Discrete and Algorithmic Geometry

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Sheet 1

Due on Tuesday, November 19, 2019

To submit your solutions to these exercises,
- ▷ create a new branch `your-awesome-team-name-sheet-1`,
- ▷ create a subdirectory `exercises/sheet1/your-awesome-team-name/`,
- ▷ and put your solutions to the exercises into a `.pdf` file into that directory.
- ▷ Now encrypt this `.pdf` using `julian.pfeifle@upc.edu.public.gpg.key`, and
- ▷ `add`, `commit` and `push` **only this encrypted pdf, not the original** `.tex`
- ▷ and create a pull request.

You will be graded collectively on these exercises, and individually in the final exam.

Exercises not submitted via this mechanism will not be graded.

Let $([n], \mathscr{I})$ be a matroid on the ground set $[n] = \{1, 2, \dots, n\}$ with independent sets $\{I : I \in \mathscr{I}\}$.

- ▷ For any proper subset $S \subset [n]$, the deletion $M \setminus S$ is the matroid on the ground set $[n] \setminus S$ whose independent sets are $\{I \subset [n] \setminus S : I \in \mathscr{I}\}$.
- ▷ The dual matroid $M^\star$ of M is the matroid on $[n]$ where I is a basis iff $[n] \setminus I$ is a basis of M.
- ▷ If $S \subset [n]$, then the contraction of M with respect to S is $M/S = (M^\star \setminus S)^\star$.
- ▷ Let G be a graph whose edges are labeled by $[n]$. The bases of the graphical matroid M_G are the sets of edges corresponding to spanning trees of G.

(1) True or false?
  (a) This notion of contraction agrees with the notion of contraction in graph theory.
  (b) $M_{G^\star} = (M_G)^\star$, if G is a planar graph and $G^\star$ its dual planar graph.

(2) Prove that if a matroid M is realizable over a ground field $\Bbbk$, then the dual matroid $M^\star$ is also realizable over $\Bbbk$. [*Hint.* Suppose that M has rank d and n elements. After a change of basis, M can be realized by the $d \times n$ matrix $A = [I|B]$, where I is the $d \times d$ identity matrix, and B has size $d \times (n-d)$. Now find a matrix that realizes $M^\star$.]

(3) Consider the matroid M realized by the columns of the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Compute a realization of $M^\star$, and some contractions of M of your choosing.

CODING

To submit your solutions to the next two exercises,

- ▷ switch to your branch `your-awesome-team-name-sheet-1`,
- ▷ create a subdirectory `exercises/sheet1/your-awesome-team-name/coding`,
- ▷ and `add` **all** files you create to your commits, **without encryption**.
- ▷ Record your results in `sheet1/results.tex`, and create a pull request.

This will make it possible to have conversations about your code in your pull requests.

(4) In the programming languages of your choice, write code that checks whether the sets of integers contained in the directory `exercises/sheet1/matroid-or-not` satisfy the matroid basis axioms or not. What is the combinatorial complexity of your code? What parameters of the data does this combinatorial complexity depend on?

(5) Write (or use, or search for and download) code that given integers $n \geq k \geq 0$ creates all $\binom{n}{k}$ combinations of an n-set. They form the set of bases of the uniform matroid of rank k on n elements. Run your code on various instances of these matroids, and plot the execution time against reasonable parameters. Is your conclusion from part (4) borne out?