CHAPTER 5

Laplace Transforms and Applications

Chapter Outline

5.1 INTRODUCTION

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations, and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions, and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing, and probability theory.

5.2 LAPLACE TRANSFORM

[Winter 2016]

If $f(t)$ is a function of t defined for all $t \geq 0$ then $\int_0^\infty e^{-st} f(t)\,dt$ is defined as the Laplace transform of $f(t)$, provided the integral exists and is denoted by $L\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

The integral is a function of the parameter s and is denoted by $F(s), \bar{f}(s)$ or $\phi(s)$.

Sufficient Conditions for Existence of Laplace Transforms

The Laplace transform of the function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous, i.e., $f(t)$ is continuous in every sub-interval and $f(t)$ has finite limits at the end points of each sub-interval.

(ii) $f(t)$ is of exponential order of α, i.e., there exists M, α such that $|f(t)| \leq Me^{\alpha t}$, for all $t \geq 0$. In other words,

$$\lim_{t\to\infty} \{e^{-\alpha t} f(t)\} = \text{finite quantity}$$

e.g.,

- $L\{\tan t\}$ does not exist since $\tan t$ is not piecewise continuous.
- $L\{e^{t^2}\}$ does not exist since e^{t^2} is not of any exponential order.

5.3 LAPLACE TRANSFORM OF ELEMENTARY FUNCTIONS

(i) $f(t) = 1$ **[Winter 2012]**

Proof: $L\{1\} = \int_0^\infty e^{-st}\,dt$

$$= \left|\frac{e^{-st}}{-s}\right|_0^\infty$$

$$= \frac{1}{s}$$

(ii) $f(t) = t^n$ **[Winter 2014, 2013; Summer 2013]**

Proof: $L\{t^n\} = \int_0^\infty e^{-st}\, t^n dt$

Putting $st = x$, $dt = \dfrac{dx}{s}$

$$L\{t^n\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x}\ x^n\ \mathrm{d}x$$

$$= \frac{\overline{|n+1}}{s^{n+1}} \qquad\qquad s > 0,\ n+1 > 0$$

If n is a positive integer, $\overline{|n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

(iii) $f(t) = e^{-at}$ **[Winter 2014; Summer 2015, 2013]**

Proof: $L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\ \mathrm{d}t$

$$= \int_0^\infty e^{-(s+a)t}\ \mathrm{d}t$$

$$= \left| \frac{e^{-(s+a)t}}{-(s+a)} \right|_0^\infty$$

$$= \frac{1}{s+a}$$

Similarly, $L\{e^{at}\} = \dfrac{1}{s-a}$

(iv) $f(t) = \sin at$

Proof: $L\{\sin at\} = \int_0^\infty e^{-st} \sin at\ \mathrm{d}t$

$$= \left| \frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at) \right|_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-a)$$

$$= \frac{a}{s^2+a^2}$$

(v) $f(t) = \cos at$

Proof: $L\{\cos at\} = \int_0^\infty e^{-st} \cos at\ \mathrm{d}t$

$$= \left| \frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at) \right|_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-s)$$

$$= \frac{s}{s^2 + a^2}$$

(vi) $f(t) = \sinh at$ **[Winter 2014, 2012; Summer 2015, 2014]**

Proof: $L\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \, dt$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2} \right) dt$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{2a}{s^2 - a^2} \right]$$

$$= \frac{a}{s^2 - a^2}$$

(vii) $f(t) = \cosh at$ **[Winter 2014]**

Proof: $L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-(s-a)t} dt + \int_0^\infty e^{-(s+a)t} dt \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{2s}{s^2 - a^2} \right]$$

$$= \frac{s}{s^2 - a^2}$$

Example 1

Find the Laplace transform of

$$f(t) = \begin{cases} 0 & 0 \le t < 3 \\ 4 & t \ge 3 \end{cases}$$

[Winter 2014]

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t$$

$$= \int_0^3 e^{-st} \cdot 0\, \mathrm{d}t + \int_3^\infty e^{-st} \cdot 4\, \mathrm{d}t$$

$$= 0 + 4 \left| \frac{e^{-st}}{-s} \right|_3^\infty$$

$$= 4 \left| \frac{0}{-s} - \frac{e^{-3s}}{-s} \right|$$

$$= \frac{4}{s} e^{-3s}$$

Example 2

Find the Laplace transform of $f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, \mathrm{d}t$$

$$= \int_0^1 e^{-st} t\, \mathrm{d}t + \int_1^\infty e^{-st} \cdot 0\, \mathrm{d}t$$

$$= \left| t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right|_0^1 + 0$$

$$= \left| -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right|_0^1$$

$$= \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right) - \left(0 - \frac{1}{s^2} \right)$$

$$= -e^{-s}\left(\frac{1}{s} + \frac{1}{s^2} \right) + \frac{1}{s^2}$$

$$= -e^{-s}\left(\frac{s+1}{s^2} \right) + \frac{1}{s^2}$$

$$= \frac{1}{s^2}[1 - e^{-s}(s+1)]$$

Example 3

Find the Laplace transform of

$$f(t) = \begin{cases} (t-2)^2 & t > 2 \\ 0 & 0 < t < 2 \end{cases}$$

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^2 e^{-st} \cdot 0\,\mathrm{d}t + \int_2^\infty e^{-st} (t-2)^2\,\mathrm{d}t \\
&= 0 + \left| \frac{e^{-st}}{-s}(t-2)^2 - \frac{e^{-st}}{s^2} 2(t-2) + \frac{e^{-st}}{-s^3} 2 \right|_2^\infty \\
&= 0 - \frac{e^{-2s}}{-s^3} 2 \\
&= \frac{2}{s^3}\, e^{-2s}
\end{aligned}
$$

Example 4

Find the Laplace transform of

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ e^t & 1 < t < 4 \\ 0 & t > 4 \end{cases}$$

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^1 e^{-st} \cdot 1\,\mathrm{d}t + \int_1^4 e^{-st} e^t\,\mathrm{d}t + \int_4^\infty e^{-st} \cdot 0\,\mathrm{d}t \\
&= \left| \frac{e^{-st}}{-s} \right|_0^1 + \left| \frac{e^{t(1-s)}}{1-s} \right|_1^4 + 0 \\
&= \frac{e^{-s} - 1}{-s} + \frac{e^{4(1-s)} - e^{(1-s)}}{1-s} \\
&= \frac{1 - e^{-s}}{s} + \frac{e^{(1-s)} - e^{4(1-s)}}{s-1}
\end{aligned}
$$

Example 5

Find the Laplace transform of $f(t)=\begin{cases}\sin t & 0<t<\pi\\ 0 & t>\pi\end{cases}$

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\\
&= \int_0^\pi e^{-st}\sin t\,\mathrm{d}t + \int_\pi^\infty e^{-st}(0)\,\mathrm{d}t\\
&= \left|\frac{e^{-st}}{s^2+1}(-s\sin t-\cos t)\right|_0^\pi + 0\\
&= \frac{e^{-\pi s}}{s^2+1}(-s\sin\pi-\cos\pi) - \left[\frac{1}{s^2+1}(0-1)\right]\\
&= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}\\
&= \frac{1+e^{-\pi s}}{s^2+1}
\end{aligned}
$$

Example 6

Find the Laplace transform of $f(t)=\begin{cases}0 & 0<t<\pi\\ \sin t & t>\pi\end{cases}$

[Summer 2015]

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\\
&= \int_0^\pi e^{-st}(0)\,\mathrm{d}t + \int_\pi^\infty e^{-st}\sin t\,\mathrm{d}t\\
&= 0 + \left|\frac{e^{-st}}{s^2+1}(-s\sin t-\cos t)\right|_\pi^\infty\\
&= 0 - \frac{e^{-\pi s}}{s^2+1}(-s\sin\pi-\cos\pi)\\
&= -\frac{e^{-\pi s}}{s^2+1}(0+1)\\
&= -\frac{e^{-\pi s}}{s^2+1}
\end{aligned}
$$

Example 7

Find the Laplace transform of $f(t)=\begin{cases}\cos t & 0<t<2\pi \\ 0 & t>2\pi\end{cases}$

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} e^{-st}\cos t\,\mathrm{d}t + \int_{2\pi}^{\infty} e^{-st}\cdot 0\,\mathrm{d}t \\
&= \left|\frac{e^{-st}}{s^2+1}(-s\cos t+\sin t)\right|_0^{2\pi} + 0 \\
&= \left[\frac{e^{-2\pi s}}{s^2+1}(-s\cos 2\pi+\sin 2\pi)\right]-\left[\frac{1}{s^2+1}(-s+0)\right] \\
&= \frac{e^{-2\pi s}}{s^2+1}(-s+0)+\frac{s}{s^2+1} \\
&= \frac{s}{s^2+1}(1-e^{-2\pi s})
\end{aligned}
$$

Example 8

Find the Laplace transform of $f(t) = \cos\left(t-\frac{2\pi}{3}\right) \quad t>\frac{2\pi}{3}$

$$= 0 \qquad t<\frac{2\pi}{3}$$

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^{\frac{2\pi}{3}} e^{-st}\cdot 0\,\mathrm{d}t + \int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)\mathrm{d}t \\
&= \int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)\mathrm{d}t
\end{aligned}
$$

Putting $t-\frac{2\pi}{3}=x,\qquad \mathrm{d}t=\mathrm{d}x$

When $\quad t = \dfrac{2\pi}{3}, \qquad x = 0$

When $\quad t \to \infty, \qquad x \to \infty$

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-s\left(x+\frac{2\pi}{3}\right)} \cos x \, \mathrm{d}x \\
&= e^{-\frac{2\pi}{3}s} \int_0^\infty e^{-xs} \cos x \, \mathrm{d}x \\
&= e^{-\frac{2\pi}{3}s} \left| \frac{e^{-xs}}{s^2+1}(-s\cos x + \sin x) \right|_0^\infty \\
&= \frac{e^{-\frac{2\pi}{3}s}}{s^2+1}(0+s) \\
&= \frac{se^{-\frac{2\pi}{3}s}}{s^2+1}
\end{aligned}
$$

Example 9

Find the Laplace transform of $\quad f(t) = \cos t \qquad 0 < t < \pi$

$\qquad\qquad\qquad\qquad\qquad\qquad = \sin t \qquad t > \pi$

Solution

$$
\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t) \, \mathrm{d}t \\
&= \int_0^\pi e^{-st} \cos t \, \mathrm{d}t + \int_\pi^\infty e^{-st} \sin t \, \mathrm{d}t \\
&= \left| \frac{e^{-st}}{s^2+1}(-s\cos t + \sin t) \right|_0^\pi + \left| \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t) \right|_\pi^\infty \\
&= \frac{1}{s^2+1}\left[e^{-\pi s}(-s\cos\pi) - (-s\cos 0) + 0 - e^{-\pi s}(-\cos\pi) \right] \\
&= \frac{1}{s^2+1}\left[e^{-\pi s}(s-1) + s \right]
\end{aligned}
$$

Example 10

Find the Laplace transform of

$$f(t) = \begin{cases} t & 0 < t < \frac{1}{2} \\ t - 1 & \frac{1}{2} < t < 1 \\ 0 & t > 1 \end{cases}$$

Solution

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\, \mathrm{d}t \\
&= \int_0^{\frac{1}{2}} e^{-st} t\, \mathrm{d}t + \int_{\frac{1}{2}}^{1} e^{-st}(t-1)\,\mathrm{d}t + \int_1^\infty e^{-st}\cdot 0\,\mathrm{d}t \\
&= \left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\cdot 1\right|_0^{\frac{1}{2}} + \left|\frac{e^{-st}}{-s}(t-1) - \frac{e^{-st}}{s^2}\cdot 1\right|_{\frac{1}{2}}^{1} + 0 \\
&= e^{-\frac{s}{2}}\left(-\frac{1}{2s} - \frac{1}{s^2}\right) - e^0\left(0 - \frac{1}{s^2}\right) - \frac{e^{-s}}{s^2} - e^{-\frac{s}{2}}\left(\frac{1}{2s} - \frac{1}{s^2}\right) \\
&= e^{-\frac{s}{2}}\left(-\frac{1}{s}\right) + \frac{1}{s^2} - \frac{e^{-s}}{s^2} \\
&= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-\frac{s}{2}}}{s}
\end{aligned}$$

Example 11

Find the Laplace transform of

$$f(t) = \begin{cases} 0 & 0 < t < \pi \\ \sin^2(t - \pi) & t > \pi \end{cases}$$

Solution

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)\, \mathrm{d}t \\
&= \int_0^\pi e^{-st}\cdot 0\,\mathrm{d}t + \int_\pi^\infty e^{-st}\sin^2(t-\pi)\,\mathrm{d}t \\
&= 0 + \int_\pi^\infty e^{-st}\left[\frac{1-\cos 2(t-\pi)}{2}\right]\mathrm{d}t \\
&= \frac{1}{2}\int_\pi^\infty e^{-st}[1 - \cos(2\pi - 2t)]\,\mathrm{d}t
\end{aligned}$$

$$= \frac{1}{2}\int_{\pi}^{\infty} e^{-st}(1-\cos 2t)\mathrm{d}t$$

$$= \frac{1}{2}\left[\int_{\pi}^{\infty} e^{-st}\mathrm{d}t - \int_{\pi}^{\infty} e^{-st}\cos 2t\ \mathrm{d}t\right]$$

$$= \frac{1}{2}\left[\left|\frac{e^{-st}}{-s}\right|_{\pi}^{\infty} - \left|\frac{e^{-st}}{s^2+4}(-s\cos 2t + 2\sin 2t)\right|_{\pi}^{\infty}\right]$$

$$= \frac{1}{2}\left[\left(0+\frac{e^{-\pi s}}{s}\right) - \left\{0 - \frac{e^{-\pi s}}{s^2+4}(-s\cos 2\pi + 2\sin 2\pi)\right\}\right]$$

$$= \frac{e^{-\pi s}}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Example 12

Find the Laplace transform of $\frac{1}{\sqrt{t}}$. **[Winter 2016]**

Solution

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^{\infty} e^{-st}\frac{1}{\sqrt{t}}\mathrm{d}t$$

$$= \int_0^{\infty} e^{-st} t^{-\frac{1}{2}}\mathrm{d}t$$

Putting $st = x$, $\mathrm{d}t = \frac{\mathrm{d}x}{s}$

When $t = 0$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$L\{f(t)\} = \int_0^{\infty} e^{-x}\left(\frac{x}{s}\right)^{-\frac{1}{2}}\frac{\mathrm{d}x}{s}$$

$$= \frac{1}{\sqrt{s}}\int_0^{\infty} e^{-x} x^{-\frac{1}{2}}\mathrm{d}x$$

$$= \frac{1}{\sqrt{s}}\int_0^{\infty} e^{-x} x^{\frac{1}{2}-1}\mathrm{d}x$$

$$= \frac{1}{\sqrt{s}}\left\lceil\frac{1}{2}\right.$$

$$= \sqrt{\frac{\pi}{s}} \qquad \left[\because \left\lceil\frac{1}{2} = \sqrt{\pi}\right.\right]$$

EXERCISE 5.1

Find the Laplace transforms of the following functions:

1. $f(t) = t \qquad 0 < t < 3$

$\qquad = 6 \qquad t > 3$

$$\left[\textbf{Ans. : } \frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-3s}\right]$$

2. $f(t) = t^2 \qquad 0 < t < 1$

$\qquad = 1 \qquad t > 1$

$$\left[\textbf{Ans. : } \frac{1}{s}(1 - e^{-s}) - \frac{2e^{-s}}{s^2} + \frac{2}{s^3}(1 - e^{-s})\right]$$

3. $f(t) = (t - a)^3 \qquad t > a$

$\qquad = 0 \qquad t < a$

$$\left[\textbf{Ans. : } \frac{6}{s^4}e^{-as}\right]$$

4. $f(t) = 0 \qquad 0 \le t \le 1$

$\qquad = t \qquad 1 < t < 2$

$\qquad = 0 \qquad t > 2$

$$\left[\textbf{Ans. : } \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} - \left(\frac{1}{s^2} + \frac{2}{s}\right)e^{-2s}\right]$$

5. $f(t) = t^2 \qquad 0 < t < 2$

$\qquad = t - 1 \qquad 2 < t < 3$

$\qquad = 7 \qquad t > 3$

$$\left[\textbf{Ans. : } \frac{2}{s^3} - \frac{e^{-2s}}{s^3}(2 + 3s + 3s^2) + \frac{e^{-3s}}{s^2}(5s - 1)\right]$$

6. $f(t) = e^t \qquad 0 < t < 1$

$\qquad = 0 \qquad t > 1$

$$\left[\textbf{Ans.}: \frac{1}{1-s}(e^{1-s} - 1)\right]$$

7. $f(t) = \cos\left(t - \frac{2\pi}{3}\right) \qquad t > \frac{2\pi}{3}$

$\qquad = 0 \qquad t < \frac{2\pi}{3}$

$$\left[\textbf{Ans.}: e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}\right]$$

8. $f(t) = \sin 2t \qquad 0 < t < \pi$

$\qquad = 0 \qquad t > \pi$

$$\left[\textbf{Ans.}: \frac{2(1-e^{-\pi s})}{s^2+4}\right]$$

5.4 BASIC PROPERTIES OF LAPLACE TRANSFORM

5.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then

$$L\{a f_1(t) + b f_2(t)\} = a F_1(s) + b F_2(s)$$

where a and b are constants.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$\begin{aligned} L\{af_1(t)+bf_2(t)\} &= \int_0^\infty e^{-st}\{af_1(t)+bf_2(t)\}\,\mathrm{d}t \\ &= a\int_0^\infty e^{-st} f_1(t)\mathrm{d}t + b\int_0^\infty e^{-st} f_2(t)\mathrm{d}t \\ &= a F_1(s) + b F_2(s) \end{aligned}$$

Example 1

Find the Laplace transform of $\left(\sqrt{t}-1\right)^2$.

Solution

$$L\left\{\left(\sqrt{t}-1\right)^2\right\} = L\left\{t - 2\sqrt{t} + 1\right\}$$

$$= L\{t\} - 2L\left\{\sqrt{t}\right\} + L\{1\}$$

$$= \frac{1}{s^2} - \frac{2\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{s}$$

$$= \frac{1}{s^2} - \frac{2 \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{s}$$

$$= \frac{1}{s^2} - \frac{\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{1}{s}$$

Example 2

Find the Laplace transform of $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$.

Solution

$$L\left\{\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{t^{\frac{3}{2}} - 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}} - t^{-\frac{3}{2}}\right\}$$

$$= L\left\{t^{\frac{3}{2}}\right\} - 3L\left\{t^{\frac{1}{2}}\right\} + 3L\left\{t^{-\frac{1}{2}}\right\} - L\left\{t^{-\frac{3}{2}}\right\}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} - \frac{3\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{3\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-\frac{1}{2}}}$$

$$= \frac{\frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} - \frac{3 \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{3\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}s^{-\frac{1}{2}}} \qquad \left[\because \begin{array}{r} \Gamma(n+1) = n\Gamma(n) \\ \Gamma(n) = \dfrac{\Gamma(n+1)}{n} \end{array}\right]$$

$$= \sqrt{\frac{\pi}{s}}\left(\frac{3}{4s^2} - \frac{3}{2s} + 3 + 2s\right)$$

Example 3

Find the Laplace transform of $t^2 + \sin 2t$.

Solution

$$L\{t^2 + \sin 2t\} = L\{t^2\} + L\{\sin 2t\}$$
$$= \frac{2}{s^3} + \frac{2}{s^2+4}$$

Example 4

Find the Laplace transform of $4t^2 + \sin 3t + e^{2t}$.

Solution

$$L\{4t^2 + \sin 3t + e^{2t}\} = 4L\{t^2\} + L\{\sin 3t\} + L\{e^{2t}\}$$
$$= 4\cdot\frac{2}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$$
$$= \frac{8}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$$

Example 5

Find the Laplace transform of $\sin 2t \sin 3t$. **[Summer 2014]**

Solution

$$L\{\sin 2t \sin 3t\} = L\left\{\frac{\cos t - \cos 5t}{2}\right\}$$
$$= \frac{1}{2}L\{\cos t\} - \frac{1}{2}L\{\cos 5t\}$$
$$= \frac{s}{2(s^2+1)} - \frac{s}{2(s^2+25)}$$

Example 6

Find the Laplace transform of $\sin^2 3t$. **[Winter 2014]**

Solution

$$L\{\sin^2 3t\} = L\left\{\frac{1-\cos 6t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 6t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+36-s^2}{s(s^2+36)}\right]$$

$$= \frac{18}{s(s^2+36)}$$

Example 7

Find the Laplace transform of $\sin^3 2t$.

Solution

$$L\{\sin^3 2t\} = L\left\{\frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t\right\}$$

$$= \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right)$$

$$= \frac{3}{2(s^2+4)} - \frac{3}{2(s^2+36)}$$

Example 8

Find the Laplace transform of $\cos^2 t$. **[Summer 2018]**

Solution

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} + L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4+s^2}{s(s^2+4)}\right]$$

$$= \frac{s^2+2}{s(s^2+4)}$$

Example 9

Find the Laplace transform of $t^2 - e^{-2t} + \cosh^2 3t$.

Solution

$$\begin{aligned} L\{t^2 - e^{-2t} + \cosh^2 3t\} &= L\{t^2\} - L\{e^{-2t}\} + L\{\cosh^2 3t\} \\ &= L\{t^2\} - L\{e^{-2t}\} + \frac{1}{2}L\{1 + \cosh 6t\} \\ &= \frac{2}{s^3} - \frac{1}{s+2} + \frac{1}{2s} + \frac{s}{2(s^2-36)} \end{aligned}$$

Example 10

Find the Laplace transform of $(\sin 2t - \cos 2t)^2$.

Solution

$$\begin{aligned} L\{(\sin 2t - \cos 2t)^2\} &= L\{\sin^2 2t + \cos^2 2t - 2\cos 2t \sin 2t\} \\ &= L\{1 - \sin 4t\} \\ &= L\{1\} - L\{\sin 4t\} \\ &= \frac{1}{s} - \frac{4}{s^2+16} \end{aligned}$$

Example 11

Find the Laplace transform of $\cos(\omega t + b)$.

Solution

$$\begin{aligned} L\{\cos(\omega t + b)\} &= L\{\cos \omega t \cos b - \sin \omega t \sin b\} \\ &= \cos b\, L\{\cos \omega t\} - \sin b\, L\{\sin \omega t\} \\ &= \cos b \cdot \frac{s}{s^2+\omega^2} - \sin b \cdot \frac{\omega}{s^2+\omega^2} \end{aligned}$$

Example 12

Find the Laplace transform of $\cos t \cos 2t \cos 3t$.

Solution

$$\begin{aligned} L\{\cos t \cos 2t \cos 3t\} &= L\left\{\frac{1}{2}(\cos 3t + \cos t)\cos 3t\right\} \\ &= \frac{1}{2}L\{\cos^2 3t + \cos t \cos 3t\} \\ &= \frac{1}{2}L\left\{\frac{1+\cos 6t}{2} + \frac{\cos 4t + \cos 2t}{2}\right\} \end{aligned}$$

$$= L\left\{\frac{1}{4}+\frac{1}{4}\cos 6t+\frac{1}{4}\cos 4t+\frac{1}{4}\cos 2t\right\}$$

$$= L\left\{\frac{1}{4}\right\}+\frac{1}{4}L\{\cos 6t\}+\frac{1}{4}L\{\cos 4t\}+\frac{1}{4}L\{\cos 2t\}$$

$$= \frac{1}{4s}+\frac{s}{4(s^2+36)}+\frac{s}{4(s^2+16)}+\frac{s}{4(s^2+4)}$$

Example 13

Find the Laplace transform of $\cosh^5 t$.

Solution

$$L\{\cosh^5 t\} = L\left\{\left(\frac{e^t+e^{-t}}{2}\right)^5\right\}$$

$$= L\left\{\frac{1}{2^5}(e^{5t}+5e^{4t}e^{-t}+10e^{3t}e^{-2t}+10\,e^{2t}e^{-3t}+5e^{t}e^{-4t}+e^{-5t})\right\}$$

$$= \frac{1}{32}L\{(e^{5t}+e^{-5t})+5(e^{3t}+e^{-3t})+10(e^{t}+e^{-t})\}$$

$$= \frac{1}{16}L\{\cosh 5t+5\cosh 3t+10\cosh t\}$$

$$= \frac{1}{16}[L\{\cosh 5t\}+5L\{\cosh 3t\}+10L\{\cosh t\}]$$

$$= \frac{1}{16}\left[\frac{s}{s^2-25}+\frac{5s}{s^2-9}+\frac{10s}{s^2-1}\right]$$

Example 14

Find the Laplace transform of $\sin\sqrt{t}$.

Solution

We know that

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

$$\sin\sqrt{t} = t^{\frac{1}{2}}-\frac{t^{\frac{3}{2}}}{3!}+\frac{t^{\frac{5}{2}}}{5!}-\cdots$$

$$L\{\sin\sqrt{t}\} = L\left\{t^{\frac{1}{2}}\right\}-\frac{1}{3!}L\left\{t^{\frac{3}{2}}\right\}+\frac{1}{5!}L\left\{t^{\frac{5}{2}}\right\}-\cdots$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}} - \cdots$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{7}{2}}} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{2\,s^{\frac{3}{2}}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{(4s)}}$$

Example 15

Find the Laplace transform of $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$.

Solution

We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-\frac{1}{2}} - \frac{t^{\frac{1}{2}}}{2!} + \frac{t^{\frac{3}{2}}}{4!} - \cdots$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\left\{t^{-\frac{1}{2}}\right\} - \frac{1}{2!}L\left\{t^{\frac{1}{2}}\right\} + \frac{1}{4!}L\left\{t^{\frac{3}{2}}\right\} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} - \cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} - \cdots$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \cdots\right]$$

$$= \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{(4s)}}$$

EXERCISE 5.2

Find the Laplace transforms of the following functions:

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

$$\left[\textbf{Ans. :}\ \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^2+9}\right]$$

2. $e^{2t} + 4t^3 - \sin 2t \cos 3t$

$$\left[\textbf{Ans. :}\ \frac{1}{s-2} + \frac{24}{s^4} - \frac{5}{2}\cdot\frac{1}{s^2+25} + \frac{1}{2(s^2+1)}\right]$$

3. $3t^2 + e^{-t} + \sin^3 2t$

$$\left[\textbf{Ans. :}\ \frac{6}{s^3} + \frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s^2+4} - \frac{3}{2}\cdot\frac{1}{s^2+36}\right]$$

4. $(t^2 + a)^2$

$$\left[\textbf{Ans. :}\ \frac{a^2 s^4 + 4as^2 + 24}{s^5}\right]$$

5. $\sin(\omega t + \alpha)$

$$\left[\textbf{Ans. :}\ \cos\alpha\cdot\frac{\omega}{s^2+\omega^2} + \sin\alpha\cdot\frac{s}{s^2+\omega^2}\right]$$

6. $\sin 2t \cos 3t$

$$\left[\textbf{Ans. :}\ \frac{2(s^2-5)}{(s^2+1)(s^2+25)}\right]$$

7. $\cos^3 2t$

$$\left[\textbf{Ans. :}\ \frac{s(s^2+28)}{(s^2+36)(s^2+4)}\right]$$

8. $\sinh^3 3t$

$$\left[\textbf{Ans. :}\ \frac{162}{(s^2-81)(s^2-8)}\right]$$

9. $\dfrac{1+2t}{\sqrt{t}}$

$$\left[\textbf{Ans. :} \sqrt{\frac{\pi}{s}}\left(1+\frac{1}{s}\right)\right]$$

10. $\sin(t+\alpha)\cos(t-\alpha)$

$$\left[\textbf{Ans. :} \frac{1}{s^2+4}+\frac{\sin 2\alpha}{s}\right]$$

5.4.2 Change of Scale

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\mathrm{d}t$$

Putting $at = x$, $\quad \mathrm{d}t = \dfrac{\mathrm{d}x}{a}$

When $t = 0$, $\quad x = 0$

When $t \to \infty$, $\quad x \to \infty$

$$L\{f(at)\} = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x)\frac{\mathrm{d}x}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)\mathrm{d}x$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

Example 1

If $L\{f(t)\} = \log\left(\dfrac{s+3}{s+1}\right)$, *find* $L\{f(2t)\}$.

Solution

$$L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$$

By change-of-scale property,

$$L\{f(2t)\} = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Example 2

If $L\{f(t)\} = \frac{1}{\sqrt{s^2+1}}$, *find* $L\{f(3t)\}$.

Solution

$$L\{f(t)\} = \frac{1}{\sqrt{s^2+1}}$$

By change-of-scale property,

$$L\{f(3t)\} = \frac{1}{3}\frac{1}{\sqrt{\left(\frac{s}{3}\right)^2+1}}$$

$$= \frac{1}{\sqrt{s^2+9}}$$

Example 3

If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}$, *find* $L\{\sin 2\sqrt{t}\}$.

Solution

$$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}$$

By change-of-scale property,

$$L\{\sin 2\sqrt{t}\} = L\{\sin\sqrt{4t}\}$$

$$= \frac{1}{4}\frac{\sqrt{\pi}}{2\cdot\frac{s}{4}\sqrt{\frac{s}{4}}}e^{-\frac{1}{4\left(\frac{s}{4}\right)}}$$

$$= \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{s}}$$

Example 4

If $L\{f(t)\} = \dfrac{1}{s\sqrt{s+1}}$, *find* $L\{f(2\sqrt{t})\}$.

Solution

$$L\{f(t)\} = \frac{1}{s\sqrt{s+1}}$$

By change-of-scale property,

$$\begin{aligned} L\{f(2\sqrt{t})\} &= L\{f(\sqrt{4t})\} \\ &= \frac{1}{4}\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} \\ &= \frac{2}{s\sqrt{s+4}} \end{aligned}$$

EXERCISE 5.3

1. If $L\{f(t)\} = \dfrac{8(s-3)}{(s^2-6s+25)^2}$, find $L\{f(2t)\}$.

$$\left[\textbf{Ans. :} \frac{32(s-6)}{(s^2-12s+100)^2}\right]$$

2. If $L\{f(t)\} = \dfrac{2}{s^3}e^{-s}$, find $L\{f(3t)\}$.

$$\left[\textbf{Ans. :} \frac{18}{s^3}e^{-\frac{s}{3}}\right]$$

3. If $L\{f(t)\} = \dfrac{s^2-s-1}{(2s+1)^2(s-1)}$, find $L\{f(2t)\}$.

$$\left[\textbf{Ans. :} \frac{s^2-2s-4}{4(s+1)^2(s-2)}\right]$$

5.4.3 First Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{e^{-at}f(t)\} = F(s+a)$.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-st} e^{-at} f(t)\mathrm{d}t$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\mathrm{d}t$$

$$= F(s+a)$$

Example 1

Find the Laplace transform of $e^{-3t} t^4$.

Solution

$$L\{t^4\} = \frac{4!}{s^5}$$

By the first shifting theorem,

$$L\{e^{-3t} t^4\} = \frac{4!}{(s+3)^5}$$

Example 2

Find the Laplace transform of $e^t t^{-\frac{1}{2}}$.

Solution

$$L\left\{t^{-\frac{1}{2}}\right\} = \frac{\left\lceil\frac{1}{2}\right.}{s^{\frac{1}{2}}}$$

$$= \sqrt{\frac{\pi}{s}}$$

By the first shifting theorem,

$$L\left\{e^t t^{-\frac{1}{2}}\right\} = \sqrt{\frac{\pi}{s-1}}$$

Example 3

Find the Laplace transform of $(t+1)^2 e^t$.

Solution

$$L\{t+1)^2\} = L\{t^2 + 2t + 1\}$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

By the first shifting theorem,

$$L\{(t+1)^2 e^t\} = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$$

Example 4

Find the Laplace transform of $e^t(1+\sqrt{t})^4$.

Solution

$$L\{(1+\sqrt{t})^4\} = L\{1+4\sqrt{t}+6(\sqrt{t})^2+4(\sqrt{t})^3+(\sqrt{t})^4\}$$

$$= L\left\{1+4t^{\frac{1}{2}}+6t+4t^{\frac{3}{2}}+t^2\right\}$$

$$= \frac{1}{s} + \frac{4\Gamma\frac{3}{2}}{s^{\frac{3}{2}}} + \frac{6\Gamma 2}{s^2} + \frac{4\Gamma\frac{5}{2}}{s^{\frac{5}{2}}} + \frac{\Gamma 3}{s^3}$$

$$= \frac{1}{s} + \frac{4\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{3}{2}}} + \frac{6}{s^2} + \frac{4\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{5}{2}}} + \frac{2}{s^3}$$

$$= \frac{1}{s} + \frac{2\sqrt{\pi}}{s^{\frac{3}{2}}} + \frac{6}{s^2} + \frac{3\sqrt{\pi}}{s^{\frac{5}{2}}} + \frac{2}{s^3}$$

By the first shifting theorem,

$$L\{e^t(1+\sqrt{t})^4\} = \frac{1}{s-1} + \frac{2\sqrt{\pi}}{(s-1)^{\frac{3}{2}}} + \frac{6}{(s-1)^2} + \frac{3\sqrt{\pi}}{(s-1)^{\frac{5}{2}}} + \frac{2}{(s-1)^3}$$

Example 5

Find the Laplace transform of $e^{2t}\sin 3t$. **[Summer 2018]**

Solution

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

By the first shifting theorem,

$$L\{e^{2t}\sin 3t\} = \frac{3}{(s-2)^2+9}$$

$$= \frac{3}{s^2-4s+13}$$

Example 6

Find the Laplace transform of $e^{-at}\cos bt$.

Solution

$$L\{\cos bt\} = \frac{s}{s^2 + b^2}$$

By the first shifting theorem,

$$L\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2 + b^2}$$

Example 7

Find the Laplace transform of $e^{-3t}(2\cos 5t - 3\sin 5t)$. **[Summer 2014]**

Solution

$$\begin{aligned} L\{2\cos 5t - 3\sin 5t\} &= 2L\{\cos 5t\} - 3L\{\sin 5t\} \\ &= \frac{2s}{s^2+25} - \frac{3(5)}{s^2+25} \\ &= \frac{2s}{s^2+25} - \frac{15}{s^2+25} \end{aligned}$$

By the first shifting theorem,

$$\begin{aligned} L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} &= \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25} \\ &= \frac{2s+6-15}{s^2+6s+9+25} \\ &= \frac{2s-9}{s^2+6s+34} \end{aligned}$$

Example 8

Find the Laplace transform of $e^{2t}\sin^2 t$. **[Summer 2017]**

Solution

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}[L\{1\} - L\{\cos 2t\}]$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= \frac{1}{2}\left[\frac{4}{s(s^2+4)}\right]$$

$$= \frac{2}{s(s^2+4)}$$

By the first shifting theorem,

$$L(e^{2t}\sin^2 t) = \frac{2}{(s-2)[(s-2)^2+4]}$$

$$= \frac{2}{(s-2)(s^2-4s+8)}$$

Example 9

Find the Laplace transform of $e^{4t}\sin^3 t$.

Solution

$$L\{\sin^3 t\} = \frac{1}{4}L\{3\sin t - \sin 3t\}$$

$$= \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$$

By the first shifting theorem,

$$L\{e^{4t}\sin^3 t\} = \frac{3}{4[(s-4)^2+1]} - \frac{3}{4[(s-4)^2+9]}$$

$$= \frac{3}{4(s^2-8s+17)} - \frac{3}{4(s^2-8s+25)}$$

$$= \frac{3[(s^2-8s+25)-(s^2-8s+17)]}{4(s^2-8s+17)(s^2-8s+25)}$$

$$= \frac{6}{(s^2-8s+7)(s^2-8s+25)}$$

Example 10

Find the Laplace transform of $e^{-2t}(\sin 4t + t^2)\cdot$ **[Winter 2014]**

Solution

$$L\{\sin 4t + t^2\} = \frac{4}{s^2+16} + \frac{2}{s^3}$$

By the first shifting theorem,

$$L\{e^{-2t}(\sin 4t + t^2)\} = \frac{4}{(s+2)^2+16} + \frac{2}{(s+2)^3}$$

Example 11

Find the Laplace transform of cosh *at* cos *at*.

Solution

$$\cosh at \cos at = \left(\frac{e^{at}+e^{-at}}{2}\right)\cos at$$

$$= \frac{1}{2}(e^{at}\cos at + e^{-at}\cos at)$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\cosh at \cos at\} = \frac{1}{2}L\ \{e^{at}\cos at + e^{-at}\cos at\}$$

By the first shifting theorem,

$$L\{\cosh at \cos at\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{s^2+2a^2-2as} + \frac{s+a}{s^2+2a^2+2as}\right]$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2+2a^2+2as)+(s+a)(s^2+2a^2-2as)}{(s^2+2a^2)^2-4a^2s^2}\right]$$

$$= \frac{s^3}{s^4+4a^4}$$

Example 12

Find the Laplace transform of $\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$.

Solution

$$\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t = \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2}\right)\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{1}{2}\left(e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right)$$

$$L\left\{\sin\frac{\sqrt{3}}{2}t\right\} = \frac{\frac{\sqrt{3}}{2}}{s^2 + \frac{3}{4}}$$

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\} = \frac{1}{2}L\left\{e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right\}$$

By the first shifting theorem,

$$L\left\{\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t\right\} = \frac{1}{2}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right]$$

$$= \frac{\sqrt{3}}{2}\frac{s}{s^4+s^2+1}$$

Example 13

Find the Laplace transform of $e^{-3t}\cosh 4t\sin 3t$.

Solution

$$e^{-3t}\cosh 4t\sin 3t = e^{-3t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\sin 3t$$

$$= \frac{1}{2}(e^{t}\sin 3t + e^{-7t}\sin 3t)$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-3t}\cosh 4t\sin 3t\} = \frac{1}{2}L\{e^{t}\sin 3t + e^{-7t}\sin 3t\}$$

By the first shifting theorem,

$$L\{e^{-3t}\cosh 4t\sin 3t\} = \frac{1}{2}\left[\frac{3}{(s-1)^2+9} + \frac{3}{(s+7)^2+9}\right]$$

$$= \frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$$

Example 14

Find the Laplace transform of $\sin 2t\cos t\cosh 2t$.

Solution

$$\sin 2t\cos t\cosh 2t = \left(\frac{\sin 3t+\sin t}{2}\right)\left(\frac{e^{2t}+e^{-2t}}{2}\right)$$

$$= \frac{1}{4}(e^{2t}\sin 3t + e^{2t}\sin t + e^{-2t}\sin 3t + e^{-2t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{\sin 2t\cos t\cosh 2t\} = \frac{1}{4}L\{e^{2t}\sin 3t + e^{2t}\sin t + e^{-2t}\sin 3t + e^{-2t}\sin t\}$$

By the first shifting theorem,

$$L\{\sin 2t\cos t\cosh 2t\} = \frac{1}{4}\left[\frac{3}{(s-2)^2+9} + \frac{1}{(s-2)^2+1} + \frac{3}{(s+2)^2+9} + \frac{1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{(s^2-4s+13)(s^2+4s+13)} + \frac{s^2+5}{(s^2-4s+5)(s^2+4s+5)}\right]$$

$$= \frac{1}{2}\left[\frac{3(s^2+13)}{s^4+10s^2+169} + \frac{s^2+5}{s^4-6s^2+25}\right]$$

Example 15

Find the Laplace transform of $\dfrac{\cos 2t\sin t}{e^{t}}$.

Solution

$$\frac{\cos 2t \sin t}{e^t} = e^{-t}\left(\frac{\sin 3t - \sin t}{2}\right)$$

$$= \frac{1}{2}(e^{-t} \sin 3t - e^{-t} \sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2} L\{e^{-t} \sin 3t - e^{-t} \sin t\}$$

By the first shifting theorem,

$$L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\frac{2s^2+4s-4}{(s^2+2s+10)(s^2+2s+2)}$$

$$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

Example 16

Find the Laplace transform of $e^{-4t} \sinh t \sin t$.

Solution

$$e^{-4t} \sinh t \sin t = e^{-4t}\left(\frac{e^t - e^{-t}}{2}\right)\sin t$$

$$= \frac{1}{2}(e^{-3t} \sin t - e^{-5t} \sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{e^{-4t} \sinh t \sin t\} = \frac{1}{2} L\{e^{-3t} \sin t - e^{-5t} \sin t\}$$

By the first shifting theorem,

$$L\{e^{-4t} \sinh t \sin t\} = \frac{1}{2}\left[\frac{1}{(s+3)^2+1} - \frac{1}{(s+5)^2+1}\right]$$

$$= \frac{1}{2}\frac{4s+16}{(s^2+6s+10)(s^2+10s+26)}$$

$$= \frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}$$

EXERCISE 5.4

Find the Laplace transforms of the following functions:

1. $t^3 e^{-3t}$

$$\left[\textbf{Ans.}: \frac{6}{(s+3)^4}\right]$$

2. $e^{-t}\cos 2t$

$$\left[\textbf{Ans.}: \frac{s+1}{s^2+2s+5}\right]$$

3. $2e^{3t}\sin 4t$

$$\left[\textbf{Ans.}: \frac{8}{s^2-6s+25}\right]$$

4. $(t+2)^2 e^t$

$$\left[\textbf{Ans.}: \frac{4s^2-4s+2}{(s-1)^3}\right]$$

5. $e^{2t}(3\sin 4t - 4\cos 4t)$

$$\left[\textbf{Ans.}: \frac{20-4s}{s^2-4s+20}\right]$$

6. $e^{-4t}\cosh 2t$

$$\left[\textbf{Ans.}: \frac{s+4}{s^2+8s+12}\right]$$

7. $(1+te^{-t})^3$

$$\left[\textbf{Ans.}: \frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}\right]$$

8. $e^{-t}(3\sinh 2t - 5\cos 2t)$

$$\left[\textbf{Ans.}: \frac{1-5s}{s^2+2s-3}\right]$$

9. $e^t \sin 2t \sin 3t$

$$\left[\textbf{Ans.}: \frac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}\right]$$

10. $e^{-3t} \cosh 5t \sin 4t$

$$\left[\textbf{Ans.}: \frac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+20)}\right]$$

11. $e^{-4t} \cosh t \sin t$

$$\left[\textbf{Ans.}: \frac{s^2+8s+18}{(s^2+6s+10)(s^2+10s+26)}\right]$$

12. $e^{2t} \sin^4 t$

$$\left[\textbf{Ans.}: \frac{3}{8(s-2)} - \frac{s-2}{2(s^2-4s+8)} + \frac{s-4}{8(s^2-8s+32)}\right]$$

5.4.4 Second Shifting Theorem **[Summer 2013]**

If $L\{f(t)\} = F(s)$

and
$$g(t) = f(t-a) \quad t > a$$
$$= 0 \quad t < a$$

then $L\{g(t)\} = e^{-as} F(s)$

Proof: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)\mathrm{d}t$$

$$= \int_0^a e^{-st} \cdot 0 \, \mathrm{d}t + \int_a^\infty e^{-st} f(t-a)\mathrm{d}t$$

Putting $t - a = x, \quad \mathrm{d}t = \mathrm{d}x$

When $t = a, \quad x = 0$

When $t \to \infty, \ x \to \infty$

$$L\{g(t)\} = \int_0^\infty e^{-s(a+x)} f(x)\mathrm{d}x$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x)\mathrm{d}x$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$= e^{-as} F(s)$$

Example 1

Find the Laplace transform of $g(t) = e^{t-a} \quad t > a$

$= 0 \quad t < a$

Solution

Let $f(t) = e^t$

$$L\{f(t)\} = F(s) = \frac{1}{s-1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{1}{s-1}$$

Example 2

Find the Laplace transform of $g(t) = \cos(t-a) \quad t > a$

$= 0 \quad t < a$

Solution

Let $f(t) = \cos t$

$$L\{f(t)\} = F(s) = \frac{s}{s^2+1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-as}\frac{s}{s^2+1}$$

Example 3

Find the Laplace transform of $g(t) = \sin\left(t - \frac{\pi}{4}\right) \quad t > \frac{\pi}{4}$

$= 0 \quad t < \frac{\pi}{4}$

Solution

Let $f(t) = \sin t$

$$L\{f(t)\} = F(s) = \frac{1}{s^2+1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-\frac{\pi s}{4}}\frac{1}{s^2+1}$$

Example 4

Find the Laplace transform of $g(t) = (t-1)^3 \quad t > 1$

$$= 0 \qquad t < 1$$

Solution

Let $f(t) = t^3$

$$L\{f(t)\} = F(s) = \frac{3!}{s^4}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-s}\frac{3!}{s^4}$$

EXERCISE 5.5

Find the Laplace transforms of the following functions:

1. $f(t) = \cos\left(t - \frac{2\pi}{3}\right) \qquad t > \frac{2\pi}{3}$

$\qquad = 0 \qquad t < \frac{2\pi}{3}$

$$\left[\text{Ans.}: e^{-\frac{2\pi s}{3}}\frac{s}{s^2+1}\right]$$

2. $f(t) = (t-2)^2 \qquad t > 2$

$\qquad = 0 \qquad t < 2$

$$\left[\text{Ans.}: e^{-2s}\frac{2}{s^3}\right]$$

3. $f(t) = 5\sin 3\left(t - \frac{\pi}{4}\right) \qquad t > \frac{\pi}{4}$

$\qquad = 0 \qquad t < \frac{\pi}{4}$

$$\left[\text{Ans.}: e^{-\frac{\pi s}{4}}\frac{1}{s^2+9}\right]$$

5.5 DIFFERENTIATION OF LAPLACE TRANSFORMS (MULTIPLICATION BY *t*)

If $L\{f(t)\} = F(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$. **[Winter 2014, 2013]**

Proof: $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

Differentiating both the sides w.r.t. s using DUIS,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} F(s) &= \frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty e^{-st} f(t)\mathrm{d}t \\
&= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\mathrm{d}t \\
&= \int_0^\infty (-t\, e^{-st}) f(t)\mathrm{d}t \\
&= \int_0^\infty e^{-st} \{-t\, f(t)\}\,\mathrm{d}t \\
&= -L\{t\, f(t)\}
\end{aligned}$$

$$L\{t\, f(t)\} = (-1)\frac{\mathrm{d}}{\mathrm{d}s} F(s)$$

Similarly, $$L\{t^2 f(t)\} = (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s)$$

In general, $$L\{t^n\, f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$$

Example 1

Find the Laplace transform of te^{-t}.

Solution

$$L\{e^{-t}\} = \frac{1}{s+1}$$

$$\begin{aligned}
L\{te^{-t}\} &= -\frac{\mathrm{d}}{\mathrm{d}s} L\{e^{-t}\} \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s+1}\right) \\
&= -\left[\frac{-1}{(s+1)^2}\right] \\
&= \frac{1}{(s+1)^2}
\end{aligned}$$

Example 2

Find the Laplace transform of t cos *at.*

Solution

$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{t\cos at\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\cos at\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s}{s^2 + a^2}\right]$$

$$= -\left[\frac{(s^2 + a^2)(1) - s(2s)}{(s^2 + a^2)^2}\right]$$

$$= -\left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2}\right]$$

$$= -\left[\frac{-s^2 + a^2}{(s^2 + a^2)^2}\right]$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

Example 3

Find the Laplace transform of t sin *at.* **[Summer 2016]**

Solution

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\{t\sin at\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin at\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{a}{s^2 + a^2}\right)$$

$$= -\left[\frac{(s^2 + a^2)(0) - a(2s)}{(s^2 + a^2)^2}\right]$$

$$= \frac{2as}{(s^2 + a^2)^2}$$

Example 4

Find the Laplace transform of $t\ \sin 2t$. **[Summer 2015]**

Solution

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\{t\sin 2t\} = -\frac{d}{ds}L\{\sin 2t\}$$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -\left[\frac{(s^2+4)(0)-2(2s)}{(s^2+4)^2}\right]$$

$$= \frac{4s}{(s^2+4)^2}$$

Example 5

Find the Laplace transform of t cosh at.

Solution

$$L\{\cosh at\} = \frac{s}{s^2-a^2}$$

$$L\{t\cosh at\} = -\frac{d}{ds}L\{\cosh at\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2-a^2}\right)$$

$$= -\left[\frac{(s^2-a^2)(1)-s(2s)}{(s^2-a^2)^2}\right]$$

$$= -\left[\frac{s^2-a^2-2s^2}{(s^2-a^2)^2}\right]$$

$$= -\left[\frac{-s^2-a^2}{(s^2-a^2)^2}\right]$$

$$= \frac{s^2+a^2}{(s^2-a^2)^2}$$

Example 6

Find the Laplace transform of $t\cos^2 t$.

Solution

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\}$$

$$= \frac{1}{2}L\{1+\cos 2t\}$$

$$= \frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$L\{t\cos^2 t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\cos^2 t\}$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}+\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{1}{2s^2}+\frac{s^2-4}{2(s^2+4)^2}$$

Example 7

Find the Laplace transform of $t\sin^2 3t$. **[Winter 2017]**

Solution

$$L\{\sin^2 3t\} = L\left\{\frac{1-\cos 6t}{2}\right\}$$

$$= \frac{1}{2}L\{1-\cos 6t\}$$

$$= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+36}\right)$$

$$L\{t\sin^2 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin^2 3t\}$$

$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}-\frac{s}{s^2+36}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}-\frac{-s^2+36}{(s^2+36)^2}\right]$$

$$= \frac{1}{2s^2}+\frac{-s^2+36}{2(s^2+36)^2}$$

Example 8

Find the Laplace transform of $t\sin^3 t$.

Solution

$$L\{\sin^3 t\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\}$$

$$= \frac{1}{4}\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right)$$

$$= \frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$L\{t\sin^3 t\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{\sin^3 t\}$$

$$= -\frac{3}{4}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$= -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{(s^2+9)^2 - (s^2+1)^2}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{s^4 + 18s^2 + 81 - s^4 - 2s^2 - 1}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\frac{16(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

$$= \frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

Example 9

Find the Laplace transform of $t\sin 2t\cosh t$.

Solution

$$L\{\sin 2t\cosh t\} = L\left\{\sin 2t\left(\frac{e^t + e^{-t}}{2}\right)\right\}$$

$$= \frac{1}{2} L\{e^t \sin 2t + e^{-t}\sin 2t\}$$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^2+4} + \frac{2}{(s+1)^2+4}\right]$$

$$= \frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}$$

$$L\{t\sin 2t\cosh t\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{\sin 2t\cosh t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2-2s+5} + \frac{1}{s^2+2s+5}\right)$$

$$= \frac{2s-2}{(s^2-2s+5)^2} + \frac{2s+2}{(s^2+2s+5)^2}$$

Example 10

Find the Laplace transform of $t \sin 3t \cos 2t$. **[Winter 2016]**

Solution

$$L\{\sin 3t\cos 2t\} = L\left\{\frac{\sin 5t + \sin t}{2}\right\}$$

$$= \frac{1}{2}L\{\sin 5t\} + \frac{1}{2}L\{\sin t\}$$

$$= \frac{5}{2(s^2+25)} + \frac{1}{2(s^2+1)}$$

$$L\{t\sin 3t\cos 2t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\cos 2t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{5}{2(s^2+25)} + \frac{1}{2(s^2+1)}\right]$$

$$= -\frac{1}{2}\left[\frac{-5(2s)}{(s^2+25)^2} - \frac{1(2s)}{(s^2+1)^2}\right]$$

$$= \frac{5s}{(s^2+25)} + \frac{s}{(s^2+1)^2}$$

Example 11

Find the Laplace transform of $t\sqrt{1+\sin t}$.

Solution

$$L\{\sqrt{1+\sin t}\} = L\left\{\sin\frac{t}{2} + \cos\frac{t}{2}\right\}$$

$$= \frac{\frac{1}{2}}{s^2 + \frac{1}{4}} + \frac{s}{s^2 + \frac{1}{4}}$$

$$= \frac{1}{2} \cdot \frac{4}{4s^2 + 1} + \frac{4s}{4s^2 + 1}$$

$$= \frac{4s + 2}{4s^2 + 1}$$

$$L\left\{t\sqrt{1 + \sin t}\right\} = -\frac{d}{ds} L\left\{\sqrt{1 + \sin t}\right\}$$

$$= -\frac{d}{ds}\left(\frac{4s + 2}{4s^2 + 1}\right)$$

$$= -\left[\frac{(4s^2 + 1)4 - (4s + 2)8s}{(4s^2 + 1)^2}\right]$$

$$= \frac{-16s^2 - 4 + 32s^2 + 16s}{(4s^2 + 1)^2}$$

$$= \frac{16s^2 + 16s - 4}{(4s^2 + 1)^2}$$

$$= \frac{4(4s^2 + 4s - 1)}{(4s^2 + 1)^2}$$

Example 12

Find the Laplace transform of $te^{-t} \cos t$.

Solution

$$L\{\cos t\} = \frac{s}{s^2 + 1}$$

$$L\{t \cos t\} = -\frac{d}{ds} L\{\cos t\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2 + 1}\right)$$

$$= -\left[\frac{(s^2 + 1)(1) - s(2s)}{(s^2 + 1)^2}\right]$$

$$= -\left[\frac{s^2 + 1 - 2s^2}{(s^2 + 1)^2}\right]$$

$$= \frac{s^2 - 1}{(s^2 + 1)^2}$$

By the first shifting theorem,

$$L\{e^{-t} t \cos t\} = \frac{(s+1)^2 - 1}{[(s+1)^2 + 1]^2}$$

Example 13

Find the Laplace transform of $t\,e^{4t} \cos 2t$. **[Summer 2017]**

Solution

$$L\{\cos 2t\} = \frac{s}{s^2 + 4}$$

By the first shifting theorem,

$$L\{e^{4t} \cos 2t\} = \frac{s-4}{(s-4)^2 + 4}$$

$$= \frac{s-4}{s^2 - 8s + 20}$$

$$L\{t\,e^{4t} \cos 2t\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{e^{4t} \cos 2t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s-4}{s^2 - 8s + 20}\right)$$

$$= -\left[\frac{(s^2 - 8s + 20)(1) - (s-4)(2s-8)}{(s^2 - 8s + 20)^2}\right]$$

$$= -\left[\frac{s^2 - 8s + 20 - 2s^2 + 8s + 8s - 32}{(s^2 - 8s + 20)^2}\right]$$

$$= -\left[\frac{-s^2 + 8s - 12}{(s^2 - 8s + 20)^2}\right]$$

$$= \frac{s^2 - 8s + 12}{(s^2 - 8s + 20)^2}$$

$$= \frac{(s-4)^2 - 4}{(s-4)^2 + 4}$$

Example 14

Find the Laplace transform of $te^{at}\sin at$. **[Winter 2013]**

Solution

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\{e^{at}\sin at\} = \frac{a}{(s-a)^2 + a^2}$$

$$= \frac{a}{s^2 - 2as + 2a^2}$$

$$L\{te^{at}\sin at\} = -\frac{d}{ds} L\{e^{at}\sin at\}$$

$$= -\frac{d}{ds}\left[\frac{a}{(s-a)^2 + a^2}\right]$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2 - 2as + 2a^2}\right)$$

$$= \frac{a}{(s^2 - 2as + 2a^2)^2}(2s - 2a)$$

$$= \frac{2a(s-a)}{(s^2 - 2as + 2a^2)^2}$$

Example 15

Find the Laplace transform of $t\left(\dfrac{\sin t}{e^t}\right)^2$.

Solution

$$t\left(\frac{\sin t}{e^t}\right)^2 = t\,e^{-2t}\sin^2 t$$

$$= t\,e^{-2t}\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2}\,t\,e^{-2t}(1-\cos 2t)$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\{t(1-\cos 2t)\} = -\frac{d}{ds}L(1-\cos 2t)$$

$$= -\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= -\left[-\frac{1}{s^2} - \frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}$$

By the first shifting theorem,

$$L\left\{\frac{1}{2}te^{-2t}(1-\cos 2t)\right\} = \frac{1}{2}\left[\frac{1}{(s+2)^2} + \frac{4-(s+2)^2}{\{(s+2)^2+4\}^2}\right]$$

Example 16

Find the Laplace transform of sin 2t – 2t cos 2t.

Solution

$$L\{\sin 2t - 2t\cos 2t\} = L\{\sin 2t\} - 2L\{t\cos 2t\}$$

$$= \frac{2}{s^2+4} - 2\left[-\frac{d}{ds}L\{\cos 2t\}\right]$$

$$= \frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{2}{s^2+4} + 2\left[\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{2}{s^2+4} + 2\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{2}{s^2+4} + 2\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{2}{s^2+4}\left[1 + \frac{4-s^2}{s^2+4}\right]$$

$$= \frac{2}{s^2+4}\left[\frac{s^2+4+4-s^2}{s^2+4}\right]$$

$$= \frac{2}{s^2+4}\left[\frac{8}{s^2+4}\right]$$

$$= \frac{16}{(s^2+4)^2}$$

Example 17

Find the Laplace transform of $t(\sin t - t\cos t)$. **[Winter 2015]**

Solution

$$L\{t(\sin t - t\cos t)\} = L\{t\sin t - t^2\cos t\}$$

$$\begin{aligned}
L\{t\sin t\} &= -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin t\}\\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{(s^2+1)}\\
&= -\left[\frac{-2s}{(s^2+1)^2}\right]\\
&= \frac{2s}{(s^2+1)^2}
\end{aligned}$$

$$\begin{aligned}
L\{t^2\cos t\} &= (-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}L\{\cos t\}\\
&= (-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{s}{s^2+1}\right)\\
&= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{s}{s^2+1}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right)\right]\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1-s^2}{(s^2+1)^2}\right]\\
&= \frac{(s^2+1)^2(-2s)-(1-s^2)\cdot 2(s^2+1)2s}{(s^2+1)^4}\\
&= \frac{(s^2+1)(-2s)-4s(1-s^2)}{(s^2+1)^3}\\
&= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}
\end{aligned}$$

$$= \frac{2s^3 - 6s}{(s^2+1)^3}$$

$$L\{t \sin t - t^2 \cos t\} = \frac{2s}{(s^2+1)^2} - \frac{2s^3 - 6s}{(s^2+1)^3}$$

$$= \frac{2s(s^2+1) - (2s^3 - 6s)}{(s^2+1)^3}$$

$$= \frac{2s^3 + 2s - 2s^3 + 6s}{(s^2+1)^3}$$

$$= \frac{8s}{(s^2+1)^3}$$

Example 18

Find the Laplace transform of $t^2 \sin \omega t$. **[Winter 2014]**

Solution

$$L\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

$$L\{t^2 \sin \omega t\} = (-1)^2 \frac{d^2}{ds^2} L\{\sin \omega t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= \frac{d}{ds}\left[-\frac{\omega(2s)}{(s^2+\omega^2)^2}\right]$$

$$= -2\omega\left[\frac{1}{(s^2+\omega^2)^2} - \frac{2s}{(s^2+\omega^2)^3}\cdot 2s\right]$$

$$= -2\omega\left[\frac{s^2+\omega^2 - 4s^2}{(s^2+\omega^2)^3}\right]$$

$$= \frac{2\omega(3s^2 - \omega^2)}{(s^2+\omega^2)^3}$$

Example 19

Find the Laplace transform of $t^2 \cosh 3t$. **[Summer 2016]**

Solution

$$L\{\cosh 3t\} = \frac{s}{s^2 - 9}$$

$$\begin{aligned}
L\{t^2 \cosh 3t\} &= (-1)^2 \frac{d^2}{ds^2} L\{\cosh 3t\} \\
&= \frac{d^2}{ds^2}\left(\frac{s}{s^2-9}\right) \\
&= \frac{d}{ds}\left[\frac{1}{s^2-9} - \frac{s}{(s^2-9)^2}(2s)\right] \\
&= -\frac{1}{(s^2-9)^2}(2s) - \frac{4s}{(s^2-9)^2} + \frac{4s^2}{(s^2-9)^3}(2s) \\
&= \frac{-(s^2-9)2s - 4s(s^2-9) + 8s^3}{(s^2-9)^3} \\
&= \frac{-2s^3 + 18s - 4s^3 + 36s + 8s^3}{(s^2-9)^3} \\
&= \frac{2s^3 + 54s}{(s^2-9)^3} \\
&= \frac{2s(s^2+27)}{(s^2-9)^3}
\end{aligned}$$

Example 20

Find the Laplace transform of $t^2 \cosh \pi t$. **[Winter 2014]**

Solution

$$L\{\cosh \pi t\} = \frac{s}{s^2 - \pi^2}$$

$$\begin{aligned}
L\{t^2 \cosh \pi t\} &= (-1)^2 \frac{d^2}{ds^2} L\{\cosh \pi t\} \\
&= \frac{d^2}{ds^2}\left(\frac{s}{s^2-\pi^2}\right) \\
&= \frac{d}{ds}\left[\frac{1}{s^2-\pi^2} - \frac{s}{(s^2-\pi^2)^2}(2s)\right]
\end{aligned}$$

$$= -\frac{1}{(s^2-\pi^2)^2}(2s) - \frac{4s}{(s^2-\pi^2)^2} + \frac{4s^2}{(s^2-\pi^2)^3}(2s)$$

$$= \frac{-2s^3 + 2\pi^2 s - 4s^3 + 4\pi^2 s + 8s^3}{(s^2-\pi^2)^3}$$

$$= \frac{2s^3 + 6\pi^2 s}{(s^2-\pi^2)^3}$$

$$= \frac{2s(s^2+3\pi^2)}{(s^2-\pi^2)^3}$$

Example 21

Find the Laplace transform of $t^2e^t \sin 4t$.

Solution

$$L\{\sin 4t\} = \frac{4}{s^2+16}$$

$$L\{t^2 \sin 4t\} = (-1)^2 \frac{d^2}{ds^2} L\{\sin 4t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{4}{s^2+16}\right)$$

$$= -\frac{d}{ds}\left[\frac{4(2s)}{(s^2+16)^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{8s}{(s^2+16)^2}\right]$$

$$= -\left[\frac{(s^2+16)^2(8) - 8s\cdot 2(s^2+16)(2s)}{(s^2+16)^4}\right]$$

$$= \frac{-8s^2 - 128 + 32s^2}{(s^2+16)^3}$$

$$= \frac{24s^2 - 128}{(s^2+16)^3}$$

$$= \frac{8(3s^2-16)}{(s^2+16)^3}$$

By the first shifting theorem,

$$L\{t^2 e^t \sin 4t\} = \frac{8\left[3(s-1)^2 - 16\right]}{\left[(s-1)^2 + 16\right]^3}$$

$$= \frac{8(3s^2 - 6s - 13)}{(s^2 - 2s + 17)^3}$$

EXERCISE 5.6

Find the Laplace transforms of the following functions:

1. $t \cos^3 t$

$$\left[\textbf{Ans. :} \frac{1}{4}\left[\frac{-s^2+9}{(s^2+9)^2} + \frac{s^2+3}{(s^2+1)^2}\right]\right]$$

2. $t \cos(\omega t - \alpha)$

$$\left[\textbf{Ans. :} \frac{(s^2-\omega^2)\cos\alpha + 2\omega s \sin\alpha}{(s^2+\omega^2)^2}\right]$$

3. $t\sqrt{1-\sin t}$

$$\left[\textbf{Ans. :} \frac{4(4s^2-4s-1)}{(4s^2+1)^2}\right]$$

4. $t \cosh 3t$

$$\left[\textbf{Ans. :} \frac{s^2+9}{(s^2-9)^2}\right]$$

5. $t \sinh 2t \sin 3t$

$$\left[\textbf{Ans. :} 3\left[\frac{s-2}{(s^2-4s+13)^2} - \frac{s-2}{(s^2+4s+13)^2}\right]\right]$$

6. $t(3\sin 2t - 2\cos 2t)$

$$\left[\textbf{Ans. :} \frac{8+12s-2s^2}{(s^2+4)^2}\right]$$

7. $t\, e^{3t} \sin 2t$

$$\left[\textbf{Ans. :} \frac{4(s-3)}{(s^2-6s+13)^2}\right]$$

8. $t\sqrt{1+\sin 2t}$

$$\left[\textbf{Ans.}: \frac{s^2+2s-1}{(s^2+1)^2}\right]$$

9. $t\,e^{2t}(\cos t - \sin t)$

$$\left[\textbf{Ans.}: \frac{s^2-6s+7}{(s^2-4s+5)^2}\right]$$

10. $(t^2-3t+2)\sin 3t$

$$\left[\textbf{Ans.}: \frac{6s^4-18s^3+126s^2-162s+432}{(s^2+9)^3}\right]$$

11. $(t+\sin 2t)^2$

$$\left[\textbf{Ans.}: \frac{2}{s^3}+\frac{s}{(s^2+1)^2}+\frac{1}{2s}-\frac{s}{2(s^2+4)}\right]$$

12. $(t\sinh 2t)^2$

$$\left[\textbf{Ans.}: \frac{1}{2}\left[\frac{1}{(s-4)^3}+\frac{1}{(s+4)^3}\right]\right]$$

13. $t^2e^{-3t}\cosh 2t$

$$\left[\textbf{Ans.}: \frac{1}{(s+1)^3}+\frac{1}{(s+5)^3}\right]$$

14. $t^2e^{-2t}\sin 3t$

$$\left[\textbf{Ans.}: \frac{18(s^2+4s+1)}{(s^2+4s+13)^2}\right]$$

15. $(t\cos 2t)^2$

$$\left[\textbf{Ans.}: \frac{1}{s^3}-\frac{s(48-s^2)}{(s^2+16)^3}\right]$$

16. $t^2\sin t\cos 2t$

$$\left[\textbf{Ans.}: \frac{9(s^2-3)}{(s^2+9)^3}+\frac{1-3s^2}{(s^2+1)^3}\right]$$

17. $t^3 \cos t$

$$\left[\textbf{Ans.:} \frac{6s^4 - 36s^2 + 6}{(s^2+9)^3}\right]$$

5.6 INTEGRATION OF LAPLACE TRANSFORMS (DIVISION BY *t*)

If $L\{f(t)\} = F(s)$ then $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^\infty F(s)\mathrm{d}s$. **[Winter 2014; Summer 2014]**

Proof: $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

Integrating both the sides w.r.t. s from s to ∞,

$$\int_s^\infty F(s)\mathrm{d}s = \int_s^\infty \int_0^\infty e^{-st} f(t)\mathrm{d}t\,\mathrm{d}s$$

Since s and t are independent variables, interchanging the order of integration,

$$\begin{aligned}
\int_s^\infty F(s)\mathrm{d}s &= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,\mathrm{d}s\right]\mathrm{d}t \\
&= \int_0^\infty \left|\frac{e^{-st}}{-t} f(t)\right|_s^\infty \mathrm{d}t \\
&= \int_0^\infty e^{-st}\frac{f(t)}{t}\mathrm{d}t \\
&= L\left\{\frac{f(t)}{t}\right\}
\end{aligned}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\mathrm{d}s$$

Example 1

Find the Laplace transform of $\dfrac{1-e^{-t}}{t}$.

Solution

$$L\{1-e^{-t}\} = \frac{1}{s} - \frac{1}{s+1}$$

$$L\left\{\frac{1-e^{-t}}{t}\right\}=\int_s^\infty L\{1-e^{-t}\}\,\mathrm{d}s$$

$$=\int_s^\infty\left(\frac{1}{s}-\frac{1}{s+1}\right)\mathrm{d}s$$

$$=\left|\log s-\log(s+1)\right|_s^\infty$$

$$=\left|\log\frac{s}{s+1}\right|_s^\infty$$

$$=\log\left|\frac{1}{1+\frac{1}{s}}\right|_s^\infty$$

$$=\log 1-\log\left(\frac{1}{1+\frac{1}{s}}\right)$$

$$=-\log\left(\frac{s}{s+1}\right)$$

$$=\log\left(\frac{s+1}{s}\right)$$

Example 2

Find the Laplace transform of $\dfrac{e^{-at}-e^{-bt}}{t}$.

Solution

$$L\{e^{-at}-e^{-bt}\}=\frac{1}{s+a}-\frac{1}{s+b}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_s^\infty L\{e^{-at}-e^{-bt}\}\,\mathrm{d}s$$

$$=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)\mathrm{d}s$$

$$=\left|\log(s+a)-\log(s+b)\right|_s^\infty$$

$$= \left|\log \frac{s+a}{s+b}\right|_s^\infty$$

$$= \left|\log \frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right|_s^\infty$$

$$= \log 1 - \log \left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}} \right)$$

$$= -\log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

Example 3

Find the Laplace transform of $\dfrac{\sinh t}{t}$.

Solution

$$L\{\sinh t\} = L\left\{\frac{e^t - e^{-t}}{2}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

$$L\left\{\frac{\sinh t}{t}\right\} = \int_s^\infty L\{\sinh t\}\,\mathrm{d}s$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s-1} - \frac{1}{s+1}\right)\mathrm{d}s$$

$$= \frac{1}{2}\left|\log(s-1) - \log(s+1)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log \frac{s-1}{s+1}\right|_s^\infty$$

$$= \frac{1}{2}\left|\log \frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right|_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{1-\frac{1}{s}}{1+\frac{1}{s}}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

Example 4

Find the Laplace transform of $\dfrac{\sin 2t}{t}$. **[Winter 2014, 2012]**

Solution

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \frac{2}{s^2+4}\,\mathrm{d}s$$

$$= 2\left|\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

$$= \tan^{-1}\left(\frac{2}{s}\right)$$

Example 5

Find the Laplace transform of $\dfrac{1-\cos 2t}{t}$. **[Winter 2017]**

Solution

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty L\{1-\cos 2t\}\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\mathrm{d}s$$

$$= \left|\log\, s - \frac{1}{2}\log(s^2+4)\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log(s^2+4) - \log s^2\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log\left(\frac{s^2+4}{s^2}\right)\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log\left(1+\frac{4}{s^2}\right)\right|_s^\infty$$

$$= -\frac{1}{2}\log\, 1 + \frac{1}{2}\log\left(1+\frac{4}{s^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

Example 6

Find the Laplace transform of $\dfrac{\cos at - \cos bt}{t}$. **[Summer 2016]**

Solution

$$L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty L\{\cos at - \cos bt\}\,\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)\mathrm{d}s$$

$$= \left|\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log\frac{s^2+a^2}{s^2+b^2}\right|_s^{\infty}$$

$$= \frac{1}{2}\left|\log\frac{1+\dfrac{a^2}{s^2}}{1+\dfrac{b^2}{s^2}}\right|_s^{\infty}$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\left(\frac{1+\dfrac{a^2}{s^2}}{1+\dfrac{b^2}{s^2}}\right)$$

$$= -\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Example 7

Find the Laplace transform of $\dfrac{e^{-t}\sin t}{t}$.

Solution

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^{\infty} L\{e^{-t}\sin t\}\,\mathrm{d}s$$

$$= \int_s^{\infty}\frac{1}{(s+1)^2+1}\,\mathrm{d}s$$

$$= \left|\tan^{-1}(s+1)\right|_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

Example 8

Find the Laplace transform of $\dfrac{\cosh 2t \sin 2t}{t}$.

Solution

$$L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = L\left\{\left(\frac{e^{2t}+e^{-2t}}{2t}\right)\sin 2t\right\}$$

$$= \frac{1}{2}\left[L\left\{\frac{e^{2t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-2t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \frac{2}{s^2+4}\,\mathrm{d}s$$

$$= \left|\tan^{-1}\frac{s}{2}\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

By the first shifting theorem,

$$L\left\{\frac{\cosh 2t \sin 2t}{t}\right\} = \frac{1}{2}\left[L\left\{e^{2t}\,\frac{\sin 2t}{t}\right\} + L\left\{e^{-2t}\,\frac{\sin 2t}{t}\right\}\right]$$

$$= \frac{1}{2}\left[\cot^{-1}\left(\frac{s-2}{2}\right) + \cot^{-1}\left(\frac{s+2}{2}\right)\right]$$

Example 9

Find the Laplace transform of $\dfrac{e^{-2t}\sin 2t \cosh t}{t}$.

Solution

$$L\left\{\frac{e^{-2t}\sin 2t \cosh t}{t}\right\} = L\left\{\frac{e^{-2t}\sin 2t}{t}\,\frac{(e^{t}+e^{-t})}{2}\right\}$$

$$= \frac{1}{2}\left[L\left\{\frac{e^{-t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-3t}\sin 2t}{t}\right\}\right]$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty L\{\sin 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \frac{2}{s^2+4}\,\mathrm{d}s$$

$$= 2\cdot\frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right)$$

$$L\left\{\frac{e^{-2t}\sin 2t\cosh t}{t}\right\} = \frac{1}{2}\left[L\left\{\frac{e^{-t}\sin 2t}{t}\right\} + L\left\{\frac{e^{-3t}\sin 2t}{t}\right\}\right]$$

$$= \frac{1}{2}\left[\cot^{-1}\left(\frac{s+1}{2}\right) + \cot^{-1}\left(\frac{s+3}{2}\right)\right]$$

Example 10

Find the Laplace transform of $\dfrac{1-\cos t}{t^2}$.

Solution

$$L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left\{\frac{1-\cos t}{t^2}\right\} = \int_s^\infty\int_s^\infty L\{1-\cos t\}\,\mathrm{d}s\,\mathrm{d}s$$

$$= \int_s^\infty\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right]\mathrm{d}s\,\mathrm{d}s$$

$$= \int_s^\infty \left|\log s - \frac{1}{2}\log(s^2+1)\right|_s^\infty \mathrm{d}s$$

$$= \int_s^\infty \left| \log \frac{s}{\sqrt{s^2+1}} \right|_s^\infty \mathrm{d}s$$

$$= \int_s^\infty \left[0 - \log \frac{s}{\sqrt{s^2+1}} \right] \mathrm{d}s$$

$$= -\int_s^\infty \log \frac{s}{\sqrt{s^2+1}} \mathrm{d}s$$

$$= \int_s^\infty \log \frac{\sqrt{s^2+1}}{s} \mathrm{d}s$$

$$= \frac{1}{2} \int_s^\infty \log \left(\frac{s^2+1}{s^2} \right) \mathrm{d}s$$

$$= \frac{1}{2} \int_s^\infty \log \left(1 + \frac{1}{s^2} \right) \mathrm{d}s$$

$$= \frac{1}{2} \left[\left| s \log \left(1 + \frac{1}{s^2} \right) \right|_s^\infty - \int_s^\infty s \frac{1}{\left(1 + \frac{1}{s^2} \right)} \left(-\frac{2}{s^3} \right) \mathrm{d}s \right]$$

$$= \frac{1}{2} \left[0 - s \log \left(1 + \frac{1}{s^2} \right) + 2 \int_s^\infty \frac{1}{s^2+1} \mathrm{d}s \right]$$

$$= -\frac{1}{2} s \log \left(1 + \frac{1}{s^2} \right) + \left| \tan^{-1} s \right|_s^\infty$$

$$= -\frac{s}{2} \log \left(\frac{s^2+1}{s^2} \right) + \frac{\pi}{2} - \tan^{-1} s$$

$$= -\frac{s}{2} \log \left(\frac{s^2+1}{s^2} \right) + \cot^{-1} s$$

Example 11

Find the Laplace transform of $\dfrac{\sin^2 t}{t^2}$.

Solution

$$L\left\{ \frac{\sin^2 t}{t^2} \right\} = L\left\{ \frac{1-\cos 2t}{2t^2} \right\}$$

$$= \frac{1}{2} L\left\{ \frac{1-\cos 2t}{t^2} \right\}$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty L\{1-\cos 2t\}\,\mathrm{d}s$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\mathrm{d}s$$

$$= \left|\log\ s - \frac{1}{2}\log(s^2+4)\right|_s^\infty$$

$$= \left|\log\ \frac{s}{\sqrt{s^2+4}}\right|_s^\infty$$

$$= \left|\log\ \frac{1}{\sqrt{1+\frac{4}{s^2}}}\right|_s^\infty$$

$$= \log 1 - \log\left(\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right)$$

$$= -\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

$$L\left\{\frac{1-\cos 2t}{t^2}\right\} = \int_s^\infty L\left\{\frac{1-\cos 2t}{t}\right\}\mathrm{d}s$$

$$= \frac{1}{2}\int_s^\infty \log\left\{\frac{s^2+4}{s^2}\right\}\mathrm{d}s$$

$$= \frac{1}{2}\left[\left|s\log\left(\frac{s^2+4}{s^2}\right)\right|_s^\infty - \int_s^\infty s\frac{s^2}{s^2+4}\left\{\frac{2s(s^2)-2s(s^2+4)}{s^4}\right\}\mathrm{d}s\right]$$

$$= \frac{1}{2}\left[-s\ \log\left(\frac{s^2+4}{s^2}\right) - \int_s^\infty -\frac{8}{s^2+4}\mathrm{d}s\right]$$

$$= \frac{1}{2}\left[-s \log\left(\frac{s^2+4}{s^2}\right) + 8 \cdot \frac{1}{2}\left|\tan^{-1}\frac{s}{2}\right|_s^\infty\right]$$

$$= \frac{1}{2}\left[-s \log\left(\frac{s^2+4}{s^2}\right) + 4\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)\right\}\right]$$

$$= \frac{1}{2}\left[-s \log\left(\frac{s^2+4}{s^2}\right) + 4\cot^{-1}\left(\frac{s}{2}\right)\right]$$

$$L\left\{\frac{\sin^2 t}{t^2}\right\} = \frac{1}{2}L\left\{\frac{1-\cos 2t}{t^2}\right\}$$

$$= \frac{1}{4}\left[-s \log\left(\frac{s^2+4}{s^2}\right) + 4 \cot^{-1}\left(\frac{s}{2}\right)\right]$$

EXERCISE 5.7

Find the Laplace transforms of the following functions:

1. $\dfrac{\sin^2 t}{t}$

$$\left[\textbf{Ans.:} \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)\right]$$

2. $\left(\dfrac{\sin 2t}{\sqrt{t}}\right)^2$

$$\left[\textbf{Ans.:} \frac{1}{4}\log\left(\frac{s^2+16}{s^2}\right)\right]$$

3. $\dfrac{\sin^3 t}{t}$

$$\left[\textbf{Ans.:} \frac{1}{4}\left(3\cot^{-1} s - \cot^{-1}\frac{s}{3}\right)\right]$$

4. $\dfrac{1-\cos at}{t}$

$$\left[\textbf{Ans.:} \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right)\right]$$

5. $\dfrac{\sin t \sin 5t}{t}$

$$\left[\textbf{Ans.}: \frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

6. $\dfrac{2\sin t \sin 2t}{t}$

$$\left[\textbf{Ans.}: \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)\right]$$

7. $\dfrac{e^{2t}\sin t}{t}$

$$\left[\textbf{Ans.}: \cot^{-1}(s-2)\right]$$

8. $\dfrac{e^{2t}\sin^3 t}{t}$

$$\left[\textbf{Ans.}: \frac{3}{4}\cot^{-1}(s-2) - \frac{1}{4}\cot^{-1}\left(\frac{s-2}{3}\right)\right]$$

5.7 LAPLACE TRANSFORMS OF DERIVATIVES

If $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = s\,F(s) - f(0)$

$$L\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0)$$

In general,

$$L\{f^n(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \ldots - f^{(n-1)}(0)$$

Proof: $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$

Integrating by parts,

$$L\{f'(t)\} = \left|e^{-st} f(t)\right|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\mathrm{d}t$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$= -f(0) + s\,L\{f(t)\}$$

Similarly, $L\{f''(t)\} = -f'(0) + s\;L\{f'(t)\}$

$$= -f'(0) + s\left[-f(0) + sL\{f(t)\}\right]$$

$$= -f'(0) - s\,f(0) + s^2L\{f(t)\}$$

In general, $L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{n-1}(0)$

Example 1

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = \dfrac{\sin t}{t}$.

Solution

$$
\begin{aligned}
L\{f(t)\} = F(s) &= L\left\{\frac{\sin t}{t}\right\} \\
&= \int_s^{\infty} L\{\sin t\}\,\mathrm{d}s \\
&= \int_s^{\infty} \frac{1}{s^2+1}\,\mathrm{d}s \\
&= \left|\tan^{-1} s\right|_s^{\infty} \\
&= \frac{\pi}{2} - \tan^{-1} s \\
&= \cot^{-1} s
\end{aligned}
$$

$$
\begin{aligned}
L\{f'(t)\} &= sF(s) - f(0) \\
&= s\cot^{-1} s - \lim_{t\to 0}\frac{\sin t}{t} \\
&= s\cot^{-1} s - 1
\end{aligned}
$$

Example 2

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of*
$$
\begin{aligned}
f(t) &= 3 \qquad & 0 \le t < 5 \\
&= 0 & t > 5
\end{aligned}
$$

Solution

$$
\begin{aligned}
L\{f(t)\} = F(s) &= \int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^{5} e^{-st}\cdot 3\mathrm{d}t + \int_5^{\infty} 0\cdot \mathrm{d}t \\
&= 3\left|\frac{e^{-st}}{-s}\right|_0^5 + 0 \\
&= \frac{-3}{s}(e^{-5s} - 1)
\end{aligned}
$$

$$= \frac{3}{s}(1 - e^{-5s})$$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$= s\frac{3}{s}(1 - e^{-5s}) - 3$$

$$= -3e^{-5s}$$

Example 3

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = e^{-5t}\sin t.$

Solution

$$L\{f(t)\} = F(s) = L\{e^{-5t}\sin t\}$$

$$= \frac{1}{(s+5)^2 + 1}$$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$= s\left(\frac{1}{s^2 + 10s + 26}\right) - e^0 \sin 0$$

$$= \frac{s}{s^2 + 10s + 26}$$

Example 4

Find $L\{f(t)\}$ *and* $L\{f'(t)\}$ *of* $f(t) = t \qquad 0 \le t < 3$

$$= 6 \qquad t > 3$$

Solution

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$= \int_0^3 e^{-st}t \,\mathrm{d}t + \int_3^\infty e^{-st}\cdot 6\,\mathrm{d}t$$

$$= \left|\frac{e^{-st}}{-s}\cdot t\right|_0^3 - \left|\frac{e^{-st}}{s^2}\right|_0^3 + 6\left|\frac{e^{-st}}{-s}\right|_3^\infty$$

$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6}{s}\,e^{-3s}$$

$$= \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

$$\begin{aligned} L\{f'(t)\} &= s\,F(s) - f(0) \\ &= s\left[\frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)\right] - 0 \\ &= \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right) \end{aligned}$$

EXERCISE 5.8

Find $L\{f'(t)\}$ of the following functions:

1. $f(t) = \left(\dfrac{1-\cos 2t}{t}\right)$

$$\left[\textbf{Ans.}: s\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$$

2. $f(t) = t+1 \quad 0 \le t \le 2$

$\quad\quad = 3 \quad\quad t > 2$

$$\left[\textbf{Ans.}: \frac{1}{s}(1-e^{-2s})\right]$$

5.8 LAPLACE TRANSFORMS OF INTEGRALS

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \dfrac{F(s)}{s}$.

Proof: $L\left\{\int_0^t f(t)\mathrm{d}t\right\} = \int_0^\infty e^{-st}\left\{\int_0^t f(t)\mathrm{d}t\right\}\mathrm{d}t$

Integrating by parts,

$$\begin{aligned} L\left\{\int_0^t f(t)\mathrm{d}t\right\} &= \left|\int_0^t f(t)\mathrm{d}t\left(\frac{e^{-st}}{-s}\right)\right|_0^\infty - \int_0^\infty\left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t)\mathrm{d}t\right)\right]\mathrm{d}t \\ &= \int_0^\infty \frac{1}{s}e^{-st} f(t)\mathrm{d}t \\ &= \frac{1}{s}L\{f(t)\} \\ &= \frac{F(s)}{s} \end{aligned}$$

Example 1

Find the Laplace transform of $\int_0^t e^{-t}\,\mathrm{d}t$.

Solution

$$L\{e^{-t}\} = \frac{1}{s+1}$$

$$L\left\{\int_0^t e^{-t}\,\mathrm{d}t\right\} = \frac{1}{s}L\{e^{-t}\}$$

$$= \frac{1}{s(s+1)}$$

Example 2

Find the Laplace transform of $\int_0^t e^{-2t}t^3\,\mathrm{d}t$.

Solution

$$L\{e^{-2t}t^3\} = \frac{3!}{(s+2)^4}$$

$$= \frac{6}{(s+2)^4}$$

$$L\left\{\int_0^t e^{-2t}t^3\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{e^{-2t}t^3\right\}$$

$$= \frac{6}{s(s+2)^4}$$

Example 3

Find the Laplace transform of $\int\limits_0^t e^{-t}\cos t\,\mathrm{d}t$. **[Summer 2013]**

Solution

$$L\{\cos t\} = \frac{s}{s^2+1}$$

$$L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1}$$

$$= \frac{s+1}{s^2+2s+2}$$

$$L\left\{\int_0^t e^{-t}\cos t\,\mathrm{d}t\right\} = \frac{1}{s}L\{e^{-t}\cos t\}$$

$$= \frac{s+1}{s^3+2s^2+2s}$$

Example 4

Find the Laplace transform of $\int_0^t e^u(u+\sin u)\mathrm{d}u.$ **[Winter 2015]**

Solution

$$L\{t+\sin t\} = L\{t\} + L\{\sin t\}$$

$$= \frac{1}{s^2} + \frac{1}{s^2+1}$$

$$L\{e^t(t+\sin t)\} = \frac{1}{(s-1)^2} + \frac{1}{(s-1)^2+1}$$

$$L\left\{\int_0^t e^u(u+\sin u)\,\mathrm{d}u\right\} = \frac{1}{s}L\{e^t(t+\sin t)\}$$

$$= \frac{1}{s}\left[\frac{1}{s^2-2s+1} + \frac{1}{s^2-2s+2}\right]$$

Example 5

Find the Laplace transform of $\int_0^t t\cosh t\,\mathrm{d}t.$

Solution

$$L\{t\cosh t\} = L\left\{t\left(\frac{e^t+e^{-t}}{2}\right)\right\}$$

$$= \frac{1}{2}L\{t\,e^t + t\,e^{-t}\}$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right]$$

$$= \frac{1}{2}\cdot\frac{2(s^2+1)}{(s^2-1)^2}$$

$$= \frac{s^2+1}{(s^2-1)^2}$$

$$L\left\{\int_0^t t\cosh t\,\mathrm{d}t\right\} = \frac{1}{s}L\{t\cosh t\}$$

$$= \frac{s^2+1}{s(s^2-1)^2}$$

Example 6

Find the Laplace transform of $\int_0^t t\,e^{-4t}\sin 3t\,\mathrm{d}t$.

Solution

$$L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\{t\,e^{-4t}\sin 3t\} = \frac{6(s+4)}{\left[(s+4)^2+9\right]^2}$$

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$L\left\{\int_0^t t\,e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\{t\,e^{-4t}\sin 3t\}$$

$$= \frac{6(s+4)}{s(s^2+8s+25)^2}$$

Example 7

Find the Laplace transform of $e^{-4t}\int_0^t t\sin 3t\,\mathrm{d}t$.

Solution

$$L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{1}{s}L\{t\sin 3t\}$$

$$= \frac{6}{(s^2+9)^2}$$

$$L\left\{e^{-4t}\int_0^t t\sin 3t\right\} = \frac{6}{\left[(s+4)^2+9\right]^2}$$

$$= \frac{6}{(s^2+8s+25)^2}$$

Example 8

Find the Laplace transform of $\cosh t \int_0^t e^t \cosh t \, \mathrm{d}t$.

Solution

$$L\{\cosh t\} = \frac{s}{s^2-1}$$

$$L\{e^t \cosh t\} = \frac{s-1}{(s-1)^2-1}$$

$$= \frac{s-1}{s^2-2s+1-1}$$

$$= \frac{s-1}{s(s-2)}$$

$$L\left\{\int_0^t e^t \cosh t \, \mathrm{d}t\right\} = \frac{1}{s}L\{e^t \cosh t\}$$

$$= \frac{s-1}{s^2(s-2)}$$

$$L\left\{\cosh t \int_0^t e^t \cosh t \, \mathrm{d}t\right\} = L\left\{\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t e^t \cosh t \, \mathrm{d}t\right\}$$

$$= \frac{1}{2}\left[L\left\{e^t \int_0^t e^t \cosh t \, \mathrm{d}t\right\} + L\left\{e^{-t}\int_0^t e^t \cosh t \, \mathrm{d}t\right\}\right]$$

$$= \frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)} + \frac{(s+1)-1}{(s+1)^2(s+1-2)}\right]$$

$$= \frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)} + \frac{s}{(s+1)^2(s-1)}\right]$$

Example 9

Find the Laplace transform of $e^{-t}\int_0^t \frac{\sin t}{t}\,\mathrm{d}t$.

Solution

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\mathrm{d}s$$

$$= \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s$$

$$= \left|\tan^{-1} s\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left\{\int_0^t \frac{\sin t}{t}\mathrm{d}t\right\} = \frac{1}{s} L\left\{\frac{\sin t}{t}\right\}$$

$$= \frac{1}{s}\cot^{-1} s$$

$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\mathrm{d}t\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$$

Example 10

Find the Laplace transform of $\int_0^t e^t \frac{\sin t}{t}\,\mathrm{d}t$. **[Winter 2016]**

Solution

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} L\{\sin t\}\,\mathrm{d}s$$

$$= \int_s^{\infty} \frac{1}{s^2+1}\,\mathrm{d}s$$

$$= \left|\tan^{-1} s\right|_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left\{e^t\,\frac{\sin t}{t}\right\} = \cot^{-1}(s-1)$$

$$L\left\{\int_0^t e^t\,\frac{\sin t}{t}\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{e^t\,\frac{\sin t}{t}\right\}$$

$$= \frac{1}{s}\cot^{-1}(s-1)$$

Example 11

Find the Laplace transform of $t\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t$.

Solution

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-4t}\sin 3t\} = \frac{3}{(s+4)^2+9}$$

$$= \frac{3}{s^2+8s+25}$$

$$L\left\{\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\{e^{-4t}\sin 3t\}$$

$$= \frac{3}{s^3+8s^2+25s}$$

$$L\left\{t\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\left\{\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\}$$

$$= -\frac{d}{ds}\left(\frac{3}{s^3+8s^2+25s}\right)$$

$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

Example 12

Find the Laplace transform of $\int_0^t t\,e^{-3t}\sin^2 t\,dt$.

Solution

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$L\{t\sin^2 t\} = -\frac{d}{ds}L\{\sin^2 t\}$$

$$= -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} - \left\{\frac{s^2+4-s(2s)}{(s^2+4)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} - \frac{s^2-4}{(s^2+4)^2}\right]$$

$$L\{t\,e^{-3t}\sin^2 t\} = \frac{1}{2}\left[\frac{1}{(s+3)^2} - \frac{(s+3)^2-4}{\{(s+3)^2+4\}^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s+3)^2} - \frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

$$L\left\{\int_0^t t\,e^{-3t}\sin^2 t\,dt\right\} = \frac{1}{s}\,L\{t\,e^{-3t}\sin^2 t\}$$

$$= \frac{1}{2s}\left[\frac{1}{(s+3)^2} - \frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

Example 13

Find the Laplace transform of $\int_0^t \int_0^t \sin at \, \mathrm{d}t \, \mathrm{d}t$. **[Summer 2013]**

Solution

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$L\left\{\int_0^t \sin at \, \mathrm{d}t\right\} = \frac{1}{s} L\{\sin at\}$$

$$= \frac{1}{s}\left(\frac{a}{s^2 + a^2}\right)$$

$$= \frac{a}{s(s^2 + a^2)}$$

$$L\left\{\int_0^t \int_0^t \sin at \, \mathrm{d}t \, \mathrm{d}t\right\} = \frac{1}{s} L\left\{\int_0^t \sin at \, \mathrm{d}t\right\}$$

$$= \frac{1}{s} \frac{a}{s(s^2 + a^2)}$$

$$= \frac{a}{s^2(s^2 + a^2)}$$

Example 14

Find the Laplace transform of $\int_0^t \int_0^t \int_0^t t \sin t \, \mathrm{d}t \, \mathrm{d}t \, \mathrm{d}t$.

Solution

$$L\{t \sin t\} = -\frac{\mathrm{d}}{\mathrm{d}s} L\{\sin t\}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2 + 1}\right)$$

$$= \frac{2s}{(s^2 + 1)^2}$$

$$L\left\{\int_0^t t \sin t \, \mathrm{d}t\right\} = \frac{1}{s} L\{t \sin t\}$$

$$L\left\{\int_0^t\int_0^t t\sin t\,dt\right\}=\frac{1}{s}L\left\{\int_0^t t\sin t\,dt\right\}$$

$$=\frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\}$$

$$L\left\{\int_0^t\int_0^t\int_0^t t\sin t\,dt\right\}=\frac{1}{s}L\left\{\int_0^t\int_0^t t\sin t\,dt\right\}$$

$$=\frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\}$$

$$=\frac{1}{s^3}\cdot\frac{2s}{(s^2+1)^2}$$

$$=\frac{2}{s^2(s^2+1)^2}$$

EXERCISE 5.9

Find the Laplace transforms of the following functions:

1. $\int_0^t e^{-t}t^4\,dt$

$$\left[\textbf{Ans.}: \frac{4!}{s(s+1)^5}\right]$$

2. $\int_0^t \frac{1+e^{-t}}{t}\,dt$

$$\left[\textbf{Ans.}: \frac{1}{s}\log\left[s(s+1)\right]\right]$$

3. $\int_0^t \frac{e^t\sin t}{t}\,dt$

$$\left[\textbf{Ans.}: \frac{1}{s}\cot^{-1}(s-1)\right]$$

4. $\int_0^t t e^{-2t}\sin 3t\,dt$

$$\left[\textbf{Ans.}: \frac{1}{s}\cdot\frac{3(2s+4)}{(s^2+4s+13)^2}\right]$$

5. $e^{-3t}\int_0^t t\sin 3t\,dt$

$$\left[\textbf{Ans.}: -\frac{6}{(s^2+6s+18)^2}\right]$$

6. $\int_0^t t^2 \sin t \, dt$

$$\left[\textbf{Ans.}: -\frac{2}{s}\frac{(1-3s^2)}{(s^2+1)^3}\right]$$

7. $\int_0^t t \cos^2 t \, dt$

$$\left[\textbf{Ans.}: \frac{1}{2s^3}+\frac{1}{2}\cdot\frac{s^2-4}{s(s^2+4)^2}\right]$$

8. $\int_0^t t e^{-3t} \cos^2 2t \, dt$

$$\left[\textbf{Ans.}: \frac{1}{2s(s+3)^2}+\frac{1}{2}\cdot\frac{s^2+6s-7}{s(s^2+6s+25)^2}\right]$$

5.9 EVALUATION OF INTEGRALS USING LAPLACE TRANSFORM

Example 1

Evaluate $\int_0^\infty e^{-3t} t^5 \, dt$.

Solution

$$\int_0^\infty e^{-st} t^5 \, dt = L\{t^5\}$$

$$= \frac{5!}{s^6}$$

$$= \frac{120}{s^6} \qquad \text{... (1)}$$

Putting $s = 3$ in Eq. (1),

$$\int_0^\infty e^{-3t} t^5 \, dt = \frac{120}{3^6}$$

$$= \frac{40}{243}$$

Example 2

Evaluate $\int_0^\infty e^{-2t} \sin^3 t \, dt$.

Solution

$$\int_0^\infty e^{-st} \sin^3 t \, dt = L\{\sin^3 t\}$$

$$= L\left\{\frac{3\sin t - \sin 3t}{4}\right\}$$

$$= \frac{3}{4}\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$$

$$= \frac{3}{4}\left[\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{6}{(s^2+1)(s^2+9)} \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t}\sin^3 t\,\mathrm{d}t = \frac{6}{(4+1)(4+9)} = \frac{6}{65}$$

Example 3

If $\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\mathrm{d}t = \frac{3}{8}$, *find* α.

Solution

$$\int_0^\infty e^{-st}\sin(t+\alpha)\cos(t-\alpha)\mathrm{d}t = \frac{1}{2}\int_0^\infty e^{-st}(\sin 2t + \sin 2\alpha)\mathrm{d}t$$

$$= \frac{1}{2}L\{\sin 2t + \sin 2\alpha\}$$

$$= \frac{1}{2}\left(\frac{2}{s^2+4} + \sin 2\alpha \cdot \frac{1}{s}\right) \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\,\mathrm{d}t = \frac{1}{2}\left(\frac{2}{4+4} + \frac{1}{2}\sin 2\alpha\right)$$

$$= \frac{1}{8} + \frac{1}{4}\sin 2\alpha$$

But $\quad \int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\mathrm{d}t = \frac{3}{8}$

$$\frac{1}{8} + \frac{1}{4}\sin 2\alpha = \frac{3}{8}$$

$$\frac{1}{4}\sin 2\alpha = \frac{1}{4}$$

$$\sin 2\,\alpha = 1$$

$$2\alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4}$$

Example 4

Evaluate $\int_0^\infty te^{-2t} \cos t \, dt$.

Solution

$$\int_0^\infty e^{-st} t \cos t \, dt = L\{t \cos t\}$$

$$= -\frac{d}{ds} L\{\cos t\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)}\right]$$

$$= -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{s^2-1}{(s^2+1)^2} \qquad ...(1)$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t} t \cos t \, dt = \frac{(2)^2 - 1}{\left[(2)^2+1\right]^2} = \frac{3}{25}$$

Example 5

Show that $\int_0^\infty e^{-2t} t^2 \sin 3t \, dt = \frac{18}{2197}$.

Solution

$$\int_0^\infty e^{-st} t^2 \sin 3t \, dt = L\{t^2 \sin 3t\}$$

$$= (-1)^2 \frac{d^2}{ds^2} L\{\sin 3t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{d}{ds}\left[-\frac{3(2s)}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{(s^2+9)^2(1) - s \cdot 2(s^2+9)2s}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right]$$

$$= \frac{-6(-3s^2+9)}{(s^2+9)^3}$$

$$= \frac{18(s^2-3)}{(s^2+9)^3} \qquad \text{... (1)}$$

Putting $s = 2$ in Eq. (1),

$$\int_0^\infty e^{-2t} t^2 \sin 3t \, dt = \frac{18(4-3)}{(4+9)^3} = \frac{18}{2197}$$

Example 6

Show that $\int_0^\infty e^{-\sqrt{2}t} \dfrac{\sin t \sinh t}{t} \, dt = \dfrac{\pi}{8}$.

Solution

$$\int_0^\infty e^{-st} \frac{\sin t \sinh t}{t} dt = L\left\{\frac{\sin t \sinh t}{t}\right\}$$

$$= L\left\{\left(\frac{e^t - e^{-t}}{2}\right)\frac{\sin t}{t}\right\}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\} ds$$

$$= \int_s^\infty \frac{1}{s^2+1} ds$$

$$= \left|\tan^{-1} s\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st} \frac{\sin t \sinh t}{t} \mathrm{d}t = \frac{1}{2}\left[L\left\{e^t \frac{\sin t}{t}\right\} - L\left\{e^{-t} \frac{\sin t}{t}\right\}\right] \quad \left[\begin{array}{l}\text{Using first shifting} \\ \text{theorem}\end{array}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right]. \qquad \text{... (1)}$$

Putting $s = \sqrt{2}$ in Eq. (1),

$$\int_0^\infty e^{-\sqrt{2}\,t} \frac{\sin t \sinh t}{t} \mathrm{d}t = \frac{1}{2}\left[\tan^{-1}(\sqrt{2}+1) - \tan^{-1}(\sqrt{2}-1)\right]$$

$$= \frac{1}{2}\tan^{-1}\left\{\frac{\sqrt{2}+1-\sqrt{2}+1}{1+(\sqrt{2}+1)(\sqrt{2}-1)}\right\}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2}{1+2-1}\right)$$

$$= \frac{1}{2}\tan^{-1} 1$$

$$= \frac{1}{2}\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

Example 7

Evaluate $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} \mathrm{d}u\, \mathrm{d}t.$

Solution

$$L\{\sin u\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin u}{u}\right\} = \int_s^\infty L\{\sin u\} \mathrm{d}s$$

$$= \int_s^\infty \frac{1}{s^2+1} \mathrm{d}s$$

$$= \left|\tan^{-1} s\right|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left\{\int_0^t \frac{\sin u}{u} \mathrm{d}u\right\} = \frac{1}{s} L\left\{\frac{\sin u}{u}\right\}$$

$$= \frac{1}{s} \cot^{-1} s$$

Now, $\displaystyle\int_0^\infty e^{-st} \int_0^t \frac{\sin u}{u} \mathrm{d}u\, \mathrm{d}t = \frac{1}{s} \cot^{-1} s$

Putting $s = 1$,

$$\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u} \mathrm{d}u\, \mathrm{d}t = \cot^{-1} 1 = \frac{\pi}{4}.$$

EXERCISE 5.10

Evaluate the following integrals using the Laplace transform:

1. $\displaystyle\int_0^\infty e^{-3t} \cos^2 t \, dt$

$$\left[\textbf{Ans.:}\ \frac{11}{39}\right]$$

2. $\displaystyle\int_0^\infty e^{-5t} \sinh^3 t \, dt$

$$\left[\textbf{Ans.:}\ \frac{1}{64}\right]$$

3. $\displaystyle\int_0^\infty e^{-3t} \cos^3 t \, dt$

$$\left[\textbf{Ans.:}\ \frac{4}{15}\right]$$

4. $\displaystyle\int_0^\infty e^{-2t} t^3 \sin t \, dt$

$$\left[\textbf{Ans.:}\ -\frac{576}{25}\right]$$

5. $\displaystyle\int_0^\infty e^{-3t} t^2 \sinh 2t \, dt$

$$\left[\textbf{Ans.:}\ \frac{124}{125}\right]$$

6. $\displaystyle\int_0^\infty e^{-2t} t \sin^2 t \, dt$

$$\left[\textbf{Ans.:}\ \frac{1}{8}\right]$$

7. $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt$

[Ans.: log 3]

8. $\int_0^\infty e^{-t}\frac{(1-\cos 2t)}{2t}dt$

$\left[\textbf{Ans.: }\frac{1}{4}\log 5\right]$

9. $\int_0^\infty e^{-t}\frac{(\cos 3t - \cos 2t)}{t}dt$

$\left[\textbf{Ans.: }\frac{1}{2}\log\frac{1}{2}\right]$

10. $\int_0^\infty e^{-t}\frac{\sin\sqrt{3}t}{t}dt$

$\left[\textbf{Ans.: }\frac{\pi}{3}\right]$

11. $\int_0^\infty e^{-2t}\frac{\sinh t}{t}dt$

$\left[\textbf{Ans.: }\frac{1}{2}\log 3\right]$

12. $\int_0^\infty e^{-t}\int_0^t t\cos^2 t\,dt\,dt$

$\left[\textbf{Ans.: }\frac{12}{50}\right]$

13. $\int_0^\infty e^{-t}\left(t\int_0^t e^{-4u}\cos u\,du\right)dt$

$\left[\textbf{Ans.: }\frac{9}{64}\right]$

14. $\int_0^\infty e^{-t}\left(\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right)dt$

$\left[\textbf{Ans.: }\frac{1}{4}\log 5 - \frac{1}{2}\cot^{-1} 2\right]$

5.10 UNIT STEP FUNCTION

Unit step function (Fig. 5.1) is defined as

$$u(t) = 0 \qquad t < 0$$
$$= 1 \qquad t > 0$$

Fig. 5.1 ***Unit step function***

The displaced (delayed) unit step function $u(t-a)$ (Fig. 5.2) represents the function $u(t)$ which is displaced by a distance 'a' to the right.

$$u(t-a) = 0 \qquad t < a$$
$$= 1 \qquad t > a$$

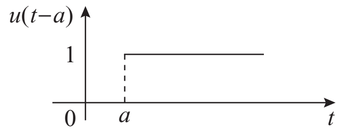


Fig. 5.2 **Delayed *unit step function***

Laplace Transforms of Unit Step Functions

(i) Laplace transform of the unit step function $u(t)$

$$u(t) = 0 \qquad t < 0$$
$$= 1 \qquad t > 0$$

$$L\{u(t)\} = \int_0^\infty e^{-st} u(t) \mathrm{d}t$$
$$= \int_0^\infty e^{-st} \mathrm{d}t$$
$$= \left| \frac{e^{-st}}{-s} \right|_0^\infty$$
$$= \frac{1}{s}$$

(ii) Laplace transform of the displaced unit step function $u(t-a)$

$$u(t-a) = 0 \qquad t < a$$
$$= 1 \qquad t > a$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) \mathrm{d}t$$
$$= \int_a^\infty e^{-st} \mathrm{d}t$$
$$= \left| \frac{e^{-st}}{-s} \right|_a^\infty$$
$$= \frac{1}{s} e^{-as}$$

(iii) Laplace transform of the function $f(t)\, u(t - a)$

$$\begin{aligned} f(t)\, u(t-a) &= 0 && t < a \\ &= f(t) && t > a \end{aligned}$$

$$L\{f(t)\, u(t-a)\} = \int_0^\infty e^{-st} f(t) u(t-a) \mathrm{d}t$$

$$= \int_a^\infty e^{-st} f(t) \mathrm{d}t$$

Putting $t - a = x$, $\mathrm{d}t = \mathrm{d}x$

When $t = a$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$L\{f(t)\, u(t-a)\} = \int_0^\infty e^{-s(x+a)} f(x+a) \mathrm{d}x$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x+a) \mathrm{d}x$$

$$= e^{-as} \int_0^\infty e^{-st} f(t+a) \mathrm{d}t$$

$$= e^{-as} L\{f(t+a)\}$$

$$= e^{-as} F(s+a)$$

(iv) Laplace transform of the function $f(t - a)\, u(t - a)$

$$\begin{aligned} f(t-a)\, u(t-a) &= 0 && t < a \\ &= f(t-a) && t > a \end{aligned}$$

$$L\{f(t-a)\, u(t-a)\} = \int_0^\infty e^{-st} f(t-a) u(t-a) \mathrm{d}t$$

$$= \int_a^\infty e^{-st} f(t-a) \mathrm{d}t$$

Putting $t - a = x$, $\mathrm{d}t = \mathrm{d}x$

When $t = a$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$L\{f(t-a)\, u(t-a)\} = \int_0^\infty e^{-s(a+x)} f(x) \mathrm{d}x$$

$$= e^{-as} \int_0^\infty e^{-sx} f(x) \mathrm{d}x$$

$$= e^{-as} L\{f(x)\}$$

$$= e^{-as} F(s)$$

Example 1

Find the Laplace transform of $e^{-3t} u(t-2)$. **[Winter 2017]**

Solution

$$L\{f(t)u(t-a) = e^{-as}L\{f(t+a)\}$$

$$L\{e^{-3t}u(t-2)\} = e^{-2s}L\{e^{-3(t+2)}\}$$

$$= e^{-2s}e^{-6}L\{e^{-3t}\}$$

$$= e^{-(2s+6)}\frac{1}{s+3}$$

Example 2

Find the Laplace transform of $t^2\, u(t-2)$. **[Winter 2014]**

Solution

$$L\{f(t)\, u(t-a)\} = e^{-as}\, L\{f(t+a)\}$$

$$L\{t^2\, u(t-2)\} = e^{-2s}\, L\{(t+2)^2\}$$

$$= e^{-2s}\, L\{t^2 + 4t + 4\}$$

$$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$$

Example 3

Find the Laplace transform of $\sin t\, u\left(t-\frac{\pi}{2}\right) - u\left(t-\frac{3\pi}{2}\right)$.

Solution

$$L\{f(t)\, u(t-a)\} = e^{-as}\, L\{f(t+a)\}$$

$$L\left\{\sin t\, u\left(t-\frac{\pi}{2}\right)-u\left(t-\frac{3\pi}{2}\right)\right\} = L\left\{\sin t\, u\left(t-\frac{\pi}{2}\right)\right\}-L\left\{u\left(t-\frac{3\pi}{2}\right)\right\}$$

$$= e^{-\frac{\pi s}{2}}L\left\{\sin\left(t+\frac{\pi}{2}\right)\right\}-\frac{e^{-\frac{3\pi s}{2}}}{s}$$

$$= e^{\frac{-\pi s}{2}}L\{\cos t\}-\frac{e^{-\frac{3\pi s}{2}}}{s}$$

$$= e^{\frac{-\pi s}{2}}\frac{s}{s^2+1}-e^{-\frac{3\pi s}{2}}\frac{1}{s}$$

Example 4

Find the Laplace transform of $e^{-t}\sin t\, u(t-\pi)$.

Solution

$$L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}$$

$$L\{e^{-t}\sin t\,u(t-\pi)\} = e^{-\pi s}\,L\{e^{-(t+\pi)}\sin(t+\pi)\}$$

$$= -e^{-\pi s}\,e^{-\pi}\,L\{e^{-t}\sin t\}$$

$$= -e^{-\pi(s+1)}\frac{1}{(s+1)^2+1}$$

$$= -e^{-\pi(s+1)}\frac{1}{s^2+2s+2}$$

Example 5

Find the Laplace transform of $(1+2t-3t^2+4t^3)\,u(t-2)$ *and, hence, evaluate* $\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\,u(t-2)\mathrm{d}t$.

Solution

$$L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}$$

$$L\{(1+2t-3t^2+4t^3)\,u(t-2)\} = e^{-2s}\,L[1+2(t+2)-3(t+2)^2+4(t+2)^3]$$

$$= e^{-2s}\,L\{1+2(t+2)-3(t^2+4t+4)$$
$$+4(t^3+6t^2+12t+8)\}$$

$$= e^{-2s}\,L\{25+38t+21t^2+4t^3\}$$

$$= e^{-2s}\left(\frac{25}{s}+38\cdot\frac{1}{s^2}+21\cdot\frac{2!}{s^3}+4\cdot\frac{3!}{s^4}\right)$$

$$= e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right)$$

Now, $$\int_0^\infty e^{-st}(1+2t-3t^2+4t^3)\,u(t-2)\,\mathrm{d}t = e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right) \quad \ldots (1)$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\,u(t-2)\mathrm{d}t = e^{-2}\left(\frac{25}{1}+\frac{38}{1^2}+\frac{42}{1^3}+\frac{24}{1^4}\right)$$

$$= \frac{129}{e^2}$$

Example 6

Find the Laplace transform of $f(t) = t^2 \quad 0 < t < 1$
$= 4t \quad t > 1$

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = t^2 u(t) - t^2 u(t-1) + 4t\, u(t-1)$$

$$L\{f(t)\} = L\{t^2 u(t) - t^2 u(t-1) + 4t\, u(t-1)\}$$

$$= L\{t^2 u(t)\} - L\{t^2 u(t-1) + 4\, L\{t\, u(t-1)\}$$

$$= \frac{2}{s^3} - e^{-s} L\{(t+1)^2\} + 4e^{-s} L\{(t+1)\}$$

$$= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + 4e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$= \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right)$$

Example 7

Find the Laplace transform of $f(t) = \sin 2t \quad 2\pi < t < 4\pi$
$= 0 \quad$ *otherwise*

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = \sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)$$

$$L\{f(t)\} = L\{\sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)\}$$

$$= L\{\sin 2t\, u(t-2\pi)\} - L\{\sin 2t\, u(t-4\pi)\}$$

$$= e^{-2\pi s} L\{\sin 2(t+2\pi)\} - e^{-4\pi s} L\{\sin 2(t+4\pi)\}$$

$$= e^{-2\pi s} L\{\sin 2t\} - e^{-4\pi s} L\{\sin 2t\}$$

$$= e^{-2\pi s}\frac{2}{s^2+4} - e^{-4\pi s}\frac{2}{s^2+4} = \frac{2}{s^2+4}\left(e^{-2\pi s} - e^{-4\pi s}\right)$$

Example 8

Find the Laplace transform of $f(t) = \cos t \quad 0 < t < \pi$
$= \sin t \quad t > \pi$

Solution

Expressing $f(t)$ in terms of the unit step function,

$$f(t) = \cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)$$

$$\begin{aligned}
L\{f(t)\} &= L\{\cos t\, u(t) - \cos t\, u(t-\pi) + \sin t\, u(t-\pi)\} \\
&= L\{\cos t\, u(t)\} - L\{\cos t\, u(t-\pi)\} + L\{\sin t\, u(t-\pi)\} \\
&= \frac{s}{s^2+1} - e^{-\pi s} L\{\cos(t+\pi)\} + e^{-\pi s} L\{\sin(t+\pi)\} \\
&= \frac{s}{s^2+1} - e^{-\pi s} L\{-\cos t\} + e^{-\pi s} L\{-\sin t\} \\
&= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos t\} - e^{-\pi s} L\{\sin t\} \\
&= \frac{s}{s^2+1} + e^{-\pi s} \cdot \frac{s}{s^2+1} - e^{-\pi s} \cdot \frac{1}{s^2+1} \\
&= \frac{1}{s^2+1}[s + e^{-\pi s}(s-1)]
\end{aligned}$$

Example 9

Find the Laplace transform of

$$f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \cos 2t & \pi < t < 2\pi \\ \cos 3t & t > 2\pi \end{cases}$$

Solution

Expressing $f(t)$ in terms of the unit step function,

$$\begin{aligned}
f(t) &= [\cos t\, u(t) - \cos t\, u(t-\pi)] + [\cos 2t\, u(t-\pi) - \cos 2t\, u(t-2\pi)] \\
&\quad + \cos 3t\, u(t-2\pi) \\
&= \cos t\, u(t) + (\cos 2t - \cos t)\, u(t-\pi) + (\cos 3t - \cos 2t)\, u(t-2\pi)
\end{aligned}$$

$$\begin{aligned}
L\{f(t)\} &= L\{\cos t\, u(t)\} + L\{(\cos 2t - \cos t)\, u(t-\pi)\} + L\{(\cos 3t - \cos 2t)\, u(t-2\pi)\} \\
&= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\} + e^{-2\pi s} L\{\cos 3(t+2\pi) \\
&\qquad - \cos 2(t+2\pi)\} \\
&= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2t + \cos t\} + e^{-2\pi s} L\{\cos 3t - \cos 2t\} \\
&= \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)
\end{aligned}$$

EXERCISE 5.11

(I) Find the Laplace transforms of the following functions:

1. $t^4\, u(t-2)$

$$\left[\textbf{Ans.: } e^{-2s}\left(\frac{16}{s} + \frac{32}{s^2} + \frac{48}{s^3} + \frac{48}{s^4} + \frac{24}{s^5}\right)\right]$$

2. $(1 + 3t - 4t^2 + 2t^3)\, u(t - 3)$

$$\left[\textbf{Ans.:}\ e^{-3s}\left(\frac{28}{s}+\frac{33}{s^2}+\frac{28}{s^3}+\frac{12}{s^4}\right)\right]$$

3. $t\, e^{-2t}\, u(t - 1)$

$$\left[\textbf{Ans.:}\quad e^{-(s+2)}\frac{s+3}{(s+2)^2}\right]$$

4. $\cos t\; u(t - 1)$

$$\left[\textbf{Ans.:}\quad e^{-s}\left(\frac{s\cos 1-\sin 1}{s^2+1}\right)\right]$$

(II) Express the following functions in terms of the unit step function and, hence, find the Laplace transform.

1. $f(t) = t \qquad 0 < t < 2$

$\quad\;\; = t^2 \qquad t > 2$

$$\left[\textbf{Ans.:}\ \frac{1}{s^2}+e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right)\right]$$

2. $f(t) = e^t \cos t \qquad 0 < t < \pi$

$\quad\;\; = e^t \sin t \qquad t > \pi$

$$\left[\textbf{Ans.:}\ \frac{s-1}{s^2-2s+2}+e^{-\pi(s-1)}\cdot\frac{s-2}{s^2-2s+2}\right]$$

3. $f(t) = \sin t \qquad 0 < t < \pi$

$\quad\;\; = \sin 2t \qquad \pi < t < 2\pi$

$\quad\;\; = \sin 3t \qquad t > 2\pi$

$$\left[\textbf{Ans.:}\ \frac{1}{s^2+1}+e^{-\pi s}\left(\frac{2}{s^2+4}+\frac{1}{s^2+1}\right)-e^{-2\pi s}\left(\frac{3}{s^2+9}+\frac{2}{s^2+4}\right)\right]$$

4. $f(t) = t - 1 \qquad 1 < t < 2$

$\quad\;\; = 3 - t \qquad 2 < t < 3$

$\quad\;\; = 0 \qquad t > 3$

$$\left[\textbf{Ans.:}\ \frac{(1-e^{-s})^2}{s^2}\right]$$

5. $f(t) = \sin t \qquad 0 < t < \pi$

$\quad\;\; = t \qquad t > \pi$

$$\left[\textbf{Ans.:}\ \frac{1+e^{-\pi s}}{s^2+1}+e^{-\pi s}\left(\frac{\pi s+1}{s^2}\right)\right]$$

5.11 DIRAC'S DELTA FUNCTION

Consider the function $f(t)$ as shown in Fig. 5.3.

$$f(t) = \frac{1}{T} \qquad -\frac{T}{2} < t < \frac{T}{2}$$
$$= 0 \qquad \text{otherwise}$$

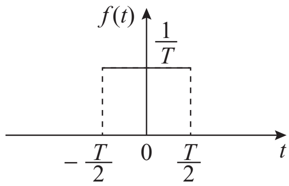


Fig. 5.3 ***A function***

The width of this function is T and its amplitude is $\frac{1}{T}$. Hence, the area of this function is one unit. As $T \to 0$, the function becomes a delta function or a unit impulse function.

$$\lim_{T \to 0} f(t) = \delta(t)$$

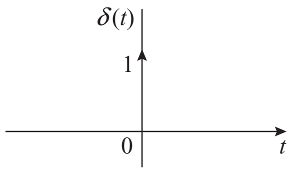


Fig. 5.4 ***Unit impulse function***

Dirac's delta, or unit impulse (Fig. 5.4), function has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude of the function is infinitely large such that the area under its curve is equal to one unit. Hence, it is defined as

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \qquad t = 0$$

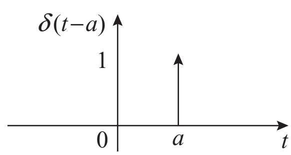


Fig. 5.5 ***Delayed unit impulse function***

The displaced (delayed) delta or unit impulse function $\delta(t - a)$ (Fig. 5.5) represents the function $\delta(t)$ which is displaced by a distance 'a' to the right.

$$\delta(t - a) = 0 \qquad t \neq a$$

and $$\int_{-\infty}^{\infty} \delta(t - a)\,\mathrm{d}t = 1 \qquad t = a$$

Properties of unit impulse functions

(i) $\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)$

(ii) $\int_{0}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)$

(iii) $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$

(iv) $\int_{0}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$

Laplace Transforms of the Unit Impulse Functions

(i) Laplace transform of $\delta(t)$

$$\delta(t) = 0 \qquad t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \qquad t = 0$$
$$L\{\delta(t)\} = \int_{0}^{\infty} e^{-st}\,\delta(t)\,\mathrm{d}t$$

$$= [e^{-st}]_{t=0}$$
$$= 1$$

(ii) Laplace transform of $\delta(t-a)$

$$\delta(t-a) = 0 \qquad t \neq a$$

$$\text{and } \int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = 1 \qquad t = a$$

$$L\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\delta(t-a)\,\mathrm{d}t$$
$$= [e^{-st}]_{t=a} \qquad \text{[From Property (iii)]}$$
$$= e^{-as}$$

(iii) Laplace transform of $f(t)\,\delta(t-a)$

$$f(t)\,\delta(t-a) = 0 \qquad t \neq a$$

$$\text{and } \int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a) \qquad t = a$$

$$L\{f(t)\,\delta(t-a)\} = \int_0^{\infty} e^{-st} f(t)\delta(t-a)\,\mathrm{d}t$$
$$= [e^{-st} f(t)]_{t=a} \qquad \text{[From Property (iii)]}$$
$$= e^{-as} f(a)$$

Example 1

Find the Laplace transform of $\sin 2t\ \delta\left(t - \frac{\pi}{4}\right) - t^2\,\delta(t-2)$.

Solution

$$L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$$

$$L\left\{\sin 2t\,\delta\left(t-\frac{\pi}{4}\right) - t^2\delta(t-2)\right\} = e^{\frac{-\pi s}{4}} \sin 2\left(\frac{\pi}{4}\right) - e^{-2s}(2)^2$$

$$= e^{\frac{-\pi s}{4}} \sin\frac{\pi}{2} - 4e^{-2s}$$

$$= e^{\frac{-\pi s}{4}} - 4e^{-2s}$$

Example 2

Find the Laplace transform of $t\,u(t-4) + t^2\,\delta(t-4)$.

Solution

$$L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$$

$$\begin{aligned} L\{t\,u(t-4) + t^2\,\delta(t-2)\} &= e^{-4s}\,L\{f(t+4)\} + L\{t^2\,\delta(t-4)\} \\ &= e^{-4s}\,L\{t+4\} + e^{-4s}\,(4)^2 \\ &= e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}\right) + 16\,e^{-4s} \\ &= e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}+16\right) \end{aligned}$$

Example 3

Find the Laplace transform of $t^2\,u(t-2) - \cosh t\;\delta(t-2)$.

Solution

$$L\{f(t)\,\delta(t-a)\} = e^{-as}\,f(a)$$

and

$$L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}$$

$$\begin{aligned} L\{t^2\,u(t-2) - \cosh t\;\delta(t-2)\} &= L\{t^2\,u(t-2)\} - L\{\cosh t\;\delta(t-2)\} \\ &= e^{-2s}\,L\{(t+2)^2\} - e^{-2s}\cosh 2 \\ &= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right) - e^{-2s}\cosh 2 \\ &= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}-\cosh 2\right) \end{aligned}$$

Example 4

Evaluate $\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)dt$.

Solution

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

$$\int_0^\infty \cos 2t\,\delta\left(t-\frac{\pi}{4}\right)dt = \cos\frac{2\pi}{4} = 0$$

Example 5

Evaluate $\int_0^\infty t^2\,e^{-t}\sin t\,\delta(t-2)\,dt$.

Solution

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

$$\int_0^\infty t^2\,e^{-t}\sin t\,\delta(t-2)\,dt = (2)^2\,e^{-2}\sin 2 = 4e^{-2}\sin 2$$

Example 6

Evaluate $\int_0^{\infty} t^m (\log t)^n \, \delta(t-3) \, dt.$

Solution

$$\int_0^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$$

$$\int_0^{\infty} t^m (\log t)^n \, \delta(t-3)\,dt = 3^m (\log 3)^n$$

EXERCISE 5.12

(I) Find the Laplace transforms of the following functions:

1. $t\,u(t-4) - t^2\,\delta(t-2)$

$$\left[\textbf{Ans.:}\ e^{-4s}\frac{1}{s^2}(1+4s) - 4e^{-2s}\right]$$

2. $\sin 2t\ \delta(t-2)$

[**Ans. :** $e^{-2s} \sin 4$]

3. $t^2\,u(t-2) - \cosh t\ \delta(t-4)$

$$\left[\textbf{Ans.:}\ \frac{2e^{-2s}}{s^3}(2s^2+2s+1) - e^{-4s}\cosh 4\right]$$

4. $t\,e^{-2t}\,\delta(t-2)$

[**Ans. :** $2e^{-(4+2s)}$]

5. $\dfrac{e^{-t}\sin t}{t}\delta(t-3)$

$$\left[\textbf{Ans.:}\ \frac{1}{3}e^{-(s+3)}\sin 3\right]$$

6. $(e^{-4t} + \log t)\ \delta(t-2)$

[**Ans. :** $(e^{-8} + \log 2)\ e^{-2s}$]

(II) Evaluate the following integrals:

1. $\int_0^{\infty} \sin 4t\,\delta\left(t - \frac{\pi}{8}\right) dt$

$$\left[\textbf{Ans.:}\ e^{\frac{-\pi s}{8}}\right]$$

2. $\int_0^\infty e^{-t} \sin t\, \delta(t-a)\, dt$

[**Ans. :** $e^{-a} (\sin a - \cos a)$]

5.12 LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

A function $f(t)$ is said to be periodic if there exists a constant $T (T > 0)$ such that $f(t + T) = f(t)$, for all values of t.

$$f(t + 2T) = f(t + T + T) = f(t + T) = f(t)$$

In general, $f(t + nT) = f(t)$ for all t, where n is an integer (positive or negative) and T is the period of the function.

If $f(t)$ is a piecewise continuous periodic function with period T then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) \mathrm{d}t$$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \mathrm{d}t = \int_0^T e^{-st} f(t) \mathrm{d}t + \int_T^\infty e^{-st} f(t) \mathrm{d}t$$

In the second integral, putting $t = x + T$, $\mathrm{d}t = \mathrm{d}x$

When $t = T$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$\begin{aligned} L\{f(t)\} &= \int_0^T e^{-st} f(t) \mathrm{d}t + \int_0^\infty e^{-s(x+T)} f(x+T) \mathrm{d}x \\ &= \int_0^T e^{-st} f(t) \mathrm{d}t + e^{-Ts} \int_0^\infty e^{-sx} f(x) \mathrm{d}x \\ &= \int_0^T e^{-st} f(t) \mathrm{d}t + e^{-Ts} \int_0^\infty e^{-st} f(t) \mathrm{d}t \\ &= \int_0^T e^{-st} f(t) \mathrm{d}t + e^{-Ts} L\{f(t)\} \end{aligned}$$

$$(1-e^{-Ts}) L\{f(t)\} = \int_0^T e^{-st} f(t) \mathrm{d}t$$

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) \mathrm{d}t$$

Example 1

Find the Laplace transform of $f(t) = e^t \quad 0 < t < 2\pi$
if $f(t) = f(t + 2\pi)$.

Solution

The function $f(t)$ is a periodic function with period 2π.

$$
\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} e^{t}\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{(1-s)t}\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2\pi s}} \left| \frac{e^{(1-s)t}}{1-s} \right|_0^{2\pi} \\
&= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{(1-s)2\pi}}{1-s} - \frac{1}{1-s} \right] \\
&= \frac{e^{(1-s)2\pi} - 1}{(1-e^{-2\pi s})(1-s)}
\end{aligned}
$$

Example 2

Find the Laplace transform of $f(t) = t^2 \qquad 0 < t < 2$
if $f(t) = f(t+2)$.

Solution

The function $f(t)$ is a periodic function with period 2.

$$
\begin{aligned}
L\{f(t)\} &= \frac{1}{1-e^{-2s}} \int_0^{2} e^{-st} f(t)\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2s}} \int_0^{2} e^{-st}\, t^2\,\mathrm{d}t \\
&= \frac{1}{1-e^{-2s}} \left| t^2 \left(\frac{e^{-st}}{-s} \right) - 2t \left(\frac{e^{-st}}{s^2} \right) + 2 \left(\frac{e^{-st}}{-s^3} \right) \right|_0^{2} \\
&= \frac{1}{1-e^{-2s}} \left(-4\frac{e^{-2s}}{s} - 4\frac{e^{-2s}}{s^2} - 2\frac{e^{-2s}}{s^3} + \frac{2}{s^3} \right) \\
&= \frac{1}{(1-e^{-2s})s^3} (2 - 2e^{-2s} - 4s\,e^{-2s} - 4s^2 e^{-2s})
\end{aligned}
$$

Example 3

Find the Laplace transform of

$$f(t) = 1 \quad 0 < t < a$$
$$= -1 \quad a < t < 2a$$

and $f(t)$ is periodic with period $2a$.

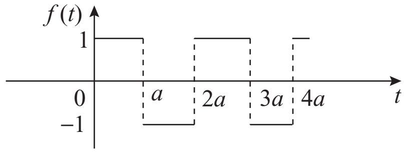


Fig. 5.6

Solution

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st}\,dt + \int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left|\frac{e^{-st}}{-s}\right|_0^a + \left|\frac{e^{-st}}{s}\right|_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left(-\frac{e^{-as}}{s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right)$$

$$= \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})}$$

$$= \frac{1-e^{-as}}{s(1+e^{-as})}$$

$$= \frac{1}{s}\cdot\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{\left(e^{\frac{as}{2}} + e^{-\frac{as}{2}}\right)}$$

$$= \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

Example 4

Find the Laplace transform of

$$f(t) = \frac{t}{T} \qquad 0 < t < T$$

if $f(t) = f(t+T)$.

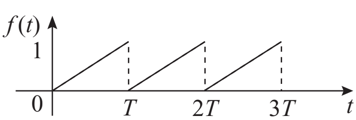


Fig. 5.7

Solution

The function $f(t)$ is a periodic function with period T.

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)\mathrm{d}t$$

$$= \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}\frac{t}{T}\mathrm{d}t$$

$$= \frac{1}{1-e^{-Ts}}\frac{1}{T}\int_0^T e^{-st}\, t\,\mathrm{d}t$$

$$= \frac{1}{T(1-e^{-Ts})}\left| t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^T$$

$$= \frac{1}{T(1-e^{-Ts})}\left(-T\frac{e^{-Ts}}{s} - \frac{e^{-Ts}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{1}{T(1-e^{-Ts})}\left[-\frac{Te^{-Ts}}{s} + \frac{1}{s^2}(1-e^{-Ts})\right]$$

$$= \frac{1}{Ts^2} - \frac{e^{-Ts}}{s(1-e^{-Ts})}$$

Example 5

Find the Laplace transform of

$$f(t) = \frac{2}{3}t \qquad 0 \le t \le 3$$

if $f(t) = f(t+3)$. **[Winter 2017]**

Solution

The function $f(t)$ is a periodic function with period 3.

$$L\{f(t)\} = \frac{1}{1-e^{-3s}}\int_0^3 e^{-st} f(t)\mathrm{d}t$$

$$= \frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\frac{2}{3}t\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-3s}}\frac{2}{3}\int_0^3 e^{-st}\, t\,\mathrm{d}t$$

$$= \frac{2}{3(1-e^{-3s})}\left| t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^3$$

$$= \frac{2}{3(1-e^{-3s})}\left(-3\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{2}{3(1-e^{-3s})}\left[-\frac{3e^{-3s}}{s}+\frac{1}{s^2}(1-e^{-3s})\right]$$

$$= \frac{2}{3s^2}-\frac{2e^{-3s}}{s(1-e^{-3s})}$$

Example 6

Find the Laplace transform of

$$f(t) = t \qquad 0<t<1$$
$$= 0 \qquad 1<t<2$$

if $f(t) = f(t+2)$.

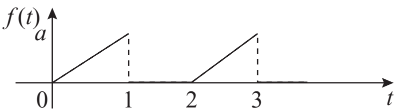


Fig. 5.8

Solution

The function $f(t)$ is a periodic function with period 2.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)\mathrm{d}t$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\,t\,\mathrm{d}t+\int_1^2 e^{-st}\cdot 0\,\mathrm{d}t\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\left|\frac{e^{-st}}{-s}t-\frac{e^{-st}}{s^2}\right|_0^1+0\right]$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}+\frac{1}{s^2}\right)$$

$$= \frac{1}{s^2(1-e^{-2s})}(1-e^{-s}-se^{-s})$$

Example 7

Find the Laplace transform of

$$f(t) = t \qquad 0<t<a$$
$$= 2a-t \qquad a<t<2a$$

if $f(t) = f(t+2a)$.

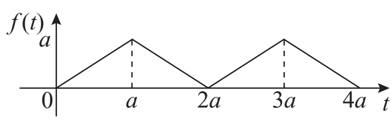


Fig. 5.9

Solution

The function $f(t)$ is a periodic function with period $2a$.

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)\mathrm{d}t$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\, t\, \mathrm{d}t + \int_a^{2a} e^{-st}(2a-t)\,\mathrm{d}t\right]$$

$$= \frac{1}{(1-e^{-2as})}\left[\left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\right|_0^a + \left|\frac{e^{-st}}{-s}(2a-t) + \frac{e^{-st}}{s^2}\right|_a^{2a}\right]$$

$$= \frac{1}{(1-e^{-2as})}\left(-\frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2}\right)$$

$$= \frac{-2e^{-as} + 1 + e^{-2as}}{s^2(1-e^{-2as})}$$

$$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$= \frac{e^{\frac{as}{2}} - e^{\frac{-as}{2}}}{s^2\left(e^{\frac{as}{2}} + e^{\frac{-as}{2}}\right)}$$

$$= \frac{\tanh\left(\frac{as}{2}\right)}{s^2}$$

Example 8

Find the Laplace transform of

$$f(t) = \sin \omega t \qquad 0 < t < \frac{\pi}{\omega}$$

$$= 0 \qquad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$$

if $f(t) = f\left(t + \frac{2\pi}{\omega}\right)$.

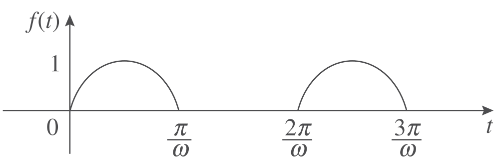


Fig. 5.10

Solution

The function $f(t)$ is a periodic function with period $\frac{2\pi}{\omega}$.

$$L\{f(t)\} = \frac{1}{1-e^{-\left(\frac{2\pi}{\omega}\right)s}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,\mathrm{d}t$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left(\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\,\mathrm{d}t + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}\cdot 0\,\mathrm{d}t\right)$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left|\frac{1}{s^2+\omega^2}\cdot e^{-st}(-s\sin\omega t - \omega\cos\omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\cdot\frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega)+\omega\right]$$

$$= \frac{\omega\left(1+e^{-\frac{\pi s}{\omega}}\right)}{\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)}\cdot\frac{1}{s^2+\omega^2}$$

$$= \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)}\cdot\frac{1}{s^2+\omega^2}$$

$$= \frac{\omega}{\left(s^2+\omega^2\right)\left(1-e^{\frac{-\pi s}{\omega}}\right)}$$

Example 9

Find the Laplace transform of

$$f(t) = |\sin\,\omega t| \qquad t \ge 0$$

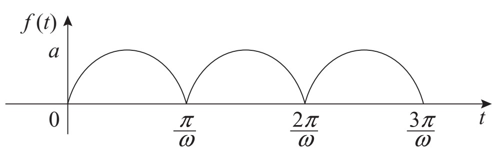


Fig. 5.11

Solution

$$f\left(t+\frac{\pi}{\omega}\right) = \left|\sin\omega\left(t+\frac{\pi}{\omega}\right)\right|$$

$$= |\sin(\omega t+\pi)|$$

$$= |-\sin\omega t|$$

$$= |\sin\omega t|$$

Hence, the function $f(t)$ is periodic with period $\frac{\pi}{\omega}$.

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} |\sin \omega t|\,dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\,dt \qquad \left[\because |\sin \omega t| = \sin \omega t \quad 0 < t < \frac{\pi}{\omega}\right]$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left|\frac{e^{-st}}{s^2+\omega^2}(-s\sin \omega t - \omega \cos \omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega) - (-\omega)\right]$$

$$= \frac{1}{s^2+\omega^2}\cdot\frac{1}{1-e^{-\frac{\pi s}{\omega}}}\omega\left(1+e^{-\frac{\pi s}{\omega}}\right)$$

$$= \frac{\omega}{s^2+\omega^2}\left(\frac{e^{\frac{\pi s}{2\omega}}+e^{-\frac{\pi s}{2\omega}}}{e^{\frac{\pi s}{2\omega}}-e^{-\frac{\pi s}{2\omega}}}\right)$$

$$= \frac{\omega}{s^2+\omega^2}\cdot \coth\left(\frac{\pi s}{2\omega}\right)$$

EXERCISE 5.13

Find the Laplace transforms of the following periodic functions:

1. $f(t) = 1 \qquad 0 < t < 1$

$\quad = 0 \qquad 1 < t < 2$

$\quad = -1 \qquad 2 < t < 3$

$f(t) = f(t+3)$

$$\left[\textbf{Ans.: } \frac{1}{s}\left(\frac{3}{1-e^{-3s}} - \frac{1}{1-e^{-s}} - 1\right)\right]$$

2. $f(t) = t \qquad 0 < t < a$

$\quad = \dfrac{2a-t}{a} \qquad a < t < 2a$

$f(t) = f(t+2a)$

$$\left[\textbf{Ans.: } \frac{1}{as^2}\tanh\frac{as}{2}\right]$$

3. $f(t) = t \qquad 0 < t < \pi$

$\quad = \pi - t \qquad \pi < t < 2\pi$

$f(t) = f(t + 2\pi)$

$$\left[\textbf{Ans.:}\ \frac{1-(1+\pi s)e^{-\pi s}}{(1+e^{-\pi s})s^2}\right]$$

4. $f(t) = |\cos \omega t| \qquad t > 0$

$$\left[\textbf{Ans.:}\ \frac{1}{s^2+\omega^2}\left(s + \omega \operatorname{cosech} \frac{\pi s}{2\omega}\right)\right]$$

5. $f(t) = \cos \omega t \qquad 0 < t < \dfrac{\pi}{\omega}$

$\quad = 0 \qquad \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega}$

$$\left[\textbf{Ans.:}\ \frac{s}{\left(1-e^{-\frac{\pi s}{\omega}}\right)(s^2+\omega^2)}\right]$$

6. $f(t) = E \qquad 0 < t < \dfrac{\pi}{2}$

$\quad = -E \qquad \dfrac{\pi}{2} < t < \pi$

$f(t) = f(t + \pi)$

$$\left[\textbf{Ans.:}\ \frac{E}{s}\tanh\left(\frac{\pi s}{4}\right)\right]$$

7. $f(t) = \left(\dfrac{\pi - t}{2}\right)^2 \qquad 0 < t < 2\pi$

$f(t) = f(t + 2\pi)$

$$\left[\textbf{Ans.:}\ \frac{1}{s^3}(2\pi s \coth \pi s - \pi^2 s^2 - 2)\right]$$

5.13 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called the inverse Laplace transform of $F(s)$ and is symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the *inverse Laplace transform operator*.

Inverse Laplace transforms of simple functions can be found from the properties of Laplace transforms.

Table of Inverse Laplace Transforms

Sr. No.	*F(s)*	*f(t)*
1	$\frac{1}{s}$	1
2	$\frac{1}{s^n}$	$\frac{t^{n-1}}{\lceil n}$
3	$\frac{1}{s-a}$	e^{at}
4	$\frac{1}{(s-a)^n}$	$e^{at}\frac{t^{n-1}}{\lceil n}$
5	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
6	$\frac{s}{s^2+a^2}$	$\cos at$
7	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
8	$\frac{s}{s^2-a^2}$	$\cosh at$
9	$\frac{1}{(s+b)^2+a^2}$	$\frac{1}{a}e^{-bt}\sin at$
10	$\frac{s+b}{(s+b)^2+a^2}$	$e^{-bt}\cos at$
11	$\frac{1}{(s+b)^2-a^2}$	$\frac{1}{a}e^{-bt}\sinh at$
12	$\frac{s+b}{(s+b)^2-a^2}$	$e^{-bt}\cosh at$

5.13.1 Linearity

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then $L^{-1}\{aF_1(s) + bF_2(s)\} = af_1(t) + bf_2(t)$ where a and b are constants.

Example 1

Find the inverse Laplace transform of $\dfrac{s^2-3s+4}{s^3}$.

Solution

Let
$$F(s) = \frac{s^2-3s+4}{s^3}$$
$$= \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$$
$$= 1 - 3t + 2t^2$$

Example 2

Find the inverse Laplace transform of $\dfrac{6s}{s^2-16}$. **[Winter 2012]**

Solution

Let
$$F(s) = \frac{6s}{s^2-16}$$
$$L^{-1}\{F(s)\} = 6L^{-1}\left\{\frac{s}{s^2-16}\right\} = 6\cosh 4t$$

Example 3

Find the inverse Laplace transform of $\dfrac{3(s^2-2)^2}{2s^5}$.

Solution

Let
$$F(s) = \frac{3(s^2-2)^2}{2s^5}$$

$$= \frac{3}{2}\frac{(s^2-2)^2}{s^5}$$

$$= \frac{3}{2}\frac{s^4-4s^2+4}{s^5}$$

$$= \frac{3}{2}\left(\frac{1}{s}-\frac{4}{s^3}+\frac{4}{s^5}\right)$$

$$L^{-1}\{F(s)\} = \frac{3}{2}\left[L^{-1}\left\{\frac{1}{s}\right\}-4L^{-1}\left\{\frac{1}{s^3}\right\}+4L^{-1}\left\{\frac{1}{s^5}\right\}\right]$$

$$= \frac{3}{2}\left[1-4\left(\frac{t^2}{2!}\right)+4\left(\frac{t^4}{4!}\right)\right]$$

$$= \frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]$$

$$= \frac{3}{2}-3t^2+\frac{t^4}{4}$$

$$= \frac{1}{4}(t^4-12t^2+6)$$

Example 4

Find the inverse Laplace transform of $\dfrac{2s+1}{s(s+1)}$.

Solution

Let $\quad F(s) = \dfrac{2s+1}{s(s+1)}$

$$= \frac{s+(s+1)}{s(s+1)}$$

$$= \frac{1}{s+1}+\frac{1}{s}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{1}{s}\right\}$$

$$= e^{-t}+1$$

Example 5

Find the inverse Laplace transform of $\dfrac{3s+4}{s^2+9}$.

Solution

Let
$$F(s) = \frac{3s+4}{s^2+9}$$
$$= \frac{3s}{s^2+9} + \frac{4}{s^2+9}$$
$$L^{-1}\{F(s)\} = 3L^{-1}\left\{\frac{s}{s^2+9}\right\} + 4L^{-1}\left\{\frac{1}{s^2+9}\right\}$$
$$= 3\cos 3t + \frac{4}{3}\sin 3t$$

Example 6

Find the inverse Laplace transform of $\dfrac{s^2+9s-9}{s^3-9s}$.

Solution

Let
$$F(s) = \frac{s^2+9s-9}{s^3-9s}$$
$$= \frac{(s^2-9)+9s}{s(s^2-9)}$$
$$= \frac{1}{s} + \frac{9}{s^2-9}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s}\right\} + 9L^{-1}\left\{\frac{1}{s^2-9}\right\}$$
$$= 1 + 3\sinh 3t$$

Example 7

Find the inverse Laplace transform of $\dfrac{4s+15}{16s^2-25}$.

Solution

Let
$$F(s) = \frac{4s+15}{16s^2-25}$$
$$= \frac{4s+15}{16\left(s^2-\frac{25}{16}\right)}$$
$$= \frac{1}{4}\,\frac{s}{s^2-\frac{25}{16}} + \frac{15}{16}\,\frac{1}{s^2-\frac{25}{16}}$$

$$L^{-1}\{F(s)\} = \frac{1}{4}\left[L^{-1}\left\{\frac{s}{s^2-\frac{25}{16}}\right\}+\frac{15}{4}L^{-1}\left\{\frac{1}{s^2-\frac{25}{16}}\right\}\right]$$

$$= \frac{1}{4}\cosh\frac{5}{4}t+\frac{3}{4}\sinh\frac{5}{4}t$$

EXERCISE 5.14

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s-5}{s^2-4}$

$$\left[\textbf{Ans.:}\ 2\cosh 2t-\frac{5}{2}\sinh 2t\right]$$

2. $\dfrac{3s-8}{4s^2+25}$

[**Ans.:** $e^{-t}+1$]

3. $\dfrac{3s-12}{s^2+18}$

$$\left[\textbf{Ans.:}\ 3\cos 2\sqrt{2}t-3\sqrt{2}\sin 2\sqrt{2}\,t\right]$$

4. $\dfrac{s+1}{s^{\frac{4}{3}}}$

$$\left[\textbf{Ans.:}\ \frac{t^{\frac{-2}{3}}+3t^{\frac{1}{3}}}{\left\lceil\frac{1}{3}\right.}\right]$$

5. $\left(\dfrac{\sqrt{s}-1}{s}\right)^2$

$$\left[\textbf{Ans.:}\ 1+t-\frac{4\sqrt{t}}{\sqrt{\pi}}\right]$$

6. $\dfrac{s^2-1}{s^5}$

$$\left[\textbf{Ans.:}\ 1-t^2-\frac{t^4}{24}\right]$$

5.13.2 Change of Scale

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right), a > 0$.

Example 1

If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t$ *then find* $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$.

Solution

Let
$$F(s) = \frac{s}{(s^2+1)^2}$$

$$F(as) = \frac{as}{(a^2s^2+1)^2}$$

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

$$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{2} \cdot \frac{1}{a} \frac{t}{a} \sin\frac{t}{a} \qquad \left[\because L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t\right]$$

$$= \frac{t}{2a^2} \sin\frac{t}{a}$$

Putting $a = 2$,

$$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\} = \frac{t}{8} \sin\frac{t}{2}$$

$$L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = 4\left(\frac{t}{8} \sin\frac{t}{2}\right)$$

$$= \frac{t}{2} \sin\frac{t}{2}$$

Example 2

If $L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = t \cos t$ *then find* $L^{-1}\left\{\frac{9s^2-1}{(9s^2+1)^2}\right\}$.

Solution

Let
$$F(s) = \frac{s^2-1}{(s^2+1)^2}$$

$$F(as) = \frac{a^2s^2 - 1}{(a^2s^2 + 1)^2}$$

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

$$L^{-1}\left\{\frac{a^2s^2 - 1}{(a^2s^2 + 1)^2}\right\} = \frac{1}{a} \cdot \frac{t}{a} \cos\frac{t}{a} \qquad \left[\because\ L^{-1}\left\{\frac{s^2 - 1}{(s^2 + 1)}\right\} = t\cos t\right]$$

Putting $a = 3$,

$$L^{-1}\left\{\frac{9s^2 - 1}{(9s^2 + 1)^2}\right\} = \frac{1}{3} \cdot \frac{t}{3} \cos\frac{t}{3}$$

$$= \frac{t}{9} \cos\frac{t}{3}$$

Example 3

Find the inverse Laplace transform of $\dfrac{s}{s^2a^2 + b^2}$.

Solution

$$\frac{s}{s^2a^2 + b^2} = \frac{1}{a} \frac{as}{(as)^2 + b^2} = \frac{1}{a} F(as), \text{ say}$$

where
$$F(as) = \frac{as}{(as)^2 + b^2}$$

Replacing as by s,

$$F(s) = \frac{s}{s^2 + b^2}$$

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

$$L^{-1}\left\{\frac{as}{s^2a^2 + b^2}\right\} = \frac{1}{a} \cos b\frac{t}{a} \qquad \left[\because\ L^{-1}\left\{\frac{s}{s^2 + b^2}\right\} = \cos bt\right]$$

$$aL^{-1}\left\{\frac{s}{s^2a^2 + b^2}\right\} = \frac{1}{a} \cos\frac{b}{a} t$$

$$L^{-1}\left\{\frac{s}{s^2a^2 + b^2}\right\} = \frac{1}{a^2} \cos\frac{b}{a} t$$

Example 4

Find the inverse Laplace transform of $\dfrac{s}{2s^2-8}$.

Solution

$$\frac{s}{2s^2-8}=\frac{2s}{4s^2-16}=\frac{2s}{(2s)^2-16}=F(2s),\text{ say}$$

Replacing $2s$ by s,

$$F(s)=\frac{s}{s^2-16}$$

$$L^{-1}\{F(2s)\}=\frac{1}{2}f\left(\frac{t}{2}\right)$$

$$L^{-1}\left\{\frac{2s}{4s^2-16}\right\}=\frac{1}{2}\cosh\frac{4t}{2} \qquad \left[\because\ L^{-1}\left\{\frac{s}{s^2-16}\right\}=\cosh 4t\right]$$

$$L^{-1}\left\{\frac{s}{2s^2-8}\right\}=\frac{1}{2}\cosh 2t$$

EXERCISE 5.15

1. Find the inverse Laplace transform of $\dfrac{3s}{9s^2+16}$.

$$\left[\textbf{Ans.}: \frac{1}{3}\cos\frac{4}{3}t\right]$$

2. If $L^{-1}\left\{\dfrac{2as}{\left(s^2+a^2\right)^2}\right\}=t\sin at$ then find $L^{-1}\left\{\dfrac{6as}{\left(9s^2+a^2\right)^2}\right\}$.

$$\left[\textbf{Ans.}: \frac{1}{9}t\sin\frac{at}{3}\right]$$

5.13.3 First Shifting Theorem

If $L^{-1}\{F(s)\}=f(t)$ then $L^{-1}\{F(s+a)\}=e^{-at}f(t)$.

Example 1

Find the inverse Laplace transform of $\dfrac{1}{(s+2)^3}$.

Solution

Let $$F(s) = \frac{1}{(s+2)^3}$$

$$L^{-1}\{F(s)\} = e^{-2t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{-2t}\, \frac{t^2}{2!}$$

$$= \frac{e^{-2t}}{2} t^2$$

Example 2

Find the inverse Laplace transform of $\dfrac{10}{(s-2)^4}$. **[Winter 2012]**

Solution

Let $$F(s) = \frac{10}{(s-2)^4}$$

$$L^{-1}\{F(s)\} = 10e^{2t}\; L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= 10e^{2t}\, \frac{t^3}{3!}$$

$$= \frac{5}{3} e^{2t}\, t^3$$

Example 3

Find the inverse Laplace transform of $\dfrac{1}{\sqrt{s+2}}$.

Solution

Let $$F(s) = \frac{1}{\sqrt{s+2}}$$

$$L^{-1}\left\{F(s)\right\} = L^{-1}\left\{\frac{1}{(s+2)^{\frac{1}{2}}}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= e^{-2t}\,\frac{t^{-\frac{1}{2}}}{\left\lceil\frac{1}{2}\right.}$$

$$= \frac{e^{-2t}}{\sqrt{\pi}}\,\frac{1}{\sqrt{t}} \qquad \left[\because\ \left\lceil\frac{1}{2}\right. = \sqrt{\pi}\right]$$

Example 4

Find the inverse Laplace transform of $\dfrac{1}{s^2+4s+4}$.

Solution

Let
$$F(s) = \frac{1}{s^2+4s+4}$$
$$= \frac{1}{(s+2)^2}$$
$$L^{-1}\{F(s)\} = e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\}$$
$$= e^{-2t}\ t$$

Example 5

Find the inverse Laplace transform of $\dfrac{s}{(2s+1)^2}$.

Solution

Let
$$F(s) = \frac{s}{(2s+1)^2}$$
$$= \frac{1}{4}\,\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2}$$
$$= \frac{1}{4}\left[\frac{1}{s+\frac{1}{2}} - \frac{1}{2}\cdot\frac{1}{\left(s+\frac{1}{2}\right)^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - \frac{1}{8}\,e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{8}\,e^{-\frac{t}{2}}t$$

$$= e^{-\frac{t}{2}}\left(\frac{1}{4} - \frac{1}{8}t\right)$$

Example 6

Find the inverse Laplace transform of $\frac{1}{\sqrt{2s+3}}$.

Solution

Let
$$F(s) = \frac{1}{\sqrt{2s+3}}$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\left(s+\frac{3}{2}\right)^{\frac{1}{2}}}$$

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{2}}\,e^{-\frac{3t}{2}}L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{\sqrt{2}}\,e^{-\frac{3t}{2}}\,\frac{t^{-\frac{1}{2}}}{\left\lceil\frac{1}{2}\right.}$$

$$= \frac{1}{\sqrt{2\pi}}t^{-\frac{1}{2}}\,e^{-\frac{3t}{2}} \qquad \left[\because \left\lceil\frac{1}{2}\right. = \sqrt{\pi}\right]$$

Example 7

Find the inverse Laplace transform of $\frac{3s+1}{(s+1)^4}$.

Solution

Let
$$F(s) = \frac{3s+1}{(s+1)^4}$$

$$= \frac{3(s-1)-2}{(s+1)^4}$$

$$= \frac{3}{(s+1)^3} - \frac{2}{(s+1)^4}$$

$$L^{-1}\{F(s)\} = 3e^{-t}L\left\{\frac{1}{s^3}\right\} - 2e^{-t}\left\{\frac{1}{s^4}\right\}$$

$$= 3e^{-t}\,\frac{t^2}{2!} - 2e^{-t}\,\frac{t^3}{3!}$$

$$= \frac{3}{2}e^{-t}t^2 - \frac{1}{3}e^{-t}t^3$$

$$= e^{-t}\left(\frac{3}{2}t^2 - \frac{1}{3}t^3\right)$$

Example 8

Find the inverse Laplace transform of $\dfrac{s+2}{s^2+4s+8}$.

Solution

Let
$$F(s) = \frac{s+2}{s^2+4s+8}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{s^2+4s+8}\right\}$$

$$= L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= e^{-2t}\cos 2t$$

Example 9

Find the inverse Laplace transform of $\dfrac{2s+2}{s^2+2s+10}$.

Solution

Let
$$F(s) = \frac{2s+2}{s^2+2s+10}$$

$$= \frac{2(s+1)}{(s+1)^2+9}$$

$$L^{-1}\{F(s)\} = 2e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\}$$
$$= 2\,e^{-t}\cos 3t$$

Example 10

Find the inverse Laplace transform of $\dfrac{s}{(s+2)^2+1}$.

Solution

Let
$$F(s) = \frac{s}{(s+2)^2+1}$$
$$= \frac{s+2-2}{(s+2)^2+1}$$
$$= \frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$
$$= e^{-2t}L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2e^{-2t}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$
$$= e^{-2t}\cos t - 2e^{-2t}\sin t$$
$$= e^{-2t}(\cos t - 2\sin t)$$

Example 11

Find the inverse Laplace transform of $\dfrac{2s+3}{s^2-4s+13}$.

Solution

Let
$$F(s) = \frac{2s+3}{s^2-4s+13}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{2s+3}{s^2-4s+13}\right\}$$
$$= L^{-1}\left\{\frac{2s+3}{(s-2)^2+13-4}\right\}$$
$$= L^{-1}\left\{\frac{2s+3}{(s-2)^2+9}\right\}$$
$$= L^{-1}\left\{\frac{2s-4+7}{(s-2)^2+9}\right\}$$

$$= L^{-1}\left\{\frac{2(s-2)}{(s-2)^2+9}\right\} + 7L^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$$

$$= 2e^{2t}L^{-1}\left\{\frac{s}{s^2+9}\right\} + 7e^{2t}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= 2e^{2t}\cos 3t + \frac{7}{3}e^{2t}\sin 3t$$

$$= \frac{1}{3}e^{2t}(6\cos 3t + 7\sin 3t)$$

Example 12

Find the inverse Laplace transform of $\dfrac{s-3}{s^2+4s+13}$.

Solution

Let
$$F(s) = \frac{s-3}{s^2+4s+13}$$

$$= \frac{s-3}{(s+2)^2+13-4}$$

$$= \frac{s-3}{(s+2)^2+9}$$

$$= \frac{s+2-5}{(s+2)^2+9}$$

$$= \frac{s+2}{(s+2)^2+9} - \frac{5}{(s+2)^2+9}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+9} - \frac{5}{(s+2)^2+9}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{s}{s^2+9}\right\} - 5L^{-1}\left\{\frac{1}{(s+2)^2+9}\right\}$$

$$= e^{-2t}\cos 3t - 5e^{-2t}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$$

$$= e^{-2t}\left(\cos 3t - \frac{5}{3}\sin 3t\right)$$

Example 13

Find the inverse Laplace transform of $\dfrac{s+7}{s^2+8s+25}$. **[Summer 2017]**

Solution

Let $$F(s)=\frac{s+7}{s^2+8s+25}$$

$$\begin{aligned}
L^{-1}\{F(s)\} &= L^{-1}\left\{\frac{s+7}{s^2+8s+16+9}\right\}\\
&= L^{-1}\left\{\frac{s+7}{(s+4)^2+9}\right\}\\
&= L^{-1}\left\{\frac{(s+4)+3}{(s+4)^2+9}\right\}\\
&= L^{-1}\left\{\frac{s+4}{(s+4)^2+9}\right\}+L^{-1}\left\{\frac{3}{(s+4)^2+9}\right\}\\
&= e^{-4t}\,L^{-1}\left\{\frac{s}{s^2+9}\right\}+e^{-4t}\,L^{-1}\left\{\frac{3}{s^2+9}\right\}\\
&= e^{-4t}\cos 3t+e^{-4t}\sin 3t\\
&= e^{-4t}(\sin 3t+\cos 3t)
\end{aligned}$$

Example 14

Find the inverse Laplace transform of $\dfrac{3s+7}{s^2-2s-3}$.

Solution

Let $$\begin{aligned}
F(s) &= \frac{3s+7}{s^2-2s-3}\\
&= \frac{3(s-1)+10}{(s-1)^2-4}\\
&= \frac{3(s-1)}{(s-1)^2-4}+10\frac{1}{(s-1)^2-4}
\end{aligned}$$

$$L^{-1}\{F(s)\} = 3e^t L^{-1}\left\{\frac{s}{s^2-4}\right\} + 10\,e^t L^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$= 3\,e^t \cosh 2t + 5\,e^t \sinh 2t$$

$$= e^t\,(3\cosh 2t + 5\sinh 2t)$$

EXERCISE 5.16

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{5}{(s+2)^5}$

$$\left[\textbf{Ans.:}\ \frac{5}{24}t^4\,e^{-2t}\right]$$

2. $\dfrac{4s+12}{s^2+8s+16}$

$$\left[\textbf{Ans.:}\ 4e^{-4t}(1-t)\right]$$

3. $\dfrac{1}{(s^2+2s+5)^2}$

$$\left[\textbf{Ans.:}\ \frac{e^{-t}}{16}(\sin 2t - 2t\cos 2t)\right]$$

4. $\dfrac{s}{(s-2)^6}$

$$\left[\textbf{Ans.:}\ e^{2t}\left(\frac{t^4}{24}+\frac{t^5}{60}\right)\right]$$

5. $\dfrac{s}{s^2+2s+2}$

$$\left[\textbf{Ans.:}\ e^{-t}(\cos t - \sin t)\right]$$

6. $\dfrac{1}{(s+2)^4}$

$$\left[\textbf{Ans.:}\ \frac{1}{6}e^{-2t}t^3\right]$$

5.13.4 Second Shifting Theorem

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{e^{-as}F(s)\} = g(t)$

where
$$\begin{aligned} g(t) &= f(t-a) \qquad & t > a \\ &= 0 & t < a \end{aligned}$$

The above result can also be expressed as

$$L^{-1}\{e^{-as}F(s)\} = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

Or $L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$

Example 1

Find the inverse Laplace transform of $\dfrac{e^{-as}}{s}$.

Solution

Let $$F(s) = \frac{1}{s}$$

$$L^{-1}\{F(s)\} = 1$$

$$L^{-1}\{e^{-as}F(s)\} = 1 \cdot u(t-a) = u(t-a)$$

Example 2

Find the inverse Laplace transform of $\dfrac{e^{-2s}}{s-3}$.

Solution

Let $$F(s) = \frac{1}{s-3}$$

$$L^{-1}\{F(s)\} = e^{3t}$$

$$L^{-1}\{e^{-2s}F(s)\} = e^{3(t-2)}u(t-2)$$

Example 3

Find the inverse Laplace transform of $e^{-s}\left(\dfrac{1+\sqrt{s}}{s^3}\right)$.

Solution

Let $$F(s) = \left(\frac{1+\sqrt{s}}{s^3}\right)$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^3} + \frac{1}{s^{\frac{5}{2}}}\right\}$$

$$= \frac{t^2}{2!} + \frac{t^{\frac{3}{2}}}{\left\lceil\frac{5}{2}\right.}$$

$$= \frac{t^2}{2} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}\,\frac{1}{2}\,\Gamma\!\left(\frac{1}{2}\right)}$$

$$= \frac{t^2}{2} + \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\{e^{-s}F(s)\} = \left[\frac{(t-1)^2}{2} + \frac{4(t-1)^{\frac{3}{2}}}{3\sqrt{\pi}}\right]u(t-1)$$

Example 4

Find the inverse Laplace transform of $\dfrac{e^{-2s}}{(s+4)^3}$.

Solution

Let $$F(s) = \frac{1}{(s+4)^3}$$

$$L^{-1}\{F(s)\} = e^{-4t}\ L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{-4t}\ \frac{t^2}{2}$$

$$L^{-1}\{e^{-2s}F(s)\} = e^{-4(t-2)}\ \frac{(t-2)^2}{2}\ u(t-2)$$

Example 5

Find the inverse Laplace transform of $\dfrac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$.

Solution

Let $$F(s) = \frac{1}{(s+4)^{\frac{5}{2}}}$$

$$L^{-1}\{F(s)\} = e^{-4t}\ L^{-1}\left\{\frac{1}{s^{\frac{5}{2}}}\right\}$$

$$= e^{-4t} \frac{t^{\frac{3}{2}}}{\left\lceil \frac{5}{2} \right.}$$

$$= \frac{e^{-4t}\ t^{\frac{3}{2}}}{\frac{3}{2} \cdot \frac{1}{2} \left\lceil \frac{1}{2} \right.}$$

$$= \frac{4e^{-4t}\ t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\{e^{4-3s} F(s)\} = \frac{e^4 \cdot 4}{3\sqrt{\pi}}\ e^{-4(t-3)}\ (t-3)^{\frac{3}{2}}\ u(t-3)$$

$$= \frac{4}{3\sqrt{\pi}}\ e^{-4(t-4)}\ (t-3)^{\frac{3}{2}} u(t-3)$$

Example 6

Find the inverse Laplace transform of $\dfrac{e^{-3s}}{s^2+4}$.

Solution

Let $$F(s) = \frac{1}{s^2+4}$$

$$L^{-1}\{F(s)\} = \frac{1}{2} \sin 2t$$

$$L^{-1}\{e^{-3s} F(s)\} = \frac{1}{2} \sin 2(t-3) u(t-3)$$

Example 7

Find the inverse Laplace transform of $\dfrac{se^{-\left(\frac{\pi}{2}\right)s}}{s^2+4}$.

Solution

Let $$F(s) = \frac{s}{s^2+4}$$

$$L^{-1}\{F(s)\} = \cos 2t$$

$$L^{-1}\left\{e^{-\left(\frac{\pi}{2}\right)s} F(s)\right\} = \cos 2\left(t - \frac{\pi}{2}\right)u\left(t - \frac{\pi}{2}\right)$$

$$= \cos(2t - \pi)u\left(t - \frac{\pi}{2}\right)$$

$$= \cos(\pi - 2t)u\left(t - \frac{\pi}{2}\right)$$

$$= -\cos 2t\, u\left(t - \frac{\pi}{2}\right)$$

Example 8

Find the inverse Laplace transform of $\dfrac{e^{-3s}}{s^2 + 8s + 25}$. **[Winter 2016]**

Solution

Let
$$F(s) = \frac{1}{s^2 + 8s + 25}$$

$$= \frac{1}{s^2 + 8s + 16 + 9}$$

$$= \frac{1}{(s+4)^2 + 9}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+4)^2 + 9}\right\}$$

$$= e^{-4t} \cdot \frac{1}{3} \sin 3t$$

$$L^{-1}\{e^{-3s} F(s)\} = \frac{1}{3} e^{-4(t-3)} \sin 3(t-3)\, u(t-3)$$

$$= \frac{1}{3} e^{12-4t} \sin(3t-9)\, u(t-3)$$

Example 9

Find the inverse Laplace transform of $\dfrac{e^{-2s}}{(s^2+2)(s^2-3)}$. **[Winter 2015]**

Solution

Let
$$F(s) = \frac{1}{(s^2+2)(s^2-3)}$$

$$L^{-1}\{F(s)\} = \frac{1}{5} L^{-1}\left\{\frac{1}{s^2-3} - \frac{1}{s^2+2}\right\}$$

$$= \frac{1}{5}\left[L^{-1}\left\{\frac{1}{s^2-3}\right\} - L^{-1}\left\{\frac{1}{s^2+2}\right\}\right]$$

$$= \frac{1}{5}\left[\frac{1}{\sqrt{3}}\sinh\sqrt{3}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right]$$

$$L^{-1}\{e^{-2s}F(s)\} = \left[\frac{1}{5\sqrt{3}}\sinh\sqrt{3}(t-2) - \frac{1}{5\sqrt{2}}\sin\sqrt{2}(t-2)\right]u(t-2)$$

Example 10

Find the inverse Laplace transform of $\dfrac{e^{-\pi s}}{s^2-2s+2}$.

Solution

Let
$$F(s) = \frac{1}{s^2-2s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= e^t\, L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^t \sin t$$

$$L^{-1}\{e^{-\pi s}F(s)\} = e^{(t-\pi)}\sin(t-\pi)u(t-\pi)$$

Example 11

Find the inverse Laplace transform of $\dfrac{(s+1)e^{-2s}}{s^2+2s+2}$.

Solution

Let
$$F(s) = \frac{s+1}{s^2+2s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{(s+1)}{(s+1)^2+1}\right\}$$

$$= e^{-t}\, L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= e^{-t}\cos t$$

$$L^{-1}\{e^{-2s}F(s)\} = e^{-(t-2)}\cos(t-2)\,u(t-2)$$

Example 12

Find the inverse Laplace transform of $\dfrac{se^{-2s}}{s^2+2s+2}$.

Solution

Let $$F(s) = \frac{s}{s^2+2s+2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+1-1}{(s+1)^2+1}\right\}$$

$$= L^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\, L^{-1}\left\{\frac{s}{s^2+1} - \frac{1}{s^2+1}\right\}$$

$$= e^{-t}(\cos t - \sin t)$$

$$L^{-1}\{e^{-2s}F(s)\} = e^{-(t-2)}\left[\cos(t-2) - \sin(t-2)\right]u(t-2)$$

EXERCISE 5.17

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{e^{-as}}{(s+b)^{\frac{5}{2}}}$

$$\left[\textbf{Ans.:}\ \frac{4}{3\sqrt{\pi}}\, e^{-b(t-a)}(t-a)^{\frac{3}{2}}\, u(t-a)\right]$$

2. $\dfrac{e^{-\pi s}}{s^2+9}$

$$\left[\textbf{Ans .: } \frac{1}{3}\sin 3\,(t-\pi)u(t-\pi)\right]$$

3. $\dfrac{e^{-\pi s}}{s^2(s^2+1)}$

$$\left[\textbf{Ans .: } [(t-\pi)+\sin(t-\pi)]u(t-\pi)\right]$$

4. $\dfrac{e^{-4s}}{\sqrt{2s+7}}$

$$\left[\textbf{Ans .: } \frac{e^{\frac{-7(t-4)}{2}}}{\sqrt{2\pi(t-4)}}\,u(t-4)\right]$$

5. $\dfrac{(s+1)e^{-s}}{s^2+s+1}$

$$\left[\textbf{Ans .: } e^{\frac{-(t-1)}{2}}\left[\cos\left(\sqrt{3}\,\frac{(t-1)}{2}\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}(t-1)}{2}\right)\right]u(t-1)\right]$$

6. $\dfrac{se^{-3s}}{s^2-1}$

$$\left[\textbf{Ans .: } \cosh(t-3)u(t-3)\right]$$

7. $\dfrac{se^{-as}}{s^2+b^2}$

$$\left[\textbf{Ans .: } \cos b(t-a)u(t-a)\right]$$

8. $e^{-s}\left\{\dfrac{1-\sqrt{s}}{s^2}\right\}^2$

$$\left[\textbf{Ans .: }\left[\frac{(t-1)^3}{6}-\frac{16}{15\sqrt{\pi}}(t-1)^{\frac{5}{2}}+\frac{(t-1)^2}{2}\right]u(t-1)\right]$$

5.13.5 Multiplication by *s*

If $L^{-1}\{F(s)\} = f(t)$ and $f(0) = 0$ then $L^{-1}\{sF(s)\} = f'(t) = \dfrac{d}{dt}\left[L^{-1}\{F(s)\}\right]$.

In general, $L^{-1}\{s^n F(s)\} = f^n(t)$, if $f(0) = 0 = f'(0) \ldots = f^{n-1}(0)$.

Example 1

Find the inverse Laplace transform of $\dfrac{s}{s^2 - a^2}$.

Solution

Let
$$F(s) = \frac{1}{s^2 - a^2}$$

$$L^{-1}\{F(s)\} = \frac{1}{a}\sinh at$$

$$L^{-1}\{sF(s)\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{\frac{s}{s^2 - a^2}\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\left\{\frac{1}{s^2 - a^2}\right\}\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{a}\sinh at\right]$$

$$= \frac{1}{a}\cosh at(a)$$

$$= \cosh at$$

Example 2

Find the inverse Laplace transform of $\dfrac{s}{2s^2 - 1}$.

Solution

Let
$$F(s) = \frac{1}{2s^2 - 1}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{2s^2 - 1}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 - \frac{1}{2}}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 - \frac{1}{2}}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 - \left(\frac{1}{\sqrt{2}}\right)^2}\right\}$$

$$= \frac{1}{2} \frac{1}{\frac{1}{\sqrt{2}}} \sinh\left(\frac{1}{\sqrt{2}} t\right)$$

$$= \frac{1}{\sqrt{2}} \sinh\left(\frac{t}{\sqrt{2}}\right)$$

$$L^{-1}\{sF(s)\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left[\frac{s}{2s^2 - 1}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\sqrt{2}} \sinh\left(\frac{t}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2}} \cosh\left(\frac{t}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} \cosh\left(\frac{t}{\sqrt{2}}\right)$$

Example 3

Find the inverse Laplace transform of $\dfrac{s}{(s+2)^4}$.

Solution

Let $$F(s) = \frac{1}{(s+2)^4}$$

$$L^{-1}\{F(s)\} = e^{-2t}\, L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= e^{-2t} \frac{1}{6} L^{-1}\left[\frac{6}{s^4}\right]$$

$$= \frac{e^{-2t}}{6} t^3$$

$$L^{-1}\{sF(s)\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{\frac{s}{(s+2)^4}\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\left\{\frac{1}{(s+2)^4}\right\}\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{e^{-2t}}{6} t^3\right]$$

$$= \frac{1}{6} \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-2t} t^3\right]$$

$$= \frac{1}{6}\left[e^{-2t}(3t^2) + t^3 e^{-2t}(-2)\right]$$

$$= \frac{1}{6}\left[3t^2 e^{-2t} - 2t^3 e^{-2t}\right]$$

$$= \frac{1}{6}t^2 e^{-2t}(3-2t)$$

Example 4

Find the inverse Laplace transform of $\dfrac{s^2}{(s-3)^2}$.

Solution

Let $$F(s) = \frac{1}{(s-3)^2}$$

$$L^{-1}\{F(s)\} = e^{3t} L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{3t} t$$

$$L^{-1}\{s^2 F(s)\} = \frac{d^2}{dt^2}\left[L^{-1}\{F(s)\}\right]$$

$$= \frac{d^2}{dt^2}\left[te^{3t}\right]$$

$$= \frac{d}{dt}\left[t(3e^{3t}) + e^{3t}(1)\right]$$

$$= \frac{d}{dt}\left[e^{3t}(3t+1)\right]$$

$$= e^{3t}(3) + (3t+1)3e^{3t}$$

$$= 3e^{3t}(3t+2)$$

EXERCISE 5.18

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{s}{(s+2)^2}$

[Ans.: $e^{-2t}(1-2t)$**]**

2. $\dfrac{s^2}{(s^2-a^2)^2}$

$$\left[\textbf{Ans.:}\ \frac{1}{2a}(\sinh at + at\cosh at)\right]$$

3. $\dfrac{s^2}{(s-1)^3}$

$$\left[\textbf{Ans.:}\ \frac{e^t}{2}(t^2+4t+2)\right]$$

4. $\dfrac{s^2}{(s+4)^3}$

$$\left[\textbf{Ans.:}\ e^{-4t}(8t^2-8t+1)\right]$$

5. $\dfrac{s-3}{s^2+4s+13}$

$$\left[\textbf{Ans.:}\ e^{-2t}\left(\cos 3t-\frac{5}{3}\sin 3t\right)\right]$$

5.13.6 Division by *s*

If $L^{-1}\{F(s)\}=f(t)$ then $L^{-1}\left\{\dfrac{F(s)}{s}\right\}=\int_0^t f(t)\,\mathrm{d}t=\int_0^t L^{-1}\{F(s)\}\,\mathrm{d}t.$

Similarly, $L^{-1}\left\{\dfrac{F(s)}{s^2}\right\}=\int_0^t\int_0^t f(t)\mathrm{d}t\,\mathrm{d}t$

Example 1

Find the inverse Laplace transform of $\dfrac{1}{s(s+2)}$.

Solution

Let

$$F(s)=\frac{1}{s+2}$$

$$L^{-1}\{F(s)\}=e^{-2t}$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\}=\int_0^t L^{-1}\{F(s)\}\,\mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s(s+2)}\right\}=\int_0^t e^{-2t}\,\mathrm{d}t$$

$$=\left|\frac{e^{-2t}}{-2}\right|_0^t$$

$$= -\frac{1}{2}(e^{-2t} - 1)$$

$$= \frac{1}{2}(1 - e^{-2t})$$

Example 2

Find the inverse Laplace transform of $\dfrac{1}{s(s^2 + a^2)}$.

Solution

Let
$$F(s) = \frac{1}{s^2 + a^2}$$

$$L^{-1}\{F(s)\} = \frac{1}{a}\sin at$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s(s^2 + a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\ \mathrm{d}t$$

$$= \frac{1}{a}\int_0^t \sin at\ \mathrm{d}t$$

$$= \frac{1}{a}\left|\frac{-\cos at}{a}\right|_0^t$$

$$= -\frac{1}{a^2}\left|\cos at\right|_0^t$$

$$= -\frac{1}{a^2}(\cos at - 1)$$

$$= \frac{1}{a^2}(1 - \cos at)$$

Example 3

Find the inverse Laplace transform of $\dfrac{1}{s(s^2 + 2s + 2)}$.

Solution

Let
$$F(s) = \frac{1}{s^2 + 2s + 2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$$
$$= L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$
$$= e^{-t}\ L^{-1}\left\{\frac{1}{s^2+1}\right\}$$
$$= e^{-t}\ \sin t$$
$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\,\mathrm{d}t$$
$$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = \int_0^t e^{-t}\sin t\,\mathrm{d}t$$
$$= \left|\frac{e^{-t}}{2}(-\sin t - \cos t)\right|_0^t$$
$$= \frac{-1}{2}\left[\left|e^{-t}(\sin t + \cos t)\right|_0^t\right]$$
$$= -\frac{1}{2}\left[e^{-t}(\sin t + \cos t) - (0+1)\right]$$
$$= \frac{1}{2}\left[1 - e^{-t}(\sin t + \cos t)\right]$$

Example 4

Find the inverse Laplace transform of $\dfrac{1}{s(s^2-3s+3)}$. **[Winter 2015]**

Solution

Let
$$F(s) = \frac{1}{s^2-3s+3}$$
$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2-3s+\frac{9}{4}+3-\frac{9}{4}}\right\}$$
$$= L^{-1}\left\{\frac{1}{\left(s-\frac{3}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= L^{-1}\left\{\frac{1}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{2}{\sqrt{3}}e^{\frac{3t}{2}}L^{-1}\left\{\frac{\frac{2}{\sqrt{3}}}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{2}{\sqrt{3}}e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s(s^2-3s+3)}\right\} = \int_0^t \frac{2}{\sqrt{3}}e^{\frac{3t}{2}}\sin\left(\frac{\sqrt{3}t}{2}\right)\mathrm{d}t$$

$$= \frac{2}{\sqrt{3}}\left|\frac{e^{\frac{3t}{2}}}{\frac{9}{4}+\frac{3}{4}}\left\{\frac{3}{2}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}t}{2}\right)\right\}\right|_0^t$$

$$= \frac{2}{3\sqrt{3}}\left|e^{\frac{3t}{2}}\left\{\frac{3}{2}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}t}{2}\right)\right\}\right|_0^t$$

$$= e^{\frac{3t}{2}}\left[\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right)\right]+\frac{2}{3\sqrt{3}}\cdot\frac{\sqrt{3}}{2}$$

$$= e^{\frac{3t}{2}}\left[\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{1}{3}\cos\left(\frac{\sqrt{3}t}{2}\right)+\frac{1}{3}\right]$$

Example 5

Find the inverse Laplace transform of $\dfrac{1}{s(s^2-1)(s^2+1)}$.

Solution

Let
$$F(s) = \frac{1}{(s^2-1)(s^2+1)}$$

$$= \frac{1}{2}\left[\frac{(s^2+1)-(s^2-1)}{(s^2-1)(s^2+1)}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s^2-1} - \frac{1}{s^2+1}\right)$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= \frac{1}{2}(\sinh t - \sin t)$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\,\mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s(s^2-1)(s^2+1)}\right\} = \int_0^t \frac{1}{2}(\sinh t - \sin t)\mathrm{d}t$$

$$= \frac{1}{2}\int_0^t (\sinh t - \sin t)\mathrm{d}t$$

$$= \frac{1}{2}\left|\cosh t + \cos t\right|_0^t$$

$$= \frac{1}{2}\left[(\cosh t + \cos t) - (1+1)\right]$$

$$= \frac{1}{2}\left[\cosh t + \cos t - 2\right]$$

Example 6

Find the inverse Laplace transform of $\dfrac{1}{s^2(1+s^2)}$.

Solution

Let $$F(s) = \frac{1}{1+s^2}$$

$$L^{-1}\{F(s)\} = \sin t$$

$$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t\int_0^t L^{-1}\left\{F(s)\right\}\mathrm{d}t\ \mathrm{d}t$$

$$L^{-1}\left\{\frac{1}{s^2(1+s^2)}\right\} = \int_0^t\int_0^t \sin t\ \mathrm{d}t\ \mathrm{d}t$$

$$= \int_0^t \left|-\cos t\right|_0^t\ \mathrm{d}t$$

$$= \int_0^t (-\cos t + 1)\mathrm{d}t$$

$$= \int_0^t (1 - \cos t)\ \mathrm{d}t$$

$$= \left| t - \sin t \right|_0^t$$
$$= (t - \sin t) - (0 - 0)$$
$$= t - \sin t$$

EXERCISE 5.19

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{1}{(s^2 + a^2)^2}$

$$\left[\textbf{Ans.:} \frac{1}{2a^3}(\sin at - at \cos at)\right]$$

2. $\dfrac{s^2 + 2}{s(s^2 + 4)}$

$$\left[\textbf{Ans.:} \frac{1}{2}(1 + \cos 2t)\right]$$

3. $\dfrac{s}{(s^2 + 4)^2}$

$$\left[\textbf{Ans.:} \frac{1}{4} t \sin 2t\right]$$

4. $\dfrac{1}{s^2(s^2 + a^2)}$

$$\left[\textbf{Ans.:} \frac{1}{a^2}\left(t - \frac{1}{a}\sin at\right)\right]$$

5. $\dfrac{s + 1}{s^2(s^2 + 1)}$

[**Ans.:** $1 + t - \cos t - \sin t$]

6. $\dfrac{1}{s(s^2 + 4s + 5)}$

$$\left[\textbf{Ans.:} \frac{1}{5}\left[1 - e^{-2t}(2 \sin t + \cos t)\right]\right]$$

5.13.7 Differentiation of Transforms

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s)\} = -\dfrac{1}{t} L^{-1}\{F'(s)\}$.

Example 1

Find the inverse Laplace transform of $\log\left(\dfrac{s + a}{s + b}\right)$. **[Summer 2013]**

Solution

Let
$$F(s) = \log\left(\frac{s+a}{s+b}\right)$$
$$= \log(s+a) - \log(s+b)$$
$$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}\, L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = -\frac{1}{t}\, L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$$
$$= -\frac{1}{t}\,(e^{-at} - e^{-bt})$$

Example 2

Find the inverse Laplace transform of $\log\left(1+\frac{\omega^2}{s^2}\right)$. **[Summer 2014]**

Solution

Let
$$F(s) = \log\left(1+\frac{\omega^2}{s^2}\right)$$
$$= \log\left(\frac{s^2+\omega^2}{s^2}\right)$$
$$= \log(s^2+\omega^2) - \log s^2$$
$$F'(s) = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}$$
$$= \frac{2s}{s^2+\omega^2} - \frac{2}{s}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}\, L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\}$$
$$= -\frac{2}{t} L^{-1}\left\{\frac{s}{s^2+\omega^2} - \frac{1}{s}\right\}$$
$$= -\frac{2}{t}\,(\cos \omega t - 1)$$
$$= \frac{2}{t}(1 - \cos \omega t)$$

Example 3

Find the inverse Laplace transform of $\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$.

Solution

Let
$$F(s) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$
$$= \log(s^2+b^2) - \log(s^2+a^2)$$
$$F'(s) = \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right\}$$
$$= -\frac{1}{t}(2\cos bt - 2\cos at)$$
$$= \frac{2}{t}(\cos at - \cos bt)$$

Example 4

Find the inverse Laplace transform of $\log\dfrac{s^2+a^2}{(s+b)^2}$.

Solution

Let
$$F(s) = \log\frac{s^2+a^2}{(s+b)^2}$$
$$= \log(s^2+a^2) - \log(s+b)^2$$
$$= \log(s^2+a^2) - 2\log(s+b)$$
$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2}{s+b}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\left\{\log\frac{s^2+a^2}{(s+b)^2}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2}{s+b}\right\}$$
$$= -\frac{1}{t}(2\cos at - 2e^{-bt})$$

$$= \frac{2}{t}(e^{-bt} - \cos at)$$

Example 5

Find the inverse Laplace transform of $\log\sqrt{\dfrac{s^2 - a^2}{s^2}}$.

Solution

Let

$$F(s) = \log\sqrt{\frac{s^2 - a^2}{s^2}}$$

$$= \log\sqrt{s^2 - a^2} - \log\sqrt{s^2}$$

$$= \frac{1}{2}\log(s^2 - a^2) - \log s$$

$$F'(s) = \frac{1}{2}\frac{2s}{s^2 - a^2} - \frac{1}{s}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\sqrt{\frac{s^2 - a^2}{s^2}}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{s}{s^2 - a^2} - \frac{1}{s}\right\}$$

$$= -\frac{1}{t}(\cosh at - 1)$$

$$= \frac{1}{t}(1 - \cosh at)$$

Example 6

Find the inverse Laplace transform of $\log\sqrt{\dfrac{s-1}{s+1}}$.

Solution

Let

$$F(s) = \log\sqrt{\frac{s-1}{s+1}}$$

$$= \log\sqrt{s-1} - \log\sqrt{s+1}$$

$$= \frac{1}{2}\log(s-1) - \frac{1}{2}\log(s+1)$$

$$F'(s) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\sqrt{\frac{s-1}{s+1}}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right\}$$

$$= -\frac{1}{t}\left(\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right)$$

$$= -\frac{1}{t}\sinh t$$

Example 7

Find the inverse Laplace transform of $\log\sqrt{\dfrac{s^2+1}{s(s+1)}}$.

Solution

Let

$$F(s) = \log\sqrt{\frac{s^2+1}{s(s+1)}}$$

$$= \frac{1}{2}\left[\log(s^2+1) - \log s - \log(s+1)\right]$$

$$F'(s) = \frac{1}{2}\left(\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right)$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\sqrt{\frac{s^2+1}{s(s+1)}}\right\} = -\frac{1}{t}L^{-1}\frac{1}{2}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= -\frac{1}{2t}(2\cos t - 1 - e^{-t})$$

Example 8

Find the inverse Laplace transform of $s\log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)$.

Solution

Let

$$F(s) = \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \log(s^2+a^2) - \log(s^2+b^2)$$

$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right\}$$

$$= -\frac{1}{t}(2\cos at - 2\cos bt)$$

$$= \frac{2}{t}(\cos bt - \cos at)$$

$$L^{-1}\{s\,F(s)\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{2}{t}(\cos bt - \cos at)\right]$$

$$= \frac{2b}{t}(-\sin bt)\,\frac{-2\cos bt}{t^2} + \frac{2a\sin at}{t} + \frac{2\cos at}{t^2}$$

$$= \frac{1}{t}\left[2(a\sin at - b\sin bt) - \frac{2(\cos bt - \cos at)}{t}\right]$$

Example 9

Find the inverse Laplace transform of $\tan^{-1} s$.

Solution

Let $$F(s) = \tan^{-1} s$$

$$F'(s) = \frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\{\tan^{-1} s\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= -\frac{1}{t}\sin t$$

Example 10

Find the inverse Laplace transform of $\tan^{-1}\left(\dfrac{s+a}{b}\right)$.

Solution

Let $$F(s) = \tan^{-1}\left(\frac{s+a}{b}\right)$$

$$F'(s) = \frac{1}{1+\left(\frac{s+a}{b}\right)^2} \cdot \frac{1}{b}$$

$$= \frac{b}{(s+a)^2 + b^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\left(\frac{s+a}{b}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{b}{(s+a)^2 + b^2}\right\}$$

$$= -\frac{1}{t} e^{-at} \sin bt$$

Example 11

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{2}{s}\right)$.

Solution

Let

$$F(s) = \tan^{-1}\left(\frac{2}{s}\right)$$

$$F'(s) = \frac{1}{1+\frac{4}{s^2}}\left(-\frac{2}{s^2}\right)$$

$$= -\frac{2}{s^2+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\frac{2}{s}\right\} = -\frac{1}{t} L^{-1}\left\{-\frac{2}{s^2+4}\right\}$$

$$= \frac{2}{t} L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{2}{t} \cdot \frac{1}{2} \sin 2t$$

$$= \frac{1}{t} \sin 2t$$

Example 12

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{2}{s^2}\right)$.

Solution

Let
$$F(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$$

$$F'(s) = \frac{1}{1+\frac{4}{s^4}}\left(-\frac{4}{s^3}\right)$$

$$= -\frac{4s}{s^4+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\} = -\frac{1}{t}L^{-1}\left\{-\frac{4s}{s^4+4}\right\}$$

$$= \frac{4}{t}L^{-1}\left\{\frac{s}{s^4+4}\right\}$$

$$= \frac{4}{t}L^{-1}\left\{\frac{s}{(s^2+2)^2-(2s)^2}\right\}$$

$$= \frac{4}{t}\cdot\frac{1}{4}L^{-1}\left\{\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right\}$$

$$= \frac{1}{t}L^{-1}\left\{\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{1}{t}(e^t \sin t - e^{-t}\sin t)$$

$$= \frac{\sin t}{t}(e^t - e^{-t})$$

$$= \frac{2}{t}\sin t \sinh t$$

Example 13

Find the inverse Laplace transform of $\cot^{-1} s$.

Solution

Let
$$F(s) = \cot^{-1} s$$

$$F'(s) = -\frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\{\cot^{-1} s\} = -\frac{1}{t} L^{-1}\left\{-\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{t} \sin t$$

Example 14

Find the inverse Laplace transform of $\cot^{-1}\left(\frac{k}{s}\right)$.

Solution

Let
$$F(s) = \cot^{-1}\left(\frac{k}{s}\right)$$

$$F'(s) = -\frac{1}{1+\frac{k^2}{s^2}}\left(-\frac{k}{s^2}\right) = \frac{k}{s^2+k^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\cot^{-1}\left(\frac{k}{s}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{k}{s^2+k^2}\right\}$$

$$= -\frac{1}{t} \sin kt$$

Example 15

Find the inverse Laplace transform of $\cot^{-1}(s + 1)$.

Solution

Let
$$F(s) = \cot^{-1}(s+1)$$

$$F'(s) = -\frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\{F'(s)\}$$

$$L^{-1}\{\cot^{-1}(s+1)\} = -\frac{1}{t} L^{-1}\left\{-\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{1}{t} e^{-t} \sin t$$

Example 16

Find the inverse Laplace transform of $2\tanh^{-1} s$.

Solution

Let
$$F(s) = 2\tanh^{-1} s$$
$$= 2\cdot\frac{1}{2}\log\frac{1+s}{1-s}$$
$$= \log(1+s) - \log(1-s)$$
$$F'(s) = \frac{1}{1+s} + \frac{1}{1-s}$$
$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\}$$
$$L^{-1}\{2\tanh^{-1} s\} = -\frac{1}{t}L^{-1}\left\{\frac{1}{1+s} + \frac{1}{1-s}\right\}$$
$$= -\frac{1}{t}(e^{-t} - e^{t})$$
$$= \frac{2}{t}\sinh t$$

EXERCISE 5.20

Find the inverse Laplace transforms of the following functions:

1. $\log\left(1+\dfrac{a^2}{s^2}\right)$

$$\left[\textbf{Ans.}: \frac{2}{t}(1-\cos at)\right]$$

2. $\log\left(1-\dfrac{1}{s^2}\right)$

$$\left[\textbf{Ans.}: \frac{2}{t}(1-\cosh t)\right]$$

3. $\log\dfrac{s^2-4}{(s-3)^2}$

$$\left[\textbf{Ans.}: \frac{2}{t}(e^{3t}-\cosh 2t)\right]$$

4. $\log\sqrt{\dfrac{s^2+1}{s^2}}$

$$\left[\textbf{Ans.}: \frac{1}{t}(1-\cos t)\right]$$

5. $\log\dfrac{(s-2)^2}{s^2+1}$

$$\left[\textbf{Ans .: } \frac{2}{t}(\cos t - e^{2t})\right]$$

6. $\log\left(\dfrac{s^2-4}{s^2}\right)^{\frac{1}{3}}$

$$\left[\textbf{Ans .: } \frac{2}{3t}(1-\cosh 2t)\right]$$

7. $\log\dfrac{1}{s}\left(1+\dfrac{1}{s^2}\right)$

$$\left[\textbf{Ans .: } \int_0^t \frac{2(1-\cos t)}{t}\,dt\right]$$

8. $\dfrac{1}{s}\log\dfrac{s+1}{s+2}$

$$\left[\textbf{Ans .: } \int_0^t \frac{e^{-2t}-e^{-t}}{t}\,dt\right]$$

9. $\tan^{-1}(s+1)$

$$\left[\textbf{Ans .: } -\frac{1}{t}e^{-t}\sin t\right]$$

10. $\tan^{-1}\left(\dfrac{s}{2}\right)$

$$\left[\textbf{Ans .: } -\frac{1}{t}\sin 2t\right]$$

11. $\cot^{-1}(as)$

$$\left[\textbf{Ans .: } \frac{1}{t}\sin\frac{t}{a}\right]$$

12. $\cot^{-1}\left(\dfrac{2}{s^2}\right)$

$$\left[\textbf{Ans .: } -\frac{2}{7}\sin t\sinh t\right]$$

5.13.8 Integration of Transforms

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s)\} = t\,L^{-1}\left[\int_s^\infty F(s)\,\mathrm{d}s\right]$.

Example 1

Find the inverse Laplace transform of $\dfrac{1}{(s+1)^2}$.

Solution

Let
$$F(s) = \frac{1}{(s+1)^2}$$

$$\int_s^\infty F(s)\,\mathrm{d}s = \int_s^\infty \frac{1}{(s+1)^2}\,\mathrm{d}s$$

$$= \left| -\left(\frac{1}{s+1}\right) \right|_s^\infty$$

$$= -\left(0 - \frac{1}{s+1}\right)$$

$$= \frac{1}{s+1}$$

$$L^{-1}\,\{F(s)\} = t\,L^{-1}\left[\int_s^\infty F(s)\,\mathrm{d}s\right]$$

$$= t\,L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= t\,e^{-t}$$

Example 2

Find the inverse Laplace transform of $\dfrac{2}{(s-a)^3}$.

Solution

Let
$$F(s) = \frac{2}{(s-a)^3}$$

$$\int_s^\infty F(s)\,\mathrm{d}s = \int_s^\infty \frac{2}{(s-a)^3}\,\mathrm{d}s$$

$$= 2\left| -\frac{1}{2(s-a)^2} \right|_s^\infty$$

$$= \frac{1}{(s-a)^2}$$

$$
\begin{aligned}
L^{-1}\{F(s)\} &= t\,L^{-1}\left[\int_s^\infty F(s)\,\mathrm{d}s\right] \\
&= t\,L^{-1}\left[\frac{1}{(s-a)^2}\right] \\
&= t\,e^{at}\,L^{-1}\left\{\frac{1}{s^2}\right\} \\
&= t\,e^{at}.\,t \\
&= t^2\,e^{at}
\end{aligned}
$$

Example 3

Find the inverse Laplace transform of $\dfrac{2s}{(s^2+1)^2}$.

Solution

Let $$F(s) = \frac{2s}{(s^2+1)^2}$$

$$
\begin{aligned}
\int_s^\infty F(s)\,\mathrm{d}s &= \int_s^\infty \frac{2s}{(s^2+1)^2}\,\mathrm{d}s \\
&= \left|\frac{-1}{s^2+1}\right|_s^\infty \\
&= 0 + \frac{1}{s^2+1} \\
&= \frac{1}{s^2+1}
\end{aligned}
$$

$$
\begin{aligned}
L^{-1}\{F(s)\} &= t\,L^{-1}\left[\int_s^\infty F(s)\,\mathrm{d}s\right] \\
&= t\,L^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= t\sin t
\end{aligned}
$$

Example 4

Find the inverse Laplace transform of $\dfrac{s}{(s^2-a^2)^2}$.

Solution

Let $$F(s) = \frac{s}{(s^2-a^2)^2}$$

$$\begin{aligned}
\int_s^\infty F(s)\,\mathrm{d}s &= \int_s^\infty \frac{s}{(s^2-a^2)^2}\,\mathrm{d}s \\
&= \frac{1}{2}\int_s^\infty \frac{2s}{(s^2-a^2)^2}\,\mathrm{d}s \\
&= \frac{1}{2}\left|-\left(\frac{1}{s^2-a^2}\right)\right|_s^\infty \\
&= \frac{1}{2}\,\frac{1}{s^2-a^2}
\end{aligned}$$

$$\begin{aligned}
L^{-1}\{F(s)\} &= t\,L^{-1}\left[\int_s^\infty F(s)\,\mathrm{d}s\right] \\
&= t\,L^{-1}\left\{\frac{1}{2}\,\frac{1}{s^2-a^2}\right\} \\
&= \frac{t}{2}\,\frac{1}{a}\sinh at \\
&= \frac{t}{2a}\sinh at
\end{aligned}$$

EXERCISE 5.21

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s}{(s^2-4)^2}$

$\left[\textbf{Ans.: } \dfrac{t}{2}\sinh 2t\right]$

2. $\dfrac{s+2}{(s^2+4s+5)^2}$

$\left[\textbf{Ans.: } \dfrac{t}{2}e^{-2t}\sin t\right]$

3. $\dfrac{s}{s^2-a^2}$

$\left[\textbf{Ans.: } \dfrac{t}{2a}\sinh at\right]$

5.13.9 Partial Fraction Expansion

Any function $F(s)$ can be written as $\dfrac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials in s. To perform partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that of $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I **Factors are linear and distinct**

$$F(s) = \frac{P(s)}{(s+a)(s+b)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case II **Factors are linear and repeated**

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \dots + \frac{B_n}{(s+b)^n}$$

Case III **Factors are quadratic and distinct**

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case IV **Factors are quadratic and repeated**

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{C_1s+D_1}{s^2+cs+d} + \frac{C_2s+D_2}{(s^2+cs+d)^2} + \dots + \frac{C_ns+D_n}{(s^2+cs+d)^n}$$

Example 1

Find the inverse Laplace transform of $\frac{1}{(s+1)(s+2)}$. **[Summer 2018]**

Solution

Let $F(s) = \frac{1}{(s+1)(s+2)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s + 2) + B(s + 1)$$

Putting $s = -1$ in Eq. (1),

$$A = 1$$

Putting $s = -2$ in Eq. (2),

$$1 = B\ (-1)$$

$$B = -1$$

$$F(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$L^{-1}\left\{F(s)\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-t} - e^{-2t}$$

Example 2

Find the inverse Laplace transform of $\dfrac{1}{(s-2)(s+3)}$. **[Summer 2015]**

Solution

Let $F(s) = \dfrac{1}{(s-2)(s+3)}$

By partial-fraction expansion,

$$F(s) = \frac{A}{s-2} + \frac{B}{s+3}$$

$$1 = A(s + 3) + B(s - 2) \qquad \text{...(1)}$$

Putting $s = 2$ in Eq. (1),

$$A = \frac{1}{5}$$

Putting $s = -3$ in Eq. (1),

$$B = -\frac{1}{5}$$

$$F(s) = \frac{1}{5}\cdot\frac{1}{s-2} - \frac{1}{5}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{5}L^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{5}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}$$

Example 3

Find the inverse Laplace transform of $\dfrac{1}{(s+\sqrt{2})(s-\sqrt{3})}$.

[Summer 2016]

Solution

Let $$f(s) = \frac{1}{(s+\sqrt{2})(s-\sqrt{3})}$$

By partial fractional expansion,

$$F(s) = \frac{A}{(s+\sqrt{2})} + \frac{B}{(s-\sqrt{3})}$$

$$1 = A(s-\sqrt{3}) + B(s+\sqrt{2}) \qquad \text{...(1)}$$

Putting $s = -\sqrt{2}$ in Eq. (1),

$$1 = A(-\sqrt{2} - \sqrt{3})$$

$$A = -\frac{1}{(\sqrt{3}+\sqrt{2})}$$

Putting $s = \sqrt{3}$ in Eq.(1),

$$1 = B(\sqrt{3}+\sqrt{2})$$

$$B = \frac{1}{(\sqrt{3}+\sqrt{2})}$$

$$F(s) = \frac{1}{(\sqrt{3}+\sqrt{2})}\left[-\frac{1}{(s+\sqrt{2})} + \frac{1}{(s-\sqrt{3})}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{3}+\sqrt{2}}\left[L^{-1}\left\{-\frac{1}{(s+\sqrt{2})} + \frac{1}{(s-\sqrt{3})}\right\}\right]$$

$$= \frac{1}{\sqrt{3}+\sqrt{2}}[-\mathrm{e}^{-\sqrt{2}t} + e^{\sqrt{3}t}]$$

$$= \frac{e^{\sqrt{3}t} - e^{-\sqrt{2}t}}{\sqrt{3} + \sqrt{2}}$$

Example 4

Find the inverse Laplace transform of $\dfrac{s+2}{s(s+1)(s+3)}$.

Solution

Let $$F(s) = \frac{s+2}{s(s+1)(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$s + 2 = A(s + 1)(s + 3) + Bs(s + 3) + Cs(s +1) \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3A$$

$$A = \frac{2}{3}$$

Putting $s = -1$ in Eq. (1),

$$1 = B(-1)(2)$$

$$B = -\frac{1}{2}$$

Putting $s = -3$ in Eq. (1),

$$-1 = C(-3)(-2)$$

$$C = -\frac{1}{6}$$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

Example 5

Find the inverse Laplace transform of $\dfrac{3s^2+2}{(s+1)(s+2)(s+3)}$.

[Winter 2014]

Solution

Let $$F(s) = \frac{3s^2+2}{(s+1)(s+2)(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$3s^2 + 2 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) \qquad ...(1)$$

Putting $s = -1$ in Eq. (1),

$$5 = 2A$$

$$A = \frac{5}{2}$$

Putting $s = -2$ in Eq. (1),

$$14 = -B$$

$$B = -14$$

Putting $s = -3$ in Eq. (1),

$$29 = 2C$$

$$C = \frac{29}{2}$$

$$F(s) = \frac{5}{2}\cdot\frac{1}{s+1} - 14\cdot\frac{1}{s+2} + \frac{29}{2}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{5}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - 14L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{29}{2}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{5}{2}e^{-t} - 14e^{-2t} + \frac{29}{2}e^{-3t}$$

Example 6

Find the inverse Laplace transform of $\dfrac{s+2}{s^2(s+3)}$.

Solution

Let $$F(s) = \frac{s+2}{s^2(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$s + 2 = As(s + 3) + B(s + 3) + Cs^2 \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3B$$

$$B = \frac{2}{3}$$

Putting $s = -3$ in Eq. (1),

$$-1 = 9C$$

$$C = -\frac{1}{9}$$

Equating the coefficients of s^2,

$$0 = A + C$$

$$A = \frac{1}{9}$$

$$F(s) = \frac{1}{9} \cdot \frac{1}{s} + \frac{2}{3} \cdot \frac{1}{s^2} - \frac{1}{9} \cdot \frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9} L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3} L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9} L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

Example 7

Find the inverse Laplace transform of $\dfrac{s}{(s+1)(s-1)^2}$. **[Winter 2014]**

Solution

Let $\quad F(s) = \dfrac{s}{(s+1)(s-1)^2}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$s = A(s-1)^2 + B(s+1)(s-1) + C(s+1) \qquad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$-1 = 4A$$

$$A = -\frac{1}{4}$$

Putting $s = 1$ in Eq. (1),

$$1 = 2C$$

$$C = \frac{1}{2}$$

Putting $s = 0$ in Eq. (1),

$$0 = A - B + C$$

$$B = A + C = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$F(s) = -\frac{1}{4} \cdot \frac{1}{s+1} + \frac{1}{4} \cdot \frac{1}{s-1} + \frac{1}{2} \cdot \frac{1}{(s-1)^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{4} L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4} L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= -\frac{1}{4} e^{-t} + \frac{1}{4} e^{t} + \frac{1}{2} te^{t}$$

Example 8

Find the inverse Laplace transform of $\dfrac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$.

Solution

Let $$F(s) = \frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$5s^2 - 15s - 11 = A(s-2)^2 + B(s+1)(s-2) + C(s+1) \qquad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$9 = 9A$$

$$A = 1$$

Putting $s = 2$ in Eq. (1),

$$-21 = 3C$$

$$C = -7$$

Equating the coefficients of s^2,

$$5 = A + B$$

$$B = 4$$

$$F(s) = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} - 7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= e^{-t} + 4e^{2t} - 7\,te^{2t}$$

Example 9

Find the inverse Laplace transform of $\dfrac{s+2}{(s+3)(s+1)^3}$.

Solution

Let $$F(s) = \frac{s+2}{(s+3)(s+1)^3}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$$s + 2 = A(s+1)^3 + B(s+3)(s+1)^2 + C(s+3)(s+1) + D(s+3) \qquad \ldots (1)$$

Putting $s = -3$ in Eq. (1),

$$-1 = -8A$$

$$A = \frac{1}{8}$$

Putting $s = -1$ in Eq. (1),

$$1 = 2D$$

$$D = \frac{1}{2}$$

Equating the coefficients of s^3,

$$0 = A + B$$

$$B = -\frac{1}{8}$$

Equating the coefficients of s^2,

$$0 = 3A + 5B + C$$

$$C = -\frac{3}{8} + \frac{5}{8} = \frac{1}{4}$$

$$F(s) = \frac{1}{8}\cdot\frac{1}{s+3} - \frac{1}{8}\cdot\frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

$$L^{-1}\{F(s)\} = \frac{1}{8}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{8}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= \frac{1}{8}e^{-3t} - \frac{1}{8}e^{-t} + \frac{1}{4}t\,e^{-t} + \frac{1}{2}\cdot\frac{t^2}{2}\cdot e^{-t}$$

$$= \frac{1}{8}\left[e^{-3t} + (2t^2 + 2t - 1)e^{-t}\right]$$

Example 10

Find the inverse Laplace transform of $\dfrac{s^3 + 6s^2 + 14s}{(s+2)^4}$.

Solution

Let $\quad F(s) = \dfrac{s^3 + 6s^2 + 14s}{(s+2)^4}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}$$

$$s^3 + 6s^2 + 14s = A(s+2)^3 + B(s+2)^2 + C(s+2) + D$$

$$= As^3 + (6A + B)s^2 + (12A + 4B + C)s + (8A + 4B + 2C + D) \qquad \text{... (1)}$$

Equating the coefficients of s^3,

$$A = 1$$

Equating the coefficients of s^2,

$$6 = 6A + B$$
$$B = 0$$

Equating the coefficients of s,

$$14 = 12A + 4B + C$$
$$C = 14 - 12 - 0 = 2$$

Equating the coefficients of s^0,

$$0 = 8A + 4B + 2C + D$$
$$D = -8 - 0 - 4 = -12$$

$$F(s) = \frac{1}{s+2} + \frac{2}{(s+2)^3} - \frac{12}{(s+2)^4}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} + 2L^{-1}\left\{\frac{1}{(s+2)^3}\right\} - 12L^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

$$= e^{-2t} + 2\cdot\frac{t^2}{2}\cdot e^{-2t} - 12\cdot\frac{t^3}{6}\cdot e^{-2t}$$

$$= e^{-2t}(1 + t^2 - 2t^3)$$

Example 11

Find the inverse Laplace transform of $\dfrac{s^2+1}{(s+1)(s-2)^2}$.

Solution

Let $$F(s)=\frac{s^2+1}{(s+1)(s-2)^2}$$

By partial fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$s^2+1=A(s-2)^2+B(s+1)(s-2)+C(s+1) \qquad \ldots(1)$$

Putting $s=-1$ in Eq. (1),

$$2=9A$$

$$A=\frac{2}{9}$$

Putting $s=2$ in Eq. (1),

$$5=3C$$

$$C=\frac{5}{3}$$

Equating the coefficients of s^2,

$$1=A+B$$

$$B=\frac{7}{9}$$

$$F(s)=\frac{2}{9}\cdot\frac{1}{s+1}+\frac{7}{9}\cdot\frac{1}{s-2}+\frac{5}{3}\cdot\frac{1}{(s-2)^2}$$

$$L^{-1}\{F(s)\}=\frac{2}{9}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{7}{9}L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{5}{3}L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$=\frac{2}{9}e^{-t}+\frac{7}{9}e^{2t}+\frac{5}{3}e^{2t}\;L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$=\frac{2}{9}e^{-t}+\frac{7}{9}e^{2t}+\frac{5}{3}te^{2t}$$

Example 12

Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s^2+1)}$.

Solution

Let $$F(s)=\frac{1}{(s+1)(s^2+1)}$$

By partial fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+1}$$

$$1 = A\,(s^2+1)+(Bs+C)\,(s+1) \qquad \ldots(1)$$

Putting $s=-1$, in Eq. (1),

$$1=2A$$

$$A=\frac{1}{2}$$

Equating the coefficients of s^2,

$$0=A+B$$

$$B=-\frac{1}{2}$$

Equating the coefficients of s,

$$0=B+C$$

$$C=\frac{1}{2}$$

$$F(s)=\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{2}\cdot\frac{s}{s^2+1}+\frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$L^{-1}\{F(s)\}=\frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{2}L^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$=\frac{1}{2}e^{-t}-\frac{1}{2}\cos t+\frac{1}{2}\sin t$$

$$=\frac{1}{2}(e^{-t}-\cos t+\sin t)$$

Example 13

Find the inverse Laplace transform of $\dfrac{3s+1}{(s+1)(s^2+2)}$.

Solution

Let $$F(s) = \frac{3s+1}{(s+1)(s^2+2)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$$

$$3s + 1 = A(s^2 + 2) + (Bs + C)(s + 1) \qquad \text{... (1)}$$

Putting $s = -1$ in Eq. (1),

$$-2 = 3A$$

$$A = -\frac{2}{3}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{2}{3}$$

Equating the coefficients of s^0,

$$1 = 2A + C$$

$$C = 1 + \frac{4}{3} = \frac{7}{3}$$

$$F(s) = -\frac{2}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{s}{s^2+2} + \frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\}$$

$$= -\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t$$

Example 14

Find the inverse Laplace transform of $\dfrac{s+4}{s(s-1)(s^2+4)}$.

Solution

Let $$F(s) = \frac{s+4}{s(s-1)(s^2+4)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$s + 4 = A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)\, s(s-1) \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$4 = -4A$$
$$A = -1$$

Putting $s = 1$ in Eq. (1),

$$5 = 5B$$
$$B = 1$$

Equating the coefficients of s^3,

$$0 = A + B + C$$
$$C = 1 - 1 = 0$$

Equating the coefficients of s,

$$1 = 4A + 4B - D$$
$$D = -4 + 4 - 1 = -1$$
$$F(s) = -\frac{1}{s} + \frac{1}{s+1} - \frac{1}{s^2+4}$$
$$L^{-1}\{F(s)\} = -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s^2+4}\right\}$$
$$= -1 + e^t - \frac{1}{2}\sin 2t$$

Example 15

Find the inverse Laplace transform of $\dfrac{1}{s^4 - 81}$. **[Summer 2016]**

Solution

Let
$$F(s) = \frac{1}{s^4 - 81} = \frac{1}{(s^2-9)(s^2+9)}$$
$$= \frac{1}{(s-3)\,(s+3)\,(s^2+9)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s-3} + \frac{B}{s+3} + \frac{Cs+D}{s^2+9}$$
$$1 = A(s+3)\,(s^2+9) + B(s-3)(s^2+9) + (Cs+D)(s^2-9) \qquad ...(1)$$

Putting $s = 3$ in Eq. (1),

$$1 = 6A\ (18)$$

$$A = \frac{1}{108}$$

Putting $s = -3$ in Eq. (1),

$$1 = (-6)\,B\,(18)$$

$$B = -\frac{1}{108}$$

Putting $s = 0$ in Eq. (1),

$$1 = 27A - 27B - 9D$$

$$9D = 27A - 27B - 1 = \frac{1}{4} + \frac{1}{4} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$D = -\frac{1}{18}$$

Equating the coefficient of s^3 from Eq. (1),

$$A + B + C = 0$$

$$\frac{1}{108} - \frac{1}{108} + C = 0$$

$$C = 0$$

$$F(s) = \frac{1}{108}\,\frac{1}{s-3} + \left(-\frac{1}{108}\right)\frac{1}{s+3} + \left(-\frac{1}{18}\right)\frac{1}{s^2+9}$$

$$L^{-1}\{F(s)\} = \frac{1}{108}\,L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{108}\,L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{18}\,L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \frac{1}{108}e^{3t} - \frac{1}{108}e^{-3t} - \frac{1}{54}\sin 3t$$

Example 16

Find the inverse Laplace transform of $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

Let $$F(s) = \frac{s}{(s^2+1)(s^2+4)}$$

$$= \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right]$$

$$= \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right]$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

Example 17

Find the inverse Laplace transform of $\dfrac{s^3}{s^4 - a^4}$. **[Summer 2018]**

Solution

Let $$F(s) = \frac{s^3}{s^4 - a^4}$$

$$= \frac{s^3}{(s^2+a^2)(s^2-a^2)}$$

$$= \frac{s}{2}\left[\frac{(s^2+a^2)+(s^2-a^2)}{(s^2+a^2)(s^2-a^2)}\right]$$

$$= \frac{s}{2}\left[\frac{1}{s^2-a^2} + \frac{1}{s^2+a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2-a^2} + \frac{s}{s^2+a^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{s}{s^2-a^2}\right\} + L^{-1}\left\{\frac{s}{s^2+a^2}\right\}\right]$$

$$= \frac{1}{2}\left[\cosh at + \cos at\right]$$

Example 18

Find the inverse Laplace transform of $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$.

Solution

Let $$F(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Let $\qquad s^2 = x$

$$G(x) = \frac{x}{(x+a^2)(x+b^2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+a^2} + \frac{B}{x+b^2}$$

$$x = A(x + b^2) + B(x + a^2) \qquad \ldots (1)$$

Putting $x = -a^2$ in Eq. (1),

$$-a^2 = A(-a^2 + b^2)$$

$$A = \frac{a^2}{a^2 - b^2}$$

Putting $x = -b^2$ in Eq. (1),

$$-b^2 = B(-b^2 + a^2)$$

$$B = -\frac{b^2}{a^2 - b^2}$$

$$G(x) = \frac{a^2}{a^2-b^2}\frac{1}{x+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{x+b^2}$$

$$F(s) = \frac{a^2}{a^2-b^2}\frac{1}{s^2+a^2} - \frac{b^2}{a^2-b^2}\frac{1}{s^2+b^2}$$

$$L^{-1}\{F(s)\} = \frac{a^2}{a^2-b^2} L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - \frac{b^2}{a^2-b^2} L^{-1}\left\{\frac{1}{s^2+b^2}\right\}$$

$$= \frac{a^2}{a^2-b^2}\frac{1}{a}\sin at - \frac{b^2}{a^2-b^2}\frac{1}{b}\sin bt$$

$$= \frac{1}{a^2-b^2}(a \sin at - b \sin bt)$$

Example 19

Find the inverse Laplace transform of $\dfrac{2s^2-1}{(s^2+1)(s^2+4)}$.

[Winter 2016]

Solution

Let $$F(s) = \frac{2s^2 - 1}{(s^2+1)(s^2+4)}$$

Let $$s^2 = x$$

$$G(x) = \frac{2x-1}{(x+1)(x+4)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+1} + \frac{B}{x+4}$$

$$2x - 1 = A(x+4) + B(x+1) \qquad \text{...(1)}$$

Putting $x = -1$ in Eq. (1),

$$-3 = 3A$$

$$A = -1$$

Putting $x = -4$ in Eq. (1),

$$-9 = B(-3)$$

$$B = 3$$

$$G(x) = -\frac{1}{x+1} + \frac{3}{x+4}$$

$$F(\mathrm{s}) = -\frac{1}{s^2+1} + \frac{3}{s^2+4}$$

$$L^{-1}\{F(s)\} = -L^{-1}\left\{\frac{1}{s^2+1}\right\} + 3\,L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= -\sin t + \frac{3}{2}\sin 2t$$

Example 20

Find the inverse Laplace transform of $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$.

[Summer 2014]

Solution

Let $$F(s) = \frac{5s+3}{(s-1)(s^2+2s+5)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

$$5s + 3 = A(s^2 + 2s + 5) + (Bs + C)(s - 1)$$

$$= s^2(A + B) + s(2A - B + C) + (5A - C)$$

Equating the coefficients of s^2, s and s^0,

$$A + B = 0$$

$$2A - B + C = 5$$

$$5A - C = 3$$

Solving these equations,

$$A = 1, \quad B = -1, \quad C = 2$$

$$F(s) = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1} - \frac{s+1-3}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+2^2} + \frac{3}{(s+1)^2+2^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + 3L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$$

Example 21

Find the inverse Laplace transform of $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$.

Solution

Let $$F(s) = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $$s^2 + 2s = x$$

$$G(x) = \frac{x+3}{(x+5)(x+2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+5} + \frac{B}{x+2}$$

$$x + 3 = A(x + 2) + B(x + 5) \qquad \ldots (1)$$

Putting $x = -5$ in Eq. (1),

$$-2 = -3A$$

$$A = \frac{2}{3}$$

Putting $x = -2$ in Eq. (1),

$$1 = 3B$$

$$B = \frac{1}{3}$$

$$G(x) = \frac{2}{3} \cdot \frac{1}{x+5} + \frac{1}{3} \cdot \frac{1}{x+2}$$

$$F(s) = \frac{2}{3} \cdot \frac{1}{(s^2+2s+5)} + \frac{1}{3} \cdot \frac{1}{(s^2+2s+2)}$$

$$= \frac{2}{3} \cdot \frac{1}{(s+1)^2+4} + \frac{1}{3} \cdot \frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\} = \frac{2}{3} L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{2}{3} e^{-t} \cdot \frac{1}{2} \sin 2t + \frac{1}{3} e^{-t} \sin t$$

$$= \frac{1}{3} e^{-t} (\sin 2t + \sin t)$$

Example 22

Find the inverse Laplace transform of $\dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)}$.

Solution

Let $$F(s) = \frac{s+2}{(s^2+4s+8)\,(s^2+4s+13)}$$

$$= \frac{s+2}{5}\left[\frac{s^2+4s+13-s^2-4s-8}{(s^2+4s+8)\,(s^2+4s+13)}\right]$$

$$= \frac{1}{5}\left[\frac{s+2}{s^2+4s+8} - \frac{s+2}{s^2+4s+13}\right]$$

$$= \frac{1}{5}\left[\frac{s+2}{(s+2)^2+4} - \frac{s+2}{(s+2)^2+9}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{5}\left[L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - L^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\}\right]$$

$$= \frac{1}{5}\left[e^{-2t}\ L^{-1}\left\{\frac{s}{s^2+4}\right\} - e^{-2t}\ L^{-1}\left\{\frac{s}{s^2+9}\right\}\right]$$

$$= \frac{1}{5}\left[e^{-2t}\ \cos 2t - e^{-2t}\ \cos 3t\right]$$

$$= \frac{e^{-2t}}{5}(\cos 2t - \cos 3t)$$

Example 23

Find the inverse Laplace transform of $\dfrac{s}{s^4+4a^4}$. **[Winter 2013]**

Solution

Let $$F(s) = \frac{s}{s^4+4a^4}$$

$$= \frac{s}{(s^4+4a^2s^2+4a^4)-4a^2s^2}$$

$$= \frac{s}{(s^2+2a^2)^2-(2as)^2}$$

$$= \frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$$

$$= \frac{1}{4a}\left[\frac{s^2+2as+2a^2-s^2+2as-2a^2}{(s^2+2as+2a^2)(s^2-2as+2a^2)}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{4a}\left[L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\} - L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[e^{at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - e^{-at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[e^{at}\cdot\frac{1}{a}\sin at - e^{-at}\cdot\frac{1}{a}\sin at\right]$$

$$= \frac{1}{2a^2}\sin at\left(\frac{e^{at}-e^{-at}}{2}\right)$$

$$= \frac{1}{2a^2}\sin at \sinh at$$

Example 24

Find the inverse Laplace transform of $\dfrac{s}{s^4+s^2+1}$.

Solution

Let $$F(s) = \frac{s}{s^4+s^2+1}$$

$$= \frac{s}{s^4+2s^2+1-s^2}$$

$$= \frac{s}{(s^2+1)^2-s^2}$$

$$= \frac{s}{(s^2+1+s)\,(s^2+1-s)}$$

$$= \frac{1}{2}\left[\frac{s^2+1+s-s^2-1+s}{(s^2+1+s)\,(s^2+1-s)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\} - L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}\right]$$

$$= \frac{1}{2}\left[e^{\frac{t}{2}} L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\} - e^{-\frac{t}{2}} L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}\right]$$

$$= \frac{1}{2}\left[e^{\frac{t}{2}} \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}} \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t\right]$$

$$= \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t\left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2}\right)$$

$$= \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \sinh\frac{t}{2}$$

Example 25

Find the inverse Laplace transform of $\dfrac{1}{s^3+1}$.

Solution

Let $F(s) = \dfrac{1}{s^3+1}$

$$= \frac{1}{(s+1)(s^2-s+1)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2-s+1}$$

$$1 = A(s^2-s+1) + (Bs+C)(s+1) \qquad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$1 = 3A$$

$$A = \frac{1}{3}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B = -\frac{1}{3}$$

Equating the coefficients of s,

$$0 = -A + B + C$$

$$C=\frac{2}{3}$$

$$F(s)=\frac{1}{3}\cdot\frac{1}{s+1}-\frac{1}{3}\cdot\frac{s}{s^2-s+1}+\frac{2}{3}\cdot\frac{1}{s^2-s+1}$$

$$=\frac{1}{3}\,\frac{1}{s+1}-\frac{1}{3}\left(\frac{s-2}{s^2-s+1}\right)$$

$$=\frac{1}{3}\,\frac{1}{s+1}-\frac{1}{3}\left[\frac{s-\frac{1}{2}-\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$=\frac{1}{3}\,\frac{1}{s+1}-\frac{1}{3}\,\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{1}{3}\cdot\frac{\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$L^{-1}\{F(s)\}=\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{3}L^{-1}\left\{\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$=\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{3}e^{\frac{t}{2}}L^{-1}\left\{\frac{s}{s^2+\frac{3}{4}}\right\}+\frac{1}{2}e^{\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}$$

$$=\frac{1}{3}e^{-t}-\frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t+\frac{1}{2}e^{\frac{t}{2}}\,\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$=\frac{1}{3}e^{-t}-\frac{1}{3}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}e^{\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t$$

Example 26

Find the inverse Laplace transform of $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$.

Solution

Let $\quad F(s)=\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$

By partial fraction expansion,

$$F(s)=\frac{As+B}{(s^2-2s+2)}+\frac{Cs+D}{(s^2-2s+2)^2}$$

$$s^3-3s^2+6s-4=(As+B)\,(s^2-2s+2)+Cs+D$$

$$=As^3+s^2(B-2A)+s(2A-2B+C)+2B+D$$

Equating the coefficients of s^3,

$$A=1$$

Equating the coefficients of s^2,

$$-3=B-2A$$
$$B=-3+2=-1$$

Equating the coefficients of s,

$$6=2A-2B+C$$
$$C=6-2-2=2$$

Equating the coefficients of s^0,

$$-4=2B+D$$
$$D=-4+2=-2$$

$$F(s)=\frac{s-1}{(s^2-2s+2)}+\frac{2s-2}{(s^2-2s+2)^2}$$

$$=\frac{s-1}{(s-1)^2+1}+\frac{2(s-1)}{\left[(s-1)^2+1\right]^2}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}+2\,L^{-1}\left\{\frac{s-1}{\left[(s-1)^2+1\right]^2}\right\}$$

$$=e^t\;L^{-1}\left\{\frac{s}{s^2+1}\right\}+2e^t\;L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$=e^t\;\cos t+2e^t\,\frac{t}{2}\sin t$$

$$=e^t\;(\cos t+t\sin t)$$

EXERCISE 5.22

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s^2-4}{(s+1)\,(s-2)\,(s-3)}$

$$\left[\textbf{Ans.: } -\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}\right]$$

2. $\dfrac{s+2}{s^2(s+3)}$

$$\left[\textbf{Ans.: } \frac{1}{9}(1+6t-e^{-3t})\right]$$

3. $\dfrac{1}{s(s+1)^2}$

$$\left[\textbf{Ans.: } 1-e^{-t}-te^{-t}\right]$$

4. $\dfrac{1}{s^2(s+3)^2}$

$$\left[\textbf{Ans.: } \frac{1}{27}(-2+3t+2e^{-3t}+3t^2e^{-3t})\right]$$

5. $\dfrac{s^2}{(s+4)^3}$

$$\left[\textbf{Ans.: } e^{-4t}(1-8t+8t^2)\right]$$

6. $\dfrac{1}{(s-2)^4(s+3)}$

$$\left[\textbf{Ans.: } \frac{e^{2t}}{6}\left(\frac{t^3}{5}-\frac{3}{25}t^2+\frac{6}{125}t-\frac{6}{625}\right)+\frac{1}{625}e^{-3t}\right]$$

7. $\dfrac{5s^2-7s+17}{(s-1)(s^2+4)}$

$$\left[\textbf{Ans.: } 3e^t+2\cos 2t-\frac{5}{2}\sin 2t\right]$$

8. $\dfrac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}$

$$\left[\textbf{Ans.: } \frac{1}{2}\sin t+\frac{1}{2}t\cos t-te^{-t}\right]$$

9. $\dfrac{1}{s^3(s-1)}$

$$\left[\textbf{Ans.: } 1-t+\frac{t^2}{2}-e^{-t}\right]$$

10. $\dfrac{s}{(s+1)^2(s^2+1)}$

$$\left[\textbf{Ans.: } \frac{1}{2}(\sin t-te^{-t})\right]$$

11. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

$$\left[\textbf{Ans.: } e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t\right]$$

12. $\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}$

$$\left[\textbf{Ans.: } \frac{1}{2}\sin t \sinh t\right]$$

13. $\dfrac{10}{s(s^2-2s+5)}$

$$\left[\textbf{Ans.: } 2 - e^t(2\cos 2t - \sin 2t)\right]$$

14. $\dfrac{s^2+8s+27}{(s+1)(s^2+4s+13)}$

$$\left[\textbf{Ans.: } 2e^{-t} + e^{-2t}(\sin 3t - \cos 3t)\right]$$

15. $\dfrac{2s-1}{s^4+s^2+1}$

$$\left[\textbf{Ans.: } \frac{1}{2}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2}t - \frac{1}{2}e^{-\frac{1}{2}}\cos\frac{\sqrt{3}}{2}t - \frac{5}{2\sqrt{3}}e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2}t\right]$$

16. $\dfrac{s}{s^4+4a^4}$

$$\left[\textbf{Ans.: } \frac{1}{2a^2}\sin at \sinh at\right]$$

17. $\dfrac{s^2}{s^4+4a^4}$

$$\left[\textbf{Ans.: } \frac{1}{2a}\left[\sinh at\cos at + \cosh at \sin at\right]\right]$$

5.14 CONVOLUTION THEOREM

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\,\mathrm{d}u$$

where $\int_0^t f_1(u) f_2(t-u)\,\mathrm{d}u = f_1(t) * f_2(t)$

$f_1(t)*f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$.

Proof: $F_1(s)\cdot F_2(s) = L\{f_1(t)\}\cdot L\{f_2(t)\}$

$$= \int_0^\infty e^{-su} f_1(u)\mathrm{d}u \cdot \int_0^\infty e^{-sv} f_2(v)\mathrm{d}v$$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f_1(u) f_2(v)\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^\infty f_1(u)\left[\int_0^\infty e^{-s(u+v)} f_2(v)\mathrm{d}v\right]\mathrm{d}u$$

Putting $u+v=t,\ \mathrm{d}v=\mathrm{d}t$

When $v=0, \quad t=u$

When $v\to\infty, \quad t\to\infty$

$$F_1(s)\cdot F_2(s) = \int_0^\infty f_1(u)\left[\int_u^\infty e^{-st} f_2(t-u)\mathrm{d}t\right]\mathrm{d}u$$

$$= \int_0^\infty \int_u^\infty e^{-st} f_1(u) f_2(t-u)\mathrm{d}t\,\mathrm{d}u$$

The region of integration is bounded by the lines $u=0$ and $u=t$ (Fig. 5.12). To change the order of integration, draw a vertical strip which starts from the line $u=0$ and terminates on the line $u=t$. Hence, u varies from 0 to t and t varies from 0 to ∞.

$$F_1(s)\cdot F_2(s) = \int_0^\infty e^{-st} \int_0^t f_1(u) f_2(t-u)\mathrm{d}u\,\mathrm{d}t$$

$$= L\left\{\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\right\}$$

Hence, $L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\mathrm{d}u$

Note: The convolution operation is commutative, i.e.,

$$L\left\{\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\right\} = L\left\{\int_0^t f_1(t-u) f_2(u)\mathrm{d}u\right\}$$

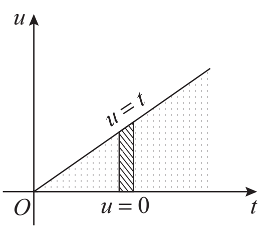


Fig. 5.12 ***Region of integration bounded by line***

Example 1

Evaluate $t * e^t$. **[Summer 2015]**

Solution

Let $f(t)=t \quad g(t)=e^t$

$$f(t)*g(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u$$

$$t*e^t = \int_0^t u\cdot e^{(t-u)}\,\mathrm{d}u$$

$$= e^t\int_0^t u\,e^{-u}\,\mathrm{d}u$$

$$= e^t\left|u\left(\frac{e^{-u}}{-1}\right)-(1)\left(\frac{e^{-u}}{1}\right)\right|_0^t$$

$$= e^t \left| -u\,e^{-u} - e^{-u} \right|_0^t$$

$$= e^t \left[-te^{-t} - e^{-t} + e^0 \right]$$

$$= -t - 1 + e^t$$

Example 2

State the convolution theorem and verify it for $f(t) = t$ and $g(t) = e^{2t}$. **[Winter 2015]**

Solution

If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$ and $L^{-1}\ \{F(s)\} = f(t)$ and $L^{-1}\ \{G(s\} = g(t)$ then

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(u)\, g(t-u)\, \mathrm{d}u = f(t) * g(t)$$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\mathrm{d}u$$

$$= \int_0^t u\, e^{2(t-u)} du$$

$$= e^{2t} \int_0^t u\, e^{-2u} \mathrm{d}u$$

$$= e^{2t} \left| u\left(\frac{e^{-2u}}{-2}\right) - (1)\left(\frac{e^{-2u}}{4}\right) \right|_0^t$$

$$= e^{2t} \left| -\frac{u}{2} e^{-2u} - \frac{e^{-2u}}{4} \right|_0^t$$

$$= e^{2t} \left[-\frac{t}{2} e^{-2t} - \frac{e^{-2t}}{4} + \frac{1}{4} \right]$$

$$= -\frac{t}{2} - \frac{1}{4} + \frac{1}{4} e^{2t}$$

$$= \frac{1}{4}(e^{2t} - 2t - 1)$$

$$
\begin{aligned}
L\{f(t) * g(t)\} &= L\left\{\frac{e^{2t}-2t-1}{4}\right\} \\
&= \frac{1}{4}\left[L\{e^{2t}\}-2L\{t\}-L\{1\}\right] \\
&= \frac{1}{4}\left(\frac{1}{s-2}-\frac{2}{s^2}-\frac{1}{s}\right) \\
&= \frac{1}{4}\left[\frac{s^2-2(s-2)-s(s-2)}{s^2(s-2)}\right] \\
&= \frac{1}{4}\left[\frac{s^2-2s+4-s^2+2s}{s^2(s-2)}\right] \\
&= \frac{1}{4}\left[\frac{4}{s^2(s-2)}\right] \\
&= \frac{1}{s^2(s-2)} \\
L\{f(t)\}\cdot L\{g(t)\} &= L\{t\}\cdot L\{e^{2t}\} \\
&= \frac{1}{s^2}\cdot\frac{1}{s-2} \\
&= \frac{1}{s^2(s-2)}
\end{aligned}
$$

Hence, convolution theorem is verified.

Example 3

Find the inverse Laplace transform of $\dfrac{1}{(s+2)(s-1)}$.

Solution

Let $F(s)=\dfrac{1}{(s+2)(s-1)}$

Let $F_1(s)=\dfrac{1}{s+2}$ $\qquad F_2(s)=\dfrac{1}{s-1}$

$f_1(t)=e^{-2t}$ $\qquad f_2(t)=e^{t}$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-2u}\ e^{t-u}\mathrm{d}u$$

$$= e^t \int_0^t e^{-3u}\,\mathrm{d}u$$

$$= e^t \left| \frac{e^{-3u}}{-3} \right|_0^t$$

$$= \frac{e^t}{3}(1 - e^{-3t})$$

Example 4

Find the inverse Laplace transform of $\dfrac{1}{s^2(s+5)}$.

Solution

Let $F(s) = \dfrac{1}{s^2(s+5)}$

Let $F_1(s) = \dfrac{1}{s^2}$ $\qquad$ $F_2(s) = \dfrac{1}{s+5}$

$f_1(t) = t$ $\qquad$ $f_2(t) = e^{-5t}$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t u\, e^{-5(t-u)}\mathrm{d}u$$

$$= \int_0^t u\, e^{-5t+5u}\mathrm{d}u$$

$$= e^{-5t}\int_0^t u\, e^{5u}\mathrm{d}u$$

$$= e^{-5t}\left| u\frac{e^{5u}}{5} - (1)\frac{e^{5u}}{25} \right|_0^t$$

$$= e^{-5t}\left[\left(t\frac{e^{5t}}{5} - \frac{e^{5t}}{25} \right) - \left(0 - \frac{1}{25} \right) \right]$$

$$= e^{-5t}\left[t\frac{e^{5t}}{5} - \frac{e^{5t}}{25} + \frac{1}{25} \right]$$

$$= \frac{t}{5} - \frac{1}{25} + \frac{e^{-5t}}{25}$$

$$= \frac{1}{25}(e^{-5t} + 5t - 1)$$

Example 5

Find the inverse Laplace transform of $\dfrac{1}{s^2(s+1)^2}$.

Solution

Let $F(s) = \dfrac{1}{s^2(s+1)^2}$

Let $F_1(s) = \dfrac{1}{(s+1)^2}$ $\qquad F_2(s) = \dfrac{1}{s^2}$

$f_1(t) = te^{-t}$ $\qquad f_2(t) = t$

By the convolution theorem,

$$\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t ue^{-u}(t-u)\,\mathrm{d}u \\
&= \int_0^t (ut-u^2)e^{-u}\mathrm{d}u \\
&= \left|(ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\right|_0^t \\
&= te^{-t} + 2e^{-t} + t - 2
\end{aligned}$$

Example 6

Find the inverse Laplace transform of $\dfrac{1}{(s-2)(s+2)^2}$.

Solution

Let $F(s) = \dfrac{1}{(s-2)(s+2)^2}$

Let $F_1(s) = \dfrac{1}{(s+2)^2}$ $\qquad F_2(s) = \dfrac{1}{s-2}$

$f_1(t) = te^{-2t}$ $\qquad f_2(t) = e^{2t}$

By the convolution theorem,

$$\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t ue^{-2u}\; e^{2(t-u)}\mathrm{d}u \\
&= e^{2t}\int_0^t ue^{-4u}\mathrm{d}u \\
&= e^{2t}\left|\frac{ue^{-4u}}{-4} - \frac{e^{-4u}}{16}\right|_0^t \\
&= e^{2t}\left[\frac{-te^{-4t}}{4} - \frac{e^{-4t}}{16} + \frac{1}{16}\right] \\
&= \frac{e^{2t}}{16} - \frac{te^{-2t}}{4} - \frac{e^{-2t}}{16}
\end{aligned}$$

$$= \frac{1}{16}(e^{2t} - e^{-2t} - 4t\, e^{-2t})$$

Example 7

Find the inverse Laplace transform of $\dfrac{1}{(s-2)^4(s+3)}$.

Solution

Let $F(s) = \dfrac{1}{(s-2)^4(s+3)}$

Let $F_1(s) = \dfrac{1}{(s-2)^4}$ $\qquad F_2(s) = \dfrac{1}{s+3}$

$f_1(t) = e^{2t}\dfrac{t^3}{6}$ $\qquad f_2(t) = e^{-3t}$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{2u}\frac{u^3}{6}\, e^{-3(t-u)}\mathrm{d}u$$

$$= \frac{e^{-3t}}{6}\int_0^t u^3 e^{5u}\mathrm{d}u$$

$$= \frac{e^{-3t}}{6}\left| u^3\frac{e^{5u}}{5} - 3u^2\frac{e^{5u}}{25} + 6u\frac{e^{5u}}{125} - 6\frac{e^{5u}}{625}\right|_0^t$$

$$= \frac{e^{-3t}}{6}\left[t^3\frac{e^{5t}}{5} - 3t^2\frac{e^{5t}}{25} + 6t\frac{e^{5t}}{125} - 6\frac{e^{5t}}{625} + \frac{6}{625}\right]$$

$$= \frac{e^{-3t}}{625} + \frac{e^{2t}}{6}\left[\frac{t^3}{5} - \frac{3t^2}{25} + \frac{6t}{125} - \frac{6}{625}\right]$$

Example 8

Find the inverse Laplace transform of $\dfrac{1}{s(s^2+4)}$.

[Winter 2014; Summer 2015]

Solution

Let $F(s) = \dfrac{1}{s(s^2+4)}$

Let $F_1(s) = \dfrac{1}{s^2+4}$ $\qquad F_2(s) = \dfrac{1}{s}$

$f_1(t) = \dfrac{1}{2}\sin 2t$ $\qquad f_2(t) = 1$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \frac{1}{2} \sin 2u \, \mathrm{d}u$$
$$= \frac{1}{2}\left|-\frac{\cos 2u}{2}\right|_0^t$$
$$= \frac{1}{4}(1 - \cos 2t)$$

Example 9

Find the inverse Laplace transform of $\dfrac{1}{s^2(s^2+1)}$.

Solution

Let $F(s) = \dfrac{1}{s^2(s^2+1)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\qquad$ $F_2(s) = \dfrac{1}{s^2}$

$f_1(t) = \sin t$ $\qquad$ $f_2(t) = t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u \, (t-u) \mathrm{d}u$$
$$= \left|(t-u)(-\cos u) - \sin u\right|_0^t$$
$$= t - \sin t$$

Example 10

Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s^2+1)}$.

Solution

Let $F(s) = \dfrac{1}{(s+1)(s^2+1)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\qquad$ $F_2(s) = \dfrac{1}{s+1}$

$f_1(t) = \sin t$ $\qquad$ $f_2(t) = e^{-t}$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u \, e^{-(t-u)} \mathrm{d}u$$

$$= \int_0^t e^{u-t} \sin u \, du$$

$$= e^{-t} \left| \frac{e^u}{2} (\sin u - \cos u) \right|_0^t$$

$$= \frac{e^{-t}}{2} \left[e^t (\sin t - \cos t) + 1 \right]$$

$$= \frac{1}{2} (\sin t - \cos t) + \frac{1}{2} e^{-t}$$

Example 11

Find the inverse Laplace transform of $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

Let $F(s) = \dfrac{s}{(s^2+1)(s^2+4)}$

Let $F_1(s) = \dfrac{1}{s^2+1}$ $\qquad F_2(s) = \dfrac{s}{s^2+4}$

$f_1(t) = \sin t$ $\qquad f_2(t) = \cos 2t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \sin u \cos 2(t-u) du$$

$$= \frac{1}{2} \int_0^t \left[\sin(2t-u) + \sin(3u-2t) \right] du$$

$$= \frac{1}{2} \left| \frac{-\cos(2t-u)}{-1} - \frac{\cos(3u-2t)}{3} \right|_0^t$$

$$= \frac{1}{2} \left| \cos(2t-u) - \frac{1}{3} \cos(3u-2t) \right|_0^t$$

$$= \frac{1}{2} \left[\left(\cos t - \frac{1}{3} \cos t \right) - \left(\cos 2t - \frac{1}{3} \cos 2t \right) \right]$$

$$= \frac{1}{2} \left[\frac{2}{3} \cos t - \frac{2}{3} \cos 2t \right]$$

$$= \frac{1}{3} (\cos t - \cos 2t)$$

Example 12

Find the inverse Laplace transform of $\dfrac{1}{(s^2+a^2)^2}$. **[Summer 2016]**

Solution

Let $$F(s)=\frac{1}{(s^2+a^2)^2}$$

Let $$F_1(s)=F_2(s)=\frac{1}{s^2+a^2}$$

$$f_1(t)=f_2(t)=\frac{1}{a}\sin at$$

By the convolution theorem,

$$\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t \frac{1}{a}\sin au\cdot\frac{1}{a}\sin a(t-u)\mathrm{d}u\\
&= \frac{1}{a^2}\int_0^t \sin au\sin a(t-u)\mathrm{d}u\\
&= \frac{1}{2a^2}\int_0^t 2\sin au\sin a(t-u)\mathrm{d}u\\
&= \frac{1}{2a^2}\int_0^t [\cos(2au-at)-\cos at]\,\mathrm{d}u\\
&= \frac{1}{2a^2}\left|\frac{\sin(2au-at)}{2a}-u\cos at\right|_0^t\\
&= \frac{1}{2a^2}\left[\left(\frac{1}{2a}\sin at-t\cos at\right)-\left(-\frac{\sin at}{2a}\right)\right]\\
&= \frac{1}{2a^2}\left[\frac{1}{2a}\sin at-t\cos at+\frac{1}{2a}\sin at\right]\\
&= \frac{1}{2a^2}\left[\frac{1}{a}\sin at-t\cos at\right]\\
&= \frac{1}{2a^3}[\sin at-at\cos at]
\end{aligned}$$

Example 13

Find the inverse Laplace transform of $\dfrac{1}{(s^2+4)^2}$. **[Summer 2018, 2017]**

Solution

Let $$F(s) = \frac{1}{(s^2+4)^2}$$

Let $$F_1(s) = F_2(s) = \frac{1}{s^2+4}$$

$$f_1(t) = f_2(t) = \frac{1}{2}\sin 2\,t$$

By the convolution theorem,

$$\begin{aligned}
L^{-1}\left\{F(s)\right\} &= \int_0^t \frac{1}{2}\sin 2u.\frac{1}{2}\sin 2(t-u)\mathrm{d}u \\
&= \frac{1}{4}\int_0^t \sin 2u \sin 2(t-u)\mathrm{d}u \\
&= \frac{1}{8}\int_0^t \left[\cos(4u-2t)-\cos 2t\right]\mathrm{d}u \\
&= \frac{1}{8}\left|\frac{\sin(4u-2t)}{4}-(\cos 2t)u\right|_0^t \\
&= \frac{1}{8}\left[\left(\frac{\sin 2t}{4}-t\cos 2t\right)-\left(\frac{-\sin 2t}{4}-0\right)\right] \\
&= \frac{1}{8}\left[\frac{\sin 2t}{4}-t\cos 2t+\frac{\sin 2t}{4}\right] \\
&= \frac{1}{8}\left[\frac{2\sin 2t}{4}-t\cos 2t\right] \\
&= \frac{1}{16}(\sin 2t-2t\cos 2t)
\end{aligned}$$

Example 14

Find the inverse Laplace transform of $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$.

[Winter 2017]

Solution

Let $F(s) = \dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Let $F_1(s) = \dfrac{s}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{s}{s^2+b^2}$

$f_1(t) = \cos at \qquad f_2(t) = \cos bt$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cos b(t-u)\mathrm{d}u$$
$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu) + \cos(au-bt+bu)\mathrm{d}u\right]$$
$$= \frac{1}{2}\int_0^t \left[\cos\{(a-b)u+bt\} + \cos\{(a+b)u-bt\}\right]\mathrm{d}u$$
$$= \frac{1}{2}\left|\frac{\sin\{bt+(a-b)u\}}{a-b} + \frac{\sin\{(a+b)u-bt\}}{a+b}\right|_0^t$$
$$= \frac{1}{2}\left[\left\{\frac{\sin(bt+at-bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b}\right\} - \left(\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)\right]$$
$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$
$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right]$$
$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right]$$
$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Example 15

Find the inverse Laplace transform of $\dfrac{s^2}{(s^2+a^2)^2}$.

Solution

Let $F(s) = \dfrac{s^2}{(s^2+a^2)^2}$

Let $F_1(s) = \dfrac{s}{s^2+a^2} \qquad F_2(s) = \dfrac{s}{s^2+a^2}$

$f_1(t) = \cos at \qquad f_2(t) = \cos at$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cos a(t-u)\mathrm{d}u$$
$$= \frac{1}{2}\int_0^t [\cos at + \cos(2au-at)]\mathrm{d}u$$
$$= \frac{1}{2}\left|u\cos at + \frac{1}{2a}\sin(2au-at)\right|_0^t$$

$$= \frac{1}{2}\left(t \cos at + \frac{1}{a} \sin at \right)$$

$$= \frac{1}{2a}(\sin at + at \cos at)$$

Example 16

Find the inverse Laplace transform of $\dfrac{s}{(s^2+a^2)^2}$.

[Winter 2016, 2014; Summer 2014]

Solution

Let $F(s) = \dfrac{s}{(s^2+a^2)^2}$

Let $F_1(s) = \dfrac{s}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{1}{s^2+a^2}$

$f_1(t) = \cos at$ $\qquad f_2(t) = \dfrac{1}{a} \sin at$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \cos au \cdot \frac{1}{a} \sin a(t-u) \mathrm{d}u$$

$$= \frac{1}{2a} \int_0^t \left[\sin at + \sin a(t-2u)\right] \mathrm{d}u$$

$$= \frac{1}{2a} \left| u \sin at + \frac{1}{2a} \cos a(t-2u) \right|_0^t$$

$$= \frac{1}{2a} t \sin at$$

Example 17

Find the inverse Laplace transform of $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$.

Solution

Let $F(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)}$

Let $F_1(s) = \dfrac{1}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{1}{s^2+b^2}$

$f_1(t) = \dfrac{1}{a} \sin at$ $\qquad f_2(t) = \dfrac{1}{b} \sin bt$

By the convolution theorem,

$$
\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t \frac{1}{a}\sin au \cdot \frac{1}{b}\sin b(t-u)\,\mathrm{d}u \\
&= \frac{1}{ab}\int_0^t \sin au \sin b(t-u)\,\mathrm{d}u \\
&= -\frac{1}{2ab}\int_0^t \left[\cos\{(a-b)u+bt\}-\cos\{(a+b)\,u-bt\}\right]\mathrm{d}u \\
&= -\frac{1}{2ab}\left|\frac{\sin\{(a-b)u+bt\}}{a-b}-\frac{\sin\{(a+b)u-bt\}}{a+b}\right|_0^t \\
&= -\frac{1}{2ab}\left[\frac{\sin at}{a-b}-\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right] \\
&= -\frac{1}{2ab}\left[2b\,\frac{\sin at}{a^2-b^2}-2a\,\frac{\sin bt}{a^2-b^2}\right] \\
&= \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}
\end{aligned}
$$

Example 18

Find the inverse Laplace transform of $\dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$.

Solution

Let $F(s) = \dfrac{s(s+1)}{(s^2+1)\,(s^2+2s+2)}$

Let $F_1(s) = \dfrac{s+1}{s^2+2s+2}$ $\qquad F_2(s) = \dfrac{s}{s^2+1}$

$= \dfrac{s+1}{(s+1)^2+1}$ $\qquad f_2(t) = \cos t$

$f_1(t) = e^{-t}\cos t$

By the convolution theorem,

$$
\begin{aligned}
L^{-1}\{F(s)\} &= \int_0^t e^{-u}\cos u\cos(t-u)\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^t e^{-u}\left[\cos t + \cos(2u-t)\right]\mathrm{d}u \\
&= \frac{1}{2}\left|-e^{-u}\cos t + \frac{e^{-u}}{5}\{-\cos(2u-t)+2\sin(2u-t)\}\right|_0^t
\end{aligned}
$$

$$= \frac{1}{2}\left[-e^{-t}\cos t + \frac{e^{-t}}{5}(-\cos t + 2\sin t) - \frac{1}{5}(-\cos t - 2\sin t) + \cos t\right]$$

$$= \frac{1}{10}\left[e^{-t}(2\sin t - 6\cos t) + (2\sin t + 6\cos t)\right]$$

Example 19

Find the inverse Laplace transform of $\dfrac{1}{(s^2+4s+13)^2}$.

Solution

Let $F(s) = \dfrac{1}{(s^2+4s+13)^2}$

Let $F_1(s) = F_2(s) = \dfrac{1}{s^2+4s+13}$

$$= \frac{1}{(s+2)^2+9}$$

$$f_1(t) = f_2(t) = \frac{e^{-2t}}{3}\sin 3t$$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t \frac{e^{-2u}}{3}\sin 3u \cdot \frac{e^{-2(t-u)}}{3}\sin 3(t-u)\,du$$

$$= \frac{e^{-2t}}{9}\int_0^t \sin 3u \sin 3(t-u)\,du$$

$$= -\frac{e^{-2t}}{18}\int_0^t \left[\cos 3t - \cos(6u-3t)\right]du$$

$$= -\frac{e^{-2t}}{18}\left|u\cos 3t - \frac{\sin(6u-3t)}{6}\right|_0^t$$

$$= -\frac{e^{-2t}}{18}\left[t\cos 3t - \frac{\sin 3t}{6} - \frac{\sin 3t}{6}\right]$$

$$= \frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

Example 20

Find the inverse Laplace transform of $\dfrac{s+2}{(s^2+4s+5)^2}$. **[Summer 2015]**

Solution

Let $$F(s) = \frac{s+2}{(s^2+4s+5)^2}$$

Let $$F_1(s) = \frac{s+2}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1} \qquad F_2(s) = \frac{1}{s^2+4s+5} = \frac{1}{(s+2)^2+1}$$

$$f_1(t) = e^{-2t}\cos t \qquad f_2(t) = e^{-2t}\sin t$$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u}\cos u \cdot e^{-2(t-u)}\sin(t-u)\,\mathrm{d}u$$

$$= e^{-2t}\int_0^t \cos u \sin(t-u)\,\mathrm{d}u$$

$$= e^{-2t}\int_0^t \frac{1}{2}\big[\sin t - \sin(-t+2u)\big]\mathrm{d}u$$

$$= \frac{e^{-2t}}{2}\int_0^t \big[\sin t - \sin(2u-t)\big]\mathrm{d}u$$

$$= \frac{e^{-2t}}{2}\left|\sin t\; u - \left\{\frac{-\cos(2u-t)}{2}\right\}\right|_0^t$$

$$= \frac{e^{-2t}}{2}\left[t\sin t + \frac{1}{2}(\cos t - \cos t)\right]$$

$$= \frac{e^{-2t}}{2}t\sin t$$

Example 21

Find the inverse Laplace transform of $\dfrac{s+2}{(s^2+4s+13)^2}$.

Solution

Let $$F(s) = \frac{s+2}{(s^2+4s+13)^2}$$

Let $$F_1(s) = \frac{s+2}{s^2+4s+13} = \frac{s+2}{(s+2)^2+9} \qquad F_2(s) = \frac{1}{s^2+4s+13} = \frac{1}{(s+2)^2+9}$$

$$f_1(t) = e^{-2t}\cos 3t \qquad f_2(t) = \frac{1}{3}e^{-2t}\sin 3t$$

By the convolution theorem,

$$L^{-1}\left\{F(s)\right\} = \int_0^t e^{-2u}\cos 3u \cdot \frac{1}{3}e^{-2(t-u)}\sin 3(t-u)\mathrm{d}u$$

$$= \frac{e^{-2t}}{3}\int_0^t \cos 3u \sin 3(t-u)\mathrm{d}u$$

$$= \frac{e^{-2t}}{3}\int_0^t \frac{1}{2}\left[\sin 3t - \sin(-3t+6u)\right]\mathrm{d}u$$

$$= \frac{e^{-2t}}{6}\int_0^t \left[\sin 3t - \sin(6u-3t)\right]\mathrm{d}u$$

$$= \frac{e^{-2t}}{6}\left|\sin 3t\ u - \left\{\frac{-\cos(6u-3t)}{6}\right\}\right|_0^t$$

$$= \frac{e^{-2t}}{6}\left[t\sin 3t + \frac{1}{6}(\cos 3t - \cos 3t)\right]$$

$$= \frac{e^{-2t}}{6}t\sin 3t$$

Example 22

Find the inverse Laplace transform of $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$.

Solution

Let $F(s) = \dfrac{(s+2)^2}{(s^2+4s+8)^2}$

Let $F_1(s) = F_2(s) = \dfrac{s+2}{s^2+4s+8}$

$$= \frac{s+2}{(s+2)^2+4}$$

$$f_1(t) = f_2(t) = e^{-2t}\cos 2t$$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u}\cos 2u\, e^{-2(t-u)}\cos 2(t-u)\,\mathrm{d}u$$

$$= e^{-2t}\int_0^t \cos 2u \cos 2(t-u)\,\mathrm{d}u$$

$$= \frac{e^{-2t}}{2}\int_0^t \left[\cos 2t + \cos(4u-2t)\right]\mathrm{d}u$$

$$= \frac{e^{-2t}}{2} \left| u \cos 2t + \frac{\sin(4u-2t)}{4} \right|_0^t$$

$$= \frac{e^{-2t}}{2} \left[t \cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4} \right]$$

$$= \frac{e^{-2t}}{4} \left[\sin 2t + 2t \cos 2t \right]$$

Example 23

Find the inverse Laplace transform of $\dfrac{1}{(s+3)(s^2+2s+2)}$.

Solution

Let $F(s) = \dfrac{1}{(s+3)(s^2+2s+2)}$

Let $F_1(s) = \dfrac{1}{s^2+2s+2}$ $\qquad F_2(s) = \dfrac{1}{s+3}$

$= \dfrac{1}{(s+1)^2+1}$ $\qquad f_2(t) = e^{-3t}$

$f_1(t) = e^{-t} \sin t$

By the convolution theorem,

$$L^{-1}\{F(s)\} = \int_0^t e^{-u} \sin u \, e^{-3(t-u)} \mathrm{d}u$$

$$= e^{-3t} \int_0^t e^{2u} \sin u \, \mathrm{d}u$$

$$= e^{-3t} \left| \frac{e^{2u}}{5} (2\sin u - \cos u) \right|_0^t$$

$$= \frac{e^{-3t}}{5} \left[e^{2t} (2\sin t - \cos t) + 1 \right]$$

$$= \frac{1}{5} \left[e^{-t} (2\sin t - \cos t) + e^{-3t} \right]$$

Example 24

Find the inverse Laplace transform of $\dfrac{1}{s(s+a)^3}$. **[Winter 2013]**

Solution

Let $$F(s)=\frac{1}{s(s+a)^3}$$

Let $$F_1(s)=\frac{1}{(s+a)^3} \qquad F_2(s)=\frac{1}{s}$$

$$f_1(t)=e^{-at}\frac{t^2}{2} \qquad f_2(t)=1$$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-au}\frac{u^2}{2}\mathrm{d}u$$

$$=\frac{1}{2}\left|\frac{u^2e^{-au}}{-a}-\frac{2u\,e^{-au}}{a^2}+\frac{2e^{-au}}{-a^3}\right|_0^t$$

$$=\frac{1}{2}\left[-\frac{t^2e^{-at}}{a}-\frac{2te^{-at}}{a^2}-\frac{2e^{-at}}{a^3}+\frac{2}{a^3}\right]$$

$$=-\frac{1}{2a}t^2e^{-at}-\frac{1}{a^2}te^{-at}-\frac{1}{a^3}e^{-at}+\frac{1}{a^3}$$

Example 25

Find the inverse Laplace transform of $\dfrac{1}{(s^2+4)(s+1)^2}$.

Solution

Let $F(s)=\dfrac{1}{(s^2+4)\,(s+1)^2}$

Considering $F(s)$ as a product of three functions,

$$F(s)=\frac{1}{(s^2+4)}\cdot\frac{1}{s+1}\cdot\frac{1}{s+1}$$

Let $$F_1(s)=\frac{1}{s^2+4} \qquad F_2(s)=\frac{1}{s+1} \qquad F_3(s)=\frac{1}{s+1}$$

$$f_1(t)=\frac{1}{2}\sin 2t \qquad f_2(t)=e^{-t} \qquad f_3(t)=e^{-t}$$

By the convolution theorem,

$$L^{-1}\{F_1(s)\cdot F_2(s)\}=\int_0^t\frac{1}{2}\sin 2u\,e^{-(t-u)}\mathrm{d}u$$

$$= \frac{e^{-t}}{2}\left|\frac{e^{u}}{5}(\sin 2u - 2\cos 2u)\right|_0^t$$

$$= \frac{e^{-t}}{10}\left[e^{t}(\sin 2t - 2\cos 2t) + 2\right]$$

$$= \frac{\sin 2t - 2\cos 2t}{10} + \frac{e^{-t}}{5}$$

$$L^{-1}\{F_1(s)F_2(s)F_3(s)\} = \int_0^t \left[\frac{\sin 2u - 2\cos 2u}{10} + \frac{e^{-u}}{5}\right] e^{-(t-u)}\mathrm{d}u$$

$$= \frac{e^{-t}}{10}\int_0^t \left[e^{u}(\sin 2u - 2\cos 2u) + 2\right]\mathrm{d}u$$

$$= \frac{e^{-t}}{10}\left|\frac{e^{u}}{5}\{(\sin 2u - 2\cos 2u) - 2(\cos 2u + 2\sin 2u)\} + 2u\right|_0^t$$

$$= \frac{e^{-t}}{10}\left[\frac{e^{t}}{5}(-3\sin 2t - 4\cos 2t) + 2t + \frac{4}{5}\right]$$

$$= \frac{2}{25}e^{-t} + \frac{te^{-t}}{5} - \frac{1}{50}(3\sin 2t + 4\cos 2t)$$

EXERCISE 5.23

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{1}{(s+3)(s-1)}$

$$\left[\textbf{Ans. :} \frac{e^{t}}{4}(1 - e^{-4t})\right]$$

2. $\dfrac{1}{s(s^2+4)}$

$$\left[\textbf{Ans. :} \frac{1}{4}(1 - \cos 2t)\right]$$

3. $\dfrac{1}{(s-3)(s+3)^2}$

$$\left[\textbf{Ans. :} \frac{1}{36}(e^{3t} - e^{-3t} - 6te^{-3t})\right]$$

4. $\dfrac{s}{(s^2+4)^2}$

$$\left[\textbf{Ans. :} \frac{1}{4}t\sin 2t\right]$$

5. $\dfrac{s^2}{(s^2-a^2)^2}$

$$\left[\textbf{Ans .: } \frac{1}{2}(\sinh at + at \cosh at)\right]$$

6. $\dfrac{1}{s(s^2-a^2)}$

$$\left[\textbf{Ans .: } \frac{1}{a^2}(\cosh at - 1)\right]$$

7. $\dfrac{1}{s^3(s^2+1)}$

$$\left[\textbf{Ans .: } \frac{t^2}{2} + \cos t - 1\right]$$

8. $\dfrac{s^2}{(s^2+4)^2}$

$$\left[\textbf{Ans .: } \frac{1}{4}(\sin 2t + 2t \cos 2t)\right]$$

9. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

$$\left[\textbf{Ans .: } \frac{1}{3}(2\sin 2t - \sin t)\right]$$

10. $\dfrac{s}{(s^2-a^2)^2}$

$$\left[\textbf{Ans .: } \frac{1}{2a}(at \cosh at + \sinh at)\right]$$

11. $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

$$\left[\textbf{Ans .: } \frac{1}{b^2-a^2}(\sin at - \sin bt)\right]$$

12. $\dfrac{s}{(s^2+a^2)^3}$

$$\left[\textbf{Ans .: } \frac{t}{8a^3}(\sin at - at \cos at)\right]$$

13. $\dfrac{s+3}{(s^2+6s+13)^2}$

$$\left[\textbf{Ans .: } \frac{1}{4}e^{-3t}\, t \sin 2t\right]$$

14. $\dfrac{s}{s^4+8s^2+16}$

$$\left[\textbf{Ans .: } \frac{1}{4} t \sin 2t\right]$$

15. $\dfrac{(s+3)^2}{(s^4+6s+5)^2}$

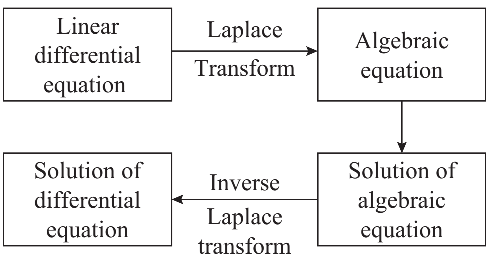
$$\left[\textbf{Ans.: } \frac{1}{4}(2t\cosh 2t + \sinh 2t)\right]$$

16. $\dfrac{1}{s(s+1)(s+2)}$

$$\left[\textbf{Ans.: } \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}\right]$$

5.15 SOLUTION OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

The Laplace transform is useful in solving linear differential equations with given initial conditions by using algebraic methods. Initial conditions are included from the very beginning of the solution.



Example 1

Solve $\dfrac{dy}{dt} = 1,\ y(0) = 0.$

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) = \frac{1}{s}$$

$$sY(s) - 0 = \frac{1}{s} \qquad [\because y(0) = 0]$$

$$Y(s) = \frac{1}{s^2}$$

Taking inverse Laplace transform of both the sides, $y(t) = t$

Example 2

Solve $y' - 3y = 1,\ y(0) = 2.$

Solution

Taking Laplace transform of both the sides,

$$s\,Y(s) - y(0) - 3\,Y(s) = \frac{1}{s}$$

$$s\,Y(s) - 2 - 3\,Y(s) = \frac{1}{s} \qquad [\because y(0) = 2]$$

$$(s-3)\,Y(s) = \frac{1}{s} + 2 = \frac{2s+1}{s}$$

$$Y(s) = \frac{2s+1}{s(s-3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{B}{s-3}$$

$$2s+1 = A(s-3) + Bs \qquad \text{...(1)}$$

Putting $s = 0$ in Eq.(1),

$$1 = -3A$$

$$A = -\frac{1}{3}$$

Putting $s = 3$ in Eq.(1),

$$7 = 3B$$

$$B = \frac{7}{3}$$

$$Y(s) = -\frac{1}{3}\frac{1}{s} + \frac{7}{3}\frac{1}{s-3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = -\frac{1}{3} + \frac{7}{3}e^{3t}$$

Example 3

Solve $\dfrac{dy}{dt} + 2y = e^{-3t},\ y(0) = 1.$

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s+3}$$

$$sY(s) - 1 + 2Y(s) = \frac{1}{s+3} \qquad [\because y = 1]$$

$$(s+2)\,Y(s) = \frac{1}{s+3} + 1 = \frac{s+4}{s+3}$$

$$Y(s) = \frac{s+4}{(s+2)(s+3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+2} + \frac{B}{s+3}$$

$$s + 4 = A(s+3) + B(s+2) \qquad \ldots (1)$$

Putting $s = -2$ in Eq. (1),

$A = 2$

Putting $s = -3$ in Eq. (1),

$B = -1$

$$Y(s) = \frac{2}{s+2} - \frac{1}{s+3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^{-2t} - e^{-3t}$$

Example 4

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}t} + y = \cos 2t,\ y(0) = 1.$

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) + Y(s) = \frac{s}{s^2+4}$$

$$sY(s) - 1 + Y(s) = \frac{s}{s^2+4} \qquad [\because y(0) = 1]$$

$$(s+1)\,Y(s) = \frac{s}{s^2+4} + 1 = \frac{s^2+s+4}{(s^2+4)}$$

$$Y(s) = \frac{s^2+s+4}{(s+1)(s^2+4)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$s^2 + s + 4 = A(s^2+4) + (Bs+C)(s+1) \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$4 = 5A$$

$$A = \frac{4}{5}$$

Equating the coefficients of s^2,

$$1 = A + B$$

$$B = 1 - \frac{4}{5} = \frac{1}{5}$$

Equating the coefficients of s^0,

$$4 = 4A + C$$

$$C = 4 - 4A = 4 - \frac{16}{5} = \frac{4}{5}$$

$$Y(s) = \frac{4}{5}\cdot\frac{1}{s+1} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{4}{5}e^{-t} + \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$$

Example 5

Solve $y'' + 6y' = 1$, $y(0) = 2$, $y'(0) = 0$. **[Winter 2012]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 6\left[sY(s) - y(0)\right] = \frac{1}{s}$$

$$\left[s^2Y(s) - 2s\right] + 6\left[sY(s) - 2\right] = \frac{1}{s}$$

$$(s^2+6s)Y(s) = 2s + 12 + \frac{1}{s}$$

$$Y(s) = \frac{2s+12}{s^2+6s} + \frac{1}{s(s^2+6s)}$$

$$= \frac{2(s+6)}{s(s+6)} + \frac{1}{s^2(s+6)}$$

$$= \frac{2}{s} + \frac{1}{s^2(s+6)}$$

By partial fraction expansion,

$$\frac{1}{s^2(s+6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+6}$$

$$1 = As\,(s + 6) + B\,(s + 6) + Cs^2 \qquad ...(1)$$

Putting $s = 0$ in Eq. (1),

$$1 = 6\,B$$

$$B = \frac{1}{6}$$

Putting $s = -6$ in Eq. (1),

$$1 = 36\,C$$

$$C = \frac{1}{36}$$

Putting $s = 1$ in Eq. (1),

$$1 = 7A + 7B + C$$

$$7A = 1 - \frac{7}{6} - \frac{1}{36}$$

$$= -\frac{7}{36}$$

$$A = -\frac{1}{36}$$

$$Y(s) = \frac{2}{s} - \frac{1}{36}\frac{1}{s} + \frac{1}{6}\frac{1}{s^2} + \frac{1}{36}\frac{1}{(s+6)}$$

$$= \frac{71}{36}\cdot\frac{1}{s} + \frac{1}{6}\cdot\frac{1}{s^2} + \frac{1}{36}\cdot\frac{1}{(s+6)}$$

Taking inverse Laplace transforms of both the sides,

$$y(t) = \frac{71}{36} + \frac{t}{6} + \frac{1}{36}e^{-6t}$$

Example 6

Solve $y'' + 4y' + 8y = 1$, $y(0) = 0$, $y'(0) = 1$.

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 8Y(s) = \frac{1}{s}$$

$$\left[s^2Y(s) - 1\right] + 4sY(s) + 8Y(s) = \frac{1}{s} \qquad \left[\because y(0) = 0,\ y'(0) = 1\right]$$

$$(s^2+4s+8)Y(s)=\frac{1}{s}+1=\frac{s+1}{s}$$

$$Y(s)=\frac{s+1}{s\,(s^2+4s+8)}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s}+\frac{Bs+C}{s^2+4s+8}$$

$$s+1=A(s^2+4s+8)+(Bs+C)s \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$1 = 8A$$

$$A=\frac{1}{8}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B=-\frac{1}{8}$$

Equating the coefficients of s,

$$1=4A+C$$

$$C=1-4A=1-\frac{1}{2}=\frac{1}{2}$$

$$Y(s)=\frac{1}{8}\cdot\frac{1}{s}-\frac{1}{8}\cdot\frac{s}{s^2+4s+8}+\frac{1}{2}\cdot\frac{1}{s^2+4s+8}$$

$$=\frac{1}{8}\cdot\frac{1}{s}-\frac{1}{8}\cdot\frac{(s+2)-2}{(s+2)^2+4}+\frac{1}{2}\cdot\frac{1}{(s+2)^2+4}$$

$$=\frac{1}{8}\cdot\frac{1}{s}-\frac{1}{8}\frac{s+2}{(s+2)^2+4}+\frac{3}{4}\frac{1}{(s+2)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{1}{8}-\frac{1}{8}e^{-2t}\cos 2t+\frac{3}{8}e^{-2t}\sin 2t$$

Example 7

Solve $y''+y=t,\ \ y(0)=1,\ \ y'(0)=0.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+Y(s)=\frac{1}{s^2}$$

$$s^2Y(s) - s + Y(s) = \frac{1}{s^2} \qquad [\because y(0) = 1, y'(0) = 0]$$

$$(s^2+1)\, Y(s) = \frac{1}{s^2} + s = \frac{s^3+1}{s^2}$$

$$Y(s) = \frac{s^3+1}{s^2(s^2+1)}$$

$$= \frac{s}{s^2+1} + \frac{1}{s^2(s^2+1)}$$

$$= \frac{s}{s^2+1} + \frac{s^2+1-s^2}{s^2(s^2+1)}$$

$$= \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \cos t + t - \sin t$$

Example 8

Solve $y'' - 3y' + 2y = 4t,\ y(0) = 1,\ y'(0) = -1.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{4}{s^2}$$

$$s^2Y(s) - s + 1 - 3sY(s) + 3 + 2Y(s) = \frac{4}{s^2} \qquad [\because y(0) = 1, y'(0) = -1]$$

$$(s^2 - 3s + 2)Y(s) - s + 4 = \frac{4}{s^2}$$

$$(s^2 - 3s + 2)Y(s) = \frac{4}{s^2} + s - 4 = \frac{4 + s^3 - 4s^2}{s^2}$$

$$Y(s) = \frac{4 + s^3 - 4s^2}{s^2\left(s^2 - 3s + 2\right)}$$

$$= \frac{s^3 - 4s^2 + 4}{s^2(s-1)(s-2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-2}$$

$$s^3 - 4s^2 + 4 = As(s-1)(s-2) + B(s-1)(s-2) + C(s^2)(s-2) + D(s^2)(s-1) \qquad \cdots(1)$$

Putting $s = 0$ in Eq. (1),

$$4 = 2B$$

$$B = 2$$

Putting $s = 1$ in Eq. (1),

$$1 - 4 + 4 = -C$$

$$C = -1$$

Putting $s = 2$ in Eq. (1),

$$8 - 16 + 4 = D(4)$$

$$D = -1$$

Equating the coefficients of s^3,

$$1 = A + C + D$$

$$A = 3$$

$$Y(s) = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{s-1} - \frac{1}{s-2}$$

Taking inverse Laplace transform of both the sides.

$$y(t) = 3 + 2t - e^t - e^{2t}$$

Example 9

Solve $(D^2 + 9)y = 18t, \;\; y(0) = 0, \;\; y\left(\frac{\pi}{2}\right) = 1.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2\, Y(s) - sy(0) - y'(0)\right] + 9Y(s) = \frac{18}{s^2}$$

Let $y'(0) = A$

$$\mathrm{s^2\, Y(s) - A + 9Y(s)} = \frac{18}{\mathrm{s^2}} \qquad [\because y(0) = 0]$$

$$(s^2 + 9)\, Y(s) = \frac{18}{s^2} + A$$

$$Y(s) = \frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}$$

$$= \frac{18}{9}\left(\frac{1}{s^2} - \frac{1}{s^2+9}\right) + \frac{A}{s^2+9}$$

$$= \frac{2}{s^2} + \frac{A-2}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2t + \frac{A-2}{3}\sin 3t$$

Putting $t = \frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right) = 1$,

$$1 = 2\cdot\frac{\pi}{2} + \frac{A-2}{3}\sin\frac{3\pi}{2}$$

$$= \pi - \frac{A-2}{3}$$

$$3 = 3\pi - A + 2$$

$$A = 3\pi - 1$$

Hence,

$$y(t) = 2t + \frac{3\pi-1-2}{3}\sin 3t$$

$$= 2t + (\pi - 1)\sin 3t$$

Example 10

Solve $y'' + y' = t^2 + 2t,\ \ y(0) = 4,\ \ y'(0) = -2.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2Y(s) - 4s + 2 + sY(s) - 4 = \frac{2}{s^3} + \frac{2}{s^2} \qquad [\because y(0) = 4,\ y'(0) = -2]$$

$$(s^2+s)\,Y(s) = \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2 = \frac{2(1+s)}{s^3} + 4s + 2$$

$$Y(s) = \frac{2(1+s)}{s^3(s^2+s)} + \frac{4s}{s^2+s} + \frac{2}{s^2+s}$$

$$= \frac{2}{s^4} + \frac{4}{s+1} + \frac{2}{s} - \frac{2}{s+1}$$

$$= \frac{2}{s^4} + \frac{2}{s} - \frac{2}{s+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{t^3}{3} + 2 + 2e^{-t}$$

Example 11

Solve $(D^2 - 2D + 1)y = e^t$, $y = 2$ *and* $Dy = -1$ *at* $t = 0$.

Solution

Taking Laplace transform of both the sides,

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] - 2\left[sY(s) - y(0)\right] + Y(s) = \frac{1}{s-1}$$

$$\left[s^2 Y(s) - 2s + 1\right] - 2\left[sY(s) - 2\right] + Y(s) = \frac{1}{s-1} \qquad [\because y(0) = 2, y'(0) = -1]$$

$$(s^2 - 2s + 1)\, Y(s) = \frac{1}{s-1} + 2s - 5$$

$$(s-1)^2\; Y(s) = \frac{1 + 2s(s-1) - 5(s-1)}{s-1}$$

$$Y(s) = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

$$2s^2 - 7s + 6 = A(s-1)^2 + B(s-1) + C \qquad \dots (1)$$

Putting $s = 1$ in Eq. (1),

$$C = 1$$

Equating the coefficients of s^2,

$$A = 2$$

Equating the coefficients of s,

$$-7 = -2A + B$$

$$B = -7 + 4 = -3$$

$$Y(s) = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^t - 3t\, e^t + \frac{t^2}{2} e^t$$

Example 12

Solve the initial-value problem using Laplace transform

$$y'' + 3y' + 2y = e^t,\ y(0) = 1,\ y'(0) = 0$$ **[Summer 2015]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{1}{s-1}$$

$$s^2Y(s) - s + 3sY(s) - 3 + 2Y(s) = \frac{1}{s-1}$$

$$(s^2 + 3s + 2)Y(s) = (s+3) + \frac{1}{(s-1)}$$

$$Y(s) = \frac{s+3}{s^2+3s+2} + \frac{1}{(s-1)(s^2+3s+2)}$$

$$= \frac{s+3}{(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)}$$

$$= \frac{s^2+2s-2}{(s-1)(s+1)(s+2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$s^2 + 2s - 2 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1) \qquad ...(1)$$

Putting $s = 1$ in Eq. (1),

$$1 = 6A$$

$$A = \frac{1}{6}$$

Putting $s = -1$ in Eq. (1),

$$-3 = -2B$$

$$B = \frac{3}{2}$$

Putting $s = -2$ in Eq. (1),

$$-2 = 3C$$

$$C = -\frac{2}{3}$$

$$Y(s) = \frac{1}{6}\cdot\frac{1}{s-1} + \frac{3}{2}\cdot\frac{1}{s+1} - \frac{2}{3}\cdot\frac{1}{s+2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{6}e^{t} + \frac{3}{2}e^{-t} - \frac{2}{3}e^{-2t}$$

Example 13

Solve $y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$. **[Winter 2013]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 3Y(s) = \frac{1}{s+1}$$

$$\left[s^2Y(s) - s - 1\right] + 4\left[sY(s) - 1\right] + 3Y(s) = \frac{1}{s+1} \qquad [\because y(0) = 1, y'(0) = 1]$$

$$(s^2 + 4s + 3)Y(s) - s - 5 = \frac{1}{s+1}$$

$$(s^2 + 4s + 3)Y(s) = s + 5 + \frac{1}{s+1}$$

$$Y(s) = \frac{s+5}{s^2 + 4s + 3} + \frac{1}{(s+1)(s^2 + 4s + 3)}$$

$$= \frac{s+5}{(s+1)(s+3)} + \frac{1}{(s+1)^2(s+3)}$$

$$= \frac{s^2 + 6s + 6}{(s+1)^2(s+3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$s^2 + 6s + 6 = A(s + 1)(s + 3) + B(s + 3) + C(s + 1)^2 \qquad \text{...(1)}$$

Putting $s = -1$ in Eq. (1),

$$1 = 2B$$

$$B = \frac{1}{2}$$

Putting $s = -3$ in Eq. (1),

$$-3 = 4C$$

$$C = -\frac{3}{4}$$

Putting $s = 0$ in Eq. (1),

$$6 = 3A + 3B + C$$

$$A = \frac{7}{4}$$

$$Y(s) = \frac{7}{4}\frac{1}{(s+1)} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{3}{4}\frac{1}{(s+3)}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t}$$

Example 14

Use Laplace transform to solve the following initial value problem

$$y'' - 3y' + 2y = 12e^{-2t},\ y(0) = 2,\ y'(0) = 6$$ **[Summer 2017]**

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{12}{s+2}$$

$$s^2Y(s) - 2s - 6 - 3sY(s) + 6 + 2Y(s) = \frac{12}{s+2}$$

$$(s^2 - 3s + 2)\,Y(s) = \frac{12}{s+2} + 2s$$

$$(s-1)(s-2)\,Y(s) = \frac{12 + 2s^2 + 4s}{s+2}$$

$$Y(s) = \frac{2s^2 + 4s + 12}{(s-1)(s-2)(s+2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$2s^2 + 4s + 12 = A(s-2)(s+2) + B(s-1)(s+2) + C(s-1)(s-2) \tag{1}$$

Putting $s = 1$ in Eq. (1),

$$2 + 4 + 12 = A(-1)(3)$$

$$18 = -3A$$

$$A = -6$$

Putting $s = 2$ in Eq. (1),

$$8 + 8 + 12 = B(1)(4)$$

$$28 = 4B$$

$$B = 7$$

Putting $s = -2$ in Eq. (1),

$$8 - 8 + 12 = C(-3)(-4)$$
$$12 = 12C$$
$$C = 1$$
$$Y(s) = -\frac{6}{s-1} + \frac{7}{s-2} + \frac{1}{s+2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = -6e^t + 7e^{2t} + e^{-2t}$$

Example 15

Solve the equation $y'' - 3y' + 2y = 4t + e^{3t}$, *when* $y(0) = 1$ *and* $y'(0) = -1$.

[Winter 2016; Summer 2016]

Solution

Taking Laplace transform of both the sides,

$$[s^2 Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{4}{s^2} + \frac{1}{s-3}$$
$$s^2 Y(s) - s + 1 - 3sY(s) + 3 + 2Y(s) = \frac{4}{s^2} + \frac{1}{s-3}$$
$$(s^2 - 3s + 2)\, Y(s) - s + 4 = \frac{4}{s^2} + \frac{1}{s-3}$$
$$(s^2 - 3s + 2)\, Y(s) = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$$
$$(s^2 - 3s + 2)\, Y(s) = \frac{4(s-3) + s^2 + s^2\,(s-3)(s-4)}{s^2\left(s-3\right)}$$
$$(s^2 - 3s + 2)\, Y(s) = \frac{4s - 12 + s^2 + s^2\,(s^2 - 7s + 12)}{s^2\left(s-3\right)}$$
$$(s-1)\,(s-2)\,Y(s) = \frac{4s - 12 + s^2 + s^4 - 7s^3 + 12s^2}{s^2\left(s-3\right)}$$
$$Y(s) = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2\left(s-1\right)(s-2)\,(s-3)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-2} + \frac{E}{s-3}$$

$$s^4 - 7s^3 + 13s^2 + 4s - 12$$
$$= As(s-1)(s-2)(s-3) + B(s-1)(s-2)(s-3) + Cs^2(s-2)(s-3) + Ds^2(s-1)(s-3) + E(s-1)(s-2)s^2 \quad \text{...(1)}$$

Putting $s = 0$ in Eq. (1),

$$-12 = B(-1)(-2)(-3)$$
$$-12 = -6B$$
$$B = 2$$

Putting $s = 1$ in Eq. (2),

$$1 - 7 + 13 + 4 - 12 = C(-1)(-2)$$
$$-1 = 2C$$
$$C = -\frac{1}{2}$$

Putting $s = 2$ in Eq. (1),

$$16 - 56 + 52 + 8 - 12 = D(4)(1)(-1)$$
$$8 = -4D$$
$$D = -2$$

Putting $s = 3$ in Eq. (1),

$$81 - 189 + 117 + 12 - 12 = 9E(2)(1)$$
$$198 - 189 = 18E$$
$$9 = 18E$$
$$E = \frac{1}{2}$$

Equating the coefficient of s^4,

$$1 = A + C + D + E$$
$$A = 1 - C - D - E$$
$$= 1 + \frac{1}{2} + 2 - \frac{1}{2}$$
$$= 3$$

$$Y(s) = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{2}\cdot\frac{1}{s-1} - 2\cdot\frac{1}{s-2} + \frac{1}{2}\cdot\frac{1}{s-3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 3 + 2t + \frac{1}{2}(e^{3t} - e^t) - 2e^{2t}$$

Example 16

Solve $y'' + 9y = \cos 2t,\ y(0) = 1,\ y\left(\dfrac{\pi}{2}\right) = -1.$

Solution

Taking Laplace transform of both the sides,

$$[s^2 Y(s) - s\,y(0) - y'(0)] + 9Y(s) = \frac{s}{s^2+4}$$

Let $y'(0) = A$

$$s^2 Y(s) - s - A + 9Y(s) = \frac{s}{s^2+4} \qquad [\because y(0) = 1]$$

$$(s^2+9)\ Y(s) = \frac{s}{s^2+4} + s + A$$

$$Y(s) = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{s}{5}\left[\frac{(s^2+9)-(s^2+4)}{(s^2+4)(s^2+9)}\right] + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{1}{5}\cdot\frac{s}{s^2+4} - \frac{1}{5}\cdot\frac{s}{s^2+9} + \frac{s}{s^2+9} + \frac{A}{s^2+9}$$

$$= \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9} + \frac{A}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

Putting $t = \dfrac{\pi}{2}$ and $y\left(\dfrac{\pi}{2}\right) = -1$,

$$-1 = -\frac{1}{5} - \frac{A}{3}$$

$$A = \frac{12}{5}$$

Hence, $y\,(t) = \dfrac{1}{5}\cos 2t + \dfrac{4}{5}\cos 3t + \dfrac{4}{5}\sin 3t$

Example 17

Solve $\dfrac{d^2y}{dt^2} + y = \sin 2t,\ y(0) = 0,\ y'(0) = 0.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{2}{s^2+4}$$

$$s^2Y(s) + Y(s) = \frac{2}{s^2+4} \qquad \left[\because y(0) = 0,\ y'(0) = 0\right]$$

$$(s^2+1)Y(s) = \frac{2}{s^2+4}$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)}$$

$$= \frac{2}{3}\left[\frac{(s^2+4)-(s^2+1)}{(s^2+4)(s^2+1)}\right]$$

$$= \frac{2}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

$$= \frac{2}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t$$

Example 18

Solve $\dfrac{d^2y}{dt^2} + y = \sin t,\ y(0) = 1,\ y'(0) = 0.$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{1}{s^2+1}$$

$$s^2Y(s) - s - 0 + Y(s) = \frac{1}{s^2+1} \qquad [\because y(0) = 1, y'(0) = 0]$$

$$(s^2+1)\,Y(s) - s = \frac{1}{s^2+1}$$

$$(s^2+1)\,Y(s) = \frac{1}{s^2+1} + s$$

$$Y(s) = \frac{1}{(s^2+1)^2} + \frac{s}{s^2+1}$$

$$L^{-1}\left\{Y(s)\right\} = L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} + L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

Let

$$F(s) = \frac{1}{(s^2+1)^2}$$

$$F_1(s) = F_2(s) = \frac{1}{s^2+1}$$

$$f_1(t) = f_2(t) = \sin t$$

$$L^{-1}\{F(s)\} = \int_0^t \sin u \sin(t-u)\,\mathrm{d}u$$

$$= \int_0^t \frac{1}{2}[\cos(2u-t) - \cos t]\,\mathrm{d}u$$

$$= \frac{1}{2}\left|\frac{\sin(2u-t)}{2} - (\cos t)u\right|_0^t$$

$$= \frac{1}{2}\left[\frac{\sin t}{2} - t\cos t - \frac{\sin(-t)}{2}\right]$$

$$= \frac{1}{2}[\sin t - t\cos t]$$

$$L^{-1}\{Y(s)\} = \frac{1}{2}(\sin t - t\cos t) + \cos t$$

Example 19

Solve $y'' + y = \sin 2t, \quad y(0) = 2, \quad y'(0) = 1.$

[Winter 2014; Summer 2018]

Solution

Taking Laplace transform of both the sides,

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{2}{s^2+4}$$

$$\left[s^2Y(s) - 2s - 1\right] + Y(s) = \frac{2}{s^2+4} \qquad [\because\ y(0)=2,\ y'(0)=1]$$

$$(s^2+1)Y(s) = 2s + 1 + \frac{2}{s^2+4}$$

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$$

$$= \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\left[\frac{(s^2+4)-(s^2+1)}{(s^2+1)(s^2+4)}\right]$$

$$= \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

$$= \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Example 20

Solve $y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, y'(0) = 6.$

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - s\ y(0) - y'(0)] - 6\ [sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$[s^2Y(s) - 2s - 6] - 6\ [sY(s) - 2] + 9Y(s) = \frac{2}{(s-3)^3} \qquad [\because y(0) = 2, y'(0) = 6]$$

$$(s^2 - 6s + 9)\ Y(s) = \frac{2}{(s-3)^3} + 2s - 6$$

$$(s-3)^2\ Y(s) = \frac{2}{(s-3)^3} + 2(s-3)$$

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2}{s-3}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = 2e^{3t}\frac{t^4}{4!} + 2e^{3t}$$

$$= \frac{1}{12}t^4e^{3t} + 2e^{3t}$$

Example 21

Solve the initial value problem **[Winter 2015]**

$$y'' - 2y' = e^t \sin t, \quad y(0) = y'(0) = 0, \text{ using Laplace transform.}$$

Solution

Taking Laplace transform of both the sides,

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] - 2[sY(s) + y(0)] = \frac{1}{(s-1)^2 + 1}$$

$$s^2 Y(s) - 2sY(s) = \frac{1}{s^2 - 2s + 2}$$

$$Y(s)\{s(s-2)\} = \frac{1}{s^2 - 2s + 2}$$

$$Y(s) = \frac{1}{s(s-2)(s^2 - 2s + 2)}$$

By partial fraction expansion,

$$Y(s) = \ = \frac{A}{s} + \frac{B}{(s-2)} + \frac{Cs + D}{s^2 - 2s + 2}$$

$$1 = A(s-2)(s^2 - 2s + 2) + Bs(s^2 - 2s + 2) + (Cs + D)\, s(s-2) \quad \text{...(1)}$$

Putting $s = 0$ in Eq. (1),

$$A = -\frac{1}{4}$$

Putting $s = 2$ in Eq. (1),

$$B = \frac{1}{4}$$

Equating the coefficients of s^3,

$$A + B + C = 0$$

$$-\frac{1}{4} + \frac{1}{4} + C = 0$$

$$C = 0$$

Equating the coefficients of s,

$$6A + 2B - 2D = 0$$

$$-\frac{3}{2} + \frac{1}{2} = 2D$$

$$-1 = 2D$$

$$D = -\frac{1}{2}$$

$$Y(s) = -\frac{1}{4}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s-2} - \frac{1}{2}\cdot\frac{1}{s^2-2s+2}$$

Taking inverse Laplace transform both the sides,

$$y(t) = -\frac{1}{4} + \frac{1}{4}e^{2t} - \frac{1}{2}e^{t}\sin t$$

Example 22

Solve $(D^2+2D+5)y = e^{-t}\sin t,\ \ y(0)=0,\ \ y'(0)=1.$

Solution

Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = \frac{1}{(s+1)^2+1}$$

$$s^2Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{1}{s^2+2s+2} \qquad [\because y(0)=0,\ y'(0)=1]$$

$$(s^2+2s+5)\ Y(s) = \frac{1}{s^2+2s+2} + 1$$

$$= \frac{s^2+2s+3}{s^2+2s+2}$$

$$Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

By partial fraction expansion,

$$Y(s) = \frac{\mathrm{As+B}}{\mathrm{s^2+2s+2}} + \frac{\mathrm{Cs+D}}{(\mathrm{s^2+2s+5})}$$

$$\mathrm{s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)}$$

$$= \mathrm{(A+C)\ s^3 + (2A+B+2C+D)s^2}$$

$$+(5A+2B+2C+2D)s + (5B+2D)$$

Equating the coefficients of s^3, s^2, s, and s^0,

$$A + C = 0$$

$$2A + B + 2C + D = 1$$

$$5A + 2B + 2C + 2D = 2$$

$$5B + 2D = 3$$

Solving these equations,

$$A = 0,\ B = \frac{1}{3},\ C = 0,\ D = \frac{2}{3}$$

$$Y(s) = \frac{1}{3}\cdot\frac{1}{s^2+2s+2} + \frac{2}{3}\cdot\frac{1}{s^2+2s+5}$$

$$= \frac{1}{3}\cdot\frac{1}{(s+1)^2+1} + \frac{2}{3}\cdot\frac{1}{(s+1)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t$$

$$= \frac{e^{-t}}{3}(\sin t + \sin 2t)$$

Example 23

Solve the following initial value problem using Laplace transform

$y''' + 2y'' - y' - 2y = 0,\ y(0) = 1,\ y'(0) = 2,\ y''(0) = 2$ **[Winter 2017]**

Solution

Taking Laplace transform of both the sides,

$$\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] + 2\left[s^2Y(s) - sy(0) - y'(0)\right] - [sY(s) - y(0)] - 2Y(s) = 0$$

$$\left[s^3Y(s) - s^2 - 2s - 2\right] + 2\left[s^2Y(s) - s - 2\right] - [sY(s) - 1] - 2Y(s) = 0$$

$$(s^3 + 2s^2 - s - 2)Y(s) = s^2 + 4s + 5$$

$$Y(s) = \frac{s^2+4s+5}{s^3+2s^2-s-2}$$

$$= \frac{s^2+4s+5}{(s-1)(s+1)(s+2)}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$s^2 + 4s + 5 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1) \qquad \text{...(1)}$$

Putting $s = 1$ in Eq. (1),

$$10 = A(2)(3)$$

$$10 = 6A$$

$$A = \frac{5}{3}$$

Putting $s = -1$ in Eq. (1),

$$2 = B(-2)(1)$$
$$2 = -2B$$
$$B = -1$$

Putting $s = -2$ in Eq. (1),

$$1 = C(-3)(-1)$$
$$1 = 3C$$
$$C = \frac{1}{3}$$
$$Y(s) = \frac{5}{3}\cdot\frac{1}{s-1} - \frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s+2}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}$$

Example 24

Solve $\dfrac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t,\ y(0) = 1.$

Solution

Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1}$$
$$sY(s) - 1 + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1} \qquad [\because y(0) = 1]$$
$$\left(s + 2 + \frac{1}{s}\right)Y(s) = \frac{1}{s^2+1} + 1$$
$$= \frac{s^2+2}{s^2+1}$$
$$\frac{s^2+2s+1}{s}Y(s) = \frac{s^2+2}{s^2+1}$$
$$Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+2s+1)}$$
$$= \frac{s(s^2+2)}{(s^2+1)(s+1)^2}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$s(s^2+2) = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 \qquad \ldots (1)$$

Putting $s = -1$ in Eq. (1),

$$-3 = 2B$$

$$B = -\frac{3}{2} \qquad \ldots (2)$$

Equating the coefficients of s^0,

$$0 = A + B + D \qquad \ldots (3)$$

Equating the coefficients of s^3,

$$1 = A + C \qquad \ldots (4)$$

Equating the coefficients of s^2,

$$0 = A + B + 2C + D \qquad \ldots (5)$$

Solving Eqs (2), (3), (4), and (5),

$$A = 1,\ C = 0,\ D = \frac{1}{2}$$

$$Y(s) = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = e^{-t} - \frac{3}{2}e^{-t}t + \frac{1}{2}\sin t$$

EXERCISE 5.24

Using Laplace transform, solve the following differential equations:

1. $y' + 4y = 1,\ y(0) = -3$

$$\left[\textbf{Ans.: } y(t) = \frac{1}{4} - \frac{13}{4}e^{-4t}\right]$$

2. $y' + 6y = e^{4t},\ y(0) = 2$

$$\left[\textbf{Ans.: } y(t) = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}\right]$$

3. $y' + 4y = \cos t,\ y(0) = 0$

$$\left[\textbf{Ans.: } y(t) = -\frac{4}{17}e^{-4t} + \frac{4}{17}\cos t + \frac{1}{17}\sin t\right]$$

4. $y' + 3y = 10\sin t,\ y(0) = 0$

$$\left[\textbf{Ans.: } y(t) = e^{-3t} - \cos t + 3\sin t\right]$$

5. $y' + 0.2y = 0.01t,\ y(0) = -0.25$

[Ans.: $y(t) = 0.05\,t - 0.25$**]**

6. $y' - 2y = 1 - t,\ y(0) = 1$

$\left[\textbf{Ans.}: y(t) = -\frac{1}{4} + \frac{1}{2}t + \frac{5}{4}e^{2t}\right]$

7. $y'' + 5y' + 4y = 0,\ y(0) = 1, y'(0) = -1$

$\left[\textbf{Ans.}: y(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}\right]$

8. $y'' + 2y' - 3y = 6e^{-2t},\ y(0) = 2, y'(0) = -14$

$\left[\textbf{Ans.}: y(t) = -2e^{-2t} + \frac{11}{2}e^{-3t} - \frac{3}{2}e^{t}\right]$

9. $y'' - 4y' + 4y = 1,\ y(0) = 1, y'(0) = 4$

$\left[\textbf{Ans.}: y(t) = \frac{1}{4} + \frac{3}{4}e^{2t} + \frac{5}{2}te^{2t}\right]$

10. $y'' - 4y' + 3y = 6t - 8,\ y(0) = 0,\ y'(0) = 0$

$\left[\textbf{Ans.}: y(t) = 2t + e^{t} - e^{3t}\right]$

11. $y'' + 2y' + y = 3te^{-t},\ y(0) = 4,\ y'(0) = 2$

$\left[\textbf{Ans.}: y(t) = 4e^{-t} + 6t\,e^{-t} + \frac{t^3}{2}e^{-t}\right]$

12. $y'' + y = \sin t \cdot \sin 2t,\ y(0) = 1,\ y'(0) = 0$

$\left[\textbf{Ans.}: y(t) = \frac{15}{16}\cos t + \frac{t}{4}\sin t + \frac{1}{16}\cos 3t\right]$

13. $y'' + y = e^{-2t}\sin t,\ y(0) = 0, y'(0) = 0$

$\left[\textbf{Ans.}: y(t) = \frac{1}{8}\sin t - \frac{1}{8}\cos t + \frac{1}{8}e^{-2t}\sin t + \frac{1}{8}e^{-2t}\cos t\right]$

14. $y'' + y = t\cos 2t,\ y(0) = 0, y'(0) = 0$

$\left[\textbf{Ans.}: y(t) = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{1}{3}t\cos 2t\right]$

15. $y' + y - 2\int_0^t y\,dt = \dfrac{t^2}{2}$, $y(0) = 1$, $y'(0) = -2$

$$\left[\textbf{Ans.}: y(t) = \frac{1}{3}e^{t} + \frac{11}{12}e^{-2t} - \frac{1}{2}t - \frac{1}{4}\right]$$

Points to Remember

Laplace Transform

If $f(t)$ is a function of t defined for all $t \geq 0$ then $\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$ is defined as the Laplace transform of $f(t)$, provided the integral exists and is denoted by $L\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

Sufficient Conditions for Existence of Laplace Transform

The Laplace transform of the function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous, i.e., $f(t)$ is continuous in every subinterval and $f(t)$ has finite limits at the end points of each subinterval.

(ii) $f(t)$ is of exponential order of α, i.e., there exists M, α such that $|f(t)| \leq \mathrm{M}e^{\alpha t}$, for all $t \geq 0$. In other words,

$$\lim_{t\to\infty}\left\{e^{-\alpha t} f(t)\right\} = \text{finite quantity}$$

Properties of Laplace Transform

(i) Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then

$$L\{a\,f_1(t) + b\,f_2(t)\} = a\,F_1(s) + b\,F_2(s)$$

where a and b are constants.

(ii) Change of Scale

If $L\left\{f(t)\right\} = F(s)$ then $L\left\{f(at)\right\} = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$.

(iii) First Shifting Theorem

If $L\left\{f(t)\right\} = F(s)$ then $L\left\{e^{-at} f(t)\right\} = F(s+a)$.

(iv) Second Shifting Theorem

If $L\{f(t)\} = F(s)$

and
$$g(t) = f(t-a) \quad t > a$$
$$= 0 \quad t < a$$

then $L\{g(t)\} = e^{-as} F(s)$

(v) Differentiation of Laplace transform (Multiplication by t)

$$\text{If } L\{f(t)\} = F(s) \text{ then } L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

(vi) Integration of Laplace Transform (Division by t)

$$\text{If } L\{f(t)\} = F(s) \text{ then } L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds.$$

(vii) Laplace Transforms of Derivatives

If $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = s\,F(s) - f(0)$

$L\{f''(t)\} = s^2 F(s) - s\,f(0) - f'(0)$

In general,

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$$

(viii) Laplace Transforms of Integrals

$$\text{If } L\{f(t)\} = F(s) \text{ then } L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}.$$

Laplace Transform of Periodic Functions

If $f(t)$ is a piecewise continuous periodic function with period T then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$$

Convolution Theorem

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then

$$L^{-1}\{F_1(s) \cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\,du$$

where $\int_0^t f_1(u) f_2(t-u)\,du = f_1(t) * f_2(t)$

$f_1(t)*f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$.

Table of Laplace Transforms

Sr. No.	$f(t)$	$F(s)$
1	k	$\frac{k}{s}$
2	t	$\frac{1}{s^2}$
3	t^n	$\frac{\lfloor n+1}{s^{n+1}}$
4	e^{at}	$\frac{1}{s-a}$
5	$\sin at$	$\frac{a}{s^2+a^2}$
6	$\cos at$	$\frac{s}{s^2+a^2}$
7	$\sinh at$	$\frac{a}{s^2-a^2}$
8	$\cosh at$	$\frac{s}{s^2-a^2}$
9	$e^{-bt}\sin at$	$\frac{a}{(s+b)^2+a^2}$
10	$e^{-bt}\cos at$	$\frac{s+b}{(s+b)^2+a^2}$
11	$e^{-bt}\sinh at$	$\frac{a}{(s+b)^2-a^2}$
12	$e^{-bt}\cosh at$	$\frac{s+b}{(s+b)^2-a^2}$
13	$u(t)$	$\frac{1}{s}$
14	$u(t-a)$	$\frac{e^{-as}}{s}$
15	$\delta(t)$	1
16	$\delta(t-a)$	e^{-as}

Multiple Choice Questions

Select the most appropriate response out of the various alternatives given in each of the following questions:

1. For a periodic function $f(t)$ with fundamental period P, its Laplace transform is **[Winter 2015]**

(a) $\dfrac{1}{1-e^{-Ps}}\displaystyle\int_0^P e^{-st} f(t)\,dt$ (b) $\dfrac{1}{1+e^{-Ps}}\displaystyle\int_0^P e^{-st} f(t)\,dt$

(c) $\dfrac{1}{1-e^{Ps}}\displaystyle\int_0^P e^{-st} f(t)dt$ (d) $\dfrac{1}{1+e^{Ps}}\displaystyle\int_0^P e^{-st} f(t)dt$

2. If $L[f(t)]=\dfrac{s}{(s-3)^2}$, then $L\{e^{-3t}f(t)\}$ is **[Winter 2015]**

(a) $\dfrac{s-3}{s^2}$ (b) $\dfrac{s+3}{s}$ (c) $\dfrac{s+3}{s^2}$ (d) $\dfrac{s-3}{s}$

3. $L\{(2t-1)^2\} =$ **[Winter 2015]**

(a) $\dfrac{8}{s^3}+\dfrac{4}{s^2}-\dfrac{1}{s}$ (b) $\dfrac{8}{s^3}-\dfrac{4}{s^2}-\dfrac{1}{3}$

(c) $\dfrac{8}{s^3}+\dfrac{4}{s^2}+\dfrac{1}{s}$ (d) $\dfrac{8}{s^3}-\dfrac{4}{s^2}+\dfrac{1}{s}$

4. $L^{-1}\left\{\dfrac{1}{(s+a)^2}\right\}=$ **[Summer 2016]**

(a) e^{-at} (b) te^{-at} (c) t^2e^{-at} (d) te^{at}

5. If $f(t)$ is a periodic function with period t, then $L\{f(t)\}$ is **[Winter 2016; Summer 2016]**

(a) $\displaystyle\int_0^\infty e^{st} f(t)\,dt$ (b) $\displaystyle\int_0^\infty e^{-st} f(t)\,dt$

(c) $\displaystyle\int_0^\infty e^{-2st} f(t)\,dt$ (d) $\displaystyle\int_0^\infty e^{2st} f(t)dt$

6 Laplace transform of $t^{-\frac{1}{2}}$ is **[Winter 2016]**

(a) $\dfrac{\pi}{5}$ (b) $\sqrt{\dfrac{\pi}{s}}$ (c) $\dfrac{\pi}{\sqrt{s}}$ (d) $\dfrac{\sqrt{\pi}}{s}$

7. If $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$, then $L\left\{\frac{\sin at}{t}\right\}$ is **[Winter 2016]**

(a) $\tan^{-1}(s)$ (b) $\tan^{-1}\left(\frac{s}{a}\right)$ (c) $\tan^{-1}\left(\frac{a}{s}\right)$ (d) $\tan^{-1}\left(\frac{1}{s}\right)$

8. If $u(t)$ is a unit step function, $L\{u(t-a)\} =$

(a) $\frac{e^{as}}{s^2}$ (b) $\frac{e^{-as}}{s^2}$ (c) $\frac{e^{-as}}{s}$ (d) $\frac{e^{as}}{s}$

9. Laplace transform of the unit impulse function $s(t-a)$ is

(a) e^{as} (b) e^{-as} (c) e^{s} (d) e^{-s}

10. $L\{f'(t)\} =$

(a) $s\,F(s) - f(0)$ (b) $s\,F(s) + f(0)$

(c) $F(s) - F(0)$ (d) $F(s) + f(0)$

11. If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s-a)\} =$

(a) $e^{-at} f(t)$ (b) $e^{t} f(t)$ (c) $e^{at} f(t)$ (d) $e^{-t} f(t)$

12. If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} =$

(a) $\frac{1}{a} f\left(\frac{t}{a}\right)$ (b) $a\,f\left(\frac{t}{a}\right)$ (c) $\frac{1}{a} f(t)$ (d) $\frac{1}{a} f(at)$

13. $L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} =$

(a) $\frac{t}{2}\sin t$ (b) $t \sin t$ (c) $t^2 \sin t$ (d) $\frac{t^2}{2}\cos t$

14. Using Laplace transform, the equation $(D^2 + 9)y = \cos 2t$ can be written as $(s^2 + 9)\,Y(s) - s\,y(0) - y'(0) =$

(a) $\frac{s}{s^2+2}$ (b) $\frac{s}{s^2+4}$ (c) $\frac{s}{s+2}$ (d) $\frac{s}{s+4}$

15. The value of $L\{e^{3t+3}\}$ is **[Summer 2017]**

(a) $\frac{e^3}{s+3}$ (b) $\frac{e^3}{s-3}$ (c) $\frac{e^3}{s}$ (d) $\frac{e^3}{s^2-3}$

Answers

1. (a)	2. (c)	3. (d)	4. (b)	5. (b)	6. (b)	7. (c)	8. (c)
9. (b)	10. (a)	11. (c)	12. (a)	13. (b)	14. (b)	15. (b)	