[Advance Counting]

Discrete Structures (CSc 511)

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Recurrence Relations

Some counting problems cannot be solved using the methods we have learnt before. One of the ways of solving counting problems is by finding relationships, called recurrence relation, between the terms of a sequence. When we represent some problem using recursive definition then we specify some initial condition and the recursive condition. We use such definition to solve the relation called recurrence relation.

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its term satisfies the recurrence relation.

Example:

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_n + 1$ for n = 1, 2, …, and suppose that $a_1 = 1$. What is the sequence?

Solution:

We have $a_1 = 1$, $a_2 = a_1 + 1 = 1 + 1 = 2$ ….. In similar way we have the set {1, 2, ….}.

Example:

For $a_n = -2a_{n-1}$, $a_0 = -1$ find $a_1, a_2, \ldots a_5$.

Solution:

We have $a_0 =- -1$

$a_1 = -2a_0 = -2.-1 = 2.$

$a_2 = -2a_1 = -2.2 = -4.$

$a_3 = -2a_2 = -2.-4 = 8.$

$a_4 = -2a_3 = -2.8 = -16.$

$a_5 = -2a_4 = -2.-16 = 32.$

Example:

Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if a) $a_n = 0$? and b) $a_n = 2n$?

Solution:

$a_n = -3a_{n-1} + 4a_{n-2}$, for $a_n = 0$ we have **a)** $a_n = 0$ so the sequence $\{a_n\}$ is a solution. b) $a_n = -3.2(n-1) + 4.2(n-2) = 6n - 6 + 8n - 16 = 14n - 22 \neq 2n$, so it is not a solution.

Example:

Find the recurrence relation satisfied by the sequence $a_n = n!$, and $a_n = 2n + 3$ (There may be more than one relation for some sequence).

Solution:

Take $a_0 = 1$ and $a_n = a_{n-1}.n$ (this is the relation for $a_n = n!$)

Take $a_1 = 5$ and $a_n = a_{n-1} + 2$ (this is the relation for $a_n = 2n + 3$, verify!!)

Example:

Find a recurrence relation for the number of bit strings of length n with an even numbers of 0s.

Solution:

Let a_n denotes the number of bit strings of length n with even numbers of 0s. There is 1 bit string of length one that is valid since among two bits we can choose only 1. Recursively we can define this in terms of bit strings of length n –1. Here we have two conditions for getting bit strings of length n-1 with even numbers of 0s. First, if the bit strings end with 1, then we can have valid bit stings of length n – 1 ending with 1 so that there are a_{n-1} numbers with even numbers of 0s. Second, if the bit strings end with 0, then we can make a bit string valid if we add the bit strings of length n-1 that have odd numbers of 0s. Since there are 2^{n-1} possible ways of getting bit strings of length n-1 and a_{n-1} is the number of valid bit strings of length n-1 we have $2^{n-1} - a_{n-1}$ numbers of invalid bit strings of length n-1. So, the total numbers of valid bit strings is $a_{n-1} + 2^{n-1} - a_{n-1} = 2^{n-1}$. Hence we have the relation $a_n = 2^{n-1}$(This is not recursive but method used was recursive). So since we have $a_1 = 1$, we can write $a_n = 2a_{n-1}$.

Solving Recurrences

We encounter different types of recurrence relations. There is no specific technique to solve all the recurrence relation. However, we solve recurrence relation with some particular forms by using the systematic methods. In this section we are going to see few of them.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$, where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$. The above relation is linear since right hand side is a sum of the multiples of previous terms of the sequence. It is homogeneous since no term occurs without being multiple if some a_js. All the coefficients of the terms are constants and degree k is due to the representation of a_n in terms of previous k terms of the sequence.

In solving the recurrence relation of the type above, the approach is to look for the solution of the form $a_n = r^n$, where r is a constant. $a_n = r^n$ is a solution of a recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if $r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_kr^{n-k}$. when we divide both sides by r^{n-k} and transpose the right hand side we have

$r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0$. Here we can say $a_n = r^n$ is a solution if and only if r is the solution if the equation $r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0$ (**characteristic equation** of the recurrence relation) and solutions to this equations are called **characteristic roots** of the recurrence relation.

Theorem 1: (without proof)

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for n = 0, 1, 2, …, where α_1 and α_2 are constants.

Example:

Solve the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$.

Solution:

Characteristic equation of the given relation is $r^2 - r - 6 = 0$. Its roots are r = 3 and r = -2 since (r - 3)(r + 2) = 0. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 3^n + \alpha_2(-2)^n$, for some constants α_1 and α_2. From the initial conditions we have $a_0 = 3 = \alpha_1 + \alpha_2$, $a_1 = 6 = 3\alpha_1 + (-2)\alpha_2$. Solving these two equations we have $\alpha_1 = 12/5$ and $\alpha_1 = 3/5$. Hence, the solution is the sequence $\{a_n\}$ with $a_n = (12.3^n + 3(-2)^n)/5$.

Theorem 2: (without proof)

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_0^n + \alpha_2nr_0^n$ for n = 0, 1, 2, …, where α_1 and α_2 are constants.

Example:

Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$.

Solution:

Characteristic equation of the given relation is $r^2 - 2r + 1 = 0$. Its only root are r = 1. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 1^n + \alpha_2 n1^n$, for some constants α_1 and α_2. From the initial conditions we have $a_0 = 3 = \alpha_1$, $a_1 = 6 = \alpha_1 + \alpha_2$. Solving these two equations we have $\alpha_1 = 3$ and $\alpha_2 = 3$. Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3(1^n + n1^n)$.

Theorem 3: (without proof)

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that $r^k - c_1r^{k-1} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n + \ldots + \alpha_kr_k^n$ for n = 0, 1, 2, …, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Example:

Solve the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$, $a_0 = 3$, $a_1 = 6$ and $a_2 = 9$.

Solution:

Characteristic equation of the given relation is $r^3 - 2r^2 - r + 2 = 0$. Its roots are r = 1, r = -1, and r = 2. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 1^n + \alpha_2(-1)^n + \alpha_3 2^n$, for some constants α_1, α_2, and α_3. From the initial conditions we have $a_0 = 3 = \alpha_1 + \alpha_2 + \alpha_3$, $a_1 = 6 = \alpha_1 - \alpha_2 + 2\alpha_3$, and $a_2 = 9 = \alpha_1 + \alpha_2 + 4\alpha_3$. Solving these two equations we have $\alpha_1 = 3/2$, $\alpha_2 = -1/2$, and $\alpha_3 = 2$. Hence, the solution is the sequence $\{a_n\}$ with $a_n = (3/2)1^n – (1/2)(-1)^n + 2.2^n$.

Theorem 4: (without proof)

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that $r^k - c_1r^{k-1} - \ldots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicity $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if

$$
\begin{aligned}
a_n = {} & (\alpha_{1,0} + \alpha_{1,1}n + \ldots + \alpha_{1,m1-1}n^{m1-1})\, r_1^n \\
& + (\alpha_{2,0} + \alpha_{2,1}n + \ldots + \alpha_{2,m2-1}n^{m2-1})\, r_2^n \\
& + \ldots + (\alpha_{t,0} + \alpha_{t,1}n + \ldots + \alpha_{t,mt-1}n^{mt-1})\, r_t^n
\end{aligned}
$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i-1$.

Example:

Solve the recurrence relation $a_n = 5a_{n-1} - 7a_{n-2} + 3a_{n-3}$ for $n \geq 3$, $a_0 = 1$, $a_1 = 9$ and $a_2 = 15$.

Solution:

Characteristic equation of the given relation is $r^3 - 5r^2 + 7r - 3 = 0$. Its roots are $r = 1$, $r = 3$, and $r = 1$. i.e. $r_1 = 1$, $m_1 = 2$ and $r_2 = 3$, $m_2 = 1$ Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = (\alpha_{1,0} + \alpha_{1,1}n)\ 1^n + (\alpha_{2,0})\ 3^n$, for some constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{2,0}$. From the initial conditions we have $a_0 = 5 = \alpha_{1,0} + \alpha_{2,0}$, $a_1 = 9 = \alpha_{1,0} + \alpha_{1,1} + 3\alpha_{2,0}$, and $a_2 = 15 = \alpha_{1,0} + 2\alpha_{1,1} + 9\alpha_{2,0}$. Solving these two equations we have $\alpha_{1,0} = 3/2$, $\alpha_{1,1} = 9$, and $\alpha_3 = -1/2$. Hence, the solution is the sequence $\{a_n\}$ with $a_n = (3/2)1^n + 9n1^n - (1/2)3^n$.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

The recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and F(n) is a function depending upon n. The recurrence relation preceding F(n) is called **associated homogeneous recurrence relation**. For example $a_n = 7a_{n-1} + 3a_{n-2} + 6n$ is a linear nonhomogeneous recurrence relation with constant coefficients.

Theorem 5: (without proof)

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$, then every solution of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$.

Example:

Find all the solutions of the recurrence relation $a_n = 4a_{n-1} + n^2$. Also find the solution of the relation with initial condition $a_1 = 1$.

Solution:

We have associated linear homogeneous recurrence relation as $a_n = 4a_{n-1}$. The root is 4, so the solutions are $a_n^{(h)} = \alpha 4^n$, where α is a constant. Since $F(n) = n^2$ is a polynomial of degree 2, a trial solution is a quadratic function in n, say, $p_n = an^2 + bn + c$, where a, b, and c are constants. To determine whether there are any solution of this form, suppose that $p_n = an^2 + bn + c$ is such solution. Then the equation $a_n = 4a_{n-1} + n^2$ becomes

$$
\begin{aligned}
an^2 + bn + c &= 4(a(n-1)^2 + b(n-1) + c) + n^2. \\
&= 4a\,n^2 - 8an + 4a + 4bn - 4b + 4c + n^2 \\
&= (4a + 1)n^2 + (-8a + 4b)n + (4a - 4b + 4c)
\end{aligned}
$$

Here $an^2 + bn + c$ is the solution if and only if $4a + 1 = a$ i.e. $a = -1/3$; $-8a + 4b = b$ i.e. $b = -8/3$; $4a - 4b + 4c = c$ i.e. $c = -28/3$. So $a_n^{(p)} = -(n^2 + 8n + 28)/3$ is a particular solution and all solutions are $a_n = \{a_n^{(p)} + a_n^{(h)}\} = -(n^2 + 8n + 28)/3 + \alpha 4^n$, where α is a constant.

For solution with $a_n = 1$, we have $a_n = 1 = -(1 + 8 + 28)/3 + \alpha 4$ i.e. $\alpha = 10/3$. Then the solution is $a_n = (10.4^n - n^2 - 8n - 28)/3$.

Theorem 6: (without proof)

Suppose that {an} satisfies the linear nonhomogeneous recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n) = (b_tn^t + b_{t-1}n^{t-1} + \ldots + b_1n + b_0)s^n$, where $b_0, b_1, \ldots, b_t$ and s are real numbers.

When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)s^n$.

When s is a root of the characteristic equation and its multiplicity is m, there is a particular solution of the form

$n^m(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)s^n$.

Example:

Find the solution of the recurrence relation $a_n = 2a_{n-1} + n.2^n$.

Solution:

We have the associated linear homogeneous recurrence relation is $a_n = 2a_{n-1}$. The characteristic equation for this would be r-2 = 0, so the root is 2 and hence the solutions are $a_n^{(h)} = \alpha 2^n$, where α is a constant. We have $F(n) = n.2^n$. (Of the form $n.s^n$) where s is the root of the characteristic equation and the multiplicity of 2 is 1 so, the particular solution has the form $n.(p_1 n)2^n = p_1 n^2 2^n$. The solution is, $a_n = \alpha 2^n + p_1 n^2 2^n$.

Recurrences Applications

One of the application areas of recurrence relations is analysis of divide and conquer algorithms.

Divide and Conquer Algorithms

Divide and conquer algorithms divide a problem of larger size to the problem of smaller size so continually such that the problem of the smallest size that has trivial solution is obtained. If f(n) represents the number of operations required to solve the problem of size n, then follows the recurrence relation f(n) = af(n/b) + g(n), called divide and conquer recurrence relation. In the relation above the problem of size n is partitioned into a parts of the problem of the size n/b and g(n) is the operations required to conquer the solutions. In this section no algorithms are presented but their recurrence relations are tried to achieve.

Example 1:Fibonacci Numbers

We know that the fibonacci numbers are generated by the formula $f_n = f_{n-1} + f_{n-2}$. Here n^{th} Fibonacci number is the sum of $(n-1)^{th}$ and $(n-2)^{nd}$ fibonacci numbers. Here for the initial conditions are $f_0 = 0$, and $f_1 = 1$. Use of the above relation does not exactly produce the recurrence relation mentioned above, however this is an example of divide and conquer algorithm since each time the problem is changed into two problems of smaller size.

Example 2: Merge Sort

In merge sorting the input sequence of n items is broken down into 2 halves (here there may be difference in 1 item between two parts). Since the list of size n need more comparisons than list of size n/2, the problem here is simplified. This process continues until all the comparisons are trivial. This problem satisfies the divide and conquer recurrence relation

$M(n) = 2M(n/2) + O(1)$.

Theorem 7: (without proof)

Let f be an increasing function that satisfies the recurrence relation $f(n) = af(n/b) + c$ whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then $f(n)$ is $\begin{cases} O(n^{\log_b a}) & \text{if } a > 1, \\ O(\log n) & \text{if } a = 1. \end{cases}$. Furthermore, when $n = b^k$, where k is a positive integer, $f(n) = C_1 n^{\log_b a} + C_2$, where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

Example:

Solve the recurrence relations $f(n) = 2f(n/2) + 4$ and $g(n) = g(n/2) + 2$ to get their upper bound.

Solution:

Using theorem 7 above we have $f(n) = O(n)$ since, $\log_b{}^a = \log_2{}^2 = 1$. Similarly we have $g(n) = O(\log n)$.

Theorem 7: Master Theorem (without proof)

Let f be an increasing function that satisfies the recurrence relation $f(n) = af(n/b) + cn^d$ whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, c is a positive real number, and d is nonnegative real number. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Example:

Solve using Master Theorem the following recurrences where each n is by 2^k.

i) $f(n) = 2f(n/2) + n^2$.

ii) $f(n) = 2f(n/2) + n$.

iii) $f(n) = 7f(n/2) + n^2$.

Solution:

Using master theorem we have

i) $f(n) = O(n^2)$, since (a)2 < $2^2(b^d)$

ii) $f(n) = O(n\log n)$, since (a)2 = $2(b^d)$

iii) $f(n) = O(n^{2.807})$, since (a)7 > $2^2(b^d)$

Inclusion and Exclusion and Applications

In the counting problems where the sets are not disjoint we extensively use inclusion exclusion principle. Given set A and set B the union of A and B is given by the formula $|A \cup B| = |A| + |B| - |A \cap B|$.

Example 1:

There are 345 students at a college who have taken a course in calculus, 212 who have taken a course in discrete mathematics, and 188 who have taken course in both calculus and discrete mathematics. How many students have taken the course in either calculus or discrete mathematics?

Solution:

Here we have $|C| = 345$ (students taking the calculus course), $|D| = 212$ (students taking the discrete mathematics course), and $|C \cap D| = 188$ (students taking both discrete mathematics and calculus courses). Number of students taking either discrete mathematics or calculus, $|C \cup D| = |C| + |D| - |C \cap D| = 345 + 212 – 188 =369$.

Theorem 8: The Principle of Inclusion – Exclusion

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then $|A_1 \cup A_2 \cup \ldots \cup A_n|$

$$= \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Proof:

We try to prove this theorem using counting technique. Suppose that the element a is a member of exactly r of the sets $A_1, A_2, \ldots, A_n$, where $1 \le r \le n$. Then this element is counted C(r, 1) times in $\sum|A_i|$. The element a is counted C(r, 2) times in $\sum|A_i \cap A_j|$. So if there are m sets C(r, m) times a is counted from the summation that contains m of the sets of A_i. Using these counts in the right hand side of the formula above we get

$C(r, 1) - C(r, 2) + C(r, 3) - \ldots + (-1)^{r+1}C(r, r)$ counts for the common element (i.e. a here)

We know that

$C(r, 0) - C(r, 1) + C(r, 2) - \ldots + (-1)^{r}C(r, r) = 0.$

So,

$-C(r, 0) + C(r, 1) - C(r, 2) - \ldots + (-1)^{r+1}C(r, r) = 0$

$C(r, 0) = C(r, 1) - C(r, 2) - \ldots + (-1)^{r+1}C(r, r) = 1$ [since C(r, 0) = 1]

From this fact we know that each element from the all sets are counted just once. Hence the proof.

Example:

How many elements are in the union of four sets if each of the set has 100 elements, each pair of the set shares 50 elements, each three of the sets share 25 elements, and there are 5 elements in all four sets?

Solution:

Let the four sets be A, B, C, and D. then we have $|A| = |B| = |C| = |D| = 100$, $|A \cap B| = |A \cap C| = |A \cap D| = |B \cap C| = |B \cap D| = |C \cap D| = 50$, $|A \cap B \cap C| = |A \cap B \cap D| = |A \cap C \cap D| = |B \cap C \cap D| = 25$ and $|A \cap B \cap C \cap D| = 5$. Principle of inclusion exclusion shows that

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$

$=100 + 100 + 100 + 100 –50 –50 –50 –50 –50 -50 + 25 +25 + +25 +25 – 5 = 195.$

Alternative form of Inclusion – Exclusion

Let A_i be the subset containing the elements that have property p_i. The number of elements with all the properties $P_{i1}, P_{i2}, \ldots, P_{ik}$ will be denoted by $N(P_{i1} P_{i2} \ldots P_{ik})$. In set notation we can write these quantities as

$|A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}| = N(P_{i1} P_{i2} \ldots P_{ik})$.

If the number of elements with none of the properties is denoted by $N(P'_1 P'_2 \ldots P'_n)$ and the number of elements in the set is denoted by N. we have

$N(P'_1 P'_2 \ldots P'_n) = N - |A_1 \cup A_2 \cup \ldots \cup A_n|$ Using inclusion exclusion principle, we have

$$N(P'_1 P'_2 \ldots P'_n) = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) - \sum_{1 \le i < j < k \le n} N(P_i P_j P_k) + \ldots + (-1)^n N(P_1 P_2 \cdots P_n).$$

Example:

Find the number of solutions of the equation $x_1 + x_2 + x_3 = 13$, where x_1, x_2 and x3 are nonnegative integer less than 6.

Solution:

Let properties P_1 be $x_1 \ge 6$, P_2 be $x_2 \ge 6$ and P_2 be $x_3 \ge 6$ then the number of solutions of the equation $x_1 + x_2 + x_3 = 13$, where x_1, x_2 and x3 are nonnegative integer less than 6 is

$N(P'_1 P'_2 P'_3) = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$

Now, the equation can be viewed as selecting 13 items where there are x_1 items of type one, x_2 items of type two, and x_3 items of type three. Here we can use the same numbers as much as we can from a set with three numbers (repetition is allowed) so we have,

$C(3 + 13 - 1, 13) = C(15,13) = 15!/(2!13!) = 15.14/2 = 105$ solutions i.e. N = 105.

Similarly, $N(P_1) = N(P_2) = N(P_3) = C(3 + 7 - 1,7) = C(9,7) = 36$. [Here for $x_i \ge 6$ we can select 6 items of the particular type and remaining 7 items are selected to have 13 items from other types.]. Number of solutions with $x_1 \ge 6$ and $x_2 \ge 6$ is $C(3 + 1 - 1,1) = 3$, number of solutions with $x_1 \ge 6$ and $x_3 \ge 6$ is $C(3 + 1 - 1,1) = 3$, number of solutions with $x_2 \ge 6$ and $x_3 \ge 6$ is $C(3 + 1 - 1,1) = 3$, and number of solutions with $x_1 \ge 6$, $x_2 \ge 6$ and $x_3 \ge 6$ is 0. so using above formula we have $N(P'_1 P'_2 P'_3) = 105 - 3.36 + 3.3 - 0 = 6$.

Applications

Example 1:(Sieve of Eratosthenes)

Find the number of primes less than 200 using the principle of inclusion exclusion.

Solution:

We know that the composite number not exceeding 200 has prime factor not exceeding $\sqrt{200} = 14.14$ i.e. not exceeding 14. Primes less than 14 are 2, 3, 5, 7, 11, and 13. Now we can say that the above 6 primes and all other numbers greater than 1 not exceeding 200 and not divisible by above 6 primes are primes. So if we take properties, P_1 that a number is divisible by 2, P_2 that a number is divisible by 3, P_3 that a number is divisible by 5, P_4 that a number is divisible by 7, P_5 that a number is divisible by 11, and P_6 that a number is divisible by 13, then prime numbers not exceeding 200 is $6 + N(P'_1 P'_2 P'_3 P'_4 P'_5 P'_6)$. We know that there are 199 integers not exceeding 200 and greater than 1 we have by principle of inclusion exclusion

$N(P'_1 P'_2 P'_3 P'_4 P'_5 P'_6) =$

$199 – N(P_1) – N(P_2) – N(P_3) – N(P_4) – N(P_5) – N(P_6) + N(P_1P_2) + N(P_1P_3) + N(P_1P_4) + N(P_1P_5) + N(P_1P_6) + N(P_2P_3) + N(P_2P_4) + N(P_2P_5) + N(P_2P_6) + N(P_3P_4) + N(P_3P_5) + N(P_3P_6) + N(P_4P_5) + N(P_4P_6) + N(P_5P_6) - N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_2P_5) - N(P_1P_2P_6) - N(P_1P_3P_4) - N(P_1P_3P_5) - N(P_1P_3P_6) - N(P_1P_4P_5) - N(P_1P_4P_6) - N(P_1P_5P_6) - N(P_2P_3P_4) - N(P_2P_3P_5) - N(P_2P_3P_6) - N(P_2P_4P_5) - N(P_2P_4P_6) - N(P_2P_5P_6) - N(P_3P_4P_5) - N(P_3P_4P_6) - N(P_3P_5P_6) - N(P_{34}P_5P_6) + N(P_1P_2P_3P_4) + N(P_1P_2P_3P_5) + N(P_1P_2P_3P_6) + N(P_1P_2P_4P_5) + N(P_1P_2P_4P_6) + N(P_1P_2P_5P_6) + N(P_1P_3P_4P_5) + N(P_1P_3P_4P_6) + N(P_1P_3P_5P_6) + N(P_1P_4P_5P_6) + N(P_2P_3P_4P_5) + N(P_2P_3P_4P_6) + N(P_2P_3P_5P_6) + N(P_2P_4P_5P_6) + N(P_3P_4P_5P_6) - N(P_1P_2P_3P_4P_5) - N(P_1P_2P_3P_4P_6) - N(P_1P_2P_3P_5P_6) - N(P_1P_2P_4P_5P_6) - N(P_1P_3P_4P_5P_6) - N(P_2P_3P_4P_5P_6) + N(P_1P_2P_3P_4P_5 P_6)$

$$= 199 - \left\lfloor \frac{200}{2} \right\rfloor - \left\lfloor \frac{200}{3} \right\rfloor - \left\lfloor \frac{200}{5} \right\rfloor - \left\lfloor \frac{200}{7} \right\rfloor - \left\lfloor \frac{200}{11} \right\rfloor - \left\lfloor \frac{200}{13} \right\rfloor + \left\lfloor \frac{200}{2.3} \right\rfloor + \left\lfloor \frac{200}{2.5} \right\rfloor + \left\lfloor \frac{200}{2.7} \right\rfloor +$$

$$\left\lfloor \frac{200}{2.11} \right\rfloor + \left\lfloor \frac{200}{2.13} \right\rfloor + \left\lfloor \frac{200}{3.5} \right\rfloor + \left\lfloor \frac{200}{3.7} \right\rfloor + \left\lfloor \frac{200}{3.11} \right\rfloor + \left\lfloor \frac{200}{3.13} \right\rfloor + \left\lfloor \frac{200}{5.7} \right\rfloor + \left\lfloor \frac{200}{5.11} \right\rfloor + \left\lfloor \frac{200}{5.13} \right\rfloor + \left\lfloor \frac{200}{7.11} \right\rfloor +$$

$$\left\lfloor \frac{200}{7.13} \right\rfloor + \left\lfloor \frac{200}{11.13} \right\rfloor - \left\lfloor \frac{200}{2.3.5} \right\rfloor - \left\lfloor \frac{200}{2.3.7} \right\rfloor - \left\lfloor \frac{200}{2.3.11} \right\rfloor - \left\lfloor \frac{200}{2.3.13} \right\rfloor - \left\lfloor \frac{200}{2.5.7} \right\rfloor - \left\lfloor \frac{200}{2.5.11} \right\rfloor - \left\lfloor \frac{200}{2.5.13} \right\rfloor -$$

$$\left\lfloor\frac{200}{2.7.11}\right\rfloor-\left\lfloor\frac{200}{2.7.13}\right\rfloor-\left\lfloor\frac{200}{2.11.13}\right\rfloor-\left\lfloor\frac{200}{3.5.7}\right\rfloor-\left\lfloor\frac{200}{3.5.11}\right\rfloor-\left\lfloor\frac{200}{3.5.13}\right\rfloor-\left\lfloor\frac{200}{3.7.11}\right\rfloor-\left\lfloor\frac{200}{3.7.13}\right\rfloor-$$

$$\left\lfloor\frac{200}{3.11.13}\right\rfloor-\left\lfloor\frac{200}{5.7.11}\right\rfloor-\left\lfloor\frac{200}{5.7.13}\right\rfloor-\left\lfloor\frac{200}{5.11.13}\right\rfloor-\left\lfloor\frac{200}{7.11.13}\right\rfloor+\left\lfloor\frac{200}{2.3.5.7}\right\rfloor+\left\lfloor\frac{200}{2.3.5.11}\right\rfloor+$$

$$\left\lfloor\frac{200}{2.3.5.13}\right\rfloor+\left\lfloor\frac{200}{2.3.7.11}\right\rfloor+\left\lfloor\frac{200}{2.3.7.13}\right\rfloor+\left\lfloor\frac{200}{2.3.11.13}\right\rfloor+\left\lfloor\frac{200}{2.5.7.11}\right\rfloor+\left\lfloor\frac{200}{2.5.7.13}\right\rfloor+$$

$$\left\lfloor\frac{200}{2.5.11.13}\right\rfloor+\left\lfloor\frac{200}{2.7.11.13}\right\rfloor+\left\lfloor\frac{200}{3.5.7.11}\right\rfloor+\left\lfloor\frac{200}{3.5.7.13}\right\rfloor+\left\lfloor\frac{200}{3.5.11.13}\right\rfloor+\left\lfloor\frac{200}{3.7.11.13}\right\rfloor+$$

$$\left\lfloor\frac{200}{5.7.11.13}\right\rfloor-\left\lfloor\frac{200}{2.3.5.7.11}\right\rfloor-\left\lfloor\frac{200}{2.3.5.7.13}\right\rfloor-\left\lfloor\frac{200}{2.3.5.11.13}\right\rfloor-\left\lfloor\frac{200}{2.3.7.11.13}\right\rfloor-\left\lfloor\frac{200}{2.5.7.11.13}\right\rfloor-$$

$$\left\lfloor\frac{200}{3.5.7.11.13}\right\rfloor+\left\lfloor\frac{200}{2.3.5.7.11.13}\right\rfloor$$

= 199 – (100 + 66 + 40 + 28 + 18 + 15) + (33 + 20 + 14 + 9 + 7 + 13 + 9 + 6 + 5 + 5 + 3 + 3 + 2 + 2 + 1) – (6 + 4 + 3 + 2 + 2 + 1 + 1 + 1 + 1 + 0 + 1 + 1 + 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0) + all others will be 0.

= 199 – 267 +132 – 24 = 40.

Now the total number of primes not exceeding 200 is 6 + 40 = 46.

Example 2:(The Number of Onto Functions)

How many onto functions are there from a set with seven elements to one with five elements?

Solution:

Suppose that the elements in codomain be a_i, i = 1 …5. Let P_i, i =1…5 be the properties that a_i's for i = 1, 2, …, 5 are not in the range of the functions, respectively. To have a number of onto function if we can exclude the functions holding the above properties we are done. So, using principle of inclusion exclusion we have,

$$N(P'_1P'_2P'_3P'_4P'_5)=N-\sum_{1\le i\le n}N(P_i)+\sum_{1\le i<j\le n}N(P_iP_j)-\sum_{1\le i<j<k\le n}N(P_iP_jP_k)+\sum_{1\le i<j<k\le l\le n}N(P_iP_jP_kP_l)-N(P_1P_2P_3P_4P_5).$$

In the formula above,

N, total number of functions = 5^7.

We can select any one element of codomain to be not in the range then selecting this element can be done in C(5,1) ways. There are 4 elements left in the codomain after not taking one element for function then for each such a removal of elements from range we can make 4^7 numbers of total functions. So total number of functions with one element missing from the range is $\sum_{1\le i\le n} N(P_i) =$ C(5,1) 4^7.

We can select any two elements of codomain to be not in the range then selecting this element can be done in C(5,2) ways. There are 3 elements left in the codomain after not taking two elements for function then for each such a removal of elements from range we can make 3^7 numbers of total functions. So total number of functions with one element missing from the range is $\sum_{1\le i<j\le n} N(P_iP_j) =$ C(5,2) 3^7.

Using similar reason as above,

$\sum_{1\le i<j\le k\le n} N(P_iP_jP_k)$ = C(5,3) 2^7.

$\sum_{1\le i<j\le k\le l\le n} N(P_iP_jP_kP_l)$ = C(5,4) 1^7.

$N(P_1P_2P_3P_4P_5) = 0$.

Then we have total number of onto functions as 5^7 - C(5,1) 4^7 + C(5,2) 3^7 - C(5,3) 2^7 + C(5,4) 1^7 – 0 = 78125 – 81920 + 21870 – 1280 + 5 = 16800.

Example 3:(Derangements)

The permutations of n elements that leave no objects in their original position are called derangements.

How many derangements are there of a set with four elements?

Solution:

Let a permutation have property P_i if it fixes an element i. Then we can say that the number of derangements is the number of permutations having none of the properties P_i for i = 1, 2, 3, 4. Then we can write $N(P'_1P'_2P'_3P'_4)$, using principle of inclusion exclusion, equals to N - $\sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) - \sum_{1\le i<j<k\le n} N(P_iP_jP_k) + N(P_1P_2P_3P_4)$., where

N is the total number of permutations i.e. 4! = 24. If one element is fixed some where in

C(4,1) place then total number of such permutations would be $\sum_{1 \le i \le n} N(P_i)$ = C(4,1) 3! = 24.

If two elements are fixed some where in C(4,2) places then the total number of such permutations would be $\sum_{1 \le i < j \le n} N(P_i P_j)$ = C(4,2) 2! = 12.

Similarly, $\sum_{1 \le i < j \le k \le n} N(P_i P_j P_k)$ = C(4,3)1! = 4 and

$N(P_1 P_2 P_3 P_4)$ = C(4,4)0! =1. Hence total number of derangements is

24 – 24 + 12 – 4 +1 = 9.

Self Studies

Read chapter 6 of your textbook such that you can cover all the read materials in the class.