Graph Coloring and Football Graph

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Contents

Abstract

Graph coloring is one of the intresting part in graph theory where elements of the graph is colored where no 2 adjacent element get same color. This problem has many real-world applications, such as scheduling tasks, and coloring maps.However,there are many algorithms and concepts that can solve the problem of coloring. Additionally, graph coloring has connections to other areas of computer science, such as computational complexity theory and algorithm design.

Keywords : Dual graph,Degree sequence,Chromatic number,Chromatic index,Football graph.

1 Introduction

1.1 Objective

if G(v,E) is a graph having n elements then our objective to use minimum color to color their elements such that no adjacent color may have same color.

1.2 Problem Statement

Let G(V,E) be a graph, consisting of a set of vertices V,faces F and edges E.Given such a graph,the vertex coloring problem seeks to assign each vertex v € V an integer c(V) € 1,2,3,.....k such that :

* c(V) c(U) v,u € Z; and
* k is minimal

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* Here integers represent different types of colors assigned to vertex.
* c(u) and c(v) represents colors assigned to adjacent elements u and v.
*Z represents elements of graph :-
1.edge for edge coloring
2.vertex for vertex coloring
3.face for face coloring.

1.3 Input and Output

Input :- G(V,E)
Output :- k (no of colors) $\longrightarrow$ κ1(v1),κ2(v2)
here ki(vj) $\longrightarrow$ assign ith color to jth vertex.

1.4 Illustrations

1.4.1 Graph coloring

Figure 1: Vertex coloring

Figure 2: Edge coloring

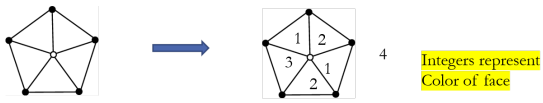


Figure 3: Football coloring with 4 colors

1.4.2 India map and its coloring

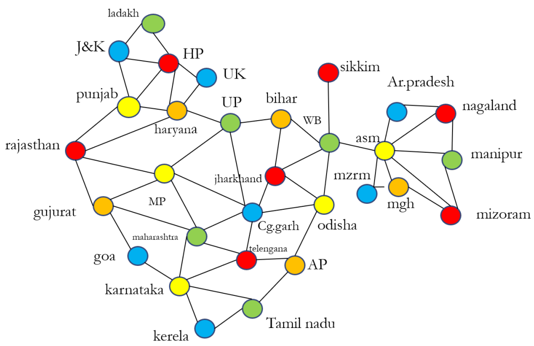


Figure 4: coloring of India map such that no adjacent state have same color

1.4.3 Coloring of a football

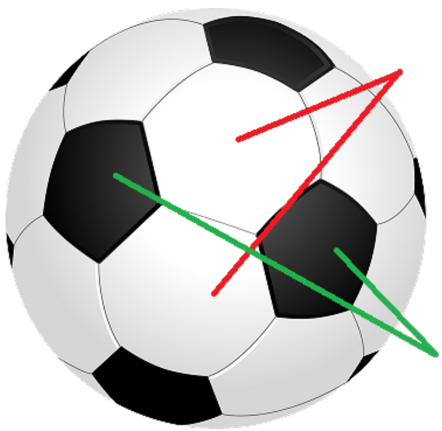


Figure 5: Football coloring with 4 colors

1.5 Practical Application

1.5.1 Scheduling

In scheduling problems, graph coloring is used to assign tasks to different time slots. The tasks are represented as vertices in a graph, and the time slots are represented as colors. The goal is to minimize the number of time slots needed to complete all tasks.

1.5.2 Frequency assignment

In communication if same tower is assigned to different tower then there may be conflict and we have to use minimum frequency so here frequency represented as vertex and tower can be represented as edge.So that by coloring we can find minimum frequency with no conflict.

1.5.3 Map coloring

In cartography, graph coloring is used to color maps of different regions such that no two adjacent regions have the same color. This is used to make maps more readable and to avoid confusion.

1.5.4 Wireless channel assignment

In wireless networks, graph coloring is used to assign different channels to different nodes in the network. The nodes are represented as vertices in a graph, and the channels are represented as colors. The goal is to minimize interference and maximize network throughput.

1.6 Research Motivation

Graph coloring is a fascinating topic in mathematics and computer science that has important application in important fields such as time scheduling,network optimization etc.
Studying graph coloring improves problem-solving,algorithm designing and critical thinking.There are also some unorderised structures which can be designed better through coloring accordingly like some sculptures etc.
Through this exploration,we hope to gain a deeper understanding and its relevance to modern science.

2 Preliminaries and Background

2.1 Keywords and their definitions:

- **Chromatic Number:** Minimum no of color to color a graph such that no 2 vertex have same color.It is denoted by X(G).
- **Chromatic Index** Minimum no of color to color a graph such that no 2 edge have same color.It is denoted by X'(G).
- **Face:** A face of the graph is a region bounded by a set of edges and vertices in the embedding. Let G(V,E) is a graph then no of face of that graph will be 1 if f(link(a))='a'.
  Where link(a) =
  $$\left\{ \begin{array}{c} =\text{'a' if a is connected to vertex 'a'} \\ =\text{(b) if a is connected to other than a ie (b)} \\ \text{where b} \neq a. \end{array} \right\}$$
- **Embedding of a graph:** Embedding of a graph is a graph in which no edges cross each other except vertices.
- **Dual of a graph:** Dual graph of a plane graph G is a graph has a vertex for each face of G. and each edge for the boundary shared by 2 faces.
- **Eulers formula:**Euler theorem is known to be one of the most important mathematical theorems named after Leonhand Euler.Eulers formula can be written as **F+V=E+2**,where F ,V ,E represents no of faces,vertices and edges respectively.
- **4 color theorem:**This theorem states that any planer graph can be colored with 4 color such that no adjacent faces have same color.

2.2 Literature Overview

In 1890,Heawood prooved 5 colrs theorem saying that every planer graph can be colored with no more than 5 colors.After that in 1976 by Kenneth Appel and Wolfgang Haken optimised the 5 colors theorem and reduced it to 4 color theorem saying that every graph now can be colored with 4 colors. The proof went back to the ideas of Heawood and Kempe and largely disregarded the intervening developments

. In 1912, George David Birkhoff introduced the chromatic polynomial to study the coloring problems, which was generalised to the Tutte polynomial by Tutte, important structures in algebraic graph theory.

In 1960, Claude Berge formulated another conjecture about graph coloring originally motivated by an information-theoretic concept called the zero-error capacity of a graph introduced by Shannon. The conjecture remained unresolved for 40 years, until it was established as the celebrated strong perfect graph theorem by Chudnovsky, Robertson, Seymour, and Thomas in 2002.

Year	Scientist	Contribution
1890	Heawood	Prooved 5 colrs theorem saying that every planer graph can be colored with no more than 5 colors
1976	Kenneth Appel and Wolfgang Haken	Optimised the 5 colors theorem and reduced it to 4 color theorem saying that every graph now can be colored with 4 colors.
1912	Claude Berge	formulated another conjecture about graph coloring originally motivated by an information-theoretic concept called the zero-error capacity of a graph introduced by Shannon

3 Our Contributions

3.1 Found some results for graph coloring

odd cycle graph :- 3 color for edge and vertex coloring
even cycle graph :- 2 color for edge and vertex coloring
bipartite graph :- 2 color for vertex and D(G) for edge coloring
odd complete graph :- n color for vertex and edge coloring
even complete graph :- n color for vertex and n-1 color for edge coloring

Here n is no of vertex and D(G) is degree of graph

3.2 Created an algorithm for face coloring

Algorithm 1 Making dual graph

1.count faces N; //Adding vertex and edges
2.for i=0 to N
3.add vertex Vi;
4.add edges Ei;
5.for k=0 to N
6. count =0;
7. for l=0 to N

8. if(adj(l,k,e)==k) // adding edges for 2 face(k) and face(l) adjacent with edge(e);
9. count e=1;
10.   add edge(l,k);
11.if(e==1)
12.break;

Algorithm 2 Coloring the final graph

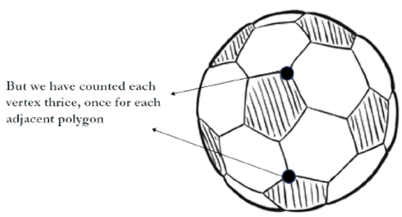
1.set color c:1,2,3,4
2.for i=0 to N
3.v[i]=null;
4.for j=0 to N
5. k=j;
6. if(adj(V[j])!=color[k])
7. v[j]=color[k];
8. else
9. k++;
10.return colors

3.3 Invented a football graph

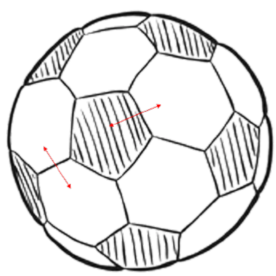
A graph having set of different polygons and have colored in such a way that no adjacent polygon have same color.

3.3.1 No of pentagon and hexagon in football

In football assume there are 'h' hexagon and 'p' pentagon so no of vertices are 6h and 5p respectively. But we are counting each vertex thrice,once for each pentagon so no of vertice will be (6h+5p)/3. Similarly we are calculating each edge twice for each polygon so, no of edges will be (6h+5p)/2.
. But According to euler's formula V-E+f=2. So,(6h+5p)/3-(6h+5p)/2+(p+h) = 2. By equating we will get p=12.
. Here,we have to note that each pentagon sorrounded by 5 hexagon and each hexagon sorrounded by 3 pentagon so no of hexagon will be 5p/3 = 20.



vertices of football



edges of football

3.3.2 Coloring of football

Here each pentagon is sorrounded by 5 hexagon through each edge of pentagon. If we assign a color let 'black' to that pentagon then we will easily colored that circular ring of hexagon over pentagon.Assume 5 hexagon as 5 vertices and the sorrounding edge representing edge so it may represented as odd cyclic graph of 5 vertex which get easily colored with 3 color as stated in sec 3.1 .

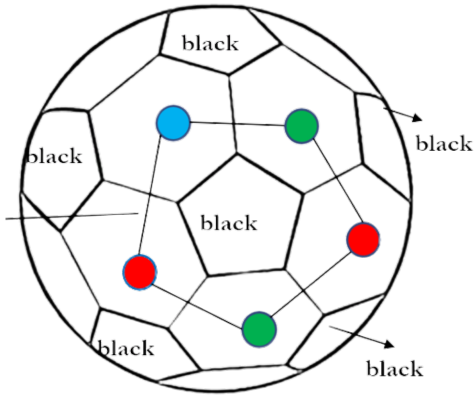


Figure 6: Football coloring with 4 colors

3.3.3 Fotball graph as Atomic Orbital Energy Model

This Football graph [refer to fig:6] with dissimilar adjacent colored graph can be illustrated as Atomic energy model where each black polygon ie..pentagon can be represented as nucleus of atom and each polygon of the circular polygonal ring can be represented as energy level of the molecular orbit.So, when energy is given to nucleus ie.. pentagon then energy level of each molecular orbit changes and get transitioned to next energy level.

3.3.4 Sirish's theorem

Note :- Football can be made using any polygon not only from pentagon and hexagon.

Theorem :- This theorem states that minimum 6 vertex or node each for each face is needed to make a cuboid or cube to a football.

Proof:-In cuboid or cube we have 6 faces which is not in rounded one.So, to make them rounded we have to add some no of vertices or node for each face .To achieve our goal we may add 1 or many vertices.But to make a shape rounded we have to make a corner and with minimum vertices which is only achieved by adding one vertex for each face which in result gives 4 triangle replacing each face.and triangle have minimum edges which in turn there will be minimum face in resulting football.

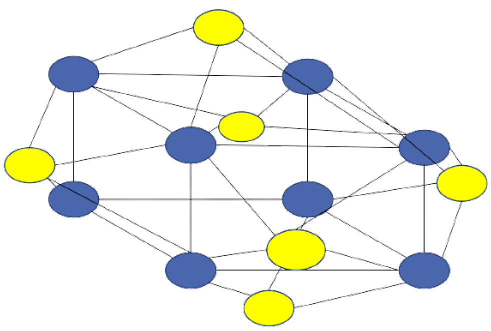


Figure 7: Making a football with adding 6 nodes for each faces

Output of this theorem:-A football with 24 triangle Which is colorable with only 4 colors such that no adjacent triangle assigned same color. **Result from Football graph:-** $X(G_f) = 4$

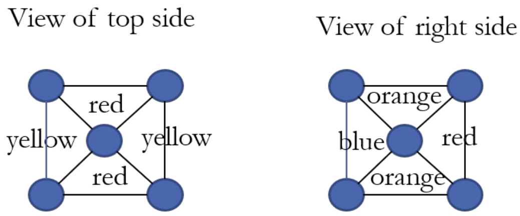


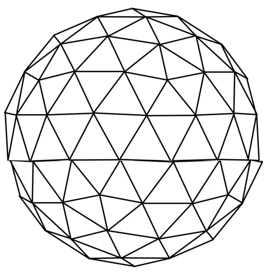
Figure 8: Top and right side view on resultant football

3.4 Sculptures that can be visualised as football graph

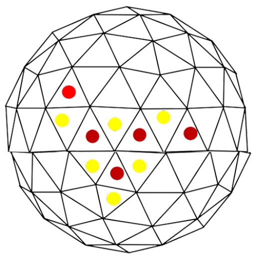
There are many sculptures that can be visualised as football graph one of these are geodesic dome .In a geodesic dome, each triangle is connected to other triangles at its edges, creating a strong and stable structure that distributes stress evenly throughout the entire dome.

3.4.1 Buckminster Fuller's formula

the inventor of the geodesic dome, the minimum number of triangles required to approximate a sphere with a radius R using geodesic triangles is given by the formula:
$N = 2(R/D)^2$ where,
N : is the number of triangles
E : is the edge length of the triangles
R : is radius of the dome.



geodesic dome



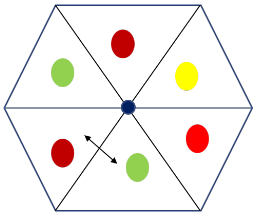
dome after colored

3.4.2 calculation of edge and column of dome:

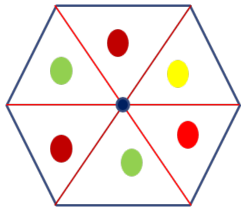
As in below figure Each vertex surrounded by 6 triangles and each edge connected to 2 triangles.
So,E = 3F / 2 = 6V / 2
F:-no of faces ie .triangles
So,according to the euler's formula : V - E + F = 2.
By, using the formula V = 2F / 3.
and By refering to next below figure (right one)Here for one vertex (blue one) there are 6 edges (red ones).
Total no of edges = 6*V = 4*F.

3.4.3 coloring of geodesic dome

By coloring all faces (triangles) we can make the dome beautiful and no many colors are used to color all faces.
Here 2 colors will be used if no of faces = even 3 colors if no of faces = odd.



geodesic dome



dome after colored

4 Conclusion

Graph coloring gives explorations on various algorithms, concepts and approximating solutions to various problems on coloring.We designed a algorithm on face coloring of a graph having time complexity O(n^2).It requires a deep understanding of computational thinking and algorithm designing.Multiple graph can be designed on this concept like we discovered football graph having 12 pentagon and 20 hexagon can be colored using only 4 colors.we have discovered some ideas upon football graph that can be useful in demonstrating energy levels of molecular orbits.There are also many challenges like NP-hardness,finding accurate chromatic number,Symmetrical graphs.

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