Discrete and Algorithmic Geometry

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Sheet 1

Due on Tuesday, December 10, 2019

You should work in your teams and complete half of the exercises on this sheet.

To submit your solutions to these exercises,
- ▷ fill out on the course wiki which exercises your team has completed.
- ▷ create a new branch `your-awesome-team-name-sheet-1`,
- ▷ create a subdirectory `exercises/sheet1/your-awesome-team-name/`,
- ▷ and put your solutions to the exercises into one or more `.tex` files into that directory.
- ▷ Now encrypt these `tex` files using `julian.pfeifle@upc.edu.public.gpg.key`,
- ▷ `add`, `commit` and `push` **only these encrypted files, not the original** `.tex`,
- ▷ and create a merge request.

You will be graded collectively on these exercises, and individually in the final exam.

Exercises not submitted via this mechanism will not be graded.

(1) Recall the following definitions for a matroid M on the ground set $[n] = \{1,2,\dots,n\}$ with family of independent sets $\{I : I \in \mathscr{I}\}$.
- ▷ For any proper subset $A \subset [n]$, the deletion $M \setminus A$ is the matroid on the ground set $[n] \setminus A$ whose independent sets are $\{I \subset [n] \setminus A : I \in \mathscr{I}\}$.
- ▷ For $a \in E$, the independent sets of the contraction M/a are $\big\{I - \{a\} : a \in I \in \mathscr{I}\big\}$.
- ▷ The dual matroid $M^\star$ of M is the matroid on $[n]$ where I is a basis iff $[n] \setminus I$ is a basis of M.

Now prove the following statements.

(a) Contraction in M as defined in class agrees with the notion of contraction in graphs.

(b) Contraction in M as defined in class agrees with $M/S := (M^\star \setminus S)^\star$ for a subset $S \subset [n]$.

(c) $M_{G^\star} = (M_G)^\star$, if G is a planar graph and $G^\star$ its dual planar graph.

(d) Consider the matroid M realized by the columns of the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Compute a realization of $M^\star$, and some contractions of M of your choosing. Compute the set of circuits and cocircuits of M and of $M^\star$.

(2) Let $A \in \mathbb{R}^{d\times n}$ be a matrix of full rank, and let the columns of $B \in \mathbb{R}^{n\times(n-d)}$ be a basis of the row space of A, so that $AB = 0$ and the rows of B are a Gale transform of the columns of A. We saw in class that $\mathsf{LinVal}(A) = \mathsf{LinDep}(B)$. Show that $\mathsf{LinVal}(B) = \mathsf{LinDep}(A)$.

(3) *The greedy algorithm always works for matroids.*

(a) Show that Kruskal's greedy algorithm always finds a maximum weight independent set in a matroid $M = (E, \mathscr{I})$, regardless of the choice of weight function $\omega : E \to \mathbb{R}_{>0}$. Recall that the greedy algorithm starts by setting $I := \emptyset$, and next repeatedly chooses $y \in E \setminus I$ with $I \cup \{y\} \in \mathscr{I}$ and with $\omega(y)$ as large as possible. It stops if no such y exists.

(b) Show that that this property characterizes independent sets of matroids among all simplicial complexes. In other words, given a simplicial complex Σ for which the greedy algorithm always works, regardless of the weight function ω, show that $\Sigma = \mathscr{I}(M)$ for some matroid M.

Hint: You only need to show that the exchange axiom I3 holds. To do this, given $I_1, I_2 \in \mathscr{I}$ with $|I_2| = |I_1| + 1 = k+1$, consider the weight function

$$\omega(E) := \begin{cases} \frac{k+1}{k+2} & \text{for } e \in I_1 \\ \frac{k}{k+1} & \text{for } e \in I_2 \setminus I_1 \end{cases}.$$

Explain why the greedy algorithm will build up I_1 first, and then at the next step, will exhibit an element of the form $I_1 \cup \{e\} \in \mathscr{I}$ with $e \in I_2 - I_1$. In particular, explain why the algorithm will not just stop after having found I_1!

(4) A famous result by Jack Edmonds states the following:

Theorem (Matroid Intersection Theorem)**.** *Let $M_1 = (E, \mathscr{I}_1)$ and $M_2 = (E, \mathscr{I}_2)$ be two matroids on the same ground set E, with rank functions*

$$r_i : 2^E \to \mathbb{N}_{>0}, \quad S \mapsto \max\{|I| : I \in \mathscr{I}_i, I \subseteq S\}, \qquad i = 1,2.$$

Then the maximum size of a common independent set in $\mathscr{I}_1 \cap \mathscr{I}_2$ is equal to

$$\min\{r_1(S) + r_2(E \setminus S) : S \subseteq E\}.$$

(a) Let $G = (V,E)$ be a bipartite graph with color classes V_1, V_2. For $i = 1,2$, let M_i be the matroid where $I \subseteq E$ is independent if and only if each vertex in V_i is covered by at most one edge in I. Use the Matroid Intersection Theorem to prove

Theorem (Kőnig's Matching Theorem)**.** *The maximum size of a matching in a bipartite graph equals the minimum size of a vertex cover.*

(A vertex cover of G is a subset of V that intersects each edge.)

(b) Let $G = (V,E)$ be a graph whose edges are partitioned into k colors, $E = E_1 \cup E_2 \cup \cdots \cup E_k$. Use the Matroid Intersection Theorem to prove that there exists a rainbow spanning tree (a spanning tree all of whose edges are colored differently) if and only if $G - F$ has at most $t+1$ connected components, for any union F of t colors, for any $t \geq 0$.

Hint: Apply the Matroid Intersection Theorem to the cycle matroid of G and the partition matroid induced by $E_1, \ldots, E_k$. Here, the cycle matroid on G is the matroid whose independent sets are the edge sets that form a forest, and the partition matroid is just the transversal matroid; it is called that because the edges are partitioned into colors. Thus, the independent sets in the partition matroid are of the form $\{i_1, \ldots, i_s\}$ for $i_j \in E_j$.

(5) Prove (a) and (b) of the following implications for matroids, and illustrate them with a well-chosen example. Part (c) is optional and depends on your knowledge of algebra.

$$\begin{tikzcd}
\text{Graphic} \arrow[dr, "(a)"] & & \\
& \text{Linear} \arrow[r, "(c)"] & \text{Algebraic} \\
\text{Transversal} \arrow[ur, "(b)"'] & &
\end{tikzcd}$$

Hints.

(a) Suppose that $G = (V, E)$. In the vector space $\mathbb{R}^V$ whose standard basis is indexed by the vertices of G, represent the element $e = (v, w)$ in E by the vector $e_v - e_w$. In other words, show that the linearly independent subsets of these vectors are indexed by forests of edges of G.

(b) Given a bipartite graph G with vertex bipartition $E \cup F$, show that you can represent its transversal matroid as follows. Let $\Bbbk(X) := \Bbbk(x_{e,f} : e \in E, f \in F)$ be a field extension of the field $\Bbbk$ by transcendentals $\{x_{e,f}\}$ indexed by all edges $\{e, f\}$ of G. Then in the vector space $\Bbbk(X)^F$ having standard basis vectors u_f indexed by the vertices f in F, represent the element $e \in E$ by the vector

$$\sum_{f \in F : \{e,f\} \in G} x_{e,f}\, u_f.$$

In other words, show that the linearly independent subsets of these vectors are indexed by the subsets of vertices in E that can be matched into F along edges of G.

(c) Given a matroid M of rank r linearly represented by a set of vectors $\{v_1, \ldots, v_n\}$ in the vector space $\Bbbk^r$, represent M algebraically by elements of the rational function field $\Bbbk(x_1, \ldots, x_r)$ as follows. If v_i has coordinates $(v_{i1}, \ldots, v_{ir})$ with respect to the standard basis for $\Bbbk^r$, then represent v_i by $f_i := \sum_{j=1}^{r} v_{ij} x_j$. In other words, show that the algebraically independent subsets of these rational functions f_i are indexed the same as the linearly independent subsets of the v_i.

(6) *The Matroid Application Treasure Hunt.* Find as many applications of matroid theory as you can, both inside and outside of mathematics. One point for every application you find that has not been found by any other team; zero points for any duplicate application. I will collect the unique applications and make them available to all participants.

(7) In each of the classes or models of matroids discussed in class (linear / graphical / transversal / algebraic matroids; hyperplane / affine hyperplane arrangements, matroid polytopes),
  (a) describe independent sets, bases, circuits, cocircuits, and flats.
  (b) For which of these entities can you rapidly see that they fulfill the corresponding axiom systems? For which does it seem mysterious?
  (c) Describe the dual matroid of a matroid in each of these situations.

(8) We have seen that a matroid can be given by its collections of independent sets, bases, circuits, cocircuits, flats, or its rank function. Find algorithms to convert between as many of these entities as you can. What is their combinatorial complexity?

(9) *Independence complexes of matroids are vertex-decomposable.* Let Δ be a simplicial complex on the vertex set E. We do not assume that every $e \in E$ is actually used as a vertex of Δ. The concept of vertex-decomposability for a simplicial complex Δ on the vertex set E is defined recursively: both the complex $\Delta = \emptyset$ having no faces at all (not even the empty face) and any complex Δ consisting of a single vertex are defined to be vertex-decomposable, and then Δ is said to be vertex-decomposable if it is pure (all facets have the same dimension), and there exists a vertex $e \in E$ for which both its deletion and link

$$\mathrm{del}_\Delta(e) := \{F \in \Delta : e \notin F\}$$
$$\mathrm{link}_\Delta(e) := \{F - \{e\} : e \in F \in \Delta\}$$

are vertex-decomposable complexes.
  (a) Show that vertex-decomposable complexes Δ are shellable.
  Hint: Obviously you want to use induction. Shell the facets in the deletion of e first, then those in the star of e, which is the cone over the link with apex e. Formally, for a face F of Δ,

$$\mathrm{star}_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta\},$$
$$\mathrm{link}_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}.$$

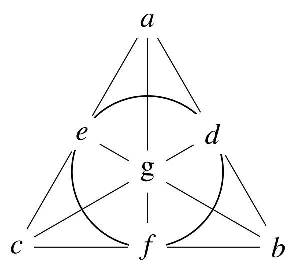
(b) Show that for a matroid M with $\Delta = \mathscr{I}(M)$ and any non-loop, non-coloop element $e \in E$,

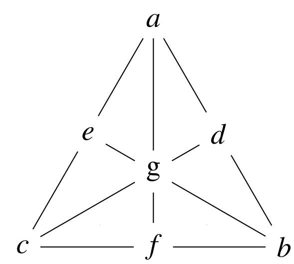
$$\begin{aligned} \mathrm{del}_\Delta(e) &= \mathscr{I}(M \setminus e), \\ \mathrm{link}_\Delta(e) &= \mathscr{I}(M/e). \end{aligned}$$

Deduce that independent set complexes $\mathscr{I}(M)$ of matroids are vertex-decomposable.

(10) *Representability of the Fano and non-Fano matroids.* Show that the Fano matroid is coordinatizable only in characteristic 2, and the non-Fano matroid is coordinatizable only in characteristic distinct from 2.

(a) First show that in any coordinatization $V = \{a,b,c,d,e,f,g\}$ of either the Fano or non-Fano matroids, with elements labelled as





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you can use the action of $GL_3(\Bbbk)$ along with scaling of individual vectors to assume that the representing matrix has columns looking like this:

$$\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & \gamma \\ 0 & 1 & 0 & 1 & 0 & \alpha & \delta \\ 0 & 0 & 1 & 0 & 1 & \beta & \varepsilon \end{bmatrix} \end{array}$$

(b) Use some of the matroid dependencies to show that $\gamma = \delta = \varepsilon$, and hence by scaling, $\gamma = \delta = \varepsilon = 1$.

(c) Use some more of the matroid dependencies to show that $\alpha = \beta$.

(d) Use the last matroid dependence in the Fano matroid (and its absence in the non-Fano matroid) to decide whether or not the characteristic of $\Bbbk$ is 2.

(11) Let $L_{n,d}$ be the set of *lattice points* (i.e., points all whose coordinates are integers) in the following $(d-1)$-dimensional simplex in $\mathbb{R}^d$:

$$\Delta_{d-1}(n) \;=\; \big\{(x_1,x_2,\dots,x_d) \in \mathbb{R}^d : x_1+x_2+\cdots+x_d = n-1 \text{ and } x_i \geq 0 \text{ for all } i\big\}.$$

For $1 \leq k \in \mathbb{N}$ a *simplex of size* k is a parallel translate of $L_{k,d}$. For any subset $I \subset L_{n,d}$, a simplex of size k is *I-saturated* if it contains exactly k points of I.

(a) Let S, S' be two I-saturated simplices with $S \cap S' \neq \emptyset$, and let $S \vee S'$ be the smallest simplex containing S and S'. Then the simplices $S \cap S'$ and $S \vee S'$ are also I-saturated.

Let $\mathscr{I}_{n,d}$ be the collection of subsets I of $L_{n,d}$ such that for every $k \in \mathbb{N}$ with $1 \leq k \leq n$, every parallel translate of $L_{n,k}$ contains at most k points of I, cf. Figure 1.

(b) Show that $\mathscr{I}_{n,d}$ is the collection of independent sets of a matroid.

(c) Let $T(n)$ be an equilateral triangle with side length n. Suppose we want to tile $T(n)$ using unit rhombi with angles equal to 60° and 120°. Show that this is impossible.

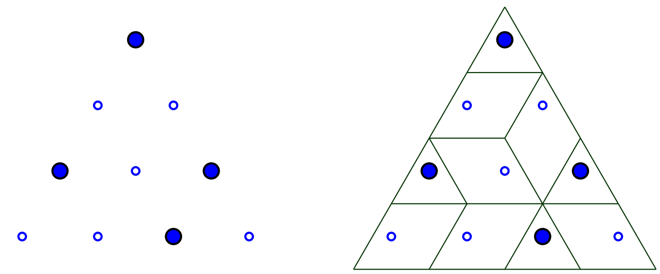

FIGURE 1. *Left:* The lattice points $L_{4,3}$ in $\Delta_2(4)$. The filled circles correspond to a basis (a maximal independent set) of $\mathscr{I}_{4,3}$. *Right:* A tesselation of $T(4)$ with 4 triangles ("holes") and the rest rhombi of angles 60° and 120°.

Hint: Cut $T(n)$ into n^2 unit equilateral triangles. How many of these point upward? How many downward?

(d) Suppose that we make n holes in the triangle $T(n)$, by cutting out n of the upward triangles. Show that it may or may not be possible to tile the remaining shape with rhombi.

(e) Show that the possible locations of n holes for which a rhombus tiling of the "holey" triangle $T(n)$ exists correspond to the bases of the matroid $\mathscr{I}_{n,3}$.

(12) For each the affine Gale diagrams of Figure 2, determine the dimension and write down the vertex sets of the facets of their corresponding polytopes. Gray points are pyramid points.

FIGURE 2. Four affine Gale diagrams

(13) Consider the convex polytopes described by the following affine Gale diagrams in Figure 3.

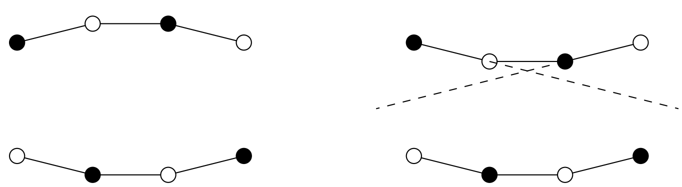

FIGURE 3. Two affine Gale diagrams

(a) Show that both convex polytopes have dimension 4 and 8 vertices.
(b) Using the Gale diagram, show that both polytopes are 2-neighborly, which means that all $\binom{8}{2}$ edges lie on the convex hull, or equivalently, that their graph is the complete graph K_8.
(c) From the Gale diagram, deduce all 2- and 3-dimensional faces of these polytopes.
(d) Draw the dual graph of these polytopes, which is the graph that has the 3-dimensional faces ("facets") as nodes, and in which two nodes are adjacent iff the corresponding 3-dimensional faces intersect in a 2-dimensional face.
(e) Show that the two polytopes are not combinatorially equivalent, which means that there is no bijection between their vertex sets that induces a bijection between the sets of faces.

Hint: The left diagram is that of the *cyclic* polytope $C_4(8)$. We met its smaller cousin, $C_4(7)$, in an in-class exercise.

(14) Consider the affine Gale diagram of Figure 4.

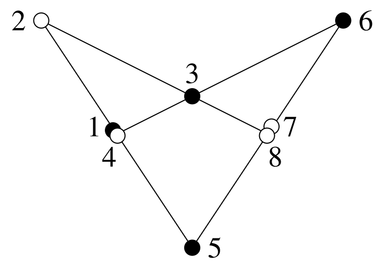

FIGURE 4. Another affine Gale diagram

(a) Show that it represents a 4-dimensional polytope with 8 vertices.
(b) Show that the polytope has 9 facets: four tetrahedra, four square pyramids, and an octahedron 235678. Write down their vertex sets.
(c) Every Gale diagram with the same positive circuits has 7 and 8 on the same point.
(d) Therefore, in every combinatorially equivalent polytope the vertices 2356 of the octahedron facet are coplanar. In consequence, the shape of the octahedron facet cannot be prescribed arbitrarily.