Solutions to Sample Problems for Midterm

Problem 1. The *dual* of a proposition is defined for which contains only $\vee, \wedge, \neg$. It is For a compound proposition that only uses $\vee, \wedge, \neg$ as operators, we obtained the *dual* replacing every $\wedge$ with an $\vee$, every $\vee$ with an $\wedge$, every T with a F and every F with a T.

Let us extend the idea of the dual by also exchanging every literal $\neg x$ by x and every literal x by $\neg x$. For a proposition p we denote it's extended dual by p^*.

So for the proposition $(p \wedge q) \vee (r \rightarrow \neg p)$ the dual $\big((p \wedge \neg q) \vee (r \rightarrow \neg p)\big)^* = (\neg p \vee q) \wedge (\neg r \rightarrow p)$.

(a) Show that $((p \rightarrow q) \rightarrow r) \iff ((p \wedge \neg q) \vee r)$ using a truth table.

Answer

p	q	r	$((p \rightarrow q)$	$\rightarrow$	$r)$	$\leftrightarrow$	$((p\ \wedge$	$\neg q)$	$\vee\ r)$
T	T	T	T	T	T	T	F	F	T
T	T	F	T	F	F	T	F	F	F
T	F	T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	T	T	T
F	T	T	T	T	T	T	F	F	T
F	T	F	T	F	F	T	F	F	F
F	F	T	T	T	T	T	F	T	T
F	F	F	T	F	F	T	F	T	F

(b) Compute $((p \wedge \neg q) \vee r)^*$.

Answer

Just exchange the $\wedge$ with $\vee$, the $\vee$ with $\wedge$, the p with $\neg p$, the $\neg q$ with q and the r with $\neg r$.

$$((p \wedge \neg q) \vee r)^* = (\neg p \vee q) \wedge \neg r$$

(c) Show that $((p \wedge \neg q) \vee r)^* \iff \neg((p \rightarrow q) \rightarrow r)$ using logical equivalences.

Answer

Since $((p \wedge \neg q) \vee r)^* = (\neg p \vee q) \wedge \neg r$ we have to show $(\neg p \vee q) \wedge \neg r \iff \neg((p \rightarrow q) \rightarrow r)$.

$$\begin{aligned}
\neg((p \rightarrow q) \rightarrow r) &\iff \neg(\neg(p \rightarrow q) \vee r) && \textit{(rewriting implication)}\\
&\iff (\neg\neg(p \rightarrow q) \wedge \neg r) && \textit{(DeMorgan's law)}\\
&\iff ((p \rightarrow q) \wedge \neg r) && \textit{(double negation)}\\
&\iff ((\neg p \vee q) \wedge \neg r) && \textit{(rewriting implication)}
\end{aligned}$$

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(d) Let q be a proposition in DNF. Argue that q^* is in CNF and that $q^* \iff \neg q$.

Answer

We know that q is a proposition in DNF. In other words q is an AND of ORs of literals. By exchanging every $\wedge$ with an $\vee$ and every $\vee$ with and $\wedge$, we convert q into and OR of ANDs of literals. Now finally we exchange every negated variable $\neg x$ with x and every positive variable x with $\neg x$, i.e. we exchange literals with other literals. Therefore the structure of q^* is still an OR of ANDs of literals and thus it is in CNF.

Now let us think about the truth values of q and q^*. Let us assume q is true, which means that at least one clause is true and thus all the literals in that clause must be true. In q^* we have almost the same clause, the only difference is that the literals are negated (x became $\neg x$ and $\neg x$ became x) and that the literals are now combined with and $\vee$. Since all the literals in that clause in q are true, all the literals in q^* are false. Hence the entire clause in q^* is false. So if a truth assignment makes a clause in q true, it will make the corresponding clause in q^* false. Since all the clauses in q^* are connected with an $\wedge$, one false clause will make all of q^* false. Therefore, whenever q is true, q^* will be false.

We can use the same argument to show that whenever q^* is false, q must be true, as q^* can only be false, if at least one clause is false. This clause would make the corresponding clause in q true and thus q would be true.

So we showed that q is true if and only if q^* is false and therefore

$$q \iff \neg q^* \quad .$$

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Problem 2. Let A, B and C be sets. Prove the following:

(a) If $C \subseteq B$ then $A - B \subseteq A - C$.

Answer

Let us assume $C \subseteq B$. Now we have to show $x \in A - B$ implies $x \in A - C$. So consider an element $x \in A - B$, that is $x \in A$ but $x \notin B$. Since $C \subseteq B$ this means x cannot be in C either and therefore $x \in A - C$.

(b) If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.

Answer

Let us assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Now we have to show that $x \in B$ if and only if $x \in C$.

Let us first consider the implication $x \in B$ implies $x \in C$ and assume $x \in B$. We can consider two cases.

(i) *$x \notin A$. Here we can observe that $x \in A \cup B$ and thus $x \in A \cup C$. However, since $x \notin A$ this implies $x \in C$.*

(ii) *$x \in A$. Here we can argue that $x \in A \cap B$. Since $A \cap B \subseteq B$ and $A \cap C = A \cap B$ we know $x \in A \cap C$ and thus $x \in C$.*

In order to show the other implication $x \in C$ implies $x \in B$, we just reverse the roles of B and C and repeat the argument from above.

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Problem 3. Let the universe of discourse consist of all possible sets. Consider the following predicates.

$$F(A) = \text{“}A \text{ is finite set”}$$
$$S(A,B) = \text{“}A \text{ is a subset of } B\text{”}$$

For each of the following statements, first translate it into predicate logical using the given predicates. Then give a proof that it is true.

(a) Not all sets are finite.

Answer

Immediately translating the statement we get

$$\neg\forall A\, F(A) \quad .$$

Pulling the negation to the inside we get the equivalent, but easier to prove statement

$$\exists A\, \neg F(A) \quad .$$

In order to give a construtive existance proof, we need to find a set that is not finite. We can consider the set of all nonnegative integers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ *which is infinite by definition and thus not finite.*

(b) Every subsets of a finite set is finite.

Answer

$\forall A \forall B\ \big(S(A,B) \wedge F(B)\big) \rightarrow F(A)$

Proof. Let two sets A and B be given. We need to prove that for those two sets

$$\big(S(A,B) \wedge F(B)\big) \rightarrow F(A) \quad .$$

Let us give a proof by contradiction we assume that B is finite, that $A \subseteq B$, but that A is not finite, i.e. infinite.

Since $A \subseteq B$ every element from A also belongs to B. However A contains inifinitely many elements and all these elements must also belong to B. Therefore B must be infinite as well. This is a contradiction to our assumption that B is finite. $\square$

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(c) For every set A there is a set B such that every set C is a subset of $A \cup B$.

Answer

$\forall A \exists B \forall C\ S(C, A \cup B)$.

Proof. Let some set A be given. Show that

$$\exists B \forall C\ S(C, A \cup B) \quad .$$

We are trying to prove that there is some set B such that *every* C will be a subset of $A \cup B$. In particular the universal set U must be a subset of $A \cup B$. So $A \cup B$ must contain all possible elements.

We can build such a B by just including all the elements that A is missing, i.e. pick $B = \overline{A}$. Now we need to verify that

$$\forall C\ S(C, A \cup \overline{A}) \quad .$$

In other words, show that for every set $C \subseteq A \cup \overline{A}$.

Since $A \cup \overline{A} = U$ and $C \subseteq U$ for every set C, we can see that $C \subseteq A \cup \overline{A}$. Thus $B = \overline{A}$ is a suitable choice for B. $\square$

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Problem 4. Let $f : A \mapsto B$ be a function from the set A to the set B, and let S and T be subsets of A. Recall that for a set $S \subseteq A$ the *image* $f(S)$ of S was defined as

$$f(S) = \{f(s) \mid s \in S\} \quad .$$

(a) Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.

Answer

Proof. We have to show that $y \in f(S \cap T)$ implies $y \in f(S) \cap f(T)$. So let us assume $y \in f(S \cap T)$. This means that $y = f(x)$ for some $x \in S \cap T$. Since $x \in S$ we have $y = f(x) \in f(S)$. At the same time we also have $x \in T$ and thus $y = f(x) \in f(T)$. Therefore we can conclude $y \in f(S) \cap f(T)$. □

(b) Give an example where $f(S \cap T) \neq f(S) \cap f(T)$.

Answer

Pick $S = \{1\}$, $T = \{2\}$ *and define* $f(1) = 1$ *and* $f(2) = 1$. *Then* $f(S \cap T) = f(\emptyset) = \emptyset$ *but* $f(S) \cap f(T) = \{f(1)\} \cap \{f(2)\} = \{1\} \cap \{1\} = \{1\}$.

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