Discrete and Algorithmic Geometry

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Sheet 2

Due on Thursday, December 19, 2019

You should work in your teams and complete half of the exercises on this sheet. Submissions work in the same way as for Sheet 1.

(1) Let $\mathscr{A} = (a_1, \dots, a_{d+2}) \subset \mathbb{R}^d$ be a full-dimensional point configuration of $d+2$ points, so that the associated vector configuration that one gets by homogenizing coordinates has rank $d+1$, and therefore $\operatorname{corank} \mathscr{A} = 1$. Let $\lambda \in \mathbb{R}^{d+2}$ be the vector of coefficients of the unique affine dependency $J = (J_+, J_-)$ among the members of $\mathscr{A}$.

(a) Show using Cramer's rule that, up to a global sign,

$$\lambda_i \;=\; (\text{volume of the simplex } \mathscr{A} \setminus a_i) \cdot \begin{cases} +1 & \text{if } i \in J_+ \\ -1 & \text{if } i \in J_- \end{cases}$$

(b) Let $T_\pm = \{\mathscr{A} \setminus a_j : j \in J_\pm\}$ be the two unique triangulations of $\mathscr{A}$, and let

$$\phi_{\mathscr{A}}(T_\pm) \;=\; \sum_{j=1}^{d+2} \operatorname{vol} \operatorname{star}_{T_\pm}(a_j)\, e_j$$

be their corresponding GKZ-vectors. Show that

$$\big(\phi_{\mathscr{A}}(T_+)\big)_i \;=\; \begin{cases} \operatorname{vol} \operatorname{conv} \mathscr{A} & \text{if } i \in J_-, \\ \operatorname{vol} \operatorname{conv} \mathscr{A} - \operatorname{vol} \operatorname{conv}(\mathscr{A} \setminus a_i) & \text{if } i \in J_+. \end{cases}$$

(c) What is the relationship between $\phi_{\mathscr{A}}(T_+)$, $\phi_{\mathscr{A}}(T_-)$, and λ?

(2) Consider a full-dimensional point configuration $\mathscr{A} \subset \mathbb{R}^d$. Show that the affine hull of the secondary polytope Σ-poly$(\mathscr{A})$ is given by

$$\sum_{\mathbf{a} \in \mathscr{A}} x_{\mathbf{a}} \;=\; d(d+1) \operatorname{vol} \operatorname{conv} \mathscr{A},$$

$$\sum_{\mathbf{a} \in \mathscr{A}} x_{\mathbf{a}}\, \mathbf{a} \;=\; \big((d+1) \operatorname{vol} \operatorname{conv} \mathscr{A}\big) \mathbf{c}_{\mathscr{A}},$$

where $\mathbf{c}_{\mathscr{A}}$ denotes the centroid of $\operatorname{conv} \mathscr{A}$.

(3) Show that every configuration $\mathscr{A}$ of $d+4$ points in dimension d has at most $O(d^4)$ regular triangulations:

(a) Think of their chamber complexes as cell decompositions of a 2-sphere.

(b) Bound the number of vertices in these chamber complexes by $O(d^4)$, showing that every vertex corresponds (not uniquely, but that is not a problem) to a pair of circuits of $\mathscr{A}$.

(c) Using Euler's formula, show that any (polyhedral) cell decomposition of a 2-sphere has about as many vertices as it has 2-cells. More precisely, these two numbers are within a factor of 2 of each other.

(4) Show that the cyclic polytope $C_{4k}(4k+4)$ has at least $\frac{1}{4} 2^k$ many non-regular triangulations:

(a) Consider first the coordinatization having a cyclically symmetric affine Gale diagram, in which k lines meet in a zero-dimensional chamber at the center.

(b) Show that by perturbing the coordinates, each of these k lines can be moved towards one or the other side of the center. Since the cyclic configuration is in general position, these perturbations produce different coordinatizations of the same oriented matroid, hence give the same polytope.

(c) Show that for all these perturbations the center of the Gale diagram corresponds to distinct triangulations. For this, use the description of a chamber as an intersection of the "dual simplicial cones" of the corresponding triangulation.

(5) Let $\mathscr{T} = \{\sigma_1, \dots, \sigma_m\}$ be a triangulation of a full-dimensional point configuration $\mathscr{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n) \subset \mathbb{R}^d$ of n points, where we consider the $\sigma_i \in \binom{[n]}{d+1}$ to be index sets. Let R^{int} be the set of interior ridges, defined to be intersections $\rho = \rho_{ij} = \sigma_i \cap \sigma_j$ of two facets of $\mathscr{T}$ such that the affine span of the points of $\mathscr{A}$ indexed by ρ has dimension $d-1$. (In the triangulations of Figure 1, they are the interior edges.)

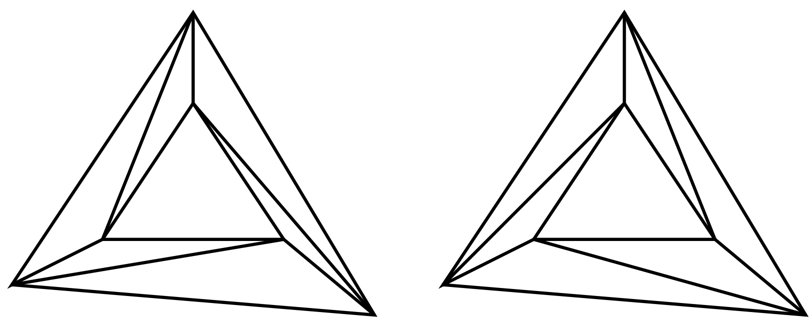

FIGURE 1. Two triangulations. View the source code for the coordinates of the points.

(a) For a vector $\omega \in \mathbb{R}^n$, lift the points in $\mathscr{A}$ to heights ω, so that $\mathscr{A}^\omega = \left(\binom{\mathbf{a}_1}{\omega_1}, \dots, \binom{\mathbf{a}_n}{\omega_n}\right)$. For each interior ridge $\rho = \sigma_i \cap \sigma_j \in R^{\text{int}}$, formulate the folding condition that expresses that ρ indexes a face of the lower convex hull of $\mathscr{A}^\omega$, in terms of the coordinates of the $\mathbf{a}_i$ and ω. Your folding condition should be an inequality that is linear in each height ω_i.

(b) Write code that takes the coordinates of the $\mathbf{a}_i$ and the facets σ_i of a triangulation as input, and outputs the set of folding conditions in a text file in LP file format.

(c) Download a linear programming software such as `gurobi`, `cplex` or `scip/soplex` and check explicitly whether there exists a choice of heights ω that induces each of the triangulations of Figure 1.

(d) Using this code, check that the triangulation of the 4-dimensional cube from [1] given by the files `4-cube.vertices` and `4-cube.triangulation` is non-regular, i.e., it does not come from a lifting to $\mathbb{R}^5$. If you like, download and play with TOPCOM.

(6) The permutohedron Π^{n-1} is the convex hull of all $n!$ permutations of the vector $(1, 2, \dots, n)$. For $n \geq 3$, show that $\dim \Pi^{n-1} = n-1$, that $f_{n-2}(\Pi^{n-1}) = 2^n - 2$, and $f_1(\Pi^{n-1}) = \frac{1}{2}(n-1)n!$. If you like, show that $f_k(\Pi^{n-1}) = k!\left\{ n \atop k \right\} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$.

REFERENCES

[1] Jesús A. De Loera. Nonregular triangulations of products of simplices. *Discrete Comput. Geom.*, 15(3):253–264, 1996.