UNIT - I
LESSON - 1
The Solution of Numerical Algebraic and Transcendental Equations

Contents:

1.0 Aims and Objectives

In this Lesson, we have discussed about the solution of equations of the form $f(x) = 0$, $f(x)$ is polynomial of degree two or three or four or more. We get the solution of the equations by using Bisection method, Newton–Raphson method and method of false position.

After reading this lesson, you should be able to

- To compute solution of equations by using the Bisection Method.
- To compute real root by Newton-Raphson Method.
- To find out the real root of equation by false position method

1.1 Introduction

The solution of the equation of the form $f(x) = 0$ occurs in the field of science, engineering and other applications. If $f(x)$ is a polynomial of degree two or more , we have formulae to find solution. But, if $f(x)$ is a transcendental function, we do not have formulae to obtain solutions. When such type of equations are there, we have some methods like Bisection method, Newton-Raphson Method and The method of false position. Those methods are solved by using a theorem in theory of equations, *i.e., If f(x) is continuous in the interval (a,b) and if f(a) and f(b) are of opposite signs, then the equation f(x) = 0 will have atleast one real root between a and b.*

1.2 Bisection Method

Let us suppose we have an equation of the form $f(x) = 0$ in which solution lies between in the range (a,b). Also $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are opposite signs, then there exist atleast one real root between a and b.

Let $f(a)$ be positive and $f(b)$ negative. Which implies atleast one root exits between a and b. We assume that root to be $x_o = (a+b)/2$. Check the sign of $f(x_o)$. If $f(x_0)$is negative , the root lies between a and x_o. If $f(x_0)$is positive , the root lies between x_o and b. Subsequently any one of this case occur.

$$x_1 = \frac{a + x_0}{Z} \text{ or } x_1 = \frac{x_0 + b}{2}$$

When $f(x_1)$ is negative , the root lies between xo and x1 and let the root be $x_2 = (x_0 + x_1) / 2$. Again $f(x_2)$ negative then the root lies between x_o and x_2, let $x_3 = (x_0+x_2)/2$ and so on. Repeat the process $x_o, x_1, x_2, \ldots$. Whose limit of convergence is the exact root.

Steps:

1. Find a and b in which f(a) and f(b) are opposite signs for the given equation using trial and error method.
2. Assume initial root as $x_o = (a+b)/2$.
3. If $f(x_0)$is negative , the root lies between a and x_o and take the root as $x_1 = (x_o+a)/2$.
4. If $f(x_o)$ is positive , then the root lies between x_o and b and take the root as $x_1 = (x_o +b)/ 2$.
5. If $f(x_1)$ is negative , the root lies between x_o and x_1 and let the root be $x_2 = (x_0 + x_1) / 2$.
6. If $f(x_2)$ is negative , the root lies between x_o and x_1 and let the root be $x_3 = (x_0 + x_2) / 2$.
7. Repeat the process until any two consecutive values are equal and hence the root.

Illustrations:

Find the positive root of $x^3 - x = 1$ correct to four decimal places by bisection method.

Solution:

Let $f(x) = x^3 - x - 1$

$f(0) = 0^3 - 0 - 1 = -1 = -ve$
$f(1) = 1^3 - 1 - 1 = -1 = -ve$
$f(2) = 2^3 - 2 - 1 = 5 = +ve$

So root lies between 1 and 2 , we can take (1+2) /2 as initial root and proceed.
i.e., f(1.5) = 0.8750 = +ve
and f(1) = -1 = -ve

So root lies between 1 and 1.5 ,
Let x_o = (1+1.5) /2 as initial root and proceed.
f(1.25) = - 0.2969

So root lies between x_1 *bet*ween 1.25 and 1.5
Now x_1 = (1.25 + 1.5) /2 = *1.3750*
f(1.375) = 0.2246 = +ve

So root lies between x_2 *bet*ween 1.25 and 1.375
Now x_2 = *(1.25 + 1.375)/2 = 1.3125*
f(1.3125) = -0.051514 = -ve

Therefore, root lies between 1.375and 1.3125
Now x_3 = (1.375 + 1.3125) / 2 = *1.3438*
f(1.3438) = 0.082832 = +ve

So root lies between 1.3125 and 1.3438
Now x_4 = (1.3125 + 1.3438) / 2 = *1.3282*
f(1.3282) = 0.014898 = +ve

So root lies between 1.3125 and 1.3282
Now x_5 = (1.3125 + 1.3282) / 2 = *1.3204*
f(1.3204) = - 0.018340 = -ve

So root lies between 1.3204 and 1.3282
Now x_6 = (1.3204 + 1.3282) / 2 = *1.3243*
f(1.3243) = -ve

So root lies between 1.3243 and 1.3282
Now x_7 = (1.3243 + 1.3282) / 2 = *1.3263*
f(1.3263) = +ve

So root lies between 1.3243 and 1.3263

Now x_8 = (1.3243 + 1.3263) / 2 = *1.3253*

f(1.3253) = *+ve*

So root lies between 1.3243 and 1.3253

Now x_9 = (1.3243 + 1.3253) / 2 = *1.3248*

f(1.3248) = *+ve*

So root lies between 1.3243 and 1.3248

Now x_{10} = (1.3243 + 1.3248) / 2 = *1.3246*

f(1.3246) = *-ve*

So root lies between 1.3248 and 1.3246

Now x_{11} = (1.3248 + 1.3246) / 2 = *1.3247*

f(1.3247) = *-ve*

So root lies between 1.3247 and 1.3248

Now x_{12} = (1.3247 + 1.3247) / 2 = *1.32475*

Therefore , the approximate root is 1.32475

Illustration :

Find the positive root of x – cos x = 0 by bisection method.

Solution :

Let f(x) = x – cos x

f(0) = 0 - cos (0) = 0 -1 = -1 = -ve

f(0.5) = 0.5 – cos (0.5) = -0.37758 = -ve

f(1) = 1 – cos (1) = 0.42970 = +ve

So root lies between 0.5 and 1

Let x_o = (0.5 +1) /2 as initial root and proceed.

f(0.75) = *0.75 – cos (0.75) = 0.018311 = +ve*

So root lies between 0.5 and 0.75

x_1 = (0.5 +0.75) /2 =0.625

f(0.625) = *0.625 – cos (0.625)* = - *0.18596*

So root lies between 0.625 and 0.750

x_2 = (0.625 +0.750) /2 = 0.6875
f(0.6875) = - *0.085335*

So root lies between 0.6875 and 0.750

x_3 = (0.6875 +0.750) /2 = 0.71875
f(0.71875) = 0.71875-cos(0.71875) = - *0.033879*

So root lies between 0.71875 and 0.750

x_4 = (0.71875 +0.750) /2 = 0.73438
f(0.73438) = -0.0078664 = - ve

So root lies between 0.73438 and 0.750

x_5 = 0.742190
f(0.742190) = 0.0051999 = + ve

x_6 = (0.73438 +0.742190) /2 = 0.73829
f(0.73829) = -0.0013305

So root lies between 0.73829 and 0.74219

x_7 = (0.73829+0.74219) = 0.7402
f(0.7402) = 0.7402-cos(0.7402) = 0.0018663

So root lies between 0.73829 and 0.7402

x_8 = 0.73925
f(0.73925) = 0.00027593
x_9 = 0.7388

The root is 0.7388.

Check Your Progress :

1. Find a positive root of the following equation by bisection method :
   (i) x^3 -4x -9 (Ans: 2.7065)

1.3 Newton-Raphson method (or Newton's method)

Let us suppose we have an equation of the form $f(x) = 0$ in which solution is lies between in the range (a,b). Also $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are opposite signs, then there exist atleast one real root between a and b.

Let $f(a)$ be positive and $f(b)$ negative. Which implies atleast one root exits between a and b. We assume that root to be either a or b, in which the value of f(a) or f(b) is very close to zero. That number is assumed to be initial root.. Then we iterate the process by using the following formula until the value is converges.

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

Steps:

1. Find a and b in which $f(a)$ and $f(b)$ are opposite signs for the given equation using trial and error method.
2. Assume initial root as $X_o = a$ i.e., *if* $f(a)$is very close to zero or $X_o = b$ *if* $f(a)$is very close to zero

3. Find X_1 by using the formula

$$X_1 = X_o - \frac{f(X_o)}{f'(X_o)}$$

4. Find X_2 by using the following formula

$$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$$

5. Find $X_3, X_4, \ldots X_n$ until any two successive values are equal

Illustration 1:

Find the positive root of $f(x) = 2x^3 - 3x-6 = 0$ by Newton – Raphson method correct to five decimal places.

Solution:

Let $f(x) = 2x^3 - 3x - 6$; $f'(x) = 6x^2 - 3$

$f(1)$ = 2-3-6 = -7 = -ve
$f(2)$ = 16 – 6-6 = 4 = +ve

So, a root between 1 and 2 . In which 4 is closer to 0 Hence we assume initial root as 2.

Consider $x_0 = 2$

So $X_1 = X_0 – f(X_0)/f'(X_0)$

$= X_0 - ((2X_0^3 - 3X_0 - 6) / 6\alpha_0 - 3) = (4X_0^3 + 6)/(6X_0^2 - 3)$

$X_{i+1} = (4X_i^3 + 6)/(6X_i^2 - 3)$

$X_1 = (4(2)^2+6)/(6(2)^2 - 3) = 38/21 = 1.809524$

$X_2 = (4(1.809524)^3+6)/(6(1.809524)^2 - 3) = 29.700256/16.646263 = 1.784200$

$X_3 = (4(1.784200)^3+6)/(6(1.784200)^2 - 3) = 28.719072/16.100218 = 1.783769$

$X_4 = (4(1.783769)^3+6)/(6(1.783769)^2 - 3) = 28.702612/16.090991 = 1.783769$

Since X_3 and X_4 are equal, hence root is 1.783769

Illustrations 2:

Using Newton's method, find the root between 0 and 1 of $x^3 = 6x – 4$ correct to 5 decimal places.

Solution :

Let $f(x) = x^3 - 6x + 4$; $f(0) = 4$ = +ve; $f(1) = -1$ = -ve

So a root lies between 0 and 1

f(1) is nearer to 0. Therefore we take initial root as $X_0 = 1$

$f'(x) = 3x^2 - 6$

$$= x - \frac{f(x)}{f'(x)}$$

$= x - (3x^3 - 6x + 4)/(3x^2 - 6)$

$= (2x^3 - 4)/(3x^2 - 6)$

$X_1 = (2X_0^3 – 4)/(3X_0^2 - 6) = (2-4)/(3-6) = 2/3 = 0.66666$

$X_2 = (2(2/3)^3 - 4)/(3(2/3)^2 - 6) = 0.73016$

$X_3 = (2(0.73015873)^3 - 4)/(3(0.73015873)^2 - 6)$
$= (3.22145837/ 4.40060469)$
$= 0.73205$

$X_4 = (2(0.73204903)^3 - 4)/(3(0.73204903)^2 - 6)$
$= (3.21539602/ 4.439231265)$
$= 0.73205$

The root is 0.73205 correct to 5 decimal places.

Check Your Progress :

Solve the following by using Newton – Raphson Method :

$x^3 - x - 1$ (Ans :1.3247)

1.4 Method of False Position (or Regula Falsi Method)

Consider the equation $f(x) = 0$ and $f(a)$ and $f(b)$ *are* of opposite signs. Also let $a < b$. The graph $y = f(x)$ *will* Meet the x-axis at some point between A(a, f(a)) and B (b,f(b)). The equation of the chord joining the two points A(a, f(a)) and B (b,f(b)) is

$$\frac{y - f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$$

The x- Coordinate of the point of intersection of this chord with the x-axis gives an approximate value for the of f(x) = 0. Taking y = 0 in the chord equation, we get

$$\frac{- f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$$

$x[f(a) - f(b)] - a f(a) + a f(b) = -a f(a) + b f(b)$
$x[f(a) - f(b)] = b f(a) - a f(b)$

This x_1 gives an approximate value of the root f(x) = 0. $(a < x_1 < b)$

Now $f(x_1)$ and f(a)are of opposite signs or $f(x_1)$ and f(b) are opposite signs.

If $f(x_1)$, $f(a) < 0$. then x_2 lies between x_1 and a.

Therefore $$x_2 = \frac{a\, f(x_1) - x_1\, f(b)}{f(x_1) - f(a)}$$

This process of calculation of ($x_3, x_4, x_5, \ldots$) is continued till any two successive values are equal and subsequently we get the solution of the given equation.

Steps:

1. Find *a* and *b* in which *f(a)* and *f(b)* are opposite signs for the given equation using trial and error method.
2. Therefore root lies between *a* and *b if f(a)* is very close to zero select and compute x_1 by using the following formula:

$$x_1 = \frac{a\, f(b) - b\, f(a)}{f(b) - f(a)}$$

3. If $f(x_1), f(a) < 0$. then root lies between x_1 and *a*. Compute x_2 by using the following formula:

$$x_2 = \frac{a\, f(x_1) - x_1\, f(b)}{f(x_1) - f(a)}$$

4. Calculate the values of ($x_3, x_4, x_5, \ldots$) by using the above formula until any two successive values are equal and subsequently we get the solution of the given equation.

Illustrations 1:

Solve for a positive root of $x^3 - 4x + 1 = 0$ by and Regula Falsi method

Solution :

Let $f(x) = x^3 - 4x + 1 = 0$

$f(0) = 0^3 - 4\,(0) + 1 = 1 = +ve$
$f(1) = 1^3 - 4(1) + 1 = -2 = -ve$

So a root lies between 0 and 1

We shall find the root that lies between 0 and 1.

Here a=0, b=1

$a\, f(b) - b\, f(a)$

x_1 = | | | | | | | | |
f(b) - f(a)

(0 x f(1) – 1 x f(0))
= | | | | | | | | | | | |
(f(1) –f(0))

- 1
= | | | | | |
(-2 - 1)

= 0.333333

$f(x_1) = f(1/3) = (1/27)-(4/3) +1 = -0.2963$

Now f(0) and f(1/3) are opposite in sign.

Hence the root lies between 0 and 1/3.

(0 x f(1/3) – 1/3 x f(0))
x_2 = | | | | | | | | | | | | | | |
(f(1/3) –f(0))

$x_2 = (-1/3)/ (-1.2963) = 0.25714$

Now $f(x_2) = f(0.25714) = - 0.011558$ = -ve

So the root lies between 0 and 0.25714

$x_3 = (0 \times f(0.25714) - 0.25714 \times f(0)) / (f(0.25714) – f(0))$

$= -0.25714/-1.011558 = 0.25420$

$f(x_3) = f(0.25420) = -0.0003742$

So the root lies between 0 and 0.25420

$x_4 = (0 \times f(0.25420)- 0.25420 \times f(0)) / (f(0.25420) – f(0))$

$= -0.25420 / -1.0003742 = 0.25410$

$f(x_4) = f(0.25410) = - 0.000012936$

The root lies between 0 and 0.25410

$x_5 = (0 \text{ x } f(0.25410) - 0.25410 \text{ x } f(0)) / (f(0.25410) – f(0))$

$= -0.25410 / -1.000012936 = 0.25410$

Hence the root is 0.25410.

Illustration 2.

Find an approximate root of $x \log_{10}x – 1.2 = 0$ by False position method.

Solution :

Let $f(x) = x \log_{10}x – 1.2$

$f(1) = -1.2 = -ve$; $f(2) = 2 \text{ x } 0.30103 -1.2 = - 0.597940$

$f(3) = 3\text{x}0.47712 – 1.2 = 0.231364 = +ve$

So, the root lies between 2 and 3.

$$x_1 = \frac{2f(3) – 3f(2)}{f(3) – f(2)} = \frac{2 \text{ x } 0.23136 – 3 \text{ x } (-0.59794)}{0.23136 + 0.597} = 2.721014$$

$f(x_1) = f(2.7210) = - 0.017104$

The root lies between x_1 and 3.

$$x_2 = \frac{x_1 \text{ x } f(3) – 3 \text{ x } f(x_1)}{f(3) – f(x_1)} = \frac{2.721014 \text{ x } 0.231364 – 3 \text{ x } (-0.017104)}{0.23136 + 0.017104} = 2.721014$$

$$= \frac{0.68084}{0.24846} = 2.740211$$

$f(x_2) = f(2.7402) = 2.7402 \times \log(2.7402) - 1.2$

$= -\ 0.00038905$

So the root lies between 2.740211 and 3

$$x_3 = \frac{2.7402 \times f(3) - 3 \times f(2.7402)}{f(3) - f(2.7402)} = \frac{2.7402 \times 0.231336 + 3 \times (0.00038905)}{0.23136 + 0.00038905}$$

$$= \frac{0.63514}{0.23175} = 2.740627$$

$f(2.7406) = 0.00011998$

So the root lies between 2.740211 and 2.740627

$$x_4 = \frac{2.7402 \times f(2.7406) - 2.7406 \times f(2.7402)}{f(2.7406) - f(2.7402)}$$

$$= \frac{2.7402 \times 0.00011998 + 2.7406 \times 0.00038905}{0.00011998 + 0.00038905}$$

$$= \frac{0.0013950}{0.00050903} = 2.7405$$

Hence the root is 2.7405.

Check Your Progress

1.Solve the following by method of false position (Regula Falsi Method) :

(i) $x^3 + 2x^2 + 10x - 20$ (Ans :1.3688)

1.5 Lesson End Activities

1.4.1 Find a positive root of the following equation by bisection method :

(i) $3x = \cos x + 1$

(ii) $x^3 + 3x - 1$

(iii) $e^x - 3x$

(iv) $\cos x - 2x + 3$

1.4.2 Solve the following by using Newton–Raphson Method :

(i) $x^4 - x - 9$

(ii) $x^3 + 2x^2 + 50x + 7$

(iii) $\cos x - x e^x$

(iv) $x - 2 \sin x$

1.4.3. Solve the following by method of false position (Regula Falsi Method):

(i) $2x - 3 \sin x = 5$

(ii) $e^x - 3x$

(iii) $e^{-x} = \text{Sin } x$

(iv) $\cos x - 2x + 3$

1.6 Let us Sum Up

In this lesson we have dealt with the following:

1. We have discussed the Bisection method to find solution of numerical algebraic and transcendental equations.
2. We have discussed iterative formulae by name Newton-Raphson method to find solution of algebraic and transcendental equations.
3. We have also discussed the method of false position, which gives solution of numerical algebraic and transcendental equations.

Model Answers

1.5.1 Answer (Bisection Method)

(i) 0.66664

(ii) 0.322

(iii) 0.6190

(iv) 0.3604

1.5.2 Answer (Newton–Raphson Method)

(i) 1.813

(ii) -0.1474

(iii) 0.5177

(iv) 1.8955

1.5.3. Answer (Regula Falsi Method)

(i) 2.8832

(ii) 6.0890

(iii) 0.5885
(iv) 1.5236

1.7 Reference:

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LESSON - 2

The Solution of Simultaneous Linear Algebraic Equation - Direct Methods

Contents:

2.0 Aims and Objectives

In this Lesson, we have discussed about the solving of simultaneous linear algebraic equations , which occurs in the field of science and engineering . Early studies of solving the equations are tedious. With help computer we solve by using numerical methods. These numerical methods are of two types namely Gauss Elimination and Gauss-Jordon method

After reading this lesson, you should be able

- To Solve the system of equations by using the Gauss Elimination method
- To Solve the system of equations by using the Gauss-Jordon method

2.1 Introduction

Simultaneous linear algebraic equations occur in various fields of science and engineering. We solve such type of equation by Cramer's rule. These methods are time consuming and tedious. To solve such equations, we go to numerical methods. The numerical methods are of two types name (i) direct (ii) iterative. Now, we will study these methods and detailed below :

2.2 Gauss Elimination Method

This is direct method based on number of unknowns, by eliminating the same by combining the equations to a triangular form. To illustrate the method consider the following system equations

Consider the n linear equations in n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots + a_{nn}x_n = b_n$$

Where a_{ij} and b_i are unknown constants and x_i 's are unknowns.
$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1$

The above system of equation is written in the matrix form as

$$AX = B$$

Now our aim is to reduce the given matrix (A,B) to upper triangular matrix.
The system of equation can be solved simply thus :

$$\begin{array}{lllll} & a_{11} & a_{12} \ldots\ldots\ldots a_{1n} & b_1 & - \\ - & a_{21} & a_{22} \ldots\ldots\ldots a_{2nn} & b_2 & - \\ - & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & - \\ - & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & - \\ - & a_{n1} & a_{n2} \ldots\ldots\ldots a_{nn} & b_n & - \end{array}$$

Now, multiply the first row of above matrix by $- a_{i1}/ a_{11}$ and add to the i th row (A,B) , where i = 2, 3, …., n. By doing this to all elements in the first column of (A, B) except first row. Now the above matrix is reduced to

$$\begin{array}{lllll} & a_{11} & a_{12} \ldots\ldots\ldots a_{1n} & b_1 & - \\ - & 0 & b_{22} \ldots\ldots\ldots b_{2nn} & c_2 & - \\ - & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & - \\ - & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & - \\ - & 0 & b_{n2} \ldots\ldots\ldots b_{nn} & c_n & - \end{array}$$

Now, multiply the Second row of the above matrix by $- b_{i1}/ b_{22}$ and add to the i^{th} row (A,B) , where i = 3,4 …., n. By doing this to all elements in the second column of (A, B) except first and second row. Now the above matrix is reduced to

$$\begin{array}{llllll} & a_{11} & a_{12} & \ldots\ldots\ldots a_{1n} & b_1 & - \\ - & 0 & b_{22} & \ldots\ldots\ldots b_{2n} & c_2 & - \\ - & 0 & 0 & c_{33} \ldots c_{3n} & d_3 & - \\ - & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & - \\ - & 0 & 0 & c_{n3} \ldots\ldots c_{nn} & d_n & - \end{array}$$

Now, multiply the third row of the above matrix by $- c_{i1}/ c_{33}$ and add to the i^{th} row (A,B) , where i = 4 …., n. By doing this to all elements in the third column of (A, B) except first , second and third row. Continuing the process, all elements below the leading diagonal elements of A, and the above matrix is reduced to

$$a_{11} \quad a_{12} \quad \ldots\ldots\ldots a_{1n} \quad b_1 \quad -$$

$$\begin{matrix} -0 & b_{22} \ldots\ldots\ldots & b_{2n} & c_2 & - \\ -0 & 0 \quad c_{33} & \ldots c_{3n} & d_3 & - \\ -\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & - \\ -0 & 0 \quad 0_3 & \ldots\ldots n_{nn} & n_n & - \end{matrix}$$

The above system of linear equations is equivalent to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n &= b_1 \\ b_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n &= c_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ n_{nn}x_n &= m_n \end{aligned}$$

Going from the bottom of these equation, we get $x_n = m_n / n_{nn}$ and subsequently we solve the equation $x_{n-1}, x_{n-2}, \quad \ldots. \; x_2, x_1.$

Steps to solve the system of three equation with three unknowns :

Let us consider the system of equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

1. To eliminate x_1 from the second equation, multiply the first row of the equation matrix by $- a_{21}/ a_{11}$ and add it to second equation. Similarly eliminate x_1 from the third equation and subsequently all other equations. We get new equation of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ + b_{22}x_2 + b_{23}x_3 &= c_2 \\ + b_{32}x_2 + b_{33}x_3 &= c_3 \end{aligned}$$

Where

$$\begin{aligned} b_{22} &= a_{22} - (a_{21} / a_{11}) \text{ X } a_{12} \\ b_{23} &= a_{23} - (a_{21} / a_{11}) \text{ X } a_{13}) \\ c_2 &= b_2 - (a_{21} / a_{11}) \text{ x } b_1 \end{aligned}$$

$$\begin{aligned} b_{32} &= a_{32} - (a_{31} / a_{31}) \text{ X } a_{12} \\ b_{33} &= a_{33} - (a_{31} / a_{11}) \text{ X } a_{13} \\ c3 &= b_3 - (a_{31} / a_{11}) \text{ x } b_1 \end{aligned}$$

2. To eliminate x_2 from the third equation, multiply the second row of the equation matrix by $- b_{32}/ b_{22}$ and add it to third equation. Similarly eliminate x_2 from the third equation and subsequently all other equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ + b_{22}x_2 + b_{23}x_3 &= c_2 \\ b_{33}x_3 &= d_3 \end{aligned}$$

where $c_{33} = b_{33} - (b_{32} / b_{22}) \text{ X } b_{23}$

$$d3 = c_3 - (b_{32} / b_{22}) \times c_2$$

3. From the above reduced system of equation substitute the values x_3 , x_2 and x_1 by backward substitution we get the solution of the given equations.

Illustration 1 :

Solve the system of equation by Gauss elimination method

$$x + 2y + z = 3$$
$$2x + 3y + 3z = 10$$
$$3x - y + 2z = 13$$

Solution :

The given system of equation is equivalent to

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{bmatrix}$$

Now, we have to make the above matrix as upper triangular

By using the following modifications

$R_2' = R_2 + (-2) R_1$; $R_3' = R_3 + (-3) R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -8 & 4 \end{bmatrix}$$

Now we have to take $b_{22} = -1$ as the key element and reduce b_{32} as 0

By using the following modifications

$R_3' = R_3 + (+7) R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{bmatrix}$$

From the above matrix

$$x + 2y + z = 3$$
$$-y + z = 4$$
$$-8z = -24$$

Therefore $z = 3$, $y = -1$, $x = 2$ by back substitution.

Illustration 2 :

Solve the system of equation by Gauss elimination method

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$

Solution :

The given system of equation is equivalent to

$$\begin{vmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{vmatrix}$$

Now, we have to eliminate x from the second and third equation

By using the following modifications

$R_2' = R_2 + (-4) R_1$; $R_3' = R_3 + (-2) R_1$

$$\begin{vmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{vmatrix}$$

Second step we eliminate y from the third equation. Taking ($b_{23} = 9/7$) as the key element multiply the second equation by key element and add it to the third equation

By using the following modifications

$R_3' = R_3 + (9/7) R_2$

$$\begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & -27 & -27 \end{vmatrix}$$

From the above matrix

$$\begin{aligned} 2x + y + 4z &= 12 \\ -7y - 14z &= -28 \\ -27z &= -27 \end{aligned}$$

By back substitution, we get the solution of the equation

$z = 1$, $y = 2$. $x = 3$

Therefore $z = 3$, $y = -1$, $x = 2$ by back substitution.

Check Your Progress

Solve the system of equation by Gauss elimination method

$$\begin{aligned} 20x + y + 4z &= 25 \\ 8x + 13y + 2z &= 23 \\ 4x - 11y + 21z &= 14 \end{aligned}$$

(Ans: x= y= z = 1)

2.3 Gauss Jordon Method

This method is a slightly modification of the above Gauss Elimination method . Here elimination is performed not only in the lower triangular but also upper triangular . This leads to unit matrix and hence solution is obtained . This is

Jordon's modification of the Gauss elimination and hence the name is Gauss-Jordon Method.

Consider the n linear equations in n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{nn}x_n = b_n$$

Where a_{ij} and b_i are unknown constants and x_i 's are unknowns.

$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1$

The above system of equation is written in the matrix form as

$$AX = B$$

Now our aim is to reduce the given matrix (A,) to unit matrix.
The system of equation can be solved simply thus :

$$\begin{array}{cccccc} a_{11} & 0 & 0 & \dots 0 & b_1 & - \\ -0 & b_{22} & 0 & \dots\ 0 & c_2 & - \\ -0 & 0 & c_{33} & \dots 0 & d_3 & - \\ -\dots\dots\dots\dots\dots & & & & & - \\ -0 & 0 & 0 & \dots\dots n_{nn} & m_n & - \end{array}$$

The above system of linear equations is equivalent to

$$a_{11}x_1 + \quad 0 \quad + \dots\dots\dots + 0 \quad = b_1$$
$$b_{22}x_2 + \dots\dots\dots + 0 \quad = c_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$n_{nn}x_n = m_n$$

From the above the equation we get solution directly..

Steps to solve the system of three equation with three unknowns :

Let us consider the system of equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

2. To eliminate x_1 from the second equation, multiply the first row of the equation matrix by $- a_{21}/ a_{11}$ and add it to second equation. Similarly eliminate x_1 from the third equation and subsequently all other equations. We get new equation of the form

   $$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$+b_{22}x_2 + b_{23}x_3 = c_2$$
$$+ b_{32}x_2 + b_{33}x_3 = c_3$$

Where $b_{22} = a_{22} - (a_{21} / a_{11}) X a_{12}$

$b_{23} = a_{23} - (a_{21} / a_{11}) X a_{13})$

$c_2 = b_2 - (a_{21} / a_{11}) x b_1$

$b_{32} = a_{32} - (a_{31} / a_{31}) X a_{12}$

$b_{33} = a_{33} - (a_{31} / a_{11}) X a_{13}$

$c3 = b_3 - (a_{31} / a_{11}) x b_1$

2. In this method, we eliminate x_2 from the first and third equation, Multiply the second row of the equation matrix by $- b_{32}/ b_{22}$ and add it to third equation. Similarly eliminate x_2 from the first equation.

$$a_{11}x_1 + 0 + a_{13}x_3 = b_1$$
$$+b_{22}x_2 + b_{23}x_3 = c_2$$
$$b_{33}x_3 = d_3$$

where $c_{33} = b_{33} - (b_{32} / b_{22}) X b_{23}$

$d_3 = c_3 - (b_{32} / b_{22}) x c_2$

$b_{13} = a_{13} - a_{12} / b_{22} x b_{23}$

\ $b'_1 = b_1 - a_{12} / b_{22} x c_2$

Similarly eliminate x_3 *from* first and second equation

3. From the above reduced system of equation the values x_1 , x_2 and x_3 are obtained.

Illustration 1 :

Solve the system of equation by Gauss-Jordon method

$$10x + y + z = 12$$
$$2x + 10y + z = 10$$
$$x + y + 5z = 13$$

Solution :

The given system of equation is rearranged for computation convenience, Interchange the first and last equation, since coefficient of the x in the last equation is unity (1) :

- 1	1	5	7 -
- 2	10	1	13-
- 10	1	1	12 -

Now, we have to make the above matrix as upper triangular
By using the following modifications

$R_2' = R_2 + (-2) R_1$; $R_3' = R_3 + (-10) R_1$

$$\begin{vmatrix} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{vmatrix}$$

Now we have to take $b_{22} = 1/8$ as the key element and reduce b_{32} as 0
By using the following modifications
$R_2' = R_2 /8$; $R_3' = R_3 + (+9/8) R_2$

$$\begin{vmatrix} 1 & 1 & 5 & 7 \\ 0 & 1 & -9/8 & -1/8 \\ 0 & 0 & -473/8 & -473/8 \end{vmatrix}$$

Now we have to make $b_{33} = 1$ as the key element and reduce b_{32} as 0
By using the following modifications
$R_3' = R_3 \text{ x } - 8 /473$; $R_1' = R_1 + (-1) R_2$

$$\begin{vmatrix} 1 & 0 & 49/8 & 57/8 \\ 0 & 1 & -9/8 & -1/8 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

Now we have to make $b_{23} = 0$ and $b_{13} = 0$.
By using the following modifications
$R_2' = R_2 + (- 9/8)R_3$; $R_1' = R_1 + (-49/8) R_3$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

Therefore $x = 1, y = 1, z= 1$.

Illustration 2 :

Solve the system of equation by Gauss-Jordon method

$$2x + y + 4 z = 12$$
$$8x- 3y + 2z = 20$$
$$4x + 11y - z = 33$$

Solution :

The given system of equation is equivalent to

$$\begin{vmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \end{vmatrix}$$

- 4 11 -1 33 -

Now, we have to eliminate x from the second and third equation

By using the following modifications

$R_2' = R_2 + (-4) R_1$; $R_3' = R_3 + (-2) R_1$

- 2	1	4	12 -
- 0	-7	-14	-28-
- 0	9	-9	9 -

Second step we eliminate y from the third equation. Taking (b_{23} = 9 /7) as the key element multiply the second equation by key element and add it to the third equation

By using the following modifications

$R_3' = R_3 + (9/7) R_2$

- 2	1	4	12 -
- 0	-7	-14	-28 -
- 0	0	-27	-27-

At this stage, we eliminate y from the first equation. Z from the first and second equation. By using following modifications ;

$R_1' = R_1 /2$; $R_2' = R_2 /-7$; $R_3' = R_{3\,/}\,(-27) R_1$

- 1	1/2	2	6 -
- 0	1	2	4 -
- 0	0	1	1 -

$R_2' = R_2 + (-2) R_3$; $R_1' = R_1 + (-1/2) R_2$;

- 1	0	1	4 -
- 0	1	0	2 -
- 0	0	1	1 -

$R_1' = R_1 + (-1) R_1$;

- 1	0	0	3 -
- 0	1	0	2 -
- 0	0	1	1 -

Therefore x = 3, y = 2, z= 1.

Check Your Progress

Solve the system of equation by Gauss-Jordon method

$$10x + y + z = 13$$
$$2x + 10y + z = 14$$
$$x + y + 15z = 32$$

(Ans: x=y=1, z =2)

2.4 Lesson End Activities

1. Solve the system of equation by Gauss Elimination method

$$3.15x - 1.96\,y + 3.85\,z = 12.95$$

$$2.13x - 5.12y \quad -2.892z = -8.61$$
$$5.92x + 3.051y + 2.15z = 6.88$$

2. *Solve the system of equation by Gauss elimination method*
$$3x + 4y + 6z = 18$$
$$2x - y + 8z = 13$$
$$5x - 2y + 7z = 20$$
3. *Solve the system of equation by Gauss-Jordon method*
$$2x + y + 4z = 9$$
$$8x - 3y + z = 12$$
$$4x + 11y - z = 18$$
4. *Solve the system of equation by Gauss-Jordon method*
$$2x - y + 4z = 5$$
$$8x - 3y + z = 6$$
$$x + 11y - z = 11$$

2.5 Let us Sum Up

In this lesson we have dealt with the following:

- We have discussed Gauss Elimination method to solve the system of linear equations, which occurs in the field of science and engineering
- We have discussed the Gauss-Jordon method to solve the system of equations.

Model Answer For Lesson End Activities

1. *(Ans : $x = 1.7089$, $y = -1.8005$, $z = 1.0488$)*
2. *(Ans: $x = 3$, $y = 1$, $z = 1$)*
3. *(Ans: $x = 2$, $y = 1$, $z = 1$)*
4. *(Ans: $1 = 1$, $y = 1$, $z = 1$)*

2.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

LESSON - 3
Gauss-Jacobi Method

Contents:

3.0 Aims and Objectives

In this Lesson, we have discussed about the solving simultaneous linear algebraic equations , which occurs in the field science and engineering . Early study of solving the equations is not applicable for all the problem, even if we apply its required tedious calculations. With help of Gauss-Jacobi iteration process, we solve by linear equations with minimal steps.

After reading this lesson, you should be able to

- To know about Gauss-Jacobi method iteration procedure.
- To Solve the system of equations by using the Gauss-Jacob method

3.1 Introduction

Early methods of study in solving algebraic linear equations are direct methods. Now we will discuss some indirect methods or iterative methods. This method is not always successful to all systems of equations. If this method is to succeed, each equation must satisfy a condition . i.e., when diagonal elements are exceeding all other elements in the respective equations. We will discuss two methods of this category namely Gauss-Jacobi and Gauss-Seidel method.

3.2 Gauss-Jacobi Method

Let us consider this method in the case of three equations in three unknowns.

Consider the 3 linear equations in 3 unknowns,

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

This method is applied only when diagonal elements are exceeding all other elements in the respective equations i.e.,

$$|a_1| > |b_1| + |c_1| = d_1$$
$$|a_2| > |b_2| + |c_2| = d_2$$
$$|a_3| > |b_3| + |c_3| = d_3$$

Let the above condition is true we apply this method or we have to rearrange the equations in the above form to fulfill the above condition.

We start with initial values of x,y and z as zero. Solve x, y ,z in terms of other variables. That is,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

The above values are initial values $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ of x , y z respectively, then

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(0)} - c_2 z^{(0)})$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(0)} - b_3 y^{(0)})$$

Again using new values of $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ of x , y z respectively, then

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(1)})$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

Repeating the process in the same way, and the r^{th} iterates are $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ and given below

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(r)} - c_2 z^{(r)}\right)$$

$$z^{(r+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(r)} - b_3 y^{(r)}\right)$$

The above iteration is continued until any two successive values are equal.

3.3 Illustration 1:

1 . Solve the system of equation by Gauss-Jacobi method

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + 6y + 54z = 110$$

Solution:

To apply this method , first we have to check the diagonal elements are dominant.

i.e., 27 > 6+ 1 ; 15 > 6+ 2 ; 54 > 1+1 . So iteration method can be applied

$$x = 1/27\,(85 - 6y + z)$$
$$y = 1/15\,(72 - 6x - 2z)$$
$$z = 1/54\,(110 - x - y)$$

First iteration : From the above equations, we start with $x = y = z = 0$

$x^{(1)} = 85/27 = 3.14815$ ……………..(1)

$y^{(1)} = 72/15 = 4.8$ ……………...(2)

$z^{(1)} = 110/54 = 2.03704$ ……………….(3)

Second iteration :Consider the new values of $y^{(1)} = 4.8$ *and* $z^{(1)} = 2.03704$ *in the first equation*

$x^{(2)} = 1/27(85 - 6 \times 4.8 + 2.03704) = 2.15693$

$y^{(2)} = 1/15(72 - 6 \times 3.14815 - 2 \times 2.03704) = 3.26913$

$z^{(2)} = 1/54(110 - 3.14815 - 4.8) = -0.515$

Fourth iteration : Consider the new values of $x^{(2)} = 2.15693$, $y^{(2)} = 3.26913$ *and* $z^{(2)} = -0.515$ *in the first equation*

$x^{(3)} = 1/27(85 - 6 \times 3.26913 + -0.515) = 2.49167$

$y^{(3)} = 1/15(72 - 6 \times 2.15693 - 2 \times 2.15693) = 3.68525$

$z^{(3)} = 1/54(110 - 2.15693 - 3.26913) = 1.93655$

Thus, we continue the iteration and result is noted below

Iteration No.	x	y	z
4	2.40093	3.54513	1.92265
5	2.43155	3.58327	1.92692
6	2.42323	3.57046	1.92565
7	2.42603	3.57395	1.92604
8	2.42527	3.57278	1.92593
9	2.42552	3.57310	1.92596
10	2.42546	3.57300	1.92595

From the above table 9 th and 10 th iterations are equal by considering the four decimal places. Hence the solution of the equation is

$x = 2.4255 \qquad y = 3.5730 \qquad z = 1.9260.$

Illustration 2 . *Solve the system of equation by Gauss-Jacobi method*

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y + 10z = 3$$

Solution:

To apply this method , first we have to check the diagonal elements are dominant.

i.e., 10 > 5+ 2 ; 10 > 4+ 3 ; 10 > 1+6 . So iteration method can be applied

$$x = 1/10\,(3 + 5y + 2z)$$
$$y = 1/10\,(3 + 4x + 3z)$$
$$z = 1/10\,(-3 - x - 6y)$$

First iteration : From the above equations, we start with $x = y = z = 0$

$x^{(1)} = 3/10 = 0.3$ ……………(1)

$y^{(1)} = 3/10 = 0.3$ ……………..(2)

$z^{(1)} = -3/10 = -0.3$ ……………..(3)

Second iteration :Consider the new values of $y^{(1)} = 0.3$ *and* $z^{(1)} = -0.3$ *in the first equation*

$$x^{(2)} = 1/10(3 + 5 \times .3 + (-0.3)) = 0.39$$

$$y^{(2)} = 1/10(3 + 4 \times 0.3 + 3 \times (-0.3)) = 0.33$$

$$z^{(2)} = 1/10[-3 \,|\, (0.3) \,|\, 6(0.3)] = -0.51$$

Third iteration : Consider the new values of $x^{(2)} = 0.39$, $y^{(2)} = 0.33$ *and* $z^{(2)} = -0.51$ *in the first equation*

$x^{(3)}$ = 1/10 [3 | 5 x0.33 +(-0.51)] = 0.363

$y^{(3)}$ = 1/10 (3 + 4 x 0.39 + 3 x (-0.51)) = 0.303

$z^{(3)}$ = 1/10[-3 | 0.39 | 6 x (0.33)] = -0.537

Thus, we continue the iteration and result is noted below

Iteration No.	X	y	z
4	0.3441	0.2841	-0.5181
5	0.33843	0.2822	-0.50487
6	0.340126	0.283911	0.503163
7	0.3413229	0.2851015	-0.5043592
8	0.34167891	0.2852214	-0.50519319
9	0.341572062	0.285113607	-0.505300731

From the above table 8 th and 9 th iterations are equal by considering the 3 decimal places. Hence the solution of the equation is

$x = 0.342$, $y = 0.285$, $z = -0.505$.

Check Your Progress

1. Solve the system of equation by Gauss-Jacobi method

$$3.15x - 1.96y + 3.85z = 12.95$$
$$2.13x - 5.12y - 2.892z = -8.61$$
$$5.92x + 3.051y + 2.15z = 6.88$$

(Ans : x =1.7089, y = -1.8005, z = 1.0488)

3.4 Lesson End Activities

Solve the following system of equations by using Gauss-Jacobi Method

1. $8x - 3y + 2z = 20$; $4x + 11y - z = 33$; $6x + 3y + 12z = 35$
2. $28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 3y + 10z = 24$
3. $5x - 2y + z = -4$; $x + 6y - 2z = -1$; $3x + y + 5z = 13$
4. $8x + y + z = 8$; $2x + 4y + z = 4$; $x + 3y + 3z = 5$

3.5 Let us Sum Up

In this lesson we have dealt with the following:

- We have discussed Gauss Jacobi method to solve the system of linear equations, which occurs in the field of science and engineering. This method is an iterative method and it is widely applied.

Model Answer For Lesson End Activities

1. *(Ans: 3.017, 1.986, 0.912)*
2. *(Ans: 0.994, 1.507, 1.849)*
3. *(Ans:-1.0, .999, 3)*

4. *(Ans: 0.83, .32, 1.07)*

3.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

LESSON - 4
Gauss-Seidel Method

Contents:

4.0 Aims and Objectives

In this Lesson, we have discussed about the solving simultaneous linear algebraic equations, which occurs in the field of science and engineering. Early study of solving the equations are time consuming when compared to this method. With help of Gauss-Seidel iteration process, we solve by linear equations with minimal iterations

After reading this lesson, you should be able

- To know about Gauss-Seidel method iteration procedure.
- To Solve the system of equations by using the Gauss-Seidel method

4.1 Introduction

Early methods of study in solving algebraic linear equations are direct methods. Now we will discuss some indirect methods or iterative methods. This method is not always successful to all systems of equations. If this method is to succeed, each equation must satisfy a condition . i.e., when diagonal elements are exceeding all other elements in the respective equations. We will discuss an iterative and self correcting method, namely Gauss-Seidel method.

4.2 Gauss-Seidel Method

This method is only an enhancement of Gauss-Jaobi Method. In the previous method

Let us consider this method in the case of three equations in three unknowns.

Consider the 3 linear equations in 3 unknowns,

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

This method is applied only when diagonal elements are exceeding all other elements in the respective equations i.e.,

$$|a_1| > |b_1| + |c_1| = d_1$$
$$|a_2| > |b_2| + |c_2| = d_2$$
$$|a_3| > |b_3| + |c_3| = d_3$$

Let the above condition is true we apply this method or we have to rearrange the equations in the above form to fulfill the above condition.

We start with initial values of x,y and z as zero. Solve x, y ,z in terms of other variables. That is,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We proceed with the initial values $y^{(0)}$, $z^{(0)}$ for y , z and get $x^{(1)}$ from the first equation, i.e.,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

When we calculate $y^{(1)}$, we use new values of x i.e., $x^{(1)}$ and $z^{(0)}$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Similarly, while we calculate $z^{(1)}$, we use new values of x,y i.e., $x^{(1)}$ and $y^{(1)}$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

Again using new values of $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ of x , y z respectively, then

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(2)} - c_2 z^{(1)})$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3 x^{(2)} - b_3 y^{(2)})$$

Repeating the process in the same way, and the r^{th} iterates are $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ and given below

$$x^{(r+1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(r)} - c_1 z^{(r)}\right)$$

$$y^{(r+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)}\right)$$

$$z^{(r+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)}\right)$$

The above iteration is continued until any two successive values are equal.

Note : 1. For all systems of equation, this method will not work
2.Iteration method is self correcting method. Any error made in computation is corrected automatically in subsequent iterations
3. Iteration is stopped when any two successive iteration values are equal

4.3 Illustration : 1 . *Solve the system of equation by Gauss-Seidel method*

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y + 10z = 3$$

Solution:

To apply this method , first we have to check the diagonal elements are dominant.

ie., 10 > 5+ 2 ; 10 > 4+ 3 ; 10 > 1+6 . So iteration method can be applied

$$x = 1/10\,(3 + 5y + 2z)$$
$$y = 1/10\,(3 + 4x + 3z)$$
$$z = 1/10\,(-3 - x - 6y)$$

***First iteration** :*

From the above equations, we start with $x = y = z = 0$

$$x^{(1)} = 3/10 \qquad = 0.3 \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

New value of x is used for further calculation ie., x = 0.3

$$y^{(1)} = 1/10\,(3 + 4 \times 0.3 + 3(0)] \qquad = 0.42 \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

New values of x and y is used for further calculation ie., x = 0.3 and y = 0.42

$$z^{(1)} = 1/10\,(-3 - 0.3 - 6(0.42) \qquad = -0.582 \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

Second iteration :

Consider the new values of $y^{(1)} = 0.42$ and $z^{(1)} = -0.582$ in the first equation

$$x^{(2)} = 1/10(\,3 + 5 \times 0.42 + (-0.582)) \qquad = 0.3936$$

$$y^{(2)} = 1/10\,(\,3 + 4 \times 0.3936 + 3 \times (-0.582)\,) \qquad = 0.28284$$

$$z^{(2)} = 1/10\,[\,-3 - (0.3936) - 6(0.28284)\,] \qquad = -0.509064$$

Third iteration : Consider the new values of $x^{(2)} = 0.3936$, $y^{(2)} = 0.28284$ *and* $z^{(2)} = -0.509064$ *in the first equation*

$$x^{(3)} = 1/10 [3 | 5 x 0.28284 +(-0.509064)] = 0.3396072$$

$$y^{(3)} = 1/10 (3 + 4 x 0.3396072 + 3 x (-0.509064) = 0.28312368$$

$$z^{(3)} = 1/10[-3 | 0.3396072 | 6 x (0.283123678)] = -0.503834928$$

Thus, we continue the iteration and result is noted below

Iteration No.	*X*	*Y*	*Z*
4	*0.34079485*	*0.28516746*	*-0.50517996*
5	*0.3415547*	*0.28506792*	*-0.505196229*
6	*0.3414947*	*0.2850390*	*-0.5051728*
7	*0.3414849*	*0.28504212*	*-0.5051737*

The values correct to 3 decimal places are

x = 0.342, y = 0.285, z = -0.505

Note : Check the above equations by substituting values of x,y and z

Illustration 2 : 1 . *Solve the system of equation by Gauss-Seidel method*

$$28x + 4y - z = 32$$
$$4x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

Solution:

To apply this method , first we have to rewrite the equation in such way that to fulfill diagonal elements are dominant.

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$4x + 3y + 10z = 24$$

ie., 28 > 4+ 1 ; 17 > 2+ 4 ; 10 > 4+3 . So iteration method can be applied

$$x = 1/28 (32 - 4y + z)$$
$$y = 1/17 (35 - 2x - 4z)$$
$$z = 1/10 (24 - x - 3y)$$

First iteration :

From the above equations, we start with $y = z = 0$, *we get*

$$x^{(1)} = 32/28 = 1.1429$$

New value of x is used for further calculation ie., x = 1.1429

$$y^{(1)} = 1/17 (35 + 1.1429 + 3(0)] = 1.9244$$

New values of x and y is used for further calculation ie., x = 1.1429 *and y = 1.9244*

$z^{(1)} = 1/10\ [24\ -1.1429\ -3(1.9244)\ ] \quad = 1.8084)$

Second iteration :

Consider the new values of $y^{(1)} = 1.9244$ *and* $z^{(1)} = 1.8084$

$x^{(2)} = \quad 1/28[\ 32 - 4\ (1.9244)\ +(\ 1.8084)] \quad = 0.9325$

$y^{(2)} = \quad 1/17\ [\ 35\ - 2\ (0\ .9325)\ - 4\ (\ 1.8084)\ ] \quad = 1.5236$

$z^{(2)} = \quad 1/10\ [\ 24\ |\ (0.9325)\ |\ 3(1.5236)\ ] \quad = 1.8497$

Third iteration :

Consider the new values of $x^{(2)} = 0.9325$, $y^{(2)} = 1.5236$ *and* $z^{(2)} = 1.8497$

$x^{(3)} = \quad 1/28[\ 32 - 4\ (1.5236)\ +(\ 1.8497)] \quad = 0.9913$

$y^{(3)} = \quad 1/17\ [\ 35\ - 2\ (0\ .9913)\ - 4\ (\ 1.8497)\ ) \quad = 1.5070$

$z^{(3)} = \quad 1/10\ [\ 24\ |\ (0.9913)\ |\ 3(1.5070)\ ] \quad = 1.8488$

Thus, we continue the iteration and result is noted below

Iteration No.	*X*	*y*	*Z*
4	*0.9936*	*1.5069*	*1.8486*
5	*0.9936*	*1.5069*	*1.8486*

Therefore x = 0.9936, y = 1.5069, z = 1.8486

Check Your Progress

1. Solve the system of equation by Gauss Seidel method

$3.15x - 1.96\,y + 3.85\,z = 12.95$

$2.13x - 5.12y \quad -2.892z = -8.61$

$5.92x + 3.051y + 2.15\,z = 6.88$

(Ans : x =1.7089, y = -1.8005, z = 1.0488)

4.4 Lesson End Activities

1. 8x -6y +z =13.67; 3x +y -2z =17.59; 2x -6y +9z =29.29
2. 30x – 2y +3z =75; 2x+ 2y +18z = 30 ; x + 17y -2z =48
3. y –x + 10z=35.61; x + z + 10y =20.08; y- z +10x =11.19
4. 10x -2y +z = 12 ; x + 9y -z =10; 2x – y + 11z = 20
5. 8x – y +z =18; 2x +5y -2z = 3; x+y – 3z = -16
6. 2x + y + z =4; x + 2y –z = 4; x + y + 2z = 4

4.5 Let us Sum Up

In this lesson we have dealt with the following:

- We have discussed Gauss-Seidel method to solve the system of linear equations, which occurs in the field of science and engineering. This method is an iterative method and it is widely applied.

Model Answer For Lesson End Activities

1. 0.83, 0.32,1.07
2. 2.5796, 2.7976, 1.0693
3. 1.321, 1.522, 3.541
4. 1.2624, 1.1591, 1.694
5. 2, 0.9998, 2.9999
6. 1, 1, 1

4.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 .

UNIT-II
LESSON - 5

Numerical Differentiation

Contents :

5.0 Aims and Objectives

In this Lesson, we have discussed about Newton's forward difference formula for finding derivatives. If the derivative occurs closer to the beginning of the table, we use this method.
After reading this lesson, you should be able

- To know about construction of the difference table.
- To find derivatives using Newton's forward difference formula.

5.1 Introduction

Let y=*f(x)* be a function taking the values $y_0, y_1, y_2, \quad y_n$ corresponding to the *values* $x_0, x_1, x_2, \ldots\ldots\ldots \; x_n$ of the independent variable x. Now we are trying to find the derivative value of $y = y_k$ for the given $x = x_k$ .. If the derivative is required at a point nearer to starting value in the table., i.e., If the value occur between x_o to x_1 or beginning of the table, we use Newton's forward interpolation formula.

5.2 Newton's forward difference formula :

Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ………	x_n
y:	y_0	y_1	y_2	y_3 ………	y_n

From the above values, we want to find the derivative of $y = f(x)$ *passing through* $(n+1)$ *points*, at a point closer to the starting value $x = x_0$

Newton's forward difference interpolation formula is given below

$$y(x_0 + uh) = y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots \qquad \ldots\ldots(1)$$

Where $y(x)$ *is a polynomial of degree n in x.*

$$\text{where} \quad u = \frac{x - x_0}{h} \quad \text{and}$$

Δy_0 , $\Delta^2 y_0$, $\Delta^3 y_0$ *are obtained from difference table*

Differentiating with respect to x, finally it reduced to the following way

$$dy/dx = Dy_0 = (1/h)\ \{\ \Delta y_0 - (1/2)\,\Delta^2 y_0 + (1/3)\ \Delta^3 y_0 + \ \ldots.. \ \}$$

$$d^2y/dx^2 = D^2 y_0 = (1/h^2)\ \{\ \Delta^2 y_0 - \Delta^3 y_0 + (11/12)\ \Delta^4 y_0 \ \ldots.. \ \}$$

5.3 Illustration 1.

Find the first two derivatives of $(x)^{(1/3)}$ *at* $x = 50$ *given table below* :

x :	50	51	52	53	54	55	56
y :	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution :

Step 1. Write down the formula :

$$dy/dx = Dy_0 = (1/h)\ \{\ \Delta y_0 - (1/2)\,\Delta^2 y_0 + (1/3)\ \Delta^3 y_0 + \ \ldots.. \ \}$$

$$d^2y/dx^2 = D^2 y_0 = (1/h^2)\ \{\ \Delta^2 y_0 - \Delta^3 y_0 + (11/12)\ \Delta^4 y_0 \ \ldots.. \ \}$$

Step 2 . Construct the difference table to find various Δ' s

Difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
50	3.6840	(3.7084-3.6840)		
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0
54	3.7798		-0.0003	

		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

By applying Newton's forward formula :

$$[\,dy/dx]_{x=x_0} = [\,dy/dx]_{u=0}$$

$$= (1/h)\ \{\ \Delta y_0 - \underline{(1/2)}\,\Delta^2 y_0 + (1/3)\ \Delta^3 y_0 + \ \ldots\ldots\ \}$$

$$= (1/1)\ [\ 0.0244 - \underline{(1/2)}\,(-0.0003) + (1/3)\ 0\ ]$$

$$= \mathbf{0.02455}$$

$$[d^2y/dx^2\,]_{x=50} = D^2 y_0 = (1/h^2)\ [\ \Delta^2 y_0 - \Delta^3 y_0 + \ldots\ ]$$

$$= 1\ [\ -0.0003]$$

$$= \mathbf{-0.0003}$$

Illustration 2. *The population of a certain town is given below. Find the rate of growth of the population in 1931 and 1941*

Year	*x* :	*1931*	*1941*	*1951*	*1961*	*1971*
Population	*y* :	*40.62*	*60.80*	*79.95*	*103.56*	*132.65*

Solution. *Construct the difference table*

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	
1951	79.95		4.46		-4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

By applying Newton's forward formula :
To find (i) *f'(1931)*

$$\textit{where}\quad u = \frac{x - x_0}{|\ |\ |\ |} = \frac{1931 - 1931}{|||||||} = 0$$

$h \qquad 10$

$[dy/dx]_{x=1931} = [dy/dx]_{u=o}$

$= (1/h)\ [\ \Delta y_0 - \underline{(1/2)}\,\Delta^2 y_0 + (1/3)\ \Delta^3 y_0 - (1/4)\ \Delta^4 y_0\ ]$

$= (1/10)\ [\ 20.18 - \underline{(1/2)}\,(-1.03) + (1/3)\,(5.49) - (-4.47)\ ]$

$= (1/10)\ [\ 20.18\ +0515+1.83+1.1175\ ]$

$= \mathbf{2.36425}$

To find (i) *f'(1941)*

$$\text{where} \quad u = \frac{x - x_0}{h} = \frac{1941 - 1931}{10} = 1$$

$[dy/dx]_{x=1941} = [dy/dx]_{u=1}$

$= (1/h)\ \{\ \Delta y_0 + [(2u-1)/2]\,\Delta^2 y_0 + [(3u^2-6u+2)/6]\ \Delta^3 y_0 + [(4u^3-18u^2+22u-6)/24\ ]\ \Delta^4 y_0\ \}$

$= (1/10)\ [\ 20.18 + \underline{(1/2)}\,(-1.03) - (1/6)\,(5.49) + (1/12)(-4.47)\ ]$

$= (1/10)\ [\ 20.18\ |\ \ 0.515|0.915|0.3725\ ]$

$= \mathbf{1.83775}$

Check Your Progress

4.Find the first two derivatives of t5he function x = 1.5 from the table below

x	*1.5*	*2.0*	*2.5*	*3.0*	*3.5*	*4.0*
y	*3.375*	*7.0*	*13.625*	*24.0*	*38.875*	*59.0*

(Ans : 4.75, 9.0)

5.4 Lesson End Activities

1. *Find the first and second derivative of the function tabulated below at x = 3*

x	:	*3.0*	*3.2*	*3.4*	*3.6*	*3.8*	*4.0*
f(x)	:	*-14*	*-10.32*	*-5.296*	*-0.256*	*6.672*	*14*

2. *Find second derivative of y at x = 0.96 from the data*

x	:	*0.96*	*0.98*	*1.00*	*1.02*	*1.04*
y	:	*0.7825*	*0.7739*	*0.7651*	*0.7563*	*0.7473*

3. Find the value of cos(1.74) from the following table.

x	*:1.7*	*1.74*	*1.78*	*1.82*	*1.86*
sin x	*:0.9916*	*0.9857*	*0.9781*	*0.9691*	*0.9584*

5.5 Let us Sum UPs

In this lesson we have dealt with the following:

- We have discussed Newton's forward difference formulae to get the derivative by using the difference table. When the value of the derivative required at a point ,which nearer to the starting value in the table.

Model Answer For Lesson End Activities

1. *f(3)' =f''(3) =18*

2. *f(0.96)' =-1.91666*

3. -0.17125

5.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

LESSON - 6

Newton's Backward Difference formula

Contents

6.0 Aims and Objectives

In this Lesson, we have discussed about Newton's backward difference formula for finding derivatives. If the derivative occur closer to the end of the table, we use this method.

After reading this lesson, you should be able to

- To find derivatives using Newton's backward difference formula.
- Computation of derivatives.

6.1 Introduction

Let $y=f(x)$ be a function taking the values $y_0, y_1, y_2, \ldots\ldots\ldots\ y_n$ corresponding to the *values* $x_0, x_1, x_2, \ldots\ldots\ldots\ x_n$ of the independent variable x. Now we are trying to find the derivative value of $y = y_k$ for the given $x = x_k$ .. If the derivative is required at a point nearer to end of the table., ie, If the value occur between x_{n-1} to x_n or end of the table, we use Newton's Backward difference formula.

6.2 Newton's backward difference formula to compute derivative:

Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ……….	x_n
y:	y_0	y_1	y_2	y_3 ……….	y_n

From the above values, we want to find the derivative of $y = f(x)$ *passing through* $(n+1)$ *points*, at a point closer to the starting value $x = x_0$

Let us consider Newton's backward interpolation formula and is given below

$$y(x_n + vh) = y_u = y_n + v \;\breve{}\; y_n + \frac{v\,(v+1)}{2!} \;\breve{}\; ^2y_n + \frac{v\,(v+1)(v+2)}{3!} \;\breve{}\; ^3y_n \qquad \ldots\ldots(1)$$

Where y(x) is a polynomial of degree n in x.

$$\text{where} \quad v = \left|\,\left|\,\frac{x_n - x}{h}\,\right|\,\right| \quad \text{and}$$

$\breve{}\; y_n\,,\ \breve{}\; ^2y_n\,,\ \breve{}\; ^3y_n$ *are obtained from difference table*

Differentiating with respect to x, finally it reduced to the following way

$$dy\,/\,dx = Dy_n = (1/\,h)\ \{\ \breve{}\; y_n + \underline{(1/2)}\;\breve{}\; ^2y_n + (1/3)\;\breve{}\;\Delta^3y_n + \ \ldots..\ \}$$

$$d^2y\,/\,dx^2 = D^2y_n = (1/\,h^2)\ \{\ \breve{}\; ^2y_n + \breve{}\; ^3y_n + (11/12)\breve{}\; ^4y_n \ldots..\ \}$$

6.3 Illustration 1.

Find the first two derivatives of $(x)^{(1/3)}$ *at x = 56 given table below* :

x	:	50	51	52	53	54	55	56
y	:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution :

Step 1. Write down the formula :

$$dy\,/\,dx = Dy_n = (1/\,h)\ \{\ \breve{}\; y_n + \underline{(1/2)}\;\breve{}\; ^2y_n + (1/3)\;\breve{}\;\Delta^3y_n + \ \ldots..\ \}$$

$$d^2y\,/\,dx^2 = D^2y_n = (1/\,h^2)\ \{\ \breve{}\; ^2y_n + \breve{}\; ^3y_n + (11/12)\breve{}\; ^4y_n \ldots..\ \}$$

Step 2 . Construct the difference table to find various Δ' s

Difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
50	3.6840	(3.7084-3.6840)		
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0

54	3.7798		-0.0003	
		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

By applying Newton’s backward difference formula :

$$[\, dy / dx]_{x=xn} \quad =[\, dy / dx]_{v=o}$$

$$= \ (1/ h) \ \{ \ \text{ˇ} \ y_n \ + \underline{(1/2)} \ \text{ˇ} \ {}^2y_n + (1/3) \ \text{ˇ} \ \Delta^3 y_n \ + \ \ldots.. \ \}$$

$$= \ (1/ 1) \ [\ 0.0299 + \underline{(1/2)} \ (-0.0003) + (1/3) \ 0\ ]$$

$$= \ \mathbf{0.02275}$$

$$[d^2y / dx^2\ ]_{x=56} \ = \ D^2y_0 \quad = \ (1/ h^2) \ [\ \text{ˇ} \ {}^2y_n + \text{ˇ} \ {}^3y_n \ +\ldots\ ]$$

$$= 1\ [\ -0.0003]$$

$$= \mathbf{-0.0003}$$

Illustration 2. *The population of a certain town is given below. Find the rate of growth of the population in 1961 and 1971*

Year	*x*	*:*	*1931*	*1941*	*1951*	*1961*	*1971*
Population	*y*	*:*	*40.62*	*60.80*	*79.95*	*103.56*	*132.65*

Solution. *Construct the difference table*

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	
1951	79.95		4.46		-4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

By applying Newton’s forward formula :

To find (i) $f'(1961)$

$$\text{where} \quad v = \frac{x_n - x}{h} = \frac{1961 - 1971}{10} = -1$$

$$[\,dy / dx]_{x=1961} = [\,dy / dx]_{v=-1}$$

$$= (1/h) \; \{ \; ˘ \; y_n + ((2v+1)/2) \; ˘ \; {}^2y_n + ((3v^2+6v+2)/6) \; ˘ \; \Delta^3 y_n + \ldots \}$$

$$= (1/10) \;\; [29.09 – (1/2)(5.48) – (1/6)(1.02) – (1/12)(-4.47)\,]$$
$$= (1/10) \;\; [29.09 – 2.74 - 0.17 + 0.3725]$$

$$= \mathbf{2.65525}$$

To find (i) $f'(1971)$

$$\text{where} \quad v = \frac{x_n - x}{h} = \frac{1971 - 1971}{10} = 0$$

$$[\,dy / dx]_{x=1971} = [\,dy / dx]_{v=0}$$

$$= (1/h) \; \{ \; ˘ \; y_n + (1/2) \; ˘ \; {}^2y_n + (1/3) \; ˘ \; \Delta^3 y_n + (1/4) \; ˘ \; \Delta^4 y_n \ldots \}$$

$$= (1/10) \;\; [29.09 + (1/2)(5.48) + (1/3)(1.02) + (1/4)(-4.47)\,]$$
$$= (1/10) \;\; [29.09 + 2.74 + 0.34 - 1.1175]$$

$$= \mathbf{2.65525}$$

6.4 Lesson End Activities

1. *find the first and second derivative of the function tabulated below at x = 4.0*

x	:	*3.0*	*3.2*	*3.4*	*3.6*	*3.8*	*4.0*
f(x)	:	*-14*	*-10.32*	*-5.296*	*-0.256*	*6.672*	*14*

2. *find second derivative of y at x = 1.04 from the data*

x	:	*0.96*	*0.98*	*1.00*	*1.02*	*1.04*
y	:	*0.7825*	*0.7739*	*0.7651*	*0.7563*	*0.7473*

3. From the table below find y' and y'' at x = 1.25

x	:	*1.00*	*1.05*	*1.10*	*1.15*	*1.20*	*1.25*	*1.30*
y	:	*1.00000*	*1.02470*	*1.04881*	*1.07238*	*1.09544*	*1.11803*	*1.14017*

4. *Find first and second derivative of x at x =23*

x_	:	15	17	19	21	23	25
x	:	3.873	4.123	4.359	4.583	4.796	5.000

Model Answer for selected Lesson End Activities

3. 0.4473, -0.1583
4.0.1041. -0.0023

6.5 Let us Sum Up

In this lesson we have dealt with the following:

- We have discussed Newton's backward difference formulae to get the derivative by using the difference table. When the value of the derivative required at the end of the table.

6.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

LESSON – 7

Numerical Integration

Contents

7.0 Aims and Objectives

In this Lesson, we have discussed about numerical integration. Given set of paired values (x_i, y_i) $I = 0, 1, 2 \ldots n$, even $f(x)$ is unknown , it is possible to evaluate integration value. We have discussed about the Trapezoidal rule to evaluate integration.

After reading this lesson, you should be able to

- To know about numerical integration.
- To evaluate integration using Trapezoidal Rule

7.1 Introduction

Let $_a{}^b$ *f(x) dx represents* the area between y = f(x), with the rand x =a and x =b. This integration is possible only when f(x) is explicitly given or otherwise it sis not possible to evaluate. In numerical integration can be detailed as follows ; Given set of (n+1) paired values of the function taking the values $y_0, y_1, y_2, \ldots\ldots\ldots\ y_n$ corresponding to the *values* $x_0, x_1, x_2, \ldots\ldots\ldots\ x_n$.. *where f(x) is not known explicitly, it is possible to compute* $_a{}^b$ *f(x) dx* by numerical integration by using various method. One of the simplest method is trapezoidal Rule that explained below

7.2 Trapezoidal Rule

Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ………	x_n
y:	y_0	y_1	y_2	y_3 ………	y_n

From the above values, we want to find the integration of *$y = f(x)$ with the range x_0 and x_0 +mh*

$$\int_{x_0}^{x_0+mh} f(x)\,dx = (h/2)\,[\,(y_0 + y_n) + 2\,(y_1 + y_2 + y_3 + \ldots + y_{n-1})\,]$$

$$= (h/2)\,[\,(\text{sum of the first and last ordinates}) + (\text{Sum of the remaining ordinates})\,]$$

7.3 Illustrations *1. Evaluate $\int_{-3}^{3} x^4\,dx$ by using Trapezoidal rule. Verify result by actual integration.*

Step 1. We are given that $f(x) = x^4$. Interval length $(b - a) = (3 - (-3)) = 6$. So we divide 6 equal intervals with $h = 6/6 = 1.0$ And tabulate the values as below

x	:	-3	-2	-1	0	1	2	3
y	:	81	16	1	0	1	16	81

Step2. Write down the trapezoidal rule and put the respective values in that rule

$$\int_{-3}^{3} f(x)\,dx = (h/2)\,[\,(y_0 + y_n) + 2\,(y_1 + y_2 + y_3 + \ldots + y_{n-1})\,]$$

$$= (h/2)\,[\,(\text{sum of the first and last ordinates}) + (\text{Sum of the remaining ordinates})\,]$$

$$= (1/2)\,[\,(81+81) + 2\,(16+1+0+1+16)\,]$$

$$= 115$$

By actual integration $\int_{-3}^{3} f(x)\,dx = \int_{-3}^{3} x^4\,dx$

$$= [\,(3^5/5) - (-3^5/5)\,]$$

$$= [\,(243/5) + (243/5)\,]$$

$$= 97.5$$

Illustration 2 :*Evaluate $\int_0^1 1/(1+x^2)\,dx$ by using Trapezoidal rule with $h = 0.2$*

Solution:

Step 1. We are given that $f(x) = 1/(1+x^2)$. Interval length $(b - a) = (1 - 0) = 1$. So we divide 6 equal intervals with $h = 0.2$ And tabulate the values as below

x :	0	0.2	0.4	0.6	0.8	1.0
y $1/(1+x^2)$:	1	0.96154	0.86207	0.73529	0.60976	0.5000

Step2. Write down the trapezoidal rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\,dx = (h/2)\,[\,(y_{0} + y_{n}) + 2\,(y_{1} + y_{2+}\, y_{3+} \;\ldots\; + y_{n-1})\,]$

= (h/2) [(sum of the first and last ordinates) +
(Sum of the remaining ordinates)]

= (0.2/2) [(1+0.5) + 2 (0.96154+0.86207+0.73529+0.60976)]

= (0.1) [(1.05) + 6.33732]

= 0.783732

Illustration 3. *Evaluate* $_{0}{}^{6}\, 1/(1+x)\,dx$ *by using Trapezoidal rule .*

Solution:

Step 1. We are given that f(x) = 1/(1+x). Interval length (b –a) = (6 – 0) = 6. So we divide 6 equal intervals with h= 1. And tabulate the values as below

x	:	0	1	2	3	4	5	6
$y1/(1+x^2)$:		1	0.5	1/3	1/4	1/5	1/6	1/7

Step2. Write down the trapezoidal rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\,dx = (h/2)\,[\,(y_{0} + y_{n}) + 2\,(y_{1} + y_{2+}\, y_{3+} \;\ldots\; + y_{n-1})\,]$

= (h/2) [(sum of the first and last ordinates) +
(Sum of the remaining ordinates)]

= (1/2) [(1+1/7) + 2 (0.5+1/3 + 1/4 + 1/5 +1/6)]

= (0.5) [(1.05) + 6.33732]

Illustration 4. *Evaluate* $_{4}{}^{5.2} \log_e x\;\; dx$ *by using Trapezoidal rule .*

Solution:

Step 1. We are given that f(x) = $\log_e x$ *Interval length (b –a) = (5.2 – 4) = 1.2. So we divide 6 equal intervals with h= 0.2. And tabulate the values as below*

x	:	4	4.2	4.4	4.6	4.8	5.0	5.2
y	:	1.39	1.44	1.48	1.53	1.57	1.61	1.65

Step2. Write down the trapezoidal rule and put the respective values of y in that rule

$\int_{-3}^{3} f(x)\, dx = (h/2)\ [\ (y_0 + y_n) + 2\ (y_1 + y_2 + y_3 + \ \ldots \ + y_{n-1})\ ]$

$= (h/2)$ [(sum of the first and last ordinates) +
(Sum of the remaining ordinates)]

$= (0.2/2)\ [\ (1.39+1.65) + 2\ (1.44 +1.48 \ + 1.53+ 1.57 +1.61)\ ]$

$= \ (0.1)\ \ [\ 3.04 + 2(7.63)\ ]$

$= 1.83$

Illustration 5. *Evaluate* $\int_{0}^{\pi} \sin x\, dx$ *by using Trapezoidal rule, by dividing the range into ten equal parts .*

Solution :

Step 1. We are given that $f(x) = \sin x$ *Interval length* $(b - a) = (\pi - 0) = \pi$.
So we divide 10 equal intervals with $h = \pi/10$ *(specified in the question itself), and tabulate the values as below*

x:	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	
Y:	0.0	0.3090	0.5878	0.8090	0.9511	
x:	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
Y:	1.0	0.9511	0.8090	0.5878	0.3090	0

Step2. Write down the trapezoidal rule and put the respective values of y in that rule

$\int_{-3}^{3} f(x)\, dx = (h/2)\ [\ (y_0 + y_n) + 2\ (y_1 + y_2 + y_3 + \ \ldots \ + y_{n-1})\ ]$

$= (h/2)$ [(sum of the first and last ordinates) +
(Sum of the remaining ordinates)]

$=(\pi/20)[(0+0)+2(0.3090+0.5878+0.8090+0.9511+1.0+ 0.9511+0.8090+0.5878+0.309)]$

$= 1.9843$

7.4 Let us Sum Up

In this lesson we have dealt with the following:

- We have discussed Trapezoidal rule for numerical integration. This method is very simple for calculation purposes.

7.5 Lesson end Activities

1. *Evaluate* $\int_1^2 1/(1+x^2)\,dx$ *taking* $h = 0.2$ *using Trapezoidal rule.*

2. *Compute the value of* $\int_1^2 dx/x$ *using Trapezoidal rule. Take* h *0.25*

3. *Evaluate* $\int_0^{\pi/2} \sin x\,dx$ *by using Trapezoidal rule, by dividing the range into ten equal parts .*

4. *Evaluate* $\int_0^1 \sqrt{\sin x + \cos x}\,dx$ *by using Trapezoidal rule, by dividing the range into seven equal parts .*

Model Answer

1. *(Ans: 0.3228)*

2. (Ans;0.6931)

3. (Ans:0.9981)

7.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

LESSON - 8

Simpson's one-third rule

Contents

8.0 Aims and Objectives

In this Lesson, we have discussed about numerical integration. Given set of paired values *(x_i, y_i) I = 0, 1,2 …n ,* even *f(x)* is unknown , it is possible to evaluate integration value. We have discussed about the Simpson's one third rule to evaluate integration. These methods more relatively accurate than the Simpson's one third rule.

After reading this lesson, you should be able to

- To know about numerical integration.
- To evaluate integration using Simpson's one third Rule

8.1 Introduction

Let $_a{}^b$ *f(x) dx represents* the area between y = f(x), with x =a and x =b. This integration is possible only when f(x) is explicitly given or otherwise it sis not possible to evaluate. The numerical integration can be detailed as follows ; Given set of (n+1) paired values of the function taking the values $y_0, y_1, y_2,$ ……….. y_n corresponding to the *values* x_0, x_1, x_2, ……….. x_{n}.. *Where f(x) is not known explicitly, it is possible to compute* $_a{}^b$ *f(x) dx* by numerical integration by using various methods. One other simplest method is Simpson's one third Rule that explained below

7.3 Simpson's one third Rule

Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ……….	x_n
y:	y_0	y_1	y_2	y_3 ……….	y_n

From the above values, we want to find the integration of $y = f(x)$ *with the range* x_0 *and* $x_0 + n^h$

$$_{x_0}{}^{x_n} f(x)\,dx = (h/3)\,[\,(y_0 + y_n) + 2\,(y_2 + y_4 + \quad) + 4\,(y_1 + y_3 + \ \ldots\ )\,]$$

= (h/3) [(sum of the first and last ordinates) +
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates)]

The above equation is called *Simpson's one third rule and* it is applicable only when **number of ordinates must be odd** (no. of pairs).

8.3 Illustrations *1. Evaluate* $_{-3}{}^{3}\, x^4\,dx$ *by using Simpson's one third rule. Verify result by actual integration.*

Step 1. We are given that $f(x) = x^4$. *Interval length* $(b - a) = (3 - (-3)) = 6$. *So we divide 6 equal intervals with* $h = 6/6 = 1.0$ *And tabulate the values as below*

x	:	-3	-2	-1	0	1	2	3
y	:	81	16	1	0	1	16	81

Step2. Write down the Simpson's one third rule and put the respective values in that rule

$$_{-3}{}^{3} f(x)\,dx\ ) = (h/3)\,[\,(y_0 + y_6) + 2\,(y_2 + y_4\ ) + 4\,(y_1 + y_3 + y_5\ )\,]$$

= (h/3) [(sum of the first and last ordinates) +
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates)]

$$= (1/3)\,[\,(81+81) + 2\,(1+1) + 4(16+1+16)\,]$$

$$= 98$$

By actual integration $_{-3}{}^{3} f(x)\,dx = {}_{-3}{}^{3}\, x^4\,dx$

$$= [\,(3^5/5) - (-3^5/5)\,]$$

$$= [\,(243/5) + (243/5)]$$

$$= 97.5$$

2. *Evaluate* $\int_{0}^{1.2} 1/(1+x^2)\,dx$ *by using Simpson's one third rule with h = 0.2*

Solution:

Step 1. We are given that $f(x) = 1/(1+x^2)$. *Interval length (b – a) = (1.2 – 0) = 1.2 . So we divide 6 equal intervals with h= 0.2 And tabulate the values as below*

x	:	0	0.2	0.4	0.6	0.8	1.0	1.2
$y\,1/(1+x^2)$	:	1	0.9615	0.8621	0.7353	0.6098	0.5000	0.4098

Step2. Write down the Simpson's one third rule and put the respective values of y in that rule

$$\int_{-3}^{3} f(x)\,dx = (h/3)\,[\,(y_0 + y_6) + 2\,(y_2 + y_4\,) + 4\,(y_1 + y_3 + y_5\,)\,]$$

$$= (h/3)\,[\,(\text{sum of the first and last ordinates}) + 2\,(\text{Sum of remaining even ordinates}) + 4\,(\text{sum of remaining odd ordinates})\,]$$

$$= (0.2/3)\,[\,(1+0.4098) + 2\,(0.8621 + 0.6098) + 4\,(0.9615+0.7353+0.5)\,]$$

$$= (0.0667)\,[\,(1.4098) + 2(1.4719) + 4\,(2.1503)\,]$$

$$= (0.0667)\,[\,1.4098 + 2.9438 + 8.6012]$$

$$\mathbf{=0.8641}$$

3. *Evaluate* $\int_{0}^{6} 1/(1+x)\,dx$ *by using Simpson's one third rule .*

Solution:

Step 1. We are given that $f(x) = 1/(1+x)$. *Interval length (b – a) = (6 – 0) = 6. So we divide 6 equal intervals with h= 1. And tabulate the values as below*

x	:	0	1	2	3	4	5	6
$y\,1/(1+x^2)$	:	1	0.5	1/3	1/4	1/5	1/6	1/7

Step2. Write down the Simpson's one third rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\, dx \;=\; (h/3)\, [\, (y_0 + y_6) + 2\,(y_2 + y_4\,) + 4\,(y_1 + y_3 + y_5\,)\,]$

= (h/3) [(sum of the first and last ordinates) +
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates)]

= (1/3) [(1+1/7) + 2 (1/3 + 1/5) + 4(0.5 +1/4 +1/6)]

= **1.9587**

4. *Evaluate* $_{4}{}^{5.2} \log_e x\; dx$ *by using Simpson's one third rule .*

Solution:

Step 1. We are given that f(x) = $\log_e x$ Interval length (b –a) = (5.2 – 4) = 1.2. So we divide 6 equal intervals with h= 0.2. And tabulate the values as below

x	:	4	4.2	4.4	4.6	4.8	5.0	5.2
y	:	1.39	1.44	1.48	1.53	1.57	1.61	1.65

Step2. Write down the Simpson's one third rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\, dx \;=\; (h/3)\, [\, (y_0 + y_6) + 2\,(y_2 + y_4\,) + 4\,(y_1 + y_3 + y_5\,)\,]$

= (h/3) [(sum of the first and last ordinates) +
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates)]

= (0.2/3) [(1.39+1.65) + 2 (1.48+ 1.57) + 4 (1.44+ 1.53++1.61)]

= (0.0667) [3.04 + 2(3.05)+ 4 (4.58)]

= 1.83

5. *Evaluate* $_{0}{}^{\pi}$ *sin x dx by using Simpson's one third rule, by dividing the range into ten equal parts .*

Solution :

Step 1. We are given that $f(x) = \sin x$ Interval length $(b - a) = (\pi - 0) = \pi$.
So we divide 10 equal intervals with $h = \pi/10$ (specified in the question itself), and tabulate the values as below

x:	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	
Y:	0.0	0.3090	0.5878	0.8090	0.9511	
x:	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
Y:	1.0	0.9511	0.8090	0.5878	0.3090	0

Step2. Write down the Simpson's one third rule and put the respective values of y in that rule

$$\int_{-3}^{3} f(x)\,dx = (h/3)\,[\,(y_0 + y_{10}) + 2\,(y_2 + y_4 + y_6 + y_8) + 4\,(y_1 + y_3 + y_5 + y_7 + y_9)\,]$$

= (h/3) [(sum of the first and last ordinates) +
+ 2 (Sum of remaining even ordinates)
+4 (sum of remaining odd ordinates

$$= (\pi/20)[(0+0)+2(0.5878+0.9511+0.9511+0.5878) + 4(0.3090+0.8090+1+0.8090+0.3090)]$$

$$= \mathbf{2.0009}$$

8.4 Lesson End Activities

2. *Evaluate $\int_1^2 1/(1+x^2)\,dx$ taking $h = 0.2$ using Simpson's one third rule.*

2. *Compute the value of $\int_1^2 dx/x$ using Simpson's one third rule. Take h 0.25*

3. *Evaluate $\int_0^{\pi/2} \sin x\,dx$ by using Simpson's one third rule, by dividing the range into ten equal parts .)*

4. *Evaluate $\int_0^1 \sqrt{\sin x + \cos x}\,dx$ by using Simpson's one third rule, by dividing the range into seven equal parts .*

8.5 Let us Sum Up

In this lesson we have dealt with the following:

We have discussed about the Simpson's one third rule to evaluate integration. This method more easier than any other method.

Model Answer for selected lesson end activities

1. **0.3228**
2. **0.6971**
3. **1.0006**

8.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

LESSON - 9

Simpson's three-eighths rule

Contents

9.0 Aims and Objectives

In this Lesson, we have discussed about numerical integration. Given set of paired values (x_i, y_i) $I = 0, 1,2 \ldots n$, even $f(x)$ is unknown , it is possible to evaluate integration value. We have discussed about the Simpson's three-eighths rule to evaluate integration. This method is relatively accurate than earlier methods of numerical integration.

After reading this lesson, you should be able to

- To know about the numerical integration.
- To evaluate integration using Simpson's three-eighth Rule

9.1 Introduction

Let $_a{}^b$ *f(x) dx represents* the area between y = f(x), with the rand x =a and x =b. This integration is possible only when f(x) is explicitly given or otherwise it sis not possible to evaluate. In numerical integration can be detailed as follows ; Given set of (n+1) paired values of the function taking the values $y_0, y_1, y_2, \ldots\ldots\ldots\ y_n$ corresponding to the values $x_0, x_1, x_2, \ldots\ldots\ldots\ x_n,$. where $f(x)$ is not known explicitly, it is possible to compute $_a{}^b f(x)\, dx$ by numerical integration by using various method. One of the simplest method is Simpson's three-eighth rule, that is explained in detail and also this method is method is more accurate than the earlier methods of study.

9.2 Simpson's three-eighth Rule

Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ……….	x_n
y:	y_0	y_1	y_2	y_3 ……….	y_n

From the above values, we want to find the integration of *y = f(x) with the range* x_0 *and* $x_0 + h$

$$\int_{x_0}^{x_n} f(x)\, dx = (3h/8)\, [\, (y_0 + y_n) + 2\,(y_3 + y_6 + y_9 + \ldots) + 3\,(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1})]$$

= (3h/8) [(sum of the first and last ordinates) +
+ 2 (Sum of multiples of three ordinates)
+3 (sum of remaining ordinates)]

The above equation is called Simpson's three-eighths rule which ***is applicable*** *only when* **n is multiple of** *3* .

9.3 Illustrations *1. Evaluate* $\int_{-3}^{3} x^4\, dx$ *by using Simpson's three-eighth rule. Verify result by actual integration.*

Step 1. We are given that $f(x) = x^4$. *Interval length (b –a) = (3 – (-3)) = 6. So we divide 6 equal intervals with h= 6/6 = 1.0 And tabulate the values as below*

x	:	-3	-2	-1	0	1	2	3
y	:	81	16	1	0	1	16	81

Step2. Write down the Simpson's three-eighth rule and put the respective values in that rule

$$\int_{-3}^{3} f(x)\, dx\ ) = (3h/8)\, [\, (y_0 + y_6) + 2\,(y_3\ ) + 3\,(y_1 + y_2 + y_4 + y_5\ )\,]$$

= (3h/8) [(sum of the first and last ordinates) +
+ 2 (Sum of multiples of three, other than last ordinates)
+3 (sum of remaining ordinates)]

$$= (3/8)\, [\, (81+81) + 2\,(0)) + 3(16+1+1+16)\,]$$

$$= 99$$

By actual integration $\int_{-3}^{3} f(x)\, dx = \int_{-3}^{3} x^4\, dx$

$$= [\, (3^5/5) - (-3^5/5)\,]$$

$=[\ (243/5) + (243/5)]$

$= 97.5$

6. *Evaluate* $\int_0^{1.2} 1/(1+x^2)\, dx$ *by using Simpson's three-eighth rule with* $h = 0.2$

Solution:

Step 1. We are given that $f(x) = 1/(1+x^2)$. *Interval length* $(b - a) = (1 - 0) = 1$. *So we divide 6 equal intervals with* $h = 0.2$ *and tabulate the values as below*

x :	0	0.2	0.4	0.6	0.8	1.0	1.2
$y1/(1+x^2)$:	1	0.9615	0.8621	0.7353	0.6098	0.5000	0.4098

Step2. Write down the Simpson's three-eighth rule and put the respective values of y in that rule

$\int_{-3}^{3} f(x)\, dx = (3h/8)\ [\ (y_0 + y_6) + 2\ (y_3\ ) + 3\ (y_1 + y_2 + y_4 + y_5\ )\ ]$

$= (3h/8)$ [(sum of the first and last ordinates) +
+ 2 (Sum of multiples of three, other than last ordinates)
+3 (sum of remaining ordinates)]

$= (3 \times 0.2 /3)\ [\ (1+0.4098) + 2\ (0.7353\ )$
$+ 3\ (0.9615+0.8621+0.6098 + 0.5\ )\ ]$

$= (0.075)\ [1.4098 + 1.4706 + 3\ (\ 2.9334)\ ]$

$= (0.075)\ [\ 1.4098 + 1.4706 + 8.8002]$

$\boldsymbol{=0.8760}$

7. *Evaluate* $\int_0^{6} 1/(1+x)\, dx$ *by using Simpson's three-eighth rule .*

Solution:

Step 1. We are given that $f(x) = 1/(1+x)$. *Interval length* $(b - a) = (6 - 0) = 6$. *So we divide 6 equal intervals with* $h = 1$. *And tabulate the values as below*

X :	0	1	2	3	4	5	6
$y = 1/(1+x^2)x$:	1	0.5	1/3	1/4	1/5	1/6	1/7

Step2. Write down the Simpson's three-eighth rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\,dx \;=\; (3h/8)\,[\,(y_0 + y_6) + 2\,(y_3\;) + 3\,(y_1 + y_2 + y_4 + y_5\;)\,]$

= (3h/8) [(sum of the first and last ordinates) +
+ 2 (Sum of multiples of three, other than last ordinates)
+3 (sum of remaining ordinates)]

$= (3/8)\,[\,(1+1/7) + 2\,(1/4) \;+\; 3\,(\,0.5 + 1/3 + 1/5 + 1/6\,)\,]$

= 1.9661

8. *Evaluate* $_{4}{}^{5.2} \log_e x\;dx$ *by using Simpson's three-eighth rule .*

Solution:

Step 1. We are given that $f(x) = \log_e x$ *Interval length (b –a) = (5.2 – 4) = 1.2. So we divide 6 equal intervals with h= 0.2. And tabulate the values as below*

x	:	4	4.2	4.4	4.6	4.8	5.0	5.2
y	:	1.39	1.44	1.48	1.53	1.57	1.61	1.65

Step2. Write down the Simpson's three-eighth rule and put the respective values of y in that rule

$_{-3}{}^{3} f(x)\,dx \;=\; (3h/8)\,[\,(y_0 + y_6) + 2\,(y_3\;) + 3\,(y_1 + y_2 + y_4 + y_5\;)\,]$

= (3h/8) [(sum of the first and last ordinates) +
+ 2 (Sum of multiples of three, other than last ordinates)
+3 (sum of remaining ordinates)]

$= (3\text{x}\;0.2\;/8\,)\,[\,(1.39+1.65) + 2\,(1.53) \;+ 3\,(1.44+ 1.48 +1.57++1.61)\,]$

$= \;(0.075\,)\;[\,3.04 + 3.06 + 3\;(6.1)\,]$

= **1.83**

9. *Evaluate* $_{0}{}^{9}\; x^2\,dx$ *by using Simpson's three-eighth rule, by dividing the range into nine equal parts and verify your answer with actual integration.*

Solution :

Step 1. We are given that $f(x) = x^2$ *Interval length (b –a) = (9 – 0) =9.*

So, we divide 9 equal intervals with h=9 /9 = 1 (specified in the question itself), and tabulate the values as below

X :	0	1	2	3	4
$Y = x^2$:	0	1	4	9	16
x:	5	6	7	8	9
Y:	25	36	49	64	81

Step2. Write down the Simpson's three-eighth rule and put the respective values of y in that rule

$\int_{-3}^{3} f(x)\,dx = (3h/8)\,[\,(y_0 + y_9) + 2\,(y_3 + y_6) + 3\,(y_1 + y_2 + y_4 + y_5 + y_7 + y_8)\,]$

= (3h/8) [(sum of the first and last ordinates) +
+ 2 (Sum of multiples of three, other than last ordinates)
+3 (sum of remaining ordinates)]

$= (3/8)\,[\,(0 + 81) + 2\,(9 + 36) + 3\,(1 + 4 + 16 + 25 + 49 + 64)\,]$

$= (.375)\,[\,81 + 90 + 477\,]$

$= \mathbf{243}$

By actual integration $\int_0^9 f(x)\,dx = \int_0^9 x^2\,dx$

$= [\,(9^3/3) - (0^3/3)\,]$

$= [\,(729/3) + 0\,]$

$= \mathbf{243}$

9.4 Lesson End Activities

1. *Evaluate* $\int_1^{2.2} 1/(1+x^2)\,dx$ *taking h = 0.2 using Simpson's three eighth rule.*

2. *Compute the value of* $\int_0^{1.2} dx/x$ *using Simpson's three eighth rule. Take h = 0.2*

3. *Evaluate* $\int_0^{\pi/2} \sin x\,dx$ *by using Simpson's three eighth rule, by dividing the range into nine equal parts .*

4. *Evaluate* $\int_0^{6} \sin x + \cos x\,dx$ *by using Simpson's three eighth rule, by dividing the range into six equal parts .*

9.5 Let us Sum Up

In this lesson we have dealt with the following:

We have discussed about the Simpson's three eighth rule to evaluate integration. This method more relatively accurate than the Simpson's one third rule

Model Answer for selected lesson end activities

1 **0.3228**
2. **0.6971**
3. **1.0006**

9.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

UNIT - III

LESSON – 10

Interpolation

Contents

10.0 Aims and Objectives

In this Lesson, we have discussed about Interpolation, which means process of computing intermediate value of a function. We have also discussed about Newton's forward interpolation formula for finding intermediate value of a function.
After reading this lesson, you should be able to

- To know about Interpolation.
- To find intermediate values using Newton's forward difference formula.

10.1 Introduction

Interpolation means the process of computing intermediate values of a function a given set of tabular values of a function. Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ……….	x_n
y:	y_0	y_1	y_2	y_3 ……….	y_n

We may require the value of $y = y_i$ for the given $x = x_i$, where x lies between x_0 to x_n

Let y = *f(x)* be a function taking the values $y_0, y_1, y_2, \ldots\ y_n$ corresponding to the *values* $x_0, x_1, x_2, \ldots\ x_n$. Now we are trying to find $y = y_i$ for the given $x = x_i$ under assumption that the function f(x) is not known. In such cases , we replace f(x) by simple fan arbitrary function and let Φ(x) denotes an arbitrary function which satisfies the set of values given in the table above . The function Φ(x) is called interpolating function or smoothing function or interpolation formula.

10.2 Newton's forward interpolation formula (or) Gregory-Newton forward interpolation formula (for equal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \ldots\ldots, y_n$ corresponding to the values $x_0, x_1, x_2 \ldots\ldots, x_n$.

Let suppose that the values of x i.e., $x_0, x_1, x_2 \ldots\ldots, x_n$ are equidistant .

$x_1 = x_0 + h$; $x_2 = x_1 + h$; *and so on* $x_n = x_{n-1} + h$;

Therefore $x_i = x_0 + i\,h$, *where* $i = 1,2, \ldots, n$

Let $P_n(x)$ be a polynomial of the n th degree in which x is such that

$y_I = f(x_i) = P_n(x_i),\ I = 0,1,2, \ldots\ n$

Let us assume $P_{n(x)}$ in the form given below

$$P_n(x) = a_0 + a_1(x - x_0)^{(1)} + a_2(x - x_0)^{(2)} + \ldots\ldots + a_r(x - x_0)^{(r)} + \ldots\ldots + \ldots\ldots + a_n(x - x_0)^{(n)} \ldots\ldots(1)$$

This polynomial contains the n + 1 constants $a_0, a_1, a_2, \ldots\ldots a_n$ can be found as follows :

$P_n(x_0) = y_0 = a_0$ (setting $x = x_0$, in (1))

Similarly $y_1 = a_0 + a_1(x_1 - x_0)$

$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)$

From these, we get the values of $a_0, a_1, a_2, \ldots\ldots a_n$

i.e.,

Therefore, $a_0 = y_0$

$\Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0)$

$= a_1 h$

$\Rightarrow a_1 = \Delta y_0 / h$

lly $\Rightarrow a_2 = (\Delta y_1 - \Delta y_0) / 2h^2 = \Delta^2 y_0 / 2!\, h^2$

lly $\Rightarrow a_3 = \Delta^3 y_0 / 3!\, h^3$

Putting these values in (1), we get

$$P_n(x) = = y_0 + (x - x_0)^{(1)} \Delta y_0 / h + (x - x_0)^{(2)} \Delta^2 y_0 / (2!\, h^2) + \ldots\ldots + (x - x_0)^{(r)} \Delta^r y_0 / (r!\, h^r) + \ldots\ldots + (x - x_0)^{(n)} \Delta^r y_0 / (n!\, h^n)$$

By substituting $\left(\dfrac{x - x_0}{h}\right) = u$ *, the above equation becomes*

$$y(x_0 + uh) = y_u = y_0 + u\,\Delta y_0 + \frac{u\,(u-1)}{2!}\,\Delta^2 y_0 + \frac{u\,(u-1)(u-2)}{3!}\,\Delta^3 y_0 + \ldots\ldots \qquad \ldots$$

By substituting $u = u^{(1)}$,

$u\,(u-1) = u^{(2)}$,

$u(u-1)(u-2) = u^{(3)}$, … *in the above equation, we get*

$$P_n\,(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\,\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y_0 + \ldots. + \frac{u^{(n)}}{n!}\Delta^n y_0$$

The above equation is known as **Gregory-Newton forward formula or Newton's forward interpolation formula.**

Note : 1. This formula is applicable only when the interval of difference is uniform.

2. This formula apply forward differences of y_0, hence this is used to interpolate the values of y nearer to beginning value of the table (i.e., x lies between x_0 to x_1 or x_1 to x_2)

10.3 Illustrations 1. Find the values of y at x = 21 from the following data.

x:	20	23	26	29
y	0.3420	0.3907	0.4384	0.4848

Solution.

Step 1. Since x = 21 is nearer to beginning of the table. Hence we apply Newton's forward formula.

Step 2. Construct the difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
20	0.3420	(0.3420-0.3907)		
		0.0487	(0.0477-0.0487)	
23	0.3907		-0.001	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

Step 3. Write down the formula and put the various values :

$$P_n\,(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\,\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y_0 + \ldots. + \frac{u^{(n)}}{n!}\Delta^n y_0$$

Where $u^{(1)} = (x - x_0) / h = (21 - 20) / 3 = 0.3333$
$\underline{u^{(2)} = u(u\text{-}1) = (0.3333)(0.6666)}$

$$P_n\,(x{=}21){=}\,y(21) = 0.3420 + (0.3333)(\,0.0487) + (0.3333)\,(-0.6666)\,(\,-0.001) + (0.3333)\,(-0.6666)(-1.6666)\,(\,-0.0003)$$

$$= \mathbf{0.3583}$$

Illustrations 2 . From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46.

Age	x:	45	50	55	60	65
Premium	y:	114.84	96.16	83.32	74.48	68.48

Solution.

Step 1. Since $x = 46$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 5$.. Hence we apply Newton's forward formula.

Step 2. Construct the difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.12		4.00		0.68
		-8.84		-1.16	
60	74.48		2.84		
		-6.00			
65	68.48				

Step 3. Write down the formula and put the various values :

$$P_n\,(x){=}\,P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y_0 + \ldots. + \frac{u^{(n)}}{n!}\Delta^n y_0$$

Where $u = (x - x_0) / h = (46 - 45) / 5 = 01/5 = 0.2$

$$P_n\,(x{=}46){=}\,y(46) = 114.84 + [0.2\,(-18.68)] + [0.2\,(-0.8)\,(5.84)/\,3] + [0.2\,(-0.8)\,(-1.8)(-1.84)/6\,] + [0.2\,(-0.8)\,(-1.8)(-2.8)(0.68)]$$

$= 114.84 – 3.7360 – 0.4672 – 0.08832 – 0.228$

$=$ **110.5257**

Illustrations 3 . From the following table , find the value of tan 45^0 15'

	x^0:	45	46	47	48	49	50
tan	x^0:	1.0	1.03553	1.07237	1.11061	1.15037	1.19175

Solution.

Step 1. Since $x = 45^o\ 15'$ is nearer to beginning of the table and the values of x is equidistant i.e., $h =1.$ Hence we apply Newton's forward formula.

Step 2. Construct the difference table to find various Δ's

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
45^0	1.0000					
		0.03553				
46^0	1.03553		0.00131			
		0.03684		0.00009		
47^0	1.07237		0.00140		0.00003	
		0.03824		0.00012		-0.00005
48^0	1.11061		0.00152		-0.00002	
		0.03976		0.00010		
49^0	1.15037		0.00162			
		0.04138				
50^0	1.19175					

Step 3. Write down the formula and substitute the various values :

$$P_n (x)= P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 +\ldots\ldots+ \frac{u^{(r)}}{r!}\Delta^r y_0 +\ldots. + \frac{u^{(n)}}{n!}\Delta^n y_0$$

Where u $= (45^o\ 15' – 45^0) / 1^0$
$= 15' / 1^0$
$= 0.25$ ……………(since $1^0 = 60$ ')

$$y\ (x=45^o\ 15'\ )= P_5\ (45^o\ 15') =1.00 + (0.25)(\ 0.03553) + (0.25)(- 0.75)(0.00131)/2$$
$$+(0.25)(- 0.75)(-1.75)(0.00009)/6$$
$$+(0.25)(- 0.75) (-1.75) (-2.75) (0.0003)/24$$
$$+(0.25)(- 0.75) (-1.75) (-2.75) (-3.75) (-0.00005)/120$$
$$= 1.000 + 0.0088825 – 0.0001228 +0.0000049$$

= ***1.00876***

4. The Population of a town is as follows.

Year x:	1941	1951	1961	1971	1981	1991
Population in lakhs y :	20	24	29	36	46	51

Estimate the population increase during the period 1946.

Step 1**.** Since $x = 46$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 5$. Hence we apply Newton's forward formula.

Step 2. Construct the difference table to find various Δ's

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
1941	20					
		4				
1951	24		1			
		5		1		
1961	29		2		0	
		7		1		-9
1971	36		3		-9	
		10		-8		
1981	46		-5			
		5				
1991	51					

Step 3. Write down the formula and substitute the various values :

$$P_n(x) = P_5\, y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \frac{u^{(4)}}{4!}\Delta^4 y_0 + \frac{u^{(5)}}{5!!}\Delta^5 y_0$$

Where u = (1946 –19 41) / 10
= 5/ 10
= 0.5

Y(1946) = 20 + (0.5) (4) + (0.5)(-0.5) (1/2) + (0.5)(-0.5)(-1.5)(1/6) +
+(0.5)(-0.5) (-1.5)(-2.5) (0/24)
+(0.5)(-0.5) (-1.5) (-2.5) (-3.5) (-9 /120)
= 20 + 2 -0.125 + 0.0625 + 0 -0.24609

= 21.69

10.4 Lesson end activities

1.From the following data find θ at x = 43 . Also express θ in terms of *x*.

x :	*40*	*50*	*60*	*70*	*80*	*90*
y :	*184*	*204*	*226*	*250*	*276*	*304*

2. From the data given below, find the number of students whose weight is between 60 and 70.

Weight in lbs. :	*0-40*	*40-60*	*60-80*	*80-100*	*100-120*
No. of students:	*250*	*120*	*100*	*70*	*50*

3. Find a polynomial of degree two which takes the values

x :	0	1	2	3	4	5	6	7
y :	1	2	4	7	11	16	22	29

4.The following data are taken from the steam table.

Temp. ^{0}C	:140	150	160	170	180
Pressure	: 3.685	4.854	6.302	8.076	10.225

Find the pressure at temperature t = 142^0. (Ans : 3.898)

10.5 Let us Sum Up:
In this lesson we have dealt with following

* We have discussed about the Newton's forward difference formula to find intermediate values. This method more useful when the function type is not exactly known.

Model Answer

1. 189.79, $0.01x^2 + 1.1 x + 124$
2. 424
3. $0.5 (x^2 + x + 2)$
4. 3.898

10.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 .

LESSON - 11

Newton Backward Interpolation Formula

Contents

11.0 Aims and Objectives

In this Lesson, we have discussed about Newton's backward interpolation formula for finding intermediate value of a function.
After reading this lesson, you should be able to

* To find intermediate values which occur end of the series using Newton's backward interpolation formula.

11.1 Introduction

Interpolation means the process of computing intermediate values of a function a given set of tabular values of a function. Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ……….	x_n
y:	y_0	y_1	y_2	y_3 ……….	y_n

We may require the value of $y = y_i$ for the given $x = x_i$, where x lies between x_{n-1} *to* x_n
Let y = $f(x)$ be a function taking the values $y_0, y_1, y_2, \ldots\ y_n$ corresponding to the *values* $x_0,\ x_1, x_2,\ \ldots\ldots\ x_n$. Now we are trying to find $y = y_i$ for the given $x = x_i$ under assumption that the function f(x) is not known. In such cases , we replace f(x) by simple an arbitrary function and let Φ(x) denote an arbitrary function which satisfies the set of values given in the table above . The function Φ(x) is called interpolating function or smoothing function or interpolation formula.

11.2 Newton's backward interpolation formula (or) Gregory-Newton backward interpolation formula (for equal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \dots\dots\dots, y_n$ corresponding to the values $x_0, x_1, x_2 \dots\dots\dots, x_n.$

Let suppose that the values of x i.e., $x_0, x_1, x_2 \dots\dots\dots, x_n$ are equidistant .

$$x_1 = x_0 + h\ ;\ \ x_2 = x_1 + h\ ;\ \ \textit{and so on}\ \ \ x_n = x_{n-1} + h\ ;$$

Therefore $x_i = x_0 + i\,h,$ *where* $i = 1,2, \dots, n$

Let $P_n(x)$ be a polynomial of the n th degree in which x is such that

$$y_I = f(x_i) = P_n(x_i),\ \ I = 0,1,2, \dots. n$$

$$P_n(x) = a_0 + a_1(x - x_n)^{(1)} + a_2(x - x_n)(x - x_{n-1})^{)} + \dots\dots\dots + a_n(x - x_n)(x - x_{n-1}) \dots(x - x_1) \dots\dots(1)$$

Let us assume $P_{n(x)}$ in the form given below

$$P_n(x) = a_0 + a_1(x - x_n)^{(1)} + a_2(x - x_n)^{(2)} + \dots\dots\dots + a_r(x - x_n)^{(r)} + \dots\dots + \dots\dots + a_n(x - x_n)^{(n)} \dots\dots(1.1)$$

This polynomial contains the n + 1 constants $a_0, a_1, a_2, \dots\dots\dots a_n$ can be found as follows :

$P_n(x_n) =$ $y_n = a_0$ (setting $x = x_n,$ in (1))

Similarly $y_{n-1} = a_0 + a_1(x_{n-1} - x_n)$

$y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)$

From these, we get the values of $a_0, a_1, a_2, \dots\dots a_n$

Therefore, $a_0 = y_n$

$\nabla y_n = y_n - y_{n-1} = a_1(x_{n-1} - x_n)$

$= a_1 h$

$=> a_1 = \nabla y_n / h$

lly $=> a_2 = (\nabla y_1 - \nabla y_n) / 2h^2 = \nabla^2 y_n / 2!\, h^2$

lly $=> a_3 = \nabla^3 / 3!\, h^3$

Putting these values in (1), we get

$$P_n(x) = = y_n + (x - x_n)^{(n)} \nabla y_n / h + (x - x_n)^{(2)} \nabla^2 y_n / (2!\, h^2) + (x - x_n)^{(r)} \nabla^r y_n / (r!\, h^r) + \dots\dots + (x - x_n)^{(n)} \nabla^r y_n / (n!\, h^n)$$

By substituting $\left| \frac{x - x_n}{h} \right| = v$ *, the above equation becomes*

$$y(x_n + vh) = y_n + v\, \nabla y_n + \underline{v\,(v+1)}\, \nabla^2 y_n + \underline{v\,(v+1)(v+2)}\, \nabla^3 y_n + \dots\dots\dots \quad \dots$$

2! *3!*

By substituting $v = v^{(1)}$,

$v\,(v+1) = v^{(2)}$,

$v(v+1)(v+2) = v^{(3)}$, *… in the above equation, we get*

$$P_n\,(x)= P_n y(x_n + vh) = y_n + v^{(1)}\nabla y_n + \frac{v^{(2)}}{2!}\nabla^2 y_n + \frac{v^{(3)}}{3!}\nabla^3 y_n + \ldots + \frac{v^{(r)}}{r!}\nabla^r y_n + \ldots + \frac{v^{(n)}}{n!}\Delta^n y_n$$

The above equation is known as **Gregory-Newton backward formula or Newton's backward interpolation formula.**

Note : 1. This formula is applicable only when the interval of difference is uniform.
2. This formula apply backward differences of y_n, hence this is used to interpolate the values of y nearer to the end of a set tabular values. (i.e., x lies between x_n to x_{n-1} and x_{n-1} to x_{n-2})

11.3 Illustrations 1. Find the values of y at x = 28 from the following data.

x:	20	23	26	29
y	0.3420	0.3907	0.4384	0.4848

Solution.

Step 1. Since x = 28 is nearer to beginning of the table. Hence we apply Newton's backward formula.

Step 2. Construct the difference table

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$
20	0.3420	(0.3420-0.3907)		
		0.0487	(0.0477-0.0487)	
23	0.3907		-0.001	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

Step 3. Write down the formula and put the various values :

$$P_3\,(x)= P_3\,y(x_n + vh) = y_n + v^{(1)}\nabla y_n + \frac{v^{(2)}}{2!}\nabla^2 y_n + \frac{v^{(3)}}{3!}\nabla^3 y_n$$

Where $v^{(1)} = (x - x_n) / h = (28 - 29) / 3 = -0.3333$

$v^{(2)}= v(v+1) =(-0.333)(0.6666)$

$v^{(3)}= v(v+1)\,(v+2) =(-0.333)(0.6666)(1.6666)$

$$P_n\,(x=28)= y(28) = 0.4848 + (-0.3333)(0.0464)+ (-0.3333)\,(0.6666)\,(-0.0013)/2$$

$$+(-0.3333)\ (0.6666)(1.6666)\ \ (-0.0003)/6$$
$$= 0.4848 - 0.015465 + 0.0001444 + 0.0000185$$

$$= \mathbf{0.4695}$$

Illustrations 2 . From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 63.

Age	x:	45	50	55	60	65
Premium	y:	114.84	96.16	83.32	74.48	68.48

Solution.

Step 1. Since $x = 63$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 5$.. Hence we apply Newton's backward formula.

Step 2. Construct the difference table

x	y	∇y_0	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.12		4.00		0.68
		-8.84		-1.16	
60	74.48		2.84		
		-6.00			
65	68.48				

Step 3. Write down the formula and put the various values :

$$P_3\ (x) = P_3\, y(x_n + vh) = y_n + v^{(1)} \nabla y_n + \frac{v^{(2)}}{2!} \nabla^2 y_n + \frac{v^{(3)}}{3!} \nabla^3 y_n + \frac{v^{(4)}}{4!} \nabla^4 y_n$$

Where

$$v^{(1)} = (x - x_n) / h = (63 - 65) / 5 = -2/5 = -0.4$$
$$v^{(2)} = v(v+1) = (-0.4)(1.6)$$
$$v^{(3)} = v(v+1)(v+2) = (-0.4)(1.6)(2.6)$$
$$v^{(4)} = v(v+1)(v+2))(v+3) = (-0.4)(1.6)(2.6)(3.6)$$

$$P_4\ (x=63) = y(63) = 68.48 + [(-0.4)(-6.0)] + [(-0.4)(1.6)(2.84)/2]$$
$$+ [(-0.4)(1.6)(2.6)(-1.16)/6]$$
$$+ [(-0.4)(1.6)(2.6)(3.6)(0.68)/24]$$

$$= 68.48 + 2.40 - 0.3408 + 0.07424 - 0.028288$$

$$= \mathbf{70.5852}$$

Illustrations 3 . From the following table , find the value of tan 49^0 15'

	x^0:	45	46	47	48	49	50
tan	x^0:	1.0	1.03553	1.07237	1.11061	1.15037	1.19175

Solution.

Step 1. Since $x = 49^o\ 45'$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 1$. Hence we apply Newton's backward formula.

Step 2. Construct the difference table to find various Δ's

x	y	˘ y_0	˘ $^2y_0$	˘ $^3y_0$	˘ $^4y_0$	˘ $^5y_0$
45^0	1.0000					
		0.03553				
46^0	1.03553		0.00131			
		0.03684		0.00009		
47^0	1.07237		0.00140		0.00003	
		0.03824		0.00012		-0.00005
48^0	1.11061		0.00152		-0.00002	
		0.03976		0.00010		
49^0	1.15037		0.00162			
		0.04138				
50^0	1.19175					

Step 3. Write down the formula and substitute the various values :

$$P_5\ (x) = P_5\, y(x_n + vh) = y_n + v^{(1)}\text{˘}\ y_n + \frac{v^{(2)}}{2!}\text{˘}\ ^2y_n + \frac{v^{(3)}}{3!}\text{˘}\ ^3y_n + \frac{v^{(4)}}{4!}\text{˘}\ ^4y_n + \frac{v^{(5)}}{5!}\text{˘}\ ^5y_n$$

Where $v = (49^o\ 45' - 50^0) / 1^0$

$= -\ 15' / 1^0$

$= -\ 0.25$ ……………(since $1^0 = 60$ ')

$v^{(2)} = v(v+1) \qquad = (-0.25)\,)(0.75)$

$v^{(3)} = v(v+1)\,(v+2) \qquad = (-0.25)\,(0.75)(1.75)$

$v^{(4)} = v(v+1)\,(v+2)\,)\,(v+3) \qquad = (-0.25)(0.75)\,(1.75)\,(2.75)$

$y\ (x=49^o\ 15'\ ) = P_5\ (49^o\ 15') = 1.19175 + (-0.25)(0.04138) + (-0.25)(0.75)\,(0.00162)/2$
$+(-0.25)\,(0.75)(1.75)\,(0.0001)/6$

$$+(-0.25)(0.75)(1.75)(2.75)(-0.0002)/24$$
$$+(-0.25)(0.75)(1.75)(2.75)(3.75)(-0.00005)/120$$
$$= 1.19175 - 0.010345 - 0.000151875 + 0.000005 + \ldots.$$
$$= \mathbf{1.18126}$$

4. The Population of a town is as follows.

Year x:	1941	1951	1961	1971	1981	1991
Population in lakhs y :	20	24	29	36	46	51

Estimate the population in the year of 1976.

Step 1**.** Since $x = 76$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 5$. Hence we apply Newton's backward formula.

Step 2. Construct the difference table to find various Δ's

x	y	∇y_0	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$	$\nabla^5 y_0$
1941	20					
		4				
1951	24		1			
		5		1		
1961	29		2		0	
		7		1		-9
1971	36		3		-9	
		10		-8		
1981	46		-5			
		5				
1991	51					

Step 3. Write down the formula and substitute the various values :

$$P_5(x) = P_5\, y(x_n + vh) = y_n + v^{(1)}\nabla y_n + \frac{v^{(2)}}{2!}\nabla^2 y_n + \frac{v^{(3)}}{3!}\nabla^3 y_n + \frac{v^{(4)}}{4!}\nabla^4 y_n + \frac{v^{(5)}}{5!}\nabla^5 y_n$$

Where $v^{(1)} = (1976 - 1991) / 10$
$= -15/10$
$= -1.5$
$v^{(2)} = v(v+1) = (-1.5)(0.5)$ and so on.,

Y(1976) = 51 + (-1.5) (5) + (-1.5)(0.5) (-5/2) + (-1.5)(0.5) (1.5) (-8/6) +
+ (-1.5)(0.5) (1.5) (2.5) (-9/24)

$$+ (-1.5)(0.5)(1.5)(2.5)(-3.5)(-9/120)$$

$= 51 – 7.5 - 1.875 – 0.5 - 0.2109 + 0.1055$
= 40.8086

11.4 Lesson End Activities

1.From the following data find θ at x = 84 . Also express θ in terms of *x*.

x :	*40*	*50*	*60*	*70*	*80*	*90*
y :	*184*	*204*	*226*	*250*	*276*	*304*

2. From the data given below, find the number of students whose weight is between 100 and 110

Weight in lbs. :	*0-40*	*40-60*	*60-80*	*80-100*	*100-120*
No. of students:	*250*	*120*	*100*	*70*	*50*

3. Find a polynomial of degree two which takes the following . Also find x= 6.5

x :	0	1	2	3	4	5	6	7
y :	1	2	4	7	11	16	22	29

4.The following data are taken from the steam table.

Temp. ^{0}C	:140	150	160	170	180
Pressure	: 3.685	4.854	6.302	8.076	10.225

Find the pressure at temperature t = 175^0.

11.5 Let us Sum Up

In this lesson we have dealt with the following :
Newton's backward interpolation formula to find intermediate values which occur at end of the series.

Model Answer for selected lesson end activities
1. *286.96, $0.01x^2+1.1x+124$*
3. 0.5 (x^2+x+2)
4. 9.100

11.6 Reference:

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

LESSON – 12

Lagrange's Interpolation Formula

Contents

12.0 Aims and Objectives

In this Lesson, we have discussed about Lagrange's interpolation formula for finding intermediate value of a function in which the values of independent variable are not equally spaced.
After reading this lesson, you should be able to

- To find intermediate values which occur anywhere else of the series using Lagrange's interpolation formula.

12.1 Introduction

Interpolation means the process of computing intermediate values of a function a given set of tabular values of a function. Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ……….	x_n
y:	y_0	y_1	y_2	y_3 ……….	y_n

We may require the value of $y = y_i$ for the given $x = x_i$, where x lies between x_0 to x_n
Let y = $f(x)$ be a function taking the values $y_0, y_1, y_2, \dots\ y_n$ corresponding to the *values* $x_0,\ x_1, x_2,\ \dots\dots\ x_n$. Now we are trying to find $y = y_i$ for the given $x = x_i$ under assumption that the function f(x) is not known. In such cases, x_i '*s* are not equally spaced we use *Lagrange's interpolation formula.*

12.2 Lagrange's interpolation formula (for unequal intervals)

Let $y = f(x)$ denote a function which takes the values y_0, y_1, y_2 ………., y_n corresponding to the values x_0, x_1, x_2 ………., x_n.

Let suppose that the values of x *i.e.*, x_0, x_1, x_2 ………., x_n. are not equidistant .

$$y_I = f(x_i) \quad I = 0,1,2, \dots N$$

Now, there are (n+1) paired values (x_i,y_i), $I = 0,1,2, \dots n$ *and hence* $f(x)$ *can be represented by a polynomial function of degree n in x.*

Let us consider f(x) as follows

$$\begin{aligned} f(x) = {} & a_0 (x-x_1)(x-x_2)(x-x_3)\dots(x-x_n) \\ & + a_1 (x-x_0)(x-x_2)(x-x_3)\dots(x-x_n) \\ & + a_2 (x-x_0)(x-x_3)(x-x_4)\dots(x-x_n) \\ & \dots\dots\dots\dots\dots\dots\dots\dots \\ & + a_n (x-x_0)(x-x_2)(x-x_3)\dots(x-x_{n-1}) \qquad \dots\dots\dots(1) \end{aligned}$$

Substituting $x= x_0, y=y_0$, *in the above equation*

$$y_0 = a_0 (x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)$$

which implies $\quad a_0 = y_0 / (x_0-x_1)(x_0-x_2)(x_0-x_3)\dots(x_0-x_n)$

Similarly $\quad a_1 = y_1 / (x_1-x_0)(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)$

$$a_2 = y_2 / (x_2-x_0)(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_n = y_n (x_n-x_0)(x_n-x_2)(x_n-x_3)\dots(x_n-x_{n-1})$$

Putting these values in (1), we get

$$\begin{aligned} y = f(x) = {} & \frac{(x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\dots(x_0-x_n)}\, y_0 \\ & + \frac{(x-x_0)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}\, y_1 \\ & + \frac{(x-x_0)(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_0)(x_2-x_2)(x_1-x_3)\dots(x_1-x_n)}\, y_2 \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \frac{(x-x_0)(x-x_2)(x-x_3)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)(x_n-x_3)\dots(x_n-x_{n-1})}\, y_n \end{aligned}$$

The above equation is called ***Lagrange's interpolation formula*** for unequal intervals.

Note : 1. This formula is will be more useful when the interval of difference is not uniform.

12.3 Illustrations 1. Using Lagrange's interpolation formula, find *y(10) from the following table*

x	:	*5*	*6*	*9*	*11*
y	:	*3*	*13*	*14*	*16*

Solution:

Step 1. Write down the Lagrange's formula :

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)\dots(x_0 - x_n)}\, y_0$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}\, y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_2)(x_1 - x_3))}\, y_2$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_2)}{(x_3 - x_0)(x_3 - x_2)(x_3 - x_3)}\, y_3$$

$$= \frac{(x - 6)\ (x - 9)\ (x - 11)}{(5 - 6)(5 - 9)(5 - 11)} \quad (\qquad 12)$$

$$+ \frac{(x - 5)\ (x - 9)\ (x - 11)}{(6 - 5)(6 - 9)(6 - 11)} \quad (13)$$

$$+ \frac{(x - 5)\ (x - 6)\ (x - 11)}{(9 - 5)(9 - 6)(9 - 11)} \quad (14)$$

$$+ \frac{(x-5)\ (x-6)\ (x-19)}{(11-5)(11-6)(11-9)} \quad (16)$$

Putting x = 10 in the above equation

$$Y(10) = f(10) = \frac{(4)(1)(-1)}{(-1)(-4)(-6)}\ (12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}\ (13)$$

$$\frac{(5)(4)(1))}{(4)(3)(-2)}\ (14) + \frac{(5)(4)(1)}{(6)(5)(2)}\ (16)$$

= ***14.6666***

Illustrations 2. Using Lagrange's interpolation formula, find *y(10) from the following table*

x	:	*7*	*8*	*9*	*10*
y	:	*3*	*1*	*1*	*9*

Step 1. Write down the Lagrange's formula :

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)}\ y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\ y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_2)(x_1-x_3))}\ y_2$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_2)}{(x_3-x_0)(x_3-x_2)(x_3-x_3)}\ y_3$$

Substitute the various values of x_i and y_i

$$= \frac{(x-8)\ (x-9)\ (x-10)}{(7-8)(7-9)(7-10)} \quad (\quad 3)$$

$$+\frac{(x-7)\ (x-9)\ (x-10)}{(8-7)\,(8-9)\,(8-10)}\qquad(1)$$

$$+\frac{(x-7)\ (x-8)\ (x-10)}{(9-7)\,(9-8)\,(9-10)}\qquad(1)$$

$$+\frac{(x-7)\ (x-8)\ (x-19)}{(10-7)\,(10-8)\,(10-9)}\qquad(9)$$

Putting x = 9.5 in the above equation

$$Y(9.5) = f(9.5) = \frac{(1.5)(0.5)\,(-0.5)}{(-1)(-2)(-3)}\ (3) \ + \ \frac{(2.5)(0.5)\,(-0.5)}{(1)(-1)(-2)}\ (1)$$

$$\frac{(2.5)(1.5)\,(-0.5)}{(2)(1)(-1))}\ (1) \ + \ \frac{(2.5)(0.5)\,(0.5)}{(3)\,(2)\,(1)}\ (9)$$

= ***3.625***

Illustrations 3. Using Lagrange's interpolation formula, find the value *f(x)* at *x=27* , *from the following table*

x	:	*14*	*17*	*31*	*35*
y	:	*68.7*	*64.0*	*44*	*39.1*

(or)Find y(27) given y(14)=68.7, y(17)=64, y(31)=44, y(35)=39.1,
Step 1. Write down the Lagrange's formula :

$$y = f(x) = \frac{(x-x_1)\,(x-x_2)\,(x-x_3)\ldots(x-x_n)}{(x_0-x_1)\,(x_0-x_2)\,(x_0-x_3)\ldots(x_0-x_n)}\ y_0$$

$$+\frac{(x-x_0)(x-x_2)\,(x-x_3)}{(x_1-x_0)\,(x_1-x_2)\,(x_1-x_3)}\ y_1$$

$$+\frac{(x-x_0)(x-x_1)\,(x-x_3)}{(x_2-x_0)\,(x_2-x_2)\,(x_1-x_3))}\ y_2$$

$$(x-x_0)(x-x_2)\,(x-x_2)$$

$$+ \frac{\qquad}{(x_3 - x_0)(x_3 - x_2)(x_3 - x_3)} \quad y_3$$

Substitute the various values of x_i and y_i

$$= \frac{(x-17)(x-31)(x-35)}{(14-17)(14-31)(14-35)} (68.7)$$

$$+ \frac{(x-14)(x-31)(x-35)}{(17-14)(17-31)(17-35)} (64.0)$$

$$+ \frac{(x-14)(x-17)(x-35)}{(31-14)(31-17)(31-35)} (44)$$

$$+ \frac{(x-14)(x-17)(x-31)}{(35-14)(35-17)(35-31)} (39.1)$$

Putting $x = 27$ in the above equation

$$Y(27) = f(27) = \frac{(10)(-4)(-8)}{(-3)(-17)(-21)} (68.7) + \frac{(13)(-4)(-8)}{(3)(-14)(-18)} (64)$$

$$+ \frac{(13)(10)(-8)}{(17)(14)(-4))} (44) + \frac{(13)(-4)(-4)}{(21)(18)(4))} (39.1)$$

= ***-20.52 + 35.22 +48.07-13.45***

= 49.3

12.4 Lesson end activities

1. Use Lagrange's formula to fit a polynomial to the data and hence find y(1)

X	:	-1	0	2	3
Y	:	-8	3	1	12

(Hint: Keep *x* in the formula as it is)

2. Using Lagrange's interpolation formula, find the value $f(x)$ at $x=20$, *from the following table*

x	:	*14*	*17*	*31*	*35*

y	:	*68.7*	*64.0*	*44*	*39.1*

3. Using Lagrange's interpolation formula, find the value *y at x=5, from the following table*

x	:	*1*	*2*	*3*	*4*	*7*
y	:	*2*	*4*	*8*	*16*	*128*

4. Find y(10) given y(5) = 12,y(6) = 13, y(9) = 14and y(11) =16.

12.5 Let us Sum Up

In this lesson we have dealt with the following :

Lagrange's interpolation formula for unequal intervals to find intermediate values which occur anywhere in the series

Model Answer for selected questions

1. $2x^3+3x-6x^2+3$, 2
3. 32.9
4. 14.7

12.6 References

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

LESSON - 13

Numerical Solution of Ordinary Differential Equations

Contents

13.0 Aims and Objectives

In this Lesson, we have discussed about row to evaluate the solution of the differential equation. In the field of Engineering and Science, some are represented by mathematical models, which are happened to be differential equations. we present Taylor Method of numerical solutions of the ordinary differential equations. These are not exact solutions, but an approximate solutions. In many cases, approximate solutions to the required accuracy are quite sufficient.

After reading this lesson, you should be able to

- To know about numerical solution of ordinary differential equation.
- To evaluate the solution by Taylor Method.

13.1 Introduction

Suppose we require to solve $dy/dx = f(x,y)$ with the initial condition $y(x_0) = y_0$. By numerical solution of the differential equation, let $y(x_0) = y_0$, $y(x_1)$, $y(x_2)$,….. be the solutions of y at $x = x_0, x_1, x_2, \ldots\ldots\ldots x_n$. *Let* $y = y(x)$ be the exact solution.. If we plot and draw the graph of $y = y(x)$, and also draw the approximate curve by plotting (x_0, y_0), (x_1, y_1), (x_2, y_2),… *we get two curves.. For each* x_i *approximate values of the dependent variable* $y(x)$ *are calculated* using a suitable recursive formula. These values are y_0, y_1,

y_2, y_3,….. y_n. Computation of these approximate values is known as numerical solution of the differential equation. Many techniques are available for the approximate solution of ordinary differential equations by numerical methods. In this lesson we consider the most frequently used Taylor Method

13.2 Taylor Method

Suppose we want to find the numerical solution of the equation

$$\frac{dy}{dx} = f(x,y)$$

Given the initial condition $y(x_0) = y_0$

$y(x)$ *can be expanded about the point* $x = x_0$ *in a Taylor's series as*

Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	x_3 ……….	x_n
y:	y_0	y_1	y_2	y_3 ……….	y_n

From the above values, we want to find the derivative of $y = f(x)$, *passing through (n+1) points*, at a point closer to the starting value $x = x_0$

$$y(x) = y(x_0) + (x - x_0)^1 [\, y'(x)]_{xo} /1! + (x - x_0)^2 [y''(x)\,]_{xo} / 2! + \ldots$$

$$y(x) = y_0 + (x - x_0)^1 \; y'_0 /1! + (x - x_0)^2 \, y_0'' / 2! + \ldots$$

Putting $x = x_1 = x_0+h$, *we get*

$$y_1 = y_0 + h\, y'_0 /1! + h^2 y_0'' / 2! + h^3 y_0''' / 3! + \ldots$$

Now $y(x)$ *can be expanded about the point* $x = x_1$ *in a Taylor's series as*

$$y_2 = y_1 + h\, y'_1 /1! + h^2 y_1'' / 2! + h^3 y_1''' / 3! + \ldots$$

Proceeding in the same way, we get

$$y_{n+1} = y_n + h\, y'_n /1! + h^2 y_n'' / 2! + h^3 y_n''' / 3! + \ldots$$

In the above equation $y^r{}_n = [\, d^ry/dx^r]_{(xn,yn)}$ *and also an infinite series and hence we have to truncate at some term to have the numerical value calculated.*
For calculation purpose, the terms unto and including h^n *and neglect terms involving* h^{n+1}, *the Taylor algorithm used is said to be of* n^{th} *order. By increasing the number of terms in the series, the error can be reduced further .*

13.3 Illustrations **1.** Solve dy/dx = x +y , given y(1) = 0, and get y(1.1) , y(1.2) by Taylor series method. Compare the result with the actual solution.

Solution :

We are given that $y(1) = 0 \Rightarrow x_0 = 1, y_0 = 0, h = 0.1$

Also $y' = x+y$	$y_0' = x_0+y_0 = 1+0 = 1$
$\Rightarrow y'' = 1 + y'$	$y_0'' = y_0' + 1 = 2$
$\Rightarrow y''' = y''$	$y_0''' = y_0'' + 2 = 2$
$\Rightarrow y^{iv} = y'''$	$y_0^{iv} = 2.$

By Taylor series, we have

$$y_1 = y_0 + h\, y'_0 /1! + h^2 y_0''/2! + h^3 y_0'''/3! + \ldots$$

$$= y(1.1) = 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2) + \frac{(0.1)^5}{120}$$

$= 0.1 + 0.01 + 0.00033 + 0.00000833 + 0.000000166$
$= 0.11033847$

Consider that $x_0 = 1.1$, $h = 0.1$, and evaluate y_2

$$y_2 = y_1 + h\, y'_1 /1! + h^2 y_1''/2! + h^3 y_1'''/3! + \ldots$$

Calculate $y_1', y_1'', y_1''', y_1^{iv, \cdots}$; $x_1 = 1.1, y_1 = 0.11033847$

$y_1' = x_1+y_1 = 1.1 + 0.11033847 = 1.21033847$

$y_1'' = 1 + y_1' = 2.21033847$

Using the above values :

$$y_2 = y(1.2) = 0.11033847 + 0.1(1.21033847) + (0.1)^2 (2.21033847)/2 + (0.1)^3 (2.21033847)/6 + (0.1)^4 (2.21033847)/24$$

$$= 0.11033847 + 0.121033847 + 2.21033847\,(0.005) + \ldots.$$

$$= 0.2461077$$

The actual solution *of* dy/dx = x + y , is

$y = -x - 1 + 2e^{x-1}$
$y(1.1) = -1.1 - 1 + 2e^{0.1} = 0.11034$
$y(1.2) = -1.2 - 1 + 2e^{0.2} = 0.2428$
$y(1.1) = 0.11033847$
$y(1.2) = 0.2461077$

Actual values : $y(1.1) = 0.110341836$
$y(1.2) = 0.242805552.$

Illustration 2. Apply Taylor series method , find correct to four decimal places, the value of y(0.1), given $dy/dx = x^2+y^2$ *and* $y(0) = 1$.

Solution :

We are given that $y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1, \ h = 0.1$

$x_1 = 0.1$, *To find* $y_1 = y(0.1)$ *by using following series :*

Also $y' = x^2+y^2$	$y_0' = x_0^2+y_0^2 = 0+1 = 1$
$\Rightarrow y'' = 2x + 2y\,y'$	$y_0'' = 2x_0 + 2y_0\,y_0' = 2$
$\Rightarrow y''' = 2+2y\,y''+2\,(y')^2$	$y_0''' = 2+2y_0\,y_0''+2\,(y_0')^2$
$\Rightarrow y^{iv} = 2y\,y'''+2y'y''+4y'y''$	$= 2+2\,(1)(2)+2\,(1)^2 = 8$
$= 2yy''' + 6y'y''$	$y_0^{iv} = 2(1)(8)+6(1)(2) = 28$

By Taylor series, we have

$$y_1 = y_0 + h\,y'_0/1! + h^2\,y_0''/2! + h^3\,y_0'''/3! + \ldots$$

$$= y(0.1) = 1 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(28) + \ldots.$$

= 1 +0.1 + 0.00133 +0.00011

= 1.11144999

= 1.11145

13.4 Lesson end activities

1. Apply Taylor series method , find correct to four decimal places, the value of and given $dy/dx = 1 - 2x^2$ *and* $y(0) = 0$.

2. Apply Taylor series method , find correct to four decimal places, the value of y(1.1) , y(1.2) and given $dy/dx = xy^{1/3}$ *and* $y(1) = 1$.

3. Using Taylor series method, find y at x = 0.1(0.1)0.4 given $dy/dx = x^2 - y$, $y(0) = 1$. *Correct to 4 decimal places.*
(Hint: calculate the values of y(0.1) , y(0.2) y(0.3) and y(0.4))

4. By means of Taylor series expansion, find y at x = 0.1, 0.2 correct to three significant digits given $dy/dx - 2y = 3e^{x}$, y(0) = 0.

13.5 Let us Sum Up

In this lesson we have dealt with the following :

- We have discussed about numerical solution of ordinary differential equation.
- We have discussed equation.about the Taylor Method to obtain solution of ordinary differential equation.

Model answer

1. 0.1948,0.3599
2. 1.1068, 1.2277
3. 0.9051, 0.8212, 0.7492, 0.6897

13.6 References

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

LESSON - 14

Euler's method

Contents

14.0 Aims and Objectives

In this Lesson, we have discussed about to evaluate the solution of the differential equation by using Euler's method and Runge-Kutta. In the field of Engineering and Science, some are represented by mathematical models, which are happened to be differential equations. Euler's Method and Runge-Kutta methods are step by step methods because the values of y are calculated by short steps. After reading this lesson, you should be able to

- To evaluate the solution of ordinary differential equation by Euler's Method.
- To evaluate the solution of ordinary differential equation by Runge-Kutta Method.

14.1 Introduction

In solving a first order differential equation by numerical method, there are two types of solution:

(i) A series solution of y in terms of x, from which the values of y can be obtained by substitution.

(ii) Values of y at specified values of x.

Earlier method of study .i.e. Taylor method belong to the first category. Whereas Euler Method and Runge-Kutta methods are coming under second category. In which ,the values of y are calculated by short steps with equal interval h of the independent variable x.

14.2 Euler' Method

Suppose we want to find the numerical solution of the equation

$$\frac{dy}{dx} = f(x,y)$$

Given the initial condition $y(x_0) = y_0.$ ………………..(1)

$y(x)$ can be expanded about the point $x = x_0$ in a Taylor's series as

Suppose the following table represents a set of values of x :

x:	x_0	x_1	x_2	x_3 ……….	x_n

In which x_i-$x_{i+1} = h$, *i.e.*, $x_i = x_0 + ih$, $i = 0,1,2,$……

Let the actual solution of differential equation be denoted by continuous line graph lies on the curve. We have to find value of y of the curve at $x = x_i$

y:	y_0	y_1	y_2	y_3 ……….	y_n

From the above values, we want to find the derivative of $y = f(x)$, *passing through (n+1) points*, at a point closer to the starting value $x = x_0$

The equation of tangent at (x0, y0) to curve is

$$y - y_0 = y'_{(x0,y0)}\,(x - x_0)$$

$$= f(x_0,y_0).\,(x - x_0)$$

Therefore $y = y_0 + f(x_0,y_0).\,(x - x_0)$

This is the value of the y coordinate of a point on the tangent. The curve is approximated by the tangent, in the interval (x_0, x_1). *Hence, the value of y on the curve is approximately equal to the value of y on the tangent at* (x_0,y_0), *corresponding to* $x = x_1.$

Therefore $y_1 = y_0 + + f(x_0,y_0)\,(x_1 - x_0)$

$$y_1 = y_0 + +h\, y_0'$$

Similarly, we approximate the curve by the line through (x_1, y_1) whose slope *is* $f(x_0,y_0)$ *we* get $y_2 = y_1 + h\, f(x_1,y_1) = y_1 + h\, y_1'$

Thus $y_{n+1} = y_n + h\, f(x_n,y_n) = y_n + h\, y_n'$; $n = 0,1,2,$ …..

This formula is called ***Euler's algorithm.***

Improved Euler method

Slight change may be included in the above mentioned algorithm ., *i.e.*, we approximate the curve by the tangent and we get improved Euler formula ;

$$y_{n+1} = y_n + (1/2)\ h[\ f(x_n, y_n) + \ f(x_n + h, y_{n+} h\ \ f(x_n, y_n)\ )$$

This equation is called ***improved Euler's method.***

Modified Euler method

Slight change may be included in the above mentioned improved Euler's method., *i.e.*, we averaged the slopes, whereas in modified Euler method, we will average the points. We get the formula for Modified Euler method **,** given by

$$y_{n+1} = y_n + h[\ f(x_n + h/2,\ y_{n+}(h/2)\ f(x_n, y_n)\ ]$$

or $$y(x+h) = y(x) + h[\ f(x + h/2,\ y_+(h/2)\ f(x,y)\ ]$$

This equation is called ***Modified Euler's method.***

Note : 1. Use the formula correctly, after understanding the problem.

14.4 Illustration 1: Solve y' = - y ,and y(0) = 1, determine the values of y at x = (0.01)(0.01)(0.04) by Euler's method.

Solution.

Step 1.

Calculate various values of x_i' s and respective y_i's

We are given that y' = -y and y(0) = 1 ; f(x,y) = -y;

x = (0.01)(0.01)(0.04) => $x_0 = 0$, $y_0 = 1$

$x_1 = 0.01$, $x_1 = 0.01$, $x_2 = 0.02$, $x_3 = 0.03$, $x_4 = 0.04$

Step 2. To find y_1, y_2, y_3, y_4. Take h = 0.01 (Specified in the problem itself)

Write down the Euler formula ,

$$y_{n+1} = y_n + h\, f(x_n, y_n) = y_n + h\, y_n' \ ;\ n = 0,1,2, \ldots..$$

$$y_1 = y_0 + h\, f(x_0, y_0) = 1 + (0.01)(-1) = 1 - 0.01 = 0.99$$

$$y_2 = y_1 + h\, f(x_1, y_1) = 0.99 + (0.01)(-y_1) = 0.99 + (0.01)(-0.99) = 0.9801$$

$$y_3 = y_2 + h\, f(x_2, y_2) = 0.9801 + (0.01)(-0.9801) = 0.9703$$

$$y_4 = y_3 + h\, f(x_3, y_3) = 0.9703 + (0.01)(-0.9703) =\text{x}\ 0.9606$$

Step 3. Flash the values in tabular form

X	*0*	*0.01*	*0.02*	*0.03*	*0.04*
Y	*1*	*0.9900*	*0.9801*	*0.9703*	*0.9606*
Exact y	*1*	*0.9900*	*0.9802*	*0.9704*	*0.9608*

Since, $y = e^{-x}$ is the exact solution.

Illustration 2: Solve $y' = x + y$ and $y(0) = 1$, determine the values of y at x = 0.0(0.2)(1.0) by Euler's method. Compare answer with actual answer.

Solution.
We are given that $h = 0.2$, $f(x,y) = x + y$

$y(0) = 1 \Rightarrow x_0 = 0$, $y_0 = 1$

$x = (0.0)(0.2)(1.0) \Rightarrow x_1 = 0.0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.4$, $x_4 = 0.8$, $x_5 = 1$

$$y_{n+1} = y_n + h\,f(x_n,y_n) = y_n + h\,y_n' \ ; n = 0,1,2, \ldots..$$

$$y_1 = y_0 + h\,f(x_0,y_0) = 1 + (0.2)\,(0+1) = 1 + 0.2 = 1.2$$

$$y_2 = y_1 + h\,f(x_1,y_1) = 1.2 + (0.2)\,() = 1.2 + (0.2)\,(0.2 + 1.2) = 1.48$$

$$y_3 = y_2 + h\,f(x_2,y_2) = 1.48 + (0.2)\,(0.4 + 1.48) = 1.856$$

$$y_4 = y_3 + h\,f(x_3,y_3) = 1.856 + (0.2)\,(0.6 + 1.856) = 2.3472$$

$$y_5 = y_4 + h\,f(x_4,y_4) = 2.3472 + (0.2)\,(0.8 + 2.3472) = 2.94664$$

Exact solution is $y = 2e^x - x - 1$. Flash the values in tabular form

X	0	0.2	0.4	0.6	0.8	1.0
Euler y	1	1.2	1.48	1.856	2.3472	2.94664
Exact y	1	1.2428	1.5836	2.0442	2.6511	3.4366

The value of y deviates from the exact values as x increases. Hence we require to use either Modified Euler or Improved Euler method for the above problem.

Illustration 3 : Solve numerically $y' = y + e^x$, $y(0) = 1$,for x= 0.2, 0.4by Improved Euler's method.

Solution :
Step 1.
We are given that $y' = y + e^x$, $y(0) = 0$; $f(x,y) = y + e^x$.

$y(0) = 0 \Rightarrow x_0 = 0$, $y_0 = 0$

$x = 0.2, 0.4 \Rightarrow x_1 = 0.0$, $x_1 = 0.2$, $x_2 = 0.4$

Step 2. Write down the formula for Improved Euler method

$y_{n+1} = y_n + (1/2)\ h[\ f(x_n,y_n) + f(x_n+h, y_n + h\ f(x_n,y_n)\ ]$

$y_1 = y_0 + (1/2)\ h[\ f(x_0,y_0) + f(x_0+h, y_0 + h\ f(x_0,y_0)\ )\ )\ ]$

$= 0 + (0.5)\ (0.2)\ [\ \overline{y_0 + e^{xo}} + y_0 + h(y_0 + e^{xo}) + e^{xo+h}\ ]$

$= 0.1\ [0+1+0+0.2(0+1)+e^{0.2}]$

$= 0.1\ (1 + 0.2 + 1.2214)$

⇨ $y(0.2) = \mathbf{0.24214}$

$y_2 = y_1 + (1/2)\ h[\ f(x_1,y_1) + f(\ x_1+h,\ \ y_1 + h\ f(x_1,y_1)\ )\ )\ ]$

where $f(x_1,y_1) = y_1 + e^{x1} = 0.24214 + e^{0.2} = 1.46354$

$y_1 + h\ f(x_1,y_1) = 0.24214 + (0.2)(1.46354) = 0.53485$

$f(\ x_1+h,\ \ y_1 + h\ f(x_1,y_1)) = f(0.4, 0.53485) = 0.53485 + e^{0.4} = 2.02667$

Substituting the above values, we get

$y(0.4) = 0.24212 + (0.5)(0.2)[1.46354 + 2.02667]$

$= \mathbf{0.59116}$

Tabulate the values and it given below:

x	0	.0.2	0.4
y	0	0.24214	0.59116

Illustration 4. Compute *y* at $x = 0.25$ by Modified Euler method given $y' = 2xy$, $y(0) = 1$.

Solution :

Step 1.

We are given that $f(x,y) = 2xy$;

$y(0) => x_0 = 0, y_0 = 1$

$h = 0.25 => x_1 = 0 + 0.25 = 0.25$

Step 2. Write down the Modified Euler formula

$y_{n+1} = y_n + h[\ f(x_n+h/2,\ y_n + (h/2)\ f(x_n,y_n)]$

$=> y_1 = y_0 + h[\ f(x_0+h/2,\ y_0 + (h/2)\ f(x_0,y_0)\ ]$

$= 1 + (0.25)[f(0.125,1]$

$= 1 + (0.25)[2 \text{ x } (0.125) \text{ x } 1]$

$= 1.0625.$

Note : By solving the equation $y(0.25) = 1.0645$ and error is only 0.002 To improve the result take $h= 0.125$ iterate twice, which incurs lot of mathematical calculation.

14.3 Runge-Kutta Method

Suppose we want to find the numerical solution of the equation

$$\frac{dy}{dx} = f(x,y)$$

Given the initial condition $y(x_0) = y_0.$ ……………….(1)

Calculate

$k_1 = h\, f(x_0\,, y_o)$

$k_2 = h\, f(x_0 + (1/2)h,\ y_o + (1/2)k_1)$

and $\Delta y == k_2,$ *where* $h = \Delta x$

The above mentioned algorithm is ***Second order Runge-Kutta Algorithm***

Calculate

$k_1 = h\, f(x_0\,, y_o)$

$k_2 = h\, f(x_0 + (1/2)h,\ y_o + (1/2)k_1)$

$k_3 = h\, f(x_0 + (1/2)h,\ y_o + (1/2)k_2)$

and $\Delta y = (1/6)[k_1 + 4k_2 + k_3)$

The above mentioned algorithm is ***Third order Runge-Kutta Algorithm***

Calculate

$k_1 = h\, f(x_0\,, y_o)$

$k_2 = h\, f(x_0 + (1/2)h,\ y_o + (1/2)k_1)$

$k_3 = h\, f(x_0 + (1/2)h,\ y_o + (1/2)k_2)$

$k_4 = h\, f(x_0 + h\,,\ y_o + k3)$

and $\Delta y = (1/6)[k_1 + 2k_2 + 2\,k_3 + k_4]$

$y(x+h) = y(x) + \Delta y$

The above mentioned algorithm is ***Fourth order Runge-Kutta Algorithm***

Where $\Delta x = h.$

Calculate

$y_1 = y_0 + \Delta y$

Now starting from (x_1,y_1) *and repeating the above process, we get* (x_2,y_2) *etc.*

Note 1: In second order Runge–Kutta method

$\Delta y_0 = k_2 = h\, f(x_0 + (1/2)h,\ y_o + (1/2)k_1)$

$\Delta y_0 = k_2 = h\, f(x_0 + (1/2)h,\ y_o + (1/2)h\, f(x_{0,y0}))\,)$

Therefore

$$y_1 = y_0 + h[\ f(x_0 + h/2,\ y_{0+}(h/2)\ f(x_0,y_0)\ ]$$

This is equivalent to modified Euler Method.

Hence, the **Runge-Kutta method of second order is nothing but the Modified Euler *Method*.**

Note 2: *if $f(x,y) = f(x)$, i.e., $f(x,y)$* is only depending on a function x alone, then the fourth order Runge-Kutta method reduces to **Simpson's one third rule**

Note 3. In all the three methods the values of $k1,k2,k3$ are same. Therefore, no need to calculate the constants while doing by all the three method.

Illustration 1*. Apply the fourth order Runge-Kutta method to find t(0.2) given that $y' = x+y$, $y(0) = 1$.*

Solution:

Step 1. We are given that $y' = x+y, y(0) = 1 \Rightarrow f(x,y) = x + y, x_0 = 0 , y_0 = 1$
Since h is not specified in the question, we take h = 0.1; $x_1 = 0.1, x_2 = 0.2$

Step 2. We have to find various constants in fourth order Runge-Kutta method

$$k_1 = h\, f(x_0 , y_o) = (0.1)(x_0 + y_0) = (0.1)(0+1) = 0.1$$
$$k_2 = h\, f(x_0 +(1/2)h,\ y_o+(1/2)k_1) = (0.1)f(0.05,1.05)$$
$$= 0.1(0.05+1.05) = 0.11$$
$$k_3 = h\, f(x_0 +(1/2)h,\ y_o+(1/2)k_2) = (0.1)f(0.05,1.055)$$
$$= 0.1(0.05+1.055) = 0.1105$$
$$k_4 = h\, f(x_0+h ,\ y_o +k3) = = 0.1f(0.1, 1.1105) = 0.12105$$

$$\text{and } \Delta y = (1/6)[k_1+ 2k_2+2 k_3+ k_4]$$
$$= (0.16666)(0.1+0.22+0.2210+0.12105)$$
$$= 0.110342.$$
$$y_1 = y_0 + \Delta y$$
$$y(0.1) = y_1 = y_0 + \Delta y = 1.110342$$

Step 3. Now starting from (x_1,y_1) and repeating the above process, we get (x_2,y_2). Again apply Runge-Kutta method replacing (x_0,y_0) by (x_1,y_1).

$$k_1 = h\, f(x_1 , y_1) = (0.1)(x_1 + y_1) = (0.1)(0.1 +1.110342) = 0.1$$
$$k_2 = h\, f(x_1 +(1/2)h,\ y_1+(1/2)k_1) = (0.1)f(0.15,1.170859)$$
$$= 0.1(0.15+1.170859) = 0.1320859$$
$$k_3 = h\, f(x_1 +(1/2)h,\ y_1+(1/2)k_2) = (0.1)f(0.15, 1.1763848)$$
$$= 0.1(0.15+1.1763848) = 0.13263848$$
$$k_4 = h\, f(x_1+h ,\ y_1 +k3) = 0.1f(0.2, 1.24298048) = 0.144298048$$
$$\text{and } \Delta y = (1/6)[k_1+ 2k_2+2 k_3+ k_4]$$
$$= (0.16666)(0.1+0.2641718+0.26527696+0.144298048)$$

$$y(0.2) = y_1 = y_0 + \Delta y = 1.110342 + (0.166666)(0.7947810008)$$
$$y(0.2) = 1.2428055 \Rightarrow \mathbf{1.2428} \text{ (Correct to four decimal places).}$$

Illustration 2. *Obtain the values of y at x= 0.1, 0.2 using R.K. method of (i) second order (ii) third order and (iii) fourth order for the differential equation y' = -y, given y(0) = 1.*

Solution:

Step 1. We are given that $y' = -y, y(0) = 1 \Rightarrow f(x,y) = -y, x_0 = 0, y_0 = 1$
Since h is clearly specified in the question, we take h = 0.1; $x_1 = 0.1, x_2 = 0.2$

Step 2. *(i)* We have to find various constants in **Second order** Runge-Kutta method

$$k_1 = h\,f(x_0, y_o) = (0.1)(-y_0) = (0.1)(-1) = -0.1$$

$$k_2 = h\,f(x_0 + (1/2)h,\ y_o + (1/2)k_1) = (0.1)f(0.05, .95)$$
$$= 0.1(-0.95) = -0.095 = \Delta y$$

$$y_1 = y_0 + \Delta y$$
$$y(0.1) = y_1 = y_0 + \Delta y = 1 - 0.095 = 0.905$$

Now starting from (x_1,y_1) *i.e.,* $(.01, 0.905)$ and repeating the above process, we get (x_2,y_2). Again apply Runge-Kutta method replacing (x_0,y_0) *by* (x_1,y_1).

$$k_1 = h\,f(x_1, y_1) = (0.1)(-y_0) = (0.1)(-0.905) = -0.0905$$

$$k_2 = h\,f(x_1 + (1/2)h,\ y_1 + (1/2)k_1) = (0.1)f(0.15, 0.85975)$$
$$= 0.1(-0.85975) = -0.85975 = \Delta y$$

$$y_1 = y_1 + \Delta y$$
$$y(0.2) = y_1 = y_1 + \Delta y = \mathbf{0.819025}$$

Step 3. *(i)* We have to find various constants in **Third order** Runge-Kutta method

$$k_1 = h\,f(x_0, y_o) = (0.1)(-y_0) = (0.1)(-1) = -0.1$$

$$k_2 = h\,f(x_0 + (1/2)h,\ y_o + (1/2)k_1) = (0.1)f(0.05, 0.95)$$
$$= 0.1(-0.95) = -0.095$$
$$k_3 = h\,f(x_0 + (1/2)h,\ y_o + (1/2)k_2) = (0.1)\,f(0.1, 0.9) = (-0.09)$$

$$\Delta y = (1/6)[k_1 + 4k_2 + k_3]$$

$$y_1 = y_0 + \Delta y$$
$$y(0.1) = y_1 = y_0 + \Delta y = 1 - 0.09 = 0.91$$

Now starting from (x_1,y_1) *i.e.,* $(.01, 0.905)$ and repeating the above process, we get (x_2,y_2). Again apply Runge-Kutta method replacing (x_0,y_0) *by* (x_1,y_1).

$$k_1 = h\,f(x_1, y_1) = (0.1)(-y_0) = (0.1)(-0.91) = -0.091$$

$$k_2 = h\,f(x_1 + (1/2)h,\ y_1 + (1/2)k_1) = (0.1)f(0.15, 0.865)$$
$$= 0.1(-0.865) = -0.865$$
$$k_3 = h\,f(x_0 + (1/2)h,\ y_o + (1/2)k_2) = (0.1)\,f(0.2, 0.828) = -0.0828$$
$$\Delta y = (1/6)[k_1 + 4k_2 + k_3]$$

$$y_2 = y_1 + \Delta y$$

$$y(0.2) = y_2 = y_1 + \Delta y = 0.91 + (0.16666)((-0.091 - 0.346 - 0.0828)$$

$$= \mathbf{0.823366}$$

Step 4. *(i)* We have to find various constants in **fourth order** Runge-Kutta method

$$k_1 = h\,f(x_0\,,y_o) = (0.1)(-y_0) = (0.1)(-1) = -0.1$$

$$k_2 = h\,f(x_0 + (1/2)h,\ y_o + (1/2)k_1) = (0.1)f(0.05, 0.95)$$
$$= 0.1(-0.95) = -0.095$$
$$k_3 = h\,f(x_0 + (1/2)h,\ y_o + (1/2)k_2) = (0.1)\,f(0.1,\ 0\,.9525) = -0.09525$$

$$k_4 = h\,f(x_0 + h\,,\ y_o + k3) \quad = 0.1f(0.1, 0.90475) = -0.090475$$

$$\text{and } \Delta y = (1/6)[k_1 + 2k_2 + 2\,k_3 + k_4]$$
$$= (0.16666)(-0.095 - 0.19 - 0.1905 - 0.090475)$$
$$= -0.0951625$$

$$y_1 = y_0 + \Delta y$$

$$y(0.1) = y_1 = y_0 + \Delta y = 1 + (0.0951625)$$

$$= \mathbf{0.9048375}$$

Now starting from (x_1,y_1) *i.e.,* $(0.1, 0.9048375)$ and repeating the above process, we get (x_2,y_2). Again apply Runge-Kutta method replacing (x_0,y_0) by (x_1,y_1).

$$k_1 = h\,f(x_1\,,y_1) = (0.1)(-y_1) = (0.1)(-0.91) = -0.09048375$$

$$k_2 = h\,f(x_1 + (1/2)h,\ y_1 + (1/2)k_1) = (0.1)f(0.15,\ 0.8595956)$$
$$= 0.1(-0.\ 0.8595956) = -0.08595956$$
$$k_3 = h\,f(x_1 + (1/2)h,\ y_1 + (1/2)k_2) = (0.1)\,f(0.15,\ 0\,.8618577) = -0.08618577$$

$$k_4 = h\,f(x_1 + h\,,\ y_1 + k3) \qquad = = 0.1f(0.2,\ 0.8186517) = -0.0818517$$

$$\text{and } \Delta y = (1/6)[k_1 + 2k_2 + 2\,k_3 + k_4]$$
$$= (0.16666)(-0.09048375 + 2(-0.08595956) +$$
$$2(-0.08618577) - 0.0818517)$$
$$= -0.086106607$$

$$y_2 = y_1 + \Delta y$$

$$y(0.2) = y_2 = y_1 + \Delta y = 0.9048375 + (-0.086106607)$$

$$= \mathbf{0.81873089}$$

Tabular values are

x	Second order	Third order	Fourth order	Exact Value
0.1	0.905	0.91	0.9048375	0.904837418
0.2	0.819025	0.823366	0.81873089	0.818730753

Note : While evaluating values in Runge-Kutta method , unless not specified in the problem apply fourth order Runge-Kutta method only.

14.4 Lesson end activities

1. Use Euler's method find y(0.4) given $y' = xy$, $y(0) = 1$.

2. Find $y(0.6)$, $y(0.8)$, $y(1)$ given $(dy/dx) = x+y$, $y(0) = 0$ taking $h = 0.2$ by improved Euler method.

3. Using Improved Euler method find $y(0.2)$, $y(0.4)$ given $dy/dx = y + x^2$, $y(0) = 1$.

4. Solve $y' = 3x^2 + y$ given $y(0) = 4$, if h=0.25 to obtain $y(0.25)$, $y(0.5)$.

5. Using Modified Euler method , find $y (0.2)$, $y(0.4)$, $y(0.6)$ given $dy/dx = y - x^2$, y(0)= 1.

6. Apply Modified Euler method and obtain y(0.2) given $dy/dx = y – x^2$, $y(0) = 1$.

7. Find y (0.2) given dy/dx = y – x , y(0) = 2 taking h = 0.1. by Runge –Kutta method.

8. Evaluate y(1.4) given dy/dx = x +y , y(1.2) = 2. By Runge-Kutta Method.

14.5 Let us Sum Up

In this lesson we have dealt with the following :

- We have discussed equation about the Euler's Method to obtain solution of ordinary differential
- We have discussed equation about the Runge-Kutta Method to obtain solution of ordinary differential

Model Answer For selected questions

1. 1.061106
2. 0.2158,0.4153,0.7027
3. 1.224,1.514)

5 1.218,1.467,1.737

6. 1.0095
7. 2.4214
8. 2.7299

14.6 References

Numerical Methods – P.Kandasamy, K.Thilagavathi, K.Gunavathi, S.Chand &Company Ltd., Revised Edition 2005 **.**

UNIT - IV

LESSON – 15

MEASURES OF CENTRAL TENDENCY

Contents:

15.0 Aims and Objectives

In this Lesson, we have discussed the measures of central tendency. In which, we have defined Mean and its characteristics. Computation of Mean for Discrete and Continuous case are explained with illustrations. Relationship among the same are explained.

After reading this lesson, you should be able to

- To compute Mean for both Discrete and Continuous case.
- Merits and Demerits of Mean

15.1 Introduction - Mean

Averages are "statistical constants which enable us to comprehend in a single effort the significance of the whole", this is definition is given by Professor Bowley; they give us useful information about the complete group. According to father of statistics R.Fisher,

"The inherent inability of the human mind to grasp in it's entirely a large body of numerical data compels us to seek relatively few constants that will adequately describe the data".
Hence a simple figure, which is used to represent the whole group, that must be a representative number, it is termed to be "s measure of central tendency or the average"

15.2.1 Characteristics of a Typical Average

According to Yule and Kendall average must satisfied the following characteristics:

1. It should be rigidly defined
2. It should be simple to calculate and easy to understand
3. It should be based on all the observations
4. It should be proficient of being used in further statistical computations
5. It should not be affected much by extreme values

15.2.2 Types of averages:

1. Arithmetic Mean
2. Median
3. Mode
4. Geometric Mean, and
5. Harmonic Mean

ARITHMETIC MEAN

Arithmetic mean of set of values is obtained by dividing the total value by the number of observations and it is also called Mean. There are two types of Arithmetic average: 1. Simple Mean 2. Weighted Average

Computation of Mean

Case 1: **Individual observations:**

(a) Direct Method:

Step 1. Add all the observations of the variables X and compute X

Step 2. Divide X by number of observation (N)

$$\overline{X} = \frac{X1+X_2+\ldots\ldots X_n}{N} \quad \text{or} \quad \overline{X} = \frac{X}{N}$$

$\overline{X}$ = Mean

X = the sum of observations
N = Number of observation

Illustration: 1

Calculate mean from the following data :

Roll No.	1	2	3	4	5	6	7	8	9	10
Mark	45	50	55	60	55	90	95	65	35	80

Solution : Calculation of Mean by using following table :

Roll Nos.	Marks
1	45
2	50
3	55
4	60
5	55
6	90
7	95
8	65
9	35
10	80
Total	630

$$\bar{X} = \frac{X}{N}$$

$$\bar{X} = \frac{630}{10}$$

$$= 63 \text{ Marks}$$

(b) Short Cut Method

The arithmetic mean can also be calculated by short cut method. This method applied to reduce the calculation. It includes the following steps:

Step 1. Assume any value as assumed mean, which is very close to the given values (Let A= Assumed Mean)
Step 2. Find out D (D= X- A)

Step 3. Add all the deviations D

Step 4. Using the formula:

$$\bar{X} = A + \frac{D}{N}$$

$\bar{X}$ = Arithmetic Mean

A = Assumed Mean

D= Sum of the Deviations

N = Number of observations.

Illustration : 2

(Solving the above problem by using short cut method)

Roll Nos	Marks	D= X-60
1	45	-15
2	50	-10
3	55	-5
4	60	0
5	55	-5
6	90	30
7	95	35
9	35	-25
10	80	20
N= 10	Total	30

$$\bar{X} = 60 + \frac{30}{10}$$

$$\bar{X} = 60 + 3 = 63$$

2. Weighted Arithmetic Mean

While we calculate the arithmetic mean ,all the items in the series have al importance. In certain cases it is not so. Importance of items depends on the weights attached to it. Weighted average can be defined as an average whose items are multiplied by certain values(weights) divided by the total weights.

Weighted Mean = WX / W

Illustrations: 3

Calculate weighted average of the following data:

Course :	BA	BSc	MA	MCA	MBA
%of Pass	70	65	75	90	99
No of Students	20	30	30	50	40

Solution:

Weighted Average = 13300 / 170
= 76.47

15.3

% of Pass X	No of Student W	XW
70	20	1400
65	30	1950
75	30	2250
90	50	4500
80	40	3200
X= 380	W = 170	W X = 13300

Computation of mean for DISCRETE SERIES

Direct Method : Computation of mean by using the following steps

Steps

1. Multiply each item (X) by frequency (f) *i.e.*, (f x X)
2. Sum up all the fX , f X
3. Divide fX by the total frequency (N)

Formula is $\overline{X} = \frac{f X}{N}$

Illustrations:

Calculate the mean of the following data:

Course :	BA	BSc	MA	MCA	MBA
Mark	40	60	80	90	100
No of Students	6	9	15	7	3

Solution:

Mean = 2910 / 40

= 72.75

(c) Short Cut Method

The arithmetic mean can also be calculated by short cut method. This method applied to reduce the calculation. It includes the following steps:

Step 1. Assume any value as assumed mean, which is very close to the given values (Let A= Assumed Mean)
Step 2. Find out D (D= X- A)

Step 3. Sum up all the products f D
Step 4. Using the formula:

$$\bar{X} = A + \frac{f D}{N}$$

$\bar{X}$ = Arithmetic Mean
A = Assumed Mean
f D = Sum of total Deviations
f = N = Total frequency.

Illustrations

Calculate Mean of the following data:

X	10	20	30	40	50
F	5	10	25	7	3

Solution:

Construction of the table to find mean:

X	F	D=X-(A=30)	fD
10	5	-20	-100
20	10	-10	-100
30	25	0	0
40	7	10	70
50	3	20	60
Total	f =N=50	Total	-70

:

$$\bar{X} = A + \frac{f D}{N}$$

$$\bar{X} = 30 + (-70 / 50)$$
$$= 30 – 1.4$$
$$= 28.6$$

15.4 Computation of mean for continuous case

In continuous frequency distribution, the mean can be computed by using the following methods after identifying mid point of each class:

1. Direct Method
2. Short cut Method

Direct Method : The following steps are used to compute mean in continuous series

Steps:

1. Find out mid point of each class (m) i.e.((Upper limit + lower limit) / 2)
4. Multiply the mid point of each class (m) by frequency (f) i.e. (f x m)
5. Sum up all the fm , f m
6. Divide fm by the total frequency (N)

Formula is $\overline{X} = \dfrac{f\,m}{N}$

Illustrations:

Calculate Mean of the following data:

X	10--20	20--30	30--40	40--50	50--60
F	5	10	25	7	3

:**Solution:**

Construction of the table to find mean:

X	Mid value m	f	fm
10-20	15	5	75
20-30	25	10	250
30-40	35	25	875
40-50	45	7	315
50-60	55	3	165
	Total	f=N=50	1680

$$\overline{X} = \frac{f\,m}{N}$$

$$\overline{X} = (1680 / 50)$$

$$= 33.6$$

3. Short cut Method:

Step 1. Find out mid point of each class (m) i.e.((Upper limit + lower limit) / 2).
2. Assume any mid value as assumed mean (A).
3. Find out deviations of the mid point of each from the assumed mean (d = m – A).
4. Multiply the deviations of each class by its frequency (fd)
5. Sum up all the fd , f d
6. Divide fd by the total frequency (N)

Formula is $\overline{X} = \dfrac{f\,d}{N}$

Illustrations:

Calculate Mean of the following data:

Marks	10--20	20--30	30--40	40--50	50--60
f	5	10	25	7	3

Solution:
Construction of the table to find mean by short cut method:

X	Mid value m	d= m- A	f	fm
10-20	15	-20	5	-100
20-30	25	-10	10	-100
30-40	35	0	25	0
40-50	45	10	7	70
50-60	55	20	3	60
		Total	f=N=50	-70

$$\overline{X} = A + \frac{f\,m}{N}$$

$$\overline{X} = 35 + (-70 / 50)$$
$$= 35 - 1.4$$
$$= 33.6$$

15.5 Merits of Mean

Arithmetic mean is the simplest measurement of central tendency of a group. It is extensively used because :

1. It is easy to calculate and easy to understand.
2. It is based on all the observations.
3. It is rigidly defined.
4. It provides good basis of comparison.
5. It can be used for further analysis and algebraic treatment.

Demerits of Mean

1. It is affected by the extreme values.
2. It may lead to a wrong conclusion..
3. It is unrealistic.
4. Arithmetic mean cannot be obtained even if single observation is missing
5. It cannot be identified observation or graphic method

15.6 Lesson end activities

1. Calculate mean from the following data :

Roll No.	1	2	3	4	5	6	7	8	9	10
Mark	40	50	55	78	58	60	73	35	43	48

2. Calculate Mean of the following data:

Value	0-4	5-9	10-19	20-29	30-39	40-49	50-59	60-69
Frequency	328	350	720	664	598	524	378	244

3. Calculate Mean of the following data:

Marks	0-10	10-20	20-30	.30-40	.40-50	.50-60	.60-70	.70-80
Frequency	15	.20	.25	.24	.12	.31	.71	52

15.7 Let us Sum Up

In this lesson we have dealt with the following:

- We have discussed about Mean, types of mean and their characteristics.
- We have discussed about the computation of mean for both discrete and continuous case

Model Answer for selected lesson end activities

4. **54**
5. **28.91**
6. **50.4**

15.8 References

Statistics –R.S.N Pillai, V.Bgavathi,, S.Chand &Company Ltd., Revised Edition and Reprint 2005 **.**

Fundamentals of Mathematical Statistics. S.C. Gupta, V.K. Kapoor. Sultan Chand and Sons.

LESSON – 16

MEDIAN

Contents:

16.0 Aims and Objectives

In this Lesson, we have discussed the measures of central tendency. In which, we have defined Median and its characteristics. Computation of Median for Discrete and Continuous case are explained with illustrations. Relationship among the same are explained.

After reading this lesson, you should be able to

- To compute Median for both Discrete and Continuous case.
- Merits and Demerits of Median

16.1 Introduction

Median is the value which divides the series into two equal parts. One half containing values greater than it and the other half containing values less than it. Hence series has to be arranged in ascending or descending order to find median.

According to Croxton and Cowden "The median is that value which divides a series so that one half or more of the items are equal to or less than it and one half or more of the items are equal to or greater than it "

16.2 Computation of Median – Individual Series

Steps: 1. Arrange the data in increasing or decreasing order.
Apply the formula

Median = Size of (N+1) / 2 th item

Illustration : Calculate Median for the following data:

Roll Nos.	Marks
1	45
2	50
3	55
4	60
5	55
6	90
7	95
8	65
9	35

Solution:
Computation of Median requires arrangement of the given data (increasing order)

Roll Nos.	Marks
9	35
1	45
2	50
3	55
5	55
4	60
8	65
6	90
7	95

Median = Size of (N+1) / 2 th item

= Size of (9+ 1) /2 th item

= Size of 5 th item

= 55

16.3 Computation of Median – Discrete case

Steps: 1. Arrange the data in increasing or decreasing order.
2. Find the cumulative frequency
3. Apply the formula

Median = Size of (N+1) / 2 th item

Illustrations:

Identify median from the following data:

Size of shirts	32	34	36	38	40	42
Nos.	10	20	25	10	7	3

Solution:
Construction of the new table to find median :

Size	f	c.f
32	10	10
34	20	10+20=30
36	25	30+25=55
38	10	55+10=65
40	7	65+7 =72
42	3	72+3 =75

Median = Size of (75+1) / 2 item
= Size of 38 th item
= Size of 38 th item = 36
Median size of shoe is 36

16.4 Computation of Median – Continuous case :

Steps:
1. Construct the table which consists of c.f
2. Find the cumulative frequency
3. Find out Median class by using N / 2

Apply the formula

$$\text{Median} = L + \frac{\{(N/2) - cf\}}{f} \times i$$

L = Lower limit of the median class
f = Frequency of the Median class
cf = Cumulative frequency of the class preceding median class
i = Class interval of median class

Illustrations:

Calculate Median of the following data:

Marks	10--20	20--30	30--40	40--50	50--60
f	5	20	25	15	5

Solution :

Construction of the table to find Median:

Marks	f	cf
10-20	5	5
20-30	20	5+20=25
30-40	25	25+25=50
40-50	15	50+15=65
50-60	5	65+5=70

Median = N/2 i.e. 70 /2 = 35

Median occurs in the Class 30-40 Marks

$$\text{Median} = L + \frac{\{(N/2) - cf\}}{f} \times i$$

L = Lower limit of the median class = 30
f = Frequency of the Median class = 25
cf = Cumulative frequency of the class preceding median class = 25
i = Class interval of median class = 10

$$\text{Median} = 30 + \frac{\{(70/2) - 25\}}{25} \times 10$$

$$= 30 + \frac{\{35 - 25\}}{25} \times 10$$

$$= 30 + (10 \times 10) / 25$$

$$= 30 + 4$$

$$= 34 \text{ Marks}$$

16.5 Merits of Median:

- It is easy to calculate and easy to understand.
- It is based on all the observations.
- It is rigidly defined.
- It eliminates the impact of extreme values.
- It can be used for further analysis and algebraic treatment.
- Median can be found out just by inspection in some cases.

Demerits of Median

- It simply ignores the extreme values.
- It may lead to a wrong conclusion. When distribution of observations is irregular.
- The median is estimated in continuous case.

16.6 Lesson End Activities

1. Calculate Median of the following data:

Marks	0-10	10-20	20-30	.30-40	.40-50	.50-60	.60-70	.70-80
Nos. of students (‘000’)	2	3	4	3	2	1	0.5	0.1

2. Calculate the median of the following data :

Marks	10-25	25-40	.40-55	.55-70	.70-85	.85-100
Frequency	6	20	44	26	3	1

3. Calculate the median of the following data

Marks	10-19	20-29	30-39	.40-49	.50-59	.60-69	.70-79	.80-89	90-99
Frequency	7	15	18	25	30	20	16	7	2

4. Calculate the median of the following data

Mid values	115	125	135	145	155	165	175	185	195
Frequency	6	25	48	72	116	60	38	22	3

5. The following table gives the marks obtained by65 students in c language in a certain examination. Calculate the median.

Marks	No of students
More than 70%	7
More than 60%	18
More than 50%	40
More than 40%	45
More than 30%	50
More than 20%	63
More than 10%	65

16.7 Let us Sum Up

In this lesson we have dealt with the following:

- We have discussed about the computation of median for both discrete and continuous case.
- We have discussed about merit and demerits of median.

Model Answer for selected lesson end activities

1. 27
2. 48.18
4. 153.79
5 53.4

16.8 References

Statistics –R.S.N Pillai, V.Bgavathi,, S.Chand &Company Ltd., Revised Edition and Reprint 2005 **.**

LESSON – 17

MODE

Contents:

17.0 Aims and Objectives

In this Lesson, we have discussed the measures of central tendency. In which, we have defined Mode and its characteristics. Computation of Mode for Discrete and Continuous case are explained with illustrations. Relationship among the same are explained.

After reading this lesson, you should be able to

- To compute Mode for both Discrete and Continuous case.
- Merits and Demerits of Mode
- Relationship among mean, median and mode

17.1 Introduction

Mode is the value which is the highest number of frequency in a group. Mode is most attractive value in distribution, because it is repeated many number of times. According to Croxton and Cowden. " The mode of a distribution the value at the point around which the item tend to be most heavily concentrated" .

17.2 Calculation of Mode – Individual observations

Mode can be found out by mere inspection in case of individual observations. The data have to be arranged and look for the highest frequency that value is known as Mode.

Example: Percentage of Top 10 Students is as follows

95, 98, 95, 80, 88, 95, 89, 91,94, 93

Percentage 95 repeats three times, therefore the mode percentage is 95 .

17.3 Computation of Mode - Continuous Case

Steps :
1. Construct the table
2. Find the Modal class
3. Find out Mode class by using N / 2
4. Apply the formula

$$Mode = L + \frac{(f_1 - f_2)}{2f_1 - f_0 - f_2} X \; i$$

L = Lower limit of the Modal class
f_1 = Frequency of the Modal class
f_0 = Frequency of the class preceding the Modal class
f_2 = Frequency of the class succeeding Modal class
i = Class interval of modal class

Illustrations:

Calculate mode of the following data:

Marks	10--20	20--30	30--40	40--50	50--60
f	5	20	25	15	5

Solution :

Construction of the table to find Mode:

Marks	F
10-20	5
20-30	20
30-40	25
40-50	15
50-60	5

Modal Class is 30 - 40 Since highest frequency occurs here
i.e., frequency of that class is = 25

$$Mode = L + \frac{(f_1 - f_0)}{2f_1 - f_0 - f_2} X \quad i$$

L = Lower limit of the Modal class = 30
f_1 = Frequency of the Modal class = 25
f_0 = Frequency of the class preceding the Modal class = 20
f_2 = Frequency of the class succeeding Modal class = 15
i = Class interval of modal class = 10

$$Mode = 30 + \frac{\{25 - 20\}}{2x25 - 20 - 15} X \quad 10$$

= 30 + (5 x 10) / 15
= 33.33

17.4 Merits of Mode:

1. It is easy to calculate and easy to understand.
2. It eliminates the impact of extreme values.
3. It can be identified by using graphical method.

Demerits of Mode

1. It is not suitable for further mathematical treatments.
2. It may lead to a wrong conclusion. When bimodal distribution.
3. It is difficult to compute in some cases.
4. Mode is influenced by length of the class interval.

Relationship between Mean , Median and Mode.

Symmetrical distribution : Mean = Median =Mode

Moderately Asymmetrical Distribution : Mean – Mode = 3(Mean – Median)

Asymmetrical positively skewed : (Mean > Median > Mode)

Asymmetrical negatively skewed : (Mode > Median > Mean)

17.5 Lesson End Activities

1. Calculate the mode of the following data

Marks	20-25	25-30	30-35	35-40	.40-45	.45-50	.50-55	.55-60
Frequency	50	70	80	180	150	120	70	50

2. Calculate the median of the following data

Marks	0-9	10-19	20-29	30-39	.40-49	.50-59	.60-69	.70-79	.80-89	90-99
Frequency	6	29	87	181	247	263	133	43	9	2

3. Calculate Mode of the following data:

Class	10-20	20-30	.30-40	.40-50	.50-60	.60-70	.70-80
Frequency	24	42	56	66	108	130	154

4. Calculate Mode of the following data:

Size	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	5	7	12	18	16	10	5

5. Calculate Mode of the following data:

Size	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	4	10	16	22	20	18	8	2

17.6 Let us Sum Up

In this lesson we have dealt with the following:

- Computation of mode for both discrete and continuous case.
- Merit and demerits of median.
- Relationship between Mean, Median and Mode.

Model Answer for lesson end activities

1. **42**
2. **54.09**
3. **71.34**
4. **37.5**
5. **36**

17.7 References

Statistics –R.S.N Pillai, V.Bgavathi,, S.Chand &Company Ltd., Revised Edition and Reprint 2005 **.**

LESSON - 18

DISPERSION

Contents:

18.0 Aims and Objectives

In this Lesson, we have discussed the measures of Dispersion. In which, we have defined Range, Mean Deviation and Standard Deviation and also their characteristics. Computations of Range, Mean Deviation and Standard Deviation for Discrete and Continuous case are explained with illustrations. Merits and demerits are discussed. Comparisons between Mean Deviation and Standard Deviation are explained.
After reading this lesson, you should be able to

- To compute Range, Mean Deviation and Standard Deviation.
- Merits and Demerits of Range, Mean Deviation and Standard deviation.
- Comparisons between Mean deviation and Standard Deviation.

18.1 Introduction

Averages give us information of concentration of the observations about the central part of the distribution. But they fail to give anything further about the data. According to George Simpson and Fritz Kafka, “An average does not tell the full story. It is hardly fully representative of a mass, unless we know the manner in which the individual items scatter around it. A further description of the series is necessary if we to gauge how representative the average is. “

Definitions:

"Dispersion is the measure of the variations of the items." - A.L Bowley

"Dispersion is the measure of extend to which individual items vary." – L.R. Connor

"The degree to which numerical data tend to spread about an average value is called variation or dispersion of the data." - Spiegel

18.2 RANGE

The range is the difference between two extreme values of the given observations

Range = Largest value – Smallest value

$$\text{Co- efficient of Range} = \frac{\text{Largest value} - \text{Smallest value}}{\text{Largest value} - \text{Smallest value}}$$

Illustration : 1

Find the range of Marks of 10 students from the following

65,35,48,99,56,88,78,20,66,53

Solution:

Range = L – S
= 99 -20
= 79

$$\text{Co- efficient of Range} = \frac{99-20}{99+20}$$

$$= \frac{79}{109}$$

$$= 0.72$$

Merits:

1. It is easy to compute and understand.
2. It gives an idea about the distribution immediately.

Demerits:

1. Calculation range depends only on the basis of extreme items, hence it is not reliable.
2. It is not applied to open end cases
3. Not suitable for mathematical treatments.

18.3 MEAN DEVIATION

Mean deviation is the arithmetic mean of the difference of a series computed from any measure of central tendency *i.e.*, Deviations from Mean or Mode or Median. All the deviation's absolute values are considered. According to Clark and Schekade, "Average deviations the average amount of scatter of the items in a distribution from either the mean or the median, ignoring the signs of the deviations. The average that is taken of scatter is an arithmetic mean, which accounts for the fact that this measure is often called "mean deviation".

Mean deviation is computed from the following formula:

- x - Mean -
M.D. (from Mean) = | | | | | | | | | |
N

- x - Median -
M.D. (from Median) = | | | | | | | | | |
N

- x - Mode -
M.D. (from Mode) = | | | | | | | | | |
N

Coefficient of Mean Deviation

Mean Deviation X 100
| | | | | | | | | | | | |
Mean or Median or Mode

Mean Deviation – Individual series

Steps
1. Calculate Mean of given series.
2. Compute the deviations of the observations from Mean. Ignoring signs and denotes this by - D - .
3. Compute the sum of these deviations i.e., - D - .
4. Divide this sum by number of observations.

Formula is given by

-D -
M.D. = | | | | | |
N

Where M.D. = Mean Deviations
-D- = Sum of the deviations.
N = Number of observations

Illustrations:

Calculate mean deviation from the following data:

100	200	300	470	500	600	700

Solution :

X	-D-= X – 410
100	310
200	210
300	110
400	0
500	110
600	210
700	310
Total	1320

Mean $= \frac{2870}{7}$

$= 410$

M.D. $= \frac{1320}{7}$

$= 188.57$

Coefficient of M.D.

$= \frac{M.D.}{X}$

$= \frac{188.57}{410}$

$= 0.46$

Mean Deviation – Discrete series

Steps. 1. Calculate Mean or Median or Mode of given series .

2. Compute the deviations of the observations from Mean. Ignoring signs and denotes this by -D-.

3. Multiply the deviation of each size-D- by its frequency and Compute the sum of these deviations i.e., f-D-.

4. Divide this total by sum of the frequency

Formula is given by

$$\boldsymbol{M.D. = \frac{f\text{-}D\text{-}}{N}}$$

Where M.D. = Mean Deviations.
f-D- = Sum of the products of frequency and respective deviations.
N = Total of the frequency.

Illustrations:

Calculate mean deviation from the following data:

X	2	4	6	8	10
f	1	3	6	3	1

Solution:

Construct the following table to compute the mean deviation (from Mean):

X	*f*	*fX*	*- X-Mean-*	*f* - ***D*** -
2	1	2	|2-6| = 4	4
4	3	12	2	6
6	7	42	0	0
8	3	24	2	6
10	1	10	4	4
Total	15	90	12	20

$$\bar{X} = \frac{90}{15} = 6$$

$$\textbf{\textit{M.D.}} = \frac{f - D -}{N}$$

$$= \frac{20}{15}$$

$$= 1.25$$

Illustration:

Calculate the mean deviation from mean for the following data:

Class interval :	2| 4	4|6	6|8	8|10
Frequency	6	8	4	2

Solution :

Construction of the table to find mean deviation

Class	Mid Value	Frequency	Fm	X-Mean	f|D|
2| 4	3	6	18	2.2	13.2
4| 6	5	8	40	0.2	1.6
6| 8	7	4	28	1.8	7.2
8| 1 0	9	2	18	3.8	7.6
	Total	20	104		29.6

$$\bar{X} = \frac{104}{20} = 5.2$$

$$M.D. = \frac{f - D -}{N}$$

$$= \frac{29.6}{20}$$

$$= 1.48$$

Merits of Mean Deviation:

1. It is simple to understand and easy to calculate
2. The computation process is based on all items of the series
3. It is less affected by the extreme items.
4. This measure is flexible, Since it can be calculated from mean, meadian, or mode.
5. This measure is rigidly defined.

Demerits of Mean Deviation:

1. This measure is not a very accurate measure of dispersion.
2. Not suitable for further mathematical calculation.
3. It is rarely used.
4. Absolute values are considered, mathematically unsound and illogical.

18.4 STANDARD DEVIATION

The famous statistician Karl Pearson introduced the concept of Standard Deviation in 1893.
This is the most accepted measure of dispersion and also widely used in many statistical applications. Standard deviation is also referred as Root-Mean Square Deviation or Mean Square error. It gives accurate results. The standard deviation is also denoted by the Greek letter (σ).

18.4.1 Calculation of standard Deviation – Individual observation

There are two method of calculating standard deviation in an individual observation:

(i) Direct Method – Deviation taken from actual mean

(ii) Short –cut Method – Deviation taken from assumed mean

(i) Direct Method

The following are the steps :

1. Find out the actual mean of the given observations.
2. Compute deviation of each observation from the mean (X-Mean).
3. Square the deviations and find out the sum *i.e.*, $\sum(X-\bar{X})^2$.
4. Divide this total by the number of observations and take square root of the quotient, the value is standard deviation.

$$\sigma = \sqrt{\frac{\sum(X-\bar{X})^2}{N}}$$

Illustration :

Calculate the standard deviation from the following data :

15, 12, 17, 10, 21, 18, 11, 16

Solution :

Calculation of S.D from Mean

Values (X)	$(X-\bar{X})$	$(X-\bar{X})^2$
15	15-15 = 0	0
12	12-15 = -3	9
17	2	4
10	-5	25
21	6	36
18	3	9
11	-4	16
16	1	1
$\sum X = 120$		$\sum(X-\bar{X})^2 = 100$

$$\bar{X} = \frac{120}{8}$$

$$= 15$$

$$\sigma = \sqrt{\frac{\sum(X-\bar{X})^2}{N}} = \sqrt{\frac{100}{8}}$$

$$= 3.53$$

Method 2 :

Standard Deviation can be found out by using variables directly

Values (X)	X^2
15	225
12	144
17	289
10	100
21	441
18	324
11	121
16	256
X = 120	X^2 = 1900

$$\sigma = \sqrt{\frac{X^2}{N} - \left(\frac{X}{N}\right)^2}$$

$$\sigma = \sqrt{\frac{1900}{8} - \left(\frac{120}{8}\right)^2}$$

$$\sigma = \sqrt{237.5 - 225} = 3.53$$

(b) Deviation taken from assumed mean

This method is used when arithmetic mean is fractional value. A deviation from fractional value leads to tedious task. To save calculation time, we apply this method. The formula is

$$\sigma = \sqrt{\frac{d^2}{N} - \left[\frac{d}{N}\right]^2}$$

Where d = Deviations from assumed mean = (X –A)
N = Number of observations

The following are the steps:

1. Assume any value which is very close to the given observations just by inspection *(A)*.
2. Find out the deviations from the assumed Mean . i.e., *(X-A)* denoted by *d*
3. Find out the sum of the deviations i.e., d
4. find out Square of the deviations; i.e., d^2
5. Apply all the values to the above mentioned formula.

Illustrations :

Compute the standard deviation of the following data:

Roll No. :	1	2	3	4	5	6	7	8	9	10
Marks :	45	50	56	63	68	78	74	66	72	33

Solution : (Construct the table to find out SD)

Roll No	Marks (X)	d = -A d = X-50	d^2
1	45	45-50= -5	25
2	50	50-50= 0	0
3	56	6	36
4	63	13	169
5	68	18	324
6	78	28	784
7	74	24	576
8	66	16	256
9	72	22	484
10	33	-17	289
N=10	Total	110	2943

$$\sigma = \sqrt{\frac{d^2}{N} - \left[\frac{d}{N}\right]^2}$$

$$\sigma = \sqrt{\frac{2943}{10} - \left[\frac{110}{10}\right]^2}$$

$$= \sqrt{294.3 - (11)^2}$$

$$= \sqrt{294.3 \quad - \ 121}$$

$$= \sqrt{173.3}$$

$$= 13.165$$

18.4.2 Computation of Standard Deviation : Discrete Case

Following are the methods used to compute standard deviation:

(i) Actual Mean method
(ii) Assumed Mean method
(iii) Step deviation Method

Actual Mean method

Steps:
1. Compute mean of the observations.
2. Compute deviation from the mean $(d = X - \bar{X})$.
3. Square the deviations (d^2) and multiply these values with respective frequencies (f) i.e., fd^2
4. Sum the products fd^2 and apply the formula

$$\sigma = \sqrt{\frac{fd^2}{f}}$$

Illustrations:

Compute standard deviation from the following data :

Marks	10	20	30	40	50
Frequency	2	8	10	8	2

Solution :

Construct the table to compute the standard deviation

Marks X	*f*	*fX*	*d = X-X X-30*	d^2	fd^2
10	2	20	-20	400	800
20	8	160	-10	100	800
30	10	300	0	0	0
40	8	320	10	100	800
50	2	100	20	400	800
	f = 30	*f X=900*			$fd^2=3200$

$$\overline{X} = (900 / 30) = 30$$

$$\sigma = \sqrt{\frac{\sum fd^2}{\sum f}}$$

$$\sigma = \sqrt{\frac{3200}{30}}$$

$$= \sqrt{106.66}$$

$$= 10.325$$

Assumed Mean Method

Steps:
1. Assume any one of the given value as assumed mean A
2. Compute deviation from the assumed mean *(d= X-A).*
3. Multiply these deviations by its frequencies fd.
4. Square the deviations (d^2) and multiply these values with respective frequencies *(f)* i.e., fd^2
5. Sum the products $\sum fd^2$ and apply the formula.

$$\sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \frac{(\sum fd)^2}{(\sum f)^2}}$$

Illustrations:

Compute standard deviation from the following data :

Marks	10	20	30	40	50
Frequency	2	8	10	8	2

Solution:.

Construct the table to compute the standard deviation

Marks X	*F*	*d = X-A X-20*	*fd*	fd^2
10	2	-10	-20	200
20	8	0	0	0
30	10	10	100	1000
40	8	20	160	3200
50	2	30	60	1800
	$\sum f = 30$		300	$\sum fd^2=6200$

$$\sigma = \sqrt{\frac{fd^2}{f} - \frac{(fd)^2}{(f)^2}}$$

$$\sigma = \sqrt{\frac{6200}{30} - \frac{300^2}{30^2}}$$

$$= \sqrt{206.66 - 100}$$

$$= \sqrt{106.66}$$

$$= 10.325$$

Step Deviation Method :

Take a common factor and divide that item by all deviations

Steps: 1. Assume any one of the given value as assumed mean A

2. Compute deviation from the assumed mean *(d = (X-A)/i).*
3. Multiply these deviations by its frequencies *fd.*
4. Square the deviations (d^2) and multiply these values with respective frequencies *(f)* i.e., fd^2
5. Sum the products fd^2 and apply the formula.

$$\sigma = \sqrt{\frac{fd^2}{f} - \frac{(fd)^2}{(f)^2}} \times i$$

Illustrations:

Solving the above problem.

Construct the table to compute the standard deviation

Marks X	*F*	*d = (X-A)/i (X-20)/10*	*fd*	fd^2
10	2	-1	-2	2
20	8	0	0	0
30	10	1	10	10
40	8	2	16	32
50	2	3	6	18
	$f = 30$		30	$fd^2=62$

$$\sigma = \sqrt{\frac{fd^2}{f} - \left(\frac{(fd)^2}{(f)^2}\right)} \; x\, i$$

$$\sigma = \sqrt{\frac{62}{30} - \left(\frac{30^2}{30^2}\right)} \; X\ 10$$

$$= \sqrt{2.0666 - 1} \quad X\ 10$$

$$= \sqrt{1.0666}\ X\ 10$$

$$= 1.0325$$

18.4.3 Calculation of Standard Deviation – Continuous Case

In continuous case, mid-values of the class intervals are to be found out and that values are used for calculations

Step : 1. Compute mid value of each class and also assume any one of the mid values as assumed mead A

2. Compute deviation of each class from the assumed mean and divide that value by class interval , it is denoted by *(d = (m-A)/i).*

3. Multiply these deviations by its frequencies *fd.*

4. Square the deviations (d^2) and multiply these values with respective frequencies *(f)* i.e., fd^2

5. Sum the products fd^2 and apply the formula.

$$\sigma = \sqrt{\frac{fd^2}{f} - \left(\frac{(fd)^2}{(f)^2}\right)} \; X\ i$$

$$d = \frac{m - A}{i} \quad \textit{where} \quad i \textit{ is class interval.}$$

Illustrations:

Compute the standard deviation from the following data.

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	7	13	14	20	16	7	3

.

Solution:

Construct the table to compute the standard deviation

Marks X	*Frequency*	*Mid value m*	*d = (X-A)/i (m-35)/10*	*fd*	*fd^2*
0-10	7	5	-3	-21	63
10-20	13	15	-2	-26	52
20-30	14	25	-1	-14	14
30-40	20	35	0	0	0
40-50	16	45	1	16	16
50-60	*7*	*55*	2	14	28
60-70	*3*	*65*	3	9	*27*
	80			-22	*200*

$$\sigma = \sqrt{\frac{fd^2}{f} - \frac{(fd)^2}{(f)^2}} \times i$$

$$\sigma = \sqrt{\frac{200}{80} - \frac{(-22)^2}{80^2}} \times 10$$

$$= \sqrt{2.5 - 0.0756} \times 10$$

$$= \sqrt{2.4244} \times 10$$

$$= 1.5555 \times 10$$

$$= 15.555$$

Illustration :

The daily temperature recorded in a place in Ooty in a year is given below :

Temperature ^oC	No. of days
-40 to -30	20
-30 to -20	38
-20 to -10	40
-10 to 0	52
0 to 10	65
10 to 20	140
20 to 30	10

Compute Mean and Standard deviation.

Solution:

Construct the table to compute mean and standard deviation.

Temperature ^oC	*Mid X (m)*	*No. of days (f)*	*d= $\frac{m-0}{10}$*	*fd*	*fd^2*
-40 to -30	-35	20	-3.5	-70	245
-30 to -20	-25	38	-2.5	-95	232.5
-20 to -10	-15	40	-1.5	-60	90
-10 to 0	-5	52	-.5	-26	13
0 to 10	5	65	.5	32.5	16.25
10 to 20	15	140	1.5	210	315
20 to 30	25	10	2.5	25	62.5
		365		16.5	974.25

$$\bar{X} = 0 + \frac{16.5 \times 10}{365} = 0.45$$

$$\sigma = \sqrt{\frac{fd^2}{f} - \frac{(fd)^2}{(f)^2}} \times i$$

$$\sigma = \sqrt{\frac{974.25}{365} - \frac{(16.5)^2}{(365)^2}} \times 10$$

$$\sigma = \sqrt{2.66 - 0.20} \times 10$$

$$\sigma = \sqrt{2.46} \times 10$$

$$= 1.57\ ^oC$$

18.5 Lesson end activities:

1. Calculate Standard Deviation for the following data:

Class Intervals	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45
Frequency	6	5	15	10	5	4	3	2

2. Calculate Standard Deviation for the following data:

Age	20-25	25-30	30-35	35-40	40-45	45-50	50-55
F	170	110	80	45	40	30	25

3. Calculate Standard Deviation for the following data giving 300 telephone calls according to their duration in seconds :

Duration (in sec.)	0-30	30-60	60-90	90-120	120-150	150-180	180-210
No of calls	9	17	43	82	81	44	24

4. Calculate the S.D of the following

Size of the item	6	7	8	9	10	11	12
Frequency	3	6	9	13	8	5	4

5. Calculate Mean Deviation of the following :

X	10	11	12	13	14
f	3	12	18	12	3

6. Calculate Mean Deviation and Standard Deviation for the following data:

Age	20-25	25-30	30-35	35-40	40-45	45-50
F	170	110	80	45	40	35

18.6 Let us Sum Up

In this lesson we have dealt with the following:

- Computation of Range.
- Computation of Mean Deviation for both discrete and continuous case.
- Computation of Standard Deviation using various methods for both discrete and continuous case.

Model Answer for lesson end activities

1. **9.1**
2. **9.05**
3. **42.42**
4. **1.6**
5. **0.75**
6. **6.36**
7. **94**

18.7 Reference

Statistics –R.S.N Pillai, V.Bgavathi,, S.Chand &Company Ltd., Revised Edition and Reprint 2005 **.**

UNIT – V

LESSON - 19

CORRELATION

Contents:

19.1 Aims and Objectives

In this Lesson, we have discussed about the correlation. In which, we have defined the measure of relationship between two variables... Computations of Correlation and its types, Karl Pearson's coefficient of Rank correlation is explained with illustrations.

After reading this lesson, you should be able to
To compute

- To Compute Karl Pearson's Coefficient of correlation between variables.
- Types of correlation
- Mathematical properties of correlation.

19.2 Introduction

Definition of Correlation:
According to Croxton and Cowden, "The relationship of quantitative nature, the appropriate statistical tool for discovering and measuring the relationship and expressing it in brief formula is known as correlation"

The relationship of any two variables is known as correlation. The correlation expresses the association or interdependence of two variables. Correlation is the numerical measure, which shows degree of correlation between two variables.

19.3 Types of correlation.

Correlation is classified by the following types and they are:

1. Positive and Negative
2. Simple and Multiple
3. Partial and total.
4. Linear and non linear.

1. Positive and Negative

Positive and negative correlation depends on the direction of change of the variables. If move variables move in the same direction i.e., when there is an increase in the value one variable influenced by an increase in the value of other variable is called positive correlation. If two variables tend to move in opposite directions so that an increase in the values of one variable is influenced by decrease in the value of the other variable, then the correlation is said to be negative correlation.

2. Simple and Multiple

When we study about only two variables then the correlation is said to be simple where as we study about more than two variables is called multiple correlation.

3. Partial and total

In multiple variable environments, some variables excluded due to some reason then it is termed as partial correlation. In total correlation, all the facts are taken into consideration.

4. Linear and non linear

If the ratios of change between two variables are uniform, then there exists uniform correlation. In non linear correlation, the quantum of change is one variable not in constant ratio.

19.3 Coefficient of Correlation

Correlation is statistical technique used for analysing the behaviour of two variables. This analysis refers with the relationship between two or more variables. Statistical measures of correlation are co-variation between series, not of functional relationship. It is not possible to obtain another variable, if the value of a one variable known in one series.

19.4 Karl Pearson’s Coefficient of Correlation

The famous statistician Karl Pearson suggested a mathematical method to compute the magnitude of linear relationship between two variables. This method is widely used and it is called Karl Pearson’s coefficient of correlation. It is denoted by the symbol *‘r’. The formula is given by*

$$r = \frac{\textit{Covariance of } xy}{\sigma_x \times \sigma_y} = \frac{xy}{N\ \sigma_x \times \sigma_y} = \frac{xy}{\sqrt{x^2 \quad y^2}}$$

Where x = $(x-\bar{X})$ $y = (y-\bar{Y})$

σ_x = *Standard deviation of series x*

σ_y = *Standard deviation of series y*

Steps:

1. Find out the mean of the two series $\bar{X}$ and $\bar{Y}$.
2. Take deviations of the two series from the respective means $\bar{X}$ and $\bar{Y}$ and denote *x* and *y.*
3. Square the deviations and find the sum of square and denote x^2 and y^2
4. Multiply deviations of x and y *i.e.*, x y
5. Substitute the values of xy, x^2 and y^2 in the formula.

19.5 Computation of Coefficient of Correlation

Illustration:
Calculate coefficient of correlation from the following data

X	3	5	7	3	2
Y	2	4	6	2	1

Solution:

Construct the table to compute correlation coefficient:

X	Y	$x = X - \bar{X}$	$y = Y - \bar{Y}$	*xy*	x^2	y^2
3	2	-1	-1	1	1	1
5	4	1	1	1	1	1
7	6	3	3	9	9	9
3	2	-1	-1	1	1	1
2	1	-2	-2	4	4	4
			Total	16	16	16

$$r = \frac{xy}{x^{2} \quad y^{2}}$$

$$r = \frac{16}{16 \quad 16}$$

$$= 1$$

Computation of correlation coefficient by another formula:

When the both series observations are small in numbers, Correlation can also be computed without taking deviation from actual mean. The formula is given by this method is:

$$r = \frac{N(XY)(X \; x \quad Y)}{N \; X^{2}\text{-}(X)^{2} \quad N \; Y^{2}\text{-}(Y)^{2}}$$

Illustration:

Compute the correlation coefficient to the following data :

X	8	7	6	4	3	2
Y	9	1	6	3	4	5

Solution:

Construct the table to compute the correlation coefficient

X	Y	X^2	Y^2	XY
8	9	64	91	72
7	1	49	1	7
6	6	36	36	36
4	3	16	9	12
3	4	9	16	12
2	5	4	25	10
X= 30	Y= 28	X^2 =178	Y^2= 178	XY= 149

$$r = \frac{N(XY)(X \; x \quad Y)}{N \; X^{2}\text{-}(X)^{2} \quad N \; Y^{2}\text{-}(Y)^{2}}$$

$$= \frac{6\ (149) - (30)(28)}{\sqrt{6(178) - (30)^2}\ \sqrt{6(178) - (28)^2}}$$

$$= \frac{6\ (149) - (30)(28)}{\sqrt{6(178) - (900)}\ \sqrt{6(178) - (784)}}$$

$$= \frac{894 - 840}{\sqrt{1068 - (900)}\ \sqrt{1068 - (784)}}$$

$$= \frac{54}{\sqrt{168}\ \sqrt{284}}$$

$$= \frac{54}{\sqrt{47712}}$$

Alternate Method : Computation of correlation by Assumed Mean

When actual mean is a fraction , the calculation of direct method will involve a lot of arithmetic calculations. To avoid tedious calculation, we use assumed mean method the formula is given by :

$$r = \frac{N\ \ dx\,dy\ -\ \ dx\ x\ \ dy}{\sqrt{N\ dx^2 - (\ dx\,)^2}\ \ \sqrt{N\ dy^2 - (\ dy\,)^2}}$$

dx = (X-A) Where A = assumed mean of series X.

dy = (Y-B) Where B is assumed mean of series Y.

N = No. of observations in a series.

dxdy = the total product of the deviations of x and y form their assumed mean .

$d x^2$ = the total of the squares of the deviations of the series X from an assumed mean.
$d y^2$ = the total of the squares of the deviations of the series Y from an assumed mean.

Steps:

1. Take the deviations of the series X from assumed mean *i.e.,* dx = X-A and get the total i.e., d x
2. Take the deviations of the series X from assumed mean i.e. dy = Y-B and get the total. dy
3. Multiply dx and dy and find the total i.e., dxdy .
4. Square dx and obtain the total $d x^2$ and square dy and obtain the total $d y^2$
5. Substitute the above values to the following formula :

$$r = \frac{N \quad dx\,dy \;-\; dx \times dy}{\sqrt{N\ dx^2-(\ d x\,)^2}\ \sqrt{N\ dy^2-(\ d y\,)^2}}$$

Illustrations:

Find out coefficient of correlation in the following data:

Mark in Maths :	65	66	67	67	68	69	71	73
Mark in Statistics:	67	68	64	68	72	70	69	70

Solution:

Construct the table to compute the correlation coefficient :

Mark in Maths	*Deviations X-67*	*Square of Deviations*	*Mark in Statistics*	*Deviations Y-68*	*Square of Deviations*	*Products*
X	*Dx*	*dx^2*	*Y*	*dy*	*dy^2*	*dxdy*
65	-2	4	67	-1	1	2
66	-1	1	68	0	0	0
67	0	0	64	-4	16	0
67	0	0	68	0	0	0
68	1	1	72	4	16	4
69	2	4	70	2	4	4
71	4	16	69	1	1	4
73	6	36	70	2	4	12
X =546	*dx = 10*	*$d x^2 = 62$*	*Y = 548*	*dy = 4*	*$d y^2 = 42$*	*dxdy=26*

$$r = \frac{N \quad dx\,dy \;-\; dx \times dy \ \ dx}{\sqrt{N\ dx^2-(\ d x\,)^2}\ \sqrt{N\ dy^2-(\ d y\,)^2}}$$

$$r = \frac{8\,(26) - (10)(4)}{\sqrt{8\,(62) - (10)^2}\;\sqrt{8(42)^2 - (4)^2}}$$

$$r = \frac{208 - 40}{\sqrt{496 - 100}\;\sqrt{336 - 16}}$$

$$r = \frac{168}{355.98}$$

$$r = 0.472$$

Correlation of grouped bi-variate data

When the numbers of observations are very high, the data is classified into two way frequency table. The class intervals for y are in column headings and for x in the stubs. The formula for calculating the correlation coefficient is given below :

$$r = \frac{N \quad fdx\,dy \;-\; fdx \; x \; fdy \;\; dx}{\sqrt{N\;\; fdx^2 - (\, fdx\,)^2}\;\;\sqrt{N\;\; fdy^2 - (\, fdy\,)^2}}$$

Steps :

1. Find the mid-points of the various classes of x and y.
2. Take step deviations of the two series from the respective means X and Y and denote *dx* and d*y*.
3. Multiply *dx*,*dy* with respective frequencies and note the figure in the left hand corner of each cell.
4. Sum up all the values as calculated in step 4 and get the total *fdx dy* .
5. Multiply *dx* by its frequencies and get *fdx* .
6. Multiply dx^2 by its frequencies and get fdx^2.
7. Multiply *dy* by its frequencies and get *f dy* .
8. Multiply dy^2 by its frequencies and get $f\,dy^2$.

Illustrations:

Calculate coefficient of correlation between the marks obtained by a batch of 100 students in accountancy and statistics as given below:

Marks in Statistics	Marks in Accountancy					
	20-30	30-40	40-50	50-60	60-70	Total
15-25	5	9	3			17
25-35		10	25	2		37
35-45		1	12	2		15
45-55			4	16	5	25
55-65				4	2	6
Total	5	20	44	24	7	100

Solution :

Let the marks in Accountancy be denoted by X and the Marks in Statistic by Y.

$$r = \frac{N\ \ fdx\ dy\ -\ \ fdx\ x\ \ fdy\ \ dx}{\sqrt{N\ \ fdx^2-(\ fdx\ )^2}\ \ \sqrt{N\ \ fdy^2-(\ fdy\ )^2}}$$

Construction of the table to compute the correlation:

X		M	20-30 25	30-40 35	40-50 45	50-60 55	60-70 65				
Y			-2	-1	0	1	2	f	fdy	fdy^2	dxdy
15-20	20	-2	5 20	9 18	3 0			17	-34	68	38
25-35	30	-1		10 10	25 0	2 -2		37	-37	37	8
35-45	40	0			1	12	2	15	0	0	0
45-55	50	1			4 0	16 16	5 10	25	25	25	26
55-65	60	2				4 8	2 8	6	12	24	16
		f	5	20	44	24	7	100	-34	-154	88
		fdx	-10	-20	0	24	14	8			
		fdx^2	20	20	0	24	28	92			
		dxdy	20	28	0	22	18	88			

$$r = \frac{100\ (88)-(8)(-34)}{\sqrt{100(92)-8^2}\ \ \sqrt{100(154) - (-34)^2}}$$

$$r = \frac{8800 + 272}{\sqrt{9200-64}\;\sqrt{15400 - 1156}}$$

$$r = \frac{9072}{\sqrt{9136}\;\sqrt{14244}}$$

$$r = 0.7953$$

Mathematical Properties of the correlation coefficient

1. Correlation always lies between -1 and +1 i.e., -1 r +1 .

2. Correlation is independent of change of origin and scale.

19.6 Lesson end Activities

Problems:

1.Find Karl Pearson's coefficient of correlation from the following data :

Wages:	100	101	102	102	100	99	97	98	96	95
Cost of living	98	99	99	97	95	92	95	94	90	91

2.Find Karl Pearson's coefficient of correlation from the following case :

Height of father	65	66	67	67	68	69	71	73
Height of son	67	68	64	68	72	70	69	70

3.Find the co-efficient of correlation between sales and expenses of the following ten firms.

Firm No:	1	2	3	4	5	6	7	8	9	10
Sales	50	50	55	60	65	65	65	60	60	50
Expenses	11	13	14	16	16	15	15	14	13	13

19.7 Let us Sum Up

In this lesson we have dealt with the following:

- Computation coefficient of correlation by using Karl Pearson's Coefficient of correlation method.
- Types of correlation
- Mathematical properties of correlation.

Model Answer for lesson end activities

7. **0.847**
8. **0.472**
3. **0.79**

19.8 Reference

Statistics –R.S.N Pillai, V.Bgavathi,, S.Chand &Company Ltd., Revised Edition and Reprint 2005 **.**

LESSON - 20

RANK CORRELATION

Contents:

20.0 Aims and Objectives

In this Lesson, we have discussed about the Rank correlation. In which, we have defined the measure of relationship between two variables by using a simple method. A computation of Rank correlation is explained with illustrations.

After reading this lesson, you should be able to
To compute

- To Compute Rank correlation between variables.
- To Compute Rank correlation with repeated observations.

20.1 Introduction

Definition of Correlation:

According to Croxton and Cowden, “The relationship of quantitative nature, the appropriate statistical tool for discovering and measuring the relationship and expressing it in brief formula is known as correlation”

The relationship of any two variables is known as correlation. The correlation expresses the association or interdependence of two variables. Correlation is the numerical measure, which shows degree of correlation between two variables.

20.2 Rank Correlation Coefficient

The formula for spearman's rank correlation is given below :

$$R = 1 - \frac{6\,D^2}{N(N^2 - 1)}$$

Where D^2 = *Sum of the squares of the differences of two ranks*. ($R_1 - R_2$).
N = *No. of paired observations.*

Steps:

1. Fix the ranks of the two series R_1 *and* R_2.
2. Find the difference between two ranks $D = (R_1 - R_2)$.
3. Square D and find sum of D^2 .
4. Substitute the above values in the formula.

Illustration:

Following are the rank obtained by 10 students in two subjects Statistics and mathematics .To what extent the knowledge of the students in the two subjects is related?

Statistics:	1	2	3	4	5	6	7	8	9	10
Maths:	2	4	1	5	3	9	7	10	6	8

Solution: Calculation Of rank correlation coefficient

Rank of Statistics R_1	Rank of Maths R_2	$D = R_1 - R_2$	D^2
1	2	-1	1
2	4	-2	4
3	1	2	4
4	5	-1	1
5	3	2	4
6	9	-3	9
7	7	0	0
8	10	-2	4
9	6	3	9
10	8	2	4
			$D^2 = 40$

$$R = 1 - \frac{6\,D^2}{N(N^2 - 1)}$$

$$6 \times 40$$

$$R = 1 - \frac{}{10(10^2 - 1)}$$

$$= 1 - \frac{240}{10(100 - 1)}$$

$$= 1 - \frac{240}{990}$$

$$= 1 - 0.24$$

$$= 0.76$$

Illustration :

A random sample of 5 college students is selected and their percentage in Economics and Business Mathematics are found to be :

Economics :	75	85	68	48	92
Mathematics :	91	73	63	55	78

Compute rank correlation coefficient.

Solution:

Mark in Economics X	Rank of R_1	Mark in Mathematics Y	Rank of Maths R_2	$D = R_1 - R_2$	D^2
75	3	91	1	2	4
85	2	73	3	-1	1
68	4	63	4	0	0
48	5	55	5	0	0
92	1	78	2	-1	1
				D^2	6

$$R = 1 - \frac{6\ D^2}{N(N^2 - 1)}$$

$$R = 1 - \frac{6 \times 6}{5(5^2 - 1)}$$

$$= 1 - \frac{36}{5(25 - 1)}$$

$$= 1 - \frac{36}{120}$$

$$= 1 - 0.3$$

$$= 0.70$$

20.3 Computation of correlation for Repeated Ranks

For the given observations, two or more observations have equal Ranks. Slightly different formula is used and it is given below :

$$R = 1 - \frac{6\{ D^2 + \frac{1}{12}(m^3 - m) + \frac{1}{12}(m^3 - m) \ldots\}}{N(N^2 - 1)}$$

Where m = the number of items whose ranks are common.

Compute rank correlation to the following data:

X	47	32	39	8	15	15	64	23	15	56
Y	12	12	23	5	14	3	19	8	5	18

Solution :

Construct the table to compute rank correlation coefficient

X	R_x	Y	R_y	$D = R_x - R_y$	D^2
47	8	12	5.5	2.5	6.25
32	6	12	5.5	0.5	0.25
39	7	23	10	-3	9.00
8	1	5	2.5	-1.5	2.25
15	3	14	7	-4	16.00
15	3	3	1	2	4.00
64	10	19	9	1	1.00
23	5	8	4	1	1.00
15	3	5	2.5	.5	0.25
56	9	18	8	1	1.00
				D^2	41

$$R = 1 - \frac{6\{\sum D^2 + \frac{1}{12}(m^3-m)+\frac{1}{12}(m^3-m) \ldots.\}}{N(N^2 - 1)}$$

$$R = 1 - \frac{6\{ 41 + \frac{1}{12}(3^3-3)+\frac{1}{12}(3^3-3)+ \frac{1}{12}(2^3-2)+ \frac{1}{12}(2^3-2)\}}{10(10^2 - 1)}$$

$$= 1 - \frac{6\{41+2+0.5+0.5\}}{990}$$

$$= \frac{990 - 264}{990}$$

$$= 0.733$$

Merits of Rank correlation Coefficient

1. It is easy to calculate and simple to understand.
2. It is much useful when the data in the case of qualitative nature.
3. When the data is given in the form of ranks, no other method is applied. except this method.

20.4 Lesson end activities

1.Calculate Pearson's Coefficient of correlation between advertisement and sales as per data given below :

Advt.('000)	80	64	54	49	48	35	32	29	20	18	15	10
Sales(lakhs) :	36	38	39	41	27	43	45	52	51	42	40	52

2. Ten competitors in voice contest are ranked by three judges in the following orders :

Judge I	1	6	5	10	3	2	4	9	7	8
Judge II	3	5	8	4	7	10	2	1	6	9
Judge III	6	4	9	8	1	2	3	10	5	7

Use the rank correlation to gauge which pair of judges have the nearest approach to common likings in voice.

20.5 Let us Sum Up

In this lesson we have dealt with the following:

- Computation of rank correlation
- Computation of rank correlation with repeated observations

Model Answer for lesson end activities

1. **-0.685**
2. **I & II = -0.212 ; II & III = -0.297 I & III = +0.636**

20.6 Reference:

Statistics –R.S.N Pillai, V.Bgavathi,, S.Chand &Company Ltd., Revised Edition and Reprint 2005 .

LESSON - 21

REGRESSION

Contents:

21.0 Aims and Objectives

In this Lesson, we have discussed about the Regression lines. In which, we can estimate the value of one variable, provided the value of the other variable is given. Regression equations are explained with illustrations.

After reading this lesson, you should be able to
To compute

- To find out Regression equation X on Y.
- To find out Regression equation Y on X.
- Difference between correlation and regression

21.1 Introduction

Regression is an estimate or predict the value of one variable from the given value of another variable when there exit some relationship among the two variables.

“Regression is the measure of the average relationship between two or more variable in terms of the original units of the data” - Blair

"It is often more important to find out what the relation actually is , in order to estimate or predict one variable and statistical technique appropriate in such a case is called Regression Analysis" - Wallis and Robert

Regression Equations:

21.2 Regression Equations by Method of least squares:

Regression equation of X on Y is defined by

$$X_e = a + b\,y$$

Where *a* and *b* is computed by using method of least squares, the formula to determine '*a*' and '*b*' is :

$$X \quad = N a \quad + b \quad Y$$
$$XY = a \quad Y \quad + b \quad Y^2$$

Regression equation of Y on X is defined by

$$Y_e = a + b\,X$$

Where *a* and *b* is computed by using method of least squares, the formula to determine '*a*' and '*b*' is :

$$Y \quad = N a \quad + b \quad X$$
$$XY = a \quad X \quad + b \quad X^2$$

The above equations are called normal equations.

Illustration:

Determine the equation n of a straight line which best fits the data :

X	10	12	13	16	17	20	25
Y	10	22	24	27	29	33	37

Solution :

Straight line $Y = a + bx$

The two normal equations are :

$$Y \quad = N a \quad + b \quad X$$
$$XY = a \quad X \quad + b \quad X^2$$

X	X^2	Y	XY
10	100	10	100
12	144	22	264
13	169	24	312
16	256	27	432
17	289	29	493
20	400	33	660
25	625	37	925
X=113	X^2=1983	y = 182	XY=3186

Substituting the values,

$$\Sigma Y = N a + b \Sigma X$$

$$182 = 7 a + b\ 113 \qquad \ldots\ldots\ldots\ldots(1)$$

$$\Sigma XY = a \Sigma X + b \Sigma X^2$$

$$3186 = a\ 113 + b\ 1983 \qquad \ldots\ldots\ldots\ldots(2)$$

Multiplying (1) by 113 we get

$$12769\ b + 791\ a = 20566 \qquad \ldots\ldots\ldots\ldots(3)$$

Multiplying (2) by.7 we get

$$13887\ b + 791\ a = 22302 \qquad \ldots\ldots\ldots\ldots(4)$$

Subtracting equation (4) from (3) we get

$$-1112b = -1736$$

$$b = -1736 / -1112 = 1.56$$

When b = 1.56 = > 7a = (182)- (1.56)(113)

a = 5.72 / 1.56

a = 0.82

The equation of straight line is

Y = 0.82 + 1.56 X

This is called regression of y on x.

21.3 Deviation taken from arithmetic mean of X and Y

Regression Equation X on Y is defined by

$$(X-\bar{X}) = r \frac{\sigma_x}{\sigma_y} (Y - \bar{Y})$$

The regression coefficient X on Y or b_{xy} is defined by

$$r \frac{\sigma_x}{\sigma_y}$$

Regression Equation Y on X is defined by

$$(Y-\bar{Y}) = r \frac{\sigma_y}{\sigma_x} (X - \bar{X})$$

The regression coefficient Y on X or b_{yx} is defined by

$$r \, \frac{\sigma_y}{\sigma_x}$$

Where

r = Correlation between X and Y.

$\bar{X}$ = mean of X .

$\bar{Y}$ = Mean of Y.

σ_x = Standard Deviation of X.

σ_y = Standard Deviation of Y.

Illustration :

Calculate the two regression equations of X on Y and Yon X from the data given below

X	9	11	12	11	15	14
Y	39	37	42	44	36	42

Estimate the Y when X is 30

Solution:

Construction of the table to compute regression :

X	x=(X– 12)	x^2	Y	y=(Y-40)	y^2	xy
9	-3	9	39	-1	1	3
11	-1	1	37	-3	9	3
12	0	0	42	2	4	0
11	-1	1	44	4	16	-4
15	3	9	36	-4	16	-12
14	2	4	42	2	4	4
X = 72	x = 0	x^{2}=24	Y = 240	y = 0	y^2= 50	x y = -6

Regression equation of X on Y

$$(X-\bar{X}) = r \, \frac{\sigma_x}{\sigma_y} \, (Y - \bar{Y})$$

$\bar{X}$ = 72/6 = 12

$$\overline{Y} = 240/6 = 40$$

$$r \frac{\sigma_x}{\sigma_y} = \frac{xy}{y} = \frac{-6}{50} = 0.12-$$

$$X - 12 = -0.12(Y - 40)$$

$$X = -0.12Y + 16.8$$

Regression equation of Y on X

$$(Y-\overline{Y}) = r \frac{\sigma_y}{\Sigma_x} (X - \overline{X})$$

$$\overline{X} = 72/6 = 12$$

$$\overline{Y} = 240/6 = 40$$

$$r \frac{\sigma_Y}{\sigma_x} = \frac{xy}{y} = \frac{-6}{24} = -0.25$$

$$Y - 40 = -0.25(X - 12)$$

$$Y = -0.25X + 43$$

When X = 30, Y will be

$$Y = -0.25X + 43$$
$$Y = -0.25(30) + 43$$
$$= \mathbf{50.5}$$

When X = 30 , the likely value of Y is 50.5

21.4 Deviation taken from assumed mean of X and Y

When the actual mean is a fractional value, this method will be applied to reduce the calculations at larger extend.

Regression Equation X on Y is defined by

$$(X-\bar{X}) = r \frac{\sigma_x}{\sigma_y} (Y - \bar{Y})$$

Regression Equation Y on X is defined by

$$(Y-\bar{Y}) = r \frac{\sigma_y}{\sigma_x} (X - \bar{X})$$

$$\text{Where} \quad r \frac{\sigma_x}{\sigma_y} = b_{xy} = \frac{\Sigma dxdy - (\Sigma dx \times \Sigma dy)/N}{\Sigma dy^2 - (\Sigma dy)^2/N}$$

$dx = X - A$

$dy = Y - B$

where A and B are assumed mean of the respective series.

$\bar{X}$ = *mean of X .*

$\bar{Y}$ = *Mean of Y.*

σ_x = *Standard Deviation of X.*

σ_y = *Standard Deviation of Y.*

Illustrations :

Price indices of rice and wheat are given below for the 12 months of a year. Obtain the equations of the lines of regression between the indices.

Price index of rice (X)	77	76	84	87	86	81	80	76	72	82
Price index of wheat (Y)	70	75	85	87	83	81	79	82	84	87

Solution :

Price index of rice (X)	*dx= X - 80*	*dx^2*	*Price index of wheat(Y)*	*dy= Y - 80*	*dy^2*	*dxdy*
77	-3	9	70	-10	100	30
76	-4	16	75	-5	25	20
84	4	16	85	5	25	20
87	7	49	87	7	49	79
86	6	36	83	3	9	18
81	1	1	81	1	1	1
80	0	0	79	-1	1	0
76	-4	16	82	2	4	-8
72	-8	64	84	4	16	-32
82	2	4	87	7	49	14
801	dx=1	dx^2=211	813	13	dy^2=279	dxdy=142

Regression Equation X on Y is defined by

$$(X-\bar{X}) = bxy\ (\bar{Y} - Y)$$

Where

$$b_{xy} = \frac{dxdy \mid (dx \;\; X \;\; dy)/N}{dy^2 \mid (dy)^2/N}$$

$$b_{xy} = \frac{142 \mid (1 \;\; X\ 13)/\ 10}{279 \mid (13)^2/10}$$

$$b_{xy} = \frac{142 \mid 1.3}{279 \mid (169/10)}$$

$$b_{xy} = \frac{140.7}{279 \mid (16.9)}$$

$$= 0.536$$

X- 80.1 = 0.536(Y – 81.3)

X = 0.536 Y - 43.5768 + 80.1

X = 0.536 Y + 36.5232

Regression Equation Y on X is defined by

$(Y-\bar{Y}) = byx\ (\bar{X} - X)$

Where *dxdy* | (*dx* X *dy*) /*N*
byx =| | | | | | | | | | | | | | | | | | | |
dx^2 | $(dx)^2 / N$

142 | (1 X 13)/ 10
b_{yx} =| | | | | | | | | | | | | | | | | | | |
211 | $(1)^2 / 10$

142 | 1.3
b_{yx} =| | | | | | | | | | | | | | | | | | | |
211 | (1 / 10)

140.7
b_{tx} =| | | | | | | | | | | | | | | | | | |
211 | (0.1)

= *0.667*

Y- 81.3 = 0.667(X – 80.1)

Y = 0.667 X - 53.4267 + 81.3

Y = 0.667 X + 27.8733.

21.5 Correlation And Regression

Correlation	Regression
Correlation is the relationship between two or more variable.	Regression is a mathematical measure showing the relationship between two variables.
Both the variables X and Y are random variables	X is a random variable and y is a fixed variable.
It finds out the degree of relationship between two variables and not the cause and effect of the variable.	It indicates the cause and effect relationship between variables.
It is used for testing and verifying the relation between two variables and gives limited information	It is used for prediction of one variable, in the relationship to the other variable
The coefficient of correlation is relative measure. The range lies between -1 and +1	Regression coefficient is an absolute figure
There may be nonsense correlation between two variables	In regression there is no such nonsense regression.
It has limited mathematical application, because it is confined only to linear relationship between the variables.	It has wider application, as it studies linear and non-linear relationship between the variables.
It is not very useful for further mathematical treatment	It is widely used for further mathematical treatment.
If the coefficient of correlation is positive, then the two variables are positively correlated.	The regression coefficient explains that the decrease in one variable is associated with the increase in the other variable.
It is immaterial whether X depends upon Y or Y depends on X	There is a functional relationship between the two variables so that we may identify between the independent and dependent variables

21.6 Lesson end activities:

1.Calculate the co-efficient of correlation and obtain the lines of regression for the following:

X	1	2	3	4	5	6	7	8	9
Y	9	8	10	12	11	13	14	16	15

Obtain an estimate of Y which should correspond to the average X=6.2

2.A panel of two judges P and Q graded dramatic performance by independently awarding marks as follows:

Performance	1	2	3	4	5	6	7
Marks by P	46	42	44	40	43	41	45
Marks by Q	40	38	36	35	39	37	41

The eighth performance which judge Q could not attend, was awarded 37 marks by judge P. If the judge Q had also been present, how many marks could be expected to have by him to eighth performance.

3.The quantity of a raw material purchased by a Company at the specified prices during the 12 months of 1992 is given:

Month	Price Per kg(Rs)	Quantity kg.
Jan	96	250
Feb	110	200
Mar	100	250
Apr	90	280
May	86	300
June	92	300
July	112	220
Aug	112	220
Sep	108	200
Oct	116	210
Nov	86	300
Dec	92	250

a. Find the regression equations based on the above data.
b. Can you estimate the approximate quantity likely to be purchased if the price shoots up to Rs.124 per Kg.?
c. Hence or otherwise obtain the co-efficient of correlation between the price prevailing and the quantity demanded.

4.Given:

N=50

Mean of Y=44

Variance of X is 9/16 of the variance of Y

Regression Equation of X on Y =3Y-5Y = -180

Find : 1) the mean of X

2) Co-efficient of correlation between X and Y.

5.From the following regression equations find the mean values of X and Y series :

8X-10Y = -66

40X -18Y = 214

Find:

1) Average values of X and Y
2) Correlation Coefficient between the two variables
3) Standard Deviation of Y

6.In a correlation analysis, between production and price of a commodity, the following consonants were obtained:

	Production Index	Price Index
Arithmetic Mean	110	98
Standard Deviation	12	5
r between production and price	0.4	

Write down the regression equation of price on production and calculate the price index when the production index is 116.

7. You are given the following results for the heights(X) and weights (Y) of 1,000 workers of a factory.

X = 68" σ_x = 2.5" γ = 0.6

Y = 150 lbs σ_y = 20 lbs

Estimate from the above data:

i) The weight of a particular factory worker who is 5' tall.

ii) The height of a particular factory worker whose weight is 200 lbs.

8. The following results were worked out from the scores in Statistics and Mathematics in a certain examination:

	Scores in Statistics (X)	Scores in Maths (Y)
Mean	39.5	47.5
Standard Deviations	10.8	17.8

Karl Pearson's correlation coefficient between X and Y = -0.42. Find both the regression lines. Use these regression equation and estimate the value of Y for X = 50 and also estimate the value of X for Y =30.

9. Given the following data, estimate the marks in Mathematics obtained by a student who has scored 60 marks in English.

Mean of Marks in Mathematics	80
Mean of Marks in English	50
SD of Marks in Mathematics	15
SD of Marks in English	10
Coefficient of correlation	0.4.

10. Calculate the coefficient of correlation and obtain the lines of regression for the following data:

X	1	2	3	4	5	6	7
Y	9	8	10	12	11	13	14

Obtain an estimate of y which should correspond to the average X=6.2.

11. From the following data of the rainfall and production of rice, find the most likely production corresponding to the rainfall of 40".

	Rainfall(inches)	Production(Quintals)
Mean	35	50
S.D.	5	8

Co-efficient of correlation = +0.8
(Ans : Y = 56.4)

12. From 10 observations on price (X) and supply(Y) of a commodity, the following summary figures were obtained (in a appropriate units).

Y = 130, y = 220, X^2= 2288, XY = 3467

Compute the line of regression of Y on X and interpret the result. Estimate the supply when a price is 16 units.
(Ans : 25.06)

13. The following table gives the various values of two variables.

X	42	44	58	55	89	98	66
Y	56	49	53	58	65	76	58

Determine the regression equations which may be associated with these values and calculate Karl Pearson's coefficient of correlation.
(Ans: r= 0.9)

Points for Discussions

In this lesson we have dealt with the following:

- Regression equations
- Estimation of a variable by using Regression equations.
- Difference between regression and correlation.

Model Answer for lesson end activities

a. Y = 13.14
b. Q = 33.5
3. X= -.26Y +164.57 Y = -3.244X + 572.73 Y= 170.474 r= 0.92
4. r = 0.8 Mean of X = 62.4
5. 1. 13,17 2. r =0.6 3. SD of Y 4
6. 97
7. Y = 111.6 X = 71.15
8. X = 35.0387 Y = 54.766
9. X = 86
10. Y = 13.044 X = .929 Y – 6.219

21.7 Let us Sum Up

It is needless to say that we have learnt regression equitation, by following method of least square, deviation from mean and deviation from assumed mean and also learnt the difference between regression and correlation.

21.8 Reference:

Statistics –R.S.N Pillai, V.Bgavathi,, S.Chand &Company Ltd., Revised Edition and Reprint 2005 .