Chapter 8

Transfer Functions and Z Transforms

8.1 Introduction

Thus far we have concerned ourselves only with continuous time systems and signals. In the case of signals, by "continuous time" we really implied "continuous time, continuous amplitude." Our signal was somehow defined or characterized over a continuous range of time, and it could take on any value, say y(t), within some continuous range of amplitudes.

In this chapter we will present a review of discrete time signals and systems. If we think in terms of a discrete time signal, this could mean that we are defining some signal x(t) only at discrete instants (usually uniformly spaced in time), say at t=0, t=T, t=2T, etc. The data could result from sampling some continuous time signal (Figure 8.1), where T represents the spacing or period between samples, and we would then form an output sample sequence $\{x(n)\} = \{x(0), x(T), x(2T),...\}$. We will discuss this time domain sampling which is an important link between continuous and discrete time. On the other hand, some data may already be described in discrete time, such as the closing daily price quotes for your favorite stock for each day of the calendar year. Depending on the number of trading days in the year, the closings might form a sample sequence of $\{x(0), x(1),...,x(M)\}$ where x(0) could represent the closing price for 2 January and x(M) might represent the closing price for 31 December of that same year.

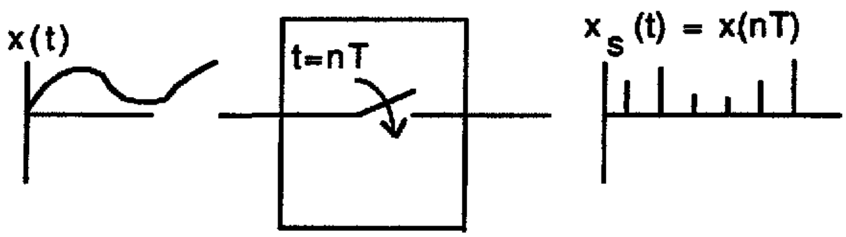


Figure 8.1. Ideal sampling of a continuous-time signal.

With respect to systems, we will learn in this chapter that as in continuous time, our linear, time invariant (LTI) systems in discrete time can be characterized by a system impulse response, step response, frequency response, or transfer function. In continuous time, recall that we could also choose to describe our LTI system in terms of a linear *differential equation* with constant coefficients. In discrete time we will characterize our systems instead with linear, constant coefficient *difference equations*.

Finally, we end the chapter with an introduction to discrete/continuous time equivalence. We will learn that in some cases we might choose to model a system, *design* a controller or filter, or simulate some system on a digital computer. For these cases it will be useful to consider an "equivalent" discrete time system for some continuous time system, or some "equivalent" continuous time system for an existing discrete time system.

8.2 Sampling

Instead of representing the sampling operation as shown in Figure 8.1, consider an equivalent representation in Figure 8.2. In this case we are multiplying the continuous time signal x(t) by a "gating" waveform g(t) to produce the output sequence $x_S(t)$.

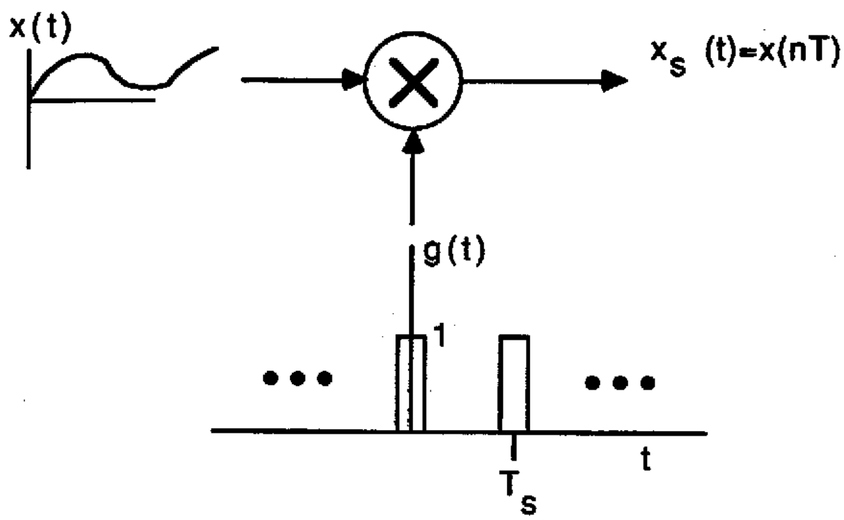


Figure 8.2. "Sampling gate" model.

Of concern to us is how this sampling operation in the time domain affects the frequency domain characteristics. More specifically, if $X(\omega)$ represents the Fourier transform of x(t) and $X_s(\omega)$ represents the Fourier Transform of $x_s(t)$, how does $X_s(\omega)$ compare to $X(\omega)$?

To answer this question we note that

$$x_s(t) = x(t)g(t) \Leftrightarrow (1/2\pi)\, X(\omega)*G(\omega) = X_s(\omega) \tag{8.1}$$

where * denotes a convolution operation, and $\Leftrightarrow$ denotes a Fourier transform pair. To understand $G(\omega)$, we can first represent the gating waveform $g(t)$

(of infinite extent) as a convolution of a single gate of width τ ($g_i(t)$, shown in Figure 8.3) with an infinite impulse train.

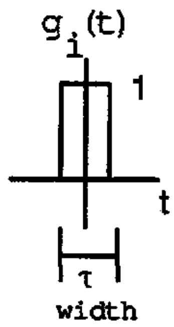


Figure 8.3. Single sample gate.

Therefore,

$$g(t) = g_i(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT) \ . \tag{8.2}$$

If we take the Fourier transform of both sides of 8.2,

$$G(\omega) = (2/\omega)\sin(\omega\tau/2) \cdot (2\pi/T) \sum_{n=-\infty}^{\infty} \delta[\omega-(2\pi n/T)] \tag{8.3}$$

$$= (2/\omega)\sin(\omega\tau/2) \cdot (\omega_S) \sum_{n=-\infty}^{\infty} \delta[\omega-n\omega_S] \ , \tag{8.4}$$

where ω_S is the sampling frequency measured in radians/sec. Returning to our expression for $X_S(\omega)$ in 8.1,

$$X_S(\omega) = (1/2\pi)X(\omega)*G(\omega)$$

$$= \frac{X(\omega)}{2\pi} * \left((2/\omega)\sin(\omega\tau/2) \cdot \omega_S \sum_{n=-\infty}^{\infty} \delta[\omega-n\omega_S] \right) \tag{8.5}$$

$$= \frac{X(\omega)}{T} * \left((2/\omega)\sin(\omega\tau/2) \cdot \sum_{n=-\infty}^{\infty} \delta[\omega - n\omega_s] \right). \tag{8.6}$$

From the sifting property of the delta function we know that $f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)$, so 8.6 simplifies to

$$X_s(\omega) = \frac{X(\omega)}{T} * \left(\frac{2\sin(n\omega_s\tau/2)}{n\omega_s} \cdot \sum_{n=-\infty}^{\infty} \delta[\omega - n\omega_s] \right) \tag{8.7}$$

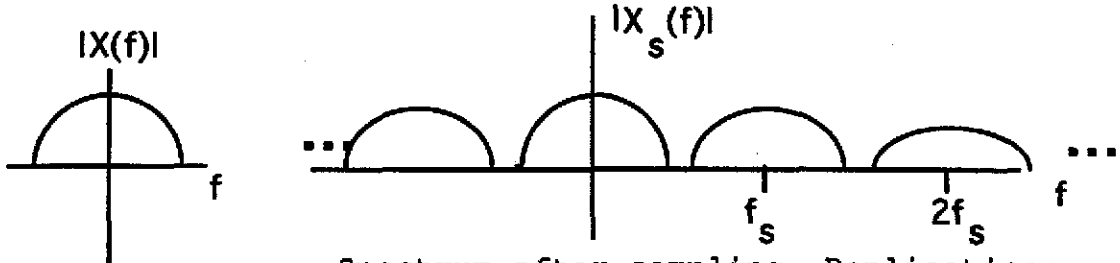
$$= \frac{2}{T} \sum_{n=-\infty}^{\infty} \left(\frac{\sin(n\omega_s\tau/2)}{n\omega_s} X(\omega - n\omega_s) \right) \tag{8.8}$$

$$= \frac{\tau}{T} \sum_{n=-\infty}^{\infty} \left(\frac{\sin(\pi f_s n\tau)}{\pi f_s n\tau} X(f - nf_s) \right) \tag{8.9}$$

$$= \frac{\tau}{T} \sum_{n=-\infty}^{\infty} \left(\mathrm{sinc}(nf_s\tau)\, X(f - nf_s) \right), \tag{8.10}$$

where $\mathrm{sinc}(x) = (\sin\pi x)/(\pi x)$.

As a matter of interest we should recognize that the coefficients multiplying $X(f\text{-}nf_s)$ in (8.10), namely $G_n = (\tau/T)\mathrm{sinc}(nf_s\tau)$ for $n=0, \pm 1, \pm 2$, etc., represent the complex exponential Fourier Series expansion for the gating or sampling function g(t). More importantly, when we sampled in the time domain with this gating function we replicated the original spectrum X(f) in the frequency domain (Figure 8.4). This is such an important result that it deserves a restatement! *Sampling in the time domain at a uniform sampling rate has the effect in the frequency domain of replicating the original spectrum at integer multiples of the sampling rate.*



Original spectrum before sampling

Spectrum after sampling- Replications occur at multiples of f_s and are scaled by the Fourier Series coefficients of the gating function.

Figure 8.4. Original spectrum and spectrum after sampling.

Sampling Theorem--How Fast Should We Sample?

Suppose we want to sample some continuous time signal, perform a filtering operation (or any linear operation) on the discrete time samples, and then return to an analog signal. (Recall this is exactly what takes place in CD technology! First an audio waveform is sampled in a recording studio, and recorded to CD. The consumer plays the disk on a CD player, which takes these samples and reconstructs the original analog audio waveform from the sample sequence.) The question is, how fast must we sample?

The best way to visualize the sampling theorem is from Figure 8.5, which shows an original lowpass spectrum, followed by the spectrum depicted after the sampling process. As already derived, inherent in the time domain sampling process is this spectral replication at integer multiples of the sampling frequency. To recover the desired "baseband image" (i.e. the spectrum from $-f_h$ to f_h), we must lowpass filter to remove all the unwanted replications. Assuming an ideal filter as shown in Figure 8.6, the output of the filter will be the recovered spectrum also shown in Figure 8.6.

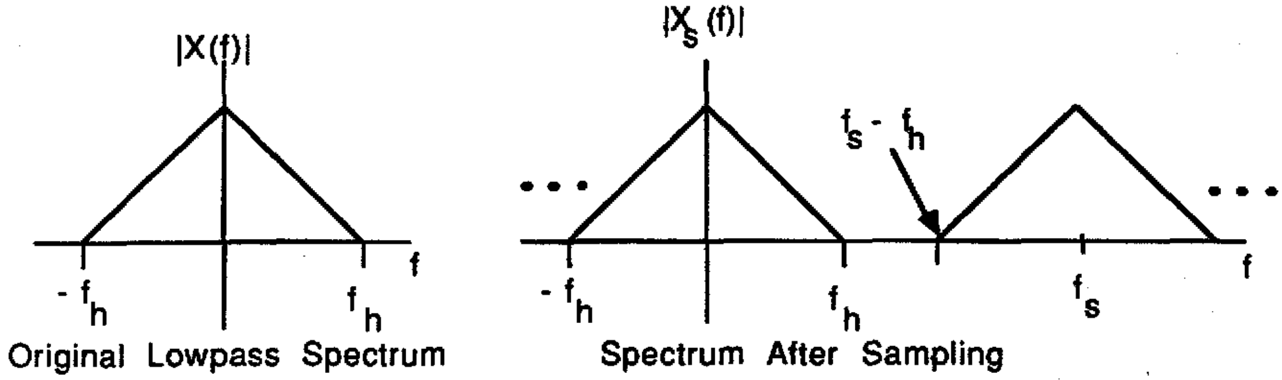


Figure 8.5. Illustration of Sampling Theorem.

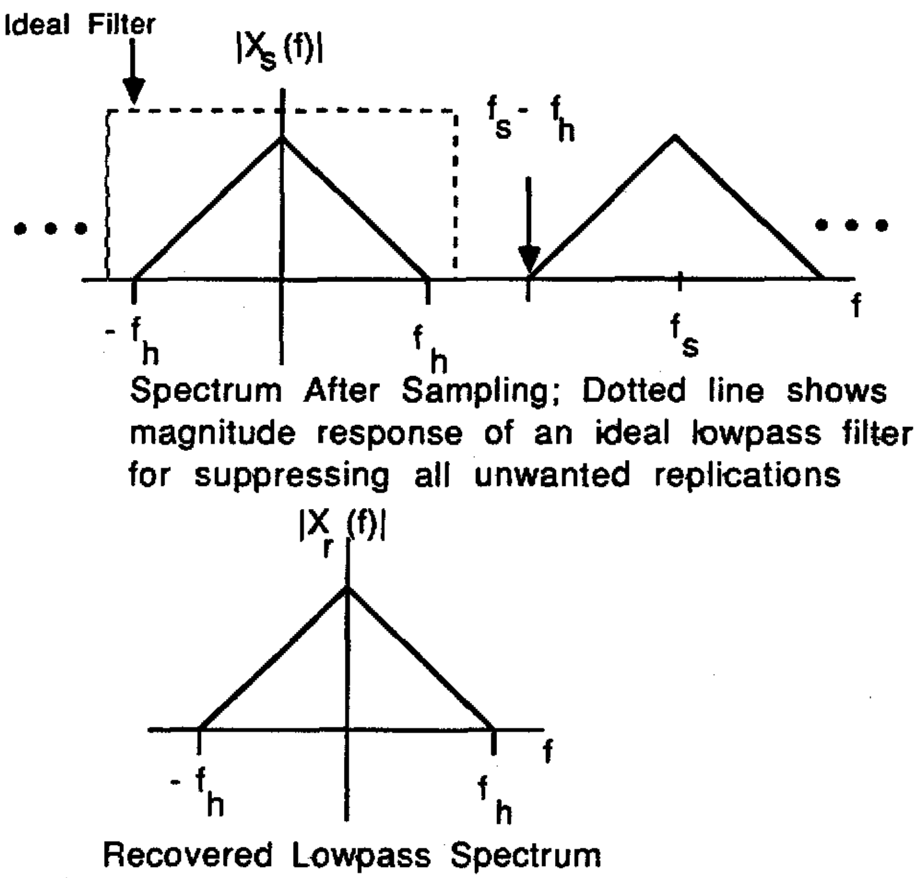


Figure 8.6. Recovery of Baseband Spectrum $X_r(f)$.

Unfortunately, we have already made a big assumption by drawing the sprectral replications due to the sampling process as being well separated.

That is, in order to guarantee that the baseband spectrum and the first replication do not overlap, refer to Figure 8.5 or 8.6 to verify that we must sample at a rate f_s sufficiently high to guarantee that

$$f_s - f_h \geq f_h, \tag{8.11}$$

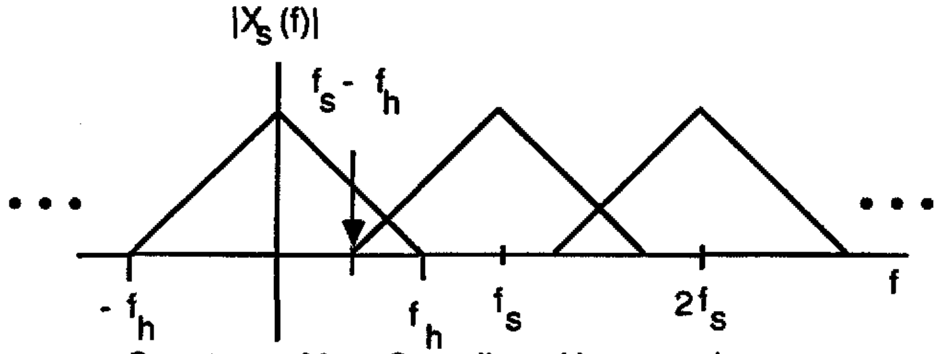
or equivalently

$$\boxed{f_s \geq 2f_h}\,. \tag{8.12}$$

Equation 8.12 is our desired result, derived more by intuition than by mathematical rigor. The interpretation is that if we start with an analog lowpass process with a highest significant frequency of f_h Hz, we must sample at a rate of $2f_h$ samples/second in order to recover the desired spectrum from all of the spectral replications.

If we ever have a situation such that $f_s < 2f_h$, we will incur *aliasing* . Aliasing, which is a distortion caused by an insufficient sampling rate, can be described both in terms of sampling theory and in terms of common intuition. In terms of sampling theory, observe in Figure 8.7 the case where we have sampled with $f_s < 2f_h$. Note that no matter how we lowpass filter the spectrum after sampling, we will always have high frequency distortion in the region of $f_s/2$. Moreover, any frequency components in the original signal at frequency $f_s/2 + \Delta$ will be "aliased" into the baseband image at frequency $f_s/2 - \Delta$. As a result, this frequency is often called the "folding frequency" since components above $f_s/2$ are "folded" back into the range 0-$f_s/2$.

A more intuitive example comes from using a strobe light, a bicycle wheel, and a piece of chalk. Suppose you draw a dot on the side of a bicycle tire, and then spin the tire at N revolutions per second. In order to see the tire turn in the proper direction and at the proper rate, you must set your strobe light to flash at a speed greater than 2N flashes per second. For example, at a rate of 4N flashes per second, you might see the dot at a sequence of positions of 12o'clock, 3o'clock, 6o'clock, 9o'clock, etc. If instead you set your sampling rate to N flashes per second (i.e. severely undersampled), the tire would not appear to move at all! Due to the sampling rate of N flashes/sec, that would imply that any frequency at $(N/2 + \Delta)$ rev/sec would appear at $(N/2-\Delta)$ rev/sec, and a frequency of N rev/sec would appear at 0 rev/sec, or would appear stationary!



Spectrum After Sampling; Here we have an insufficient sampling rate, and aliasing occurs about the region fs/2.

Figure 8.7. Illustration of insufficient sampling rate.

Example 8.1

Given the signal $x(t) = 5 + 2\cos(2200\pi t + \pi/6) + 4\sin(3600\pi t - \pi/4)$, calculate the theoretical minimum sampling rate that will allow us to perfectly reconstruct $x(t)$ from its sample sequence $\{x(n)\}$.

Solution:

$x(t)$ has one component at DC, one component at 1100Hz, and one component at 1800Hz. The highest significant frequency in $x(t)$ is 1800Hz. The sampling theorem tells us that in order to reconstruct a waveform from its sample sequence, we should sample at a rate

$f_s \geq 2f_h$, or

$f_s \geq 3600\text{Hz}$.

The Generic Sampled-Data System

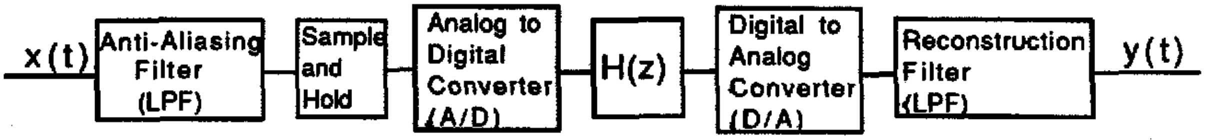


Figure 8.8. Generic system for converting from continuous-time to discrete time, then back to continuous time.

Figure 8.8 depicts a "generic" sampled-data system. By "generic," we mean that it could really represent portions of most any sampled data system of interest. For instance, the first four blocks might represent the conversion of an audio signal to data suitable for recording to CD. Similarly, the final three blocks might represent the reading of data from CD and conversion to a continuous time audio signal by a CD player. If we understand the function of each of these generic building blocks, it will help us to understand more complex, specific systems of interest.

Anti-Aliasing Filter - This is usually an analog lowpass filter. Its primary function is to help guarantee that the analog signal that enters the sampling process is bandlimited to some highest significant frequency f_h. As its name suggests, it is used to help prevent aliasing resulting from the sampling process.

Sample and Hold (S/H) - Again, as its name suggests, its purpose is to sample an analog waveform at a specific instant and maintain that voltage long enough for the A/D converter in the next block to "process" the sample. At the output of the S/H we have a "discrete time" signal, meaning that we have exactly represented its voltage value, but we have defined it only at specific instants in time (t=nT). This process is precisely controlled by a system clock.

Analog to Digital Converter (A/D) - Assume we have a K bit A/D converter. Its purpose is to convert the voltage value taken from the S/H to a K bit digital word. It performs two functions in order to accomplish this. First, the A/D converter *quantizes* each sample to one of 2^K possible quantization levels. Second, it *encodes* each sample as a K bit digital word for future representation of that sample. The output data sequence from the A/D converter is a *digital* data sequence.

H(z) - This represents any digital system. It could be a digital controller or a filter, or any process that may be described in terms of a transfer function.

Digital to Analog Converter (D/A)- This device accepts a K bit digital word from our system H(z) and creates an output voltage proportional to the value of that K bit digital word. The analog output from the D/A often looks like a "staircase" type waveform as shown in Figure 8.9.

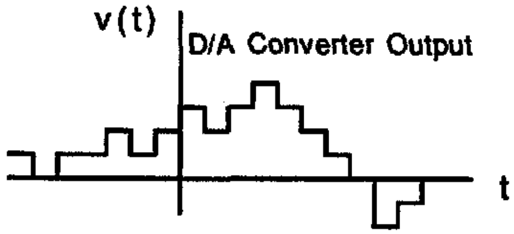


Figure 8.9. Example of D/A converter "staircase" output.

Reconstruction Filter - This is usually an analog lowpass filter that is very similar to the anti-aliasing LPF discussed above. As viewed from the time domain, this filter's purpose is to smooth the crude analog approximation obtained from the D/A converter. As viewed from the frequency domain, this filter's purpose is to pass only the baseband spectral image, thereby eliminating all remaining spectral replications resulting from the sampling process. This filter's passband should extend no further than that of the anti-aliasing filter.

8.3 Difference Equations, Impulse Response, Step Response, and Discrete Time Convolution

A discrete time, linear, time invariant system can be characterized by its difference equation, its system transfer function, its system impulse or step response, or its system frequency response.

Difference Equations

A difference equation, just like a continuous-time differential equation, describes the current system output in terms of present and past values of the system input, and in terms of past values of the system output. Difference equations describe any discrete time process! This month's savings account balance (current system output) is a function of last month's balance (multiple of past output depending on interest rate), and a function of any deposits or withdrawals this month. If we designate the system input and output at time $t=nT$ as $x(n)$ and $y(n)$ respectively, then a system difference equation describing a single-input single-output system might look like

$$y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M)$$

or

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k) \tag{8.13}$$

Equation 8.13 is an example of an Nth order difference equation, since the present value of the output is a function of the N previous values of the output. As a result, to solve an Nth order difference equation completely, we must be given these N initial conditions for y(n-1), y(n-2),...,y(n-N). Equation 8.13 also represents a *recursive* system. By recursive, we mean that the current value of the output depends on past values of the output. Of course a non-recursive system is one where the current output value is not a function of past output values.

Finally, we should recognize that equation 8.13 characterizes a *causal* system as well. By causal, we mean that the system output y(n) does not depend on future values of the input or output, or it is non-anticipatory. That is, y(n) is not a function of x(n+k) or y(n+k), k>0.

Example 8.2

Given a system that is described by

$$y(n) = 2\ x(n) + 3\ x(n-1) - 0.9\ y(n-1),$$

calculate the system output y(n) for n=0, 1, ... ,4 given that the system input is known to be x(n) = δ(n). (δ(n)=1 for n=0, and δ(n)=0 for $n \neq 0$.) Assume y(-1)=0.

<u>Solution</u>: Since x(n) = δ(n), the only system input occurs at n=0.

$$\begin{aligned} y(0) &= 2\ x(0) + 3\ x(-1) - 0.9\ y(-1) \\ &= (2)(1) + (3)(0) - (0.9)(0) \\ &= 2 \end{aligned}$$

Similarly,

$$\begin{aligned} y(1) &= (2)(0) + (3)(1) - (0.9)(2) \\ &= 1.2 \\ y(2) &= (2)(0) + (3)(0) - (0.9)(1.2) \\ &= -1.08 \\ y(3) &= -(0.9)(-1.08) \\ &= 0.972 \\ y(4) &= -(0.9)(.972) \\ &= -.8748 \end{aligned}$$

Example 8.3

Given a system that is described by

$$y(n) = x(n) + 0.9\ y(n-1),$$

calculate the system output y(n) for n=0, 1, ... ,4 given that the system input is known to be $x(n) = \delta(n)$. Assume $y(-1)=0$.

Solution: Since $x(n) = \delta(n)$, again the only system input occurs at n=0.

$$\begin{aligned} y(0) &= 1\ x(0) + 0.9\ y(-1) \\ &= 1 \end{aligned}$$

Similarly,

$$\begin{aligned} y(1) &= x(1) + 0.9\ y(0) \\ &= 0 + 0.9 \\ &= 0.9 \end{aligned}$$

$$\begin{aligned} y(2) &= x(2) + 0.9\ y(1) \\ &= 0 + 0.81 \\ &= 0.81 \end{aligned}$$

$$y(3) = 0.729$$

$$y(4) = 0.6561$$

In general, the output sequence is $y(n)=(0.9)^n\ u(n)$.

System Impulse Response and Step Response

Examples 8.2 and 8.3 are the perfect review for the concept of system impulse response. Recall that a linear, time invariant system can be characterized by a system impulse response. This "system impulse response" simply describes how a system reacts to a very simple input, namely $x(n) = \delta(n)$. To give you a continuous-time example, think about how you decide if an automobile suspension system needs to have its shock absorbers replaced. (Answer: You jump on and off the bumper a couple of times to see how your suspension system reacts to these "impulses.") If the car bounces up and down several times after you remove the "system input," it tells you that the system is not properly damped, and must have new shock absorbers! (A trip to the garage for new shocks will never be quite the same now!)

The systems described in Examples 8.2 and 8.3 are fully characterized by their impulse responses. That is, given initial conditions and a system input, one way to calculate the system output is by "turning the crank" on the difference equation as we did above. The impulse response is simply the output of the system with zero initial conditions and $x(n) = \delta(n)$. We will designate the impulse response as h(n). Assuming a *causal* or non-anticipatory system, we would certainly expect h(n) to be zero for n<0. (Your car did not bounce before you jumped on the bumper, right?)

Another way to characterize a system is by its system step response. Again, this is just the system's response to the special input x(n)=u(n). If we know the system impulse response h(n), the system step response follows easily. Recall that the impulse response is (by definition) the response due to an impulse input. The step input u(n) is really

$$x(n) = u(n) = 1 \text{ for } n \geq 0 \text{ and } u(n) = 0 \text{ for } n < 0$$

$$= \sum_{k=0}^{\infty} \delta(n-k) \quad . \tag{8.14}$$

By linearity and time invariance we can say the output due to this step input is

$$s(n) = \sum_{k=0}^{\infty} h(n-k) \quad , \tag{8.15}$$

and if we assume that our system is causal (i.e. h(n)=0 for n<0),

$$s(n) = \sum_{k=0}^{n} h(n-k) \quad . \tag{8.16}$$

Note that the discrete time step response is the summation of the impulse response. This is equivalent to the continuous time step response being the integral of the continuous time impulse response.

Therefore the step response can be easily derived from the impulse response as

s(0)=h(0),
s(1)=h(0)+h(1),
s(2)=h(0)+h(1)+h(2),
s(3)=h(0)+h(1)+h(2)+h(3),
etc.

Similarly, from the equations above, an impulse response h(n) can be derived from the step response as

h(0)=s(0),
h(1)=s(1)-s(0),
h(2)=s(2)-s(1),
etc.,

or in general,

$$h(n)=s(n)-s(n-1).$$

Again this is equivalent to the continuous time impulse response being the derivative of the continuous time step response.

Example 8.4

Given the same system described in Example 8.3, namely

$$y(n) = x(n) + 0.9\ y(n-1),$$

calculate the system output $y(n)$ for $n \geq 0$, given that the system input is known to be $x(n) = u(n)-u(n-4)$. Assume $y(-1)=0$.

Solution: We will solve this problem simply from our knowledge of linear time invariant systems. First, recall from Example 8.3 that this system impulse response is $h(n)=(0.9)^n\ u(n)$. Second, note that our system input $x(n)$ can be expressed as

$$\begin{aligned} x(n) &= u(n)-u(n-4) \\ &= \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3). \end{aligned}$$

Since we have a time invariant system, a unit delay of the input means a unit delay of the output. This means that if an input $x(n)=\delta(n)$ produces an output $y(n)=h(n)$, then an input of $x(n)= \delta(n-k)$ produces an output of $h(n-k)$. Therefore,

$$\begin{aligned} y(n) &= h(n) + h(n-1) + h(n-2) + h(n-3) \\ &= (0.9)^n\ u(n) + (0.9)^{n-1}\ u(n-1) + (0.9)^{n-2}\ u(n-2) + (0.9)^{n-3}\ u(n-3) \\ &= \delta(n)+1.9\delta(n-1)+2.71\delta(n-2)+\left[(0.9)^n+(0.9)^{n-1}+(0.9)^{n-2}+(0.9)^{n-3}\right]u(n-3) \\ &= \delta(n) +1.9\ \delta(n-1) + 2.71\ \delta(n-2) \\ &\qquad +\left[0.729(0.9)^{n-3}+ 0.81(0.9)^{n-3}+0.9(0.9)^{n-3}+(0.9)^{n-3}\right]\ u(n-3) \end{aligned}$$

and finally,

$$y(n) = \delta(n) +1.9\ \delta(n-1) + 2.71\ \delta(n-2) + \left[3.439(0.9)^{n-3}\right]\ u(n-3)$$

Example 8.5

Given the same system described in Example 8.3, namely

$$y(n) = x(n) + 0.9\ y(n-1),$$

calculate the system step response s(n). In addition, calculate once again the system output y(n) for input x(n) = u(n)-u(n-4) by noting that for an input x(n) = u(n)-u(n-4), y(n) can be calculated as y(n)=s(n)-s(n-4).
Assume y(-1)=0.

<u>Solution</u>: From Example 8.3 we know $h(n)=(0.9)^n\, u(n)$.

$$s(n) = \sum_{k=0}^{n} h(n-k)\ , \text{ so}$$

$$s(n) = \sum_{k=0}^{n} (0.9)^{n-k}\, u(n-k)$$

$$= (0.9)^n \sum_{k=0}^{n} \left(\frac{1}{0.9}\right)^k .$$

If we invoke a well-known formula for the sum of a geometric series as

$$\sum_{m=L}^{U} \rho^L = \frac{\left(\rho^L - \rho^{U+1}\right)}{1 - \rho},$$

we have

$$s(n) = (0.9)^n \frac{\left[1 - \left(\frac{1}{0.9}\right)^{n+1}\right]}{1 - \left(\frac{1}{0.9}\right)}$$

which says our step response is

$$s(n) = 10 \left[1 - (0.9)^{n+1}\right] u(n).$$

Now we calculate y(n) as y(n)=s(n)-s(n-4), so

$$y(n) = 10\left[1-(0.9)^{n+1}\right]u(n) - 10\left[1-(0.9)^{n-3}\right]u(n-4)$$

$$= 10\left[1-(0.9)^{n+1}\right]u(n) - 10\left[1-(0.9)^{n-3}\right]u(n-3)$$

where we note that we have not changed the second term by using u(n-3) instead of u(n-4). Therefore,

$$y(n) = \delta(n) + 1.9\,\delta(n-1) + 2.71\,\delta(n-2) + 10\left[1-0.9^{n+1} - 1 + 0.9^{n-3}\right]u(n-3)$$

$$= \delta(n) + 1.9\,\delta(n-1) + 2.71\,\delta(n-2) + 10\left[0.9^{n-3} - (0.9)^4\,0.9^{n-3}\right]u(n-3)$$

$$= \delta(n) + 1.9\,\delta(n-1) + 2.71\,\delta(n-2) + 10\left[1 - (0.9)^4\right](0.9)^{n-3}\;u(n-3)$$

$$= \delta(n) + 1.9\,\delta(n-1) + 2.71\,\delta(n-2) + 3.439\,(0.9)^{n-3}\;u(n-3)\;.$$

Note that y(n) calculated in this way gives us exactly the same answer as for Example 8.4.

Discrete Time Convolution

Up until this point, we have calculated the system output y(t) by patiently "turning the crank" on the difference equation that described the system. While this method is certainly valid, we introduce a more compact way of calculating the system output.

Assuming zero initial conditions, we know that the input $\delta(n)$ produces the output h(n). Furthermore, because we have a time invariant system, an input of $\delta(n-k)$ would produce the output h(n-k). By linearity, an input of $K\delta(n-k)$ would produce the output Kh(n-k). Therefore,

$$x(k)\delta(n-k) \rightarrow x(k)h(n-k)\;, \tag{8.17}$$

where $\rightarrow$ means "produces the output." Note that in general we can represent any input x(n) as

$$x(n) = \sum_{k=0}^{\infty} x(k)\,\delta(n-k), \tag{8.18}$$

so if we represent any causal input x(n) in this fashion, we have

$$\sum_{k=0}^{\infty} x(k)\,\delta(n-k) = x(n) \rightarrow y(n) = \sum_{k=0}^{\infty} x(k)\,h(n-k) \quad . \tag{8.19}$$

That is, given zero initial conditions, a system impulse response h(n), and a system input x(n), the output y(n) may be calculated as the sum

$$\boxed{y(n) = \sum_{k=0}^{\infty} x(k)\,h(n-k)} \,. \tag{8.20}$$

Equation 8.20 is called a convolution sum. A shorthand notation for 8.20 is

$$y(n) = x(n) * h(n) \ . \tag{8.21}$$

By a simple change of variables, it is straightforward to show that 8.20 may also be written as

$$\boxed{y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)} \,. \tag{8.22}$$

Example 8.6

Given the system input x(n) and system impulse response h(n) as shown below in Figure 8.10, calculate the system output, y(n).

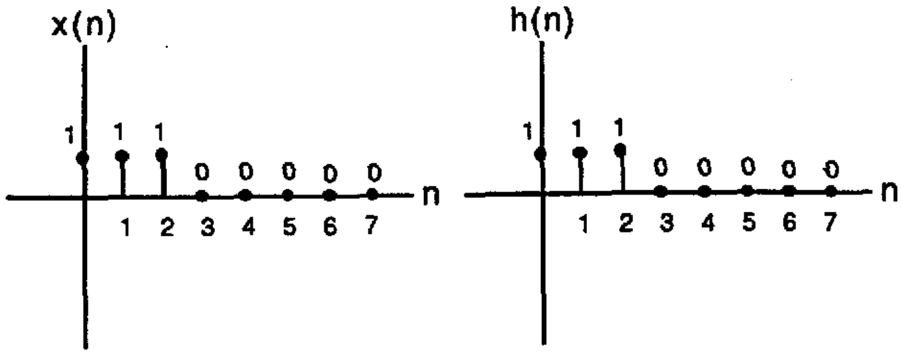


Figure 8.10. System input and impulse response for Example 8.6.

<u>Solution</u>:

The convolution sum in 8.20 says we should "flip" the h(k) sequence about the y axis to form h(-k), shift it n time units, multiply by the x(k) sequence, and add the result for each possible shift n.

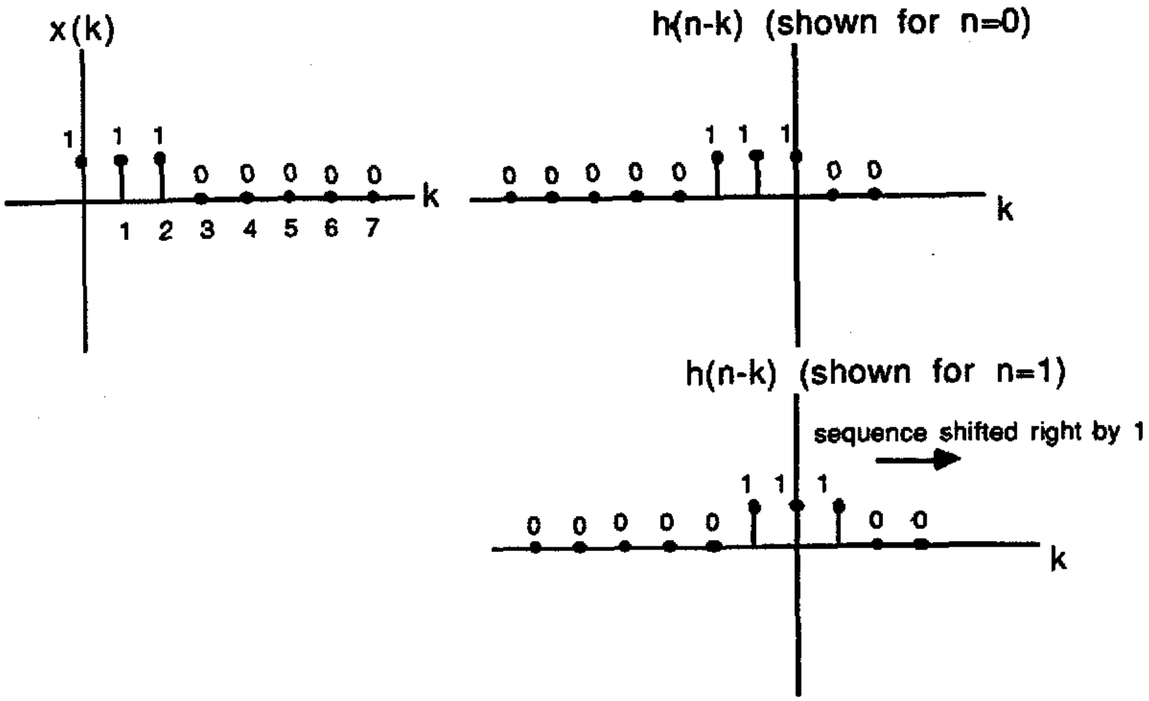


Figure 8.11. Illustration of convolution for Example 8.6.

Clearly h(n-k) for k negative or for k>4 will provide for no overlap, so the product of x(k) with h(n-k) would be zero, and resulting sum would be zero. Therefore, y(n)=0 for n<0, and y(n)=0 for n>4. For n=0, there is an overlap of one sample at the origin, hence

$$y(0) = \sum_{k=0}^{0} x(k)\,h(0-k) = x(0)\ h(0)$$
$$= 1.$$

For n=1, there is an overlap of two samples, hence

$$y(1) = \sum_{k=0}^{1} x(k)\,h(1-k) = x(0)\ h(1) + x(1)\ h(0)$$
$$= 2.$$

In a similar way we perform calculations for n=2, n=3, n=4, etc. to obtain the graph of the final output in Figure 8.12.

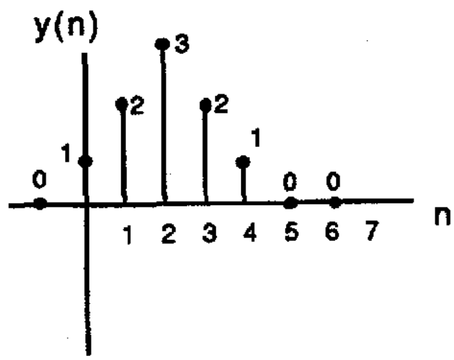


Figure 8.12. Final output y(n) for Example 8.6.

Another View of Convolution

The summation in 8.20, namely

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k) \quad , \tag{8.23}$$

may also be written in matrix form. Convince yourself that for a causal system as described by 8.23 we may also write the system output

$$y = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & 0 & 0 & \cdots \\ h(1) & h(0) & 0 & 0 & 0 & \cdots \\ h(2) & h(1) & h(0) & 0 & 0 & \cdots \\ h(3) & h(2) & h(1) & h(0) & 0 & \cdots \\ h(4) & h(3) & h(2) & h(1) & h(0) & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Example 8.7

Given the system and input from Example 8.6, calculate the system output y(n) using matrix multiplication methods.

<u>Solution</u>:

$$\mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & 0 & 0 & \cdots \\ h(1) & h(0) & 0 & 0 & 0 & \cdots \\ h(2) & h(1) & h(0) & 0 & 0 & \cdots \\ h(3) & h(2) & h(1) & h(0) & 0 & \cdots \\ h(4) & h(3) & h(2) & h(1) & h(0) & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \\ 0 \\ 0 \\ \cdot \end{bmatrix} .$$

System Stability Requirements on the Impulse Response

Without assuming anything about the causality of $x(n)$ or $h(n)$, we can calculate the system output $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad . \tag{8.24}$$

If we restrict $x(n)$ to be a bounded input (i.e. $|x(n)|<M$ where M is some finite upper bound), then we can establish necessary and sufficient conditions on the impulse response for bounded input/bounded output (BIBO) stability. More specifically,

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \right| < \sum_{k=-\infty}^{\infty} M\, |h(n-k)| = M \sum_{k=-\infty}^{\infty} |h(n-k)| \tag{8.25}$$

and if $y(n)$ is to be finite for all n, then we shall insist that

$$|y(n)| < M \sum_{k=-\infty}^{\infty} |h(n-k)| < \infty \text{ for all } n,$$

or equivalently we insist that

$$\sum_{k=-\infty}^{\infty} |h(n)| < \infty \ . \tag{8.26}$$

A system is BIBO stable if and only if 8.26 is satified.

8.4 The Z Transform

For continuous time systems, our primary tool for systems analysis in the frequency domain was the Laplace Transform. The Z Transform will be our analysis tool for discrete time signals and systems.

Beginning in Section 8.2 we modeled the sampling process as the multiplication of the input x(t) by some gating function called g(t). That is,

$$x_s(t) = x(t)g(t), \tag{8.27}$$

where

$$g(t) = g_i(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT) \ , \quad \text{and}$$

(8.28)

$g_i(t)$ is defined by Figure 8.3. Furthermore, recall from Section 8.2 that the only effect that the "shape" or "width" of $g_i(t)$ had was to scale the amplitudes of the spectral replications of $X_s(f)$ in Equation 8.10 and in Figure 8.4 by the Fourier series coefficients associated with g(t). Had we instead chosen $g_i(t)=\delta(t)$, we would have obtained spectral replications at integer multiples f_s, however they would have all been at the same amplitude. Since we are less interested in the exact amplitudes of these spectral replications and more interested in their positions, let us simplify our model of the sampling process by saying

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) \ . \tag{8.29}$$

Let us assume x(t)=0 for t<0 (i.e. a "one-sided" signal). This implies

$$x_s(t) = x(t) \sum_{n=0}^{\infty} \delta(t-nT) \ . \qquad (8.30)$$

one-sided sampled signal

From the sifting property of delta functions this becomes

$$x_s(t) = \sum_{n=0}^{\infty} x(nT)\, \delta(t-nT) \ . \qquad (8.31)$$

By taking the Laplace transform of both sides of 8.31 and invoking the sifting property of the delta function once again, we obtain

$$\int_0^{\infty} x_s(t)\, e^{-st}\, dt = \int_0^{\infty} \sum_{n=0}^{\infty} x(nT)\, \delta(t-nT)\, e^{-snT}\, dt \ . \qquad (8.32)$$

By interchanging the order of summation and integration in 8.32 we get

$$\int_0^{\infty} x_s(t)\, e^{-st}\, dt = \sum_{n=0}^{\infty} x(nT)\, e^{-snT} \int_0^{\infty} \delta(t-nT)\ \ dt \ , \qquad (8.33)$$

or

$$X_S(s) = \sum_{n=0}^{\infty} x(nT)\, e^{-snT} \ . \qquad (8.34)$$

For convenience, let us define a new variable, $z = e^{sT}$, and let us refer to $x(nT)$ simply as $x(n)$. Substituting into 8.34 we have

$$\boxed{X_S(z) = \sum_{n=0}^{\infty} x(n)\ z^{-n}} \ . \qquad (8.35)$$

Equation 8.35 is the definition of the one-sided z-transform. It is simply the Laplace transform of our sample sequence from 8.31, evaluated at $z = e^{sT}$.

This mapping $z = e^{sT}$ deserves some additional comment. (In fact we devote an entire section to this mapping in Chapter 12, so we defer the major portion of that discussion until then.) Recall that "stable poles" in the s plane were those in the left half plane, where Re{s}=σ<0. These stable poles map to stable poles in the z plane, where

$$z = e^{sT} = e^{(\sigma+j\omega)T} = e^{\sigma T} e^{j\omega T} = \left|e^{\sigma T}\right| \angle\omega T, \tag{8.36}$$

and $\left|e^{\sigma T}\right| < 1$ since $\sigma<0$. Thus the entire left half of the s plane maps to the interior of the z plane unit circle. Likewise, the right half of the s plane maps to the exterior of the z plane unit circle. From 8.36 we observe that the s plane $j\omega$ axis (i.e. when $\sigma=0$) corresponds to the unit circle itself in the z plane. Recall from continuous time systems, we evaluate the frequency response as H(s) evaluated at $s=j\omega$ (i.e. along the $j\omega$ axis), and this is the definition of the Fourier Transform. It will probably come as no surprise in Section 8.5 when we define discrete time frequency response as H(z) evaluated at $z = e^{j\omega T}$, which is the z transform evaluated around the unit circle!

Example 8.8

Calculate the z-transform of the sequence $x(n) = 4\ \delta(n-2)$.

<u>Solution</u>:

$$X(z) = \sum_{n=0}^{\infty} x(n)\ z^{-n}\ , \text{ so}$$

$$\begin{aligned} X(z) &= 0 + 0 + 4\,z^{-2} + 0 + 0 + \dots \\ &= 4\,z^{-2},\ |z| > 0\,. \end{aligned}$$

Example 8.9

Calculate the z-transform of the sequence x(n) = u(n).

<u>Solution</u>:

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \text{ , so}$$

$$X(z) = 1 + z^{-1} + z^{-2} + \ldots$$

$$= \sum_{n=0}^{\infty} z^{-n} \, .$$

If we invoke a well-known formula for the sum of a geometric series as

$$\sum_{m=L}^{U} \rho^{L} = \frac{\left(\rho^{L} - \rho^{U+1}\right)}{1 - \rho},$$

we have (with $\rho = z^{-1}$)

$$X(z) = \frac{1}{1 - z^{-1}} \qquad \text{provided} \qquad |z| > 1 \, .$$

Table 8.1. Common z transform pairs

Sequence	Z Transform	Region of Convergence
$\delta(n)$	1	all z
$\delta(n-m)$, $m>0$	z^{-m}	$\lvert z\rvert > 0$
$\delta(n+m)$, $m>0$	z^{m}	$\lvert z\rvert < I$
$u(n)$	$\dfrac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
$a^{n}\ u(n)$	$\dfrac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$n\,a^{n}\ u(n)$	$\dfrac{az^{-1}}{\left(1 - az^{-1}\right)^{2}}$	$\lvert z\rvert > \lvert a\rvert$
$r^{n}\ \cos(n\omega_0 T)\ u(n)$	$\dfrac{1 - r\cos(\omega_0 T)\,z^{-1}}{1 - 2r\left(\cos(\omega_0 T)\right)z^{-1} + r^{2}z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$r^{n}\ \sin(n\omega_0 T)\ u(n)$	$\dfrac{r\sin(\omega_0 T)\,z^{-1}}{1 - 2r\left(\cos(\omega_0 T)\right)z^{-1} + r^{2}z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$

Table 8.2. Common z transform properties

Sequence	Z Transform
1) $x(n)$	$X(z)$
2) $Ax(n) + By(n)$	$AX(z) + BY(z)$
3) $x(n-k)$	$z^{-k}\ X(z)$
4) $x(n)*y(n)$	$X(z)\ Y(z)$
5) $y(n)=x^*(n)$	$Y(z) = X^*(z^*)$
6) $x(-n)$	$X(1/z)$
7) $a^n x(n)$	$X(z/a)$
8) $nx(n)$	$-z\dfrac{dX(z)}{dz}$

$$\lim_{z\to\infty} X(z) = x(0)$$ (Initial Value Theorem)

$$\lim_{z\to 1} (1 - z^{-1})\, X(z) = x(\infty)$$ (Final Value Theorem)

In section 8.3 we learned that the system output $y(n)$ for zero initial conditions, a given impulse response $h(n)$, and some input $x(n)$ (assume $x(n)=0$ for $n<0$) may be calculated as

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k) \quad . \tag{8.37}$$

If we take the z transform of both sides of this equation, we see from 8.37 that

$$Y(z) = \sum_{n=0}^{\infty} y(n)\, z^{-n} \tag{8.38}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} x(k)\, h(n-k) \quad z^{-n} \tag{8.39}$$

$$= \sum_{k=0}^{\infty} x(k) \sum_{n=0}^{\infty} h(n-k)\, z^{-n} \quad , \tag{8.40}$$

where we have shifted the order of summation, and pulled $x(k)$ outside the second summation because it is not a function of the summing index n. By change of variables $(m=n-k)$ equation in 8.40 becomes

$$Y(z) \quad = \sum_{k=0}^{\infty} x(k) \sum_{m=-k}^{\infty} h(m)\, z^{-m-k} \tag{8.41}$$

Since $h(m)=0$ for $m<0$, and since $k\geq 0$, we can sum from $m=0$ instead of $m=-k$ in the second summation. Equation 8.41 then simplifies to

$$Y(z) = \sum_{k=0} x(k)\, z^{-k} \sum_{m=0}^{\infty} h(m)\, z^{-m} , \tag{8.42}$$

which is

$$\boxed{Y(z) \;\; = X(z)\; H(z)} \; . \tag{8.43}$$

Therefore, convolution in the time domain implies a multiplication of z-transforms in the frequency domain.

Example 8.10

Given the system input x(n) and system impulse response h(n) from Example 8.6 (and as repeated below for convenience) calculate the system output, y(n) by using properties of the z transform.

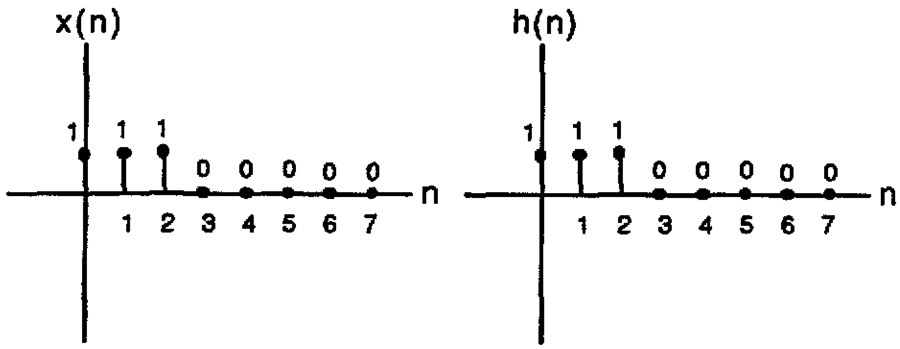


Figure 8.13. System input and impulse response for Example 8.10.

<u>Solution</u>: Since $y(n)=x(n)*h(n)$, $Y(z)=X(z)H(z)$. From the definition of the z transform (equation 8.35),

$$X(z) = 1 + z^{-1} + z^{-2} \quad |z|>0,$$

and

$$H(z) = 1 + z^{-1} + z^{-2} \quad |z|>0 .$$

This means

$$Y(z) = (1 + z^{-1} + z^{-2})(1 + z^{-1} + z^{-2}) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} \;, |z|>0 .$$

From the definition of the z transform, we can solve for y(n) by inspection since

$$Y(z) = \sum_{n=0}^{\infty} y(n)\, z^{-n} = y(0) + y(1)z^{-1} + y(2)z^{-2} + y(3)z^{-3} + \ldots,$$

or

$$y(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 2\delta(n-3) + \delta(n-4)\ .$$

Inverting the Z-Transform

There are several methods used to invert the z transform, however the more common methods are listed below:

1) By inspection-
2) Table look-up-
3) Long division-
4) Partial fraction expansion, then table look-up-
5) Contour integral evaluated by Cauchy Residue Theorem-

*********** TO BE DEVELOPED FURTHER***********

8.5 Transfer Functions and Frequency Response for Linear Time Invariant Systems

Transfer Functions

In Section 8.3 we discussed that a single input/single output, linear, time-invariant system could be described by the constant coefficient, linear difference equation

$$y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M). \quad (8.44)$$

By taking the z transform of both sides,

$$Y(z) + a_1 Y(z)z^{-1} + \ldots + a_N Y(z)z^{-N} = b_0 X(z) + b_1 X(z)z^{-1} + \ldots + b_M X(z)z^{-M}, \quad (8.45)$$

or

$$Y(z)\left[1 + a_1 z^{-1} + \ldots + a_N z^{-N}\right] = X(z)\left[b_0 z^{-1} + b_1 z^{-1} + \ldots + b_M z^{-M}\right]. \quad (8.46)$$

From 8.43 we know that the z transform of the impulse response, H(z), is given by $H(z)=\frac{Y(z)}{X(z)}$, so we solve for $\frac{Y(z)}{X(z)}$ as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 z^{-1} + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} . \tag{8.47}$$

At this point let us define H(z) as the system *transfer function.*

Just as we were able to judge the stability of a system by verifying that

$$\sum_{k=-\infty}^{\infty} |h(n)| < \infty , \tag{8.26}$$

we can judge the stability of a system by examining the transfer function H(z) itself. Consider a system described by

$$H(z) = \frac{K_0}{1 - p_0 z^{-1}} + \frac{K_1}{1 - p_1 z^{-1}} + \frac{K_2}{1 - p_2 z^{-1}} + \ldots + \frac{K_M}{1 - p_M z^{-1}} \tag{8.48}$$

where the pole locations are $\{p_0, p_1, \ldots, p_M\}$. (Note that $\{p_0, p_1, \ldots, p_M\}$ may in general be complex.) Assuming we have a causal system, h(n) is given as

$$h(n) = K_0 \left(p_0\right)^n u(n) + K_1 \left(p_1\right)^n u(n) + \ldots + K_M \left(p_M\right)^n u(n) . \tag{8.49}$$

(Do not be alarmed. Even though $\{p_0, p_1, \ldots, p_M\}$ may be complex, if we insist that any complex poles appear in conjugate pairs, we will still end up with a real h(n).) If we require a stable system in that 8.26 must be satisfied, then from 8.49 we must insist that

$$|p_i| < 1, \; i=0,1,\ldots,M . \tag{8.50}$$

This is a fundamental result that deserves restatement. *Assuming we have a causal system, this system is stable if and only if all poles of H(z) lie <u>inside</u> the z plane unit circle.*

Frequency Response

In Section 8.3 we learned that the system output y(n) could be computed for a given x(n), h(n), and zero initial conditions, by the convolution formula

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k) \quad . \tag{8.22}$$

We know that if we choose one very simple input, namely $x(n)=\delta(n)$, our output is $h(n)$, or the system impulse response. Another very simple input of interest is

$$x(n) = Ae^{jn\omega T} , \tag{8.51}$$

which we will call a simple, elemental sinusoid of amplitude A. (You may argue that a single sine or cosine should be a simple sinusoid, however $\sin(n\omega T) = \frac{1}{2j}\left(e^{jn\omega T} - e^{-jn\omega T}\right)$, which is really the sum of two elemental sinusoids as we have defined in 8.43!) With this simple test input as defined in 8.51,

$$y(n) = \sum_{k=0}^{\infty} A\ h(k)\ e^{j\omega(n-k)T} \tag{8.52}$$

$$= Ae^{j\omega nT}\sum_{k=0}^{\infty} h(k)\ e^{-j\omega kT} . \tag{8.53}$$

$$= Ae^{j\omega nT}\ H(z)\ \Big|_{z=e^{j\omega T}} \tag{8.54}$$

$$y(n) = Ae^{j\omega nT}\ H'(\omega) , \tag{8.55}$$

where we adopt the shorthand notation of

$$\boxed{H'(\omega) = H(z)\ \Big|_{z=e^{j\omega T}}} . \tag{8.56}$$

Equation 8.56 is the definition of frequency response. It is the transfer function $H(z)$ evaluated at $z=e^{j\omega T}$, or $H(z)$ evaluated around the unit circle.

A bit more interpretation of 8.55 and 8.56 is in order. Consider that we started with a linear system characterized by $h(n)$ or $H(z)$. The output $y(n)$ from 8.55 is just our original input $x(n) = Ae^{jn\omega T}$ as modified by the way our system responds to the input elemental sinusoid at frequency ω. This fundamental characteristic is only true of linear systems. This "modification" is better expressed as

$$H'(\omega) = |H'(\omega)| \angle \left[\tan^{-1} \frac{\mathrm{Im}(H'(\omega))}{\mathrm{Re}(H'(\omega))} \right] \tag{8.57}$$

$$= M(\omega) \angle \theta(\omega) \ , \tag{8.58}$$

where we define $M(\omega)$ as the magnitude response and $\theta(\omega)$ as the phase response of our system. This says,

$$y(n) = Ae^{j\omega nT} \ M(\omega) \angle \theta(\omega) \ , \tag{8.59}$$

so that the output due to this elemental sinusoid is just an amplified or attenuated, phase shifted version of the input. Note that no additional frequency components appeared at the output. Again, this is an important characteristic of a linear system.

Another View of Frequency Response

We can predict frequency response (and gain insight) by treating frequency response as a problem in vector multiplication from a pole/zero diagram. A linear time invariant system has a transfer function that may be represented as a ratio of products such as

$$H(z) = \frac{\prod_{i=0}^{M} (z - z_i)}{\prod_{i=0}^{K} (z - p_i)}, \tag{8.60}$$

where the z_i are the system zeros and the p_i are the system poles. If we evaluate this $H(z)$ at $z=e^{j\omega T}$ we have

$$H(z) = \frac{\prod_{i=0}^{M} \left(e^{j\omega T} - z_i \right)}{\prod_{i=0}^{K} \left(e^{j\omega T} - p_i \right)}. \tag{8.61}$$

vector from the pole location drawn from p_i to the point on the unit circle at which the frequency response is being evaluated. A similar case holds for the product of vectors in the numerator. Figure 8.13(b) shows how a frequency response with two zeros and two poles might be evaluated. The total magnitude response (to within a gain constant for H(z)) can be evaluated as

$$|H'(\omega)| = \frac{\prod_{i=0}^{M} M_{z_i}(\omega)}{\prod_{i=0}^{K} M_{p_i}(\omega)}. \tag{8.62}$$

where $M_{z_i}(\omega)$ is the length of the vector $\left(e^{j\omega T} - z_i\right)$, and $M_{p_i}(\omega)$ is the length of the vector $\left(e^{j\omega T} - p_i\right)$. Similarly the phase response can be evaluated as

$$\angle H'(\omega)| = \sum_{i=0}^{M} \theta_{z_i}(\omega) - \sum_{i=0}^{K} \theta_{p_i}(\omega) , \tag{8.63}$$

where $\theta_{z_i}(\omega)$ and $\theta_{p_i}(\omega)$ are the angles (with respect to the positive real z axis) associated with the vectors $\left(e^{j\omega T} - z_i\right)$ and $\left(e^{j\omega T} - p_i\right)$ respectively.

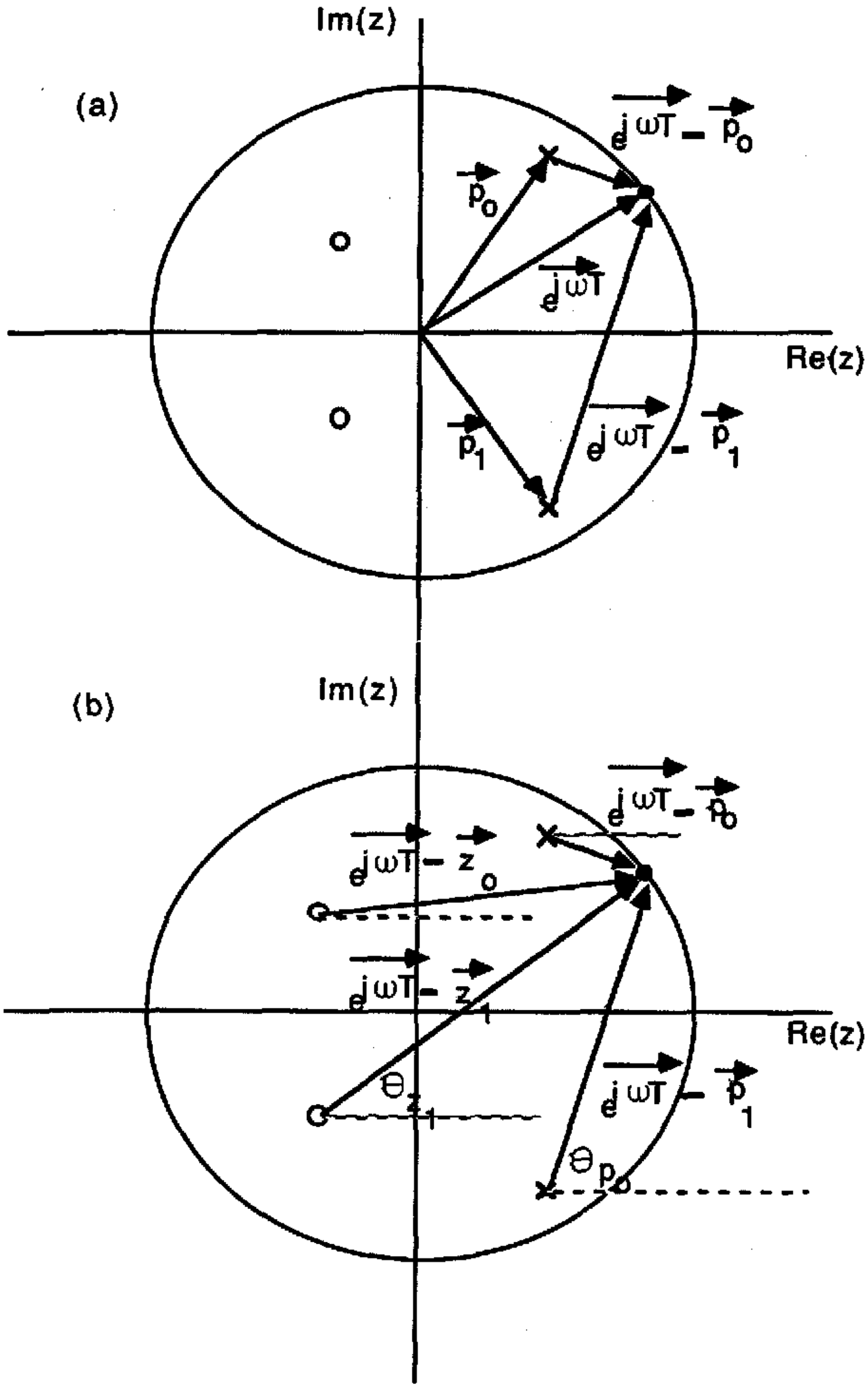


Figure 8.13. (a) Illustration of how vector subtraction produces a vector from each of two poles to the point on unit circle at which frequency response is being evaluated. (b) Illustration of vectors from each pole and zero to a point on the unit circle. These vectors are used in the calculation of frequency response.

Example 8.11

Given the pole/zero diagram shown below in Figure 8.14, give a rough sketch of the predicted magnitude response for this system.

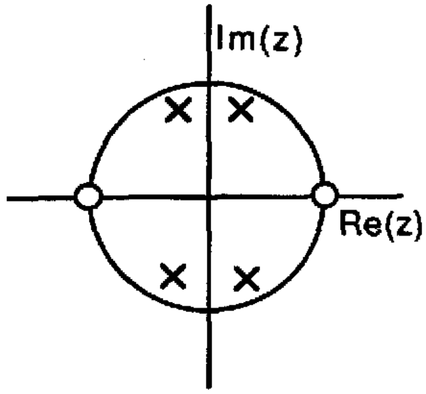


Figure 8.14. Pole/zero diagram for system in Example 8.11.

Solution: To evaluate frequency response we need only travel around the unit circle from z=1 (DC) to $z=e^{j\pi} = -1$, since traveling from z=-1 to z=1 on the lower side of the unit circle will just give us a mirror image. We will expect our response to tend to zero in the vicinity of z=1 and z=-1 since we have zeros on the unit circle at z=1 and z=-1. Furthermore, we should expect a peak in the response as we travel around the unit circle and come in close proximity to each of the two poles near the upper part of the unit circle. The response should be drawn symmetric about $\pi/2T$ since the poles and zeros appear to have symmetry about that frequency.

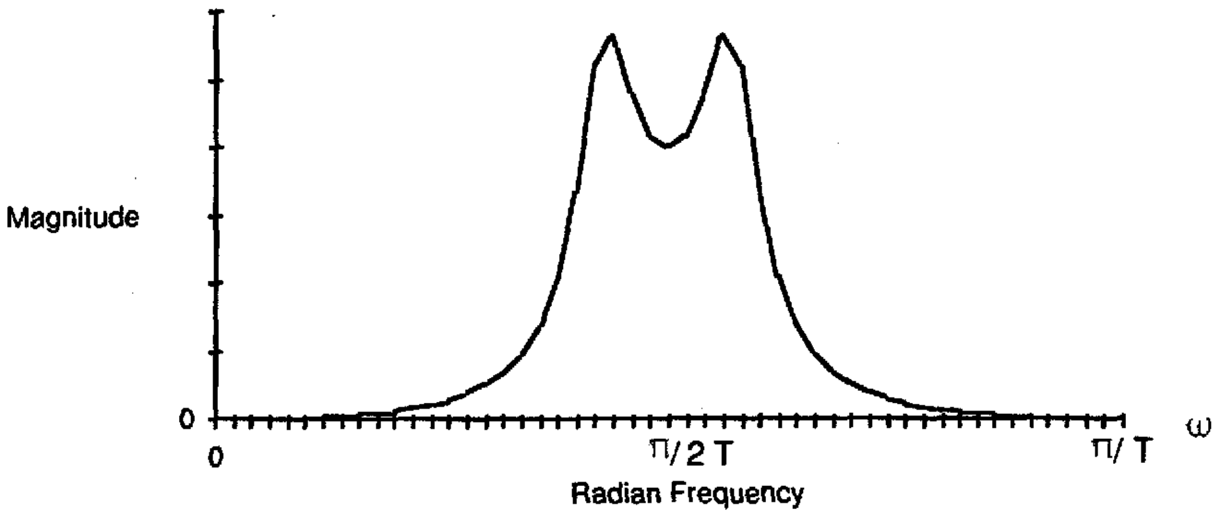
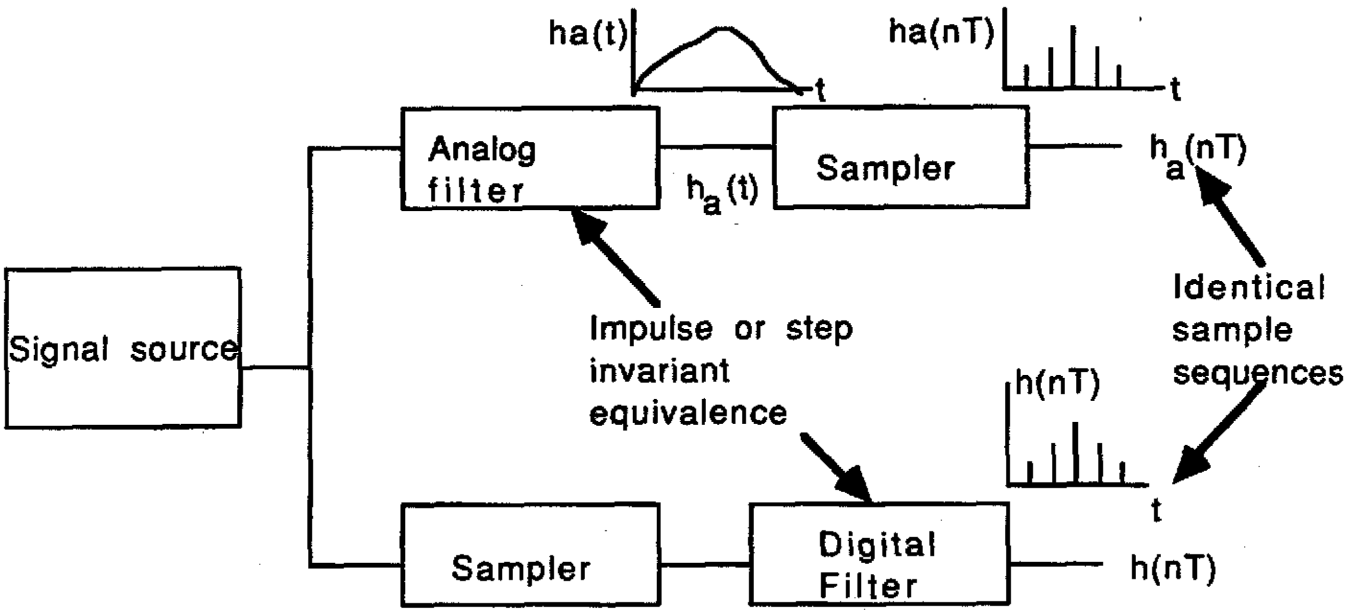


Figure 8.15. Approximate frequency response for system in Example 8.11.

8.6 Digital Filters: Block Diagrams and Structures

********************TO BE DEVELOPED *********************

8.7 Discrete/Continuous Time Equivalence: A Preview



********************** TO BE DEVELOPED ********************

Problems

8.1. Compact disk technology currently records/reads data from CD at a rate of 44,100 samples per second (44.1kHz).

a) In order to avoid aliasing, what is the highest significant audio frequency we should allow into the digital sampling system at the recording studio? $f_s \geq 2 f_{max}$ $f_{max} = 44.1k/2$ Hz

b) Suppose we insist on a minimum of 40dB anti-aliasing protection, and suppose we want a relatively flat audio response for our CD recording system out to approximately 15kHz.

i) Calculate the required order for a Chebyshev anti-aliasing lowpass filter, given that we will allow 2dB of passband ripple.

ii) If we choose not to tolerate the ripple in part (i) and opt for a Butterworth filter, calculate the required order.

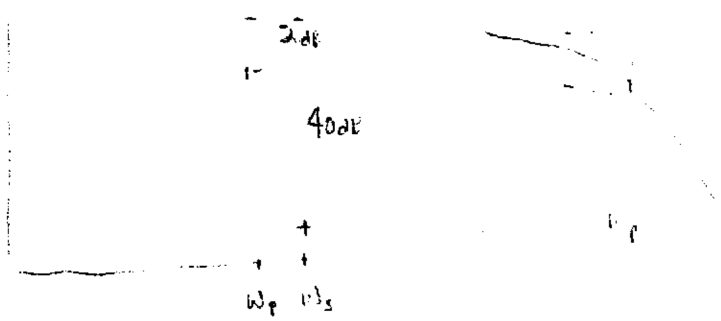
8.2. A system is known to behave according to the first order difference equation,

$$y(n) = 5\ x(n) + 0.9\ y(n-1)\ .$$

a) Calculate and sketch the system impulse response, h(n).

b) Calculate and sketch the system step response, s(n). For large n, what value does s(n) asymptotically approach?

c) Calculate the system output for the input x(n)=u(n)-u(n-4). (Assume that y(-1)=0.)



8.3. You are given the system impulse response h(n) and system input x(n) shown in Figure P8.1.

a) Calculate and sketch the system step response s(n).

b) Calculate and sketch the output y(n).

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = x[n] * h[n]$

$u[n] * h[n] = s[n]$

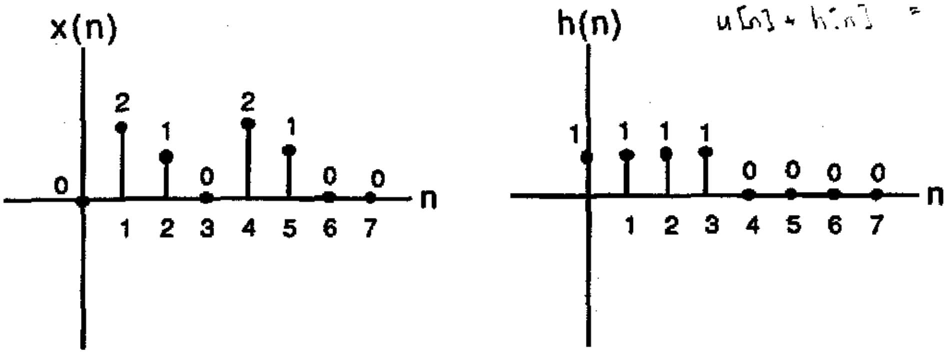


Figure P8.1. System input and impulse response for problem 8.3.

8.4. Suppose the difference equation for a some linear time invariant (LTI) system is

$$y(n) = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M) \ .$$

a) Is this a recursive or non-recursive system? Explain.

b) Without any specific knowledge of the b coefficients (other than being finite), can you determine whether or not this system remains stable for any bounded input? Explain.

c) Determine the impulse response for this system. Does this system have a finite impulse response (FIR) or infinite impulse response (IIR)? Explain.

8.5. You are given two systems that are described by the following two difference equations:

<u>System #1</u>: $y(n) = \frac{1}{4}[x(n) - x(n-4)] + y(n-1)$

<u>System #2</u>: $y(n) = \frac{1}{4}[x(n) + x(n-1) + x(n-2) + x(n-3)]$

a) Calculate and sketch the impulse response for system #1, $h_1(n)$. What is the function of this filter?

b) Does system #1 have a finite impulse response (FIR) or an infinite impulse response (IIR)?

c) Does system #1 represent a recursive or non-recursive implementation? Explain.

d) Calculate and sketch the impulse response for system #2, $h_2(n)$. (Your answer should be the same as your answer for system #1!)

e) Does system #2 have a finite impulse response (FIR) or an infinite impulse response (IIR)?

f) Does system #2 represent a recursive or non-recursive implementation? Explain.

g) Based on your observations above, in general, is it necessary for two systems to have identical difference equations in order to have the same impulse response?

8.6. You are given the system impulse response h(n) and system input x(n) shown in Figure P8.2.

a) Calculate and sketch the system step response s(n).

b) Calculate and sketch the output y(n).

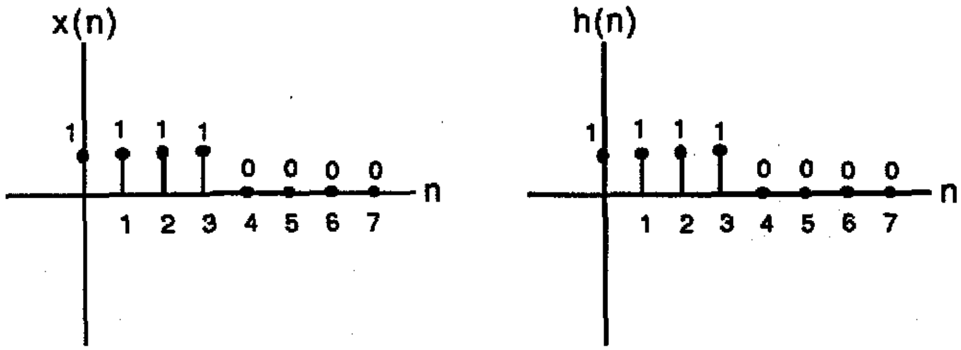


Figure P8.2. System input and impulse response for problem 8.6.

8.7. <u>Using the definition of the z-transform</u>, calculate the z-transforms of the following causal sequences:

a) $x(n) = u(n) - u(n-5)$
b) $x(n) = 0.5^n\ u(n)$
c) $x(n) = 0.5^n\ u(n-2)$
d) $y(n)=x(4n)$, where x(n) is defined in (c) above

8.8. Prove the following z-transform properties:

a) Property 2, Table 8.2
b) Property 5, Table 8.2
c) Property 7, Table 8.2
d) Property 8, Table 8.2

8.9. Use the relationship $\sin(n\omega_0 T) = \frac{1}{2j}\left(e^{jn\omega_0 T} - e^{-jn\omega_0 T}\right)$ together with methods like those used used in Example 8.9 to prove that if $x(n) = \sin(n\omega_0 T)\ u(n)$, then X(z) is given as

$$X(z) = \frac{\sin(\omega_0 T)\, z^{-1}}{1 - 2\left(\cos(\omega_0 T)\right) z^{-1} + z^{-2}} \qquad |z| > |1|$$

8.10. Given $x(n)$ is causal, prove that

$$\lim_{z \to \infty} X(z) = x(0)$$

8.11 In speech processing, a prefilter is often used to pre-emphasize or boost the high frequency components. Assume the desired difference equation relating the output $y_1(n)$ to the input $x(n)$ for this filter is given by $y_1(n) = x(n) - 0.8x(n-1)$.

a) Calculate both $H_1(z)$ and $h_1(n)$ for this system.

b) Does $H_1(z)$ represent a stable system? Justify.

c) Based on your answer in (a), sketch the frequency response $|H_1(e^{j\omega T})|$ from DC to one half the sampling frequency.

d) Suppose the programmer goofs in his microprocessor algorithm and he actually calculates $y_2(n) = 0.8x(n) - x(n-1)$.

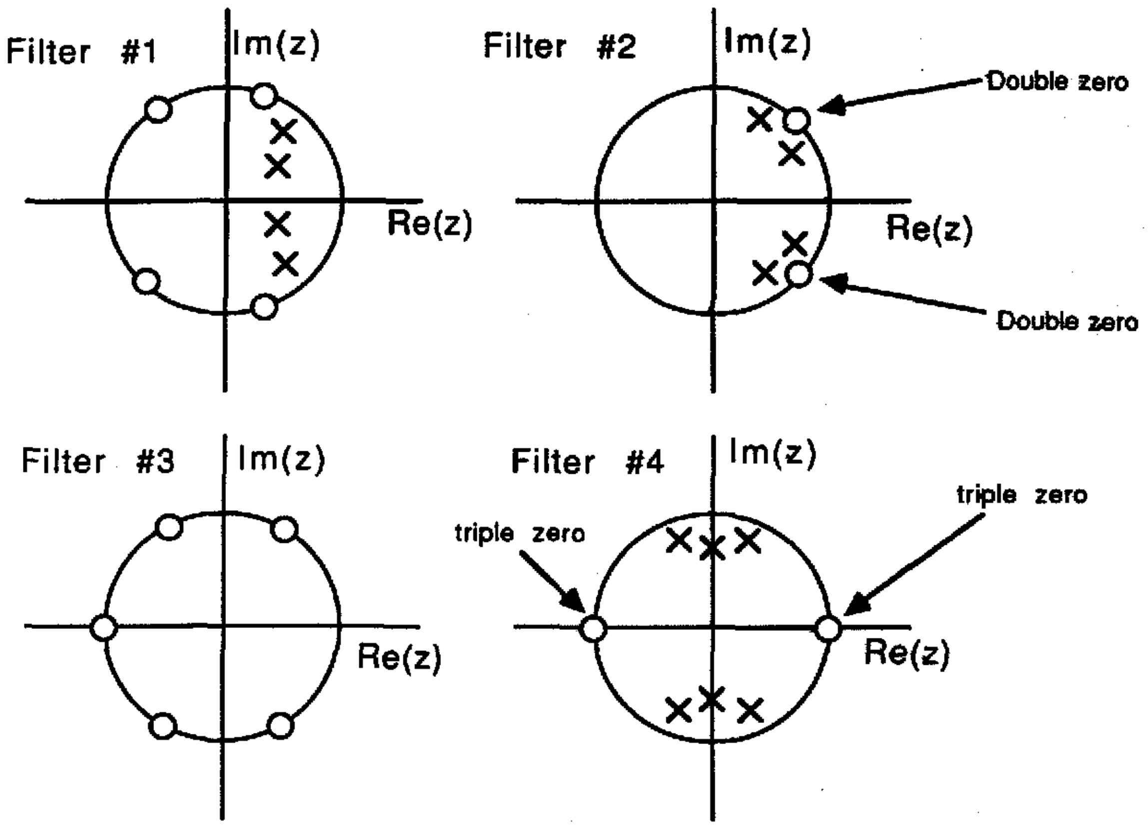
i) Calculate both $H_2(z)$ and $h_2(n)$ for this system..

ii) Based on your answer for $H_2(z)$, sketch the frequency response $|H_2(e^{j\omega T})|$ or $|H'_2(\omega)|$ from DC to one half the sampling frequency. How does this compare to your answer for $|H_1(e^{j\omega T})|$?

8.12 You are given the pole/zero diagrams below for four linear, time-invariant, causal, discrete-time filters. In each case calculate/sketch/determine the following:

a) Sketch the frequency response from 0 to $\frac{f_s}{2}$, and be sure to label on your frequency axis any specific points of interest (i.e. notch or bandpass center frequencies, or LP or HP cutoff frequencies, etc.).

b) For each case, what type of filter is depicted?



8.13. It is well known in digital signal processing that a zero order hold D/A converter has a frequency response that looks like a sinc function $\frac{\sin \pi\omega/\omega_s}{\pi\omega/\omega_s}$, whose "nulls" are at integer multiples of the sampling frequency. These "nulls" help to suppress the replication images in the frequency domain caused by the sampling process. However, one problem with the D/A converter is that its sinc transfer function distorts the lower frequency components of our desired signal, since a sinc function is not perfectly flat over these low frequencies, and may cause unwanted spectral shaping. In order to compensate for this effect, it is sometimes necessary to design a digital filter to place just prior to the D/A converter, which will compensate for this effect. Suppose you have already done laborious calculations, and you know that a simple FIR filter that can be used to perform the equalization is

$$H_1(z) = 1.125 - 0.1406\, z^{-1} + 0.0176\, z^{-2}$$

Write a simple program in the language or spreadsheet of your choice that will perform the following:

a) Graph the frequency response of the D/A converter.
b) Graph the frequency response of the equalizing filter.
c) Graph the frequency response of the cascade of the D/A converter plus equalizing filter.

Does the equalizing filter perform the desired task?

8.14. Write a simple program in the language or spreadsheet of your choice that will calculate the impulse, step, magnitude, and phase response for the transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_8 z^{-8}}{a_0 + a_1 z^{-1} + \ldots + a_8 z^{-8}},$$

for any values of a_n and b_n entered. (Assume a causal h(n).)