9

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

9.1 INTRODUCTION

A differential equation in which the dependent variable, $y(x)$ and its derivatives, say, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc. occur in the first degree and are not multiplied together is called **linear differential equation**. Then, we classify them as linear differential equation with constant co-efficients and the other with variable coefficients.

The linear differential equations with *constant coefficients* generally arises in practical problems related to the study of mechanical, acoustical and electrical vibrations, whereas linear differential equations with *variable coefficients* arise generally in mathematical modeling of physical problems. Some of the important linear differential equations with variable coefficients are Bessel equation, Legendre's equation, Chebyshev equation etc.

The solution of linear differential equations with constant coefficients are generally found in terms of known standard functions while there exists no such procedure in case of differential equations with variable coefficients and their solutions many a times results in the form of an infinite series.

The general form of the nth order linear differential with constant coefficients is

$$k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} \frac{dy}{dx} + k_n y = X(x) \qquad \dots(1)$$

where $k_0, k_1, k_2, \dots, k_n$ are constants and X is a function of 'x' only.

The general linear differential equation with variable coefficients is written as

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = X(x) \qquad \dots(2)$$

where P_0 ($\neq 0$), $P_1, P_2, \dots, P_n$ and X are function of 'x' only.

If X(x) = 0 in (1) and (2), then they are called linear homogeneous differential equations with constant coefficients and variable coefficients respectively.

9.2 SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

In equation (1), x varies on some interval of definition, say I, which may be open, semi open, closed or infinite and the differential equation may be valid for all $x \in (0, \infty), (-\infty, 0), (-\infty, \infty)$. If $y_1(x)$ is a solution of the equation (1), then it must satisfy the equation identically and whence $y_1(x)$ must be continuously differentiable $(n - 1)$ times and $\frac{d^n}{dx^n} y_1(x)$ must be continuous in that interval.

Further, if coefficients $P_0(x)$, $P_1(x)$,…….., $P_n(x)$, $P_0(x) \neq 0$, in the linear homogeneous equation (2) are continuous on some interval of def, say I, then this equation has n linearly independent solutions. If $y_1(x)$, $y_2(x)$,…, $y_n(x)$ are n linearly independent solutions, then the general solution is their combination.

i.e. $$y(x) = c_1y_1 + c_2y_2 + \ldots + c_ny_n \qquad \ldots (3)$$

Theorem: If $y_1(x)$, $y_2(x)$,…, $y_n(x)$ be n linearly independent solutions of

$$k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_{n-1} \frac{dy}{dx} + k_n y = 0,$$

where $k_0, k_1 ,\ldots, k_n$ are all constants, then $y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is also a solution of (1). *This is called the Principle of Superposition or Principle of linearity*.

Proof: Putting $y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ into left hand side of equation (1), we get

$$k_0 \frac{d^n}{dx^n}\left[c_1y_1 + c_2y_2 + \ldots + c_ny_n\right] + k_1 \frac{d^{n-1}}{dx^{n-1}}\left[c_1y_1 + c_2y_2 + \ldots + c_ny_n\right]$$

$$+ \ldots + k_{n-1} \frac{d}{dx}\left[c_1y_1 + c_2y_2 + \ldots + c_ny_n\right] + k_n\left[c_1y_1 + c_2y_2 + \ldots + c_ny_n\right]$$

$$= c_1\left[k_0 \frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \ldots + k_{n-1} \frac{dy_1}{dx} + k_n y_1\right]$$

$$+ c_2\left[k_0 \frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \ldots + k_{n-1} \frac{dy_2}{dx} + k_n y_2\right]$$

$$+ \ldots + c_n\left[k_0 \frac{d^n y_n}{dx^n} + k_1 \frac{d^{n-1} y_n}{dx^{n-1}} + \ldots + k_{n-1} \frac{dy_n}{dx} + k_n y_n\right]$$

$$= c_1[0] + c_2[0] + \ \ldots \ + c_n[0] = 0, \text{ since } y_1(x), y_2(x), \ldots, y_n(x)$$

are solution of the linear equation. This proves the theorem.

Remarks: The above n linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$ are called the fundamental solutions of equation (1) and the set comprising them forms a basis of the nth order linear homogeneous equations.

The solution (3), $y = c_1y_1 + c_2y_2 + \ldots + c_ny_n = u(x)$ is a combination of n linearly independent solutions containing n arbitrary constants $c_1, c_2, \ldots, c_n$. It is called the **general solution** of equation (1). It is also known as the **complementary function** (C.F.).

Further, if $y = v(x)$ be a solution of the non–homogeneous equation containing no arbitrary constant is called it's **particular solution** (P.I.). Therefore, $y = u(x) + v(x) = C.F. + P.I.$ is called the **complete solution** of equation (1). Hence, in order to solve equation (1), first find the general solution (C.F.) and then find the particular solution (P.I.).

Linear Independence and Dependence of Solutions

Functions $y_1(x)$, $y_2(x)$,..., $y_n(x)$ are said to **linearly independent** on some interval of definition, say I, if the relation (3) viz. $c_1y_1 + c_2y_2 + \ldots + c_ny_n = 0$. Implies $c_1 = 0 = c_2 = \ldots = c_n$. This system (or set) of **linearly independent solutions** is called fundamental system (or set) of solution (or integrals).

However, these functions are said to be dependent on the interval of definition, say I, if relation (3) holds for $c_1, c_2 ,\ldots, c_n$ not all zero. In this case, one or more functions can be expressed as a linear combination of the remaining functions.

e.g. if $c_1 \neq 0$, then $y_1 = -\frac{1}{c_1}\left[c_2y_2 + c_3y_3 + \ldots + c_ny_n\right]$ ...(4)

Conversely, if any of $y_{i's}$ can be expressed as the linear combination of the remaining functions $y_1, y_2 ,\ldots, y_{i-1}, y_{i+1} ,\ldots, y_n$ then the given set of functions are **linearly dependent**.

Theorem: The necessary and sufficient condition that n integrals $y_1, y_2, \ldots, y_n$ of the linear differential equation (2) viz. $P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_ny = 0$, where $P_0, P_1, P_2 ,\ldots, P_n$ $(P_0 \neq 0)$ are continuous functions of x on a common interval I or constants, be linearly independent is that the determinant, W^* (Wronskian),

$$\begin{vmatrix} y_1 & y_2 & \ldots & \ldots & \ldots & \ldots & \ldots & y_n \\ y_1' & y_2' & \ldots & \ldots & \ldots & \ldots & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ y_1^{n-1} & y_2^{n-1} & \ldots & \ldots & \ldots & \ldots & \ldots & y_n^{n-1} \end{vmatrix} \text{ does not vanish identically on I.}$$

Proof: Necessary Condition: If $y_1(x)$, $y_2(x)$,..., $y_n(x)$ are not linearly independent then there are constants $c_1, c_2 ,\ldots, c_n$ not all zero such that $c_1y_1 + c_2y_2 \ldots + c_ny_n = 0$.

Also $c_1y_1^{(i)} + c_2y_2^{(i)} + \ldots + c_ny_n^{(i)} = 0$, $\quad i = 1, 2 ,\ldots, n - 1$.

It follows the determinant

$$W^*(x) = \begin{vmatrix} y_1 & y_2 & \ldots & \ldots & \ldots & \ldots & \ldots & y_n \\ y_1' & y_2' & \ldots & \ldots & \ldots & \ldots & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ y_1^{n-1} & y_2^{n-1} & \ldots & \ldots & \ldots & \ldots & \ldots & y_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} c_1y_1 + c_2y_2 + \ldots\ldots\ldots\ldots + c_ny_n & y_2 \ldots\ldots\ldots y_n \\ c_1y_1' + c_2y_2' + \ldots\ldots\ldots\ldots + c_ny_n' & y_2' \ldots\ldots\ldots y_n' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ \ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ c_1y_1^{n-1} + c_2y_2^{n-1} + \ldots + c_ny_n^{n-1} & y_2^{n-1}\ldots\ldots\ldots y_n^{n-1} \end{vmatrix}$$

$$= \frac{1}{c_1}\begin{vmatrix} 0 & y_2 & \dots & \dots & \dots & \dots & \dots & y_n \\ 0 & y_2' & \dots & \dots & \dots & \dots & \dots & y_n' \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & y_2^{n-1} & \dots & \dots & \dots & \dots & \dots & y_n^{n-1} \end{vmatrix} = 0$$

Thus, if $y_1(x)$, $y_2(x)$,..., $y_n(x)$ are not linearly independent, then the determinant W^* (Wronskian) will be identically zero.

Sufficient Condition: If $W^*(x) = 0$, then there can be found constants $c_1, c_2, \dots, c_n$ not all zero such that $c_1y_1 + c_2y_2 + \dots + c_ny_n = 0$;

whence $c_1y_1^{(i)} + c_2y_2^{(i)} + \dots + c_ny_n^{(i)} = 0,\ i = 1, 2, ,\dots, n-1.$

Now $W^* = 0$ implies that the determinant must be reducible to the form wherein all the elements of one row (or one column) are zero or in other words there must be certain multiplier $\lambda_1, \lambda_2 ,\dots, \lambda_n$ such that

$$\left.\begin{array}{llll} \lambda_1 y_1 + \lambda_2 y_2 + & \dots\ \dots\ \dots\ \dots\ \dots & +\lambda_n y_n = 0 & \dots(i) \\ \lambda_1 y_1' + \lambda_2 y_2' + & \dots\ \dots\ \dots\ \dots\ \dots & +\lambda_n y_n' = 0 & \dots(ii) \\ \dots & \dots\ \dots\ \dots\ \dots\ \dots\ \dots & & \dots \\ \dots & \dots\ \dots\ \dots\ \dots\ \dots\ \dots & & \dots \\ \lambda_1 y_1^{n-1} + \lambda_2 y_2^{n-1} + & \dots\ \dots\ \dots\ \dots\ \dots & +\lambda_n y_n^{n-1} = 0 & \dots(n) \end{array}\right\} \quad \dots(5)$$

Differentiating each of these equations and subtracting from the next following equations, we get

$$\left.\begin{array}{llll} \lambda_1' y_1 + \lambda_2' y_2 + & \dots\ \dots\ \dots\ \dots\ \dots & +\lambda_n' y_n = 0 & \dots(i) \\ \lambda_1' y_1' + \lambda_2' y_2' + & \dots\ \dots\ \dots\ \dots\ \dots & +\lambda_n' y_n' = 0 & \dots(ii) \\ \dots & \dots\ \dots\ \dots\ \dots\ \dots\ \dots & & \dots \\ \dots & \dots\ \dots\ \dots\ \dots\ \dots\ \dots & & \dots \\ \lambda_1' y_1^{n-1} + \lambda_2' y_2^{n-1} + & \dots\ \dots\ \dots\ \dots\ \dots & +\lambda_n' y_n^{n-1} = 0 & \dots(n) \end{array}\right\} \quad \dots(6)$$

Ist operation is explained as: $d(\lambda_1y_1 + \lambda_2y_2 + \dots + \lambda_ny_n) = 0$ implies

$$(\lambda_1y_1' + \lambda_2y_2' + \dots + \lambda_ny_n') + (\lambda_1'y_1 + \lambda_2'y_2 + \dots + \lambda_n'y_n) = 0$$

and on subtraction this from $(\lambda_1y_1 + \lambda_2y_2 + \dots + \lambda_ny_n) = 0$ results in

$$(l_1y_1 + l_2y_2 + \dots + l_ny_n) = 0 \text{ and so on.}$$

Equation (6) in matrix form is equivalent to $\begin{vmatrix} y_1 & y_2 & \dots & \dots & \dots & \dots & \dots & y_n \\ y_1' & y_2' & \dots & \dots & \dots & \dots & \dots & y_n' \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_1^{n-1} & y_2^{n-1} & \dots & \dots & \dots & \dots & \dots & y_n^{n-1} \end{vmatrix} = 0$

If one of the determinant, say formed by omitting the rth column, then there will be a relation, $a_1y_1 + a_2y_2 + ---- + a_{r-1}y_{r-1} + a_{r+1}y_{r+1} + --- + a_ny_n = 0$

In general, if one of these determinants vanishes, then from (6) and the first $(n-1)$ equations in (5), we get

$$\frac{\lambda_1'}{\lambda_1} = \frac{\lambda_2'}{\lambda_2} = \ldots = \frac{\lambda_n'}{\lambda_n} = \mu\,(\text{say}),$$

implying $$\frac{\lambda_1'}{\lambda_1} = \mu \quad \text{or} \quad \frac{d(\lambda_1)}{\lambda_1} = \mu \quad \text{or} \quad \log\lambda_1 = \int \mu\, dx \ \text{ or } \ \lambda_1 = c_1 e^{\int \mu dx}$$

Similarly, $\lambda_2 = c_2 e^{\int \mu dx}, \ldots, \lambda_n = c_n e^{\int \mu dx}$; $c_1, c_2, \ldots, c_n$ being constants. On substituting these λ_i's in the equation (5(*i*)), we get $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$.

Hence, if the condition holds for $(n-1)$ functions, it also holds for n. Whence the necessary and sufficient condition that $y_1, y_2, y_3, \ldots, y_n$ forms a system of linearly independent integral is that determinant W^* does not vanish indentically.

9.3 OPERATOR 'D' AND COMPLEMENTARY FUNCTION

To solve the equation

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + K_{n-1} \frac{dy}{dx} + K_n y = 0 \qquad \ldots(1)$$

We first define the operator 'D'

Operator D: If we write $\frac{dy}{dx} = Dy$ so that $D = \frac{d}{dx}$ is an operator which when applied to a function of 'x', differentiates that function with respect to 'x'. Likewise, $\frac{d^n y}{dx^n} = D^n y$ and so on. Further, $\frac{d^n}{dx^n}\left(\frac{d^m y}{dx^m}\right) = \frac{d^{m+n} y}{dx^{m+n}}$ or $D^m D^n = D^n D^m = D^{m+n}$ Means the differential operator, 'D' fully obeys the laws of algebra.

Whence, the linear differential equation (1) in symbolic form may be written as

$$D^n y + k_1 D^{n-1} y + \ldots + k_{n-1} D y + k_n y = X(x) \qquad \ldots(2)$$

Or more precisely, $f(D)y = X$ where in $f(D)$ can be treated much the same as an algebraic expression in D.

Further, $$(D^n + K_1 D^{n-1} + \ldots + k_{n-1} D + k_n) = 0 \quad \text{or} \quad f(D) = 0 \qquad \ldots(3)$$

is called **auxiliary equation** to (2) with m_1, m_2, --------, m_n as its n roots, means further discussion on solution of (1), depends on the nature of these n roots.

Case1: When all the roots are real and distinct:

In this case, equation (2) will become

$$(D - m_1)(D - m_2) \ldots (D - m_n) y = 0 \qquad \ldots(4)$$

It will be satisfied by the solutions of

$$(D - m_1)y = 0; \quad (D - m_2)y = 0; \quad \ldots\ldots; \ (D - m_n)y = 0,$$

where $(D - m_1)y = 0$ means $\dfrac{dy}{dx} - m_1 y = 0$ which is a Leibnitz linear equation

Here, $\text{I.F.} = e^{-m_1 x}$ and $y.e^{-m_1 x} = c_1$ or $y = c_1 e^{m_1 x}$. Like wise others.

Therefore, general solution becomes

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x} \qquad \ldots (5)$$

Case 2: When two roots are real and equal:
In this case, equation (2) corresponding to the two repeated roots becomes

$$(D - m_1)(D - m_1)y = 0$$

Let $(D - m_1)y = z$, so that above equation reduces to

$$(D - m_1)z = 0 \quad \text{or} \quad Dz - m_1 z = 0$$

Again a Leibnitz Linear, resulting in $z = c_1 e^{-m_1 x}$ so that $(D - m_1)y = z$ becomes $(D - m_1)y = c_1 e^{m_1 x}$ or $\dfrac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$, a Leibnitz Linear with $P = -m_1,\ Q = c_1 e^{m_1 x}$ and its I.F. is $e^{-m_1 x}$ and solution becomes $ye^{-m_1 x} = \int \left(c_1 e^{m_1 x}\right) e^{-m_1 x} dx = c_1 x + c_2$ or $y = \left(c_1 x + c_2\right) e^{m_1 x}$

Thus, the general solution of equation (2), corresponding to two repeated roots becomes

$$y = \left(c_1 x + c_2\right) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \qquad \ldots (6)$$

Case 3: (*i*) When one pair of roots is imaginary (complex):
Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ and the remaining roots are real, then the general solution of equation (1) becomes

$$\begin{aligned}
y &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \\
&= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \\
&= e^{\alpha x}\left[c_1\left(\cos\beta x + i\sin\beta x\right) + c_2\left(\cos\beta x - i\sin\beta x\right)\right] + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \\
&= e^{\alpha x}\left[\left(c_1 + c_2\right)\cos\beta x + i\left(c_1 - c_2\right)\sin\beta x\right] + c_3 e^{m_3 x} + \ldots + C_n e^{m_m x} \\
&= e^{\alpha x}\left[C_1\cos\beta x + C_2\sin\beta x\right] + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x} \qquad \ldots (7)
\end{aligned}$$

where $(c_1 + c_2) = C_1,\ i(c_1 - c_2) = C_2,\ c_3 = C_3$ etc.

Case 3: (*ii*) When one pair of roots is *equal and imaginary*:
(*complex roots equal means* $m_1 = \alpha + i\beta = m_2,\quad m_3 = \alpha - i\beta = m_4$)
In this case general solution becomes.

$$y = e^{\alpha x}\left\{\left(c_1 x + c_2\right)\cos\beta x + \left(c_3 x + c_4\right)\sin\beta x\right\} + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x} \qquad \ldots (8)$$

Example 1: Find the solution of the differential equation $4y''' + 4y'' + y' = 0$.

Solution: The given equation in symbolic form is written as $(4D^3 + 4D^2 + D)y = 0$ and its auxiliary equation is $4D^3 + 4D^2 + D = 0$

$$D(4D^2 + 4D + 1) = 0 \quad D(2D + 1)^2 = 0 \quad \text{i.e.} \quad D = 0, -\frac{1}{2}, -\frac{1}{2}$$

Whence $\quad y(C.F.) = c_1 e^0 + (c_2 x + c_3) e^{-\frac{x}{2}}.$

Example 2: Solve $\dfrac{d^3y}{dx^3} + y = 0$

Solution: The given equation in symbolic form is given by $(D^3 + 1)y = 0$ and its auxiliary equation is $D^3 + 1 = 0$ i.e. $(D + 1)(D^2 - D + 1) = 0$ or $D = -1, \dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

Whence $\quad C.F. = c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right)$

Example 3: $\dfrac{d^4x}{dt^4} = m^4 x$, **show that** $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$.

Solution: The given equation in symbolic form is $(D^4 - m^4)x = 0$

The auxiliary equation becomes $(D^2 - m^2)(D^2 + m^2) = 0$

Which implies either $D^2 - m^2 = 0$ or $D^2 + m^2 = 0$

i.e. $\quad D = \pm m$ and $D = \pm im$

Hence the required solution is

$$x = \left[ae^{mt} + be^{-mt}\right] + \left[ce^{imt} + de^{-imt}\right]$$

$$= \left[a\frac{(\cosh mt + \sinh mt)}{2}\right] + \left[b\frac{(\cosh mt - \sinh mt)}{2}\right]$$

$$+ \left[c\frac{(\cos mt + i\sin mt)}{2} + d\frac{(\cos mt - i\sin mt)}{2}\right]$$

$$= \left[\frac{(a+b)}{2}\cosh mt + \frac{(a-b)}{2}\sinh mt\right] + \left[\frac{(c+d)}{2}\cos mt + i\frac{(c-d)}{2}\sin mt\right]$$

$$\Rightarrow \quad x = \left[C_1 \cosh mt + C_2 \sinh mt\right] + \left[C_3 \cos mt + C_4 \sin mt\right]$$

where $\quad C_1 = \dfrac{a+b}{2}, C_2 = \dfrac{a-b}{2}, C_3 = \dfrac{c+d}{2}$ and $C_4 = \dfrac{i(c-d)}{2}$

Example 4: Solve $\dfrac{d^4y}{dx^4} + a^4 y = 0$ **[NIT Kurukshetra, 2007; NIT Jalandhar, 2007]**

Solution: Symbolic form of the given equation is

$$(D^4 + a^4)y = 0 \qquad \text{.... (1)}$$

Therefore, the auxiliary equation becomes

$$D^4 + a^4 = 0 \quad \text{or} \quad D^4 + a^4 + 2a^2D^2 - 2a^2D^2 = 0$$

or

$$\left(D^2 + a^2\right)^2 - \left(\sqrt{2}\, aD\right)^2 = 0$$

$$\left(D^2 + a^2 - \sqrt{2}aD\right)\left(D^2 + a^2 + \sqrt{2}aD\right) = 0$$

Which implies, either $D^2 + a^2 - \sqrt{2}aD = 0$ .... (2)

or $D^2 + a^2 + \sqrt{2}aD = 0$ ....(3)

On solving the two quadratic equations (2) and (3), we get

$$D = \frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}} \quad \text{and} \quad D = \frac{-a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}} \ \text{respectively}$$

Hence the required solution is

$$y = e^{\frac{a}{\sqrt{2}}x}\left(c_1 \cos\tfrac{a}{\sqrt{2}}x + c_2 \sin\tfrac{a}{\sqrt{2}}x\right) + e^{-\frac{a}{\sqrt{2}}x}\left(c_3 \cos\tfrac{a}{\sqrt{2}}x + c_4 \sin\tfrac{a}{\sqrt{2}}x\right) \qquad \text{... (4)}$$

Alternately: $D^4 = -a^4$ i.e. $D = (-1)^{\frac{1}{4}}a = a\ cis\dfrac{(2n+1)\pi}{4}, n = 0, 1, 2, 3$

Using, *D*. Moivre`s Theorem,

$$n = 0, \quad D_1 = a\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = a\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{a}{\sqrt{2}}(1+i),$$

$$n = 1, \quad D_2 = a\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = a\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -\frac{a}{\sqrt{2}}(1-i),$$

$$n = 2, \quad D_3 = a\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = a\left(-\frac{1}{\sqrt{2}} + -i\frac{1}{\sqrt{2}}\right) = -\frac{a}{\sqrt{2}}(1+i),$$

$$n = 3, \quad D_4 = a\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = a\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{a}{\sqrt{2}}(1-i)$$

Now roots are comparable to $(\alpha \pm i\beta)$, $(-\alpha \pm i\beta)$, so that complementary function becomes,

$$y = e^{\alpha x}c_1(\cos\beta x + i\sin\beta x) + e^{-\alpha x}c_2(\cos\beta x - i\sin\beta x)$$

$$= e^{\frac{a}{\sqrt{2}}x}\left(c_1\cos\frac{a}{\sqrt{2}}x + c_2\sin\frac{a}{\sqrt{2}}x\right) + e^{\frac{-a}{\sqrt{2}}x}\left(c_3\cos\frac{a}{\sqrt{2}}x + c_4\sin\frac{a}{\sqrt{2}}x\right)$$

ASSIGNMENT 1

Solve (*i*) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0,$ (*ii*) $\left(D^2 + D + 1\right)y = 0,$

(*iii*) $\dfrac{d^3y}{dx^3} - y = 0,$ (*iv*) $\dfrac{d^4y}{dx^4} - 8\dfrac{d^2y}{dx^2} + 16y = 0$

(*v*) $\dfrac{d^2y}{dx^2} - 2y\dfrac{dy}{dx} + 10y = 0, \quad y(0) = 4; \quad y'(0) = 1$

(*vi*) $4y''' + 4y'' + y' = 0$ (*vii*) $\left(D^4 + 2n^2D^2 + n^4\right)y = 0$

(*viii*) $l\dfrac{d^2\theta}{dt^2} + g\theta = 0,$ with $\left.\begin{matrix}\theta = \alpha \\ \theta = 0\end{matrix}\right\}$ at $t = 0$

9.4 INVERSE OPERATOR

(*a*) **Def^n:** $\dfrac{1}{f(D)}X(x)$ is that function of '*x*' independent of arbitrary constants which when operated on by $f(D)$ gives $X(x)$. Hence $\dfrac{1}{f(D)}$ is the inverse operator of $f(D)$.

Further, $\dfrac{1}{f(D)}X(x)$ is the particular integral of the equation $f(D)y = X(x)$.

(*b*) $\dfrac{1}{D}X = \int X dx,$ no arbitrary constant is being added to.

Let $\dfrac{1}{D}X(x) = y \;\Rightarrow\; D.\dfrac{1}{D}X(x) = Dy$ or $X = \dfrac{dy}{dx}$ or $dy = X dx$

Integrating both sides, $y = \int X dx.$

(*c*) $\dfrac{1}{D-a}X(x) = e^{ax}\int e^{-ax}X \; dx$

Let, $\dfrac{1}{D-a}X(x) = y \;\Rightarrow\; (D-a)\dfrac{1}{(D-a)}X(x) = (D-a)y$

$$X = Dy - ay \quad \text{or} \quad \frac{dy}{dx} - ay = X$$

which is a Leibnitz linear equation whose, integrating factor is e^{-ax} and its solution becomes $y.e^{-ax} = \int e^{-ax}X\,dx$ or $y = e^{ax}\int e^{-ax}X\,dx$

Problems based on use of Inverse operator in finding particular integral.

Example 5: Solve the differential equation $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$

Solution: Equation in its symbolic form: $(D^2 + a^2)y = \sec ax$ … (1)

Thus auxiliary equation, $D^2 + a^2 = 0$ i.e. $D = \pm\, ia$ … (2)

$\therefore$ $C.F. = e^{ox}\,(c_1\cos ax + c_2\sin ax$ … (3)

$$P.I. = \frac{1}{D^2+a^2}\sec ax = \frac{1}{D^2-(-a^2)}\sec ax = \frac{1}{(D+ia)(D-ia)}\sec ax$$

$$= \frac{1}{2ia}\left(\frac{1}{D-ia} - \frac{1}{D+ia}\right)\sec ax, \text{ (By Partial Fractions)} \qquad \dots (4)$$

Now, $\dfrac{1}{D-ia}\sec ax = e^{iax}\displaystyle\int \sec ax\, e^{-iax}dx,$ $\left(\text{using } \dfrac{1}{D-a}X(x) = e^{ax}\displaystyle\int X e^{-ax}dx\right)$

$$= e^{iax}\int \sec ax(\cos ax - i\sin ax)dx = e^{iax}\int (1 - i\tan ax)dx$$

$$= e^{iax}\left[x - \frac{i(-\log\cos ax)}{a}\right] = e^{iax}\left[x + i\frac{\log\cos ax}{a}\right] \qquad \dots (5)$$

On the same lines, changing i to $-i$, we get

$$\frac{1}{D+ia}\sec ax = e^{-iax}\left[x - i\frac{\log\cos ax}{a}\right] \qquad \dots (6)$$

Using (5) and (6), (4) becomes,

$$P.I. = \frac{1}{2ia}\left[e^{iax}\left\{x + i\frac{\log\cos ax}{a}\right\} - e^{-iax}\left\{x - i\frac{\log\cos ax}{a}\right\}\right]$$

$$\therefore \quad = \frac{1}{2ia}\left[x\left(e^{iax} - e^{-iax}\right) + i\frac{\log\cos ax}{a}\left(e^{iax} + e^{-iax}\right)\right]$$

$$\therefore \quad = \frac{x}{a}.\sin ax + \frac{\log\cos ax}{a^2}.\cos ax \qquad \dots (7)$$

$$\left(\text{since } \frac{e^{iax} - e^{-iax}}{2i} = \sin ax \text{ and } \frac{e^{iax} + e^{iax}}{2} = \cos ax\right)$$

Hence the complete solution is

$$y = (c_1\cos ax + c_2\sin ax) + \left(\frac{x}{a}\sin ax + \frac{\log(\cos ax)}{a^2}\cos ax\right)$$

Note: If $a = 1$, then differential equation becomes $\dfrac{d^2y}{dx^2} + y = secx$ and its solution is $(c_1\cos x + c_2\sin x) + (x\sin x + \cos x \log\cos x)$

Example 6: Solve the differential equations $\frac{d^2y}{dx^2} + a^2y = \operatorname{cosec} ax$

Solution: The given equation in its symbolic form is $(D^2 + a^2)y = \operatorname{cosec} ax$

Thus, the auxiliary equation becomes $D^2 + a^2 = 0$ *i.e.* $D = \pm ia$

$\therefore \qquad C.F. = (c_1 \cos ax + c_2 \sin ax)$

For

$$P.I. = \frac{1}{D^2 + a^2}\operatorname{cosec} ax$$

$$= \frac{1}{D^2 - (-a^2)}\operatorname{cosec} ax$$

$$= \frac{1}{(D + ia)(D - ia)}\operatorname{cosec} ax$$

$$= \frac{1}{2ia}\left(\frac{1}{D - ia} - \frac{1}{D + ia}\right)\operatorname{cosec} ax \quad \text{(By Partial Fractions)}$$

Now, $\frac{1}{D - ia}\operatorname{cosec} ax = e^{iax}\int e^{-iax}\operatorname{cosec} ax\, dx \qquad \left(u\sin g\ \frac{1}{D - a}X = e^{ax}\int e^{-ax} X\, dx\right)$

$$= e^{iax}\int(\cos ax - i\sin ax)\operatorname{cosec} ax\, dx$$

$$= e^{iax}\int(\cot ax - i)\, dx$$

$$e^{iax}\int(\cot ax - i)\, dx$$

$$= e^{iax}\left[\frac{\log(\sin ax)}{a} - ix\right] \qquad \dots (1)$$

On the same lines, changing i to $-i$, we get

$$\frac{1}{D + ia}\operatorname{cosec} ax = e^{-iax}\left[\frac{\log(\sin ax)}{a} + ix\right] \qquad \dots (2)$$

Using (1) and (2),

$$P.I. = \frac{1}{2ia}\left[e^{iax}\left\{x + i\frac{\log\cos ax}{a}\right\} - e^{-iax}\left\{x - \frac{\log\cos ax}{a}\right\}\right]$$

$$= \frac{1}{2ia}\left[\frac{\log(\sin ax)}{a}(e^{iax} - e^{-iax}) - ix(e^{iax} + e^{-iax})\right]$$

$$= \frac{1}{a}\left[\frac{\log\sin ax}{a}\sin ax - x\cos ax\right] \qquad \dots (3)$$

$$\left(\text{since } \frac{e^{iax} - e^{-iax}}{2i} = \sin ax \text{ and } \frac{e^{iax} + e^{-iax}}{2} = \cos ax\right)$$

Hence the complete solution is

$$y = (c_1 \cos ax + c_2 \sin ax) + \frac{1}{a}\left(\frac{\log(\sin ax)}{a}\sin ax - x\cos ax\right)$$

Note: If $a = 1$, then differential equation reduces to $\frac{d^2y}{dx^2} + y = \sec x$ and the corresponding solution becomes $y = (c_1 \cos x + c_2 \sin x) + \sin x(\log \sin x) - x\cos x$. However, these above two problems are easier if followed under the method of variation of parameter, discussed in subsequent articles.

9.5 GENERAL PROCEDURE FOR FINDING PARTICULAR INTEGRAL

Consider the equation,

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_{n-1}\frac{dy}{dx} + k_n y = X(x) \quad \ldots (1)$$

Which in symbolic form becomes

$$(D^n + k_1D^{n-1} + \ldots + k_{n-1}D + k_n)y = X(x) \text{ or } f(D)y = X(x) \quad \ldots(2)$$

Thus the particular integral, $y(x) = \frac{1}{f(D)}X(x)$

Case 1: when $\boldsymbol{X(x) = e^{ax}}$, then

$$\boldsymbol{P.I. = \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}, \text{ provided } f(a) \neq 0} \quad \ldots (3)$$

Since,
$$\left.\begin{array}{l} De^{ax} = ae^{ax} \\ D^2e^{ax} = a^2e^{ax} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ D^ne^{ax} = a^ne^{ax} \end{array}\right\}$$

$\therefore$
$$(D^n + k_1D^{n-1} + \ldots + k_{n-1}D + k_n)e^{ax} = (a^n + k_1a^{n-1} + \ldots + k_{n-1}a + k_n)e^{ax}$$
$$f(D)e^{ax} = f(a)e^{ax}$$

Operating on both sides $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}\cdot f(D)e^{ax} = f(a)\frac{1}{f(D)}e^{ax} \quad \text{or} \quad \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)},\ f(a) \neq 0$$

If $\boldsymbol{f(a)} = 0$, it is a case of failure and then we proceed as

$$\frac{1}{f(D)}e^{ax} = x\frac{1}{\frac{d}{dD}[f(D)]}e^{ax} = x\frac{e^{ax}}{f'(a)}, \text{ provided } f'(a) \neq 0 \quad \ldots (3a)$$

If $f(a) = 0$, then $(D - a)$ must be a factor of $f(D)$

i.e. $$f(D) = (D-a)\phi(D) \text{ and } \frac{d}{dD}\left[f(D)\right] = (D-a)\phi'(D) + 1.\phi(D),$$

which means, $$f'(a) = \frac{d}{dD}\left[f(D)\right]_{at\ D=a} = \phi(a), \text{ provided } \phi(a) \neq 0$$

So that $$\frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)\phi(D)}e^{ax} = \frac{1}{\phi(a)}\frac{1}{(D-a)}e^{ax} = \frac{1}{\phi(a)}e^{ax}\int e^{ax}e^{-ax}dx$$

$$= \frac{e^{ax}}{\phi(a)}x = x\frac{e^{ax}}{f'(a)}, \quad \text{provided} \quad f'(a) \neq 0$$

Further if $f'(a) = 0$ then again it is case of failure and

$$\frac{1}{f(D)}e^{ax} = x^2\frac{e^{ax}}{f''(a)}, f''(a) \neq 0 \qquad \dots (3b)$$

Example7: Solve the differential equation $(D^2 - 2D - 3)y = 3e^{2x}$

Solution: The given equation $(D^2 - 2D - 3)y = 3e^{2x}$ … (1)
has auxiliary equation as: $D^2 - 2D - 3 = 0$ or $D = 3, -1$ ….(2)
Thus the complementary function, C.F. $= c_1e^{3x} + c_2e^{-x}$ … (3)

For particular integral,

$$P.I. = \frac{1}{D^2 - 2D - 3}3e^{2x} \quad \left(comparable\ to \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}, f(a) \neq 0\right)$$

$$= \frac{3}{(2)^2 - 2(2) - 3}e^{2x} = -e^{2x} \qquad \dots (4)$$

Whence the complete solution, $y = (c_1e^{3x} + c_2e^{-x}) - e^{-2x}$

Example 8: Solve the differential equation $(D^3 - 3D^2 + 4D - 2)y = e^x$

Solution: Auxiliary equation for the given equation becomes

$$D^3 - 3D^2 + 4D - 2 = 0 \quad \text{or} \quad (D-1)(D^2 - 2D + 2) = 0$$

Implying $D = 1, \dfrac{2 + \sqrt{4-8}}{2}$ *i.e.* $1, (1 \pm i)$

Whence $y_{C.F.}(x) = c_1e^x + e^x(c_2\cos x + c_3\sin x)$

And $$y_{P.I.}(x) = \frac{1}{D^3 - 3D^2 + 4D - 2}e^x = \frac{1}{1 - 3 + 4 - 2}e^x$$

A case of failure as, $f(a) = 0$.

$$y_{PI} = x\frac{1}{3D^2 - 6D + 4}e^x, \quad \left(x\frac{1}{f'(D)}e^{ax}, \text{ at } D = a \text{ provide } f'(a) \neq 0\right)$$

$$= x\frac{1}{3(1)^2 - 6(1) + 4}e^x = x\,e^x$$

Example 9: **Solve** $\dfrac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$ **[VTU, 2000; KUK, 2003–04]**

Solution: Symbolic form of the equation: $(D^2 - 4)y = \cosh(2x - 1) + 3^x$

Its auxiliary equation becomes,

$$(D^2 - 4) = 0 \;\Rightarrow\; D^2 = 4 \quad \text{or} \quad D = \pm 2$$

So that $y_{C.F.}(x) = c_1e^{2x} + c_2e^{-2x}$

And $$y(x)_{P.I} = \frac{1}{D^2-4}\cosh(2x-1) + \frac{1}{D^2-4}3^x$$

implying $$y_{P.I} = \frac{1}{D^2-4}\left[\frac{e^{(2x-1)} + e^{-(2x-1)}}{2}\right] + \frac{1}{D^2-4}3^x$$

$$= \frac{e^{-1}}{2}\frac{1}{D^2-4}e^{2x} + \frac{e}{2}\frac{1}{D^2-4}e^{-2x} + \frac{e^{x\log 3}}{D^2-4}, \quad [3^x \textit{ is comparable to } a^z = e^{z.\log a}]$$

$$= \frac{1}{2e}x\frac{1}{2D}e^{2x} + \frac{e}{2}x\frac{1}{2D}e^{-2x} + \frac{e^{(\log 3)x}}{(\log 3)^2 - 4}, \quad [3^x = e^{x.\log 3} = e^{(\log 3)x}]$$

$$= \frac{x}{2e}\frac{e^{2x}}{4} - \frac{x.e}{2.}\frac{e^{-2x}}{4} + \frac{3^x}{(\log 3)^2 - 4}, \quad (\text{Rewrite,} \quad e^{(\log 3)x} = 3^x)$$

$$= \frac{x}{4}\left[\frac{e^{2x-1} - e^{-(2x-1)}}{2}\right] + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

$\therefore$ Complete Solution, $y = c_1e^{2x} + c_2e^{-2x} + \dfrac{x}{4}\sinh(2x-1) + \dfrac{3^x}{(log\,3)^2 - 4}$

Case 2: When $\boldsymbol{X(x) = \sin(ax + b)}$ **or** $\boldsymbol{\cos(ax + b)}$, then

$$\boldsymbol{y_{P.I.} = \frac{1}{f(D^2)}\sin(ax+b) = \frac{\sin(ax+b)}{f(-a^2)}, f(-a^2) \neq 0}$$

Since
$$D\sin(ax + b) = a\cos(ax + b)$$
$$D^2\sin(ax + b) = -a^2\sin(ax + b)$$
$$D^3\sin(ax + b) = -a^3\cos(ax + b)$$
$$D^4\sin(ax + b) = a^4\sin(ax + b)$$

or
$$(D^2)^2\sin(ax + b) = (-a^2)^2\sin(ax + b)$$

....

....

In general, $(D^2)^r\sin(ax + b) = (-a^2)^r\sin(ax + b)$

Adding all the above expressions,

$$\left((D^2)^n + k_1(D^2)^{n-1} + \ldots + k_{n-1}D^2 + k_n\right)\sin(ax+b)$$

$$= \left((-a^2)^n + k_1(-a^2)^{n-1} + \ldots + k_{n-1}(-a^2) + k_n\right)\sin(ax+b)$$

or $$f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$$

Operating on both sides the inverse operator $\dfrac{1}{f(D^2)}$

$$\frac{1}{f(D^2)} f(D^2)\sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)\sin(ax+b)$$

or $$\frac{1}{f(D^2)}\sin(ax+b) = \frac{\sin(ax+b)}{f(-a^2)},\ f\left(-a^2\right) \neq 0$$

In case of failure where $f(-a^2) = 0$, we proceed as:

$$\frac{1}{f(D^2)}\sin(ax+b) = x\frac{\sin(ax+b)}{f'(-a^2)}, \quad \text{provided } f'(-a^2) \neq 0 \qquad \ldots (4a)$$

Similarly, if $f'(-a^2) = 0$ then

$$\frac{1}{f(D^2)}\sin(ax+b) = x^2\frac{\sin(ax+b)}{f''(-a^2)},\ f''(-a^2) \neq 0 \text{ and so on.} \qquad \ldots (4b)$$

Example 10: Solve the equation $(2D^2 + D - 1)y = 16\cos 2x$ **[Jammu Univ. 2002]**

Solution: Given equation $(2D^2 + D - 1)y = 16\cos 2x$ … (1)

Its auxiliary equation as: $2D^2 + D - 1 = 0$ *i.e.* $D = \dfrac{1}{2}, -1$ …(2)

Thus complementary function. $(C.F.) = c_1 e^{\frac{x}{2}} + c_2 e^{-x}$ … (3)

$$P.I. = \frac{1}{(2D^2 + D - 1)}16\cos 2x \quad \left(\because \frac{1}{f(D^2)}\cos(ax+b) = \frac{\cos(ax+b)}{f(-a^2)};\ f(-a^2) \neq 0\right)$$

$$= \frac{16}{2(-4) + D - 1}\cos 2x \text{ (as here } a = 2)$$

$$= \frac{16}{D-9}\cos 2x = \frac{16(D+9)}{(D-9)(D+9)}\cos 2x$$

$$= \frac{16}{D^2 - 81}(D+9)\cos 2x$$

$$= \frac{16}{-4-81}[D\cos 2x + 9\cos 2x] = \frac{16}{85}(2\sin 2x - 9\cos 2x) \qquad \ldots (4)$$

Hence, the complete solution becomes

$$y = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-x}\right) + \frac{16}{85}(2\sin 2x - 9\cos 2x)$$

Example 11: Solve the equation $\dfrac{d^2y}{dx^2} + 4y = \sin 2x.$

Solution: Symbolic form of the given equation, $(D^2 + 4)y = \sin 2x$

Corresponding auxiliary equation, $D^2 + 4 = 0$ *i.e.* $D = \pm 2i$

Thus, $y(C.F) = (c_1 \cos 2x + c_2 \sin 2x)$

$$y(P.I.) = \frac{1}{D^2 + 4}\sin 2x; \text{ Replace } D^2 = -a^2 = -4, \text{ but here } f(-a^2) = 0$$

$$\left[\text{In case of failure, } \frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b)\right]$$

Implying $y(P.I.) = x\dfrac{1}{2.D}\sin 2x = \dfrac{x}{2}\left(\dfrac{-\cos 2x}{2}\right) = -\dfrac{x\cos 2x}{4}.$

Hence the complete solution, $y = (c_1 \cos 2x + c_2 \sin 2x) - \dfrac{x\cos 2x}{4}$.

Case 3: When $X(x) = x^m$ then, $y_{P.I.} = \dfrac{1}{f(D)}x^m = [1 \pm \phi(D)]^{-1} x^m$, where m is a positive integer.

From $f(D)$, take the lowest degree term outside so that the remaining expression in $f(D)$ becomes $[1 \pm \phi(D)]$. Now take $[1 \pm \phi(D)]$ to the numerator and expand it by binomial theorem upto D^m so that $(m + 1)^{\text{th}}$ and higher order derivatives of x^m are zero.

Note: *Some useful results:*

(*i*) $(1 - D)^{-1} = 1 + D + D^2 + \ldots\ldots\ldots\ldots \infty$

(*ii*) $(1 + D)^{-1} = 1 - D + D^2 + \ldots\ldots\ldots\ldots \infty$

(*iii*) $(1 - D)^{-2} = 1 + 2D + 3D^2 + \ldots\ldots\ldots\ldots \infty$

(*iv*) $(1 - D)^{-3} = 1 + 3D + 6D^2 + \ldots\ldots\ldots\ldots \infty$

Example 12: Solve $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = 1 + x^2$

Solution: Symbolic form becomes $(D^3 - D^2 - 6D)y = 1 + x^2$

Auxiliary Equation, $D^3 - D^2 - 6D = 0$

Implying $D(D + 2)(D - 3) = 0$ or $D = 0, 3, -2$

Whence $y_{C.F.} = (c_1 + c_2 e^{3x} + c_3 e^{-2x})$

And $$y_{P.I.} = \frac{1}{D^3 - D^2 - 6D}(1 + x^2) = \frac{1}{-6D\left(1 + \frac{D}{6} - \frac{D^2}{6}\right)}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}(1+x^2)$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2+----+\right](1+x^2)$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\frac{D^2}{36}\right](1+x^2)$$

$$= -\frac{1}{6D}\left[(1+x^2)-\frac{D}{6}(1+x^2)+\frac{7}{36}D^2(1+x^2)\right]$$

$$= -\frac{1}{6D}\left[(1+x^2)-\frac{x}{3}+\frac{7}{36}(2)\right] = -\frac{1}{6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{1}{3}\frac{x^2}{2}+\frac{25}{18}x\right] = -\frac{1}{108}(6x^3-3x^2+25x)$$

Therefore $$y(x) = (c_1+c_2e^{3x}+c_3e^{-2x}) - \frac{1}{108}(6x^3-3x^2+25x)$$

Example 13: Solve the differential equation $(D^4 + 16D^2)y = x^2 + 5$.
[Jammu Univ, 2002; NIT Kurukshetra, 2006]

Solution: Given $(D^4 + 16D^2)y = x^2 + 5$. …(1)

A.E., $D^4 + 16D^2 = 0$ *i.e.* $D^2(D^2 + 16) = 0$ or $D = 0, 0, \pm 4i$ … (2)

Thus, the complementary function,

$$C.F. = (c_1 + c_2x) + (c_3\cos 4x + c_4\sin 4x) \quad \ldots (3)$$

For particular integral,

$$P.I. = \frac{1}{D^4+16D^2}(x^2+5) \qquad \left(\textit{comparable to}\ \frac{1}{f(D)}x^m = f(D)^{-1}x^m\right)$$

$$= \frac{1}{D^2(D^2+16)}(x^2+5) = \frac{1}{D^2}\frac{1}{16\left(1+\frac{1}{16}D^2\right)}(x^2+5)$$

$$= \frac{1}{16D^2}\left(1+\frac{1}{16}D^2\right)^{-1}(x^2+5)$$

$$= \frac{1}{16D^2}\left(1-\frac{1}{16}D^2\right)(x^2+5)$$

$$= \frac{1}{16D^2}\left(x^2 + 5 - \frac{1}{16}.2\right) = \frac{1}{16}\frac{1}{D}\int\left(x^2 + \frac{39}{8}\right)dx$$

$$= \frac{1}{16}\frac{1}{D}\left(\frac{x^3}{3} + \frac{39}{8}x\right) = \frac{1}{16}\left[\frac{x^4}{12} + \frac{39x^2}{16}\right]$$

Hence complete solution becomes,

$$y = (c_1 + c_2 x) + (c_3 \cos 4x + c_4 \sin 4x) + \frac{x^2}{64}\left[\frac{x^2}{3} + \frac{39}{4}\right]$$

Case 4: When $X(x) = e^{ax}V(x)$, then, $\dfrac{1}{f(D)}e^{ax}V(x) = e^{ax}\dfrac{1}{f(D+a)}V(x)$

For some $u(x)$,

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D + a)u$$

$$D^2(e^{ax}u) = e^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u = e^{ax}(D + a)^2u$$

$$\dots \dots \dots$$

$$\dots \dots \dots$$

$$D^n(e^{ax}u) = e^{ax}(D + a)^n u$$

Adding all,

$$D^n(e^{ax}u) + k_1D^{n-1}(e^{ax}u) + \dots + k_{n-1}D(e^{ax}u) + k_n(e^{ax}u)$$

$$= e^{ax}(D + a)^n u + k_1e^{ax}(D + a)^{n-1}u + \dots + k_{n-1}(D + a)u + k_n u$$

Implying $f(D)(e^{ax}u) = e^{ax}f(D + a)u$

Operating $f(D)$ on both sides, $\dfrac{1}{f(D)}f(D)\left(e^{ax}u\right) = \dfrac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$

or
$$e^{ax}u = \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$$

Now put $f(D + a)u = V$ *i.e.* $u = \dfrac{1}{f(D+a)}V$

So that
$$e^{ax}\frac{1}{f(D+a)}V = \frac{1}{f(D)}\left(e^{ax}V\right)$$

Example 14: Solve $(D^4 - 1)y = e^x \cos x$

Solution: The corresponding Auxiliary Equation is

$$D^4 - 1 = 0 \quad i.e. \quad (D^2 - 1)(D^2 + 1) = 0 \quad \text{or} \quad D = \pm 1, \pm i$$

Hence the complementary function

$$y(C.F.) = (c_1e^x + c_2e^{-x}) + (c_3\cos x + c_4\sin x)$$

For particular Integral,

$$y(P.I.) = \frac{1}{(D^4 - 1)}e^x \cos x = e^x\frac{1}{(D+1)^4 - 1}\cos x$$

$$= e^x \frac{1}{\left(D^4 + 4D^3 + 6D^2 + 4D + 1\right) - 1} \cos x$$

$$= e^x \frac{1}{\left(D^2\right)^2 + 4D\left(D^2\right) + 6\left(D^2\right) + 4D} \cos x$$

$$= e^x \frac{1}{1 + 4D(-1) + 6(-1) + 4D} \cos x \quad \text{(replace } D^2 = -a^2 = -1\text{)}$$

$$= e^x \frac{\cos x}{-5}$$

Hence the complete solution is

$$y = \left(c_1 e^x + c_2 e^{-x}\right) + \left(c_3 \cos x + c_4 \sin x\right) - \frac{1}{5} e^x \cos x$$

Note: *For problem $(D^4 - 1)y = \cos x . \cosh x$, rewrite,* $\left(D^4 - 1\right)y = \frac{1}{2}\left(e^x \cos x + e^{-x} \cos x\right)$

Here, $PI_1 = \frac{e^x \cos x}{-5}$ and $PI_2 = \frac{e^{-x} \cos x}{-5}$

Example 15: Solve the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

[KUK, 2003; NIT Kurukshetra 2010]

Solution: Symbolic form of the given equation, $(D^2 + 3D + 2)y = e^{e^x}$

Auxiliary equation becomes $D^2 + 3D + 2 = 0$ or $D = -1, -2.$

$C.F. = c_1 e^{-x} + c_2 e^{-2x}$

And $y(P.I.) = \frac{1}{D^2 + 3D + 2} e^{e^x} = \frac{1}{D^2 + 3D + 2} e^{-x}\left(e^x e^{e^x}\right)$

$$\left(\text{Comparable to } \frac{1}{f(D)} e^{ax} V(x) = e^{ax} \frac{1}{f(D+a)} V(x) \quad \text{with} \quad a = -1 \quad \text{and} \quad V(x) = e^x e^{e^x}\right)$$

$$= e^{-x} \frac{1}{(D-1)^2 + 3(D-1) + 2}\left(e^x e^{e^x}\right)$$

$$= e^{-x} \frac{1}{D^2 + D} D\left(e^{e^x}\right), \quad \left[\text{as} \quad D\left(e^{e^x}\right) = e^{e^x} D\left(e^x\right) = e^{e^x} e^x\right]$$

$$= e^{-x} \frac{1}{(D+1)} \frac{1}{D} D\left(e^{e^x}\right)$$

$$= e^{-x}\left[\frac{1}{(D+1)} e^{(e^x)}\right] = e^{-x}\left[e^{-x} \int e^x . e^{e^x} dx\right], \quad \left(\text{using } \frac{1}{D-a} X(x) = e^{ax} \int e^{-ax} X dx\right)$$

$$= e^{-x}\left[e^{-x}\int d\left(e^{e^x}\right)dx\right] = e^{-x}\left[\left(e^{-x}e^{e^x}\right)\right] = e^{-2x}e^{e^x}$$

Alternately:

$$y_{P.I.} = \frac{1}{D^2+3D+2}e^{e^x} = \frac{1}{(D+1)(D+2)}\left(e^{e^x}\right)$$

$$= \frac{1}{D+1}e^{e^x} - \frac{1}{D+2}e^{e^x} \quad \text{(by partial fraction)}$$

$$= e^{-x}\int e^x e^{e^x}dx - e^{-2x}\int e^{2x}e^{e^x}dx$$

$$= e^{-x}e^{e^x} - e^{-2x}\left(e^x e^{e^x} - \int e^{e^x}e^x dx\right) = e^{-2x}\int d\left(e^{e^x}\right)dx$$

$$= e^{-2x}e^{e^x}.$$

Example 16: Solve the differential equation $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x$
[PTU, 2003; NIT Kurukshetra, 2008]

Solution: The auxiliary equation given by $D^3 + 2D^2 + D = 0$ *i.e.* $D = 0, -1, -1$

$\therefore$ $C.F. = c_1 + (c_2x + c_3)e^{-x}$

For $$P.I. = \frac{1}{D^3+2D^2+D}\left(x^2e^{2x} + \sin^2 x\right) = P.I._1 + P.I._2$$

$$P.I._1 = \frac{1}{D^3+2D^2+D}x^2e^{2x}$$

$$= e^{2x}\frac{1}{(D+2)^3+2(D+2)^2+(D+2)}x^2$$

$$= e^{2x}\frac{1}{18\left[1+\left(\frac{21}{18}D+\frac{8}{18}D^2+\frac{1}{18}D^3\right)\right]}x^2$$

$$= \frac{e^{2x}}{18}\left[1+\left(\frac{7}{6}D+\frac{4}{9}D^2+\dots\dots\right)\right]^{-1}x^2$$

$$= \frac{e^{2x}}{18}\left[1-\left(\frac{7}{6}D+\frac{4}{9}D^2+\dots\dots\right)+\left(\frac{7}{6}D\right)^2+\dots\dots\right]x^2$$

$$= \frac{e^{2x}}{18}\left[1-\frac{7}{6}D+\frac{33}{36}D^2\right]x^2 = \frac{e^{2x}}{18}\left[x^2-\frac{7}{6}2x+\frac{33}{36}.2\right]$$

$$= \frac{e^{2x}}{18}\left[x^2-\frac{7}{3}x+\frac{11}{6}\right]$$

$$P.I._2 = \frac{1}{D^3 + 2D^2 + D}\sin^2 x$$

$$= \frac{1}{D^3 + 2D^2 + D}\frac{1 - \cos 2x}{2}$$

$$= \frac{1}{2D}\frac{1}{1 + (2D + D^2)}e^{0x} - \frac{1}{2}\frac{1}{D(D^2) + 2D^2 + D}\cos 2x$$

$$= \frac{1}{2D} - \frac{1}{2}\frac{\cos 2x}{(-4D - 8 + D)}$$

$$= \frac{1}{2}\frac{1}{D} + \frac{1}{2}\frac{1}{(3D + 8)}\frac{(3D - 8)}{(3D - 8)}\cos 2x$$

$$= \frac{1}{2}x + \frac{1}{2}\frac{3D - 8}{9D^2 - 64}\cos 2x = \frac{x}{2} + \frac{1}{2}\frac{(3D - 8)\cos 2x}{(-36 - 64)}$$

$$= \frac{x}{2} - \frac{1}{200}(3D\cos 2x - 8\cos 2x)$$

$$= \frac{x}{2} + \frac{3\sin 2x + 4\cos 2x}{100}$$

Hence the complete solution is

$$y = c_1 + (c_2 x + c_3)e^{-x} + \frac{x}{2} + \frac{3\sin 2x + 4\cos 2x}{100} + \frac{e^{2x}}{18}\left[x^2 - \frac{7}{3}x + \frac{11}{6}\right]$$

Case 5: When $\boldsymbol{X = xV}$, V being a function of x, then

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left(\frac{d}{dD}\frac{1}{f(D)}\right)V$$

By Leibnitz's Theorem on successive differentiation, we have

$$D^n(Xx) = {}^nc_0(D^nX).x + {}^nc_1 D^{n-1}X.1$$

$$\Rightarrow \quad D^n(xX) = x D^n X + n D^{n-1} X$$

$$\Rightarrow \quad D^n(xX) = x D^n X + \left(\frac{d}{dD}D^n\right)X, \qquad \left(\because \frac{d}{dD}D^n = nD^{n-1}\right) \qquad \ldots (1)$$

or $$f(D)(xX) = x\, f(D)X + \left(\frac{d}{dD}f(D)\right)X \qquad \ldots (2)$$

(if D^n represents a polynomial of nth degree in D)

Putting $f(D)X = V$, we get $\dfrac{1}{f(D)}(f(D)X) = \dfrac{1}{f(D)}V$ or $X = \dfrac{1}{f(D)}V$ … (3)

With the help of (3), (2) becomes

$$f(D)\left(x.\frac{1}{f(D)}V\right) = x.V + f'(D)\left(\frac{1}{f(D)}V\right)$$

Operating $\dfrac{1}{f(D)}$ on both sides of above equation, we get

$$\left(x.\frac{1}{f(D)}V\right) = \frac{1}{f(D)}(x.V) + \frac{1}{f(D)}\frac{f'(D)}{f(D)}V$$

or $$x.\frac{1}{f(D)}V - \frac{f'(D)}{\left[f(D)\right]^2}V = \frac{1}{f(D)}(x.V)$$

or $$x.\frac{1}{f(D)}V - \left(\frac{d}{dD}\frac{1}{f(D)}\right)V = \frac{1}{f(D)}(x.V), \quad \left(\text{since } \frac{d}{dD}\left(f(D)\right)^{-1} = -\frac{f'(D)}{\left[f(D)\right]^2}\right)$$

Example 17: Solve $\dfrac{d^2y}{dx^2} + 4y = x\sin x$ [Raipur, 2004]

Solution: Symbolic form, $(D^2 + 4)y = x\sin x$

A.E., $D^2 + 4 = 0$ implying $D = \pm 2i$ so that

$y_{C.F.}(x) = (c_1\cos 2x + c_2\sin 2x)$

$$y(P.I.) = \frac{1}{D^2+4}(x\sin x) \qquad \text{(Comparable to case 5.)}$$

$$= x\frac{1}{D^2+4}\sin x - \frac{2D}{\left(D^2+4\right)^2}\sin x$$

$$= x\frac{1}{(-1)+4}\sin x - \frac{2D}{(-1+4)^2}\sin x \qquad \text{Replace } f(D^2) = f(-a^2)$$

$$= \frac{x\sin x}{3} - \frac{2}{9}D(\sin x) = \frac{1}{3}x\sin x - \frac{2}{9}\cos x.$$

$\therefore$ $$y(x) = y_{C.F.} + y_{P.I.} = \left(c_1\cos 2x + c_2\sin 2x\right) + \frac{1}{9}\left(3x\sin x - 2\cos x\right)$$

Example 18: Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x \sin x$ **[VTU, 2001; Osmania, 2003]**

Solution: Here A.E. is $D^2 + 3D + 2 = 0 \quad \Rightarrow \quad D = -1, -2$

$$y_{(C.F.)} = c_1e^{-x} + c_2e^{-2x}$$

$$y_{(P.I.)} = \frac{1}{D^2 + 3D + 2}x \quad e^x \sin x = e^x \frac{1}{(D+1)^2 + 3(D+1) + 2}(x \sin x)$$

$$= e^x \frac{1}{D^2 + 5D + 6}(x \sin x)$$

$$= e^x \left[x \frac{1}{D^2 + 5D + 6} \sin x + \left(\frac{d}{dD} \frac{1}{D^2 + 5D + 6} \right) \sin x \right],$$

$$= e^x \left[x \frac{1}{-1 + 5D + 6} \sin x - \frac{2D + 5}{(D^2 + 5D + 6)^2} \sin x \right]$$

$$= e^x \left[x \frac{1}{5(D+1)} \sin x - \frac{2D + 5}{(-1 + 5D + 6)^2} \sin x \right]$$

$$= e^x \left[\frac{x}{5} \frac{D+1}{D^2 - 1} \sin x - \frac{2D + 5}{25(D+1)^2} \sin x \right]$$

$$= e^x \left[-\frac{x}{10}(D \sin x + \sin x) - \frac{1}{25} \frac{(2D + 5)}{(-1 + 2D + 1)} \sin x \right]$$

$$= e^x \left[-\frac{x}{10}(D \sin x + \sin x) - \frac{1}{25} \frac{2D + 5}{2D} \sin x \right]$$

$$= e^x \left[-\frac{x}{10}(\cos x + \sin x) - \frac{1}{25}(\sin x + \frac{5}{2}\frac{1}{D} \sin x) \right]$$

$$= e^x \left[-\frac{x}{10}(\sin x + \cos x) - \frac{1}{50}(2 \sin x - 5 \cos x) \right]$$

Complete solution, $y(x) = \left(c_1e^{-x} + c_2e^{-2x}\right) - e^{-2x}\left[\frac{x}{10}(\sin x + \cos x) + \frac{1}{50}(2 \sin x - 5 \cos x)\right]$

Example 19: Solve $(D^2 - 4D + 4)y = 8x^2e^{2x}\sin^2 x$

[Delhi, 2002; JNTU, 2006; NIT Kurukshetra, 2007; UP Tech, 2007]

Solution: A.E. $D^2 - 4D + 4 = 0$ or $D = 2, 2$

Whence, $y_{C.F.}(x) = (c_1 + c_2x)e^{2x}$ … (1)

$$y_{PI}(x) = \frac{1}{D^2 - 4D + 4}\, 8x^2 e^{2x} \sin 2x$$

$$= 8\,e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x$$

$$= 8\,e^{2x} \frac{1}{D^2} x^2 \sin 2x = 8e^{2x} \frac{1}{D}\frac{1}{D}\left(x^2 \sin 2x\right) \quad \dots (2)$$

$$= 8\,e^{2x} \frac{1}{D} \int x^2 \sin 2x\, dx,$$

$$= 8\,e^{2x} \frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right], \text{ integration by parts}$$

$$= 8\,e^{2x}\left[-\frac{1}{2}\int x^2 \cos 2x\, dx + \frac{1}{2}\int x \sin 2x\, dx + \frac{1}{4}\int \cos 2x\, dx\right]$$

$$= 8\,e^{2x}\left[-\frac{1}{2}\left\{x^2 \frac{\sin 2x}{2} - \int 2x \frac{\sin 2x}{2} dx\right\} + \frac{1}{2}\int x \sin 2x\, dx + \frac{1}{4}\int \cos 2x dx\right]$$

$$= 8\,e^{2x}\left[-\frac{x^2 \sin 2x}{4} + \int x \sin 2x\, dx + \frac{1}{4}\int \cos 2x\, dx\right]$$

$$= 8\,e^{2x}\left[-\frac{x^2 \sin 2x}{4} + x\frac{(-\cos 2x)}{2} - \int 1 \frac{(-\cos 2x)}{2} dx + \frac{1}{4}\int \cos 2x\, dx\right]$$

$$= 8\,e^{2x}\left[-\frac{x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{4}\int \cos 2x\, dx\right]$$

$$= 8\,e^{2x}\left[-\frac{x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right] \quad \dots (3)$$

Therefore,

$$y(x) = y_{CF} + y_{PI} = (c_1 + c_2 x)e^{2x} + 8\,e^{2x}\left[-\frac{x^2}{4}\sin 2x + \frac{3}{8}\sin 2x - \frac{x}{2}\cos 2x\right]$$

$$= \left[c_1 + c_2 x - (2x^2 - 3)\sin 2x - 4x\cos 2x\right] e^{2x}$$

Alternately: From equation (2) onwards,

$$y_{\cdot P.I.} = \text{Imag. Part of } 8\,e^{2x} \frac{1}{D^2} x^2 e^{2ix} = I.P.\ 8\,e^{2x} e^{2ix} \frac{1}{(D+2i)^2} x^2$$

$$= imag\ 8e^{2x}e^{2ix}\frac{1}{\left[2i\left(1+\frac{D}{2i}\right)\right]^2}x^2 = imag.\ 8e^{2x}e^{2ix}\frac{1}{-4}\left(1-\frac{iD}{2}\right)^{-2}x^2$$

$$= imag.\ \frac{8}{-4}e^{2x}e^{2ix}\left[1+2\frac{iD}{2}+3\frac{(iD)^2}{4}+\ldots\right]x^2$$

$$= imag. -2e^{2x}e^{2ix}\left[x^2+2ix-\frac{3}{2}\right]$$

$$= imag.\ e^{2x}(\cos 2x + i\sin 2x)\left[(3-2x^2)-4ix\right]$$

$$= 8e^{2x}\left[-\frac{x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right]$$

ASSIGNMENT 2

Solve 1. $\dfrac{d^2y}{dx^2} + a^2y = x\cos ax,$ 2. $\dfrac{d^2y}{dx^2} - y = x^2\cos x,$

3. $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x,$ [MDU, 2002] 4. $\dfrac{d^4y}{dx^4} - y = x\sin x,$ **[KUK, 2005-06]**

5. $(D^2 + a^2)y = \tan ax,$ **[MDU, 2002; VTU, 2005]**

6. $(D^2 + 4)y = e^x\sin^2 x$ 7. $(D^2 + 4D + 3)y = e^{-x}\sin x + x,$

8. $(D^2 - 1)y = x\sin x + (1 + x^2)e^x,$ **[KUK, 2010]**

9. $(D^2 - 4)y = x\sinh x,$ **[Madras, 2002; KUK, 2007]**

10. $(D^4 + 1)y = x^2\cos x,$ **[Osmania, 2002]**

11. $\dfrac{d^4y}{dx^4} - y = \cos x\cosh x,$

12. $(D^2 - 4D + 3)y = \sin 3x\cos 2x,$ **[Madras, 2000]**

13*. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x\sin x,$ 14. $(D^2 + n^2)y = x^2e^x,$ **[*UP Tech; 2009]**

[VTU, 2001; Osmania, 2003; UPTech., 2005; JNTU, 2006; SVTU, 2007; PTU, 2009]

$\left[\text{Hint: } y_{PI} = \dfrac{1}{f(0)}e^xV(x) = e^x\dfrac{1}{f(D+1)}V(x) = e^x\dfrac{1}{D^2}(x\sin x);\ \text{integration by parts}\right]$

9.6 SPECIAL METHODS FOR FINDING PARTICULAR INTEGRAL

I. Method of Variation of Parameters

The method of variation of parameters which is due to J.L Langrange (1736–1813), enables us to find the general solution of any second order non homogeneous linear differential equation (including both, with variable coefficients as well as with constant coefficients) whose complementary function is known. Under this method, complementary function of the reduced equation with no $X(x)$ is used by replacing the arbitrary constants with functions so chosen that a particular integral is obtained.

Consider the equation, $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = X(x)$ … (1)

If y_1 and y_2 are two linearly independent solutions of the above equation and $W^* = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the **Wronskian** of y_1 and y_2 (named after the noted **Polish Mathematician and Philosopher *Hoene Wronsky*** (1778–1853)), then *particular integral*.

$$y_{P.I.} = u_1 y_1 + u_2 y_2 = y_1\int -y_2\left(\frac{X}{W^*}\right)dx + y_2\int y_1\left(\frac{X}{W^*}\right)dx \quad \text{… (2)}$$

Proof: Let general solution of the equation (1) be,

$$y_{C.F.} = c_1 y_1 + c_2 y_2, \quad \text{… (3)}$$

where c_1 & c_2 are arbitrary constants. The idea behind this method is that assume the arbitrary constants of the general solution c_1 & c_2 as functions of x *i.e.* $c_1 = u_1(x)$ & $c_2 = u_2(x)$, so that

$$y_{P.I.} = u_1(x)y_1 + u_2(x)y_2 \quad \text{… (4)}$$

Differentiating (4), $\dfrac{dy}{dx} = \left(u_1 y_1' + u_2 y_2'\right) + \left(u_1' y_1 + u_2' y_2\right)$

On assumption that $u_1' y_1 + u_2' y_2 = 0$ …(6)

We get $\dfrac{dy}{dx} = \left(u_1 y_1' + u_2 y_2'\right)$ … (5)

Further, differentiating (6),

$$\frac{d^2y}{dx^2} = \left(u_1 y_1'' + u_2 y_2''\right) + \left(u_1' y_1' + u_2' y_2'\right) \quad \text{… (7)}$$

On substituting values of $y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ from (4), (6), (7) respectively into (1), we see

$$u_1' y_1' + u_2' y_2' = X \quad \text{… (8)}$$

Now equations (5) and (8) constitute two linear equations in u_1' and u_2', which on solving, results in

$$u_1' = -y_2\left(\frac{X}{W^*}\right) \quad \text{and} \quad u_2' = y_1\left(\frac{X}{W^*}\right), \text{ where } W^* = y_1 y_2' - y_1' y_2$$

From above $u_1(x)$ *and* $u_2(x)$ are obtained by integration and, whence the particular integral and complete solution, $y = y_{C.F.} + y_{P.I.}$.

Observations:

1. Here constant of integration is not required, since we find particular integral, and any function $u_1(x)$ and $u_2(x)$ satisfying the required conditions will take care of it.
2. Though this method is commonly applied for 2nd order equations but it can easily be extended to equations of any order.

e.g. Consider 3rd order equation $P_0(x)\dfrac{d^3y}{dx^3} + P_1(x)\dfrac{d^2y}{dx^2} + P_2(x)\dfrac{dy}{dx} + P_3(x)y = R(x)$

with $y_{C.F.}(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x)$, where $y_1(x)$, $y_2(x)$ and $y_3(x)$ are three linearly independent solutions corresponding to homogeneous equation and c_1, c_2, c_3 are arbitrary constants. In this case, $y_{P.I.}(x) = u_1(x)\,y_1 + u_2(x)\,y_2 + u_3(x)\,y_3$:

Here also we follow the same procedure as discussed earlier and obtain the required corresponding equations for solving u_1', u_2', u_3' as

$$\left.\begin{aligned} &u_1'y_1 + u_2'y_2 + u_3'y_3 = 0 \\ &u_1'y_1' + u_2'y_2' + u_3'y_3' = 0 \\ &u_1'y_1'' + u_2'y_2'' + u_3'y_3'' = \frac{R(x)}{P_0(x)} = X(x) \end{aligned}\right\}$$

Here, Wronskian (a non-zero entity) becomes, $W^* = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

Example 20: Solve $\dfrac{d^2y}{dx^2} + y = \tan x$ by the method of variation of parameters.

[KUK, 2004–05; PU, 2003-04; Raipur, 2004; NIT Kurukshetra, 2008; UP Tech, 2009]

Solution: Rewrite the given equation as $(D^2 + 1)y = \tan x$

Here A.E $D^2 + 1 = 0$ or $D = \pm i$

Thus $y_{C.F.}(x) = c_1y_1 + c_2y_2 = c_1\cos x + c_2 \sin x$

Assume $y_{P.I.}(x) = u_1(x)\,y_1 + u_2(x)\,y_2 = u_1\cos x + u_2 \sin x$

By method of variation of parameter, we have

$$\left.\begin{aligned} u_1'(x) &= -y_2\left(\frac{X}{W^*}\right) \\ u_2'(x) &= \ \ y_1\left(\frac{X}{W^*}\right) \end{aligned}\right\} \text{ and } W^* = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1;\ X = \tan x$$

On integration,

$$u_1(x) = \int -\sin x \tan x\, dx = -\int \sin x \frac{\sin x}{\cos x}\, dx = \int \frac{\cos^2 x - 1}{\cos x}\, dx$$

$$= \int (\cos x - \sec x)\, dx = \sin x - \log(\sec x + \tan x)$$

And $u_2(x) = \int \cos x \tan x\, dx = \int \sin x\, dx = -\cos x$

Whence $y_{P.I.}(x) = \cos x[\sin x - \log(\sec x + \tan x)] + \sin x(-\cos x)$

$= -\cos x \log(\sec x + \tan x)$

And therefore the complete solution becomes,

$$y = (c_1\cos x + c_2\sin x) - (\cos x)\log(\sec x + \tan x)$$

Example 21: Solve $\frac{d^2y}{dx^2} + y = x\sin x$, using method of variation of parameter.

[SVTU 2007; JNTU, 2005]

Solution: Symbolic form of the given equation, $(D^2 + 1)\, y = x\sin x$

Auxiliary equation, $D^2 + 1 = 0$ i.e. $D = \pm i$

Thus $C.F. = c_1y_1 + c_2y_2 = c_1\cos x + c_2\sin x$ … (1)

Let $P.I. = u_1(x)y_1 + u_2(x)y_2;$

Now by method of variation of parameter,

$$\left.\begin{aligned} u_1'(x) &= -y_2\left(\frac{X}{W^*}\right) \\ u_2'(x) &= y_1\left(\frac{X}{W^*}\right) \end{aligned}\right\} \text{ and } W^* = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1; \qquad \text{…(3)}$$

So that
$$u_1(x) = -\int y_2 \frac{X}{W^*}\,dx = -\int \sin x\,(x\sin x)\,dx = -\int x\,\frac{(1-\cos 2x)}{2}\,dx$$

$$= \left(-\frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}\right) \qquad \text{… (4)}$$

$$u_2(x) = \int y_1 \frac{X}{W^*}\,dx = \int \cos x\,(x\sin x)\,dx = \frac{1}{2}\int x\sin 2x\,dx \qquad \text{… (5)}$$

$$= \left(-\frac{x}{4}\cos 2x + \frac{\sin 2x}{8}\right)$$

Thus,
$$P.I. = \cos x\left(-\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x\right) + \sin x\left(-\frac{x}{4}\cos 2x + \frac{\sin 2x}{8}\right)$$

$$= -\frac{x^2}{4}\cos x + \frac{x}{4}(\sin 2x\cos x - \sin x\cos 2x) + \frac{1}{8}(\cos x\cos 2x + \sin x\sin 2x)$$

$$= -\frac{x^2}{4}\cos x + \frac{x}{4}\sin x + \frac{1}{8}\cos x \qquad \text{… (6)}$$

Therefore complete solution

$$y = (c_1\cos x + c_2\sin x) + \frac{1}{8}\cos x + \frac{x}{4}\sin x - \frac{x^2}{4}\cos x$$

Example 22: Solve $y'' - 2y' + 2y = e^x \tan x$ by variation of parameter. **[Burdwan, 2003]**

Solution: Rewrite the given equation, $(D^2 - 2D + 2)y = e^x\tan x$.

Here A.E becomes, $D^2 - 2D + 2 = 0$ $D = 1 \pm i$

Thus, $y_{C.F.}(x) = c_1y_1 + c_2y_2 = e^x[c_1\cos x + c_2\sin x]$ … (1)

Assume: $y_{P.I.}(x) = u_1(x)y_1 + u_2(x)y_2$ … (2)

By method of variation of parameter,

$$\left. u_1'(x) = -y_2\left(\frac{X}{W^*}\right),\ u_2'(x) = y_1\left(\frac{X}{W^*}\right)\right\};$$

$$W^* = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x\cos x & e^x\sin x \\ -e^x\sin x + e^x\cos x & e^x\sin x + e^x\cos x \end{vmatrix}$$

$$= \begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix}$$

Thereby on integration of u_1' & u_2', we get

$$y_{P.I.}(x) = y_1\int -y_2\frac{X}{W^*}\,dx + y_2\int y_1\frac{X}{W^*}\,dx$$

$$= y_1\int -e^x\sin x\frac{e^x\tan x}{e^{2x}}\,dx + y_2\int e^x\cos x\frac{e^x\tan x}{e^{2x}}\,dx$$

$$= -y_1\int \sin x\tan x\,dx + y_2\int\cos x\tan x\,dx = -y_1I_1 + y_2I_2 \quad \ldots (3)$$

$$I_1 = \int\tan x\sin x\,dx = -\tan x\cos x + \int\sec^2 x\cos x\,dx$$, integration by parts.

$$= -\sin x + \log(\sec x + \tan x) \quad \ldots (4)$$

$$I_2 = \int\sin x\,dx = -\cos x \quad \ldots (5)$$

On putting the values of integrals, (4) and (5) again into (3),

$$y_{PI} = -e^x\cos x[-\sin x + \log(\sec x + \tan x)] - e^x\sin x\cos x$$

$$= -e^x\cos x\log(\sec x + \tan x)$$

Hence the complete solution is

$$y = e^x(c_1\cos x + c_2\sin x) - e^x\cos x\log(\sec x + \tan x).$$

Example 23: Solve $(D^2 - 1)y = e^{-2x}\sin e^{-x}$ using method of variation of parameter.

Solution: The given equation $(D^2 - 1)y = e^{-2x}\sin e^{-x}$ … (1)

Auxiliary form $D^2 - 1 = 0$ or $D = \pm 1$

So that $y_{C.F.}(x) = c_1y_1 + c_2y_2 = (c_1e^x + c_2e^{-x})$ … (2)

Assume $y_{PI}(x) = u_1(x)y_1 + u_2(x)y_2$ … (3)

Now by the method of variation of parameter,

$$u_1' = -y_2\left(\frac{X}{W^*}\right),\ u_2' = -y_1\left(\frac{X}{W^*}\right);\ \text{where } X = e^{2x}\sin e^{-x}$$

And $$W^* = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Thereby giving

$$u_1 = -\int e^{-x} \frac{e^{-2x}\sin e^{-x}}{-2}\,dx = -\int \frac{t.t^2\sin t}{-2}\frac{dt}{-t} \qquad \text{(taking } e^{-x} = t\text{)}$$

$$= -\frac{1}{2}\int t^2(\sin t)\,dt = \frac{-1}{2}\left[t^2(-\cos t) - \int 2t(-\cos t)\,dt\right]$$

$$= -\frac{1}{2}\left[-t^2\cos t + 2t(\sin t) - 2\int(\sin t)\,dt\right]$$

$$= \frac{t^2}{2}\cos t - t\sin t - \cos t$$

$$= \frac{e^{-2x}}{2}\cos e^{-x} - e^{-x}\sin e^{-x} - \cos e^{-x} \qquad \dots (4)$$

And $$u_2 = -\int e^x \frac{e^{-2x}\sin e^{-x}}{-2}\,dx = \frac{1}{2}\int e^{-x}\sin e^{-x}dx$$

$$= \frac{1}{2}\int \sin t\,dt = \frac{-1}{2}\cos t = \frac{-1}{2}\cos e^{-x} \qquad \dots (5)$$

Whence using (4) and (5) in (3), we get

$$y_{P.I.}(x) = e^x\left\{\frac{e^{-2x}}{2}\cos e^{-x} - e^{-x}\sin e^{-x} - \cos e^{-x}\right\} + e^{-x}\left(\frac{-1}{2}\cos e^{-x}\right)$$

$$= -(\sin e^{-x} + e^x\cos e^{-x}) \qquad \dots (6)$$

And therefore complete solution becomes

$$y(x) = y_{CF}(x) + y_{PI}(x) = (c_1e^x + c_2e^{-x}) - (\sin e^{-x} + e^x\cos e^{-x}) \qquad \dots (7)$$

Example 24: Solve by the method of variation of parameter, $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$.

[VTU 2005; Hisar, 2005]

Solution: Here auxiliary equation becomes $D^2 - 1 = 0$, *means* $D = \pm 1$.

Thus $$y_{C.F.}(x) = c_1y_1 + c_2y_2 = c_1e^x + c_2e^{-x} \qquad \dots (1)$$

By method of variation of parameter, we assume the particular integral as

$$y_{PI}(x) = u_1y_1 + u_2y_2 = u_1e^x + u_2e^{-x}, \qquad \dots (2)$$

$$W^* = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

and $$u_1'(x) = -y_2\left(\frac{X}{W^*}\right),\ u_2'(x) = y_1\left(\frac{X}{W^*}\right), \qquad \dots (3)$$

Implying $u_1(x) = \int -e^{-x}\left(\frac{2}{1+e^x}\cdot\frac{1}{-2}\right)dx = \int \frac{e^{-x}}{1+e^x}\,dx$ (Taking $e^x = t,\ e^x dx = dt$)

$$= \int \frac{1}{t^2(1+t)}\,dt = \int \frac{1}{t^2}\,dt - \int \frac{1}{t}\,dt + \int \frac{1}{1+t}\,dt \qquad \text{(By partial fractions)}$$

$$= -\frac{1}{t} - \log t + \log(1+t) = \log\left(\frac{1+e^x}{e^x}\right) - e^{-x}$$

and $u_2 = -\int \frac{e^x}{1+e^x}\,dx = -\int \frac{f'(x)}{f(x)}\,dx = -\log\left(1+e^x\right)$

Whence, $y_{P.I.}(x) = \left[\log\left(\frac{1+e^x}{e^x}\right) - e^{-x}\right]e^x + \left[-\log\left(1+e^x\right)\right]e^{-x}$

And the complete solution becomes

$$y(x) = y_{C.F.} + y_{P.I.} = \left(c_1 e^x + c_2 e^{-x}\right) + e^x \log\left(\frac{1+e^x}{e^x}\right) - e^{-x}\log\left(1+e^x\right) - 1.$$

Example 25: It is given that $y_1 = x$ and $y_2 = \frac{1}{x}$ are two linear independent solution of the associated homogeneous equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x,\ x \neq 0.$ Find particular integral and the general solution. [NIT Kurukshetra, 2010]

Solution: Rewrite the given equation as $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x}, \quad x \neq 0.$ … (1)

Further, we are given $y_{C.F.}(x) = c_1 y_1 + c_2 y_2 = c_1 x + c_2 \frac{1}{x}$ … (2)

By method of Variation of Parameter,

$$y_{P.I.}(x) = u_1(x)y_1 + u_2(x)y_2 = u_1 x + u_2\frac{1}{x} \qquad \text{… (3)}$$

The Wronskian of $y_1 = x$ and $y_2 = \frac{1}{x}$ is $W^* = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x},\ x \neq 0$

$$u_1'(x) = -y_2\left(\frac{X}{W^*}\right) \quad \text{and} \quad u_2'(x) = y_1\left(\frac{X}{W^*}\right) \quad \text{with} \quad X = \frac{1}{x}$$

Implying $u_1(x) = -\int \frac{1}{x}\left(\frac{1}{x}\cdot\frac{x}{-2}\right)dx = \frac{1}{2}\log x$

$$\left.\begin{aligned} u_2(x) &= \int x\left(\frac{1}{x}\cdot\frac{x}{-2}\right)dx = -\frac{x^2}{4} \\ \text{Whence}\quad y_{P.I.}(x) &= \left(\frac{1}{2}\log x\right)x + \left(\frac{-x^2}{4}\right)\left(\frac{1}{x}\right) \end{aligned}\right\} \quad \ldots (4)$$

And complete solution,

$$y(x) = \left(c_1 x + c_2\frac{1}{x}\right) + \left\{\frac{x}{2}\log x - \frac{x}{4}\right\} = c_1^* x + \frac{c_2}{x} + \frac{x}{2}\log x,\ c_1^* = \left(c_1 - \frac{1}{4}\right).$$

II Method Of Undetermined Coefficients:

Under this method *which is applicable only to equations with constant coefficients*, we find $y_{\text{P.I.}}(x)$ of $f(D)y = X(x)$ *by assuming a trial solution containing unknown constants* which are determined by substituting it and its derivatives in the given equation. **Here, assumed** $y_{P.I.}(x)$ **depends on the form of $X(x)$ in each case**. In cases, when the right hand side of the equation viz. $X(x)$ is of the form containing *exponential, polynomial, cosine and sine functions, sums or product of these functions*, then particular integral can easily be found by this method. The accompanying table illustrates the procedure as:

$X(x)$	Assumed $y_{P.I.}(x)$
e^{ax}	Ae^{ax}
$\cos\beta x$ or $\sin\beta x$	$(A\sin\beta x + B\cos\beta x)$
x^n	$(A_0x^n + A_1x^{n-1} + \ldots\ldots + A_{n-1}x + A_n)$
$e^{ax}x^n$	$e^{ax}(A_0x^n + A_1x^{n-1} + \ldots\ldots + A_{n-1}x + A_n)$
$e^{ax}\cos\beta x$ or $e^{ax}\sin\beta x$	$e^{ax}(A\sin\beta x + B\cos\beta x)$

This method holds so long as no term of the assumed Particular Integral appears in Complementary Function. If any term of the assumed expression (P.I.) is present in the complementary function, then multiply the assumed particular integral by x repeatedly until no terms of product is present in the complementary function. It fails, when $X(x) = \tan x$, $\sec x$, $\cot x$, since the differentiation of $X(x)$ results in infinite number of terms.

Example 26: Using the method of undetermined coefficients, solve the differential equation: $(D^2 - 2D + 3)y = x^3 + \cos x$

Solution: The corresponding auxiliary equation is

$$D^2 - 2D + 3 = 0 \quad\Rightarrow\quad D = \frac{2 \pm \sqrt{4-12}}{2} \quad i.e.\quad D = 1 \pm i\sqrt{2}$$

$$\therefore \qquad C.F. = e^x\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right) \quad \ldots (1)$$

Supposing the particular integral

$$y = \left(a_1x^3 + a_2x^2 + a_3x + a_4\right) + \left(a_5\cos x + a_6\sin x\right) \quad \ldots (2)$$

So that
$$\frac{dy}{dx} = \left(3a_1x^2 + 2a_2x + a_3\right) + \left(-a_5\sin x + a_6\cos x\right), \quad \ldots (3)$$

$$\frac{d^2y}{dx^2} = (6a_1x + 2a_2) + (-a_5 \cos x - a_6 \sin x), \qquad \ldots (4)$$

On substituting above values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$\left[(6a_1x + 2a_2) + (-a_5 \cos x - a_6 \sin x)\right] - 2\left[(3a_1x^2 + 2a_2x + a_3) + (-a_5 \sin x + a_6 \cos x)\right]$$

$$+3\left[(a_1x^3 + a_2x^2 + a_3x + a_4) + (a_5 \cos x + a_6 \sin x)\right] = x^3 + \cos x \qquad \ldots (5)$$

Comparing the coefficients of x^3, x^2, x, constant term, $\cos x$ and that of $\sin x$ on both sides of equation (5) we get

$3a_1 = \frac{1}{3}$	$3a_2 - 6a_1 = 0$	$6a_1 - 4a_2 + 3a_3 = 0$	$2a_2 - 2a_3 + 3a_4 = 0$	$2a_5 - 2a_6 = 1,$
$\Rightarrow a_1 = \frac{1}{3}$	$\Rightarrow a_2 = \frac{2}{3}$	$\Rightarrow \frac{6}{3} - \frac{8}{3} + 3a_3 = 0$	$\Rightarrow \frac{4}{3} - \frac{4}{9} + 3a_4 = 0$	$a_5 + a_6 = 0$
		i.e. $a_3 = \frac{2}{9}$	i.e. $a_4 = -\frac{8}{27}$	i.e. $a_5 = \frac{1}{4}, a_6 = -\frac{1}{4}$

With the above values of a_1, a_2, a_3, a_4, a_5 and a_6, particular integral becomes

$$y(P.I.) = \left(\frac{x^2}{3} + \frac{2}{3}x^2 + \frac{2}{9}x - \frac{8}{27}\right) + \left(\frac{1}{4}\cos x - \frac{1}{4}\sin x\right)$$

$$= \frac{1}{27}(9x^2 + 18x^2 - 6x - 8) + \frac{1}{4}(\cos x - \sin x) \qquad \ldots (6)$$

Hence the complete solution,

$$y = e^x(c_1 \cos\sqrt{2x} + c_2 \sin\sqrt{2}x) + \frac{1}{27}(9x^2 + 18x^2 - 6x - 8) + \frac{1}{4}(\cos x - \sin x)$$

Example 27: Solve by Method of undetermined, the equation $(D^2 - 2D)y = e^x \sin x$
[VTU, 2006; NIT Kurukshetra, 2009]

Solution: Here $y_{C.F.}(x) = (c_1 + c_2 e^{2x})$ $\qquad \ldots (1)$

Assume $\quad y_{PI}(x) = e^x(A \cos x + B \sin x)$ $\qquad \ldots (2)$

$$Dy = \frac{dy}{dx} = e^x(-A \sin x + B \cos x) + e^x(A \cos x + B \sin x) \qquad \ldots (3)$$

$$= e^x\left[(A+B)\cos x + (B-A)\sin x\right]$$

$$D^2y = \frac{d^2y}{dx^2} = e^x\left[-(A+B)\sin x + (B-A)\cos x\right] + e^x\left[(A+B)\cos x + (B-A)\sin x\right]$$

$$= e^x(2B\cos x - 2A\sin x) \qquad \dots (4)$$

On substituting values D^2y, Dy and y in the given equation, we get

$$e^x(2B\cos x - 2A\sin x) - 2e^x\left[(A+B)\cos x + (B-A)\sin x\right] = e^x \sin x \qquad \dots (5)$$

Comparing the coefficient of $e^x\cos x$, $e^x\sin x$ on both sides,

$$\left.\begin{aligned} &2B - 2(A+B) = 0 \quad \Rightarrow \quad A = 0 \\ \text{and} \quad &-2A - 2(B-A) = 1 \quad \Rightarrow \quad B = -\frac{1}{2} \end{aligned}\right\} \qquad \dots (6)$$

Whence, $y_{PI}(x) = -\frac{1}{2}e^x \sin x$ and complete solution is

$$y(x) = \left(c_1 + c_2 e^{2x}\right) - \frac{1}{2}e^x \sin x$$

Example 28: Solve $(D^2 - 2D + 1)y = e^x$ by method of undetermined coefficients.
[NIT Kurukshetra, 2006]

Solution: Here $y_{C.F.} = c_1e^x + c_2xe^x$ Since the right hand member of the complete equation is $X(x) = e^x$, therefore the first possible assumption for particular integral would be $y_{P.I.} = Ae^x$, but this solution has already occurred in $y_{C.F.}$ We, therefore, $y_{P.I.}(x) = Axe^x$. Still this solution also has appeared in $y_{C.F.}$ We, therefore, again multiply by x and try $y_{P.I.}(x) = Ax^2e^x$

Thus $$\frac{dy}{dx} = A\left[2xe^x + x^2e^x\right] = A(2x + x^2)e^x$$

And $$\frac{d^2y}{dx^2} = A\left[\left(2e^x + 2xe^x\right) + \left(2xe^x + x^2e^x\right)\right] = A\left(x^2 + 4x + 2\right)e^x$$

On substituting values of D^2y, Dy and y in the given equation, we get

$$A(x^2 + 4x + 2)e^x - 2A(2x + x^2)e^x + Ax^2e^x = e^x$$

or $2Ae^x = e^x$ implying $A = \frac{1}{2}$ $(e^x \neq 0)$

Whence, $y_{P.I.}(x) = \frac{1}{2}x^2e^x$

and complete solution becomes $y(x) = c_1e^x + c_2e^{-x} + \frac{1}{2}x^2e^x$.

III The Complete Solution in Terms of a known Integral

OR

Solution: By Method of Reducing the Order of the Equation

If an integral included in the complementary function of a second order equation,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = X \qquad \dots (1)$$

be known, then the complete solution can be found.

Let $y = u(x)$ be a known solution (i.e. **complementary function**) of the homogeneous equation corresponding to (1).

$$\therefore \qquad \frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = 0 \qquad \ldots (2)$$

Now, let $y = u(x)\,v(x)$ **be the general solution** of (1), so that

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} \qquad \ldots (3)$$

On substituting these values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ into equation (1) and re-arranging the terms,

we get. $$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = X$$

or $$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} = X \text{, using (2)}$$

or $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{X}{u} \qquad \ldots (4)$$

If we take $\frac{dv}{dx} = p,$ equation (4) reduces to

$$\frac{dp}{dx} + \left(P + \frac{2}{u}\frac{du}{dx}\right)p = \frac{X}{u} \qquad \ldots (5)$$

Clearly, this equation being a Leibnitz Linear differential equation of 1st order can be solved for 'p', where $p = \frac{dv}{dx}$ which on integration w.r.t. x results in an expression for v.

$\therefore$ $y = uv$ gives a solution of equation (1).

Note: Complete solution corresponding to equation $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = X(x)$ can be found by the following rules:

1. $y = x$ is the part of the C.F. if $P + xQ = 0$
2. $y = x^2$ is the part of the C.F. if $2 + 2xP + x^2Q = 0$
3. $y = e^x$ is the part of the C.F. if $1 + P + Q = 0$
4. $y = e^{-x}$ is the part of the C.F. if $1 - P + Q = 0$
5. $y = e^{ax}$ is the part of the C.F. if $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$

Example 29: Solve $y'' - 2y' + y = e^x \sin x$ given that $y = e^x$ is a solution corresponding to homogeneous equation.

Solution: The given equation is comparable to $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=X$ with $P=-2$, $Q=1$ so that $1+P+Q=0$, means $y=e^x$ is a part of the complementary function. Let $y=ve^x$ be a solution of the given equation.

Thus, $$\frac{dy}{dx}=ve^x+e^x\frac{dv}{dx},\quad \frac{d^2y}{dx^2}=ve^x+2e^x\frac{dv}{dx}+e^x\frac{d^2v}{dx^2}$$

On using these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the given equation,

$$\left(ve^x+2e^x\frac{dv}{dx}+e^x\frac{d^2v}{dx^2}\right)-2\left(ve^x+e^x\frac{dv}{dx}\right)+ve^x=e^x\sin x$$

On simplification, $e^x\frac{d^2v}{dx^2}=e^x\sin x$ or $\frac{d^2v}{dx^2}=\sin x\ (e^x\neq 0)$

Which on integration implies, $\frac{dv}{dx}=-\cos x+c_1$

Further, $v=-\sin x+c_1x+c_2$

$\therefore$ The complete solution, $y=e^x(-\sin x+c_1x+c_2)$

Example 30: Apply Method of Variation of Parameter to solve the equation

$$(1-x)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=(1-x^2)$$

Solution: Rewrite the given equation as

$$\frac{d^2y}{dx^2}+\frac{x}{(1-x)}\frac{dy}{dx}-\frac{1}{(1-x)}y=(1+x) \quad \ldots (1)$$

Here $P+xQ=0$, with $P=\frac{x}{1-x}$ and $Q=-\frac{1}{1-x}$; means $y=x$ is a part of the complementary function. Thus for finding complementary function of the equation (1),

Put $y=vx$ so that equation (1), reduces to

$$\frac{d^2v}{dx^2}+\left(\frac{x}{(1-x)}+\frac{2}{x}\right)\frac{dv}{dx}=0 \quad \ldots (2)$$

or $$\frac{dp}{dx}-\left(\frac{-x}{1-x}-\frac{2}{x}\right)p=0,\quad \text{where}\quad p=\frac{dv}{dx}$$

or $$\frac{dp}{dx}-\left(1+\frac{1}{1-x}-\frac{2}{x}\right)p=0 \quad\Rightarrow\quad \frac{dp}{p}=\left(1+\frac{1}{x-1}-\frac{2}{x}\right)dx$$

(case of variable-separable)

On integration, $\log p=x+\log(x-1)-2\log x+\log c_1$

implying $$p = \frac{c_1(x-1)e^x}{x^2} \qquad \ldots(3)$$

i.e. $$\frac{dv}{dx} = c_1\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) \text{ again a case of variable-separable,} \qquad \ldots(4)$$

$$v = c_1\left(\int \frac{e^x}{x}dx \; - \int \frac{e^x}{x^2}dx\right) + c_2 = \left(\frac{c_1}{x}e^x + c_2\right) \qquad \ldots (5)$$

$\therefore$ $\qquad y(C.F) = vx = (c_1e^x + c_2x)$ $\qquad \ldots (6)$

Now let, $\quad y(P.I.) = u_1y_1 + u_2y_2$,

where u_1 and u_2 are functions of x and $y_1 = e^x$, $y_2 = x$.

Now under this method $u_1{}' = -y_2\left(\frac{X}{W^*}\right)$ and $u_2{}' = y_1\left(\frac{X}{W^*}\right)$

where $$W^* = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x(1-x); \quad \text{and} \quad X = (1-x)$$

$\therefore$ $$u_1 = -\int y_2\left(\frac{X}{W^*}\right)dx = -\int x \cdot \frac{(1-x)}{e^x(1-x)}dx = \int xe^{-x}dx = (x+1)e^{-x} \qquad \ldots (7)$$

And $$u_2 = \int y_1\left(\frac{X}{W^*}\right)dx = \int e^x \cdot \frac{(1-x)}{e^x(1-x)}dx = x \qquad \ldots (8)$$

$\therefore$ $\qquad y_{(P.I.)} = u_1y_1 + u_2y_2 = (x+1)e^{-x}e^x + x^2 = (1 + x + x^2)$

And complete solution $y = (c_1e^x + c_2x) + (1 + x + x^2)$

Example 31: Solve $\dfrac{d^2y}{dx^2} + (1 - \cot x)\dfrac{dy}{dx} - y\cot x = \sin^2 x$.

Solution: The given equation is comparable to

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = \sin^2 x \qquad \ldots (1)$$

Here, $\qquad P = 1 - \cot x, \quad Q = -\cot x \quad$ and $\quad 1 - P + Q = 0.$

Therefore, $y = ve^{-x}$ is complementary solution of the equation,

$$\frac{d^2y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y\cot x = 0 \qquad \ldots (2)$$

Take, $\qquad y = ve^{-x}$ so that $$\left.\begin{aligned} \frac{dy}{dx} &= e^{-x}\left(\frac{dv}{dx} - v\right) \\ \frac{d^2y}{dx^2} &= e^{-x}\left(\frac{d^2v}{dx^2} - 2\frac{dv}{dx} + v\right) \end{aligned}\right\}$$

By this substitution, the given equation reduces to

$$\frac{d^2v}{dx^2} = (1+\cot x)\frac{dv}{dx} \quad \text{or} \quad \frac{dz}{dx} = (1+\cot x)z \quad \text{for} \quad \frac{dv}{dx} = z \qquad \ldots (3)$$

Implying $\displaystyle\int \frac{dz}{z} = \int (1+\cot x)\,dx$ or $\log(c_1 z) = x + \log\sin x$

or
$$c_1 \frac{dv}{dx} = (e^x \sin x) \qquad \ldots (4)$$

Integrating, $c_1\int dv = \int e^x \sin x\,dx$ or $v = \dfrac{e^x}{2c_1}(\sin x - \cos x) + \dfrac{c_2}{c_1}$ $\qquad \ldots (5)$

$\therefore$
$$\frac{y}{e^{-x}} = \left[\frac{e^x}{2c_1}(\sin x - \cos x) + \frac{c_2}{c_1}\right], \quad \text{as } y = ve^{-x}$$

or
$$y_{C.F.} = a_1(\sin x - \cos x) + a_2 e^{-x}; \quad a_1 = \frac{1}{2c}, \; a_2 = \frac{c_2}{c_1} \qquad \ldots (6)$$

Let $y_{P.I.}(x) = u_1y_1 + u_2y_2$, *where* u_1 *and* u_2 are functions of 'x'

(i.e. replace arbitrary constants in C.F by $u_1(x)$ and $u_2(x)$.)

Then by method of variation of parameter,

$$u_1' = -y_2\left(\frac{X}{W^*}\right); \; u_2' = y_1\left(\frac{X}{W^*}\right), \qquad \ldots (7)$$

$$W^* = \begin{vmatrix} \sin x - \cos x & e^{-x} \\ \cos x + \sin x & -e^{-x} \end{vmatrix} = -2e^{-x}\sin x \quad \text{and} \quad X = \sin^2 x \qquad \ldots (8)$$

Now
$$u_1 = -\int y_2 \frac{X}{W^*}\,dx = -\int e^{-x}\frac{\sin^2 x}{-2e^{-x}\sin x}\,dx = \frac{1}{2}\int \sin x\,dx = \frac{\cos x}{2} \qquad \ldots (9)$$

And,
$$u_2 = \int y_1 \frac{X}{W^*}\,dx = \int (\sin x - \cos x)\frac{\sin^2 x}{-2e^{-x}\sin x}\,dx$$

$$= -\frac{1}{2}\int e^x(\sin^2 x - \sin x \cos x)\,dx$$

$$= -\frac{1}{2}\int \frac{(1-\cos 2x)}{2}e^x dx + \frac{1}{4}\int e^x \sin 2x\,dx$$

$$= \frac{e^x}{20}(\cos 2x + 2\sin 2x) - \frac{e^x}{20}(2\cos 2x - \sin 2x) - \frac{-e^x}{4} \qquad \ldots (10)$$

On substituting values of u_1, u_2, and y_1, y_2 particular integral becomes.

$$y_{P.I.} = \frac{\cos x}{2}(\sin x - \cos x) + e^{-x}\left[\frac{e^x}{20}(\cos 2x + 2\sin 2x) - \frac{e^x}{20}(2\cos 2x - \sin 2x) - \frac{-e^x}{4}\right]$$

and, $\quad y(complete) = y_{C.F.} + y_{P.I.}$

ASSIGNMENT 3

1. Using Method of Variation of Parameter, solve the following equations

(*i*) $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x^2$; (*ii*) $(D^2 - 2D + 2)y = e^x \operatorname{cosec} x$;

(*iii*) $\left(D^2 - 4D + 4\right)y = 8x^2 e^x \sin 2x$; (*iv*) $y'' - 2y' + 2y = e^x \tan x$;

(*v*) $(D^2 - D - 2)y = e^{(e^x + 3x)}$; (*vi*) $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$; **[PTU, 2006]**

2. Apply Method of variation of parameter to the following equations:

(*i*) $x^2y'' - 4xy' + 6y = x^2 \log x$ (*ii*) $x^2y_2 + xy_1 - y = x^2e^x$

[**Hint**: Either follow example 27 or reduce the given Cauchy equation with variable coefficient to equations with constant coefficients by putting $x = e^t$ and then apply Method of variation].

3. Solve by Method of variation of parameter, the equation

$$x^2 \frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

[**Hint:** $y = x$ is a part of C.F., take $y = vx$. as y (C.F.)]

4. Using method of undetermined coefficients, solve the following equations:

(*i*) $(D^2 - 3D + 2)y = 10$; (*ii*) $\frac{d^2y}{dx^2} + y = x$;

(*iii*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = -4e^x$; (*iv*) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = xe^x + \sin x$;

5. Use Method of undetermined coefficient to find solution of

(*i*) $(D^2 - 3D + 2)y = x^2 + e^x$ (*ii*) $(D^2 - 1)y = x \sin 3x + \cos x$

[KUK, 2008]

6. Solve $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1 - \cot x)y = e^x \sin x$; by reducing the order of integration .

[Type III, complete solution in terms of known integral.]

9.7 SOLUTION OF EULER–CAUCHY AND LEGENDRE: LINEAR EQUATIONS

These are two special type of homogeneous linear differential equations *with variable coefficients* which on application of certain transformation reduce to homogeneous linear differential equations with constant coefficients. Cauchy–Euler equation is more commonly known as Cauchy's equation. The details of the two are as follows:

I Cauchy's Linear Equation

The differential equation of the form

$$K_0x^n \frac{d^ny}{dx^n} + K_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + K_{n-1}x\frac{dy}{dx} + K_ny = X \quad \ldots (1)$$

where $K_0, K_1, \ldots\ldots, K_n$ are constants and X is a function of x only, known as Cauchy's linear equation. Such equations can be reduced to linear differential equation with constant coefficients by putting.

$$x = e^t \quad \text{or} \quad t = \log x \quad \text{so that} \quad \frac{dt}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x} \quad \text{or} \quad x\frac{dy}{dx} = \frac{dy}{dt} = Dy. \quad \text{if} \quad \frac{d}{dt} = D.$$

Again, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\frac{dt}{dx}\left(\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

or $$x^2\frac{d^2y}{dx^2} = D^2y - Dy = D(D-1)y \quad \text{and so on.}$$

By substituting all these values in (1), we obtain linear equation with constant coefficients, which has already been discussed.

Example 32: Solve the differential equation **[Raipur, 2004]**

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x), \qquad x > 0$$

Solution: The equation $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x),\ x > 0$ represents Cauchy's Homogeneous Linear Equation. In this case, we substitute, $x = e^t$

so that $$x\frac{dy}{dx} = Dy;\ x^2\frac{d^2}{dx^2}y = D(D-1)y \quad \text{and so on. for } \frac{d}{dt} = D;$$

The given equation becomes,

$$D(D-1)(D-2)y + 3D(D-1)y + Dy + 8y = 65\cos t$$

$$\Rightarrow \qquad [D^3 + 8]y = 65\cos t$$

A.E. $\qquad D^3 + 8 = 0 \qquad \Rightarrow \qquad D = -2,\ 1 \pm \sqrt{3}i$

Hence the complementary function $(C.F) = c_1e^{-2t} + e^t\left(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t\right)$

Particular Integral, $P.I. = \frac{1}{(D^3+8)}65\cos t = \frac{1}{-D+8}65\cos t$

$$= 65\frac{1}{8-D}\cos t = 65\frac{8+D}{64-D^2}\cos t$$

$$= 65\frac{(8+D)\cos t}{64+1} = (8\cos t - \sin t),\ t = \log x$$

Example 33: **Solve** $\boldsymbol{x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2}$. **[SVTU, 2007]**

Solution: Here we substitute $x = e^t$ so that $\log x = t$ and $\frac{dt}{dx} = \frac{1}{x}$

If $\frac{d}{dt} = D$; $x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = D(D-1)y$ and so on.

Therefore the given equation reduces to $[D(D-1) - 3D + 4]y = (1 + e^t)^2$

or $(D^2 - 4D + 4)y = 1 + 2e^t + e^{2t}$. …(1)

A.E. $D^2 - 4D + 4 = 0$ *i.e.* $D = 2, 2$ so that

$$y_{C.F.}(x) = (c_1 + c_2 t)e^{2t} = (c_1 + c_2 \log x)x^2 \quad \ldots(2)$$

and
$$y_{PI}(x) = \frac{1}{D^2 - 4D + 4}\left(1 + 2e^t + e^{2t}\right) = PI_1 + PI_2 + PI_3$$

$$PI_1 = \frac{1}{D^2 - 4D + 4}e^{0t} = \frac{1}{4}, \quad \text{(replace } D \text{ by 0)}$$

$$PI_2 = 2\frac{1}{(D-2)^2}e^t = 2\frac{1}{(1-2)^2}e^t = 2e^t = 2x.$$

$$PI_3 = \frac{1}{(D-2)^2}e^{2t} = t\frac{1}{2(D-2)}e^{2t} = \frac{t^2}{2}\frac{e^{2t}}{1} = \frac{(\log x)^2 x^2}{2}.$$

Whence complete solution becomes,

$$y(x) = y_{C.F.} + y_{P.I.} = (c_1 + c_2 \log x)x^2 + \frac{1}{4} + 2x + \frac{x^2}{2}(\log x)^2$$

Example 34: **Solve** $\boldsymbol{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = (\log x)\sin(\log x), x > 0}$

[KUK 2006, Madras 2006; Kerala, 2005; Raipur, 2005]

Solution: This is a Cauchy's linear homogeneous equation, where

$$x\frac{dy}{dx} = Dy, \qquad x^2\frac{d^2y}{dx^2} = D(D-1)y, \ldots \text{ with } D = \frac{d}{dt}$$

Given equation reduces to

$$[D(D-1) + D + 1]y = t\sin t \quad \text{or} \quad (D^2 + 1)y = t\sin t \quad \ldots(1)$$

$$\therefore \quad y_{C.F.}(x) = (c_1\cos t + c_2 \sin t) \quad \ldots(2)$$

And, $$y_{P.I.}(x) = \frac{1}{D^2+1}t\sin t = \text{Imaginary } \frac{1}{D^2+1}te^{it} = \text{Imaginary } e^{it}\frac{1}{(D+i)^2+1}t$$

$$= \text{Img. } e^{it} \frac{1}{2iD\left(1+\frac{D}{2i}\right)} t = \text{Img. } \frac{e^{it}}{2i}\frac{1}{D}\left(1+\frac{D}{2i}\right)^{-1} t$$

$$= \text{Img.} \frac{e^{it}}{2i}\frac{1}{D}\left(t-\frac{1}{2i}\right) = \text{Img.} \frac{i\,e^{it}}{-2}\left(\frac{t}{D}-\frac{1}{D}\frac{1}{2i}\right)$$

$$= \text{Img } \frac{i \quad e^{it}}{-2}\left(\frac{t^2}{2}+\frac{i^2 t}{2i}\right) = \text{Img } \frac{i\,e^{it}}{-2}\left(\frac{t^2}{2}+\frac{it}{2}\right)$$

$$= \text{Img } \frac{i(\cos t + i\sin t)}{-4}\left(t^2+it\right)$$

$$= \frac{t}{4}(\sin t - t\cos t) \qquad \ldots(3)$$

Therefore, complete solution

$$y(x) = y_{CF} + y_{PI} = \left(c_1 \cos t + c_2 \sin t\right) + \frac{t}{4}(\sin t - t\cos t),\ t = \log x \qquad \ldots(4)$$

II Legendre's Linear Differential Equation:

The differential equations of the form

$$K_0\left(ax+b\right)^n \frac{d^n y}{dx^n} + K_1\left(ax+b\right)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots\ldots + K_{n-1}\left(ax+b\right)\frac{dy}{dx} + K_n y = X$$

where $K_0, K_1, \ldots, K_n$ are constants and X is a function of x only, known as Legendre's Equation. Such equation can be reduced to linear differential equation with constant coefficients by putting

$$(ax + b) = e^t \quad \text{or} \quad t = \log(ax + b) \text{ so that } \frac{dt}{dx} = \frac{a}{ax+b}.$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{a}{ax+b} \quad \text{or} \quad (ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy, \quad \text{if } \frac{d}{dt} = D$$

Again,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right) = -\frac{a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\frac{dt}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a^2}{(ax+b)^2}\frac{d^2y}{dt^2}$$

or $$(ax+b)^2\frac{d^2y}{dx^2} = a^2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) = a^2D(D-1)y \quad \text{and} \quad \text{so on}$$

By substituting all these values in (2), we obtain linear equation with constant coefficients which has already been discussed.

Example 35: Solve $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=4\cos(\log(1+x))$

[MDU, 2005; NIT Kurukshetra, 2010]

Solution: As the given equation is Legendre's Linear equation. Here we take $(1 + x) = e^t$

$$\left.\begin{aligned}&(1+x)\frac{d}{dx}y = Dy,\\&(1+x)^2\frac{d^2}{dx^2}y = D(D-1)y,\\&(1+x)^2\frac{d^3}{dx^3}y=(D)(D-1)(D-2)y,\ldots\text{so on}\end{aligned}\right\}\quad\text{where}\quad D=\frac{d}{dt}$$

The given equation reduces to

$$D(D-1)y + Dy + y = 4\cos t \quad\Rightarrow\quad (D^2+1)y = 4\cos t,\quad t=\log(1+x)$$

A.E. $\quad D^2 + 1 = 0 \quad\Rightarrow\quad D = \pm i$

$\therefore\quad C.F. = (c_1\cos t + c_2\sin t),$

$$P.I.=\frac{1}{D^2+1}4\cos t=4t\frac{1}{2D}\cos t=2t\frac{D}{D^2}\cos t=2t\frac{D}{-1}\cos t=2t\sin t$$

Complete solution, $y = (c_1\cos t + c_2\sin t) + 2t\sin t,\quad t = \log(1 + x).$

Example 36: Solve $[(3x + 2)^2D^2 + 3(3x + 2)D - 36]y = (3x^2 + 4x + 1)$ **[Sambalpur, 2002]**

Solution: Legendre's equation $((3\mathbf{x} + 2)^2D^2 + 3(3x + 2)D - 36)y = (3x^2 + 4x + 1)$ … (1)

Take $\quad (3x + 2) = e^t \quad$ so that $\quad t = \log(3x + 2)\quad$ and $\quad \dfrac{dt}{dx}=\dfrac{3}{3x+2}$ … (2)

$$\therefore\quad \frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{3}{3x+2}\frac{dy}{dt}=\frac{3}{3x+2}\delta y,\quad\text{where}\quad \delta=\frac{d}{dt}$$

or $\quad (3x+2)\dfrac{dy}{dx}=3\delta y$ … (3)

Similarly $\quad \dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\dfrac{3}{3x+2}.\dfrac{dy}{dt}\right)=\left(\dfrac{-9}{(3x+2)^2}\right)\dfrac{dy}{dt}+\dfrac{9}{(3x+2)}\dfrac{d^2y}{dt^2}$

or $\quad (3x+2)^2\dfrac{d^2y}{dx^2}=9\left(\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\right)=9\delta(\delta-1)y$ … (4)

Therefore, the given equation reduces to

$$9\delta(\delta-1)y+3(3\delta y)-36y=\frac{1}{3}(9x^2+12x+3)$$

Implying $\quad (9\delta^2-36)y=\dfrac{(3x+2)^2-1}{3}$

$$9\left(\delta^2 - 4\right)y = \frac{e^{2t} - 1}{3}; \text{ as } \left(3x + 2\right) = e^t \qquad \dots (5)$$

A.E. $\delta^2 - 4 = 0$ or $\delta = \pm 2$ and thus

$$y_{C.F.}(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{2t} + c_2 e^{-2t}$$

And $$y_{PI}(t) = \frac{1}{f(\delta)} \frac{e^{2t} - 1}{3} = \frac{1}{3} \cdot \frac{1}{9}\left[\frac{1}{\left(\delta^2 - 4\right)} e^{2t} - \frac{1}{\delta^2 - 4} e^{0t}\right] = \frac{1}{27}\left[t \frac{e^{2t}}{2\delta} - \frac{e^{0t}}{-4}\right]$$

$$= \frac{1}{27}\left[\frac{t \quad e^{2t}}{4} + \frac{1}{4}\right] = \frac{te^{2t} + 1}{108}, \quad t = \log(3x + 2)$$

Complete solution, $y(t) = \left(c_1 e^{2t} + c_2 e^{-2t}\right) + \dfrac{te^{2t} + 1}{108}$

Example 37: Solve $\left(\mathbf{2x + 3}\right)^2 \dfrac{\boldsymbol{d^2y}}{\boldsymbol{dx^2}} - \left(\mathbf{2x + 3}\right) \dfrac{\boldsymbol{dy}}{\boldsymbol{dx}} - \mathbf{12y = 6x}$

[VTU, 2003, 2007; Kerala, 2005; KUK, 2005; NIT Kurukshetra, 2008]

Solution: Take $2x + 3 = e^t$, $t = \log(2x + 3)$ so that the given equation reduced to

$$[4D(D - 1) - 2D - 12]y = 3(e^t - 3) \text{ as } 6x = 3(2x) = 3(e^t - 3)$$

or $$2(2D^2 - 3D - 6)y = 3(e^t - 3)$$

A.E $2D^2 - 3D - 6 = 0$ or $D = \dfrac{3 \pm \sqrt{57}}{4} = m_1, m_2$

$$y_{CF}(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} = c_1 (e^t)^{m_1} + c_2 (e^t)^{m_2} = c_1 (e^t)^{\frac{3+\sqrt{57}}{4}} + c_2 (e^t)^{\frac{3-\sqrt{57}}{4}}$$

$$P.I. = \frac{1}{4D^2 - 6D - 12} 3\left(e^t - 3\right)$$

$$= 3 \frac{1}{4D^2 - 6D - 12} e^t - 9 \frac{1}{4D^2 - 6D - 12} e^{0t} = -\frac{3}{14} e^t + \frac{3}{4}$$

Whence $$y(x) = y_{CF}(x) + y_{PI}(x) = c_1 \left(2x + 3\right)^{m_1} + c_2 \left(2x + 3\right)^{m_2} - \frac{3}{14}\left(2x + 3\right) + \frac{3}{4}.$$

ASSIGNMENT 4

Solve the following equations

1. (*i*) $x^2 \dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + 6y = x^2;$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} - 4y = x^4;$

(*iii*) $x^2 \dfrac{d^2y}{dx^2} - \dfrac{2y}{x} = x + \dfrac{1}{x^2};$ (*iv*) $\dfrac{d^2y}{dx^2} + \dfrac{1}{x} \dfrac{dy}{dx} = \dfrac{12 \log x}{x^2}, \quad x > 0;$

(*v*) $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right);$ **[SVTU, 2006; PTU, 03]** (*vi*) $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2\log x$; $x > o$ **[Bhopal, 2003; KUK, 2010]**

(*vii*) The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2 \dfrac{d^2u}{dr^2} + r\dfrac{du}{dr} - u + kr^3 = 0,$ where k is a constant. Solve the equation under the conditions $u = 0$, when $r = 0$; $u = 0$ when $r = a$.

(*viii*) $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin(\log(1+x))$

(*ix*) $((1 + 2x)^2D^2 - 6(1 + 2x)D + 16)y = 8(1 + 2x)^2$

(x) $((x + 1)^2D^2 + (x + 1)D)y = (2x + 3)(2x + 4)$

2. Establish the Euler`s Cauchy equation of IIIrd order whose general solution is
$$y = Ax + Bx^2 + Cx^3.$$
3. Establish the Euler-Cauchy equation of IIIrd order whose general solution is
$$y = Ax + Bx(\log x) + Cx(\log x)^2$$

9.8 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Linear differential equations consisting of two or more dependent variables and single independent variable say $x(t)$, $y(t)$ etc. are called simultaneous differential equations. To solve these type of equations, eliminate one of the dependent variable and then solve the resulting equation which gives the solution of the dependent variable in terms of the independent variable. Repeat the process for 2nd dependent variable. In this section, we shall restrict our discussion to solutions of linear equations with constant coefficients only.

Example 38: Solve $\boldsymbol{\dfrac{dx}{dt} + 2y = e^t,\ \dfrac{dy}{dt} - 2x = e^{-t}.}$

Solution: The given equation in symbolic form can be written as

$$Dx + 2y = e^t \qquad \ldots(1)$$

$$Dy - 2x = e^{-t} \qquad \ldots(2)$$

Operate D on (2) and add to it 2 time of (1), we get

$$(D^2 + 4)y = 2e^t - e^{-t} \qquad \ldots (3)$$

Here A.E. is $D^2 + 4 = 0$ *i.e.* $D = \pm 2i$

$$\therefore \quad y_{C.F.}(t) = (c_1\cos 2t + c_2\sin 2t) \qquad \ldots (4)$$

And $$y_{P.I.}(t) = \frac{1}{D^2+4}\left(2e^t - e^{-t}\right) = 2\frac{1}{D^4+4}e^t - \frac{1}{D^2+4}e^{-t} = 2\frac{e^t}{5} - \frac{e^{-t}}{5} \qquad \ldots (5)$$

Hence $$y(t) = \left(c_1\cos 2t + c_2\sin t\right) + \left(\frac{2}{5}e^t - \frac{e^{-t}}{5}\right) \qquad \ldots(6)$$

Now from (2), $x = \frac{1}{2}\left[Dy - e^{-t}\right] = \frac{1}{2}\left[D\left(c_1 \cos 2t + c_2 \sin 2t\right) + D\left(\frac{2}{5}e^t - \frac{e^{-t}}{5}\right) - e^t\right]$

$$= \frac{1}{2}\left(-2c_1 \sin 2t + 2c_2 \cos 2t + \frac{2}{5}e^t + \frac{e^{-t}}{5} - e^{-t}\right)$$

$$= -c_1 \sin 2t + c_2 \cos 2t + \frac{e^t}{5} - \frac{2}{5}e^{-t} \qquad \dots (7)$$

Example 39: Solve $(D-1)x + Dy = 2t + 1,\ (2D+1)x + 2Dy = t$

Solution: For elimination of y, take difference of 2 time of 1st from 2nd i.e.

$$((2D + 1)x + 2Dy) - 2((D - 1)x + Dy) = t - 2(2t + 1)$$

or $\quad 3x = -3t - 2 \quad$ or $\quad x(t) = -t - \frac{2}{3} \quad$ Implying $\quad \frac{dx}{dt} = -1.$

Now using above values of $x(t)$ and $\frac{dx}{dt}$ in Ist equation, we get

$$Dx - x + Dy = 2t + 1 \quad \text{or} \quad -1 - \left(-t - \frac{2}{3}\right) + Dy = 2t + 1$$

Implying $\quad Dy = t + \frac{4}{3} \quad$ i.e. $\quad y(t) = \frac{t^2}{2} + \frac{4}{3}t + c$

where c is a constant of integration.

Example 40: Solve $\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0$

Solution: The given equations in symbolic form are written as:

$$(D - 7)x + y = 0 \qquad \dots(1)$$

$$-2x + (D - 5)y = 0 \qquad \dots(2)$$

To eliminate y, operate $(D-5)$ on (1) and add the two equations to get

$$(D - 5)(D - 7)x + 2x = 0 \quad \text{or} \quad (D^2 - 12D + 37)x = 0 \qquad \dots (3)$$

So that A.E. is $D^2 - 12D + 37 = 0$ or $D = 6 \pm i$

$\therefore \qquad x_{C.F.}(t) = e^{6t}(c_1 \cos t + c_2 \sin t) \qquad \dots (4)$

Implying $\quad \frac{dx}{dt} = e^{6t}\left(-c_1 \sin t + c_2 \cos t\right) + 6e^{6t}\left(c_1 \cos t + c_2 \sin t\right) \qquad \dots (5)$

On substituting values of $x(t)$ and Dx from (4) and (5) respectively in equation (1), we get

$$e^{6t}(-c_1 \sin t + c_2 \cos t) + 6e^{6t}(c_1 \cos t + c_2 \sin t) - 7e^{6t}(c_1 \cos t + c_2 \sin t) + y = 0$$

or $\quad y = e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t]$

$$= e^{6t}[C \cos t + D \sin t] \qquad \dots (6)$$

where, $\quad C = c_1 - c_2 \quad$ and $\quad D = c_1 + c_2$

Example 41: Solve the simultaneous differential equations

$$t\frac{dx}{dt}+y=0,\ t\frac{dy}{dt}+x=0 \text{ with conditions: } x(1)=1 \text{ and } y(-1)=0$$

Solution: $t\dfrac{dx}{dt}+y=0$ … (1)

$$t\frac{dy}{dt}+x=0 \quad \dots (2)$$

Differentiating (1) with recept to 't',

$$t\frac{d^2x}{dt^2}+\frac{dx}{dt}+\frac{dy}{dt}=0 \quad \text{or} \quad t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}+t\frac{dy}{dt}=0 \quad \dots (3)$$

Substituting the value of $t\dfrac{dy}{dt}$ from (2), equation (3) reduces to

$$t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}-x=0 \quad \dots(4)$$

Which is Cauchy's homogeneous linear differential equation in t.

Put $t=e^p$ or $\log t=p$ so that $\dfrac{dx}{dt}=\dfrac{dx}{dp}\dfrac{dp}{dt}=\dfrac{1}{t}\dfrac{dx}{dp}$ or $t\dfrac{dx}{dt}=\dfrac{dx}{dp}=Dx$

Likewise $t^2\dfrac{d^2x}{dt^2}=D(D-1)x$ and so on

Therefore (4) reduces to, $D(D-1)x+Dx-x=0$ or $(D^2-1)x=0$ …(5)

Whence $x_{C.F.}(p)=c_1e^p+c_2e^{-p}$ or $x(t)=\left(c_1t+c_2\dfrac{1}{t}\right)$ … (6)

Implying $\dfrac{dx}{dt}=c_1-\dfrac{c_2}{t^2}$ … (7)

From (1), we get $y=-t\dfrac{dx}{dt}=-t\left(c_1-\dfrac{c_2}{t^2}\right)=\left(-c_1t+\dfrac{c_2}{t}\right)$ … (8)

On using the conditions, $x(1)=1$ and $y(-1)=0$ in (6) & (7), we get $c_1=\dfrac{1}{2}=c_2$

Example 42: To small Oscillations of a certain system with two degrees of freedom are given by the equations

$$\left.\begin{array}{l} D^2x+3x-2y=0 \\ D^2x+D^2y-3x+5y=0 \end{array}\right\},\quad \text{where} \quad D=\frac{d}{dt}$$

If $x=0$, $y=0$; $Dx=0$, $Dy=2$ when $t=0$, find x and y when $t=1/2$.

Solution: Re–write the given set of simultaneous equations as

$$(D^2+3)x-2y=0 \quad \dots(1)$$

$$(D^2 - 3)x + (D^2 + 5)y = 0 \qquad \ldots(2)$$

In order to make above equations separately in $x(t)$ and $y(t)$ i.e. *to eliminate x*, first operate these equations by $(D^2 - 3)$ and $(D^2 + 3)$ respectively and then subtract (1) from (2), we get

$$[(D^2 + 3)(D^2 + 5) + 2(D^2 - 3)]y = 0$$

On simplification, $(D^4 + 10D^2 + 9)y = 0$ ... (3)

Corresponding auxiliary equation becomes

$D^4 + 10D^2 + 9 = 0$ or $(D^2 + 1)(D^2 + 9) = 0$ i.e. $D = \pm i, \pm 3i$

Thus, $y(C.F.) = (a_1 \cos t + a_2 \sin t) + (a_3 \cos 3t + a_4 \sin 3t)$ ... (4)

To find *x*, eliminate y from (1) and (2).

Operate (1) by $(D^2 + 5)$ and multiply (2) by 2 and then add the two,

$(D^4 + 10D^2 + 9)x = 0$ (an equation identical to (3)) i.e. $D = \pm i, \pm 3i$ ...(5)

Thus, $x(C.F.) = (b_1 \cos t + b_2 \sin t) + (b_3 \cos 3t + b_4 \sin 3t)$ ...(6)

To find the relation between constants involved in (4) and (6)

Substitute values of x and y in either of the given equations, say in (1), we get

$$2(b_1 - a_1)\cos t + 2(b_2 - a_2)\sin t - 2(3b_3 + a_3)\cos 3t - 2(3b_4 + a_4)\sin 3t = 0 \qquad \ldots(7)$$

Which must holds for all t.

On equating co–efficient of $\cos t$, $\sin t$, $\cos 3t$, $\sin 3t$, we get

$$b_1 = a_1, \qquad b_2 = a_2, \qquad b_3 = -\frac{a_3}{3}, \quad b_4 = -\frac{a_4}{3} \qquad \ldots(8)$$

Thus on replacing b_i's by respective a_i's, equation (6) becomes

$$x = (a_1 \cos t + a_2 \sin t) - \frac{1}{3}(a_3 \cos 3t + a_4 \sin 3t) \qquad \ldots(9)$$

Clearly, relations (4) and (9) are the solutions of the set of given equation.

In order to find out these constants, make use of the initial conditions.

Using $x = y = 0$ when $t = 0$, equation (4) and (9) gives,

$$\left.\begin{aligned} a_1 + a_3 &= 0 \\ a_1 - \frac{1}{3}a_3 &= 0 \end{aligned}\right\} \text{ implying } a_1 = a_3 = 0.$$

With above values of a_1 and a_3, equation (9) and (4) reduces to

$$\left.\begin{aligned} x &= a_2 \sin t - \frac{a_4}{3}\sin 3t \\ y &= a_2 \sin t + a_4 \sin 3t \end{aligned}\right\} \text{ and } \left.\begin{aligned} Dx &= a_2 \cos t - a_4 \cos 3t \\ Dy &= a_2 \cos t + 3a_4 \cos 3t \end{aligned}\right\} \qquad \ldots(10)$$

With the conditions $Dx = 3$ and $Dy = 2$ when $t = 0$, equations (10) give

$$\left.\begin{aligned} 3 &= a_2 - a_4 \\ 2 &= a_2 + 3a_4 \end{aligned}\right\} \text{ implying } a_2 = \frac{11}{4},\ a_4 = -\frac{1}{4}.$$

Hence from equation (10),

$$\left.\begin{aligned} x &= \frac{1}{4}\left(11\sin t + \frac{1}{3}\sin 3t\right) \\ y &= \frac{1}{4}(11\sin t - \sin 3t) \end{aligned}\right\}$$

Further when $t = \frac{1}{2}$, $\left.\begin{aligned} x &= \frac{1}{4}\left[11\sin(0.5) + \frac{1}{3}\sin(1.5)\right] = 1.4015 \\ y &= \frac{1}{4}\left[11\sin(0.5) - \sin(1.5)\right] = 1.069 \end{aligned}\right\}$

ASSIGNMENT 5

Solve the following simultaneous equations:

(*i*) $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$; [UP Tech, 2008] [KUK, 2005]

(*ii*) $\frac{dx}{dt} + 2x + 3y = 0$, $3x + \frac{dy}{dt} + 2y = 2e^{2t}$; [Delhi, 2002]

(*iii*) $(D+1)x + (2D+1)y = e^t$, $(D-1)x + (D+1)y = 1$;

(*iv*) $t\frac{dx}{dt} + y = 0$, $t\frac{dy}{dt} + x = 0$; given $\left.\begin{aligned} x(1) &= 1 \\ y(-1) &= 0 \end{aligned}\right\}$

(*v*) $\frac{d^2x}{dt^2} + y = \sin t$, $\frac{d^2y}{dt^2} + x = \cos t$; [NIT Jalandhar, 2006]

(*vi*) $\frac{dx}{dt} = 2y$, $\frac{dy}{dt} = 2z$, $\frac{dz}{dt} = 2x$

(*vii*) A mechanical system with two degrees of freedom satisfies the equation

$$2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} = 4, \quad 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$$

Obtain expression for x and y in terms of t, given x, $\frac{dy}{dx}$, $\frac{dy}{dt}$ all vanish at $t = 0$.

ANSWERS

Assignment 1

(*i*) $y = (c_1 + c_2x)e^{3x}$,

(*ii*) $y = e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$

(*iii*) $y = c_1e^x + e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$

(*iv*) $y = (c_1 + c_2x)e^{2x} + (c_3 + c_4x)e^{-2x}$

(*v*) $y = e^x\left(c_1\cos 3x + c_2\sin 3x\right),\ \left.\begin{matrix} c_1 = 4 \\ c_2 = -3 \end{matrix}\right\},$ (*vi*) $y = c_1 + (c_2 + c_3x)e^{-x/2}$

(*vii*) $y = (c_1 + c_2x)\cos nx + (c_3 + c_4x)\sin nx,$ (*viii*) $\theta = \alpha\cos\sqrt{g/\ell}$

Assignment 2

1. $y = \left(c_1\cos ax + c_2\sin ax\right) + \dfrac{x}{4a^2}\left(\cos ax + ax\sin ax\right)$
2. $\left(c_1e^x + c_2e^{-x}\right) - \dfrac{1}{2}\left(x^2 - 1\right)\cos x + x\sin x$
3. $(c_1\cos x + c_2\sin x) + [\sin\log(\sin x) - x\cos x]$
4. $y = \left(c_1\cos x + c_2\sin x\right) + \left(c_3e^x + c_4e^{-x}\right) + \dfrac{x}{8}(x\cos x - 3x\sin x)$
5. $y = \left(c_1\cos ax + c_2\sin ax\right) - \dfrac{1}{a^2}\cos ax\,\log\left(\sec ax + \tan ax\right)$
6. $y = \left(c_1\cos 2x + c_2\sin 2x\right) + \dfrac{e^x}{2}\left[\dfrac{1}{5} - \dfrac{1}{17}\left(4\sin 2x + \cos 2x\right)\right]$
7. $y = c_1e^{-x} + c_2e^{-3x} - \dfrac{e^{-x}}{5}(2\cos x + \sin x) + \dfrac{1}{3}\left(x - \dfrac{4}{3}\right)$
8. $y = \left(c_1e^x + c_2e^{-x}\right) - \dfrac{1}{2}(x\sin x + \cos x) + \dfrac{xe^x}{4}\left(x^2 - x + 3\right)$
9. $y = \left(c_1e^{2x} + c_2e^{-2x}\right) - \dfrac{x}{3}\sinh x - \dfrac{2}{7}\cosh x$
10. $y = e^{\frac{x}{\sqrt{2}}}\left(c_1\cos\dfrac{x}{\sqrt{2}} + c_2\sin\dfrac{x}{\sqrt{2}}\right) + e^{-\frac{x}{\sqrt{2}}}\left(c_3\cos\dfrac{x}{\sqrt{2}} + c_4\sin\dfrac{x}{\sqrt{2}}\right) + \dfrac{x^2 - 2}{2}\cos x - 2x\sin x$
11. $y = \left(c_1e^x + c_2e^{-x}\right) + \left(c_3\cos x + c_4\sin x\right) - \dfrac{1}{5}\cos x\cosh x$
12. $y = \left(c_1e^x + c_2e^{3x}\right) + \dfrac{1}{884}(10\cos 5x - 11\sin 5x) + \dfrac{1}{20}(\sin x + 2\cos x)$
13. $y = (c_1 + c_2x)e^x - e^x(x\sin x + \cos x)$
14. $y = \left(c_1\cos nx + c_2\sin nx\right) + \dfrac{e^x}{1 + n^2}\left\{x^2 + \dfrac{4x - 2}{1 + n^2} + \dfrac{8}{\left(1 + n^2\right)^2}\right\}.$

Assignment 3

1. (*i*) $y = (c_1\cos x + c_2\sin x) - x\cos x + \sin x(\log\sin x)$

(*ii*) $y = (c_1\cos x + c_2\sin x)e^x - xe^x\cos x + e^x\sin(\log\sin x)$

(*iii*) $y = (c_1 + c_2x)e^{2x} - e^{2x}[4x\cos 2x + (2x^2 - 3)\sin x]$

(*iv*) $y = e^x(c_1\cos x + c_2\sin x) - e^x(\cos x)\log(\sec x + \tan x)$

(*v*) $y = \left(c_1e^{2x} + c_2e^{-x}\right) + e^{e^x}\left(3e^x - 6 + e^{-x}\right)$

(*vi*) $y = (c_1 + c_2x)e^{3x} - e^{3x}\log x$

2. (*i*) $\left(c_1x^2 + c_2x^3\right) - \dfrac{x^2}{2}(\log x)^2 - x^2(\log x)$ (*ii*) $\left(c_1x + \dfrac{c_2}{x}\right) + \left(e^x - \dfrac{e^x}{x}\right)$

3. $y = \left(c_1x\,e^{2x} + c_2x\right) - \dfrac{x^2}{2}$

4. (*i*) $y = (c_1e^x + c_2e^{2x}) + 5;$ (*ii*) $y = (c_1\cos x + c_2\sin x) + x;$

(*iii*) $y = (c_1 + c_2x)e^x - 2x^2\lambda e^x;$

(*iv*) $\left(c_1 + c_2x\right)e^{-2x} + \left(\dfrac{1}{9}x - \dfrac{2}{27}\right)e^x + \left(\dfrac{3}{25}\sin x - \dfrac{4}{25}\cos x\right)$

5. (*i*) $y = \left(c_1e^x + c_2e^{2x}\right) + \dfrac{1}{2}\left(x^2 + 3x + \dfrac{7}{2} - 2xe^x\right)$

(*ii*) $y = -\dfrac{1}{10}\left(x\sin 3x + \dfrac{3}{5}\cos 3x\right) - \dfrac{\cos x}{2}$

6. $y = -\dfrac{e^x\cos x}{2} - \dfrac{c_1e^{-x}}{5}(2\sin x + \cos x) + c_2e^x.$

Assignment 4

1. (*i*) $y = c_1x^2 + c_2x^3 - x^2\log x;$ (*ii*) $y = c_1x^4 + \dfrac{c_2}{x} + \dfrac{x^4}{5}\log x$

(*iii*) $y = c_1x^2 + \dfrac{c_2}{x} + \dfrac{1}{3}\left(x^2 - \dfrac{1}{x}\right)\log x;$ (*iv*) $y = 2(\log x)^3 + c_1\log x + c_2.$

(*v*) $y = \dfrac{c_1}{x} + x\left[c_2\cos(\log x) + c_3\sin(\log x)\right] + 5x + 10\dfrac{\log x}{x};$

(*vi*) $y = \dfrac{c_1}{x} + c_2x^4 - \dfrac{x^2}{6} - \dfrac{1}{2}\log(x) + \dfrac{3}{8};$

(*vii*) $u = \dfrac{kr}{8}\left(a^2 - r^2\right)$

(*viii*) $y = c_1\cos\left(\log(1+x)\right) + c_2\sin\left(\log(1+x)\right) - \log(1+x)\cos\left(\log(1+x)\right)$

(ix) $y = (1+2x)^2 \left\{ \left[\log(1+2x)\right]^2 + \left(c_1 \log(1+2x)\right) + c_2 \right\}$

(x) $y = \left[c_1 + c_2 \log(x+1)\right] + \left[\left(\log(x+1)\right)^2 + x^2 + 8x\right]$

2. $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$

3. $x^3 y''' + xy' - y = 0$

Assignment 5

(i) $x = -\dfrac{1}{27}(1+6t)e^{-3t} + \dfrac{1}{27}(1+3t), \quad y = -\dfrac{2}{27}(2+3t)e^{-3t} + \dfrac{2}{27}(2-3t)$

(ii) $x = e^t + e^{-t}, \; y = e^{-t} - e^t + \sin t$

(iii) $x = c_1 e^t + c_2 e^{-2t} + 2e^{-t}, \quad y = 3c_1 e^t + 2c_2 e^{-2t} + 3e^{-t}$

(iv) $x = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right), \quad y = \dfrac{1}{2}\left(-t + \dfrac{1}{t}\right)$

(v) $x = \left(c_1 e^t + c_2 e^{-t}\right) + \left(c_3 \cos t + c_4 \sin t\right) - \dfrac{t}{4}\cos t + \dfrac{t}{4}\sin t,$

$y = \left(-c_1 e^t - c_2 e^{-t}\right) + \left(c_3 \cos t + c_4 \sin t\right) + \dfrac{1}{4}(2+t)(\sin t - \cos t)$

(vi) $x = c_1 e^{2t} + c_2 e^{-t} \cos\left(\sqrt{3}t - c_3\right), \quad y = c_1 e^{2t} + c_2 e^{-t} \cos(\sqrt{3}t - c_3 + \dfrac{2\pi}{3})$

$z = c_1 e^{2t} + c_2 e^{-t} \cos(\sqrt{3}t - c_3 + \dfrac{4\pi}{3})$

(vii) $x = \dfrac{8}{9}\left(1 - \cos\dfrac{3}{2}t\right), \quad y = \dfrac{4}{3}t - \dfrac{8}{9}\sin\dfrac{3}{2}t$