ANNOTATED BIBLIOGRAPHY
DISCRETE AND ALGORITHMIC GEOMETRY, UPC, 2019

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References

[BLS+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter Ziegler. *Oriented matroids. 2nd ed.*, volume 46. Cambridge: Cambridge University Press, 2nd ed. edition, 1999.

This book is a comprehensive exposition of oriented matroids consisting of ten chapters. It is intended for graduate students as an introduction (where some knowledge in discrete mathematics is of advantage) and for researchers who need a thorough reference work. The book starts with two orientation sessions which illustrate different aspects of oriented matroids. Since oriented matroids can be thought of as combinatorial abstraction of directed graphs, of real hyperplane arrangements, of point configurations, and of convex polytopes, these mathematical theories are used to motivate oriented matroids. In these two chapters a number of examples is presented and simultaneously the main concepts and terminology of oriented matroids is introduced. In the third chapter a formal introduction is given by presenting four basic axiom systems for oriented matroids: circuit axioms, orthogonality axioms, chirotopes, and vector axioms. These axiom systems arise from directed graphs, orthogonal pairs of real vector subspaces, point configurations and convex polytopes, and real hyperplane arrangements, respectively. Other topics discussed in this chapter include minors, duality, and local realizability. Chapter 4 is entitled "From Face Lattices to Topology". Face lattices are studied in this chapter in a general axiomatized version. The lattices are formed by the covectors of an oriented matroid under a natural partial ordering. The connection to topology is established by showing that the covector lattice of an oriented matroid uniquely determines a regular cell decomposition of a sphere. In Chapter 5 topological models for oriented matroids are presented. The core of this chapter is the Topological Representation Theorem. It says that general oriented matroids similarly correspond to arrangements of generalized hyperplanes, each obtained from a flat hyperplane by tame topological deformation. The rank 3 case of the Topological Representation Theorem can easily be visualized. In the projective version, this identifies rank 3 oriented matroids with arrangements of pseudolines. These arrangements are the topic of Chapter 6. In Chapter 7 it is discussed how oriented matroids can be extended, deformed, locally perturbed, flipped, glued together, and how the old and the newly obtained oriented matroids are related. The realizability of oriented matroids is the topic of Chapter 8. To this for the space R(M) of all vector realizations of a fixed oriented matroid M is introduced. The realizability problem for M now becomes the question whether the semialgebraic variety R(M) is empty or not. Chapter 9 is devoted to the combinatorial theory of convex polytopes. Several new results on polytopes as well as new simplified proofs for known results could be found with the help of oriented matroid theory. Chapter 10 gives an introduction to linear programming on oriented matroids. A geometric access to the fundamental ideas of oriented matroid programming, as developed by Bland is given. A list of open problems and exercises is included after each of the ten chapters. In the second edition an appendix has been added about some current frontiers of research. Here, the progress made since the first edition was published is summarized. Also the already excellent bibliography of the first edition has been greatly expanded.

[Bok06] Jürgen G. Bokowski. *Computational oriented matroids. Equivalence classes of matrices within a natural framework.* Cambridge: Cambridge University Press, 2006.

This book is simultaneously a thoroughgoing introduction to certain computational aspects of oriented matroids as developed by the author, his coauthors and students, and others over more than the last quarter of a century, and a platform upon which to begin to learn the Haskell programming language. The book describes data structures derived from various definitions for "oriented matroid" and Haskell routines to navigate from one data structure to another, as well as routines to produce the dual or minors of a given oriented matroid, to produce the graphic oriented matroid of a given graph, and so forth. Then these tools are employed to study questions of combinatorial topology and geometry, such as "Given a (particular) triangulation of the 3-dimensional sphere, can it be realized as the poset of faces of the boundary of a convex 4-dimensional convex polytope?" Several instances of this question are answered, with the aid of more specialized Haskell code. For the sphere problem and many others, a two-step process is used: First decide whether or not the sphere triangulation can be realized as an oriented matroid polytope; then, if so, decide whether or not the oriented matroid(s) found can be realized by matrices. Other applications are considered; in particular, a later chapter considers the realizability of subdivisions of the torus and of surfaces of higher genus by geometric polygons in space. The complete Haskell code is included, documented, and used to help with explanations of the mathematical content. A short Haskell tutorial is included in an appendix. For illustrative purposes, the book contains not only a large number of colorful pictures of oriented matroids, but also pictures of artwork, by various artists, all having oriented matroid themes; the book, itself, is perhaps a piece of art.

[BR15] Matthias Beck and Sinai Robins. *Computing the continuous discretely. Integer-point enumeration in polyhedra. With illustrations by David Austin. 2nd edition.* New York, NY: Springer, 2nd edition edition, 2015.
This book is an outstanding book on counting integer points of polytopes (usually with integer vertices). The book is divided into two parts. In part I the authors discuss the basics of theory of Ehrhart polynomials and various preliminary subjects related to polytopes, e.g., face structure of the polytopes, Dedekind summation, magic squares, etc. Second part of the book contains recent results and approaches to integer-point enumeration in polyhedra. The book contains main results of Ehrhart theory related to Dedekind summation, finite Fourier analysis, various questions regarding triangulation of polytopes and decompositions into its cones. Numerous important examples are exhaustively studied. The reader will find here both basic examples of certain polyhedra (simplices, cubes, cross-polytopes and pyramids) and more advance series of examples (zonotopes, and in particular, permutahedra, etc.). The book contains lots of exercises with very helpful hints. Another essential feature of the book is a vast collection of open problems on different aspects of integer point counting and related areas. To my opinion it is very important to have such a list of open problems for further references. The book is reader-friendly written, self-contained and contains numerous beautiful illustrations. The reader is always accompanied with deep research jokes by famous researchers and valuable historical notes.

[CLO15] David A. Cox, John Little, and Donal O'Shea. *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra. 4th revised ed.* Cham: Springer, 4th revised ed. edition, 2015.
In the 80th of the last century, B. Buchberger invented the study of Gröbner bases into the field of commutative algebra and algebraic geometry. With the development of the personal computers it became a powerful technique for computational aspects in various computer algebra systems. It was the intention of the authors to provide a comprehensive easily accessible introduction to important concepts of computational commutative algebra in the first edition of their book see [Zbl 0756.13017]. It became a well accepted source for students and researchers as an introduction to the subject (see reviews about the second [Zbl 0861.13012] and third edition [Zbl 1118.13001]. In each of the new editions the authors were interested to incorporate new developments, simplifications of arguments as well as further applications. Thanks to the authors this is also the case in the present fourth edition. In fact the authors provide the following substantial changes: (1) They define standard representations and lcm representations

in Chapter 2. (2) They give two new proofs of the extension theorem, one based on resultants and one with Gröbner bases (inspired by *P. Schauenburg* [J. Symb. Comput. 42, No. 9, 859–870 (2007; Zbl 1137.13018)]). (3) In Chapter 4, they present a proof of the weak Nullstellensatz by the use of Gröbner bases (see also [*L. Glebsky*, "A proof of Hilbert's Nullstellensatz based on Groebner bases", arXiv:1204.3128]), they include saturations in addition to ideal quotients and prove the closure theorem by Gröbner bases techniques (see also Schauenburg [loc.cit]). (4) In Chapter 5, they add a section about Noether normalization, in particular for the application in dimension theory. (5) The results on Gröbner bases under specializations in Chapter 6 have been supplemented by the concept of Gröbner covers (see *A. Montes* and *M. Wibmer* [J. Symb. Comput. 45, No. 12, 1391–1425 (2010; Zbl 1207.13018)]). (6) There is a new Chapter 10 about the progress over 25 years development about methods for computing Gröbner bases. This includes Traverso's Hilbert driven Buchberger algorithm, Faugère's F4-algorithm and an introduction to the signature-based family of algorithms. This, in particular illustrates the close bridge between theory and practice in computational commutative algebra. (7) Following the previous editions, the authors discuss software for computational aspects (Maple, Mathematica, Sage, CoCoa, Macaulay 2, and Singular) and several other systems that can be used in courses based on the text in the book. (8) In an Appendix, one may find 14 students projects substantially updated with new ideas. (9) The bibliography has been completed and expanded to reflect some of the new developments of the subject. Thanks to the continuously updating the textbook will remain an excellent source for the computational commutative algebra for students as well as for researchers interested in learning the subject.

[DRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations. Structures for algorithms and applications.*, volume 25. Berlin: Springer, 2010.
This book masterfully presents the theory of triangulations of (the convex hull of) a point set alongside many appealing applications in algebra, computer science, combinatorics, and optimization. The authors "firmly believe that understanding the fundamentals of geometry and combinatorics pays up for algorithms and applications" of triangulations, and this interplay is beautifully developed. After introductory examples, setting up language and basic constructions, and a concrete chapter on computational geometry in dimension two, the book discusses Gelfand–Kapranov–Zelevinsky's central theorem on secondary polytopes (whose face lattices realize the posets of regular subdivisions of point configurations), followed by important families of configurations and triangulations (e.g., cyclic polytopes, product of simplices, 0/1-polytopes), a chapter on computation and algorithms, and a chapter on further topics (including fiber polytopes, mixed subdivisions, lattice polytopes, Gröbner bases and polytopal complexes). The writing is thorough and engaging, assisted by clear (and numerous) illustrations, and many exercises for the reader. Graduate students and researchers in any area in which triangulations of points set configurations play a role will find this book a comprehensive and most useful reference.

[Mat02] Jiří Matoušek. *Lectures on discrete geometry.*, volume 212. New York, NY: Springer, 2002.
This is an introduction to the field of discrete geometry understood as the investigation of combinatorial properties of configurations of (usually finitely many) geometric objects, like arrangements of points or hyperplanes or, more generally, convex sets. Some emphasis is laid on asymptotic results which are relevant for estimating the complexity of geometric algorithms. Certain areas of discrete geometry are neglected intentionally, such as the theory of packing and covering. Besides fundamental facts on convex sets, lattice points, and so on, the book deals with the following themes: combinatorial complexity of geometric configurations, intersection patterns and transversals of convex sets, geometric Ramsey theory, polyhedral combinatorics and high-dimensional convexity (including for instance a discussion of volumes in high dimensions and Dvoretzky's theorem on almost spherical sections), embedding of finite metric spaces into normed spaces. All needed concepts and tools are well explained, in particular the more advanced ones like Gale diagrams, Davenport-Schinzel sequences, and Vapnik-Chervonenkis dimension. Even some of the more complicated proofs are given in full detail. At the end of each section a comprehensive

survey on historical and recent results is presented. There are also many well-selected exercises, classified according to their difficulty. For some of the exercises hints are given in an appendix. The book is written in a lively and stimulating but very precise style and contains many figures. It gives a good impression of the richness and the relevance of the field. At the end of the book there is a short summary of each chapter.

[Rei05] Vic Reiner. Lecture notes for the ACE Summer School 2005 in Geometric Combinatorics. `http://www-users.math.umn.edu/~reiner/Talks/Vienna05/index.html`, 2005.
The lecture notes themselves are at `http://www-users.math.umn.edu/~reiner/Talks/Vienna05/Lectures.pdf`; the web page contains further references. Much of the lectures and exercises in class have been based on these notes.

[Ric11] Jürgen Richter-Gebert. *Perspectives on projective geometry. A guided tour through real and complex geometry.* Berlin: Springer, 2011.
The author of this very well written and detailed book is an expert in projective geometry, especially in computational projective geometry, as the books [Geometries. (Geometriekalküle.) Springer-Lehrbuch. Berlin: Springer (2009; Zbl 1181.00003)], [Die interaktive Geometrie-Software Cinderella. Version 1. 2. Einzelplatzversion. Berlin: Springer (1999; Zbl 0969.51001)], and numerous scientific articles show. The reader of the present book on classical projective geometry is expected to be familiar with elementary linear algebra, apart from students this book is accessible to mathematicians as well as computer scientists and physicists. The author presents the rich interplay of geometric structures and their algebraic counterparts, thus following the philosophy of J. Plücker (1801–1868) and F. Klein (1849–1925). Definitions and theorems never stand alone, they are patiently explained and always discussed from the view of computer implementation. The emphasis is on structures in order to express the fundamental objects and operations in a most elegant way. The book points out the advantages of projective geometry with respect to Euclidean, hyperbolic, and elliptic geometry. Each chapter of the book starts with a motto, for instance chapter 17 on the complex projective line is introduced as follows: "The shortest route between two truths in the real domain passes through the complex domain" (J. S. Hadamard (1865–1963)). Many of the approximately 230 very aesthetical, computer generated, partially colored illustrations show that beside rigorous abstractions also visualizations can play a helpful role in the study of geometric objects. Except for chapter 1, the book is divided into three parts. Chapter 1. Pappos's theorem: nine proofs and three variations: among the nine are those via the Pascal theorem, the Cayley-Bacharach-Chasles theorem, and the Miquel theorem. **Part I Projective Geometry.** Chapter 2. Projective planes: drawings and perspectives, the axioms, the smallest projective plane. Chapter 3. Homogeneous coordinates: the real projective plane, joins and meets, parallelism, duality, projective transformations, basic facts on finite projective planes. Chapter 4. Lines and cross-ratio: the real projective line, elementary properties of the cross-ratio. Chapter 5. Calculating with points on lines: harmonic points, projective scales, from geometry to real numbers, the fundamental theorem, a note on other fields, von Staudt's original constructions, Pappos's theorem. Chapter 6. Determinants: (this chapter demonstrates the importance of determinants and multihomogeneous bracket polynomials in expressing projectively invariant properties) the "determinantal" point of view, Plücker's μ, invariant properties, the Grassmann-Plücker relations. Chapter 7. More on bracket algebra: from points to determinants and back, a glimpse of invariant theory, projectively invariant functions, the bracket algebra. **Part II Working and playing with Geometry.** Chapter 8. Quadrilateral sets and liftings: symmetry and generalizations of quadrilateral sets, involutions and quadrilateral sets. Chapter 9. Conics and their duals: equation of a conic, polars and tangents, dual quadratic forms, transformation of conics, degenerate conics, primal-dual pairs. Chapter 10. Conics and perspectivity: conic through five points, conics and cross-ratio, perspective generation of a conic, transformations and conics, Hesse's "Übertragungsprinzip", the theorems of Pascal and Brianchon, harmonic points on a conic. Chapter 11. Calculating with conics: splitting a degenerate conic, the necessity of "if" Operations, intersecting a conic and a line, intersecting two conics, the role of complex numbers, one tangent and four points. Chapter 12. Projective d-space: elements at infinity, homogeneous

coordinates and transformations, points, planes, and lines in 3-space, joins and meets (a universal system and how to use it). Chapter 13. Diagram techniques: (shows that diagrams formed by geometric objects and ε -tensors form invariants under projective transformations) from points, lines, and matrices to tensors, tensor diagrams, how transformations work, the δ- and the ε-tensors, the ε-δ rule, transforming ε -tensors, invariants of line and point configurations. Chapter 14. Working with diagrams: (gives more advanced applications of tensors and diagrams) a trace condition, Pascal's theorem, closed ε-cycles, conics quadratic forms and tangents, diagrams in the real projective 3-space, the ε-δ rule in rank 4, co- and contravariant lines in rank 4, tensors versus Plücker coordinates. Chapter 15. Configurations, theorems, and bracket expressions: Desargues's theorem, binomial proofs, chains and cycles of cross-ratios, Ceva and Menelaus, gluing Ceva and Menelaus configurations, **Part III Measurements.** Chapter 16. Complex numbers: a primer: historical background, fundamental theorem, geometry of complex numbers, Euler's formula, complex conjugation. Chapter 17. The complex projective line: geometric properties, projective transformations, inversions and Möbius reflections, Grassmann-Plücker relations, intersection angles, stereographic projection. Chapter 18. Euclidean geometry: the circular points of plane Euclidean geometry (Reviewer's remark: The mentioned name is avoided by the author probably in view of the later chapters about Cayley-Klein geometries.), cocircularity, the robustness of the cross-ratio, projective, affine, similarity, Euclidean transformations, perpendicularity, Laguerre's formula, distances. Chapter 19. Euclidean structures from a projective perspective: (Euclidean geometry is projective geometry together with the pair of circular points) mirror images, angle bisectors, center of a circle, constructing the foci of a conic, constructing a conic by foci, triangle theorems, hybrid thinking (demonstrated with the nine-point circle). Chapter 20. Cayley-Klein geometries: (concentrates on the planar case) interpretation of the circular points of plane Euclidean geometry as degenerate dual conic, distance and angle measurement, hyperbolic, elliptic, and parabolic measurements along a line, an investigation of distances and angles in the hyperbolic plane with 7 helpful and well described figures, the seven types of planar Cayley-Klein geometries, coarser and finer classifications. Chapter 21. Measurements and transformations: (focusses on transformations, their projective invariants, and the behavior of measurements under these transformations) measurements versus oriented measurements, comparing measurements, reflections and pole/polar pairs, rotations. Chapter 22. Cayley-Klein geometries at work: orthogonality, constructive versus implicit representations, commonalities and differences, midpoints and angle bisector, trigonometry. Chapter 23. Circles and cycles: circles via distances, relation to the fundamental conic, centers at infinity, organizing principle (cycles as limiting cases, duality of circles, curves of constant curvature), cycles in Galilean geometry (= plane isotropic geometry). Chapter 24. Non-Euclidean geometry: A historical interlude: the inner geometry of a space, Euclid's postulates, Gauss, Bolyai, and Lobachevsky, Beltrami and Klein, the Beltrami-Klein model, Poincaré. Chapter 25. Hyperbolic geometry: hyperbolic transformations, angles and boundaries, the Poincaré disk, transformations of the complex projective line and the Poincaré disk, angles and distances in the Poincaré disk. Chapter 26. Selected topics in hyperbolic geometry: circles and cycles in the Poincaré disk, area and angle defect, (hyperbolic) Thales and (hyperbolic) Pythagoras, constructing regular n-gons, symmetry groups (accompanied among others by a figure showing a pentagonal hyperbolic checkerboard). Chapter 27. What we did not touch: (gives a brief overview of a loose selection of topics) algebraic projective geometry (cubics, Cayley-Bacharach-Chasles theorem, Bézout's theorem, special points, duality), projective geometry and discrete mathematics (arrangement of pseudolines), projective geometry and quantum theory, a dynamic projective geometry.

[Sch03] Alexander Schrijver. *Combinatorial optimization. Polyhedra and efficiency (3 volumes).*, volume 24. Berlin: Springer, 2003.
Schrijver's 3 volumes on combinatorial optimization reflect the current state of the art in this field, in particular from the viewpoint of polyhedral combinatorics and efficient algorithms. Less emphasis is put on advanced data structures, approximative, randomized and parallel algorithms, semidefinite programming and graph decomposition. The book offers a masterly introduction with many interesting historical remarks as well as an in-depth survey of combinatorial optimization.

It is divided into eight main parts with 83 chapters. The main parts are (I) paths and flows, (II) bipartite matching and covering, (III) nonbipartite matching and covering, (IV) matroids and submodular functions, (V) trees, branchings and connectors, (VI) cliques, stable sets and colouring, (VII) multiflows and disjoint paths and, finally, (VIII) hypergraphs. Volume A contains an introduction, preliminaries and the first three main parts. Volume B contains the main parts (IV)-(VI), and Volume C contains the last two parts, a survey of open problems, over 4500 references for further research and name and subject indices. The reader is supposed to have a basic knowledge of graph theory and linear as well as integer programming. The author gives short and elegants proof to all main results. In order to keep the overall size of the volumes under control, the topics applications and modelling of optimization problems, and computational methods for NP-hard problems are not treated. The first part of the book, devoted to paths and flows, starts with a description of various shortest paths methods. Thereafter it discusses disjoint paths giving several proofs for Menger's theorem, and continues with maximum flows, circulations and transshipments. Furthermore, the following topics are treated: path and flow polyhedra and total unimodularity, Dilworth's theorem and connectivity algorithms, in particular the Gomory-Hu tree. The second part starts with bipartite matchings (giving several proofs of Königs and Frobenius' theorems). Then the linear assignment problem is discussed with many historical remarks. The bipartite matching polytope, bipartite edge cover and stable sets, bipartite edge-colouring and transversals are further topics of part (II). A particular emphasis is put on b-matching and transportation problems with interesting information about early investigations in the former Soviet Union. The third part is devoted to nonbipartite matching and covering and discusses in detail the pioneering work of Tutte and Edmonds. The first chapter in this part is devoted to non-bipartite cardinality matching, again with detailed historical remarks. Then the matching polytope is studied and algorithms for the weighted nonbipartite matching problem are treated. Further chapters deal with edge-colourings (Vizing's theorem), the Chinese postman problem, T-cuts and T-joins, 2-matching, 2-cover and 2-factor problems, b-matchings, bidirected graphs, the perfect matching polytope and the perfect matching lattice. Volume B starts with matroids. Again detailed historical remarks are given which demonstrate the development of this notion. The chapter finds its classical continuation with the greedy algorithm and the independent set polytope. The next chapters treat matroid intersection problems and algorithms, matroid union and matroid matching. Then submodular functions and polymatroids are introduced and a strongly polynomial-time algorithm for finding the minimum of a submodular function is described. Finally, the author deals with polymatroid intersection, Dilworth truncation and closes this part with generalizations of submodular functions based on lattice families, intersecting families and crossing families. Part (V) is devoted to trees, branchings and connectors. It starts with a chapter on shortest spanning trees and the corresponding historic background. Then the packing and covering of trees is treated. Moreover, trees in directed graphs are considered, namely branchings and arborescences. Further chapters deal with biconnectors and bibranchings, minimum directed cut covers and the packing of directed cuts as well as with strong connectors. Chapter 58 describes the basics of the travelling salesman problem together with extensive historical notes and a detailed list of further references. Finally, matching forests and submodular functions on directed graphs are investigated. Further chapters deal with graph orientation problems, network synthesis and connectivity augmentation. Part (VI) is devoted to problems which are NP-hard in general,namely cliques, stable sets and colourings. In the case of perfect graphs these problems become polynomially solvable. Therefore perfect graphs are treated in detail and many interesting historical remarks are provided. Moreover, it is shown that a maximum-weighted stable set and a minimum weight clique cover in a perfect graph can even be found in strongly polynomial time. Finally, T-perfect graphs and claw-free graphs are considered. Volume C starts with part (VII) on multicommodity flows and disjoint paths. After discussing the basic properties, the two commodity case is treated. Then it is investigated to which extent Hu's theorem can be generalized to three or more commodities. In the next chapter the T-path problem and Mader's theorem is analyzed. This problem becomes particularly interesting in planar graphs which are treated in Chapter 74. In particular, Okamura's theorem is proved. Further chapters deal with cuts, odd circuits and multicommodity flows and graphs on surfaces

where an interesting homotopic circulation theorem is proved. The last part of this compendium is devoted to hypergraphs and their relation to matching, vertex cover, edge cover and the stable set problem. Starting from packing and blocking problems in hypergraphs, the author describes ideal, Mengerian and binary hypergraphs and gives the short proof of Guenin for Seymour's characterization of binary Mengerian hypergraphs. Then a survey on relations between matroids and multicommodity flows in hypergraphs is given. Moreover, stable sets and edge covers in hypergraphs are studied. Finally, balanced and unimodular hypergraphs are introduced and studied. These three volumes contain an immense richness of results up to 2002 and will prove to be indispensible for any further research in the field of combinatorial optimization.

[Seg04] A.J.M. Segers. Algebraic Attacks from a Gröbner Basis Perspective. Master's thesis, Technische Universiteit Eindhoven, 2004. `https://www.win.tue.nl/~henkvt/images/ReportSegersGB2-11-04.pdf`.
Recently, a special kind of cryptanalysis coined as the algebraic attack has gained a lot of attention. In this thesis, we clarify this attack and discuss the threat to common ciphers. Among the known attacks, one can roughly distinguish between two classes. The first consists of structural attacks that focus on specific properties of a certain cipher. The second includes inversion attacks, which are general purpose algorithms that solve multivariate systems of equations. In this thesis we focus on the latter. The different methods appear to be reducible to Gröbner Basis techniques, an area of algebraic geometry that is understood reasonably well. This report focusses on three topics. Firstly, we introduce a variety of common ciphers and discuss advanced techniques based on Gröbner Bases to cryptanalyze these ciphers. These are implemented in Magma and presented to the reader. Secondly, the much discussed cryptanalytic tool XL is shown to be related to methods based on Gröbner Bases. And lastly, this report discusses a complexity approximation of XL based on Hilbert series.

[Tho06] Rekha R. Thomas. *Lectures in geometric combinatorics.*, volume 33. Providence, RI: American Mathematical Society (AMS); Princeton, NJ: Institute for Advanced Studies, 2006.
The text under review is an introduction to some actual topics of the modern geometric combinatorics. Prepared for an advanced undergraduate course, it is aimed to present recent developments in the theory of convex polytopes, which are centered around secondary and state polytopes arising from point configurations. The text consists of 14 chapters. Chapters 1–3 are devoted to some fundamental geometric and algebraic notions from the theory of polytopes. Chapters 4–6 deal with the representations of n-polytopes in terms of the well-known Schlegel diagrams and Gale diagrams. Point configurations in $\mathbb{R}^n$ and their triangulations are analyzed in details in Chapter 7. The next Chapter 8, the soul of the book, is devoted to the secondary polytopes of point configurations. As a concrete example of secondary polytopes, the permutahedron is described in Chapter 9. Fundamental relations between secondary polytopes and state polytopes of toric ideals of point configurations are established in Chapters 10–14. The theory of Gröbner bases, where the notion of state polytopes arises naturally, is developed in Chapters 10–12. Connections between Gröbner bases of toric ideals and regular triangulation of point configurations defining ideals are discussed in Chapter 13. In the final Chapter 14 the author demonstrates how to construct state polytopes of toric ideals naturally related to secondary polytopes. The theory of secondary and state polytopes is a recently developed direction of the modern geometry, which has numerous applications to combinatorics, algebraic geometry, discrete geometry, etc. Moreover, the text itself is clearly written and self-contained, numerous illustrations and exercises are included. Thus, this "Lectures in geometric combinatorics" may be undoubtedly recommended for undergraduate and graduate students as a textbook on modern aspects of the theory of polytopes.

[Zie95] Günter M. Ziegler. *Lectures on polytopes.*, volume 152. Berlin: Springer-Verlag, 1995.
This book introduces into the world of convex polytopes in d-dimensional affine spaces. It is developed from a course for advanced studies, and the reader can notice the author's successful didactical effort and enthusiasm for this subject. Famous in the theory of polytopes is the classical

treatment of *L. Schläfli* [‘Theorie der vielfachen Kontinuität’ (1850-1852), Gesammelte Math. Abh. Vol. 1, Birkhäuser Basel, pp. 167-387 (1949; Zbl 0035.21902)]. The modern theory was established by *B. Grünbaum* [‘Convex polytopes’, Interscience, London (1967; Zbl 0163.16603)]. Using these concepts and results here, now the author gives an excellent introduction to some basic methods and furthermore modern tools of polytope theory. Except for the basics in chapter 0 and 3 the lectures of the book are essentially independent from each other and require only a basic background knowledge of real affine geometry and of vector spaces. Especially, we find a concentration on combinatorial aspects of the theory. In every chapter the author shows important and interesting figures to illustrate his text, and he also tries to visualize the geometry of higher dimensional polytopes. At the end of each of the ten chapters there are interesting notations, problems, exercises, and historical comments. A good book to study geometry of polytopes and to give pleasure in this field! This expressed aim of the book has been doubtlessly achieved. It will be useful for both students and lecturers, for it can serve as an easy to read but nevertheless profound textbook on polytopes. The titles of the ten chapters (and some subchapters) are: 0. Introduction and examples. 1. Polytopes, polyhedra and cones (Fourier- Motzkin elimination, Farkas lemma, Carathéodory’s theorem). 2. Faces of polytopes (Face lattice, polarity, simplicial and simple polytopes). 3. Graphs of polytopes. 4. Steinitz’ theorem for 3-polytopes. 5. Schlegel diagrams for 4-polytopes. 6. Duality, Gale diagrams and applications (Vector configurations, oriented matroids, polytopes with few vertices, rigidity, universality theorem). 7. Fans, arrangements, zonotopes, and tilings (Minkowski sums, nonrealizable oriented matroids, zonotopal tilings). 8. Shellability and the upper bound theorem (Euler-Poincaré formula, Dehn-Sommerville equations, upper bound theorem from McMullen, some extremal set theory, Kruskal-Katona theorem, Macaulay’s theorem). 9. Fiber polytopes, and beyond. At the end of the book we find references to nearly 500 titles on this subject.