# CONTENTS AND SHORT BIBLIOGRAPHY
# DISCRETE AND ALGORITHMIC GEOMETRY, UPC, 2019

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Due to my teaching (and grading) load this semester, I have not had time to prepare lecture notes for this class. But since I more or less directly copied the content from various sources, I hope to make your job of studying the material easier by explicitly listing the chapters I used. A big thanks to Moritz Otth for pushing me do compile this list!

The overarching theme are realizations of oriented matroids.

## 1. Matroids

This material is directly copied from [Rei05, Lecture 1].

*Examples*: Vector and point configurations; algebraic matroids; transversal matroids; graphical matroids

*Axiom systems*: Independent sets; bases; circuits; cocircuits; rank function; flats; hereditarily pure simplicial complexes; universally shellable simplicial complexes; greedily optimizable independent set systems

*Operations*: Direct sum; deletion; contraction

*Oriented Matroids*: Axiom systems: Circuits, cocircuits

I didn't end up introducing Coxeter matroids, or talk much about matroid base polytopes. Also, Chirotopes weren't introduced until later.

For the direct sum, deletion and contraction I followed [Rei05, Lecture 2]. I did not use [BLS$^+$99] except for its Theorem 7.4.2, even though it's great as a reference.

The exercises in this section were taken from [Rei05], [Sch03], [Bok06], [AB07], [Zie95].

## 2. Oriented matroid / Gale duality

LinVal($A$); LinDep($A$); AffVal($A$); AffDep($A$). Radon's Lemma. Dictionary of linear/affine Gale transform: Faces of convex hull; convex position. Chirotopes. Cyclic polytopes and neighborliness. Asymptotic Upper Bound Theorem for simplicial polytopes. Arrangements of real algebraic varieties and the Milnor-Thom-Oleinik-Petrovski theorem.

The oriented matroid / Gale duality construction follows [Zie95, Chapter 6], as does the discussion of Radon partitions, (affine) Gale diagrams, cyclic polytopes and neighborliness. The example of a non-rational polytope is [Zie95, Example 6.21]. The Asymptotic Upper Bound Theorem and the presentation of the Milnor-Thom-Oleinik-Petrovski theorem are from [Mat02, Chapter 5], which also has a good introduction to polytopes and Gale duality.

## 3. Regular triangulations and the secondary polytope

Regular polyhedral subdivisions from projections of lower faces. The Union and Intersection properties. The refinement partial order. Examples. The GKZ vector of a triangulation. The secondary polytope and its vertices and affine hull.

I initially tried to follow [DRS10, Chapter 5], but found it to be too verbose for presentation in class. For a leisurely introduction it works great, though. In the end, I used [Tho06, Chapters 7,8]. An additional source is [Zie95, Chapter 9].

The exercises were taken from [DRS10] and [Zie95].

## 4. Gröbner bases

Motivation: the Apollonius Circle Theorem. Monomial/term/polynomial/support/ideal/variety. Hilbert basis theorem. Radical of an ideal. Hilbert's Nullstellensatz. Example: Algebraic attack on a small block cipher. Monomial orders. Division algorithm. Elimination Theorem.

The primary sources here are [CLO15, Chapter 2, §§1–8 and Chapter 3, §1] and [Tho06, Chapters 10–12]. A secondary source is [Ric11]. The example on the key sniffing attack is [Seg04, Section 3.1].

## 5. The Grassmannian and flags

Plücker coordinates of a point configuration. The Grassmannian and Flag variety. The matroid associated to a generic line arrangement via a scaffolding flag. The initial monomials of the Plücker ideal as the incomparable pairs in the straightening poset.

The material on Plücker coordinates and the flag variety is from [MS05, Sections 14.1 and 14.2]. The matroid associated to a generic line arrangement (which also makes an appearance in the exercises) is due to [AB07].

## 6. Slack realization space of polytopes and oriented matroids

Slack matrix, symbolic slack matrix, generalized slack matrix, and slack ideal of a polytope or matroid. Ideal quotient and saturation with respect to a principal ideal. Realization spaces of polytopes; Mnëv's Universality Theorem. Relationship between slack variety and generalized slack matrices. Simplification of slack ideal via row and column rescaling. Examples.

The relevant papers are [GMTW19] and [BW19].

## References


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