The Pumping Lemma

Lemma: (Pumping Lemma) If L is a regular language, then there exists a positive integer p (the pumping length) such that every string $s \in L$, $|s| \geq p$, can be partitioned into three pieces, $s = xyz$, such that the following conditions hold:

- $|y| > 0$,
- $|xy| \leq p$, and
- for each $i \geq 0$, $xy^i z \in L$,

Strategy for showing that a language L is not regular

1. Assume that L is regular (with the aim of reaching a contradiction).
2. Choose a string $s \in L$, such that $|s| \geq p$ where p is the pumping length given by the pumping lemma.
   - The pumping lemma then says that s can be partitioned into three pieces $s = xyz$ where $|y| > 0$ and $|xy| \leq p$, such that $xy^iz \in L$ for all $i \geq 0$.
3. Show that for ALL POSSIBLE partitions of $s = xyz$ satisfying $|y| > 0$ and $|xy| \leq p$ there exists an i such that $xy^iz \notin L$.
   - If we succeed to show this, then we have a contradiction with the pumping lemma and our assumption that L is regular is wrong.

The standard example

$L = \{a^n b^n \mid n \geq 0\}$ is not regular.

Proof:

1. Assume that L is regular (with the goal of reaching a contradiction).
2. Choose $s = a^p b^p$ where p is the pumping length given by the pumping lemma.
   - $s \in L$ and $|s| \geq p$, so the pumping lemma says that s can be partitioned into three pieces $s = xyz$ where $|y| > 0$ and $|xy| \leq p$, such that $xy^iz \in L$ for all $i \geq 0$.
3. $s = a^p b^p = xyz$ where $|y| > 0$ and $|xy| \leq p$ implies that for all such partitions of s, y is a string of a's of length at least 1. Choose $i = 2$, xy^2z contains more a's than b's, thus, $xy^2z \notin L$.
   - This contradicts the pumping lemma, hence, our assumption that L is regular is wrong and consequently L is not regular.

Primes represented in unary

$L = \{1^n \mid n \text{ is a prime number}\}$ is not regular.

Proof: Assume that L is regular. Let p be the pumping length for L given by the pumping lemma.

Choose $s = 1^n$ where n is a prime and $n > p + 1$. $s \in L$ and $|s| \geq p$, hence, s can be partitioned into xyz satisfying the conditions in the pumping lemma. $|xy| \leq p$ implies $|z| > 1$. Since $|z| > 1$ we have $|xz| > 1$. Let $i = |xz|$, then $|xy^iz| = |xz| + |y||xz| = (1 + |y|)|xz|$. Since both $(1 + |y|)$ and $|xz|$ are at least 2, their product cannot be a prime. Thus, $|xy^iz|$ is not a prime. Contradiction, the assumption is wrong and L is not regular.

Pumping down

$L = \{0^i 1^j \mid i > j\}$ is not regular.

Proof: Assume that L is regular. Let p be the pumping length for L given by the pumping lemma. Choose $s = 0^{p+1}1^p$. $s \in L$ and $|s| \geq p$, hence, s can be partitioned into xyz, satisfying the conditions in the pumping lemma. The condition $|xy| \leq p$ implies that y consists only of 0's.

The pumping lemma says that $xy^iz \in L$ even if $i = 0$, $xy^0z = xz$ which gives $xz \in L$, and since $|y| > 0$ we know that xz contains fewer 0's than $xyz = s$. Since s only has one more 0 than 1's, we conclude that xz does not contain more 0's than 1's and $xz \notin L$.

Contradiction (between $xz \in L$ and $xz \notin L$), the assumption must be wrong and L is not regular.