3 + 18y_1^2y_2$$
$$= (1 + 3y^2)\,y_4 + 24yy_1y_3 + 36y_1^2\,y_2 + 18yy_2^2 \qquad \ldots (6)$$

Putting $x = 0$, $y_4\,(0) = 0$,

Differentiating Eq. (6) w.r.t. x,

$$0 = (1 + 3y^2)\,y_5 + 6yy_1y_4 + 24y_1^2\,y_3 + 24yy_2y_3 + 24yy_1y_4 + 72y_1y_2^2$$
$$+ 36{y_1}^2y_3 + 36yy_2y_3 + 18y_1{y_2}^2$$

Putting $x = 0$,

$$0 = y_5\,(0) + 24\,(-6) + 36\,(-6)$$
$$y_5\,(0) = 360 \text{ and so on.}$$

Substituting in Eq. (1),

$$y = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!}(-6) + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot 360 + \cdots\cdots$$
$$= x - x^3 + 3x^5 + \cdots\cdots$$

By Standard Expansion

Example 1

Obtain the expansion of $\dfrac{1+x^2}{1+x^4}$.

Solution

$$\frac{1+x^2}{1+x^4} = (1+x^2)(1+x^4)^{-1}$$
$$= (1 + x^2)\,(1 - x^4 + x^8 - x^{12} + x^{16} - \ldots)$$
$$= 1 + x^2 - x^4 - x^6 + x^8 + x^{10} - \ldots$$

Example 2

If $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots,$ *prove that*
$y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots\cdots$ *and conversely.*

Solution

$$x = \log(1+y)$$
$$1 + y = e^x$$
$$y = e^x - 1$$
$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

Conversely,

$$y = e^x - 1$$
$$e^x = 1 + y$$
$$x = \log(1+y)$$
$$= y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots$$

Example 3

Expand $\sqrt{1 + \sin x}.$

Solution

$$\sqrt{1+\sin x} = \sin\frac{x}{2} + \cos\frac{x}{2}$$
$$= \left[\frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right] + \left[1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4}\left(\frac{x}{2}\right)^4 - \cdots\right]$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \cdots$$

Example 4

Prove that $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots.$

Solution

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right]$$

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots$$

Example 5

Show that $\sin^2 x = x^2 - \dfrac{x^4}{3} + \dfrac{2}{45}x^6 + \cdots$.

Solution

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$= \frac{1}{2}\left(1 - 1 + \frac{4x^2}{2!} - \frac{16x^4}{4!} + \frac{64x^6}{6!} - \cdots\right)$$

$$= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \cdots$$

Example 6

Prove that $\cosh^3 x = \dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n} + 3}{(2n)!} x^{2n}$.

Solution

$$\cosh^3 x = \frac{1}{4}(\cosh 3x + 3\cosh x)$$

$$= \frac{1}{4}\left[\left(1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \cdots\right) + 3\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\right]$$

$$= \frac{1}{4}\left[(1 + 3) + \frac{3^2 + 3}{2!}x^2 + \frac{3^4 + 3}{4!}x^4 + \cdots\right]$$

$$= \frac{1}{4}\sum_{n=0}^{\infty} \frac{3^{2n} + 3}{(2n)!} x^{2n}$$

Example 7

Prove that $\sin x \sinh x = x^2 - \dfrac{8}{6!}x^6 + \dfrac{32}{10!}x^{10} - \ldots$.

Solution

$$\sin x \sinh x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\right)$$

$$= x^2 + x^6\left[\frac{2}{5!} - \frac{1}{(3!)^2}\right] + x^{10}\left[\frac{2}{9!} - \frac{2}{7!3!} + \frac{1}{(5!)^2}\right] + \ldots$$

$$= x^2 - \frac{8}{6!}x^6 + \frac{32}{10!}x^{10} - \ldots$$

Example 8

Prove that $\cos x \cosh x = 1 - \dfrac{2^2 x^4}{4!} + \dfrac{2^2 x^8}{8!} - \cdots$.

Solution

$$\cos x \cosh x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\right)$$

$$= 1 + x^4\left(\frac{2}{4!} - \frac{1}{(2!)^2}\right) + x^8\left[\frac{2}{8!} - \frac{2}{6!2!} + \frac{1}{(4!)^2}\right] + \cdots$$

$$= 1 - \frac{2^2 x^4}{4!} + \frac{2^2 x^8}{8!} - \cdots$$

Example 9

Expand $\sin x \cosh x$ *in ascending powers of x up to* x^5.

Solution

$$\sin x \cosh x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$= x\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right) - \frac{x^3}{3!}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^5}{5!}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots$$

$$= x + \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^3}{6} - \frac{x^5}{6.2} + \frac{x^5}{120} + \cdots$$

[Considering the terms only up to x^5]

$$= x + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

Example 10

Prove that $\log x = \log 2 + \left(\dfrac{x}{2} - 1\right) - \dfrac{1}{2}\left(\dfrac{x}{2} - 1\right)^2 + \dfrac{1}{3}\left(\dfrac{x}{2} - 1\right)^3 + \cdots$.

Solution

$$\log x = \log\left(2 \cdot \frac{x}{2}\right)$$

$$= \log 2 + \log \frac{x}{2}$$

$$= \log 2 + \log\left[1 + \left(\frac{x}{2} - 1\right)\right]$$

$$= \log 2 + \left(\frac{x}{2} - 1\right) - \frac{1}{2}\left(\frac{x}{2} - 1\right)^2 + \frac{1}{3}\left(\frac{x}{2} - 1\right)^3 - \ldots$$

Example 11

Expand $\log(1 + x + x^2 + x^3)$ *up to* x^8.

Solution

$$\log(1 + x + x^2 + x^3) = \log[(1 + x)(1 + x^2)]$$

$$= \log(1 + x) + \log(1 + x^2)$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \ldots\right] + \left[x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \frac{(x^2)^4}{4} + \ldots\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3}{4}x^4 + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3}{8}x^8 + \ldots$$

Example 12

Prove that $\log(1 + x + x^2 + x^3 + x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$.

Solution

$$\log(1 + x + x^2 + x^3 + x^4) = \log\left(\frac{1 - x^5}{1 - x}\right) \qquad \text{[Using sum of G.P.]}$$

$$= \log(1 - x^5) - \log(1 - x)$$

$$= \left(-x^5 - \frac{x^{10}}{2} - \frac{x^{15}}{3} - \frac{x^{20}}{4} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots\right)$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$$

Example 13

Prove that $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$.

Solution

$$\log(1+\sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} - \frac{\sin^4 x}{4} + \cdots$$
$$= \left(x - \frac{x^3}{3!} + \cdots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{3!} + \cdots\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{3!} + \cdots\right)^4 + \cdots$$
$$= x - \frac{x^2}{2} + x^3\left(-\frac{1}{6} + \frac{1}{3}\right) + x^4\left(\frac{1}{6} - \frac{1}{4}\right) + \cdots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$$

Example 14

Prove that $\log(1+\tan x) = x - \dfrac{x^2}{2} + \dfrac{2x^3}{3} + \cdots$.

Solution

$$\log(1+\tan x) = \tan x - \frac{\tan^2 x}{2} + \frac{\tan^3 x}{3} - \cdots$$
$$= \left(x + \frac{x^3}{3} + \cdots\right) - \frac{1}{2}\left(x + \frac{x^3}{3} + \cdots\right)^2 + \frac{1}{3}(x + \cdots)^3 - \cdots$$
$$= x - \frac{x^2}{2} + x^3\left(\frac{1}{3} + \frac{1}{3}\right) + \cdots$$
$$= x - \frac{x^2}{2} + \frac{2}{3}x^3 - \cdots$$

Example 15

Prove that $\log\left(\dfrac{\tan x}{x}\right) = \dfrac{x^3}{3} + \dfrac{7}{90}x^4 + \cdots$.

Solution

$$\log\left(\frac{\tan x}{x}\right) = \log\left[\frac{1}{x}\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots\right)\right]$$
$$= \log\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \cdots\right)$$
$$= \left(\frac{x^2}{3} + \frac{2}{15}x^4 + \cdots\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \cdots\right)^2 + \cdots$$

$$= \frac{x^2}{3} + x^4\left(\frac{2}{15} - \frac{1}{18}\right) + \cdots$$

$$= \frac{x^2}{3} + \frac{7}{90}x^4 + \cdots$$

Example 16

Prove that $\log\left(\frac{\sinh x}{x}\right) = \frac{x^2}{6} - \frac{x^4}{180} + \ldots.$

Solution

$$\log\left(\frac{\sinh x}{x}\right) = \log\left[\frac{1}{x}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)\right]$$

$$= \log\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)$$

$$= \left(\frac{x^2}{3!} + \frac{x^4}{5!}\cdots\right) - \frac{1}{2}\left(\frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)^2 + \cdots\cdots$$

$$= x^2 + x^4\left(\frac{1}{120} - \frac{1}{72}\right) + \cdots$$

$$= \frac{x^2}{6} - \frac{x^4}{180} + \cdots\cdots$$

Example 17

Prove that $\log(x\cot x) = -\frac{x^2}{3} - \frac{7}{90}x^4 + \cdots$.

Solution

$$\log(x\cot x) = -\log\left(\frac{1}{x\cot x}\right)$$

$$= -\log\left(\frac{\tan x}{x}\right)$$

$$= -\log\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)$$

$$= -\left[\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots.\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)^2 + \ldots\right]$$

$$= -\left[\frac{x^2}{3} + x^4\left(\frac{2}{15} - \frac{1}{18}\right) + \ldots\right]$$

$$= -\frac{x^2}{3} - \frac{7}{90}x^4 + \ldots$$

Example 18

Expand $\left[\log(1+x)\right]^2$ *in ascending powers of x.*

Solution

$$\begin{aligned}
\left[\log(1+x)\right]^2 &= \left[\log(1+x)\right]\cdot\left[\log(1+x)\right] \\
&= \left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right) \\
&= x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-\frac{x^2}{2}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right) \\
&\quad +\frac{x^3}{3}\left(x-\frac{x^2}{2}+\cdots\right)-\frac{x^4}{4}(x-\cdots)
\end{aligned}$$

[Considering the terms only up to x^5]

$$\begin{aligned}
&= x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}-\frac{x^3}{2}+\frac{x^4}{4}-\frac{x^5}{6}+\frac{x^4}{3}-\frac{x^5}{6}-\frac{x^5}{4}+\cdots \\
&= x^2-x^3+\frac{11}{12}x^4-\frac{10}{12}x^5+\cdots
\end{aligned}$$

Example 19

Prove that $\log\left(\frac{1+e^{2x}}{e^x}\right)=\log 2+\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{45}-\cdots.$

Solution

$$\begin{aligned}
\log\left(\frac{1+e^{2x}}{e^x}\right) &= \log\left(e^{-x}+e^x\right) \\
&= \log(2\cosh x) \\
&= \log 2+\log\cosh x \\
&= \log 2+\log\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots\right) \\
&= \log 2+\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots\right)-\frac{1}{2}\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)^2+\frac{1}{3}\left(\frac{x^2}{2!}+\ldots\right)^3+\ldots \\
&= \log 2+\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots\right)-\frac{1}{2}\left(\frac{x^4}{4}+2\cdot\frac{x^6}{48}+\ldots\right)+\frac{1}{3}\left(\frac{x^6}{8}+\ldots\right)+\ldots \\
&= \log 2+\frac{x^2}{2}+x^4\left(\frac{1}{24}-\frac{1}{8}\right)+x^6\left(\frac{1}{720}-\frac{1}{48}+\frac{1}{24}\right)+\ldots \\
&= \log 2+\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{45}-\ldots
\end{aligned}$$

Example 20

Expand $\log\left(\dfrac{xe^x}{e^x-1}\right)$ *in ascending powers of x up to the terms in* x^4.

Solution

$$\begin{aligned}
\log\left(\frac{xe^x}{e^x-1}\right) &= -\log\left(\frac{e^x-1}{xe^x}\right)\\
&= -\log\left(\frac{1-e^{-x}}{x}\right)\\
&= -\log\left[\frac{1}{x}\left\{1-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\cdots\right)\right\}\right]\\
&= -\log\left[\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{24}+\frac{x^5}{120}+\cdots\right)\right]\\
&= -\log\left[1-\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\frac{x^4}{120}+\cdots\right)\right]
\end{aligned}$$

$$= -\left[-\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\frac{x^4}{120}+\cdots\right)-\frac{1}{2}\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\cdots\right)^2-\frac{1}{3}\left(\frac{x}{2}-\frac{x^2}{6}+\cdots\right)^3\right.$$

$$\left.-\frac{1}{4}\left(\frac{x}{2}-\cdots\right)^4-\cdots\right] \qquad \text{[Considering the terms only up to } x^4\text{]}$$

$$=\left(\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\frac{x^4}{120}+\cdots\right)+\frac{1}{2}\left[\frac{x^2}{4}+\frac{x^4}{36}+2\left(\frac{x}{2}\right)\left(-\frac{x^2}{6}\right)+2\left(\frac{x}{2}\right)\left(\frac{x^3}{24}\right)+\cdots\right]$$

$$+\frac{1}{3}\left[\frac{x^3}{8}+3\left(\frac{x}{2}\right)^2\left(-\frac{x^2}{6}\right)+\cdots\right]+\frac{1}{4}\cdot\frac{x^4}{16}+\cdots$$

$$=\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\frac{x^4}{120}+\frac{x^2}{8}+\frac{x^4}{72}-\frac{x^3}{12}+\frac{x^4}{48}+\frac{x^3}{24}-\frac{x^4}{24}+\frac{x^4}{64}+\cdots$$

$$=\frac{x}{2}-\frac{x^2}{24}+\frac{x^4}{2880}+\cdots$$

Example 21

Prove that $\log(1+e^x)=\log 2+\dfrac{x}{2}+\dfrac{x^2}{8}-\dfrac{x^4}{192}+\cdots$.

Solution

$$\log(1+e^x) = \log\left(1+1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)$$
$$= \log\left[2\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)\right]$$
$$= \log 2+\log\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)$$
$$= \log 2+\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)-\frac{1}{2}\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\ldots\right)^2+\frac{1}{3}\left(\frac{x}{2}+\frac{x^2}{4}+\ldots\right)^3$$
$$-\frac{1}{4}\left(\frac{x}{2}+\ldots\right)^4+\ldots$$
$$= \log 2+\left(\frac{x}{2}\right)+x^2\left(\frac{1}{4}-\frac{1}{8}\right)+x^3\left(\frac{1}{12}-\frac{1}{8}+\frac{1}{24}\right)+x^4\left(\frac{1}{48}-\frac{1}{32}-\frac{1}{24}+\frac{1}{16}-\frac{1}{64}\right)+\ldots$$
$$= \log 2+\frac{x}{2}+\frac{x^2}{8}+0+\left(-\frac{1}{192}\right)x^4+\ldots$$
$$= \log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots$$

Example 22

Prove that $\log\left[\log(1+x)^{\frac{1}{x}}\right] = -\frac{x}{2}+\frac{5x^2}{24}-\frac{x^3}{8}+\frac{251}{2880}x^4+\ldots.$

Solution

$$\log(1+x)^{\frac{1}{x}} = \frac{1}{x}\log(1+x)$$
$$= \frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right)$$
$$= 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\cdots$$
$$= 1-\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\frac{x^4}{5}+\cdots\right)$$
$$= 1-y$$

Now, $$\log\left[\log(1+x)^{\frac{1}{x}}\right] = \log(1-y)$$

$$= -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots$$

$$= -\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} - \frac{x^4}{5} + \ldots\right) - \frac{1}{2}\cdot\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} - \ldots\right)^2$$

$$-\frac{1}{3}\left(\frac{x}{2} - \frac{x^2}{3} - \ldots\right)^3 - \frac{1}{4}\left(\frac{x}{2} - \frac{x^2}{3} + \ldots\right)^4 - \ldots$$

$$= -\frac{x}{2} + x^2\left(\frac{1}{3} - \frac{1}{8}\right) - x^3\left(\frac{1}{4} - \frac{1}{6} + \frac{1}{24}\right) + x^4\left(\frac{1}{5} - \frac{1}{18} - \frac{1}{8} + \frac{1}{12} - \frac{1}{64}\right) + \ldots$$

$$= -\frac{x}{2} + \frac{5x^2}{24} - \frac{x^3}{8} + \frac{251}{2880}x^4 + \ldots$$

Example 23

Expand $\left(\frac{1+e^x}{2e^x}\right)^{\frac{1}{2}}$ *up to the term containing* x^2.

Solution

$$\left(\frac{1+e^x}{2e^x}\right)^{\frac{1}{2}} = \left(\frac{1}{2}e^{-x} + \frac{1}{2}\right)^{\frac{1}{2}}$$

$$= \left[\frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \cdots\right) + \frac{1}{2}\right]^{\frac{1}{2}}$$

$$= \left(1 - \frac{1}{2}x + \frac{x^2}{4} - \cdots\right)^{\frac{1}{2}}$$

$$= \left[1 - \left(\frac{x}{2} - \frac{x^2}{4} + \cdots\right)\right]^{\frac{1}{2}}$$

$$= 1 - \frac{1}{2}\left(\frac{x}{2} - \frac{x^2}{4} + \cdots\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(\frac{x}{2} - \frac{x^2}{4} + \cdots\right)^2 - \cdots$$

$$= 1 - \frac{x}{4} + \frac{x^2}{8} - \frac{1}{8}\cdot\frac{x^2}{4} + \cdots$$

$$= 1 - \frac{x}{4} + \frac{3}{32}x^2 + \cdots$$

Example 24

Expand $e^{\cos x}$ *up to* x^4.

Solution

$$
\begin{aligned}
y &= e^{\cos x}\\
&= e^{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)}\\
&= e\, e^{\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)}\\
&= e\left[1+\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)+\frac{1}{2!}\left(-\frac{x^2}{2!}+\cdots\right)^2+\cdots\right]\\
&= e\left(1-\frac{x^2}{2!}+\frac{x^4}{24}+\frac{x^4}{8}+\cdots\right)\\
&= e\left(1-\frac{x^2}{2!}+\frac{x^4}{6}-\cdots\right)
\end{aligned}
$$

Example 25

Prove that $e^x \cos x = 1 + x - \dfrac{x^3}{3} - \dfrac{x^4}{6} + \cdots$.

Solution

$$
\begin{aligned}
e^x \cos x &= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)\\
&= 1\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+\frac{x^2}{2!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\\
&\quad+\frac{x^3}{3!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)+\frac{x^4}{4!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+\cdots\\
&= 1-\frac{x^2}{2}+\frac{x^4}{24}+x-\frac{x^3}{2}+\frac{x^2}{2}-\frac{x^4}{4}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots
\end{aligned}
$$

[Considering the terms only up to x^4]

$$
= 1+x-\frac{x^3}{3}-\frac{x^4}{6}+\cdots
$$

Example 26

Prove that $e^{x\cos x} = 1 + x + \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{11}{24}x^4 - \dfrac{x^5}{5} + \cdots$.

Solution

$$
\begin{aligned}
e^{x\cos x} &= e^{x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)} \\
&= 1 + \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \ldots\right) + \frac{1}{2!}\left(x - \frac{x^3}{2!} + \ldots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{2!} + \ldots\right)^3 \\
&\qquad + \frac{1}{4!}\left(x - \frac{x^3}{2!} - \ldots\right)^4 + \frac{1}{5!}\left(x - \frac{x^3}{2!} - \ldots\right)^5 \\
&= 1 + x + \frac{x^2}{2} + x^3\left(-\frac{1}{2} + \frac{1}{6}\right) + x^4\left(-\frac{1}{2} + \frac{1}{24}\right) + x^5\left(\frac{1}{24} - \frac{1}{4} + \frac{1}{120}\right) + \ldots \\
&= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 - \frac{x^5}{5} + \ldots
\end{aligned}
$$

Example 27

Prove that $e^{x\sin x} = 1 + x^2 + \dfrac{x^4}{3} + \dfrac{x^6}{120} + \cdots$.

Solution

$$
\begin{aligned}
e^{x\sin x} &= \left[1 + x\sin x + \frac{(x\sin x)^2}{2!} + \frac{(x\sin x)^3}{3!} + \cdots\right] \\
&= 1 + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \frac{x^2}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \frac{x^3}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^3 + \cdots \\
&= 1 + x^2 + x^4\left(-\frac{1}{6} + \frac{1}{2}\right) + x^6\left(\frac{1}{120} - \frac{1}{6} + \frac{1}{6}\right) + \cdots \\
&= 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \cdots
\end{aligned}
$$

Example 28

Prove that $e^{e^x} = e\left(1 + x + x^2 + \dfrac{5x^3}{6} + \cdots\right)$.

Solution

$$
\begin{aligned}
e^{e^x} &= e^{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)} \\
&= ee^{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots} \\
&= e\left[1+\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)+\frac{1}{2!}\left(x+\frac{x^2}{2!}+\cdots\right)^2+\frac{1}{3!}(x+\ldots)^3+\cdots\right] \\
&= e\left[1+x+x^2\left(\frac{1}{2}+\frac{1}{2}\right)+x^3\left(\frac{1}{6}+\frac{1}{2}+\frac{1}{6}\right)+\cdots\right] \\
&= e\left(1+x+x^2+\frac{5}{6}x^3+\cdots\right)
\end{aligned}
$$

Example 29

Expand $(1 + x)^x$ *in a series up to the term in* x^4.

Solution

$$
\begin{aligned}
(1+x)^x &= e^{\log(1+x)^x} \\
&= e^{x\log(1+x)} \\
&= e^{x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)} \\
&= e^{\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)} \\
&= 1+\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)+\frac{1}{2!}\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)^2+\cdots \\
&= 1+\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)+\frac{1}{2}\left(x^4+\cdots\right)+\cdots
\end{aligned}
$$

[Considering the terms only up to x^4]

$$
= 1+x^2-\frac{x^3}{2}+\frac{5}{6}x^4+\cdots
$$

Example 30

Prove that $(1+x)^{\frac{1}{x}} = e-\frac{e}{2}x+\frac{11e}{24}x^2+\cdots\cdots.$

Solution

$$
(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)}
$$

$$= e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right)}$$

$$= e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\dots\right)}$$

$$= ee^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\dots\right)}$$

$$= e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\cdots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots\right)^2+\cdots\right]$$

$$= e\left[1-\frac{x}{2}+x^2\left(\frac{1}{3}+\frac{1}{8}\right)+\cdots\right]$$

$$= e-\frac{e}{2}x+\frac{11e}{24}x^2+\cdots$$

Example 31

Expand $(1+x)^{(1+x)}$ *up to the term containing* x^3.

Solution

$$(1+x)^{(1+x)} = e^{\log(1+x)^{(1+x)}}$$

$$= e^{(1+x)\log(1+x)}$$

$$= e^{(1+x)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}$$

$$= e^{\left(x-\frac{x^2}{2}+\frac{x^3}{3}+x^2-\frac{x^3}{2}+\cdots\right)}$$ [Considering the terms only up to x^3]

$$= e^{\left(x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots\right)}$$

$$= 1+\left(x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots\right)+\frac{1}{2!}\left(x+\frac{x^2}{2}-\cdots\right)^2+\frac{1}{3!}(x+\cdots)^3+\cdots$$

[Considering the terms only up to x^3]

$$= 1+\left(x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots\right)+\frac{1}{2}\left(x^2+2\cdot x\cdot\frac{x^2}{2}+\cdots\right)+\frac{1}{6}\left(x^3+\cdots\right)+\cdots$$

$$= 1+x+x^2+\frac{x^3}{2}+\cdots$$

Example 32

Prove that $\sin(e^x - 1) = x + \dfrac{x^2}{2} - \dfrac{5}{24}x^4 + \cdots.$

Solution

$$\begin{aligned}
\sin(e^x - 1) &= \sin\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \\
&= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \frac{1}{3!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3 + \cdots \\
&= x + \frac{x^2}{2} + x^3\left(\frac{1}{6} - \frac{1}{6}\right) + x^4\left(\frac{1}{24} - \frac{1}{4}\right) + \cdots \\
&= x + \frac{x^2}{2} - \frac{5}{24}x^4 + \cdots
\end{aligned}$$

Example 33

Prove that $x \operatorname{cosec} x = 1 + \dfrac{x^2}{6} + \dfrac{7}{360}x^4 + \cdots$

Solution

$$\begin{aligned}
x \operatorname{cosec} x &= \frac{x}{\sin x} \\
&= \frac{x}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} \\
&= \frac{1}{\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots\right)} \\
&= \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} - \cdots\right)\right]^{-1} \\
&= \left[1 + \left(\frac{x^2}{6} - \frac{x^4}{120} - \cdots\right) + \left(\frac{x^3}{6} - \cdots\right)^2 + \cdots\right]
\end{aligned}$$

[Using $(1-x)^{-1} = 1 + x + x^2 + \cdots$]

$$= 1 + \left(\frac{x^2}{6} - \frac{x^4}{120} - \cdots\right) + \left(\frac{x^4}{36} - \cdots\right) + \cdots$$

[Considering the terms only up to x^4]

$$= 1 + \frac{x^2}{6} + \left(-\frac{1}{120} + \frac{1}{36}\right)x^4 + \cdots$$

$$= 1 + \frac{x^2}{6} + \frac{7}{360}x^4$$

Example 34

Expand $\dfrac{x}{e^x - 1}$ *up to* x^4 *and hence, prove that*

$$\frac{x}{2}\frac{e^x + 1}{e^x - 1} = 1 + \frac{x^2}{12} - \frac{x^4}{720} + \cdots.$$

Solution

$$\frac{x}{e^x - 1} = \frac{x}{\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right) - 1\right]}$$

$$= \frac{x}{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)}$$

$$= \left[1 + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right)\right]^{-1}$$

$$= 1 - \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots\right) + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots\right)^2 - \left(\frac{x}{2} + \frac{x^2}{6} + \cdots\right)^3 + \left(\frac{x}{2} + \cdots\right)^4$$

$$= 1 - \frac{x}{2} + x^2\left(-\frac{1}{6} + \frac{1}{4}\right) + x^3\left(-\frac{1}{24} + \frac{1}{6} - \frac{1}{8}\right) + x^4\left(-\frac{1}{120} + \frac{1}{36} + \frac{1}{24} - \frac{1}{8} + \frac{1}{16}\right) + \cdots$$

$$= 1 - \frac{x}{2} + \frac{x^2}{12} + x^3(0) - \frac{x^4}{720} + \cdots \qquad \ldots(1)$$

$$\frac{x}{2}\frac{e^x + 1}{e^x - 1} = \frac{x}{2}\left(1 + \frac{2}{e^x - 1}\right)$$

$$= \frac{x}{2} + \frac{x}{e^x - 1}$$

$$= \frac{x}{2} + 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots \qquad \text{[Using Eq. (1)]}$$

$$= 1 + \frac{x^2}{12} - \frac{x^4}{720} + \cdots$$

Example 35

Prove that

$$\tan^{-1}\left(\frac{x\sin\theta}{1 - x\cos\theta}\right) = x\sin\theta + \frac{x^2}{2}\sin 2\theta + \frac{x^3}{3}\sin 3\theta + \cdots.$$

Solution

Let
$$y = \tan^{-1}\left(\frac{x\sin\theta}{1 - x\cos\theta}\right)$$

$$\tan y = \frac{x\sin\theta}{1 - x\cos\theta}$$

$$\frac{e^{iy} - e^{-iy}}{i(e^{iy} + e^{-iy})} = \frac{x\sin\theta}{1 - x\cos\theta}$$

$$\frac{e^{iy} - e^{-iy}}{e^{iy} + e^{-iy}} = \frac{ix\sin\theta}{1 - x\cos\theta}$$

Applying componendo–dividendo,

$$\frac{e^{iy}}{e^{-iy}} = \frac{1 - x(\cos\theta - i\sin\theta)}{1 - x(\cos\theta + i\sin\theta)}$$

$$e^{2iy} = \frac{1 - xe^{-i\theta}}{1 - xe^{i\theta}}$$

$$2iy = \log(1 - xe^{-i\theta}) - \log(1 - xe^{i\theta})$$

$$= \left(-xe^{-i\theta} - \frac{x^2e^{-2i\theta}}{2} - \frac{x^3e^{-3i\theta}}{3} - \dots\right) - \left(-xe^{i\theta} - \frac{x^2e^{2i\theta}}{2} - \frac{x^3e^{3i\theta}}{3} - \dots\right)$$

$$= x(e^{i\theta} - e^{-i\theta}) + \frac{x^2}{2}(e^{2i\theta} - e^{-2i\theta}) + \frac{x^3}{3}(e^{3i\theta} - e^{-3i\theta}) + \dots$$

$$= x \cdot 2i\sin\theta + \frac{x^2}{2} \cdot 2i\sin 2\theta + \frac{x^3}{3} \cdot 2i\sin 3\theta + \dots$$

$$y = x\sin\theta + \frac{x^2}{2}\sin 2\theta + \frac{x^3}{3}\sin 3\theta + \dots$$

Example 36

Prove that $e^{ax}\cos bx = 1 + ax + \dfrac{(a^2-b^2)}{2!}x^2 + \dfrac{a(a^2-3b^2)}{3!}x^3 + \cdots$

and hence, deduce $e^{x\cos\alpha}\cos(x\sin\alpha) = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}\cos n\alpha.$

Solution

$$e^{ax}\cos bx = e^{ax}\cdot \text{Real Part of } (e^{ibx})$$

$$= \text{RP of } e^{(a+ib)x}$$

$$= \text{RP of}\left[1 + (a+ib)x + \frac{(a+ib)^2}{2!}x^2 + \frac{(a+ib)^3}{3!}x^3 + \cdots\right]$$

$$= \text{RP of}\left[1 + (a+ib)x + \frac{(a^2-b^2+2aib)}{2!}x^2 + \frac{(a^3-ib^3+3ia^2b-3ab^2)}{3!}x^3 + \ldots\right]$$

$$= 1 + ax + \frac{(a^2-b^2)}{2!}x^2 + \frac{a(a^2-3b^2)}{3!}x^3 + \ldots$$

Putting $a = \cos\alpha$ and $b = \sin\alpha$,

$$e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{(\cos^2\alpha - \sin^2\alpha)}{2!}x^2 + \frac{\cos^3\alpha - 3\cos\alpha\cdot\sin^2\alpha}{3!}x^3 + \ldots$$

$$= 1 + x\cos\alpha + \frac{\cos 2\alpha}{2!}x^2 + \frac{\cos^3\alpha - 3\cos\alpha(1-\cos^2\alpha)}{3!}x^3 + \ldots$$

$$= 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \frac{x^3}{3!}\cos 3\alpha + \ldots.$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\cos n\alpha$$

Example 37

Prove that $e^x = 1 + \tan x + \dfrac{1}{2!}\tan^2 x - \dfrac{1}{3!}\tan^3 x - \dfrac{7}{4!}\tan^4 x + \cdots.$

Solution

Let $$e^x = a_0 + a_1\tan x + a_2\tan^2 x + a_3\tan^3 x + a_4\tan^4 x + \cdots\cdots\cdots \qquad \ldots (1)$$

$$= a_0 + a_1\left(x + \frac{x^3}{3} + \ldots\right) + a_2\left(x + \frac{x^3}{3} + \ldots\right)^2 + a_3\left(x + \frac{x^3}{3} + \ldots\right)^3 + a_4\left(x + \frac{x^3}{3} + \ldots\right)^4 + \ldots\ldots\ldots$$

$$= a_0 + a_1\left(x + \frac{x^3}{3} + \ldots\right) + a_2\left(x^2 + \frac{2x^4}{3} + \ldots\right) + a_3(x^3 + \ldots) + a_4(x^4 + \ldots) + \ldots\ldots\ldots\ldots$$

$$= a_0 + a_1 x + a_2 x^2 + \left(\frac{a_1}{3} + a_3\right)x^3 + \left(\frac{2}{3}a_2 + a_4\right)x^4 + \ldots\ldots\ldots\ldots \qquad \ldots(2)$$

But
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots \qquad \ldots(3)$$

From Eqs (2) and (3),

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots = a_0 + a_1 x + a_2 x^2 + \left(\frac{a_1}{3} + a_3\right)x^3 + \left(\frac{2}{3}a_2 + a_4\right)x^4 + \ldots\ldots\ldots$$

Comparing coefficients of x, x^2, x^3 and x^4 on both the sides,

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2!} = \frac{1}{2}, \frac{a_1}{3} + a_3 = \frac{1}{3!} = \frac{1}{6}$$

$$a_3 = \frac{1}{6} - \frac{a_1}{3} = \frac{1}{6} - \frac{1}{3} = -\frac{1}{6} = -\frac{1}{3!}$$

$$\frac{2}{3}a_2 + a_4 = \frac{1}{4!} = \frac{1}{24}, \quad a_4 = \frac{1}{24} - \frac{2}{3}\cdot\frac{1}{2} = -\frac{7}{24} = -\frac{7}{4!}$$

Substituting in Eq. (1),

$$e^x = 1 + \tan x + \frac{1}{2!}\tan^2 x - \frac{1}{3!}\tan^3 x - \frac{7}{4!}\tan^4 x + \ldots$$

Example 38

Find the values of a and b such that the expansion of

$\log(1+x) - \dfrac{x(1+ax)}{1+bx}$ *in ascending powers of x begins with the term x^4 and prove that this term is* $-\dfrac{x^4}{36}$.

Solution

Let $\quad f(x) = \log(1+x) - \dfrac{x(1+ax)}{1+bx}$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-(x+ax^2)(1+bx)^{-1}$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-(x+ax^2)(1-bx+b^2x^2-b^3x^3+\cdots)$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-(x-bx^2+b^2x^3-b^3x^4+ax^2-abx^3+ab^2x^4-ab^3x^5+\cdots\cdots)$$

$$=\left(-\frac{1}{2}+b-a\right)x^2+\left(\frac{1}{3}-b^2+ab\right)x^3+\left(-\frac{1}{4}+b^3-ab^2\right)x^4+\cdots\cdots$$

If the expansion begins with the term x^4, the coefficients of x^2 and x^3 must be zero.

$$-\frac{1}{2}+b-a=0, \quad b=a+\frac{1}{2} \quad \text{and} \quad \frac{1}{3}-b^2+ab=0 \qquad \text{... (1)}$$

Substituting b in Eq. (1),

$$\frac{1}{3}-\left(a+\frac{1}{2}\right)^2+a\left(a+\frac{1}{2}\right)=0$$

$$\frac{1}{3}-a^2-\frac{1}{4}-a+a^2+\frac{1}{2}a=0$$

$$\frac{1}{2}a=\frac{1}{12}, \quad a=\frac{1}{6}$$

$$b=\frac{1}{6}+\frac{1}{2}=\frac{4}{6}=\frac{2}{3}$$

Coefficient of $x^4=-\frac{1}{4}+b^3-ab^2=-\frac{1}{4}+\left(\frac{2}{3}\right)^3-\frac{1}{6}\left(\frac{2}{3}\right)^2=-\frac{1}{36}$

Hence, the expansion begins with the term $-\frac{x^4}{36}$.

EXERCISE 6.2

1. Expand $e^x \sec x$ in powers of x using Maclaurin's series.

[Ans.: $1 + x + x^2 + \ldots$**]**

2. Using Maclaurin's series, prove that $e^{\sin x}=1+x+\frac{x^2}{2}+\cdots$

3. Using Maclaurin's series, prove that $a^x=1+x\log a+\frac{x^2}{2!}(\log a)^2+\cdots$

4. Prove that $\sin^2 x=x^2-\frac{x^4}{3}+\frac{2}{45}x^6+\cdots$

5. Prove that $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \cdots$

$$\left[\textbf{Hint:}\ \sec x = \frac{1}{\cos x} = (\cos x)^{-1} = \left[1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\right)\right]^{-1}\right]$$

6. Prove that $e^x \sin 2x = 2x + 2x^2 - \dfrac{x^3}{3} + \cdots$

7. Prove that $e^x \cos x = 1 + x - \dfrac{x^3}{3} + \cdots$

8. Prove that $\cos x \cosh x = 1 - \dfrac{2^2 x^4}{4!} + \dfrac{2^4 x^8}{8!} - \cdots$

9. Prove that $\sin(e^x - 1) = x + \dfrac{x^2}{2} - \dfrac{5}{24}x^4 + \cdots$

10. Prove that $\cos^n x = 1 - n \cdot \dfrac{x^2}{2!} + n(3n-2) \cdot \dfrac{x^4}{4!} - \cdots$

Hence, deduce that $\cos^3 x = 1 - \dfrac{3x^2}{2} + \dfrac{15x^4}{48} - \cdots$

11. Prove that $\sinh^3 x = \sum \dfrac{(3^n - 3) - [1 - (-1)^n]x^n}{8 \cdot n!}$.

12. Prove that $e^{x \sin x} = 1 + x^2 + \dfrac{x^4}{3} + \dfrac{x^6}{120} + \cdots\cdots\cdots$

13. Prove that $\log x = (x-1) - \dfrac{1}{2}(x-1)^2 + \dfrac{1}{3}(x-1)^3 + \cdots$

14. Prove that $\log(1 - x + x^2) = -x + \dfrac{x^2}{2} + \dfrac{2x^3}{3} - \ldots$

15. Prove that $\log \cosh x = \dfrac{1}{2}x^2 - \dfrac{1}{12}x^4 + \dfrac{1}{45}x^6 - \ldots$

16. Prove that $\log(1 + \tan x) = x - \dfrac{x^2}{2} + \dfrac{2x^3}{3} + \cdots\cdots\cdots$

17. Prove that $\log\left(\dfrac{\sin x}{x}\right) = -\left(\dfrac{x^2}{6} + \dfrac{x^4}{180} + \dfrac{x^6}{2835} + \cdots\right)$

18. Prove that $\log\left(\dfrac{\tan x}{x}\right) = \dfrac{x^3}{3} + \dfrac{7}{90}x^4 + \cdots\cdots\cdots$

19. Prove that $e^x \log(1 + x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots$

20. Expand $\log\tan\left(\frac{\pi}{4}+x\right)$ upto x^5.

$$\left[\textbf{Hint}: \log\tan\left(\frac{\pi}{4}+x\right)=\log\left(\frac{1+\tan x}{1-\tan x}\right)=\log(1+\tan x)-\log(1-\tan x)\right]$$

$$\left[\textbf{Ans.}: 2x+\frac{4}{3}x^3+\frac{4}{3}x^5+\ldots\right]$$

21. Prove that $x=y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+\cdots$ if $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

By Differentiation and Integration

Example 1

Prove that $\log(\sec x)=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\cdots.$ **[Summer 2017]**

Solution

Let

$$y=\log(\sec x)$$

$$\frac{dy}{dx}=\frac{1}{\sec x}\cdot\sec x\tan x$$

$$=\tan x$$

$$=x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\cdots\cdots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y=c+\frac{x^2}{2}+\frac{x^4}{12}+\frac{2}{15}\cdot\frac{x^6}{6}+\ldots\ldots..$$

$$\log(\sec x)=c+\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\ldots\ldots..$$

Putting $x=0$,

$$\log(\sec 0)=c+0$$

$$c=\log 1, \quad c=0$$

Hence,

$$\log(\sec x)=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\cdots\cdots\cdots$$

Example 2

Prove that $\sin^{-1}x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\cdots.$

Solution

Let $$y = \sin^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$= (1-x^2)^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \dots$$

$$= 1 + \frac{x^2}{2} + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \dots \qquad \dots (1)$$

Integrating Eq. (1),

$$y = c + x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \dots\dots\dots$$

$$\sin^{-1} x = c + x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \dots\dots\dots$$

Putting $x = 0$,

$$\sin^{-1} 0 = c$$

$$c = 0$$

Hence, $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \dots\dots\dots$

Example 3

Prove that $\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots\right)$.

Solution

Let $$y = \cos^{-1} x$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Proceeding as in Example 2,

$$\cos^{-1} x = c - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \dots\right)$$

Putting $x = 0$,

$$\cos^{-1} 0 = c$$

$$c = \frac{\pi}{2}$$

Hence, $$\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots \right)$$

Example 4

Prove that $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$.

Solution

Let $$y = \tan^{-1} x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\tan^{-1} x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots..$$

Putting $x = 0$,

$$\tan^{-1} 0 = c$$

$$c = 0$$

Hence, $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Example 5

Prove that $\sinh^{-1} x = x - \dfrac{x^3}{6} + \dfrac{3x^5}{40} + \cdots$.

Solution

Let $$y = \sin\mathrm{h}^{-1} x = \log\left(x + \sqrt{x^2+1} \right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x + \sqrt{x^2+1}} \left(1 + \frac{2x}{2\sqrt{x^2+1}} \right)$$

$$= \frac{1}{\sqrt{x^2+1}}$$

$$= (1+x^2)^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(x^2)^2 - \dots$$

$$= 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \dots \qquad \dots (1)$$

Integrating Eq. (1),

$$y = c + x - \frac{x^3}{6} + \frac{3}{8}\cdot\frac{x^5}{5} - \dots$$

$$\sinh^{-1} x = c + x - \frac{x^3}{6} + \frac{3x^5}{40} - \dots$$

Putting $x = 0$,

$$\sinh^{-1} 0 = c, \ c = 0$$

$$\sinh^{-1} x = x - \frac{x^3}{6} + \frac{3x^5}{40} - \dots$$

Example 6

Prove that $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots.$

Solution

Let $$y = \tanh^{-1} x = \frac{1}{2}\log\frac{(1+x)}{(1-x)} = \frac{1}{2}[\log(1+x) - \log(1-x)]$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

$$= (1-x^2)^{-1}$$

$$= 1 + x^2 + x^4 + x^6 + \cdots \qquad \dots(1)$$

Integrating Eq. (1),

$$y = c + x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$\tanh^{-1} x = c + x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Putting $x = 0$,

$$\tanh^{-1} 0 = c, \ c = 0$$

Hence, $$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Example 7

If $x = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + ...$, *find y in a series of x.*

Solution

$$x = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + ...$$
$$= \cos y$$
$$y = \cos^{-1} x$$

Proceeding as in Example 3,

$$y = \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots\right)$$

Example 8

Show that $\tan^{-1}\sqrt{\frac{1-x}{1+x}} = \frac{\pi}{4} - \frac{1}{2}\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots\right)$

Solution

Let $$y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Putting $x = \cos 2\theta$,

$$y = \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$$
$$= \tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}$$
$$= \tan^{-1}\tan\theta$$
$$= \theta$$
$$= \frac{1}{2}\cos^{-1} x$$
$$= \frac{1}{2}\left[\frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots\right)\right]$$

By Substitution

Example 1

Expand $\sin^{-1}(3x - 4x^3)$ *in ascending powers of x.*

Solution

Let $$y = \sin^{-1}(3x - 4x^3)$$

Putting $x = \sin\theta$,

$$\begin{aligned} y &= \sin^{-1}(3\sin\theta - 4\sin^3\theta) \\ &= \sin^{-1}(\sin 3\theta) \\ &= 3\theta \\ &= 3\sin^{-1} x \\ &= 3\left(x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots\right) \end{aligned}$$

Example 2

Prove that $\sinh^{-1}(3x + 4x^3) = 3\left(x - \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots\right)$.

Solution

Let $$y = \sinh^{-1}(3x + 4x^3)$$

Putting $x = \sinh\theta$,

$$\begin{aligned} y &= \sinh^{-1}(3\sinh\theta + 4\sinh^3\theta) \\ &= \sinh^{-1}(\sinh 3\theta) \\ &= 3\theta \\ &= 3\sinh^{-1} x \\ &= 3\left(x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots\right) \end{aligned}$$

Example 3

Prove that $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$.

Solution

Let $$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Putting $x = \tan\theta$,

$$\begin{aligned} y &= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \\ &= \sin^{-1}(\sin 2\theta) \\ &= 2\theta \\ &= 2\tan^{-1} x \\ &= 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) \end{aligned}$$

Example 4

Expand $\sec^{-1}\left(\dfrac{1}{1-2x^2}\right)$.

Solution

Let $$y = \sec^{-1}\left(\frac{1}{1-2x^2}\right)$$

Putting $x = \sin\theta$,

$$\begin{aligned} y &= \sec^{-1}\left(\frac{1}{1-2\sin^2\theta}\right) \\ &= \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) \\ &= \sec^{-1}(\sec 2\theta) \\ &= 2\theta \\ &= 2\sin^{-1} x \\ &= 2\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots\right) \end{aligned}$$

Example 5

Prove that $\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right) = \pi - 2\left(n\pi + x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots\right)$.

Solution

Let $$y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

Putting $x = \tan\theta$,

$$\begin{aligned}
y &= \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right) \\
&= \cos^{-1}(-\cos 2\theta) \\
&= \cos^{-1}[-\cos(2n\pi + 2\theta)] \\
&\qquad\qquad [\text{Considering general value of } \cos 2\theta] \\
&= \cos^{-1}[\cos\{\pi - (2n\pi + 2\theta)\}] \\
&= \pi - 2(n\pi + \theta) \\
&= \pi - 2(n\pi + \tan^{-1} x) \\
&= \pi - 2\left(n\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)
\end{aligned}$$

Example 6

Prove that $\cos^{-1}[\tanh(\log x)] = \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$.

Solution

Let

$$\begin{aligned}
y &= \cos^{-1}[\tanh(\log x)] \\
&= \cos^{-1}\left(\frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}\right) \\
&= \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right) \\
&= \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)
\end{aligned}$$

Putting $x = \tan\theta$,

$$\begin{aligned}
y &= \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right) \\
&= \cos^{-1}(-\cos 2\theta) \\
&= \cos^{-1}[\cos(\pi - 2\theta)] \\
&= \pi - 2\theta \\
&= \pi - 2\tan^{-1} x \\
&= \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)
\end{aligned}$$

Example 7

Prove that $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)=\dfrac{1}{2}\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots\right).$ **[Winter 2013]**

Solution

Let $$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

Putting $x=\tan\theta$,

$$\begin{aligned}
y &= \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)\\
&= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)\\
&= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)\\
&= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)\\
&= \tan^{-1}\left(\tan\frac{\theta}{2}\right)\\
&= \frac{\theta}{2}\\
&= \frac{1}{2}\tan^{-1}x\\
&= \frac{1}{2}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right)
\end{aligned}$$

Example 8

Prove that $\tan^{-1}\left(\dfrac{p-qx}{q+px}\right)=\tan^{-1}\dfrac{p}{q}-\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots\right).$

Solution

Let $$y=\tan^{-1}\left(\frac{\frac{p}{q}-x}{1+\frac{p}{q}x}\right)$$

Putting $x=\tan\theta,\ \dfrac{p}{q}=\tan A$

$$\begin{aligned} y &= \tan^{-1}\left(\frac{\tan A - \tan\theta}{1+\tan A\cdot\tan\theta}\right) \\ &= \tan^{-1}[\tan(A-\theta)] \\ &= A-\theta \\ &= \tan^{-1}\frac{p}{q} - \tan^{-1}x \\ &= \tan^{-1}\frac{p}{q} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) \end{aligned}$$

EXERCISE 6.3

1. Prove that $\dfrac{\tan^{-1}x}{1+x^2} = x - \dfrac{4}{3}x^3 + \dfrac{23}{15}x^5 - \dots\dots$

2. Prove that $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = 2\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots\right)$.

3. Prove that $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) = 3\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots\right)$.

4. Prove that $\cot^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) = \dfrac{\pi}{2} - 3\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots\right)$.

5. Prove that $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = x + \dfrac{1}{6}x^3 + \dfrac{3}{40}x^5 + \dots\dots$.

6. Prove that $\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{\pi}{4} - \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots\right)$.

7. Prove that $\tan^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right) = \dfrac{\pi}{4} - \dfrac{1}{2}\left(x + \dfrac{x^3}{6} + \dfrac{3x^5}{40} + \dots\right)$.

8. Prove that $\cot^{-1}x = \dfrac{\pi}{2} - \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots\right)$.

9. Prove that $\cos^{-1}(4x^3 - 3x) = 3\left[\dfrac{\pi}{2} - \left(x + \dfrac{x^3}{6} + \dfrac{3}{40}x^5 + \dots\right)\right]$.

10. Prove that $\sec^{-1}\left(\sqrt{1+x^2}\right) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots\dots$

11. Prove that $\tan^{-1}\left(\dfrac{2-3x}{3+2x}\right) = \tan^{-1}\dfrac{2}{3} - \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots\right)$.

Points to Remember

Taylor's Series

(i) $f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2!}f''(x) + \dfrac{h^3}{3!}f'''(x) + \dots + \dfrac{h^n}{n!}f^n(x) + \dots$

(ii) $f(x) = f(a) + (x-a)f'(a) + \dfrac{(x-a)^2}{2!}f''(a) + \dfrac{(x-a)^3}{3!}f'''(a)$

$$+ \dots + \frac{(x-a)^n}{n!}f^n(a) + \dots$$

Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^n(0) + \dots$$

List of Expansion of Some Standard Functions

(i) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots$

(ii) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots$

(iii) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots$

(iv) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dots$

(v) $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \dots$

(vi) $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \dots$

(vii) $\tanh x = x - \dfrac{x^3}{3} + \dfrac{2x^5}{15} - \dots$

(viii) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} \dots$

(ix) $(1+x)^m = 1 + mx + \dfrac{m(m-1)}{2!}x^2 + \dots$

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. The expansion of log $(1 + x)$ in Maclaurin's series is

(a) $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ (b) $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$

(c) $x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$ (d) $x-\frac{x^2}{2}+\frac{x^3}{3!}-\frac{x^4}{4!}+\cdots$

2. The expansion of tan x in Maclaurin's series is

(a) $x+\frac{x^3}{2!}+\frac{x^5}{3!}+\cdots$ (b) $x-\frac{x^3}{3}+\frac{2}{15}x^5-\cdots$

(c) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$ (d) $x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots$

3. The Maclaurin's series of sin x is

(a) $x-\frac{x^3}{3!}+\frac{x^5}{5!}x^5-\cdots$ (b) $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$

(c) $1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$ (d) $x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$

4. The nth term in Maclaurin's series expansion is

(a) $\frac{f^n(x)}{n!}$ (b) $\frac{f^n(0)}{n!}$ (c) $\frac{f(x)}{n!}$ (d) $\frac{f(0)}{n!}$

5. Taylor's series expansion of $y=\frac{1}{x}$ about $x = 1$ is

(a) $1-(x-1)+(x-1)^2-(x-1)^3+\cdots$

(b) $1+(x-1)+(x-1)^2+(x-1)^3+\cdots$

(c) $1-(x-1)+\frac{(x-1)^2}{2!}-\frac{(x-1)^3}{3!}+\cdots$

(d) $1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\cdots$

6. Taylor's series expansion of $y = \sin x$ about $x=\frac{\pi}{2}$ is

(a) $1-\left(x-\frac{\pi}{2}\right)^2+\left(x-\frac{\pi}{2}\right)^4-\cdots$

(b) $1-\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4-\cdots$

(c) $\left(x-\frac{\pi}{2}\right)-\frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3+\frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5-\cdots$

(d) $\left(x-\frac{\pi}{2}\right)-\left(x-\frac{\pi}{2}\right)^3+\left(x-\frac{\pi}{2}\right)^5-\cdots$

7. Maclaurin's series of $f(x)$ is

(a) $f(x)+\frac{x}{1!}f'(x)+\frac{x^2}{2!}f''(x)+\cdots+\frac{x^n}{n!}f^n(x)+\cdots$

(b) $f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^n}{n!}f^n(0)+\cdots$

(c) $1+\frac{x}{1!}f'(1)+\frac{x^2}{2!}f''(1)+\cdots+\frac{x^n}{n!}f^n(1)+\cdots$

(d) $1+\frac{x}{1!}f'(x)+\frac{x^2}{2!}f''(x)+\cdots+\frac{x^n}{n!}f^n(x)+\cdots$

8. The Taylor's series expansion of log x in $(x-1)$ is

(a) $(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\cdots$

(b) $(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\cdots$

(c) $1+(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\cdots$

(d) $1-(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{3}(x-1)^3+\cdots$

9. Which of the following is the coefficient of x^{11} in the expansion of e^x?

(a) $\frac{1}{11}$ (b) $11!$ (c) $-\frac{1}{11}$ (d) $\frac{1}{11!}$

10. The coefficient of x^5 in the expansion of cos x is

(a) 0 (b) $\frac{1}{5!}$ (c) $-\frac{1}{5!}$ (d) $\frac{1}{5}$

11. The coefficient of x^{100} in the expansion of log $(1-x)^2$ is

(a) $\frac{1}{100}$ (b) $-\frac{1}{100}$ (c) $-\frac{1}{50}$ (d) $\frac{1}{50}$

12. The constant term in the expansion of $7+(x+2)+3(x+2)^3+(x+2)^4$ is

(a) 40 (b) 48 (c) 49 (d) 50

13. The coefficient of x^2 in the expansion of $e^x \cos x$ is

(a) $\frac{1}{2}$ (b) -1 (c) 0 (d) $-\frac{1}{2}$

14. The coefficients of x^4 and x^5 respectively, in the expansion of $(x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5(x-1) - 100$ are

(a) (1, 1) (b) (0, 0) (c) (0, 1) (d) (1, 0)

15. The expansion of $x^4 - 3x^3 + 2x^2 - x + 1$ about 3 is

(a) $16 + 38(x+3) + 29(x+3)^2 + 9(x+3)^3 + (x+3)^4$

(b) $16 + 38x + 29x^2 + 9x^3 + x^4$

(c) $16 + 38(x-3) + 29(x-3)^2 + 9(x-3)^3 + (x-3)^4$

(d) $16 - 38(x+3) + 29(x+3)^2 - 9(x+3)^3 + (x+3)^4$

16. The Maclaurin's series of $\sin x$ is **[Summer 2015]**

(a) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ (d) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

17. The Maclaurin's series of e^{-x} is **[Winter 2015]**

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ (c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$

18. The series $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ represent expansion of **[Summer 2016]**

(a) $\sin x$ (b) $\cos x$ (c) $\sinh x$ (d) $\cosh x$

19. The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ represent expansion of **[Summer 2017]**

(a) e^x (b) $\log(1+x)$ (c) $\sin x$ (d) $\cos x$

20. The coefficient of x^5 in the expansion of e^x is **[Winter 2016]**

(a) $\frac{1}{5}$ (b) $\frac{1}{4!}$ (c) $\frac{1}{5!}$ (d) 5

Answers

1. (a)	2. (d)	3. (a)	4. (b)	5. (a)	6. (b)	7. (b)	8. (b)	9. (d)
10. (a)	11. (c)	12. (c)	13. (c)	14. (d)	15. (c)	16. (b)	17. (b)	18. (c)
19. (b)	20. (c)							