# Differential Calculus I

Chapter 2

## 2.1 INTRODUCTION

Differential calculus is the study of derivative, i.e., the study of change of functions w.r.t. the change in inputs. It is the mathematical study of change, motion, growth or decay, etc. In this chapter, we will study successive differentiation, mean value theorems, such as Rolle's theorem, Lagrange's mean value theorem, Cauchy' mean value theorem, expansion of functions and indeterminate forms.

## 2.2 SUCCESSIVE DIFFERENTIATION

If $y = f(x)$ be a differentiable function of $x$, then its derivative $\frac{dy}{dx}$ is called the first order derivative of $y$ and is in general a function of $x$. If $\frac{dy}{dx}$ is differentiable, then its derivative is called the second order derivative of $y$ and is denoted by $\frac{d^2y}{dx^2}$. Similarly, the derivative of $\frac{d^2y}{dx^2}$ is called the third order derivative of $y$ and is denoted by $\frac{d^3y}{dx^3}$ and so on.

The successive differential coefficients of the function $y = f(x)$ are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}, \ldots$$

Alternative methods of writing the differential coefficients are

$Dy, D^2y, D^3y, \ldots\ldots, D^ny, \ldots$ where $D = \frac{d}{dx}$

$f'(x), f''(x), f'''(x), \ldots, f^n(x), \ldots$

$y'(x), y''(x), y'''(x), \ldots, y^n(x), \ldots$

$y_1(x), y_2(x), y_3(x), \ldots, y_n(x), \ldots$

The value of $n^{\text{th}}$ differential coefficient at $x = a$ is denoted by

$$\left(\frac{d^n y}{dx^n}\right)_{x=a} \quad \text{or} \quad (y_n)_a \quad \text{or} \quad f^n(a) \quad \text{or} \quad y^n(a).$$

## 2.2.1 $n^{\text{th}}$ Order Derivative of Some Standard Functions

**1. $y = (ax + b)^m$, where $m$ is any real number.**

**Proof:** $y = (ax + b)^m$

Differentiating w.r.t. $x$ successively,

$$y_1 = ma\,(ax + b)^{m-1}$$
$$y_2 = m\,(m - 1)\,a^2\,(ax + b)^{m-2}$$
$$y_3 = m\,(m - 1)\,(m - 2)\,a^3\,(ax + b)^{m-3},$$

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$$y_n = m\,(m - 1)\,(m - 2)\,\ldots\,(m - n + 1)\,a^n\,(ax + b)^{m-n}$$

Hence, $$\frac{d^n}{dx^n}(ax+b)^m = m\,(m - 1)\,(m - 2)\,\ldots\,(m - n + 1)\,a^n\,(ax + b)^{m-n}$$

$$= \frac{m(m-1)\ldots(m-n+1)[(m-n)(m-n-1)\ldots 3\cdot 2\cdot 1]a^n(ax+b)^{m-n}}{(m-n)(m-n-1)\ldots 3\cdot 2\cdot 1}$$

$$= \frac{a^n m!(ax+b)^{m-n}}{(m-n)!}, \qquad \text{if } n < m$$

$$= n!\,a^n, \qquad \text{if } n = m$$

$$= 0, \qquad \text{if } n > m$$

**2. $y = (ax + b)^{-m}$, where $m$ is any positive integer.**

**Proof:** $y = (ax + b)^{-m}$

Differentiating w.r.t. $x$ successively,

$$y_1 = (-1)\,ma\,(ax + b)^{-m-1}$$
$$y_2 = (-1)^2\,m\,(m + 1)\,a^2\,(ax + b)^{-m-2}$$
$$y_3 = (-1)^3\,m\,(m + 1)\,(m + 2)\,a^3\,(ax + b)^{-m-3}$$

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$$y_n = (-1)^n\,m\,(m + 1)\,(m + 2)\,\ldots\,(m + n - 1)\,a^n\,(ax + b)^{-m-n}$$

$$= (-1)^n\,\frac{(m+n-1)\ldots m(m-1)(m-2)\ldots 2\cdot 1}{(m-1)(m-2)\ldots 2\cdot 1}\,\frac{a^n}{(ax+b)^{m+n}}$$

Hence, $$\frac{d^n}{dx^n}(ax+b)^{-m} = (-1)^n\,\frac{(m+n-1)!}{(m-1)!}\,\frac{a^n}{(ax+b)^{m+n}}.$$

**Corollary 1:** Putting $m = 1$, we get $\dfrac{d^n}{dx^n}(ax+b)^{-1} = (-1)^n n!\dfrac{a^n}{(ax+b)^{1+n}}$.

**3.** $\boldsymbol{y = \log(ax+b)}$

**Proof:** $y = \log(ax+b)$

Differentiating w.r.t. $x$,

$$y_1 = \frac{a}{ax+b}$$

Differentiating $(n-1)$ times w.r.t. $x$,

$$\frac{d^{n-1}}{dx^{n-1}}y_1 = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{a}{ax+b}\right)$$

$$\frac{d^{n-1}}{dx^{n-1}}\left(\frac{dy}{dx}\right) = \frac{a(-1)^{n-1}(n-1)!a^{n-1}}{(ax+b)^n}$$

Hence, $$\frac{d^n}{dx^n}\log(ax+b) = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

**4.** $\boldsymbol{y = e^{ax}}$

**Proof:** $y = e^{ax}$

Differentiating w.r.t. $x$ successively,

$$y_1 = ae^{ax}$$
$$y_2 = a^2e^{ax}$$
$$y_3 = a^3e^{ax}$$

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$$y_n = a^ne^{ax}$$

Hence, $$\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$$

**5.** $\boldsymbol{y = a^{mx}}$

**Proof:** $y = a^{mx}$

Differentiating w.r.t. $x$ successively,

$$y_1 = ma^{mx}\log a$$
$$y_2 = m^2a^{mx}(\log a)^2$$
$$y_3 = m^3a^{mx}(\log a)^3$$

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$$y_n = m^na^{mx}(\log a)^n$$

Hence, $$\frac{d^n}{dx^n}(a^{mx}) = m^n a^{mx}(\log a)^n$$

**6.** $\boldsymbol{y = \sin(ax + b)}$

**Proof:** $y = \sin(ax + b)$

Differentiating w.r.t. $x$ successively,

$$y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2} + ax + b\right)$$

$$y_2 = a^2\cos\left(\frac{\pi}{2} + ax + b\right) = a^2\sin\left(\frac{2\pi}{2} + ax + b\right)$$

$$y_3 = a^3\cos\left(\frac{2\pi}{2} + ax + b\right) = a^3\sin\left(\frac{3\pi}{2} + ax + b\right)$$

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$$y_n = a^n\sin\left(\frac{n\pi}{2} + ax + b\right)$$

Hence, $\dfrac{d^n}{dx^n}[\sin(ax+b)] = a^n\sin\left(\dfrac{n\pi}{2} + ax + b\right)$

**Corollary 2:** $\dfrac{d^n}{dx^n}[\cos(ax+b)] = a^n\cos\left(\dfrac{n\pi}{2} + ax + b\right)$

**7.** $\boldsymbol{y = e^{ax}\cos(bx + c)}$

**Proof:** $y = e^{ax}\cos(bx + c)$

Differentiating w.r.t. $x$,

$$y_1 = ae^{ax}\cos(bx+c) + (-1)\,b\,e^{ax}\sin(bx+c)$$

$$y_1 = e^{ax}[a\cos(bx+c) - b\sin(bx+c)]$$

Let $a = r\cos\theta$, $b = r\sin\theta$

Then $$r = \sqrt{a^2+b^2},\ \theta = \tan^{-1}\frac{b}{a}$$

$$y_1 = e^{ax}[r\cos\theta\cos(bx+c) - r\sin\theta\sin(bx+c)]$$

$$= re^{ax}\cos(bx+c+\theta)$$

Differentiating w.r.t. $x$,

$$y_2 = ae^{ax}\,r\cos(bx+c+\theta) - b\,e^{ax}\,r\sin(bx+c+\theta)$$

$$= re^{ax}[r\cos\theta\cos(bx+c+\theta) - r\sin\theta\sin(bx+c+\theta)]$$

$$= r^2e^{ax}\cos(bx+c+2\theta)$$

Differentiating $n$ times w.r.t. $x$,

$$y_n = r^ne^{ax}\cos(bx+c+n\theta),$$

where $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\dfrac{b}{a}$.

Hence, $\dfrac{d^n}{dx^n}\,[e^{ax}\cos(bx+c)] = r^n e^{ax}\cos(bx+c+n\theta)$,

where $r=\sqrt{a^2+b^2}, \theta = \tan^{-1}\dfrac{b}{a}$.

**Corollary 3:** $\dfrac{d^n}{dx^n}\,[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx+c+n\theta)$,

where $r=\sqrt{a^2+b^2}$, $\theta = \tan^{-1}\dfrac{b}{a}$.

**Example 1: Find $y_n$ if $y=\dfrac{x^n-1}{x-1}$.**

**Solution:** $y=\dfrac{x^n-1}{x-1}=\dfrac{(x-1)(x^{n-1}+x^{n-2}+x^{n-3}+\ldots\ldots\ldots+1)}{(x-1)}$

$= x^{n-1}+x^{n-2}+x^{n-3}+\ldots\ldots\ldots+1$

Differentiating $n$ times w.r.t. $x$,

$$y_n=\frac{d^n}{dx^n}(x^{n-1}+x^{n-2}+x^{n-3}+\ldots\ldots\ldots+1)$$

$$=0 \qquad \left[\because \frac{d^n}{dx^n}(ax+b)^m=0, \text{ if } n>m\right]$$

**Example 2: Prove that $\dfrac{d^{2n}}{dx^{2n}}(x^2-1)^n=(2n)!$.**

**Solution:** $(x^2-1)^n=(x^2)^n-{}^nC_1\,(x^2)^{n-1}+{}^nC_2\,(x^2)^{n-2}-\ldots\ldots\ldots\ (-1)^{n-1}$

$= x^{2n}-{}^nC_1\,x^{2n-2}+{}^nC_2\,x^{2n-4}-\ldots\ldots\ldots$

Differentiating $2n$ times w.r.t. $x$,

$$\frac{d^{2n}}{dx^{2n}}(x^2-1)^n=\frac{d^{2n}}{dx^{2n}}(x^{2n}-{}^nC_1\,x^{2n-2}+{}^nC_2\,x^{2n-4}-\ldots\ldots\ldots)$$

$$=(2n)! \qquad \left[\because \frac{d^n}{dx^n}(ax+b)^m \begin{array}{ll} =m!a^n, & \text{if } n=m \\ =0, & \text{if } n>m \end{array}\right]$$

**Example 3: Find $y_n$, if $y=\dfrac{x}{(x+1)^4}$.**

**Solution:**

$$y=\frac{x}{(x+1)^4}=\frac{(x+1)-1}{(x+1)^4}$$

$$=\frac{1}{(x+1)^3}-\frac{1}{(x+1)^4}$$

Differentiating $n$ times w.r.t. $x$,

$$y_n = \frac{(-1)^n (n+2)!}{2!\,(x+1)^{n+3}} - \frac{(-1)^n (n+3)!}{3!\,(x+1)^{n+4}}$$

$$= \frac{(-1)^n (n+2)!}{2!\,(x+1)^{n+3}}\left[1 - \frac{n+3}{3(x+1)}\right]$$

$$= \frac{(-1)^n (n+2)!}{2!\,(x+1)^{n+3}}\left[\frac{3x-n}{3(x+1)}\right].$$

**Example 4: Find $y_n$, if $y = \dfrac{x^2+4x+1}{x^3+2x^2-x-2}$.**

**Solution:**

$$y = \frac{x^2+4x+1}{x^3+2x^2-x-2} = \frac{(x^2+4x+1)}{x^2(x+2)-(x+2)}$$

$$= \frac{x^2+4x+1}{(x+2)(x^2-1)} = \frac{x^2+4x+1}{(x+2)(x+1)(x-1)}$$

$$= \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{x-1}$$

[By partial fraction expansion]

$$(x^2+4x+1) = A(x+1)(x-1) + B(x+2)(x-1) + C(x+2)(x+1)$$

Putting $x = -2$, $A = -1$

Putting $x = -1$, $B = 1$

Putting $x = 1$, $C = 1$

$$y = \frac{-1}{x+2} + \frac{1}{x+1} + \frac{1}{x-1}$$

Differentiating $n$ times w.r.t. $x$,

$$y_n = -\frac{(-1)^n n!}{(x+2)^{n+1}} + \frac{(-1)^n n!}{(x+1)^{n+1}} + \frac{(-1)^n n!}{(x-1)^{n+1}}$$

[Using Cor.1]

**Example 5: Find $y_n$, where $y = \dfrac{x^2+4}{(2x+3)(x-1)^2}$.**

**Solution:** $y = \dfrac{x^2+4}{(2x+3)(x-1)^2} = \dfrac{(x-1)^2+(2x+3)}{(2x+3)(x-1)^2}$

$$= \frac{1}{2x+3} + \frac{1}{(x-1)^2}$$

Differentiating $n$ times w.r.t. $x$,

$$y_n = (-1)^n \left[ \frac{(n!)2^n}{(2x+3)^{n+1}} + \frac{(n+1)!}{(x-1)^{n+2}} \right]$$

**Example 6: Find $y_n$, where $y = \dfrac{x^4}{(x-1)(x-2)}$.**

**Solution:** $y = \dfrac{x^4}{(x-1)(x-2)} = \dfrac{x^4}{x^2-3x+2}$

$$= x^2+3x+7+\frac{15x-14}{x^2-3x+2} \qquad \text{[By dividing]}$$

$$= x^2+3x+7+\frac{15x-14}{(x-1)(x-2)}$$

$$= x^2+3x+7+\frac{A}{x-1}+\frac{B}{x-2} \qquad \text{[By partial fraction expansion]}$$

$$= x^2+3x+7+\left(\frac{-1}{x-1}\right)+\frac{16}{x-2}$$

Differentiating $n$ times w.r.t. $x$,

$$y_n = -\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{16(-1)^n n!}{(x-2)^{n+1}}$$

$$= (-1)^n n! \left[ \frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right]$$

**Example 7: If $y = \dfrac{x^3}{x^2-1}$, then prove that $(y_n)_0 = \begin{cases} -(n!) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$.**

**Solution:** $y = \dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1}$

$$= x + \frac{x}{(x-1)(x+1)} = x + \frac{1}{2}\left[\frac{x-1+x+1}{(x-1)(x+1)}\right] = x + \frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)$$

Differentiating $n$ times w.r.t. $x$,

$$y_n = \frac{1}{2}(-1)^n\, n!\left[\frac{1}{(x+1)^{n+1}} + \frac{1}{(x-1)^{n+1}}\right] \qquad \text{[Using Cor.1]}$$

$$(y_n)_0 = \frac{1}{2}(-1)^n\, n!\left[\frac{1}{(1)^{n+1}} + \frac{1}{(-1)^{n+1}}\right]$$

$$= -(n!), \quad \text{if } n \text{ is odd.}$$

$$= 0, \quad \text{if } n \text{ is even.}$$

**Example 8: Find $y_n$, where $y = \dfrac{1}{1+x+x^2}$.**

**Solution:**

$$y = \frac{1}{1+x+x^2}$$

$$= \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)} = \frac{\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)-\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}{\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x+\frac{1}{2}-i\frac{\sqrt{3}}{2}} - \frac{1}{x+\frac{1}{2}+i\frac{\sqrt{3}}{2}}\right)$$

Differentiating $n$ times w.r.t. $x$,

$$y_n = \frac{1}{i\sqrt{3}}(-1)^n n!\left[\frac{1}{\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{n+1}}\right] \qquad \text{[Using Cor.1]}$$

Let $\left(x+\frac{1}{2}\right)+i\frac{\sqrt{3}}{2} = re^{i\theta}, \left(x+\frac{1}{2}\right)-i\frac{\sqrt{3}}{2} = re^{-i\theta}$

where $r = \sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}, \quad \theta = \tan^{-1}\dfrac{\frac{\sqrt{3}}{2}}{\left(x+\frac{1}{2}\right)}$

$$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\cot\theta$$

Substituting in $r$,

$$r = \sqrt{\frac{3}{4}\cot^2\theta+\frac{3}{4}} = \frac{\sqrt{3}}{2}\operatorname{cosec}\theta$$

Hence,

$$y_n = \frac{(-1)^n n!}{i\sqrt{3}}\left[\frac{1}{(re^{-i\theta})^{n+1}} - \frac{1}{(re^{i\theta})^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{i\sqrt{3}}\left[\frac{e^{i(n+1)\theta}-e^{-i(n+1)\theta}}{r^{n+1}}\right]$$

$$= \frac{(-1)^n n!\, 2i \sin(n+1)\theta}{i\sqrt{3}\left(\frac{\sqrt{3}}{2}\operatorname{cosec}\theta\right)^{n+1}}$$

$$= \frac{(-1)^n n!\, 2^{n+2} \sin^{n+1}\theta \sin(n+1)\theta}{3^{\frac{n+2}{2}}}, \quad \text{where } \theta = \tan^{-1}\frac{\sqrt{3}}{(2x+1)}$$

**Example 9: If $y = \dfrac{1}{x^4 - a^4}$, find $y_n$.**

**Solution:** $y = \dfrac{1}{x^4 - a^4} = \dfrac{1}{(x^2+a^2)(x^2-a^2)} = \dfrac{1}{2a^2} \cdot \dfrac{(x^2+a^2-x^2+a^2)}{(x^2+a^2)(x^2-a^2)}$

$$= \frac{1}{2a^2}\left[\frac{1}{(x^2-a^2)} - \frac{1}{(x^2+a^2)}\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{(x+a)\,(x-a)} - \frac{1}{(x+ia)\,(x-ia)}\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\left\{\frac{(x+a)-(x-a)}{(x+a)(x-a)}\right\} - \frac{1}{2ia}\left\{\frac{(x+ia)-(x-ia)}{(x+ia)(x-ia)}\right\}\right]$$

$$= \frac{1}{4ia^3}\left(\frac{1}{x+ia} - \frac{1}{x-ia}\right) - \frac{1}{4a^3}\left(\frac{1}{x+a} - \frac{1}{x-a}\right)$$

Differentiating $n$ times w.r.t. $x$,

Hence, $$y_n = \frac{(-1)^n n!}{4ia^3}\left[\frac{1}{(x+ia)^{n+1}} - \frac{1}{(x-ia)^{n+1}}\right] - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

Let $x + ia = re^{i\theta}, x - ia = re^{-i\theta}$

where $r = \sqrt{x^2+a^2}, \theta = \tan^{-1}\left(\dfrac{a}{x}\right)$.

$$y_n = \frac{(-1)^n n!}{4ia^3}\left[\frac{1}{(re^{i\theta})^{n+1}} - \frac{1}{(re^{-i\theta})^{n+1}}\right] - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{4ia^3} \cdot \frac{1}{r^{n+1}}\left[e^{-i(n+1)\theta} - e^{i(n+1)\theta}\right] - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{4ia^3} \cdot \frac{1}{(x^2+a^2)^{\frac{n+1}{2}}}\left[-2i\sin(n+1)\theta\right] - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

$$= \frac{(-1)^{n+1} n!}{2a^3(x^2+a^2)^{\frac{n+1}{2}}}\sin(n+1)\theta - \frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x+a)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$$

**Example 10: Find $n^{\text{th}}$ order derivatives of**

**(i) $y = \sin 2x \sin 3x \cos 4x$** **(ii) $y = \cos^4 x$**

**(iii) $y = \sin^5 x \cos^3 x$** **(iv) $y = e^x (\sin x + \cos x)$**

**(v) $y = e^{x \cos\alpha} \cos (x \sin \alpha)$.**

**Solution:**

(i)
$$y = \sin 2x \sin 3x \cos 4x$$
$$= \frac{1}{2}(\cos x - \cos 5x) \cos 4x$$
$$= \frac{1}{2}(\cos x \cos 4x - \cos 5x \cos 4x)$$
$$= \frac{1}{4}(\cos 5x + \cos 3x - \cos 9x - \cos x)$$

Differentiating $n$ times w.r.t. $x$,

$$y_n = \frac{1}{4}\left[5^n \cos\left(5x + \frac{n\pi}{2}\right) + 3^n \cos\left(3x + \frac{n\pi}{2}\right) - 9^n \cos\left(9x + \frac{n\pi}{2}\right) - \cos\left(x + \frac{n\pi}{2}\right)\right] \quad \text{[Using Cor. 2]}$$

(ii)
$$y = \cos^4 x$$
$$= \left(\frac{2\cos^2 x}{2}\right)^2$$
$$= \frac{1}{4}(1 + \cos 2x)^2$$
$$= \frac{1}{4}(1 + \cos^2 2x + 2\cos 2x)$$
$$= \frac{1}{4}\left(1 + \frac{1 + \cos 4x}{2} + 2\cos 2x\right)$$
$$= \frac{1}{8}(3 + \cos 4x + 4\cos 2x)$$

Differentiating $n$ times w.r.t. $x$,

$$y_n = \frac{1}{8}\left[4^n \cos\left(4x + \frac{n\pi}{2}\right) + 4.2^n \cos\left(2x + \frac{n\pi}{2}\right)\right] \quad \text{[Using Cor. 2]}$$

(iii)
$$y = \sin^5 x \cos^3 x$$
$$= \sin^2 x (\sin x \cos x)^3$$
$$= \frac{\sin^2 x}{2^3} \sin^3 2x$$

$$= \frac{1}{8}\left(\frac{1-\cos 2x}{2}\right)\left(\frac{3\sin 2x - \sin 6x}{4}\right)$$

$$= \frac{1}{2^6}(3\sin 2x - \sin 6x - 3\cos 2x \sin 2x + \cos 2x \sin 6x)$$

$$= \frac{1}{2^6}\left[3\sin 2x - \sin 6x - \frac{3}{2}\sin 4x + \frac{1}{2}(\sin 8x + \sin 4x)\right]$$

$$= \frac{1}{2^6}\left[3\sin 2x - \sin 4x - \sin 6x + \frac{1}{2}\sin 8x\right]$$

Differentiating *n* times w.r.t. *x*,

$$y_n = \frac{1}{2^6}\left[2^n \cdot 3\sin\left(2x + \frac{n\pi}{2}\right) - 4^n \sin\left(4x + \frac{n\pi}{2}\right) - 6^n \sin\left(6x + \frac{n\pi}{2}\right) + \frac{1}{2}8^n \sin\left(8x + \frac{n\pi}{2}\right)\right] \quad \text{[Using result (6)]}$$

(iv) $y = e^x (\sin x + \cos x)$

Differentiating *n* times w.r.t. *x*,

$$y_n = (1+1)^{\frac{n}{2}} \cdot e^x [\sin(x + n\tan^{-1} 1) + \cos(x + n\tan^{-1} 1)] \quad \text{[Using result (7) and Cor. 3]}$$

$$= 2^{\frac{n}{2}} e^x \left[\sin\left(x + \frac{n\pi}{4}\right) + \cos\left(x + \frac{n\pi}{4}\right)\right]$$

$$= 2^{\frac{n}{2}} e^x \left[\sin\left(x + \frac{n\pi}{4}\right) + \sin\left(\frac{\pi}{2} + x + \frac{n\pi}{4}\right)\right]$$

$$= 2^{\frac{n}{2}} e^x \cdot 2\sin\left(\frac{2x + \frac{\pi}{2} + \frac{n\pi}{2}}{2}\right)\cos\frac{\pi}{4}$$

$$= 2^{\frac{n}{2}} e^x \cdot \frac{2}{\sqrt{2}}\sin\left[x + \frac{\pi}{4}(n+1)\right]$$

$$= 2^{\frac{n+1}{2}} e^x \sin\left[x + (n+1)\frac{\pi}{4}\right]$$

(v) $y = e^{x\cos\alpha} \cos(x \sin \alpha)$

Differentiating *n* times w.r.t. *x*,

$$y_n = (\cos^2\alpha + \sin^2\alpha)^{\frac{n}{2}} e^{x\cos\alpha} \cdot \cos\left(x\sin\alpha + n\tan^{-1}\frac{\sin\alpha}{\cos\alpha}\right) \quad \text{[Using result (7)]}$$

$$= e^{x\cos\alpha}\cos(x\sin\alpha + n\alpha)$$

**Example 11: If $y(x) = \sin px + \cos px$, prove that $y_n(x) = p^n[1+(-1)^n \sin 2px]^{\frac{1}{2}}$. Hence, find $y_8(\pi)$, when $p = \frac{1}{4}$.**

**Solution:** $y(x) = \sin px + \cos px$

Differentiating $n$ times w.r.t. $x$,

$$y_n(x) = p^n\left[\sin\left(px+\frac{n\pi}{2}\right)+\cos\left(px+\frac{n\pi}{2}\right)\right] \qquad \text{[Using result (6) and Cor. 2]}$$

$$= p^n\left[\left\{\sin\left(px+\frac{n\pi}{2}\right)+\cos\left(px+\frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= p^n\left[\sin^2\left(px+\frac{n\pi}{2}\right)+\cos^2\left(px+\frac{n\pi}{2}\right)+2\sin\left(px+\frac{n\pi}{2}\right)\cos\left(px+\frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$= p^n[1+\sin(2px+n\pi)]^{\frac{1}{2}}$$

$$= p^n[1+\sin 2px\,\cos n\pi+\cos 2px\sin n\pi]^{\frac{1}{2}}$$

$$= p^n[1+(-1)^n\sin 2px]^{\frac{1}{2}}$$

Putting $n = 8,\ p = \frac{1}{4}$ and $x = \pi$,

$$y_8(\pi) = \left(\frac{1}{4}\right)^8\left[1+(-1)^8\sin 2\left(\frac{\pi}{4}\right)\right]^{\frac{1}{2}}$$

$$= \left(\frac{1}{4}\right)^8\left[1+\sin\left(\frac{\pi}{2}\right)\right]^{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{31}{2}}$$

**Example 12: If $y = \tan^{-1}\left(\frac{x}{a}\right)$, prove that $y_n = \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\sin^n\theta$, where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$.**

**Solution:** $y = \tan^{-1}\left(\frac{x}{a}\right)$

Differentiating w.r.t. $x$,

$$y_1 = \frac{1}{1+\frac{x^2}{a^2}}\cdot\frac{1}{a} = \frac{a}{a^2+x^2} = \frac{a}{(x+ia)(x-ia)}$$

$$= \frac{1}{2i}\left[\frac{(x+ia)-(x-ia)}{(x+ia)(x-ia)}\right]$$

$$= \frac{1}{2i}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

Differentiating $(n-1)$ times w.r.t. $x$,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(x-ia)^n} - \frac{1}{(x+ia)^n}\right] \qquad \text{[Using Cor. 1]}$$

Let $x + ia = re^{i\theta}$, $x - ia = re^{-i\theta}$

where, $r = \sqrt{x^2+a^2}$, $\theta = \tan^{-1}\left(\frac{a}{x}\right)$, $\tan\theta = \frac{a}{x}$, $x = a\cot\theta$

$$r = \sqrt{a^2\cot^2\theta + a^2} = a\operatorname{cosec}\theta$$

Hence,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(re^{-i\theta})^n} - \frac{1}{(re^{i\theta})^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\frac{1}{r^n}\left(e^{in\theta} - e^{-in\theta}\right)$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\cdot\frac{2i\sin n\theta}{r^n}$$

$$= (-1)^{n-1}(n-1)!\,\frac{\sin n\theta}{r^n}$$

$$= (-1)^{n-1}(n-1)!\,\frac{\sin n\theta}{a^n\operatorname{cosec}^n\theta}$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\sin^n\theta.$$

where, $\theta = \tan^{-1}\left(\frac{a}{x}\right)$.

**Example 13: If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, prove that**

**$y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\theta = \cot^{-1}x$.**

**Solution:** $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Putting $x = \tan\phi$,

$$y = \tan^{-1}\left(\frac{\sec\phi - 1}{\tan\phi}\right) = \tan^{-1}\left(\frac{1-\cos\phi}{\sin\phi}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\phi}{2}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}\right) = \tan^{-1}\tan\frac{\phi}{2} = \frac{\phi}{2}$$

$$y = \frac{1}{2}\tan^{-1} x$$

Differentiating w.r.t. $x$,

$$y_1 = \frac{1}{2}\cdot\frac{1}{(1+x^2)}$$

$$= \frac{1}{2}\cdot\frac{1}{(x+i)(x-i)} = \frac{1}{4i}\cdot\frac{(x+i)-(x-i)}{(x+i)(x-i)}$$

$$= \frac{1}{4i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

Differentiating $(n - 1)$ times w.r.t. $x$,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{4i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$

Let $\quad x + i = re^{i\theta}, x - i = re^{-i\theta}$

where, $\quad r = \sqrt{x^2+1}, \quad \theta = \tan^{-1}\left(\frac{1}{x}\right), \quad \tan\theta = \frac{1}{x}, \quad x = \cot\theta$

$$r = \sqrt{\cot^2\theta + 1} = \operatorname{cosec}\theta$$

Hence, $\quad y_n = \dfrac{(-1)^{n-1}(n-1)!}{4i}\left[\dfrac{1}{(re^{-i\theta})^n} - \dfrac{1}{(re^{i\theta})^n}\right]$

$$= \frac{(-1)^{n-1}(n-1)!}{4i}\frac{1}{r^n}\left(e^{in\theta} - e^{-in\theta}\right)$$

$$= \frac{(-1)^{n-1}(n-1)!}{4i}\cdot\frac{2i\sin n\theta}{r^n}$$

$$= \frac{(-1)^{n-1}(n-1)!\sin n\theta}{2r^n}$$

$$= \frac{(-1)^{n-1}(n-1)!\sin n\theta}{2\operatorname{cosec}^n\theta}$$

$$= \frac{1}{2}(-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta, \text{ where } \theta = \tan^{-1}\left(\frac{1}{x}\right).$$

**Example 14: If $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, prove that $y_n = (-1)^{n-1}\ (n-1)\ !\ \sin^n\theta \sin n\theta$ where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$.**

**Solution:** $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

Putting $x = \tan\phi$,

$$y = \tan^{-1}\left(\frac{1+\tan\phi}{1-\tan\phi}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\phi}{1 - \tan\frac{\pi}{4}\cdot\tan\phi}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \phi\right)\right] = \frac{\pi}{4} + \phi = \frac{\pi}{4} + \tan^{-1} x$$

Proceeding as in Example 13, we get

$$y_n = (-1)^{n-1}\ (n-1)!\ \sin^n\theta \sin n\theta.$$

**Example 15: Find the $n^{\text{th}}$ order derivative of $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$.**

**Solution:** $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right) = \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)$

Putting $x = \tan\phi$,

$$y = \cos^{-1}\left(\frac{\tan^2\phi - 1}{\tan^2\phi + 1}\right) = \cos^{-1}(-\cos 2\phi)$$

$$= \cos^{-1}[\cos(\pi - 2\phi)] = \pi - 2\phi = \pi - 2\tan^{-1} x$$

Proceeding as in Example 13, we get

$$y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right).$$

**Example 16: If $y = x \log(1+x)$, prove that $y_n = \frac{(-1)^{n-2}(n-2)!(x+n)}{(x+1)^n}$.**

**Solution:** $y = x \log(1+x)$

Differentiating w.r.t. $x$,

$$y_1 = \log(1+x) + \frac{x}{1+x} = \log(1+x) + 1 - \frac{1}{1+x}$$

Differentiating $(n-1)$ times w.r.t. $x$,

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} + 0 - \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} \qquad \text{[Using result (3) and Cor. 1]}$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}\left[\frac{1}{(x+1)^{-1}} - \frac{(-1)^1(n-1)}{1}\right]$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}(x+1+n-1)$$

$$= \frac{(-1)^{n-2}(n-2)!\,(x+n)}{(x+1)^n}.$$

**Example 17: If $y = x\log\left(\dfrac{x-1}{x+1}\right)$, prove that $y_n = (-1)^n(n-2)!\left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right]$, $n \geq 2$.**

**Solution:** $y = x\log\left(\dfrac{x-1}{x+1}\right)$

$$= x\log(x-1) - x\log(x+1)$$

Differentiating $y$ w.r.t. $x$,

$$y_1 = \log(x-1) + \frac{x}{x-1} - \frac{x}{x+1} - \log(x+1)$$

$$= \log(x-1) + 1 + \frac{1}{x-1} - 1 + \frac{1}{x+1} - \log(x+1)$$

Differentiating $(n-1)$ times w.r.t. $x$,

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} \; [\text{if } n-2 \geq 0]$$

[Using result (3) and Cor. 1]

$$= \frac{(-1)^n(n-2)!}{(x-1)^n}\left[\frac{(-1)^{-2}}{(x-1)^{-1}} + \frac{(-1)^{-1}(n-1)}{1}\right]$$

$$+ \frac{(-1)^n(n-2)!}{(x+1)^n}\left[\frac{(-1)^{-1}(n-1)}{1} - \frac{(-1)^{-2}}{(x+1)^{-1}}\right]$$

$$= \frac{(-1)^n(n-2)!}{(x-1)^n}(x-1-n+1) + \frac{(-1)^n(n-2)!}{(x+1)^n}(-n+1-x-1)$$

$$= (-1)^n(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right], \; n \geq 2.$$

**Example 18: If $y = x \coth^{-1} x$, prove that**
$$y_n = \frac{(-1)^n (n-2)!}{2}\left[\frac{x+n}{(x+1)^n} - \frac{x-n}{(x-1)^n}\right], n \ge 2.$$

**Solution:**
$$y = x \coth^{-1} x$$
$$= x \tanh^{-1}\left(\frac{1}{x}\right)$$
$$= x \cdot \frac{1}{2} \log \frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$
$$= \frac{x}{2} \log\left(\frac{x+1}{x-1}\right)$$
$$y = \frac{1}{2}[x \log(x+1) - x \log(x-1)]$$

Proceeding as in Example 17, we get
$$y_n = \frac{(-1)^n (n-2)!}{2}\left[\frac{x+n}{(x+1)^n} - \frac{x-n}{(x-1)^n}\right], n \ge 2 \cdot$$

**Example 19: If $y = (x-1)^n$, prove that $y + \frac{y_1}{1!} + \frac{y_2}{2!} + \frac{y_3}{3!} + \cdots + \frac{y_n}{n!} = x^n$.**

**Solution:** $y = (x-1)^n$

Differentiating w.r.t. $x$ successively,
$$y_1 = n(x-1)^{n-1}$$
$$y_2 = n(n-1)(x-1)^{n-2}$$
$$y_3 = n(n-1)(n-2)(x-1)^{n-3}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n = n!$$

Hence, $y + \frac{y_1}{1!} + \frac{y_2}{2!} + \frac{y_3}{3!} + \cdots + \frac{y_n}{n!}$
$$= (x-1)^n + \frac{n}{1!}(x-1)^{n-1} + \frac{n(n-1)}{2!}(x-1)^{n-2} + \frac{n(n-1)(n-2)}{3!}(x-1)^{n-3} + \cdots + \frac{n!}{n!}$$
$$= (x-1)^n + {}^nC_1(x-1)^{n-1} + {}^nC_2(x-1)^{n-2} + {}^nC_3(x-1)^{n-3} + \cdots + {}^nC_n$$
$$= [1 + (x-1)]^n$$
$$= x^n. \qquad \text{[Using Binomial Expansion]}$$

**Example 20: If $I_n = \dfrac{d^n}{dx^n}(x^n \log x)$, prove that $I_n = n\,I_{n-1} + (n-1)\,!$**

**Hence, prove that $I_n = n!\left(\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}\right)$.**

**Solution:**

$$I_n = \frac{d^n}{dx^n}(x^n \log x)$$

For $n = 1$

$$I_1 = \frac{d}{dx}(x \log x) = \log x + 1$$

$$I_n = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n \log x)\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(nx^{n-1}\log x + x^n \frac{1}{x}\right)$$

$$= n\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$I_n = nI_{n-1} + (n-1)!$$

$$\frac{I_n}{n!} = \frac{I_{n-1}}{(n-1)!} + \frac{1}{n} \qquad \text{... (1)}$$

Putting $n = 2, 3, 4, \ldots\ldots.$ in Eq. (1),

$$\frac{I_2}{2!} = \frac{I_1}{1!} + \frac{1}{2}$$

$$\frac{I_3}{3!} = \frac{I_2}{2!} + \frac{1}{3}$$

$$\frac{I_4}{4!} = \frac{I_3}{3!} + \frac{1}{4}$$

..................

..................

$$\frac{I_{n-1}}{(n-1)!} = \frac{I_{n-2}}{(n-2)!} + \frac{1}{n-1}$$

From Eq. (1),

$$\frac{I_n}{n!} = \frac{I_{n-1}}{(n-1)!} + \frac{1}{n}$$

Adding all the above equations,

$$\frac{I_n}{n!} = I_1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

$$I_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}\right)$$

**Example 21: Prove that** $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{n-1}e^{\frac{1}{x}}\right) = \dfrac{(-1)^n e^{\frac{1}{x}}}{x^{n+1}}$.

**Solution:** $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{n-1}e^{\frac{1}{x}}\right)$

$$= \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{n-1}\left(1 + \frac{1}{x} + \frac{1}{2!x^2} + \frac{1}{3!x^3} + \ldots + \frac{1}{(n-1)!x^{n-1}} + \frac{1}{n!x^n} + \frac{1}{(n+1)!x^{n+1}} + \frac{1}{(n+2)!x^{n+2}} + \ldots\right)\right]$$

$$= \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{n-1} + x^{n-2} + \frac{x^{n-3}}{2!} + \ldots + \frac{1}{(n-1)!} + \frac{1}{x(n!)} + \frac{1}{x^2(n+1)!} + \frac{1}{x^3(n+2)!} + \ldots\right]$$

$$= 0 + \frac{1(-1)^n n!}{n!\, x^{n+1}} + \frac{(-1)^n (n+1)!}{(n+1)!\,(1!)x^{n+2}} + \frac{(-1)^n (n+2)!}{(n+2)!\,(2!)x^{n+3}} + \ldots. \quad \text{[Using result (1) and (2)]}$$

$$= \frac{(-1)^n}{x^{n+1}}\left(1 + \frac{1}{x} + \frac{1}{2!x^2} + \ldots\right)$$

$$= \frac{(-1)^n}{x^{n+1}} e^{\frac{1}{x}}.$$

## Exercise 2.1

**1.** Find the $n^{\text{th}}$ order derivative of

(i) $y = \dfrac{x+1}{x^2-4}$ (ii) $y = \dfrac{x}{1-4x^2}$.

[**Ans.:** (i) $\dfrac{3}{4}\dfrac{(-1)^n n!}{(x-2)^{n+1}} + \dfrac{1}{4}\dfrac{(-1)^n n!}{(x+2)^{n+1}}$

(ii) $\dfrac{1}{4}\left[\dfrac{(-1)^n n!\,(-2)^n}{(1-2x)^{n+1}} - \dfrac{(-1)^n n!\,2^n}{(1+2x)^{n+1}}\right]$]

**2.** Find the $n^{\text{th}}$ order derivative of $y = \dfrac{x}{(x-1)(x-2)(x-3)}$.

$$\left[\textbf{Ans.: } (-1)^n n!\left[\frac{1}{2(x-1)^{n+1}} - \frac{2}{(x-2)^{n+1}} + \frac{3}{2(x-3)^{n+1}}\right]\right]$$

**3.** Find the $n^{\text{th}}$ order derivative of $y = \dfrac{x}{1+3x+2x^2}$.

$$\left[\textbf{Ans.: } (-1)^n n!\left[\frac{1}{(x+1)^{n+1}} - \frac{2^n}{(2x+1)^{n+1}}\right]\right]$$

**4.** Find the $n^{\text{th}}$ order derivative of $y = \dfrac{x^2}{(x+2)(2x+3)}$.

$$\left[\textbf{Ans.: } \frac{-4(-1)^n n!}{(x+2)^{n+1}} + \frac{9(-1)^n n!\,(2)^{n-1}}{(2x+3)^{n+1}}\right]$$

**5.** Find the $n^{\text{th}}$ order derivative of $y = \dfrac{2x+3}{(x-1)^2}$.

$$\left[\textbf{Ans.: } \frac{2(-1)^n n!}{(x-1)^{n+1}} + \frac{5(-1)^n (n+1)!}{(x-1)^{n+2}}\right]$$

**6.** Find the $n^{\text{th}}$ order derivative of $y = \dfrac{x}{(x-1)^2}$.

$$\left[\textbf{Ans.: } \frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)^n (n+1)!}{(x-1)^{n+2}}\right]$$

**7.** Find the $n^{\text{th}}$ order derivative of $y = \dfrac{x+1}{(x-1)^n}$.

$$\left[\textbf{Hint: } y = \frac{(x-1)+2}{(x-1)^n} = \frac{1}{(x-1)^{n-1}} + \frac{2}{(x-1)^n}\right]$$

$$\left[\textbf{Ans.: } (-1)^n\left[\frac{(2n-2)!}{(x-1)^{2n-1}} + \frac{(2n-1)!}{(x-1)^{2n}}\right]\right]$$

**8.** Find the $n^{\text{th}}$ order derivative of $y = \dfrac{4x}{(x-1)^2(x+1)}$.

$$\left[\textbf{Ans.: } (-1)^n n!\left[\frac{1}{(x-1)^{n+1}} + \frac{2(n+1)}{(x-1)^{n+2}} - \frac{n!}{(x+1)^{n+1}}\right]\right]$$

**9.** Find the $n^{\text{th}}$ order derivative of $y = \dfrac{1}{(3x-2)(x-3)^2}$.

$$\left[\textbf{Ans.: } (-1)^n n!\left[\frac{3^{n+2}}{49(3x-2)^{n+1}} - \frac{3}{49(x-3)^{n+1}} + \frac{(n+1)}{7(x-3)^{n+2}}\right]\right]$$

**10.** If $y = \dfrac{x^2}{2x^2+7x+6}$, find $y_n$.

$$\left[\textbf{Hint: } \text{Divide } x^2 \text{ by } 2x^2+7x+6,\quad y = \frac{1}{2} - \frac{7x+6}{2(x+2)(2x+3)}\right]$$

$$\left[\textbf{Ans.: } (-1)^n n!\left[-\frac{8}{(x+2)^{n+1}} + \frac{9(2)^n}{(2x+3)^{n+1}}\right]\right]$$

**11.** Prove that $\dfrac{d^4}{dx^4}\left(\dfrac{x^3}{x^2-1}\right)_{x=0} = 0$.

**12.** If $y = \dfrac{x}{x^2+a^2}$, prove that $y_n = (-1)^n\, n!\, a^{-n-1} (\sin\theta)^{n+1} \cos(n+1)\,\theta$.

**13.** If $y = \dfrac{x}{x^2+1}$, prove that

$y_n = (-1)^n\, n!\, \sin^{n+1}\theta \cos(n+1)\,\theta$

where $\theta = \tan^{-1}\left(\dfrac{1}{x}\right)$.

**14.** Find the $n^{\text{th}}$ order derivative of

$$y = \frac{1}{1+x+x^2+x^3}.$$

$$\left[\textbf{Hint:}\ y = \frac{1}{(1+x)(1+x^2)} = \frac{1}{(1+x)(x+i)(x-i)}\right]$$

$$\left[\textbf{Ans.:}\ \frac{(-1)^n n!}{2}\left[\frac{1}{(1+x)^{n+1}} + \frac{1}{2r^{n+1}}\{\sin(n+1)\theta - \cos(n+1)\theta\}\right]\right]$$

**15.** Find the $n^{\text{th}}$ order derivative of

$$y = \frac{x}{1+x+x^2}.$$

$$\left[\textbf{Ans.:}\ \frac{(-1)^n n!}{r^{n+1}}\left[\cos(n+1)\theta - \frac{1}{\sqrt{3}}\sin(n+1)\theta\right], \text{ where } r = \sqrt{x^2+x+1},\ \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2x+1}\right)\right]$$

**16.** Prove that $\dfrac{d^n}{dx^n}\tan^{-1} x$

$$= (-1)^{n-1}(n-1)!\frac{\sin\left(n\tan^{-1}\dfrac{1}{x}\right)}{(x^2+1)^{\frac{n}{2}}}.$$

**17.** Find the $n^{\text{th}}$ order derivatives of

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

$$\left[\textbf{Ans.:}\ 2(-1)^{n-1}(n-1)!\,(\sin\theta)^n\ \sin n\theta, \text{ where } \theta = \tan^{-1}\left(\frac{1}{x}\right).\right]$$

**18.** If $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, prove that

$$y_n = 2(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta,$$

where $\theta = \tan^{-1}\left(\dfrac{1}{x}\right)$.

**19.** If $y = \sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$, prove that

$$y_n = 2(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta.$$

**20.** Find the $n^{\text{th}}$ order derivative w.r.t. $x$ of

(i) $\sin^4 x$ (ii) $\sin^7 x$

$$\left[\textbf{Hint:}\ \sin^7 x = \left[\frac{1}{2i}(e^{ix} - e^{-ix})\right]^7, \text{ expand using binomial expansion}\right]$$

(iii) $\sin^3 x\cos^2 x$ (iv) $\sin^3 3x$.

$$\left[\begin{aligned}&\textbf{Ans.:}\ \text{(i)}\ -2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right) + 2^{2n-3}\cos\left(4x+\frac{n\pi}{2}\right)\\ &\text{(ii)}\ -\frac{1}{64}\left[7^n\sin\left(7x+\frac{n\pi}{2}\right) - 7.5^n\sin\left(5x+\frac{n\pi}{2}\right) + 21.3^n\sin\left(3x+\frac{n\pi}{2}\right) - 35\sin\left(x+\frac{n\pi}{2}\right)\right]\\ &\text{(iii)}\ \frac{1}{16}\left[2\sin\left(x+\frac{n\pi}{2}\right) + 3^n\sin\left(3x+\frac{n\pi}{2}\right) - 5^n\sin\left(5x+\frac{n\pi}{2}\right)\right]\\ &\text{(iv)}\ \frac{3^{n+1}}{4}\sin\left(3x+\frac{n\pi}{2}\right) - \frac{1}{4}\cdot 3^{2n}\sin\left(9x+\frac{n\pi}{2}\right)\end{aligned}\right]$$

**21.** Find the $n^{\text{th}}$ order derivative w.r.t. $x$ of

(i) $\sin 2x\cos 6x$ (ii) $\sin x\cos 3x$

(iii) $\cos x\cos 2x\cos 3x$.

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{1}{2}\left[8^n \sin\left(8x+\frac{n\pi}{2}\right) - 4^n \sin\left(4x+\frac{n\pi}{2}\right)\right]\right.$$

$$\text{(ii)}\ \frac{1}{2}\left[4^n \sin\left(4x+\frac{n\pi}{2}\right) - 2^n \sin\left(2x+\frac{n\pi}{2}\right)\right]$$

$$\left.\text{(iii)}\ \frac{1}{4}\left[6^n \cos\left(6x+\frac{n\pi}{2}\right) + 4^n \cos\left(4x+\frac{n\pi}{2}\right) + 2^n \cos\left(2x+\frac{n\pi}{2}\right)\right]\right]$$

**22.** Find the $n^{\text{th}}$ order derivative w.r.t. $x$ of

(i) $e^{5x} \cos x \cos 3x$

(ii) $e^x \cos x \cos 2x$

(iii) $e^{ax} \cos^2 x \sin x$

(iv) $2^x \sin^2 x \cos x$

(v) $2^x \sin (3x + 1)$.

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{1}{2}e^{5x}\left[(41)^{\frac{n}{2}} \cos\left(4x+n\tan^{-1}\frac{4}{5}\right) + (29)^{\frac{n}{2}} \cos\left(2x+n\tan^{-1}\frac{2}{5}\right)\right]\right.$$

$$\text{(ii)}\ \frac{1}{2}e^x\left[(10)^{\frac{n}{2}} \cos(3x + n\tan^{-1}3) + (2)^{\frac{n}{2}} \cos\left(x+\frac{n\pi}{4}\right)\right]$$

$$\text{(iii)}\ \frac{e^{ax}}{4}\left[(a^2+9)^{\frac{n}{2}} \sin\left(3x+n\tan^{-1}\frac{3}{a}\right) + (a^2+1)^{\frac{n}{2}} \sin\left(x+n\tan^{-1}\frac{1}{a}\right)\right]$$

$$\text{(iv)}\ -\frac{1}{4}r_1^n 2^x \cos(3x+n\phi_1) + \frac{1}{4}r_2^n 2^x \cos(x+n\phi_2)$$

$$\text{where } r_1 = \sqrt{(\log 2)^2 + 3^2},\quad \phi_1 = \tan^{-1}\left(\frac{3}{\log 2}\right),$$

$$r_2 = \sqrt{(\log 2)^2 + 1},\quad \phi_2 = \tan^{-1}\left(\frac{1}{\log 2}\right)$$

$$\left.\text{(v)}\ 2^x[(\log 2)^2 + 9]^{\frac{n}{2}} \sin\left(3x+1+n\tan^{-1}\frac{3}{\log 2}\right)\right]$$

## 2.3 LEIBNITZ'S THEOREM

**Statement:** If $y = uv$, where $u$ and $v$ are two functions of $x$ whose $n^{\text{th}}$ derivatives are known, then

$$y_n = (uv)_n = u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3 + \ldots + {}^nC_n u v_n.$$

**Proof:** $$y = uv$$

Differentiating w.r.t. $x$ successively,

$$y_1 = u_1 v + u v_1$$
$$= {}^1C_0 u_1 v + {}^1C_1 u v_1$$
$$y_2 = u_2 v + 2\, u_1 v_1 + u v_2$$
$$= {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2$$

This shows that theorem is true for $n = 1$ and $n = 2$.
Let the theorem is true for $n = m$.

$$y_m = (uv)_m = {}^mC_0 u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots\ldots\ldots\ldots + {}^mC_m u v_m .$$

Differentiating $y_m$ w.r.t. $x$,

$$\frac{\mathrm{d}}{\mathrm{d}x} y_m = \frac{\mathrm{d}}{\mathrm{d}x}(uv)_m = {}^mC_0 (u_{m+1} v + u_m v_1) + {}^mC_1 (u_m v_1 + u_{m-1} v_2) + {}^mC_2 (u_{m-1} v_2 + u_{m-2} v_3)$$
$$+ \ldots\ldots\ldots\ldots + {}^mC_m (u_1 v_m + u v_{m+1})$$
$$= {}^mC_0 u_{m+1} v + \left({}^mC_0 + {}^mC_1\right) u_m v_1 + \left({}^mC_1 + {}^mC_2\right) u_{m-1} v_2 + \ldots\ldots\ldots\ldots + {}^mC_m u v_{m+1}$$
$$y_{m+1} = (uv)_{m+1} = {}^{m+1}C_0 u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2$$
$$+ \ldots + {}^{m+1}C_{m+1} u v_{m+1}. \qquad \left[\because {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r\right]$$

This shows that theorem is true for $n = m + 1$ also.
By mathematical induction, theorem is true for all integer values of $n$.

Hence, $y_n = (uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3 + \ldots + {}^nC_n u v_n .$

**Example 1: If $y = x^2 e^{2x}$, prove that $(y_n)_0 = 2^{n-2}\, n\, (n-1)$.**

**Solution:** $y = x^2 e^{2x}$

Differentiating $n$ times using Leibnitz's Theorem,

$$y_n = x^2\, 2^n e^{2x} + n \cdot 2x\, 2^{n-1} e^{2x} + \frac{n(n-1)}{2!} \cdot 2 \cdot 2^{n-2} e^{2x}$$

Putting $x = 0$,

$$(y_n)_0 = 2^{n-2}\, n\, (n-1)$$

**Example 2: If $u$ is a function of $x$, and $y = e^{ax} u$, prove that $D^n y = e^{ax} (D + a)^n u$, where $D = \frac{d}{dx}$.**

**Solution:** $y = e^{ax} u$

$$D^n (e^{ax} u) = (D^n e^{ax})\, u + {}^nC_1 (D^{n-1} e^{ax})(Du) + {}^nC_2 (D^{n-2} e^{ax})(D^2 u)$$
$$+ {}^nC_3 (D^{n-3} e^{ax})(D^3 u) + \ldots\ldots\ldots + e^{ax} (D^n u)$$

$$= (a^n e^{ax})\, u + {}^nC_1\, (a^{n-1} e^{ax})\, (\mathrm{D}u) + {}^nC_2\, (a^{n-2} e^{ax})\, (\mathrm{D}^2u)$$
$$+ {}^nC_3\, (a^{n-3} e^{ax})\, (\mathrm{D}^3u) + \ldots\ldots\ldots + e^{ax}\, (\mathrm{D}^nu)$$
$$= e^{ax}\, [a^n + {}^nC_1\, a^{n-1}\, (\mathrm{D}) + {}^nC_2\, a^{n-2}\, (\mathrm{D}^2) + {}^nC_3\, a^{n-3}\, (\mathrm{D}^3) + \ldots + \mathrm{D}^n]\, u$$
$$= e^{ax}\, (\mathrm{D} + a)^n\, u, \text{ where } \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x} \qquad \text{[Using Binomial Expansion]}$$

**Example 3: Find the $n^{\text{th}}$ order derivative of**

**(i) $y = x^2 e^{ax}$ (ii) $x^3 \cos x$ (iii) $x^2 e^x \cos x$ (iv) $x^2 \tan^{-1} x$.**

**Solution:** (i) $y = x^2 e^{ax}$

Let $u = e^{ax}$, $v = x^2$

Differentiating $n$ times using Leibnitz's Theorem,

$$y_n = u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots + {}^nC_n\, u\, v_n$$
$$= a^n e^{ax} \cdot x^2 + n\, a^{n-1} e^{ax} \cdot 2x + \frac{n(n-1)}{2\,!} a^{n-2} e^{ax} \cdot 2$$
$$= e^{ax} [x^2 a^n + 2nxa^{n-1} + n(n-1) a^{n-2}]$$

(ii) $y = x^3 \cos x$

Let $u = \cos x$, $v = x^3$

Differentiating $n$ times using Leibnitz's Theorem,

$$y_n = u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots + {}^nC_n\, u\, v_n$$
$$= \cos\left(x + \frac{n\pi}{2}\right) x^3 + n \cos\left[x + \frac{(n-1)\pi}{2}\right] 3x^2$$
$$+ \frac{n(n-1)}{2!} \cos\left[x + \frac{(n-2)\pi}{2}\right] 6x + \frac{n(n-1)(n-2)}{3!} \cos\left[x + \frac{(n-3)\pi}{2}\right] 6$$
$$= x^3 \cos\left(x + \frac{n\pi}{2}\right) + 3nx^2 \cos\left[x + \frac{(n-1)\pi}{2}\right]$$
$$+ 3x\, n(n-1) \cos\left[x + \frac{(n-2)\pi}{2}\right] + n(n-1)(n-2) \cos\left[x + \frac{(n-3)\pi}{2}\right]$$

(iii) $y = x^2 e^x \cos x$

Let $u = e^x \cos x$, $v = x^2$.

$$u_n = e^x (2)^{\frac{n}{2}} \cos\left(x + \frac{n\pi}{4}\right) \qquad \text{[Using result (7)]}$$

Differentiating $n$ times using Leibnitz's Theorem,

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots + {}^nC_n u\, v_n$$
$$= e^x (2)^{\frac{n}{2}} \cos\left(x + \frac{n\pi}{4}\right) x^2 + ne^x (2)^{\frac{n-1}{2}} \cos\left[x + \frac{(n-1)\pi}{4}\right] 2x$$

$$+\frac{n(n-1)}{2!}(2)^{\frac{n-2}{2}}\cos\left[x+\frac{(n-2)\pi}{4}\right]2$$

$$= e^x(2)^{\frac{n}{2}}\cos\left(x+\frac{n\pi}{4}\right)x^2 + nxe^x(2)^{\frac{n+1}{2}}\cos\left[x+\frac{(n-1)\pi}{4}\right]$$

$$+\frac{n(n-1)}{2!}(2)^{\frac{n}{2}}\cos\left[x+\frac{(n-2)\pi}{4}\right].$$

(iv) $y = x^2 \tan^{-1} x$

Let $u = \tan^{-1} x$, $v = x^2$.

$$u_n = (-1)^{n-1}(n-1)!\frac{\sin n\theta}{r^n}, \text{where } \theta = \tan^{-1}\frac{1}{x}, r = \sqrt{1+x^2} \quad [\text{Refer Ex.12, page 12}]$$

Differentiating $n$ times using Leibnitz's Theorem,

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots + {}^nC_n u\, v_n$$

$$= (-1)^{n-1}(n-1)!\frac{\sin n\theta}{r^n}\cdot x^2 + n(-1)^{n-2}(n-2)!\frac{\sin(n-1)\theta}{r^{n-1}}\cdot 2x$$

$$+\frac{n(n-1)}{2!}(-1)^{n-3}(n-3)!\frac{\sin(n-2)\theta}{r^{n-2}}\cdot 2$$

$$= (-1)^{n-1}(n-1)!\frac{\sin n\theta}{r^n}\cdot x^2 + 2nx(-1)^{n-2}(n-2)!\frac{\sin(n-1)\theta}{r^{n-1}}$$

$$+n(n-1)(-1)^{n-3}(n-3)!\frac{\sin(n-2)\theta}{r^{n-2}}$$

**Example 4: Find $n^{\text{th}}$ order derivative of $y = x^2 e^x$ and hence, prove that**

$$\boldsymbol{y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)\,y_1 + \frac{1}{2}(n-1)(n-2)y\,.}$$

**Solution:** $y = x^2 e^x$

Let $u = e^x$, $v = x^2$

Differentiating $n$ times using Leibnitz's Theorem,

$$y_n = e^x\cdot x^2 + n\,e^x\cdot 2x + \frac{n(n-1)}{2!}e^x\cdot 2$$

$$= e^x[x^2 + 2nx + n(n-1)] \qquad \ldots (1)$$

Putting $n = 1$ and 2 successively in Eq. (1),

$$y_1 = e^x(x^2+2x), \qquad y_2 = e^x(x^2+4x+2)$$

Consider, $\frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y$

$$= \frac{1}{2} n(n-1) e^x (x^2 + 4x + 2) - n(n-2) e^x (x^2 + 2x) + \frac{1}{2} (n-1)(n-2) x^2 e^x$$
$$= e^x [x^2 + 2nx + n(n-1)] = y_n$$

**Example 5: If $y = x^n \log x$, prove that $y_{n+1} = \dfrac{n!}{x}$.**

**Solution:** $y = x^n \log x$

Differentiating w.r.t. $x$,

$$y_1 = x^n \frac{1}{x} + nx^{n-1} \cdot \log x$$
$$xy_1 = x^n + nx^n \log x$$
$$= x^n + ny$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$xy_{n+1} + ny_n = n\,! + ny_n$$
$$y_{n+1} = \frac{n!}{x}.$$

**Example 6: If $y = (x^2 - 1)^n$, prove that $(x^2 - 1)\, y_{n+2} + 2x\, y_{n+1} - n\,(n + 1)\, y_n = 0$.**

**Solution:** $y = (x^2 - 1)^n$

Differentiating w.r.t. $x$,

$$y_1 = n(x^2 - 1)^{n-1} \cdot 2x$$
$$(x^2 - 1)\, y_1 = n\,(x^2 - 1)^n\, 2x = 2nyx$$

Differentiating again w.r.t. $x$,

$$(x^2 - 1)\, y_2 + 2xy_1 = 2\,(ny_1 x + ny)$$
$$(x^2 - 1)\, y_2 + (2x - 2nx)\, y_1 = 2ny$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(x^2 - 1) y_{n+2} + n \cdot 2xy_{n+1} + \frac{n(n-1)}{2!} \cdot 2y_n + (2x - 2nx) y_{n+1} + n.2(1-n) y_n = 2ny_n$$
$$(x^2 - 1) y_{n+2} + 2xy_{n+1} - n(n+1) y_n = 0.$$

**Example 7: If $y = \sin [\log (x^2 + 2x + 1)]$, prove that $(x + 1)^2\, y_{n+2} + (2n + 1)\,(x + 1) y_{n+1} + (n^2 + 4)\, y_n = 0$.**

**Solution:** $y = \sin [\log (x^2 + 2x + 1)] = \sin [\log (x + 1)^2] = \sin [2 \log (x + 1)]$

Differentiating w.r.t. $x$,

$$y_1 = \cos[2 \log(x+1)] \cdot \frac{2}{x+1}$$
$$(x+1) y_1 = 2 \cos[2 \log(x+1)]$$

Differentiating again w.r.t. $x$,

$$(x+1)y_2 + y_1 = -2\sin[2\log(x+1)]\cdot\frac{2}{(x+1)}$$

$$(x+1)^2 y_2 + (x+1) y_1 = -4y$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(x+1)^2 y_{n+2} + n\cdot 2(x+1)y_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + (x+1)y_{n+1} + n\cdot y_n = -4y_n$$

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0.$$

**Example 8: If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.**

**Solution:** $y = a\cos(\log x) + b\sin(\log x)$

Differentiating w.r.t. $x$,

$$y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again w.r.t. $x$,

$$xy_2 + y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

$$x^2y_2 + xy_1 = -y$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$x^2 y_{n+2} + n\cdot 2x\,y_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + xy_{n+1} + ny_n = -y_n$$

$$x^2y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+1)y_n = 0.$$

**Example 9: If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$.**

**Solution:** $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n = n\log\left(\frac{x}{n}\right)$

$$\frac{y}{b} = \cos\left[n\log\left(\frac{x}{n}\right)\right]$$

$$y = b\cos\left[n\log\left(\frac{x}{n}\right)\right]$$

Differentiating w.r.t. $x$,

$$y_1 = -b\sin\left[n\log\left(\frac{x}{n}\right)\right]\cdot n\cdot\frac{1}{\frac{x}{n}}\cdot\frac{1}{n} = \frac{-bn}{x}\sin\left[n\log\left(\frac{x}{n}\right)\right]$$

$$xy_1 = -bn\sin\left[n\log\left(\frac{x}{n}\right)\right]$$

Differentiating again w.r.t. $x$,

$$xy_2 + y_1 = -bn\cos\left[n\log\left(\frac{x}{n}\right)\right] n\cdot\frac{1}{\frac{x}{n}}\cdot\frac{1}{n}$$

$$= \frac{-bn^2}{x}\cos\left[n\log\left(\frac{x}{n}\right)\right]$$

$$x^2y_2 + xy_1 = -bn^2\cos\left[n\log\left(\frac{x}{n}\right)\right] = -n^2y$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$x^2y_{n+2} + n\cdot 2xy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + xy_{n+1} + ny_n = -n^2y_n$$

$$x^2\,y_{n+2} + (2n+1)\,xy_{n+1} + 2n^2\,y_n = 0$$

**Example 10: If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that $(x^2-1)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2-m^2)\,y_n = 0$.**

**Solution:**

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$$

$$y^{\frac{1}{m}} + \frac{1}{y^{\frac{1}{m}}} = 2x$$

$$y^{\frac{2}{m}} + 1 = 2x\,y^{\frac{1}{m}}$$

$$y^{\frac{2}{m}} - 2x\,y^{\frac{1}{m}} + 1 = 0, \text{ equation is quadratic in } y^{\frac{1}{m}}.$$

Hence,

$$y^{\frac{1}{m}} = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$$y = \left(x \pm \sqrt{x^2-1}\right)^m$$

Differentiating w.r.t. $x$,

$$y_1 = m\left(x \pm \sqrt{x^2-1}\right)^{m-1}\left(1 \pm \frac{2x}{2\sqrt{x^2-1}}\right)$$

$$= m\left(x \pm \sqrt{x^2-1}\right)^{m-1}\frac{\left(\sqrt{x^2-1} \pm x\right)}{\sqrt{x^2-1}}$$

$$= \frac{m}{\sqrt{x^2-1}}\left(x \pm \sqrt{x^2-1}\right)^m$$

$$y_1\sqrt{x^2-1} = my$$

$$(x^2-1)y_1^2 = m^2y^2$$

Differentiating again w.r.t. $x$,

$$(x^2 - 1)\, 2y_1 y_2 + 2x\, y_1^2 = m^2\, 2y\, y_1$$
$$(x^2 - 1)\, y_2 + xy_1 = m^2 y$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(x^2 - 1)y_{n+2} + n \cdot 2x\, y_{n+1} + \frac{n(n-1)}{2!} \cdot 2y_n + x\, y_{n+1} + ny_n = m^2 y_n$$
$$(x^2 - 1)\, y_{n+2} + (2n + 1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$$

**Example 11: If $x = \cosh\left(\frac{1}{m}\log y\right)$, prove that $(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0$.**

**Solution:**

$$x = \cosh\left(\frac{1}{m}\log y\right)$$
$$\cosh^{-1} x = \frac{1}{m}\log y$$
$$\log y = m\log\left(x + \sqrt{x^2 - 1}\right) = \log\left(x + \sqrt{x^2 - 1}\right)^m$$
$$y = \left(x + \sqrt{x^2 - 1}\right)^m$$

Differentiating w.r.t. $x$,

$$y_1 = m\left(x + \sqrt{x^2 - 1}\right)^{m-1}\left(1 + \frac{2x}{2\sqrt{x^2 - 1}}\right)$$
$$= m\left(x + \sqrt{x^2 - 1}\right)^{m-1}\frac{\left(\sqrt{x^2 - 1} + x\right)}{\sqrt{x^2 - 1}}$$
$$= \frac{m\left(x + \sqrt{x^2 - 1}\right)^m}{\sqrt{x^2 - 1}} = \frac{my}{\sqrt{x^2 - 1}}$$
$$y_1\sqrt{x^2 - 1} = my$$
$$(x^2 - 1)y_1^2 = m^2 y^2$$

Differentiating again w.r.t. $x$,

$$(x^2 - 1)\, 2y_1 y_2 + 2x\, y_1^2 = m^2\, 2y\, y_1$$
$$(x^2 - 1)\, y_2 + xy_1 = m^2 y$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(x^2-1)y_{n+2}+n\cdot 2x\,y_{n+1}+\frac{n(n-1)}{2!}\cdot 2y_n+x\,y_{n+1}+ny_n=m^2y_n$$

$$(x^2-1)\,y_{n+2}+2nx\,y_{n+1}+n^2\,y_n-ny_n+x\,y_{n+1}+ny_n=m^2y_n$$

$$(x^2-1)\,y_{n+2}+(2n+1)\,x\,y_{n+1}+(n^2-m^2)\,y_n=0.$$

**Example 12: If $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, prove that $(1-x^2)\,y_{n+1}-(2n+1)\,xy_n-n^2\,y_{n-1}=0$.**

**Solution:**

$$y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$y\sqrt{1-x^2}=\sin^{-1}x$$

$$y^2(1-x^2)=(\sin^{-1}x)^2$$

Differentiating the above equation w.r.t. $x$,

$$2yy_1(1-x^2)+y^2(-2x)=2\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}=2y$$

$$(1-x^2)\,y_1-xy=1$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(1-x^2)y_{n+1}+n(-2x)\,y_n+\frac{n(n-1)}{2!}(-2)y_{n-1}-xy_n-ny_{n-1}=0$$

$$(1-x^2)\,y_{n+1}-(2n+1)\,xy_n-n^2\,y_{n-1}=0.$$

**Example 13: If $y=\sec^{-1}x$, prove that**
**$x\,(x^2-1)\,y_{n+2}+[(2+3n)\,x^2-(n+1)]\,y_{n+1}+n\,(3n+1)\,xy_n+n^2\,(n-1)\,y_{n-1}=0.$**

**Solution:** $y=\sec^{-1}x$

Differentiating w.r.t. $x$,

$$y_1=\frac{-1}{x\sqrt{x^2-1}}$$

$$x^2\,(x^2-1)\,y_1^{\,2}=1$$

Differentiating again w.r.t. $x$,

$$2x\,(x^2-1)\,y_1^{\,2}+x^2\cdot 2x\,y_1^{\,2}+x^2\,(x^2-1)\cdot 2y_1\,y_2=0$$

$$(x^2-1)\,y_1+x^2\,y_1+x\,(x^2-1)\,y_2=0$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(x^2-1)y_{n+1}+n\cdot 2x\,y_n+\frac{n(n-1)}{2!}\cdot 2y_{n-1}+x^2y_{n+1}+n\cdot 2x\,y_n+\frac{n(n-1)}{2!}\cdot 2y_{n-1}$$

$$+x(x^2-1)\,y_{n+2}+n(3x^2-1)\,y_{n+1}+\frac{n(n-1)}{2!}\cdot 6x\,y_n+\frac{n(n-1)(n-2)}{3!}\cdot 6y_{n-1}=0$$

$$x\,(x^2-1)\,y_{n+2}+[(2+3n)\,x^2-(n+1)]\,y_{n+1}+n\,(3n+1)\,xy_n+n^2\,(n-1)\,y_{n-1}=0$$

**Example 14: If $y=\sinh^{-1}x$, prove that $(1+x^2)\,y_{n+2}+(2n+1)\,xy_{n+1}+n^2y_n=0$.**

**Solution:** $y=\sinh^{-1}x=\log\left(x+\sqrt{x^2+1}\right)$

Differentiating w.r.t. $x$,

$$y_1=\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)=\frac{1}{x+\sqrt{x^2+1}}\cdot\frac{\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}}$$

$$(x^2+1)\,y_1^{\,2}=1$$

Differentiating again w.r.t. $x$,

$$(x^2+1)\,2y_1\,y_2+2xy_1^{\,2}=0$$

$$(x^2+1)\,y_2+xy_1=0$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(x^2+1)\,y_{n+2}+n\cdot 2y_{n+1}+\frac{n(n-1)}{2!}\cdot 2y_n+xy_{n+1}+ny_n=0$$

$$(x^2+1)\,y_{n+2}+(2n+1)\,xy_{n+1}+n^2y_n=0$$

**Example 15: If $y=e^{a\sin^{-1}x}$, prove that $(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+a^2)\,y_n=0$.**

**Solution:** $y=e^{a\sin^{-1}x}$

Differentiating w.r.t. $x$,

$$y_1=e^{a\sin^{-1}x}\cdot\frac{a}{\sqrt{1-x^2}}$$

$$y_1\sqrt{1-x^2}=ay$$

$$(1-x^2)\,y_1^{\,2}=a^2y^2$$

Differentiating again w.r.t. $x$,

$$(1-x^2)\,2y_1\,y_2+(-2x)\,y_1^{\,2}=a^2\,2y\,y_1$$
$$(1-x^2)\,y_2-xy_1=a^2y$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(1-x^2)\,y_{n+2}+n(-2x)\,y_{n+1}+\frac{n(n-1)}{2!}(-2)y_n-xy_{n+1}-ny_n=a^2y_n$$
$$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+a^2)\,y_n=0.$$

**Example 16: If $y=\tan^{-1}\left(\dfrac{a+x}{a-x}\right)$, prove that $(a^2+x^2)\,y_{n+2}+2\,(n+1)\,x\,y_{n+1}+n\,(n+1)\,y_n=0.$**

**Solution:** $$y=\tan^{-1}\left(\frac{a+x}{a-x}\right)$$

Putting $x=a\tan\theta$,

$$y=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)=\tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right)$$
$$=\frac{\pi}{4}+\theta=\frac{\pi}{4}+\tan^{-1}\left(\frac{x}{a}\right)$$

Differentiating w.r.t. $x$,

$$y_1=\frac{1}{1+\dfrac{x^2}{a^2}}\cdot\frac{1}{a}=\frac{a}{x^2+a^2}$$

$$(x^2+a^2)\,y_1=a$$
$$(x^2+a^2)\,y_2+2x\,y_1=0$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(x^2+a^2)y_{n+2}+n\cdot 2x\,y_{n+1}+\frac{n(n-1)}{2!}\cdot 2\,y_n+2x\,y_{n+1}+n\cdot 2\,y_n=0$$
$$(x^2+a^2)\,y_{n+2}+2(n+1)\,x\,y_{n+1}+n\,(n+1)\,y_n=0.$$

**Example 17: If $y=\sqrt{\dfrac{1+x}{1-x}}$, prove that $y=(1-x^2)\,y_1$ and hence, deduce that $(1-x^2)\,y_n-[2\,(n-1)\,x+1]\,y_{n-1}-(n-1)\,(n-2)\,y_{n-2}=0.$**

**Solution:** $$y=\sqrt{\frac{1+x}{1-x}}$$

$$\log y=\log\sqrt{\frac{1+x}{1-x}}$$
$$=\frac{1}{2}\left[\log(1+x)-\log(1-x)\right]$$

Differentiating w.r.t. $x$,

$$\frac{1}{y}\cdot y_1 = \frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right) = \frac{1}{1-x^2}$$

$$(1-x^2)\, y_1 = y$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(1-x^2)y_{n+1} + n(-2x)y_n + \frac{n(n-1)}{2!}(-2)y_{n-1} = y_n$$

Replacing $n$ by $n-1$,

$$(1-x^2)y_n - [2(n-1)x+1]y_{n-1} - (n-1)(n-2)y_{n-2} = 0.$$

**Example 18: If $f(x) = \tan x$, prove that**

$$f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) + \ldots\ldots = \sin\frac{n\pi}{2}.$$

**Solution:**

$$f(x) = \tan x$$

$$= \frac{\sin x}{\cos x}$$

$$\cos x \cdot f(x) = \sin x$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$\cos x\, f^n(x) + {}^nC_1(-\sin x) f^{n-1}(x) + {}^nC_2(-\cos x) f^{n-2}(x) + {}^nC_3(\sin x) f^{n-3}(x)$$

$$+ {}^nC_4(\cos x) f^{n-4}(x) + \ldots\ldots\ldots + f(x)\cdot\cos\left(x+\frac{n\pi}{2}\right) = \sin\left(x+\frac{n\pi}{2}\right)$$

Putting $x = 0$,

$$f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) + \ldots\ldots\ldots\ldots + f(0)\cos\left(\frac{n\pi}{2}\right) = \sin\frac{n\pi}{2}.$$

**Example 19: If $y = \left[\log\left(x+\sqrt{1+x^2}\right)\right]^2$, prove that $y_{n+2}(0) = -n^2 y_n(0)$.**

**Solution:** $y = \left[\log\left(x+\sqrt{1+x^2}\right)\right]^2$

Differentiating w.r.t. $x$,

$$y_1 = 2\left[\log\left(x+\sqrt{1+x^2}\right)\right]\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right)$$

$$= 2\log\left(x+\sqrt{1+x^2}\right)\cdot\frac{1}{\sqrt{1+x^2}}$$

$$y_1\sqrt{1+x^2} = 2\log\left(x+\sqrt{1+x^2}\right)$$

$$(1+x^2)y_1^{\,2} = 4\left[\log\left(x+\sqrt{1+x^2}\right)\right]^2 = 4y$$

Differentiating again w.r.t. $x$,

$$(1+x^2)\,2y_1\,y_2 + 2xy_1^{\,2} = 4y_1$$

$$(1+x^2)\,y_2 + xy_1 = 2$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(1+x^2)y_{n+2} + n\cdot 2xy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + xy_{n+1} + ny_n = 0$$

$$(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + n^2\,y_n = 0$$

Putting $x = 0$,

$$(y_{n+2})_0 = -n^2\,(y_n)_0.$$

**Example 20: If $y = \left(x+\sqrt{1+x^2}\right)^m$, prove that**

**(i) $(y_{2n})_0 = [m^2-(2n-2)^2]\,[m^2-(2n-4)^2]\,\ldots\,[m^2-2^2]\,m^2$.**

**(ii) $(y_{2n+1})_0 = [m^2-(2n-1)^2]\,[m^2-(2n-3)^2]\,\ldots\,[m^2-1^2]\,m$.**

**Solution:** $y = \left(x+\sqrt{1+x^2}\right)^m$

Differentiating w.r.t. $x$,

$$y_1 = m\left(x+\sqrt{1+x^2}\right)^{m-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$\sqrt{1+x^2}\cdot y_1 = m\left(x+\sqrt{1+x^2}\right)^m = my \qquad \ldots(1)$$

$$(1+x^2)\,y_1^{\,2} = m^2\,y^2$$

Differentiating again w.r.t. $x$,

$$(1+x^2)\,2y_1\,y_2 + 2xy_1^{\,2} = m^2\,2yy_1$$

$$(1+x^2)\,y_2 + xy_1 = m^2\,y \qquad \ldots(2)$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(1+x^2)y_{n+2} + n\cdot 2xy_{n+1} + \frac{n(n-1)}{2!}\cdot 2y_n + xy_{n+1} + ny_n = m^2\,y_n$$

$$(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0 \qquad \ldots(3)$$

Putting $x = 0$ in Eqs (1), (2) and (3),

$$(y_1)_0 = m,\ (y_2)_0 = m^2 \qquad [\because\ y(0) = 1]$$

and

$$(y_{n+2})_0 = (m^2-n^2)\,y_n(0) \qquad \ldots(4)$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (4),

$$y_3(0) = (m^2 - 1^2)\, y_1(0) = (m^2 - 1^2)\, m$$
$$y_4(0) = (m^2 - 2^2)\, y_2(0) = (m^2 - 2^2)\, m^2$$
$$y_5(0) = (m^2 - 3^2)\, y_3(0) = (m^2 - 3^2)(m^2 - 1^2)\, m$$
$$y_6(0) = (m^2 - 4^2)\, y_4(0) = (m^2 - 4^2)(m^2 - 2^2)\, m^2 \text{ and so on.}$$

In general,

Even derivative, $y_{2n}(0) = [m^2 - (2n-2)^2]\,[m^2 - (2n-4)^2] \ldots (m^2 - 2^2)\, m^2$

Odd derivative, $y_{2n+1}(0) = [m^2 - (2n-1)^2]\,[m^2 - (2n-3)^2] \ldots (m^2 - 1^2)\, m.$

**Example 21: If $y = \log\left(x + \sqrt{x^2+1}\right)$, prove that**

**$y_{2n}(0) = 0$ and $y_{2n+1}(0) = (-1)^n\,[1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2]$.**

**Solution:** $y = \log\left(x + \sqrt{x^2+1}\right)$

Differentiating w.r.t. $x$,

$$y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$$

$$\sqrt{1+x^2} \cdot y_1 = 1 \qquad \ldots (1)$$

$$(x^2+1)\, y_1^2 = 1$$

Differentiating again w.r.t. $x$,

$$(x^2+1)\, 2y_1 y_2 + 2xy_1^2 = 0$$

$$(x^2+1)\, y_2 + xy_1 = 0 \qquad \ldots (2)$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(x^2+1)\, y_{n+2} + n \cdot 2xy_{n+1} + \frac{n(n-1)}{2!} \cdot 2y_n + xy_{n+1} + ny_n = 0$$

$$(x^2+1)\, y_{n+2} + (2n+1)\, xy_{n+1} + n^2 y_n = 0 \qquad \ldots (3)$$

Putting $x = 0$ in Eqs (1), (2) and (3),

$$y_1(0) = 1 \text{ and } y_2(0) = 0$$

$$\text{and } y_{n+2}(0) = -n^2\, y_n(0) \qquad \ldots (4)$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (4),

$$y_3(0) = -1^2\, y_1(0) = -1 \cdot 1$$
$$y_4(0) = -2^2\, y_2(0) = 0$$
$$y_5(0) = -3^2\, y_3(0) = -3^2(-1) \cdot 1^2\, (-1)^2 \cdot 1^2 \cdot 3^2$$
$$y_6(0) = 0$$
$$y_7(0) = -5^2\, y_5(0) = -5^2\,(-1)^2 \cdot 1^2 \cdot 3^2 = (-1)^3 \cdot 1^2 \cdot 3^2 \cdot 5^2 \text{ etc.}$$

In general,

Even derivative, $y_{2n}(0) = 0$

Odd derivative, $y_{2n+1}(0) = (-1)^n [1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-1)^2]$

**Example 22: If $y = (\sin^{-1} x)^2$, prove that (i) $(1 - x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0$. (ii) $y_{2n+1}(0) = 0$ and $y_{2n}(0) = 2^{2n-1}\,[(n-1)!]^2$.**

**Solution:** (i) $y = (\sin^{-1} x)^2$

Differentiating w.r.t. $x$,

$$y_1 = 2\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} \qquad \text{... (1)}$$

$$(1 - x^2)y_1^2 = 4 (\sin^{-1} x)^2 = 4y$$

Differentiating again w.r.t. $x$,

$$(1 - x^2)2y_1 y_2 - 2xy_1^2 = 4y_1$$

$$(1 - x^2)y_2 - xy_1 = 2 \qquad \text{... (2)}$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(1-x^2)y_{n+2} + n\cdot(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - xy_{n+1} - ny_n = 0$$

$$(1 - x^2)y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0 \qquad \text{... (3)}$$

(ii) Putting $x = 0$ in Eq. (1), (2) and (3),

$$y_1(0) = 0,\ y_2(0) = 2 = 2^{2-1}\,[(1-1)!]^2$$

$$y_{n+2}(0) = n^2\, y_n(0)$$

Putting $n = 1, 2, 3, 4, \ldots,$ in Eq. (4) ... (4)

$$y_3(0) = 1^2\, y_1(0) = 0$$

$$y_4(0) = 2^2\, y_2(0) = 2^2 \cdot 2 = 2^3 = 2^{4-1}\,[(2-1)!]^2$$

$$y_5(0) = 0$$

$$y_6(0) = 4^2\, y_4(0) = 4^2 \cdot 2^3 = 2^{6-1}\,[(3-1)!]^2 \text{ etc.}$$

In general,

Even derivative, $y_{2n}(0) = 2^{2n-1}\,[(n-1)!]^2$

Odd derivative, $y_{2n+1}(0) = 0.$

**Example 23: If $y = \sin(m \sin^{-1} x)$, prove that**

**(i) $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 - m^2)\, y_n = 0.$**

**(ii) $(y_n)_0 = [(n-2)^2 - m^2]\,[(n-4)^2 - m^2] \ldots (1 - m^2)\, m$, if $n$ is odd**
**$= 0$, if $n$ is even.**

**Solution:** (i) $y = \sin(m \sin^{-1} x)$

Differentiating w.r.t. $x$,

$$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}} \qquad \text{... (1)}$$

$$\sqrt{1-x^2} \cdot y_1 = m \cos(m \sin^{-1} x)$$

$$(1-x^2)\,y_1^{\,2} = m^2 \cos^2 (m \sin^{-1} x)$$
$$= m^2 [1 - \sin^2 (m \sin^{-1} x)]$$
$$(1-x^2)\,y_1^{\,2} = m^2 (1 - y^2)$$

Differentiating again w.r.t. $x$,

$$(1-x^2)\,2y_1 y_2 + y_1^{\,2}(-2x) = m^2 (-2yy_1)$$
$$(1-x^2)\,y_2 - xy_1 = -m^2 y \qquad \ldots (2)$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - xy_{n+1} - ny_n = -m^2 y_n$$

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2 - m^2)\,y_n = 0 \qquad \ldots (3)$$

(ii) Putting $x = 0$ in Eqs (1), (2) and (3),

$$y_1(0) = \cos\left(m\sin^{-1} 0\right)\cdot\frac{m}{\sqrt{1-0}} = m$$

$$y_1(0) = m$$

$$y_2(0) = -\,m^2\,y(0) = 0$$

also, $$y_{n+2}(0) = (n^2 - m^2)\,y_n(0) \qquad \ldots (4)$$

Putting $n = 1, 2, 3, \ldots$ in Eq. (4),

$$y_3(0) = (1^2 - m^2)\,y_1(0) = (1^2 - m^2)\,m$$
$$y_4(0) = (2^2 - m^2)\,y_2(0) = 0$$
$$y_5(0) = (3^2 - m^2)\,y_3(0) = (3^2 - m^2)(1^2 - m^2)\,m$$
$$y_6(0) = (4^2 - m^2)\,y_4(0) = 0 \text{ etc.}$$

In general,

$$y_n(0) = [(n-2)^2 - m^2]\,[(n-4)^2 - m^2] \ldots (1^2 - m^2)\,m, \text{ if } n \text{ is odd}$$
$$= 0, \text{ if } n \text{ is even.}$$

**Example 24: If $y = \tan^{-1} x$, prove that (i) $(x^2+1)\,y_{n+1} + 2nxy_n + n(n-1)\,y_{n-1} = 0$ (ii) $y_n(0) = 0$, if $n$ is even**

$$= (-1)^{\frac{n-1}{2}}\,(n-1)!, \text{ if } n \text{ is odd.}$$

**Solution:** (i) $y = \tan^{-1} x$

Differentiating w.r.t. $x$,

$$y_1 = \frac{1}{1+x^2}$$

$$(x^2+1)\,y_1 = 1 \qquad \ldots (1)$$

Differentiating again w.r.t. $x$,

$$(x^2+1)\,y_2 + 2\,x\,y_1 = 0 \qquad \ldots (2)$$

Differentiating Eq. (1) $n$ times using Leibnitz's Theorem,

$$(x^2+1)y_{n+1}+n\cdot 2xy_n+\frac{n(n-1)}{2!}\cdot 2y_{n-1}=0$$

$$(x^2+1)\,y_{n+1}+2nxy_n+n\,(n-1)\,y_{n-1}=0 \qquad \text{... (3)}$$

(ii) Putting $x = 0$ in Eqs (1), (2) and (3),

$$y_1(0)=1,\ y_2(0)=0$$

$$y_{n+1}(0)=-n(n-1)\,y_{n-1}(0) \qquad \text{... (4)}$$

Putting $n = 2, 3, 4, \ldots$ in Eq. (4),

$$y_3(0)=-2(2-1)\,y_1(0)=-2=-(2!)=(-1)^{\frac{3-1}{2}}(2!)$$

$$y_4(0)=-3(3-1)\,y_2(0)=0$$

$$y_5(0)=-4(4-1)\,y_3(0)=-4(3)(-2)=(-1)^2(4!)=(-1)^{\frac{5-1}{2}}(4!)$$

$$y_6(0)=-5(4)\,y_4(0)=0$$

$$y_7(0)=-6(5)\,y_5(0)=-6(5)(-1)^2(4!)=(-1)^3(6!)=(-1)^{\frac{7-1}{2}}(6!)$$

In general,

$$y_n=0,\text{ if } n \text{ is even}$$

$$=(-1)^{\frac{n-1}{2}}(n-1)!,\text{ if } n \text{ is odd.}$$

**Example 25: If $y=e^{m\cos^{-1}x}$, find $(y_n)(0)$.**

**Solution:** $y=e^{m\cos^{-1}x}$

Differentiating w.r.t. $x$,

$$y_1=e^{m\cos^{-1}x}\left(\frac{-m}{\sqrt{1-x^2}}\right) \qquad \text{... (1)}$$

$$(1-x^2)\,y_1^2=m^2y^2$$

Differentiating again w.r.t. $x$

$$(1-x^2)\,2y_1y_2-2xy_1^2=2m^2yy_1$$

$$(1-x^2)\,y_2-xy_1=m^2y \qquad \text{... (2)}$$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$(1-x^2)\,y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{2!}(-2)y_n-xy_{n+1}-ny_n=m^2y_n$$

$$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+m^2)\,y_n=0 \qquad \text{... (3)}$$

Putting $x = 0$ in Eqs (1), (2) and (3),

$$y_1(0)=-me^{m\cos^{-1}0}=-me^{\frac{m\pi}{2}},\quad y_2(0)=m^2\,y(0)=m^2e^{\frac{m\pi}{2}}$$

$$y_{n+2}(0)=(n^2+m^2)\,y_n(0) \qquad \text{... (4)}$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (4),

$$y_3(0) = (1^2 + m^2)\, y_1(0) = -me^{\frac{m\pi}{2}}(1^2 + m^2)$$

$$y_4(0) = (2^2 + m^2)\, y_2(0) = m^2 e^{\frac{m\pi}{2}}(2^2 + m^2)$$

$$y_5(0) = (3^2 + m^2)\, y_3(0) = -me^{\frac{m\pi}{2}}(1^2 + m^2)(3^2 + m^2)$$

$$y_6(0) = (4^2 + m^2)\, y_4(0) = m^2 e^{\frac{m\pi}{2}}(2^2 + m^2)(4^2 + m^2)$$

In general,

Even derivative, $\quad y_{2n}(0) = m^2 e^{\frac{m\pi}{2}}(2^2 + m^2)(4^2 + m^2)\ldots[(2n-2)^2 + m^2]$

Odd derivative, $\quad y_{2n+1}(0) = -me^{\frac{m\pi}{2}}(1^2 + m^2)(3^2 + m^2)\ldots[(2n-1)^2 + m^2]$

**Example 26: If $x + y = 1$, prove that**

$$\frac{d^n}{dx^n}(x^n y^n) = n!\Big[y^n - \left({}^nC_1\right)^2 y^{n-1}x + \left({}^nC_2\right)^2 y^{n-2}x^2 - \left({}^nC_3\right)^2 y^{n-3}x^3 + \ldots\ldots\ldots + (-1)^n x^n\Big].$$

**Solution:** $x + y = 1,\ y = 1 - x,\ y_1 = -1$

Differentiating $n$ times using Leibnitz's Theorem,

$$\frac{d^n}{dx^n}x^n y^n = \frac{d^n}{dx^n}[x^n(1-x)^n]$$

$$= n!(1-x)^n + n\cdot\frac{n!}{1!}x\cdot n(1-x)^{n-1}(-1) + \frac{n(n-1)}{2!}\frac{n!}{2!}x^2 n(n-1)(1-x)^{n-2}(-1)^2$$
$$+\frac{n(n-1)(n-2)}{3!}\frac{n!}{3!}x^3 n(n-1)(n-2)(1-x)^{n-3}(-1)^3 + \ldots\ldots\ldots + (-1)^n x^n$$

$$\left[\because \frac{d^n(ax+b)^m}{dx^n} = \frac{a^n m!(ax+b)^{m-n}}{(m-n)!}\right]$$

$$= n!\Big[y^n - \left({}^nC_1\right)^2 y^{n-1}x + \left({}^nC_2\right)^2 y^{n-2}x^2 - \left({}^nC_3\right)^2 y^{n-3}x^3 + \ldots\ldots\ldots + (-1)^n x^n\Big]$$

$$[\because (1-x) = y]$$

**Example 27: By finding two different ways the $n^{\text{th}}$ derivative of $x^{2n}$, prove that**

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2\cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2} + \ldots\ldots\ldots\ldots\ldots = \frac{(2n)!}{(n!)^2}.$$

**Solution:** Let $y = x^{2n}$

$$y_n = \frac{(2n)!}{(2n-n)!}x^{2n-n} \qquad \left[\because \frac{d^n(ax+b)^m}{dx^n} = \frac{a^n m!(ax+b)^{m-n}}{(m-n)!}\right]$$

$$= \frac{(2n)!}{n!}x^n \qquad \text{... (1)}$$

Now, $y = x^n \cdot x^n$

Differentiating the above equation $n$ times using Leibnitz's Theorem,

$$y_n = x^n \cdot n! + n \cdot nx^{n-1} \cdot \frac{n!}{(n-n+1)!}x^{n-(n-1)} + \frac{n(n-1)}{2!}n(n-1)x^{n-2}\frac{n!}{(n-n+2)!}x^{n-(n-2)}$$
$$+ \frac{n(n-1)(n-2)}{3!}n(n-1)(n-2)x^{n-3}\frac{n!}{(n-n+3)!}x^{n-(n-3)} + \ldots\ldots\ldots\ldots$$
$$= x^n \cdot n! + n^2 \cdot \frac{n!}{1!}x^n + \frac{n^2(n-1)^2}{(2!)^2}n!x^n + \frac{n^2(n-1)^2(n-2)^2}{(3!)^2}n!x^n + \ldots\ldots\ldots\ldots$$
$$= x^n \cdot n!\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 \cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \ldots\ldots\ldots\ldots\right] \qquad \text{... (2)}$$

Equating coefficients of $x^n$ in Eqs (1) and (2),

$$n!\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 \cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \ldots\ldots\ldots\ldots\right] = \frac{(2n)!}{(n!)}$$

Hence, $1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 \cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \ldots\ldots\ldots\ldots\ldots\ldots y = \frac{(2n)!}{(n!)^2}$

**Example 28: If $y = \frac{\log x}{x}$, prove that $y_n = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right]$.**

**Solution:** $y = \frac{\log x}{x}$

Differentiating $n$ times using Leibnitz's Theorem,

$$y_n = \frac{(-1)^n n!}{x^{n+1}} \cdot \log x + n\frac{(-1)^{n-1}(n-1)!}{x^n} \cdot \frac{1}{x} + \frac{n(n-1)}{2!}\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\left(-\frac{1}{x^2}\right)$$
$$+ \frac{n(n-1)(n-2)}{3!} \cdot \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}\left(\frac{2}{x^3}\right) + \ldots\ldots\ldots\ldots + \frac{1}{x} \cdot \frac{(-1)^{n-1}(n-1)!}{x^n}$$

[Using result (2) and (3)]

$$= \frac{(-1)^n n!}{x^{n+1}} \cdot \log x - \frac{(-1)^n n!}{x^{n+1}} - \frac{(-1)^n n!}{2x^{n+1}} - \frac{(-1)^n n!}{3x^{n+1}} - \ldots\ldots\ldots\ldots - \frac{(-1)^n n!}{nx^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}} \left[ \log x - \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right) \right].$$

## Exercise 2.2

**1.** Find the $n^{\text{th}}$ order derivative w.r.t. $x$

(i) $xe^x$ (ii) $x^2 e^{2x}$

(iii) $x \log (x + 1)$ (iv) $x^3 \sin 2x$

(v) $y = x^2 \sin x$.

**Ans.:** (i) $e^x (x+n)$

(ii) $e^{2x} [2^n x^2 + 2^n nx + n(n-1)2^{n-1}]$

(iii) $\dfrac{(-1)^{n-2}(n-2)!(x+n)}{(x+1)^n}$

(iv) $2^n x^3 \sin\left(2x + \dfrac{n\pi}{2}\right) +$

$$3nx^2 2^{n-1} \sin\left[2x + (n-1)\frac{\pi}{2}\right]$$

$$+3n(n-1)x2^{n-2} \sin\left[2x + (n-2)\frac{\pi}{2}\right]$$

$$+n(n-1)(n-2)2^{n-3}$$

$$\sin\left[2x + (n-3)\frac{n\pi}{2}\right]$$

(v) $x^2 \sin\left(x + \dfrac{n\pi}{2}\right)$

$$+2nx \sin\left[x + (n-1)\frac{\pi}{2}\right]$$

$$+(n^2 - n)\sin\left[x + (n-2)\frac{\pi}{2}\right]$$

**2.** If $y = 7^x x^7$, find $y_5$.

**Ans.:** $(\log 7)^5 7^x x^7$

$$+35(\log 7)^4 7^x x^6$$

$$+420(\log 7)^3 7^x x^5$$

$$+2100(\log 7)^2 7^x x^4$$

$$+4200(\log 7) 7^x x^3$$

$$+2520 7^x x^2$$

**3.** If $y = e^{ax} [a^2 x^2 - 2nax + n(n+1)]$, prove that $y_n = a^{n+2} x^2 e^{ax}$.

**4.** If $y = x^2 \sin x$, prove that

$$y_n = (x^2 - n^2 + n)\sin\left(x + \frac{n\pi}{2}\right) - 2nx\cos\left(x + \frac{n\pi}{2}\right).$$

**5.** If $x = \tan \log y$, prove that

$$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0.$$

[**Hint:** $\log y = \tan^{-1} x,\ y = e^{\tan^{-1} x}$]

**6.** If $y = \cos (m \sin^{-1} x)$, prove that $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0$

Hence, obtain $y_n(0)$.

**Ans.:** $y_n(0) = (n^2 - m^2)\ldots\ldots\ldots\ldots (4^2 - m^2)(2^2 - m^2)(-m^2)$

**7.** If $x = \sin \theta$, $y = \sin 2\theta$, prove that $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} - (n^2 - 4) y_n = 0$.

[**Hint:** $y = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$]

**8.** If $y = x^2 e^x$, prove that

$$y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y.$$

## 2.4 MEAN VALUE THEOREM

The roots of the given function as well as equality or inequality of any two or more than two functions can be determined using Mean Value Theorems. These theorems are Rolle's Theorem, Lagrange's Mean Value Theorem, and Cauchy's Mean Value Theorem. For better understanding of these theorems, we shall first learn two type of functions.

### 2.4.1 Continuous and Differentiable Functions

A function $f(x)$ is said to be continuous at $x = a$ if

(i) $f(a)$ is finite, i.e., $f(a)$ exists.

(ii) $\lim_{x\to a} f(x) = \lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = f(a)$

A function $f(x)$ is said to be differentiable at $x = a$, if Right Hand Derivative (RHD) and Left Hand Derivative (LHD) exists and RHD = LHD.

i.e.
$$\lim_{x\to a+} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a-} \frac{f(x)-f(a)}{x-a}$$

or
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(a-h)-f(a)}{h}$$

**Note:**

(i) If function $f(x)$ is finitely differentiable (derivative is finite) in the interval $(a, b)$, then it is continuous in the interval $[a, b]$, i.e., every differentiable function is continuous in the given interval. But converse is not necessarily true, i.e., a function may be continuous for a value of $x$ without being differentiable for that value. The interval $(a, b)$ is also called as domain of the function.

(ii) Algebraic, trigonometric, inverse trigonometric, logarithmic and exponential function are ordinarily continuous and differentiable (with some exceptions).

(iii) Addition, subtraction, product and quotient of two or more continuous and differentiable functions are also continuous and differentiable.

(iv) $f(x)$ and $f'(x)$ are differentiable and continuous if $f''(x)$ exists.

(v) A function is said to be differentiable if its derivative is neither indeterminate nor infinite.

## 2.5 ROLLE'S THEOREM

**Statement:** If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$

(ii) differentiable in the open interval $(a, b)$

(iii) $f(a) = f(b)$

then there exists at least one point $c$ in the open interval $(a, b)$ such that $f'(c) = 0$.

**Proof:** Since function $f(x)$ is continuous in the closed interval $[a, b]$, it attains its maxima and minima at some points in the interval. Let $M$ and $m$ be the maximum and minimum values of $f(x)$ respectively at some points $c$ and $d$ respectively in the interval $[a, b]$.

$$f(c) = M \qquad \text{and} \qquad f(d) = m$$

Now two cases arise:

**Case I:** If $M = m$

$f(x) = M = m$ for all $x$ in $[a, b]$

$f(x) =$ constant for all $x$ in $[a, b]$

$f'(x) = 0$ for all $x$ in $[a, b]$

Hence, the theorem is true.

**Case II:** If $M \neq m$

Since $f(a) = f(b)$, either $M$ or $m$ must be different from $f(a)$ and $f(b)$.

Let $M$ is different from $f(a)$ and $f(b)$.

$f(c)$ is different from $f(a)$ and $f(b)$.

$$f(c) \neq f(a) \qquad \therefore c \neq a$$

Also, $$f(c) \neq f(b) \qquad \therefore c \neq b$$

Hence, $$a < c < b$$

Now, since $f(x)$ is differentiable in the open interval $(a, b)$, $f'(c)$ exists.
By definition,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Since $$f(c) = M,\ f(c+h) \leq f(c)$$

$$\frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{for} \quad h > 0 \qquad \text{... (1)}$$

and $$\frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{for} \quad h < 0 \qquad \text{... (2)}$$

As $h \to 0$, Eq. (1) gives $f'(c) \leq 0$ and Eq. (2) gives $f'(c) \geq 0$

Since $f(x)$ is differentiable, $f'(c)$ must be unique.

Hence, $$f'(c) = 0 \quad \text{for} \quad a < c < b$$

Similarly, it can be proved that $f'(c) = 0$ for $a < c < b$ if $m$ is different from $f(a)$ and $f(b)$.

**Note:**

(i) There may be more than one point $c$, such that, $f'(c) = 0$.

(ii) The converse of the theorem is not true, i.e., for some function $f(x)$, $f'(c) = 0$ but $f(x)$ may not satisfy the conditions of Rolle's theorem.

e.g.,
$$f(x) = 1 - 3\,(x-1)^{\frac{2}{3}} \text{ in } 0 \le x \le 10$$
$$f'(x) = 1 - \frac{2}{(x-1)^{\frac{1}{3}}}$$

$f'(c) = 0$ at $c = 9$. But $f'(x)$ does not exist at $x = 1$, i.e., not differentiable at $x = 1$. Hence, $f(x)$ does not satisfy the conditions of Rolle's theorem.

### 2.5.1 Another Form of Rolle's Theorem

If a function $f(x)$ is

(i) continuous in the closed interval $[a, a+h]$
(ii) differentiable in the open interval $(a, a+h)$
(iii) $f(a) = f(a+h)$, then there exists at least one real number $\theta$ between 0 and 1 such that $f'(a+\theta h) = 0$, for $0 < \theta < 1$.

### 2.5.2 Geometrical Interpretation of Rolle's Theorem

Let $y = f(x)$ represents a curve with $A\ [a, f(a)]$ and $B\ [b, f(b)]$ as end points and $C\ [c, f(c)]$ be any point between $A$ and $B$.

$f'(c)$ = slope of the tangent at point $C$

Thus, geometrically the theorem states that if

(i) curve is continuous at the points $A$, $B$ and at every point between $A$ and $B$, i.e., in the interval $[a, b]$.
(ii) possesses unique tangent at every point between $A$ and $B$.
(iii) ordinates of the points $A$ and $B$ are same, i.e., $f(a) = f(b)$, then there exists at least one point $C\ [c, f(c)]$ on the curve between $A$ and $B$, tangent at which is parallel to $x$-axis.

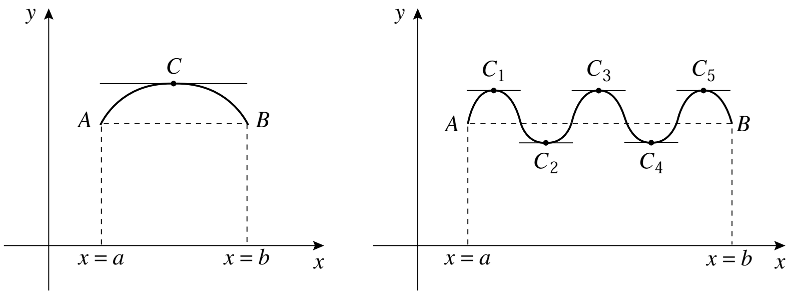



**Fig. 2.1**

### 2.5.3 Algebraic Interpretation of Rolle's Theorem

Let $f(x)$ be a polynomial in $x$. If $f(x) = 0$ satisfies all the conditions of Rolle's theorem and $x = a$, $x = b$ be the roots of the equation $f(x) = 0$, then at least one root of the equation $f'(x) = 0$ lies between $a$ and $b$.

**Example 1: Verify Rolle's theorem for the following functions:**

**(i)** $f(x) = (x-a)^m (x-b)^n$ **in** $[a, b]$**, where** $m, n$ **are positive integers.**

**(ii)** $f(x) = x(x+3)e^{-\frac{x}{2}}$ **in** $-3 \le x \le 0$

**(iii)** $f(x) = |x|$ **in** $[-1, 1]$

**(iv)** $f(x) = \dfrac{\sin x}{e^x}$ **in** $[\theta, \pi]$

**(v)** $f(x) = e^x(\sin x - \cos x)$ **in** $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

**(vi)** $f(x) = \log\left[\dfrac{x^2+ab}{(a+b)x}\right]$ **in** $[a,b]$, $a > 0,\ b > 0$

**(vii)** $f(x) = x^2 + 1 \qquad 0 \le x \le 1$

$\qquad = 3 - x \qquad 1 \le x \le 2$

**(viii)** $f(x) = x^2 - 2 \qquad -1 \le x \le 0$

$\qquad = x - 2 \qquad 0 \le x \le 1.$

**Solution:** (i) $f(x) = (x-a)^m (x-b)^n$ in $[a, b]$, where $m, n$ are positive integers.

(a) Since $m$ and $n$ are positive integers,
$f(x) = (x-a)^m (x-b)^n$, being a polynomial, is continuous in $[a, b]$.

(b) $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)]$

$= (x-a)^{m-1}(x-b)^{n-1}[(m+n)x - (mb+na)]$

exists for every value of $x$ in $(a, b)$. Therefore, $f(x)$ is differentiable in $(a, b)$.

(c) $f(a) = f(b) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that $f'(c) = 0$.

$$(c-a)^{m-1}(c-b)^{n-1}[(m+n)c - (mb+na)] = 0$$

$$c = \frac{mb+na}{m+n}$$

which represents a point that divides the interval $[a, b]$ internally in the ratio of $m:n$. Thus, $c$ lies in $(a, b)$.
Hence, theorem is verified.

(ii) $f(x) = x(x+3)e^{-\frac{x}{2}}$ in $-3 \le x \le 0$

(a) $f(x) = x(x+3)e^{-\frac{x}{2}}$, being composite function of continuous function, is continuous in $[-3, 0]$.

(b) $f'(x) = (x+3)e^{-\frac{x}{2}} + xe^{-\frac{x}{2}} - \dfrac{x(x+3)}{2}e^{-\frac{x}{2}}$

exists for every value of $x$ in (–3, 0). Therefore, $f(x)$ is differentiable in (–3, 0).

(c) $f(-3) = f(0) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point $c$ in (–3, 0) such that $f'(c) = 0$

$$(c+3)e^{-\frac{c}{2}} + ce^{-\frac{c}{2}} - \frac{c(c+3)}{2}e^{-\frac{c}{2}} = 0$$

$$2(c+3) + 2c - c(c+3) = 0 \qquad \left[\because e^{-\frac{c}{2}} \neq 0 \text{ for any finite value of } c\right]$$

$$-c^2 + c + 6 = 0,$$

$$c = -2, 3$$

$$c = -2 \text{ lies in } (-3, 0)$$

Hence, theorem is verified.

(iii) $f(x) = |x|$ in [–1, 1].

$$|x| = -x, \quad -1 \le x \le 0$$
$$= x, \quad 0 \le x \le 1$$

(a) $f(x)$ is continuous in [–1, 1].

(b) Left hand derivative at $x = 0$

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{-x-0}{x} = -1$$

Right hand derivative at $x = 0$

$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x-0}{x} = 1$$

$$f'(0^-) \neq f'(0^+)$$

Thus, function is not differentiable at $x = 0$ and hence, Rolle's theorem is not applicable.

(iv) $f(x) = \dfrac{\sin x}{e^x} = e^{-x} \sin x$

(a) $f(x) = e^{-x} \sin x$, being product of continuous functions, is continuous in $[0, \pi]$

(b) $f'(x) = -e^{-x} \sin x + e^{-x} \cos x$
$= e^{-x} (\cos x - \sin x)$

exists for every value of $x$ in $(0, \pi)$. Therefore, $f(x)$ is differentiable in $(0, \pi)$.

(c) $f(0) = f(\pi) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one $c$ in $(0, \pi)$ such that $f'(c) = 0$.

$$f'(c) = e^{-c}(\cos c - \sin c) = 0$$

$$\cos c - \sin c = 0 \qquad [\because e^{-c} \neq 0 \text{ for any finite value of } c]$$

$$\cos c = \sin c$$

$$\tan c = 1, \ c = n\pi + \frac{\pi}{4}, \text{ where } n \text{ is an integer.}$$

Putting $n = 0, 1, 2, \ldots$

$$c = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$$

$$c = \frac{\pi}{4} \text{ lies in the interval } (0, \pi).$$

Hence, theorem is verified.

(v) $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

(a) $f(x) = e^x(\sin x - \cos x)$, being composite function of continuous functions, is continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

(b) $f'(x) = e^x (\sin x - \cos x) + e^x (\cos x + \sin x)$
$= 2e^x \sin x$

exists for every value of $x$ in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. Therefore, $f(x)$ is differentiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(c) $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point $c$ in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f'(c) = 0$

$$2e^c \sin c = 0$$

$$\sin c = 0 \qquad [\because e^c \neq 0 \text{ for any finite value of } x]$$

$$c = 0, \pi, 2\pi, \ldots\ldots\ldots\ldots\ldots$$

$$c = \pi \text{ lies in } \left(\frac{\pi}{4}, \frac{5\pi}{4}\right).$$

Hence, theorem is verified.

(vi) $f(x) = \log\left[\frac{x^2 + ab}{(a+b)x}\right]$ in $[a,b]$, $a > 0, \ b > 0$

(a) $f(x) = \log(x^2 + ab) - \log x - \log(a + b)$, being composite function of continuous functions, is continuous in $[a, b]$.

(b) $f'(x) = \dfrac{2x}{x^2 + ab} - \dfrac{1}{x}$

exists for every value of $x$ in $(a, b)$ [$\because a > 0, b > 0$]. Therefore, $f(x)$ is differentiable in $(a, b)$.

(c) $f(a) = \log(a^2 + ab) - \log a - \log(a + b)$

$= \log a + \log(a + b) - \log a - \log(a + b)$

$= 0$

$f(b) = \log(b^2 + ab) - \log b - \log(a + b)$

$= \log b + \log(b + a) - \log b - \log(a + b)$

$= 0$

$f(a) = f(b)$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that $f'(c) = 0$

$$\frac{2c}{c^2 + ab} - \frac{1}{c} = 0$$

$$2c^2 - c^2 - ab = 0$$

$$c^2 - ab = 0, \ c = \pm\sqrt{ab}$$

Since $c = \sqrt{ab}$ lies between $a$ and $b$ [being geometric mean of $a$ and $b$]. Hence, theorem is verified.

(vii) $f(x) = x^2 + 1 \qquad 0 \le x \le 1$

$= 3 - x \qquad 1 \le x \le 2$

(a) $f(x) = x^2 + 1 \qquad 0 \le x \le 1$

$= 3 - x \qquad 1 \le x \le 2$

is defined every where in [0, 2] and hence, continuous in [0, 2].

(b) Left hand derivative at $x = 1$

$$f'(1^-) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 + 1 - 2}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} (x + 1) = 1 + 1 = 2$$

Right hand derivative at $x = 1$

$$f'(1^+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{3 - x - 2}{x - 1} = -1$$

$$f'(1^-) \ne f'(1^+)$$

Thus, function is not differentiable at $x = 1$ and hence, Rolle's theorem is not applicable.

(viii) $f(x) = x^2 - 2 \qquad -1 \le x \le 0$
$\quad = x - 2 \qquad 0 \le x \le 1$

(a) $f(x) = x^2 - 2 \qquad -1 \le x \le 0$
$\quad = x - 2 \qquad 0 \le x \le 1$

is defined everywhere in [–1, 1], and hence, is continuous in [–1, 1].

(b) Left hand derivative at $x = 0$

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x^2-2-(-2)}{x}$$
$$= \lim_{x\to 0^-} \frac{x^2-2+2}{x} = \lim_{x\to 0^-} \frac{x^2}{x} = 0$$

Right hand derivative at $x = 0$

$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{(x-2)-(-2)}{x}$$
$$= \lim_{x\to 0^+} \frac{x}{x} = 1$$
$$f'(0^-) \ne f'(0^+)$$

Thus, function is not differentiable at $x = 0$ and hence, Rolle's theorem is not applicable.

**Example 2: Prove that between any two roots of $e^x \sin x = 1$ there exists at least one root of $e^x \cos x + 1 = 0$.**

**Solution:** Let $f(x) = 1 - e^x \sin x$

(a) $f(x)$, being composite function of continuous functions, is continuous in a finite interval.

(b) $f'(x) = -(e^x \sin x + e^x \cos x) = -(1 + e^x \cos x)$ $\qquad [\because e^x \sin x = 1]$
exists for every finite value of $x$. Therefore, $f(x)$ is differentiable in a finite interval.

(c) Let $\alpha$ and $\beta$ are two roots of the equation, $f(x) = 1 - e^x \sin x = 0$
Then $f(\alpha) = f(\beta) = 0$
Thus, $f(x)$ satisfies all the conditions of Rolle's theorem in $[\alpha, \beta]$. Therefore, there exists at least one point $c$ in $(\alpha, \beta)$ such that $f'(c) = 0$

$$1 + e^c \cos c = 0$$

This shows that $c$ is the root of the equation $e^x \cos x + 1 = 0$ which lies between the root $\alpha$ and $\beta$ of the equation $1 - e^x \sin x = 0$.

**Example 3: Prove that the equation $2x^3 - 3x^2 - x + 1 = 0$ has at least one root between 1 and 2.**

**Solution:** Let us consider a function $f(x) = \frac{x^4}{2} - x^3 - \frac{x^2}{2} + x$ [obtained by integrating the given equation]

(a) $f(x)$, being an algebraic function, is continuous in [1, 2]
(b) $f'(x) = 2x^3 - 3x^2 - x + 1$ exists for every value of $x$ in (1, 2). Therefore, $f(x)$ is differentiable in (1, 2).
(c) $f(1) = f(2) = 0$
Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point $c$ in (1, 2) such that $f'(c) = 0$

$$2c^3 - 3c^2 - c + 1 = 0$$

This shows that $c$ is the root of the equation $2x^3 - 3x^2 - x + 1 = 0$ which lies between 1 and 2.

**Example 4: Prove that if $a_0, a_1, a_2, ....... a_n$ are real numbers such that $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + ....... + \frac{a_{n-1}}{2} + a_n = 0$, then there exists at least one real number $x$ between 0 and 1 such that $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ....... + a_n = 0$.**

**Solution:** Let us consider a function $f(x) = \frac{a_0 x^{n+1}}{n+1} + \frac{a_1 x^n}{n} + \frac{a_2 x^{n-1}}{n-1} + ....... + a_n x$ defined in [0, 1].

(a) $f(x)$, being an algebraic function, is continuous in [0, 1].
(b) $f'(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ....... + a_n$
exists for every value of $x$ in (0, 1). Therefore, $f(x)$ is differentiable in (0, 1).
(c) $f(0) = 0$

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + ........ + \frac{a_{n-1}}{2} + a_n = 0 \qquad \text{[given]}$$

$$f(0) = f(1)$$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point $c$ in (0, 1) such that $f'(c) = 0$

$$a_0 c^n + a_1 c^{n-1} + a_2 c^{n-2} + .......... + a_n = 0$$

Replacing $c$ by $x$,

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ......... + a_n = 0$$

**Example 5: If $f(x)$, $\phi(x)$, $\Psi(x)$ are differentiable in $(a, b)$, prove that there exists at least one point $c$ in $(a, b)$ such that $\begin{vmatrix} f(a) & \phi(a) & \Psi(a) \\ f(b) & \phi(b) & \Psi(b) \\ f'(c) & \phi'(c) & \Psi'(c) \end{vmatrix} = 0$.**

**Solution:** Let us consider a function $F(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f(x) & \phi(x) & \psi(x) \end{vmatrix}$

(a) Since $f(x)$, $\phi(x)$, $\Psi(x)$ are differentiable in $(a, b)$, therefore, will be continuous in $[a, b]$. $F(x)$, being composite function of continuous functions, is continuous in $[a, b]$.

(b) $F'(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(x) & \phi'(x) & \psi'(x) \end{vmatrix}$ exists for every value of $x$ in $(a, b)$. Therefore, $f(x)$ is differentiable in $(a, b)$.

(c) $f(a) = f(b) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that $f'(c) = 0$

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0.$$

**Example 6: If $f(x) = x(x + 1)(x + 2)(x + 3)$, prove that $f'(x) = 0$ has three real roots.**

**Solution:** $f(x) = x(x + 1)(x + 2)(x + 3)$

(a) $f(x)$, being polynomial is continuous, in the intervals $[-3, -2]$, $[-2, -1]$, $[-1, 0]$.

(b) $f'(x) = (x + 1)(x + 2)(x + 3) + x(x + 2)(x + 3) + x(x + 1)(x + 3) + x(x + 1)(x + 2)$ exists for every value of $x$ in $[-3, -2]$, $[-2, -1]$ and $[-1, 0]$. Therefore, $f(x)$ is differentiable in $[-3, -2]$, $[-2, -1]$ and $[-1, 0]$.

(c) $f(-3) = f(-2) = f(-1) = f(0) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one $c_1$ in $(-3, -2)$, $c_2$ in $(-2, -1)$ and $c_3$ in $(-1, 0)$ such that $f'(c_1) = f'(c_2) = f'(c_3) = 0$

Thus $c_1$, $c_2$ and $c_3$ are the roots of $f'(x) = 0$

Hence $f'(x) = 0$ has at least 3 real roots.

**Example 7: If $k$ is a real constant, prove that the equation $x^3 - 6x^2 + c = 0$ cannot have distinct roots in $[0, 4]$.**

**Solution:** Let $f(x) = x^3 - 6x^2 + c = 0$ has distinct roots $a$ and $b$ between 0 and 4 i.e.

$$0 \le a < b \le 4$$

Then $$f(a) = 0 = f(b)$$

Also, $f(x)$ being polynomial is, continuous in $[a, b]$ and $f'(x) = 3x^2 - 12x$ exists for every value of $x$ in $[a, b]$. Therefore, $f(x)$ is differentiable in $(a, b)$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\begin{aligned} f'(c) &= 0 \\ 3c^2 - 12c &= 0 \\ 3c\,(c-4) &= 0 \\ c &= 0, 4 \end{aligned}$$

But these values of $c$ lies outside the interval $(a, b)$. This is a contradiction to Rolle's theorem. Thus, our assumption is wrong.

Hence, $f(x) = 0$ cannot have distinct roots in $[0, 4]$.

## Exercise 2.3

**1.** Verify Rolle's Theorem for the following functions:

(i) $x^3 - 12x$ in $\left[0, 2\sqrt{3}\right]$ $\qquad$ $[\textbf{Ans.:}\ c = 2]$

(ii) $x^3 - 4x$ in $[-2, 2]$ $\qquad$ $\left[\textbf{Ans.:}\ c = \pm\dfrac{\sqrt{2}}{3}\right]$

(iii) $2x^3 + x^2 - 4x - 2$ in $\left[-\sqrt{2}, \sqrt{2}\right]$ $\qquad$ $\left[\textbf{Ans.:}\ c = \dfrac{2}{3}, -1\right]$

(iv) $x^2$ in $[1, 2]$ $\qquad$ $\left[\textbf{Ans.:}\ f(1) \neq f(2), \text{ theorem is not applicable}\right]$

(v) $2 + (x-1)^{\frac{2}{3}}$ in $[0, 2]$ $\qquad$ $\left[\textbf{Ans.:}\ \text{not differentiable at } x = 1, \text{ theorem is not applicable}\right]$

(vi) $1 - (x-3)^{\frac{2}{3}}$ in $[2, 4]$ $\qquad$ $\left[\textbf{Ans.:}\ \text{not differentiable at } x = 3, \text{ theorem is not applicable}\right]$

(vii) $\dfrac{x^2 - 4x}{x+2}$ in $[0, 4]$ $\qquad$ $\left[\textbf{Ans.:}\ c = 2\left(\sqrt{3} - 1\right)\right]$

(viii) $(x+2)^3 (x-3)^4$ in $[-2, 3]$ $\qquad$ $\left[\textbf{Ans.:}\ c = \dfrac{1}{7}\right]$

(ix) $\log\left(\dfrac{x^2 + 6}{5x}\right)$ in $[2, 3]$ $\qquad$ $[\textbf{Ans.:}\ c = \sqrt{6}]$

(x) $\cos^2 x$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ $\qquad$ $[\textbf{Ans.:}\ c = 0]$

(xi) $\sin x$ in $[0, 2\pi]$ $\qquad$ $\left[\textbf{Ans.:}\ c = \dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$

(xii) $|\cos x|$ in $[0, \pi]$ $\qquad$ $\left[\textbf{Ans.:}\ \text{not differentiable at } x = \dfrac{\pi}{2}, \text{ theorem is not applicable}\right]$

(xiii) $|\sin x|$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

$$\left[\begin{array}{l}\textbf{Ans.:} \text{ not differentiable} \\ \quad \text{at } x = 0, \text{ theorem} \\ \quad \text{is not applicable}\end{array}\right]$$

(xiv) $f(x) = 1 \qquad x = 0$
$\qquad = x \qquad 0 < x \le 1$

$$\left[\begin{array}{l}\textbf{Ans.:} \text{ discontinuous} \\ \quad \text{at } x = 0, \text{ theorem} \\ \quad \text{is not applicable}\end{array}\right]$$

(xv) $1 - \sqrt[3]{(x-1)^2}$ in $[0, 2]$

$$\left[\begin{array}{l}\textbf{Ans.:} \text{ not differentiable} \\ \quad \text{at } x = 1, \text{ theorem} \\ \quad \text{is not applicable}\end{array}\right]$$

(xvi) $f(x) = x^2 + 2 \quad -1 \le x \le 0$
$\qquad = x + 2 \quad 0 \le x \le 1$

$$\left[\begin{array}{l}\textbf{Ans.:} \text{ not differentiable} \\ \quad \text{at } x = 0, \text{ theorem} \\ \quad \text{is not applicable}\end{array}\right]$$

**2.** Prove that one root of the equation $x \log x - 2 + x = 0$ lies in $(1, 2)$.
[**Hint :** Consider $f(x) = (x - 2) \log x$]

**3.** Prove that the equation $\tan x = 1 - x$ has a real root in the interval $(0, 1)$.

**4.** If $c$ is a real constant, prove that the equation $x^3 - 12x + c = 0$ cannot have two distinct roots in the interval $[0, 4]$.

**5.** If $c$ is a real constant, prove that the equation $x^3 + 3x + c = 0$ cannot have more than one real root.

**6.** If $f(x) = a + b - 3bx^2 - 4ax^3$, $a \ne 0$, $b \ne 0$, prove that there exists at least one value $c$ in $(0, 1)$ such that $f'(c) = 0$.

**7.** Prove that one root of the equation $\dfrac{\sin\theta}{\theta} = \cos x\theta$ lies between 0 and 1.

## 2.6 LAGRANGE'S MEAN VALUE THEOREM (L.M.V.T.)

**Statement:** If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$,
(ii) differentiable in the open interval $(a, b)$,

then there exists at least one point $c$ in the open interval $(a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

**Proof:** Consider a function $\phi(x) = f(x) + Ax$ where $A$ is a constant to be determined, such that $\phi(a) = \phi(b)$.

$$f(a) + Aa = f(b) + Ab$$
$$A = -\frac{f(b) - f(a)}{b - a}$$

Now

(i) $\phi(x)$ is continuous in the closed interval $[a, b]$, since $f(x)$, $x$ and $A$ are continuous.
(ii) $\phi(x)$ is differentiable in the open interval $(a, b)$, since $f(x)$, $x$ and $A$ are differentiable.

(iii) $\phi(a) = \phi(b)$ [by assumption]

$\phi(x)$ satisfies all the conditions of Rolle's mean value theorem. Therefore, there exists at least one point $c$ in the open interval $(a, b)$ such that $\phi'(c) = 0$

$$f'(c) + A = 0$$

$$f'(c) = -A$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

### 2.6.1 Another Form of Lagrange's Mean Value Theorem

If a function $f(x)$ is

(i) continuous in the closed interval $[a, a + h]$,

(ii) differentiable in the open interval $(a, a + h)$,

then there exists at least one number $\theta$ between 0 and 1( $0 < \theta < 1$) such that

$$f'(a + \theta h) = \frac{f(a + h) - f(a)}{h}$$

$$f(a + h) = f(a) + h f'(a + \theta h).$$

### 2.6.2 Geometrical Interpretation of Lagrange's Mean Value Theorem

Let $y = f(x)$ represents a curve with $A[a, f(a)]$ and $B[b, f(b)]$ as end points and $C[c, f(c)]$ be any point between $A$ and $B$. Then

$\dfrac{f(b) - f(a)}{b - a}$ = slope of the chord $AB$ and $f'(c)$ = slope of the tangent at point $C$.

Thus, geometrically theorem states that if

(i) curve is continuous at the points $A$, $B$ and at every point between $A$ and $B$.

(ii) possesses unique tangent at every point between $A$ and $B$, then there exists at least one point $c$ on the curve between $A$ and $B$, tangent at which is parallel to the chord $AB$.

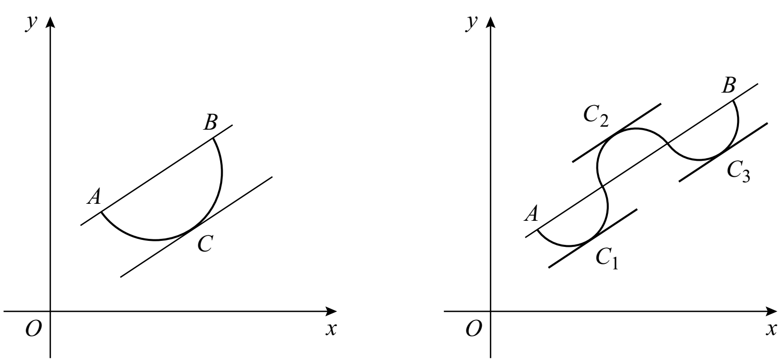



**Fig. 2.2**

## 2.6.3 Algebraic Interpretation of Lagrange's Mean Value Theorem

If a function $f(x)$ is defined in the interval $[a, b]$, then $f(b) - f(a)$ is the change in the function $f(x)$ from $x = a$ to $x = b$ and therefore, $\dfrac{f(b)-f(a)}{b-a}$ is the average rate of change of the function $f(x)$ in the interval $[a, b]$. Also, $f'(c)$ is the actual rate of change of the function at $x = c$. Thus, according to the Lagrange's Mean Value Theorem, average rate of change of a function over an interval is equal to the actual rate of change of the function at some point in the interval.

## 2.6.4 Deductions from Lagrange's Mean Value Theorem

### *Increasing Function*

**Statement:** If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$,
(ii) differentiable in the open interval $(a, b)$,
(iii) $f'(x) > 0$ throughout the interval $(a, b)$, then $f(b) > f(a)$, i.e., $f(x)$ is strictly (monotonically) increasing function in the closed interval $[a, b]$.

**Proof:** By Lagrange's Mean Value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c) \qquad \text{... (1)}$$

Let $f'(x) > 0$ for all $x$ in $(a, b)$.

Then,

$$f'(c) > 0, \qquad a < c < b \qquad \text{[Using (1)]}$$

$$\frac{f(b)-f(a)}{b-a} > 0$$

$$f(b) - f(a) > 0 \qquad [\because b - a > 0 \text{ being length of the interval}]$$

$$f(b) > f(a), \qquad b > a$$

$f(x)$ is strictly (monotonically) increasing function in the closed interval $[a, b]$.

In general,

$$f(x_2) > f(x_1) \text{ for } x_2 > x_1, \text{ for every value of } x_1, x_2 \text{ in } [a, b].$$

### *Decreasing Function*

**Statement:** If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$,
(ii) differentiable in the open interval $(a, b)$,
(iii) $f'(x) < 0$ throughout the interval $(a, b)$, then $f(b) < f(a)$, i.e., $f(x)$ is strictly (monotonically) decreasing function in the closed interval $[a, b]$.

**Proof:** By Lagrange's Mean Value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Let $f'(x) < 0$ for all $x$ in $(a, b)$.

Then,
$$f'(c) < 0 \qquad a < c < b$$

$$\frac{f(b)-f(a)}{b-a} < 0$$

$$f(b) - f(a) < 0 \qquad [\because b - a > 0 \text{ being length of the interval}]$$

$$f(b) < f(a), \qquad b > a$$

$f(x)$ is strictly (monotonically) decreasing function in the closed interval $[a, b]$.
In general,

$$f(x_2) < f(x_1) \text{ for } x_2 > x_1, \text{ for every value of } x_1, x_2 \text{ in } [a, b].$$

**Example 1: Verify Lagrange's Mean Value Theorem for the following functions:**

**(i)** $f(x) = x^3$ **in** $[-2, 2]$
**(ii)** $f(x) = lx^2 + mx + n$ **in** $[a, b]$
**(iii)** $f(x) = x^{\frac{2}{3}}$ **in** $[-8, 8]$
**(iv)** $f(x) = e^x$ **in** $[0, 1]$
**(v)** $f(x) = \log x$ **in** $[1, e]$.

**Solution:** (i) $f(x) = x^3$ in $[-2, 2]$

(a) $f(x) = x^3$, being an algebraic function, is continuous in $[-2, 2]$.
(b) $f'(x) = 3x^2$ exists for every value of $x$ in $(-2, 2)$. Therefore, $f(x)$ is differentiable in $(-2, 2)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point $c$ in $(-2, 2)$ such that

$$\frac{f(2)-f(-2)}{2-(-2)} = f'(c)$$

$$\frac{(2)^3-(-2)^3}{2-(-2)} = 3c^2$$

$$4 = 3c^2$$

$$c = \pm\frac{2}{\sqrt{3}}$$

$$c = \pm\frac{2}{\sqrt{3}} \text{ lies in } (-2, 2).$$

Hence, theorem is verified.

(ii) $f(x) = lx^2 + mx + n$ in $[a, b]$.

(a) $f(x) = lx^2 + mx + n$, being an algebraic function, is continuous in $[a, b]$.

(b) $f'(x) = 2lx + m$, exists for every value of $x$ in $(a, b)$. Therefore, $f(x)$ is differentiable in $(a, b)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{(lb^2+mb+n)-(la^2+ma+n)}{b-a} = 2lc+m$$

$$l(b+a)+m = 2lc+m$$

$c = \dfrac{b+a}{2}$ lies in $(a, b)$ being arithmetic mean of $a$ and $b$.

Hence, theorem is verified.

(iii) $f(x) = x^{\frac{2}{3}}$ in $[-8, 8]$.

(a) $f(x) = x^{\frac{2}{3}}$, being an algebraic function, is continuous in $[-8, 8]$.

(b) $f'(x) = \dfrac{2}{3}x^{-\frac{1}{3}} = \dfrac{2}{3x^{\frac{1}{3}}}$ which does not exists at $x = 0$.

Hence, theorem is not applicable.

(iv) $f(x) = e^x$ in $[0, 1]$.

(a) $f(x) = e^x$, being an exponential function, is continuous in $[0, 1]$.

(b) $f'(x) = e^x$, exists for every value of $x$ in $(0, 1)$. Therefore, $f(x)$ is differentiable in $(0, 1)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point $c$ in $(0, 1)$ such that

$$\frac{f(1)-f(0)}{1-0} = f'(c)$$

$$\frac{e^1-e^0}{1-0} = e^c$$

$$e^c = e - 1$$

$$c = \log(e-1) = 0.5413 < 1$$

$$c = 0.5413 \text{ lies in } (0, 1).$$

Hence, theorem is verified.

(v) $f(x) = \log x$ in $[1, e]$.

(a) $f(x) = \log x$, being a logarithmic function, is continuous in $[1, e]$.

(b) $f'(x) = \dfrac{1}{x}$, exists for every value of $x$ in $[1, e]$. Therefore, $f(x)$ is differentiable in $[1, e]$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point $c$ in $[1, e]$ such that

$$\frac{f(e)-f(1)}{e-1} = f'(c)$$

$$\frac{\log e-\log 1}{e-1} = \frac{1}{c}$$

$$c = e - 1$$

$$\because \quad 2 < e < 3$$

$$\therefore \quad 1 < e - 1 < 2$$

Thus, $c = e - 1$ lies in $(1, e)$.
Hence, theorem is verified.

**Example 2: If $a$, $b$ are real numbers, prove that there exists at least one real number $c$ such that $b^3 + ab^2 + a^2b + a^3 = 4c^3$, $a < c < b$.**

**Solution:** Let $f(x) = x^4$ is defined in $[a, b]$.

(a) $f(x) = x^4$, being algebraic function, is continuous in $[a, b]$.
(b) $f'(x) = 4x^3$ exists for every value of $x$ in $(a, b)$. Therefore, $f(x)$ is differentiable in $(a, b)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{b^4-a^4}{b-a} = 4c^3$$

$$\frac{(b^2-a^2)(b^2+a^2)}{b-a} = 4c^3$$

$$\frac{(b-a)(b+a)(b^2+a^2)}{b-a} = 4c^3$$

$$\frac{(b-a)(b^3+ab^2+a^2b+a^3)}{b-a} = 4c^3$$

Hence, $$b^3+ab^2+a^2b+a^3 = 4c^3,\ a<c<b.$$

**Example 3: Using Lagrange's Mean Value theorem, prove that**

$$\frac{\cos a\theta-\cos b\theta}{\theta} \le (b-a) \text{ if } \theta \ne 0.$$

**Solution:** Let $f(x) = \cos x\theta$ is defined in the interval $[a, b]$.

(a) $f(x)$, being trigonometric function, is continuous in $[a, b]$.
(b) $f'(x) = -\theta \sin x\theta$ exists for all values of $x$ in $(a, b)$. Therefore, $f(x)$ is differentiable in $(a, b)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\cos b\theta - \cos a\theta}{b-a} = -\theta \sin c\theta$$

$$\frac{\cos a\theta - \cos b\theta}{(b-a)\theta} = \sin c\theta \le 1, \quad \text{if} \quad \theta \ne 0 \qquad [\because \sin x \le 1]$$

$$\frac{\cos a\theta - \cos b\theta}{\theta} \le (b-a), \quad \text{if} \quad \theta \ne 0.$$

**Example 4: Find the point on the curve $y = \log x$, tangent at which is parallel to the chord joining the points (1, 0) and ($e$, 1).**

**Solution:** Let $c$ be the point on curve $y = \log x$, tangent at which is parallel to the chord joining the points (1, 0) and ($e$, 1).

By Lagrange's Mean Value theorem,

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Here,

$$a = x\text{-coordinate of } (1, 0) = 1$$
$$b = x\text{-coordinate of } (e, 1) = e$$
$$f(a) = \log 1 = 0, f(b) = \log e = 1$$
$$f'(x) = \frac{1}{x}$$
$$\frac{1-0}{e-1} = \frac{1}{c}$$
$$c = e-1.$$

**Example 5: At what point is the tangent to the curve $y = x^n$ parallel to the chord joining (0, 0) and ($k$, $k^n$)?**

**Solution:** Let $c$ be the point on curve $y = x^n$ tangent at which is parallel to the chord joining the points (0, 0) and ($k$, $k^n$).

By Lagrange's Mean Value theorem,

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Here,

$$a = 0,\ b = k,\ f(a) = 0,\ f(b) = k^n$$
$$f'(x) = nx^{n-1}$$
$$\frac{k^n - 0}{k-0} = nc^{n-1}$$
$$c = \frac{k}{n^{\frac{1}{n-1}}}.$$

**Example 6: Prove that for any quadratic functions $f(x) = px^2 + qx + r$ in $[a, a + h]$, the value of $\theta$ is always $\frac{1}{2}$ whatever $p, q, r, a, h$ may be.**

**Solution:** (a) $f(x) = px^2 + qx + r$ is continuous in $[a, a + h]$ being an algebraic function.

(b) $f'(x) = 2px + q$, exists for every value of $x$ in $(a, a + h)$. Therefore, $f(x)$ is differentiable in (0, 1).

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one number $\theta$ between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$$

$$f(a + h) - f(a) = h f'(a + \theta h)$$

$$p(a + h)^2 + q(a + h) + r - pa^2 - qa - r = h[2p(a + \theta h) + q]$$

$$ph^2 + 2pah + qh = h[2pa + 2p\theta h + q]$$

$\theta = \frac{1}{2}$ which is a constant and does not depend on $p, q, r, a, h$.

Hence, value of $\theta$ is always $\frac{1}{2}$ whatever $p, q, r, a, h$ may be.

**Example 7: Apply Lagrange's Mean Value theorem to the function $f(x) = \log x$ in $[a, a + h]$ and determine $\theta$ in terms of $a$ and $h$. Hence, deduce that $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$.**

**Solution:** (a) $f(x) = \log x$, being logarithmic function, is continuous in $[a, a + h]$.

(b) $f'(x) = \frac{1}{x}$, exists for every value of $x$ in $(a, a + h)$. Therefore, $f(x)$ is differentiable in $(a, a + h)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one number $\theta$ between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$$

$$f(a+h)-f(a) = h f'(a+\theta h)$$

$$\log(a+h) - \log a = \frac{h}{a+\theta h}$$

$$\log\left(1+\frac{h}{a}\right) = \frac{h}{a+\theta h}$$

$$a+\theta h=\frac{h}{\log\left(1+\frac{h}{a}\right)}$$

$$\theta=\frac{1}{\log\left(1+\frac{h}{a}\right)}-\frac{a}{h}$$

Putting $h = x$ and $a = 1$,

$$\theta=\frac{1}{\log(1+x)}-\frac{1}{x}$$

$$\because \qquad 0<\theta<1$$

$$0<\frac{1}{\log(1+x)}-\frac{1}{x}<1.$$

**Example 8: Prove that $\log_{10}(x+1)=\dfrac{x\log_{10}e}{1+\theta x}$, where $x > 0$ and $0<\theta<1$.**

**Solution:** Let $f(x)=\log_{10}(x+1)=\dfrac{\log_e(x+1)}{\log_e 10}$ is defined in $[a, a + h]$.

(a) $f(x)=\dfrac{\log_e(x+1)}{\log_e 10}$, being logarithmic function, is continuous in $[a, a + h]$.

(b) $f'(x)=\dfrac{1}{(x+1)\log_e 10}$, exists for every value of $x$ in $(a, a + h)$. Therefore, $f(x)$ is differentiable in $(a, a + h)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one number $\theta$ between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{h}=f'(a+\theta h)$$

$$f(a+h)-f(a)=h\,f'(a+\theta h)$$

$$\frac{\log_e(a+h+1)}{\log_e 10}-\frac{\log_e(a+1)}{\log_e 10}=h\frac{1}{(a+\theta h+1)\log_e 10}$$

$$\log_e(a+h+1)-\log_e(a+1)=h\frac{1}{(a+\theta h+1)}$$

Putting $a = 0$, $h = x$,

$$\log_e(0+x+1)-\log_e(0+1)=x\frac{1}{(0+\theta x+1)}$$

$$\log_e(x+1)=\frac{x}{\theta x+1}$$

$$\frac{\log_{10}(x+1)}{\log_{10}e}=\frac{x}{\theta x+1}$$

Hence, $$\log_{10}(x+1)=\frac{x\log_{10}e}{1+\theta x}.$$

**Example 9: Separate the interval in which $f(x)=x+\frac{1}{x}$ is increasing or decreasing.**

**Solution:** $$f(x)=x+\frac{1}{x}$$

$$f'(x)=1-\frac{1}{x^2}=\frac{(x-1)(x+1)}{x^2}$$

(i) $f(x)$ is an increasing function if

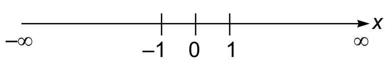
$$f'(x)>0, \frac{(x-1)(x+1)}{x^2}>0, \text{ i.e., } (x-1)\,(x+1)>0$$

Now, $(x-1)\,(x+1)>0$ if

**Case I:** $(x-1)>0$ and $(x+1)>0$ i.e., $x>1$

**Case II:** $(x-1)<0$ and $(x+1)<0$ i.e., $x<-1$

Hence, $f(x)$ is increasing in $(-\infty, -1)$ and $(1, \infty)$.




(ii) $f(x)$ is a decreasing function if $f'(x)<0$, $\frac{(x-1)(x+1)}{x^2}<0$, i.e. $(x-1)(x+1)<0$

Now, $(x-1)\,(x+1)<0$ if

**Case I:** $(x-1)<0$ and $(x+1)>0$ i.e., $-1<x<1$

**Case II:** $(x-1)>0$ and $(x+1)<0$ i.e., $x>1$ and $x<-1$ but this is not possible.

Hence, $f(x)$ is decreasing in $(-1, 1)$.

**Example 10: If $0<a<b$, prove that $\left(1-\frac{a}{b}\right)<\log\frac{b}{a}<\left(\frac{b}{a}-1\right)$. Hence, prove that $\frac{1}{6}<\log(1.2)<\frac{1}{5}$ and $\frac{1}{2}<\log 2<1$.**

**Solution:** Let $f(x) = \log x$ is defined in $[a, b]$ where $0 < a < b$.

(a) $f(x) = \log x$, being logarithmic function, is continuous in $[a, b]$.

(b) $f'(x) = \dfrac{1}{x}$ exists for every value of $x$ in $(a, b)$. Therefore, $f(x)$ is differentiable in $(a, b)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$, such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\log b - \log a}{b-a} = \frac{1}{c}$$

$$\log\frac{b}{a} = \frac{b-a}{c} \qquad \ldots (1)$$

We have,

$$a < c < b$$

$$\frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

$$\frac{b-a}{a} > \frac{b-a}{c} > \frac{b-a}{b} \qquad [\because\ b > a]$$

$$\frac{b}{a} - 1 > \log\frac{b}{a} > 1 - \frac{a}{b} \qquad \text{[Using Eq. (1)]}$$

$$1 - \frac{a}{b} < \log\frac{b}{a} < \frac{b}{a} - 1 \qquad \ldots (2)$$

(i) Putting $b = 6$, $a = 5$ in Eq. (2),

$$1 - \frac{5}{6} < \log\frac{6}{5} < \frac{6}{5} - 1$$

$$\frac{1}{6} < \log 1.2 < \frac{1}{5}$$

(ii) Putting $b = 2$, $a = 1$ in Eq. (2),

$$1 - \frac{1}{2} < \log 2 < \frac{2}{1} - 1$$

$$\frac{1}{2} < \log 2 < 1.$$

**Example 11: Prove that if $0 < a < 1$, $0 < b < 1$ and $a < b$, then $\dfrac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \dfrac{b-a}{\sqrt{1-b^2}}$ and hence, deduce that**

**(i)** $\dfrac{\pi}{6} - \dfrac{1}{2\sqrt{3}} < \sin^{-1}\dfrac{1}{4} < \dfrac{\pi}{6} - \dfrac{1}{\sqrt{15}}$ **(ii)** $\dfrac{\pi}{6} + \dfrac{\sqrt{3}}{15} < \sin^{-1}\dfrac{3}{5} < \dfrac{\pi}{6} + \dfrac{1}{8}$.

**Solution:** Let $f(x) = \sin^{-1} x$ defined in $[a, b]$.

(a) $f(x) = \sin^{-1} x$, being a trigonometric function, is continuous in $[a, b]$.

(b) $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$, exists for every value of $x$ in $(a, b)$. Therefore, $f(x)$ is differentiable in $(a, b)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\sin^{-1} b - \sin^{-1} a}{b-a} = \frac{1}{\sqrt{1-c^2}}$$

$$\sin^{-1} b - \sin^{-1} a = \frac{b-a}{\sqrt{1-c^2}} \qquad \text{... (1)}$$

We have,

$$a < c < b$$

$$a^2 < c^2 < b^2 \qquad [\because a > 0, b > 0]$$

$$-a^2 > -c^2 > -b^2$$

$$1 - a^2 > 1 - c^2 > 1 - b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2} \qquad [\because 0 < a < 1 \text{ and } 0 < b < 1]$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} < \frac{b-a}{\sqrt{1-c^2}} < \frac{b-a}{\sqrt{1-b^2}} \qquad [\because b-a > 0]$$

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}} \qquad \text{... (2) [Using Eq. (1)]}$$

(i) Putting $b = \frac{1}{4}$ and $a = \frac{1}{2}$ in Eq. (2),

$$\frac{-\frac{1}{4}}{\frac{\sqrt{3}}{2}} < \sin^{-1}\frac{1}{4} - \sin^{-1}\frac{1}{2} < \frac{-\frac{1}{4}}{\frac{\sqrt{15}}{4}}$$

$$-\frac{1}{2\sqrt{3}} < \sin^{-1}\frac{1}{4} - \frac{\pi}{6} < -\frac{1}{\sqrt{15}}$$

$$\frac{\pi}{6} - \frac{1}{2\sqrt{3}} < \sin^{-1}\frac{1}{4} < \frac{\pi}{6} - \frac{1}{\sqrt{15}}.$$

(ii) Putting $b = \frac{3}{5}$ and $a = \frac{1}{2}$ in Eq. (2),

$$\frac{\frac{1}{10}}{\frac{\sqrt{3}}{2}} < \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2} < \frac{\frac{1}{10}}{\frac{4}{5}}$$

$$\frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{5} - \frac{\pi}{6} < \frac{1}{8}$$

$$\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{5} < \frac{\pi}{6} + \frac{1}{8}$$

$$\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}\frac{3}{5} < \frac{\pi}{6} + \frac{1}{8}.$$

**Example 12: Using Lagrange's Mean Value theorem, prove that** $\boldsymbol{\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}}$ **and hence, deduce that** $\boldsymbol{\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}}$**.**

**Solution:** Let $f(x) = \tan^{-1} x$ is defined in $[a, b]$ where $a > 0$, $b > 0$.

(a) $f(x) = \tan^{-1} x$, being a trigonometric function is continuous in $[a, b]$.

(b) $f'(x) = \frac{1}{1+x^2}$, exists for every value of $x$ in $(a, b)$. Therefore, $f(x)$ is differentiable in $(a, b)$.

Thus, $f(x)$ satisfies all the conditions of Lagrange's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\tan^{-1} b - \tan^{-1} a}{b - a} = \frac{1}{1+c^2} \qquad \text{... (1)}$$

We have,

$$a < c < b$$

$$a^2 < c^2 < b^2 \qquad [\because a > 0, b > 0]$$

$$1 + a^2 < 1 + c^2 < 1 + b^2$$

$$1 + b^2 > 1 + c^2 > 1 + a^2$$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < \frac{b-a}{1+c^2} < \frac{b-a}{1+a^2} \qquad [\because b - a > 0]$$

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2} \qquad \text{... (2) [Using Eq. (1)]}$$

Putting $b = \frac{4}{3}$, $a = 1$ in Eq. (2),

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

**Example 13: Prove that $\tan^{-1}x > \dfrac{x}{1+\frac{x^2}{3}}$ if $0 < \tan^{-1}x < \dfrac{\pi}{2}$.**

**Solution:** Let $f(x) = \tan^{-1}x - \dfrac{x}{1+\frac{x^2}{3}}$

If
$$0 < \tan^{-1}x < \frac{\pi}{2}$$

$$\tan 0 < \tan(\tan^{-1}x) < \tan\frac{\pi}{2}$$

$$0 < x < \infty$$

Now,
$$f'(x) = \frac{1}{1+x^2} - \left[\frac{1+\frac{x^2}{3} - x\cdot\frac{2x}{3}}{\left(1+\frac{x^2}{3}\right)^2}\right]$$

$$= \left[\frac{\left(1+\frac{x^2}{3}\right)^2 - \left(1-\frac{x^2}{3}\right)(1+x^2)}{(1+x^2)\left(1+\frac{x^2}{3}\right)^2}\right]$$

$$= \frac{1+\frac{x^4}{9}+\frac{2x^2}{3}-1-x^2+\frac{x^2}{3}+\frac{x^4}{3}}{(1+x^2)\left(1+\frac{x^2}{3}\right)^2}$$

$$= \frac{4x^4}{9(1+x^2)\left(1+\frac{x^2}{3}\right)^2} > 0, \text{ for every value of } x \text{ in } (0, \infty),$$

i.e., $f'(x) > 0$ for every value of $x$ in $(0, \infty)$

Hence, $f(x)$ is strictly increasing function in $(0, \infty)$.

$$f(x) > f(0) \quad \text{for} \quad x > 0$$

$$f(x) > 0 \quad \text{for} \quad x > 0 \qquad [\because f(0) = 0]$$

$$\tan^{-1} x - \frac{x}{1+\frac{x^2}{3}} > 0$$

$$\tan^{-1} x > \frac{x}{1+\frac{x^2}{3}}.$$

**Example 14: Prove that $x^2 - 1 > 2x \log x > 4(x-1) - 2\log x$, for all $x > 1$.**

**Solution:** (i) Let $f(x) = x^2 - 1 - 2x \log x$

$$f'(x) = 2x - 2\log x - 2$$

It is difficult to decide about the sign of $f'(x)$ therefore differentiating again w.r.t. $x$,

$$f''(x) = 2 - \frac{2}{x} = \frac{2(x-1)}{x} > 0 \qquad [\because x > 1]$$

Hence, $f'(x)$ is strictly increasing function for $x > 1$.

$$f'(x) > f'(1) \quad \text{for} \quad x > 1$$

$$f'(x) > 0 \quad \text{for} \quad x > 1 \qquad [\because f'(1) = 2 - 2\log 1 - 2 = 0]$$

Hence, $f(x)$ is strictly increasing function for $x > 1$.

$$f(x) > f(1) \quad \text{for} \quad x > 1$$

$$f(x) > 0 \quad \text{for} \quad x > 1 \qquad [\because f(1) = 1 - 1 - 2\log 1 = 0]$$

$$x^2 - 1 - 2x\log x > 0 \quad \text{for} \quad x > 1$$

$$x^2 - 1 > 2x\log x \quad \text{for} \quad x > 1 \qquad \ldots (1)$$

(ii) Let $f(x) = 2x \log x - 4(x-1) + 2\log x$

$$f'(x) = 2\log x + 2 - 4 + \frac{2}{x}$$

It is difficult to decide about the sign of $f'(x)$. Therefore, differentiating again w.r.t. $x$,

$$f''(x) = \frac{2}{x} - \frac{2}{x^2}$$

$$= \frac{2(x-1)}{x^2} > 0 \qquad [\because x > 1]$$

Hence, $f'(x)$ is strictly increasing function for $x > 1$.

$$f'(x) > f'(1) \quad \text{for} \quad x > 1$$

$$f'(x) > 0 \quad \text{for} \quad x > 1 \qquad \left[\because f'(1) = 2\log 1 + 2 - 4 + \frac{2}{1} = 0\right]$$

Hence, $f(x)$ is an increasing function for $x > 1$.

$$f(x) > f(1) \quad \text{for} \quad x > 1$$

$$f(x) > 0 \quad \text{for} \quad x > 1 \qquad [\because f(1) = 2\log 1 - 4(1-1) + 2\log 1 = 0]$$

$$2x \log x - 4(x-1) + 2\log x > 0 \quad \text{for} \quad x > 1$$

$$2x \log x > 4(x-1) - 2\log x \quad \text{for} \quad x > 1 \qquad \dots (2)$$

From Eqs (1) and (2), we get

$$x^2 - 1 > 2x \log x > 4(x-1) - 2\log x \quad \text{for} \quad x > 1.$$

**Example 15: Prove that $2x < \log \dfrac{1+x}{1-x} < 2x\left[1 + \dfrac{1}{3}\left(\dfrac{x^2}{1-x^2}\right)\right]$ in (0, 1).**

**Solution:** (i) Let $f(x) = 2x - \log \dfrac{1+x}{1-x}$

$$= 2x - \log(1+x) + \log(1-x)$$

$$f'(x) = 2 - \frac{1}{1+x} - \frac{1}{1-x} = \frac{2 - 2x^2 - 1 + x - 1 - x}{(1-x^2)}$$

$$= \frac{-2x^2}{1-x^2} < 0 \qquad [\because 0 < x < 1, \therefore 1 - x^2 > 0]$$

Hence, $f(x)$ is a decreasing function in (0, 1).

$$f(x) < f(0) \quad \text{for} \quad x > 0$$

$$f(x) < 0 \quad \text{for} \quad x > 0 \qquad [\because f(0) = 0]$$

$$2x - \log\frac{1+x}{1-x} < 0$$

$$2x < \log\frac{1+x}{1-x} \qquad \dots (1)$$

(ii) Let $f(x) = \log\dfrac{1+x}{1-x} - 2x\left[1 + \dfrac{1}{3}\left(\dfrac{x^2}{1-x^2}\right)\right]$

$$= \log(1+x) - \log(1-x) - 2x\left[1 + \frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right]$$

$$f'(x) = \frac{1}{1+x} + \frac{1}{1-x} - 2\left[1 + \frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right] - 2x\left[0 + \frac{2x}{3(1-x^2)} + \frac{2x^3}{3(1-x^2)^2}\right]$$

$$= \frac{1-x+1+x}{1-x^2} - 2\left[\frac{3-3x^2+x^2}{3(1-x^2)}\right] - 2x\left[\frac{2x-2x^3+2x^3}{3(1-x^2)^2}\right]$$

$$= \frac{4x^2}{3(1-x^2)} - \frac{4x^2}{3(1-x^2)^2}$$

$$= \frac{4x^2(1-x^2) - 4x^2}{3(1-x^2)^2}$$

$$= \frac{-4x^4}{3(1-x^2)^2} < 0$$

Hence, $f(x)$ is a decreasing function in (0, 1).

$$f(x) < f(0) \quad \text{for} \quad x > 0$$

$$f(x) < 0 \quad \text{for} \quad x > 0 \qquad [\because f(0) = 0]$$

$$\log\frac{1+x}{1-x} - 2x\left[1+\frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right] < 0$$

$$\log\frac{1+x}{1-x} < 2x\left[1+\frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right] \qquad \text{... (2)}$$

From Eqs (1) and (2), we get

$$2x < \log\frac{1+x}{1-x} < 2x\left[1+\frac{1}{3}\left(\frac{x^2}{1-x^2}\right)\right].$$

## Exercise 2.4

**1.** Verify Lagrange's Mean Value theorem for the following functions:

(i) $\sqrt{x^2-4}$ in $[2, 3]$

$\left[\textbf{Ans.: } c = \sqrt{5}\right]$

(ii) $\frac{1}{x}$ in $[-1, 1]$

[**Ans.:** Discontinuous at $x = 0$, theorem not applicable]

(iii) $x + \frac{1}{x}$ in $\left[\frac{1}{2}, 3\right]$

$\left[\textbf{Ans.: } c = \sqrt{\frac{3}{2}}\right]$

(iv) $\log_e x$ in $\left[\frac{1}{2}, 2\right]$

$[\textbf{Ans.: } c = 1.08]$

(v) $(x-1)(x-2)$ in $[0, 4]$

$[\textbf{Ans.: } c = 2]$

(vi) $(x-1)(x-2)(x-3)$ in $[0, 4]$

$\left[\textbf{Ans.: } c = 2 \pm 2\sqrt{3}\right]$

(vii) $\tan^{-1}x$ in $[0, 1]$

$\left[\textbf{Ans.: } c = \frac{\sqrt{16-\pi^2}}{\pi}\right]$

(viii) $x^{\frac{1}{3}}$ in $[-1, 1]$

[**Ans.:** not differentiable at $x = 0$, theorem is not applicable]

(ix) $x - x^3$ in $[-2, 1]$

$[\textbf{Ans.: } c = -1]$

(x) $\sin^{-1}x$ in $[0, 1]$

$\left[\textbf{Ans.: } c = \frac{\sqrt{\pi^2-4}}{\pi}\right]$

(xi) $\cos x$ in $\left[0, \frac{\pi}{2}\right]$

$\left[\textbf{Ans.: } c = \sin^{-1}\frac{2}{\pi}\right]$

**2.** Test whether the Lagrange's Mean Value theorem holds for $f(x) = 2x^2 - 7x - 10$ in the interval [2, 5] and if so, find the value of $c$.

$\left[\textbf{Ans.: } \text{yes, } c = \frac{7}{2}\right]$

**3.** Prove that $x^3 - 3x^2 + 3x + 2$ is strictly increasing in every interval.

$\left[\textbf{Hint: } f'(x) = 3(x-1)^2 > 0 \text{ for all values of } x \text{ except } x = 1\right]$

**4.** Prove that $x - \sin x$ is strictly increasing in every interval.

**5.** Separate the intervals in which the polynomial $x^3 - 6x^2 - 36x + 7$ is increasing or decreasing:

$\left[\textbf{Ans.: } \text{Increasing in } (6, \infty), (-\infty, -2) \text{ and decreasing in } (-2, 6)\right]$

**6.** Separate the intervals in which the following polynomials are increasing or decreasing:

(i) $x^3 - 3x^2 + 24x - 31$

(ii) $2x^3 - 15x^2 - 36x + 40$

(iii) $2x^3 - 9x^2 + 12x + 5$

[**Ans.:** (i) Increasing in $(-\infty, 4)$, $(2, \infty)$ and decreasing in $(-2, 4)$.
(ii) Increasing in $(-\infty, -1)$, $(6, \infty)$ and decreasing in $(-1, 6)$.
(iii) Increasing in $(-\infty, 1)$, $(2, \infty)$ and decreasing in $(1, 2)$.]

**7.** Find the value of $\theta$ in Lagrange's Mean Value theorem for the following:

(i) $ax^2 + bx + c$ at $x = 0$

(ii) $f(x) = x^3$, $1 < x < 2$

$\left[\textbf{Ans.} \text{ (i) interval is } (0, h), \ \theta = \frac{1}{2} \quad \text{(ii) } -1 + \frac{\sqrt{7}}{3}\right]$

**8.** Prove that the following functions are increasing in the given interval:

(i) $x^3 - 3x^2 + 3x + 1$, $(-\infty, \infty)$

(ii) $\log x$, $(a, \infty)$, where $a > 0$

(iii) $e^x$, $(-\infty, \infty)$

(iv) $\sin x$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(v) $\cos x$, $(\pi, 2\pi)$

(vi) $\tan x$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

[**Hint:** Prove that $f'(x) > 0$ in the given interval]

**9.** Prove that the following functions are decreasing in the given interval:

(i) $e^{-x^2}$, $(0, \infty)$

(ii) $\cos x$, $(0, \pi)$

(iii) $\operatorname{cosec} x$, $\left(0, \frac{\pi}{2}\right)$

(iv) $\sin x$, $\left(\frac{\pi}{2}, \pi\right)$

(v) $\sec x$, $\left(-\frac{\pi}{2}, 0\right)$

(vi) $\cot x$, $(0, \pi)$

[**Hint :** Prove that $f'(x) < 0$ in the given interval]

**10.** Prove the following:

(i) $\log(1+x) < x$ for all $x > 1$

(ii) $1 - \frac{1}{x} < \log x < x - 1$ for all $x > 1$

(iii) $\log x < x < \tan x$ for all $x > 1$

(iv) $e^x > 1 + x$ for all $x > 0$

(v) $0 < -\log(1-x) < \frac{x}{1-x}$ for $0 < x < 1$

(vi) $\frac{1}{1+x^2} < \frac{\tan^{-1} x}{x} < 1$ for $x > 0$

(vii) $1 < \dfrac{\sin^{-1} x}{x} < \dfrac{1}{\sqrt{1-x^2}}$

for $0 \le x < 1$

(viii) $0 < \dfrac{1}{x}\log\left(\dfrac{e^x - 1}{x}\right) < 1$

**11.** Find the point on the curve $y = x^2$, tangent at which is parallel to the chord joining the points (1, 1) and (3, 9).

[**Ans. :** $c = 2$]

**12.** Prove that for the curve $y = x^2 + 2k_1x + k_2$, the chord joining the points $x = a$ and $x = b$ is parallel to the tangent at $x = \dfrac{a+b}{2}$.

**13.** Prove that the chord joining the points $x = 2, x = 3$ on the curve $y = x^3$ is parallel to the tangent to the curve at $x = \sqrt{\dfrac{19}{3}}$.

**14.** Prove that on the curve $y = 2 \sin x + \cos 2x$, there is a point $P$ between (0, 1) and $\left(\dfrac{\pi}{2}, 1\right)$ such that the tangent at $P$ is parallel to the $x$-axis.

Find the abscissa of $P$.

$\left[\textbf{Ans.: } c = \dfrac{\pi}{6}\right]$

**15.** Prove that $\log(x+y) < \log x + \dfrac{y}{x}$ if $x > 0, y > 0$.

$$\left[\begin{array}{l}\textbf{Hint: } f(z) = \log z \text{ in } [x, x+y], \\ f'(z) = \dfrac{1}{z} > 0, \\ \dfrac{f(x+y) - f(x)}{(x+y) - x} = f'(c), \\ \dfrac{\log(x+y) - \log x}{y} = \dfrac{1}{c} < \dfrac{1}{x} \\ \qquad\qquad (\because c > x) \\ \log(x+y) - \log x < \dfrac{y}{x}, \\ \log(x+y) < \log x + \dfrac{y}{x}\end{array}\right]$$

**16.** If $a$, $b$ are real numbers, prove that there exists at least one real number $c$ such that $b^2 + ab + a^2 = 3c^2$, $a < c < b$

[**Hint:** Let $f(x) = x^3$]

## 2.7 CAUCHY'S MEAN VALUE THEOREM (C.M.V.T.)

**Statement:** If two functions $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, b]$,

(ii) differentiable in the open interval $(a, b)$,

(iii) $g'(x) \neq 0$ for any $x$ in the open interval $(a, b)$, then there exists at least one point $c$ in the open interval $(a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

**Proof:** Consider a function $\phi(x) = f(x) + Ag(x)$, where $A$ is a constant to be determined such that $\phi(a) = \phi(b)$.

$$f(a) + Ag(a) = f(b) + Ag(b)$$

$$A = -\frac{f(b) - f(a)}{g(b) - g(a)}$$

Now since $\phi(a) = \phi(b)$ and $\phi(x)$ being the combination of two continuous and differentiable functions is also continuous in the closed interval $[a, b]$ and differentiable in the open interval $(a, b)$.

Thus, $\phi(x)$ satisfies all the conditions of Rolle's mean value theorem. Therefore, there exists at least one point $c$ in the open interval $(a, b)$ such that $\phi'(c) = 0$

$$f'(c) + Ag'(c) = 0$$

$$f'(c) = -Ag'(c)$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}, \qquad \text{[Substituting value of } A\text{]}$$

where $a < c < b$ and $g'(c) \neq 0$

## 2.7.1 Another Form of Cauchy's Mean Value Theorem

If two functions $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, a + h]$,
(ii) differentiable in the open interval $(a, a + h)$,
(iii) $g'(x) \neq 0$ for any $x$ in the open interval $(a, a + h)$, then there exists at least one number $\theta$ lying between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, \qquad \text{where } 0 < \theta < 1.$$

**Example 1: Verify Cauchy's Mean Value Theorem for the following functions:**

**(i) $x^2$ and $x^4$ in $[a, b]$, where $a > 0, b > 0$**

**(ii) sin $x$ and cos $x$ in $\left[0, \frac{\pi}{2}\right]$.**

**Solution:** (i) Let $f(x) = x^2$, $g(x) = x^4$

(a) $f(x)$ and $g(x)$, both being algebraic functions, are continuous in the closed interval $[a, b]$.
(b) $f'(x) = 2x$ and $g'(x) = 4x^3$ exists for all values of $x$ in the open interval $(a, b)$. Therefore, $f(x)$ and $g(x)$ are differentiable in $(a, b)$, and $g'(x) = 4x^3 \neq 0$ for any $x$ in $(a, b)$ since $a > 0, b > 0$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{b^2-a^2}{b^4-a^4} = \frac{2c}{4c^3} = \frac{1}{2c^2}$$

$$\frac{(b^2-a^2)}{(b^2+a^2)(b^2-a^2)} = \frac{1}{2c^2}$$

$$2c^2 = b^2 + a^2$$

$$c = \pm \sqrt{\frac{b^2 + a^2}{2}}$$

$$c = \sqrt{\frac{b^2 + a^2}{2}}$$

which lies between $a$ and $b$.
Hence, theorem is verified.

(ii) Let $f(x) = \sin x$, $g(x) = \cos x$

(a) $f(x)$ and $g(x)$, both being trigonometric functions, are continuous in $\left[0, \frac{\pi}{2}\right]$.

(b) $f'(x) = \cos x$, $g'(x) = -\sin x$ exists for all values of $x$ in $\left(0, \frac{\pi}{2}\right)$ and $g'(x) = -\sin x \neq 0$ for any $x$ in $\left(0, \frac{\pi}{2}\right)$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem.

Therefore, there exists at least one point $c$ in $\left(0, \frac{\pi}{2}\right)$ such that

$$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{g\left(\frac{\pi}{2}\right) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sin\frac{\pi}{2} - \sin 0}{\cos\frac{\pi}{2} - \cos 0} = \frac{\cos c}{-\sin c}$$

$$\frac{1-0}{0-1} = -\cot c$$

$$-1 = -\cot c$$

$$\cot c = 1, \ c = \frac{\pi}{4}$$

which lies between 0 and $\frac{\pi}{2}$.

Hence, theorem is verified.

**Example 2: If $f(x) = \frac{1}{x^2}$, and $g(x) = \frac{1}{x}$, prove that $c$ of Cauchy's Mean Value theorem is the harmonic mean between $a$ and $b$, $a > 0$, $b > 0$.**

**Solution:** (a) $f(x)$ and $g(x)$ are continuous in the closed interval $[a, b]$ for $a > 0$, $b > 0$.

(b) $f'(x) = -\frac{2}{x^3}$ and $g'(x) = -\frac{1}{x^2}$ exists for all $x$ in $(a, b)$ and $g'(x) \neq 0$ for any $x$ in $(a, b)$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}} = \frac{-\frac{2}{c^3}}{-\frac{1}{c^2}}$$

$$\frac{(a^2-b^2)(ab)}{(a^2b^2)(a-b)} = \frac{2}{c}$$

$$\frac{a+b}{ab} = \frac{2}{c}$$

$$\frac{2}{c} = \frac{1}{b}+\frac{1}{a}$$

$$\frac{1}{c} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

Hence, $c$ is the harmonic mean between $a$ and $b$.

**Example 3: If $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$, prove that $c$ of Cauchy's Mean Value theorem is geometric mean between $a$ and $b$, $a > 0, b > 0$.**

**Solution:** (a) $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ are continuous in $[a, b]$ for $a > 0, b > 0$.

(b) $f'(x) = \frac{1}{2\sqrt{x}}, \; g'(x) = -\frac{1}{2(x)^{\frac{3}{2}}}$ exists for all $x$ in $(a, b)$ and $g'(x) \neq 0$ for any $x$ in $(a, b)$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2(c)^{\frac{3}{2}}}}$$

$$\frac{\left(\sqrt{a}-\sqrt{b}\right)\sqrt{ab}}{\left(\sqrt{a}-\sqrt{b}\right)} = c$$

$$c = \sqrt{ab}$$

Hence, $c$ is the geometric mean between $a$ and $b$.

**Example 4: If $f(x) = e^x$ and $g(x) = e^{-x}$, prove that $c$ of Cauchy's Mean Value theorem is arithmetic mean between $a$ and $b$, $a > 0, b > 0$.**

**Solution:** (a) $f(x)$ and $g(x)$, being exponential functions, are continuous in $[a, b]$.

(b) $f'(x) = e^x$, $g'(x) = -e^{-x}$ exists for all $x$ in $(a, b)$ and $g'(x) \neq 0$ for any $x$ in $(a, b)$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^{2c}$$

$$\frac{-(e^a - e^b)\, e^b e^a}{(e^a - e^b)} = -e^{2c}$$

$$e^{a+b} = e^{2c}$$

$$a + b = 2c \qquad \text{[By comparing ]}$$

$$c = \frac{a+b}{2}$$

Hence, $c$ is the arithmetic mean between $a$ and $b$.

**Example 5: If $1 < a < b$, prove that there exists $c$ satisfying $a < c < b$ such that $\log\frac{b}{a} = \frac{b^2 - a^2}{2c^2}$.**

**Solution:** Let $f(x) = \log x$, $g(x) = x^2$ are defined in $(a, b)$.

(a) $f(x)$, being logarithmic function and $g(x)$, being algebraic function, are continuous in $[a, b]$ for $a > 1, b > 1$.

(b) $f'(x) = \frac{1}{x}$, $g'(x) = 2x$ exists for all $x$ in $(a, b)$ and $g'(x) \neq 0$ for any $x$ in $(a, b)$ since $a > 1, b > 1$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\log b - \log a}{b^2 - a^2} = \frac{\frac{1}{c}}{2c}$$

Hence,

$$\log\frac{b}{a} = \frac{b^2 - a^2}{2c^2}.$$

**Example 6: Using appropriate mean value theorem, prove that** $\dfrac{\sin b - \sin a}{e^b - e^a} = \dfrac{\cos c}{e^c}$ **for $a < c < b$. Hence, deduce that $e^c \sin x = (e^x - 1) \cos c$.**

**Solution:** Let $f(x) = \sin x$, $g(x) = e^x$ are defined in $(a, b)$.

(a) $f(x)$, being trigonometric function and $g(x)$, being exponential function, are continuous in $[a, b]$.

(b) $f'(x) = \cos x$, $g'(x) = e^x$ exists for all $x$ in $(a, b)$ and $g'(x) \neq 0$ for any $x$ in $(a, b)$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sin b - \sin a}{e^b - e^a} = \frac{\cos c}{e^c} \qquad \text{... (1)}$$

Putting $b = x$, $a = 0$ in Eq. (1),

$$\frac{\sin x - \sin 0}{e^x - e^0} = \frac{\cos c}{e^c}$$

$$e^c \sin x = (e^x - 1) \cos c.$$

**Example 7: Using Cauchy's Mean Value theorem, prove that there exists a number $c$ such that $0 < a < c < b$ and $f(b) - f(a) = c f'(c) \log\left(\dfrac{b}{a}\right)$. By putting $f(x) = x^{\frac{1}{n}}$, deduce that $\lim\limits_{n\to\infty} n\left(b^{\frac{1}{n}} - 1\right) = \log b$.**

**Solution:** Let $g(x) = \log x$ is defined in $[a, b]$.

(a) Let $f(x)$ is continuous in $[a, b]$ and differentiable in $(a, b)$. Also $g(x)$, being a logarithmic function, is continuous in $[a, b]$ for $0 < a < b$. $g'(x) = \dfrac{1}{x}$ exists for all $x$ in $(a, b)$ since $0 < a < b$ and $g'(x) \neq 0$ for any $x$ in $(a, b)$.

Thus, $f(x)$ and $g(x)$ satisfies all the conditions of Cauchy's Mean Value theorem. Therefore, there exists at least one point $c$ in $(a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(b) - f(a)}{\log b - \log a} = \frac{f'(c)}{\frac{1}{c}}$$

Hence,
$$f(b) - f(a) = cf'(c) \log\left(\frac{b}{a}\right)$$

Putting $f(x) = x^{\frac{1}{n}}$, $f'(x) = \frac{1}{n} x^{\frac{1}{n}-1}$ in Eq. (1),

$$(b)^{\frac{1}{n}} - (a)^{\frac{1}{n}} = c \cdot \frac{1}{n}(c)^{\frac{1}{n}-1} \log\left(\frac{b}{a}\right)$$

$$n\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right) = c^{\frac{1}{n}} \log\left(\frac{b}{a}\right)$$

$$\lim_{n\to\infty} n\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right) = \lim_{n\to\infty} c^{\frac{1}{n}} \log\left(\frac{b}{a}\right)$$

$$= c^0 \log\frac{b}{a} = \log\frac{b}{a}$$

Putting $a = 1$,

$$\lim_{n\to\infty} n\left(b^{\frac{1}{n}} - 1\right) = \log b.$$

## Exercise 2.5

1. Verify Cauchy's Mean Value theorem for the following functions :

   (i) $f(x) = 3x + 2$, $g(x) = x^2 + 1$ in $[1, 4]$

   (ii) $f(x) = x^2 + 2$, $g(x) = x^3 - 1$ in $[1, 2]$

   (iii) $f(x) = 2x^3$, $g(x) = x^6$ in $[a, b]$

   (iv) $f(x) = \log x$, $g(x)\ \frac{1}{x}$ in $[1, e]$

   [**Ans.:** (i) $c = \frac{5}{2}$ (ii) $c = \frac{14}{9}$ (iii) $c = \left(\frac{a^3 + b^3}{2}\right)^{\frac{1}{3}}$ (iv) $c = \frac{e}{e-1}$]

2. Using Cauchy's Mean Value theorem, find $\lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{\log x}$

   [**Hint:** Consider $f(x) = \cos\frac{\pi x}{2}$, $g(x) = \log x$ in the interval $(x, 1)$]

   [**Ans.:** $-\frac{\pi c}{2}$]

3. If $f(x)$ is continuous in $[a, b]$, $f'(x)$ exists in $(a, b)$, prove that there exists a point $c$ in $(a, b)$ such that

   (i) $\frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(c)}{2c}$

   (ii) $\frac{f(b) - f(a)}{b^3 - a^3} = \frac{f'(c)}{3c^2}$

   [**Hint :** (i) $g(x) = x^2$ (ii) $g(x) = x^3$]

4. Using appropriate mean value theorem, prove that

   $$\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c,\ a < c < b.$$

5. If $f(x) = \sin x$ and $g(x) = \cos x$ in $[a, b]$, prove that $c$ of Cauchy's Mean

Value theorem is the arithmetic mean of $a$ and $b$.

6. If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in $(a, b)$, $g(a) \neq g(b)$ and $g'(x) \neq 0$ in $(a, b)$, then there exists at least one $c$ between $a$ and $b$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(c)-f(a)}{g(b)-g(c)}, \; a<c<b$$

$$\left[\begin{array}{l}\textbf{Hint:} \text{ Let } P(x) = f(x)\,g(x) \\ \quad \text{and } Q(x) = f(x)\,g(b) \\ \quad\quad + g(x)\,g(a), \\ \quad \text{apply CMVT}\end{array}\right]$$

7. If $0 \le x \le 1$, prove that

$$\sqrt{\frac{1-x}{1+x}} < \frac{\log(1+x)}{\sin^{-1} x} < 1$$

$$\left[\begin{array}{l}\textbf{Hint:} f(x) = \log(1+x),\; g(x) \\ \quad = \sin^{-1} x, \\ \quad \text{apply CMVT in } [0, x] \\ \quad 0<c<x<1,\; \frac{1}{c} > \frac{1}{x} > \frac{1}{1}, \\ \quad \frac{1-c}{1+c} > \frac{1-x}{1+x} > \frac{1-1}{1+1}\end{array}\right]$$

8. If $f(x)$, $g(x)$, $h(x)$ are three functions differentiable in the interval $(a, b)$, prove that there exists a point $c$ in $(a, b)$ such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Hence, deduce Lagrange's and Cauchy's Mean Value theorem.

$$\left[\begin{array}{l}\textbf{Hint:} \text{ Consider } F(x) \\ \quad = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \\ \text{Apply Rolle's theorem. For} \\ \text{deduction of Lagrange's} \\ \text{MVT } g(x) = x,\; h(x) = 1, \text{ Keep} \\ f(x) \text{ as it is in result. For} \\ \text{deduction of Cauchy's MVT} \\ \text{take } h(x) = 1, \text{ keep } f(x) \text{ and} \\ g(x) \text{ as it is in the result.}\end{array}\right]$$

## 2.8 TAYLOR'S SERIES

**Statement:** If $f(x + h)$ be a given function of $h$ which can be expanded into a convergent series of positive ascending integral powers of $h$, then

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots + \frac{h^n}{n!} f^n(x) + \ldots\ldots\ldots$$

**Proof:** Let $f(x + h)$ be a function of $h$ which can be expanded into positive ascending integral powers of $h$, then

$$f(x+h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 + \ldots\ldots\ldots\ldots \qquad \ldots (1)$$

Differentiating w.r.t. $h$ successively,

$$f'(x+h) = a_1 + a_2 \cdot 2h + a_3 \cdot 3h^2 + a_4 \cdot 4h^3 + \ldots\ldots\ldots\ldots\ldots \qquad \ldots (2)$$

$$f''(x+h) = a_2 \cdot 2 + a_3 \cdot 6h + a_4 \cdot 12h^2 + \ldots\ldots\ldots\ldots\ldots \qquad \ldots (3)$$

$$f'''(x+h) = a_3 \cdot 6 + a_4 \cdot 24h + \ldots\ldots\ldots\ldots\ldots \qquad \ldots (4)$$

and so on

Putting $h = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(x)$$
$$a_1 = f'(x)$$
$$a_2 = \frac{1}{2!} f''(x)$$
$$a_3 = \frac{1}{3!} f'''(x) \text{ and so on}$$

Substituting $a_0$, $a_1$, $a_2$ and $a_3$ in Eq. (1),

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots. + \frac{h^n}{n!} f^n x + \ldots\ldots.$$

This is known as **Taylor's Series.**

Putting $x = a$ and $h = x - a$ in above series, we get Taylor's Series in powers of $(x - a)$ as

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots\ldots + \frac{(x-a)^n}{n!} f^n(a) + \ldots\ldots$$

**Example 1: Prove that** $\boldsymbol{f(mx) = f(x) + (m-1)x\, f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + \ldots .}$

**Solution:** $f(mx) = f(mx - x + x) = f[x + (m-1)x]$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots\ldots$$

Putting $\quad h = (m-1)x$,

$$f[x + (m-1)x] = f(mx) = f(x) + (m-1)x\, f'(x) + \frac{(m-1)^2}{2!} x^2 f''(x) + \ldots\ldots\ldots$$

**Example 2: Prove that**

$$\boldsymbol{f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!(1+x)^2} f''(x) - \frac{x^3}{3!(1+x)^3} f'''(x) + \ldots .}$$

**Solution:** $\quad \dfrac{x^2}{1+x} = x - \dfrac{x}{1+x},$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \ldots$$

Putting $\qquad h = -\dfrac{x}{1+x}$,

$$f\left(x - \frac{x}{1+x}\right) = f\left(\frac{x^2}{1+x}\right)$$

$$= f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!\,(1+x)^2} f''(x) - \frac{x^3}{3!\,(1+x)^3} f'''(x) + \ldots$$

**Example 3: Expand $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$ in powers of $(x - 1)$ and find $f(0.99)$.**

**Solution:** $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$

By Taylor's series,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots$$

Putting $\quad a = 1$,

$$f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$$

$$= f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1)$$

$$+ \frac{(x-1)^4}{4!} f^{iv}(1) + \frac{(x-1)^5}{5!} f^{v}(1) + \ldots\ldots \qquad \ldots (1)$$

$$f(1) = 1 - 1 + 1 - 1 + 1 - 1 = 0$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$f'(x) = 5x^4 - 4x^3 + 3x^2 - 2x + 1, \qquad f'(1) = 5 - 4 + 3 - 2 + 1 = 3$

$f''(x) = 20x^3 - 12x^2 + 6x - 2, \qquad f''(1) = 20 - 12 + 6 - 2 = 12$

$f'''(x) = 60x^2 - 24x + 6, \qquad f'''(1) = 60 - 24 + 6 = 42$

$f^{iv}(x) = 120x - 24, \qquad f^{iv}(1) = 120 - 24 = 96$

$f^{v}(x) = 120, \qquad f^{v}(1) = 120$

Substituting in Eq. (1),

$$f(x) = 0 + (x-1)3 + \frac{(x-1)^2}{2!}(12) + \frac{(x-1)^3}{3!}(42) + \frac{(x-1)^4}{4!}(96) + \frac{(x-1)^5}{5!}(120)$$

$$= 3(x-1) + 6(x-1)^2 + 7(x-1)^3 + 4(x-1)^4 + (x-1)^5$$

Putting $\quad x = 0.99$,

$$f(0.99) = 3\,(0.99 - 1) + 6\,(0.99 - 1)^2 + 7\,(0.99 - 1)^3 + 4\,(0.99 - 1)^4 + (0.99 - 1)^5$$

$$= 3\,(-0.01) + 6\,(-0.01)^2 + 7\,(-0.01)^3 + 4\,(-0.01)^4 + (-0.01)^5$$

$$= -0.02939$$

**Example 4: Prove that $\dfrac{1}{1-x}=\dfrac{1}{3}+\dfrac{(x+2)}{3^2}+\dfrac{(x+2)^2}{3^3}+\dfrac{(x+2)^3}{3^4}+\ldots$.**

**Solution:** Let $f(x)=\dfrac{1}{1-x}$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots\ldots\ldots\ldots$$

Putting $a=-2$,

$$f(x)=\frac{1}{1-x}=f(-2)+(x+2)f'(-2)+\frac{(x+2)^2}{2!}f''(-2)+\frac{(x+2)^3}{3!}f'''(-2)+\ldots\ldots\ldots \quad \ldots (1)$$

$$f(-2)=\frac{1}{3}$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x)=\frac{1}{(1-x)^2},\quad f'(-2)=\frac{1}{3^2}$$

$$f''(x)=\frac{2}{(1-x)^3},\quad f''(-2)=\frac{2!}{3^3}$$

$$f'''(x)=\frac{2.3}{(1-x)^4},\, f'''(-2)=\frac{3!}{3^4} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x)=\frac{1}{1-x}=\frac{1}{3}+\frac{(x+2)}{3^2}+\frac{(x+2)^2}{3^3}+\frac{(x+2)^3}{3^4}+\ldots\ldots\ldots\ldots$$

**Example 5: Expand log (cos $x$) about $\dfrac{\pi}{3}$.**

**Solution:** Let $f(x)=\log(\cos x)$

By Taylor's series,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots\ldots$$

Putting $a=\dfrac{\pi}{3}$,

$$f(x)=\log(\cos x)$$

$$=f\left(\frac{\pi}{3}\right)+\left(x-\frac{\pi}{3}\right)f'\left(\frac{\pi}{3}\right)+\frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 f''\left(\frac{\pi}{3}\right)+\frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 f'''\left(\frac{\pi}{3}\right)+\ldots \quad \ldots (1)$$

$$f\left(\frac{\pi}{3}\right)=\log\left(\cos\frac{\pi}{3}\right)=\log\left(\frac{1}{2}\right)=-\log 2$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x, \qquad f'\left(\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$$

$$f''(x) = -\sec^2 x, \qquad f''\left(\frac{\pi}{3}\right) = -\sec^2\frac{\pi}{3} = -4$$

$$f'''(x) = -2\sec^2 x \tan x, \qquad f'''\left(\frac{\pi}{3}\right) = -2\sec^2\frac{\pi}{3}\tan\frac{\pi}{3} = -2(4)\sqrt{3} = -8\sqrt{3} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x) = \log(\cos x) = -\log 2 + \left(x - \frac{\pi}{3}\right)\left(-\sqrt{3}\right) + \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2(-4) + \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3\left(-8\sqrt{3}\right) + \ldots$$

$$= -\log 2 - \sqrt{3}\left(x - \frac{\pi}{3}\right) - 2\left(x - \frac{\pi}{3}\right)^2 - \frac{4\sqrt{3}}{3}\left(x - \frac{\pi}{3}\right)^3 - \ldots$$

**Example 6: Obtain $\tan^{-1} x$ in powers of $(x - 1)$.**

**Solution:** Let $f(x) = \tan^{-1} x$

By Taylor's series,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots$$

Putting $a = 1$,

$$f(x) = \tan^{-1} x = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \ldots \qquad \ldots (1)$$

$$f(1) = \tan^{-1} 1 = \frac{\pi}{4}$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x) = \frac{1}{1+x^2}, \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}, \qquad f''(1) = -\frac{2}{4} = -\frac{1}{2} \text{ and so on}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}, \quad f'''(1) = \frac{1}{2}$$

Substituting in Eq. (1),

$$f(x) = \tan^{-1} x = \frac{\pi}{4} + (x-1)\left(\frac{1}{2}\right) + \frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3!}\left(\frac{1}{2}\right) + \cdots\cdots$$

$$= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 \cdots$$

**Example 7: Prove that**

$$\boldsymbol{\log[\sin(x+h)] = \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \cdots\cdots\cdots}$$

**Solution:** Let $f(x) = \log(\sin x)$, $f(x+h) = \log[\sin(x+h)]$

By Taylor's series,

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots\cdots\cdots\cdots \quad \ldots (1)$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x) = \frac{1}{\sin x}\cos x = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x$$

$$f'''(x) = 2\operatorname{cosec}^2 x \cot x = \frac{2\cos x}{\sin^3 x} \quad \text{and so on}$$

Substituting in Eq. (1),

$$f(x+h) = \log[\sin(x+h)]$$

$$= \log\sin x + h\cot x - \frac{h^2}{2!}\operatorname{cosec}^2 x + \frac{h^3}{3!}\frac{2\cos x}{\sin^3 x} + \cdots\cdots\cdots$$

$$= \log\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\frac{\cos x}{\sin^3 x} + \cdots\cdots\cdots$$

**Example 8: Expand $\tan^{-1}(x+h)$ in powers of $h$ and hence, find the value of $\tan^{-1}(1.003)$ up to 5 places of decimal.**

**Solution:**

Let $f(x) = \tan^{-1} x$ , $f(x+h) = \tan^{-1}(x+h)$

By Taylor's series,

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots\cdots\cdots\cdots \quad \ldots (1)$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x) = \frac{1}{1+x^2}, \qquad f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{2x \cdot 4x}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3} \text{ and so on}$$

Substituting in Eq. (1),

$$f(x+h) = \tan^{-1}(x+h) = \tan^{-1}(x+h) \cdot \frac{1}{1+x^2} + \frac{h^2}{2!}\left[-\frac{2x}{(1+x^2)^2}\right] + \frac{h^3}{3!}\left[\frac{2(3x^2-1)}{(1+x^2)^3}\right] + \dots\dots\dots$$

Putting $x = 1, h = 0.0003$,

$$\tan^{-1}(1+0.003) = \tan^{-1}(1.0003)$$

$$= \tan^{-1} 1 + \frac{0.0003}{2} + \frac{(0.0003)^2}{2!}\left(-\frac{2}{4}\right) + \frac{(0.0003)^3}{3!}\left(\frac{1}{2}\right) + \dots\dots\dots$$

$$= \frac{\pi}{4} + 0.00015 - 2.25 \times 10^{-8} + 2.25 \times 10^{-12} \quad \text{[Considering first 4 terms]}$$

$$= 0.78540$$

**Example 9: Prove that** $\sqrt{1+x+2x^2} = 1 + \frac{x}{2} + \frac{7x^2}{8} - \frac{7x^3}{16} + \dots\dots\dots$ .

**Solution:** Let $f(x) = \sqrt{x},\ f(x+h) = \sqrt{x+h}$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots\dots\dots$$

Putting $x = 1, h = x + 2x^2$,

$$f(x+h) = \sqrt{x+h} = \sqrt{1+x+2x^2}$$

$$= f(1) + (x+2x^2) f'(1) + \frac{(x+2x^2)^2}{2!} f''(1) + \frac{(x+2x^2)^3}{3!} f'''(1) + \dots\dots\dots \quad \dots (1)$$

$$f(x) = \sqrt{x}, \qquad f(1) = 1$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(1) = \frac{1}{2}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{\frac{3}{2}}}, \qquad f''(1) = -\frac{1}{4}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{x^{\frac{5}{2}}}, \qquad f'''(1) = \frac{3}{8} \text{ and so on}$$

Substituting in Eq. (1),

$$\sqrt{1+x+2x^2} = 1+\frac{1}{2}(x+2x^2)-\frac{1}{4}\frac{(x^2+4x^3+4x^4)}{2}+\frac{3}{8}\frac{(x^3+\ldots)}{6}+\ldots\ldots\ldots$$

$$= 1+\frac{x}{2}+\frac{7x^2}{8}-\frac{7x^3}{16}+\ldots$$

**Example 10: Expand $\sqrt{1+x+2x^2}$ in powers of $(x-1)$.**

**Solution:** $\sqrt{1+x+2x^2} = \sqrt{4+2(x-1)^2+5(x-1)}$ [Expressing in terms of $(x-1)$]

Let $f(x) = \sqrt{x},\ f(x+h) = \sqrt{x+h}$

By Taylor's series,

$$f(x+h) = f(x)+h\,f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\ldots\ldots$$

Putting $x = 4$, $h = 2(x-1)^2+5(x-1)$,

$$f(x+h) = \sqrt{x+h} = \sqrt{4+2(x-1)^2+5(x-1)}$$

$$= f(4)+[2(x-1)^2+5(x-1)]\,f'(4)+\frac{[2(x-1)^2+5(x-1)]^2}{2!}f''(4)+\ldots \qquad \ldots (1)$$

$$f(x) = \sqrt{x}, \qquad f(4) = 2$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(4) = \frac{1}{4}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{x^{\frac{3}{2}}}, \qquad f''(4) = -\frac{1}{32} \text{ and so on}$$

Substituting in Eq. (1),

$$\sqrt{4+2(x-1)^2+5(x-1)} = 2+[2(x-1)^2+5(x-1)]\frac{1}{4}$$

$$+\frac{[2(x-1)^2+5(x-1)]^2}{2!}\left(-\frac{1}{32}\right)+\ldots\ldots\ldots$$

$$\sqrt{1+x+2x^2} = 2+\frac{5}{4}(x-1)+\frac{7}{64}(x-1)^2+\ldots\ldots\ldots\ldots$$

**Example 11: Using Taylor's theorem, evaluate up to 4 places of decimals:**

**(i)** $\sqrt{1.02}$ **(ii)** $\sqrt{25.15}$

**(iii)** $\sqrt{9.12}$ **(iv)** $\sqrt{10}$

**Solution:** Let $f(x)=\sqrt{x},\ f(x+h)=\sqrt{x+h}$

By Taylor's series,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\ldots\ldots \qquad \ldots (1)$$

(i) Putting $x=1,\ h=0.02$,

$$f(x+h)=\sqrt{x+h}=\sqrt{1+0.02}$$

$$=f(1)+(0.02)f'(1)+\frac{(0.02)^2}{2!}f''(1)+\ldots\ldots\ldots \qquad \ldots (2)$$

$$f(x)=\sqrt{x}, \qquad f(1)=1$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x)=\frac{1}{2\sqrt{x}}, \qquad f'(1)=\frac{1}{2}$$

$$f''(x)=-\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(1)=-\frac{1}{4} \text{ and so on}$$

Substituting in Eq. (2) and considering only first 3 terms,

$$\sqrt{1.02}=1+(0.02)\frac{1}{2}+\frac{(0.02)^2}{2!}\left(-\frac{1}{4}\right)$$

$$=1.0099 \text{ approx.}$$

(ii) Putting $x=25,\ h=0.15$ in Eq. (1),

$$f(x+h)=\sqrt{x+h}=\sqrt{25+0.15}$$

$$=f(25)+(0.15)f'(25)+\frac{(0.15)^2}{2!}f''(25)+\ldots \qquad \ldots (3)$$

$$f(x)=\sqrt{x}, \qquad f(25)=5$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x)=\frac{1}{2\sqrt{x}}, \qquad f'(25)=\frac{1}{10}=0.1$$

$$f''(x)=-\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(25)=-\frac{1}{500}=-0.002 \text{ and so on}$$

Substituting in Eq. (2) and considering only first 3 terms,

$$\sqrt{25.15}=5+(0.15)(0.1)+\frac{(0.15)^2}{2}(-0.002)$$

$$=5.0150 \text{ approx.}$$

(iii) Putting $x = 9$, $h = 0.12$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+0.12}$$

$$= f(9) + (0.12)\, f'(9) + \frac{(0.12)^2}{2!} f''(9) + \ldots \qquad \text{... (3)}$$

$$f(x) = \sqrt{x}, \qquad f(9) = 3$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(9) = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}, \qquad f''(9) = -\frac{1}{108} \text{ and so on}$$

Substituting in Eq. (2) and considering only first 3 terms,

$$\sqrt{9.12} = 3 + (0.12)\left(\frac{1}{6}\right) + \frac{(0.12)^2}{2}\left(-\frac{1}{108}\right)$$

$$= 3 + 0.02 - (0.12)(0.06)(0.0093)$$

$$= 3.0199 \text{ approx.}$$

(iv) Putting $x = 9$, $h = 1$ in Eq. (1),

$$f(x+h) = \sqrt{x+h} = \sqrt{9+1} = f(9) + f'(9) + \frac{1}{2!} f''(9) + \ldots \qquad \text{... (4)}$$

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{216} \qquad \text{[refer (iii)]}$$

$$= 3.1620 \text{ approx.}$$

**Example 12: Find the value of tan (43°).**

**Solution:** Let $f(x) = \tan x$, $f(x+h) = \tan(x+h)$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots$$

Putting $x = 45^\circ$, $h = -2^\circ = -\frac{2\pi}{180} = -\frac{\pi}{90} = -0.0349$,

$$\tan(x+h) = \tan(45^\circ - 2^\circ) = \tan 43^\circ$$

$$= f(45^\circ) + (-0.0349) f'(45^\circ) + \frac{(-0.0349)^2}{2!} f''(45^\circ) + \ldots\ldots\ldots \qquad \text{... (1)}$$

$$f(x) = \tan x, \qquad f(45^\circ) = \tan(45^\circ) = 1$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x) = \sec^2 x, \quad f'(45^\circ) = \sec^2 45^\circ = 2$$

$$f''(x) = 2\sec^2 x \tan x, \quad f''(45^\circ) = 2\sec^2 45^\circ \tan 45^\circ = 4 \qquad \text{and so on}$$

Substituting in Eq. (1) and considering only first 3 terms,

$$\tan 43^\circ = 1 + (-0.0349)(2) + \frac{(-0.0349)^2}{2!}(4)$$

$$= 0.9326 \text{ approx.}$$

**Example 13: Find cosh (1.505) given sinh (1.5) = 2.1293 and cosh (1.5) = 2.3524.**

**Solution:** Let $f(x) = \cosh x$

By Taylor's series,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots\ldots$$

Putting $x = 1.5$, $h = 0.005$,

$$f(x+h) = \cosh(x+h) = \cosh(1.5 + 0.005)$$

$$= f(1.5) + (0.005) f'(1.5) + \frac{(0.005)^2}{2!} f''(1.5) + \frac{(0.005)^3}{3!} f'''(1.5) + \cdots \qquad \ldots (1)$$

$$f(x) = \cosh x, \; f(1.5) = \cosh(1.5) = 2.3524$$

Differentiating $f(x)$ w.r.t. $x$ successively,

$$f'(x) = \sinh x, \; f'(1.5) = \sinh(1.5) = 2.1293$$

$$f''(x) = \cosh x, \; f''(1.5) = \cosh(1.5) = 2.3524 \qquad \text{and so on}$$

Substituting in Eq. (1) and considering only first 3 terms,

$$\cosh(1.505) = \cosh(1.5) + (0.005)\sinh(1.5) + \frac{(0.005)^2}{2!}\cosh(1.5) + \ldots$$

$$= 2.3524 + (0.005)(2.1293) + (12.5)(10^{-6})(2.3524)$$

$$= 2.3631 \text{ approx.}$$

## Exercise 2.6

**1.** Expand $e^x$ in powers of $(x - 1)$.

$$\left[\textbf{Ans.: } e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right)\right]$$

**2.** Expand $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$.

[**Ans.:** $45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$]

**3.** Expand $x^5 - 5x^4 + 6x^3 - 7x^2 + 8x - 9$ in powers of $(x - 1)$.

$$\left[\textbf{Ans.: } -6 - 3(x-1) - 9(x-1)^2 - 4(x-1)^3 + (x-1)^5\right]$$

**4.** Expand $x^4 - 3x^3 + 2x^2 - x + 1$ in powers of $(x - 3)$.

$$\left[\textbf{Ans.: } 16 + 38(x-3) + 29(x-3)^2 + 9(x-3)^3 + (x-3)^4\right]$$

5. Expand $x^3 - 2x^2 + 3x - 5$ in power of $(x - 2)$.

[**Ans.:** $11 + 7(x - 2) + 4(x - 2)^2 + (x - 2)^3$]

6. Expand $2x^3 + 3x^2 - 8x + 7$ in terms of $(x - 2)$.

[**Ans.:** $19 + 28(x - 2) + 15(x - 2)^2 + 2(x - 2)^3$]

7. Expand $\sqrt{x}$ in powers of $(x - a)$.

$$\left[\textbf{Ans.: } \sqrt{a} + \frac{(x-a)}{2\sqrt{a}} - \frac{(x-a)^3}{8a\sqrt{a}} - \ldots\right]$$

8. Expand $\sqrt{1 + x + 2x^2}$ in powers of $(x - 1)$.

$$\left[\textbf{Ans.: } 2 + \frac{5}{4}(x-1) + \frac{7}{32}(x-1)^2 + \ldots\right]$$

9. Expand $\sin x$ in powers of $(x - a)$.

$$\left[\begin{aligned}&\textbf{Ans.: } \sin a + (x-a)\\ &\quad \cos a - \frac{(x-a)^2}{2!}\sin a\\ &\quad - \frac{(x-a)^3}{3!}\cos a + \ldots\end{aligned}\right]$$

10. Expand $\cos x$ in powers of $\left(x - \frac{\pi}{2}\right)$.

$$\left[\begin{aligned}&\textbf{Ans.: } -\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3\\ &\quad - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5 + \ldots\end{aligned}\right]$$

11. Expand $\tan x$ in powers of $\left(x - \frac{\pi}{4}\right)$.

$$\left[\textbf{Ans.: } 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \ldots\right]$$

12. Expand $\sin\left(\frac{\pi}{6} + x\right)$ in powers of $x$ upto $x^4$.

$$\left[\begin{aligned}&\textbf{Ans.: } \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{1}{2}\cdot\frac{x^2}{2!}\\ &\quad - \frac{\sqrt{3}}{2}\cdot\frac{x^3}{3!} + \frac{1}{2}\cdot\frac{x^4}{4!} + \ldots\end{aligned}\right]$$

13. Expand $\tan\left(\frac{\pi}{4} + x\right)$ in powers of $x$ upto $x^4$ and hence find the value of $\tan(46^\circ\, 36')$.

$$\left[\begin{aligned}&\textbf{Ans.: } \Big(1 + 2x + 2x^2 + \frac{8}{3}x^3\\ &\quad + \frac{10}{3}x^4 + \ldots\Big),\ 1.0574\end{aligned}\right]$$

14. Using Taylor's theorem find approximate value of $\cos 64^\circ$.

[**Ans.:** 0.4384]

15. Using Taylor's theorem find approximate value of $\sin(30^\circ\, 30')$.

[**Ans.:** 0.5073]

16. Expand $\log x$ in powers of $(x - 2)$.

$$\left[\begin{aligned}&\textbf{Ans.: } \log 2 + \frac{1}{2}(x-2) - \frac{1}{2!}\cdot\frac{(x-2)^2}{4}\\ &\quad + \frac{1}{3!}\cdot\frac{(x-2)^3}{4} + \ldots\end{aligned}\right]$$

17. Expand $\log \sin x$ in powers of $(x - 2)$.

$$\left[\begin{aligned}&\textbf{Ans.: } \log\sin 2 + (x-2)\cot 2\\ &\quad - \frac{1}{2}(x-2)^2\operatorname{cosec}^2 x + \ldots\end{aligned}\right]$$

18. Expand $\log\tan\left(\frac{\pi}{4} + x\right)$ in powers of $x$.

$$\left[\textbf{Ans.: } 2x + \frac{4}{3}x^3 + \frac{4}{3}x^5 + \ldots\right]$$

19. Arrange in powers of $x$, by Taylor's theorem, $7 + (x + 2) + 3(x + 2)^3 + (x + 2)^4$.

[**Ans.:** $49 + 69x + 42x^2 + 11x^3 + x^4$]

**20.** Arrange in powers of $x$, by Taylor's theorem, $17 + 6(x + 2) + 3(x + 2)^3 + (x + 2)^4 - (x + 2)^5$.

[**Ans.:** $37 - 6x - 38x^2 - 29x^3 - 9x^4 - x^5$]

**21.** Arrange in powers of $(x + 1)$, by Taylor's theorem, $(x + 2)^4 + 5(x + 2)^3 + 6(x + 2)^2 + 7(x + 2) + 8$.

$$\left[\begin{array}{l}\textbf{Hint:}\, f(x) = x^4 + 5x^3 + 6x^2 + 7x + 8, \\ \quad f[(x+1)+1] = f(1) \\ \quad + (x+1)f'(1) + \dfrac{(x+1)^2}{2!} f''(1) + \ldots\end{array}\right]$$

$$\left[\begin{array}{l}\textbf{Ans.:}\ 27 + 38(x+1) + 27(x+1)^2 \\ \qquad + 9(x+1)^3 + (x+1)^4\end{array}\right]$$

**22.** Prove that $\sinh(x + a) = \sinh a + x \cosh a + \dfrac{x^2}{2!}\sinh a + \ldots$

Given $\sinh(1.5) = 2.1293$, $\cosh(1.5) = 2.3524$, find the value of $\sinh(1.505)$.

[**Ans.** 2.1411]

## 2.9 MACLAURIN'S SERIES

**Statement:** If $f(x)$ be a given function of $x$ which can be expanded in positive ascending integral powers of $x$, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots\ldots$$

**Proof:** Let $f(x)$ be a function of $x$ which can be expanded into positive ascending integral powers of $x$, then

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots\ \ldots \qquad \ldots (1)$$

Differentiating w.r.t. $x$ successively,

$$f'(x) = a_1 + a_2 \cdot 2x + a_3 \cdot 3x^2 + a_4 \cdot 4x^3 + \ldots\ldots\ \ldots\ldots\ldots \qquad \ldots (2)$$

$$f''(x) = a_2 \cdot 2 + a_3 \cdot 6x + a_4 \cdot 12x^2 + \ldots\ldots\ \ldots\ldots\ldots \qquad \ldots (3)$$

$$f'''(x) = a_3 \cdot 6 + a_4 \cdot 24x + \ldots\ldots\ \ldots\ldots\ldots \qquad \ldots (4)$$

and so on

Putting $x = 0$ in Eq. (1), (2), (3) and (4),

$$a_0 = f(0)$$

$$a_1 = f'(0)$$

$$a_2 = \frac{1}{2!} f''(0)$$

$$a_3 = \frac{1}{3!} f'''(0) \quad \text{and so on.}$$

Substituting $a_0$, $a_1$, $a_2$ and $a_3$ in Eq. (1),

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots\ldots$$

This is known as **Maclaurin's Series.**
This series can also be written as,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots\ldots + \frac{x^n}{n!}y_n(0) + \ldots\ldots\ldots$$

## 2.9.1 Standard Expansions

Using Maclaurin's series, expansion of some standard functions can be obtained. These expansions can be directly used while solving the examples.

**(1) Expansion of $e^x$ (Exponential series)**

**Proof:** Let $y = e^x$, $y(0) = e^0 = 1$

Now $y_n = \frac{d^n}{dx^n}(e^x) = e^x$, $y_n(0) = e^0 = 1$ for all values of $n$.

Substituting in Maclaurin's series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots$$

This series is known as the exponential series.

**Note:** In the above series

(i) Replacing $x$ by $-x$,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\ldots$$

(ii) Replacing $x$ by $ax$,

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \ldots\ldots\ldots$$

**(2) Expansion of sin $x$ (Sine series)**

**Proof:** Let $y = \sin x$, $y(0) = \sin 0 = 0$

Now
$$y_n = \frac{d^n}{dx^n}(\sin x) = \sin\left(x + \frac{n\pi}{2}\right)$$

$$y_n(0) = \sin\left(\frac{n\pi}{2}\right)$$

Putting $n = 1, 2, 3, 4, 5, \ldots\ldots$

$y_1(0) = 1, y_2(0) = 0, y_3(0) = -1, y_4(0) = 0, y_5(0) = 1$, and so on.

Substituting in Maclaurin's series,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots$$

This series is known as the sine series.

**(3) Expansion of cos $x$ (Cosine series)**

**Proof:** Let $y = \cos x$, $y(0) = \cos 0 = 1$

Now
$$y_n = \frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{n\pi}{2}\right)$$
$$y_n(0) = \cos\left(\frac{n\pi}{2}\right)$$

Putting $n = 1, 2, 3, 4, \ldots\ldots$

$$y_1(0) = 0,\ y_2(0) = -1,\ y_3(0) = 0,\ y_4(0) = 1,$$ and so on.

Substituting in Maclaurin's series,
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\ldots$$

This series is known as the cosine series.

**(4) Expansion of tan $x$ (Tangent series)**

**Proof:** Let $y = \tan x$, $\qquad y(0) = 0$

$$\begin{aligned}
y_1 &= \sec^2 x = 1 + \tan^2 x = 1 + y^2, & y_1(0) &= 1\\
y_2 &= 2yy_1, & y_2(0) &= 2y(0)y_1(0) = 2(0)(1) = 0\\
y_3 &= 2y_1^2 + 2yy_2, & y_3(0) &= 2(1)^2 + 2(0)(0) = 2\\
y_4 &= 4y_1y_2 + 2y_1y_2 + 2yy_3 & y_4(0) &= 6(1)(0) + 2(0)(2)\\
&= 6y_1y_2 + 2yy_3, & &= 0\\
y_5 &= 6y_2^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4 & y_5(0) &= 0 + 8(1)(2) + 0\\
&= 6y_2^2 + 8y_1y_3 + 2yy_4, & &= 16
\end{aligned}$$

Substituting in Maclaurin's series,
$$\tan x = x + \frac{x^3}{3!}(2) + \frac{x^5}{5!}(16) + \ldots\ldots\ldots$$
$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots\ldots$$

This series is known as the tangent series.

**Note:** This series can also be obtained by dividing the sine and cosine series since $\tan x = \dfrac{\sin x}{\cos x}$.

**(5) Expansion of sinh $x$**

**Proof:** We have $\sinh x = \dfrac{e^x - e^{-x}}{2}$

Substituting $e^x$ and $e^{-x}$ from above exponential series,
$$\sinh x = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)}{2}$$
$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots$$

**(6) Expansion of cosh *x***

**Proof:** We have $\sinh x = \dfrac{e^x + e^{-x}}{2}$

Substituting exponential series $e^x$ and $e^{-x}$,

$$\cosh x = \frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)+\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\right)}{2}$$

$$= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots\ldots$$

**(7) Expansion of tanh *x***

**Proof:** Expansion of tanh $x$ can be obtained by dividing the series of sinh $x$ and cosh $x$.

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots}{1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots}$$

$$= x-\frac{x^2}{3}+\frac{2}{15}x^5-\ldots\ldots\ldots$$

**Note:** This series can also be obtained by using Maclaurin's series (refer tangent series)

**(8) Expansion of log (1 + *x*) (Logarithmic series)**

**Proof:** Let $y = \log(1+x)$, $y(0) = \log 1 = 0$

Now $$y_n = \frac{d^n}{dx^n}[\log(1+x)] = (-1)^{n-1}\cdot\frac{(n-1)!}{(x+1)^n}$$

$$y_n(0) = (-1)^{n-1}\cdot(n-1)!$$

Putting $n = 1, 2, 3, 4, \ldots..$

$$y_1(0) = 1,\ y_2(0) = -1,\ y_3(0) = 2! \text{ and so on}$$

Substituting in Maclaurin's series,

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\ldots\ldots$$

This series is known as the Logarithmic series and is valid for $-1 < x < 1$.

**Note:** In above series replacing $x$ by $-x$, we get expansion of log $(1-x)$

$$\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\ldots$$

**(9) Expansion of (1 + *x*)$^m$ (Binomial series)**

**Proof:** Let $y = (1+x)^m$, $y(0) = (1+0)^m = 1$

Now $$y_n = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)(1+x)^{m-n}$$
$$y_n(0) = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)$$

Putting $n = 1, 2, 3, 4, \ldots..$

$$y_1(0) = m,\ y_2(0) = m(m-1),\ y_3(0) = m(m-1)(m-2) \text{ and so on}$$

Substituting in Maclaurin's series,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots\ldots\ldots$$

This series is known as the Binomial series and is valid for $-1 < x < 1$.

## *By Definition*

**Example 1: Expand $5^x$ up to the first three non-zero terms of the series.**

**Solution:** Let $$f(x) = 5^x, f(0) = 5^0 = 1$$
$$f'(x) = 5^x \log 5,\ f'(0) = 5^0 \log 5 = \log 5$$
$$f''(x) = 5^x (\log 5)^2, f''(0) = 5^0 (\log 5)^2 = (\log 5)^2$$

Substituting in Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots\ldots$$

$$5^x = 1 + x\log 5 + \frac{x^2}{2!}(\log 5)^2 + \ldots\ldots\ldots$$

*Aliter:* $f(x) = 5^x = e^{\log 5^x} = e^{x \log 5}$

$$= 1 + x\log 5 + \frac{(x\log 5)^2}{2!} + \ldots\ldots\ldots \qquad \text{[Using Exponential series]}$$

**Example 2: Obtain the series log (1 + $x$) and find the series $\log\left(\frac{1+x}{1-x}\right)$ and hence, find the value of $\log_e\left(\frac{11}{9}\right)$.**

**Solution:** Let $y = \log(1+x)$

$$y_1 = \frac{1}{1+x}, y_2 = -\frac{1}{(1+x)^2}, y_3 = \frac{(2!)}{(1+x)^3}, y_4 = -\frac{(3!)}{(1+x)^4} \text{ etc.}$$

At $x = 0, y = 0,\ y_1 = 1,\ y_2 = -1,\ y_3 = 2!,\ y_4 = -(3!)$ etc.

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \ldots\ldots\ldots$$

$$= 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!}(2!) - \frac{x^4}{4!}(3!) + \ldots\ldots\ldots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$$

Replacing $x$ by $-x$,

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\dots\dots$$

Now,

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

Putting $x = \frac{1}{10}$, and considering first three terms,

$$\log_e\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\cdot\frac{1}{(10)^3} + \frac{1}{5}\cdot\frac{1}{(10)^5}\right] = 0.20067$$

**Example 3: If $x^3 + y^3 + xy - 1 = 0$, prove that $y = 1 - \frac{x}{3} - \frac{26}{81}x^3 - \dots$**

**Solution:** $x^3 + y^3 + xy - 1 = 0$,

Putting $x = 0$, $y(0) = 1$

Differentiating w.r.t. $x$,

$$3x^2 + 3y^2 y_1 + xy_1 + y = 0 \qquad \dots(1)$$

Putting $x = 0$, $y_1(0) = \frac{-1}{3}$

Differentiating Eq. (1) w.r.t. $x$,

$$6x + 6yy_1^2 + 3y^2y_2 + 2y_1 + xy_2 = 0 \qquad \dots(2)$$

Putting $x = 0$, $6\left(-\frac{1}{3}\right)^2 + 3y_2(0) + 2\left(-\frac{1}{3}\right) = 0$

$y_2(0) = 0$

Differentiating Eq. (2) w.r.t. $x$,

$$6 + 6y_1^3 + 12yy_1y_2 + 3y^2y_3 + 6yy_1y_2 + 3y_2 + xy_3 = 0$$

Putting $x = 0$,

$$6 + 6\left(\frac{-1}{27}\right) + 0 + 3y_3(0) = 0$$

$$y_3(0) = \frac{-52}{27} \quad \text{and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots\cdots$$

$$y = 1 - \frac{x}{3} + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\left(\frac{-52}{27}\right) + \cdots\cdots$$

$$= 1 - \frac{x}{3} - \frac{26}{81}x^3 - \cdots\cdots$$

**Example 4: If $x^3 + 2xy^2 - y^3 + x - 1 = 0$, expand $y$ in ascending powers of $x$.**

**Solution:** $$x^3 + 2xy^2 - y^3 + x - 1 = 0$$

Putting $x = 0$, $y(0) = -1$

Differentiating w.r.t. $x$,

$$3x^2 + 2y^2 + 4xyy_1 - 3y^2y_1 + 1 = 0 \quad \dots (1)$$

Putting $x = 0$,

$$2 - 3y_1(0) + 1 = 0$$
$$y_1(0) = 1$$

Differentiating Eq. (1) w.r.t. $x$,

$$6x + 4yy_1 + 4yy_1 + 4xy_1^2 + 4xyy_2 - 6yy_1^2 - 3y^2y_2 = 0$$

Putting $x = 0$,

$$-8 + 6 - 3y_2(0) = 0$$
$$y_2(0) = -\frac{2}{3} \text{ and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \cdots\cdots$$
$$y = -1 + x + \frac{x^2}{2!}\left(-\frac{2}{3}\right) + \cdots\cdots$$
$$= -1 + x - \frac{x^2}{3} + \cdots\cdots$$

**Example 5: If $x = y(1 + y^2)$, prove that $y = x - x^3 + 3x^5 + \dots$.**

**Solution:** $$x = y(1 + y^2)$$

Putting $x = 0$, $y(0) = 0$

Differentiating w.r.t. $x$,

$$1 = y_1 + 3y^2y_1 \quad \dots (1)$$

Putting $x = 0$,

$$1 = y_1(0)$$
$$y_1(0) = 1$$

Differentiating Eq. (1) w.r.t. $x$,

$$0 = y_2 + 6yy_1^2 + 3y^2y_2 \quad \dots (2)$$

Putting $x = 0$, $y_2(0) = 0$,

Differentiating Eq. (2) w.r.t. $x$,

$$0 = y_3 + 12yy_1y_2 + 6y_1^3 + 6yy_1y_2 + 3y^2y_3$$
$$0 = y_3(1 + 3y^2) + 18yy_1y_2 + 6y_1^3 \quad \dots (3)$$

Putting $x = 0$,

$$0 = y_3(0) + 6$$
$$y_3(0) = -6$$

Differentiating Eq. (3) w.r.t. $x$,

$$0 = (1 + 3y^2)\, y_4 + 6yy_1y_3 + 18y_1^2\, y_2 + 18yy_2^2 + 18yy_1y_3 + 18y_1^2y_2$$

$$= (1 + 3y^2)\, y_4 + 24yy_1y_3 + 36y_1^2y_2 + 18yy_2^2 \qquad \ldots (4)$$

Putting $x = 0$, $y_4(0) = 0$,

Differentiating Eq. (4) w.r.t. $x$,

$$0 = (1 + 3y^2)\, y_5 + 6yy_1y_4 + 24y_1^2\, y_3 + 24yy_2y_3 + 24yy_1y_4 + 72y_1y_2^2$$
$$+ 36y_1^2y_3 + 36yy_2y_3 + 18y_1y_2^2$$

Putting $x = 0$,

$$0 = y_5(0) + 24\,(-6) + 36\,(-6)$$

$$y_5(0) = 360 \text{ and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \cdots\cdots$$

$$= 0 + x.1 + \frac{x^2}{2!}\cdot 0 + \frac{x^3}{3!}(-6) + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\cdot 360 + \cdots\cdots$$

$$= x - x^3 + 3x^5 + \cdots\cdots$$

## *By Standard Expansion*

**Example 1: Obtain the expansion of $\dfrac{1+x^2}{1+x^4}$.**

**Solution:**

$$\frac{1+x^2}{1+x^4} = (1+x^2)(1+x^4)^{-1}$$

$$= (1 + x^2)\,(1 - x^4 + x^8 - x^{12} + x^{16} - \ldots)$$

$$= 1 + x^2 - x^4 - x^6 + x^8 + x^{10} - \ldots$$

**Example 2: If $x = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \dfrac{y^4}{4} + \ldots..$, prove that $y = x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots\ldots..$ and conversely.**

**Solution:**

$$x = \log(1+y)$$

$$1 + y = e^x$$

$$y = e^x - 1$$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots.$$

Conversely,

$$y = e^x - 1$$

$$e^x = 1 + y$$

$$x = \log(1+y)$$

$$= y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots..$$

**Example 3: Expand $\sqrt{1+\sin x}$.**

**Solution:**

$$\sqrt{1+\sin x} = \sin\frac{x}{2} + \cos\frac{x}{2}$$

$$= \left[\frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right] + \left[1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \cdots$$

**Example 4: Prove that $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \ldots..$.**

**Solution:**

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$= \frac{1}{2}\left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right]$$

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots$$

**Example 5: Prove that $\cosh^3 x = \frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n}+3}{(2n)!}x^{2n}$.**

**Solution:**

$$\cosh^3 x = \frac{1}{4}(\cosh 3x + 3\cosh x)$$

$$= \frac{1}{4}\left[\left(1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \cdots\right) + 3\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\right]$$

$$= \frac{1}{4}\left[(1+3) + \frac{3^2+3}{2!}x^2 + \frac{3^4+3}{4!}x^4 + \cdots\right]$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n}+3}{(2n)!}x^{2n}$$

**Example 6: Prove that $\sin x \sinh x = x^2 - \frac{8}{6!}x^6 + \frac{32}{10!}x^{10} - \ldots$.**

**Solution:** $$\sin x \sinh x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\right)\cdot\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\right)$$

$$= x^2 + x^6\left[\frac{2}{5!} - \frac{1}{(3!)^2}\right] + x^{10}\left[\frac{2}{9!} - \frac{2}{7!3!} + \frac{1}{(5!)^2}\right] + \ldots$$

$$= x^2 - \frac{8}{6!}x^6 + \frac{32}{10!}x^{10} - \ldots$$

**Example 7: Expand log $(1+x+x^2+x^3)$ up to a term in $x^8$.**

**Solution:** $\log(1+x+x^2+x^3) = \log[(1+x)(1+x^2)]$

$$= \log(1+x) + \log(1+x^2)$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \ldots\right] + \left[x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \frac{(x^2)^4}{4} + \ldots\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3}{4}x^4 + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3}{8}x^8 + \ldots$$

**Example 8: Prove that log $(1+x+x^2+x^3+x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$.**

**Solution:** $\log(1+x+x^2+x^3+x^4) = \log\left(\frac{1-x^5}{1-x}\right)$ [Using sum of G.P.]

$$= \log(1-x^5) - \log(1-x)$$

$$= \left(-x^5 - \frac{x^{10}}{2} - \frac{x^{15}}{3} - \frac{x^{20}}{4} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\cdots\right)$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \cdots$$

**Example 9: Prove that $\log x = \log 2 + \left(\frac{x}{2} - 1\right) - \frac{1}{2}\left(\frac{x}{2} - 1\right)^2 + \frac{1}{3}\left(\frac{x}{2} - 1\right)^3 + \ldots$ .**

**Solution:** $\log x = \log\left(2 \cdot \frac{x}{2}\right)$

$$= \log 2 + \log\frac{x}{2}$$

$$= \log 2 + \log\left[1 + \left(\frac{x}{2} - 1\right)\right]$$

$$= \log 2 + \left(\frac{x}{2} - 1\right) - \frac{1}{2}\left(\frac{x}{2} - 1\right)^2 + \frac{1}{3}\left(\frac{x}{2} - 1\right)^3 - \ldots$$

**Example 10: Prove that $\log\left(\frac{\sinh x}{x}\right) = \frac{x^2}{6} - \frac{x^4}{180} + \ldots$ .**

**Solution:** $\log\left(\frac{\sinh x}{x}\right) = \log\left[\frac{1}{x}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)\right] = \log\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)$

$$= \left(\frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)^2 + \ldots..$$

$$= \frac{x^2}{6} + x^4\left(\frac{1}{120} - \frac{1}{72}\right) + \ldots...$$

$$= \frac{x^2}{6} - \frac{x^4}{180} + \ldots..$$

**Example 11: Prove that $\log(x\cot x) = -\frac{x^2}{3} - \frac{7}{90}x^4 + .....$**

**Solution:** $\log(x\cot x) = -\log\left(\frac{1}{x\cot x}\right)$

$$= -\log\left(\frac{\tan x}{x}\right)$$

$$= -\log\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + ...\right)$$

$$= -\left[\left(\frac{x^2}{3} + \frac{2}{15}x^4 + ....\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + ...\right)^2 + ...\right]$$

$$= -\left[\frac{x^2}{3} + x^4\left(\frac{2}{15} - \frac{1}{18}\right) + ...\right]$$

$$= -\frac{x^2}{3} - \frac{7}{90}x^4 + ...$$

**Example 12: Prove that $\log\left(\frac{1+e^{2x}}{e^x}\right) = \log 2 + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \cdots.$**

**Solution:** $\log\left(\frac{1+e^{2x}}{e^x}\right) = \log(e^{-x} + e^x) = \log(2\cosh x)$

$$= \log 2 + \log\cosh x$$

$$= \log 2 + \log\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...\right)$$

$$= \log 2 + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...\right) - \frac{1}{2}\left(\frac{x^2}{2!} + \frac{x^4}{4!} + ...\right)^2 + \frac{1}{3}\left(\frac{x^2}{2!} + ...\right)^3 + ...$$

$$= \log 2 + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...\right) - \frac{1}{2}\left(\frac{x^4}{4} + 2\cdot\frac{x^6}{48} + ...\right) + \frac{1}{3}\left(\frac{x^6}{8} + ...\right) + ...$$

$$= \log 2 + \frac{x^2}{2} + x^4\left(\frac{1}{24} - \frac{1}{8}\right) + x^6\left(\frac{1}{720} - \frac{1}{48} + \frac{1}{24}\right) + ...$$

$$= \log 2 + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - ...$$

**Example 13: Prove that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + ......$**

**Solution:** $\log(1+e^x) = \log\left(1 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$

$$= \log\left[2\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)\right]$$

$$= \log 2 + \log\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)$$

$$= \log 2 + \left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)-\frac{1}{2}\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\ldots\right)^2$$

$$+\frac{1}{3}\left(\frac{x}{2}+\frac{x^2}{4}+\ldots\right)^3-\frac{1}{4}\left(\frac{x}{2}+\ldots\right)^4+\ldots$$

$$= \log 2 + \left(\frac{x}{2}\right)+x^2\left(\frac{1}{4}-\frac{1}{8}\right)+x^3\left(\frac{1}{12}-\frac{1}{8}+\frac{1}{24}\right)$$

$$+x^4\left(\frac{1}{48}-\frac{1}{32}-\frac{1}{24}+\frac{1}{16}-\frac{1}{64}\right)+\ldots$$

$$= \log 2 + \frac{x}{2}+\frac{x^2}{8}+0+\left(-\frac{1}{192}\right)x^4+\ldots$$

$$= \log 2 + \frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots$$

**Example 14: Prove that** $\log\left[\log(1+x)^{\frac{1}{x}}\right] = -\frac{x}{2}+\frac{5x^2}{24}-\frac{x^3}{8}+\frac{251}{2880}x^4+\ldots$ **.**

**Solution:**

$$\log(1+x)^{\frac{1}{x}} = \frac{1}{x}\log(1+x)$$

$$= \frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right)$$

$$= 1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\cdots$$

$$= 1-\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\frac{x^4}{5}+\cdots\right)$$

$$= 1-y$$

Now, $\log\left[\log(1+x)^{\frac{1}{x}}\right] = \log(1-y) = -y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}-\ldots$

$$= -\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\frac{x^4}{5}+\ldots\right)-\frac{1}{2}\cdot\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\ldots\right)^2$$

$$-\frac{1}{3}\left(\frac{x}{2}-\frac{x^2}{3}-\ldots\right)^3-\frac{1}{4}\left(\frac{x}{2}-\frac{x^2}{3}+\ldots\right)^4-\ldots$$

$$= -\frac{x}{2} + x^2\left(\frac{1}{3} - \frac{1}{8}\right) - x^3\left(\frac{1}{4} - \frac{1}{6} + \frac{1}{24}\right) + x^4\left(\frac{1}{5} - \frac{1}{18} - \frac{1}{8} + \frac{1}{12} - \frac{1}{64}\right) + \ldots$$

$$= -\frac{x}{2} + \frac{5x^2}{24} - \frac{x^3}{8} + \frac{251}{2880}x^4 + \ldots$$

**Example 15: Expand $\left(\frac{1+e^x}{2e^x}\right)^{\frac{1}{2}}$ up to the term containing $x^2$.**

**Solution:**

$$\left(\frac{1+e^x}{2e^x}\right)^{\frac{1}{2}} = \left(\frac{1}{2}e^{-x} + \frac{1}{2}\right)^{\frac{1}{2}}$$

$$= \left[\frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \cdots\right) + \frac{1}{2}\right]^{\frac{1}{2}}$$

$$= \left(1 - \frac{1}{2}x + \frac{x^2}{4} - \cdots\right)^{\frac{1}{2}}$$

$$= \left[1 - \left(\frac{x}{2} - \frac{x^2}{4} + \cdots\right)\right]^{\frac{1}{2}}$$

$$= 1 - \frac{1}{2}\left(\frac{x}{2} - \frac{x^2}{4} + \ldots\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(\frac{x}{2} - \frac{x^2}{4} + \ldots\right)^2 - \ldots$$

$$= 1 - \frac{x}{4} + \frac{x^2}{8} - \frac{1}{8}\cdot\frac{x^2}{4} + \ldots$$

$$= 1 - \frac{x}{4} + \frac{3}{32}x^2 + \ldots$$

**Example 16: Prove that $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 - \frac{x^5}{5}$.**

**Solution:** $e^{x\cos x} = e^{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)}$

$$= 1 + \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \ldots\right) + \frac{1}{2!}\left(x - \frac{x^3}{2!} + \ldots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{2!} + \ldots\right)^3$$

$$+ \frac{1}{4!}\left(x - \frac{x^3}{2!} - \ldots\right)^4 + \frac{1}{5!}\left(x - \frac{x^3}{2!} - \ldots\right)^5$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(-\frac{1}{2} + \frac{1}{6}\right) + x^4\left(-\frac{1}{2} + \frac{1}{24}\right) + x^5\left(\frac{1}{24} - \frac{1}{4} + \frac{1}{120}\right) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 - \frac{x^5}{5} + \ldots$$

**Example 17: Prove that** $e^{e^x} = e\left(1 + x + x^2 + \dfrac{5x^3}{6} + \cdots\right)$.

**Solution:** $e^{e^x} = e^{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}$

$$= ee^{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$$

$$= e\left[1 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + \frac{1}{2!}\left(x + \frac{x^2}{2!} + \cdots\right)^2 + \frac{1}{3!}(x + \ldots)^3 + \cdots\right]$$

$$= e\left[1 + x + x^2\left(\frac{1}{2} + \frac{1}{2}\right) + x^3\left(\frac{1}{6} + \frac{1}{2} + \frac{1}{6}\right) + \cdots\right]$$

$$= e\left(1 + x + x^2 + \frac{5}{6}x^3 + \cdots\right)$$

**Example 18: Prove that** $(1+x)^{\frac{1}{x}} = e - \dfrac{e}{2}x + \dfrac{11e}{24}x^2 + \ldots\ldots$.

**Solution:** $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)}$

$$= e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)}$$

$$= e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)}$$

$$= ee^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\ldots\right)}$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)^2 + \cdots\right]$$

$$= e\left[1 - \frac{x}{2} + x^2\left(\frac{1}{3} + \frac{1}{8}\right) + \cdots\right]$$

$$= e - \frac{e}{2}x + \frac{11e}{24}x^2 + \cdots$$

**Example 19: Prove that** $\sin(e^x - 1) = x + \dfrac{x^2}{2} - \dfrac{5}{24}x^4 + \ldots\ldots$.

**Solution:** $\sin(e^x - 1) = \sin\left(x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots\right)$

$$= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \frac{1}{3!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3 + \cdots$$

$$= x + \frac{x^2}{2} + x^3\left(\frac{1}{6} - \frac{1}{6}\right) + x^4\left(\frac{1}{24} - \frac{1}{4}\right) + \cdots$$

$$= x + \frac{x^2}{2} - \frac{5}{24}x^4 + \cdots$$

**Example 20: Expand $\frac{x}{e^x - 1}$ up to $x^4$ and hence, prove that**

$$\frac{x}{2}\frac{e^x + 1}{e^x - 1} = 1 + \frac{x^2}{12} - \frac{x^4}{720} + \cdots.$$

**Solution:**
$$\frac{x}{e^x - 1} = \frac{x}{\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right) - 1\right]}$$

$$= \frac{x}{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)}$$

$$= \left[1 + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right)\right]^{-1}$$

$$= 1 - \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots\right) + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots\right)^2$$
$$- \left(\frac{x}{2} + \frac{x^2}{6} + \cdots\right)^3 + \left(\frac{x}{2} + \cdots\right)^4$$

$$= 1 - \frac{x}{2} + x^2\left(-\frac{1}{6} + \frac{1}{4}\right) + x^3\left(-\frac{1}{24} + \frac{1}{6} - \frac{1}{8}\right)$$
$$+ x^4\left(-\frac{1}{120} + \frac{1}{36} + \frac{1}{24} - \frac{1}{8} + \frac{1}{16}\right) + \cdots$$

$$= 1 - \frac{x}{2} + \frac{x^2}{12} + x^3(0) - \frac{x^4}{720} + \cdots \qquad \ldots(1)$$

$$\frac{x}{2}\frac{e^x + 1}{e^x - 1} = \frac{x}{2}\left(1 + \frac{2}{e^x - 1}\right)$$

$$= \frac{x}{2} + \frac{x}{e^x - 1}$$

$$= \frac{x}{2} + 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots \qquad \text{[Using Eq. (1)]}$$

$$= 1 + \frac{x^2}{12} - \frac{x^4}{720} + \cdots$$

**Example 21: Prove that**

$$\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)=x\sin\theta+\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta+\ldots.$$

**Solution:** Let

$$y=\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)$$

$$\tan y=\frac{x\sin\theta}{1-x\cos\theta}$$

$$\frac{e^{iy}-e^{-iy}}{i(e^{iy}+e^{-iy})}=\frac{x\sin\theta}{1-x\cos\theta}$$

$$\frac{e^{iy}-e^{-iy}}{e^{iy}+e^{-iy}}=\frac{ix\sin\theta}{1-x\cos\theta}$$

Applying componendo–dividendo,

$$\frac{e^{iy}}{e^{-iy}}=\frac{1-x(\cos\theta-i\sin\theta)}{1-x(\cos\theta+i\sin\theta)}$$

$$e^{2iy}=\frac{1-xe^{-i\theta}}{1-xe^{i\theta}}$$

$$2iy=\log(1-xe^{-i\theta})-\log(1-xe^{i\theta})$$

$$=\left(-xe^{-i\theta}-\frac{x^2e^{-2i\theta}}{2}-\frac{x^3e^{-3i\theta}}{3}-\ldots\right)-\left(-xe^{i\theta}-\frac{x^2e^{2i\theta}}{2}-\frac{x^3e^{3i\theta}}{3}-\ldots\right)$$

$$=x(e^{i\theta}-e^{-i\theta})+\frac{x^2}{2}(e^{2i\theta}-e^{-2i\theta})+\frac{x^3}{3}(e^{3i\theta}-e^{-3i\theta})+\ldots$$

$$=x\cdot 2i\sin\theta+\frac{x^2}{2}\cdot 2i\sin 2\theta+\frac{x^3}{3}\cdot 2i\sin 3\theta+\ldots$$

$$y=x\sin\theta+\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta+\ldots$$

**Example 22: Prove that** $e^{ax}\cos bx=1+ax+\dfrac{(a^2-b^2)}{2!}x^2+\dfrac{a(a^2-3b^2)}{3!}x^3+\ldots$

**and hence, deduce** $e^{x\cos\alpha}\cos(x\sin\alpha)=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}\cos n\alpha$.

**Solution:** $e^{ax}\cos bx=e^{ax}$. Real Part of $(e^{ibx})$

$$=\text{R.P. of } e^{(a+ib)x}$$

$$=\text{R.P. of }\left[1+(a+ib)x+\frac{(a^2+ib)^2}{2!}x^2+\frac{(a+ib)^3}{3!}x^3+\cdots\right]$$

$$= \text{R.P}\left[1+(a+ib)x+\frac{(a^2-b^2+2aib)}{2!}x^2+\frac{(a^3-ib^3+3ia^2b-3ab^2)}{3!}x^3+\ldots\right]$$

$$=1+ax+\frac{(a^2-b^2)}{2!}x^2+\frac{a(a^2-3b^2)}{3!}x^3+\ldots$$

Putting $a=\cos\alpha$ and $b=\sin\alpha$,

$$e^{x\cos\alpha}\cos(x\sin\alpha)=1+x\cos\alpha+\frac{(\cos^2\alpha-\sin^2\alpha)}{2!}x^2+\frac{\cos^3\alpha-3\cos\alpha\cdot\sin^2\alpha}{3!}x^3+\ldots$$

$$=1+x\cos\alpha+\frac{\cos 2\alpha}{2!}x^2+\frac{\cos^3\alpha-3\cos\alpha(1-\cos^2\alpha)}{3!}x^3+\ldots$$

$$=1+x\cos\alpha+\frac{x^2}{2!}\cos 2\alpha+\frac{x^3}{3!}\cos 3\alpha+\ldots.$$

$$=\sum_{n=0}^{\infty}\frac{x^n}{n!}\cos n\alpha$$

**Example 23: Prove that** $\boldsymbol{e^x=1+\tan x+\frac{1}{2!}\tan^2 x-\frac{1}{3!}\tan^3 x-\frac{7}{4!}\tan^4 x+\ldots\ldots.}$

**Solution:** Let $e^x=a_0+a_1\tan x+a_2\tan^2 x+a_3\tan^3 x+a_4\tan^4 x+\ldots\ldots\ldots$ ...(1)

$$=a_0+a_1\left(x+\frac{x^3}{3}+\ldots\right)+a_2\left(x+\frac{x^3}{3}+\ldots\right)^2+a_3\left(x+\frac{x^3}{3}+\ldots\right)^3+a_4\left(x+\frac{x^3}{3}+\ldots\right)^4+\ldots\ldots\ldots$$

$$=a_0+a_1\left(x+\frac{x^3}{3}+\ldots\right)+a_2\left(x^2+\frac{2x^4}{3}+\ldots\right)+a_3(x^3+\ldots)+a_4(x^4+\ldots)+\ldots\ldots\ldots.$$

$$=a_0+a_1x+a_2x^2+\left(\frac{a_1}{3}+a_3\right)x^3+\left(\frac{2}{3}a_2+a_4\right)x^4+\ldots\ldots\ldots. \quad \text{...(2)}$$

But
$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\ldots \quad \text{...(3)}$$

Thus from Eqs (2) and (3)

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots..=a_0+a_1x+a_2x^2+\left(\frac{a_1}{3}+a_3\right)x^3+\left(\frac{2}{3}a_2+a_4\right)x^4+\ldots\ldots.$$

Comparing coefficients of $x$, $x^2$, $x^3$ and $x^4$ on both the sides,

$$a_0=1, a_1=1, a_2=\frac{1}{2!}=\frac{1}{2}, \frac{a_1}{3}+a_3=\frac{1}{3!}=\frac{1}{6}$$

$$a_3 = \frac{1}{6} - \frac{a_1}{3} = \frac{1}{6} - \frac{1}{3} = -\frac{1}{6} = -\frac{1}{3!}$$

$$\frac{2}{3}a_2 + a_4 = \frac{1}{4!} = \frac{1}{24}, \quad a_4 = \frac{1}{24} - \frac{2}{3}\cdot\frac{1}{2} = -\frac{7}{24} = -\frac{7}{4!}$$

Substituting in Eq. (1),

$$e^x = 1 + \tan x + \frac{1}{2!}\tan^2 x - \frac{1}{3!}\tan^3 x - \frac{7}{4!}\tan^4 x + \ldots$$

**Example 24: Find the values of $a$ and $b$ such that the expansion of $\log(1+x) - \dfrac{x(1+ax)}{1+bx}$ in ascending powers of $x$ begins with the term $x^4$ and prove that this term is $-\dfrac{x^4}{36}$.**

**Solution:** Let $f(x) = \log(1+x) - \dfrac{x(1+ax)}{1+bx}$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - (x + ax^2)(1+bx)^{-1}$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - (x + ax^2)(1 - bx + b^2x^2 - b^3x^3 + \ldots)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - (x - bx^2 + b^2x^3 - b^3x^4 + ax^2 - abx^3 + ab^2x^4 - ab^3x^5 + \ldots\ldots.)$$

$$= \left(-\frac{1}{2} + b - a\right)x^2 + \left(\frac{1}{3} - b^2 + ab\right)x^3 + \left(-\frac{1}{4} + b^3 - ab^2\right)x^4 + \ldots\ldots..$$

If the expansion begins with the term $x^4$, the coefficients of $x^2$ and $x^3$ must be zero.

$$-\frac{1}{2} + b - a = 0, \quad b = a + \frac{1}{2} \quad \text{and} \quad \frac{1}{3} - b^2 + ab = 0 \qquad \ldots (1)$$

Substituting $b$ in Eq. (1),

$$\frac{1}{3} - \left(a + \frac{1}{2}\right)^2 + a\left(a + \frac{1}{2}\right) = 0$$

$$\frac{1}{3} - a^2 - \frac{1}{4} - a + a^2 + \frac{1}{2}a = 0$$

$$\frac{1}{2}a = \frac{1}{12}, \quad a = \frac{1}{6}$$

$$b = \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

Coefficient of $x^4 = -\frac{1}{4} + b^3 - ab^2 = -\frac{1}{4} + \left(\frac{2}{3}\right)^3 - \frac{1}{6}\left(\frac{2}{3}\right)^2 = -\frac{1}{36}$

Hence, the expansion begins with the term $-\frac{x^4}{36}$.

## Exercise 2.7

**1.** Expand $e^x \sec x$ in powers of $x$ using Maclaurin's series.

[**Ans.:** $1 + x + x^2 + ...$]

**2.** Using Maclaurin's series, prove that

$$e^{\sin x} = 1 + x + \frac{x^2}{2} + \ldots$$

**3.** Using Maclaurin's series, prove that

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \ldots$$

**4.** Prove that

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \ldots$$

**5.** Prove that $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \ldots$

$$\left[\textbf{Hint: } \sec x = \frac{1}{\cos x} = (\cos x)^{-1} = \left[1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\right)\right]^{-1}\right]$$

**6.** Prove that

$$x\operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$$

**7.** Prove that

$$e^x \sin 2x = 2x + 2x^2 - \frac{x^3}{3} + \ldots$$

**8.** Prove that $e^x \cos x = 1 + x - \frac{x^3}{3} + \ldots$

**9.** Prove that

$$\cos x \cosh x = 1 - \frac{2^2 x^4}{4!} + \frac{2^4 x^8}{8!} - \ldots$$

**10.** Prove that

$$\sin(e^x - 1) = x + \frac{x^2}{2} - \frac{5}{24}x^4 + \ldots$$

**11.** Prove that

$$\cos^n x = 1 - n\cdot\frac{x^2}{2!} + n(3n-2)\cdot\frac{x^4}{4!} - \ldots$$

Hence, deduce that

$$\cos^3 x = 1 - \frac{3x^2}{2} + \frac{15x^4}{48} - \ldots$$

**12.** Prove that

$$\sinh^3 x = \sum \frac{(3^n - 3) - [1 - (-1)^n]\, x^n}{8\cdot n!}$$

**13.** Prove that

$$e^{x\sin x} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} + \ldots\ldots\ldots$$

**14.** Prove that

$$(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} - \ldots$$

$$\left[\textbf{Hint: } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\right]$$

**15.** Prove that

$$(1+x)^{1+x} = 1 + x + x^2 + \frac{x^3}{3} + \ldots$$

Hence, find approximate value of $(1.01)^{1.01}$

[**Ans.:** 1.0101]

**16.** Prove that

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \ldots$$

**17.** Prove that

$$\log(1-x+x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} - \ldots$$

**18.** Prove that

$$[\log(1+x)]^2 = x^2 - x^3 + \frac{11}{12}x^4 - \ldots$$

**19.** Prove that

$$\log\cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{45}x^6 - \ldots$$

**20.** Prove that

$$\log(1+\tan x) = x - \frac{x^2}{2} + \frac{2x^3}{3} + \ldots\ldots\ldots$$

**21.** Prove that

$$\log\left(\frac{\sin x}{x}\right) = -\left(\frac{x^2}{6} + \frac{x^4}{180} + \frac{x^6}{2835} + \ldots\right)$$

**22.** Prove that

$$\log\left(\frac{\tan x}{x}\right) = \frac{x^3}{3} + \frac{7}{90}x^4 + \ldots\ldots\ldots$$

**23.** Prove that

$$e^x \log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

**24.** Prove that

$$\log\left(\frac{xe^x}{e^x-1}\right) = \frac{x}{2} - \frac{x^2}{24} + \frac{x^4}{2880} - \ldots$$

**25.** Expand $\log\tan\left(\frac{\pi}{4}+x\right)$ upto $x^5$.

$$\left[\textbf{Hint:}\ \log\tan\left(\frac{\pi}{4}+x\right) = \log\left(\frac{1+\tan x}{1-\tan x}\right) = \log(1+\tan x) - \log(1-\tan x)\right]$$

$$\left[\textbf{Ans.:}\ 2x + \frac{4}{3}x^3 + \frac{4}{3}x^5 + \ldots\right]$$

**26.** Prove that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots$

if $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

## *By Differentiation and Integration*

**Example 1: Prove that $\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots\ldots$.**

**Solution:** Let $y = \log(\sec x)$

$$\frac{dy}{dx} = \frac{1}{\sec x}\cdot \sec x \tan x = \tan x$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots \qquad \ldots(1)$$

Integrating Eq. (1),

$$y = c + \frac{x^2}{2} + \frac{x^4}{12} + \frac{2}{15}\cdot\frac{x^6}{6} + \ldots\ldots$$

$$\log(\sec x) = c + \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots\ldots$$

Putting $x = 0$,

$$\log(\sec 0) = c + 0$$

$$c = \log 1, \quad c = 0$$

Hence, $$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots\ldots..$$

**Example 2: Prove that** $\boldsymbol{\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots}$ **.**

**Solution:** Let $y = \sin^{-1} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \ldots$$

$$= 1 + \frac{x^2}{2} + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots..$$

$$\sin^{-1} x = c + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots..$$

Putting $x = 0$,

$$\sin^{-1} 0 = c$$

$$c = 0$$

Hence, $$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots..$$

**Example 3: Prove that** $\boldsymbol{\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots\right)}$**.**

**Solution:** Let $y = \cos^{-1} x$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Proceeding as in Ex. 2, we get

$$\cos^{-1} x = c - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\right)$$

Putting $x = 0$,

$$\cos^{-1} 0 = c$$

$$c = \frac{\pi}{2}$$

Hence,
$$\cos^{-1} x = \frac{\pi}{2} - \left( x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots \right)$$

**Example 4: Prove that** $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots..$

**Solution:** Let $y = \tan^{-1} x$

$$\frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\tan^{-1} x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots..$$

Putting $x = 0$,

$$\tan^{-1} 0 = c$$

$$c = 0$$

Hence,
$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

**Example 5: Prove that** $\sinh^{-1} x = x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots\ldots..$

**Solution:** Let
$$y = \sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{2x}{2\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

$$= (1+x^2)^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(x^2)^2 - \ldots$$

$$= 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \ldots \qquad \ldots (1)$$

Integrating Eq. (1),

$$y = c + x - \frac{x^3}{6} + \frac{3}{8}\cdot\frac{x^5}{5} - \ldots$$

$$\sinh^{-1} x = c + x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots$$

Putting $x = 0$, $\sinh^{-1} 0 = c,\ c = 0$

$$\sinh^{-1} x = x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots$$

**Example 6: If $x = 1 - \dfrac{y^2}{2!} + \dfrac{y^4}{4!} - \dfrac{y^6}{6!} + \ldots\ldots$, find $y$ in a series of $x$.**

**Solution:**

$$x = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots\ldots$$

$$= \cos y$$

$$y = \cos^{-1} x$$

Proceeding as in Ex. 3, we get

$$y = \frac{\pi}{2} - \left( x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots \right)$$

## *By Substitution*

**Example 1: Prove that $\sinh^{-1}(3x + 4x^3) = 3\left( x - \dfrac{x^3}{6} + \dfrac{3}{40}x^5 + \cdots \right)$.**

**Solution:** Let $y = \sinh^{-1}(3x + 4x^3)$

Putting $x = \sinh\theta$,

$$y = \sinh^{-1}(3\sinh\theta + 4\sinh^3\theta)$$

$$= \sinh^{-1}(\sinh 3\theta)$$

$$= 3\theta$$

$$= 3\sinh^{-1} x$$

$$= 3\left( x - \frac{x^3}{6} + \frac{3x^5}{40} - \ldots \right)$$

**Example 2: Prove that $\sin^{-1}\left( \dfrac{2x}{1 + x^2} \right) = 2\left( x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots \right)$.**

**Solution:** Let $y = \sin^{-1}\left( \dfrac{2x}{1+x^2} \right)$

Putting $x = \tan\theta$,

$$y = \sin^{-1}\left( \frac{2\tan\theta}{1 + \tan^2\theta} \right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1} x$$

$$= 2\left( x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \right)$$

**Example 3: Expand** $\sec^{-1}\left(\dfrac{1}{1-2x^2}\right)$.

**Solution:** Let $y = \sec^{-1}\left(\dfrac{1}{1-2x^2}\right)$

Putting $x = \sin\theta$,

$$y = \sec^{-1}\left(\frac{1}{1-2\sin^2\theta}\right)$$
$$= \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$
$$= \sec^{-1}(\sec 2\theta)$$
$$= 2\theta = 2\sin^{-1} x$$
$$= 2\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots\right)$$

**Example 4: Prove that** $\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right) = \pi - 2\left(n\pi + x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots\right)$.

**Solution:** Let $y = \cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

Putting $x = \tan\theta$,

$$y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right)$$
$$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}[-\cos(2n\pi + 2\theta)] \text{ [Considering general value of } \cos 2\theta]$$
$$= \cos^{-1}[\cos\{\pi - (2n\pi + 2\theta)\}]$$
$$= \pi - 2(n\pi + \theta)$$
$$= \pi - 2(n\pi + \tan^{-1} x)$$
$$= \pi - 2\left(n\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right)$$

**Example 5: Prove that** $\cos^{-1}[\tanh(\log x)] = \pi - 2\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots\right)$.

**Solution:** Let $y = \cos^{-1}[\tanh(\log x)]$

$$= \cos^{-1}\left(\frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}\right)$$

$$= \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$$

$$= \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)$$

Putting $x = \tan\theta$,

$$y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right)$$

$$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}[\cos(\pi - 2\theta)]$$

$$= \pi - 2\theta$$

$$= \pi - 2\tan^{-1} x$$

$$= \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$$

**Example 6: Prove that** $\tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$.

**Solution:** Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right)$

Putting $x = \tan\theta$,

$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta} - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1 - \cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x$$

$$= \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$$

**Example 7: Prove that** $\tan^{-1}\left(\dfrac{p-qx}{q+px}\right)=\tan^{-1}\dfrac{p}{q}-\left(x-\dfrac{x^3}{3}+\dfrac{3x^5}{5}-\dfrac{x^7}{7}+\ldots\right)$.

**Solution:** Let $y=\tan^{-1}\left(\dfrac{\dfrac{p}{q}-x}{1+\dfrac{p}{q}x}\right)$

Putting $x=\tan\theta,\ \dfrac{p}{q}=\tan A$

$$y=\tan^{-1}\left(\frac{\tan A-\tan\theta}{1+\tan A\cdot\tan\theta}\right)=\tan^{-1}[\tan(A-\theta)]$$

$$=A-\theta=\tan^{-1}\frac{p}{q}-\tan^{-1}x$$

$$=\tan^{-1}\frac{p}{q}-\left(x-\frac{x^3}{3}+\frac{3x^5}{5}-\frac{x^7}{7}+\ldots\right)$$

## Exercise 2.8

1. Prove that

$$\frac{\tan^{-1}x}{1+x^2}=x-\frac{4}{3}x^3+\frac{23}{15}x^5-\ldots.$$

2. Prove that $\sin^{-1}(3x-4x^3)$

$$=3\left(x+\frac{x^3}{6}+\frac{3x^5}{40}+\ldots\right).$$

3. Prove that $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$$=2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right).$$

4. Prove that $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

$$=3\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right).$$

5. Prove that $\cot^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

$$=\frac{\pi}{2}-3\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right).$$

6. Prove that $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

$$=x+\frac{1}{6}x^3+\frac{3}{40}x^5+\ldots.$$

7. Prove that $\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

$$=\frac{\pi}{4}-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right).$$

8. Prove that $\tan^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right)$

$$=\frac{\pi}{4}-\frac{1}{2}\left(x+\frac{x^3}{6}+\frac{3x^5}{40}+\ldots\right).$$

9. Prove that $\cot^{-1}x=\dfrac{\pi}{2}$

$$-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right).$$

**10.** Prove that $\cos^{-1}(4x^3 - 3x)$

$$= 3\left[\frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3}{40}x^5 + \dots\right)\right].$$

**11.** Prove that

$$\sec^{-1}\left(\sqrt{1+x^2}\right) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots .$$

**12.** Prove that $\tan^{-1}\left(\frac{2-3x}{3+2x}\right)$

$$= \tan^{-1}\frac{2}{3} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right).$$

## *By Leibnitz's Theorem*

**Example 1: If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $y = x + \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \dots\dots\dots$ .**

**Solution:** Let

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} \qquad \dots (1)$$

$$y\sqrt{1-x^2} = \sin^{-1} x$$

$$y^2(1-x^2) = (\sin^{-1} x)^2$$

Differentiating w.r.t. $x$,

$$2yy_1(1-x^2) - 2xy^2 = \frac{2\sin^{-1} x}{\sqrt{1-x^2}} = 2y$$

$$(1-x^2)\, y_1 - xy = 1 \qquad \dots (2)$$

Differentiating again w.r.t. $x$,

$$(1-x^2)\, y_2 - 2xy_1 - xy_1 - y = 0$$

$$(1-x^2)\, y_2 - 3x\, y_1 - y = 0 \qquad \dots (3)$$

Differentiating $n$ times w.r.t. $x$ using Leibnitz's theorem,

$$(1-x^2)\, y_{n+2} - 2nxy_{n+1} - n(n-1)\, y_n - 3x\, y_{n+1} - 3ny_n - y_n = 0$$

$$(1-x^2)\, y_{n+2} - (2n+3)\, xy_{n+1} - (n^2 + 2n + 1)y_n = 0$$

$$(1-x^2)\, y_{n+2} - (2n+3)\, xy_{n+1} - (n+1)^2 y_n = 0 \qquad \dots (4)$$

Putting $x = 0$, in Eqs (1), (2), (3) and (4),

$$y(0) = 0,\ y_1(0) = 1,\ y_2(0) = 0,$$

$$y_{n+2}(0) = (n+1)^2 y_n(0) \qquad \dots (5)$$

Putting $n = 1, 2, 3, 4, \dots$ in Eq. (5),

$$y_3(0) = 2^2\, y_1(0) = 4$$

$$y_4(0) = 3^2 y_2(0) = 0$$
$$y_5(0) = 4^2 y_3(0) = 4^3$$
$$y_6(0) = 5^2 y_4(0) = 0$$
$$y_7(0) = 6^2 y_5(0) = 6^2 \cdot 4^3 \text{ and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \frac{x^5}{5!}y_5(0) + \ldots\ldots$$
$$= 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot 4 + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot 4^3 + \frac{x^6}{6!} \cdot 0 + \frac{x^7}{7!} \cdot 6^2 \cdot 4^3 + \ldots\ldots\ldots$$
$$= x + \frac{2}{3}x^3 + \frac{2 \cdot 4}{3 \cdot 5}x^5 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}x^7 + \ldots\ldots\ldots$$

**Example 2: Prove that**

$$\sin(m\sin^{-1}x) = mx + \frac{m(1^2 - m^2)}{3!}x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!}x^5 + \ldots.$$

**Solution:** Let $y = \sin(m \sin^{-1}x)$ ... (1)

Differentiating w.r.t. $x$,

$$y_1 = \cos(m\sin^{-1}x) \cdot \frac{m}{\sqrt{1-x^2}} \quad \text{... (2)}$$
$$(1-x^2)y_1^2 = m^2[1 - \sin^2(m\sin^{-1}x)]$$
$$(1-x^2)y_1^2 = m^2(1-y^2)$$

Differentiating again w.r.t. $x$,

$$(1 - x^2)\, 2y_1 y_2 - 2xy_1^2 = m^2(-2yy_1)$$
$$(1 - x^2)\, y_2 - xy_1 = -m^2 y \quad \text{... (3)}$$

Differentiating $n$ times w.r.t. $x$ using Leibnitz's theorem,

$$(1 - x^2)\, y_{n+2} - 2nxy_{n+1} - n(n-1)\, y_n - xy_{n+1} - ny_n = -m^2 y_n$$
$$(1 - x^2)\, y_{n+2} - x(2n + 1)\, y_{n+1} = (-m^2 + n^2)\, y_n \quad \text{... (4)}$$

Putting $x = 0$, in Eqs (1), (2), (3) and (4),

$$y(0) = 0,\ y_1(0) = m,\ y_2(0) = 0$$
$$y_{n+2}(0) = (-m^2 + n^2)\, y_n(0) \quad \text{... (5)}$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (5),

$$y_3(0) = (-m^2 + 1^2)\, m$$
$$y_4(0) = 0$$
$$y_5(0) = (-m^2 + 3^2)(-m^2 + 1^2)\, m$$
$$y_6(0) = 0 \text{ and so on}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \ldots\ldots$$

$$= mx + \frac{m(1^2 - m^2)}{3!} x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!} x^5 + \ldots .$$

**Example 3: Prove that** $e^{a\sin^{-1}x} = 1 + ax + \dfrac{a^2x^2}{2!} + \dfrac{a(a^2+1)}{3!} x^3 + \ldots\ldots.$

**Hence, deduce that** $e^{\theta} = 1 + \sin\theta + \dfrac{1}{2!}\sin^2\theta + \dfrac{2}{3!}\sin^3\theta + \dfrac{5}{4!}\sin^4\theta + \ldots.$

**Solution:** Let $y = e^{a\sin^{-1}x}$ ... (1)

Differentiating w.r.t. $x$,

$$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}} \qquad \text{... (2)}$$

$$(1 - x^2)\, y_1^2 = a^2 y^2$$

Differentiating again w.r.t. $x$,

$$(1 - x^2)\, 2y_1y_2 - 2xy_1^2 = a^2 \cdot 2yy_1$$
$$(1 - x^2)\, y_2 - xy_1 = a^2y \qquad \ldots (3)$$

Differentiating $n$ times w.r.t. $x$ using Leibnitz theorem,

$$(1 - x^2)\, y_{n+2} - 2nxy_{n+1} - n(n-1)\, y_n - xy_{n+1} - ny_n = a^2y_n$$
$$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} = (a^2 + n^2)y_n \qquad \ldots (4)$$

Putting $x = 0$, in Eqs (1), (2), (3) and (4),

$$y(0) = 1,\ y_1(0) = a,\ y_2(0) = a^2$$
$$y_{n+2}(0) = (a^2 + n^2)\, y_n(0) \qquad \ldots (5)$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (5),

$$y_3(0) = (a^2 + 1^2)\, a$$
$$y_4(0) = (a^2 + 2^2)\, a^2 \text{ and so on.}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \ldots\ldots\ldots$$

$$e^{a\sin^{-1} x} = 1 + ax + \frac{a^2}{2!} x^2 + \frac{a(a^2+1)}{3!} x^3 + \frac{a^2(a^2+2^2)}{4!} x^4 + \ldots\ldots\ldots \quad \ldots (6)$$

Let $\sin^{-1} x = \theta$

$x = \sin\theta$

Putting $a = 1$ and $x = \sin\theta$ in Eq. (6),

$$e^{\theta} = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \frac{5}{4!}\sin^4\theta + \ldots$$

**Example 4: If $y = e^{m\tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots$,**

**prove that (i) $y = 1 + mx + \frac{m^2}{2!} x^2 + \frac{m(m^2-2)}{3!} x^3 + \ldots$**

**(ii) $(n+1)\, a_{n+1} + (n-1)\, a_{n-1} = ma_n$.**

**Solution:** (i) Let $y = e^{m\tan^{-1} x}$ $\quad \ldots (1)$

Differentiating w.r.t. $x$,

$$y_1 = e^{m\tan^{-1} x} \cdot \frac{m}{1+x^2} \quad \ldots (2)$$

$$(1+x^2)\, y_1 = my$$

Differentiating again w.r.t. $x$,

$$(1+x^2)\, y_2 + 2xy_1 = my_1 \quad \ldots (3)$$

Differentiating $n$ times w.r.t. $x$ using Leibnitz's theorem,

$$(1+x^2)\, y_{n+2} + 2nxy_{n+1} + n(n-1)\, y_n + 2xy_{n+1} + 2ny_n = my_{n+1}$$

$$(1+x^2)\, y_{n+2} + 2(n+1)\, xy_{n+1} - my_{n+1} + (n^2+n)\, y_n = 0 \quad \ldots (4)$$

Putting $x = 0$, in Eqs (1), (2), (3) and (4),

$$y(0) = 1,\ y_1(0) = m,\ y_2(0) = m^2$$

$$y_{n+2}(0) = m\, y_{n+1}(0) - (n^2+n)\, y_n(0) \quad \ldots (5)$$

Putting $n = 1, 2, 3, 4, \ldots$ in Eq. (5),

$$y_3(0) = m^3 - 2m = m(m^2-2) \text{ and so on}$$

Substituting in Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots\ldots\ldots$$

$$= 1 + mx + \frac{m^2}{2!} x^2 + \frac{m(m^2-2)}{3!} x^3 + \ldots\ldots\ldots$$

(ii) Given $y = a_0 + a_1x + a_2x^2 + \dots\dots\dots + a_nx^n + \dots\dots\dots$

By Maclaurin's series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \dots\dots\dots\dots + \frac{x^n}{n!}y_n(0) + \dots\dots\dots$$

Comparing coefficient of $x^n$ in both the expressions,

$$a_n = \frac{y_n(0)}{n!}$$

$$y_n(0) = n!a_n$$

$$y_{n+1}(0) = (n+1)!a_{n+1}$$

and $$y_{n-1}(0) = (n-1)!\,a_{n-1}$$

Replacing $n$ by $(n-1)$ in Eq. (5),

$$y_{n+1}(0) = my_n(0) - n(n-1)\,y_{n-1}(0)$$

Substituting $y_{n+1}(0), y_n(0), y_{n-1}(0)$ in above equation,

$$(n+1)!\,a_{n+1} = m(n!)\,a_n - n(n-1)[(n-1)!]\,a_{n-1}$$

$$= n!\,[ma_n - (n-1)\,a_{n-1}]$$

Hence, $(n+1)\,a_{n+1} + (n-1)\,a_{n-1} = ma_n$

## Exercise 2.9

**1.** If $y^{\frac{1}{m}} - y^{-\frac{1}{m}} = 2x$, prove that

$$y = 1 + mx + \frac{m^2}{2!}x^2 + \frac{m^2(m^2-1^2)}{3!}x^3 + \frac{m^2(m^2-2^2)}{4!}x^4 + \dots$$

**2.** Prove that

$$\log\left(x+\sqrt{1+x^2}\right) = x - \frac{x^3}{3!}1^2 + \frac{x^5}{5!}(3^2\cdot 1^2) - \dots$$

**3.** Prove that

$$\sin(2\sin^{-1}x) = 2x - x^3 - \frac{x^5}{4} + \dots$$

**4.** Prove that

$$y = e^{\cos^{-1}x} = e^{\frac{\pi}{2}}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \dots\right).$$

**5.** Prove that

$$e^{\tan^{-1}x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

**6.** Prove that

$$e^{m\cos^{-1}x} = e^{\frac{m\pi}{2}}\left[1 - mx + \frac{m^2}{2!}x^2 - \frac{m(1^2+m^2)}{3!}x^3 + \dots\right].$$

**7.** Prove that

$$\frac{\sinh^{-1}x}{\sqrt{1+x^2}} = x - \frac{2^2}{3!}x^3 + \frac{2^2\cdot 4^2}{5!}x^5 - \dots$$

**8.** Prove that $e^x = 1 + \tan x + \dfrac{\tan^2 x}{2!} - \dfrac{\tan^3 x}{3!} - \dfrac{7\tan^4 x}{4!} - \dots$

## 2.10 INDETERMINATE FORMS

We have studied certain rules to evaluate the limits. But some limits cannot be evaluated by using these rules. These limits are known as indeterminate forms. There are seven types of indeterminate forms given as:

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $0\times\infty$ (iv) $\infty-\infty$

(v) $1^{\infty}$ (vi) $0^{\circ}$ (vii) $\infty^{\circ}$

These limits can be evaluated by using L'Hospital's Rule.

### 2.10.1 L'Hospital's Rule

**Statement:** If $f(x)$ and $g(x)$ are two functions of $x$ which can be expanded by Taylor's series in the neighbourhood of $x = a$ and

if $\lim\limits_{x\to a} f(x) = f(a) = 0$, $\lim\limits_{x\to a} g(x) = g(a) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

**Proof:** Let $x = a + h$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{h\to 0}\frac{f(a+h)}{g(a+h)}$$

$$= \lim_{h\to 0}\frac{f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+.......}{g(a)+hg'(a)+\frac{h^2}{2!}g''(a)+.......} \qquad \text{[By Taylor's theorem]}$$

$$= \lim_{h\to 0}\frac{hf'(a)+\frac{h^2}{2!}f''(a)+.......}{hg'(a)+\frac{h^2}{2!}g''(a)+.......} \qquad [\because f(a)=0,\ g(a)=0]$$

$$= \lim_{h\to 0}\frac{f'(a)+\frac{h}{2!}f''(a)+.......}{g'(a)+\frac{h}{2!}g''(a)+.......}$$

$$= \frac{f'(a)}{g'(a)}$$

$$= \lim_{x\to a}\frac{f'(x)}{g'(x)}, \text{ provided } g'(a)\neq 0.$$

## 2.10.2 Standard Limits

Following standard limits can be used to solve the problems:

(1) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

(2) $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$

(3) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log a$

(4) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$

(5) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$

(6) $\lim\limits_{x\to \infty}\left(1+\dfrac{1}{x}\right)^x=e$

(7) $\lim\limits_{x\to 0}\dfrac{\sin^{-1} x}{x}=1$

(8) $\lim\limits_{x\to 0}\dfrac{\sinh x}{x}=1$

(9) $\lim\limits_{x\to 0}\dfrac{\tan^{-1} x}{x}=1$

## 2.10.3 Type 1 : $\left(\dfrac{0}{0}\right)$

Problems under this type are solved by using L'Hospital's rule considering the fact that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a} f(x)=0 \text{ and } \lim_{x\to a} g(x)=0.$$

**Example 1: Evaluate $\lim\limits_{x\to 0}\dfrac{xe^x-\log(1+x)}{x^2}$.**

**Solution:** Let

$$l=\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{e^x+xe^x-\dfrac{1}{1+x}}{2x} \qquad \left[\frac{0}{0}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{e^x+e^x+xe^x+\dfrac{1}{(1+x)^2}}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\frac{3}{2}.$$

**Example 2: Evaluate $\lim\limits_{x\to 0}\dfrac{\log(1+x^3)}{\sin^3 x}$.**

**Solution:** Let

$$l=\lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x^3}\cdot 3x^2}{3\sin^2 x\cos x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2 \frac{1}{(1+x^3)\cos x}$$

$$= 1 \qquad \left[\lim_{x\to 0}\frac{x}{\sin x} = 1\right]$$

**Example 3: Evaluate** $\lim_{x\to\frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2xe}$**.**

**Solution:** Let $l = \lim_{x\to\frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2xe} \qquad \left[\frac{0}{0}\right]$

$$= \lim_{x\to\frac{1}{2}} \frac{2\cos \pi x(-\pi\sin \pi x)}{2e^{2x} - 2e} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to\frac{1}{2}} \frac{-\pi\sin 2\pi x}{2(e^{2x} - e)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\frac{1}{2}} \frac{-2\pi^2\cos 2\pi x}{2\cdot 2e^{2x}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\pi^2}{2e}.$$

**Example 4: Evaluate** $\lim_{x\to y} \frac{x^y - y^x}{x^x - y^y}$**.**

**Solution:** Let $l = \lim_{x\to y} \frac{x^y - y^x}{x^x - y^y} \qquad \left[\frac{0}{0}\right]$

$$= \lim_{x\to y} \frac{yx^{y-1} - y^x\log y}{x^x(1+\log x) - 0} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{y^y - y^y\log y}{y^y(1+\log y)} = \frac{(1-\log y)}{(1+\log y)}$$

**Example 5: Evaluate** $\lim_{x\to 0} \frac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1}$**.**

**Solution:** Let $$l = \lim_{x\to 0} \frac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2^x \log 2}{\frac{1}{2}(1+x)^{-\frac{1}{2}}} = 2\log 2.$$

**Example 6: Evaluate** $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin(x\cos x)}{\cos(x\sin x)}$.

**Solution:** Let $$l = \lim_{x\to\frac{\pi}{2}} \frac{\sin(x\cos x)}{\cos(x\sin x)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos(x\cos x)(\cos x - x\sin x)}{-\sin(x\sin x)(\sin x + x\cos x)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\pi}{2}.$$

**Example 7: Prove that** $\lim\limits_{\theta\to\alpha} \dfrac{1-\cos(\theta-\alpha)}{(\sin\theta - \sin\alpha)^2} = \dfrac{1}{2}\sec^2\alpha.$

**Solution:** Let $$l = \lim_{\theta\to\alpha} \frac{1-\cos(\theta-\alpha)}{(\sin\theta-\sin\alpha)^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta\to\alpha} \frac{\sin(\theta-\alpha)}{2(\sin\theta-\sin\alpha)\cos\theta} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{\theta\to\alpha} \frac{\sin(\theta-\alpha)}{(\sin 2\theta - 2\sin\alpha\cos\theta)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{\theta\to\alpha} \frac{\cos(\theta-\alpha)}{2\cos 2\theta + 2\sin\alpha\sin\theta} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{\cos 0}{2\cos 2\alpha + 2\sin\alpha\sin\alpha}$$

$$= \frac{1}{2(1-2\sin^2\alpha) + 2\sin^2\alpha} = \frac{1}{2-2\sin^2\alpha}$$

$$= \frac{1}{2\cos^2\alpha} = \frac{1}{2}\sec^2\alpha.$$

**Example 8: Evaluate** $\lim\limits_{x\to 0} \dfrac{2x^2 - 2e^{x^2} + 2\cos(x^{\frac{3}{2}}) + \sin^3 x}{x^2}$.

**Solution:** Let $l = \lim\limits_{x \to 0} \dfrac{2x^2 - 2e^{x^2} + 2\cos x^{\frac{3}{2}} + \sin^3 x}{x^2}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{4x - 2e^{x^2}(2x) - 2\sin x^{\frac{3}{2}}\left(\frac{3}{2}x^{\frac{1}{2}}\right) + 3\sin^2 x\cos x}{2x} \quad \left[\frac{0}{0}\right]$$

[Applying L'Hospital's rule]

$$= \lim_{x \to 0} \frac{4 - 4(e^{x^2} + xe^{x^2}\cdot 2x) - 3\left(\sqrt{x}\cos x^{\frac{3}{2}}\cdot\frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2\sqrt{x}}\sin x^{\frac{3}{2}}\right) + 6\sin x\cos^2 x - 3\sin^3 x}{2}$$

[Applying L'Hospital's rule]

$$= \frac{4 - 4 - \lim\limits_{x \to 0} \dfrac{\sin x^{\frac{3}{2}}}{2\sqrt{x}}\cdot\dfrac{x}{x}}{2}$$

$$= \frac{-1}{2}\cdot\frac{1}{2}\lim_{x \to 0} \frac{\sin x^{\frac{3}{2}}}{x^{\frac{3}{2}}}\cdot x \qquad \left[\because \lim_{x \to 0}\frac{\sin x^{\frac{3}{2}}}{x^{\frac{3}{2}}} = 1\right]$$

$$= 0$$

**Example 9: Evaluate $\lim\limits_{x \to 0} \dfrac{x^{\frac{1}{2}}\tan x}{(e^x - 1)^{\frac{3}{2}}}$.**

**Solution:** Let $l = \lim\limits_{x \to 0} \dfrac{\sqrt{x}\tan x}{(e^x - 1)^{\frac{3}{2}}}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{x\sqrt{x}}{(e^x - 1)^{\frac{3}{2}}}\cdot\frac{\tan x}{x}$$

$$= \lim_{x \to 0} \frac{x\sqrt{x}}{(e^x - 1)^{\frac{3}{2}}}\cdot\lim_{x \to 0}\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x \to 0}\left(\frac{x}{e^x - 1}\right)^{\frac{3}{2}} \qquad \left[\because \lim_{x \to 0}\frac{\tan x}{x} = 1\right]$$

Now, $\lim\limits_{x \to 0} \dfrac{x}{e^x - 1} = \lim\limits_{x \to 0} \dfrac{1}{e^x} = 1$ [Applying L'Hospital's rule]

Hence, $\lim\limits_{x \to 0}\left(\dfrac{x}{e^x - 1}\right)^{\frac{3}{2}} = (1)^{\frac{3}{2}} = 1$

**Example 10: Evaluate** $\lim\limits_{x\to 0} \dfrac{\log_{\sec x} \cos \frac{x}{2}}{\log_{\sec \frac{x}{2}} \cos x}$.

**Solution:** Let

$$l = \lim_{x\to 0} \frac{\log_{\sec x} \cos \frac{x}{2}}{\log_{\sec \frac{x}{2}} \cos x}$$

$$= \lim_{x\to 0} \frac{\log \cos \frac{x}{2}}{\log \sec x} \cdot \frac{\log \sec \frac{x}{2}}{\log \cos x}$$

$$= \lim_{x\to 0} \frac{\log \cos \frac{x}{2}}{(-\log \cos x)} \cdot \frac{\left(-\log \cos \frac{x}{2}\right)}{\log \cos x}$$

$$= \lim_{x\to 0} \left(\frac{\log \cos \frac{x}{2}}{\log \cos x}\right)^2 \qquad \left[\frac{0}{0}\right]$$

Now,

$$\lim_{x\to 0} \frac{\log \cos \frac{x}{2}}{\log \cos x} = \lim_{x\to 0} \frac{\frac{1}{\cos \frac{x}{2}} \cdot \left(-\frac{1}{2} \sin \frac{x}{2}\right)}{\frac{1}{\cos x}(-\sin x)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{\tan \frac{x}{2}}{2 \tan x}$$

$$= \lim_{x\to 0} \frac{1}{4} \left(\frac{\tan \frac{x}{2}}{\frac{x}{2}}\right) \cdot \left(\frac{x}{\tan x}\right)$$

$$= \frac{1}{4} \qquad \left[\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

$$\lim_{x\to 0} \left(\frac{\log \cos \frac{x}{2}}{\log \cos x}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

**Example 11: Prove that** $\lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}} - e}{x} = -\dfrac{e}{2}$.

**Solution:** Let
$$l = \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)} - e}{x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)}\left[-\frac{1}{x^2}\log(1+x) + \frac{1}{x(1+x)}\right]}{1}$$

[Applying L'Hospital's rule]

$$= \lim_{x\to 0}(1+x)^{\frac{1}{x}} \lim_{x\to 0} \frac{\left[-\{\log(1+x)\}(1+x) + x\right]}{x^2(1+x)} \qquad \left[\frac{0}{0}\right]$$

$$= e \lim_{x\to 0}\left[\frac{-\log(1+x) - 1 + 1}{2x + 3x^2}\right] \qquad \left[\frac{0}{0}\right]$$

[Applying L'Hospital's rule]

$$= e \lim_{x\to 0}\left(\frac{-\frac{1}{1+x}}{2+6x}\right) = -\frac{e}{2}.$$

**Example 12: Prove that** $\lim_{x\to 0} \dfrac{\left(\sqrt{1-x}-1\right)^{2n}}{(1-\cos x)^n} = 2^{-n}$**.**

**Solution:** Let
$$l = \lim_{x\to 0} \frac{\left(\sqrt{1-x}-1\right)^{2n}}{(1-\cos x)^n} \cdot \frac{\left(\sqrt{1-x}+1\right)^{2n}}{\left(\sqrt{1-x}+1\right)^{2n}}$$

$$= \lim_{x\to 0} \frac{(1-x-1)^{2n}}{\left(2\sin^2\frac{x}{2}\right)^n \left(\sqrt{1-x}+1\right)^{2n}}$$

$$= \lim_{x\to 0} \frac{(-x)^{2n}}{2^n\left(\sin\frac{x}{2}\right)^{2n}\left(\sqrt{1-x}+1\right)^{2n}} \cdot \frac{2^n}{2^n}$$

$$= \lim_{x\to 0} \left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^{2n} \frac{2^n}{\left(\sqrt{1-x}+1\right)^{2n}} \qquad \left[\because (-x)^{2n} = \left\{(-x)^2\right\}^n = x^{2n}\right]$$

$$= \frac{1}{2^n}.$$

**Example 13: If $\lim_{x\to 0}\dfrac{\sin 2x + p\sin x}{x^3}$ is finite, find the value of $p$ and hence, the limit.**

**Solution:** Let $l = \lim_{x\to 0}\dfrac{\sin 2x + p\sin x}{x^3}$, where $l$ is finite

$$l = \lim_{x\to 0}\frac{\sin 2x + p\sin x}{x^3} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2\cos 2x + p\cos x}{3x^2} = \frac{2+p}{0} \qquad \text{[Applying L'Hospital's rule]}$$

But limit is finite, therefore, numerator must be zero.

$$2 + p = 0,\ p = -2$$

Thus,

$$l = \lim_{x\to 0}\frac{2\cos 2x - 2\cos x}{3x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x} \quad \left[\frac{0}{0}\right] \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{-8\cos 2x + 2\cos x}{6}$$

$$= -1$$

Hence, $p = -2$ and $l = -1$

**Example 14: Find the values of $a$ and $b$ such that $\lim_{x\to 0}\dfrac{a\sin^2 x + b\log\cos x}{x^4} = \dfrac{1}{2}$.**

**Solution:** $\dfrac{1}{2} = \lim_{x\to 0}\dfrac{a\sin^2 x + b\log\cos x}{x^4}$

$$= \lim_{x\to 0}\frac{a\sin^2 x + b\log\cos x}{x^4}$$

$$= \lim_{x\to 0}\frac{a\cdot 2\sin x\cos x + b\cdot\dfrac{1}{\cos x}(-\sin x)}{4x^3} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{a\sin 2x - b\tan x}{4x^3} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2a\cos 2x - b\sec^2 x}{12x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2a-b}{0}$$

But limit is finite, therefore, numerator must be zero.

$$2a - b = 0$$
$$b = 2a$$

Thus, $\dfrac{1}{2} = \lim\limits_{x \to 0} \dfrac{2a\cos 2x - 2a\sec^2 x}{12x^2} \qquad \left[\dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{-4a\sin 2x - 4a\sec^2 x\tan x}{24x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} \left( \frac{-a}{3} \frac{\sin 2x}{2x} - \frac{a}{6}\sec^2 x \cdot \frac{\tan x}{x} \right)$$

$$\frac{1}{2} = -\frac{a}{3} - \frac{a}{6} = -\frac{a}{2} \qquad \left[\because \lim_{x \to 0} \frac{\sin 2x}{2x} = 1 \text{ and } \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

Hence, $a = -1,\ b = -2$

**Example 15: Find $a$ and $b$ if $\lim\limits_{x \to 0} \dfrac{a\sin x - \sin 2x}{\tan^3 x} = b$.**

**Solution:**

$$b = \lim_{x \to 0} \frac{a\sin x - \sin 2x}{\tan^3 x} = \lim_{x \to 0} \frac{a\sin x - \sin 2x}{x^3 \left(\dfrac{\tan x}{x}\right)^3}$$

$$= \lim_{x \to 0} \frac{a\sin x - \sin 2x}{x^3} \qquad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

$$= \lim_{x \to 0} \frac{a\cos x - 2\cos 2x}{3x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a-2}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - 2 = 0,\ a = 2$$

Thus, $b = \lim\limits_{x \to 0} \dfrac{2\cos x - 2\cos 2x}{3x^2} \qquad \left[\dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{-2\sin x + 4\sin 2x}{6x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} \left[ -\frac{2}{6}\left(\frac{\sin x}{x}\right) + \frac{4}{3}\left(\frac{\sin 2x}{2x}\right) \right]$$

$$= -\frac{2}{6} + \frac{4}{3} = 1 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

Hence, $a = 2, \; b = 1$

**Example 16: Find $a, b, c$ if $\lim\limits_{x \to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2.$**

**Solution:**

$$2 = \lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x}$$

$$= \lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \cdot x\left(\dfrac{\sin x}{x}\right)}$$

$$= \lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \frac{a - b + c}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - b + c = 0 \qquad \text{... (1)}$$

Thus,

$$2 = \lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{ae^x + b\sin x - ce^{-x}}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a - c}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a - c = 0, \quad a = c \qquad \text{... (2)}$$

$$2 = \lim_{x \to 0} \frac{ae^x + b\sin x - ae^{-x}}{2x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{ae^x + b\cos x + ae^{-x}}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{a + b + a}{2}$$

$$2a + b = 4 \qquad \text{... (3)}$$

From Eqs (1) and (2), we have

$$2a - b = 0 \qquad \text{... (4)}$$

Solving Eqs (3) and (4),

$$a = 1, \quad b = 2, \text{ and } c = 1$$

## Exercise 2.10

1. Prove that
$$\lim_{x\to a}\frac{x^2\log a-a^2\log x}{x^2-a^2}=\log a-\frac{1}{2}.$$

2. Prove that $\displaystyle\lim_{x\to 0}\frac{\log(1-x^2)}{\log\cos x}=2.$

3. Prove that
$$\lim_{x\to 0}\frac{e^x+\log\left(\frac{1-x}{e}\right)}{\tan x-x}=-\frac{1}{2}.$$

4. Prove that $\displaystyle\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}=2.$

5. Prove that $\displaystyle\lim_{x\to 0}\frac{x^2+2\cos x-2}{x\sin x}=0.$

6. Prove that $\displaystyle\lim_{x\to 0}\frac{e^x-\sqrt{1+2x}}{\log(1+x^2)}=1.$

7. Prove that
$$\lim_{x\to 0}\frac{6\sin x-6x+x^3}{2x^2\log(1+x)-2x^3+x^4}=\frac{3}{40}.$$

8. Evaluate $\displaystyle\lim_{x\to a}\frac{\sqrt{a+x}\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a-x}}.$

$$\left[\textbf{Hint: } \lim_{x\to a}(x+a)\frac{\tan^{-1}\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} \quad \text{as } x\to a,\ a-x\to 0\right]$$

[**Ans.:** $2a$]

9. Find the values of $a$ and $b$, such that
$$\lim_{x\to 0}\frac{a\sin 2x-b\tan x}{x^3}=1\cdot$$

$\left[\textbf{Ans.: } a=-\frac{1}{2}, b=-1\right]$

10. Find $a$ and $b$ if
$$\frac{x(1+a\cos x)-b\sin x}{x^3}=1.$$

$\left[\textbf{Ans.: } a=-\frac{5}{2},\ b=-\frac{3}{2}\right]$

11. Find the values of $a$, $b$ and $c$ so that
$$\lim_{x\to 0}\frac{x(a+b\cos x)-c\sin x}{x^3}=1.$$

[**Ans.:** $a=0$, $b=-3$, $c=-3$]

12. Find the values of $a$ and $b$ so that
$$\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}=\frac{1}{3}.$$

$\left[\textbf{Ans.: } a=\frac{1}{2},\ b=-\frac{1}{2}\right]$

13. Find the values of $a$, $b$ and $c$ such that $\displaystyle\lim_{x\to 0}\frac{ae^x-be^{-x}-cx}{x-\sin x}=4.$

[**Ans.:** $a=2$, $b=2$, $c=4$]

14. Evaluate $\displaystyle\lim_{x\to 0}\frac{e^x+\log_e\frac{1-x}{e}}{\tan x-x}.$

$\left[\textbf{Ans.: } -\frac{1}{2}\right]$

15. Evaluate $\displaystyle\lim_{x\to 1}\frac{1-x+\log x}{1-\sqrt{2x-x^2}}.$

[**Ans.:** $-1$]

16. Prove that $\displaystyle\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}=\frac{3}{2}.$

17. Prove that $\displaystyle\lim_{x\to 0}\frac{e^x-\sqrt{1+2x}}{\log(1+x^2)}=1.$

18. Prove that $\displaystyle\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}=2.$

19. Prove that $\displaystyle\lim_{x\to 3}\frac{\sqrt{3x}-\sqrt{12-x}}{2x-3\sqrt{19-5x}}=\frac{8}{69}.$

20. Prove that $\displaystyle\lim_{x\to 1}\frac{a^{\log x}-x}{\log x}=\log\frac{a}{e}.$

21. Prove that
$$\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}=\frac{1}{\sqrt{2a}}.$$

22. Prove that $\displaystyle\lim_{x\to 0}\frac{\tan x-x}{x^3}=\frac{1}{3}.$

## 2.10.4 Type 2: $\left(\dfrac{\infty}{\infty}\right)$

Problems under this type are also solved by using L'Hospital's rule considering the fact that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a} f(x)=\infty \text{ and } \lim_{x\to a} g(x)=\infty.$$

**Example 1: Prove that** $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}=0.$

**Solution:** Let
$$l=\lim_{x\to\frac{\pi}{2}}\frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{x-\frac{\pi}{2}}}{\sec^2 x} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{x-\frac{\pi}{2}} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{2\cos x(-\sin x)}{1}=0 \qquad \text{[Applying L'Hospital's rule]}$$

**Example 2: Prove that** $\displaystyle\lim_{x\to a}\frac{\log(x-a)}{\log(a^x-a^a)}=1.$

**Solution:** Let $\displaystyle l=\lim_{x\to a}\frac{\log(x-a)}{\log(a^x-a^a)} \qquad \left[\frac{\infty}{\infty}\right]$

$$=\lim_{x\to a}\frac{\frac{1}{(x-a)}}{\frac{1}{a^x-a^a}\cdot a^x\log a} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to a}\left(\frac{a^x-a^a}{x-a}\right)\cdot\lim_{x\to a}\frac{1}{a^x\log a}$$

$$=\lim_{x\to a}\frac{a^x\log a}{1}\cdot\frac{1}{a^a\log a} \qquad \text{[Applying L'Hospital's rule for first term]}$$

$$=a^a\log a\cdot\frac{1}{a^a\log a}=1$$

**Example 3: Prove that** $\lim\limits_{x\to\infty}\dfrac{\sinh^{-1}x}{\cosh^{-1}x}=1$.

**Solution:** Let $l=\lim\limits_{x\to\infty}\dfrac{\sinh^{-1}x}{\cosh^{-1}x}$

$$=\lim_{x\to\infty}\frac{\log\left(x+\sqrt{x^2+1}\right)}{\log\left(x+\sqrt{x^2-1}\right)}\qquad\left[\frac{\infty}{\infty}\right]$$

$$l=\lim_{x\to\infty}\frac{\dfrac{1}{\left(x+\sqrt{x^2+1}\right)}\cdot\left(1+\dfrac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{\dfrac{1}{\left(x+\sqrt{x^2-1}\right)}\cdot\left(1+\dfrac{1}{2\sqrt{x^2-1}}\cdot 2x\right)}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to\infty}\frac{\dfrac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}}{\dfrac{\sqrt{x^2-1}+x}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{\sqrt{1-\dfrac{1}{x^2}}}{\sqrt{1+\dfrac{1}{x^2}}}=1$$

**Example 4: Prove that** $\lim\limits_{x\to 0}\log_x\sin x=1$.

**Solution:** Let $l=\lim\limits_{x\to 0}\log_x\sin x$

$$=\lim_{x\to 0}\frac{\log\sin x}{\log x}\qquad\left[\frac{\infty}{\infty}\right]\qquad\text{[Change of base property]}$$

$$=\lim_{x\to 0}\frac{\dfrac{1}{\sin x}\cdot\cos x}{\dfrac{1}{x}}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\cos x$$

$$=1\qquad\left[\because\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

**Example 5: Prove that** $\lim\limits_{x\to\infty} \dfrac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{x}} + \ldots + e^{\frac{x}{x}}}{x} = e - 1.$

**Solution:** Let $l = \lim\limits_{x\to\infty} \dfrac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{x}} + \ldots + e^{\frac{x}{x}}}{x}$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}\left[1 - \left(e^{\frac{1}{x}}\right)^x\right]}{1 - e^{\frac{1}{x}}} \cdot \frac{1}{x} \qquad \text{[Sum of G.P]}$$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}(e-1)}{e^{\frac{1}{x}} - 1} \cdot \frac{1}{x}$$

Putting $\dfrac{1}{x} = y$, when $x \to \infty, y \to 0$

$$l = \lim_{y\to 0} \frac{(e-1)e^{y}\, y}{e^{y} - 1} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{y\to 0} \frac{(e-1)(ye^{y} + e^{y})}{e^{y}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= e - 1$$

**Example 6: Prove that** $\lim\limits_{x\to\infty} \dfrac{x^n}{e^{kx}} = 0.$

**Solution:** Let $l = \lim\limits_{x\to\infty} \dfrac{x^n}{e^{kx}} \qquad \left[\dfrac{\infty}{\infty}\right]$

$$= \lim_{x\to\infty} \frac{nx^{n-1}}{ke^{kx}} \qquad \left[\frac{\infty}{\infty}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{k^2 e^{kx}} \qquad \left[\frac{\infty}{\infty}\right] \qquad \text{[Applying L'Hospital's rule]}$$

Applying L'Hospital's rule $n$ times,

$$l = \lim_{x\to\infty} \frac{n(n-1)(n-2)\ldots 2.1}{k^n e^{kx}} = \lim_{x\to\infty} \frac{n!}{k^n e^{kx}} = 0 \qquad \left[\because \lim_{x\to\infty} e^{kx} = \infty\right]$$

**Example 7: Prove that** $\lim\limits_{x\to\infty} \dfrac{1^2+2^2+3^2+...+x^2}{x^3} = \dfrac{1}{3}$.

**Solution:** Let $l = \lim\limits_{x\to\infty} \dfrac{1^2+2^2+3^2+...+x^2}{x^3}$

$$= \lim_{x\to\infty} \frac{x(x+1)(2x+1)}{6x^3} \qquad \left[\because \sum n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{x\to\infty} \frac{2x^3+3x^2+x}{6x^3} = \lim_{x\to\infty} \frac{2+\frac{3}{x}+\frac{1}{x^2}}{6}$$

$$= \frac{2}{6} = \frac{1}{3}.$$

**Example 8: Prove that** $\lim\limits_{x\to\infty} \dfrac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} = e^{\frac{1}{2}}$.

**Solution:** Let $l = \lim\limits_{x\to\infty} \dfrac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} \qquad \left[\dfrac{\infty}{\infty}\right]$

$$= \lim_{x\to\infty} \frac{e^x}{\left(1+\frac{1}{x}\right)^{x^2}}$$

Taking logarithm on both the sides,

$$\log l = \lim_{x\to\infty}\left[\log e^x - \log\left(1+\frac{1}{x}\right)^{x^2}\right] = \lim_{x\to\infty}\left[x - x^2\log\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{x\to\infty} x^2\left[\frac{1}{x} - \log\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty} \frac{\frac{1}{x}-\log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\infty} \frac{-\frac{1}{x^2} - \frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{1+\frac{1}{x}}}{\frac{2}{x}}$$

$$= \frac{1}{2} \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{2}$$

Hence, $$l = e^{\frac{1}{2}}$$

## Exercise 2.11

1. Prove that $\lim_{x\to\infty} \frac{\log x}{x^n} = 0 \, (n > 0)$.

2. Prove that $\lim_{x\to 0} \frac{\log x}{\cot x} = 0.$

3. Prove that $\lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x} = 0.$

4. Prove that $\lim_{x\to 0} \frac{\log_{\sin x} \cos x}{\log_{\sin\frac{x}{2}} \cos\frac{x}{2}} = 4.$

5. Prove that $\lim_{x\to 0} \log_{\tan x} \tan 2x = 1.$

6. Prove that $\lim_{x\to 0} \log_{\sin x} \sin 2x = 1.$

7. Prove that $\lim_{x\to\infty} \frac{\log(1+e^{3x})}{x} = 3.$

8. Prove that $\lim_{x\to 0} \frac{\log x^2}{\cot x^2} = 0.$

   [**Hint :** Put $x^2 = y$]

9. Prove that $\lim_{x\to\infty} \frac{x^m}{e^x} = 0 \, (m > 0)$.

10. Prove that

$$\lim_{n\to\infty} \frac{\left(\frac{1}{e}\right)+\left(\frac{1}{e}\right)^2+\left(\frac{1}{e}\right)^3+\ldots\ldots+\left(\frac{1}{e}\right)^n}{n} = 0.$$

## 2.10.5 Type 3 : $(0 \times \infty)$

To solve the problems of the type

$$\lim_{x\to a} [f(x)\cdot g(x)], \text{ when } \lim_{x\to a} f(x) = 0, \ \lim_{x\to a} g(x) = \infty \text{ (i.e. } 0\times\infty \text{ form)}$$

We write $\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$.

These new forms are of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively, which can be solved using L'Hospital's rule.

**Example 1: Prove that $\lim_{x\to 0} \sin x \log x = 0$.**

**Solution:** Let $l = \lim_{x\to 0} \sin x \log x \qquad [0\times\infty]$

$$= \lim_{x\to 0}\frac{\log x}{\operatorname{cosec} x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0}\frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\lim_{x\to 0} \sin x \cdot \frac{\tan x}{x}$$

$$= -\lim_{x\to 0} \sin x \cdot \lim_{x\to 0}\frac{\tan x}{x}$$

$$= 0 \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

**Example 2:** $\lim\limits_{x\to\infty} 2^x \cdot \sin\left(\frac{a}{2^x}\right) = a.$

**Solution:** Let $l = \lim\limits_{x\to\infty} 2^x \cdot \sin\left(\frac{a}{2^x}\right)$

Taking $2^x = \frac{1}{t},\ t = \frac{1}{2^x},$

when $x \to \infty,\ 2^x \to \infty, \quad t \to 0$

$$l = \lim_{t\to 0}\frac{\sin at}{t} = \lim_{t\to 0}\frac{a\sin at}{at} = a\cdot 1 = a \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

**Example 3: Prove that** $\lim\limits_{x\to\infty}\left(a^{\frac{1}{x}} - 1\right)x = \log a$.

**Solution:** Let $l = \lim\limits_{x\to\infty}\left(a^{\frac{1}{x}} - 1\right)\cdot x \qquad [0\times\infty]$

$$= \lim_{x\to\infty}\frac{\left(a^{\frac{1}{x}} - 1\right)}{\frac{1}{x}} \qquad \left[\frac{0}{0}\right]$$

Taking $\frac{1}{x} = t$, when $x \to \infty,\ t \to 0$

$$l = \lim_{t\to 0}\frac{a^t - 1}{t} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{t\to 0}\frac{a^t \log a}{1} \qquad \text{[Applying L'Hospital's rule]}$$

$$= a^0 \log a = \log a$$

**Example 4:** $\lim\limits_{x\to 1} \tan^2\left(\dfrac{\pi x}{2}\right)(1+\sec \pi x) = -2.$

**Solution:** Let $l = \lim\limits_{x\to 1} \tan^2\left(\dfrac{\pi x}{2}\right)(1+\sec \pi x)$ $\qquad [\infty \times 0]$

$$= \lim_{x\to 1} \frac{1+\sec \pi x}{\cot^2\left(\dfrac{\pi x}{2}\right)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{\pi \sec \pi x \tan \pi x}{2\cot\left(\dfrac{\pi x}{2}\right)\left(-\operatorname{cosec}^2 \dfrac{\pi x}{2}\right)\dfrac{\pi}{2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -\left(\lim_{x\to 1} \frac{\sec \pi x}{\operatorname{cosec}^2 \dfrac{\pi x}{2}}\right)\left(\lim_{x\to 1} \frac{\tan \pi x}{\cot \dfrac{\pi x}{2}}\right)$$

$$= -\left(\frac{\sec \pi}{\operatorname{cosec}^2 \dfrac{\pi}{2}}\right)\lim_{x\to 1} \frac{\tan \pi x}{\cot \dfrac{\pi x}{2}} \qquad \left[\frac{0}{0}\right]$$

$$= -(-1)\lim_{x\to 1} \frac{\pi \sec^2 \pi x}{\left(-\operatorname{cosec}^2 \dfrac{\pi x}{2}\right)\dfrac{\pi}{2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= -2\frac{\sec^2 \pi}{\operatorname{cosec}^2 \dfrac{\pi}{2}} = -2$$

**Example 5:** $\lim\limits_{x\to a} \sin^{-1}\sqrt{\dfrac{a-x}{a+x}} \operatorname{cosec}\sqrt{a^2-x^2} = \dfrac{1}{2a}.$

**Solution:** Let $l = \lim\limits_{x\to a} \sin^{-1}\sqrt{\dfrac{a-x}{a+x}} \operatorname{cosec}\sqrt{a^2-x^2}$ $\qquad [0\times\infty]$

$$= \lim_{x\to a} \frac{\sin^{-1}\sqrt{\dfrac{a-x}{a+x}}}{\sin\sqrt{a^2-x^2}}$$

Here applying L'Hospital's rule will make the expression complicated, so we rearrange the terms to apply the limits directly.

Let $\sqrt{\dfrac{a-x}{a+x}} = \alpha, \sqrt{a^2 - x^2} = \beta$

when $x \to a, \alpha \to 0$ and $\beta \to 0$

Hence,
$$l = \lim_{\alpha \to 0} \sin^{-1}\alpha \lim_{\beta \to 0} \frac{1}{\sin \beta}$$

$$= \left[ \lim_{\alpha \to 0} \left( \frac{\sin^{-1}\alpha}{\alpha} \right) \cdot \alpha \right] \left[ \lim_{\beta \to 0} \left( \frac{\beta}{\sin \beta} \right) \cdot \frac{1}{\beta} \right]$$

$$= \lim_{\alpha \to 0} \alpha \cdot \lim_{\beta \to 0} \frac{1}{\beta} \qquad \left[ \because \lim_{x \to 0} \left( \frac{\sin^{-1} x}{x} \right) = 1 \text{ and } \lim_{x \to 0} \left( \frac{\sin x}{x} \right) = 1 \right]$$

$$= \lim_{x \to a} \sqrt{\frac{a-x}{a+x}} \cdot \frac{1}{\sqrt{a^2 - x^2}} \qquad \text{[Resubstituting } \alpha \text{ and } \beta]$$

$$= \lim_{x \to a} \sqrt{\frac{a-x}{a+x}} \cdot \frac{1}{\sqrt{a+x}\sqrt{a-x}}$$

$$= \lim_{x \to a} \frac{1}{a+x} = \frac{1}{2a}$$

**Example 6: Evaluate $\lim\limits_{x \to 0} x^m (\log x)^n$, where $m$ and $n$ are positive integers.**

**Solution:** Let $l = \lim\limits_{x \to 0} x^m (\log x)^n \qquad [0 \times \infty]$

$$= \lim_{x \to 0} \frac{(\log x)^n}{\dfrac{1}{x^m}} \qquad \left[ \frac{\infty}{\infty} \right]$$

$$= \lim_{x \to 0} \frac{n(\log x)^{n-1} \dfrac{1}{x}}{-m(x)^{-m-1}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{(-1)^1 n(\log x)^{n-1}}{m(x)^{-m}} \qquad \left[ \frac{\infty}{\infty} \right]$$

$$= \lim_{x \to 0} \frac{(-1)^1 n(n-1)(\log x)^{n-2} \cdot \dfrac{1}{x}}{m(-m)^1 (x)^{-m-1}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{(-1)^2 n(n-1)(\log x)^{n-2}}{m^2 (x)^{-m}} \qquad \left[ \frac{\infty}{\infty} \right]$$

Applying L'Hospital's rule $(n-2)$ times in the above expression,

$$l = \lim_{x\to 0} \frac{(-1)^n n!(\log x)^0}{m^n (x)^{-m}}$$

$$= \lim_{x\to 0} \frac{(-1)^n n!}{m^n} \cdot x^m = 0$$

## Exercise 2.12

1. Prove that $\lim\limits_{x\to 0} x\log x = 0.$
2. Prove that $\lim\limits_{x\to\infty} x^2 e^{-x} = 0.$
3. Prove that $\lim\limits_{x\to\infty} x^2\left(1-e^{-\frac{2gy}{x^2}}\right) = 2gy.$
4. Prove that $\lim\limits_{x\to 0} \tan x \log x = 0.$
5. Prove that
$$\lim_{x\to 1}(x^2-1)\tan\left(\frac{\pi x}{2}\right) = -\frac{4}{\pi}.$$
6. Prove that
$$\lim_{x\to 1}(1+\sec \pi x)\tan\frac{\pi x}{2} = 0.$$
7. Prove that
$$\log\left(2-\frac{x}{a}\right)\cot(x-a) = -\frac{1}{a}.$$
8. Prove that
$$\lim_{x\to 1}\log(1-x)\cot\left(\frac{\pi x}{2}\right) = 0.$$
9. Prove that $\lim\limits_{x\to 0}\log\left(\frac{1+x}{1-x}\right)\cot x = 2.$
10. Prove that
$$\lim_{x\to a}\sqrt{\frac{a+x}{a-x}}\tan^{-1}\sqrt{a^2-x^2} = 2a.$$
11. Prove that
$$\lim_{x\to 2}\sqrt{\frac{2+x}{2-x}}\tan^{-1}\sqrt{4-x^2} = 4.$$

## 2.10.6 Type 4 : $(\infty - \infty)$

To evaluate the limits of the type $\lim\limits_{x\to a}[f(x)-g(x)]$, when $\lim\limits_{x\to a} f(x) = \infty$ and, $\lim\limits_{x\to a} g(x) = \infty$ [i.e., $(\infty-\infty)$ form], we reduce the expression in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking LCM or by rearranging the terms and then appling L'Hospital's rule.

**Example 1: Prove that $\lim\limits_{x\to\infty}(\cosh^{-1} x - \log x) = \log 2.$**

**Solution:** Let $l = \lim\limits_{x\to\infty}(\cosh^{-1} x - \log x) \quad [\infty-\infty]$

$$= \lim_{x\to\infty}\left[\log\left(x+\sqrt{x^2-1}\right) - \log x\right]$$

$$= \lim_{x\to\infty}\log\left(\frac{x+\sqrt{x^2-1}}{x}\right)$$

$$= \lim_{x\to\infty}\log\left(1+\sqrt{1-\frac{1}{x^2}}\right)$$

$$= \log\left(1+\sqrt{1-0}\right) = \log 2$$

**Example 2: Prove that** $\lim\limits_{x \to 1}\left(\frac{1}{\log x} - \frac{x}{x-1}\right) = -\frac{1}{2}$.

**Solution:** Let $l = \lim\limits_{x \to 1}\left(\frac{1}{\log x} - \frac{x}{x-1}\right)$ $\qquad [\infty - \infty]$

$$= \lim_{x \to 1}\left[\frac{x-1-x\log x}{(x-1)\log x}\right] \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 1}\frac{1 - x\cdot\frac{1}{x} - \log x}{(x-1)\cdot\frac{1}{x} + \log x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x \to 1}\frac{-\log x}{1 - \frac{1}{x} + \log x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 1}\frac{-\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = -\frac{1}{2} \qquad \text{[Applying L'Hospital's rule]}$$

**Example 3: Prove that** $\lim\limits_{x \to 0}\left(\frac{a}{x} - \cot\frac{x}{a}\right) = 0.$

**Solution:** Let $l = \lim\limits_{x \to 0}\left(\frac{a}{x} - \cot\frac{x}{a}\right)$ $\qquad [\infty - \infty]$

Taking $\frac{x}{a} = y$, when $x \to 0$, $y \to 0$

$$l = \lim_{y \to 0}\left(\frac{1}{y} - \cot y\right)$$

$$= \lim_{y \to 0}\left(\frac{1}{y} - \frac{1}{\tan y}\right) \qquad [\infty - \infty]$$

$$= \lim_{y \to 0}\left(\frac{\tan y - y}{y\tan y}\right) \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{y \to 0}\left(\frac{\tan y - y}{y^2}\right)\cdot\lim_{y \to 0}\frac{1}{\left(\frac{\tan y}{y}\right)}$$

$$= \lim_{y \to 0}\frac{\tan y - y}{y^2}\cdot 1 \qquad \left[\frac{0}{0}\right]\left[\because \lim_{y \to 0}\frac{\tan y}{y} = 1\right]$$

$$= \lim_{y\to 0} \frac{\sec^2 y - 1}{2y} \qquad \left[\frac{0}{0}\right] \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{y\to 0} \frac{2\sec y \cdot \sec y \tan y}{2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

**Example 4: Prove that** $\lim\limits_{x\to 0}\left[\dfrac{1}{2x} - \dfrac{1}{x(e^{\pi x}+1)}\right] = \dfrac{\pi}{4}$.

**Solution:** Let $l = \lim\limits_{x\to 0}\left[\dfrac{1}{2x} - \dfrac{1}{x(e^{\pi x}+1)}\right] \qquad [\infty - \infty]$

$$= \lim_{x\to 0} \frac{e^{\pi x} + 1 - 2}{2x(e^{\pi x}+1)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\pi e^{\pi x}}{2\left[(e^{\pi x}+1) + x(\pi e^{\pi x})\right]}$$

$$= \frac{\pi}{2}\frac{e^0}{(e^0+1)} = \frac{\pi}{4}$$

**Example 5: Prove that** $\lim\limits_{x\to \infty}\left(x + \dfrac{1}{2}\right)\left[\log\left(x+\dfrac{1}{2}\right) - \log x\right] = \dfrac{1}{2}$.

**Solution:** Let $l = \lim\limits_{x\to \infty}\left(x + \dfrac{1}{2}\right)\left[\log\left(x+\dfrac{1}{2}\right) - \log x\right] \qquad [\infty - \infty]$

$$= \lim_{x\to\infty}\left(x+\frac{1}{2}\right)\log\left(\frac{x+\frac{1}{2}}{x}\right)$$

$$= \lim_{x\to\infty}\left[x\log\left(1+\frac{1}{2x}\right) + \frac{1}{2}\log\left(1+\frac{1}{2x}\right)\right]$$

$$= \lim_{x\to\infty}\frac{1}{2}\log\left(1+\frac{1}{2x}\right)^{2x} + \frac{1}{2}\lim_{x\to\infty}\log\left(1+\frac{1}{2x}\right)$$

$$= \frac{1}{2}\log e + \frac{1}{2}\log 1 \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax} = e\right]$$

$$= \frac{1}{2}.$$

**Example 6:** If $\lim\limits_{x\to 0}\left(\dfrac{a\cot x}{x}+\dfrac{b}{x^2}\right)=\dfrac{1}{3}$, find $a$ and $b$.

**Solution:**

$$\frac{1}{3}=\lim_{x\to 0}\left(\frac{a\cot x}{x}+\frac{b}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{a}{x\tan x}+\frac{b}{x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{ax+b\tan x}{x^2\tan x}\right) \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{(ax+b\tan x)}{(x^2\cdot x)\left(\dfrac{\tan x}{x}\right)}$$

$$=\lim_{x\to 0}\frac{ax+b\tan x}{x^3}\cdot\lim_{x\to 0}\left(\frac{x}{\tan x}\right)$$

$$=\lim_{x\to 0}\frac{ax+b\tan x}{x^3}\cdot 1 \qquad \left[\frac{0}{0}\right]\left[\because \lim_{x\to 0}\frac{x}{\tan x}=1\right]$$

$$=\lim_{x\to 0}\left(\frac{a+b\sec^2 x}{3x^2}\right) \qquad \text{[Applying L'Hospital's rule]}$$

$$=\frac{a+b\sec 0}{0}=\frac{a+b}{0}$$

But limit is finite, therefore, numerator must be zero.

$$a+b=0,\ a=-b \qquad \ldots (1)$$

Thus,

$$\frac{1}{3}=\lim_{x\to 0}\frac{-b+b\sec^2 x}{3x^2} \qquad \left[\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{b\cdot 2\sec x\sec x\tan x}{6x} \qquad \text{[Applying L'Hospital's rule]}$$

$$=\left(\lim_{x\to 0}\frac{b}{3}\sec^2 x\right)\cdot\left(\lim_{x\to 0}\frac{\tan x}{x}\right)$$

$$=\frac{b}{3}\sec 0\cdot 1 \qquad \left[\because \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

$$\frac{1}{3}=\frac{b}{3},\ b=1$$

From Eq. (1), $a=-b=-1$

Hence, $a=-1,\ b=1.$

## Exercise 2.12

1. Prove that $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=0$.

2. Prove that
$$\lim_{x\to a}\left[\frac{1}{x-a}-\cot(x-a)\right]=0.$$

3. Prove that
$$\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)=0.$$

4. Prove that
$$\lim_{x\to\frac{\pi}{2}}\left(\tan x-\frac{2x\sec x}{\pi}\right)=\frac{2}{\pi}.$$

5. Prove that
$$\lim_{x\to 0}\left[\frac{1}{x-a}-\frac{1}{\log(x+1-a)}\right]=-\frac{1}{2}.$$

6. Prove that
$$\lim_{x\to 3}\left[\frac{1}{x-3}-\frac{1}{\log(x-2)}\right]=-\frac{1}{2}.$$

7. Prove that $\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=\frac{1}{2}$.

8. Prove that $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\frac{1}{2}$.

## 2.10.7 Type 5: $1^\infty$, $\infty^0$, $0^0$

To evaluate the limits of the type $\lim_{x\to a}[f(x)]^{g(x)}$ which takes any one of the above form, we proceed as follows:

Let
$$l=\lim_{x\to a}[f(x)]^{g(x)}$$
$$\log l=\lim_{x\to a}[g(x)\cdot\log f(x)] \qquad [\text{if } f(x)>0]$$

which takes the form $\infty\times 0$, i.e., type 3 form.

**Example 1: Prove that $\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}=ae$.**

**Solution:** Let $l=\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}$ $\qquad [1^\infty]$

$$\log l=\lim_{x\to 0}\frac{1}{x}\cdot\log(a^x+x)$$
$$=\lim_{x\to 0}\frac{\log(a^x+x)}{x} \qquad \left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{\frac{1}{a^x+x}(a^x\log a+1)}{1} \qquad \text{[Applying L'Hospital's rule]}$$
$$=\frac{a^0\log a+1}{a^0+0}=\frac{\log_e a+\log_e e}{1}$$

Hence,
$$\log l = \log ae$$
$$l = ae$$

**Example 2: Prove that $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \sqrt{ab}$.**

**Solution:** Let $l = \lim\limits_{x\to 0}\left(\dfrac{a^x+b^x}{2}\right)^{\frac{1}{x}}$ $\quad [1^\infty]$

$$\log l = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x+b^x}{2}\right)$$
$$= \lim_{x\to 0}\frac{\log\left(\frac{a^x+b^x}{2}\right)}{x} \qquad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to 0}\left(\frac{2}{a^x+b^x}\right)\cdot\frac{(a^x\log a + b^x\log b)}{2} \qquad \text{[Applying L'Hospital's rule]}$$
$$= \left(\frac{2}{a^0+b^0}\right)\frac{(a^0\log a + \mathrm{b}^0\log b)}{2}$$
$$= \frac{1}{2}\cdot\log ab$$

Hence,
$$\log l = \log(ab)^{\frac{1}{2}}$$
$$l = \sqrt{ab}$$

**Example 3: Prove that $\lim\limits_{x\to\infty}\left(\dfrac{a^{\frac{1}{x}}+b^{\frac{1}{x}}+c^{\frac{1}{x}}+d^{\frac{1}{x}}}{4}\right)^{x} = (abcd)^{\frac{1}{4}}$.**

**Solution:** Let $l = \lim\limits_{x\to\infty}\left(\dfrac{a^{\frac{1}{x}}+b^{\frac{1}{x}}+c^{\frac{1}{x}}+d^{\frac{1}{x}}}{4}\right)^{x}$

Taking $\dfrac{1}{x} = y$, when $x\to\infty$, $y\to 0$

$$l = \lim_{y\to 0}\left(\frac{a^y+b^y+c^y+d^y}{4}\right)^{\frac{1}{y}} \qquad [1^\infty]$$

$$\log l = \lim_{y\to 0} \frac{1}{y} \log\left(\frac{a^y + b^y + c^y + d^y}{4}\right)$$

$$= \lim_{y\to 0} \frac{\log\left(\frac{a^y + b^y + c^y + d^y}{4}\right)}{y} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{y\to 0} \left(\frac{4}{a^y + b^y + c^y + d^y}\right)\left(\frac{a^y \log a + b^y \log b + c^y \log c + d^y \log d}{4}\right)$$

[Applying L'Hospital's rule]

$$= \frac{\log a + \log b + \log c + \log d}{4}$$

$$= \frac{1}{4} \log(abcd)$$

Hence,
$$\log l = \log(abcd)^{\frac{1}{4}}$$

$$l = (abcd)^{\frac{1}{4}}$$

**Example 4: Prove that** $\lim_{x\to\infty} \left(\frac{ax+1}{ax-1}\right)^x = e^{\frac{2}{a}}$ **.**

**Solution:** Let
$$l = \lim_{x\to\infty} \left(\frac{ax+1}{ax-1}\right)^x$$

$$= \lim_{x\to\infty} \left(\frac{1 + \frac{1}{ax}}{1 - \frac{1}{ax}}\right)^x \qquad [1^\infty]$$

$$\log l = \lim_{x\to\infty} x \log\left(\frac{1 + \frac{1}{ax}}{1 - \frac{1}{ax}}\right)$$

$$= \lim_{x\to\infty} \frac{ax}{a}\left[\log\left(1 + \frac{1}{ax}\right) - \log\left(1 - \frac{1}{ax}\right)\right]$$

$$= \lim_{x\to\infty} \frac{1}{a}\left[\log\left(1 + \frac{1}{ax}\right)^{ax} + \log\left(1 - \frac{1}{ax}\right)^{-ax}\right]$$

$$= \frac{1}{a}(\log e + \log e) = \frac{1}{a}(1+1) = \frac{2}{a} \qquad \left[\because \lim_{x\to\infty}\left(1 + \frac{1}{ax}\right)^{ax} = e\right]$$

Hence, $$\log l = \frac{2}{a}$$

$$l = e^{\frac{2}{a}}$$

**Example 5: Prove that** $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{\frac{2}{\pi}}$.

**Solution:** Let $l = \lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$ $\quad [1^\infty]$

$$\log l = \lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right)$$

$$= \lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to a}\frac{1}{\left(2-\frac{x}{a}\right)}\left(-\frac{1}{a}\right)\frac{1}{\left(-\operatorname{cosec}^2\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2}{\pi}$$

Hence, $$\log l = \frac{2}{\pi}$$

$$l = e^{\frac{2}{\pi}}$$

**Example 6: Prove that** $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$.

**Solution:** Let $l = \lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$ $\quad [1^\infty] \quad \left[\because \lim\limits_{x\to 0}\dfrac{\tan x}{x} = 1\right]$

$$\log l = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{x}{\tan x}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)\cdot\frac{1}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to0}\frac{x\sec^2 x-\tan x}{2x^3} \qquad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{x\to0}\frac{x}{\tan x}=1\right]$$

$$= \lim_{x\to0}\frac{\sec^2 x + x\cdot 2\sec^2 x\tan x-\sec^2 x}{6x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to0}\frac{\sec^2 x}{3}\cdot\frac{\tan x}{x}=\frac{1}{3}$$

Hence, $$\log l = \frac{1}{3}$$

$$l = e^{\frac{1}{3}}\cdot$$

**Example 7: Prove that $\lim\limits_{x\to0}\left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{6}}$.**

**Solution:** Let $l=\lim\limits_{x\to0}\left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}}$ $\qquad [1^\infty] \qquad \left[\because \lim\limits_{x\to0}\dfrac{\sinh x}{x}=1\right]$

$$\log l = \lim_{x\to0}\frac{1}{x^2}\log\left(\frac{\sinh x}{x}\right)$$

$$= \lim_{x\to0}\frac{\log\left(\frac{\sinh x}{x}\right)}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to0}\frac{x}{\sinh x}\left(\frac{x\cosh x-\sinh x}{x^2}\right)\cdot\frac{1}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to0}\frac{x\cosh x-\sinh x}{2x^3} \qquad \left[\frac{0}{0}\right] \qquad \left[\because \lim_{x\to0}\frac{x}{\sinh x}=1\right]$$

$$= \lim_{x\to0}\frac{x\sinh x+\cosh x-\cosh x}{6x^2} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to0}\frac{1}{6}\cdot\frac{\sinh x}{x}=\frac{1}{6} \qquad \left[\because \lim_{x\to0}\frac{\sinh x}{x}=1\right]$$

Hence, $$\log l = \frac{1}{6}$$

$$l = e^{\frac{1}{6}}$$

**Example 8: Prove that $\lim\limits_{x\to0}\left(\dfrac{1}{x}\right)^{1-\cos x} = 1$.**

**Solution:** Let $l=\lim\limits_{x\to0}\left(\dfrac{1}{x}\right)^{1-\cos x}$ $\qquad [\infty^0]$

$$\log l = \lim_{x\to 0}(1-\cos x)\log\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\left(2\sin^2\frac{x}{2}\right)(-\log x)$$

$$= \lim_{x\to 0}\frac{2\left(\sin\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^2}{\left(\frac{x}{2}\right)^2}(-\log x)$$

$$= \lim_{x\to 0}\frac{x^2(-\log x)}{2} \qquad \left[\because \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)=1\right]$$

$$= \lim_{x\to 0}\frac{1}{2}\frac{(-\log x)}{\left(\frac{1}{x^2}\right)} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\frac{-1}{x}}{\frac{-2}{x^3}}\right) \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{x^2}{2}\right) = 0$$

Hence, $\log l = 0$

$$l = e^0 = 1$$

**Example 9: Prove that $\lim\limits_{x\to\infty} e^{\frac{\sinh^{-1}x}{\cosh^{-1}x}} = e$.**

**Solution:** Let $l = \lim\limits_{x\to\infty} e^{\frac{\sinh^{-1}x}{\cosh^{-1}x}}$

$$= \lim_{x\to\infty}\left(e^{\sinh^{-1}x}\right)^{\frac{1}{\cosh^{-1}x}} \qquad [\infty^0]$$

$$\log l = \lim_{x\to\infty}\frac{\sinh^{-1}x}{\cosh^{-1}x}\cdot\log e$$

$$= \lim_{x\to\infty}\frac{\log(x+\sqrt{x^2+1})}{\log(x+\sqrt{x^2-1})} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{\dfrac{1}{x+\sqrt{x^2+1}}\left(1+\dfrac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{\dfrac{1}{x+\sqrt{x^2-1}}\left(1+\dfrac{1}{2\sqrt{x^2-1}}\cdot 2x\right)} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x^2}}} = 1$$

Hence,
$$\log l = 1$$
$$l = e^1 = e\,.$$

**Example 10: Prove that $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)^{\frac{1}{x}} = 1$.**

**Solution:** Let $l = \lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)^{\frac{1}{x}}$ $\quad [0^0]$

$$\log l = \lim_{x\to\infty} \frac{1}{x}\log\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{-\log x}{x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= -\lim_{x\to\infty} \frac{\dfrac{1}{x}}{1} \quad \text{[Applying L'Hospital's rule]}$$

$$= 0$$

Hence, $\log l = 0$

$$l = e^0 = 1$$

**Example 11: Prove that $\lim\limits_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}} = e$.**

**Solution:** Let $l = \lim\limits_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ $\quad [0^0]$

$$\log l = \lim_{x\to 1} \frac{1}{\log(1-x)}\log(1-x^2) \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 1} \frac{\dfrac{-2x}{(1-x^2)}}{\dfrac{1}{(1-x)}(-1)} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 1} \frac{2x(1-x)}{(1-x)(1+x)} = \lim_{x\to 1} \frac{2x}{1+x} = 1$$

Hence, $\log l = 1$

$$l = e$$

**Example 12: Prove that** $\lim\limits_{x\to 0} \dfrac{e^x}{\left[\left(1+\dfrac{1}{x}\right)^x\right]^x} = 1$ **.**

**Solution:** Let $l = \lim\limits_{x\to 0}\left[\left(1+\dfrac{1}{x}\right)^x\right]^x$

$$= \lim_{x\to 0}\left(1+\frac{1}{x}\right)^{x^2} \qquad (\infty^0)$$

$$\log l = \lim_{x\to 0} x^2 \log\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(1+\dfrac{1}{x}\right)}{\dfrac{1}{x^2}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0}\frac{\dfrac{1}{1+\dfrac{1}{x}}\left(-\dfrac{1}{x^2}\right)}{-\dfrac{2}{x^3}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{x}{2\left(1+\dfrac{1}{x}\right)} = \lim_{x\to 0}\frac{x^2}{2(x+1)} = 0$$

Hence, $\log l = 0$

$$l = e^0 = 1$$

$$\lim_{x\to 0}\left[\left(1+\frac{1}{x}\right)^x\right]^x = 1$$

Hence, $\lim\limits_{x\to 0}\dfrac{e^x}{\left[\left(1+\dfrac{1}{x}\right)^x\right]^x} = \dfrac{e^0}{1} = \dfrac{1}{1} = 1\cdot$

**Example 13: Prove that** $\lim\limits_{x\to 0} \dfrac{1-x^{\sin x}}{x\log x} = -1$.

**Solution:** Let $l_1 = \lim\limits_{x\to 0} x^{\sin x}$ $\quad [0^0]$

$$\log l_1 = \lim_{x\to 0} \sin x \cdot \log x = \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} -\frac{\sin^2 x}{x\cos x} = -\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \frac{x}{\cos x} = 0 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$\log l_1 = 0,\ l_1 = e^0 = 1,\ \therefore\ \lim_{x\to 0} x^{\sin x} = 1 \qquad \ldots(1)$$

Let $l_2 = \lim\limits_{x\to 0} x\log x$ $\quad [0\times\infty]$

$$= \lim_{x\to 0} \frac{\log x}{\frac{1}{x}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}(-x) = 0 \qquad \ldots(2)$$

Let $l = \lim\limits_{x\to 0} \dfrac{1-x^{\sin x}}{x\log x}$ $\quad \left[\dfrac{0}{0}\right]$ $\quad$ [Using Eqs (1) and (2)]

$$= \lim_{x\to 0} \frac{1-e^{\sin x \log x}}{x \log x}$$

$$\lim_{x\to 0}\frac{1-x^{\sin x}}{x\log x} = \lim_{x\to 0} \frac{-e^{\sin x\log x}\left(\frac{\sin x}{x} + \cos x\cdot\log x\right)}{1+\log x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{-x^{\sin x}\left[\left(\frac{\sin x}{x}\right)\cdot\frac{1}{\log x} + \cos x\right]}{\frac{1}{\log x}+1} \qquad \ldots(3)$$

[Dividing numerator and denominator by log $x$]

$$= -\frac{1\left(1\cdot\frac{1}{\infty}+\cos 0\right)}{\frac{1}{\infty}+1} = -1 \qquad \left[\text{Using Eq. (1) and } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

## Exercise 2.13

1. Prove that
$$\lim_{x\to 0}\left(\frac{a_1^x+a_2^x+\dots+a_n^x}{n}\right)^{\frac{1}{x}} = (a_1a_2\dots a_n)^{\frac{1}{n}}.$$

2. Prove that
$$\lim_{x\to\infty}\left(\frac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}{4}\right)^{4x} = 24.$$

3. Prove that $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{2x}} = e^{\frac{1}{2}}$.

4. Prove that $\lim\limits_{x\to 1}(x)^{\frac{1}{1-x}} = \frac{1}{e}$.

5. Prove that $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{\tan x} = 1$.

6. Prove that $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}$.

7. Prove that $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$.

8. Prove that $\lim\limits_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x = e^2$.

9. Prove that $\lim\limits_{x\to\infty}\left(\frac{2x+1}{2x-1}\right)^x = e$.

10. Prove that $\lim\limits_{x\to 0}(1+\sin x)^{\operatorname{cosec} x} = e$.

11. Prove that $\lim\limits_{x\to 0}(1+\sin x)^{\cot x} = e$.

12. Prove that $\lim\limits_{x\to 0}(1+\tan x)^{\cot x} = e$.

13. Prove that $\lim\limits_{x\to 0}(1-\tan x)^{\frac{1}{x}} = \frac{1}{e}$.

14. Prove that $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}}$.

15. Prove that $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = 0$.

16. Prove that $\lim\limits_{x\to 0}(\cos ax)^{\operatorname{cosec}^2 bx} = e^{-\frac{a^2}{2b^2}}$.

17. Prove that $\lim\limits_{x\to 0}(\cos x)^{\cot^2 x} = \frac{1}{\sqrt{e}}$.

18. Prove that $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x} = 1$.

19. Prove that $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2$.

20. Prove that $\lim\limits_{x\to a}\left(2-\frac{x}{a}\right)^{\cot(x-a)} = e^{-\frac{1}{a}}$.

21. Prove that $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\cos 2x} = 1$.

22. Prove that $\lim\limits_{x\to\frac{\pi}{2}}(\sec x)^{\cot x} = 1$.

23. Prove that $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{2\sin x} = 1$.

24. Prove that $\lim\limits_{x\to 0}(\sin x)^{\tan x} = 1$.

25. Prove that $\lim\limits_{x\to 1}(1-x^n)^{\frac{1}{\log(1-x)}} = e$.

26. Prove that
$$\lim_{x\to a}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}} = 1.$$

27. Prove that $\lim\limits_{x\to\infty}\left(x\sin\frac{1}{x}\right)^{x^2} = e^{-\frac{1}{6}}$.

28. Prove that $\lim\limits_{x\to 0} x^{\tan\left(\frac{\pi x}{2}\right)} = e$.

29. Prove that
$$\lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)} = e^{-\frac{a^2}{b^2}}.$$

30. Prove that $\lim\limits_{x\to 0}(e^{3x}-5x)^{\frac{1}{x}} = e^{-2}$.

31. Prove that $\lim\limits_{x\to 0}(\cos 2x)^{\left(\frac{3}{x^2}\right)} = e^{-6}$.

32. Prove that $\lim\limits_{x\to 0}(\cot x)^{\sin x} = 1$.

## 2.10.8 Type 6 : Using Expansion

In some cases, it is difficult to differentiate the numerator or denominator, or in some cases, power of $x$ in the denominator is very large. In such cases, we use expansion of the function to find the limit.

**Example 1: Prove that** $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x - x}{x^3} = \dfrac{1}{6}$.

**Solution:** Let $l = \lim\limits_{x\to 0}\dfrac{\sin^{-1}x - x}{x^3}$

$$= \lim_{x\to 0}\frac{\left(x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots\right) - x}{x^3}$$

$$= \lim_{x\to 0}\left(\frac{1}{6} + \frac{3}{40}x^2 + \ldots\right) = \frac{1}{6}.$$

**Example 2: Evaluate** $\lim\limits_{x\to 0}\dfrac{\sin x - \tan^{-1}x}{x^2\log(1+x)}$.

**Solution:** Let $l = \lim\limits_{x\to 0}\dfrac{\sin x - \tan^{-1}x}{x^2\log(1+x)}$

$$= \lim_{x\to 0}\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)}{x^2\left(1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)}$$

$$= \lim_{x\to 0}\frac{x^3\left(\frac{1}{6} - \frac{23}{120}x^2 + \ldots\right)}{x^3\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)} = \frac{1}{6}.$$

**Example 3: Evaluate** $\lim\limits_{x\to 0}\dfrac{\sin x\sin^{-1}x - x^2}{x^6}$.

**Solution:** Let
$$l = \lim_{x\to 0}\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots\right) - x^2}{x^6}$$

$$= \lim_{x\to 0}\frac{\left(x^2 + \frac{x^4}{6} + \frac{3}{40}x^6 - \frac{x^4}{6} - \frac{x^6}{36} - \frac{x^8}{80} + \frac{x^6}{120} + \frac{x^8}{720} + \ldots\right) - x^2}{x^6}$$

$$= \lim_{x \to 0} \frac{\frac{x^6}{18} + \text{Higher powers of } x}{x^6}$$

$$= \frac{1}{18}.$$

**Example 4: Evaluate $\lim_{x \to 0} \frac{\tan x \tan^{-1} x - x^2}{x^6}$.**

**Solution:** Let $l = \lim_{x \to 0} \frac{\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) - x^2}{x^6}$

$$= \lim_{x \to 0} \frac{\left(x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \frac{x^4}{3} - \frac{x^6}{9} + \frac{x^8}{15} + \frac{2}{15}x^6 - \frac{2}{45}x^8 + \dots\right) - x^2}{x^6}$$

$$= \lim_{x \to 0} \frac{x^6\left(\frac{1}{5} - \frac{1}{9} + \frac{2}{15}\right) + \text{Higher powers of } x \text{ more than } 6}{x^6}$$

$$= \frac{2}{9}.$$

**Example 5: Prove that $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)} = -\frac{2}{3}$.**

**Solution:** Let $l = \lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^3}{3!} + \dots\right) - x - x^2}{x^2 + x\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\right)}$$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + x^2 - \frac{x^4}{3!} + \frac{x^3}{2!} - \frac{x^5}{2!3!} + \frac{x^4}{3} - \frac{x^6}{3!3!} + \dots\right) - x - x^2}{x^2 - x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \dots}$$

$$= \lim_{x \to 0} \frac{\frac{x^3}{3} - \frac{x^5}{12} + \dots}{-\frac{x^3}{2} - \frac{x^4}{3} - \dots}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3} - \frac{x^2}{12} + \ldots}{-\frac{1}{2} - \frac{x}{3} - \ldots}$$

$$= \frac{\frac{1}{3}}{-\frac{1}{2}} = -\frac{2}{3}.$$

**Example 6: Prove that** $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{e}{2}$.

**Solution:** Let
$$l = \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)} - e}{x}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)} - e}{x}$$

$$= \lim_{x\to 0} \frac{e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \ldots\right)} - e}{x}$$

$$= \lim_{x\to 0} \frac{ee^{\left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right)} - e}{x}$$

$$= \lim_{x\to 0} \frac{e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right) + \frac{1}{2}\left(-\frac{x}{2} + \ldots\right)^2 + \ldots\right] - e}{x}$$

$$= \lim_{x\to 0} e\left(-\frac{1}{2} + \frac{x}{3} - \ldots\right)$$

$$= -\frac{e}{2}.$$

**Example 7: Prove that** $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e + \frac{ex}{2}}{x^2} = \frac{11e}{24}$.

**Solution:** Let $l = \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e + \frac{ex}{2}}{x^2}$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{ee^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\cdots\right)^2+\cdots\right]-e+\frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(e-\frac{ex}{2}+\frac{ex^2}{3}+\frac{ex^2}{8}-\frac{ex^3}{4}-\frac{ex^3}{6}+\dots\right)-e+\frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0}\left(\frac{11e}{24}-\frac{5e}{12}x+\dots\right) = \frac{11e}{24}$$

**Example 8: Prove that $\lim_{x\to 0}\left[2\left(\frac{\cosh x - 1}{x^2}\right)\right]^{\frac{1}{x^2}} = e^{\frac{1}{12}}$.**

**Solution:** Let $l = \lim_{x\to 0}\left[2\left(\frac{\cosh x-1}{x^2}\right)\right]^{\frac{1}{x^2}}$

$$\log l = \lim_{x\to 0}\frac{1}{x^2}\log\left[2\left(\frac{\cosh x-1}{x^2}\right)\right]$$

$$= \lim_{x\to 0}\frac{1}{x^2}\log\left[2\left\{\frac{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\right)-1}{x^2}\right\}\right]$$

$$= \lim_{x\to 0}\frac{1}{x^2}\cdot\log\left[2\left\{\frac{\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+\dots}{x^2}\right\}\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\cdot \log\left[1+\left(\frac{x^2}{12}+\frac{x^4}{360}+\ldots\right)\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\cdot \left[\left(\frac{x^2}{12}+\frac{x^4}{360}+\ldots\right)-\frac{1}{2}\left(\frac{x^2}{12}+\frac{x^4}{360}+\ldots\right)^2+\ldots\right]$$

$$= \lim_{x\to 0} \left(\frac{1}{12}+\frac{x^2}{360}-\frac{x^2}{288}+\ldots\right)$$

$$= \frac{1}{12}$$

Hence, $\log l = \dfrac{1}{12}$

$$l = e^{\frac{1}{12}}.$$

**Example 9: Prove that** $\displaystyle \lim_{x\to 0} \frac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1} = 2\log 2.$

**Solution:** Let $\displaystyle l = \lim_{x\to 0} \frac{2^x - 1}{(1+x)^{\frac{1}{2}} - 1} = \lim_{x\to 0} \frac{e^{x\log 2} - 1}{(1+x)^{\frac{1}{2}} - 1}$

$$= \lim_{x\to 0} \frac{\left[1+x\log 2+\frac{x^2}{2!}(\log 2)^2+\ldots\right]-1}{\left[1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\ldots\right]-1}$$

$$= \lim_{x\to 0} \frac{\log 2+\frac{x}{2!}(\log 2)^2+\ldots}{\frac{1}{2}+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{x}{2!}+\ldots}$$

$$= \frac{\log 2}{\frac{1}{2}} = 2\log 2.$$

**Example 10: Prove that** $\displaystyle \lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = 1.$

**Solution:** Let $\displaystyle l = \lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

$$
\begin{aligned}
e^x - e^{\sin x} &= e^x - e^{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)} \\
&= e^x - e^x e^{\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)} \\
&= e^x - e^x e^z \qquad \text{where } z = -\frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
&= e^x \left[1 - \left(1 + z + \frac{z^2}{2!} + \dots\right)\right] \\
&= e^x \left(-z - \frac{z^2}{2!} - \dots\right)
\end{aligned}
$$

$$
x - \sin x = x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) = \left(\frac{x^3}{3!} - \frac{x^5}{5!} + \dots\right) = -z
$$

$$
\begin{aligned}
l &= \lim_{x\to 0} \frac{-e^x\left(z + \frac{z^2}{2!} + \dots\right)}{-z} \\
&= \lim_{x\to 0} e^x \left(1 + \frac{z}{2!} + \dots\right) \\
&= e^0 = 1 \qquad \left[\because \lim_{x\to 0} z = 0\right]
\end{aligned}
$$

**Example 11: Prove that** $\displaystyle \lim_{x\to 0} \frac{x\sin(\sin x) - \sin^2 x}{x^6} = \frac{1}{18}$**.**

**Solution:** Let $\displaystyle l = \lim_{x\to 0} \frac{x\sin(\sin x) - \sin^2 x}{x^6}$

$$
\begin{aligned}
\sin(\sin x) &= \sin x - \frac{\sin^3 x}{3!} + \frac{\sin^5 x}{5!} - \dots \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - \frac{1}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^3 + \frac{1}{5!}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^5 - \dots \\
&= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) - \frac{1}{6}\left(x^3 - 3x^2 \cdot \frac{x^3}{6} + \dots\right) + \frac{1}{120}(x^5 - \dots) \\
&= x - \frac{x^3}{3} + \frac{1}{10}x^5 - \dots
\end{aligned}
$$

$$
x\sin(\sin x) = x^2 - \frac{x^4}{3} + \frac{x^6}{10} - \dots
$$

$$
\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^2
$$

$$= x^2 + \frac{x^6}{36} - 2x\,\frac{x^3}{6} + 2x \cdot \frac{x^5}{120} + \dots$$

$$= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \dots$$

$$x\sin(\sin x) - \sin^2 x = \frac{x^6}{10} - \frac{2}{45}x^6 - \dots$$

$$= \frac{1}{18}x^6 - \dots$$

Hence,

$$l = \lim_{x\to 0}\frac{\frac{1}{18}x^6 - \text{Higher powers of } x}{x^6} = \frac{1}{18}.$$

**Example 12: Find $a, b, c$ if $\lim_{x\to 0}\dfrac{x(a+b\cos x) - c\sin x}{x^5} = 1$.**

**Solution:** $1 = \lim_{x\to 0}\dfrac{x(a+b\cos x) - c\sin x}{x^5}$

$$= \lim_{x\to 0}\frac{x\left[a + b\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)\right] - c\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)}{x^5}$$

$$= \lim_{x\to 0}\frac{(a+b-c)x + x^3\left(-\frac{b}{2} + \frac{c}{6}\right) + x^5\left(\frac{b}{4!} - \frac{c}{5!}\right) + x^7\left(-\frac{b}{6!} + \frac{c}{7!}\right) + \dots}{x^5}$$

$$= \lim_{x\to 0}\frac{(a+b-c) + x^2\left(-\frac{b}{2} + \frac{c}{6}\right) + x^4\left(\frac{b}{4!} - \frac{c}{5!}\right) + x^6\left(-\frac{b}{6!} + \frac{c}{7!}\right) + \dots}{x^4}$$

But limit is given as 1.

$$a + b - c = 0,\ -\frac{b}{2} + \frac{c}{6} = 0,\ \frac{b}{24} - \frac{c}{120} = 1,$$

$$a + b - c = 0,\ -3b + c = 0,\ 5b - c = 120 \cdot$$

Solving all the equations, we get $a = 120$, $b = 60$, $c = 180$.

## Exercise 2.13

**1.** Prove that $\lim_{x\to 0}\dfrac{x - \sin x}{\tan^3 x} = \dfrac{1}{6}$.

**2.** Prove that $\lim_{x\to 0}\dfrac{\tan x - x}{x^2 \tan x} = \dfrac{1}{3}$.

**3.** Prove that $\lim_{x\to 0}\dfrac{\tan^2 x - x^2}{x^2 \tan^2 x} = \dfrac{2}{3}$.

**4.** Prove that

$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{1}{120}.$$

**5.** Prove that $\lim\limits_{x\to 0}\dfrac{2\sinh x-2x}{x^2\sin x}=\dfrac{1}{3}$.

**6.** Prove that $\lim\limits_{x\to 0}\dfrac{\log(1+x^3)}{\sin^3 x}=1$.

**7.** Prove that

$$\lim_{x\to 0}\frac{\tanh x-2\sin x+x}{x^5}=\frac{7}{60}.$$

**8.** Prove that $\lim\limits_{x\to 0}\dfrac{e^x-e^{x\cos x}}{x-\sin x}=3$.

**9.** Prove that $\lim\limits_{x\to 0}\dfrac{\sin x-\tan^{-1}x}{x^2\log(1+x)}=\dfrac{1}{3}$.

**10.** Prove that

$$\lim_{x\to 0}\frac{e^{x\sin x}-\cosh\left(x\sqrt{2}\right)}{x^4}=\frac{1}{6}.$$

**11.** Prove that

$$\lim_{x\to 0}\frac{2x^2-2e^{x^2}+2\cos\left(x^{\frac{3}{2}}\right)+\sin^3 x}{x^4}=-1\cdot$$

**12.** Prove that $\lim\limits_{x\to 0}\dfrac{\sinh x-x}{\sin x-x\cos x}=\dfrac{1}{2}$.

## FORMULAE

*$n^{th}$ Order Derivative of Some Standard Functions*

(i) $\dfrac{d^n}{dx^n}(ax+b)^m$

$$=\frac{a^n\, m!(ax+b)^{m-n}}{(m-n)!},\ \text{if } n<m$$

$$=n!\,a^n,\qquad \text{if } n=m$$

$$=0,\qquad \text{if } n>m$$

(ii) $\dfrac{d^n}{dx^n}(ax+b)^{-m}$

$$=(-1)^n\frac{(m+n-1)!}{(m-1)!}\frac{a^n}{(ax+b)^{m+n}}$$

$$\frac{d^n}{dx^n}(ax+b)^{-1}$$

$$=(-1)^n\,n!\,\frac{a^n}{(ax+b)^{1+n}}$$

(iii) $\dfrac{d^n}{dx^n}\log(ax+b)$

$$=\frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

(iv) $\dfrac{d^n}{dx^n}e^{ax}=a^n e^{ax}$

(v) $\dfrac{d^n}{dx^n}a^{mx}=m^n a^{mx}(\log a)^n$

(vi) $\dfrac{d^n}{dx^n}[\sin(ax+b)]$

$$=a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$$

(vii) $\dfrac{d^n}{dx^n}[\cos(ax+b)]$

$$=a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$$

(viii) $\dfrac{d^n}{dx^n}[e^{ax}\sin(bx+c)]$

$$=r^n e^{ax}\sin(bx+c+n\theta),$$

where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\dfrac{b}{a}$.

(ix) $\dfrac{d^n}{dx^n}[e^{ax}\cos(bx+c)]$

$$=r^n e^{ax}\cos(bx+c+n\theta),$$

where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\dfrac{b}{a}$.

*Leibnitz's Theorem*

$$y_n=(uv)_n$$
$$=u_n v+{}^nC_1 u_{n-1}v_1+{}^nC_2 u_{n-2}v_2$$
$$+{}^nC_3 u_{n-3}v_3+\ldots+{}^nC_n uv_n.$$

*Taylor's Series*

(i) $f(x+h)$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x)$$

$$+ \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^n(x) + \ldots$$

(ii) $f(x) = f(a) + (x-a)f'(a)$

$$+ \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}$$

$$f'''(a) + \ldots + \frac{(x-a)^n}{n!}f^n(a) + \ldots$$

*Maclaurin's Series*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$$

$$+ \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

*List of Expansion of Some Standard Functions*

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(iv) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

(v) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$

(vi) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

(vii) $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots$

(viii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

(ix) $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots$

*L'Hospital's Rule*

If $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

## MULTIPLE CHOICE QUESTIONS

*Choose the correct alternative in each of the following:*

1. If $f(x) = 4x^2$, then the value of $c$ in the interval $]-1, 3[$ for which $f'(c)$ $\frac{f(3)-f(-1)}{4}$ is

   (a) 0 (b) 1
   (c) 2 (d) 3

2. If $f(x) = \sum_{n=0}^{x} 2^n x^n$, then $f^{33}(0)$ is

   (a) $(33!)3^{33}$ (b) $(32!)3^{33}$
   (c) $(3!)2^{32}$ (d) $(33!)2^{33}$

3. If $f(x) = \tan^{-1} x$, then $f^{99}(0)$ is

   (a) 97 ! (b) –98 !
   (c) 99 ! (d) none of these

4. If $f(x) = \frac{1}{x^2+x+1}$ then $f^{36}(0)$ is

   (a) –36 ! (b) 36 !
   (c) $2^{36}$ (d) none of these

5. If $f(x) = e^{x^2}$ then $f^{100}(0)$ is

   (a) 100 ! (b) $2^{100}$
   (c) $\frac{100!}{50!}$ (d) 100 ! × 50 !

6. If $f(x) = x^\alpha \log x$ and $f(0) = 0$, then the value of $\alpha$ for which Rolle's theorem can be applied in [0, 1] is

   (a) – 2 (b) – 1
   (c) 0 (d) $\frac{1}{2}$

7. The abscissae of the points of the curve $y = x^3$ in the interval [–2, 2], where the slope of the tangents can

be obtained by mean value theorem for the interval [– 2, 2] are

(a) $\pm\frac{2}{\sqrt{3}}$ (b) $\pm\sqrt{3}$

(c) $\pm\frac{\sqrt{3}}{2}$ (d) 0

8. For which interval does the function $\frac{x^2 - 3x}{x - 1}$ satisfy all the conditions of Rolle's theorem?
(a) [0, 3] (b) [–3, 0]
(c) [1.5, 3] (d) for no interval

9. The function $f$ defined by
$$f(x) = (x + 2)\, e^{-x} \text{ is}$$
(a) decreasing for all $x$
(b) decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$
(c) increasing for all $x$
(d) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$

10. $y = [x(x - 3)]^2$ increases for all values of $x$ lying in the interval
(a) $0 < x < \frac{3}{2}$ (b) $0 < x < \infty$
(c) $-\infty < x < 0$ (d) $1 < x < 3$

11. The value of $a$ in order that $f(x) = \sqrt{3}\, \sin x - \cos x - 2ax + b$ decreases for all real values of $x$, is given by,
(a) $a < 1$ (b) $a \geq 1$
(c) $a \geq \sqrt{2}$ (d) $a < \sqrt{2}$

12. As $x$ is increased from $-\infty$ to $\infty$, the function $f(x) = \frac{e^x}{1+e^x}$
(a) monotonically increases
(b) monotonically decreases
(c) increases to a maximum value and then decreases
(d) decreases to a minimum value and then increases

13. The value of $c$ in the mean value theorem of $f(b) - f(a) = (b - a) f'(c)$ for $f(x) = a_1x^2 + a_2x + a_3$ in $(a, b)$ is
(a) $b + a$ (b) $b - a$
(c) $\frac{b+a}{2}$ (d) $\frac{(b-a)}{2}$

14. The Taylor series expansion of $\frac{\sin x}{x - \pi}$ at $x = \pi$ is given by
(a) $1 + \frac{(x-\pi)^2}{3!} + \ldots$
(b) $-1 - \frac{(x-\pi)^2}{3!} + \ldots$
(c) $1 - \frac{(x-\pi)^2}{3!} + \ldots$
(d) $-1 + \frac{(x-\pi)^2}{3!} + \ldots$

15. Which of the following functions would have only odd powers of $x$ in its Taylor series expansion about the point $x = 0$?
(a) $\sin(x^3)$ (b) $\sin(x^2)$
(c) $\cos(x^3)$ (d) $\cos(x^2)$

16. In the Taylor series expansion of $e^x + \sin x$ about the point $x = \pi$, the coefficient of $(x - \pi)^2$ is
(a) $e^\pi$ (b) $0.5\, e^\pi$
(c) $e^\pi + 1$ (d) $e^\pi - 1$

17. The limit of the following series as $x$ approaches $\frac{\pi}{2}$ is
$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) 1

18. If $f$ and $F$ be both continuous in $[a, b]$, and derivable in $(a, b)$ and if $f'(x) = F'(x)$ for all $x$ in $[a, b]$ then $f(x)$ and $F(x)$ differ
(a) by 1 in $[a, b]$
(b) by $x$ in $[a, b]$
(c) by a constant in $[a, b]$

(d) none of these

19. Consider the following statements:
    1. Rolle's theorem ensures that there is a point on the curve, the tangent at which is parallel to the $x$-axis
    2. Lagrange's mean value therom ensures that there is a point on the curve, the tangent at which is parallel to the $y$-axis
    3. Cauchy's mean value thorem can be deduced from Lagrange's mean value theorem.
    4. Rolle's mean value therom can be deduced from Lagrange's mean value theorem.

    Which of the above statement (s) is/ are correct?

    (a) 1 and 4 (b) 2 and 4
    (c) 1 alone (d) 1, 2 and 3

20. $\lim\limits_{\theta \to 0} \dfrac{\sin(\theta/2)}{\theta}$ is equal to

    (a) 0.5 (b) 1
    (c) 2 (d) not defined

21. $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{x}$ is equal to

    (a) 0 (b) $\infty$
    (c) 1 (d) –1

22. $\lim\limits_{x \to 1} \dfrac{(x^2-1)}{(x-1)}$ is equal to

    (a) $\infty$ (b) 0
    (c) 2 (d) 1

23. $\lim\limits_{x \to \infty} \dfrac{x^3 - \cos x}{x^2 + (\sin x)^2}$ is equal to

    (a) $\infty$
    (b) 0
    (c) 2
    (d) does not exist

24. $\lim\limits_{x \to 3} \dfrac{2x^2 - 7x + 3}{5x^2 - 12x - 9}$ is equal to

    (a) $-\dfrac{1}{3}$ (b) $\dfrac{5}{18}$
    (c) 0 (d) $\dfrac{2}{5}$

25. $\lim\limits_{n \to \infty} n^{\frac{1}{n}}$ is equal to

    (a) 0 (b) 1
    (c) $\infty$ (d) $-\infty$

26. $\lim\limits_{n \to \infty} \left(1+\dfrac{1}{n}\right)^{\frac{1}{n}} \left(1+\dfrac{2}{n}\right)^{\frac{1}{n}} \ldots \left(1+\dfrac{n}{n}\right)^{\frac{1}{n}}$ is equal to

    (a) 1 (b) $\dfrac{2}{e}$
    (c) $\dfrac{3}{e}$ (d) $\dfrac{4}{e}$

27. It is given that $f(x) = \dfrac{ax+b}{x+1}$, $\lim\limits_{x \to 0} f(x) = 2$ and $\lim\limits_{x \to \infty} f(x) = 1$, then value of $f(-2)$ is

    (a) 0 (b) 1
    (c) $e$ (d) $\infty$

28. $\lim\limits_{x \to 1} \dfrac{f(x)-2}{f(x)+2} = 0$, then $\lim\limits_{x \to 1} f(x)$ is equal to

    (a) 1 (b) –1
    (c) –2 (d) 2

29. $\lim\limits_{n \to 0} e^{-\frac{n}{\log n}}$ is equal to

    (a) 1 (b) 0
    (c) – 1 (d) does not exist

30. If $\lim\limits_{x \to 0} \dfrac{x(1-\cos x) - ax^2 \sin x}{x^5}$ exists and is finite, then the value of $a$ must be

    (a) 1 (b) $\dfrac{1}{2}$
    (c) $\dfrac{1}{3}$ (d) $\dfrac{1}{4}$

31. The function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is an increasing function in the interval
    (a) $-2 < x < -1$ (b) $-2 < x < 1$
    (c) $-1 < x < 2$ (d) $1 < x < 2$
32. Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 2$, such that $f(0) = 4, f(2) = 8$, $g(0) = 0$ and $f'(x) = g'(x)$ for all $x$ in $[0, 2]$, then the value of $g(2)$ must be
    (a) 2 (b) –2
    (c) 4 (d) –4
33. If the function $f$ and $g$ be defined and continuous on $[l, m]$, and be differentiable on $(l, m)$ then which one of the following is not correct ?
    (a) When $f(l) = f(m)$, there is $p \in (l, m)$, such that $f'(p) = 0$
    (b) There is $p \in (l, m)$ such that $f(m) - f(l) = f'(p)(m - l)$
    (c) There is $p \in (l, m)$ such that $f(m) - f(l) = f'(p)\, g(m) - g(l)]$
    (d) There is $p \in (l, m)$ such that $\dfrac{f(m) - f(l)}{g(m) - g(l)} = \dfrac{f'(p)}{g'(p)}$ where $g(m) = g(l)$ and $f'(p)$, $g'(p)$ are not simultaneously zero.
34. Let $f(x) = x^2 - 4x + 3$. The following statements are associated with $f$:
    1. $f$ is increasing in $(2, \infty)$
    2. $f$ is decreasing in $(-\infty, -2)$
    3. $f$ has a stationary point at $x = 2$

    Which of these statements are correct?
    (a) 1 and 2 (b) 1 and 3
    (c) 2 and 3 (d) 1, 2 and 3
35. The expansion of $\tan x$ in powers of $x$ by Maclaurin's theorem is valid in the interval.
    (a) $(-\infty, \infty)$ (b) $\left(\dfrac{-3\pi}{2}, \dfrac{3\pi}{2}\right)$
    (c) $(-\pi, \pi)$ (d) $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$
36. The first three terms in the power series for $\log(1 + \sin x)$ are
    (a) $x - \dfrac{1}{2}x^3 + \dfrac{1}{4}x^5$
    (b) $x + \dfrac{1}{2}x^3 + \dfrac{1}{4}x^5$
    (c) $-x - \dfrac{1}{2}x^3 + \dfrac{1}{4}x^5$
    (d) $x - \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3$

**Answers**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1. (b) | 2. (d) | 3. (b) | 4. (b) | 5. (c) | 6. (d) | 7. (a) |
| 8. (b) | 9. (d) | 10. (a) | 11. (b) | 12. (a) | 13. (c) | 14. (d) |
| 15. (a) | 16. (b) | 17. (d) | 18. (c) | 19. (a) | 20. (c) | 21. (a) |
| 22. (c) | 23. (a) | 24. (b) | 25. (b) | 26. (a) | 27. (a) | 28. (d) |
| 29. (d) | 30. (b) | 31. (a) | 32. (c) | 33. (d) | 34. (b) | 35. (a) |
| 36. (d) | | | | | | |