Differential Equations

Chapter 10

10.1 INTRODUCTION

Differential equations are very important in engineering mathematics. A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders. It provides the medium for the interaction between mathematics and various branches of science and engineering. Most common differential equations are radioactive decay, chemical reactions, Newton's law of cooling, series *RL*, *RC* and *RLC* circuits, simple harmonic motions, etc.

10.2 DIFFERENTIAL EQUATION

A differential equation is an equation which involves variables (dependent and independent) and their derivatives, e.g.,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \qquad \ldots (1)$$

$$\left(\frac{d^2y}{dx^2}\right)^2 - \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^3 = 0 \qquad \ldots (2)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \qquad \ldots (3)$$

Equations (1) and (2) involve ordinary derivatives and hence called "ordinary differential equations" whereas Eq. (3) involves partial derivatives and hence called "partial differential equation".

10.2.1 Order

Order of a differential equation is the order of the highest derivative present in the equation, e.g., the order of Eqs. (1) and (2) is 2.

10.2.2 Degree

Degree of a differential equation is the power of the highest order derivative after clearing the radical sign and fraction, e.g., the degree of Eq. (1) is 1 and the degree of Eq. (2) is 2.

10.2.3 Solution or Primitive

Solution of a differential equation is a relation between the dependent and independent variables (excluding derivatives), which satisfies the equation.

Solution of a differential equation is not always unique. It may have more than one solution or sometimes no solution.

General solution of a differential equation of order n contains n arbitrary constants.

Particular solution of a differential equation is obtained from the general solution by giving particular values to the arbitrary constants.

10.3 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation which contains first order and first degree derivative of y (dependent variable) and known functions of x (independent variable) and y is known as ordinary differential equation of first order and first degree. The general form of this equation can be written as

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

or in explicit form as

$$\frac{dy}{dx} = f(x, y) \text{ or } M(x, y)dx + N(x, y)dy = 0$$

Solution of the differential equation can be obtained by classifying them as follows:

(i) Variable separable
(ii) Homogeneous differential equations
(iii) Non homogeneous differential equations
(iv) Exact differential equations
(v) Non-exact differential equations reducible to exact form
(vi) Linear differential equations
(vii) Non-linear differential equations reducible to linear form

10.3.1 Variable Separable

A differential equation of the form

$$M(x)dx + N(y)dy = 0 \qquad \dots (1)$$

where M (x) is the function of x only and N (y) is the function of y only, is called a differential equation with variables separable as in Eq. (1) function of x and function of y can be separated easily.

Integrating Eq. (1) we get the solution as

$$\int M(x)dx + \int N(y)dy = c$$

or

$$\int g(y)dy = \int f(x)dx + c$$

where c is the arbitrary constant.

Example 1: Solve $y(1+x^2)^{\frac{1}{2}}dy + x\sqrt{1+y^2}dx = 0$.

Solution: $y(1+x^2)^{\frac{1}{2}}dy = -x\sqrt{1+y^2}dx$

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx + c$$

$$\frac{1}{2}\int (1+y^2)^{-\frac{1}{2}}(2y)dy = -\frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}(2x)dx + c$$

$$\frac{1}{2}\cdot\frac{(1+y^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}\cdot\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

Example 2: Solve $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$.

Solution:

$$e^y\frac{dy}{dx} = e^x + x^2$$

$$\int e^y dy = \int (e^x + x^2)dx$$

$$e^y = e^x + \frac{x^3}{3} + c$$

Example 3: Solve $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$.

Solution: $\sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$

$$\int \frac{\sec^2 x}{\tan x}dx = -\int \frac{\sec^2 y}{\tan y}dy + c$$

$$\log \tan x = -\log \tan y + c \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

$$\log \tan x + \log \tan y = c$$

$$\log(\tan x \tan y) = c$$

$$\tan x \tan y = e^c = k$$

$$\tan x \tan y = k$$

Example 4: Solve $(4x+y)^2 \frac{dx}{dy} = 1$.

Solution: $\frac{dy}{dx} = (4x+y)^2$ … (1)

Let $4x + y = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Substituting in Eq. (1),

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{dx} = t^2 + 4$$

$$\int \frac{dt}{t^2 + 4} = \int dx + c$$

$$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y}{2}\right) = x + c$$

Example 5: Solve $\frac{dy}{dx} = 1 + \tan(y - x)$.

Solution: Let $y - x = t$

$$\frac{dy}{dx} - 1 = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} + 1$$

Substituting in the given equation,

$$\frac{dt}{dx} + 1 = 1 + \tan t$$

$$\frac{dt}{\tan t} = dx$$

Integrating both the sides,

$$\int \frac{\cos t}{\sin t} dt = \int dx + c$$

$$\log \sin t = x + c \qquad \left[\because \int \frac{f'(x)}{f(x)} dx = \log f(x)\right]$$

$$\log\sin(y-x) = x+c$$
$$\sin(y-x) = e^{x+c}$$

Example 6: Solve $\frac{dy}{dx} = (4x+y+1)^2,\ y(0)=1.$

Solution: Let $4x+y+1=t$

$$4+\frac{dy}{dx} = \frac{dt}{dx},\quad \frac{dy}{dx} = \frac{dt}{dx}-4$$

Substituting in the given equation,

$$\frac{dt}{dx}-4 = t^2$$
$$\frac{dt}{dx} = t^2+4$$
$$\frac{dt}{t^2+4} = dx$$

Integrating both the sides,

$$\int\frac{dt}{t^2+4} = \int dx + c$$
$$\frac{1}{2}\tan^{-1}\frac{t}{2} = x+c$$
$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x+c \qquad \dots (1)$$

Given $y(0)=1$

Substituting $x=0, y=1$ in Eq. (1),

$$\frac{1}{2}\tan^{-1}(1) = 0+c,\ c=\frac{\pi}{8}$$

Hence, solution is

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x+\frac{\pi}{8}$$

Example 7: Solve $\left(x\frac{dy}{dx}-y\right)\cos\left(\frac{y}{x}\right)+x=0.$

Solution: Let $\frac{y}{x}=t$

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y = \frac{dt}{dx}$$
$$x\frac{dy}{dx}-y = x^2\frac{dt}{dx}$$

Substituting in the given equation,

$$x^2 \frac{dt}{dx} \cdot \cos t + x = 0$$

$$\cos t \, dt = -\frac{dx}{x}$$

Integrating both the sides,

$$\int \cos t \, dt = -\int \frac{dx}{x} + c$$

$$\sin t = -\log x + c$$

$$\sin \frac{y}{x} = -\log x + c$$

Example 8: Solve $2x^2 y \dfrac{dy}{dx} = \tan(x^2 y^2) - 2xy^2$.

Solution: Let $x^2 y^2 = t$

$$2xy^2 + x^2 \cdot 2y \frac{dy}{dx} = \frac{dt}{dx}$$

$$2x^2 y \frac{dy}{dx} + 2xy^2 = \frac{dt}{dx}$$

Substituting in the given equation,

$$\frac{dt}{dx} = \tan t$$

$$\frac{dt}{\tan t} = dx$$

Integrating both the sides,

$$\int \cot t \, dt = \int dx$$

$$\log \sin t = x + c$$

$$\log \sin(x^2 y^2) = x + c$$

$$\sin(x^2 y^2) = e^{x+c} = e^x e^c = e^x \cdot k$$

$$\sin(x^2 y^2) = ke^x$$

Example 9: Solve $(x \log x) \dfrac{dy}{dx} = 2y,\ y(2) = (\log 2)^2$.

Solution: $$\frac{dy}{2y} = \frac{dx}{x \log x}$$

Integrating both the sides,

$$\int \frac{dy}{2y} = \int \frac{1}{\log x} \cdot \frac{1}{x} dx$$

$$\frac{1}{2}\log y = \log(\log x) + \log c \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c\right]$$

$$\log y^{\frac{1}{2}} = \log(c \log x)$$

$$y^{\frac{1}{2}} = c \log x \qquad \ldots (1)$$

Given, $y(2) = (\log 2)^2$

Putting $x = 2, y = (\log 2)^2$ in Eq. (1),

$$(\log 2) = c \log 2$$

$$c = 1$$

Hence, solution is

$$y^{\frac{1}{2}} = \log x$$

$$y = (\log x)^2$$

Example 10: Solve $\boldsymbol{(x+y)^2\left(x\frac{dy}{dx} + y\right) = xy\left(1 + \frac{dy}{dx}\right)}$.

Solution:
$$\frac{x\frac{dy}{dx} + y}{xy} = \frac{1 + \frac{dy}{dx}}{(x+y)^2}$$

$$d(\log xy) = d\left(-\frac{1}{x+y}\right)$$

Integrating both the sides,

$$\log xy = -\frac{1}{x+y} + c$$

Exercise 10.1

Solve the following differential equations:

1. $y^2\frac{dy}{dx} + x^2 = 0$.

$\left[\textbf{Ans.:}\ x^3 + y^3 = c\right]$

2. $(1+x)y - (1+y)x\frac{dy}{dx} = 0,\ x > 0,\ y > 0$.

$\left[\textbf{Ans.:}\ x - y + \log\left(\frac{x}{y}\right) = c\right]$

3. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$.

$[\textbf{Ans.:}\ (e^y + 1)\sin x = c]$

4. $y\frac{dy}{dx} = xe^{-x}\sqrt{1-y^2}$.

$\left[\textbf{Ans.:}\ \sqrt{1-y^2} = (x+1)e^{-x} + c\right]$

5. $x(e^{4y} - 1)\frac{dy}{dx} + (x^2 - 1)e^{2y} = 0,\ x > 0$.

$\left[\textbf{Ans.:}\ \cosh(2y) = \log x - \frac{x^2}{2} + c\right]$

6. $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$.

$\left[\textbf{Ans.:}\ y\sin y = x^2 \log x + c\right]$

7. $\dfrac{dy}{dx} = \dfrac{\sin x + \dfrac{\log x}{x}}{\cos y - \sec^2 y}$.

$\left[\begin{array}{l}\textbf{Ans.:}\ \sin y - \tan y = -\cos x \\ \quad + \dfrac{1}{2}(\log x)^2 + c\end{array}\right]$

8. $y\sec^2 x + (y+7)\tan x \dfrac{dy}{dx} = 0$.

$\left[\textbf{Ans.:}\ y^7 \tan x = ce^{-y}\right]$

9. $(x+1)\left(\dfrac{dy}{dx} - 1\right) = 2(y-x)$.

$\left[\textbf{Ans.:}\ y - x = c(x+1)^2\right]$

10. $\cos(x+y)dy = dx$.

$\left[\textbf{Ans.:}\ y - \tan\left(\dfrac{x+y}{2}\right) = c\right]$

11. $\dfrac{dy}{dx} = \dfrac{y-x}{y-x+2}$.

$\left[\textbf{Ans.:}\ (y-x)^2 = c - 4y\right]$

12. $x\dfrac{dy}{dx} = y + x^2 \tan\left(\dfrac{y}{x}\right)$.

$\left[\textbf{Ans.:}\ \sin\left(\dfrac{y}{x}\right) = ce^x\right]$

13. $x\dfrac{dy}{dx} = e^{-xy} - y$.

$\left[\textbf{Ans.:}\ e^{xy} = x + c\right]$

14. $(1+x^3)dy - x^2 y\, dx = 0,\ y(1) = 2$.

$\left[\textbf{Ans.:}\ y^3 = 4(1+x^3)\right]$

15. $\dfrac{dy}{dx} = \dfrac{x}{y} - \dfrac{x}{1+y},\ y(0) = 2$.

$\left[\textbf{Ans.:}\ 3y^2 + 2y^3 = 3x^2 + 28\right]$

16. $\dfrac{dy}{dx} + 2y = x^2 y,\ y(0) = 1$.

$\left[\textbf{Ans.:}\ y = e^{\frac{x^3}{3} - 2x}\right]$

17. $e^y\left(\dfrac{dy}{dx} + 1\right) = 1,\ y(0) = 1$.

$\left[\textbf{Ans.:}\ e^y = 1 - (1-e)e^{-x}\right]$

18. $\dfrac{dy}{dx} = 2y\sin^2 x,\ y\left(\dfrac{\pi}{2}\right) = 1$.

$\left[\textbf{Ans.:}\ \log y = x - \dfrac{1}{2}\sin 2x - \dfrac{\pi}{2}\right]$

19. $\cos y\, dx + (1 + e^{-x})\sin y\, dy = 0$,

$y(0) = \dfrac{\pi}{4}$.

$\left[\textbf{Ans.:}\ (1+e^x)\sec y = 2\sqrt{2}\right]$

20. $\dfrac{dy}{dx} = y^2 \sin x,\ y(2\pi) = 1$.

$\left[\textbf{Ans.:}\ y\cos x = 1\right]$

10.3.2 Homogeneous Differential Equation

A differential equation of the form

$$\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)} \qquad \ldots (1)$$

is called a homogeneous equation if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree, i.e., degree of the R.H.S. of Eq. (1) is zero.

Equation (1) can be reduced to variable separable form by putting $y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Equation (1) reduces to

$$v + x\frac{dv}{dx} = \frac{M(x, vx)}{N(x, vx)} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

Above equation is in variable separable form and can be solved by integrating

$$\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + c$$

After integrating and replacing v by $\frac{y}{x}$, we get the solution of Eq. (1).

Note: Homogeneous functions: A function $f(x, y, z)$ is said to be a homogeneous function of degree n, if for any positive number t,

$$f(xt, yt, zt) = t^n f(x, y, z),$$

where n is a real number.

Example 1: Solve $x(x - y)dy + y^2dx = 0$.

Solution: $\frac{dy}{dx} = \frac{-y^2}{x^2 - xy} = \frac{M(x, y)}{N(x, y)}$ ... (1)

The equation is homogeneous since M and N are of the same degree 2.
Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2(1 - v)} = \frac{-v^2}{1 - v}$$

$$x\frac{dv}{dx} = \frac{-v^2}{1 - v} - v = \frac{-v}{1 - v}$$

$$\left(\frac{v - 1}{v}\right)dv = \frac{dx}{x}$$

$$\left(1 - \frac{1}{v}\right)dv = \frac{dx}{x}$$

Integrating both the sides,

$$\int\left(1-\frac{1}{v}\right)\mathrm{d}v = \int\frac{\mathrm{d}x}{x}$$

$$v - \log v = \log x + \log c$$

$$v = \log v + \log cx = \log c\,xv$$

$$\frac{y}{x} = \log cy$$

$$y = x\log cy$$

Example 2: Solve $\dfrac{\mathbf{d}y}{\mathbf{d}x} = \dfrac{y+\sqrt{x^2+y^2}}{x}$.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{\mathrm{M}(x,y)}{\mathrm{N}(x,y)} \qquad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 1.
Let $y = vx$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

Substituting in Eq. (1),

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx+\sqrt{x^2+v^2x^2}}{x} = v+\sqrt{1+v^2}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{1+v^2}$$

$$\frac{\mathrm{d}v}{\sqrt{1+v^2}} = \frac{\mathrm{d}x}{x}$$

Integrating both the sides,

$$\int\frac{\mathrm{d}v}{\sqrt{1+v^2}} = \int\frac{\mathrm{d}x}{x}$$

$$\log\left(v+\sqrt{v^2+1}\right) = \log x + \log c = \log cx$$

$$v+\sqrt{v^2+1} = cx$$

$$\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1} = cx$$

$$y+\sqrt{y^2+x^2} = cx^2$$

Example 3: Solve $2ye^{\frac{x}{y}}dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$.

Solution:
$$\frac{dy}{dx} = \frac{-2ye^{\frac{x}{y}}}{y - 2xe^{\frac{x}{y}}} = \frac{M(x, y)}{N(x, y)} \qquad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 1.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{-2vxe^{\frac{x}{vx}}}{vx - 2xe^{\frac{x}{vx}}}$$

$$x\frac{dv}{dx} = \frac{-2ve^{\frac{1}{v}}}{v - 2e^{\frac{1}{v}}} - v = \frac{-v^2}{v - 2e^{\frac{1}{v}}}$$

$$\frac{v - 2e^{\frac{1}{v}}}{-v^2}dv = \frac{dx}{x}$$

Integrating both the sides,

$$-\int \frac{1}{v}dv - 2\int e^{\frac{1}{v}}\left(-\frac{1}{v^2}\right)dv = \int \frac{dx}{x}$$

$$-\log v - 2e^{\frac{1}{v}} = \log x + c \qquad \left[\because \int e^{f(x)} f'(x)dx = e^{f(x)}\right]$$

$$-\log \frac{y}{x} - 2e^{\frac{x}{y}} = \log x + c$$

$$-\log y + \log x - 2e^{\frac{x}{y}} = \log x + c$$

$$\log y + 2e^{\frac{x}{y}} + c = 0$$

Example 4: Solve $\frac{y}{x}\cos\frac{y}{x}dx - \left(\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}\right)dy = 0$.

Solution:
$$\frac{dy}{dx} = \frac{\frac{y}{x}\cos\frac{y}{x}}{\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}} = \frac{M(x, y)}{N(x, y)} \qquad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 0.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{v\cos v}{\frac{1}{v}\sin v + \cos v}$$

$$x\frac{dv}{dx} = \frac{v\cos v}{\frac{1}{v}\sin v + \cos v} - v = \frac{-\sin v \cdot v}{\sin v + v\cos v}$$

$$\left(\frac{\sin v + v\cos v}{-v\sin v}\right)dv = \frac{dx}{x}$$

$$\left(\frac{1}{v} + \cot v\right)dv = -\frac{dx}{x}$$

Integrating both the sides,

$$\int\left(\frac{1}{v} + \cot v\right)dv = -\int\frac{dx}{x}$$

$$\log v + \log\sin v = -\log x + \log c$$

$$v\sin v = \frac{c}{x}$$

$$\frac{y}{x}\sin\frac{y}{x} = \frac{c}{x}$$

$$y\sin\frac{y}{x} = c$$

Example 5: Solve $\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\frac{y}{x} = 0,\ y(1) = 0.$

Solution: $\frac{dy}{dx} = \frac{y}{x} - \text{cosec}\frac{y}{x}$ … (1)

The equation is homogeneous since degree of each term is same.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = v - \text{cosec } v$$

$$x\frac{dv}{dx} = -\text{cosec } v$$

$$\sin v\ dv = -\frac{dx}{x}$$

Integrating both the sides,

$$\int \sin v \, dv = -\int \frac{dx}{x}$$

$$-\cos v = -\log x + c$$

$$\log x - \cos v = c$$

$$\log x - \cos \frac{y}{x} = c \qquad \ldots (2)$$

Given $y(1) = 0$

Putting $x = 1, y = 0$ in Eq. (2),

$$\log 1 - \cos 0 = c$$

$$c = -1$$

Hence, solution is

$$\log x - \cos \frac{y}{x} = -1$$

Example 6: Solve $\left[x(x^2 - y^2)^{-\frac{1}{2}} + e^{\frac{y}{x}} \right] x \dfrac{dy}{dx} = x + \left[x(x^2 - y^2)^{-\frac{1}{2}} + e^{\frac{y}{x}} \right] y, \; y(1) = 1.$

Solution:

$$\frac{dy}{dx} = \frac{x + \left[x(x^2 - y^2)^{-\frac{1}{2}} + e^{\frac{y}{x}} \right] y}{\left[x(x^2 - y^2)^{-\frac{1}{2}} + e^{\frac{y}{x}} \right] x} = \frac{M(x, y)}{N(x, y)} \qquad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 1.

Let $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x \frac{dv}{dx} = \frac{x + \left[x(x^2 - v^2 x^2)^{-\frac{1}{2}} + e^{\frac{vx}{x}} \right] vx}{\left[x(x^2 - v^2 x^2)^{-\frac{1}{2}} + e^{\frac{vx}{x}} \right] x}$$

$$x \frac{dv}{dx} = \frac{1 + \left[(1 - v^2)^{-\frac{1}{2}} + e^v \right] v}{\left[(1 - v^2)^{-\frac{1}{2}} + e^v \right]} - v = \frac{1}{\left[(1 - v^2)^{-\frac{1}{2}} + e^v \right]}$$

$$\left[(1 - v^2)^{-\frac{1}{2}} + e^v \right] dv = \frac{dx}{x}$$

Integrating both the sides,

$$\int\left[\frac{1}{\sqrt{1-v^2}}+e^v\right]dv=\int\frac{dx}{x}$$

$$\sin^{-1}v+e^v=\log x+c$$

$$\sin^{-1}\frac{y}{x}+e^{\frac{y}{x}}=\log x+c \qquad \dots (2)$$

Given $y(1)=1$

Putting $x=1, y=1$ in Eq. (2),

$$\sin^{-1}1+e=\log 1+c$$

$$\frac{\pi}{2}+e=c$$

Hence, solution is

$$\sin^{-1}\frac{y}{x}+e^{\frac{y}{x}}=\log x+\frac{\pi}{2}+e$$

Exercise 10.2

Solve the following differential equations:

1. $x(y-x)\frac{dy}{dx}=y(y+x).$

$\left[\textbf{Ans.: } \frac{y}{x}-\log xy=c\right]$

2. $\frac{dy}{dx}=\frac{3xy+y^2}{3x^2}.$

$\left[\textbf{Ans.: } 3x+y\log x+cy=0\right]$

3. $x\frac{dy}{dx}=y(\log y-\log x+1).$

$\left[\textbf{Ans.: } \log\frac{y}{x}=cx\right]$

4. $y\,dx+x\log\frac{y}{x}dy-2x\,dy=0.$

$\left[\textbf{Ans.: } y=c\left(1+\log\frac{x}{y}\right)\right]$

5. $\left(xe^{\frac{y}{x}}-y\sin\frac{y}{x}\right)dx+x\sin\frac{y}{x}dy=0.$

$\left[\textbf{Ans.: } \log x^2-e^{-\frac{y}{x}}\left(\sin\frac{y}{x}+\cos\frac{y}{x}\right)=c\right]$

6. $x\sin\frac{y}{x}dy=\left(y\sin\frac{y}{x}-x\right)dx.$

$\left[\textbf{Ans.: } \cos\frac{y}{x}=\log x+c\right]$

7. $x\frac{dy}{dx}=y+x\sec\left(\frac{y}{x}\right).$

$\left[\textbf{Ans.: } \sin\frac{y}{x}=\log(cx)\right]$

8. $\left(x\tan\frac{y}{x}-y\sec^2\frac{y}{x}\right)dx+x\sec^2\frac{y}{x}dy=0.$

$\left[\textbf{Ans.: } x\tan\frac{y}{x}=c\right]$

9. $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0.$

$\left[\textbf{Ans.:}\ x + ye^{\frac{x}{y}} = c\right]$

10. $(3xy + y^2)dx + (x^2 + xy)dy = 0,$
$y(1) = 1.$

$\left[\textbf{Ans.:}\ x^2 y\ (2x + y) = 3\right]$

11. $2x\ (x + y)\frac{dy}{dx} = 3y^2 + 4xy,\ y\ (1) = 1.$

$\left[\textbf{Ans.:}\ y^2 + 2xy = 3x^3\right]$

12. $3x\frac{dy}{dx} - 3y + (x^2 - y^2)^{\frac{1}{2}} = 0, y\ (1) = 1.$

$\left[\textbf{Ans.:}\ 3\cos^{-1}\left(\frac{y}{x}\right) - \log x = 0\right]$

13. $(x^3 - 3xy^2)dx + (y^3 - 3x^2 y)dy = 0,$
$y(0) = 1.$

$\left[\textbf{Ans.:}\ x^4 - 6x^2 y^2 + y^4 = 1\right]$

14. $xy \log\frac{x}{y} dx + \left(y^2 - x^2 \log\frac{x}{y}\right) dy = 0,$

$y\ (1) = e.$

$\left[\begin{aligned}\textbf{Ans.:}\ &\frac{x^2}{2y^2}\log\frac{x}{y} - \frac{x^2}{4y^2} + \log y \\ &= 1 - \frac{3}{4e^2}\end{aligned}\right]$

10.3.3 Non-Homogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \qquad \ldots (1)$$

is called non-homogeneous equation where a_1, b_1, c_1, a_2, b_2, c_2 are all constants. These equations are classified into two parts and can be solved by following methods:

Case I: If
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$$
$$a_1 = a_2 m,\ b_1 = b_2 m,$$

then Eq. (1) reduces to

$$\frac{dy}{dx} = \frac{m(a_2 x + b_2 y) + c_1}{a_2 x + b_2 y + c_2} \qquad \ldots (2)$$

Putting $a_2 x + b_2 y = t,\ a_2 + b_2\frac{dy}{dx} = \frac{dt}{dx}$, Eq. (2) reduces to variable-separable form and can be solved using the method of variable-separable equation.

Case II: If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then substituting

$x = X + h, y = Y + k$ in the Eq. (1),

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$$= \frac{(a_1 X + b_1 Y) + (a_1 h + b_1 k + c_1)}{(a_2 X + b_2 Y) + (a_2 h + b_2 k + c_2)} \qquad \ldots (3)$$

Choosing h, k such that

$$a_1h+b_1k+c_1=0,\ a_2h+b_2k+c_2=0,$$

then Eq. (3) reduces to

$$\frac{\mathrm{d}Y}{\mathrm{d}X}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$$

which is a homogeneous equation and can be solved using the method of homogeneous equation. Finally substituting $X=x-h$, $Y=y-k$, we get the solution of Eq. (1).

Problems based on Case I: $\frac{a_1}{b_2}=\frac{b_1}{b_2}$

Example 1: Solve $(x+y-1)\mathrm{d}x+(2x+2y-3)\mathrm{d}y=0$.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x+y-1}{2x+2y-3}=\frac{-x-y+1}{2x+2y-3} \qquad \ldots (1)$$

The equation is non-homogeneous and $\frac{a_1}{a_2}=\frac{b_1}{b_2}=-\frac{1}{2}$

Let $x+y=t$

$$1+\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}t}{\mathrm{d}x},\quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}t}{\mathrm{d}x}-1$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}t}{\mathrm{d}x}-1=\frac{-t+1}{2t-3}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{-t+1}{2t-3}+1=\frac{-t+1+2t-3}{2t-3}=\frac{t-2}{2t-3}$$

$$\left(\frac{2t-3}{t-2}\right)\mathrm{d}t=\mathrm{d}x$$

$$\left(2+\frac{1}{t-2}\right)\mathrm{d}t=\mathrm{d}x$$

Integrating both the sides,

$$\int\left(2+\frac{1}{t-2}\right)\mathrm{d}t=\int\mathrm{d}x$$

$$2t+\log(t-2)=x+c$$

$$2(x+y)+\log(x+y-2)=x+c$$

$$x+2y+\log(x+y-2)=c$$

Example 2: Solve $(x+y)\mathrm{d}x+(3x+3y-4)\mathrm{d}y=0,\ y(1)=0$.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-x-y}{3x+3y-4} \qquad \ldots (1)$$

The equation is non-homogeneous and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{-1}{3}$

Let $x + y = t$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

Substituting in Eq. (1),

$$\frac{dt}{dx} - 1 = \frac{-t}{3t-4}$$

$$\frac{dt}{dx} = \frac{-t}{3t-4} + 1 = \frac{-t+3t-4}{3t-4} = \frac{2t-4}{3t-4}$$

$$\left(\frac{3t-4}{2t-4}\right) dt = dx$$

$$\frac{1}{2}\left(3 + \frac{2}{t-2}\right) dt = dx$$

Integrating both the sides,

$$\frac{1}{2}\int\left(3 + \frac{2}{t-2}\right) dt = \int dx$$

$$\frac{1}{2}\left[3t + 2\log|(t-2)|\right] = x + c$$

$$3(x+y) + 2\log|(x+y-2)| = 2x + 2c$$

$$x + 3y + 2\log|(x+y-2)| = k, \text{ where } 2c = k$$

Given $y(1) = 0$

Putting $x = 1, y = 0$ in the above equation,

$$1 + 2\log|-1| = k$$

$$1 + 2\log 1 = k$$

$$k = 1$$

Hence, solution is

$$x + 3y + 2\log|x + y - 2| = 1$$

Problem Based on Case II: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Example 1: Solve $(x+2y)dx + (y-1)dy = 0$.

Solution:
$$\frac{dy}{dx} = \frac{-x-2y}{y-1} \qquad \ldots (1)$$

The equation is non-homogeneous and $\frac{-1}{0} \neq \frac{-2}{1}$

Let $x = X + h,\qquad y = Y + k$

$\mathrm{d}x = \mathrm{d}X,\qquad \mathrm{d}y = \mathrm{d}Y$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X}$$

Substituting in Eq. (1),

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{-(X+h)-2(Y+k)}{(Y+k)-1} = \frac{(-X-2Y)+(-h-2k)}{Y+(k-1)} \qquad \dots (2)$$

Choosing h, k such that

$$-h-2k = 0,\ k-1 = 0 \qquad \dots (3)$$

Solving these equations,

$$k = 1,\ h = -2$$

Substituting Eq. (3) in Eq. (2),

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{-X-2Y}{Y} \qquad \dots (4)$$

which is a homogeneous equation.
Let $Y = vX$

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = v + X\frac{\mathrm{d}v}{\mathrm{d}X}$$

Substituting in Eq. (4),

$$v + X\frac{\mathrm{d}v}{\mathrm{d}X} = \frac{-X-2vX}{vX} = \frac{-1-2v}{v}$$

$$X\frac{\mathrm{d}v}{\mathrm{d}X} = \frac{-1-2v}{v} - v = \frac{-1-2v-v^2}{v} = \frac{-(v+1)^2}{v}$$

$$\frac{v}{(v+1)^2}\mathrm{d}v = -\frac{\mathrm{d}X}{X}$$

$$\left[\frac{1}{v+1} - \frac{1}{(v+1)^2}\right]\mathrm{d}v = -\frac{\mathrm{d}X}{X}$$

Integrating both the sides,

$$\int\frac{1}{v+1}\mathrm{d}v - \int\frac{1}{(v+1)^2}\mathrm{d}v = -\int\frac{\mathrm{d}X}{X}$$

$$\log(v+1) + \frac{1}{v+1} = -\log X + c$$

$$\log\left(\frac{Y}{X}+1\right) + \frac{1}{\frac{Y}{X}+1} = -\log X + c$$

$$\log\left(\frac{Y+X}{X}\right)+\frac{X}{Y+X}=-\log X+c$$

$$\log(Y+X)-\log X+\frac{X}{Y+X}=-\log X+c$$

$$\log(Y+X)+\frac{X}{Y+X}=c$$

Now,
$$X = x - h = x + 2$$
$$Y = y - k = y - 1$$

Hence, solution is

$$\log(x+y+1)+\left(\frac{x+2}{x+y+1}\right)=c$$

Example 2: Solve $\dfrac{dy}{dx}=\dfrac{2x-5y+3}{2x+4y-6}$.

Solution: The equation is non-homogeneous and $\dfrac{2}{2}\neq\dfrac{-5}{4}$

Let $x = X + h, \quad y = Y + k$

$dx = dX, \quad dy = dY$

$$\frac{dy}{dx}=\frac{dY}{dX}$$

Substituting in the given equation,

$$\frac{dY}{dX}=\frac{2(X+h)-5(Y+k)+3}{2(X+h)+4(Y+k)-6}=\frac{(2X-5Y)+(2h-5k+3)}{(2X+4Y)+(2h+4k-6)} \quad \dots (1)$$

Choosing h, k such that

$$2h - 5k + 3 = 0, \qquad 2h + 4k - 6 = 0 \quad \dots (2)$$

Solving the equations,

$$h = k = 1$$

Substituting Eq. (2) in Eq. (1),

$$\frac{dY}{dX}=\frac{2X-5Y}{2X+4Y} \quad \dots (3)$$

which is a homogeneous equation.

Let $Y = vX$

$$\frac{dY}{dX}=v+X\frac{dv}{dX}$$

Substituting in Eq. (3),

$$v+X\frac{dv}{dX}=\frac{2X-5vX}{2X+4vX}=\frac{2-5v}{2+4v}$$

$$X\frac{\mathrm{d}v}{\mathrm{d}X}=\frac{2-5v}{2+4v}-v=\frac{2-5v-2v-4v^2}{2+4v}=\frac{-4v^2-7v+2}{2+4v}$$

$$\frac{2+4v}{4v^2+7v-2}\mathrm{d}v=-\frac{\mathrm{d}X}{X}$$

$$\frac{2+4v}{(4v-1)(v+2)}\mathrm{d}v=-\frac{\mathrm{d}X}{X} \qquad \dots (4)$$

Now,

$$\frac{2+4v}{(4v-1)(v+2)}\mathrm{d}v=\frac{A}{4v-1}+\frac{B}{v+2}$$

$$\begin{aligned}2+4v &= A(v+2)+B(4v-1)\\ &=(A+4B)v+(2A-B)\end{aligned}$$

Comparing coefficients on both the sides,

$$A+4B=4, \qquad 2A-B=2$$

$$A=\frac{4}{3}, \qquad B=\frac{2}{3}$$

$$\frac{2+4v}{(4v-1)(v+2)}=\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}$$

Substituting in Eq. (4),

$$\left[\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right]\mathrm{d}v=-\frac{\mathrm{d}X}{X}$$

Integrating both the sides,

$$\int\left\{\frac{4}{3(4v-1)}+\frac{2}{3(v+2)}\right\}\mathrm{d}v=-\int\frac{\mathrm{d}X}{X}$$

$$\frac{4}{3}\frac{\log(4v-1)}{4}+\frac{2}{3}\log(v+2)=-\log X+\log c$$

$$\frac{1}{3}\log(4v-1)(v+2)^2=\log\frac{c}{X}$$

$$\log(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}}=\log\frac{c}{X}$$

$$(4v-1)^{\frac{1}{3}}(v+2)^{\frac{2}{3}}=\frac{c}{X}$$

$$\left(\frac{4Y}{X}-1\right)^{\frac{1}{3}}\left(\frac{Y}{X}+2\right)^{\frac{2}{3}}=\frac{c}{X}$$

$$(4Y-X)^{\frac{1}{3}}(Y+2X)^{\frac{2}{3}}=c$$

$$(4Y-X)(Y+2X)^2=c^3=k$$

Now,
$$X = x - h = x - 1$$
$$Y = y - k = y - 1$$

Hence, solution is
$$(4y - x - 3)(y + 2x - 3)^2 = k$$

Exercise 10.3

Solve the following differential equations:

1. $(x + 2y)\mathrm{d}x + (3x + 6y + 3)\mathrm{d}y = 0.$

$\left[\textbf{Ans.: } x + 3y - 3\log|x + 2y + 3| = c\right]$

2. $(6x - 4y + 1)\mathrm{d}y - (3x - 2y + 1)\mathrm{d}x = 0.$

$\left[\textbf{Ans.: } 4x - 8y - \log(12x - 8y + 1) = c\right]$

3. $(x + y + 3)\mathrm{d}y = (x + y - 3)\mathrm{d}x.$

$\left[\textbf{Ans.: } -x + y - 3\log(x + y) = c\right]$

4. $(x + y + 3)\mathrm{d}x - (2x + 2y - 1)\mathrm{d}y = 0.$

$\left[\textbf{Ans.: } -3x + 6y - 7\log|3x + 3y + 2| = c\right]$

5. $(2x + 6y + 1)\mathrm{d}y - (x + 3y - 2)\mathrm{d}x = 0.$

$\left[\textbf{Ans.: } -x + 2y + \log|x + 3y - 1| = c\right]$

6. $(y - x + 2)\mathrm{d}y = (y - x)\mathrm{d}x.$

$\left[\textbf{Ans.: } (y - x)^2 + 4y = c\right]$

7. $(4x + 2y + 5)\mathrm{d}y - (2x + y - 1)\mathrm{d}x = 0.$

$\left[\textbf{Ans.: } 10y - 5x + 7\log|10x + 5y + 9| = c\right]$

8. $(2x - 4y + 5)\mathrm{d}y - (x - 2y + 3)\mathrm{d}x = 0.$

$\left[\textbf{Ans.: } x^2 - 4xy + 4y^2 + 6x - 10y = c\right]$

9. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{2x - y + 1}{x + y}.$

$$\left[\begin{aligned}&\textbf{Ans.: } \log\left[2\left(x + \frac{1}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2\right]\\ &\quad + \sqrt{2}\tan^{-1}\left[\frac{3y - 1}{\sqrt{2}(3x + 1)}\right] = c\end{aligned}\right]$$

10. $(x + y - 1)\mathrm{d}x - (x - y - 1)\mathrm{d}y = 0.$

$$\left[\begin{aligned}&\textbf{Ans.: } \log[(x - 1)^2 + y^2]\\ &\quad - 2\tan^{-1}\left(\frac{y}{x - 1}\right) = c\end{aligned}\right]$$

11. $(3x - 2y + 4)\mathrm{d}x - (2x + 7y - 1)\mathrm{d}y = 0.$

$\left[\textbf{Ans.: } 3x^2 - 4xy - 7y^2 + 8x + 2y = c\right]$

12. $(x - y - 1)\mathrm{d}x + (4y + x - 1)\mathrm{d}y = 0.$

$$\left[\begin{aligned}&\textbf{Ans.: } \log[4y^2 + (x - 1)^2\\ &\quad + \tan^{-1}\left(\frac{2y}{x - 1}\right) = c\end{aligned}\right]$$

13. $(x - y - 1)\mathrm{d}x + (x + y + 5)\mathrm{d}y = 0.$

$$\left[\begin{aligned}&\textbf{Ans.: } \log\left[(y + 3)^2 + (x + 2)^2\right]\\ &\quad + 2\tan^{-1}\left(\frac{y + 3}{x + 2}\right) = c\end{aligned}\right]$$

14. $(y - x + 2)\mathrm{d}x + (x + y + 6)\mathrm{d}y = 0.$

$$\left[\begin{aligned}&\textbf{Ans.: } (y + 4)^2 + 2(x + 2)(y + 4)\\ &\quad - (x + 2)^2 = c\end{aligned}\right]$$

15. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y + x - 2}{y - x - 4}.$

$$\left[\begin{aligned}&\textbf{Ans.: } (x + 1)^2 - (y - 3)^2\\ &\quad + 2(x + 1)(y - 3) = c\end{aligned}\right]$$

16. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x + 9y - 20}{6x + 2y - 10}.$

$\left[\textbf{Ans.: } (2x - y)^2 = c(x + 2y - 5)\right]$

17. $(3x+2y+3)\mathrm{d}x-(x+2y-1)\mathrm{d}y=0$, $y(-2)=1$.

$\left[\textbf{Ans :}\ (2x+2y+1)(3x-2y+9)^4=-1\right]$

18. $(x+y+2)\mathrm{d}x-(x-y-4)\mathrm{d}y=0$, $y(1)=0$.

$\left[\textbf{Ans.:}\ \log\left[(x-1)^2+(y+3)^2\right]+2\tan^{-1}\left(\frac{x-1}{y+3}\right)=2\log 3\right]$

10.3.4 Exact Differential Equation

Any first order differential equation which is obtained by differentiation of its general solution without any elimination or reduction of terms is known as exact differential equation.

If $f(x, y) = c$ is the general solution,

then

$$\mathrm{d}f=0$$

$$\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y=0$$

$$\mathrm{M}(x,y)\mathrm{d}x+\mathrm{N}(x,y)\mathrm{d}y=0 \qquad \ldots (1)$$

represents an exact differential equation

where $$\mathrm{M}(x,y)=\frac{\partial f}{\partial x},\qquad \mathrm{N}(x,y)=\frac{\partial f}{\partial y}$$

$$\frac{\partial \mathrm{M}}{\partial y}=\frac{\partial^2 f}{\partial y\partial x},\qquad \frac{\partial \mathrm{N}}{\partial x}=\frac{\partial^2 f}{\partial x\partial y}$$

But $$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$$

Therefore, $$\frac{\partial \mathrm{M}}{\partial y}=\frac{\partial \mathrm{N}}{\partial x}$$

Thus, necessary condition for a differential equation to be an exact differential equation is

$$\frac{\partial \mathrm{M}}{\partial y}=\frac{\partial \mathrm{N}}{\partial x}$$

Solution of Eq. (1) can be written as

$$\int_{y\text{ constant}}\mathrm{M}(x,y)\mathrm{d}x+\int(\text{terms of N not containing }x)\mathrm{d}y=c$$

Sometimes, integration of M w.r.t. x is tedious whereas N can be integrated easily w.r.t. y. In this case solution can be written as

$$\int(\text{terms of M not containing }y)\mathrm{d}x+\int_{x\text{ constant}}\mathrm{N}(x,y)\mathrm{d}y=c$$

Example 1: Solve $(y^2 - x^2)dx + 2xydy = 0$.

Solution: $\quad M = y^2 - x^2, \qquad N = 2xy$

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = 2y$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y \text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int (y^2 - x^2)\,dx + \int 0\,dy = c$$

$$xy^2 - \frac{x^3}{3} = c$$

Example 2: Solve $\left(x\sqrt{1-x^2y^2} - y\right)dy + \left(x + y\sqrt{1-x^2y^2}\right)dx = 0$.

Solution: $\quad N = x\sqrt{1-x^2y^2} - y, \qquad M = x + y\sqrt{1-x^2y^2}$

$$\frac{\partial N}{\partial x} = \sqrt{1-x^2y^2} + x\left[\frac{-2xy^2}{2\sqrt{1-x^2y^2}}\right], \qquad \frac{\partial M}{\partial y} = \sqrt{1-x^2y^2} + y\left[\frac{-2x^2y}{2\sqrt{1-x^2y^2}}\right]$$

$$= \sqrt{1-x^2y^2} - \frac{x^2y^2}{\sqrt{1-x^2y^2}} \qquad = \sqrt{1-x^2y^2} - \frac{x^2y^2}{\sqrt{1-x^2y^2}}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y \text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int \left(x + y\sqrt{1-x^2y^2}\right)dx + \int (-y)\,dy = c$$

$$\frac{x^2}{2} + y^2\int\left(\sqrt{\frac{1}{y^2} - x^2}\right)dx - \frac{y^2}{2} = c$$

$$\frac{x^2}{2} + y^2\left[\left|\frac{x}{2}\sqrt{\frac{1}{y^2} - x^2}\right| + \frac{1}{2y^2}\sin^{-1}\left(\frac{x}{\frac{1}{y}}\right)\right] - \frac{y^2}{2} = c$$

$$\frac{x^2 - y^2}{2} + \frac{xy}{2}\sqrt{1-x^2y^2} + \frac{1}{2}\sin^{-1}(xy) = c$$

$$x^2 - y^2 + xy\sqrt{1-x^2y^2} + \sin^{-1}(xy) = 2c = k$$

Example 3: Solve $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0$.

Solution: $M = 2xy\cos x^2 - 2xy + 1$, $\quad N = \sin x^2 - x^2$

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x, \qquad \frac{\partial N}{\partial x} = (\cos x^2)(2x) - 2x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y \text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int (2xy\cos x^2 - 2xy + 1)\,dx + \int 0\,dy = c$$

$$y\sin x^2 - x^2 y + x = c \qquad \left[\because \int \{\cos f(x)\} f'(x)\,dx = \sin f(x)\right]$$

Example 4: Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$.

Solution: $(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

$$M = y\cos x + \sin y + y, \qquad N = \sin x + x\cos y + x$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1, \qquad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y \text{ constant}} M\,dx + \int_{\text{terms not contaning } x} N\,dy = c$$

$$\int (y\cos x + \sin y + y)dx + \int 0\,dy = c$$

$$y\sin x + x(\sin y + y) = c$$

Example 5: Solve $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$, $y(0) = 4$.

Solution: $M = 1 + e^{\frac{x}{y}}$, $\quad N = e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right), \qquad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{1}{y}\right)\left(1-\frac{x}{y}\right)+e^{\frac{x}{y}}\left(-\frac{1}{y}\right)$$

$$= \frac{-x}{y^2}e^{\frac{x}{y}}, \qquad = -\frac{x}{y^2}e^{\frac{x}{y}}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y \text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int\left(1+e^{\frac{x}{y}}\right)dx + \int 0\,dy = c$$

$$1+\frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$1+ye^{\frac{x}{y}} = c \qquad \dots (1)$$

Given $y(0) = 4$

Substituting in Eq. (1),

$$1+4e^0 = c$$

$$5 = c$$

Hence, solution is

$$1+ye^{\frac{x}{y}} = 5$$

Example 6: Solve $\left[\log(x^2+y^2)+\frac{2x^2}{x^2+y^2}\right]dx+\frac{2xy}{x^2+y^2}dy = 0.$

Solution: $M = \left[\log(x^2+y^2)+\frac{2x^2}{x^2+y^2}\right], \qquad N = \frac{2xy}{x^2+y^2}$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2+y^2}\cdot 2y - \frac{2x^2}{(x^2+y^2)^2}\cdot 2y, \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2+y^2} - \frac{2xy}{(x^2+y^2)^2}\cdot 2x$$

$$= \frac{2y}{x^2+y^2} - \frac{4x^2y}{(x^2+y^2)^2} \qquad = \frac{2y}{x^2+y^2} - \frac{4x^2y}{(x^2+y^2)^2}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{\text{terms not containing } y} M\,dx \quad + \quad \int\limits_{x \text{ constant}} N\,dy = c$$

$$\int 0\,dx + \int \frac{2xy}{x^2 + y^2}\,dy = c$$

$$x\log(x^2 + y^2) = c$$

Example 7: For what values of *a* and *b*, the differential equation $(y + x^3)dx + (ax + by^3)dy = 0$ is exact. Also find the solution of the equation.

Solution: $M = y + x^3, \qquad N = ax + by^3$

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = a$$

Equation will be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = a$$

Hence, equation is exact for $a = 1$ and for all values of b.
Substituting $a = 1$ in the equation, $(y + x^3)\,dx + (x + by^3)\,dy = 0$, which is exact.
Hence, solution is

$$\int\limits_{y \text{ constant}} M\,dx + \int\limits_{\text{terms not containing } x} N\,dy = c$$

$$\int (y + x^3)\,dx + \int by^3\,dy = c$$

$$xy + \frac{x^4}{4} + \frac{by^4}{4} = c$$

Example 8: Solve $(\cos x + y\sin x)\,dx = (\cos x)dy,\ y(\pi) = 0$.

Solution: $(\cos x + y\sin x)\,dx - (\cos x)\,dy = 0$

$$M = \cos x + y\sin x, \qquad N = -\cos x$$

$$\frac{\partial M}{\partial y} = \sin x, \qquad \frac{\partial N}{\partial x} = \sin x$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} \mathrm{M}\,\mathrm{d}x + \int_{\text{terms not containing } x} \mathrm{N}\,\mathrm{d}y = c$$

$$\int (\cos x + y\sin x)\,\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$\sin x - y\cos x = c \qquad \dots (1)$$

Given $y(\pi) = 0$

Substituting $x = \pi,\ y = 0$ in Eq. (1),

$$\sin \pi - 0 = c$$

$$0 = c$$

Hence, solution is

$$\sin x - y\cos x = 0$$

$$y = \tan x$$

Example 9: Solve $(ye^{xy} + 4y^3)\,\mathrm{d}x + (xe^{xy} + 12xy^2 - 2y)\,\mathrm{d}y = 0,\ y(0) = 2.$

Solution: $\mathrm{M} = ye^{xy} + 4y^3,\qquad \mathrm{N} = xe^{xy} + 12xy^2 - 2y$

$$\frac{\partial \mathrm{M}}{\partial y} = e^{xy} + ye^{xy}\cdot x + 12y^2,\qquad \frac{\partial \mathrm{N}}{\partial x} = e^{xy} + xe^{xy}\cdot y + 12y^2$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} \mathrm{M}\,\mathrm{d}x + \int_{\text{terms not containing } x} \mathrm{N}\,\mathrm{d}y = c$$

$$\int (ye^{xy} + 4y^3)\,\mathrm{d}x + \int -2y\,\mathrm{d}y = c$$

$$y\frac{e^{xy}}{y} + 4y^3x - y^2 = c$$

$$e^{xy} + 4xy^3 - y^2 = c \qquad \dots (1)$$

Given $y(0) = 2$

Substituting $x = 0, y = 2$ in Eq. (1)

$$e^0 + 0 - 4 = c,\qquad -3 = c$$

Hence, solution is

$$e^{xy} + 4xy^3 - y^2 = -3$$

Exercise 10.4

Solve the following differential equations:

1. $(2x^3+3y)\,dx+(3x+y-1)\,dy=0.$

$$\left[\textbf{Ans.: } x^4+6xy+y^2-2y=c\right]$$

2. $(1+e^x)\,dx+y\,dy=0.$

$$\left[\textbf{Ans.: } x+e^x+\frac{y^2}{2}=c\right]$$

3. $\sinh x\cos y\,dx-\cosh x\sin y\,dy=0.$

$$\left[\textbf{Ans.: } \cosh x\cos y=c\right]$$

4. $xe^{x^2+y^2}dx+y(1+e^{x^2+y^2})dy=0,$
$y(0)=0.$

$$\left[\textbf{Ans.: } y^2+e^{x^2+y^2}=1\right]$$

5. $\left(4x^3y^3+\frac{1}{x}\right)dx+\left(3x^4y^2-\frac{1}{y}\right)dy=0,$
$y(1)=1.$

$$\left[\textbf{Ans.: } x^4y^3+\log\left(\frac{x}{y}\right)=1\right]$$

6. $(4x^3y^3dx+3x^4y^2dy)$
$-(2xy\,dx+x^2dy)=0.$

$$\left[\textbf{Ans.: } x^4y^3-x^2y=c\right]$$

7. $2x(ye^{x^2}-1)dx+e^{x^2}dy=0.$

$$\left[\textbf{Ans.: } ye^{x^2}-x^2=c\right]$$

8. $\left(1+x^2\sqrt{y}\right)y\,dx+\left(x^2\sqrt{y}+2\right)x\,dy=0.$

$$\left[\textbf{Ans.: } 2xy+\frac{2}{3}x^3y^{\frac{3}{2}}=c\right]$$

9. $(e^y+1)\cos x\,dx+e^y\sin x\,dy=0.$

$$\left[\textbf{Ans.: } \sin x(e^y+1)=c\right]$$

10. $(x^2+1)\frac{dy}{dx}=x^3-2xy+x.$

$$\left[\textbf{Ans.: } x^4-4x^2y+2x^2-4y=c\right]$$

11. $\frac{dy}{dx}=\frac{x^2-2xy}{x^2-\sin y}.$

$$\left[\textbf{Ans.: } x^3-3(x^2y+\cos y)=c\right]$$

12. $\frac{dy}{dx}=\frac{y+1}{(y+2)e^y-x}.$

$$\left[\textbf{Ans.: } (y+1)(x-e^y)=c\right]$$

13. $(x-y\cos x)dx-\sin x\,dy=0,$
$y\left(\frac{\pi}{2}\right)=1.$

$$\left[\textbf{Ans.: } x^2-2y\sin x=\frac{\pi^2}{4}-2\right]$$

14. $(2xy+e^y)\,dx+(x^2+xe^y)\,dy=0,$
$y(1)=1.$

$$\left[\textbf{Ans.: } x^2y+xe^y=e+1\right]$$

10.3.5 Non-Exact Differential Equations Reducible to Exact Form

Sometimes a differential equation is not exact but can be made exact by multiplying with a suitable function. This function is known as Integrating factor (I.F.). There may exists more than one integrating factor to a differential equation.

Here, we will discuss different methods to find an I.F. to a non exact differential equation,

$$\mathrm{M}\,\mathrm{d}x + \mathrm{N}\,\mathrm{d}y = 0$$

Case I: If $\dfrac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = f(x)$, (function of x alone), then $\text{I.F.} = e^{\int f(x)\mathrm{d}x}$

After multiplication with the I.F. the equation becomes exact and can be solved using the method of exact differential equations.

Example 1: Solve $(x^2 + y^2 + 1)\,\mathrm{d}x - 2xy\,\mathrm{d}y = 0$.

Solution:

$$\mathrm{M} = x^2 + y^2 + 1, \qquad \mathrm{N} = -2xy$$

$$\frac{\partial \mathrm{M}}{\partial y} = 2y, \qquad \frac{\partial \mathrm{N}}{\partial x} = -2y$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial \mathrm{M}}{\partial y} - \dfrac{\partial \mathrm{N}}{\partial x}}{\mathrm{N}} = \frac{2y-(-2y)}{-2xy} = -\frac{2}{x}$$

$$\text{I.F.} = e^{\int -\frac{2}{x}\mathrm{d}x} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x^2}(x^2 + y^2 + 1)\mathrm{d}x - \frac{1}{x^2}2xy\,\mathrm{d}y = 0$$

$$\left(1 + \frac{y^2+1}{x^2}\right)\mathrm{d}x - \frac{2y}{x}\mathrm{d}y = 0$$

$$\mathrm{M}_1 = 1 + \frac{y^2+1}{x^2}, \qquad \mathrm{N}_1 = -\frac{2y}{x}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{2y}{x^2}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{2y}{x^2}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}} \mathrm{M}_1\,\mathrm{d}x + \int\limits_{\text{terms not containing }x} \mathrm{N}_1\,\mathrm{d}y = c$$

$$\int\left(1+\frac{y^2+1}{x^2}\right)dx+\int 0\,dy=c$$

$$x-\frac{y^2+1}{x}=c$$

$$x^2-y^2-1=cx$$

Example 2: Solve $\left(xy^2-e^{\frac{1}{x^3}}\right)dx-x^2y\,dy=0.$

Solution:

$$M=xy^2-e^{\frac{1}{x^3}},\qquad N=-x^2y$$

$$\frac{\partial M}{\partial y}=2xy,\qquad \frac{\partial N}{\partial x}=-2xy$$

Since $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{2xy-(-2xy)}{-x^2y}=-\frac{4}{x}$$

$$\text{I.F.}=e^{\int-\frac{4}{x}dx}=e^{-4\log x}=e^{\log x^{-4}}=x^{-4}=\frac{1}{x^4}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x^4}(xy^2-e^{\frac{1}{x^3}})\,dx-\frac{1}{x^4}(x^2y)\,dy=0$$

$$\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)dx-\frac{y}{x^2}dy=0$$

$$M_1=\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4},\qquad N_1=-\frac{y}{x^2}$$

$$\frac{\partial M_1}{\partial y}=\frac{2y}{x^3},\qquad \frac{\partial N_1}{\partial x}=\frac{2y}{x^3}$$

Since $\dfrac{\partial M_1}{\partial y}=\dfrac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1\,dx+\int_{\text{terms not containing }x} N_1\,dy=c$$

$$\int \left(\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4} \right) dx + \int 0\, dy = c$$

$$-\frac{y^2}{2x^2} + \frac{1}{3} \int e^{\frac{1}{x^3}} \left(-\frac{3}{x^4} \right) dx = c$$

$$-\frac{y^2}{2x^2} + \frac{1}{3} e^{\frac{1}{x^3}} = c \qquad \left[\because \int e^{f(x)} f'(x) dx = e^{f(x)} + c \right]$$

Example 3: **Solve $(2x \log x - xy)\,dy + 2y\,dx = 0$.**

Solution: $2y\,dx + (2x \log x - xy)dy = 0$

$$M = 2y, \qquad N = 2x \log x - xy$$

$$\frac{\partial M}{\partial y} = 2, \qquad \frac{\partial N}{\partial x} = 2 \log x + 2 - y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 - (2\log x + 2 - y)}{2x \log x - xy}$$

$$= \frac{-(2\log x - y)}{x(2\log x - y)} = -\frac{1}{x}$$

$$\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}}$$

$$= x^{-1} = \frac{1}{x}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x}(2y)\,dx + \frac{1}{x}(2x \log x - xy)\,dy = 0$$

$$\frac{2y}{x} dx + (2\log x - y)dy = 0$$

$$M_1 = \frac{2y}{x}, \qquad N_1 = 2\log x - y$$

$$\frac{\partial M_1}{\partial y} = \frac{2}{x}, \qquad \frac{\partial N_1}{\partial x} = \frac{2}{x}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1\,dx + \int_{\text{terms not containing } x} N_1\,dy = c$$

$$\int \frac{2y}{x}\,dx + \int (-y)\,dy = c$$

$$2y\log x - \frac{y^2}{2} = c$$

Example 4: **Solve** $x\sin x\dfrac{dy}{dx} + y(x\cos x - \sin x) = 2.$

Solution:

$$x\sin x\,dy + (xy\cos x - y\sin x - 2)\,dx = 0$$

$$(xy\cos x - y\sin x - 2)\,dx + x\sin x\,dy = 0$$

$$M = xy\cos x - y\sin x - 2 \qquad N = x\sin x$$

$$\frac{\partial M}{\partial y} = x\cos x - \sin x \qquad \frac{\partial N}{\partial x} = \sin x + x\cos x$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{(x\cos x - \sin x) - (\sin x + x\cos x)}{x\sin x}$$

$$= -\frac{2\sin x}{x\sin x} = -\frac{2}{x}$$

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x^2}(xy\cos x - y\sin x - 2)\,dx + \frac{1}{x^2}(x\sin x)\,dy = 0$$

$$\left(\frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}\right)dx + \frac{1}{x}\sin x\,dy = 0$$

$$M_1 = \frac{y}{x}\cos x - \frac{y}{x^2}\sin x - \frac{2}{x^2}, \qquad N_1 = \frac{\sin x}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{\cos x}{x} - \frac{\sin x}{x^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{\text{terms not containing } y} \mathrm{M}_1\,\mathrm{d}x + \int_{x \text{ constant}} \mathrm{N}_1\,\mathrm{d}y = c$$

$$\int -\frac{2}{x^2}\,\mathrm{d}x + \int \frac{\sin x}{x}\,\mathrm{d}y = c$$

$$\frac{2}{x} + \left(\frac{\sin x}{x}\right) y = c$$

$$\frac{2}{x} + \frac{y\sin x}{x} = c$$

Case II: If $\dfrac{\dfrac{\partial \mathrm{N}}{\partial x} - \dfrac{\partial \mathrm{M}}{\partial y}}{\mathrm{M}} = f(y)$, (function of y alone),

then $\text{I.F.} = e^{\int f(y)\mathrm{d}y}$

After multiplying with the I.F., the equation becomes exact and can be solved using the method of exact differential equation.

Example 1: Solve $(y^4 + 2y)\mathrm{d}x + (xy^3 + 2y^4 - 4x)\mathrm{d}y = 0$.

Solution:

$$\mathrm{M} = y^4 + 2y, \qquad \mathrm{N} = xy^3 + 2y^4 - 4x$$

$$\frac{\partial \mathrm{M}}{\partial y} = 4y^3 + 2, \qquad \frac{\partial \mathrm{N}}{\partial x} = y^3 - 4$$

Since $\dfrac{\partial \mathrm{M}}{\partial y} \neq \dfrac{\partial \mathrm{N}}{\partial x}$, equation is not exact.

$$\frac{\dfrac{\partial \mathrm{N}}{\partial x} - \dfrac{\partial \mathrm{M}}{\partial y}}{\mathrm{M}} = \frac{y^3 - 4 - (4y^3 + 2)}{y^4 + 2y} = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = -\frac{3}{y}$$

$$\text{I.F.} = e^{\int -\frac{3}{y}\mathrm{d}y} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

Multiplying D.E. by I.F.,

$$\frac{1}{y^3}(y^4 + 2y)\mathrm{d}x + \frac{1}{y^3}(xy^3 + 2y^4 - 4x)\,\mathrm{d}y = 0$$

$$\left(y + \frac{2}{y^2}\right)\mathrm{d}x + \left(x + 2y - \frac{4x}{y^3}\right)\mathrm{d}y = 0$$

$$M_1 = y + \frac{2}{y^2}, \qquad N_1 = x + 2y - \frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y} = 1 - \frac{4}{y^3}, \qquad \frac{\partial N_1}{\partial x} = 1 - \frac{4}{y^3}$$

Since, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 dx + \int_{\text{terms not containing } x} N_1 dy = c$$

$$\int \left(y + \frac{2}{y^2} \right) dx + \int 2y\, dy = c$$

$$\left(y + \frac{2}{y^2} \right) x + y^2 = c$$

Example 2: Solve $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$.

Solution: $M = 2xy^4e^y + 2xy^3 + y,\qquad N = x^2y^4e^y - x^2y^2 - 3x$

$$\frac{\partial M}{\partial y} = 2x(y^4e^y + 4y^3e^y + 3y^2) + 1, \qquad \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(2xy^4e^y - 2xy^2 - 3) - (2xy^4e^y + 8xy^3e^y + 6xy^2 + 1)}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-4(2xy^3e^y + 2xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)} = -\frac{4}{y}$$

$$\text{I.F.} = e^{\int -\frac{4}{y} dy} = e^{-4\log y} = e^{\log y^{-4}} = y^{-4} = \frac{1}{y^4}$$

Multiplying D.E. by I.F.,

$$\frac{1}{y^4}(2xy^4e^y + 2xy^3 + y)dx + \frac{1}{y^4}(x^2y^4e^y - x^2y^2 - 3x)dy = 0$$

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3} \right) dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4} \right) dy = 0$$

$$M_1 = 2xe^y + \frac{2x}{y} + \frac{1}{y^3}, \qquad N_1 = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}$$

$$\frac{\partial M_1}{\partial y} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}, \qquad \frac{\partial N_1}{\partial x} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 \mathrm{d}x + \int_{\text{terms not containing } x} N_1 \mathrm{d}y = c$$

$$\int \left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right) \mathrm{d}x + \int 0 \, \mathrm{d}y = c$$

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

Example 3: Solve $xe^x(\mathrm{d}x - \mathrm{d}y) + e^x \mathrm{d}x + ye^y \mathrm{d}y = 0$.

Solution: $(xe^x + e^x)\mathrm{d}x + (ye^y - xe^x)\mathrm{d}y = 0$

$$M = xe^x + e^x, \qquad N = ye^y - xe^x$$

$$\frac{\partial M}{\partial y} = 0, \qquad \frac{\partial N}{\partial x} = -e^x - xe^x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-e^x(x+1) - 0}{e^x(x+1)} = -1$$

$$\text{I.F.} = e^{\int -\mathrm{d}y} = e^{-y}$$

Multiplying D.E. by I.F.,

$$e^{-y}(xe^x + e^x)\mathrm{d}x + e^{-y}(ye^y - xe^x) = 0$$

$$M_1 = e^{-y}(xe^x + e^x), \qquad N_1 = y - xe^{x-y}$$

$$\frac{\partial M_1}{\partial y} = -e^{-y}(xe^x + e^x), \qquad \frac{\partial N_1}{\partial x} = -e^{-y}(xe^x + e^x)$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 \mathrm{d}x + \int_{\text{terms not containing } x} N_1 \mathrm{d}y = c$$

$$\int e^{-y}(xe^x + e^x)\mathrm{d}x + \int y\,\mathrm{d}y = c$$

$$e^{-y}(xe^x - e^x + e^x) + \frac{y^2}{2} = c$$

$$xe^{x-y} + \frac{y^2}{2} = c$$

Example 4: Solve $\left(\frac{y}{x}\sec y - \tan y\right)\mathrm{d}x + (\sec y \log x - x)\mathrm{d}y = 0.$

Solution: $\mathrm{M} = \frac{y}{x}\sec y - \tan y, \qquad \mathrm{N} = \sec y \log x - x$

$$\frac{\partial \mathrm{M}}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \tan y - \sec^2 y, \qquad \frac{\partial \mathrm{N}}{\partial x} = \frac{\sec y}{x} - 1$$

Since $\frac{\partial \mathrm{M}}{\partial y} \neq \frac{\partial \mathrm{N}}{\partial x}$, equation is not exact.

$$\frac{\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y}}{\mathrm{M}} = \frac{\frac{\sec y}{x} - 1 - \frac{\sec y}{x} - \frac{y}{x}\sec y \tan y + \sec^2 y}{\frac{y}{x}\sec y - \tan y}$$

$$= \frac{-\frac{y}{x}\sec y \tan y + \tan^2 y}{\frac{y}{x}\sec y - \tan y} = -\tan y$$

$$\text{I.F.} = e^{\int -\tan y\,\mathrm{d}y} = e^{-\log \sec y} = e^{\log(\sec y)^{-1}} = (\sec y)^{-1} = \cos y$$

Multiplying D.E. by I.F.,

$$\cos y\left(\frac{y}{x}\sec y - \tan y\right)\mathrm{d}x + \cos y(\sec y \log x - x)\,\mathrm{d}y = 0$$

$$\left(\frac{y}{x} - \sin y\right)\mathrm{d}x + (\log x - x\cos y)\,\mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{y}{x} - \sin y, \qquad \mathrm{N}_1 = \log x - x\cos y$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = \frac{1}{x} - \cos y, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = \frac{1}{x} - \cos y$$

Since $\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\partial \mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} \mathrm{M}_1 \mathrm{d}x + \int_{\text{terms not containing } x} \mathrm{N}_1 \mathrm{d}y = c$$

$$\int \left(\frac{y}{x} - \sin y \right) \mathrm{d}x + \int 0 \, \mathrm{d}y = c$$

$$y \log x - x \sin y = c$$

Case III: If differential equation is of the form

$$f_1(xy) y \, \mathrm{d}x + f_2(xy) x \, \mathrm{d}y = 0, \text{ then}$$

$$\text{I.F.} = \frac{1}{\mathrm{M}x - \mathrm{N}y}, \quad \text{where } \mathrm{M} = f_1(xy)y, \ \mathrm{N} = f_2(xy)x$$

provided $\mathrm{M}x - \mathrm{N}y \neq 0$

After multiplying with the I.F., the equation becomes exact and can be solved using the method of exact differential equation.

Example 1: Solve $y(1 + xy + x^2y^2)\mathrm{d}x + x(1 - xy + x^2y^2)\mathrm{d}y = 0$.

Solution: Equation is of the form

$$f_1(xy) y \, \mathrm{d}x + f_2(xy) x \, \mathrm{d}y = 0$$

$$\text{I.F.} = \frac{1}{\mathrm{M}x - \mathrm{N}y} = \frac{1}{(xy + x^2y^2 + x^3y^3) - (xy - x^2y^2 + x^3y^3)}$$

$$= \frac{1}{2x^2y^2}$$

Multiplying D.E. by I.F.,

$$\frac{y}{2x^2y^2}(1 + xy + x^2y^2)\mathrm{d}x + \frac{x}{2x^2y^2}(1 - xy + x^2y^2)\mathrm{d}y = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2} \right) \mathrm{d}x + \left(\frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2} \right) \mathrm{d}y = 0$$

$$\mathrm{M}_1 = \frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}, \qquad \mathrm{N}_1 = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = -\frac{1}{2x^2y^2} + \frac{1}{2}, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = -\frac{1}{2x^2y^2} + \frac{1}{2}$$

Since $\frac{\partial \mathrm{M}_1}{\partial y} = \frac{\partial \mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 dx + \int_{\text{terms not containing } x} N_1 dy = c$$

$$\int \left(\frac{1}{2x^2 y} + \frac{1}{2x} + \frac{y}{2} \right) dx + \int -\frac{1}{2y} dy = c$$

$$-\frac{1}{2xy} + \frac{1}{2}\log x + \frac{xy}{2} - \frac{1}{2}\log y = c$$

$$-\frac{1}{2xy} + \frac{xy}{2} + \frac{1}{2}\log\frac{x}{y} = c$$

Example 2: Solve $(xy \sin xy + \cos xy)y\,dx + (xy \sin xy - \cos xy)x\,dy = 0$.

Solution: $M = xy^2 \sin xy + y \cos xy$, $N = x^2 y \sin xy - x \cos xy$

The equation is in the form

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$$

$$\text{I.F.} = \frac{1}{Mx - Ny}$$

$$= \frac{1}{x^2y^2 \sin xy + xy \cos xy - x^2y^2 \sin xy + xy \cos xy} = \frac{1}{2xy \cos xy}$$

Multiplying D.E. by I.F.,

$$\frac{1}{2xy \cos xy}(xy \sin xy + \cos xy)y\,dx + \frac{1}{2xy \cos xy}(xy \sin xy - \cos xy)x\,dy = 0$$

$$\left(\frac{y \tan xy}{2} + \frac{1}{2x} \right) dx + \left(\frac{x \tan xy}{2} - \frac{1}{2y} \right) dy = 0$$

$$M_1 = \frac{y \tan xy}{2} + \frac{1}{2x}, \qquad N_1 = \frac{x \tan xy}{2} - \frac{1}{2y}$$

$$\frac{\partial M_1}{\partial y} = \frac{\tan xy + xy \sec^2 xy}{2}, \qquad \frac{\partial N_1}{\partial x} = \frac{\tan xy + xy \sec^2 xy}{2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 dx + \int_{\text{terms not containing } x} N_1 dy = c$$

$$\frac{1}{2}\int \left(y \tan xy + \frac{1}{x} \right) dx + \int -\frac{1}{2y} dy = c$$

$$\frac{1}{2}\left(\frac{y}{y}\log\sec xy+\log x\right)-\frac{1}{2}\log y=c$$

$$\log(x\sec xy)-\log y=2c$$

$$\log\left(\frac{x}{y}\sec xy\right)=2c$$

$$\frac{x}{y}\sec xy=e^{2c}=k,\quad \frac{x}{y}\sec xy=k$$

Case IV: If differential equation $\mathrm{M}dx+\mathrm{N}dy=0$ is homogeneous equation in x and y (degree of each term is same), then $\text{I.F.}=\dfrac{1}{\mathrm{M}x+\mathrm{N}y}$ provided $\mathrm{M}x+\mathrm{N}y\neq 0$.

After multiplying with the I.F., the equation becomes exact and can be solved using the method of exact differential equations.

Example 1: Solve $(x^2y-2xy^2)\mathrm{d}x-(x^3-3x^2y)\mathrm{d}y=0$.

Solution: $\mathrm{M}=x^2y-2xy^2,\quad \mathrm{N}=-x^3+3x^2y$

Differential equation is homogeneous as each term is of degree 3.

$$\text{I.F.}=\frac{1}{\mathrm{M}x+\mathrm{N}y}=\frac{1}{x^3y-2x^2y^2-x^3y+3x^2y^2}=\frac{1}{x^2y^2}$$

Multiplying D.E. by I.F.,

$$\frac{1}{x^2y^2}(x^2y-2xy^2)\mathrm{d}x-\frac{1}{x^2y^2}(x^3-3x^2y)\mathrm{d}y=0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)\mathrm{d}x-\left(\frac{x}{y^2}-\frac{3}{y}\right)\mathrm{d}y=0$$

$$\mathrm{M}_1=\frac{1}{y}-\frac{2}{x},\qquad \mathrm{N}_1=-\frac{x}{y^2}+\frac{3}{y}$$

$$\frac{\partial \mathrm{M}_1}{\partial y}=-\frac{1}{y^2},\qquad \frac{\partial \mathrm{N}_1}{\partial x}=-\frac{1}{y^2}$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y}=\dfrac{\partial \mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}}\mathrm{M}_1\mathrm{d}x+\int\limits_{\text{terms not containing }x}\mathrm{N}_1\mathrm{d}y=c$$

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)\mathrm{d}x+\int\frac{3}{y}\mathrm{d}y=c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 2: Solve $x\frac{dy}{dx} + \frac{y^2}{x} = y$.

Solution: $x^2 dy + y^2 dx = xy\, dx$

$$(y^2 - xy)dx + x^2 dy = 0$$

$$M = y^2 - xy, \quad N = x^2$$

Differential equation is homogeneous as each term is of degree 2.

$$\text{I.F.} = \frac{1}{Mx + Ny}$$

$$= \frac{1}{xy^2 - x^2y + x^2y} = \frac{1}{xy^2}$$

Multiplying D.E. by I.F.,

$$\frac{1}{xy^2}(y^2 - xy)dx + \frac{x^2}{xy^2}dy = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right)dx + \frac{x}{y^2}dy = 0$$

$$M_1 = \frac{1}{x} - \frac{1}{y}, \qquad N_1 = \frac{x}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{y^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{1}{y^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 dx + \int_{\text{terms not containing } x} N_1 dy = c$$

$$\int\left(\frac{1}{x} - \frac{1}{y}\right)dx + \int 0\, dy = c$$

$$\log x - \frac{x}{y} = c$$

Example 3: Solve $3x^2y^4 dx + 4x^3y^3 dy = 0,\ y(1) = 1.$

Solution: $M = 3x^2y^4,\quad N = 4x^3y^3$

Differential equation is homogeneous as each term is of degree 6.

$$\text{I.F.} = \frac{1}{Mx + Ny}$$

$$= \frac{1}{3x^3y^4 + 4x^3y^4} = \frac{1}{7x^3y^4}$$

Multiplying D.E. by I.F.,

$$\frac{1}{7x^3y^4}(3x^2y^4)dx + \frac{1}{7x^3y^4}(4x^3y^3)dy = 0$$

$$\frac{3}{7x}dx + \frac{4}{7y}dy = 0$$

$$M_1 = \frac{3}{7x}, \qquad N_1 = \frac{4}{7y}$$

$$\frac{\partial M_1}{\partial y} = 0, \qquad \frac{\partial N_1}{\partial x} = 0$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M_1 dx + \int_{\text{terms not containing } x} N_1 dy = c$$

$$\int \frac{3}{7x}dx + \int \frac{4}{7y}dy = \log c$$

$$\frac{3}{7}\log x + \frac{4}{7}\log y = \log c$$

$$\log x^{\frac{3}{7}} + \log y^{\frac{4}{7}} = \log c$$

$$\log\left(x^{\frac{3}{7}}y^{\frac{4}{7}}\right) = \log c$$

$$x^{\frac{3}{7}}y^{\frac{4}{7}} = c \qquad \ldots (1)$$

Given $y(1) = 1$

Substituting $x = 1, y = 1$ in Eq. (1),

$$(1)^{\frac{3}{7}} \cdot (1)^{\frac{4}{7}} = c, \qquad 1 = c$$

Hence, solution is

$$x^{\frac{3}{7}} y^{\frac{4}{7}} = 1$$

Case V: If the differential equation is of the type

$$x^{m_1} y^{n_1} (a_1 y \, dx + b_1 x \, dy) + x^{m_2} y^{n_2} (a_2 y \, dx + b_2 x \, dy) = 0,$$

then I.F. $= x^h y^k$

where $$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

and $$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

Solving these two equations, we get the values of h and k.

Example 1: Solve $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0$.

Solution: $xy^0(4y\,dx + 2x\,dy) + x^0 y^3 (3y\,dx + 5x\,dy) = 0$

$$m_1 = 1, n_1 = 0, a_1 = 4, b_1 = 2, m_2 = 0, n_2 = 3, a_2 = 3, b_2 = 5$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{1 + h + 1}{4} = \frac{0 + k + 1}{2}$$

$$2h + 4 = 4k + 4$$

$$h = 2k \qquad \text{... (1)}$$

and $$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

$$\frac{0 + h + 1}{3} = \frac{3 + k + 1}{5}$$

$$5h + 5 = 3k + 12$$

$$5h - 3k = 7 \qquad \text{... (2)}$$

Solving Eqs. (1) and (2),

$$h = 2, k = 1$$

$$\text{I.F.} = x^2 y$$

Multiplying D.E. by I.F.,

$$x^3 y(4y\,dx + 2x\,dy) + x^2 y^4 (3y\,dx + 5x\,dy) = 0$$

$$(4x^3 y^2 + 3x^2 y^5)\,dx + (2x^4 y + 5x^3 y^4)\,dy = 0$$

$$M = 4x^3y^2 + 3x^2y^5, \qquad N = 2x^4y + 5x^3y^4$$

$$\frac{\partial M}{\partial y} = 8x^3y + 15x^2y^4, \qquad \frac{\partial N}{\partial x} = 8x^3y + 15x^2y^4$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y\text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int (4x^3y^2 + 3x^2y^5)\,dx + \int 0\,dy = c$$

$$x^4y^2 + x^3y^5 = c$$

Example 2: Solve $(x^7y^2 + 3y)\,dx + (3x^8y - x)\,dy = 0$.

Solution:

$$M = x^7y^2 + 3y, \qquad N = 3x^8y - x$$

$$\frac{\partial M}{\partial y} = 2x^7y + 3, \qquad \frac{\partial N}{\partial x} = 24x^7y - 1$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, equation is not exact.

Rewriting the equation,

$$x^7y^2\,dx + 3x^8y\,dy + 3y\,dx - x\,dy = 0$$

$$x^7y(y\,dx + 3x\,dy) + (3y\,dx - x\,dy) = 0$$

$$m_1 = 7,\ n_1 = 1,\ a_1 = 1,\ b_1 = 3,\ m_2 = 0,\ n_2 = 0,\ a_2 = 3,\ b_2 = -1$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{7 + h + 1}{1} = \frac{1 + k + 1}{3}$$

$$3h + 24 = k + 2$$

$$3h - k = -22 \qquad \text{... (1)}$$

and

$$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

$$\frac{0 + h + 1}{3} = \frac{0 + k + 1}{-1}$$

$$-h - 1 = 3k + 3$$

$$h + 3k = -4 \qquad \text{... (2)}$$

Solving Eqs. (1) and (2),

$$h = -7,\ k = 1$$

$$\text{I.F.} = x^{-7} y$$

Multiplying D.E. by I.F.,

$$x^{-7} y(x^7 y^2 + 3y)\mathrm{d}x + x^{-7} y(3x^8 y - x)\mathrm{d}y = 0$$

$$(y^3 + 3x^{-7} y^2)\mathrm{d}x + (3xy^2 - x^{-6} y)\mathrm{d}y = 0$$

$$\mathrm{M}_1 = y^3 + 3x^{-7} y^2, \qquad \mathrm{N}_1 = 3xy^2 - x^{-6} y$$

$$\frac{\partial \mathrm{M}_1}{\partial y} = 3y^2 + 6x^{-7} y, \qquad \frac{\partial \mathrm{N}_1}{\partial x} = 3y^2 + 6x^{-7} y$$

Since $\dfrac{\partial \mathrm{M}_1}{\partial y} = \dfrac{\partial \mathrm{N}_1}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}} \mathrm{M}_1 \mathrm{d}x + \int\limits_{\text{terms not containing } x} \mathrm{N}_1 \mathrm{d}y = c$$

$$\int (y^3 + 3x^{-7} y^2)\,\mathrm{d}x + \int 0\,\mathrm{d}y = c$$

$$xy^3 + \frac{3x^{-6} y^2}{-6} = c$$

$$xy^3 - \frac{x^{-6} y^2}{2} = c$$

Case VI: *Integrating Factors by Inspection* Sometimes integrating factor can be identified by regrouping the terms of the differential equation. The following table helps in identifying the I.F. after regrouping the terms.

Sr. No.	Group of Terms	Integrating Factor	Exact Differential Equation
1.	$\mathrm{d}x \pm \mathrm{d}y$	$\dfrac{1}{x \pm y}$	$\dfrac{\mathrm{d}x \pm \mathrm{d}y}{x \pm y} = \mathrm{d}[\log(x \pm y)]$
2.	$y\,\mathrm{d}x + x\,\mathrm{d}y$	$\dfrac{1}{2xy}$	$y\,\mathrm{d}x + x\,\mathrm{d}y = \mathrm{d}(xy)$ $2x^2 y\,\mathrm{d}y + 2xy^2 \mathrm{d}x = \mathrm{d}(x^2 y^2)$
		$\dfrac{1}{xy}$	$\dfrac{y\,\mathrm{d}x + x\,\mathrm{d}y}{xy} = \mathrm{d}[\log(xy)]$
		$\dfrac{1}{(xy)^n}$	$\dfrac{y\,\mathrm{d}x + x\,\mathrm{d}y}{(xy)^n} = \mathrm{d}\left[\dfrac{(xy)^{1-n}}{1-n}\right],\ n \neq 1$

Contd.

3.	$y\,dx - x\,dy$	$\dfrac{1}{y^2}$	$\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
		$\dfrac{1}{x^2+y^2}$	$\dfrac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$
		$\dfrac{1}{x^2}$	$\dfrac{y\,dx - x\,dy}{x^2} = d\left(-\dfrac{y}{x}\right)$
		$\dfrac{1}{xy}$	$\dfrac{y\,dx - x\,dy}{xy} = d\left[\log\left(\dfrac{x}{y}\right)\right]$
4.	$x\,dx \pm y\,dy$	2	$2x\,dx \pm 2y\,dy = d(x^2 \pm y^2)$
		$\dfrac{1}{(x^2 \pm y^2)}$	$\dfrac{2x\,dx \pm 2y\,dy}{x^2 \pm y^2} = d[\log(x^2 \pm y^2)]$
		$\dfrac{1}{(x^2 \pm y^2)^n}$	$\dfrac{2x\,dx \pm 2y\,dy}{(x^2 \pm y^2)^n} = d\left[\dfrac{(x^2 \pm y^2)^{1-n}}{2(1-n)}\right]$
5.	$2y\,dx + x\,dy$	x	$2xy\,dx + x^2\,dy = d(x^2y)$
6.	$y\,dx + 2x\,dy$	y	$y^2\,dx + 2xy\,dy = d(xy^2)$
7.	$2y\,dx - x\,dy$	$\dfrac{x}{y^2}$	$\dfrac{2xy\,dx - x^2dy}{y^2} = d\left(\dfrac{x^2}{y}\right)$
8.	$2x\,dy - y\,dx$	$\dfrac{y}{x^2}$	$\dfrac{2xy\,dy - y^2dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$

Example 1: Solve $x\,dy - y\,dx + 2x^3dx = 0$.

Solution: Dividing the equation by x^2,

$$\frac{x\,dy - y\,dx}{x^2} + 2x\,dx = 0$$

$$d\left(\frac{y}{x}\right) + d(x^2) = 0$$

Integrating both the sides,

$$\frac{y}{x} + x^2 = c$$

Example 2: Solve $x\,dx + y\,dy + 2(x^2 + y^2)\,dx = 0$.

Solution: Dividing the equation by $x^2 + y^2$,

$$\frac{x\,dx + y\,dy}{x^2 + y^2} + 2\,dx = 0$$

$$\frac{1}{2}d[\log(x^2 + y^2)] + 2\,dx = 0$$

Integrating both the sides,

$$\frac{1}{2}\log(x^2 + y^2) + 2x = c$$

Example 3: Solve $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$.

Solution: $y\,dx + xy^2\,dx + x\,dy - x^2y\,dy = 0$

Regrouping the terms,

$$(y\,dx + x\,dy) + (xy^2dx - x^2y\,dy) = 0$$

Dividing the equation by x^2y^2,

$$\frac{y\,dx + x\,dy}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

$$d\left(-\frac{1}{xy}\right) + \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating both the sides,

$$-\frac{1}{xy} + \log x - \log y = c$$

$$-\frac{1}{xy} + \log\frac{x}{y} = c$$

Example 4: Solve $x\,dy - y\,dx = 3x^2(x^2 + y^2)\,dx$.

Solution: Dividing the equation by $(x^2 + y^2)$,

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = 3x^2dx$$

$$d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = d(x^3)$$

Integrating both the sides,

$$\tan^{-1}\frac{y}{x} = x^3 + c$$

Example 5: Solve $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0$.

Solution: $xy\,dx - 2y^2dx - x^2dy + 3xy\,dy = 0$

Regrouping the terms,

$$x(y\,dx - x\,dy) - 2y^2\,dx + 3xy\,dy = 0$$

Dividing the equation by xy^2,

$$\frac{y\,dx - x\,dy}{y^2} - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

$$d\left(\frac{x}{y}\right) - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

Integrating both the sides,

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} - \log x^2 + \log y^3 = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

Example 6: Solve $y(2xy + e^x)\,dx = e^x\,dy$.

Solution: $2xy^2 dx + e^x y\,dx - e^x dy = 0$

Dividing the equation by y^2,

$$2x\,dx + \frac{ye^x\,dx - e^x\,dy}{y^2} = 0$$

$$2x\,dx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating both the sides,

$$x^2 + \frac{e^x}{y} = c$$

Example 7: Solve $y\,dx + x(x^2y - 1)\,dy = 0$.

Solution: $y\,dx + x^3 y\,dy - x\,dy = 0$

Regrouping the terms,

$$y\,dx - x\,dy + x^3 y\,dy = 0$$

Dividing the equation by $\frac{x^3}{y}$,

$$\frac{y^2 \mathrm{d}x - xy\,\mathrm{d}y}{x^3} + y^2 \mathrm{d}y = 0$$

$$\frac{1}{2}\left(\frac{y^2 \cdot 2x\,\mathrm{d}x - x^2 \cdot 2y\,\mathrm{d}y}{x^4}\right) + y^2 \mathrm{d}y = 0$$

$$\frac{1}{2}\mathrm{d}\left(-\frac{y^2}{x^2}\right) + y^2 \mathrm{d}y = 0$$

Integrating both the sides,

$$-\frac{1}{2}\frac{y^2}{x^2} + \frac{y^3}{3} = c$$

$$-\frac{y^2}{2x^2} + \frac{y^3}{3} = c$$

Example 8: Solve $y(x^3 e^{xy} - y)\mathrm{d}x + x(y + x^3 e^{xy})\mathrm{d}y = 0$.

Solution: $x^3 ye^{xy}\mathrm{d}x - y^2\mathrm{d}x + xy\,\mathrm{d}y + x^4 e^{xy}\mathrm{d}y = 0$

Regrouping the terms,

$$x^3 ye^{xy}\mathrm{d}x + x^4 e^{xy}\mathrm{d}y - y^2\mathrm{d}x + xy\,\mathrm{d}y = 0$$

Dividing the equation by x^3,

$$ye^{xy}\mathrm{d}x + xe^{xy}\mathrm{d}y - \frac{1}{2}\left(\frac{y^2 \cdot 2x\,\mathrm{d}x - x^2 \cdot 2y\,\mathrm{d}y}{x^4}\right) = 0$$

$$\mathrm{d}(e^{xy}) + \frac{1}{2}\mathrm{d}\left(\frac{y^2}{x^2}\right) = 0$$

Integrating both the sides,

$$e^{xy} + \frac{1}{2}\frac{y^2}{x^2} = c$$

Example 9: If x^n is an integrating factor of $(y - 2x^3)\mathrm{d}x - x(1 - xy)\mathrm{d}y = 0$, then find n and solve the equation.

Solution: If x^n is an I.F., then after multiplication with x^n, the equation becomes exact.

$$(x^n y - 2x^{n+3})\mathrm{d}x - x^{n+1}(1 - xy)\mathrm{d}y = 0 \text{ is an exact D.E.}$$

where $\qquad \mathrm{M} = x^n y - 2x^{n+3},\ \mathrm{N} = -x^{n+1} + x^{n+2} y$

and $\qquad \dfrac{\partial \mathrm{M}}{\partial y} = \dfrac{\partial \mathrm{N}}{\partial x}$

$$x^n = -(n+1)x^n + (n+2)x^{n+1}y$$
$$(n+2)x^n(1+xy) = 0$$
$$n+2 = 0$$
$$n = -2$$

Putting $n = -2$ in the equation,

$$(x^{-2}y - 2x)\mathrm{d}x - x^{-1}(1-xy)\mathrm{d}y = 0$$
$$\left(\frac{y}{x^2} - 2x\right)\mathrm{d}x - \left(\frac{1}{x} - y\right)\mathrm{d}y = 0$$
$$\mathrm{M} = \frac{y}{x^2} - 2x, \qquad \mathrm{N} = -\frac{1}{x} + y$$
$$\frac{\partial \mathrm{M}}{\partial y} = \frac{1}{x^2}, \qquad \frac{\partial \mathrm{N}}{\partial x} = \frac{1}{x^2}$$

Since $\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$, equation is exact.

Hence, solution is

$$\int_{y \text{ constant}} \mathrm{M}\,\mathrm{d}x + \int_{\text{terms not containing } x} \mathrm{N}\,\mathrm{d}y = c$$
$$\int\left(\frac{y}{x^2} - 2x\right)\mathrm{d}x + \int y\,\mathrm{d}y = c$$
$$-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$$

Example 10: If $(x + y)^n$ is an integrating factor of $(4x^2 + 2xy + 6y)$ dx + $(2x^2 + 9y + 3x) = 0$, then find n and solve the equation.

Solution: Since $(x + y)^n$ is an I.F., after multiplication with $(x + y)^n$, the equation becomes exact.

i.e., $$(x+y)^n(4x^2 + 2xy + 6y)\mathrm{d}x + (x+y)^n(2x^2 + 9y + 3x)\mathrm{d}y = 0 \qquad \text{... (1)}$$

is an exact D.E. where

$$\mathrm{M} = (x+y)^n(4x^2 + 2xy + 6y), \qquad \mathrm{N} = (x+y)^n(2x^2 + 9y + 3x)$$

and $$\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$$

$$(x+y)^{n-1}(4nx^2 + 2nxy + 6ny + 2x^2 + 2xy + 6x + 6y) = (x+y)^{n-1}(2nx^2 + 9ny + 3nx + 4x^2 + 4xy + 3x + 3y)$$

$$2nx^2 + 2nxy - 3ny - 3nx = 2x^2 + 2xy - 3y - 3x$$
$$n(2x^2 + 2xy - 3y - 3x) = 2x^2 + 2xy - 3y - 3x$$
$$n = 1$$

Putting $n = 1$ in Eq. (1),

$$(x+y)(4x^2 + 2xy + 6y)\mathrm{d}x + (x+y)(2x^2 + 9y + 3x)\mathrm{d}y = 0$$
$$(4x^3 + 6x^2y + 2xy^2 + 6xy + 6y^2)\mathrm{d}x + (2x^3 + 12xy + 3x^2 + 2x^2y + 9y^2)\mathrm{d}y = 0$$
$$\mathrm{M} = 4x^3 + 6x^2y + 2xy^2 + 6xy + 6y^2, \qquad \mathrm{N} = 2x^3 + 12xy + 3x^2 + 2x^2y + 9y^2$$
$$\frac{\partial \mathrm{M}}{\partial y} = 6x^2 + 4xy + 6x + 12y \qquad \frac{\partial \mathrm{N}}{\partial x} = 6x^2 + 12y + 6x + 4xy$$

Since $\frac{\partial \mathrm{M}}{\partial y} = \frac{\partial \mathrm{N}}{\partial x}$, equation is exact.

Hence, solution is

$$\int\limits_{y\text{ constant}} \mathrm{M}\,\mathrm{d}x + \int\limits_{\text{terms not containing } x} \mathrm{N}\,\mathrm{d}y = c$$
$$\int (4x^3 + 6x^2y + 2xy^2 + 6xy + 6y^2)\mathrm{d}x + \int 9y^2\,\mathrm{d}y = c$$
$$x^4 + 2x^3y + x^2y^2 + 3x^2y + 2y^3 + 3y^3 = c$$

Exercise 10.5

Solve the following differential equations:

1. $(x^2 + y^2 + x)\,\mathrm{d}x + xy\,\mathrm{d}y = 0.$

$\left[\textbf{Ans.: } 3x^4 + 4x^3 + 6x^2y^2 = c\right]$

2. $(y - 2x^3)\,\mathrm{d}x - (x - x^2y)\,\mathrm{d}y = 0.$

$\left[\textbf{Ans.: } xy^2 - 2y - 2x^3 = cx\right]$

3. $(2xy^4e^y + 2xy^3 + y)\,\mathrm{d}x + (x^2y^4e^y - x^2y^2 - 3x)\mathrm{d}y = 0.$

$\left[\textbf{Ans.: } x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c\right]$

4. $\left(2x\sinh\frac{y}{x} + 3y\cosh\frac{y}{x}\right)\mathrm{d}x - \left(3x\cosh\frac{y}{x}\right)\mathrm{d}y = 0.$

$\left[\textbf{Ans.: } -3\sinh\frac{y}{x} = cx^{\frac{2}{3}}\right]$

5. $(e^x x^4 - 2mxy^2)\,\mathrm{d}x + 2mx^2y\,\mathrm{d}y = 0.$

$\left[\textbf{Ans.: } x^2e^x + my^2 = cx^2\right]$

6. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)\mathrm{d}x + \frac{1}{4}(x + xy^2)\,\mathrm{d}y = 0.$

$\left[\textbf{Ans.: } x^6 + 3x^4y + x^4y^3 = c\right]$

7. $(x\sec^2 y - x^2\cos y)\,\mathrm{d}y = (\tan y - 3x^4)dx.$

$\left[\textbf{Ans.: } \frac{\tan y}{x} + x^3 - \sin y = c\right]$

8. $(4xy+3y^2-x)\,\mathrm{d}x+x(x+2y)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } x^4y+x^3y^2-\frac{x^4}{4}=c\right]$$

9. $(x^2+y^2+2x)\,\mathrm{d}x+2y\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } e^x(x^2+y^2)=c\right]$$

10. $(3x^2y^4+2xy)\,\mathrm{d}x+(2x^3y^3-x^2)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } x^3y^3+x^2=cy\right]$$

11. $y(xy+e^x)\,\mathrm{d}x-e^x\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } \frac{x^2}{2}+\frac{e^x}{y}=c\right]$$

12. $(3x^2y^3e^y+y^3+y^2)\mathrm{d}x + (x^3y^3e^y-xy)dy=0.$

$$\left[\textbf{Ans.: } x^3e^y+x+\frac{x}{y}=c\right]$$

13. $y(x^2y+e^x)\,\mathrm{d}x-e^x\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } \frac{x^3}{3}+\frac{e^x}{y}=c\right]$$

14. $(xy^3+y)\mathrm{d}x+2(x^2y^2+x+y^4)\,dy=0.$

$$\left[\textbf{Ans.: } 3x^2y^4+6xy^2+2y^6=c\right]$$

15. $(2x^2y+e^x)y\,\mathrm{d}x-(e^x+y^3)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } 4x^3y-3y^3+6e^x=cy\right]$$

16. $y\log y\,\mathrm{d}x+(x-\log y)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } 2x\log y=c+(\log y)^2\right]$$

17. $(x-y^2)\,\mathrm{d}x+2xy\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } \frac{y^2}{x}+\log x=c\right]$$

18. $2xy\,\mathrm{d}x+(y^2-x^2)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } x^2+y^2=cy\right]$$

19. $(1+xy)y\,\mathrm{d}x+(1-xy)x\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } \log\left(\frac{x}{y}\right)=c+\frac{1}{xy}\right]$$

20. $(1+xy+x^2y^2+x^3y^3)y\,\mathrm{d}x + (1-xy-x^2y^2+x^3y^3)x\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } xy-\frac{1}{xy}-\log y^2=c\right]$$

21. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{x^2y^3+2y}{2x-2x^3y^2}.$

$$\left[\textbf{Ans.: } \frac{1}{3}\log\frac{x}{y^2}-\frac{1}{3x^2y^2}=c\right]$$

22. $y(\sin xy+xy\cos xy)\,\mathrm{d}x + x(xy\cos xy-\sin xy)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } \frac{x\sin(xy)}{y}=c\right]$$

23. $y(x+y)\,\mathrm{d}x-x(y-x)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } \log\sqrt{xy}-\frac{y}{2x}=c\right]$$

24. $x^2y\,\mathrm{d}x-(x^3+y^3)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } y=ce^{\frac{x^3}{3y^3}}\right]$$

25. $3y\,\mathrm{d}x+2x\,\mathrm{d}y=0,\; y(1)=1.$

$$\left[\textbf{Ans.: } yx^{\frac{3}{2}}=1\right]$$

26. $(y^2+2x^2y)\,\mathrm{d}x+(2x^3-xy)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } -\frac{2}{3}x^{-\frac{3}{2}}y^{\frac{3}{2}}+4x^{\frac{1}{2}}y^{\frac{1}{2}}=c\right]$$

27. $(2x^2y^2+y)\,\mathrm{d}x-(x^3y-3x)\,\mathrm{d}y=0.$

$$\left[\textbf{Ans.: } \frac{7}{5}x^{\frac{10}{7}}y^{-\frac{5}{7}}-\frac{7}{4}x^{-\frac{4}{7}}y^{-\frac{12}{7}}=c\right]$$

28. If y^n is an integrating factor of
$(2xy^4e^y + 2xy^3 + y)\,dx$
$+ (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$, find n
and solve the equation.

$\left[\begin{array}{l}\textbf{Ans.:}\\ n = -4,\ x^2y^3e^y + x^2y^2 + x = cy^3\end{array}\right]$

10.3.6 Linear Differential Equations

If each term in a differential equation including derivative is linear in terms of dependent variable, then the equation is called linear.
A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (1)}$$

where P and Q are functions of x, is called linear differential equation and is linear in y.
To solve Eq. (1), obtain the integrating factor (I.F.) as

$$\text{I.F.} = e^{\int P dx}$$

Multiplying Eq. (1) by I.F.,

$$e^{\int P dx}\frac{dy}{dx} + Pe^{\int P dx}y = Qe^{\int P dx}$$

$$\frac{d}{dx}\left[e^{\int P dx}y\right] = Qe^{\int P dx}$$

Integrating w.r.t x,

$$e^{\int P dx}y = \int Qe^{\int P dx}dx + c$$

or

$$(\text{I.F.})\,y = \int (\text{I.F.})\,Q + c \qquad \text{... (2)}$$

Eq. (2) is the solution of differential Eq. (1).

Example 1: Solve $\dfrac{dy}{dx} + \dfrac{3y}{x} = \dfrac{\sin x}{x^3}$.

Solution: The equation is linear in y.

$$P = \frac{3}{x}, \quad Q = \frac{\sin x}{x^3}$$

$$\text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$$

Hence, solution is

$$x^3 y = \int x^3 \frac{\sin x}{x^3} \mathrm{d}x + c$$

$$= \int \sin x \, \mathrm{d}x + c = -\cos x + c$$

$$y = -\frac{\cos x}{x^3} + \frac{c}{x^3}$$

Example 2: Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{4x}{1+x^2} y = \dfrac{1}{(x^2+1)^3}$.

Solution: The equation is linear in y.

$$\mathrm{P} = \frac{4x}{1+x^2}, \quad \mathrm{Q} = \frac{1}{(x^2+1)^3}$$

$$\text{I.F.} = e^{\int \frac{4x}{1+x^2} \mathrm{d}x} = e^{2\log(1+x^2)} = e^{\log(1+x^2)^2} = (1+x^2)^2$$

Hence, solution is

$$(1+x^2)^2 y = \int (1+x^2)^2 \cdot \frac{1}{(x^2+1)^3} \mathrm{d}x + c$$

$$= \int \frac{1}{x^2+1} \mathrm{d}x + c = \tan^{-1} x + c$$

Example 3: Solve $(1-x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x\sqrt{1-x^2}$.

Solution: Rewriting the equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x}{1-x^2}\right) y = \frac{x}{\sqrt{1-x^2}}$$

The equation is linear in y.

$$\mathrm{P} = \frac{2x}{1-x^2}, \quad \mathrm{Q} = \frac{x}{\sqrt{1-x^2}}$$

$$\text{I.F.} = e^{\int \frac{2x}{1-x^2} \mathrm{d}x} = e^{-\log(1-x^2)} = e^{\log(1-x^2)^{-1}} = (1-x^2)^{-1} = \frac{1}{1-x^2}$$

Hence, solution is

$$\left(\frac{1}{1-x^2}\right) y = \int \left(\frac{1}{1-x^2}\right)\left(\frac{x}{\sqrt{1-x^2}}\right) \mathrm{d}x + c$$

$$= \int \frac{x}{(1-x^2)^{\frac{3}{2}}} \mathrm{d}x + c$$

$$= -\frac{1}{2}\int (1-x^2)^{-\frac{3}{2}}(-2x)\,dx + c$$

$$= -\frac{1}{2}\cdot\frac{(1-x^2)^{-\frac{1}{2}}}{-\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\frac{y}{1-x^2} = (1-x^2)^{-\frac{1}{2}} + c$$

$$y = \sqrt{1-x^2} + c(1-x^2)$$

Example 4: Solve $x\log x\dfrac{dy}{dx} + y = 2\log x$.

Solution: Rewriting the equation,

$$\frac{dy}{dx} + \left(\frac{1}{x\log x}\right)y = \frac{2}{x}$$

The equation is linear in y.

$$P = \frac{1}{x\log x}, \quad Q = \frac{2}{x}$$

$$\text{I.F.} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$$

Hence, solution is

$$(\log x)y = \int (\log x)\cdot\frac{2}{x}\,dx + c = 2\frac{(\log x)^2}{2} + c \qquad \left[\because \int f(x)\cdot f'(x)\,dx = \frac{[f(x)]^2}{2}\right]$$

$$= (\log x)^2 + c$$

$$y\log x = (\log x)^2 + c$$

Example 5: Solve $(1 + x + xy^2)dy + (y + y^3)dx = 0$.

Solution: Rewriting the equation,

$$(1+x+xy^2) + (y+y^3)\frac{dx}{dy} = 0$$

$$\frac{dx}{dy} + \frac{(1+y^2)x}{y+y^3} + \frac{1}{y+y^3} = 0$$

$$\frac{dx}{dy} + \left(\frac{1}{y}\right)x = -\frac{1}{y(1+y^2)} \qquad \text{... (1)}$$

The equation is linear in x.

$$P = \frac{1}{y}, \quad Q = -\frac{1}{y(1+y^2)}$$

$$\text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Hence, solution is

$$yx = \int y\left[-\frac{1}{y(1+y^2)}\right]dy + c = -\int \frac{1}{1+y^2}\,dy + c$$

$$= -\tan^{-1} y + c$$

$$xy = c - \tan^{-1} y$$

Example 6: Solve $y \log y\, dx + (x - \log y)\, dy = 0$.

Solution: Rewriting the equation,

$$y \log y \frac{dx}{dy} + x - \log y = 0$$

$$\frac{dx}{dy} + \left(\frac{1}{y \log y}\right)x = \frac{1}{y}$$

The equation is linear in x.

$$P = \frac{1}{y \log y}, \quad Q = \frac{1}{y}$$

$$\text{I.F.} = e^{\int \frac{1}{y \log y} dy} = e^{\log(\log y)} \qquad \left[\because \int \frac{f'(y)}{f(y)}\,dy = \log f(y) + c\right]$$

$$= \log y$$

Hence, solution is

$$(\log y)x = \int (\log y)\frac{1}{y}\,dy + c$$

$$x \log y = \frac{(\log y)^2}{2} + c$$

Example 7: Solve $(1 + \sin y)\,dx = (2y \cos y - x \sec y - x \tan y)\,dy$.

Solution: Rewriting the equation,

$$(1 + \sin y)\frac{dx}{dy} = 2y \cos y - (\sec y + \tan y)x$$

$$(1 + \sin y)\frac{dx}{dy} + \left(\frac{1 + \sin y}{\cos y}\right)x = 2y \cos y$$

$$\frac{dx}{dy} + \left(\frac{1}{\cos y}\right)x = \frac{2y \cos y}{1 + \sin y}$$

The equation is linear in x.

$$P = \frac{1}{\cos y}, \quad Q = \frac{2y\cos y}{1+\sin y}$$

$$\text{I.F.} = e^{\int \frac{1}{\cos y} dy} = e^{\int \sec y \, dy} = e^{\log(\sec y + \tan y)} = \sec y + \tan y$$

Hence, solution is

$$(\sec y + \tan y)x = \int (\sec y + \tan y)\left(\frac{2y\cos y}{1+\sin y}\right) dy + c$$

$$= 2\int \left(\frac{1+\sin y}{\cos y}\right)\left(\frac{y\cos y}{1+\sin y}\right) dy + c = 2\int y\, dy + c$$

$$(\sec y + \tan y)x = y^2 + c$$

Example 8: Solve $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$.

Solution: Rewriting the equation,

$$(1+y^2)\frac{dx}{dy} = \tan^{-1} y - x$$

$$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1} y}{1+y^2}$$

The equation is linear in x.

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1} y}{1+y^2}$$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Hence, solution is

$$(e^{\tan^{-1} y})x = \int e^{\tan^{-1} y}\left(\frac{\tan^{-1} y}{1+y^2}\right) dy + c$$

Let $\tan^{-1} y = t$

$$\frac{1}{1+y^2} dy = dt$$

$$(e^{\tan^{-1} y})x = \int e^t t\, dt + c = te^t - e^t + c$$

$$= e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$$

$$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$$

Example 9: Solve $\mathrm{d}r + (2r\cot\theta + \sin 2\theta)\,\mathrm{d}\theta = 0$.

Solution: Rewriting the equation,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + (2\cot\theta)r = -\sin 2\theta$$

The equation is linear in r.

$$\mathrm{P} = 2\cot\theta, \quad \mathrm{Q} = -\sin 2\theta$$

$$\text{I.F.} = e^{\int 2\cot\theta\,\mathrm{d}\theta} = e^{2\log\sin\theta} = e^{\log\sin^2\theta} = \sin^2\theta$$

Hence, solution is

$$\sin^2\theta \cdot r = \int \sin^2\theta(-\sin 2\theta)\,\mathrm{d}\theta + c$$

$$= -2\int \sin^3\theta\cos\theta\,\mathrm{d}\theta + c = -2\frac{\sin^4\theta}{4} + c \quad \left[\because \int [f(\theta)]^n f'(\theta)\,\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$r\sin^2\theta = -\frac{\sin^4\theta}{2} + c$$

Example 10: Solve $(4r^2s - 6)\,\mathrm{d}r + r^3\,\mathrm{d}s = 0$.

Solution: $4r^2s - 6 + r^3\dfrac{\mathrm{d}s}{\mathrm{d}r} = 0$

$$\frac{\mathrm{d}s}{\mathrm{d}r} + \frac{4s}{r} = \frac{6}{r^3}$$

The equation is linear in s.

$$\mathrm{P} = \frac{4}{r}, \quad \mathrm{Q} = \frac{6}{r^3}$$

$$\text{I.F.} = e^{\int \frac{4}{r}\mathrm{d}r} = e^{4\log r} = e^{\log r^4} = r^4$$

Hence, solution is

$$r^4 \cdot s = \int r^4 \cdot \frac{6}{r^3}\,\mathrm{d}r + c = 6\int r\,\mathrm{d}r + c$$

$$= 6\frac{r^2}{2} + c = 3r^2 + c$$

$$s = \frac{3}{r^2} + \frac{c}{r^4}.$$

Example 11: Solve $\cosh x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cosh^2 x\sinh x - y\sinh x$.

Solution: $\dfrac{\mathrm{d}y}{\mathrm{d}x} + (\tanh x)y = 2\cosh x\sinh x$

The equation is linear in y.

$$\mathrm{P} = \tanh x, \quad \mathrm{Q} = 2\cosh x \sinh x$$

$$\text{I.F.} = e^{\int \tanh x\, dx} = e^{\int \frac{\sinh x}{\cosh x} dx} = e^{\log \cosh x} = \cosh x$$

Hence, solution is

$$(\cosh x) y = \int \cosh x (2\cosh x \sinh x)\, dx + c$$

$$= 2\int \cosh^2 x \cdot \sinh x\, dx + c = 2\frac{\cosh^3 x}{3} + c$$

$$y \cosh x = \frac{2}{3}\cosh^3 x + c$$

Example 12: Solve $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$.

Solution: $\frac{dy}{dx} - \frac{(x-2)}{x(x-1)} y = \frac{x^2(2x-1)}{(x-1)}$

The equation is linear in y.

$$\mathrm{P} = -\frac{x-2}{x(x-1)}, \qquad \mathrm{Q} = \frac{x^2(2x-1)}{x-1}$$

$$= -\left(\frac{2}{x} - \frac{1}{x-1}\right)$$

$$\text{I.F.} = e^{\int\left(-\frac{2}{x}+\frac{1}{x-1}\right)dx} = e^{-2\log x + \log(x-1)} = e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

Hence, solution is

$$\left(\frac{x-1}{x^2}\right)\cdot y = \int\left(\frac{x-1}{x^2}\right)\cdot x^2\left(\frac{2x-1}{x-1}\right) dx + c = x^2 - x + c$$

$$y = \frac{x^3(x-1)}{x-1} + \frac{cx^2}{x-1}$$

$$y = x^3 + \frac{cx^2}{x-1}$$

Example 13: Solve $(x^2-1)\sin x \frac{dy}{dx} + [2x\sin x + (x^2-1)\cos x]y = (x^2-1)\cos x$.

Solution: $\frac{dy}{dx} + \left(\frac{2x}{x^2-1} + \cot x\right) y = \cot x$

The equation is linear in y.

$$\mathrm{P} = \frac{2x}{x^2-1} + \cot x, \quad \mathrm{Q} = \cot x$$

10.4 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = 0 \qquad \ldots (1)$$

where a_0, a_1, a_2, …, a_n are constants, is known as homogeneous linear differential equation of order n with constant coefficients. This equation is known as linear since degree of dependent variable y and all its differential coefficients is one.

Equation (1) can also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = 0$$

$$f(D)y = 0$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$.

Here $D \equiv \frac{d}{dx}$ is known as differential operator.

The operator D obeys the laws of algebra.

10.4.1 General Solution of Homogeneous Linear Differential Equation

The homogeneous equation

$$f(D)y = 0 \qquad \ldots (2)$$

can be solved by replacing D by m in $f(D)$ and solving the auxiliary equation (A.E.)

$$f(m) = 0 \qquad \ldots (3)$$

The general solution of Eq. (2) depends upon the nature of the roots of auxiliary Eq. (3).

If m_1, m_2, m_3, …, m_n are n roots of the A.E., following cases arise:

Case I: Real and distinct roots: If roots m_1, m_2, m_3, …, m_n are real and distinct, then the solution of Eq. (1) is given as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

Case II: Real and repeated roots: If two roots m_1, m_2 are real and equal and remaining $(n - 2)$ roots m_3, m_4, …, m_n are all real and distinct, then the solution of Eq. (1) is given as

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Note: If, however, r roots m_1, m_2, m_3, …, m_r are equal and remaining $(n - r)$ roots m_{r+1}, m_{r+2}, …, m_n are all real and distinct, then the solution of Eq. (1) is given as

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_r x^{r-1})e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \ldots + c_n e^{m_n x}$$

Case III: Imaginary roots: If two roots m_1, m_2 are imaginary say, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ (conjugate pair) and remaining $(n - 2)$ roots $m_3, m_4, \ldots, m_n$ are real and distinct, then the solution of Eq. (1) is given as

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Here, α is the real part and β is the imaginary part of the conjugate pair of complex roots.

Note: If, however, two pair of imaginary roots m_1, m_2 and m_3, m_4 are equal, say, $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ and remaining $(n - 4)$ roots $m_5, m_6, \ldots, m_n$ are real and distinct, then the solution of Eq. (1) is given as

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$$

Remark:

(i) In all the above cases, $c_1, c_2, \ldots, c_n$ are arbitrary constants.

(ii) In the general solution of a homogeneous equation, the number of arbitrary constants is always equal to the order of that homogeneous equation.

Example 1: Solve $2D^2y + Dy - 6y = 0$.

Solution: The equation can be written as

$$(2D^2 + D - 6)y = 0$$

Auxiliary equation is

$$2m^2 + m - 6 = 0$$
$$(2m - 3)(m + 2) = 0$$
$$m = -2, \frac{3}{2}$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}$$

Example 2: Solve $(D^3 + D^2 - 2D)y = 0$.

Solution: Auxiliary equation is

$$m^3 + m^2 - 2m = 0$$
$$m(m^2 + m - 2) = 0$$
$$m(m - 1)(m + 2) = 0$$
$$m = 0, 1, -2$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^{0x} + c_2 e^{x} + c_3 e^{-2x}$$
$$y = c_1 + c_2 e^{x} + c_3 e^{-2x}$$

Example 3: Solve $2D^2y - 2Dy - y = 0$.

Solution: The equation can be written as

$$(2D^2 - 2D - 1)y = 0$$

Auxiliary equation is

$$2m^2 - 2m - 1 = 0$$

$$m = \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm 2\sqrt{3}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

$$m = \frac{1+\sqrt{3}}{2}, \ \frac{1-\sqrt{3}}{2}$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^{\left(\frac{1+\sqrt{3}}{2}\right)x} + c_2 e^{\left(\frac{1-\sqrt{3}}{2}\right)x}$$

Example 4: Solve $D^2y + 6Dy + 9y = 0$.

Solution: The equation can be written as

$$(D^2 + 6D + 9)y = 0$$

Auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$m = -3, \ -3$$

The roots are repeated twice.
Hence, solution is

$$y = (c_1 + c_2 x)e^{-3x}$$

Example 5: Solve $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$.

Solution: Auxiliary equation is

$$m^4 - 6m^3 + 12m^2 - 8m = 0$$

$$m(m^3 - 6m^2 + 12m - 8) = 0$$

$$m(m-2)(m^2 - 4m + 4) = 0$$

$$m(m-2)(m-2)^2 = 0$$

$$m = 0, \ 2, \ 2, \ 2$$

The root $m = 2$ is repeated three times.
Hence, solution is

$$y = c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2)e^{2x}$$

$$y = c_1 + (c_2 + c_3 x + c_4 x^2)e^{2x}$$

Example 6: Solve $(D^4 - 6D^3 + 13D^2 - 12D + 4)y = 0$.

Solution: Auxiliary equation is

$$m^4 - 6m^3 + 13m^2 - 12m + 4 = 0$$
$$(m-1)^2(m-2)^2 = 0$$
$$m = 1,\ 1,\ 2,\ 2$$

The roots $m = 1$ and $m = 2$ are repeated twice.
Hence, solution is

$$y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{2x}$$

Example 7: Solve $(D^4 + 4D^2)y = 0$.

Solution: Auxiliary equation is

$$m^4 + 4m^2 = 0$$
$$m^2(m^2 + 4) = 0$$
$$m = 0,\ 0 \text{ and } m^2 = -4,\ m = \pm 2i$$

The root $m = 0$ is real and repeated twice and two roots are imaginary with $\alpha = 0,\ \beta = 2$.
Hence, solution is

$$y = (c_1 + c_2 x)e^{0x} + c_1 \cos 2x + c_2 \sin 2x$$
$$= c_1 + c_2 x + c_1 \cos 2x + c_2 \sin 2x$$

Example 8: Solve $(D^4 + 4)y = 0$.

Solution: Auxiliary equation is

$$m^4 + 4 = 0$$
$$m^4 + 4 + 4m^2 - 4m^2 = 0$$
$$(m^2 + 2)^2 - (2m)^2 = 0$$
$$(m^2 + 2 + 2m)(m^2 + 2 - 2m) = 0$$
$$(m^2 + 2m + 2)(m^2 - 2m + 2) = 0$$
$$m = -1 \pm i \text{ and } m = 1 \pm i$$

The roots are imaginary with $\alpha_1 = -1,\ \beta_1 = 1$ and $\alpha_2 = 1,\ \beta_2 = 1$.
Hence, solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^x(c_3 \cos x + c_4 \sin x)$$

Example 9: Solve $(D^3 - 5D^2 + 8D - 4)y = 0$.

Solution: Auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$
$$(m-1)(m^2 - 4m + 4) = 0$$
$$(m-1)(m-2)^2 = 0$$
$$m = 1,\ 2,\ 2$$

The roots are real and distinct, but second root $m = 2$ is repeated twice.
Hence, solution is

$$y = c_1 e^x + (c_2 + c_3 x)e^{2x}$$

Example 10: Solve $(D^4 + 8D^2 + 16)y = 0$.

Solution: Auxiliary equation is

$$m^4 + 8m^2 + 16 = 0$$
$$(m^2 + 4)^2 = 0$$
$$m = \pm 2i,\ \pm 2i$$

The pair of roots is imaginary and repeated twice with $\alpha = 0,\ \beta = 2$.
Hence, solution is

$$y = e^{0x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$$
$$= (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

Example 11: Solve $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$.

Solution: Auxiliary equation is

$$(m^2 + 1)^3 (m^2 + m + 1)^2 = 0$$
$$m^2 + 1 = 0,\ m^2 + m + 1 = 0$$
$$m = \pm i, m = \frac{-1 \pm i\sqrt{3}}{2}$$

Both pair of roots are imaginary and first pair is repeated thrice with $\alpha = 0,\ \beta = 1$ and second pair is repeated twice with $\alpha = -\frac{1}{2},\ \beta = \frac{\sqrt{3}}{2}$.
Hence, solution is

$$y = e^{0x}[(c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x]$$

$$+ e^{-\frac{x}{2}}\left[(c_7 + c_8 x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10} x)\sin\frac{\sqrt{3}}{2}x\right]$$

Example 12: Solve $(D^3 - 2D^2 - 5D + 6)y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.

Solution: Auxiliary equation is

$$m^3 - 2m^2 - 5m + 6 = 0$$
$$(m-1)(m^2 - m - 6) = 0$$
$$(m-1)(m+2)(m-3) = 0$$
$$m = 1, \ -2, \ 3$$

The roots are real and distinct.
Hence, solution is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 e^{3x} \qquad \text{... (1)}$$

Differentiating Eq. (1),

$$y' = c_1 e^x - 2c_2 e^{-2x} + 3c_3 e^{3x} \qquad \text{... (2)}$$

Differentiating Eq. (2),

$$y'' = c_1 e^x + 4c_2 e^{-2x} + 9c_3 e^{3x} \qquad \text{... (3)}$$

Putting $x = 0$ in Eqs. (1), (2) and (3),

$$y(0) = c_1 + c_2 + c_3$$
$$0 = c_1 + c_2 + c_3$$
$$c_1 + c_2 + c_3 = 0 \qquad \text{... (4)}$$
$$y'(0) = c_1 - 2c_2 + 3c_3$$
$$0 = c_1 - 2c_2 + 3c_3$$
$$c_1 - 2c_2 + 3c_3 = 0 \qquad \text{... (5)}$$
$$y''(0) = c_1 + 4c_2 + 9c_3$$
$$1 = c_1 + 4c_2 + 9c_3$$
$$c_1 + 4c_2 + 9c_3 = 1 \qquad \text{... (6)}$$

Solving Eqs. (4), (5) and (6),

$$c_1 = -\frac{1}{6}, \ c_2 = \frac{1}{15}, \ c_3 = \frac{1}{10}$$
$$y = -\frac{1}{6}e^x + \frac{1}{15}e^{-2x} + \frac{1}{10}e^{3x}$$

Example 13: Solve $(D^3 + \pi^2 D)y = 0$, $y(0) = 0$, $y(1) = 0$, $y'(0) + y'(1) = 0$.

Solution: Auxiliary equation is

$$m^3 + \pi^2 m = 0, \qquad m(m^2 + \pi^2) = 0$$
$$m = 0, \qquad m = \pm\, i\pi$$

First root is real and second pair of roots is imaginary with $\alpha = 0$, $\beta = \pi$.
Hence, solution is

$$y = c_1 + c_2 \cos \pi x + c_3 \sin \pi x \qquad \text{... (1)}$$

Differentiating Eq. (1),

$$y' = 0 - c_2 \cdot \pi \sin \pi x + c_3 \cdot \pi \cos \pi x \quad \text{... (2)}$$

Putting $x = 0$ in Eqs. (1) and (2) and using given initial conditions,

$$y(0) = 0, \quad c_1 + c_2 = 0 \quad \text{... (3)}$$

$$y(1) = 0, \quad c_1 - c_2 = 0 \quad \text{... (4)}$$

$$y'(0) + y'(1) = 0$$

$$\pi c_3 - \pi c_3 = 0$$

Solving Eqs. (3) and (4),

$c_1 = 0$, $c_2 = 0$ and c_3 cannot be determined.

Hence, solution is

$y = c_3 \sin \pi x$, where c_3 is arbitrary constant.

Exercise 10.8

Solve the following differential equations:

1. $(D^2 + D - 2)y = 0.$ $\left[\textbf{Ans.: } y = c_1 e^{-2x} + c_2 e^{x}\right]$

2. $(4D^2 + 8D - 5y) = 0.$ $\left[\textbf{Ans.: } y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{5x}{2}}\right]$

3. $(D^2 - 4D - 12)y = 0.$ $\left[\textbf{Ans.: } y = c_1 e^{6x} + c_2 e^{-2x}\right]$

4. $(D^2 + 2D - 8)y = 0.$ $\left[\textbf{Ans.: } y = c_1 e^{2x} + c_2 e^{-4x}\right]$

5. $(D^2 + 4D + 1)y = 0.$ $\left[\textbf{Ans.: } y = c_1 e^{(-2+\sqrt{3})x} + c_2 e^{(-2-\sqrt{3})x}\right]$

6. $(4D^2 + 4D + 1)y = 0.$ $\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{-\frac{x}{2}}\right]$

7. $(D^2 + 2\pi D + \pi^2)y = 0.$ $\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{-\pi x}\right]$

8. $(9D^2 - 12D + 4)y = 0.$ $\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{\frac{2x}{3}}\right]$

9. $(25D^2 - 20D + 4)y = 0.$ $\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{\frac{2x}{5}}\right]$

10. $(9D^2 - 30D + 25)y = 0.$ $\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{\frac{5x}{3}}\right]$

11. $(D^2 - 6D + 25)y = 0.$ $\left[\textbf{Ans.: } y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)\right]$

12. $(D^2 + 6D + 11)y = 0.$ $\left[\textbf{Ans.: } y = e^{-3x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)\right]$

13. $[D^2 - 2aD + (a^2 + b^2)y] = 0.$ $\left[\textbf{Ans.: } y = e^{ax}(c_1 \cos bx + c_2 \sin bx)\right]$

14. $(D^3 - 9D)y = 0.$ $\left[\textbf{Ans.: } y = c_1 + c_2 e^{3x} + c_3 e^{-3x}\right]$

15. $(D^3 - 3D^2 - D + 3)y = 0.$ $\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}\right]$

16. $(D^3 - 6D^2 + 11D - 6)y = 0.$ $\left[\textbf{Ans.: } y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}\right]$

17. $(D^3 - 6D^2 + 12D - 8)y = 0.$ $\left[\textbf{Ans.: } y = (c_1 + c_2 x + c_3 x^2)e^{2x}\right]$

18. $(D^3 + D)y = 0.$ $\left[\textbf{Ans.: } y = c_1 + c_2 \cos x + c_3 \sin x\right]$

19. $(D^3+5D^2+8D+6)y=0.$

$$\left[\textbf{Ans.:}\ y=c_1e^{-3x}+e^{-x}(c_2\cos x+c_3\sin x)\right]$$

20. $(8D^4-6D^3-7D^2+6D-1)y=0.$

$$\left[\textbf{Ans.:}\ y=c_1e^{\frac{x}{4}}+c_2e^{\frac{x}{2}}+c_3e^{x}+c_4e^{-x}\right]$$

21. $(D^4-2D^3+D^2)y=0.$

$$\left[\textbf{Ans.:}\ y=c_1+c_2x+(c_3+c_4x)e^{x}\right]$$

22. $(D^4-3D^3+3D^2-D)y=0.$

$$\left[\textbf{Ans.:}\ y=c_1+(c_2+c_3x+c_4x^2)e^{x}\right]$$

23. $(D^4+8D^2-9)y=0.$

$$\left[\textbf{Ans.:}\ y=c_1e^{x}+c_2e^{-x}+c_3\cos 3x+c_4\sin 3x\right]$$

24. $(D^4+D^3+14D^2+16D-32)y=0.$

$$\left[\textbf{Ans.:}\ y=c_1e^{x}+c_2e^{-2x}+c_3\cos 4x+c_4\sin 4x\right]$$

25. $(D^4+2D^3-9D^2-10D+50)y=0.$

$$\left[\textbf{Ans.:}\ y=e^{2x}(c_1\cos x+c_2\sin x)+e^{-3x}(c_3\cos x+c_4\sin x)\right]$$

26. $(D^4+18D^3+81)y=0.$

$$\left[\textbf{Ans.:}\ y=(c_1+c_2x)\cos 3x+(c_3+c_4x)\sin 3x\right]$$

27. $(D^4-4D^3+14D^2-20D+25)y=0.$

$$\left[\textbf{Ans.:}\ y=e^{x}[(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x]\right]$$

28. $(D^2+D-2)y=0, y(0)=4, y'(0)=-5.$

$$\left[\textbf{Ans.:}\ y=e^{x}+3e^{-2x}\right]$$

29. $(4D^2+12D+9)y=0,$
$y(0)=-1,\ y'(0)=2.$

$$\left[\textbf{Ans.:}\ y=\left(\frac{x}{2}-1\right)e^{-\frac{3x}{2}}\right]$$

30. $(D^2-4D+5)y=0,$
$y(0)=2,\ y'(0)=-1.$

$$\left[\textbf{Ans.:}\ y=e^{2x}(2\cos x-5\sin x)\right]$$

31. $(9D^2-6D+1)y=0,$
$y(1)=e^{\frac{1}{3}},\ y(2)=1.$

$$\left[\textbf{Ans.:}\ y=\left[\left(e^{-\frac{2}{3}}-1\right)x+\left(2-e^{-\frac{2}{3}}\right)\right]e^{\frac{x}{3}}\right]$$

32. $(4D^3-4D^2-9D+9)y=0,$
$y(0)=1,\ y'(0)=0,\ y''(0)=0.$

$$\left[\textbf{Ans.:}\ y=\frac{1}{5}\left(9e^{x}-5e^{\frac{3x}{2}}+e^{\frac{-3x}{2}}\right)\right]$$

33. $(D^3+D^2-2)y=0,\ y(0)=2,$
$y'(0)=2,\ y''(0)=-3.$

$$\left[\textbf{Ans.:}\ y=e^{x}+e^{-x}(\cos x+2\sin x)\right]$$

34. $(D^4-3D^3)=0,\ y(0)=2,$
$y'(0)=5,\ y''(0)=15,\ y'''(0)=27.$

$$\left[\textbf{Ans.:}\ y=1+2x+3x^2+e^{3x}\right]$$

35. $(D^4-3D^3+2D^2)y=0,\ y(0)=2,$
$y'(0)=0,\ y''(0)=2, y'''(0)=2.$

$$\left[\textbf{Ans.:}\ y=2(e^{x}-x)\right]$$

10.5 NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-2}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = Q(x) \qquad \ldots (1)$$

where a_0, a_1, a_2, ..., a_n are constants and Q is a function of x, is known as Non-homogeneous linear differential equation with constant coefficients.

Equation (1) can also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n) y = Q(x) \qquad \ldots (2)$$

$$f(D) y = Q(x)$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$

10.5.1 General Solution of Non-Homogeneous Linear Differential Equation

A general solution of Eq. (1) is obtained in two parts as

General solution = complimentary function + particular integral

$$y = \text{C.F} + \text{P.I.}$$

Complimentary function (C.F.) is the general solution of the homogeneous equation obtained by putting $Q(x) = 0$ in Eq. (1).

Particular integral (P.I.) is any particular solution of the non-homogeneous Eq. (1) and contains no arbitrary constants.

Inverse Operator and Particular Integral

$f(D)$ is known as differential operator and $\frac{1}{f(D)}$ is known as inverse differential operator.

$$f(D)\left[\frac{1}{f(D)} Q(x)\right] = Q(x)$$

This shows that $\frac{1}{f(D)} Q(x)$ satisfies the equation $f(D) y = Q(x)$ and since $\frac{1}{f(D)} Q(x)$ does not contain any arbitrary constants, gives the P.I. of the equation $f(D) y = Q(x)$.

Hence, $$\text{P.I.} = \frac{1}{f(D)} Q(x)$$

(i) If $f(D) = D$, then

$$\text{P.I.} = \frac{1}{D} Q(x) = \int Q(x)\, dx$$

(ii) If $f(\mathrm{D}) = \mathrm{D} - a$, then equation $f(\mathrm{D})y = \mathrm{Q}(x)$ becomes

$$(\mathrm{D} - a)y = \mathrm{Q}(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - ay = \mathrm{Q}(x)$$

is a first order linear differential equation.

$$\text{I.F.} = e^{\int -a\mathrm{d}x} = e^{-ax}$$

Solution is

$$ye^{-ax} = \int e^{-ax}\mathrm{Q}(x)\,\mathrm{d}x + c$$

$$y = e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x + ce^{ax}$$

Here, ce^{ax} is the complimentary function since it contains arbitrary constant c and $e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x$ is the particular integral.

Hence, $$\text{P.I.} = \frac{1}{\mathrm{D} - a}\mathrm{Q}(x) = e^{ax}\int \mathrm{Q}(x)e^{-ax}\mathrm{d}x$$

10.5.2 Direct (Short-cut) Method of Obtaining Particular Integral (P.I)

This method depends on the nature of $\mathrm{Q}(x)$ in Eq. (1). Particular Integral by this method can be obtained when $\mathrm{Q}(x)$ has the following forms:

(i) $\mathrm{Q}(x) = e^{ax+b}$

(ii) $\mathrm{Q}(x) = \sin(ax+b)$ or $\cos(ax+b)$

(iii) $\mathrm{Q}(x) = x^m$ or polynomial in x

(iv) $\mathrm{Q}(x) = e^{ax}v(x)$

(v) $\mathrm{Q}(x) = xv(x)$

Case I: $\mathrm{Q}(x) = e^{ax+b}$:

$$f(\mathrm{D})y = e^{ax+b}$$

Now, $\mathrm{D}(e^{ax+b}) = ae^{ax+b}$, $\mathrm{D}^2(e^{ax+b}) = a^2e^{ax+b}$, ..., $\mathrm{D}^n e^{ax+b} = a^n e^{ax+b}$

Consider

$$f(\mathrm{D})(e^{ax+b}) = (a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \ldots + a_n)e^{ax+b}$$

$$= (a_0a^n + a_1a^{n-1} + \ldots + a_n)e^{ax+b} = f(a)e^{ax}$$

Operating both the sides with $\frac{1}{f(\mathrm{D})}$,

$$\frac{1}{f(\mathrm{D})}\left[f(\mathrm{D})\left(e^{ax+b}\right)\right] = \frac{1}{f(\mathrm{D})}\left[f(a)e^{ax+b}\right]$$

$$e^{ax+b} = f(a)\,\frac{1}{f(\mathrm{D})}e^{ax+b}$$

$$\frac{1}{f(a)}e^{ax+b} = \frac{1}{f(\mathrm{D})}e^{ax+b}, \qquad f(a) \neq 0$$
$$\frac{1}{f(\mathrm{D})}e^{ax+b} = \frac{1}{f(a)}e^{ax+b}, \qquad f(a) \neq 0$$

Hence, $\text{P.I.} = \frac{1}{f(a)}e^{ax+b}$ if $f(a) \neq 0$

Note: If $f(a) = 0$, then $(\mathrm{D} - a)$ is a factor of $f(\mathrm{D})$ and hence, above rule fails.
Let $f(\mathrm{D}) = (\mathrm{D} - a)\phi(\mathrm{D})$, where $\phi(a) \neq 0$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(\mathrm{D})}e^{ax+b} = \frac{1}{(\mathrm{D}-a)\phi(\mathrm{D})}e^{ax+b} \\
&= \frac{1}{\phi(a)} \cdot \frac{1}{(\mathrm{D}-a)}e^{ax+b} \\
&= \frac{1}{\phi(a)} \cdot e^{ax}\int e^{-ax}e^{ax+b}\mathrm{d}x = \frac{1}{\phi(a)} \cdot e^{ax} \cdot xe^{b} \\
&= x\frac{1}{\phi(a)}e^{ax+b} \qquad \ldots (3)
\end{aligned}$$

Since
$$\begin{aligned}
f(\mathrm{D}) &= (\mathrm{D}-a)\phi(\mathrm{D}) \\
f'(\mathrm{D}) &= (\mathrm{D}-a)\phi'(\mathrm{D}) + \phi(\mathrm{D}) \\
f'(a) &= \phi(a)
\end{aligned}$$

Substituting in Eq. (3),

$$\frac{1}{f(\mathrm{D})}e^{ax+b} = x \cdot \frac{1}{f'(a)}e^{ax+b} \quad \text{where } f'(a) \neq 0$$

If $f'(a) = 0$, then repeating the above process,

$$\begin{aligned}
\frac{1}{f(\mathrm{D})}e^{ax+b} &= x \cdot x \cdot \frac{1}{f''(a)}e^{ax+b} \\
&= x^2 \frac{1}{f''(a)}e^{ax+b} \qquad \text{where } f''(a) \neq 0
\end{aligned}$$

In general if $(\mathrm{D} - a)^r$ is a factor of $f(\mathrm{D})$, then

$$\frac{1}{f(\mathrm{D})}e^{ax} = x^r \frac{1}{f^{(r)}(a)}e^{ax+b}$$

Hence,
$$\text{P.I.} = x^r \frac{1}{f^{(r)}(a)}e^{ax+b}.$$

Example 1: Solve $(\mathrm{D}^2 - 3\mathrm{D} + 2)y = e^{3x}$.

Solution: Auxiliary equation is

$$\begin{aligned}
m^2 - 3m + 2 &= 0 \\
m &= 1, 2
\end{aligned}$$

$$\text{C.F.} = c_1 e^x + c_2 e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} e^{3x} = \frac{1}{3^2 - 3(3) + 2} e^{3x} = \frac{1}{2} e^{3x}$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{2x} + \frac{1}{2} e^{3x}$$

Example 2: Solve $(D^2 + 6D + 9)y = 5^x - \log 2$.

Solution: Auxiliary equation is

$$m^2 + 6m + 9 = 0, \quad (m+3)^2 = 0, \qquad m = -3, -3$$

$$\text{C.F.} = (c_1 + c_2 x)e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 6D + 9}(5^x - \log 2)$$

$$= \frac{1}{(D+3)^2}(e^{x\log 5}) - \frac{1}{(D+3)^2}(\log 2)e^{0\cdot x}$$

$$= \frac{1}{(\log 5 + 3)^2} e^{x \log 5} - \log 2 \cdot \frac{1}{(0+3)^2} e^{0 \cdot x}$$

$$= \frac{5^x}{(\log 5 + 3)^2} - \frac{\log 2}{9}$$

Hence, the general solution is

$$y = (c_1 + c_2 x)e^{-3x} + \frac{5^x}{(\log 5 + 3)^2} - \frac{\log 2}{9}$$

Example 3: Solve $(D^3 - D^2 + 4D - 4)y = e^x$.

Solution: Auxiliary equation is

$$m^3 - m^2 + 4m - 4 = 0, \; (m-1)(m^2 + 4) = 0$$

$$m - 1 = 0, \qquad m^2 + 4 = 0$$

$$m = 1, \qquad m = \pm 2i$$

$$\text{C.F.} = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^3 - D^2 + 4D - 4} e^x = x \cdot \frac{1}{3D^2 - 2D + 4} e^x = x \frac{1}{3 - 2 + 4} e^x = \frac{x}{5} e^x$$

Hence, the general solution is

$$y = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x + \frac{xe^x}{5}$$

Example 4: Solve $(D^6 - 64)y = e^x \cosh 2x$.

Solution: Auxiliary equation is

$$m^6 - 64 = 0$$

$$(m^3)^2 - (8)^2 = 0, \qquad (m^3 + 8)(m^3 - 8) = 0$$

$$(m+2)(m^2 - 2m + 4)(m-2)(m^2 + 2m + 4) = 0$$

$$m + 2 = 0,\ m^2 - 2m + 4 = 0,$$

$$m - 2 = 0,\ m^2 + 2m + 4 = 0$$

$$m = -2,\quad m = 1 \pm i\sqrt{3}, \qquad m = 2, \quad m = -1 \pm i\sqrt{3}$$

Two roots are real and two pair of roots are complex.

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{2x} + e^x\left(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x\right) + e^{-x}\left(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x\right)$$

Now, $$e^x \cosh 2x = e^x\left(\frac{e^{2x} + e^{-2x}}{2}\right) = \frac{1}{2}(e^{3x} + e^{-x})$$

$$\text{P.I.} = \frac{1}{D^6 - 64} \cdot e^x \cosh 2x = \frac{1}{D^6 - 64} \cdot \frac{1}{2}(e^{3x} + e^{-x})$$

$$= \frac{1}{2}\left[\frac{1}{3^6 - 64} e^{3x} + \frac{1}{(-1)^6 - 64} e^{-x}\right] = \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^x(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x)$$
$$+ e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x) + \frac{1}{2}\left(\frac{e^{3x}}{665} - \frac{e^{-x}}{63}\right)$$

Example 5: Solve $(D^3 - 4D)y = 2\cosh^2(2x)$.

Solution: Auxiliary equation is

$$m^3 - 4m = 0, \qquad m(m^2 - 4) = 0$$

$$m = 0, \quad m = \pm 2$$

$$\text{C.F.} = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-2x} = c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

Now,

$$X = 2\cosh^2 2x = 2\left(\frac{e^{2x} + e^{-2x}}{2}\right)^2 = \frac{1}{2}(e^{4x} + e^{-4x} + 2)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^3-4D}\cdot 2\cosh^2 2x \\
&= \frac{1}{D^3-4D}\cdot\frac{1}{2}(e^{4x}+e^{-4x}+2) = \frac{1}{2}\cdot\frac{1}{D^3-4D}(e^{4x}+e^{-4x}+2e^{0x}) \\
&= \frac{1}{2}\left[\frac{1}{4^3-16}e^{4x}+\frac{1}{(-4)^3+16}e^{-4x}+x\frac{1}{3D^2-4}2e^{0x}\right] \\
&= \frac{1}{2}\left[\frac{1}{48}\cdot e^{4x}+\frac{1}{(-48)}\cdot e^{-4x}+x\cdot\frac{1}{0-4}\cdot 2e^{0x}\right] \\
&= \frac{1}{2}\left[\frac{e^{4x}-e^{-4x}}{48}-\frac{x}{2}\right] = \frac{\sinh 4x}{48}-\frac{x}{4}
\end{aligned}$$

Hence, the general solution is

$$y = c_1 + c_2e^{2x} + c_3e^{-2x} + \frac{\sinh 4x}{48} - \frac{x}{4}$$

Example 6: Solve $(4D^2-4D+1)y = e^{\frac{x}{2}}$.

Solution: Auxiliary equation is

$$4m^2-4m+1=0,\ (2m-1)^2=0, \qquad m=\frac{1}{2},\frac{1}{2}$$

$$\text{C.F.} = (c_1+c_2x)e^{\frac{x}{2}}$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{4D^2-4D+1}e^{\frac{x}{2}} = x\cdot\frac{1}{8D-4}e^{\frac{x}{2}} = x^2\cdot\frac{1}{8}e^{\frac{x}{2}} \\
&= \frac{x^2}{8}e^{\frac{x}{2}}
\end{aligned}$$

Hence, the general solution is

$$y = (c_1+c_2x)e^{\frac{x}{2}} + \frac{x^2}{8}e^{\frac{x}{2}}$$

Example 7: Solve $(D^3-5D^2+8D-4)y = e^{2x}+2e^x+3e^{-x}+2$.

Solution: Auxiliary equation is

$$m^3-5m^2+8m-4=0$$

$$(m-1)(m^2-4m+4)=0,\ (m-1)(m-2)^2=0$$

$$m=1, \quad m=2,\ 2$$

$$\text{C.F.} = c_1e^x + (c_2+c_3x)e^{2x}$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^3-5D^2+8D-4}(e^{2x}+2e^x+3e^{-x}+2e^{0\cdot x}) \\
&= x\cdot\frac{1}{3D^2-10D+8}\cdot e^{2x} + x\cdot\frac{1}{3D^2-10D+8}2e^x + \frac{1}{-1-5-8-4}3e^{-x} + \frac{1}{-4}2e^{0\cdot x}
\end{aligned}$$

$$= x^2 \cdot \frac{1}{6\text{D}-10} e^{2x} + x \frac{1}{3-10+8} 2e^x - \frac{1}{18} \cdot 3e^{-x} - \frac{1}{2}$$

$$= x^2 \frac{1}{12-10} e^{2x} + 2xe^x - \frac{1}{6} e^{-x} - \frac{1}{2}$$

$$= \frac{x^2}{2} e^{2x} + 2xe^x - \frac{1}{6} e^{-x} - \frac{1}{2}$$

Hence, the general solution is

$$y = c_1 e^x + (c_2 + c_3 x) e^{2x} + \frac{x^2}{2} e^{2x} + 2xe^x - \frac{1}{6} e^{-x} - \frac{1}{2}$$

Case II: $\text{Q}(x) = \sin(ax+b)$ or $\cos(ax+b)$

(i) If $\text{Q}(x) = \sin(ax + b)$, then Eq. (2) reduces to

$$f(\text{D}) y = \sin(ax+b)$$

Now

$$\text{D}[\sin(ax+b)] = a\cos(ax+b)$$

$$\text{D}^2[\sin(ax+b)] = (-a^2)\sin(ax+b)$$

$$\text{D}^3[\sin(ax+b)] = -a^3\cos(ax+b)$$

$$\text{D}^4[\sin(ax+b)] = a^4\sin(ax+b)$$

$$(\text{D}^2)^2[\sin(ax+b)] = (-a^2)^2\sin(ax+b)$$

In general,

$$(\text{D}^2)^r[\sin(ax+b)] = (-a^2)^r\sin(ax+b)$$

This shows that

$$\phi(\text{D}^2)\sin(ax+b) = \phi(-a^2)\sin(ax+b)$$

Operating both the sides with $\dfrac{1}{\phi(\text{D}^2)}$,

$$\frac{1}{\phi(\text{D}^2)}\left[\phi(\text{D}^2)\sin(ax+b)\right] = \frac{1}{\phi(\text{D}^2)}\left[\phi(-a^2)\sin(ax+b)\right]$$

$$\sin(ax+b) = \phi(-a^2)\frac{1}{\phi(\text{D}^2)}\sin(ax+b)$$

$$\frac{1}{\phi(-a^2)}\sin(ax+b) = \frac{1}{\phi(\text{D}^2)}\sin(ax+b)$$

$$\frac{1}{\phi(\text{D}^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b)$$

If $f(\text{D}) = \phi(\text{D}^2)$, then

$$\text{P.I.} = \frac{1}{f(\text{D})}\sin(ax+b)$$

$$= \frac{1}{\phi(\text{D}^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b), \text{ if } \phi(-a^2) \neq 0$$

If $\phi(-a^2)=0,$ then (D^2+a^2) is a factor of $\phi(D^2)$ and hence, above rule fails.

$$\text{P.I.}=\frac{1}{\phi(D^2)}\sin(ax+b)=\frac{1}{\phi(D^2)}\left[\text{I.P. of } e^{i(ax+b)}\right]=\text{I.P. of }\frac{1}{\phi(D^2)}e^{i(ax+b)}$$

$$=\text{I.P. of } x\cdot\frac{1}{\phi'(D^2)}e^{i(ax+b)} \qquad \left[\because \phi(i^2a^2)=\phi(-a^2)=0\right]$$

$$=\text{I.P. of } x\cdot\frac{1}{\phi'(i^2a^2)}e^{i(ax+b)}=\text{I.P. of } x\cdot\frac{1}{\phi'(-a^2)}e^{i(ax+b)}$$

$$=x\cdot\frac{1}{\phi'(-a^2)}\sin(ax+b)$$

If $\phi'(-a^2)=0,$ then

$$\frac{1}{\phi(D^2)}\sin(ax+b)=x^2\cdot\frac{1}{\phi''(-a^2)}\sin(ax+b), \qquad \text{where } \phi''(-a^2)\neq 0$$

In general, if $\phi^{(r)}(-a^2)=0,$ then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\sin(ax+b)=x^{(r+1)}\frac{1}{\phi^{(r+1)}(-a^2)}\sin(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2)\neq 0$$

(ii) Similarly, if $Q(x)=\cos(ax+b)$

$$\text{P.I.}=\frac{1}{\phi(D^2)}\cos(ax+b)=\frac{1}{\phi(-a^2)}\cos(ax+b),\ \phi(-a^2)\neq 0$$

If $\phi(-a^2)=0,$ then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\cos(ax+b)=x\cdot\frac{1}{\phi'(-a^2)}\cos(ax+b)$$

If $\phi'(-a^2)=0,$ then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\cos(ax+b)=x^2\cdot\frac{1}{\phi''(-a^2)}\cos(ax+b), \qquad \text{where } \phi''(-a^2)\neq 0$$

In general, if $\phi^{(r)}(-a^2)=0,$ then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\cos(ax+b)=x^{r+1}\frac{1}{\phi^{(r+1)}(-a^2)}\cos(ax+b), \qquad \text{where } \phi^{(r+1)}(-a^2)\neq 0$$

Note: If after replacing D^2 by $-a^2$, $f(D)$ contains terms of D, then denominator is rationalised to obtained the even powers of D.

Example 1: Solve $(D^2+9)y=\sin 4x$.

Solution: Auxiliary equation is

$$m^2+9=0, \qquad m=\pm 3i \text{ (imaginary)}$$

$$\text{C.F.}=c_1\cos 3x+c_2\sin 3x$$

$$\text{P.I.}=\frac{1}{D^2+9}\sin 4x=\frac{1}{-4^2+9}\sin 4x=-\frac{1}{7}\sin 4x$$

Hence, the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{7}\sin 4x$$

Example 2: Solve $(D^4 + 2a^2D^2 + a^4)y = 8\cos ax$.

Solution: Auxiliary equation is

$$m^4 + 2a^2m^2 + a^4 = 0$$

$$(m^2 + a^2)^2 = 0, \quad m = \pm ia, \pm ia \text{ (imaginary and repeated twice)}$$

$$\text{C.F.} = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax$$

$$\text{P.I.} = \frac{1}{D^4 + 2a^2D^2 + a^4}\cdot 8\cos ax = x\cdot\frac{1}{4D^3 + 4a^2D}\cdot 8\cos ax$$

$$= x^2\cdot\frac{1}{12D^2 + 4a^2}\cdot 8\cos ax = x^2\cdot\frac{1}{12(-a^2) + 4a^2}\cdot 8\cos ax$$

$$= -\frac{x^2}{a^2}\cos ax$$

Hence, the general solution is

$$y = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax - \frac{x^2}{a^2}\cos ax$$

Example 3: Solve $(D^2 + 3D + 2)y = \sin 2x$.

Solution: Auxiliary equation is

$$m^2 + 3m + 2 = 0, \quad (m+1)(m+2) = 0$$

$$m = -1, -2 \qquad \text{(Real and distinct)}$$

$$\text{C.F.} = c_1e^{-x} + c_2e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2}\cdot\sin 2x = \frac{1}{-4 + 3D + 2}\cdot\sin 2x$$

$$= \frac{1}{3D - 2}\cdot\sin 2x = \frac{1}{(3D-2)}\cdot\frac{(3D+2)}{(3D+2)}\sin 2x$$

$$= \frac{(3D+2)}{9D^2 - 4}\sin 2x = \frac{3D+2}{9(-4) - 4}\sin 2x$$

$$= \frac{3D+2}{-40}\sin 2x = -\frac{3}{40}(D\sin 2x) - \frac{1}{20}\sin 2x$$

$$= -\frac{3}{40}\cdot 2\cos 2x - \frac{1}{20}\sin 2x = -\frac{1}{20}(3\cos 2x + \sin 2x)$$

Hence, the general solution is

$$y = c_1e^{-x} + c_2e^{-2x} - \frac{1}{20}(3\cos 2x + \sin 2x)$$

Example 4: Solve $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$.

Solution: Auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$(m-1)(m^2 - 2m + 2) = 0$$

$$m - 1 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = 1, \qquad m = 1 \pm i \qquad \text{(imaginary)}$$

$$\text{C.F.} = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x)$$

$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x)$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4} e^x + \frac{1}{D(-1^2) - 3(-1^2) + 4D - 2} \cos x$$

$$= x \frac{1}{3 - 6 + 4} \cdot e^x + \frac{1}{3D + 1} \cos x = xe^x + \frac{1}{(3D+1)} \cdot \frac{(3D-1)}{(3D-1)} \cos x$$

$$= xe^x + \frac{3D - 1}{9D^2 - 1} \cos x = xe^x + \frac{3D - 1}{9(-1^2) - 1} \cos x$$

$$= xe^x - \frac{1}{10}(3D \cos x - \cos x) = xe^x - \frac{1}{10}(-3 \sin x - \cos x)$$

$$= xe^x + \frac{1}{10}(3 \sin x + \cos x)$$

Hence, the general solution is

$$y = (c_1 + c_2 \cos x + c_3 \sin x)e^x + xe^x + \frac{1}{10}(3 \sin x + \cos x)$$

Example 5: Solve $(D-1)^2(D^2+1)y = e^x + \sin^2 \frac{x}{2}$.

Solution: Auxiliary equation is

$$(m-1)^2(m^2+1) = 0$$

$$(m-1)^2 = 0, \qquad m^2 + 1 = 0$$

$$m = 1,\ 1 \text{ (repeated twice)}, \quad m = \pm i \text{ (imaginary)}$$

$$\text{C.F.} = (c_1 + c_2 x)e^x + c_3 \cos x + c_4 \sin x$$

Now, $$Q(x) = e^x + \sin^2 \frac{x}{2} = e^x + \frac{1 - \cos x}{2}$$

$$\text{P.I.} = \frac{1}{(D-1)^2(D^2+1)}\left(e^x + \frac{e^{0x}}{2} - \frac{\cos x}{2}\right)$$

$$= \frac{1}{(D-1)^2}\cdot\frac{1}{(1^2+1)}e^x + \frac{1}{(0-1)^2(0+1)}\cdot\frac{e^{0x}}{2} - \frac{1}{(D^2+1)(D^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$= x\cdot\frac{1}{2(D-1)}\cdot\frac{e^x}{2} + \frac{1}{2} - \frac{1}{(D^2+1)(-1^2-2D+1)}\cdot\frac{\cos x}{2}$$

$$= \frac{x^2}{2}\cdot\frac{e^x}{2} + \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{(D^2+1)}\frac{1}{D}\cos x = \frac{x^2e^x}{4} + \frac{1}{2} + \frac{1}{4}\frac{1}{(D^2+1)}\int \cos x\, dx$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{D^2+1}\sin x = \frac{x^2e^x}{4} + \frac{1}{2} + \frac{1}{4}x\frac{1}{2D}\sin x$$

$$= \frac{x^2e^x}{4} + \frac{1}{2} + \frac{x}{8}\int \sin\ x\ dx = \frac{x^2e^x}{4} + \frac{1}{2} + \frac{x}{8}(-\cos x)$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^x + c_3\cos x + c_4\sin x + \frac{x^2e^x}{4} + \frac{1}{2} - \frac{x\cos x}{8}$$

Case III: $Q(x) = x^m$

In this case, Eq. (2) reduces to $f(D)y = x^m$.

Hence, $\quad \text{P.I.} = \frac{1}{f(D)}x^m$

$$= [f(D)]^{-1}x^m = [1+\phi(D)]^{-1}x^m$$

Expanding in ascending powers of D up to D^m using Binomial Expansion, since $D^nx^m = 0$ when $n > m$,

$$\text{P.I.} = (a_0 + a_1D + a_2D^2 + \ldots + a_mD^m)x^m$$

Example 1: Solve $(D^2+2D+1)y = x$.

Solution: Auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0, \qquad m = -1, -1 \text{ (real and repeated twice)}$$

$$\text{C.F.} = (c_1 + c_2x)e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2+2D+1}\cdot x = \frac{1}{(1+D)^2}\cdot x$$

$$= (1+D)^{-2}x = (1-2D+3D^2-\ldots)x = x - 2Dx + 3D^2x - \ldots$$

$$= x - 2 + 0 = x - 2$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{-x} + x - 2$$

Example 2: Solve $(D^4 - 2D^3 + D^2)y = x^3$.

Solution: Auxiliary equation is

$$m^4 - 2m^3 + m^2 = 0, \qquad m^2(m^2 - 2m + 1) = 0$$

$$m^2(m-1)^2 = 0, \qquad m = 0,\ 0,\ m = 1, 1$$

Both the roots are real and repeated twice.

$$\text{C.F.} = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^x = c_1 + c_2 x + (c_3 + c_4 x)e^x$$

$$\text{P.I.} = \frac{1}{D^4 - 2D^3 + D^2} \cdot x^3 = \frac{1}{D^2(D^2 - 2D + 1)} x^3$$

$$= \frac{1}{D^2(1-D)^2} \cdot x^3 = \frac{1}{D^2}(1-D)^{-2} x^3 = \frac{1}{D^2}(1 + 2D + 3D^2 + 4D^3 + 5D^4 + \ldots)x^3$$

$$= \frac{1}{D^2}(x^3 + 2Dx^3 + 3D^2x^3 + 4D^3x^3 + 5D^4x^3 + \ldots)$$

$$= \frac{1}{D^2}(x^3 + 2 \cdot 3x^2 + 3 \cdot 6x + 4 \cdot 6 + 0) = \frac{1}{D^2}(x^3 + 6x^2 + 18x + 24)$$

$$= \int\left[\int (x^3 + 6x^2 + 18x + 24)\mathrm{d}x\right]\mathrm{d}x = \int\left(\frac{x^4}{4} + 6\frac{x^3}{3} + 18\frac{x^2}{2} + 24x\right)\mathrm{d}x$$

$$= \frac{x^5}{20} + 2\frac{x^4}{4} + 9\frac{x^3}{3} + 24\frac{x^2}{2} = \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Hence, the general solution is

$$y = c_1 + c_2 x + (c_3 + c_4 x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Example 3: Solve $(D^3 - D^2 - 6D)y = 1 + x^2$.

Solution: Auxiliary equation is

$$m^3 - m^2 - 6m = 0, m(m^2 - m - 6) = 0$$

$$m(m-3)(m+2) = 0, \qquad m = 0,\ 3,\ -2 \qquad \text{(real and distinct)}$$

$$\text{C.F.} = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x} = c_1 + c_2 e^{3x} + c_3 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D}(1 + x^2) = \frac{1}{-6D\left[1 - \dfrac{D^2 - D}{6}\right]}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D^2 - D}{6}\right)\right]^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + \left(\frac{D^2 - D}{6}\right) + \left(\frac{D^2 - D}{6}\right)^2 + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + \frac{D^2 - D}{6} + \frac{D^4 - 2D^3 + D^2}{36} + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{7D^2}{36} + \text{Higher powers of D}\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[(1 + x^2) - \frac{1}{6}D(1 + x^2) + \frac{7}{36}D^2(1 + x^2)\right]$$

$$= -\frac{1}{6D}\left[1 + x^2 - \frac{1}{6}(2x) + \frac{7}{36}(2)\right]$$

$$= -\frac{1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right] = -\frac{1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx$$

$$= -\frac{1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right)$$

Hence, the general solution is

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{18}\left(x^3 - \frac{x^2}{2} + \frac{25}{6}x\right)$$

Example 4: Solve $(D^3 - 2D + 4)y = x^4 + 3x^2 - 5x + 2$.

Solution: Auxiliary equation is

$$m^3 - 2m + 4 = 0$$

$$(m + 2)(m^2 - 2m + 2) = 0$$

$$m + 2 = 0, \qquad m^2 - 2m + 2 = 0$$

$$m = -2 \text{ (real)}, \qquad m = 1 \pm i \text{ (complex)}$$

$$\text{C.F.} = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$$

$$\text{P.I.} = \frac{1}{(D^3 - 2D + 4)}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left(1 + \frac{D^3 - 2D}{4}\right)^{-1}(x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 - \left(\frac{D^3 - 2D}{4}\right)^3\right.$$

$$\left. + \left(\frac{D^3 - 2D}{4}\right)^4 - \ldots\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^3 - 2D}{4}\right) + \frac{4D^2}{16} - \frac{4D^4}{16} + \frac{8D^3}{64}\right.$$

$$\left. + \frac{16D^4}{256} + \text{ higher powers of D}\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 + \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} - \frac{3D^4}{16} + \text{higher powers of D}\right](x^4 + 3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[(x^4 + 3x^2 - 5x + 2) + \frac{1}{2}D(x^4 + 3x^2 - 5x + 2) + \frac{1}{4}D^2(x^4 + 3x^2 - 5x + 2)\right.$$

$$\left. - \frac{1}{8}D^3(x^4 + 3x^2 - 5x + 2) - \frac{3}{16}D^4(x^4 + 3x^2 - 5x + 2)\right]$$

$$= \frac{1}{4}\left[(x^4 + 3x^2 - 5x + 2) + \frac{1}{2}(4x^3 + 6x - 5) + \frac{1}{4}(12x^2 + 6) - \frac{1}{8}(24x) - \frac{3}{16}(24)\right]$$

$$= \frac{1}{4}\left(x^4 + 2x^3 + 6x^2 - 5x - \frac{7}{2}\right)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x) + \frac{1}{4}\left(x^4 + 2x^3 + 6x^2 - 5x - \frac{7}{2}\right)$$

Example 5: Solve $(D^4 + 2D^3 - 3D^2)y = x^2 + 3e^{2x}$.

Solution: Auxiliary equation is

$$m^4 + 2m^3 - 3m^2 = 0$$

$$m^2(m^2 + 2m - 3) = 0, \qquad m^2(m-1)(m+3) = 0$$

$$m = 0,\ 0 \text{ (repeated twice)}, \quad m = 1, -3 \text{ (real and distinct)}$$

$$\text{C.F.} = (c_1 + c_2 x)e^{0x} + c_3 e^x + c_4 e^{-3x} = c_1 + c_2 x + c_3 e^x + c_4 e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^4 + 2D^3 - 3D^2} x^2 + \frac{1}{D^4 + 2D^3 - 3D^2} 3e^{2x}$$

$$= \frac{1}{-3D^2\left(1 - \frac{D^2 + 2D}{3}\right)} x^2 + \frac{1}{16 + 16 - 12} 3e^{2x}$$

$$= -\frac{1}{3D^2}\left(1 - \frac{D^2 + 2D}{3}\right)^{-1} x^2 + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D^2}\left[1 + \frac{D^2 + 2D}{3} + \left(\frac{D^2 + 2D}{3}\right)^2 + \ldots\right] x^2 + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D^2}\left(1 + \frac{D^2 + 2D}{3} + \frac{4D^2}{9} + \text{higher powers of D}\right) x^2 + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3D^2}\left(x^2 + \frac{2}{3}Dx^2 + \frac{7}{9}D^2x^2\right) + \frac{3}{20}e^{2x} = -\frac{1}{3D^2}\left[x^2 + \frac{2}{3}(2x) + \frac{7}{9}(2)\right] + \frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\int\left[\int\left(x^2+\frac{4}{3}x+\frac{14}{9}\right)dx\right]dx+\frac{3e^{2x}}{20} = -\frac{1}{3}\int\left(\frac{x^3}{3}+\frac{2}{3}x^2+\frac{14}{9}x\right)dx+\frac{3e^{2x}}{20}$$

$$= -\frac{1}{3}\left(\frac{x^4}{12}+\frac{2x^3}{9}+\frac{7x^2}{9}\right)+\frac{3e^{2x}}{20}$$

Hence, the general solution is

$$y = c_1 + c_2x + c_3e^x + c_4e^{-3x} - \frac{x^2}{9}\left(\frac{x^2}{4}+\frac{2x}{3}+\frac{7}{3}\right)+\frac{3e^{2x}}{20}$$

Example 6: Solve $(D^2+2)y = x^3+x^2+e^{-2x}+\cos 3x$.

Solution: Auxiliary equation is

$$m^2+2=0, \qquad m = \pm i\sqrt{2} \text{ (imaginary)}$$

$$\text{C.F.} = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$$

$$\text{P.I.} = \frac{1}{D^2+2}(x^3+x^2+e^{-2x}+\cos 3x)$$

$$= \frac{1}{2\left(1+\frac{D^2}{2}\right)}(x^3+x^2)+\frac{1}{D^2+2}e^{-2x}+\frac{1}{D^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1+\frac{D^2}{2}\right)^{-1}(x^3+x^2)+\frac{1}{4+2}e^{-2x}+\frac{1}{-3^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1-\frac{D^2}{2}+\frac{D^4}{4}-\ldots\right)(x^3+x^2)+\frac{e^{-2x}}{6}-\frac{\cos 3x}{7}$$

$$= \frac{1}{2}(x^3+x^2)-\frac{1}{4}D^2(x^3+x^2)+\frac{e^{-2x}}{6}-\frac{\cos 3x}{7}$$

$$= \frac{1}{2}(x^3+x^2)-\frac{1}{4}(6x+2)+\frac{e^{-2x}}{6}-\frac{\cos 3x}{7}$$

Hence, the general solution is

$$y = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{1}{2}(x^3+x^2-3x-1)+\frac{e^{-2x}}{6}-\frac{\cos 3x}{7}$$

Case IV: $Q = e^{ax}V$, where V is a function of x.

In this case, Eq. (2) reduces to $f(D)y = e^{ax}V$.

Let u be a function of x.

Then $\quad D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$

$$D^2(e^{ax}u) = D\left[e^{ax}(D+a)u\right] = ae^{ax}(D+a)u + e^{ax}(D^2+aD)u$$

$$= e^{ax}(D^2+2aD+a^2)u = e^{ax}(D+a)^2u$$

In general,

$$D^r(e^{ax}u) = e^{ax}(D+a)^r u$$

Let $D^r = f(D)$, $\qquad (D+a)^r = f(D+a)$

$$f(D)(e^{ax}u) = e^{ax} f(D+a)u$$

Operating both the sides with $\dfrac{1}{f(D)}$,

$$\frac{1}{f(D)}\left[f(D)(e^{ax}u)\right] = \frac{1}{f(D)}\left[e^{ax} f(D+a)u\right]$$

$$e^{ax}u = \frac{1}{f(D)}\left[e^{ax} f(D+a)u\right]$$

Putting $f(D+a)u = V$, $u = \dfrac{1}{f(D+a)}V$

$$e^{ax}\cdot\frac{1}{f(D+a)}V = \frac{1}{f(D)}(e^{ax}V)$$

Hence, $$\text{P.I.} = \frac{1}{f(D)}\cdot e^{ax}V = e^{ax}\cdot\frac{1}{f(D+a)}V$$

Example 1: Solve $(D^2 - 2D - 1)y = e^x \cos x$.

Solution: Auxiliary equation is

$$m^2 - 2m - 1 = 0$$

$$m = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2} \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}$$

$$\text{P.I.} = \frac{1}{D^2 - 2D - 1} e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) - 1}\cos x$$

$$= e^x \frac{1}{(D^2 - 2)}\cos x = e^x \frac{1}{-1^2 - 2}\cos x = -\frac{e^x \cos x}{3}$$

Hence, the general solution is

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x} - \frac{e^x \cos x}{3}$$

Example 2: Solve $(D^3 + 3D^2 - 4D - 12)y = 12xe^{-2x}$.

Solution: Auxiliary equation is

$$m^3 + 3m^2 - 4m - 12 = 0$$

$$m^2(m+3) - 4(m+3) = 0$$

$$(m+3)(m^2 - 4) = 0$$

$$m = -3, -2, 2 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{2x}$$

$$\text{P.I.} = \frac{1}{(\mathrm{D}+3)(\mathrm{D}+2)(\mathrm{D}-2)} 12xe^{-2x}$$

$$= 12e^{-2x} \frac{1}{(\mathrm{D}-2+3)(\mathrm{D}-2+2)(\mathrm{D}-2-2)} x$$

$$= 12e^{-2x} \frac{1}{(\mathrm{D}+1)\mathrm{D}(\mathrm{D}-4)} x = 12e^{-2x} \frac{1}{\mathrm{D}(\mathrm{D}^2-3\mathrm{D}-4)} x$$

$$= 12e^{-2x} \frac{1}{-4\mathrm{D}\left(1+\dfrac{3\mathrm{D}-\mathrm{D}^2}{4}\right)} x$$

$$= -3e^{-2x} \frac{1}{\mathrm{D}} \left(1+\frac{3\mathrm{D}-\mathrm{D}^2}{4}\right)^{-1} x$$

$$= -3e^{-2x} \frac{1}{\mathrm{D}} \left(1-\frac{3\mathrm{D}-\mathrm{D}^2}{4} + \text{Higher powers of D}\right) x$$

$$= -3e^{-2x} \frac{1}{\mathrm{D}} \left[x - \frac{3}{4}\mathrm{D}(x)\right] = -3e^{-2x} \frac{1}{\mathrm{D}} \left(x - \frac{3}{4}\right)$$

$$= -3e^{-2x} \int \left(x - \frac{3}{4}\right) \mathrm{d}x = -3e^{-2x} \left(\frac{x^2}{2} - \frac{3}{4}x\right)$$

Hence, the general solution is

$$y = c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{2x} - 3e^{-2x} \left(\frac{x^2}{2} - \frac{3}{4}x\right)$$

Case V: $\mathrm{Q} = x\mathrm{V}$, where V is a function of x.
In this case Eq. (2) reduces to $f(\mathrm{D})y = x\mathrm{V}$.
Let u be a function of x.

Then $\quad \mathrm{D}(xu) = x\mathrm{D}u + u$

$$\mathrm{D}^2(xu) = \mathrm{D}(x\mathrm{D}u + u) = x\mathrm{D}^2u + \mathrm{D}u + \mathrm{D}u = x\mathrm{D}^2u + 2\mathrm{D}u$$

$$\mathrm{D}^3(xu) = \mathrm{D}(x\mathrm{D}^2u + 2\mathrm{D}u) = x\mathrm{D}^3u + \mathrm{D}^2u + 2\mathrm{D}^2u = x\mathrm{D}^3u + 3\mathrm{D}^2u$$

In general,

$$\mathrm{D}^r(xu) = x\mathrm{D}^r u + r\mathrm{D}^{r-1}u = x\mathrm{D}^r u + \left[\frac{\mathrm{d}}{\mathrm{dD}}(\mathrm{D}^r)\right] u$$

Let $\quad \mathrm{D}^r = f(\mathrm{D})$

$$f(\mathrm{D})(xu) = x\,f(\mathrm{D})u + \left[\frac{\mathrm{d}}{\mathrm{dD}} f(\mathrm{D})\right] u$$

$$= xf(\mathrm{D})u + f'(\mathrm{D})u$$

Putting $f(\text{D})u = \text{V}$, $u = \dfrac{1}{f(\text{D})}\text{V}$ in above equation,

$$f(\text{D})\left[x\frac{1}{f(\text{D})}\text{V}\right] = x\text{V} + f'(\text{D})\left[\frac{1}{f(\text{D})}\text{V}\right]$$

$$x\text{V} = f(\text{D})\left[x\frac{1}{f(\text{D})}\text{V}\right] - f'(\text{D})\left[\frac{1}{f(\text{D})}\text{V}\right]$$

Operating both the sides with $\dfrac{1}{f(\text{D})}$,

$$\frac{1}{f(\text{D})}x\text{V} = \frac{1}{f(\text{D})}\left[f(\text{D})\left(x\frac{1}{f(\text{D})}\text{V}\right)\right] - \frac{1}{f(\text{D})}\left[f'(\text{D})\left(\frac{1}{f(\text{D})}\right)\text{V}\right]$$

$$= x\frac{1}{f(\text{D})}\text{V} - \frac{f'(\text{D})}{[f(\text{D})]^2}\text{V}$$

Hence, $$\text{P.I.} = \frac{1}{f(\text{D})}x\text{V} = x\frac{1}{f(\text{D})}\text{V} - \frac{f'(\text{D})}{[f(\text{D})]^2}\text{V}$$

Example 1: Solve $(\mathbf{D}^2 - 5\mathbf{D} + 6)y = x\cos 2x$.

Solution: Auxiliary equation is

$$m^2 - 5m + 6 = 0, \quad (m-2)(m-3) = 0$$

$$m = 2,\ 3 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1e^{2x} + c_2e^{3x}$$

$$\text{P.I.} = \frac{1}{\text{D}^2 - 5\text{D} + 6}x\cos 2x$$

$$= x\frac{1}{\text{D}^2 - 5\text{D} + 6}\cos 2x - \frac{2\text{D} - 5}{(\text{D}^2 - 5\text{D} + 6)^2}\cos 2x$$

$$= x\frac{1}{-4 - 5\text{D} + 6}\cos 2x - \frac{2\text{D} - 5}{(-4 - 5\text{D} + 6)^2}\cos 2x$$

$$= x\frac{1}{(2 - 5\text{D})}\cdot\frac{(2 + 5\text{D})}{(2 + 5\text{D})}\cos 2x - \frac{2\text{D} - 5}{(4 - 20\text{D} + 25\text{D}^2)}\cos 2x$$

$$= x\frac{(2 + 5\text{D})}{4 - 25\text{D}^2}\cos 2x - \frac{2\text{D} - 5}{(4 - 20\text{D} - 100)}\cos 2x$$

$$= x\frac{(2 + 5\text{D})}{4 + 100}\cos 2x + \frac{2\text{D} - 5}{4(5\text{D} + 24)}\cos 2x$$

$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{2\text{D} - 5}{4(5\text{D} + 24)}\cdot\frac{(5\text{D} - 24)}{(5\text{D} - 24)}\cos 2x$$

$$= \frac{x}{104}(2\cos 2x - 10\sin 2x) + \frac{(10D^2 - 73D + 120)}{4(25D^2 - 576)}\cos 2x$$

$$= \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{(10D^2 - 73D + 120)}{4(-100 - 576)} \cdot \cos 2x$$

$$= \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{1}{2704}(-40\cos 2x + 146\sin 2x + 120\cos 2x)$$

Hence, the general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{1}{1352}(40\cos 2x + 73\sin 2x)$$

Example 2: **Solve $(D^2 + 3D + 2)y = xe^x \sin x$.**

Solution: Auxiliary equation is

$$m^2 + 3m + 2 = 0, \quad (m+1)(m+2) = 0$$

$$m = -1, \ -2 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{(D+1)(D+2)} xe^x \sin x = e^x \frac{1}{(D+1+1)(D+1+2)} x\sin x$$

$$= e^x \frac{1}{(D+2)(D+3)} x\sin x = e^x \frac{1}{D^2 + 5D + 6} x\sin x$$

$$= e^x \left[x\frac{1}{D^2 + 5D + 6}\sin x - \frac{2D+5}{(D^2 + 5D + 6)^2}\sin x \right]$$

$$= e^x \left[x\frac{1}{-1 + 5D + 6}\sin x - \frac{2D+5}{(-1 + 5D + 6)^2}\sin x \right]$$

$$= e^x \left[x\frac{1}{5(D+1)} \cdot \frac{(D-1)}{(D-1)}\sin x - \frac{2D+5}{25(D^2 + 2D + 1)}\sin x \right]$$

$$= e^x \left[\frac{x}{5} \cdot \frac{(D-1)}{(D^2 - 1)}\sin x - \frac{2D+5}{25(-1 + 2D + 1)}\sin x \right] = e^x \left[\frac{x}{5} \cdot \frac{(D-1)}{(-1-1)}\sin x - \frac{2D+5}{25(2D)}\sin x \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(1 + \frac{5}{2D}\right)\sin x \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(\sin x + \frac{5}{2}\int \sin x \, dx\right) \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\left(\sin x - \frac{5}{2}\cos x\right) \right]$$

Hence, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{e^x}{5}\left[\frac{x}{2}(\cos x - \sin x) + \frac{1}{5}\left(\sin x - \frac{5}{2}\cos x\right)\right]$$

Example 3: Solve $(4D^2 + 8D + 3)y = xe^{-\frac{x}{2}}\cos x$.

Solution: Auxiliary equation is

$$4m^2 + 8m + 3 = 0,\ (2m+1)(2m+3) = 0$$

$$m = -\frac{1}{2},\ -\frac{3}{2} \qquad \text{(real and distinct)}$$

$$\text{C.F.} = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{3x}{2}}$$

$$\text{P.I.} = \frac{1}{4D^2 + 8D + 3} xe^{-\frac{x}{2}}\cos x = e^{-\frac{x}{2}} \frac{1}{4\left(D - \frac{1}{2}\right)^2 + 8\left(D - \frac{1}{2}\right) + 3} x\cos x$$

$$= e^{-\frac{x}{2}} \frac{1}{4\left(D^2 + \frac{1}{4} - D\right) + 8D - 4 + 3} x\cos x = e^{-\frac{x}{2}} \frac{1}{(4D^2 + 4D)} x\cos x$$

$$= \frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left(\frac{1}{D+1}x\cos x\right) = \frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left[x\cdot\frac{1}{D+1}\cos x - \frac{1}{(D+1)^2}\cos x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left(x\cdot\frac{D-1}{D^2-1}\cos x - \frac{1}{D^2+2D+1}\cos x\right)$$

$$= \frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left[x\frac{(D-1)}{(-1-1)}\cos x - \frac{1}{-1+2D+1}\cos x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left[-\frac{x}{2}(D\cos x - \cos x) - \frac{1}{2}\int \cos x\, dx\right]$$

$$= \frac{e^{-\frac{x}{2}}}{4}\cdot\frac{1}{D}\left[-\frac{x}{2}(-\sin x - \cos x) - \frac{1}{2}\sin x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}\left[\int x(\sin x + \cos x)dx + \int \sin x\ dx\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}\left[x(-\cos x + \sin x) - (-\sin x - \cos x) - \cos x\right]$$

$$= \frac{e^{-\frac{x}{2}}}{8}\left[x(\sin x - \cos x) + \sin x\right]$$

Hence, the general solution is

$$y = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{3x}{2}} + \frac{e^{-\frac{x}{2}}}{8}[x(\sin x - \cos x) + \sin x]$$

Exercise 10.9

Solve the following differential equations:

1. $(D^2 + D + 2)y = e^{\frac{x}{2}}$.

$$\left[\textbf{Ans. : } y = e^{-\frac{x}{2}}\left[c_1 \cos\left(\frac{\sqrt{7}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{7}x}{2}\right)\right] + \frac{4}{11}e^{\frac{x}{2}}\right]$$

2. $(D^2 - 4)y = (1 + e^x)^2$.

$$\left[\textbf{Ans. : } y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}\right]$$

3. $(D^2 + D + 1)y = e^{3x} + 6e^x - 3e^{-2x} + 5$.

$$\left[\textbf{Ans. : } y = e^{-\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{3}x}{2} + c_2 \sin\frac{\sqrt{3}x}{2}\right) + \frac{e^{3x}}{13} + 2e^x - e^{-2x} + 5\right]$$

4. $(D^2 + 4D + 5)y = -2\cosh x + 2^x$.

$$\left[\textbf{Ans. : } y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2} + \frac{2^x}{(\log 2)^2 + 4(\log 2) + 5}\right]$$

5. $(D^3 + D^2 + D + 1)y = \sin 2x$.

$$\left[\textbf{Ans. : } y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{15}(2\cos 2x - \sin 2x)\right]$$

6. $(3D^2 - 7D + 2)y = \sin x + \cos x$.

$$\left[\textbf{Ans. : } y = c_1 e^{2x} + c_2 e^{\frac{x}{3}} + \frac{1}{25}(3\cos x - 4\sin x)\right]$$

7. $(D^3 - 2D^2 + 4D)y = e^{2x} + \sin 2x$.

$$\left[\textbf{Ans. : } y = c_1 + e^x\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right) + \frac{1}{8}(e^{2x} + \sin 2x)\right]$$

8. $(D^3 + 2D^2 + D)y = \sin^2 x$.

$$\left[\textbf{Ans. : } y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{x}{2} + \frac{1}{100}(3\sin 2x + 4\cos 2x)\right]$$

9. $(D^2 + D - 6)y = e^{2x}$.

$$\left[\textbf{Ans. : } y = c_1 e^{2x} + c_2 e^{-3x} + \frac{xe^{2x}}{5}\right]$$

10. $(9D^2 + 6D + 1)y = e^{-\frac{x}{3}}$.

$$\left[\textbf{Ans. : } y = (c_1 + c_2 x)e^{-\frac{x}{3}} + \frac{x^2}{18}e^{-\frac{x}{3}}\right]$$

11. $(D^2 + 4)y = e^x + \sin 2x$.

$$\left[\textbf{Ans. : } y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{5} - \frac{x}{4}\cos 2x\right]$$

12. $(D^2-4)y = x^2.$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}\left(x^2+\frac{1}{2}\right)\right]$$

13. $(D^2+D)y = x^2+2x+4.$

$$\left[\textbf{Ans.: } y = c_1 + c_2e^{-x} + \frac{x^3}{3} + 4x\right]$$

14. $(D^2+1)y = e^{2x} + \cosh 2x + x^3.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = c_1\cos x + c_2\sin x\\ &+\frac{e^{2x}}{5}+\frac{1}{5}\cosh 2x + x^3 - 6x\end{aligned}\right]$$

15. $(D-1)^2(D+1)^2 y = \sin^2\frac{x}{2} + e^x + x.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = (c_1+c_2x)e^x + (c_3+c_4x)e^{-x}\\ &\qquad +\frac{1}{2}-\frac{1}{8}\cos x + \frac{x^2}{8}e^x + x\end{aligned}\right]$$

16. $(D^2-3D+2)y = 2e^x\cos\frac{x}{2}.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = c_1e^x + c_2e^{2x} - \frac{8}{5}e^x\\ &\qquad\left(2\sin\frac{x}{2}+\cos\frac{x}{2}\right)\end{aligned}\right]$$

17. $(D^2-2D+10)y = 16e^x\cos 3x + 24e^x\sin 3x.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = e^x(c_1\cos 3x + c_2\sin 3x)\\ &\qquad+\frac{xe^x}{3}(8\sin 3x - 12\cos 3x)\end{aligned}\right]$$

18. $(D^3-4D^2+9D-10)y = 24e^x\sin 2x.$

$$\left[\begin{aligned}&\textbf{Ans.:}\\ &y = c_1e^{2x} + e^x(c_2\cos 2x + c_3\sin 2x)\\ &\qquad-\frac{6xe^x}{5}(2\sin 2x - \cos 2x)\end{aligned}\right]$$

19. $(4D^3-12D^2+13D-10)y = 16e^{\frac{x}{2}}\cos x.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = c_1e^{2x} + e^{\frac{x}{2}}(c_2\cos x + c_3\sin x)\\ &\qquad-\frac{4xe^{\frac{x}{2}}}{13}(2\cos x + 3\sin x)\end{aligned}\right]$$

20. $(D^2+4D+8)y = 12e^{-2x}\sin x\sin 3x.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = e^{-2x}(c_1\cos 2x + c_2\sin 2x)\\ &\qquad+\frac{1}{2}e^{-2x}(3x\sin 2x + \cos 4x)\end{aligned}\right]$$

21. $(4D^2+9D+2)y = xe^{-2x}.$

$$\left[\begin{aligned}&\textbf{Ans.:}\\ &y = c_1e^{-2x} + c_2e^{-\frac{x}{4}} - \frac{1}{98}(7x^2+8x)e^{-2x}\end{aligned}\right]$$

22. $(D^2+4)y = x\sin x.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = c_1\cos 2x + c_2\sin 2x\\ &\qquad+\frac{1}{9}(3x\sin x - 2\cos x)\end{aligned}\right]$$

23. $(D^2+9)y = xe^{2x}\cos x.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = c_1\cos 3x + c_2\sin 3x\\ &+\frac{e^{2x}}{400}[(30x-11)\cos x + (10x-2)\sin x]\end{aligned}\right]$$

24. $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = (c_1+c_2x)e^{2x} - e^{2x}[4x\cos 2x\\ &\qquad+(2x^2-3)\sin 2x]\end{aligned}\right]$$

25. $(D^2-1)y = x\sin x + (1+x^2)e^x.$

$$\left[\begin{aligned}&\textbf{Ans.: } y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x\\ &\qquad+\cos x) + \frac{1}{12}xe^x(2x^2-3x+9)\end{aligned}\right]$$

10.5.3 General Method of Obtaining Particular Integral (P.I.)

In a linear differential equation

$$f(\mathrm{D})y = \mathrm{Q}(x)$$

if $\mathrm{Q}(x)$ is not in any of the standard forms discussed in the previous section, then particular integral is obtained using the general method described below.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(\mathrm{D})}\mathrm{Q}(x) \\
&= \frac{1}{(\mathrm{D}-m_1)(\mathrm{D}-m_2)\ldots(\mathrm{D}-m_n)}\mathrm{Q}(x) \\
&= \left(\frac{\mathrm{A}_1}{\mathrm{D}-m_1}+\frac{\mathrm{A}_2}{\mathrm{D}-m_2}+\ldots+\frac{\mathrm{A}_n}{\mathrm{D}-m_n}\right)\mathrm{Q}(x) \qquad \text{[Using Partial Fraction]} \\
&= \mathrm{A}_1\cdot\frac{1}{\mathrm{D}-m_1}\mathrm{Q}(x)+\mathrm{A}_2\cdot\frac{1}{\mathrm{D}-m_2}\mathrm{Q}(x)+\ldots+\mathrm{A}_n\cdot\frac{1}{\mathrm{D}-m_n}\mathrm{Q}(x) \\
&= \mathrm{A}_1e^{m_1x}\int \mathrm{Q}(x)\cdot e^{-m_1x}\mathrm{d}x+\mathrm{A}_2e^{m_2x}\int \mathrm{Q}(x)\cdot e^{-m_2x}\mathrm{d}x+\ldots+\mathrm{A}_ne^{m_nx}\int \mathrm{Q}(x)e^{-m_nx}\mathrm{d}x
\end{aligned}$$

This method can be applied for any form of $\mathrm{Q}(x)$. But sometimes integration of the terms become complicated and lengthy therefore direct (short-cut) methods are preferred to find the P.I. and general method is used only if direct method can not be applied.

Example 1: Solve $(\mathrm{D}^2+3\mathrm{D}+2)y = e^{e^x}$.

Solution: Auxiliary equation is

$$m^2+3m+2=0, \quad (m+1)(m+2)=0,$$

$$m = -1, -2 \text{ (real and distinict)}$$

$$\text{C.F.} = c_1e^{-x}+c_2e^{-2x}$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{\mathrm{D}^2+3\mathrm{D}+2}e^{e^x} = \frac{1}{(\mathrm{D}+2)(\mathrm{D}+1)}e^{e^x} \\
&= \frac{1}{(\mathrm{D}+2)}\left(e^{-x}\int e^{e^x}e^x\mathrm{d}x\right) = \frac{1}{\mathrm{D}+2}\left(e^{-x}e^{e^x}\right) \qquad \left[\because \int e^{f(x)}f'(x)\,\mathrm{d}x = e^{f(x)}\right] \\
&= e^{-2x}\int e^{-x}e^{e^x}e^{2x}\mathrm{d}x \\
&= e^{-2x}\int e^{e^x}e^x\mathrm{d}x = e^{-2x}e^{e^x}
\end{aligned}$$

Hence, the general solution is

$$y = c_1e^{-x}+c_2e^{-2x}+e^{-2x}e^{e^x}$$

Example 2: Solve $(D^2-1)y=(1+e^{-x})^{-2}$.

Solution: Auxiliary equation is

$m^2-1=0,\ m=\pm 1$ (real and distinct)

$$\text{C.F.}=c_1e^{x}+c_2e^{-x}$$

$$\text{P.I.}=\frac{1}{D^2-1}(1+e^{-x})^{-2}=\frac{1}{(D-1)}\cdot\frac{1}{D+1}\frac{1}{(1+e^{-x})^2}=\frac{1}{2}\left(\frac{1}{D-1}-\frac{1}{D+1}\right)\frac{1}{(1+e^{-x})^2}$$

$$\frac{1}{D+1}\cdot\frac{1}{(1+e^{-x})^2}=e^{-x}\int\frac{1}{(1+e^{-x})^2}\cdot e^{x}dx=e^{-x}\int\frac{e^{2x}}{(e^{x}+1)^2}\,e^{x}dx$$

Let $1+e^{x}=t,\ e^{x}dx=dt$

$$\frac{1}{(D+1)}\cdot\frac{1}{(1+e^{-x})^2}=e^{-x}\int\frac{(t-1)^2}{t^2}dt=e^{-x}\int\left(1-\frac{2}{t}+\frac{1}{t^2}\right)dt$$

$$=e^{-x}\left(t-2\log t-\frac{1}{t}\right)$$

$$=e^{-x}\left[1+e^{x}-2\log(1+e^{x})-\frac{1}{1+e^{x}}\right]$$

$$=e^{-x}+1-2e^{-x}\log(1+e^{x})-\frac{e^{-x}}{1+e^{x}}$$

$$\frac{1}{(D-1)}\cdot\frac{1}{(1+e^{-x})^2}=e^{x}\int\frac{1}{(1+e^{-x})^2}\cdot e^{-x}dx$$

Let $1+e^{-x}=t,\ -e^{-x}dx=dt$

$$\frac{1}{(D-1)}\cdot\frac{1}{(1+e^{-x})^2}=e^{x}\int\frac{1}{t^2}(-dt)=e^{x}\left(\frac{1}{t}\right)$$

$$=\frac{e^{x}}{1+e^{-x}}$$

$$\text{P.I.}=\frac{1}{2}\left[\frac{e^{x}}{1+e^{-x}}-e^{-x}-1+2e^{-x}\log(1+e^{x})+\frac{e^{-x}}{1+e^{x}}\right]$$

Hence, the general solution is

$$y=c_1e^{x}+c_2e^{-x}+\frac{1}{2}\left[\frac{e^{x}}{1+e^{-x}}-e^{-x}-1+2e^{-x}\log(1+e^{x})+\frac{e^{-x}}{1+e^{x}}\right]$$

Example 3: Solve $(D^2+a^2)y=\sec ax$.

Solution: Auxiliary equation is

$$m^2+a^2=0,\qquad m=\pm ia \text{ (complex)}$$

$$\text{C.F.}=c_1\cos ax+c_2\sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax$$
$$= \frac{1}{(D - ia)(D + ia)} \sec ax$$
$$= \frac{1}{2ia}\left(\frac{1}{D - ia} - \frac{1}{D + ia}\right)\sec ax$$
$$\frac{1}{D - ia} \sec ax = e^{iax} \int \sec ax \cdot e^{-iax} \mathrm{d}x$$
$$= e^{iax} \int \frac{2}{e^{iax} + e^{-iax}}\, e^{-iax} \mathrm{d}x$$
$$= e^{iax} \int \frac{2}{1 + e^{-2iax}}\, e^{-2iax} \mathrm{d}x$$

Let $1 + e^{-2iax} = t, -2iae^{-2iax}\mathrm{d}x = \mathrm{d}t$

$$\frac{1}{D - ia} \sec ax = e^{iax} \int \frac{2}{t}\left(-\frac{\mathrm{d}t}{2ia}\right) = -\frac{e^{iax}}{ia} \log t$$
$$= -\frac{e^{iax}}{ia} \log(1 + e^{-2iax})$$
$$= -\frac{e^{iax}}{ia} \log(1 + \cos 2ax - i \sin 2ax)$$
$$= -\frac{e^{iax}}{ia} \log(2\cos^2 ax - 2i \sin ax \cos ax)$$
$$= -\frac{e^{iax}}{ia} \log(2\cos ax)(\cos ax - i \sin ax)$$
$$= -\frac{e^{iax}}{ia} [\log(2\cos ax) + \log e^{-iax}]$$
$$\frac{1}{D - ia} \sec ax = -\frac{e^{iax}}{ia} [\log(2\cos ax) - iax] \quad \ldots (1)$$

Replacing i by $-i$ in Eq. (1),

$$\frac{1}{D + ia} \sec ax = \frac{e^{-iax}}{ia} [\log(2\cos ax) + iax]$$
$$\text{P.I.} = \frac{1}{2ia}\left[-\frac{e^{iax}}{ia}\{\log(2\cos ax) - iax\} - \frac{e^{-iax}}{ia}\{\log(2\cos ax) + iax\}\right]$$
$$= \frac{1}{2ia}\left[-\frac{\log(2\cos ax)}{ia}(e^{iax} + e^{-iax}) + x(e^{iax} - e^{-iax})\right]$$
$$= \frac{\log(2\cos ax)}{2a^2}(2\cos ax) + \frac{x}{2ia}(2i \sin ax)$$
$$= \frac{1}{a^2}[\log(2\cos ax)]\cos ax + \frac{x \sin ax}{a}$$

Hence, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2}[\log(2\cos ax)]\cos ax + \frac{x \sin ax}{a}$$

Example 4: Solve $(D^2 + 5D + 6)y = e^{-2x} \sec^2 x(1 + 2\tan x)$.

Solution: Auxiliary equation is

$$m^2 + 5m + 6 = 0, \quad (m+2)(m+3) = 0$$

$$m = -2, -3 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 5D + 6} e^{-2x} \sec^2 x(1 + 2\tan x)$$

$$= \frac{1}{(D+2)(D+3)} e^{-2x} \sec^2 x(1 + 2\tan x)$$

$$= \left(\frac{1}{D+2} - \frac{1}{D+3}\right) e^{-2x} \sec^2 x(1 + 2\tan x)$$

$$\frac{1}{D+2} e^{-2x} \sec^2 x(1 + 2\tan x) = e^{-2x} \int e^{-2x} \sec^2 x(1 + 2\tan x) \cdot e^{2x} dx$$

$$= e^{-2x} \int \sec^2 x(1 + 2\tan x) dx = \frac{e^{-2x}}{2} \cdot \frac{(1 + 2\tan x)^2}{2}$$

$$\frac{1}{D+3} e^{-2x} \sec^2 x(1 + 2\tan x) = e^{-3x} \int e^{-2x} \sec^2 x(1 + 2\tan x) \cdot e^{3x} dx$$

$$= e^{-3x} \int e^x \sec^2 x(1 + 2\tan x) dx$$

$$= e^{-3x} \left(\int e^x \sec^2 x \, dx + \int e^x \sec^2 x \cdot 2\tan x \, dx\right)$$

$$= e^{-3x} \left(e^x \sec^2 x - \int e^x \cdot 2\sec x \cdot \sec x \tan x \, dx \right.$$

$$\left. + \int e^x \sec^2 x \cdot 2\tan x \, dx\right)$$

$$= e^{-3x} e^x \sec^2 x = e^{-2x} \sec^2 x$$

$$\text{P.I.} = \frac{e^{-2x}}{4}(1 + 2\tan x)^2 - e^{-2x}\sec^2 x$$

$$= \frac{e^{-2x}}{4}(1 + 4\tan^2 x + 4\tan x) - e^{-2x}(1 + \tan^2 x) = \frac{e^{-2x}}{4}(4\tan x - 3)$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^{-2x}}{4}(4\tan x - 3)$$

Exercise 10.10

Solve the following differential equations:

1. $(D^2+3D+2)y = \sin e^x$.

$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{-2x} - e^{-2x}\sin e^x\right]$

2. $(D^2+1)y = \operatorname{cosec} x$.

$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x - x\cos x + \sin x\log(\sin x)\right]$

3. $(D^2+4)y = \tan 2x$.

$\left[\textbf{Ans.: } y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x)\right]$

4. $(D^2+1)y = x - \cot x$.

$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x - x\cos^2 x + x\sin^2 x - \sin x\log(\operatorname{cosec} x - \cot x)\right]$

5. $(D^2+D)y = \dfrac{1}{1+e^x}$.

$\left[\textbf{Ans.: } y = c_1 + c_2e^{-x} - e^{-x}[e^x\log(e^{-x}+1) + \log(e^x+1)]\right]$

6. $(D^2-2D+2)y = e^x\tan x$.

$\left[\textbf{Ans.: } y = e^x(c_1\cos x + c_2\sin x) - e^x\cos x\log(\sec x + \tan x)\right]$

7. $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$.

$\left[\textbf{Ans.: } y = (c_1+c_2x)e^{2x} - e^{2x}(2x^2\sin 2x + 4x\cos 2x - 3\sin 2x)\right]$

8. $(D^2+2D+1)y = e^{-x}\log x$.

$\left[\textbf{Ans.: } y = (c_1+c_2x)e^{-x} + \frac{x^2}{2}e^{-x}\left(\log x - \frac{3}{2}\right)\right]$

10.6 HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

In this section we will discuss two types of differential equations with variable coefficients. These differential equations have variable coefficients and can be solved by reducing to linear differential equation with constant coefficients form.

10.6.1 Cauchy's Linear Equation

An equation of the form

$$a_0x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}x\frac{dy}{dx} + a_ny = Q(x) \qquad \text{... (1)}$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called Cauchy's linear equation.

To solve Eq. (1),

Let $x = e^z$, $1 = e^z\dfrac{dz}{dx}$, $\dfrac{dz}{dx} = \dfrac{1}{e^z} = \dfrac{1}{x}$

Now, $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx} = \frac{dy}{dz}, \ x\mathrm{D}y = \mathcal{D}y, \quad \text{where } \mathcal{D} \equiv \frac{d}{dz} \text{ and } \mathrm{D} = \frac{d}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{1}{x}$$

$$x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \text{ or } x^2\mathrm{D}^2y = \mathcal{D}(\mathcal{D}-1)y$$

Similarly, $$x^3\mathrm{D}^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

....

....

$$x^n\mathrm{D}^ny = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (1),

$$[a_0\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1) + a_1\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2)$$
$$+\ldots + a_{n-1}\mathcal{D} + a_n]y = \mathrm{Q}(e^z)$$

which is a linear differential equation with constant coefficients and can be solved by usual methods described in previous section.

Example 1: Solve $(4x^2\mathrm{D}^2 + 16x\mathrm{D} + 9)y = 0$.

Solution: Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D}-1) + 16\mathcal{D} + 9]y = 0, \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2 + 12\mathcal{D} + 9)y = 0$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$4m^2 + 12m + 9 = 0$$

$$(2m+3)^2 = 0, \qquad m = -\frac{3}{2}, \ -\frac{3}{2} \text{ (real and repeated twice)}$$

$$\text{C.F.} = (c_1 + c_2z)e^{-\frac{3}{2}z} = (c_1 + c_2\log x)x^{-\frac{3}{2}}$$

Since $\mathrm{Q}(e^z) = 0$, P.I. = 0

Hence, the general solution is

$$y = (c_1 + c_2\log x)x^{-\frac{3}{2}}$$

Example 2: Solve $(4x^2D^2+1)y = 19\cos(\log x) + 22\sin(\log x)$.

Solution: Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D}-1)+1]y = 19\cos z + 22\sin z, \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(4\mathcal{D}^2 - 4\mathcal{D} + 1)y = 19\cos z + 22\sin z$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0, \qquad m = \frac{1}{2} \text{ (real and repeated twice)}$$

$$\text{C.F.} = (c_1 + c_2 z)e^{\frac{1}{2}z} = (c_1 + c_2\log x)x^{\frac{1}{2}}$$

$$\text{P.I.} = \frac{1}{4\mathcal{D}^2 - 4\mathcal{D} + 1}(19\cos z + 22\sin z)$$

$$= \frac{1}{-4-4\mathcal{D}+1}(19\cos z + 22\sin z) = \frac{1}{-(4\mathcal{D}+3)}\cdot\frac{(4\mathcal{D}-3)}{(4\mathcal{D}-3)}(19\cos z + 22\sin z)$$

$$= \frac{4\mathcal{D}-3}{-(16\mathcal{D}^2-9)}(19\cos z + 22\sin z) = \frac{4\mathcal{D}-3}{-(-16-9)}(19\cos z + 22\sin z)$$

$$= \frac{1}{25}\left[4(-19\sin z + 22\cos z) - 3(19\cos z + 22\sin z)\right]$$

$$= \frac{1}{25}(31\cos z - 142\sin z) = \frac{1}{25}\left[31\cos(\log x) - 142\sin(\log x)\right]$$

Hence, the general solution is

$$y = (c_1 + c_2\log x)x^{\frac{1}{2}} + \frac{1}{25}\left[31\cos(\log x) - 142\sin(\log x)\right]$$

Example 3: Solve $(x^3D^3 + x^2D^2 - 2)y = x + \frac{1}{x^3}$.

Solution: Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + \mathcal{D}(\mathcal{D}-1) - 2\right]y = e^z + e^{-3z} \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2)y = e^z + e^{-3z}$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^3 - 2m^2 + m - 2 = 0$$

$$(m-2)(m^2+1) = 0, \; m = 2, \; m = \pm i \text{ (imaginary)}$$

$$\text{C.F.} = c_1e^{2z} + c_2\cos z + c_3\sin z$$

$$= c_1x^2 + c_2\cos(\log x) + c_3\sin(\log x)$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^3 - 2\mathcal{D}^2 + \mathcal{D} - 2}(e^z + e^{-3z})$$

$$= \frac{1}{1-2+1-2}e^z + \frac{1}{(-3)^3 - 2(-3)^2 - 3 - 2}e^{-3z}$$

$$= -\frac{1}{2}e^z - \frac{1}{50}e^{-3z} = -\frac{1}{2}x - \frac{1}{50}(x)^{-3} = -\frac{1}{2}x - \frac{1}{50x^3}$$

Hence, the general solution is

$$y = c_1 x^2 + c_2 \cos(\log x) + c_3 \sin(\log x) - \frac{1}{2}x - \frac{1}{50x^3}$$

Example 4: Solve $\left(x\mathrm{D}^2 + \mathrm{D} - \frac{1}{x}\right)y = -ax^2$.

Solution: Multiplying the given equation by x,

$$(x^2\mathrm{D}^2 + x\mathrm{D} - 1)y = -ax^3$$

which is Cauchy's linear equation.
Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) + \mathcal{D} - 1\right]y = -ae^{3z} \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 1)y = -ae^{3z}$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 - 1 = 0, \qquad m = \pm 1 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^z + c_2 e^{-z} = c_1 x + c_2 x^{-1} = c_1 x + \frac{c_2}{x}$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2 - 1}(-ae^{3z}) = -a\frac{1}{8}e^{3Z} = -\frac{a}{8}x^3$$

Hence, the general solution is

$$y = c_1 x + \frac{c_2}{x} - \frac{a}{8}x^3$$

Example 5: Solve $\left(\mathrm{D} + \frac{1}{x}\right)^2 y = \frac{1}{x^4}$.

Solution:

$$\left(\mathrm{D} + \frac{1}{x}\right)^2 y = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)^2 y = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)y$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x^2}$$

$$= \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x}$$

Substituting in the given equation,

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}=\frac{1}{x^4}$$

$$\left(D^2+\frac{2}{x}D\right)y=\frac{1}{x^4}$$

Multiplying the equation by x^2,

$$(x^2D^2+2xD)y=\frac{1}{x^2}$$

which is Cauchy's equation.

Putting $x=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+2\mathcal{D}]y=\frac{1}{e^{2z}}$$

$$(\mathcal{D}^2+\mathcal{D})y=e^{-2z}$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2+m=0, \qquad m=0,-1 \text{ (real and distinct)}$$

$$\text{C.F.}=c_1+c_2e^{-z}=c_1+c_2x^{-1}=c_1+\frac{c_2}{x}$$

$$\text{P.I.}=\frac{1}{\mathcal{D}^2+\mathcal{D}}e^{-2z}=\frac{1}{4-2}e^{-2z}=\frac{1}{2}e^{-2z}=\frac{1}{2}(x)^{-2}=\frac{1}{2x^2}$$

Hence, the general solution is

$$y=c_1+\frac{c_2}{x}+\frac{1}{2x^2}$$

Example 6: Solve $(x^2D^2+xD-1)y=\dfrac{x^3}{1+x^2}$.

Solution: Putting $x=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+\mathcal{D}-1]y=\frac{e^{3z}}{1+e^{2z}}, \quad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$(\mathcal{D}^2-1)y=\frac{e^{3z}}{1+e^{2z}}$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2-1=0, \qquad m=\pm1 \text{ (real and distinct)}$$

$$\text{C.F.}=c_1e^z+c_2e^{-z}=c_1x+\frac{c_2}{x}$$

$$\text{P.I.} = \frac{1}{(\mathcal{D}^2-1)}\left(\frac{e^{3z}}{1+e^{2z}}\right) = \frac{1}{(\mathcal{D}+1)(\mathcal{D}-1)}\left(\frac{e^{3z}}{1+e^{2z}}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\mathcal{D}-1} - \frac{1}{\mathcal{D}+1}\right)\left(\frac{e^{3z}}{1+e^{2z}}\right)$$

$$= \frac{1}{2}\left[\frac{1}{\mathcal{D}-1}\left(\frac{e^{3z}}{1+e^{2z}}\right) - \frac{1}{\mathcal{D}+1}\left(\frac{e^{3z}}{1+e^{2z}}\right)\right]$$

$$= \frac{1}{2}\left[e^z\int\frac{e^{3z}}{1+e^{2z}}e^{-z}\mathrm{d}z - e^{-z}\int\frac{e^{3z}}{1+e^{2z}}\cdot e^z\mathrm{d}z\right]$$

$$= \frac{1}{2}\left[e^z\int\frac{e^{2z}}{1+e^{2z}}\mathrm{d}z - e^{-z}\int\frac{e^{4z}}{1+e^{2z}}\mathrm{d}z\right]$$

Putting $1+e^{2z} = t,\ 2e^{2z}\mathrm{d}z = \mathrm{d}t$

$$\text{P.I.} = \frac{1}{2}\left[e^z\int\frac{1}{t}\cdot\frac{\mathrm{d}t}{2} - e^{-z}\int\left(\frac{t-1}{t}\right)\frac{\mathrm{d}t}{2}\right]$$

$$= \frac{1}{2}\left[\frac{e^z}{2}\log t - \frac{e^{-z}}{2}(t-\log t)\right]$$

$$= \frac{1}{4}\left[e^z\log(1+e^{2z}) - e^{-z}\{1+e^{2z} - \log(1+e^{2z})\}\right]$$

$$= \frac{1}{4}\left[x\log(1+x^2) - (x)^{-1}\{1+x^2 - \log(1+x^2)\}\right]$$

$$= \frac{x}{4}\log(1+x^2) - \frac{1}{4x} - \frac{x}{4} + \frac{1}{4x}\log(1+x^2)$$

Hence, the general solution is

$$y = c_1x + \frac{c_2}{x} + \frac{x}{4}\log(1+x^2) - \frac{1}{4x} - \frac{x}{4} + \frac{1}{4x}\log(1+x^2)$$

$$= c_1'x + \frac{c_2'}{x} + \frac{x}{4}\log(1+x^2) + \frac{1}{4x}\log(1+x^2)$$

where $\quad c_1' = c_1 - \frac{1}{4}, \qquad c_2' = c_2 - \frac{1}{4}$

Example 7: Solve $(x^2D^2 - 4xD + 6)y = -x^4\sin x$.

Solution: Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) - 4\mathcal{D} + 6\right]y = -e^{4z}\sin e^z \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 5\mathcal{D} + 6)y = -e^{4z}\sin e^z$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0, \qquad m = 2, 3 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{3z} = c_1 x^2 + c_2 x^3$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2 - 5\mathcal{D} + 6}(-e^{4z} \sin e^z) = \frac{1}{(\mathcal{D}-2)(\mathcal{D}-3)}(-e^{4z} \sin e^z)$$

$$= \left(\frac{1}{\mathcal{D}-3} - \frac{1}{\mathcal{D}-2}\right)(-e^{4z} \sin e^z)$$

$$= \frac{1}{\mathcal{D}-2}(e^{4z} \sin e^z) - \frac{1}{\mathcal{D}-3}(e^{4z} \sin e^z)$$

$$= e^{2z} \int e^{4z} \sin e^z \cdot e^{-2z} \mathrm{d}z - e^{3z} \int e^{4z} \sin e^z \cdot e^{-3z} \mathrm{d}z$$

$$= e^{2z} \int \sin e^z \cdot e^{2z} \mathrm{d}z - e^{3z} \int \sin e^z \cdot e^z \mathrm{d}z$$

Putting $e^z = t,\ e^z \mathrm{d}z = \mathrm{d}t$

$$\text{P.I.} = e^{2z} \int \sin t \cdot t \mathrm{d}t - e^{3z} \int \sin t \, \mathrm{d}t$$

$$= e^{2z}(-t \cos t + \sin t) - e^{3z}(-\cos t)$$

$$= e^{2z}(-e^z \cos e^z + \sin e^z) + e^{3z} \cos e^z$$

$$= e^{2z} \sin e^z$$

$$= x^2 \sin x$$

Hence, the general solution is

$$y = c_1 x^2 + c_2 x^3 + x^2 \sin x$$

Example 8: Solve $(x^2\mathrm{D}^2 - x\mathrm{D} + 2)y = 6,\ y(1) = 1,\ y'(1) = 2.$

Solution: Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 2\right] y = 6, \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 2) y = 6$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 - 2m + 2 = 0, \quad m = 1 \pm i \text{ (imaginary)}$$

$$\text{C.F.} = e^z (c_1 \cos z + c_2 \sin z)$$

$$= x[c_1 \cos(\log x) + c_2 \sin(\log x)]$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 2} 6e^{0z} = \frac{1}{2} \cdot 6 = 3$$

Hence, the general solution is

$$y = x\left[c_1 \cos(\log x) + c_2 \sin(\log x)\right] + 3 \qquad \text{... (1)}$$

$$y' = \left[c_1 \cos(\log x) + c_2 \sin(\log x)\right] + x\left[-c_1 \sin(\log x)\cdot\frac{1}{x} + c_2 \cos(\log x)\cdot\frac{1}{x}\right]$$

$$y' = (c_1 + c_2)\cos(\log x) + (c_2 - c_1)\sin(\log x) \qquad \text{... (2)}$$

Given $y(1) = 1,\ y'(1) = 2$

Putting $x = 1, y = 1$ and $y' = 2$ in Eqs. (1) and (2),

$$1 = c_1 \cos(0) + c_2 \sin(0) + 3 = c_1 + 3$$

$$c_1 = -2$$

and

$$2 = (c_1 + c_2)\cos(0) + (c_2 - c_1)\sin 0$$

$$2 = c_1 + c_2$$

$$c_2 = 4$$

Hence, the general solution is

$$y = -2x\cos(\log x) + 4x\sin(\log x) + 3$$

Example 9: Solve $(4x^2D^2 + 1)y = \log x,\ x > 0,\ y(1) = 0,\ y(e) = 5.$

Solution: Putting $x = e^z$,

$$[4\mathcal{D}(\mathcal{D} - 1) + 1]y = z, \qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(4\mathcal{D}^2 - 4\mathcal{D} + 1)y = z$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$$(2m - 1)^2 = 0, \qquad m = \frac{1}{2}, \frac{1}{2} \qquad \text{(real and repeated)}$$

$$\text{C.F.} = (c_1 + c_2 z)e^{\frac{1}{2}z} = (c_1 + c_2 \log x)x^{\frac{1}{2}}$$

$$\text{P.I.} = \frac{1}{4\mathcal{D}^2 - 4\mathcal{D} + 1}z = \frac{1}{(2\mathcal{D} - 1)^2}z = \frac{1}{(1 - 2\mathcal{D})^2}z = (1 - 2\mathcal{D})^{-2}z$$

$$= (1 + 4\mathcal{D} + 6\mathcal{D}^2 + \ldots)z = z + 4\mathcal{D}z + 6\mathcal{D}^2 z + \ldots$$

$$= z + 4 + 0 = \log x + 4$$

Hence, the general solution is

$$y = (c_1 + c_2 \log x)\sqrt{x} + \log x + 4 \qquad \text{... (1)}$$

Given $y\ (1) = 0,\ y\ (e) = 5$

Putting $x = 1, y = 0$ and then $x = e, y = 5$ in Eq. (1),

$$0 = c_1 + c_2 \log 1 + \log 1 + 4 = c_1 + 4$$

$$c_1 = -4$$

and
$$5 = (c_1 + c_2 \log e)\sqrt{e} + \log e + 4 = \sqrt{e}(-4 + c_2) + 1 + 4$$
$$c_2 = 4$$

Hence, the general solution is
$$y = (-4 + 4\log x)\sqrt{x} + \log x + 4$$

Example 10: Solve $(x^2D^2 + 5xD + 3)y = \dfrac{\log x}{x^2}$.

Solution: Putting $x = e^z$,
$$[\mathcal{D}(\mathcal{D}-1) + 5\mathcal{D} + 3]\,y = \frac{z}{e^{2z}}, \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$
$$(\mathcal{D}^2 + 4\mathcal{D} + 3)y = e^{-2z}z$$

which is a linear equation with constant coefficients.

Auxiliary equation is
$$m^2 + 4m + 3 = 0$$
$$(m+1)(m+3) = 0, \qquad m = -1, -3 \text{ (real and distinct)}$$
$$\text{C.F.} = c_1e^{-z} + c_2e^{-3z} = c_1(x)^{-1} + c_2(x)^{-3} = \frac{c_1}{x} + \frac{c_2}{x^3}$$
$$\text{P.I.} = \frac{1}{\mathcal{D}^2 + 4\mathcal{D} + 3}e^{-2z}z = e^{-2z}\frac{1}{(\mathcal{D}-2)^2 + 4(\mathcal{D}-2) + 3}z$$
$$= e^{-2z}\frac{1}{\mathcal{D}^2 - 1}z = -e^{-2z}(1 - \mathcal{D}^2)^{-1}z = -e^{-2z}(1 + \mathcal{D}^2 + \mathcal{D}^4 + \ldots)z$$
$$= -e^{-2z}(z + \mathcal{D}^2z + \mathcal{D}^4z + \ldots) = -e^{-2z}(z + 0) = -(x)^{-2}(\log x) = -\frac{\log x}{x^2}$$

Hence, the general solution is
$$y = \frac{c_1}{x} + \frac{c_2}{x^3} - \frac{\log x}{x^2}$$

Example 11: Solve $(x^2D^2 + xD + 1)y = \log x \sin(\log x)$.

Solution: Putting $x = e^z$,
$$[\mathcal{D}(\mathcal{D}-1) + \mathcal{D} + 1]y = z\sin z, \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$
$$(\mathcal{D}^2 + 1)y = z\sin z$$

which is a linear equation with constant coefficients.

Auxiliary equation is
$$m^2 + 1 = 0, \qquad m = \pm i \quad \text{(imaginary)}$$
$$\text{C.F.} = c_1\cos z + c_2\sin z = c_1\cos(\log x) + c_2\sin(\log x)$$
$$\text{P.I.} = \frac{1}{\mathcal{D}^2 + 1}z\sin z = \frac{1}{\mathcal{D}^2 + 1}z \text{ (Imaginary part of } e^{iz})$$

$$= \text{I.P.}\left[\frac{1}{\mathcal{D}^2+1} z e^{iz}\right] = \text{I.P.}\left[e^{iz}\cdot\frac{1}{(\mathcal{D}+i)^2+1} z\right]$$

$$= \text{I.P.}\left[e^{iz}\cdot\frac{1}{\mathcal{D}^2+2i\mathcal{D}} z\right] = \text{I.P.}\left[e^{iz}\cdot\frac{1}{2i\mathcal{D}\left(1+\frac{\mathcal{D}}{2i}\right)}\cdot z\right]$$

$$= \text{I.P.}\left[\frac{e^{iz}}{2i}\cdot\frac{1}{\mathcal{D}}\left(1+\frac{\mathcal{D}}{2i}\right)^{-1} z\right]$$

$$= \text{I.P.}\left[\frac{e^{iz}}{2i}\cdot\frac{1}{\mathcal{D}}\left(1-\frac{\mathcal{D}}{2i}+\frac{\mathcal{D}^2}{4i^2}-\ldots\right)z\right] = \text{I.P.}\left[\frac{e^{iz}}{2i}\cdot\frac{1}{\mathcal{D}}\left(z-\frac{1}{2i}+0\right)\right]$$

$$= \text{I.P.}\left[\frac{e^{iz}}{2i}\int\left(z-\frac{1}{2i}\right)dz\right] = \text{I.P.}\left[\frac{-ie^{iz}}{2}\left(\frac{z^2}{2}-\frac{z}{2i}\right)\right]$$

$$= \text{I.P.}\left[\frac{-i(\cos z+i\sin z)(iz^2-z)}{4i}\right]$$

$$= \frac{-z^2\cos z+z\sin z}{4}$$

$$= -\frac{(\log x)^2}{4}\cos(\log x)+\frac{\log x}{4}\sin(\log x)$$

Hence, the general solution is

$$y = c_1\cos(\log x)+c_2\sin(\log x)-\frac{(\log x)^2}{4}\cos(\log x)+\frac{\log x}{4}\sin(\log x)$$

Exercise 10.11

Solve the following differential equations:

1. $(x^2D^2+xD-1)y=0.$

$\left[\textbf{Ans.: } y = c_1x+\frac{c_2}{x}\right]$

2. $(9x^2D^2+3xD+10)y=0.$

$\left[\textbf{Ans.: } y = x^{\frac{1}{3}}\left[c_1\cos(\log x)+c_2\sin(\log x)\right]\right]$

3. $(x^3D^3-2xD+4)y=0.$

$\left[\textbf{Ans.: } y = \frac{c_1}{x}+(c_2+c_3\log x)x^2\right]$

4. $(x^3D^3+3x^2D^2+14xD+34)y=0.$

$\left[\textbf{Ans.: } \frac{c_1}{x^2}+x[c_2\cos(4\log x)+c_3\sin(4\log x)]\right]$

5. $(x^2D^2-3xD+4)y=x^3.$

$\left[\textbf{Ans.: } y=(c_1+c_2\log x)x^2+x^3\right]$

6. $(x^3D^3+6x^2D^2-12)y=\frac{12}{x^2}.$

$\left[\textbf{Ans.: } y=c_1x^2+\frac{c_2}{x^2}+\frac{c_3}{x^3}-\frac{3}{x^2}\log x\right]$

7. $(4x^3D^3+12x^2D^2+xD+1)y = 50\sin(\log x)$.

$$\left[\textbf{Ans. : } y=(c_1+c_2\log x)x^{\frac{1}{2}}+\frac{c_3}{x} + \sin(\log x)+7\cos(\log x)\right]$$

8. $(x^2D^2-3xD+3)y = 2+3\log x$.

$$\left[\textbf{Ans. : } y=c_1x+c_2x^3+\log x+2\right]$$

9. $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+y=\dfrac{\sin(\log x)+1}{x}$.

$$\left[\textbf{Ans. : } y=x^2\left[c_1\cosh\left(\sqrt{3}\log x\right)+c_2\sinh\left(\sqrt{3}\log x\right)\right]+\frac{1}{6x}+\frac{1}{61x}[5\sin(\log x)+6\cos(\log x)]\right]$$

10. $(x^2D^2-3xD+5)y = x^2\sin(\log x)$.

$$\left[\textbf{Ans. : } y=x^2[c_1\cos(\log x)+c_2\sin(\log x)]-\frac{x^2}{2}\log x\cos(\log x)\right]$$

11. $(x^2D^3+3xD^2+D)y = x^2\log x$.

$$\left[\textbf{Ans. : } c_1+c_2\log x+c_3(\log x)^2+\frac{x^3}{27}(\log x-1)\right]$$

12. $(x^3D^3+2x^2D^2+2)y = 10\left(x+\dfrac{1}{x}\right)$.

$$\left[\textbf{Ans. : } y=\frac{c_1}{x}+x[c_2\cos(\log x)+c_3\sin(\log x)]+5x+\frac{2}{x}\log x\right]$$

13. $(x^2D^2-2xD+2)y = (\log x)^2-\log x^2$.

$$\left[\textbf{Ans. : } y=c_1x+c_2x^2+\frac{1}{2}[(\log x)^2+\log x]+\frac{1}{4}\right]$$

14. $(x^2D^2+3xD+1)y = \dfrac{1}{(1-x)^2}$.

$$\left[\textbf{Ans. : } y=\frac{1}{x}(c_1+c_2\log x)+\frac{1}{x}\log\frac{x}{x-1}\right]$$

15. $(x^2D^2+3xD+10)y = 9x^2, y(1)=\dfrac{5}{2}$, $y'(1)=8$.

$$\left[\textbf{Ans. : } y=\frac{1}{x}[2\cos(3\log x)+3\sin(3\log x)]+\frac{x^2}{2}\right]$$

16. $(2x^2D^2+3xD-1)y = x$, $y(1)=1$, $y(4)=\dfrac{41}{16}$.

$$\left[\textbf{Ans. : } y=\frac{1}{4}\left(\sqrt{x}+\frac{1}{x}\right)+\frac{x}{2}\right]$$

10.6.2 Legendre's Linear Equation

An equation of the form

$$a_0(a+bx)^n\frac{d^ny}{dx^n}+a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2(a+bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\ldots$$

$$\ldots+a_{n-1}(a+bx)\frac{dy}{dx}+a_ny=Q(x) \qquad \text{... (1)}$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called Legendre's linear equation.

Let $(a+bx) = e^z$

$$b = e^z \frac{dz}{dx}, \qquad \frac{dz}{dx} = \frac{b}{e^z} = \frac{b}{a+bx}$$

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{b}{(a+bx)}$$

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz}$$

$$(a+bx)Dy = b\mathcal{D}y \qquad \text{where } D \equiv \frac{d}{dx} \text{ and } \mathcal{D} \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+bx}\cdot\frac{dy}{dz}\right)$$

$$= -\frac{b}{(a+bx)^2}\cdot b\cdot\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d^2y}{dz^2}\left(\frac{b}{a+bx}\right)$$

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$(a+bx)^2 D^2 y = b^2(\mathcal{D}^2 - \mathcal{D})y = b^2\mathcal{D}(\mathcal{D}-1)y$$

Similarly, $$(a+bx)^3 D^3 y = b^3\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

....

....

$$(a+bx)^n D^n y = b^n\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (1), we get

$$[a_0 b^n\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1) + a_1 b^{n-1}\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2) + \ldots + a_{n+1}\mathcal{D} + a_n]y$$

$$= Q\left(\frac{e^z - a}{b}\right)$$

which is a linear differential equation with constant coefficients and can be solved by usual methods described in previous section.

Example 1: **Solve $[(x+1)^2 D^2 + (x+1)D]y = (2x+3)(2x+4)$.**

Solution: Here $a = 1$, $b = 1$

Putting $x + 1 = e^z$,

$$[\mathcal{D}(\mathcal{D}-1) + \mathcal{D}]y = [2(e^z-1)+3][2(e^z-1)+4], \qquad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$\mathcal{D}^2 y = 4e^{2z} + 6e^z + 2$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 = 0,\ \ m = 0, 0,\quad \text{(real and repeated twice)}$$

$$\text{C.F.} = (c_1 + c_2 z)\, e^{0z} = c_1 + c_2 z = c_1 + c_2 \log(x+1)$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2}(4e^{2z} + 6e^z + 2) = 4\cdot\frac{1}{\mathcal{D}^2}\cdot e^{2z} + 6\cdot\frac{1}{\mathcal{D}^2}e^z + 2\cdot\frac{1}{\mathcal{D}^2}e^{0\cdot z}$$

$$= 4\cdot\frac{1}{2^2}e^{2z} + 6\cdot\frac{1}{1^2}e^z + 2z\cdot\frac{1}{2\mathcal{D}}e^{0\cdot z}$$

$$= e^{2z} + 6e^z + 2z^2\cdot\frac{1}{2}e^{0\cdot z} = e^{2z} + 6e^z + z^2$$

$$= (x+1)^2 + 6(x+1) + [\log(x+1)]^2 = x^2 + 8x + 7 + [\log(x+1)]^2$$

Hence, the general solution is

$$y = c_1 + c_2 \log(x+1) + x^2 + 8x + 7 + [\log(x+1)]^2$$

Aliter: After putting $x + 1 = e^z$,

$$\mathcal{D}^2 y = 4e^{2z} + 6e^z + 2$$

$$y = \frac{1}{\mathcal{D}^2}(4e^{2z} + 6e^z + 2) = \int\left[\int (4e^{2z} + 6e^z + 2)\,\mathrm{d}z\right]\mathrm{d}z$$

$$= \int (2e^{2z} + 6e^z + 2z + \mathrm{A})\,\mathrm{d}z = e^{2z} + 6e^z + z^2 + \mathrm{A}z + \mathrm{B}$$

$$= (x+1)^2 + 6(x+1) + [\log(x+1)]^2 + \mathrm{A}\log(x+1) + \mathrm{B}$$

$$= x^2 + 8x + 7 + [\log(x+1)]^2 + \mathrm{A}\log(x+1) + \mathrm{B}$$

Example 2: Solve $\boldsymbol{(2+3x)^2 \frac{d^2y}{dx^2} + 3(2+3x)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.}$

Solution: Here $a = 2$, $b = 3$

Putting $2 + 3x = e^z$,

$$\left[9\mathcal{D}(\mathcal{D}-1) + 3\cdot 3\mathcal{D} - 36\right]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1,\qquad \text{where } \mathcal{D} \equiv \frac{\mathrm{d}}{\mathrm{d}z}$$

$$(9\mathcal{D}^2 - 36)y = \frac{e^{2z} - 4e^z + 4 + 4e^z - 8 + 3}{3}$$

$$(\mathcal{D}^2 - 4)y = \frac{1}{27}(e^{2z} - 1)$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^2 - 4 = 0,\qquad m = \pm 2\ \text{(real and distinct)}$$

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{-2z} = c_1(2+3x)^2 + c_2(2+3x)^{-2}$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^2 - 4}\cdot\frac{1}{27}(e^{2z} - 1) = \frac{1}{27}\left(\frac{1}{\mathcal{D}^2-4}e^{2z} - \frac{1}{\mathcal{D}^2-4}e^{0\cdot z}\right)$$

$$= \frac{1}{27}\left(z \cdot \frac{1}{2\mathcal{D}} e^{2z} - \frac{1}{0-4} e^{0\cdot z}\right) = \frac{1}{27}\left(z \cdot \frac{1}{4} e^{2z} + \frac{1}{4}\right)$$

$$= \frac{1}{108}(ze^{2z} + 1) = \frac{1}{108}[\log(2+3x)\cdot(2+3x)^2 + 1].$$

Hence, the general solution is

$$y = c_1(2+3x)^2 + \frac{c^2}{(2+3x)^2} + \frac{1}{108}[(2+3x)^2 \log(2+3x) + 1]$$

Example 3: Solve $(x-1)^3 \frac{d^3y}{dx^3} + 2(x-1)^2 \frac{d^2y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y = 4\log(x-1)$.

Solution: Here $a = -1$, $b = 1$

Putting $(x - 1) = e^z$,

$$[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + 2\mathcal{D}(\mathcal{D}-1) - 4\mathcal{D} + 4]y = 4z$$

$$(\mathcal{D}^3 - \mathcal{D}^2 - 4\mathcal{D} + 4)y = 4z$$

which is a linear equation with constant coefficients.

Auxiliary equation is

$$m^3 - m^2 - 4m + 4 = 0$$

$$(m^2 - 4)(m-1) = 0, \qquad m = \pm 2,\ 1 \text{ (real and distinct)}$$

$$\text{C.F.} = c_1 e^z + c_2 e^{2z} + c_3 e^{-2z} = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2}$$

$$\text{P.I.} = \frac{1}{\mathcal{D}^3 - \mathcal{D}^2 - 4\mathcal{D} + 4} \cdot 4z = \frac{1}{4\left(1 - \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)} \cdot 4z$$

$$= \left(1 - \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^{-1} z$$

$$= \left[1 + \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4} + \left(\frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^2 + \ldots\right] z$$

$$= z + \mathcal{D}(z) + (\text{Higher powers of } \mathcal{D})z$$

$$= z + 1 + 0 = \log(x-1) + 1.$$

Hence, the general solution is

$$y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x-1) + 1$$

Exercise 10.12

Solve the following differential equations:

1. $[(1+x)^2 D^2 + (1+x)D + 1]y = 2\sin\log(x+1)$.

$$\left[\textbf{Ans.: } y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \log(1+x)\cos\log(1+x)\right]$$

2. $[(x+2)^2 D^2 - (x+2)D + 1]y = 3x + 4$.

$$\left[\textbf{Ans.: } y = [c_1 + c_2 \log(x+2)](x+2) + \frac{3}{2}[\log(x+2)]^2 (x+2) - 2\right]$$

3. $[(x-1)^3 D^3 + 2(x-1)^2 D^2 - 4(x-1) D + 4] y = 4 \log (x-1)$.

$$\left[\textbf{Ans.: } y = c_1 (x-1) + c_2 (x-1)^2 + c_3 (x-1)^{-2} + \log (x-1) + 1\right]$$

4. $[(x+2)^2 D^2 - (x+2) D + 1] y = 3x + 4$

$$\left[\textbf{Ans.: } y = (x+2)\left[c_1 + c_2 \log (x+2) + \frac{3}{2}\left\{\log (x+2)^2\right\}\right] - 2\right]$$

5. $[(2x+1)^2 D^2 - 2(2x+1) D - 12] y = 6x$.

$$\left[\textbf{Ans.: } y = c_1 (2x+1)^{-1} + c_2 (2x+1)^3 - \frac{3}{8} x + \frac{1}{16}\right]$$

6. $[(x+a)^2 D^2 - 4D + 6] y = x$.

$$\left[\textbf{Ans.: } y = c_1 (x+a)^3 + c_2 (x+a)^2 + \frac{1}{6}(3x+2a)\right]$$

7. $[3x+1)^2 D^2 - 3(3x+1) D - 12] y = 9x$.

$$\left[\textbf{Ans.: } y = (3x+1)\left[c_1 (3x+1)^{\sqrt{\frac{7}{12}}} + c_2 (3x+1)^{-\sqrt{\frac{7}{12}}}\right] - 3\left[\frac{3x+1}{7} + \frac{1}{4}\right]\right]$$

8. $[(2x+5)^2 D^2 - 6D + 8] y = 6x$.

$$\left[\textbf{Ans.: } y = (2x+5)^2\left[c_1 (2x+5)^{\sqrt{2}} + c_2 (2x+5)^{-\sqrt{2}}\right] - \frac{3}{2} x - \frac{45}{8}\right]$$

9. $[(2+3x)^2 D^2 + 5(2+3x) D - 3] y = x^2 + x + 1$.

$$\left[\textbf{Ans.: } c_1 (2+3x)^{\frac{1}{3}} + c_2 (2+3x)^{-1} + \frac{1}{27}\left[\frac{1}{15}(2+3x)^2 + \frac{1}{4}(2+3x) - 7\right]\right]$$

10. $[(2x-1)^3 D^3 + (2x-1) D - 2] y = 0$.

$$\left[\textbf{Ans.: } y = c_1 (2x-1) + (2x-1)\left[c_2 (2x-1)^{\frac{\sqrt{3}}{2}} + c_3 (2x-1)^{-\frac{\sqrt{3}}{2}}\right]\right]$$

10.7 METHOD OF VARIATION OF PARAMETERS

This method is used to find the particular integral if complimentary function is known. In this method, the particular integral is obtained by varying the arbitrary constants of the complimentary function and hence known as variation of parameters method.

Consider a linear non-homogeneous differential equation of second order with constant coefficients

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = Q(x) \qquad \text{... (1)}$$

Let the complimentary function is

$$\text{C.F.} = c_1 y_1 + c_2 y_2 \qquad \text{... (2)}$$

where y_1, y_2 are the solution of

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 \qquad \text{... (3)}$$

Let the particular integral is

$$y = v_1(x)y_1 + v_2(x)y_2 \qquad \text{... (4)}$$

where v_1 and v_2 are unknown functions of x.

Differentiating Eq. (4) w.r.t. x,

$$y' = v_1y_1' + v_2y_2' + v_1'y_1 + v_2'y_2$$

Let v_1, v_2 satisfy the equation

$$v_1'y_1 + v_2'y_2 = 0 \qquad \text{... (5)}$$

Then $\quad y' = v_1y_1' + v_2y_2'$

Differentiating Eq. (5) w.r.t. x,

$$y'' = v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'$$

Substituting y'', y' and y in Eq. (1),

$$v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2' + a_1(v_1y_1' + v_2y_2') + a_2(v_1y_1 + v_2y_2) = Q(x)$$

$$v_1(y_1'' + a_1y_1' + a_2y_1) + v_2(y_2'' + a_1y_2' + a_2y_2) + v_1'y_1' + v_2'y_2' = Q(x)$$

Since y_1 and y_2 satisfy Eq. (3), we get

$$v_1'y_1' + v_2'y_2' = Q \qquad \text{... (6)}$$

Solving Eqs. (5) and (6) by using Cramer's rule,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ Q & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2Q}{y_1y_2' - y_1'y_2}$$

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & Q \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_1Q}{y_1y_2' - y_1'y_2}$$

$$v_1 = \int -\frac{y_2Q}{y_1y_2' - y_1'y_2}\,dx = \int -\frac{y_2Q}{W}\,dx \qquad \text{... (7)}$$

$$v_2 = \int \frac{y_1Q}{y_1y_2' - y_1'y_2}\,dx = \int \frac{y_1Q}{W}\,dx \qquad \text{... (8)}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is known as Wronskian of y_1, y_2.

Hence, the required general solution of Eq. (1) is,

$$\text{general solution} = \text{C.F.} + \text{P.I.}$$

$$= c_1y_1 + c_2y_2 + v_1y_1 + v_2y_2$$

where v_1, v_2 are obtained using formulas (7) and (8).

Note: The above method can also be extended for third order differential equation. Let complementary function of a third order differential equation is

$$\text{C.F.} = c_1y_1 + c_2y_2 + c_3y_3$$

Let P.I. $= v_1(x)y_1 + v_2(x)y_2 + v_3(x)y_3$

where
$$v_1(x) = \int \frac{(y_2y_3' - y_3y_2')\text{Q}}{W}\text{d}x$$
$$v_2(x) = \int \frac{(y_3y_1' - y_1y_3')\text{Q}}{W}\text{d}x$$
$$v_3(x) = \int \frac{(y_1y_2' - y_2y_1')\text{Q}}{W}\text{d}x$$

Wronskian,
$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Example 1: Solve $(D^2 + 1)y = \text{cosec } x$.

Solution: Auxiliary equation is

$$m^2 = 1, m = \pm i \quad \text{(imaginary)}$$
$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$
$$y_1 = \cos x, y_2 = \sin x$$

Wronskian,
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$
$$= 1$$

Let particular integral is

$$\text{P.I.} = x_1(x)\cos x + v_2(x)\sin x$$

where
$$v_1 = \int -\frac{y_2\text{Q}}{W}\text{d}x$$
$$= \int \frac{\sin x \text{ cosec } x}{1}\text{d}x = -x$$
$$v_2 = \int \frac{y_1\text{Q}}{W}\text{d}x$$
$$= \int \frac{\cos x \text{ cosec } x}{1}\text{d}x$$
$$= \int \cot x \, \text{d}x$$
$$= \log \sin x$$
$$\text{P.I.} = -x \cos x + (\log \sin x) \sin x.$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log \sin x$$

Example 2: Solve $(D^2 - 1)y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$.

Solution: Auxiliary equation is

$$m^2 - 1, = 0, \quad m = \pm 1 \qquad \text{(real and distinct)}$$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$y_1 = e^x, \quad y_2 = e^{-x}$$

Wronskian,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -2.$$

Let particular integral is

$$\text{P.I.} = v_1(x) e^x + v_2(x) e^{-x}$$

where,

$$v_1 = \int -\frac{y_2 Q}{W} dx$$

$$= -\int \frac{e^{-x}[e^{-x} \sin(e^{-x}) + \cos(e^{-x})}{-2} dx$$

Putting $e^{-x} = t$, $-e^{-x} dx = dt$

$$v_1 = \frac{1}{2}\int -(t \sin t + \cos t)\, dt$$

$$= -\frac{1}{2}[t(-\cos t) - (-\sin t) + \sin t)]$$

$$= \frac{1}{2} t \cos t - \sin t = \frac{1}{2} e^{-x} \cos(e^{-x}) - \sin(e^{-x})$$

$$v_2 = \int \frac{y_1 Q}{W} = \int \frac{e^x[e^{-x} \sin(e^{-x}) + \cos(e^{-x})]}{-2} dx$$

$$= -\frac{1}{2}\int [\sin(e^{-x}) + e^x \cos(e^{-x})] dx$$

$$= -\frac{1}{2}\int d[e^x \cos(e^{-x})] = \frac{1}{2} e^x \cos(e^{-x})$$

$$\text{P.I.} = \frac{1}{2}\cos(e^{-x}) - e^x \sin(e^{-x}) - \frac{1}{2}\cos(e^{-x})$$

$$= -e^x \sin(e^{-x}).$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x}).$$

Example 3: Solve $(D^3 - 6D^2 + 12D - 8)y = \dfrac{e^{2x}}{x}$

Solution: Auxiliary equation is

$$m^2 - 6m^2 + 12m - 8 = 0$$
$$(m-2)^3 = 0$$
$$m = 2, 2, 2 \quad \text{(repeated thrice)}$$

$$\text{C.F.} = (c_1 + c_2 x + c_3 x_2)\, e^{2x}$$
$$y_1 = e^{2x},\ y_2 = xe^{2x},\ y_3 = x^2 e^{2x}$$

Wronskian,
$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$
$$= \begin{vmatrix} e^{2x} & xe^{2x} & x^2 e^{2x} \\ 2e^{2x} & (2x+1)e^{2x} & (2x^2+2x)e^{2x} \\ 4e^{2x} & 4(x+1)e^{2x} & (4x^2+8x+2)e^{2x} \end{vmatrix} = 2e^{6x}$$

Let particular integral is

$$\text{P.I.} = v_1(x)e^{2x} + v_2(x)\cdot xe^{2x} + v_3(x)\cdot x^2 e^{2x}$$

where,
$$v_1 = \int \frac{(y_2 y_3' - y_3 y_2')\,Q}{W}\,\mathrm{d}x$$
$$= \int \frac{[xe^{2x}\cdot(2x^2+2x)e^{2x} - x^2e^{2x}\cdot(2x+1)e^{2x}]}{2e^{6x}}\cdot\frac{e^{2x}}{x}\,\mathrm{d}x$$
$$= \int \frac{x}{2}\,\mathrm{d}x = \frac{x^2}{4}$$
$$v_2 = \int \frac{(y_3 y_1' - y_1 y_3')\,\mathrm{Q}}{W}\,\mathrm{d}x$$
$$= \int \frac{[x^2e^{2x}\cdot 2e^{2x} - e^{2x}\cdot(2x^2+2x)e^{2x}]}{2e^{6x}}\cdot\frac{e^{2x}}{x}\,\mathrm{d}x$$
$$= \int -\,\mathrm{d}x = -x$$
$$v_3 = \int \frac{(y_1 y_2' - y_2 y_1')\,\mathrm{Q}}{W}\,\mathrm{d}x$$
$$= \int \frac{e^{2x}\cdot(2x+1)e^{2x} - xe^{2x}\cdot 2e^{2x}}{2e^{6x}}\cdot\frac{e^{2x}}{x}\,\mathrm{d}x$$
$$= \int \frac{1}{2x}\,\mathrm{d}x$$
$$= \frac{1}{2}\log x$$

$$\text{P.I.} = \frac{x^2}{4}e^{2x} - x^2e^{2x} + \frac{1}{2}\log x\cdot x^2e^{2x}$$
$$= -\frac{3x^2}{4}e^{2x} + \frac{x^2}{2}e^{2x}\log x$$

Hence, the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)\, e^{2x} - \frac{3x^2}{4} e^{2x} + \frac{x^2}{2} e^{2x} \log x.$$

Exercise 10.13

Solve the following differential equations using variation of parameter method:

1. $(D^2 + 1)y = \tan x.$

$\left[\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)\right]$

2. $(D^2 + 4)y = \sec^2 2x.$

$\left[\textbf{Ans.: } y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} + \frac{\sin 2x}{4} \log(\sec 2x + \tan 2x)\right]$

3. $(D^2 + 1)y = \operatorname{cosec} \cot x.$

$\left[\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x - \cos x \log|\sin x| - x \sin x - \sin x \cot x\right]$

4. $(D^2 + 1)y = \dfrac{1}{1 + \sin x}.$

$\left[\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x - (1 - \sin x + x \cos x) + \sin x \log(1 + \sin x)\right]$

5. $(D^2 - 1)y = \dfrac{2}{1 - e^x}.$

$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{-x} + e^x \log(1 + e^{-x}) - e^x - 1 - e^{-x} \log(1 + e^x)\right]$

6. $(D^2 - 6D + 9)y = \dfrac{e^{3x}}{x^2}.$

$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^{3x} - (1 + \log x)e^{3x}\right]$

7. $(D^2 - 1)y = 2(1 - e^{-2x})^{-\frac{1}{2}}.$

$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{-x} - e^x \sin^{-1}(e^{-x}) - (e^{2x} - 1)^{\frac{1}{2}} e^{-x}\right]$

8. $(D^2 - 2D)y = e^x \sin x.$

$\left[\textbf{Ans.: } y = c_1 + c_2 e^{2x} - \frac{e^x}{2} \sin x\right]$

9. $(D^2 + 3D + 2)y = e^x + x^2.$

$\left[\textbf{Ans.: } y = c_1 e^{-x} + c_2 e^{-2x} + \frac{e^x}{6} + \left(\frac{x^2}{2} - \frac{3x}{2} + \frac{7}{4}\right)\right]$

10. $(D^2 - 2D + 1)y = x^{\frac{3}{2}} e^x.$

$\left[\textbf{Ans.: } y = (c_1 + c_2 x)e^x + \frac{4}{35} x^{\frac{7}{2}} e^x\right]$

11. $(D^2 - 3D + 2)y = xe^x + 2x.$

$\left[\textbf{Ans.: } y = c_1 e^x + c_2 e^{2x} - \frac{x^2}{2} e^x - xe^{-x} + x + \frac{3}{2}\right]$

12. $(D^2 + 1)y = x \cos 2x.$

$\left[\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x - \frac{x}{2} \cos 2x + \frac{4}{9} \sin 2x\right]$

13. $(D^2 + 1)y = \log \cos x.$

$\left[\textbf{Ans.: } y = c_1 \cos x + c_2 \sin x + (\log \cos x - 1) + \sin x \log(\sec x + \tan x)\right]$

14. $(D^2 + 4D + 8)y = 16e^{-2x} \operatorname{cosec}^2 2x.$

$\left[\textbf{Ans.: } y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + 4e^{-2x} \cos 2x \log|\operatorname{cosec} x + \cot 2x| - 4e^{-2x}\right]$

10.8 METHOD OF UNDETERMINED COEFFICIENTS

This method can be used to find the particular integral only if linearly independent derivatives of Q(x) are finite in number. This restriction implies that Q(x) can only have the terms such as k, x^n, e^{ax}, sin ax, cos ax and combinations of such terms where k, a are constants and n is a positive integer. However, when $Q(x) = \frac{1}{x}$ or tan x or sec$x$, etc., this method fails, since each function has an infinite number of linearly independent derivatives.

In this method, particular integral is assumed as a linear combination of the terms in Q(x) and all its linearly independent derivatives. Some of the choices of particular integral are given below.

Sr. No.	Q(x)	Particular Integral
1.	ke^{ax}	Ae^{ax}
2.	$k\sin(ax+b)$ or $k\cos(ax+b)$	$A\sin(ax+b) + B\cos(ax+b)$
3.	$ke^{ax}\sin(bx+c)$ or $ke^{ax}\cos(bx+c)$	$Ae^{ax}\sin(bx+c) + Be^{ax}\cos(bx+c)$
4.	kx^n $n = 0, 1, 2, ...$	$A_nx^n + A_{n-1}x^{n-1} + ... + A_2x^2 + A_1x + A_0$
5.	$kx^n e^{ax}$ $n = 0, 1, 2, ...$	$e^{ax}(A_nx^n + A_{n-1}x^{n-1} + ... + A_2x^2 + A_1x + A_0)$
6.	$kx^n\sin(ax+b)$ or $kx^n\cos(ax+b)$	$x^n[A_n\sin(ax+b) + B_n\cos(ax+b)] + x^{n-1}$ $[A_{n-1}\sin(ax+b) + B_{n-1}\cos(ax+b)] + ...$ $+ x[A_1\sin(ax+b) + B_1\cos(ax+b)]$ $+ [A_0\sin(ax+b) + B_0\cos(ax+b)]$
7.	$kx^n e^{ax}\sin(bx+c)$ or $kx^n e^{ax}\cos(bx+c)$	$e^{ax}[x^n\{A_n\sin(ax+b) + B_n\cos(ax+b)\}$ $+ x^{n-1}\{A_{n-1}\sin(ax+b) + B_{n-1}\cos(ax+b)\}$ $+ ..... + x\{A_1\sin(ax+b) + B_1\cos(ax+b)$ $+ \{A_0\sin(ax+b) + B_0\cos(ax+b)\}]$

In the table, $A_0, A_1, A_2, \ldots, A_n$ are coefficients to be determined. To obtain the values of these coefficients, we use the fact that the particular integral satisfies the given differential equation.

However, before assuming the particular integral it is necessary to compare the terms of Q(x) with the complimentary function. While comparing the terms following different cases arise.

Case I: If no terms of Q(x) occurs in the complimentary function, then particular integral is assumed from the table depending on the nature of Q(x).

Case II: If a term u of Q(x) is also a term of the complimentary function corresponding to an r-fold root, then assumed particular integral corresponding to u should be multiplied by x^r.

Case III: If x^su is a term of Q(x) and only u is a term of complimentary function corresponding to an r-fold root, then assumed particular integral corresponding to x^su should be multiplied by x^r.

Note: In case (ii) and (iii) initially similar type of terms appear in complimentary function and in assumed particular integral. After multiplication by x^r the terms of particular integral changes. Hence this method avoids the repetition of similar terms in complimentary function and particular integral.

Example 1: Solve $(D^2 - 2D + 5)y = 25x^2 + 12$

Solution: Auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \quad \text{(imaginary)}$$

$$\text{C.F.} = e^x(c_1 \cos 2x + c_2 \sin 2x)$$

$$Q = 25x^2 + 12$$

Let particular integral is

$$y = A_1x^2 + A_2x + A_3$$

$$Dy = 2A_1x + A_2$$

$$D^2y = 2A_1$$

Substituting these derivatives in the given equation,

$$2A_1 - 2(2A_1x + A_2) + 5(A_1x^2 + A_2x + A_3) = 25x^2 + 12$$

$$5A_1x^2 + (-4A_1 + 5A_2)x + (2A_1 - 2A_2 + 5A_3) = 25x^2 + 12$$

Comparing coefficients on both the sides,

$$5A_1 = 25, \qquad A_1 = 5$$

$$-4A_1 + 5A_2 = 0, \qquad A_2 = \frac{4}{5}A_1 = 4$$

$$2A_1 - 2A_2 + 5A_3 = 12, \qquad A_3 = \frac{1}{5}(12 - 10 + 8) = 2$$

$$\text{P.I.} = 5x^2 + 4x + 2$$

Hence, the general solution is

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + 2$$

Example 2: Solve $(D^2 - 2D + 3)y = x^3 + \sin x$.

Solution: Auxiliary equation is

$$m^2 - 2m + 3 = 0$$

$$m = 1 \pm i\sqrt{2} \qquad \text{(imaginary)}$$

$$\text{C.F.} = e^x\left(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x\right)$$

$$Q = x^3 + \sin x$$

Let particular integral is

$$y = A_1x^3 + A_2x^2 + A_3x + A_4 + A_5 \sin x + A_6 \cos x$$
$$Dy = 3A_1x^2 + 2A_2x + A_3 + A_5 \cos x - A_6 \sin x$$
$$D^2y = 6A_1x + 2A_2 - A_5 \sin x - A_6 \cos x$$

Substituting these derivatives in the given equation,

$$(6A_1x + 2A_2 - A_5 \sin x - A_6 \cos x) - 2(3A_1x^2 + 2A_2x + A_3 + A_5 \cos x$$
$$- A_6 \sin x) + 3(A_1x^3 + A_2x^2 + A_3x + A_4 + A_5 \sin x + A_6 \cos x) = x^3 + \sin x$$
$$3A_1x^3 + (-6A_1 + 3A_2)x^2 + (6A_1 - 4A_2 + 3A_3)x + (2A_2 - 2A_3 + 3A_4)$$
$$- 2(A_5 - A_6)\cos x + 2(A_5 + A_6)\sin x = x^3 + \sin x$$

Comparing coefficients on both the sides,

$$3A_1 = 1, \qquad A_1 = \frac{1}{3}$$
$$-6A_1 + 3A_2 = 0, \qquad A_2 = 2A_1 = \frac{2}{3}$$
$$6A_1 - 4A_2 + 3A_3 = 0, \qquad A_3 = \frac{1}{3}(4A_2 - 6A_1) = \frac{2}{9}$$
$$2A_2 - 2A_3 + 3A_4 = 0, \qquad A_4 = \frac{2}{3}(A_3 - A_2) = -\frac{8}{27}$$
$$2(A_5 - A_6) = 0, \qquad A_5 = A_6$$
$$2(A_5 + A_6) = 1, \qquad 2(A_5 + A_5) = 1, \qquad A_5 = \frac{1}{4},\ A_6 = \frac{1}{4}$$

$$\text{P.I.} = \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Hence, the general solution is

$$y = e^x(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{1}{3}x^3 + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27} + \frac{1}{4}(\sin x + \cos x)$$

Example 3: Solve $(D^2 - 9)y = x + e^{2x} - \sin 2x$.

Solution: Auxiliary equation is

$$m^2 - 9 = 0$$
$$m = \pm 3 \quad \text{(real and distinct)}$$
$$\text{C.F.} = c_1e^{3x} + c_2e^{-3x}$$
$$Q = x + e^{2x} - \sin 2x$$

Let particular integral is

$$y = A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x$$
$$Dy = A_1 + 2A_3 e^{2x} + 2A_4 \cos 2x - 2A_5 \sin 2x$$
$$D^2 y = 4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x$$

Substituting these derivatives in the given equation,

$$4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x - 9(A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x)$$
$$= x + e^{2x} - \sin 2x$$
$$(-5A_3)e^{2x} - 9A_1 x - 9A_2 + \sin 2x(-13A_4) + \cos 2x(-13A_5) = x + e^{2x} - \sin 2x$$

Comparing coefficients on both the sides,

$$-5A_3 = 1, \qquad A_3 = -\frac{1}{5}$$
$$-9A_1 = 1, \qquad A_1 = -\frac{1}{9}$$
$$-9A_2 = 0, \qquad A_2 = 0$$
$$-13A_4 = -1, \qquad A_4 = \frac{1}{13}$$
$$-13A_5 = 0, \qquad A_5 = 0$$
$$\text{P.I.} = -\frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{x}{9} - \frac{e^{2x}}{5} + \frac{\sin 2x}{13}$$

Example 4: Solve $(D^2 - 2D)y = e^x \sin x$.

Solution: Auxiliary equation is

$$m^2 - 2m = 0$$
$$m = 0, \ -2 \qquad \text{(real and distinct)}$$
$$\text{C.F.} = c_1 + c_2 e^{2x}$$
$$Q = e^x \sin x$$

Let particular integral is

$$y = A_1 e^x \sin x + A_2 e^x \cos x$$
$$Dy = A_1(e^x \sin x + e^x \cos x) + A_2(e^x \cos x - e^x \sin x)$$
$$= (A_1 - A_2)e^x \sin x + (A_1 + A_2)e^x \cos x$$
$$D^2 y = (A_1 - A_2)(e^x \sin x + e^x \cos x) + (A_1 + A_2)(e^x \cos x - e^x \sin x)$$
$$= -2A_2 e^x \sin x + 2A_1 e^x \cos x$$

Substituting these derivatives in the given equation,

$$-2A_2e^x \sin x + 2A_1e^x \cos x - 2(A_1 - A_2)e^x \sin x - 2(A_1 + A_2)e^x \cos x = e^x \sin x$$

$$-2A_1e^x \sin x - 2A_2e^x \cos x = e^x \sin x$$

Comparing coefficients on both the sides,

$$-2A_1 = 1, \qquad A_1 = -\frac{1}{2}$$

$$2A_2 = 0, \qquad A_2 = 0$$

$$\text{P.I.} = -\frac{1}{2}e^x \sin x$$

Hence, the general solution is

$$y = c_1 + c_2e^{2x} - \frac{1}{2}e^x \sin x$$

Example 5: Solve $(D^3 + 3D^2 + 2D)y = x^2 + 4x + 8$.

Solution: Auxiliary equation is

$$m^3 + 3m^2 + 2m = 0$$

$$m(m+1)(m+2) = 0$$

$$m = 0, -1, -2 \qquad \text{(real and distinct)}$$

$$\text{C.F.} = c_1 + c_2e^{-x} + c_3e^{-2x}$$

$$Q = x^2 + 4x + 8$$

Let particular integral is

$$y = A_1x^2 + A_2x + A_3$$

Since constant occurs in $Q(x)$ and is also a part of C.F. corresponding to 1-fold root $m = 0$, multiplying assumed particular integral by x.

$$y = A_1x^3 + A_2x^2 + A_3x$$

$$Dy = 3A_1x^2 + 2A_2x + A_3$$

$$D^2y = 6A_1x + 2A_2$$

$$D^3y = 6A_1$$

Substituting these derivatives in the given equation,

$$6A_1 + 3(6A_1x + 2A_2) + 2(3A_1x^2 + 2A_2x + A_3) = x^2 + 4x + 8$$

$$6A_1x^2 + (18A_1 + 4A_2)x + (6A_1 + 6A_2 + 2A_3) = x^2 + 4x + 8$$

Comparing coefficients on both the sides,

$$6A_1 = 1, \qquad A_1 = \frac{1}{6}$$

$$18A_1 + 4A_2 = 4, \qquad A_2 = \frac{1}{4}(4-3) = \frac{1}{4}$$

$$6A_1+6A_2+2A_3=8, \qquad A_3=\frac{1}{2}(8-6A_1-6A_2)=\frac{1}{2}\left(8-1-\frac{3}{2}\right)=\frac{11}{4}$$

$$\text{P.I.}=\frac{1}{6}x^3+\frac{1}{4}x^2+\frac{11}{4}x$$

Hence, the general solution is

$$y=c_1+c_2e^{-x}+c_3e^{-2x}+\frac{x^3}{6}+\frac{x^2}{4}+\frac{11x}{4}$$

Example 6: Solve $(D^2+1)y=4x\cos x-2\sin x$.

Solution: Auxiliary equation is

$$m^2+1=0$$
$$m=\pm i$$
$$\text{C.F.}=c_1\cos x+c_2\sin x$$
$$Q=4x\cos x-2\sin x$$

Let particular integral is

$$y=A_1x\sin x+A_2x\cos x+A_3\sin x+A_4\cos x$$

Since $x\cos x$ and its derivatives occur in Q(x) and $\cos x$ is a part of C.F. corresponding to 1 fold pair of complex root $m=\pm i$, multiplying assumed particular integral by x,

$$y=A_1x^2\sin x+A_2x^2\cos x+A_3x\sin x+A_4x\cos x$$

$$\begin{aligned}Dy&=A_1x^2\cos x+2A_1x\sin x-A_2x^2\sin x+2A_2x\cos x+A_3x\cos x\\&\quad+A_3\sin x-A_4x\sin x+A_4\cos x\\&=(A_1\cos x-A_2\sin x)x^2+(2A_1-A_4)x\sin x+(2A_2+A_3)x\cos x\\&\quad+A_3\sin x+A_4\cos x\end{aligned}$$

$$\begin{aligned}D^2y&=(-A_1\sin x-A_2\cos x)x^2+(A_1\cos x-A_2\sin x)(2x)+(2A_1-A_4)\sin x\\&\quad+(2A_1-A_4)x\cos x+(2A_2+A_3)\cos x-(2A_2+A_3)x\sin x+A_3\cos x-A_4\sin x\\&=-A_1x^2\sin x-A_2x^2\cos x+(4A_1-A_4)x\cos x\\&\quad-(4A_2+A_3)x\sin x+2(A_1-A_4)\sin x+2(A_2+A_3)\cos x\end{aligned}$$

Substituting these derivatives in given equation,

$$\begin{aligned}&-A_1x^2\sin x-A_2x^2\cos x+(4A_1-A_4)x\cos x-(4A_2+A_3)x\sin x\\&+2(A_1-A_4)\sin x+2(A_2+A_3)\cos x+A_1x^2\sin x+A_2x^2\cos x\\&+A_3x\sin x+A_4x\cos x=4x\cos x-2\sin x\\&4A_1x\cos x-4A_2x\sin x+2(A_1-A_4)\sin x+2(A_2+A_3)\cos x=+4x\cos x-2\sin x\end{aligned}$$

Comparing coefficients on both the sides,

$$4A_1 = 4, \qquad A_1 = 1$$
$$-4A_2 = 0, \qquad A_2 = 0$$
$$2(A_1 - A_4) = -2, \qquad A_4 = A_1 + 1 = 2$$
$$2(A_2 + A_3) = 0, \qquad A_3 = 0$$
$$\text{P.I.} = x^2 \sin x + 2x \cos x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + x^2 \sin x + 2x \cos x$$

Example 7: Solve $(D^3 - D^2 - 4D + 4)y = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$.

Solution: Auxiliary equation is

$$m^3 - m^2 - 4m + 4 = 0$$
$$(m-1)(m^2 - 4) = 0$$
$$m = 1, \pm 2 \text{ (real and distinct)}$$
$$\text{C.F.} = c_1e^x + c_2e^{2x} + c_3e^{-2x}$$
$$Q = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$$

Let particular integral is

$$y = A_1x^2 + A_2x + A_3 + A_4x^2e^{2x} + A_5xe^{2x} + A_6e^{2x}$$

Since x^2e^{2x} and its derivatives occur in $Q(x)$ and e^{2x} is a part of C.F. corresponding to 1-fold root $m = 2$, multiplying assumed particular integral corresponding to x^2e^{2x} by x,

$$y = A_1x^2 + A_2x + A_3 + A_4x^3e^{2x} + A_5x^2e^{2x} + A_6xe^{2x}$$

$$Dy = 2A_1x + A_2 + 2e^{2x}(A_4x^3 + A_5x^2 + A_6x) + e^{2x}(3A_4x^2 + 2A_5x + A_6)$$
$$= 2A_1x + A_2 + e^{2x}[2A_4x^3 + (3A_4 + 2A_5)x^2 + (2A_5 + 2A_6)x + A_6]$$

$$D^2y = 2A_1 + 2e^{2x}[2A_4x^3 + (3A_4 + 2A_5)x^2 + (2A_5 + 2A_6)x + A_6]$$
$$+ e^{2x}[6A_4x^2 + (3A_4 + 2A_5)2x + (2A_5 + 2A_6)]$$
$$= 2A_1 + e^{2x}[4A_4x^3 + (12A_4 + 4A_5)x^2 + (6A_4 + 8A_5 + 4A_6)x + (2A_5 + 4A_6)]$$

$$D^3y = 2e^{2x}[4A_4x^3 + (12A_4 + 4A_5)x^2 + (6A_4 + 8A_5 + 4A_6)x + (2A_5 + 4A_6)]$$
$$+ e^{2x}[12A_4x^2 + (12A_4 + 4A_5)2x + (6A_4 + 8A_5 + 4A_6)]$$
$$= e^{2x}[8A_4x^3 + (36A_4 + 8A_5)x^2 + (36A_4 + 24A_5 + 8A_6)x + (6A_4 + 12A_5 + 12A_6)]$$

Substituting these derivatives in the given equation,

$$e^{2x}[(8A_4 - 4A_4 - 8A_4 + 4A_4)x^3 + (36A_4 + 8A_5 - 12A_4 - 4A_5 - 12A_4$$
$$- 8A_5 + 4A_5)x^2 + (36A_4 + 24A_5 + 8A_6 - 6A_4 - 8A_5 - 4A_6 - 8A_5$$
$$- 8A_6 + 4A_6)x + (6A_4 + 12A_5 + 12A_6 - 2A_5 - 2A_6 - 4A_6)]$$

$$-2A_1 - 8A_1x - 4A_2 + 4A_1x^2 + 4A_2x + 4A_3 = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$$

$$12A_4x^2e^{2x} + (30A_4 + 8A_5)xe^{2x} + (6A_4 + 10A_5 + 6A_6)e^{2x} + 4A_1x^2$$
$$+ (4A_2 - 8A_1)x + (-2A_1 - 4A_2 + 4A_3) = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$$

Comparing coefficients on both the sides,

$$12A_4 = 2, \qquad A_4 = \frac{1}{6}$$

$$30A_4 + 8A_5 = 5, \qquad A_5 = \frac{1}{8}(5-5) = 0$$

$$6A_4 + 10A_5 + 6A_6 = -1, \qquad A_6 = \frac{1}{6}(1-1-0) = 0$$

$$4A_1 = 2, \qquad A_1 = \frac{1}{2}$$

$$4A_2 - 8A_1 = -4, \qquad A_2 = -1 + 2A_1 = -1 + 1 = 0$$

$$-2A_1 - 4A_2 + 4A_3 = -1, \qquad A_3 = -\frac{1}{4} + \frac{A_1}{2} + A_2 = -\frac{1}{4} + \frac{1}{4} + 0 = 0$$

Particular integral is

$$y = \frac{1}{2}x^2 + \frac{1}{6}x^3e^{2x}$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{-2x} + \frac{1}{2}x^2 + \frac{1}{6}x^3e^{2x}$$

Example 8: Solve $(D-3)^2(D+2)y = x^2e^{3x} + x^2$.

Solution: Auxiliary equation is

$$(m-3)^2(m+2) = 0$$
$$m = 3 \quad \text{(repeated twice)}, \; m = -2$$
$$\text{C.F.} = (c_1 + c_2x)e^{3x} + c_3e^{-2x}$$
$$Q = x^2e^{3x} + x^2$$

Let particular integral is

$$y = A_1x^2e^{3x} + A_2xe^{3x} + A_3e^{3x} + A_4x^2 + A_5x + A_6$$

Since x^2e^{3x} occurs in Q(x) and e^{3x} is a part of C.F. corresponding to a 2 fold root $m = 3$, multiplying assumed particular integral corresponding to x^2e^{3x} by x^2,

$$y = A_1x^4e^{3x} + A_2x^3e^{3x} + A_3x^2e^{3x} + A_4x^2 + A_5x + A_6$$

$$Dy = 3e^{3x}(A_1x^4 + A_2x^3 + A_3x^2) + e^{3x}(4A_1x^3 + 3A_2x^2 + 2A_3x) + 2A_4x + A_5$$
$$= e^{3x}[3A_1x^4 + (3A_2 + 4A_1)x^3 + (3A_3 + 3A_2)x^2 + 2A_3x] + 2A_4x + A_5$$

$$
\begin{aligned}
D^2y &= 3e^{3x}[3A_1x^4 + (3A_2 + 4A_1)x^3 + (3A_3 + 3A_2)x^2 + 2A_3x] \\
&\quad + e^{3x}[12A_1x^3 + 3(3A_2 + 4A_1)x^2 + 2(3A_3 + 3A_2)x + 2A_3] + 2A_4 \\
&= e^{3x}[9A_1x^4 + (24A_1 + 9A_2)x^3 + (12A_1 + 18A_2 + 9A_3)x^2 \\
&\quad + (6A_2 + 12A_3)x + 2A_3] + 2A_4
\end{aligned}
$$

$$
\begin{aligned}
D^3y &= 3e^{3x}[9A_1x^4 + (24A_1 + 9A_2)x^3 + (12A_1 + 18A_2 + 9A_3)x^2 \\
&\quad + (6A_2 + 12A_3)x + 2A_3] + e^{3x}[36A_1x^3 + 3(24A_1 + 9A_2)x^2 \\
&\quad + 2(12A_1 + 18A_2 + 9A_3)x + 6A_2 + 12A_3] \\
&= e^{3x}[27A_1x^4 + (108A_1 + 27A_2)x^3 + (108A_1 + 81A_2 + 27A_3)x^2 \\
&\quad + (24A_1 + 54A_2 + 54A_3)x + 6A_2 + 12A_3]
\end{aligned}
$$

Substituting these derivatives in the given equation,

$$(D-3)^2(D+2)y = x^2e^{3x} + x^2$$

$$(D^3 - 4D^2 - 3D + 18)y = x^2e^{3x} + x^2$$

$$
\begin{aligned}
&(27A_1 - 39A_1 - 9A_1 + 18A_1)x^4e^{3x} + (108A_1 + 27A_2 - 96A_1 - 36A_2 \\
&\quad -12A_1 - 9A_2 + 18A_2)x^3e^{3x} + (108A_1 + 81A_2 + 27A_3 - 48A_1 - 72A_2 \\
&\quad -36A_3 - 9A_3 - 9A_2 + 18A_3)x^2e^{3x} + (24A_1 + 54A_2 + 54A_3 - 24A_2 \\
&\quad -48A_3 - 6A_3)xe^{3x} + (6A_2 + 12A_3 - 8A_3)e^{3x} - 8A_4 - 3A_5 \\
&\quad +18A_6 + (-6A_4 + 18A_5)x + 18A_4x^2 = x^2e^{3x} + x^2
\end{aligned}
$$

$$
\begin{aligned}
&(60A_1)x^2e^{3x} + (24A_1 + 30A_2)xe^{3x} + (6A_2 + 4A_3)e^{3x} \\
&\quad + 18A_4x^2 + (-6A_4 + 18A_5)x + (-8A_4 - 3A_5 + 18A_6) = x^2e^{3x} + x^2
\end{aligned}
$$

Comparing coefficients on both the sides,

$$60A_1 = 1, \qquad A_1 = \frac{1}{60}$$

$$24A_1 + 30A_2 = 0, \qquad A_2 = -\frac{1}{75}$$

$$6A_2 + 4A_3 = 0, \qquad A_3 = -\frac{3A_2}{2} = \frac{1}{50}$$

$$18A_4 = 1, \qquad A_4 = \frac{1}{18}$$

$$-6A_4 + 18A_5 = 0, \qquad A_5 = \frac{A_4}{3} = \frac{1}{54}$$

$$-8A_4 - 3A_5 + 18A_6 = 0, \qquad A_6 = \frac{1}{18}(8A_4 + 3A_5) = \frac{1}{36}$$

Particular integral is

$$\text{P.I.} = \frac{1}{60}x^4e^{3x} - \frac{1}{75}x^3e^{3x} + \frac{1}{50}x^2e^{3x} + \frac{1}{18}x^2 + \frac{1}{54}x + \frac{1}{36}$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{3x} + c_3e^{-2x} + \left(\frac{1}{60}x^4 - \frac{7}{150}x^3 + \frac{7}{100}x^2\right)e^{3x} + \frac{1}{18}x^2 + \frac{1}{54}x + \frac{1}{36}$$

Exercise 10.14

Solve the following differential equations using method of undetermined coefficients:

1. $(D^2 + 6D + 8)y = e^{-3x} + e^x$.

$$\left[\textbf{Ans.: } y = c_1e^{-2x} + c_2e^{-4x} - e^{-3x} + \frac{e^x}{15}\right]$$

2. $(4D^2 - 1)y = e^x + e^{3x}$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= c_1e^{\frac{x}{2}} + c_2e^{-\frac{x}{2}}\\ &+ \frac{1}{105}(35e^x + 3e^{3x})\end{aligned}\right]$$

3. $(D^2 + D - 6)y = 39\cos 3x$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= c_1e^{2x} + c_2e^{-3x}\\ &+ \frac{1}{2}(\sin 3x - 5\cos 3x)\end{aligned}\right]$$

4. $(D^2 + 2D + 5)y = 6\sin 2x + 7\cos 2x$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= e^{-x}(c_1\cos 2x + c_2\sin 2x)\\ &+ 2\sin 2x - \cos 2x\end{aligned}\right]$$

5. $(D^2 + 4D - 5)y = 34\cos 2x - 2\sin 2x$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= c_1e^x + c_2e^{-5x}\\ &+ 2(\sin 2x - \cos 2x)\end{aligned}\right]$$

6. $(D^3 - D^2 + D - 1)y = 6\cos 2x$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= c_1e^x + c_2\cos x + c_3\sin x\\ &+ \frac{2}{5}(\cos 2x - 2\sin 2x)\end{aligned}\right]$$

7. $(2D^2 - D - 3)y = x^3 + x + 1$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= c_1e^{-x} + c_2e^{\frac{3x}{2}}\\ &- \frac{1}{27}(9x^3 - 9x^2 + 51x - 20)\end{aligned}\right]$$

8. $(D^2 + 4)y = 8x^2$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= c_1\cos 2x + c_2\sin 2x\\ &+ 2x^2 - 1\end{aligned}\right]$$

9. $(3D^2 + 2D - 1)y = e^{-2x} + x$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= c_1e^{-x} + c_2e^{\frac{x}{3}}\\ &+ \frac{1}{7}(e^{-2x} - 7x - 14)\end{aligned}\right]$$

10. $(D^2 - 2D + 3)y = x^2 + \sin x$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= e^x\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right)\\ &+ \frac{1}{27}(9x^2 + 6x - 8) + \frac{1}{4}(\sin x + \cos x)\end{aligned}\right]$$

11. $(D^4 - 1)y = x^4 + 1$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= c_1e^x + c_2e^{-x} + c_3\cos x\\ &+ c_4\sin x - x^4 - 25\end{aligned}\right]$$

12. $(D^2 - 1)y = e^{3x}\cos 2x - e^{2x}\sin 3x$.

$$\left[\begin{aligned}\textbf{Ans.: } y &= c_1e^x + c_2e^{-x}\\ &+ \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x)\\ &+ \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)\end{aligned}\right]$$

13. $(D^2 + 3D + 2)y = 12e^{-x}\sin^3 x.$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{-2x} + \frac{e^{-x}}{10}[(\cos 3x + 3\sin 3x) - 45(\cos x + \sin x)]\right]$$

14. $(D^2 + 4D + 3)y = 6e^{-x}.$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{-3x} + 3xe^{-x}\right]$$

15. $(D^2 - D - 6)y = 5e^{-2x} + 10e^{3x}.$

$$\left[\textbf{Ans.: } y = c_1e^{3x} + c_2e^{-2x} + 2xe^{3x} - xe^{-2x}\right]$$

16. $(D^2 + 16)y = 16\sin 4x.$

$$\left[\textbf{Ans.: } y = c_1\cos 4x + c_2\sin 4x - 2x\cos 4x\right]$$

17. $(D^2 + 25)y = 50\cos 5x + 30\sin 5x.$

$$\left[\textbf{Ans.: } y = c_1\cos 5x + c_2\sin 5x - x(3\cos 5x - 5\sin 5x)\right]$$

18. $(D^3 - 2D^2 + 4D - 8)y = 8(x^2 + \cos 2x)$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2\cos 2x + c_3\sin 2x - (x^2 + x) - \frac{x}{2}(\cos 2x + \sin 2x)\right]$$

19. $(D^2 - 4D + 5)y = 16e^{2x}\cos x.$

$$\left[\textbf{Ans.: } y = e^{2x}(c_1\cos x + c_2\sin x) + 8xe^{2x}\sin x\right]$$

20. $(D^2 - 6D + 13)y = 6e^{3x}\sin x\cos x.$

$$\left[\textbf{Ans.: } y = e^{3x}(c_1\cos 2x + c_2\sin 2x) - \frac{3x}{4}e^{3x}\cos 2x\right]$$

21. $(D^3 + 2D^2 - D - 2)y = e^x + x^2.$

$$\left[\textbf{Ans.: } y = c_1e^{x} + c_2e^{-x} + c_3e^{-2x} + \frac{1}{6}xe^x - \frac{x^2}{2} + \frac{x}{2} - \frac{5}{4}\right]$$

22. $(D^2 - 4D + 4)y = x^3e^{2x} + xe^{2x}.$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{2x} + \left(\frac{x^5}{20} + \frac{x^3}{6}\right)e^{2x}\right]$$

23. $(D^2 - 3D + 2)y = xe^{2x} + \sin x.$

$$\left[\textbf{Ans.: } y = c_1e^{x} + c_2e^{2x} + \left(\frac{x^2}{2} - x\right)e^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x\right]$$

24. $(D^2 + 1)y = \sin^3 x.$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x + \frac{1}{32}\sin 3x - \frac{3}{8}x\cos x\right]$$

25. $(D^2 + 2D + 1)y = x^2e^{-x}.$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{-x} + \frac{x^4}{12}e^{-x}\right]$$

26. $(D^3 - D^2 - 4D + 4)y = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}.$

$$\left[\textbf{Ans.: } y = c_1e^{x} + c_2e^{2x} + c_3e^{-2x} + \frac{x^2}{2} + \frac{x^3}{6}e^{2x}\right]$$

27. $(D^2 - 5D - 6)y = e^{3x},$
$y(0) = 2, \qquad y'(0) = 1$

$$\left[\textbf{Ans.: } y = \frac{10}{21}e^{6x} + \frac{45}{28}e^{-x} - \frac{1}{12}e^{3x}\right]$$

28. $(D^2 - 5D + 6)y = e^x(2x - 3),$
$y(0) = 1,\ y'(0) = 3.$

$$\left[\textbf{Ans.: } y = e^{2x} + xe^{x}\right]$$

29. $(D^3 - D)y = 4e^{-x} + 3e^{2x},$
$y(0) = 0,\ y'(0) = -1,\ y''(0) = 2.$

$$\left[\textbf{Ans.:}\ y = c_1 + c_2 e^{x} + c_3 e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}\right]$$

30. $(D^3 - 2D^2 + D)y = 2e^x + 2x,$
$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0.$

$$\left[\textbf{Ans.:}\ y = x^2 + 4x + 4 + e^x(x^2 - 4)\right]$$

10.9 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Sometimes we come across the linear differential equation with more than one dependent variable and a single independent variable. Such equations are called simultaneous linear differential equations with constant coefficients and can be solved by eliminating one of the dependent variable. This method is known as elimination method. These equations can also be solved by using Laplace transform method, matrices method, short cut operator method. Here, we will discuss the elimination method only.

Note: The total number of arbitrary constants in the general solution is equal to the order of the differential equation of that dependent variable which is obtained first. If total number of arbitrary constants are more than the order of the differential equation (degree of auxiliary equation), then arbitrary constants are obtained by putting dependent variables and their derivatives (as required) in the given simultaneous equation.

Example 1: Solve $\dfrac{dx}{dt} = 5x + y,\ \dfrac{dy}{dt} = y - 4x.$

Solution: Putting $\dfrac{d}{dt} \equiv D$, equations reduce to

$$(D-5)x - y = 0 \qquad \text{... (1)}$$

$$4x + (D-1)y = 0 \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by (D - 1) and then adding,

$$(D-5)(D-1)x + 4x = 0$$
$$(D^2 - 6D + 9)x = 0$$
$$(D-3)^2 x = 0$$

Auxiliary equation,

$$(m-3)^2 = 0$$
$$m = 3, 3 \quad \text{(repeated twice)}$$
$$\text{C.F.} = (c_1 + c_2 t)e^{3t},\ \text{P.I.} = 0$$

Hence, $x = (c_1 + c_2 t)e^{3t}$

$$D(x) = c_2 e^{3t} + (c_1 + c_2 t)3e^{3t}$$
$$= (c_2 + 3c_1 + 3c_2 t)e^{3t}$$

Putting the value of x and Dx in Eq. (1),

$$y = (c_2 + 3c_1 + 3c_2 t)e^{3t} - 5(c_1 + c_2 t)e^{3t}$$
$$= (-2c_1 + c_2 - 2c_2 t)e^{3t}$$

Hence, the general solution is

$$x = (c_1 + c_2 t)e^{3t}$$
$$y = (-2c_1 + c_2 - 2c_2 t)e^{3t}$$

Example 2: Solve $\frac{dx}{dt} - 3x - 6y = t^2,\ \frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t$.

Solution: Putting $\frac{d}{dt} \equiv D$, equations reduce to

$$(D-3)x - 6y = t^2 \qquad \text{... (1)}$$
$$Dx + (D-3)y = e^t \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by (D – 3) and multiplying Eq. (2) by 6 and then adding,

$$(D-3)^2 x + 6Dx = (D-3)t^2 + 6e^t$$
$$(D^2+9)x = 2t - 3t^2 + 6e^t$$

Auxiliary equation,

$$m^2 + 9 = 0, \qquad m = \pm 3i$$
$$\text{C.F.} = c_1 \cos 3t + c_2 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^2+9}(2t-3t^2) + \frac{1}{D^2+9}.6e^t = \frac{1}{9}\left(1+\frac{D^2}{9}\right)^{-1}(2t-3t^2) + \frac{6e^t}{10}$$
$$= \frac{1}{9}\left(1 - \frac{D^2}{9} + \frac{D^4}{81} - \ldots\right)(2t-3t^2) + \frac{3}{5}e^t = \frac{1}{9}\left[(2t-3t^2) - \frac{1}{9}(0-6) + 0\right] + \frac{3}{5}e^t$$
$$= -\frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t$$

Hence,
$$x = c_1 \cos 3t + c_2 \sin 3t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t$$
$$Dx = -3c_1 \sin 3t + 3c_2 \cos 3t - \frac{2t}{3} + \frac{2}{9} + \frac{3}{5}e^t$$

Putting the value of x and Dx in Eqs. (1),

$$-3c_1 \sin 3t + 3c_2 \cos 3t - \frac{2t}{3} + \frac{2}{9} + \frac{3}{5}e^t$$
$$-3\left(c_1 \cos 3t + c_2 \sin 3t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t\right) - t^2 = 6y$$
$$y = -\frac{1}{2}(c_1 + c_2)\sin 3t + \frac{1}{2}(c_2 - c_1)\cos 3t - \frac{2t}{9} - \frac{1}{5}e^t$$

Hence, the general solution is

$$x = c_1 \cos 3t + c_2 \sin 3t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t$$

$$y = -\frac{1}{2}(c_1 + c_2)\sin 3t + \frac{1}{2}(c_2 - c_1)\cos 3t - \frac{2t}{9} - \frac{1}{5}e^t$$

Example 3: Solve $\mathbf{D^2 y = x - 2, D^2 x = y + 2}$.

Solution:

$$D^2 y - x = -2 \qquad \text{... (1)}$$

$$-y + D^2 x = 2 \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (2) by D^2 and then adding,

$$-x + D^4 x = -2 + D^2(2)$$

$$(D^4 - 1)x = -2$$

Auxiliary equation

$$m^4 - 1 = 0$$

$$m = 1,\ -1,\ i,\ -i$$

$$\text{C.F.} = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

$$\text{P.I.} = \frac{1}{D^4 - 1}(-2) = \frac{1}{D^4 - 1}(-2e^{0t})$$

$$= 2$$

Hence, $x = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + 2$

$$Dx = c_1 e^t - c_2 e^t - c_3 \sin t + c_4 \cos t$$

$$D^2 x = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t$$

Putting $D^2 x$ in Eq. (2),

$$y = D^2 x - 2$$

$$= c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t - 2$$

Hence, the general solution is

$$x = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + 2$$

$$y = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t - 2$$

Example 4: Solve $\mathbf{(D^2 + 4)x - 3Dy = 0, 3Dx + (D^2 + 4)y = 0}$.

Solution:

$$(D^2 + 4)x - 3Dy = 0 \qquad \text{... (1)}$$

$$3Dx + (D^2 + 4)y = 0 \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by $(D^2 + 4)$ and Eqs. (2) by 3D and then adding,

$$(D^2+4)^2 x + 9D^2 x = 0$$
$$(D^4+17D^2+16)x = 0$$

Auxiliary equation,

$$m^4+17m^2+16=0$$
$$(m^2+1)(m^2+16)=0$$
$$m = \pm i, \ \pm 4i$$
$$\text{C.F.} = c_1 \cos t + c_2 \sin t + c_3 \cos 4t + c_4 \sin 4t$$

$$\text{P.I.} = \frac{1}{D^4+17D^2+16} \cdot 0 = 0$$

Hence,
$$x = c_1 \cos t + c_2 \sin t + c_3 \cos 4t + c_4 \sin 4t$$
$$Dx = -c_1 \sin t + c_2 \cos t - 4c_3 \sin 4t + 4c_4 \cos 4t$$
$$D^2 x = -c_1 \cos t - c_2 \sin t - 16c_3 \cos 4t - 16c_4 \sin 4t$$

Putting x and D^2x in Eq. (1),

$$3Dy = -c_1 \cos t - c_2 \sin t - 16c_3 \cos 4t - 16c_4 \sin 4t$$
$$+ 4(c_1 \cos t + c_2 \sin t + c_3 \cos 4t + c_4 \sin 4t)$$
$$Dy = \frac{1}{3}(3c_1 \cos t + 3c_2 \sin t - 12c_3 \cos 4t - 12c_4 \sin 4t)$$

Integrating w.r.t. t,

$$y = c_1 \sin t - c_2 \cos t - c_3 \sin 4t + c_4 \cos 4t + k,$$

Putting Dx, D^2y, and y in Eq. (2), we get $k_1 = 0$
Hence, the general solution is

$$x = c_1 \cos t + c_2 \sin t + c_3 \cos 4t + c_4 \sin 4t$$
$$y = c_1 \sin t - c_2 \cos t - c_3 \sin 4t + c_4 \cos 4t$$

Example 5: Solve $(D^2 + D + 1)x + (D^2 + 1)\, y = e^t$, $(D^2 + D)x + D^2\, y = e^{-t}$.

Solution:
$$(D^2+D+1)x + (D^2+1)y = e^t \qquad \text{... (1)}$$
$$(D^2+D)x + D^2 y = e^{-t} \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by D^2 and Eq. (2) by $(D^2 + 1)$ and then subtracting,

$$D^2(D^2+D+1)x - (D^2+1)(D^2+D)x = D^2 e^t - (D^2+1)e^{-t}$$
$$-Dx = e^t - e^{-t} - e^{-t}$$
$$Dx = -e^t + 2e^{-t} \qquad \text{... (3)}$$

Integrating w.r.t. t,

$$x = -e^t - 2e^{-t} + c_1$$

$$D^2 x = -e^t - 2e^{-t}$$

Putting Dx, D^2x in Eq. (2),

$$\begin{aligned} D^2 y &= e^{-t} - D^2 x - Dx \\ &= e^{-t} + e^t + 2e^{-t} + e^t - 2e^{-t} \\ D^2 y &= e^{-t} + 2e^t \end{aligned}$$

Integrating w.r.t. t,

$$Dy = -e^{-t} + 2e^t + k_1$$

Integrating again w.r.t. t,

$$y = e^{-t} + 2e^t + k_1 t + k_2$$

Since the order of the Eq. (3) is one, there should be only one arbitrary constant in the general solution.
Putting x, Dx, D^2x, y, D^2y in Eq. (1),

$$\begin{aligned} (D^2 + D + 1)x + (D^2 + 1)y &= e^t \\ (-e^t - 2e^{-t} - e^t + 2e^{-t} - e^t - 2e^{-t} + c_1) + (e^{-t} + 2e^t + e^{-t} + 2e^t + k_1 t + k_2) &= e^t \\ e^t + c_1 + k_1 t + k_2 &= e^t \\ k_1 t + k_2 &= -c_1 \end{aligned}$$

Therefore, $y = e^{-t} + 2e^t - c_1$

Hence, the general solution is

$$x = -e^t - 2e^{-t} + c_1$$

$$y = 2e^t + e^{-t} - c_1$$

Example 6: Solve $2D^2 x + 3Dy - 4 = 0, 2D^2 y - 3Dx = 0$ where $x = y + Dx = Dy = 0$ at $t = 0$.

Solution:

$$2D^2 x + 3Dy - 4 = 0 \qquad \text{... (1)}$$

$$-3Dx + 2D^2 y = 0 \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by 2D and multiplying Eq. (2) by 3 and then subtracting,

$$4D^3 x + 9Dx - 2D4 = 0$$

$$D(4D^2 + 9)x = 0$$

Auxiliary equation,

$$D(4D^2 + 9) = 0$$

$$D = 0, \ \pm\frac{3i}{2}$$

$$\text{C.F.} = c_1 e^{0t} + c_2 \cos\frac{3}{2}t + c_3 \sin\frac{3}{2}t$$

$$\text{P.I.} = \frac{1}{4D^3 + 9D}\cdot 0 = 0$$

Hence,

$$x = c_1 + c_2 \cos\frac{3}{2}t + c_3 \sin\frac{3}{2}t$$

$$Dx = -\frac{3}{2}c_2 \sin\frac{3}{2}t + \frac{3}{2}c_3 \cos\frac{3}{2}t$$

At $t = 0, x = 0$ and $Dx = 0$

$$c_1 + c_2 = 0 \text{ and } c_1 = 0$$

$$x = c_1 - c_1 \cos\frac{3}{2}t$$

$$Dx = \frac{3}{2}c_1 \sin\frac{3}{2}t$$

Putting the value of Dx in Eq. (2),

$$2D^2 y = 3\cdot\frac{3}{2}c_1 \sin\frac{3}{2}t$$

$$D^2 y = \frac{9}{4}\left(c_1 \sin\frac{3}{2}t\right)$$

Integrating w.r.t. t,

$$Dy = \frac{9}{4}\left(-c_1\frac{\cos\frac{3}{2}t}{\frac{3}{2}}\right) + k_1 = \frac{-3c_1}{2}\cos\frac{3}{2}t + k_1$$

Integrating again w.r.t. t,

$$y = \frac{-3c_1}{2}\left(\frac{\sin\frac{3}{2}t}{\frac{3}{2}}\right) + k_1 t + k_2$$

$$y = -c_1 \sin\frac{3}{2}t + k_1 t + k_2$$

At $t = 0$, $y = 0$ and D$y = 0$

$$k_2 = 0 \quad \text{and} \quad k_1 = \frac{3c_1}{2}$$

Hence,
$$y = -c_1 \sin\frac{3}{2}t + \frac{3c_1}{2}t$$

$$\text{D}y = -\frac{3c_1}{2}\cos\frac{3}{2}t + \frac{3c_1}{2}$$

Also,
$$\text{D}^2 x = \frac{9}{4}c_1 \cos\frac{3}{2}t$$

Putting the value of D^2x and Dy in Eq. (1),

$$\frac{9}{2}c_1 \cos\frac{3}{2}t - \frac{9}{2}c_1 \cos\frac{3}{2}t + \frac{9}{2}c_1 - 4 = 0$$

$$c_1 = \frac{8}{9}$$

Hence, the general solution is

$$x = \frac{8}{9}\left(1 - \cos\frac{3}{2}t\right)$$

$$y = \frac{4}{3}\left(t - \frac{2}{3}\sin\frac{3}{2}t\right)$$

Example 7: Solve $\mathbf{D^2x + 2Dy + 8x = 32t}$, $\mathbf{D^2y + 3Dx - 2y = 60e^{-t}}$ where at $\mathbf{t = 0}$, $\mathbf{x = 6}$, $\mathbf{Dx = 8}$, $\mathbf{y = -24}$ and $\mathbf{Dy = 0}$.

Solution:
$$(\text{D}^2 + 8)x + 2\text{D}y = 32t \qquad \text{... (1)}$$

$$3\text{D}x + (\text{D}^2 - 2)y = 60e^{-t} \qquad \text{... (2)}$$

Eliminating y from Eqs. (1) and (2) by operating Eq. (1) by $(\text{D}^2 - 2)$ and Eq. (2) by 2D and then subtracting,

$$(\text{D}^2 - 2)(\text{D}^2 + 8)x - 6\text{D}^2x = (\text{D}^2 - 2)32t - 2\text{D}(60e^{-t})$$

$$(\text{D}^4 - 16)x = -64t + 120e^{-t} \qquad \text{... (3)}$$

Auxiliary equation

$$m^4 - 16 = 0$$

$$m = -2,\ 2,\ -2i,\ 2i$$

$$\text{C.F.} = c_1e^{-2t} + c_2e^{2t} + c_3\cos 2t + c_4 \sin 2t$$

$$\text{P.I.} = \frac{1}{D^4 - 16}(-64t + 120e^{-t})$$

$$= \frac{-64}{-16}\left(1 - \frac{D^4}{16}\right)^{-1} t + \frac{120e^{-t}}{1-16} = 4\left(1 + \frac{D^4}{16} + \ldots\right) t - 8e^{-t}$$

$$= 4t - 8e^{-t}$$

Hence, $$x = c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos 2t + c_4 \sin 2t + 4t - 8e^{-t}$$

$$Dx = -2c_1 e^{-2t} + 2c_2 e^{2t} - 2c_3 \sin 2t + 2c_4 \cos 2t + 4 + 8e^{-t}$$

$$D^2 x = 4c_1 e^{-2t} + 4c_2 e^{2t} - 4c_3 \cos 2t - 4c_4 \sin 2t - 8e^{-t}$$

Putting x and D^2x in Eq. (1),

$$2Dy = 32t - D^2 x - 8x$$

$$= 32t - 4(c_1 e^{-2t} + c_2 e^{2t} - c_3 \cos 2t - c_4 \sin 2t - 2e^{-t})$$

$$- 8(c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos 2t + c_4 \sin 2t + 4t - 8e^{-t})$$

$$Dy = 36e^{-t} - 6c_1 e^{-2t} - 6c_2 e^{2t} - 2c_3 \cos 2t - 2c_4 \sin 2t$$

Integrating w.r.t. t,

$$y = -36e^{-t} + 3c_1 e^{-2t} - 3c_2 e^{2t} - c_3 \sin 2t + c_4 \cos 2t + k_1$$

Since the order of the Eq. (3) is 4, there should be only 4 arbitrary constants in the general solution.
Putting Dx, y, D^2y in Eq. (2),

$$D^2 y + 3Dx - 2y = 60e^{-t}$$

$$-36e^{-t} + 12 + 24e^{-t} + 72e^{-t} - 2k_1 = 60e^{-t}$$

Comparing costant term on both the sides, $k_1 = 6$

Hence, $y = -36e^{-t} + 3c_1 e^{-2t} - 3c_2 e^{2t} - c_3 \sin 2t + c_4 \cos 2t + 6$

At $t = 0$, $x = 6$, $Dx = 8$ and $y = -24$, $Dy = 0$

$$6 = c_1 + c_2 + c_3 - 8 \qquad \ldots (4)$$

$$c_1 + c_2 + c_3 = 14$$

and $$8 = -2c_1 + 2c_2 + 2c_4 + 4 + 8$$

or $$-c_1 + c_2 + c_4 = -2 \qquad \ldots (5)$$

$$-24 = -36 + 3c_1 - 3c_2 + c_4 + 6$$

or $$3c_1 - 3c_2 + c_4 = 6 \qquad \ldots (6)$$

and $$0 = 36 - 6c_1 - 6c_2 - 2c_3$$

or $$3c_1 + 3c_2 + c_3 = 18 \qquad \ldots (7)$$

Solving Eqs. (4), (5), (6) and (7), we get

$$c_1 = 2,\ c_2 = 0,\ c_3 = 12,\ c_4 = 0$$

Hence, the general solution is

$$x = 2e^{-2t} + 12\cos 2t + 4t - 8e^{-t}$$

$$y = -36e^{-t} + 6e^{-2t} - 12\sin 2t + 6$$

Exercise 10.15

Solve the following differential equations:

1. $\dfrac{dx}{dt} = 3x + 8y, \quad \dfrac{dy}{dt} = -x - 3y$

$$\left[\begin{aligned}\textbf{Ans.: } & x = -4c_1e^{t} - 2c_2e^{-t}, \\ & y = c_1e^{t} - c_2e^{-t}\end{aligned}\right]$$

2. $\dfrac{dx}{dt} = 2y - 1, \quad \dfrac{dy}{dt} = 1 + 2x$

$$\left[\begin{aligned}\textbf{Ans.: } & x = c_1e^{2t} + c_2e^{-2t} - \frac{1}{2}, \\ & y = c_1e^{2t} - c_2e^{-2t} + \frac{1}{2}\end{aligned}\right]$$

3. $(D + 6)y - Dx = 0,\ (3 - D)x - 2\,Dy = 0$ with $x = 2,\ y = 3$ at $t = 0$

$$\left[\begin{aligned}\textbf{Ans.: } & x = 4e^{2t} - 2e^{-3t}, \\ & y = e^{2t} + 2e^{-3t}\end{aligned}\right]$$

4. $\dfrac{dx}{dt} + y - 1 = \sin t, \quad \dfrac{dy}{dt} + x = \cos t$

$$\left[\begin{aligned}\textbf{Ans.: } & x = c_1e^{t} + c_2e^{-t}, \\ & y = 1 + \sin t - c_1e^{t} + c_2e^{-t}\end{aligned}\right]$$

5. $(D + 5)x + (D + 7)y = 2e^{t}$,
$(2D + 1)x + (3D + 1)y = e^{t}$

$$\left[\begin{aligned}\textbf{Ans.: } & x = \frac{1}{1+5t}\left\{(2 - 8c_2)e^{t} + \frac{5}{2}c_1e^{-2t}\right\}, \\ & y = c_1e^{-2t} + c_2e^{t}\end{aligned}\right]$$

6. $\dfrac{d^2x}{dt^2} + y = \sin t, \quad \dfrac{d^2y}{dt^2} + x = \cos t$

$$\left[\begin{aligned}\textbf{Ans.: } & x = c_1e^{t} + c_2e^{-t} + c_3\cos t \\ & \quad + c_4\sin t - \frac{t}{4}\cos t + \frac{t}{4}\sin t \\ & y = -c_1e^{t} - c_2e^{-t} + c_3\cos t \\ & \quad + c_4\sin t + \frac{1}{4}(2 + t)(\sin t - \cos t)\end{aligned}\right]$$

7. $D^2x + 3x - 2y = 0,\ D^2x + D^2y - 3x + 5y = 0$ with $x = 0,\ y = 0,\ Dx = 3,\ Dy = 2$ when $t = 0$

$$\left[\begin{aligned}\textbf{Ans.: } & x = \frac{1}{4}\left(11\sin t + \frac{1}{3}\sin 3t\right), \\ & y = \frac{1}{4}(11\sin t - \sin 3t)\end{aligned}\right]$$

10.10 APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

10.10.1 Orthogonal Trajectories

Two families of curves are called orthogonal trajectories of each other if every curve of one family cuts each curve of another family at right angles.

Working rule:

(a) Cartesian curve $f(x, y, c) = 0$

(i) Obtain the differential equation $F\left(x, y, \frac{dy}{dx}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(x, y, -\frac{dx}{dy}\right) = 0.$

(iii) Solve the differential equation $F\left(x, y, -\frac{dx}{dy}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

(b) Polar curve $f(r, \theta, c) = 0$

(i) Obtain the differential equation $F\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(r, \theta, \ -r^2 \frac{d\theta}{dr}\right) = 0.$

(iii) Solve the differential equation $F\left(r, \theta, \ -r^2 \frac{d\theta}{dr}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

Example 1: Find the orthogonal trajectories of the family of semicubical parabolas $ay^2 = x^3$.

Solution: The equation of the family of curves is

$$ay^2 = x^3 \qquad \dots (1)$$

Differentiating Eq. (1) w.r.t. x,

$$a \cdot 2y \frac{dy}{dx} = 3x^2$$

Substituting $a = \frac{x^3}{y^2}$ from Eq. (1),

$$\frac{x^3}{y^2} \cdot 2y \frac{dy}{dx} = 3x^2$$

$$\frac{2x}{y} \frac{dy}{dx} = 3 \qquad \dots (2)$$

This is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (2),

$$\frac{-2x}{y}\frac{dx}{dy} = 3 \qquad \text{... (3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int -2x\,dx = \int 3y\,dy$$

$$-x^2 = \frac{3y^2}{2} + c$$

$$-2x^2 = 3y^2 + 2c$$

$$2x^2 + 3y^2 + 2c = 0$$

which is the equation of the required orthogonal trajectories.

Example 2: Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is a parameter.

Solution: The equation of the family of curves is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$$\frac{x}{a^2} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0 \qquad \text{... (2)}$$

$$\frac{y}{b^2+\lambda} = -\frac{x}{a^2\left(\frac{dy}{dx}\right)}$$

$$\frac{y^2}{b^2+\lambda} = -\frac{xy}{a^2\left(\frac{dy}{dx}\right)} \qquad \text{... (3)}$$

Substituting Eq. (3) in Eq. (1),

$$\frac{x^2}{a^2} - \frac{xy}{a^2\left(\frac{dy}{dx}\right)} = 1$$

$$(x^2 - a^2)\frac{dy}{dx} = xy \qquad \text{... (4)}$$

This is the differential equation of given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (4),

$$(a^2 - x^2)\frac{dx}{dy} = xy \qquad \text{... (5)}$$

This is the differential equation of the orthogonal trajectories.
Separating the variables and integrating Eq. (5),

$$\int y\,dy = \int \frac{a^2 - x^2}{x}\,dx + c$$

$$\frac{1}{2}y^2 = a^2 \log x - \frac{1}{2}x^2 + c$$

$$x^2 + y^2 = 2a^2 \log x + 2c$$

which is the equation of the required orthogonal trajectories.

Example 3: Find the equation of the family of all orthogonal trajectories of the family of circles, which pass through the origin (0, 0) and have centres on the y-axis.

Solution: The equation of the family of circles passing through (0, 0) and having centres on y-axis is

$$x^2 + y^2 + 2fy = 0 \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. x,

$$2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y+f} \qquad \text{... (2)}$$

From Eq. (1),

$$f = -\frac{x^2 + y^2}{2y}$$

$$y + f = y - \frac{x^2 + y^2}{2y} = \frac{y^2 - x^2}{2y}$$

Substituting in Eq. (2),

$$\frac{dy}{dx} = \frac{-2xy}{y^2 - x^2} \qquad \text{... (3)}$$

This is the differential equation of the given family of circles.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (3),

$$\frac{dx}{dy} = \frac{2xy}{y^2 - x^2}$$

This is the differential equation of the family of orthogonal trajectories.

$$(y^2 - x^2)\mathrm{d}x - 2xy\,\mathrm{d}y = 0 \qquad \ldots (4)$$

$$M = y^2 - x^2, \qquad N = -2xy$$

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation is not exact.

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{4y}{-2xy} = -\frac{2}{x}$$

$$\text{I.F.} = e^{\int -\frac{2}{x}\mathrm{d}x} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying Eq. (4) by $\frac{1}{x^2}$,

$$\left(\frac{y^2}{x^2} - 1\right)\mathrm{d}x - \frac{2y}{x}\mathrm{d}y = 0$$

$$M_1 = \frac{y^2}{x^2} - 1, \qquad N_1 = -\frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} = \frac{2y}{x^2},$$

The equation is exact.

Hence, solution is

$$\int_{y\,\text{constant}} \left(\frac{y^2}{x^2} - 1\right)\mathrm{d}x - \int 0\,\mathrm{d}y = c$$

$$\frac{-y^2}{x} - x = c$$

$$x^2 + y^2 + cx = 0$$

which is the equation of the required orthogonal trajectories representing the equation of the family of the circles with centre on x-axis and passing through origin.

Example 4: Show that the family of confocal conics $\frac{x^2}{a} + \frac{y^2}{a-b} = 1$ is self-orthogonal. Here a is the parameter and b is the constant.

Solution: The equation of the family of curves is

$$\frac{x^2}{a} + \frac{y^2}{a-b} = 1 \qquad \ldots (1)$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{2x}{a}+\frac{2y}{a-b}\frac{dy}{dx}=0$$

$$\frac{yy'}{a-b}=-\frac{x}{a}, \qquad \text{where } y'=\frac{dy}{dx}$$

$$ayy'=-ax+bx$$

$$a(x+yy')=bx$$

$$a=\frac{bx}{x+yy'}$$

Putting value of a in Eq. (1),

$$\frac{x^2(x+yy')}{bx}+\frac{y^2}{\frac{bx}{x+yy'}-b}=1$$

$$\frac{x(x+yy')}{b}+\frac{y^2(x+yy')}{-byy'}=1$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'} \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing y' by $-\frac{1}{y'}$ in Eq. (2),

$$\frac{-\frac{x}{y'}-y}{-\frac{1}{y'}}=\frac{b}{x+\left(-\frac{y}{y'}\right)}$$

$$x+yy'=\frac{by'}{xy'-y}$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'}$$

which is same as Eq. (2). Therefore, differential equation of the family of orthogonal trajectories is the same as differential equation of the family of curves. Hence, the given family of curves is self orthogonal.

Example 5: Find the orthogonal trajectories of the family of the curves $r^n \sin n\theta = a^n$.

Solution: The family of the curves is given by the equation

$$r^n \sin n\theta = a^n \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$nr^{n-1}\frac{dr}{d\theta}\cdot\sin n\theta + r^n n\cos n\theta = 0$$

$$\frac{dr}{d\theta} = -r\cot n\theta \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in Eq. (2),

$$-r^2\frac{d\theta}{dr} = -r\cot n\theta$$

$$r\frac{d\theta}{dr} = \cot n\theta \qquad \text{... (3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int \tan n\theta\, d\theta = \int \frac{dr}{r}$$

$$\frac{\log\sec n\theta}{n} = \log r + \log c$$

$$\log\sec n\theta = n\log rc = \log(rc)^n$$

$$\sec n\theta = c^n r^n$$

$$r^n\cos n\theta = k \text{ where } k = \frac{1}{c^n}$$

which is the equation of the required orthogonal trajectories.

Example 6: Find the orthogonal trajectories of the family of the curves $r = 4a\sec\theta\tan\theta$.

Solution: The equation of the family of curves is

$$r = 4a\sec\theta\,\tan\theta \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{dr}{d\theta} = 4a(\sec\theta\tan\theta\,\tan\theta + \sec\theta\,\sec^2\theta)$$

Substituting $4a = \frac{r}{\sec\theta\tan\theta}$ from Eq. (1),

$$\frac{dr}{d\theta} = r(\tan\theta + \cot\theta\sec^2\theta) \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in Eq. (2),

$$-r^2\frac{d\theta}{dr} = r(\tan\theta + \cot\theta\sec^2\theta)$$

$$-r\frac{d\theta}{dr} = \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta\sin\theta}$$

$$-r\frac{d\theta}{dr} = \frac{\sin^2\theta + 1}{\cos\theta\sin\theta} \qquad \text{... (3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$-\frac{1}{2}\int\frac{2\cos\theta\sin\theta}{\sin^2\theta+1}d\theta = \int\frac{dr}{r}$$

$$-\frac{1}{2}\log(1+\sin^2\theta) = \log r - \log c$$

$$-\log(1+\sin^2\theta) = 2\log r - 2\log c = \log r^2 - \log c^2$$

$$\log r^2(1+\sin^2\theta) = \log c^2$$

$$r^2(1+\sin^2\theta) = c^2$$

which is the equation of the family of orthogonal trajectories.

Example 7: Find the orthogonal trajectories of the family of the curves $r = a\,(1 + \sin^2\theta)$.

Solutin: The equation of the family of the curves is

$$r = a\,(1 + \sin^2\theta) \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$\frac{dr}{d\theta} = a\cdot 2\sin\theta\cos\theta$$

Substituting $a = \frac{r}{1+\sin^2\theta}$ from Eq. (1),

$$\frac{dr}{d\theta} = \frac{r}{1+\sin^2\theta}\cdot 2\sin\theta\cos\theta \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in Eq. (2),

$$-r^2\frac{d\theta}{dr} = \frac{r}{1+\sin^2\theta}\cdot 2\sin\theta\cos\theta$$

$$-r\frac{d\theta}{dr} = \frac{2\sin\theta\cos\theta}{1+\sin^2\theta} \qquad \text{... (3)}$$

This is the differential equation of the family of orthogonal trajectories.

Separating the variables and integrating Eq. (3),

$$\int\left(\frac{1+\sin^2\theta}{2\sin\theta\cos\theta}\right)\mathrm{d}\theta = -\int\frac{\mathrm{d}r}{r}$$

$$\int\left(\operatorname{cosec}\,2\theta+\frac{\tan\theta}{2}\right)\mathrm{d}\theta = -\int\frac{\mathrm{d}r}{r}$$

$$\frac{\log\left(\operatorname{cosec}\,2\theta-\cot 2\theta\right)}{2}+\frac{\log\sec\theta}{2} = -\log r+\log c$$

$$\log\left[\sec\theta\left(\frac{1-\cos 2\theta}{\sin 2\theta}\right)\right] = -2\log r+2\log c = -\log r^2+\log c^2$$

$$\log\left[\sec\theta\cdot\frac{2\sin^2\theta}{2\sin\theta\cos\theta}\right] = \log\frac{c^2}{r^2}$$

$$\sec\theta\tan\theta = \frac{c^2}{r^2}$$

$$r^2 = c^2\cos\theta\cot\theta$$

which is the equation of the family of orthogonal trajectories.

Exercise 10.16

1. Find the orthogonal trajectories of the family of following curves:

(i) $y^2 = 4\,ax$

(ii) $x^2 - y^2 = ax$

(iii) $y^2 = \dfrac{x^3}{a-x}$

(iv) $x^2 + y^2 + 2ay + b = 2$

(v) $(a+x)y^2 = x^2(3a-x)$

[**Ans.:** (i) $2x^2+y^2=c$
(ii) $y(y^2+3x^2)=c$
(iii) $(x^2+y^2)^2=c(2x^2+y^2)$
(iv) $x^2+y^2+2cx-b=0$
(v) $(x^2+y^2)^5=cy^3(5x^2+y^2)$]

2. Show that the family of confocal conics $\dfrac{x^2}{a^2+c}+\dfrac{y^2}{b^2+c}=1$ is self orthogonal. Here a and b are constants and c is the parameter.

3. Find the value of the constant d such that the parabolas $y = c_1x^2 + d$ are the orthogonal trajectories of the family of the ellipses $x^2+2y^2-y=c_2$

[**Ans.:** $d=\dfrac{1}{4}$]

4. Find the orthogonal trajectories of the family of following curves:

(i) $r = a(1+\cos\,\theta)$

(ii) $r = \dfrac{2a}{1+\cos\theta}$

(iii) $r^2 = a\,\sin 2\,\theta$

(iv) $r^n = a^n\cos n\theta$

(v) $r = a\,(\sec\theta+\tan\theta)$

(vi) $r = ae^{\theta}$

[**Ans.:** (i) $r=c(1-\cos\theta)$
(ii) $r=\dfrac{c}{1-\cos\theta}$
(iii) $r^2=c^2\cos 2\theta$
(iv) $r^n=c^n\sin n\theta$
(v) $\log r=-\sin\theta+c$
(vi) $r=ce^{-\theta}$]

10.10.2 Electrical Circuit

A simple electric circuit consists of a voltage source, resistor, inductor and capacitor. To find current, voltage or change in an electric circuit, a differential equation is formed using Kirchhoff's voltage Law (KVL) which states that the algebraic sum of all the voltages in a closed loop or circuit is zero. The voltage across resistor, inductor and capacitor are given by,

$$v_R = \mathrm{R}\, i$$

$$v_L = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

$$v_C = \frac{1}{C}\int i\, \mathrm{d}t$$

R-*L* circuit

The figure shows a simple *R*-*L* circuit.
Applying Kirchhoff's voltage law to the circuit,

$$Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} = e(t)$$

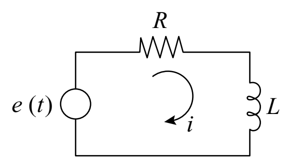

Fig. 10.1

The differential equation is

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R}{L}i = \frac{e(t)}{L}$$

R-*C* circuit

The figure shows a simple *R*-*C* circuit.
Applying Kirchhoff's voltage Law to the circuit,

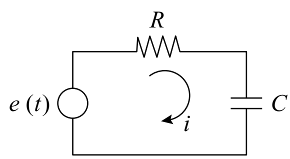

Fig. 10.2

$$Ri + \frac{1}{C}\int i\mathrm{d}t = e(t)$$

Differentiating the equation,

$$R\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{i}{C} = \frac{\mathrm{d}}{\mathrm{d}t}e(t)$$

The differential equation is

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{RC}i = \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

Example 1: A circuit consisting of a resistance *R* and inductance *L* is connected in series with a voltage *E*. (a) Find the value of the current at any time *t*. Given that $i = 0$ at $t = 0$. (b) Show that the current builds up to half its maximum value in $\frac{L}{R}\log 2$ seconds.

Solution: Applying Kirchhoff's law to series *R*-*L* circuit,

$$Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} = E$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R}{L}i = \frac{E}{L}$$

The equation is linear in i.

$$P = \frac{R}{L},\ Q = \frac{E}{L}$$

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

Solution is

$$i.e^{\frac{R}{L}t} = \int \frac{E}{L} e^{\frac{R}{L}t} dt + c = \frac{E}{L} \cdot \frac{L}{R} e^{\frac{R}{L}t} + c$$

$$i = \frac{E}{R} + c\, e^{-\frac{R}{L}t}$$

At $t = 0$, $i = 0$

$$0 = \frac{E}{R} + c$$

$$c = \frac{-E}{R}$$

Hence,

$$i = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L}t} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

(b) The current reaches its maximum value as $t \to \infty$

$$i(\infty) = \frac{E}{R} = I_{\max}$$

When

$$i = \frac{I_{\max}}{2} = \frac{E}{2R}$$

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\frac{1}{2} = 1 - e^{-\frac{R}{L}t}$$

$$e^{-\frac{R}{L}t} = \frac{1}{2}$$

$$e^{\frac{R}{L}t} = 2$$

$$\frac{R}{L}t = \log 2$$

$$t = \frac{L}{R}\log 2$$

Example 2: The current in a circuit containing an inductance L, resistance R and voltage E sin ωt is given by $L\dfrac{di}{dt}+Ri=E\sin\omega t$. If initially there is no current in the circuit show that $i=\dfrac{E}{\sqrt{R^2+\omega^2L^2}}\left[\sin(\omega t-\phi)+\sin\phi\cdot e^{\frac{R}{L}t}\right]$ where $\tan\phi=\dfrac{\omega L}{R}$.

Solution:

$$L\frac{dt}{dt}+Ri=E\sin\omega t$$

$$\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}\sin\omega t$$

The equation is linear in i.

$$P=\frac{R}{L},\ Q=\frac{E}{L}\sin\omega t$$

$$\text{I.F.}=e^{\int\frac{R}{L}dt}=e^{\frac{R}{L}t}$$

Solution is

$$i.e^{\frac{R}{L}t}=\int\frac{E}{L}\sin\omega t\cdot e^{\frac{R}{t}}+c=\frac{E}{L}\frac{e^{\frac{R}{L}t}}{\omega^2+\frac{R^2}{L^2}}\left(\frac{R}{L}\sin\omega t-\omega\cos\omega t\right)+c$$

$$i=\frac{E}{R^2+\omega^2L^2}(R\sin\omega t-\omega L\cos\omega t)+ce^{-\frac{R}{L}t}$$

At $t=0$, $i=0$

$$0=-E\frac{\omega L}{R^2+\omega^2L^2}+c$$

$$c=E\cdot\frac{\omega L}{R^2+\omega^2L^2}$$

Hence,

$$i=\frac{E}{R^2+\omega^2L^2}(R\ \sin\omega t-\omega L\cos\omega t)+e^{-\frac{R}{L}t}\frac{E\omega L}{R^2+\omega^2L^2}$$

$$=E\cdot\frac{1}{\sqrt{R^2+\omega^2L^2}}\left(\frac{R}{\sqrt{R^2+\omega^2L^2}}\sin\omega t-\frac{\omega L}{\sqrt{R^2+\omega^2L^2}}\cos\omega t\right)$$

$$+e^{-\frac{R}{L}t}\frac{E}{\sqrt{R^2+\omega^2L^2}}\cdot\frac{\omega L}{\sqrt{R^2+\omega^2L^2}}$$

Putting $\dfrac{R}{\sqrt{R^2+\omega^2 L^2}} = \cos\phi$ and $\dfrac{\omega L}{\sqrt{R^2+\omega^2 L^2}} = \sin\phi$

$$i = \frac{E}{\sqrt{R^2+\omega^2 L^2}}\left[\sin(\omega t-\phi)+e^{-\frac{R}{L}t}\sin\phi\right]$$

Example 3: A circuit consisting of resistance *R* and a condenser of capacity *C* is connected in series with a voltage *E*. Assuming that there is no charge on condenser at *t* = 0, find the value of current *i*, voltage and charge *q* at any time *t*.

Solution: Applying Kirchoff's law to series R-C circuit,

$$Ri+\frac{1}{C}\int i\,\mathrm{d}t = E$$

But
$$i = \frac{\mathrm{d}q}{\mathrm{d}t}$$

$$R\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{q}{C} = E$$

$$\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{1}{RC}q = \frac{E}{R}$$

The equation is linear in q.

$$\mathrm{P} = \frac{1}{RC},\ \mathrm{Q} = \frac{E}{R}$$

$$\text{I.F.} = e^{\int\frac{1}{RC}\mathrm{d}t} = e^{\frac{t}{RC}}$$

Solution is

$$q\,e^{\frac{t}{RC}} = \int e^{\frac{t}{RC}}\frac{E}{R}\mathrm{d}t+k$$

$$q\,e^{\frac{t}{RC}} = \frac{E}{R}\frac{e^{\frac{t}{RC}}}{\frac{1}{RC}}+k = CE\,e^{\frac{t}{RC}}+k$$

$$q = CE+ke^{-\frac{t}{RC}}$$

At $t = 0$, $q = 0$

$$0 = CE+k$$
$$k = -CE$$

Hence, $$q = CE - CE\, e^{-\frac{t}{RC}} = CE\left(1 - e^{-\frac{t}{RC}}\right) = CE\left(1 - e^{-\frac{t}{RC}}\right)$$

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(CE - CE\, e^{-\frac{t}{RC}}\right)$$

$$= CE\frac{d}{dt}\left(1 - e^{-\frac{t}{RC}}\right) = \frac{CE}{RC}e^{-\frac{t}{RC}}$$

$$= \frac{E}{R}e^{-\frac{t}{RC}}$$

$$e = \frac{1}{C}\int i\ dt = \frac{1}{C}\int \frac{E}{R}e^{-\frac{t}{RC}} = -E\ e^{-\frac{t}{RC}} + k$$

At $t = 0,\ e = 0$

$$0 = -E + k$$
$$k = E$$

$$e = -Ee^{-\frac{t}{RC}} + E$$

$$= E\left(1 - e^{-\frac{t}{RC}}\right)$$

Example 4: An emf $e = 200\, e^{-5t}$ is applied to a series circuit consisting of 20 ohm resistor and 0.01 F capacitor. Find the charge and current at any time assuming that there is no initial charge on capacitor.

Solution: Applying Kirchoff's law to series *R*-*C* circuit,

$$Ri + \frac{1}{C}\int i\ dt = e(t)$$

But $$i = \frac{dq}{dt}$$

$$R\frac{dq}{dt} + \frac{q}{C} = e(t)$$

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{e}{R}$$

Putting the values of *R*, *C* and *e* (*t*),

$$\frac{dq}{dt} + \frac{1}{20 \times 0.01}\ q = \frac{200e^{-5t}}{20}$$

$$\frac{dq}{dt} + 5q = 10e^{-5t}$$

Equation is linear in q.

$$P = 5,\ Q = 10e^{-5t}$$

$$\text{I.F.} = e^{\int 5\,dt} = e^{5t}$$

Solution is

$$qe^{5t} = \int e^{5t}10e^{-5t}dt + k = \int 10\,dt + k = 10\,t + k$$

$$q = 10te^{-5t} + ke^{-5t}$$

At $t = 0$, $q = 0$

$0 = 0 + k$

$k = 0$

Hence,

$$q = 10\,te^{-5t}$$

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(10\,te^{-5t}\right) = 10\,e^{-5t} - 50\,te^{-5t}.$$

Exercise 10.17

1. A coil having a resistance of 15 ohms and an inductance of 10 henry is connected to 90 volt supply. Determine the value of current after 2 seconds.

[**Ans. :** 5.985 amp]

2. If a voltage of 20 cos 5t is applied to a series circuit consisting of 10 ohm resistor and 2 henry inductor, determine the current at any time $t > 0$.

[**Ans. :** $i = \cos 5t + \sin 5t - e^{-5t}$]

3. A capacitor of C farad with voltage v_0 is discharged through a resistance of R ohms. Show that if q coulomb is the charge on capacitor, i ampere is the current and v volt is the voltage at time t, show that $v = v_0 e^{-\frac{t}{RC}}$.

4. Find the current in series $R - C$ circuit with $R = 10\ \Omega$, $c = 0.1\ F$, $e\,(t) = 110 \sin 314\,t$, $i\,(0) = 0$.

$\left[\textbf{Ans. :}\ i(t) = 0.035(\cos 314\,t + 314 \sin 314\,t - e^{-t})\right]$

5. Determine the charge and current at any time t in a series $R - C$ circuit with $R = 10\ \Omega$, $C = 2 \times 10^{-4} F$ and $E = 100\ V$. Given that $q(0) = 0$.

$\left[\textbf{Ans.:}\ q(t) = \dfrac{1 - e^{-500t}}{50},\ i(t) = 10e^{-500t}\right]$

10.10.3 Mechanical System

If a body moves in a straight line starting from a fixed point O and covers a distance x at any instant t, then velocity of the body is given by

$$v = \frac{dx}{dt}$$

and acceleration of the body is given by

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

or
$$a = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v = v\frac{dv}{dx}$$

If mass of the body is m and force acting on it is F, then by Newton's second law of motion

$$F = ma = m\frac{dv}{dt}$$

or
$$F = mv\frac{dv}{dx}$$

This is the equation of the motion of the particle.

Example 1: A chain coiled up near the edge of a smooth table just starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\frac{dv}{dx} + v^2 = gx$. Show that $V = \sqrt{\frac{2gx}{3}}$.

Solution: The equation of the motion is given by

$$xv\frac{dv}{dx} + v^2 = gx$$

$$2v\frac{dv}{dx} + \frac{2v^2}{x} = 2g \qquad \text{... (1)}$$

Putting $v^2 = z$, $2v\frac{dv}{dx} = \frac{dz}{dx}$

Substituting in Eq. (1),

$$\frac{dz}{dx} + \frac{2}{x}z = 2g \qquad \text{... (2)}$$

The equation is linear in z.

$$P = \frac{2}{x}, \ Q = 2g$$

$$\text{I.F.} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Solution of Eq. (2) is

$$zx^2 = \int 2g \cdot x^2\, dx + c$$

$$x^2 z = 2g\frac{x^3}{3} + c$$

$$x^2 v^2 = 2g\frac{x^3}{3} + c$$

Initially, $x = 0$ when $v = 0$

$0 = 0 + c, \qquad c = 0$

Hence,

$$x^2v^2 = \frac{2gx^3}{3}$$

$$v^2 = \frac{2gx}{3}$$

$$v = \sqrt{\frac{2gx}{3}}$$

Example 2: A particle of mass m moves in a horizontal straight line with acceleration $\frac{mk}{x^3}$ directed towards the origin at a distance x from the origin. If initially the particle was at rest at a distance a from the origin, show that it will be at a distance $\frac{a}{2}$ from the origin at $t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$.

Solution: Since acceleration is directed towards the origin, equation of motion is given by

$$mv\frac{dv}{dx} = -\frac{mk}{x^3}$$

$$v\frac{dv}{dx} = -\frac{k}{x^3}$$

Separating the variables and integrating,

$$\int v\,dv = \int -\frac{k}{x^3}\,dx$$

$$\frac{v^2}{2} = \frac{k}{2x^2} + c$$

Initially, when $v = 0$, $x = a$

$$0 = \frac{k}{2a^2} + c$$

$$c = -\frac{k}{2a^2}$$

Hence,

$$\frac{v^2}{2} = \frac{k}{2x^2} - \frac{k}{2a^2}$$

$$v^2 = \frac{k(a^2 - x^2)}{a^2x^2}$$

$$v = \pm\sqrt{k}\,\frac{\sqrt{a^2 - x^2}}{ax}$$

$$\frac{dx}{dt} = -\sqrt{k}\,\frac{\sqrt{a^2 - x^2}}{ax}$$ (Negative sign is taken since x is decreasing with time)

Separating the variables and integrating,

$$\int \frac{ax}{\sqrt{a^2 - x^2}}\,dx = -\int \sqrt{k}\,dt$$

$$-\frac{a}{2}\int (a^2 - x^2)^{-\frac{1}{2}}(-2x)\,dx = -\sqrt{k}\int dt$$

$$-\frac{a}{2}\cdot 2(a^2 - x^2)^{\frac{1}{2}} = -\sqrt{k}t + c' \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$-a(a^2 - x^2)^{\frac{1}{2}} = -\sqrt{k}t + c'$$

At $t = 0,\ x = a$

$c' = 0$

Hence, $$-a(a^2 - x^2)^{\frac{1}{2}} = -\sqrt{k}t$$

When $$x = \frac{a}{2}$$

$$a\left(a^2 - \frac{a^2}{4}\right)^{\frac{1}{2}} = \sqrt{k}t$$

$$t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$$

Example 3: A particle is projected with velocity v_0 along a smooth horizontal plane in the medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance covered by the particle in time t is $\frac{1}{\mu v_0}\left[\sqrt{1+\mu v_0^2 t} - 1\right]$.

Solution: Resistance per unit mass = μv^3

where v is the velocity at any instant t.
By Newton's second law,

$$v\frac{dv}{dx} = -\mu v^3$$

$$\frac{dv}{v^2} = -\mu\,dx$$

Separating the variables and integrating,

$$\int \frac{dv}{v^2} = \int -\mu\, dx$$

$$-\frac{1}{v} = -\mu x + c$$

Initially, $v = v_0$, $x = 0$

$$c = -\frac{1}{v_0}$$

Hence,
$$-\frac{1}{v} = -\mu x - \frac{1}{v_0}$$

$$\frac{1}{v} = \frac{\mu v_0 x + 1}{v_0}$$

$$v = \frac{v_0}{\mu v_0 x + 1}$$

$$\frac{dx}{dt} = \frac{v_0}{\mu v_0 x + 1}$$

where x is the distance travelled at any instant t.
Separating the variables and integrating,

$$\int (\mu v_0 x + 1) dx = \int v_0\, dt$$

$$\mu v_0 \frac{x^2}{2} + x = v_0 t + k$$

At $t = 0$, $x = 0$ $k = 0$

Hence, $\mu v_0 x^2 + 2x - 2v_0 t = 0$

$$x = \frac{-2 \pm \sqrt{4 + 4\mu v_0^2 t}}{2\mu v_0} = \frac{-1 \pm \sqrt{1 + \mu v_0^2 t}}{\mu v_0}$$

But distance is always positive.

Hence,
$$x = \frac{-1 + \sqrt{1 + \mu v_0^2 t}}{\mu v_0}$$

Example 4: A body of mass m, falling from rest, is subjected to the force of gravity and an air resistance proportional to the square of the velocity (i.e. kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that $\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}$ where $mg = ka^2$.

Solution: The forces acting on the body are

(i) its weight mg acting downwards
(ii) air resistance kv^2 acting upwards

Net force acting upon the body $= mg - kv^2 = ka^2 - kv^2 = k(a^2 - v^2)$

By Newton's second law,

$$mv\frac{\mathrm{d}v}{\mathrm{d}x} = k(a^2 - v^2)$$

$$\frac{v}{a^2 - v^2}\mathrm{d}v = \frac{k}{m}\mathrm{d}x$$

Integrating both the sides,

$$-\frac{1}{2}\log(a^2 - v^2) = \frac{k}{m}x + c$$

At $x = 0$, $v = 0$

$$-\frac{1}{2}\log a^2 = c$$

Hence,
$$-\frac{1}{2}\log(a^2 - v^2) = \frac{k}{m}x - \frac{1}{2}\log a^2$$

$$\frac{2\,kx}{m} = \log a^2 - \log(a^2 - v^2)$$

$$= \log\frac{a^2}{a^2 - v^2}$$

Example 5: A moving body is opposed by a force per unit mass of value k_1x and resistance per unit mass of valve k_2v^2, where x and v are the displacement and velocity of the body at some instant. If equation of motion of the moving body is given as $v\frac{\mathrm{d}v}{\mathrm{d}x} = -k_1x - k_2v^2$, find the velocity of the body in terms of x, if it starts from the rest.

Solution: The equation of the motion of the moving body is given by,

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -k_1x - k_2v^2$$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} + k_2v^2 = -k_1x \qquad \text{... (1)}$$

Putting $v^2 = z$, $\quad 2v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x}$

Substituting in Eq. (1),

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 2k_2z = -2k_1x$$

Equation is linear in z.

$$P = 2k_2,\ Q = -2k_1 x$$

$$\text{I.F.} = e^{\int 2k_2 \mathrm{d}x} = e^{2k_2 x}$$

Solution is

$$ze^{2k_2 x} = -\int 2k_1 x e^{2k_2 x}\mathrm{d}x + c = -2k_1\left[x\cdot\frac{e^{2k_2 x}}{2k_2} - (1)\frac{e^{2k_2 x}}{4k_2^2}\right] + c$$

$$= -\frac{k_1 x}{k_2}e^{2k_2 x} + \frac{k_1}{2k_2^2}e^{2k_2 x} + c$$

$$v^2 e^{2k_2 x} = -\frac{k_1}{k_2}xe^{2k_2 x} + \frac{k_1}{2k_2^2}e^{2k_2 x} + c$$

$$v^2 = -\frac{k_1}{k_2}x + \frac{k_1}{2k_2^2} + c\,e^{-2k_2 x}$$

At $x = 0$, $v = 0$

$$\frac{k_1}{2k_2^2} + c = 0$$

$$c = -\frac{k_1}{2k_2^2}$$

Hence,

$$v^2 = -\frac{k_1}{k_2}x + \frac{k_1}{2k_2^2} - \frac{k_1}{2k_2^2}e^{-2k_2 x}$$

$$= \frac{k_1}{2k_2^2}(1 - e^{-2k_2 x}) - \frac{k_1 x}{k_2}$$

Exercise 10.18

1. A moving body is opposed by a force per unit mass of value Cx and resistance per unit mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.

$$\left[\textbf{Ans.}: v = \frac{1}{b}\sqrt{\frac{C}{2}(1 - 2bx - e^{-2bx})}\right]$$

2. When a bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand tank with velocity v_0?

$$\left[\textbf{Ans.}: t = \frac{2}{k}\sqrt{v_0}\right]$$

3. A particle of mass m is projected vertically with velocity v. If the air resistance is directly proportional to the velocity, then show that the particle will reach maximum height in time $\frac{m}{k}\log\left(1 + \frac{kv^2}{mg}\right)$.

4. A body of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is mk times the velocity where k is a constant. Find the terminal velocity of the body and also the time required to attain one half of its terminal velocity.

   Hint: Terminal velocity is velocity at $t \to \infty$.

   $\left[\textbf{Ans.: } v = \dfrac{g}{k},\ t = \dfrac{1}{k}\log 2\right]$

5. A particle is moving in a straight line with acceleration $k\left(x + \dfrac{a^4}{x^3}\right)$ directed towards origin. If it starts from rest at a distance a from the origin, show that it will reach at the origin at the end of time $\dfrac{\pi}{4\sqrt{k}}$.

6. A vehicle starts from rest and its acceleration is given by $k\left(1 - \dfrac{t}{T}\right)$, where k and T are constants. Find the maximum speed and the distance travelled when the maximum speed is attained.

   $\left[\textbf{Ans.: } v_{\max} = \dfrac{kT}{2},\ x = \dfrac{kT^2}{3}\right]$

7. The distance x descended by a parachuter satisfies the differential equation $\left(\dfrac{dx}{dt}\right)^2 = k^2\left[1 - e^{\frac{-2gx}{k^2}}\right]$, where k and g are constants. Show that $x = \dfrac{k^2}{g}\log\ \cosh\left(\dfrac{gt}{k}\right)$ if $x = 0$ at $t = 0$.

10.10.4 Rate of Growth or Decay

If the rate of change of a quantity y at any instant t is directly proportional to the quantity present at that time, then the differential equation of

(i) growth is

$$\frac{dy}{dt} = ky$$

(ii) decay is

$$\frac{dy}{dt} = -ky$$

Example 1: In a culture of yeast, at each instant, the time rate of change of active ferment is proportional to the amount present. If the active ferment doubles in two hours, how much can be expected at the end of 8 hours at the same rate of growth. Find also, how much time will elapse, before the active ferment grows to eight times its initial value.

Solution: Let y quantity of active ferment be present at any time t.

The equation of fermentation of yeast is

$$\frac{dy}{dt} = ky, \quad \text{where } k \text{ is a constant}$$

Separating the variables and integrating,

$$\int \frac{dy}{y} = \int k \, dt$$

$$\log y = kt + c$$

Let at $t = 0$, $y = y_0$

$$\log y_0 = c$$

Hence,

$$\log y = kt + \log y_0$$

$$\log\left(\frac{y}{y_0}\right) = kt \qquad \ldots (1)$$

The active ferment doubles in two hours.
Therefore, at $t = 2$, $y = 2y_0$

$$\log\left(\frac{2y_0}{y_0}\right) = k(2)$$

$$k = \frac{1}{2}\log 2$$

Substituting in Eq. (1),

$$\log\left(\frac{y}{y_0}\right) = \frac{t}{2}\log 2$$

$$y = y_0 e^{\frac{t}{2}\log 2}$$

(i) When $t = 8$

$$y = y_0 e^{4\log 2} = y_0 e^{\log 2^4} = y_0 \cdot 2^4$$

$$y = 16y_0$$

Hence, active ferment grows 16 times of its initial value at the end of 8 hours.

(ii) When $y = 8y_0$

$$8y_0 = y_0 e^{\frac{t}{2}\log 2}$$

$$\log 8 = \frac{t}{2}\log 2$$

$$\log 2^3 = \frac{t}{2}\log 2$$

$$3\log 2 = \frac{t}{2}\log 2$$

$$t = 6 \text{ hours}$$

Hence, active ferment grows 8 times its initial value at the end of 6 hours.

Example 2: Find the half-life of uranium, which disintegrates at a rate proportional to the amount present at any instant. Given that m_1 and m_2 grams of uranium are present at time t_1 and t_2 respectively.

Solution: Let m grams of uranium be present at any time t. The equation of disintegration of uranium is

$$\frac{dm}{dt} = -km \qquad \text{where } k \text{ is a constant}$$

$$\frac{dm}{m} = -k\, dt$$

Integrating both the sides,

$$\log m = -kt + c$$

At $t = 0$, $m = m_0$

$$\log m_0 = c$$

Hence,

$$\log m = -kt + \log m_0$$

$$kt = \log m_0 - \log m \qquad \dots (1)$$

At $t = t_1$, $m = m_1$ and at $t = t_2$, $m = m_2$

$$kt_1 = \log m_0 - \log m_1 \qquad \dots (2)$$

$$kt_2 = \log m_0 - \log m_2 \qquad \dots (3)$$

Subtracting Eqs. (2) from (3),

$$k(t_2 - t_1) = \log m_1 - \log m_2$$

$$k = \frac{\log\left(\frac{m_1}{m_2}\right)}{t_2 - t_1}$$

Let T be the half-life of uranium, i.e., at $t = T$, $m = \frac{1}{2} m_0$

From Eq. (1),

$$kT = \log m_0 - \log \frac{m_0}{2} = \log 2$$

$$T = \frac{\log 2}{k} = \frac{(t_2 - t_1)\log 2}{\log\left(\frac{m_1}{m_2}\right)}$$

Exercise 10.19

1. If the population of a country doubles in 50 years, in how many years will it triple under the assumption that the rate of increase is proportional to the number of inhabitants?

[**Ans. :** 79 years]

2. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour, what would be the value of N after $1\frac{1}{2}$ hours?

[**Ans. :** 605]

3. A radioactive substance disintegrates at a rate proportional to its mass. When mass is 10 mgm, the rate of disintegration is 0.051 mgm per day. How long will it take for the mass to be reduced from 10 mgm to 5 mgm?

[**Ans. :** 136 days]

4. Radium decomposes at a rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will remain at the end of 21 years?

$\left[\textbf{Ans.: } \left(1-\frac{1}{p}\right)^{21} \text{ times the initial amount}\right]$

5. Find the time required for the sum of money to double itself at 5% per annum compounded continuously.

[**Ans. :** 13.9 years]

10.10.5 Newton's Law of Cooling

It states that rate of change of temperature of a body is directly proportional to the difference between the temperature of the body and that of the surrounding medium.

If T is the temperature of the body and T_0 is the temperature of the surrounding medium at any time t then its differential equation is

$$\frac{dT}{dt} = -k(T - T_0) \qquad \text{where } k \text{ is a constant.}$$

Example 1: According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 40°C and the substance cools from 80°C to 60°C in 20 minutes, what will be the temperature of the substance after 40 minutes?

Solution: Let T be the temperature of the substance at the time t.

$$\frac{dT}{dt} = -k(T - 40)$$

Separating the variables and integrating,

$$\int \frac{dT}{T-40} = \int -k\,dt$$

$$\log(T-40) = -kt + c$$

At $t = 0, \quad T = 80$

$$\log 40 = c$$

Hence,

$$kt = \log 40 - \log(T-40)$$

At $t = 20, \quad T = 60$

$$20k = \log 40 - \log 20 = \log 2$$

$$k = \frac{1}{20}\log 2$$

Hence,
$$t \cdot \frac{1}{20}\log 2 = \log 40 - \log(T - 40)$$

At $t = 40$,

$$40 \cdot \frac{1}{20}\log 2 = \log 40 - \log(T - 40)$$

$$2\log 2 = \log\frac{40}{T - 40}$$

$$\log 4 = \log\frac{40}{T - 40}$$

$$4 = \frac{40}{T - 40}$$

$$T = 50°\text{C}$$

Exercise 10.20

1. Water at temperature 100°C cools in 10 minutes to 88°C in a room of temperature 25°C. Find the temperature of water after 20 minutes.
[**Ans. :** 77.9°C]

2. If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C.
[**Ans. :** 52.5 minutes]

10.11 APPLICATIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

10.11.1 Simple Harmonic Motion

If a particle moves in a straight line with an acceleration directly proportional to its displacement from a fixed point O and is always directed towards O, then the motion is said to be simple harmonic motion.

Let the displacement of the particle from a fixed point O at some instant t is x, then

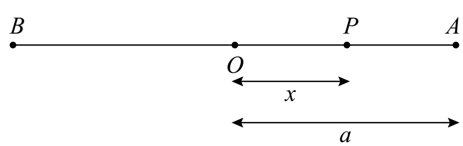

Fig. 10.3

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$(D^2 + \omega^2)x = 0 \qquad \text{... (1)}$$

where $$D \equiv \frac{d}{dt}$$

Solution of differential Eq. (1) is

$$x = c_1 \cos \omega t + c_2 \sin \omega t \qquad \text{... (2)}$$

Velocity of the particle at point P is

$$v = \frac{dx}{dt} = -\omega c_1 \sin \omega t + \omega c_2 \cos \omega t \qquad \text{... (3)}$$

Let the particle starts from rest at distance a from the fixed point O.

Then at $t = 0$, $x = a$, $v = 0$

From Eq. (2)

$$a = c_1$$

From Eq. (3)

$$0 = \omega c_2$$
$$c_2 = 0$$

Hence, displacement $x = a \cos \omega t$

and velocity, $v = -a\omega \sin \omega t = -a\omega\sqrt{1 - \cos^2 \omega t} = -a\omega\sqrt{1 - \frac{x^2}{a^2}}$

$$v = -\omega\sqrt{a^2 - x^2}$$

Nature of motion: The particle starts from rest and moves towards O and attains its maximum velocity at O.

Hence, $|v_{\max}| = a\omega$

At O acceleration is zero but velocity is maximum. Hence, particle moves further and comes to rest at B such that OA = OB. Then it retraces its path and oscillates between A and B.

(i) The amplitude (maximum displacement from O) = a

(ii) The time period (time for a complete oscillation) $= \frac{2\pi}{\omega}$

(iii) The frequency (number of oscillations per second) $= \frac{1}{\text{time period}} = \frac{\omega}{2\pi}$.

Example 1: A Particle is executing simple harmonic motion with amplitude 5 metres and time 4 seconds. Find the time required by the particle in passing between points which are at distances 4 and 2 metres from the centre of force and arc on the same side of it.

Solution: Amplitude, $a = 5$ meter

Time period, $$\frac{2\pi}{\omega} = 4\,\text{sec}$$

$$\omega = \frac{\pi}{2}$$

Let particle is at distances 4 and 2 meter from the centre at time t_1 and t_2 seconds respectively.

Since
$$x = a\cos\omega t$$
$$4 = 5\cos\left(\frac{\pi}{2}t_1\right)$$
$$t_1 = \frac{2}{\pi}\cos^{-1}\frac{4}{5} = 23.47 \text{ sec}$$
and
$$2 = 5\cos\left(\frac{\pi}{2}t_2\right)$$
$$t_2 = \frac{2}{\pi}\cos^{-1}\frac{2}{5} = 42.29 \text{ sec}$$

Time required in passing between the points at distances 4 and 2 meter $= t_2 - t_1 = 18.82$ seconds.

Example 2: A particle of mass 4 g executing S.H.M. has velocities 8 cm/sec and 6 cm/sec when it is at distances 3 cm and 4 cm from the centre of its path. Find its period and amplitude. Find also the force acting on the particle when it is at a distance 1 cm from the centre.

Solution: Velocity of the particle when it is at a distance x from the centre is
$$v^2 = \omega^2(a^2 - x^2)$$

At $x = 3$, $v = 8$ and at $x = 4$, $v = 6$
$$(8)^2 = \omega^2\left[a^2 - (3)^2\right]$$
$$64 = \omega^2(a^2 - 9) \quad \text{... (1)}$$
and
$$(6)^2 = \omega^2\left[a^2 - (4)^2\right]$$
$$36 = \omega^2(a^2 - 16) \quad \text{... (2)}$$

Dividing Eqs. (1) and (2),
$$\frac{64}{36} = \frac{a^2 - 9}{a^2 - 16}$$
$$a^2 = 25$$
$$a = 5$$

Hence, amplitude = 5 cm.
Putting $a = 5$ in Eq. (1)
$$64 = \omega^2(25 - 9)$$
$$\omega^2 = 4$$
$$\omega = 2$$

Hence, time period $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$ sec

Acceleration $= -\omega^2 x$

At $x = 1$ acceleration $= -\omega^2 = -4$

Force = mass × acceleration

$= 4(-4) = -16$ dynes

Negative sign indicates that the force is acting towards the centre.

Exercise 10.21

1. A particle is executing simple harmonic motion with amplitude 20 cm. and time 4 sec. Find the time required by the particle in passing between points which are at distance 15 cm and 5 cm from the centre of force and are the same side of it.

   [**Ans. :** 0.38 sec]

2. A particle of mass 4 gm vibrates through one cm on each side of the centre making 330 complete vibrations per minute. Assuming its motion to be S.H.M, show that the maximum force upon the particle is 484 π^2 dynes.

3. Find the time of a complete oscillation in a simple harmonic motion if $x = x_1$, $x = x_2$ and $x = x_3$ when $t = 1$ sec, $t = 2$ sec, $t = 3$ sec respectively.

   $$\left[\textbf{Ans. :} \frac{2\pi}{\theta} \text{ where } \cos\theta = \frac{x_1 + x_3}{2x_2}\right]$$

10.11.2 Simple Pendulum

A simple pendulum consists of a heavy mass m called bob attached to one end of a light inextensible string with other end fixed. The mass of the string is negligible as compared to the mass m (bob).

Let the pendulum is suspended from a fixed point O. Let l be the length of the light string and m be the mass of the bob. Let P be the position of the bob at any instant t. Let arc $AP = s$ and θ is the angle which OP makes with vertical line OA, then $s = l\theta$.

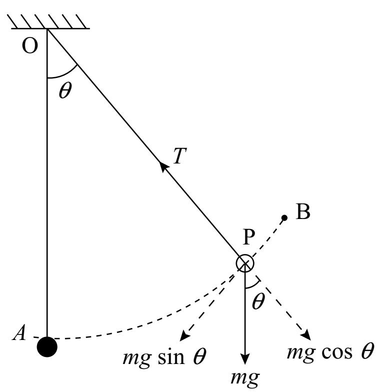

Fig. 10.4

The equation of motion of the bob along the tangent is

$$m\frac{d^2 s}{dt^2} = -mg\sin\theta$$

$$\frac{d^2}{dt^2}(l\theta) = -g\sin\theta$$

$$l\frac{d^2\theta}{dt^2} = -g\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

For sufficiently small θ, higher powers of θ can be neglected.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad \text{where } \omega^2 = \frac{g}{l}$$

This shows that the motion of the bob is simple harmonic motion.

Time period $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{l}{g}}$.

The motion of the bob from one extreme position to another extreme position completes half an oscillation and is called a beat or a swing.

Hence, time of one beat $= \pi\sqrt{\dfrac{l}{g}}$.

Change in the number of beats: If a simple pendulum of length l makes n beats in time t, then

$$t = n\pi\sqrt{\frac{l}{g}}$$

$$n = \frac{t}{\pi}\sqrt{\frac{g}{l}}$$

$$\log n = \log\frac{t}{\pi} + \frac{1}{2}(\log g - \log l)$$

Differentiating both sides,

$$\frac{dn}{n} = \frac{1}{2}\left(\frac{dg}{g} - \frac{dl}{l}\right)$$

This gives the change in number of beats as g and (or) l changes.

(i) If l is constant and g changes,

$$\frac{dn}{n} = \frac{1}{2}\frac{dg}{g}$$

(ii) If l changes and g is constant,

$$\frac{dn}{n} = -\frac{1}{2}\frac{dl}{l}.$$

Example 1: A clock with a seconds pendulum is gaining 2 minutes a day. Prove that the length of the pendulum must be decreased by 0.0028 of its original length to make it go correctly.

Solution: Total number of beats per day,

$$n = 24 \times 60 \times 60 = 86400 \text{ sec}$$

gain per day,
$$\mathrm{d}n = 2 \text{ minutes}$$
$$= 120 \text{ sec}$$

Let l be the original length and dl be the change in length.
Assuming g to be constant

$$\frac{\mathrm{d}n}{n} = -\frac{1}{2}\frac{\mathrm{d}l}{l}$$

$$\frac{\mathrm{d}l}{l} = -\frac{2 \times 120}{86400} = -0.0028$$

$$\mathrm{d}l = -0.0028\, l$$

Hence, length must be decreased by 0.0028 of its original length.

Example 2: The differential equation of a simple pendulum is $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = F \sin nt$, where ω and F are constants. If at $t = 0, x = 0, \frac{\mathrm{d}x}{\mathrm{d}t} = 0$, determine the motion when $n = \omega$.

Solution: The differential equation is given as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = F \sin nt \qquad \text{... (1)}$$

$$(D^2 + \omega^2)\, x = F \sin nt$$

Auxiliary equation is

$$m^2 + \omega^2 = 0$$
$$m = \pm\, i\omega$$
$$\text{C.F.} = c_1 \cos \omega t + c_2 \sin \omega t$$
$$\text{P.I.} = \frac{1}{D^2 + \omega^2} F \sin nt$$

If $n = \omega$

$$\text{P.I.} = \frac{1}{D^2 + \omega^2} F \sin \omega t = Ft \cdot \frac{1}{2D} \sin \omega t$$
$$= \frac{Ft}{2} \int \sin \omega t \, \mathrm{d}t = -\frac{Ft}{2\omega} \cos \omega t$$

Hence, the general solution of Eq. (1) is

$$x = \text{C.F} + \text{P.I.}$$

$$= c_1 \cos \omega t + c_2 \sin \omega t - \frac{Ft}{2\omega} \cos \omega t$$

$$\frac{dx}{dt} = -c_1 \omega \sin \omega t + c_2 \omega \cos \omega t - \frac{F}{2\omega}(\cos \omega t - t\omega \sin \omega t)$$

At $t = 0, x = 0$ and $\frac{dx}{dt} = 0$

$$0 = c_1$$

and

$$0 = c_2 \omega - \frac{F}{2\omega}$$

$$c_2 = \frac{F}{2\omega^2}$$

Hence, the equation of motion is

$$x = \frac{F}{2\omega^2} \sin \omega t - \frac{Ft}{2\omega} \cos \omega t.$$

Exercise 10.22

1. A clock loses five seconds a day, find alteration required in the length of its pendulum in order to keep correct time.

   [**Ans. :** Shortened by $\frac{1}{8640}$ of its original length]

2. A seconds pendulum which gains 10 seconds per day at one place loses 10 seconds per day at another. Compare the acceleration due to gravity at the two places.

   [**Ans. :** $\frac{4321}{4319}$]

3. If a pendulum clock loses 9 minutes per week, what change is required in the length of the pendulum in order to keep correct time?

   [**Ans. :** 1.7 mm]

10.11.3 Oscillation of Spring

Consider a spring suspended vertically from a fixed point support. Let a mass m attached to the lower end A of the spring stretches the spring by a length e called elongation and comes to rest at B. This position is called as static equilibrium.

Now mass is set in motion from equilibrium position. Let at any time t the mass is at P such that $BP = x$. The mass m experience the following forces:

(i) gravitational force mg acting downwards
(ii) restoring force $k(e + x)$ due to displacement of spring acting upwards

(iii) damping (frictional or resistance) force $c\dfrac{dx}{dt}$ of the medium opposing the motion (acting upward)

(iv) external force $F(t)$ considering downward direction as positive.

By Newton's second law, the differential equation of motion of the mass m is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) - c\frac{dx}{dt} + F(t)$$

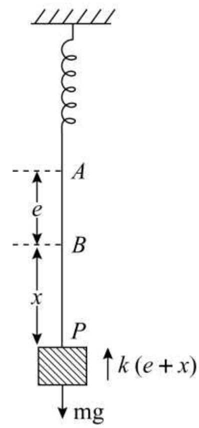

Fig. 10.5

At equilibrium position B,

$$mg = ke$$

Hence,
$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F(t)$$

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = F(t)$$

Let $\dfrac{c}{m} = 2\lambda$ and $\dfrac{k}{m} = \omega^2$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t) \qquad \dots (1)$$

which represent the equation of motion and its solution gives the displacement x of the mass m at any instant t.

Let us consider the different cases of motion.

(a) Free Oscillation If the external force $F(t)$ is absent and damping force is negligible, then Eq. (1) reduces to

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

which represents the equation of simple harmonic motion.
Hence, motion of mass m is S.H.M.

$$\text{Time period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$\text{Frequency} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

(b) Free Damped Oscillations If the external force $F(t)$ is absent and damping is present, then Eq. (1) reduces to

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

(c) Forced Undamped Oscillation If an external periodic force $F(t) = Q\cos nt$ is applied to the support of the spring and damping force is negligible then Eq. (1) reduces to

$$\frac{d^2x}{dt^2} + \omega^2 x = Q\cos nt$$

$$(D^2 + \omega^2)x = Q\cos nt \qquad \text{... (2)}$$

$$\text{C.F.} = c_1 \cos \omega t + c_2 \sin \omega t$$

$$\text{P.I.} = \frac{1}{D^2 + \omega^2} Q\cos nt$$

Hence general solution of Eq. (2) is

$$x = \text{C.F.} + \text{P.I.}$$

If frequency of the external force $\left(\frac{n}{2\pi}\right)$ and the natural frequency $\left(\frac{\omega}{2\pi}\right)$ are same i.e. $\omega = n$, then resonance occurs.

(d) Forced Damped Oscillation If an external periodic force $F(t) = Q \cos nt$ is applied to the support of the spring and damping force is present then Eq. (1) reduces to

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = Q\cos nt$$

Auxiliary Equation is $p^2 + 2\lambda p + \omega^2 = 0$ ... (3)

The general solution is

$$x = \text{C.F.} + \text{P.I.} = x_c + x_p$$

The x_c is known as transient term and tends to zero as $t \to \infty$. Thus term represent damped oscillations. The x_p is known as steady-state term. This term represent simple harmonic motion of period $\frac{2\pi}{n}$.

$$p = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

The motion of the mass depends on the nature of the roots of the Eq. (3), i.e., on discriminant $\lambda^2 - \omega^2$.

Case I: If $\lambda^2 - \omega^2 > 0$ then the roots of Eq. (3) are real and distinct.

$$x_c = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}t}\right).$$

$$x_c \to 0 \text{ as } t \to \infty.$$

This shows that in this case damping is so large that no oscillation can occur. Hence, the motion is called over damped or dead-beat motion.

Case II: If $\lambda^2 - \omega^2 = 0$. then roots of Eq. (2) are equal and real.

$$x_c = (c_1 + c_2 t)e^{-\lambda t}$$

$$x_c \to 0 \text{ as } t \to \infty.$$

In this case damping is just enough to prevent oscillation. Hence the motion is called critically damped.

Case III: If $\lambda^2 - \omega^2 < 0$ then the roots of Eq. (2) are imaginary

$$p = -\lambda \pm i\sqrt{\omega^2 - \lambda^2}$$

Hence, $$x_c = e^{-\lambda t}\left[c_1 \cos\left(\sqrt{\omega^2 - \lambda^2}\right)t + c_2 \sin\left(\sqrt{\omega^2 - \lambda^2}\right)t\right]$$

In this case motion is oscillatory due to the presence of the trigonometric factor. Such a motion is called damped oscillatory motion.

Free oscillation

Example 1: A body weighing 20 kg is hung from a spring. A pull of 40 kg weight will stretch the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time *t* seconds, the maximum velocity and the period of oscillation.

Solution: Since a pull of 40 kg weight stretches the spring to 10 cm,. i.e., 0·1 m

$$40 = k \times 0 \cdot 1$$

$$k = 400 \text{ kg/m}$$

Weight of the body, $W = 20$ kg

$$m = \frac{W}{g} = \frac{20}{9.8}$$

The equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \text{ where } \omega^2 = \frac{k}{m} = 196$$

$$\frac{d^2x}{dt^2} + 196x = 0$$

$$(D^2 + 196)x = 0 \qquad \text{... (1)}$$

Auxiliary equation is

$$m^2 + 196 = 0$$

$$m = \pm 14i$$

Hence, the general solution of Eq. (1) is

$$x = c_1 \cos 14t + c_2 \sin 14t$$

$$\frac{dx}{dt} = -14c_1 \sin 14t + 14c_2 \cos 14t$$

At $t = 0$, $x = 20 \text{ cm} = 0.2 \text{ m}$, $v = \frac{dx}{dt} = 0$,

$0.2 = c_1$ and $0 = 14c_2$, $c_2 = 0$

(i) Hence, displacement of the body from its equilibrium position at time t is given by

$$x = 0{\cdot}2\cos 14t$$

(ii) Amplitude = 20 cm = 0.2 m

maximum velocity $= \omega \times \text{amplitude} = 14 \times 0\cdot 2 = 2\cdot 8$ m/sec

(iii) Period of oscillation $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{14} = 0.45$ sec

Free Damped Oscillation

Example 2: A 3 lb weight on a spring stretches it to 6 inches. Suppose a damping force λv is present ($\lambda > 0$). Show that the motion is (a) critically damped if $\lambda = 1.5$ (b) overdamped if $\lambda > 1.5$ (c) oscillatory if $\lambda < 1.5$.

Solution: A 3 lb weight stretches the spring to 6 inches, i.e., $\dfrac{1}{2}\,ft$

$$3 = k \times \frac{1}{2}$$

$$k = 6\,\text{lb/ft}$$

$$\text{weight} = 3\text{ lb}$$

$$\text{mass} = \frac{3}{g} = \frac{3}{32}$$

$$\text{damping force} = \lambda v = \lambda\frac{\mathrm{d}x}{\mathrm{d}t} \qquad \text{where } \lambda > 0$$

The equation of motion is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx + \lambda\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

$$\frac{3}{32}\mathrm{D}^2x + 6x + \lambda\mathrm{D}x = 0$$

$$\left(\mathrm{D}^2 + \frac{32}{3}\lambda\,\mathrm{D} + 64\right)x = 0 \qquad \text{where } \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}$$

Auxiliary equation is

$$m^2 + \frac{32}{3}\lambda m + 64 = 0 \qquad \text{... (1)}$$

$$m = \frac{-\dfrac{32}{3}\lambda \pm \sqrt{\left(\dfrac{32}{3}\lambda\right)^2 - 256}}{2}$$

$$= \frac{-32\lambda \pm \sqrt{1024\lambda^2 - 2304}}{6}$$

(a) The motion is critically damped when roots of Eq. (1) are equal, i.e., $1024\lambda^2 - 2304 = 0$.

$$\lambda = 1{\cdot}5.$$

(b) The motion is overdamped when roots of Eq. (1) are real and distinct, i.e., $1024\lambda^2 - 2304 > 0\cdot$

$$\lambda > 1.5.$$

(c) The motion is oscillatory when roots of Eq. (1) are imaginary, i.e., $1024\lambda^2 - 2304 < 0$.

$$\lambda < 1.5.$$

Forced Undamped Oscillation

Example 3: Determine whether resonance occurs in a system consisting of a weight 32 lb attached to a spring with constant $k = 4$ lb/ft and external force 16 sin 2t and no damping force present. Initially $x = \frac{1}{2}$ and $\frac{dx}{dt} = -4$.

Solution: The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 16\sin 2t$$

$$\frac{32}{g}\frac{d^2x}{dt^2} + 4x = 16\sin 2t$$

$$(D^2 + 4)x = 16\sin 2t \qquad \left[\because g = 32 \text{ ft/sec}^2\right] \qquad \dots (1)$$

Auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

C.F. $= c_1\cos 2t + c_2 \sin 2\, t$

$$\text{P.I.} = \frac{1}{D^2 + 4}16\sin 2t$$

$$= 16t\frac{1}{2D}\sin 2t$$

$$= 8t\int \sin 2t \; dt = 8t\left(-\frac{\cos 2t}{2}\right)$$

$$= -4t\cos 2t$$

Hence, general solution of Eq. (1) is

$$x = c_1\cos 2t + c_2\sin 2t - 4t\cos 2t$$

$$\frac{dx}{dt} = -2c_1\sin 2t + 2c_2\cos 2t - 4\cos 2t + 8t\sin 2t$$

Initially at $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = -4$

$$\frac{1}{2} = c_1$$

and $$-4 = 2c_2 - 4$$
$$c_2 = 0$$

Hence, $x = \frac{1}{2}\cos 2t - 4t\cos 2t$

$$\omega^2 = \frac{k}{m} = \frac{4}{1}$$
$$\omega = 2$$

Also, $n = 2$

Frequency of the external force $= \frac{n}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$ cycles/sec

Natural frequency $= \frac{\omega}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$ cycles/sec

Since both the frequencies are same, resonance occurs in the system.

Forced Damped Oscillations

Example 4: Determine the transient and steady-state solutions of mechanical system with weight 6 lb, stiffness constant 12 lb/ft, damping force 1.5 times instantaneous velocity, external force 24 cos 8*t* and initial conditions $x = \frac{1}{3}ft$, $\frac{dx}{dt} = 0$.

Solution: Weight = 6 lb, $k = 12$ lb/ft

$$m = \frac{6}{g} = \frac{6}{32}$$

Damping force $= 1.5\frac{dx}{dt}$

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx + 1.5\frac{dx}{dt} = 24\cos 8t$$
$$\frac{6}{32}\frac{d^2x}{dt^2} + 12x + 1.5\frac{dx}{dt} = 24\cos 8t$$
$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 64x = 128\cos 8t$$
$$(D^2 + 8D + 64)x = 128\cos 8t$$

Auxiliary equation is

$$m^2 + 8m + 64 = 0$$
$$m = \frac{-8 \pm \sqrt{64 - 256}}{2} = -4 \pm i4\sqrt{3}$$
$$\text{C.F.} = e^{-4t}(c_1\cos 4\sqrt{3}t + c_2\sin 4\sqrt{3}t)$$

which gives the transient solution

$$\text{P.I.} = \frac{1}{D^2+8D+64}\cdot 128\cos 8t = 128\cdot\frac{1}{-64+8D+64}\cos 8t$$

$$= 16\int \cos 8t\ dt = 16\frac{\sin 8t}{8} = 2\sin 8t$$

P.I. = 2 sin 8t

which gives the steady-state solution.

Exercise 10.23

1. A body weighing 4·9 kg is hung from a spring. A pull of 10 kg will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds, the maximum velocity and the period of oscillation.

$$\left[\begin{array}{l}\textbf{Ans.:}\\ 0.06\cos 20t,\ 1.2\text{ m/sec},\ 0.314\text{ sec}\end{array}\right]$$

2. A mass of 200 gm is tied at the end of a spring which extends to 4 cm under a force of 196, 000 dynes. The spring is pulled 5 cm and released. Find the displacement t seconds after released, if there be a damping force of 2000 dynes per cm per second. What should be the damping force for the dead beat motion?

$$\left[\begin{array}{l}\textbf{Ans.:}\\ e^{-5t}\left(5\cos\sqrt{220}t+\dfrac{25}{\sqrt{220}}\sin\sqrt{220}t\right),\\ 6261\end{array}\right]$$

3. A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of W lb at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by $x = ct \sin 2t$, and find the values of W and c.

$$\left[\textbf{Ans.:}\ W = 6g, c = \frac{1}{12}\right]$$

4. Find the steady-state and transient oscillations of the mechanical system corresponding to the differential equation $\ddot{x} + 2\dot{x} + 2x = \sin 2t - 2\cos 2t$, $x(0) = \dot{x}(0) = 0$.

[**Ans.:** $-0.5 \sin 2t$, $e^{-t} \sin t$]

5. If weight $W = 16$ lb, spring constant k = 10 lb/ft, damping force $2\dfrac{d\dot{x}}{dt}$, external force $F(t)$ is 5 cos 2t, find the motion of the weight given $x(0) = \dot{x}(0) = 0$. Write the transient and steady-state solutions.

$$\left[\begin{array}{l}\textbf{Ans.:}\ x(t) = e^{-2t}\left(-\dfrac{3}{8}\sin 4t-\dfrac{1}{2}\cos 4t\right)\\ \qquad +\dfrac{1}{2}\cos 2t+\dfrac{1}{4}\sin 2t\\ \text{Transient solution}:\ \dfrac{5e^{-2t}}{8}\cos(4t-0.64)\\ \text{Transient solution}:\ \dfrac{\sqrt{5}}{4}\cos(2t-0.46)\end{array}\right]$$

10.11.4 Deflection of Beams

When a beam made up of fibres is bent the fibres of the upper half are compressed and of lower half are stretched. In between there is a region where the fibres are neither compressed nor stretched. This region is called the neutral surface of the beam and the curve of any particular fibre on neutral surface is called elastic curve or deflection curve of the beam. The line at which any plane section of the beam cuts the neutral surface is called neutral axis of that section.

The bending moment M created by the forces acting above and below the neutral surface in opposite direction is

$$M = \frac{EI}{R}$$

where E = modulus of elasticity of the beam

I = moment of inertia of the section about neutral axis

R = radius of curvature of elastic curve at any point $P(x, y)$

$$= \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$, where y represent deflection at a distance x from one end.

Assuming deflection to be very small, $\left(\frac{dy}{dx}\right)^2$ can be neglected

$$R = \frac{1}{\frac{d^2y}{dx^2}}$$

$$M = EI\frac{d^2y}{dx^2} \qquad \text{... (1)}$$

which is the differential equation of the elastic curve.

Boundary Conditions

The arbitrary constants in the solution of Eq. (1) can be found using following boundary conditions:

(i) End freely supported: $y = 0$, $\frac{d^2y}{dx^2} = 0$

(ii) End fixed horizontally: $y = 0$, $\frac{dy}{dx} = 0$

(iii) End perfectly free: $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$

Example 1: A light horizontal strut *AB* is freely pinned at *A* and *B*. It is under the action of equal and opposite compressive forces *P* at its ends and it carries a

load W at its centre. Then for $0 < x < \frac{l}{2}$, $EI\frac{d^2y}{dx^2} + Py + \frac{1}{2}Wx = 0$. Also $y = 0$ at $x = 0$ and $\frac{dy}{dx} = 0$ at $x = \frac{l}{2}$. Prove that $y = \frac{W}{2P}\left(\frac{\sin nx}{n\cos\frac{nl}{2}} - x\right)$ where $n^2 = \frac{P}{EI}$.

Solution: The equation of action of forces is

$$EI\frac{d^2y}{dx^2} + Py + \frac{1}{2}Wx = 0$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y + \frac{W}{2EI}x = 0$$

$$(D^2 + n^2)y = -\frac{W}{2EI}x, \text{ where } \frac{P}{EI} = n^2 \text{ and } D = \frac{d}{dx}$$

$$(D^2 + n^2)y = -\frac{n^2W}{2P}x \qquad \ldots (1)$$

Auxiliary equation is

$$m^2 + n^2 = 0$$

$$m = \pm in$$

$$\text{C.F.} = c_1 \cos nx + c_2 \sin nx$$

$$\text{P.I.} = \frac{1}{D^2 + n^2}\left(-\frac{n^2W}{2P}x\right) = -\frac{n^2W}{2P}\cdot\frac{1}{n^2}\left(1 + \frac{D^2}{n^2}\right)^{-1}x$$

$$= -\frac{W}{2P}\left(1 - \frac{D^2}{n^2} + \frac{D^4}{n^4} - \ldots\right)x$$

$$= -\frac{Wx}{2P}$$

Hence, the general solution of Eq. (1) is

$$y = c_1 \cos nx + c_2 \sin nx - \frac{Wx}{2P}$$

$$\frac{dy}{dx} = -c_1 n \sin nx + c_2 n \cos nx - \frac{W}{2P}$$

At $x = 0$, $y = 0$

$$0 = c_1$$

At $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$

$$0 = c_2 n \cos\frac{nl}{2} - \frac{W}{2P}$$

$$c_2 = \frac{W}{2P}\cdot\frac{1}{n\cos\frac{nl}{2}}$$

Hence, $$y = \frac{W}{2P}\left(\frac{\sin nx}{n\cos\frac{nl}{2}} - \frac{Wx}{2P}\right) \quad \text{where} \quad n^2 = \frac{P}{EI}.$$

Example 2: Find the equation of the elastic curve and its maximum deflection for the simply supported beam of length 2*l*, having uniformly distributed load *W* per unit length.

Solution: Consider the segment $AP = x$.

The forces acting on the segment AP are

(i) The upward thrust Wl at A

(ii) The load Wx at the midpoint of AP.

$$\text{Moment } M = Wlx - Wx\frac{x}{2} = Wlx - \frac{Wx^2}{2} \qquad \text{... (1)}$$

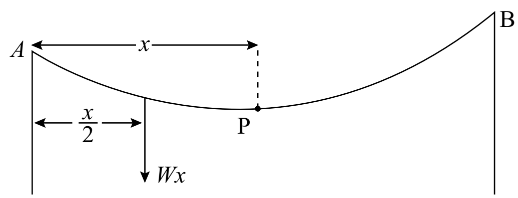

Fig. 10.6

The equation of elastic curve is

$$M = EI\frac{d^2y}{dx^2} \qquad \text{... (2)}$$

Equating equations (1) and (2),

$$EI\frac{d^2y}{dx^2} = Wlx - \frac{Wx^2}{2}$$

Integrating w.r.t. x,

$$EI\frac{dy}{dx} = Wl\frac{x^2}{2} - \frac{Wx^3}{6} + c_1$$

Integrating again w.r.t. x,

$$EI\,y = \frac{Wlx^3}{6} - \frac{Wx^4}{24} + c_1x + c_2 \qquad \text{... (3)}$$

Since ends A and B are freely supported, at A, $x = 0$, $y = 0$ and at B, $x = 2l$, $y = 0$

Putting in Eq. (3),

$$0 = c_2$$

and $$0 = \frac{Wl}{6}(2l)^3 - \frac{W}{24}(2l)^4 + c_1(2l)$$

$$c_1 = \frac{-Wl^3}{3}$$

Hence,
$$EI\,y = \frac{Wlx^3}{6} - \frac{Wx^4}{24} - \frac{Wl^3x}{3}$$
$$y = \frac{Wx}{EI}\left(\frac{lx^2}{6} - \frac{x^3}{24} - \frac{l^3}{3}\right)$$

Deflection of the beam is max. at $x = l$ (mid point)
$$y_{\max} = \frac{Wl}{EI}\left(\frac{l^3}{6} - \frac{l^3}{24} - \frac{l^3}{3}\right) = -\frac{5Wl^4}{24EI}$$

Exercise 10.24

1. A horizontal tie-rod of length $2l$ with concentrated load W at the centre and ends freely hinged, satisfies the differential equation $EI\dfrac{d^2y}{dx^2} = Py - \dfrac{W}{2}x$. With conditions $x = 0$, $y = 0$ and $x = l$, $\dfrac{dy}{dx} = 0$, prove that the deflection δ and the bending moment M at the centre ($x = l$) are given by $\delta = \dfrac{W}{2Pn}(nl - \tanh nl)$ and $M = -\dfrac{W}{2n}\tanh nl$ where $n^2EI = P$.

2. The shape of a strut of length l subjected to an end thrust P and lateral load W per unit length, when the ends are built in, is given by $EI\dfrac{d^2y}{dx^2} + Py = \dfrac{Wx^2}{2} - \dfrac{Wlx}{2} + M$, where M is the moment at a fixed end. Find y in terms of x, given that $y = 0$, $\dfrac{dy}{dx} = 0$ at $x = 0$ and $\dfrac{dy}{dx} = 0$ at $x = \dfrac{l}{2}$.

$$\left[\textbf{Ans.: } y = \frac{Wl}{2Pn}\operatorname{cosec}\frac{nl}{2}\cos\left(nx - \frac{nl}{2}\right) - \frac{Wl}{2Pn}\cot\frac{nl}{2} + \frac{W}{2P}(x^2 - lx)\right]$$

3. A uniform horizontal strut of length l freely supported at both ends, carries a uniformly distributed load W per unit length. If the thrust at each end is P, prove that the maximum deflection is $\dfrac{W}{Pa^2}\left(\sec\dfrac{al}{2} - 1\right) - \dfrac{Wl^2}{8P}$ where $\dfrac{P}{EI} = a^2$. Prove also that the maximum bending moment is of magnitude $\dfrac{W}{a^2}\left(\sec\dfrac{al}{2} - 1\right)$.

4. A horizontal tie-rod is freely pined at each end. It carries a uniform load W per unit length and has a horizontal pull P. Find the central deflection and the maximum bending moment taking the origin at one of its ends.

$$\left[\textbf{Ans.: } \frac{W}{Pa^2}\left(\operatorname{sech}\frac{al}{2} - 1\right) + \frac{Wl^2}{8P},\ \frac{W}{a^2}\left(\operatorname{sech}\frac{al}{2} - 1\right)\right]$$

10.11.5 Electrical Circuit

A second order electrical circuit consists of a resistor, an inductor and a capacitor in series with an emf $e(t)$ as shown in the figure.
Applying Kirchhoff's voltage law to the circuit

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e(t) \qquad \text{... (1)}$$

But $$i = \frac{dq}{dt}$$

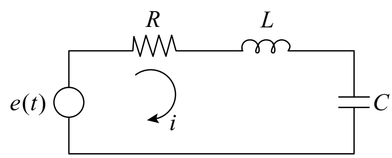

Fig. 10.7

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t) \qquad \text{... (2)}$$

Differentiating Eq. (1) w.r.t. t

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{de(t)}{dt} \qquad \text{... (3)}$$

The Eqs. (2) and (3) are both second order linear non homogeneous ordinary differential equations.

Example 1: A circuit consists of an inductance of 2 henrys, a resistance of 4 ohms and capacitance of 0.05 farads. If $q = i = 0$ at $t = 0$, (a) find $q(t)$ and $i(t)$ when there is a constant emf of 100 volts (b) Find steady state solutions.

Solution: (a) The differential equation of the *RLC* circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = e(t)$$

$$2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + \frac{q}{0.05} = 100$$

$$\frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 10q = 50$$

$$(D^2 + 2D + 10)q = 50$$

Auxiliary equation is

$$m^2 + 2m + 10 = 0$$

$$m = -1 \pm 3i$$

$$\text{C.F.} = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 10}(50e^{0t})$$

$$= \frac{1}{10} \cdot 50 = 5$$

General solution is

$$q = e^{-t}(c_1 \cos 3t + c_2 \sin 3t) + 5 \qquad \text{... (1)}$$

At $t = 0, \quad q = 0$

$$0 = c_1 + 5$$
$$c_1 = -5$$

Differentiating Eq. (1) w.r.t. t,

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = e^{-t}(-3c_1 \sin 3t + 3c_2 \cos 3t) - e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

At $t = 0, i = 0$

$$0 = 3c_2 - c_1$$
$$3c_2 = c_1$$
$$c_2 = -\frac{5}{3}$$

Hence, $q(t) = 5 + e^{-t}(-5\cos 3t - \frac{5}{3}\sin 3t) = 5 - \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$

and $i(t) = e^{-t}(15\sin 3t - 5\cos 3t) + e^{-t}(5\cos 3t + \frac{5}{3}\sin 3t)$

$$= -\frac{5}{3}e^{-t}(3\cos 3t - 9\sin 3t) + \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$$

(b) The steady state solution is obtained by putting $t = \infty$.

$$q(t) = 5$$
$$i(t) = 0$$

Example 2: (a) Determine q and i in an RLC circuit with $L = 0.5$ H, $R = 6\ \Omega$, $C = 0.02$ F, $e = 24 \sin 10t$ and initial conditions $q = i = 0$ at $t = 0$. (b) Find steady state and transient solutions.

Solution: The differential equation of the RLC circuit is

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = e$$

$$0.5\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 6\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{0.02} = 24\sin 10t$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 12\frac{\mathrm{d}q}{\mathrm{d}t} + 100q = 48\sin 10t$$

$$(\mathrm{D}^2 + 12\mathrm{D} + 100)q = 48\sin 10t$$

Auxiliary solution is

$$m^2 + 12m + 100 = 0$$
$$m = -6 \pm 8i$$
$$\text{C.F.} = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t)$$

$$\text{P.I.} = \frac{1}{D^2 + 12D + 100} 48 \sin 10t$$

$$= 48 \cdot \frac{1}{-10^2 + 12D + 100} \sin 10t = \frac{48}{12} \int \sin 10t \, dt$$

$$= 4\left(-\frac{\cos 10t}{10}\right) = -\frac{2}{5}\cos 10t$$

General solution is

$$q = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) - \frac{2}{5}\cos 10t \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. t

$$i = \frac{dq}{dt} = -6e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) + e^{-6t}(-8c_1 \sin 8t + 8c_2 \cos 8t) + \frac{2}{5} \cdot 10 \sin 10t$$

$$= e^{-6t}[(-6c_1 + 8c_2)\cos 8t - (6c_2 + 8c_1)\sin 8t] + 4 \sin 10t$$

At $t = 0$, $q = 0$, $i = 0$

$$0 = c_1 - \frac{2}{5}$$

$$c_1 = \frac{2}{5}$$

and
$$0 = -6c_1 + 8c_2$$

$$c_2 = \frac{6c_1}{8} = \frac{3}{10}$$

Hence,
$$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right) - \frac{2}{5}\cos 10t$$

and
$$i(t) = e^{-6t}(-5 \sin 8t) + 4 \sin 10t$$

The steady state solution is obtained by putting $t = \infty$.

$$q(t) = -\frac{2}{5}\cos 10t$$

$$i(t) = 4 \sin 10t$$

The transient solution

$$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right)$$

$$i(t) = e^{-6t}(-5 \sin 8t)$$

Exercise 10.25

1. A circuit consists of resistance of 5 ohms, inductance of 0.05 henrys and capacitance of 4×10^{-4} farads. If $q(0) = 0$, $i(0) = 0$, find $q(t)$ and $i(t)$ when an emf of 110 volts is applied.

$$\left[\begin{aligned} &\textbf{Ans. :}\ q(t) = e^{-50t}\left(-\frac{11}{250}\cos 50\sqrt{19}t - \frac{11\sqrt{19}}{4750}\sin 50\sqrt{19}t\right) + \frac{11}{250}, \\ &i(t) = \frac{44}{\sqrt{19}}e^{-50t}\sin 50\sqrt{19}t \end{aligned}\right]$$

2. Determine the charge on the capacitor at any time t in series circuit having a resistor of 2 Ω, inductor of 0.1 H capacitor of $\frac{1}{260}$F and $e(t) = 100 \sin 60t$. If the initial current and initial charge on capacitor are both zero, find steady state solution.

$$\left[\begin{aligned} &\textbf{Ans. :}\ q(t) = \frac{6e^{-10t}}{61}(6\sin 50t + 5\cos 50t) - \frac{5}{\sqrt{61}}(5\sin 60t + 6\cos 60t), \text{ Steady state solution:} \\ &q(t) = -\frac{5}{61}(5\sin 60t + 6\cos 60t). \end{aligned}\right]$$

FORMULAE

First Order Differential Equation

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
1.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Exact i.e. $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$	-	(i) $\int M(x, y)\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M \text{ not containing } y)\,dx + \int N(x, y)\,dy = c$
2.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-Exact and $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$,	I.F. $= e^{\int f(x)\,dx}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
3.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-Exact and $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$	I.F. $= e^{\int f(y)\,dy}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
4.	$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$,	Non-Exact	I.F. $= \frac{1}{Mx - Ny}$,	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$
5.	$M(x, y)\,dx + N(x, y)\,dy = 0$	Non-Exact and Homogeneous	I.F. $= \frac{1}{Mx + Ny}$	(i) $\int M_1(x, y)\,dx + \int (\text{terms of } N_1 \text{ not containing } x)\,dy = c$ (ii) $\int (\text{terms of } M_1 \text{ not containing } y)\,dx + \int N_1(x, y)\,dy = c$

Sr. No.	Differential Equation	Type	Integrating Factor	Solution
6.	$x^{m_1} y^{n_1}(a_1 y\,dx + b_1 x\,dy)$ $+ x^{m_2} y^{n_2}(a_2 y\,dx + b_2 x\,dy) = 0$	Non-Exact	I.F. $= x^h y^k$ where $\dfrac{m_1+h+1}{a_1} = \dfrac{n_1+k+1}{b_1}$ and $\dfrac{m_2+h+1}{a_2} = \dfrac{n_2+k+1}{b_2}$	(i) $\int M_1(x, y)\,dx + \int$(terms of N_1 not containing x) $dy = c$ (ii) $\int$(terms of M_1 not containing y) $dx + \int N_1(x, y)\,dy = c$
7.	$\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x	Linear in y	I.F. $= e^{\int P\,dx}$	$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$
8.	$\dfrac{dy}{dx} + Py = Qy^n$	Non linear	I.F. $= e^{\int P_1\,dx}$ where $P_1 = (1-n)v$ and $v = y^{1-n}$	$ve^{\int P_1\,dx} = \int Q_1 e^{\int P_1\,dx}\,dx + c$ where $Q_1 = (1-n)Q$
9.	$f'(y)\dfrac{dy}{dx} + Pf(y) = Q$	Non linear	I.F. $= e^{\int P\,dx}$	$ve^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$ where $f(y) = v$

Note: In the cases 1 to 6 after multiplication by I.F., differential equation reduces to $M_1(x, y)\,dx + N_1(x, y)\,dy = 0$

Higher Order Differential Equations

Homogeneous Linear Differential Equations with constant coefficients

Sr. No.	Roots	Complimentary Function(C.F.)
1.	Real and distinct roots $(m_1, m_2, \dots m_n)$	$y = c_1e^{m_1x} + c_2e^{m_2x} + c_2e^{m_3x} + \dots + c_ne^{m_nx}$
2.	Real and repeated roots $(m_1 = m_2)$	$y = (c_1 + c_2x)e^{m_1x} + c_3e^{m_3x} + c_4e^{m_4x} + \dots + c_ne^{m_nx}$
3.	Imaginary roots $(m_1 = \alpha + i\beta, m_2 = \alpha - i\beta)$	$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + c_4e^{m_4x} + \dots + c_ne^{m_nx}$
4.	Imaginary and repeated roots $(m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta)$	$y = e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x] + c_5e^{m_5x} + c_6e^{m_6x} + \dots + c_ne^{m_nx}$

Non-Homogeneous Linear Differential Equations with constant coefficients

Sr. No.	Q(x)	Particular Integral (P.I.)
1.	e^{ax+b}	(i) $\frac{1}{f(a)}e^{ax+b}$ if $f(a) \neq 0$ (ii) $x^r\frac{1}{f^{(r)}(a)}e^{ax+b}$ if $f^{(r-1)}(a) = 0$ and $f^{(r)}(a) \neq 0$
2.	$\sin(ax+b)$ or $\cos(ax+b)$	(i) $\frac{1}{\phi(-a^2)}\sin(ax+b)$ or $\frac{1}{\phi(-a^2)}\cos(ax+b)$ if $\phi(-a^2) \neq 0$ (ii) $x^r\frac{1}{\phi^{(r)}(-a^2)}\cos(ax+b)$, if $\phi^{(r-1)}(-a^2) = 0$ and $\phi^{(r)}(-a^2) \neq 0$
3.	x^m	$[f(D)]^{-1}x^m = [1 + \phi(D)]^{-1}x^m = (a_0 + a_1D + a_2D^2 + \dots + a_mD^m)x^m$
4.	$e^{ax}V$	$e^{ax}\cdot\frac{1}{f(D+a)}V$
5.	xV	$x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$

If Q(x) is not in any of the above 5 forms, then the solution of the differential equation can be obtained by the following methods:

(i) f(D) is factorized as linear factors of D and P.I. is obtained using the formula

$$\frac{1}{D-a} Q(x) = e^{ax}\int Q(x)e^{-ax}dx$$

(ii) Variation of parameters: If C.F. $= c_1y_1 + c_2y_2$, assume P.I. $= y = v_1(x)y_1 + v_2(x)y_2$ where $v_1 = \int \frac{-y_2Q}{W}dx$, $v_2 = \int \frac{y_1Q}{W}dx$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative in each of the following:

1. The order of the differential equation whose general solution is given by $y=(c_1+c_2)\cos(x+c_3) - c_4e^{x+c_5}$, where c_1, c_2, c_3, c_4, c_5 are arbitrary constants is
   (a) 5 (b) 4
   (c) 3 (d) 2
2. The value of n so that e^{ny^2} is an integrating factor of the differential equation $(e^{\frac{y^2}{2}} - xy)dy - dx = 0$ is
   (a) –1 (b) 1
   (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$
3. The solution of the differential equation $y\frac{dy}{dx} = x - 1$ satisfying $y(1) = 1$ is
   (a) $y^2 = x^2 - 2x + 2$
   (b) $y^2 = 2x^2 - x - 1$
   (c) $y = x^2 - 2x + 2$
   (d) none of these
4. The order of the differential equation $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$ is
   (a) 1 (b) 2
   (c) 3 (d) none of above
5. Match each differential equation in Group I to its family of solution curves from Group II.

Group I	Group II
(P) $\frac{dy}{dx} = \frac{y}{x}$	1. Circles
(Q) $\frac{dy}{dx} = -\frac{y}{x}$	2. Straight lines
(R) $\frac{dy}{dx} = \frac{x}{y}$	3. Hyperbolas
(S) $\frac{dy}{dx} = -\frac{x}{y}$	

   (a) P-2, Q-3, R-3, S-1
   (b) P-1, Q-3, R-2, S-1
   (c) P-2, Q-1, R-3, S-3
   (d) P-3, Q-2, R-1, S-2
6. Which of the following is a solution to the differential equation $\frac{dx}{dt} + 3x = 0$?
   (a) $x = 3e^{-t}$ (b) $x = 2e^{-3t}$
   (c) $x = -\frac{3}{2}t^2$ (d) $x = 3t^2$
7. The solution of the differential equation $k^2\frac{d^2y}{dx^2} = y - y_2$ under the boundary conditions (i) $y = y_1$ at $x = 0$ and (ii) $y = y_2$ at $x = \infty$, where k, y_1 and y_2 are constants, is
   (a) $y = (y_1 - y_2)e^{\left(\frac{-x}{k^2}\right)} + y_2$
   (b) $y = (y_2 - y_1)e^{\left(\frac{-x}{k}\right)} + y_1$

(c) $y = (y_1 - y_2)\sinh\left(\frac{x}{k}\right) + y_1$

(d) $y = (y_1 - y_2)\, e^{\left(\frac{-x}{k}\right)} + y_2$

8. A solution of the following differential equation is given by

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

(a) $y = e^{2x} + e^{-3x}$
(b) $y = e^{2x} + e^{3x}$
(c) $y = e^{-2x} + e^{3x}$
(d) $y = e^{-2x} + e^{-3x}$

9. The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

(a) $(1 + x)e^{x^2}$ (b) $(1 + x)e^{-x^2}$
(c) $(1 - x)e^{x^2}$ (d) $(1 - x)e^{-x^2}$

10. For $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, the particular integral is

(a) $\frac{1}{15}e^{2x}$ (b) $\frac{1}{5}e^{2x}$
(c) $3e^{2x}$ (d) $c_1e^{-x} + c_2e^{-3x}$

11. The solution of the differential equation $\frac{dy}{dx} + y^2 = 0$ is

(a) $y = \frac{1}{x + c}$ (b) $y = -\frac{x^3}{3} + c$
(c) $c\,e^x$
(d) unsolvable as equation is non-linear

12. The differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \sin y = 0 \text{ is}$$

(a) linear (b) non-linear
(c) homogeneous (d) of degree 2

13. The particular solution for the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\cos x \text{ is}$$

(a) $0.5\cos x + 1.5\sin x$
(b) $1.5\,\text{cox}\,x + 0.5\sin x$
(c) $1.5\sin x$
(d) $0.5\cos x$

14. The general solution of the differential equation $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ is

(a) $Ax + Bx^2$
(b) $Ax + B\log x$
(c) $Ax + Bx^2\log x$
(d) $Ax + Bx\log x$
where A and B are constants

15. If $x^2\frac{dy}{dx} + 2xy = \frac{2\log x}{x}$ and $y(1) = 0$, then what is $y(e)$?

(a) e (b) 1
(c) $\frac{1}{e}$ (d) $\frac{1}{e^2}$

16. The family of conics represented by the solution of differential equation $(4x+3y+1)\,dx+(3x+2y+1)dy=0$ is

(a) circles (b) parabolas
(c) hyperbolas (d) ellipses

17. Which one of the following does not satisfy the differential equation

$$\frac{d^3y}{dx^3} - y = 0?$$

(a) e^x
(b) e^{-x}
(c) $e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$
(d) $e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$

18. The orthogonal trajectory of family of curve $y = ax^2$ is

(a) $x^2 + 2y^2 = c$
(b) $x^2 + y^2 = c$
(c) $x^2 - y^2 = c$
(d) $2x^2 + y^2 = c$

19. If e^{-x} and $x\,e^{-x}$ are the fundamental solution of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + y = 0$, the value of a is

(a) 1 (b) 3
(c) 2 (d) 4

20. On conversion $\frac{dy}{dx} = \frac{xy^2 - y}{x}$ into exact equation the differential equation becomes

(a) $\frac{xy-1}{x^2y}dx - \frac{1}{xy^2}dy = 0$

(b) $\frac{x-1}{xy}dx - \frac{1}{xy}dy = 0$

(c) $\frac{1}{x}dx - \frac{1}{y}dy = 0$

(d) None of above

21. The orthogonal trajectories of the cardioid $r = k(1 - \cos\theta)$, where k is a parameter, is
(a) $r = c(1 + \cos\theta)$
(b) $r = c(1 - \sin\theta)$
(c) $r(1 + \cos\theta) = c$
(d) $r(1 - \sin\theta) = c$

22. If $D \equiv \frac{d}{dz}$ and $z = \log x$, then the differential equation
$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 6x$ becomes
(a) $D(D-1)y = 6e^z$
(b) $D(D-1)y = 6e^{2z}$
(c) $D(D+1)y = 6e^{2z}$
(d) $D(D+1)y = 6e^z$

23. The solution of the equation $\frac{d^2y}{dx^2} - y = k$ (k = a non zero constant) which vanishes when $x = 0$ and which tends to a finite limit as x tends to infinity, is
(a) $y = k(1 + e^{-x})$
(b) $y = k(e^{-x} - 1)$
(c) $y = k(e^x + e^{-x} - 2)$
(d) $y = k(e^x - 1)$

24. If the rate of growth is proportional to the amount x of the substance present and $\frac{dx}{dt} = kx$, then x is equal to (with c_1 constant)
(a) c_1e^{-kt} (b) c_1e^{kt}
(c) c_1e^{-2kt} (d) c_1e^{2kt}

25. The solution of differential equation
$$\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)} \text{ is}$$
(a) $\phi\left(\frac{y}{x}\right) = kx$

(b) $\phi\left(\frac{y}{x}\right) = ky$

(c) $x\phi\left(\frac{y}{x}\right) = k$

(d) $y\phi\left(\frac{y}{x}\right) = k$

for some constant k.

26. $m = 2$ is a double root and $m = -1$ is another root of the auxiliary equation of a homogeneous differential with constant coefficient. The differential equation is
(a) $(D^3 + 3D^2 + 4)y = 0$
(b) $(D^3 + 3D^2 - 4)y = 0$
(c) $(D^3 - 3D^2 + 4)y = 0$
(d) $(D^3 - 3D^2 - 4)y = 0$

Answers

1. (c)	2. (c)	3. (a)	4. (b)	5. (a)	6. (b)	7. (d)
8. (b)	9. (a)	10. (b)	11. (a)	12. (b)	13. (a)	14. (d)
15. (d)	16. (c)	17. (b)	18. (a)	19. (c)	20. (a)	21. (a)
22. (c)	23. (b)	24. (b)	25. (a)	26. (c)		