[Combinatorics]

Discrete Structures (CSc 511)

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Elementary Combinatorics

Combinatorics is the study of arrangements or possible combination of objects. We come up with different situations where we need to identify the number of elements having similar features, number of steps required to solve the problem, amount of storage required, etc.

Basics of Counting

There are two basic counting principles that can be used to solve the counting problems. We define those two principles below:

Sum rule: **The principle of disjunctive counting.**

If the first task can be done in m ways and the second task can be done in n ways and if both the tasks cannot be done at a time, then there are m + n ways to do one of the task. We can generalize this rule as, if a set X is union of disjoint nonempty subsets $S_1, S_2, \ldots, S_n$, then $|X| = |S_1| + |S_2| + \ldots + |S_n|$.

Remember: the set must be disjoint, for overlapping set we use different principle called inclusion exclusion principle (will be covered later).

Example 1:

In how many ways we can draw a heart or a diamond from an ordinary deck of playing cards?

Solution:

There are total 13 cards of heart and 13 card of diamond. So, by sum rule total number of ways of picking heart or diamond is 13 + 13 = 26.

Example 2:

How many ways we can get a sum of 4 or of 8 when two distinguishable dice (say one die is red and the other is white) are rolled?

Solution:

Since dice are distinguishable outcome (1, 3) is different form (3, 1) so to get 4 as sum we have the pairs (1, 3), (3, 1), (2, 2), so total of 3 ways. And similarly getting 8 can be from pairs (2, 6), (6, 2), (3, 5), (5, 3), (4, 4), so total 5 ways. Hence getting sum of 4 or 8 is 3 + 5 = 8.

Product Rule: Principle of sequential counting.

If a work can be done in m ways and another work can be done after the completion of first work in n ways, then there are $m \times n$ ways to do the task that consists both the work. Generalizing the rule, if $S_1, S_2, \ldots, S_n$ are non empty sets, then the number of elements in the Cartesian product $S_1 \times S_2 \times \ldots \times S_n$, is the product $\prod_{i=1}^{n} |S_i|$ i.e. $|S_1 \times S_2 \times \ldots \times S_n| = \prod_{i=1}^{n} |S_i|$.

Example 1:

An office building contains 27 floors and has 37 offices on each floor. How many offices are there are in the building?

Solution:

By the product rule there are 27.37 = 999 offices in the building.

Example 2:

How many different three-letter initials with none of the letters can be repeated can people have?

Solution:

Here the first letter can be chosen in 26 ways, since the first letter is assigned we can choose second letter in 25 ways and in the same manner we can choose third letter in 24 ways. So by product rule number of different three-letter initials are 26.25.24 = 15600.

More Examples on Basics:

Example 1:

How many strings are there of four lowercase letters that have the letter x in them?

Solution:

There are total 26.26.26.26 strings of four lowercase letters, by product rule. In the same way we can say that there are 25.25.25.25 strings of four lowercase letters without x, since without x there will be a set of 25 characters only. So there are total of 26.26.26.26 - 25.25.25.25 = 66351 four lowercase letter strings with x in them. This is true because we are decrementing total numbers of strings with the number of strings that do not contain x in them so at least one x will be in the strings.

Example 2:

How many functions are there from the set {1, 2, …, n}, where n is a positive integer, to the set {0, 1}.

Solution:

Each element from the set {1, 2, …, n}can map the set {0, 1} in 2 ways. Since there are n elements in the first set by the product rule number of possible functions are 2.2.2…..n^{th} term i.e. 2^n.

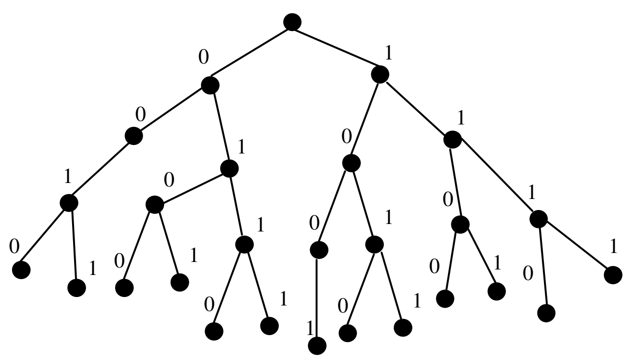
Tree Diagrams

We can use a tree diagram to solve the counting problem (don't worry we will study tree in detail later).

Example:

Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

Solution:



From the above tree we can get that there are total number of 13 bit strings of length four with no three consecutive zeroes. For this we can explain as if a bit string start with 1 then there is only one bit string that can have three consecutive 0s (1000), the total number of bit string of length starting with 1 and have no three consecutive 0s is thus 2.2.2 –1 =7, similarly if the bit string start with 0 then there is a possibility that the next two bits may be 0 so the possible bit strings of length four with consecutive 0s starting with 0 are 0001 and 0000, so the total number of bit string of length starting with 1 and have no three consecutive 0s is thus 2.2.2 –2 =6. Using the sum rule the total number of such bit strings is 13.

Pigeonhole principle

The pigeonhole principle states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons. The concept of pigeons can be extended to any objects.

Theorem 1: The pigeonhole principle

If k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof:

We use proof by contradiction here. Suppose that k+1 or more boxes are placed into k boxes and no boxes contain more than one object in it. If there are k boxes then there must be k objects such that there are no two objects in a box. This contradicts our assumption. So there is at least one box containing two or more of the objects.

Example:

Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

Proof:

There are 30 students in the class and we have 26 letters in English alphabet that can be used in beginning of the last name. Since there are only 26 letters and 30 students, by pigeonhole principle at least two students have the last name that begins with the same letter.

Theorem 2: The generalized pigeonhole principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof:

Suppose N objects are placed into k boxes and there is no box containing more than $\lceil N/k \rceil$-1 objects. So the total number of objects is at most

$k(\lceil N/k \rceil-1) < k((N/k + 1) -1) = N$. This is the contradiction that N objects are placed into k boxes (since we showed that there are total number of objects less than N). Hence, the proof.

Example:

If a class has 24 students, what is the maximum number of possible grading that must be done to ensure that there at least two students with the same grade.

Solution:

There are total 24 students and the class and at least two students must have same grade. If the number of possible grades is k then by pigeonhole principle we have $\lceil 24/k \rceil = 2$. Here the largest value that k can have is 23 since 24 = 23.1 + 1. So the maximum number of possible grading to ensure that at least two of the students have same grading is 23.

Applications: Pigeonhole principles

Example 1:

How many numbers must be selected from the set {1, 3, 5, 7, 9, 11, 13, 15} to guarantee that at least one pair of these numbers add up to 16?

Solution:

The pairs of numbers that sum 16 are (1,15), (3, 13), (5, 11), (7, 9) i.e. 4 pairs of numbers are there that add to 16. If we select 5 numbers then by pigeonhole principle there are at least $\lceil 5/4 \rceil$ = 2 numbers, that are from the set of selected 5 numbers, that constitute a pair. Hence 5 numbers must be selected.

Example 2:

Find the least number of cables required to connect eight computers to four printers to guarantee that four computers can directly access four different printers. Justify your answer.

Solution:

If we connect first 4 computers directly to each of the 4 printers and the other 4 computers are connected to all the printers, then the number of connection required is 4 + 4.4 = 20. To verify that 20 is the least number of cables required we have if there may be less than 20 cables then we would have 19 cables, then some printers would be connected by at most $\lfloor 19/4 \rfloor$ = 4 cables to the computers. Then the other 3 printers would have to connect the other 4 computers here all the computers cannot simultaneously access different printer. So if we use 20 cables, then at least $\lceil 20/4 \rceil$ = 5 cables connects a printer to a computer directly. So the remaining 3 printers are required to connect only 3 computers. Hence the least number of cables required is 20.

Example 3:

Among n + 1 different integral powers of an integer a, there are at least two of them that have same remainder when divided by the positive integer n.

Proof:

Let a^1, a^2, …, a^{n+1}, be n+1 different integral powers of integer a. when these numbers are divided by n then the set of possible remainders is {0, 1, 2, …, n-1). Since there are n remainders and n+1 numbers by pigeonhole principle at least 2 of the reminders must be same.

Self Studies

Read chapter 4.1 and 4.2 of your textbook such that you can cover all the read materials in the class.