Chapter 6

Band Pass and Notch Filters

6.1 Introduction

In this chapter we consider the design of both band pass and band reject (notch) filters. Band pass filters, by definition, will "pass" signal components that lie within some continuous range of frequencies, and attenuate signal components that fall above and below that range. Similarly, band reject filters (or notch filters) will attenuate signal components that lie within some continuous range of frequencies, and "pass" signal components that lie outside that range. In either case we may specify that the filter be a Butterworth or Chebyshev type, depending on whether we require the response to be maximally flat (Butterworth), or whether we can tolerate some ripple (Chebyshev).

Our strategy in design of band pass and notch filters makes use of the low pass and high pass filter design techniques discussed in Chapters 4 and 5. More specifically, we will generate the band pass response by mapping a low pass magnitude response ("centered" at DC) to a band pass response which is centered at the geometric mean of the pass band limits. Likewise for notch filters, we will start with a high pass prototype filter, and map its response to a notch filter response as shown in Figure 6.1. This mapping of the frequency axis will be extended to mapping the entire low pass (or high pass) prototype complex (S) plane to the band pass (or notch) filter complex (s) plane. The mapping will be "one to two" in that an n^{th} order low pass prototype filter will map to a $2n^{th}$ order band pass filter, meaning that a single low pass prototype pole will map to two band pass poles.

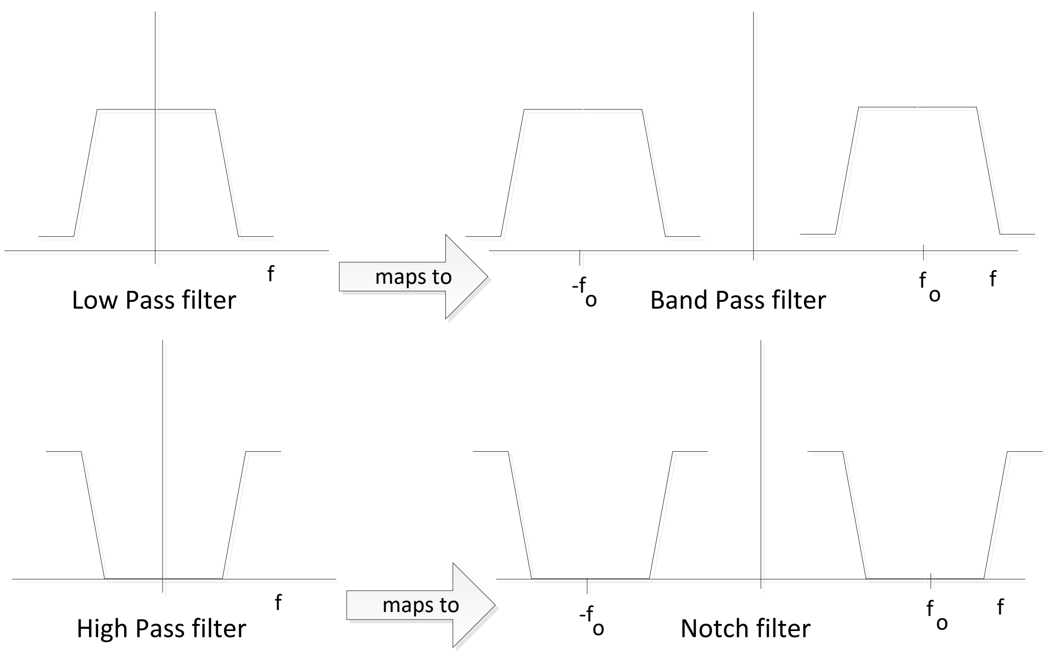

Figure 6.1. Frequency response mappings (low pass to band pass, and high pass to notch).

The big problem in this mapping is calculating the band pass or notch filter pole locations from the low pass (or high pass) pole locations. Closed form solutions to this problem are presented in this chapter, however they are best accomplished with the aid of a short computer program. For purposes of visualization, we present a method that uses a computer generated chart of this mapping, where the low pass "S" (upper case) plane has been mapped onto the band pass "s" (lower case) plane.

Later in the chapter we present circuits to synthesize second order filter sections for a given Q and resonant frequency, and we consider some specific examples. As in Chapter 4, we also analyze the effects of finite op-amp gain bandwidth product. Here we observe similar effects as in the low pass or high pass case, when poorer gain-bandwidth product op-amps are used for these circuits. However, since the Q's of band pass filters are generally higher, these effects (pole movements from desired locations) will be more pronounced at lower frequencies than in the low pass case.

Finally we consider an innovative alternative for producing a 4th order circuit which uses the cascade of two identical 2nd order sections, together with negative feedback. This structure has been implemented in an integrated circuit (MF8) by National Semiconductor, making the design of band pass filters (to 20kHz) quite simple.

6.2 Band Pass Response

In Chapter 2 we learned a general expression for a second order band pass filter transfer function (with resonant frequency ω_o=1), namely

$$H_{BP}(s) = \frac{\left(\frac{1}{Q}\right)s}{s^2 + \left(\frac{1}{Q}\right)s + 1} \tag{6.1}$$

As we will soon see, when we design band pass filters using graphical techniques, we will determine the resonant frequency (radius) of each second order section by measuring the radius from the s-plane origin of each pole in the s-plane. This radius will be interpreted as the resonant frequency relative to the center frequency. *Therefore, for all calculations in this chapter, the center frequency will be normalized to unity, and other frequencies will be interpreted relative to this center frequency.* For example, a "normalized" frequency of f=1.2 will be interpreted as an *actual* frequency of 1.2 times the center frequency of that second order section.

At this point we calculate the magnitude squared response of 6.1 as

$$H_{BP}(j\omega)H_{BP}(-j\omega) = \frac{\left(\frac{\omega}{Q}\right)^2}{\left(1-\omega^2\right)^2 + \left(\frac{\omega}{Q}\right)^2} = \frac{1}{Q^2\left(\frac{1}{\omega}-\omega\right)^2 + 1} \quad , \tag{6.2}$$

so

$$A(\omega)=10\log_{10}\left[1+Q^2\left(\frac{1}{\omega}-\omega\right)^2\right] \tag{6.3}$$

Recall from Chapter 2 where we saw that the magnitude response of a second order band pass filter is symmetric on a logarithmic scale about the center frequency. Observe from 6.2 and 6.3 for example, that for a center frequency of unity, the response at ω is the same as that at $1/\omega$. If we define $\Delta\omega$ as the difference between these frequencies, namely

$$\Delta\omega=\frac{1}{\omega}-\omega, \tag{6.4}$$

from 6.3 we have

$$A(\Delta\omega)=10\log_{10}\left[1+Q^2(\Delta\omega)^2\right] \tag{6.5}$$

as the attenuation in dB at each of these two frequencies separated by $\Delta\omega$. (Note once again from 6.3 the symmetry about frequency $\omega=1$ on a logarithmic scale.) In Chapter 2 we defined Q as the half power (or 3dB) bandwidth relative to the center frequency. In 6.5, $A(\Delta\omega)$ is clearly 3dB when $\Delta\omega=\frac{1}{Q}$, and if we define this 3dB bandwidth relative to the center frequency as $\Delta\omega_o$, and then substitute into 6.5, we obtain

$$A(\Delta\omega)=10\log_{10}\left[1+\left(\frac{\Delta\omega}{\Delta\omega_o}\right)^2\right] \tag{6.6}$$

Now we are in a position to compare this result with something we have seen in Chapter 4. More specifically, consider the transfer function of a first order (n=1) low pass filter with half power bandwidth of Ω_o (equation 4.3 from Chapter 4),

$$H_{LP}(S)=\frac{\Omega_o}{S+\Omega_o}, \tag{6.7}$$

where we calculated the attenuation equation (equation 4.3, n=1, from Chapter 4) as

$$A(\Omega)=10\log_{10}\left[1+\left(\Omega/\Omega_o\right)^2\right]. \tag{6.8}$$

Note that the response of the second order band pass filter (6.6) is exactly the same as that of a first order low pass filter (6.8) when we substitute $\Delta\omega$ for Ω and $\Delta\omega_o$ for Ω_o! This could be extended to higher order filters as well. For example, we could write the attenuation of an nth order Butterworth band pass filter as

$$A(\Delta\omega) = 10\log_{10}\left[1 + \left(\frac{\Delta\omega}{\Delta\omega_o}\right)^n\right] \tag{6.9}$$

and similarly for Chebyshev band pass filters. One important point is that the magnitude response will always be symmetric about the center frequency on a logarithmic frequency axis, so that the response at twice the center frequency is the same magnitude as the response at half the center frequency, etc.

Rather than attempting to substitute s/j for ω in equations such as 6.9, and solve for the poles as we did in the previous two chapters, here we will develop expressions that will relate poles (and zeros) of low pass "prototype" filters to those of band pass filters. Setting 6.1 and 6.7 equal,

$$H_{LP}(S) = H_{BP}(s)$$

so

$$\frac{\Omega_o}{S + \Omega_o} = \frac{\left(\frac{1}{Q}\right)s}{s^2 + \left(\frac{1}{Q}\right)s + 1} \tag{6.10}$$

Solving 6.10 for S,

$$S + \Omega_o = \left(\frac{s^2 + \left(\frac{1}{Q}\right)s + 1}{\left(\frac{1}{Q}\right)s}\right)\Omega_o \tag{6.11}$$

so

$$S = \left(\frac{s^2 + 1}{s}\right)Q\Omega_o . \tag{6.12}$$

Recall that Q is the center frequency relative to the 3dB bandwidth. Just as in the case of Chebyshev low pass filters in Chapter 5, we will set (normalize) the low pass prototype pass band limit frequency (Ω_p) to unity. That means Ω_o is the 3dB bandwidth relative to Ω_p (the pass bandwidth as defined by Amax). Their product ($Q\Omega_o$) is simply the center frequency relative to the pass band width, or

$$\begin{aligned} Q\Omega_o &= \left(\frac{center\ \ freq}{3dB\ \ bandwidth}\right)\left(\frac{3dB\ \ bandwidth}{bandwidth\ \ defined\ \ by\ \ A_{\max}}\right) \\ &= \left(\frac{center\ \ freq}{bandwidth\ \ defined\ \ by\ \ A_{\max}}\right) \\ &= \left(\frac{1}{\Delta\omega_p}\right). \end{aligned} \tag{6.13}$$

Substituting into 6.12 we obtain

$$
\begin{aligned}
S &= \frac{1}{\Delta\omega_p}\left(\frac{s^2+1}{s}\right) \\
&= \frac{1}{b}\left(\frac{s^2+1}{s}\right)
\end{aligned}
\tag{6.14}
$$

where "b" is defined as the pass band width (defined by A_{max}) relative to the center frequency. Although our derivation thus far has used an example of mapping a first order low pass prototype to a second order band pass filter, the mathematical mapping in 6.14 is valid for filters of arbitrary order. Substituting $j\Omega$ for S and $j\omega$ for s in 6.14 results in a frequency mapping of

$$\Omega = \frac{1}{b}\left(\omega - \frac{1}{\omega}\right). \tag{6.15}$$

Since "b" is the pass band width, if ω_p (>1) is the upper pass band limit

$$b = \left(\omega_p - \frac{1}{\omega_p}\right), \tag{6.16}$$

and from 6.15 and 6.16 it follows that $\omega=\omega_p$ and $\omega=1/\omega_p$ map to $\Omega=1$ and $\Omega=-1$, respectively. These mappings are illustrated in Figure 6.2.

The determination of the minimum order to satisfy a given set of specifications follows directly from the relationships shown in Figure 6.2. The ratio of the stopband width to the pass band width in a band pass filter is the same as the ratio of the stop band limit to the pass band limit (normalized to 1) of the low pass prototype, or

$$\frac{\Delta\omega_s}{\Delta\omega_p} = \frac{\Omega_s}{\Omega_p} = \Omega_s\,. \tag{6.17}$$

As a result, the minimum order of the low pass prototype (or the number of <u>second</u> order sections in the band pass filter) is determined as in Chapters 4 and 5 for Butterworth and Chebyshev band pass filters respectively, with $\frac{\Delta\omega_s}{\Delta\omega_p}$ substituted for $\frac{\omega_s}{\omega_p}$ in all order calculation equations.

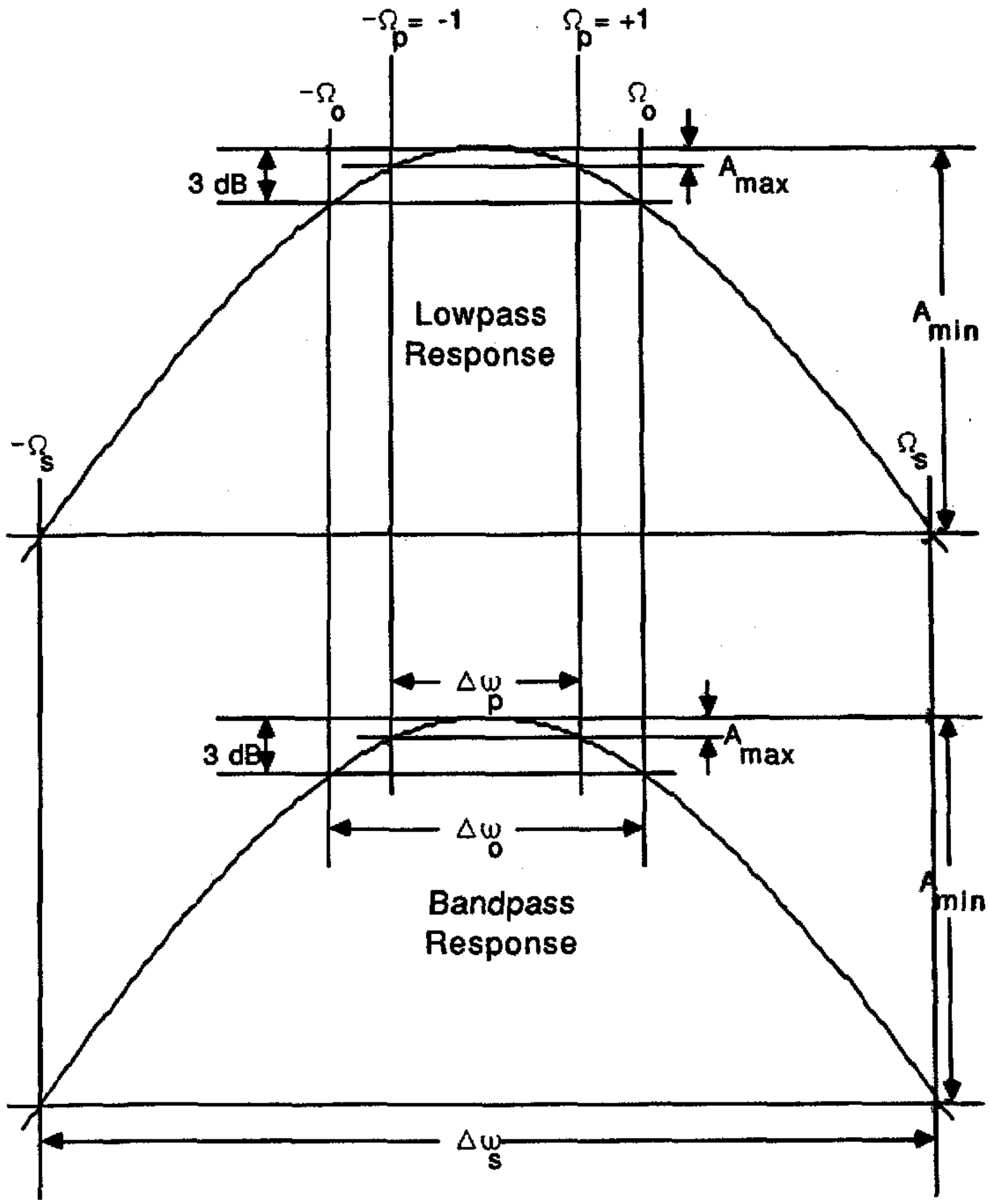

Figure 6.2. Comparison of low pass and band pass frequency responses.

Example 6.1

Determine the minimum order Butterworth and Chebyshev band pass filters to satisfy the following specifications:

A_{max} = 1.0dB
A_{min} = 30dB
Pass band limits = 1kHz to 2kHz
Stop band limits = 500Hz to 4kHz

Solution: Because the filter specifications must be symmetric about the center frequency of the filter, we must verify that these pass band and stop band limits are indeed symmetric about the same center frequency. The center frequency based on pass band limits is $\sqrt{1000 \times 2000}$ = 1.414kHz, and center frequency based on stop band limits is $\sqrt{500 \times 4000}$ = 1.414kHz. Since these are the same, we may then calculate the ratio of stop bandwidth to pass bandwidth as

$$\frac{\Delta\omega_s}{\Delta\omega_p} = \frac{(4000-500)\times 2\pi}{(2000-1000)\times 2\pi} = 3.5 \tag{6.18}$$

Substituting this ratio in equation 4.11 for a Butterworth filter,

$$n = \frac{\ln\left(\dfrac{10^{A_{min}/10}-1}{10^{A_{max}/10}-1}\right)}{2\ln\left(\dfrac{\Delta\omega_s}{\Delta\omega_p}\right)} = 3.29\text{, which we round to n=4.} \tag{6.19}$$

In a similar fashion from equation 5.29, we obtain the order for a Chebyshev filter as

$$n = \frac{\cosh^{-1}\left(\sqrt{\dfrac{10^{A_{min}/10}-1}{10^{A_{max}/10}-1}}\right)}{\cosh^{-1}\left(\dfrac{\Delta\omega_s}{\Delta\omega_p}\right)} = 2.32\text{, which we round to n=3.} \tag{6.20}$$

The orders "n" calculated in 6.19 and 6.20 are the required orders for the LOW PASS PROTOTYPE filters needed to meet the desired specifications. Our Butterworth and Chebyshev band pass filters would actually be twice the orders calculated by 6.19 and 6.20, which means our Butterworth band pass filter would be 8th order (4 second order sections), and our Chebyshev band pass filter would be 6th order (3 second order sections).

Example 6.2

Determine the minimum order Butterworth band pass filter to satisfy the following specifications:

A_{max} = 0.5dB
A_{min} = 40dB
Pass band limits = 800Hz to 1.25kHz
Stop band limits = 400Hz to 5kHz

<u>Solution</u>: First observe that the pass band and stop bands are not symmetric about the same center frequency. What this means is that we must change the specifications such that symmetry is enforced, while still meeting the original desired filter specifications. On the surface, it would appear that we might have four choices:

Choice 1: Force the response to be symmetric about the geometric mean of the pass band limits (1kHz) by moving lower stop band *lower* in frequency:
Pass band limits = 800Hz to 1.25kHz
Stop band limits = 200Hz to 5kHz

Choice 2: Force the response to be symmetric about the geometric mean of the pass band limits (1kHz) by moving upper stop band *lower* in frequency:
Pass band limits = 800Hz to 1.25kHz
Stop band limits = 400Hz to 2.5kHz

Choice 3: Force the response to be symmetric about the geometric mean of the stop band limits (1.414kHz) by moving upper pass band *higher* in frequency:
Pass band limits = 800Hz to 2.5kHz
Stop band limits = 400Hz to 5kHz

Choice 4: Force the response to be symmetric about the geometric mean of the stop band limits (1.414kHz) by moving lower pass (and perhaps also the upper pass band) *higher* in frequency:
Pass band limits = 1.25kHz to 1.6kHz
Stop band limits = 400Hz to 5kHz

We can immediately eliminate choices #1 and #4. Choice #1 is eliminated because we moved the stop band limit DOWN in frequency to 200Hz, which violates our specification that we want 40dB attenuation from 400Hz down to DC. Similarly, Choice #4 is eliminated because we originally insisted that our minimal desired pass band range was from 800Hz to 1.25kHz, and Choice#4 does not include that complete frequency range.

We are left with two legitimate choices, namely Choice#2 and Choice#3. In Choice#2, the ratio of stop band limits to pass band limits is

$\dfrac{2500-400}{1250-800} = 4.67$, which from equation 4.11 and 6.19 results in an order (for low pass prototype) of 3.67, which we round to 4, so an 8th order band pass filter is required. In Choice#3, the ratio of stop band limits to pass band limits is

$$\frac{5000-400}{2500-800} = 2.71$$, which from equation 4.11 and 6.19 results in an order (for low pass prototype) of 5.67, which we round to 6, so a 12th order band pass filter is required.

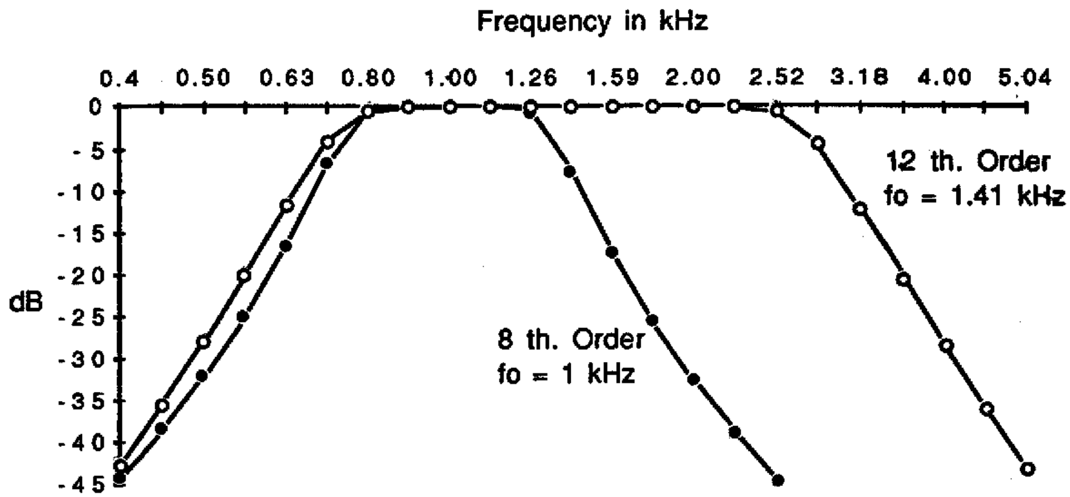

Figure 6.3. Comparison of 8th and 12th order band pass responses.

Observe from Figure 6.3 and equations 4.11, 6.19, and 6.20, that the minimum required order decreases as the ratio of stop band width to pass band width increases. This suggests that for band pass filter design, we can always satisfy the specifications with a lower order filter if we force the center frequency to be the geometric mean of the pass band limits (and move a stop band limit), and our example above supports this idea. A somewhat more physical explanation is that by picking "Choice#2," the bandwidth relative to the center frequency is smaller, which implies our filter sections will have higher Q's and steeper roll-off characteristics between pass band and stop band. The magnitude responses of these two filters are compared in Figure 6.3.

6.3 Band Pass Poles

Thus far we have considered the problem of mapping a low pass frequency response to a band pass response, and the problem of determining the minimum order for a band pass design. In order to complete the design process for a band pass (and notch) filter, we will also need to determine the complete transfer function, and more specifically, we will need to determine the Q's and resonant frequencies of each second order section.

We return to equation 6.14, which illustrates the mapping between low pass prototype poles and band pass poles as

$$S = \frac{1}{b}\left(\frac{s^2+1}{s}\right) \tag{6.21}$$

where "S" represents a location in our low pass prototype complex plane, "s" represents a location in our band pass complex plane, and "b" is the pass band width (defined by A_{max}) relative to the center frequency. Solving 6.21 for the band pass poles (which inherently give us resonant frequencies and Q's for later design),

$$(bS)s = s^2 + 1 \tag{6.22}$$

or

$$s^2 - (bS)s + 1 = 0\,. \tag{6.23}$$

Therefore, if we have a low pass prototype pole at location "S", the band pass poles appear at

$$\boxed{s = \left(\frac{bS}{2}\right) \pm \sqrt{\frac{(bS)^2}{4} - 1}}\;, \tag{6.24}$$

where one low pass prototype pole maps to TWO band pass pole locations.

Example 6.3. Determine the pole locations of a 4th order Butterworth band pass filter with 3dB bandwidth of $\sqrt{2}$ relative to the center frequency.

Solution: Based on our work in Chapter 4 (equation 4.6) we know that the frequency of the low pass prototype poles is given as

$$\Omega_o = \frac{\Omega_p}{\left(10^{A_{max}/10} - 1\right)^{\frac{1}{2n}}} \tag{4.6}$$

where A_{max}=3dB, n=2, and Ω_p is the normalized frequency at which A_{max} has been specified (Ω_p=1). Substituting these values into (4.6) yields Ω_o=1, and since n=2 we know the low pass pole angles are $\pm 45^\circ$ (Q=0.707) relative to the negative real S-plane axis. This means that

$$H_{LP}(S) = \frac{1}{S^2 + \sqrt{2}S + 1} \tag{6.25}$$

so

$$H_{BP}(s) = H_{LP}\left(S = \frac{1}{b}\left(\frac{s^2+1}{s}\right)\right). \tag{6.26}$$

Since the 3dB bandwidth relative to the center frequency (i.e. "b") was given,

$$H_{BP}(s) = H_{LP}\left(S = \frac{1}{\sqrt{2}}\left(\frac{s^2+1}{s}\right)\right) \tag{6.27}$$

so

$$H_{BP}(s) = \frac{1}{\frac{1}{2}\left(\frac{s^2+1}{s}\right)^2 + \left(\frac{s^2+1}{s}\right) + 1} \tag{6.28}$$

and

$$H_{BP}(s) = \frac{2s^2}{s^4 + 2s^3 + 4s^2 + 2s + 1} \tag{6.29}$$

Certainly we could find the 4 roots of this denominator by using a Matlab command such as "roots"

```
roots([1 2 4 2 1])
```

to obtain the 4 band pass pole locations $s = -0.7429 \pm j1.5291$ and $s = -0.2571 \pm j0.5291$, but perhaps a more straightforward way to obtain the band pass poles is to map each low pass prototype pole to a pair of band pass poles. In this case, given a low pass prototype pole at $S = \frac{\sqrt{2}}{2}(-1+j)$, we calculate the band pass pole locations by substituting in for "S" in 6.24 (where $b = \sqrt{2}$), to obtain

$$\boxed{s = \left(\frac{bS}{2}\right) \pm \sqrt{\frac{(bS)^2}{4} - 1} = \quad -0.2571 - j0.5291 \quad and \quad -0.7429 + j1.5291}$$

Similarly, given a low pass prototype pole at $S = \frac{\sqrt{2}}{2}(-1-j)$, we calculate the band pass pole locations by substituting in for "S" in 6.24 (where $b = \sqrt{2}$), to obtain

$$\boxed{s = \left(\frac{bS}{2}\right) \pm \sqrt{\frac{(bS)^2}{4} - 1} = \quad -0.2571 + j0.5291 \quad and \quad -0.7429 - j1.5291}$$

At this point, we make some observations:

(1) Each pair of band pass poles is at angles of $\pm 64.1^\circ$ (Q=1.144) with respect to the negative real axis.
(2) Their resonant frequencies (pole radii) are at 1.70 and 0.588, and are symmetric on a logarithmic scale about the center frequency of 1. That is, (1.70)(0.588)=1.0. (As we will later observe, this will be true for any two pairs of band pass poles resulting from a second order low pass prototype. Therefore, once we have determined one of the four poles, we know locations for the other three poles.)
(3) One can create a very convenient function that performs the low pass to band pass mapping in Matlab (Figure 6.4).

```
function [bp_poles] = lp_to_bp(b, lp_poles)

%   Parameters:  b:          passband width relative to center frequency
%                lp_poles:   low pass prototype poles (with wp = 1)
%
%   Returns:     bp_poles:   pole locations of band pass filter relative
%                to center frequency
%
%   Description:     Implement equation 6.22

bp_poles1=(b*lp_poles)/2+sqrt((b*lp_poles).^2/4 - 1);
bp_poles2=(b*lp_poles)/2-sqrt((b*lp_poles).^2/4 - 1);
bp_poles=[bp_poles1 bp_poles2];
return
```
Figure 6.4. MATLAB function for low pass to band pass mapping.

6.4 Graphical Techniques

In order to ease the burden of calculating band pass poles from low pass prototype poles, and also to visualize the nature of the mapping from the low pass prototype "S" plane to the band pass "s" plane, we have developed computer generated charts of the mapping. Note that in 6.24, only the product bS appears in the equation, and not the individual values "b" and "S." As a result, we enter the mapping chart with this product in the form of a magnitude ($b\Omega_o$) and the angle of the low pass prototype pole relative to the negative real S plane axis. We have produced the chart in two versions. Figure 6.5a is used for more narrow band filters $\left(b\Omega_o \le 1.2\right)$, and Figure 6.5b is used for wider band filters $\left(1.2 \le b\Omega_o \le 3.4\right)$. Very wide band filters $\left(b\Omega_o > 3.4\right)$ are generally constructed as the cascade of low and high pass filters, and are designed accordingly. Using these charts to design filters proceeds as follows:

1) Determine the minimum order of the low pass prototype filter as described in Section 6.2.

2) Determine the angle (or Q) and the frequency $\left(\Omega_o\right)$ of each low pass prototype pole. Again, the pass band (defined by A_{max}) limit is normalized to 1. For Chebyshev filters, this is merely the radius measured on the chart for plotting Chebyshev low pass poles (Figure 5.6). For Butterworth filters, it is merely equation 4.6 with the pass band limit set to 1, or

$$\Omega_o = \frac{1}{\left(10^{A_{max}/10} - 1\right)^{\frac{1}{2N_{LP}}}}, \tag{6.30}$$

where N_{LP} is the order of the low pass prototype.

3) Plot the band pass poles at the intersections of the curves for the low pass prototype angles (lines radiating outward from j1 in Figure 6.5a or Figure 6.5b) and the curves for $b\Omega_o$. The frequency of each band pass pole relative to the center frequency is its radius, and its angle and Q are determined by extending a line from the origin through the pole to the outer two scales (where we read angle and Q). Typically for second order low pass sections, it is only necessary to plot the band pass pole of higher frequency, since the other pair has the same Q and reciprocal frequency (relative to center frequency = 1).

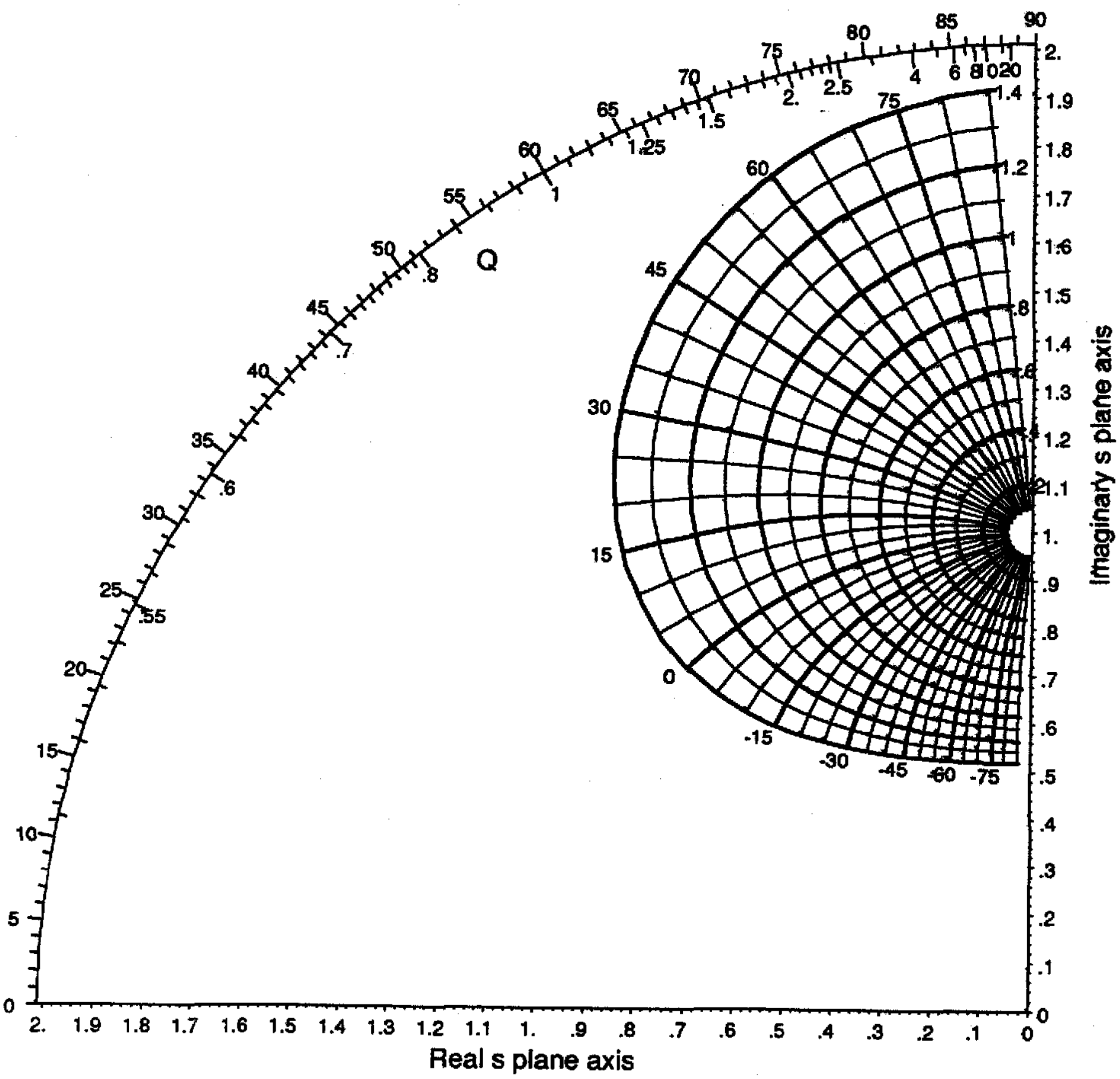

Figure 6.5a. Low pass to band pass mapping (narrow band), $(b\Omega_o \leq 1.2)$.

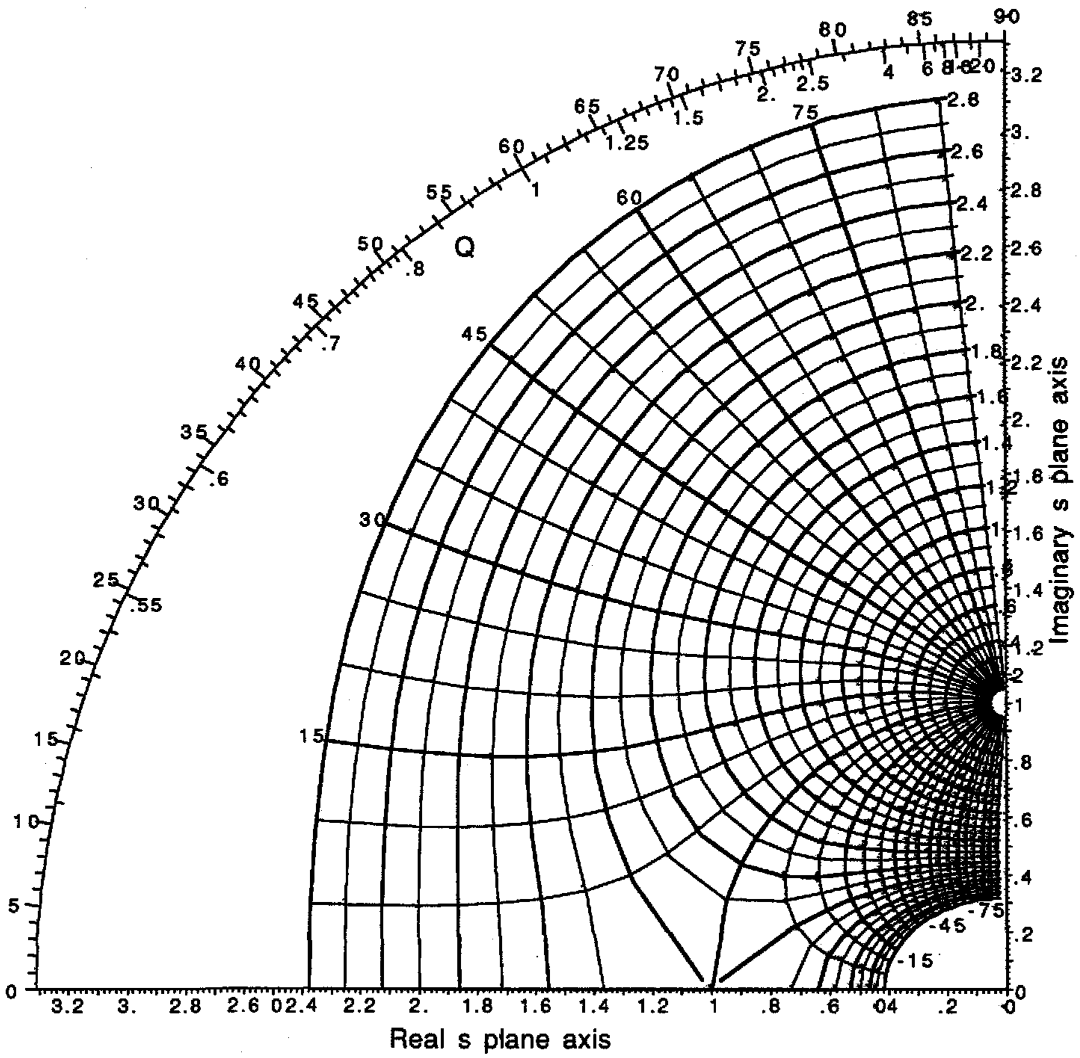

Figure 6.5b. Low pass to band pass mapping (wide band), $\left(b\Omega_o \leq 3.4\right)$.

To illustrate the use of the charts in Figures 6.5a and 6.5b, we will consider some examples.

Example 6.4

Graphically determine the pole locations for a fourth order Butterworth band pass filter, with a 3dB bandwidth of $\sqrt{2}$ relative to the center frequency (i.e. specifications from Example 6.3).

<u>Solution</u>: A_{max}=3dB, N_{LP}=2, Ω_p=1, and $b=\sqrt{2}$. The low pass prototype was a second order Butterworth low pass filter with a resonant frequency of

$$\Omega_o = \frac{1}{\left(10^{A_{max}/10} - 1\right)^{\frac{1}{2N_{LP}}}} = 1 \,,$$

so we plot our band pass poles at the intersection of the $b\Omega_o = \sqrt{2}$ line, and the $\pm 45^\circ$ line (Figure 6.6). Note that these results are identical to those found in Example 6.3. Note that there are two additional complex conjugate poles (below the negative real axis) not shown. Resulting poles are at "reciprocal radii" (ω_o=1.7 and ω_o =0.59) relative to center frequency, at Q=1.14.

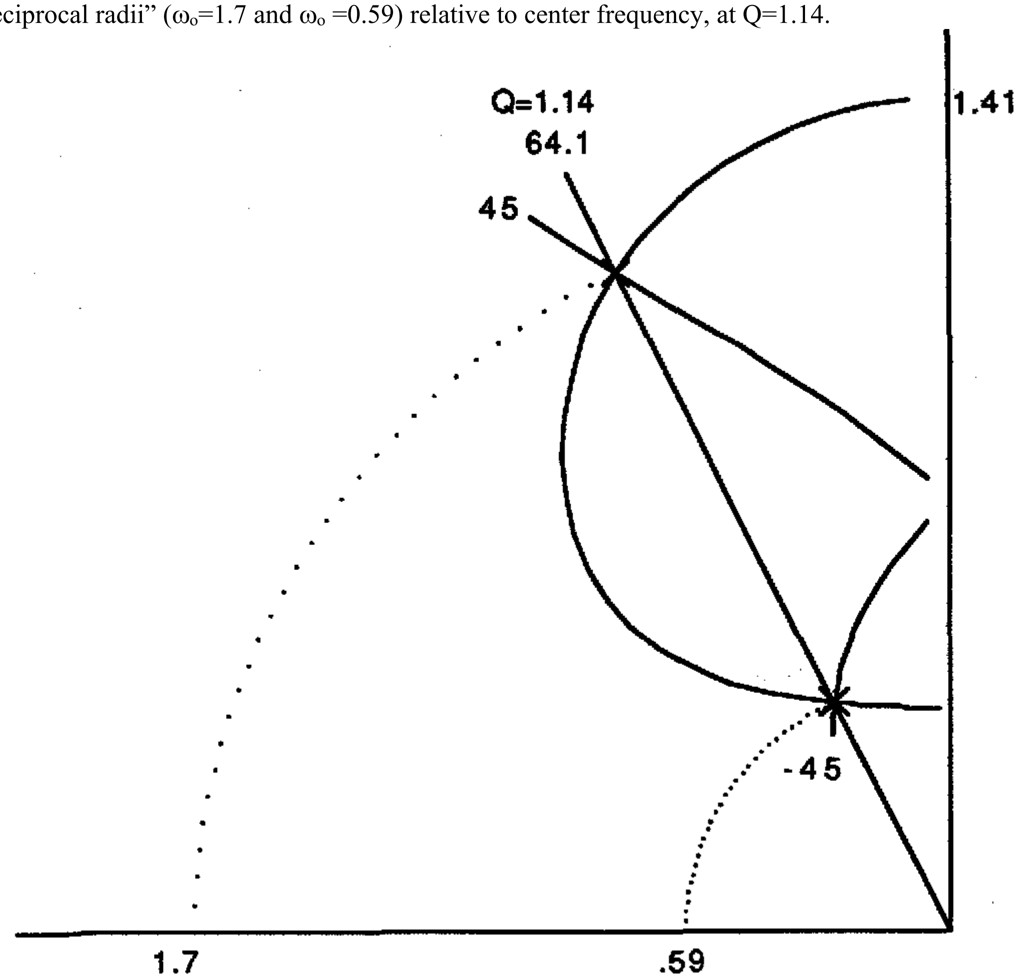

Figure 6.6. Plot of band pass pole locations for Example 6.4.

Example 6.5

Graphically determine the poles of the Butterworth and Chebyshev band pass filters from Example 6.1, where

A_{max} = 1.0dB
A_{min} = 30dB
Pass band limits = 1kHz to 2kHz
Stop band limits = 500Hz to 4kHz

<u>Solution</u>: Note that the specifications are symmetric about a center frequency of 1.414kHz. Recall from (6.19) that the required low pass prototype order N_{LP} = 4. Using this prototype order,

$$\Omega_o = \frac{1}{\left(10^{A_{max}/10} - 1\right)^{\frac{1}{2N_{LP}}}} = 1.18 \text{ (for Butterworth low pass prototype).}$$

The bandwidth relative to the center frequency (b) is given by $b = \dfrac{2kHz - 1kHz}{1.414kHz} = 0.707$.

The poles are plotted at the intersections of the $b\Omega_o = (0.707 \times 1.18) = 0.837$ curve, and the $\pm$22.5 and $\pm$67.5 degree curves. These poles are shown in Figure 6.7. Note that these band pass poles are at angles of $\pm 67.5^\circ$ (Q=1.3) at frequencies 1.19 and 0.84 (or 1/1.19) relative to the center frequency (of $\omega_o = 2\pi \times 1414 = 8.88 \times 10^3 rad/\sec$), and at $\pm 81^\circ$ (Q=3.3) and frequencies 1.46 and 0.68 (or 1/1.46) relative to the center frequency. For the four second order sections we have:

Section#1:

$\omega_{o_1} = 1.19 \times (2\pi \times 1414) = 1.06 \times 10^4 rad/\sec$, and $Q_1 = 1.3$ (using charts),

or $\omega_{o_1} = 1.1882 \times \left(2\pi \times \sqrt{1000 \times 2000}\right) = 10557.78 rad/\sec$ and $Q_1 = 1.312$ (using computer).

Section#2:

$\omega_{o_2} = \left(\dfrac{1}{1.19}\right) \times (2\pi \times 1414) = 7.46 \times 10^3 rad/\sec$, and $Q_2 = 1.3$ (using charts),

or

$\omega_{o_2} = \left(\dfrac{1}{1.1882}\right) \times \left(2\pi \times \sqrt{1000 \times 2000}\right) = 7478.54 rad/\sec$ and $Q_2 = 1.312$ (computer).

Section#3:

$\omega_{o_3} = 1.46 \times (2\pi \times 1414) = 1.30 \times 10^4 rad/\sec$, and $Q_3 = 3.3$ (charts),

or

$\omega_{o_3} = (1.4649) \times \left(2\pi \times \sqrt{1000 \times 2000}\right) = 13016.65 rad/\sec$ and $Q_3 = 3.351$ (computer).

Section#4:

$\omega_{o_4} = \left(\dfrac{1}{1.46}\right) \times (2\pi \times 1414) = 6.04 \times 10^3 rad/\sec$, and $Q_4 = 3.3$ (charts),

or

$\omega_{o_4} = \left(\dfrac{1}{1.4649}\right) \times \left(2\pi \times \sqrt{1000 \times 2000}\right) = 6065.83 rad/\sec$ and $Q_4 = 3.351$ (computer).

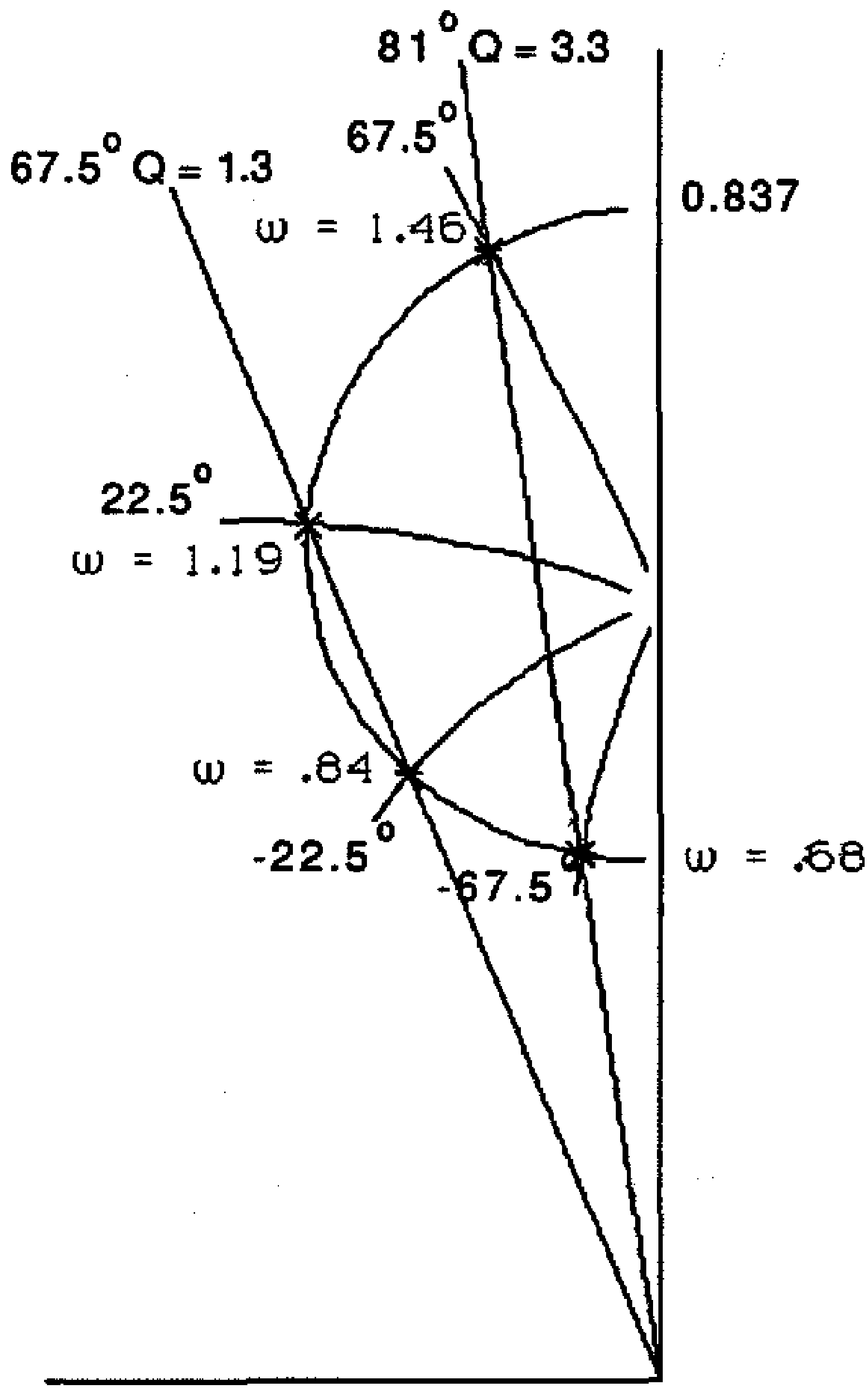

Figure 6.7. Butterworth filter poles for Example 6.5.

For the Chebyshev filter, the low pass prototype order N_{LP} is 3, and from equation 5.48 or from Figure 5.4, the low pass prototype poles are on an ellipse with minor axis a=0.49. A plot of these poles is shown in Figure 6.8. There is one real pole at S=-0.49, and a complex conjugate pair at $\pm 76^{\circ}$ *and* $\Omega_o = 1$.

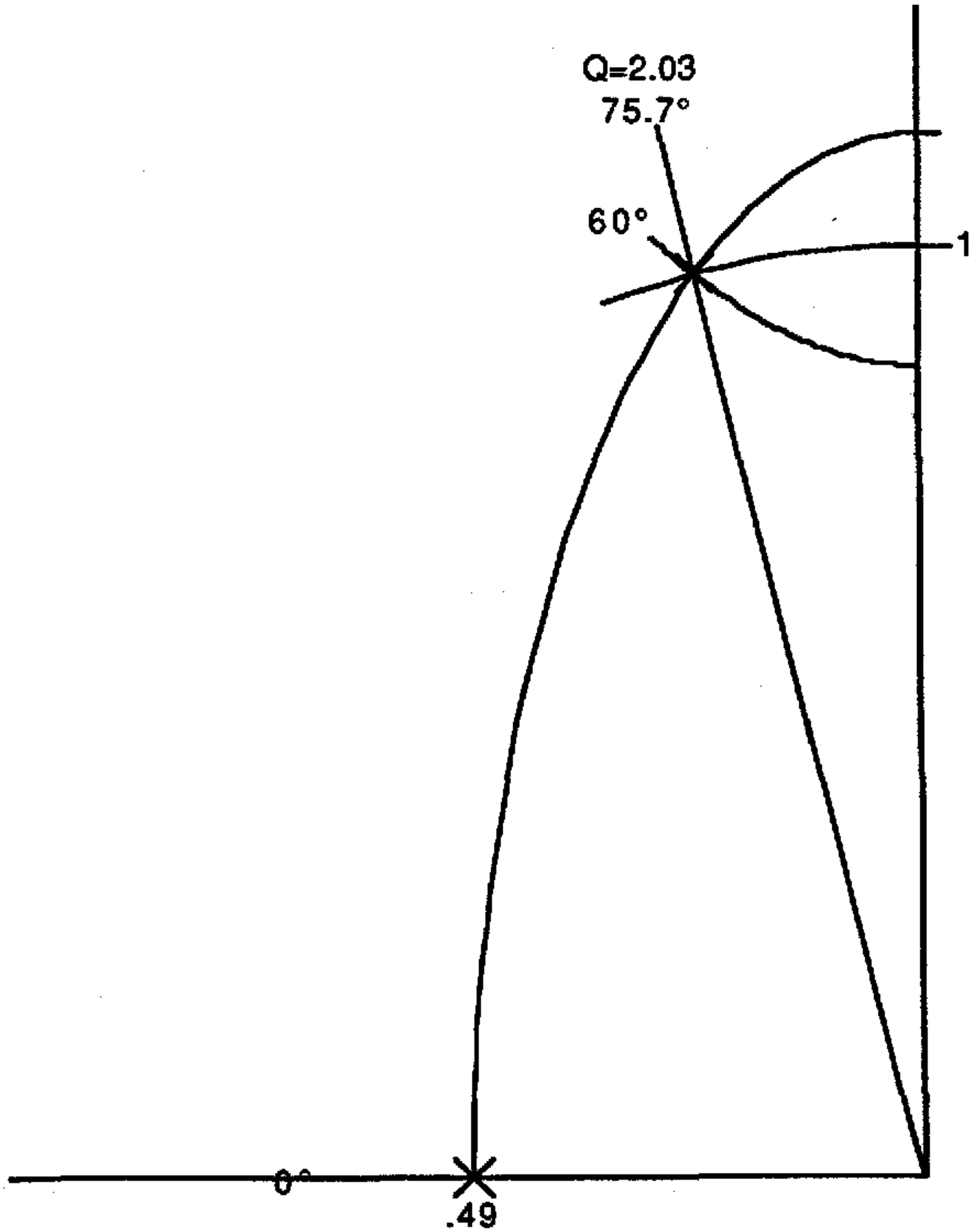

Figure 6.8. Low pass prototype poles for Chebyshev filter in Example 6.5.

One set of band pass poles (shown in Figure 6.9) is at the intersection of the 0 degree curve and the $b\Omega_o = (0.707 \times 0.49) = 0.35$ curve. Another set of band pass poles is at the intersection of the $b\Omega_o = (0.707 \times 1.0) = 0.707$ curve and the $\pm 76^\circ$ curve. The band pass filter then consists of a single second order section at the center frequency $(8.88 \times 10^3 rad/\sec)$ at $\pm 79.9^\circ$, and a pair of second order sections at $\pm 85.4^\circ$ (Q=6.05) and at frequencies of 1.4 and (1/1.4) times the center frequency (corresponding to $1.24 \times 10^4 rad/\sec \quad and \quad 6.3 \times 10^3 rad/\sec$).

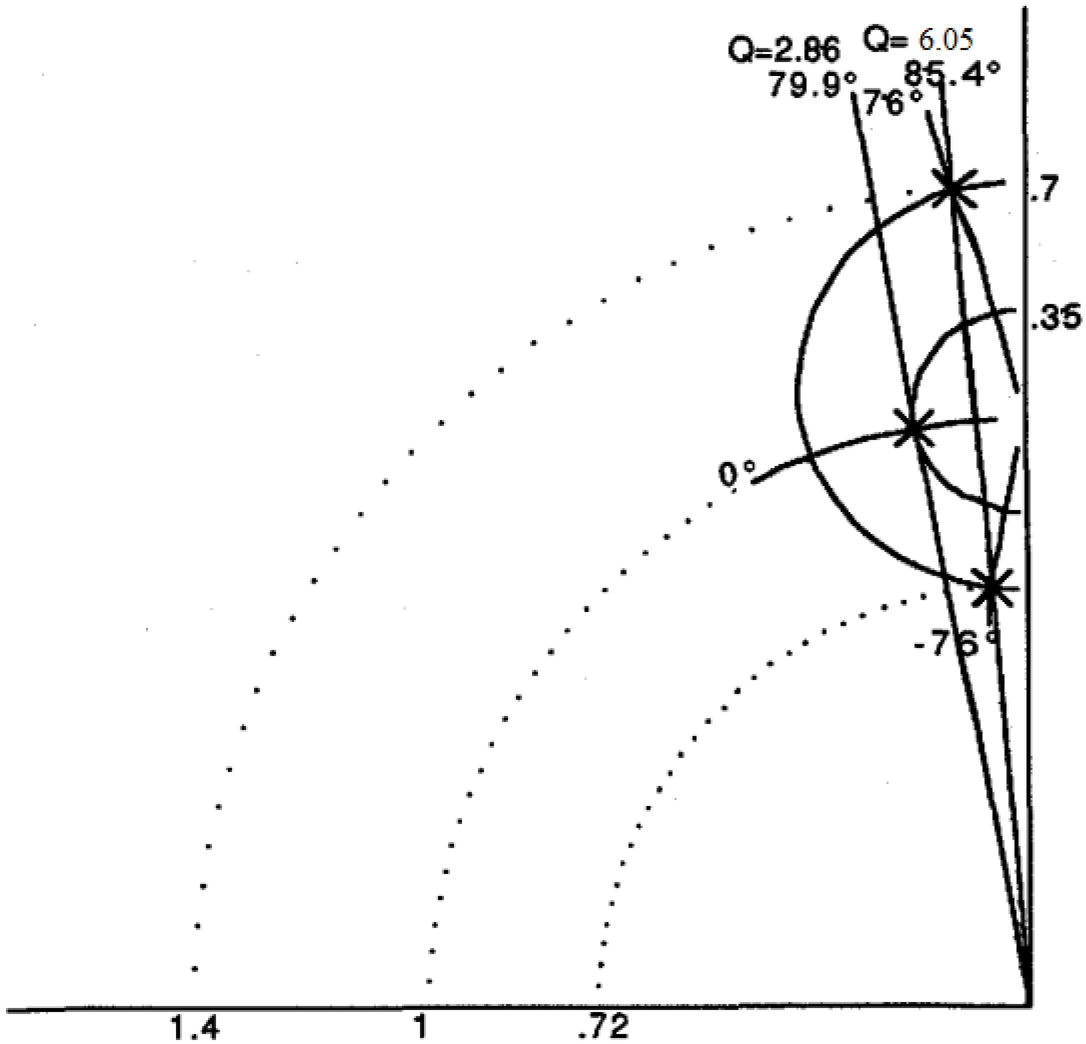

Figure 6.9. Chebyshev filter poles for Example 6.5.

6.5. Band Pass Circuits

Now that we understand the steps for band pass filter design, we will consider two circuits for implementation of these filters. The first (Case I) is similar to the Sallen Key low pass circuit introduced in Chapter 4, and is shown in Figure 6.10.

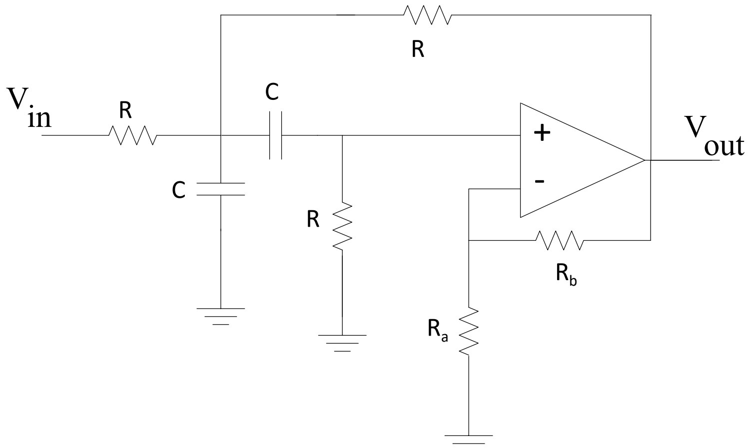

Figure 6.10. Second order band pass filter section (Case I).

Just as in our analysis of the Sallen Key circuit from Chapter 4 (Figures 4.6 and 4.7), here we will replace the combination of the op-amp and negative feedback with a non-inverting amplifier labeled A(s) in Figure 6.11. For design purposes in this section, we will assume ideal op-amp characteristics, and A(s) will be considered to have constant gain (i.e. $A(s) = 1 + \frac{R_b}{R_a}$). Later we will model A(s) as a first order low pass device and analyze the effects of finite gain-bandwidth product on our desired filter characteristics.

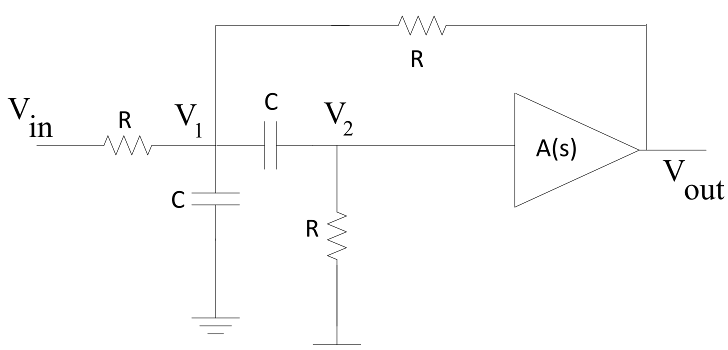

Figure 6.11. Simplified model of second order band pass filter section.

To analyze the circuit in Figure 6.11 we will first write the node voltage equation at node V_1 and the voltage divider equation relating V_2 and V_1.

$$\frac{V_1 - V_{in}}{R} + (V_1 - V_2)sC + \frac{V_1 - V_{out}}{R} + sCV_1 = 0 \tag{6.31}$$

and

$$V_2 = \left(\frac{RCs}{RCs+1}\right)V_1 \ . \tag{6.32}$$

Solving for V_1 in terms of V_2,

$$V_1 = \left(1 + \frac{1}{RCs}\right)V_2 \tag{6.33}$$

Collecting terms in 6.31, we have

$$\left(\frac{2}{R} + 2sC\right)V_1 - sCV_2 = \frac{V_{in} + V_{out}}{R}, \tag{6.34}$$

and substituting for V_1 in 6.34 we get that

$$\left(\frac{2}{R} + 2sC\right)\left(1 + \frac{1}{RCs}\right)V_2 - sCV_2 = \frac{V_{in} + V_{out}}{R} \quad . \tag{6.35}$$

After collecting terms in 6.35 we see

$$\left(\frac{4}{R} + sC + \frac{2}{R^2 sC}\right)V_2 = \frac{V_{in} + V_{out}}{R}, \tag{6.36}$$

and noting that

$$V_2 = \frac{V_{out}}{A(s)} \ , \tag{6.37}$$

we use 6.37 in 6.36 to eliminate V_2, which says

$$\left(\frac{4}{R} + sC + \frac{2}{R^2 sC}\right)\frac{V_{out}}{A(s)} = \frac{V_{in} + V_{out}}{R} \ . \tag{6.38}$$

At this point we multiply both sides of 6.38 by $\left(s\dfrac{A(s)}{C}\right)$ to obtain

$$\left[s^2 + \left(\frac{4 - A(s)}{RC}\right)s + \frac{2}{R^2C^2}\right]V_{out} = \frac{A(s)s}{RC}V_{in} \tag{6.39}$$

Now we will define

$$\omega_o = \frac{\sqrt{2}}{RC} \qquad (6.40)$$

so that 6.39 becomes

$$\left\{s^2 + \frac{1}{\sqrt{2}}\left[4 - A(s)\right]\omega_o s + \omega_o^2\right\}V_{out} = \frac{A(s)}{\sqrt{2}}\omega_o s V_{in} \qquad (6.40)$$

or

$$\frac{V_{out}}{V_{in}} = \frac{\frac{A(s)}{\sqrt{2}}\omega_o s}{s^2 + \frac{1}{\sqrt{2}}\left[4 - A(s)\right]\omega_o s + \omega_o^2} \quad . \qquad (6.41)$$

Note first that this is in the general form for a second order band pass filter. If A(s) is a constant (A), then

$$Q = \frac{\sqrt{2}}{4 - A} \qquad (6.42)$$

or

$$A = 4 - \frac{\sqrt{2}}{Q}\,, \qquad (6.43)$$

and the center frequency gain (<u>at this section's center frequency of ω_o</u>) is

$$\frac{A}{4 - A} = \frac{\left(4 - \frac{\sqrt{2}}{Q}\right)}{\left(\frac{\sqrt{2}}{Q}\right)} = 2\sqrt{2}Q - 1\,. \qquad (6.44)$$

From this point the design procedure is relatively straightforward. Once we have determined the center frequency and Q of each section in order to meet the desired filter specifications, we need only refer to two equations in order to determine component values. Equation 6.40 is used to select R and C such that

$$\boxed{RC = \frac{\sqrt{2}}{\omega_o}} \qquad (6.45)$$

and from equation 6.43 we know that

$$A = 1 + \frac{R_b}{R_a} = 4 - \frac{\sqrt{2}}{Q}$$

so

$$\boxed{\frac{R_b}{R_a} = 3 - \frac{\sqrt{2}}{Q}}\,. \tag{6.46}$$

Example 6.6

Design a Butterworth band pass filter to satisfy the specifications of Example 6.1 with the additional requirement that the center frequency gain is 0dB. These specifications are:

A_{max} = 1.0dB
A_{min} = 30dB
Pass band limits = 1kHz to 2kHz
Stop band limits = 500Hz to 4kHz

Solution:

From Example 6.1 we saw that an 8th order band pass filter is required. We solved for the Q's and resonant frequencies of each section as part of Example 6.5 and determined that:

Section#1:

$\omega_{o_1} = 1.19 \times (2\pi \times 1414) = 1.06 \times 10^4\, rad/\sec$, and $Q_1 = 1.3$ (using charts),

or $\omega_{o_1} = 1.1882 \times \left(2\pi \times \sqrt{1000 \times 2000}\right) = 10557.78\, rad/\sec$ and $Q_1 = 1.312$ (from computer).

Section#2:

$\omega_{o_2} = \left(\frac{1}{1.19}\right) \times (2\pi \times 1414) = 7.46 \times 10^3\, rad/\sec$, and $Q_2 = 1.3$ (charts),

or

$\omega_{o_2} = \left(\frac{1}{1.1882}\right) \times \left(2\pi \times \sqrt{1000 \times 2000}\right) = 7478.54\, rad/\sec$ and $Q_2 = 1.312$ (computer).

Section#3:

$\omega_{o_3} = 1.46 \times (2\pi \times 1414) = 1.30 \times 10^4\, rad/\sec$, and $Q_3 = 3.3$ (charts),

or

$\omega_{o_3} = (1.4649) \times \left(2\pi \times \sqrt{1000 \times 2000}\right) = 13016.65\, rad/\sec$ and $Q_3 = 3.351$ (computer).

Section#4:

$\omega_{o_4} = \left(\frac{1}{1.46}\right) \times (2\pi \times 1414) = 6.04 \times 10^3\, rad/\sec$, and $Q_4 = 3.3$ (charts),

or

$\omega_{o_4} = \left(\frac{1}{1.4649}\right) \times \left(2\pi \times \sqrt{1000 \times 2000}\right) = 6065.83\, rad/\sec$ and $Q_4 = 3.351$ (computer).

If we assume some reasonable value of capacitance (say 10nF for each capacitor), the resistors are given by

$$R = \frac{\sqrt{2}}{C\omega_o} = \frac{\sqrt{2} \times 10^8}{\omega_o} \tag{6.47}$$

Results for the four sections, including resistor values, are given in Table 6.1.

Section #	ω_o	R(kΩ)	Q	$\frac{R_b}{R_a}=3-\frac{\sqrt{2}}{Q}$
1	$\omega_{o_1}=1.06\times10^4 rad/\sec$	13.3	$Q_1=1.3$	1.91
2	$\omega_{o_2}=7.46\times10^3 rad/\sec$	19.0	$Q_2=1.3$	1.91
3	$\omega_{o_3}=1.30\times10^4 rad/\sec$	10.9	$Q_3=3.3$	2.57
4	$\omega_{o_4}=6.04\times10^3 rad/\sec$	23.4	$Q_4=3.3$	2.57

Table 6.1. Summary of results for Example 6.6.

At this point it might appear that our design is complete, however we do need to consider center frequency gain. In order to design for a center frequency gain of 0dB, we first need to know the gain of each second order section <u>at the overall center frequency of the entire 8th order band pass filter</u>. (One might initially be tempted to say that the overall gain is the product of the center frequency gains of each of the four stages. Unfortunately this is incorrect since these individual center frequency gains occur at different frequencies!) Since sections #1 and #2 have the same Q and are equidistant from the center frequency (on a logarithmic frequency scale), they have the same gain at the overall center frequency. The same is true for sections #3 and #4. Therefore, if we know the gain of one of these pairs at the center frequency, we know the other. In equation 6.3 we expressed the attenuation of a second order band pass filter with a center frequency of unity and a center frequency gain of 0dB as

$$A(\omega)=10\log_{10}\left[1+Q^2\left(\frac{1}{\omega}-\omega\right)^2\right]. \tag{6.3}$$

By combining 6.3 with 6.44 we can express the gain of a single second order section at any frequency ω as

$$G(\omega)=-10\log_{10}\left[1+Q^2\left(\frac{1}{\omega}-\omega\right)^2\right]+20\log_{10}\left(2\sqrt{2}Q-1\right) \tag{6.48}$$

Section#1's center frequency is $\omega_{o_1}=1.19\times CF$ (where CF is the overall filter's center frequency in rad/sec). To calculate Section#1's gain at the overall center frequency (CF) for our band pass filter, we must calculate equation 6.48 for $\omega=\frac{1}{1.19}$ which implies

$$G\left(\frac{1}{1.19}\right)=-10\log_{10}\left[1+(1.3)^2\left(1.19-\frac{1}{1.19}\right)^2\right]+20\log_{10}\left(2\sqrt{2}(1.3)-1\right)=7.74dB\,.$$

Similarly, Section#2's center frequency is $\omega_{o_2}=\frac{1}{1.19}\times CF$ (where CF is the overall filter's center frequency in rad/sec). To calculate Section#2's gain at the overall center frequency (CF) for our band pass filter, we must calculate 6.42 for $\omega=1.19$ which implies

$$G(1.19) = -10\log_{10}\left[1 + (1.3)^2\left(\frac{1}{1.19} - 1.19\right)^2\right] + 20\log_{10}\left(2\sqrt{2}(1.3) - 1\right) = 7.74dB\,.$$

Similarly we calculate gains for Sections#3 and #4 as

$$G\left(\frac{1}{1.46}\right) = -10\log_{10}\left[1 + (3.3)^2\left(1.46 - \frac{1}{1.46}\right)^2\right] + 20\log_{10}\left(2\sqrt{2}(3.3) - 1\right) = 9.64dB\,,$$

and

$$G(1.46) = -10\log_{10}\left[1 + (3.3)^2\left(\frac{1}{1.46} - 1.46\right)^2\right] + 20\log_{10}\left(2\sqrt{2}(3.3) - 1\right) = 9.64dB\,.$$

As a result, our 8th order filter would have a center frequency gain of 34.8dB if our design were not somehow modified. Using a similar technique as we used for Chebyshev low pass filters in Chapter 5, we replace the resistor at the input to each section with a voltage divider, and "spread the attenuation" across the four stages. The parallel combination of the resistors for each stage will be the value listed in Table 6.1, and the gain of each section will be adjusted to give 0dB gain at the overall center frequency. For example, for the first section we require that

$$\frac{R_1 R_2}{R_1 + R_2} = 13.3k\Omega$$

and

$$\frac{R_2}{R_1 + R_2} = 10^{\left(\frac{-7.74}{20}\right)} = 0.41$$

This voltage divider network is shown in Figure 6.12 and the final circuit is shown in Figure 6.13. The magnitude response of each individual section is shown in Figure 6.14.

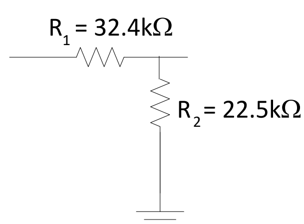

Figure 6.12. Voltage divider network for stage#1 in Example 6.6.

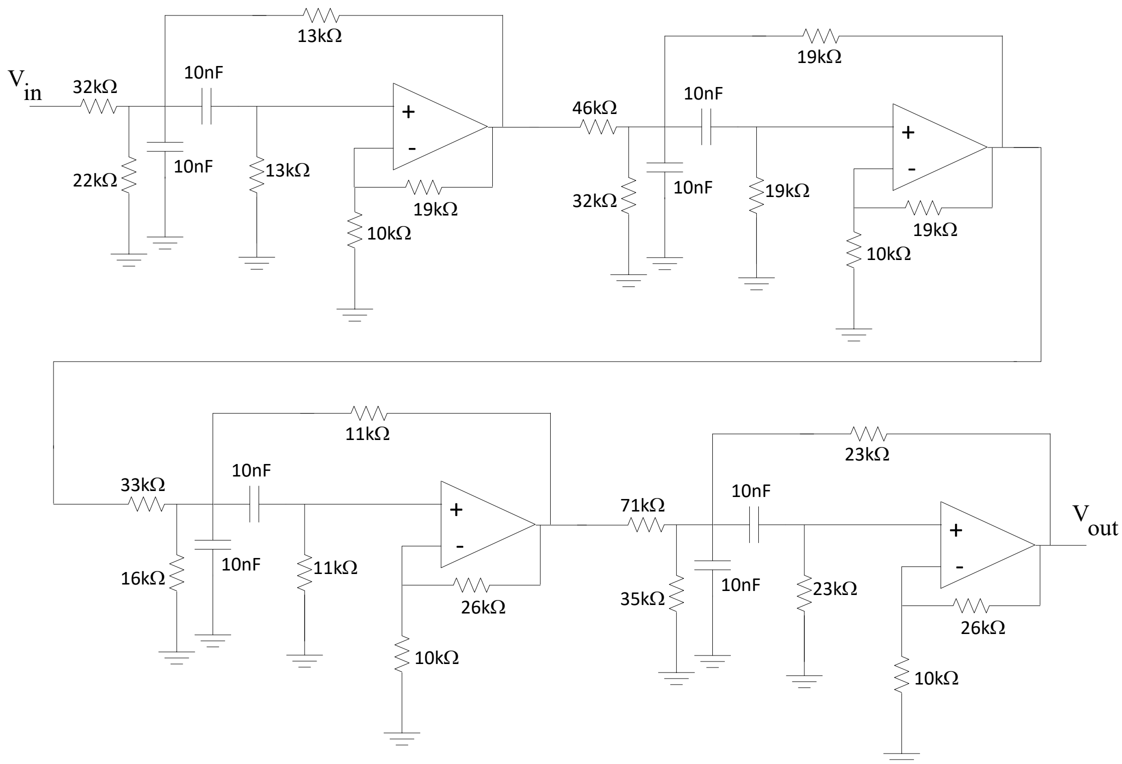

Figure 6.13. Completed circuit design for Example 6.6.

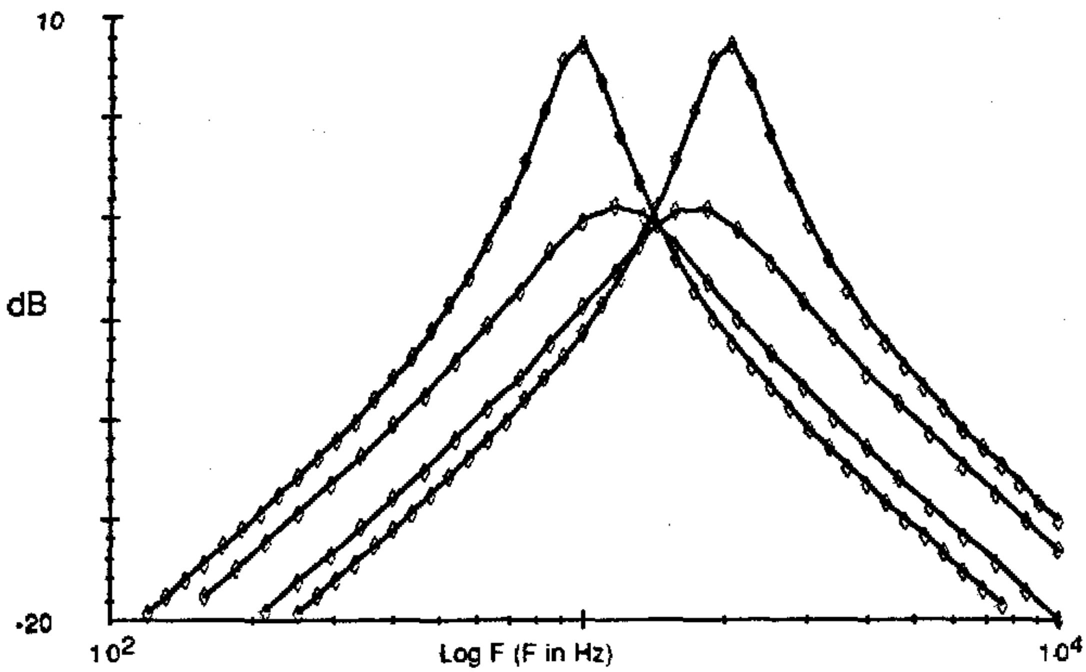

Figure 6.14. Magnitude response of individual second order sections for Example 6.6.

At this point one might question why we should "spread the attenuation" over all four stages. Certainly we could have elected to reduce the gain by 34.8dB with a single voltage divider in any single stage. The reason for spreading the attenuation across the four sections is to preserve as much dynamic range as possible. If we were to attenuate an incoming signal by 34.8dB at the first stage, our signals could easily be dropped down to the millivolt level before entering the first stage of our amplifier/filter. In general this is a very poor practice. If we instead were to attenuate by 34.8dB in the last section, the earlier stages could easily be driven into saturation at small signal levels. For these reasons, we typically "spread the gain" (or attenuation) across all stages.

Case 2

A second band pass circuit is shown in Figure 6.15.

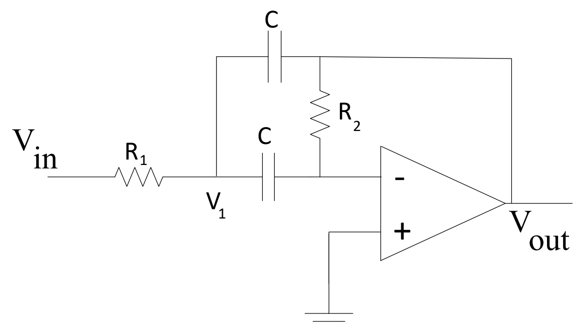

Figure 6.15. Second order band pass filter section (Case 2).

To analyze this circuit we will assume an ideal op-amp and that the inverting terminal is a virtual ground (as first presented in Chapter 3). Writing the node voltage equation at V_1, we get

$$\frac{V_1 - V_{in}}{R_1} + V_1 sC + \left(V_1 - V_{out}\right)sC_1 = 0 \ . \tag{6.49}$$

Collecting terms in 6.49 we have

$$\left(\frac{1}{R_1} + 2sC\right)V_1 - V_{out}sC = \frac{V_{in}}{R_1} \ . \tag{6.50}$$

Now let's write the node voltage equation at the "V-" node:

$$\left(V_- - V_1\right)sC + \frac{\left(V_- - V_{out}\right)}{R_2} = 0 \tag{6.51}$$

and if we recall that $V_- \cong 0$, we see that

$$V_1 sC + \frac{V_{out}}{R_2} = 0 \tag{6.52}$$

or

$$V_1 = \frac{-V_{out}}{sCR_2} \tag{6.53}$$

Substituting 6.53 into 6.50 we have

$$-\left[\left(\frac{1}{R_1} + 2sC\right)\left(\frac{1}{sCR_2}\right) + sC\right]V_{out} = \frac{V_{in}}{R_1} \tag{6.54}$$

or

$$-\left(\frac{1}{sR_1R_2C} + \frac{2}{R_2} + sC\right)V_{out} = \frac{V_{in}}{R_1} \quad . \tag{6.55}$$

Multiplying both sides of 6.55 by $\left(\frac{s}{C}\right)$ we simplify to

$$-\left(s^2 + \frac{2s}{R_2C} + \frac{1}{R_1R_2C^2}\right)V_{out} = \frac{sV_{in}}{R_1C} \tag{6.56}$$

and if we define ω_o as

$$\omega_o = \frac{1}{C\sqrt{R_1R_2}} = \frac{1}{CR_{eq}} \tag{6.57}$$

the transfer function becomes

$$\frac{V_{out}}{V_{in}} = \frac{-\sqrt{\left(\frac{R_2}{R_1}\right)}\omega_o s}{s^2 + 2\sqrt{\left(\frac{R_1}{R_2}\right)}\omega_o s + \omega_o^2} \quad . \tag{6.58}$$

Therefore

$$Q = \frac{1}{2}\sqrt{\left(\frac{R_2}{R_1}\right)} \tag{6.59}$$

and our design equations become

$$\boxed{R_2 = (2Q)R_{eq}} \tag{6.60}$$

and

$$\boxed{R_1 = \left(\frac{1}{2Q}\right)R_{eq}}. \tag{6.61}$$

From 6.58 we also see that the center frequency gain (magnitude) is

$$G_{cf} = \left. \frac{\sqrt{\left(\frac{R_2}{R_1}\right)}\omega_o s}{s^2 + 2\sqrt{\left(\frac{R_1}{R_2}\right)}\omega_o s + \omega_o^2} \right|_{s=j\omega_o} \tag{6.62}$$

$$= \frac{\sqrt{\left(\frac{R_2}{R_1}\right)}}{2\sqrt{\frac{R_1}{R_2}}} \tag{6.63}$$

so

$$\boxed{G_{cf} = \frac{R_2}{2R_1} = 2Q^2} \tag{6.64}$$

Example 6.7

Design a Chebyshev band pass filter to satisfy the specifications of Example 6.1, with the additional requirement that the maximum gain should 0dB. The specifications are:

A_{max} = 1.0dB
A_{min} = 30dB
Pass band limits = 1kHz to 2kHz
Stop band limits = 500Hz to 4kHz

Solution:

As we saw from Example 6.1, a 3rd order low pass prototype is required, which implies that we need a 6th order band pass filter in order to meet these specifications. The Q's and resonant frequencies of each section were determined as part of Example 6.5 and are repeated for convenience in Table 6.2. If we choose 10nF for each capacitor, the resistors are given by equations 6.57, 6.60 and 6.61.

Section #	ω_o	R_{eq}(kΩ)	Q	R_1(kΩ)	R_2(kΩ)
1	$\omega_{o_1} = 8.88\times10^3 rad/\sec$	11.3	$Q_1 = 2.9$	1.95	66
2	$\omega_{o_2} = 6.3\times10^3 rad/\sec$	15.9	$Q_2 = 6.05$	1.31	192
3	$\omega_{o_3} = 1.24\times10^4 rad/\sec$	8.06	$Q_3 = 6.05$	0.67	98

Table 6.2. Summary of results for Example 6.5.

Since the low pass Chebyshev prototype was 3rd order, its DC gain was a maximum, and therefore the maximum gain of our filter does occur at the center frequency of our filter. Therefore, if we set the center frequency gain to be 0dB, we are setting the maximum gain to be 0dB. (Recall for Chebyshev filters, this is not always the case due to pass band ripple!) Just as we did for the case of the Butterworth band pass filter in Example 6.6, in order to constrain the center frequency gain to be 0dB, we need to know the gain of each section at the overall filter center frequency. Similar to the case of equation 6.48 in the previous example, we find that the gain in dB of an individual section at the overall filter center frequency is given as

$$G(\omega) = -10\log_{10}\left[1+Q^2\left(\frac{1}{\omega}-\omega\right)^2\right] + 20\log_{10}\left(2Q^2\right), \tag{6.65}$$

or for this example, stages #2 and #3 each have a gain of

$$-10\log_{10}\left[1+(6.05)^2\left(\frac{1}{1.4}-1.4\right)^2\right] + 20\log_{10}\left(2\times(6.05)^2\right) = 24.7dB$$

at the filter center frequency, and section #1 has a gain of

$$20\log_{10}\left(2\times 2.9^2\right) = 24.5dB$$

at the filter center frequency.

Once again we will use a voltage divider at the input to each section, so that the gain of each section at the overall filter center frequency is 0dB. We replace the resistor at the input to each section (R_1) with a voltage divider, and "spread the attenuation" across the four stages. The parallel combination of the resistors for each stage will be the value listed for (R_1) in Table 6.2, and the gain of each section will be adjusted to give 0dB gain at the overall center frequency. For example, for the first section we require that

$$\frac{R_1R_2}{R_1+R_2} = 1.95k\Omega$$

and

$$\frac{R_2}{R_1+R_2} = 10^{\left(\frac{-24.5}{20}\right)} = 0.0596$$

This voltage divider network is shown in Figure 6.16 and the final circuit is shown in Figure 6.17. The magnitude response of each individual section is shown in Figure 6.18.

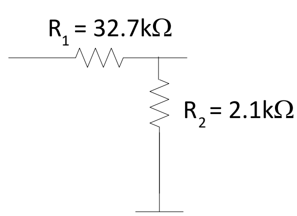

Figure 6.16. Voltage divider network for stage#1 in Example 6.7.

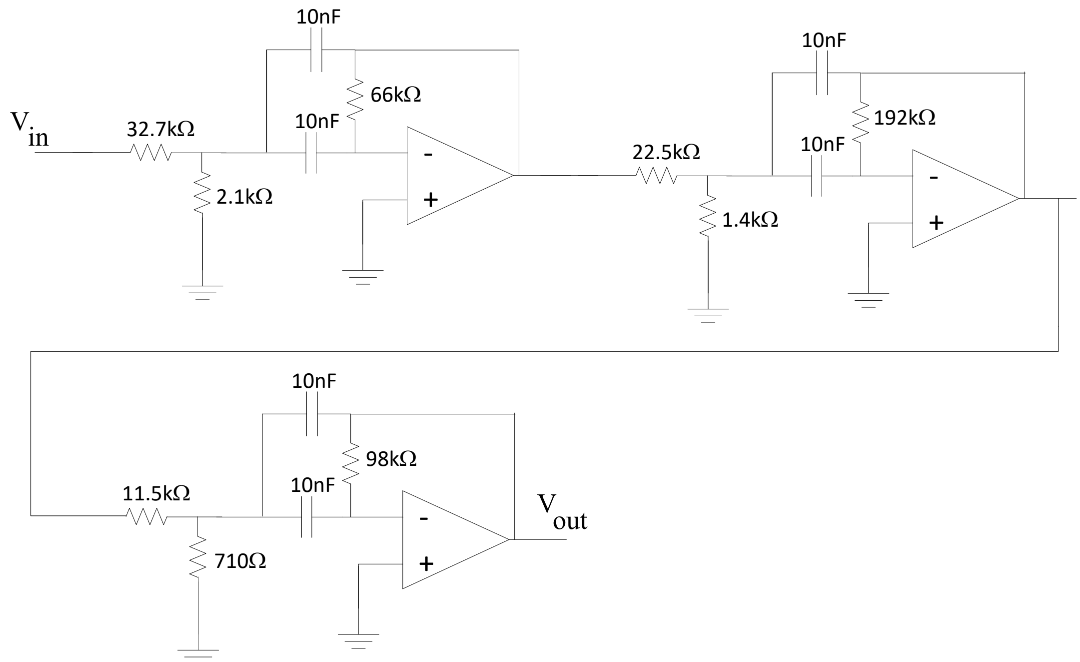

Figure 6.17. Final design for Example 6.7.

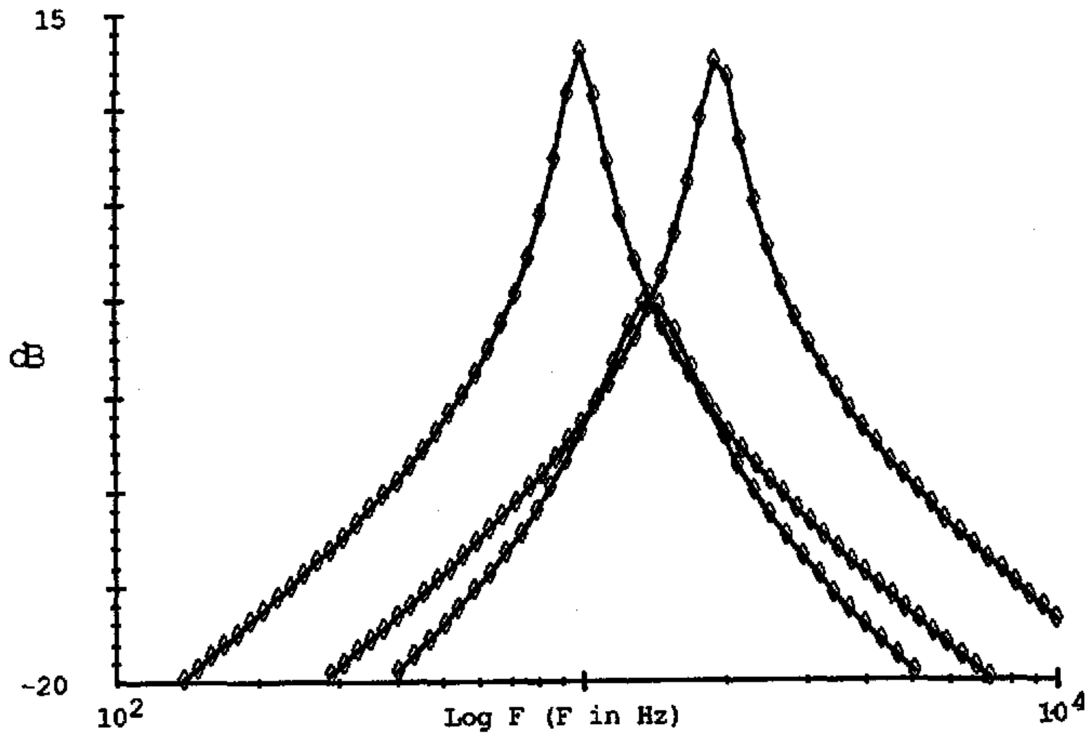

Figure 6.18. Frequency responses of individual second order sections of Example 6.7

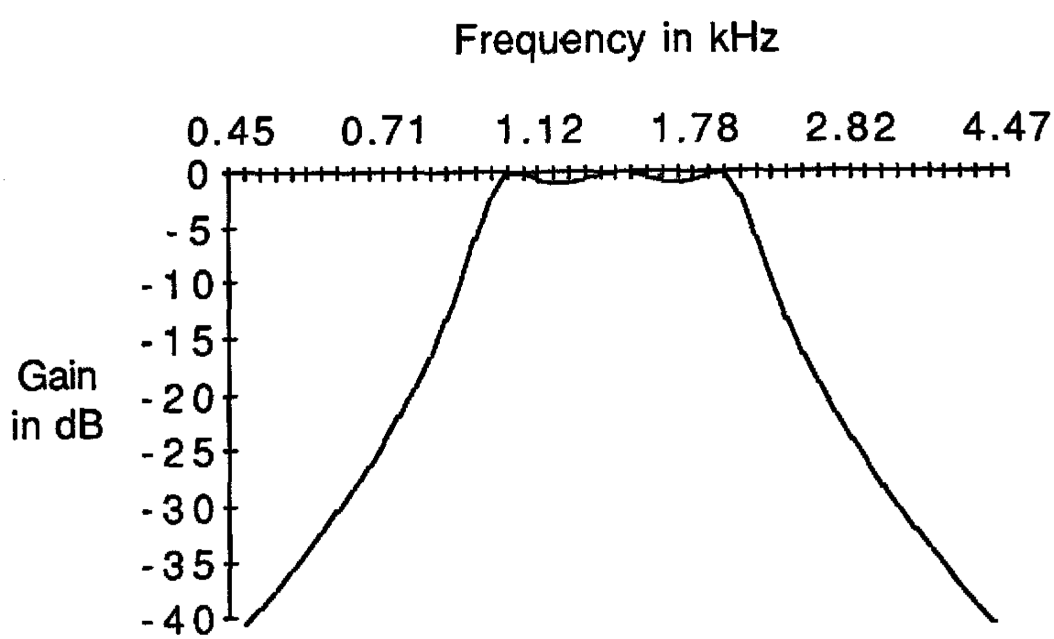

Figure 6.19. Composite frequency response for filter design in Example 6.7.

6.6. Notch Filters

The relationship between notch filters and high pass prototypes is exactly the same as that between band pass filters and low pass prototypes. This relationship is illustrated in Figure 6.20.

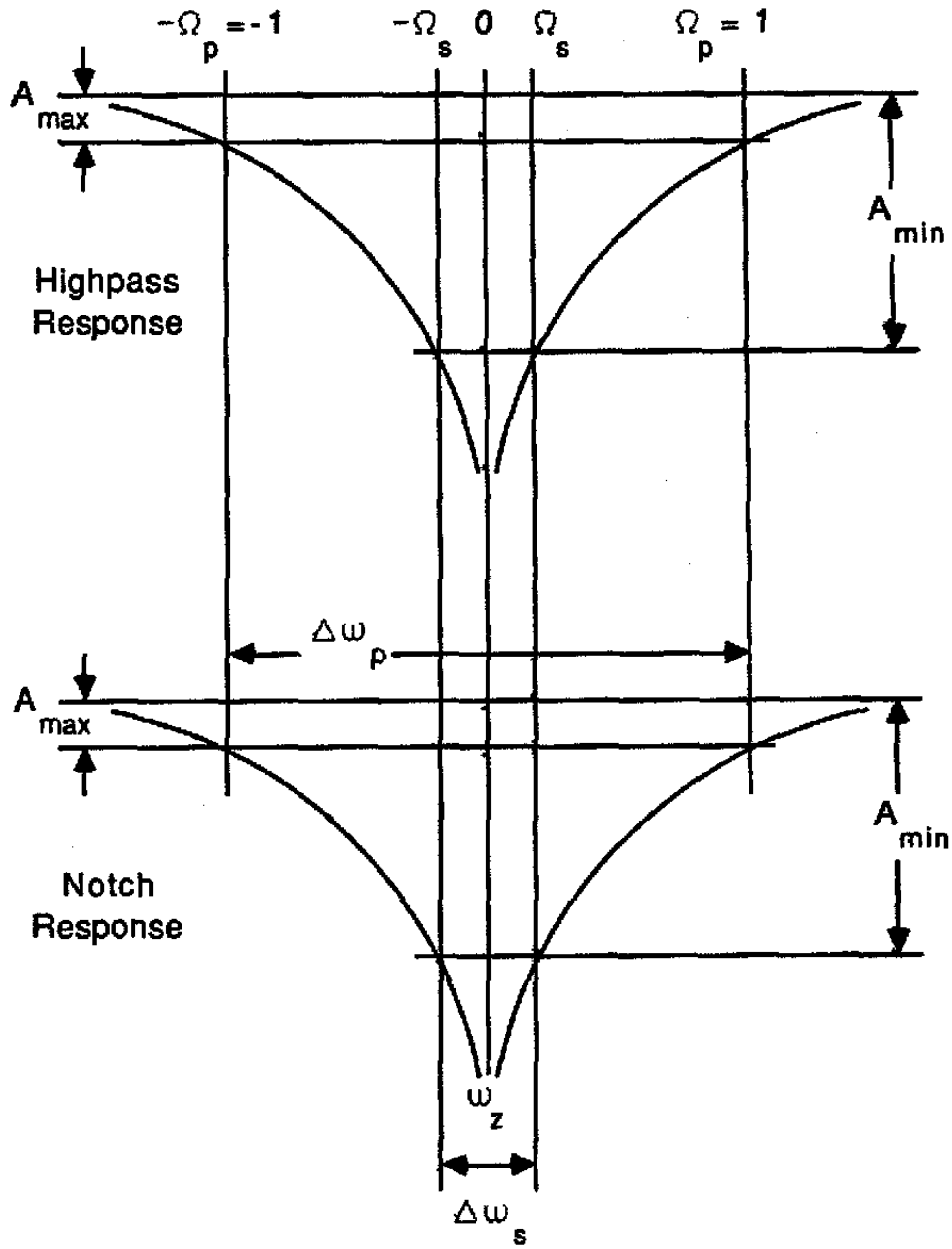

Figure 6.20. Frequency response relationships between high pass prototype and notch filter response.

Again we will set the pass band limit of or (high pass) prototype equal to unity. The poles will map from our high pass "S" plane to the notch "s" plane exactly as poles mapped from the low pass "S" plane to the band pass "s" plane. In addition, using 6.17 (or Figures 6.5a or 6.5b) the zeros at S=0 in our high pass prototypes will map to

$$s = \left(\frac{bS}{2}\right) \pm \sqrt{\frac{(bS)^2}{4} - 1}\,\Bigg|_{S=0} \tag{6.66}$$

$$= \pm\sqrt{-1}$$
$$= \pm j \text{ (normalized to } \omega_z=1\text{).}$$

Since the zero at S=0 in our high pass prototype maps to a zero at $\pm j$ in our notch filter, it means we need to implement filters with zeros on the s-plane jω axis and with complex conjugate poles. This implies the general form of the notch filter transfer function will be

$$\frac{V_{out}}{V_{in}} = \frac{A\left(s^2 + \omega_z^2\right)}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2} \tag{6.67}$$

Example 6.8.

Determine the minimum order and the pole/zero locations for a Butterworth notch filter to satisfy the following specifications:

A_{max}=1dB
A_{min}=15dB
Maximum gain = 0dB
Pass band limits = 500Hz and 2kHz
Stop band limits = 800Hz to 1250Hz

Solution:
The pass band and stop band limits are both geometrically symmetric about the same frequency (1kHz). Since this is a "notch filter" design, we must calculate the ratio of $\frac{\Delta\omega_p}{\Delta\omega_s}$ (instead of $\frac{\Delta\omega_s}{\Delta\omega_p}$ as we had in our band pass filter design)

$$\frac{\Delta\omega_p}{\Delta\omega_s} = \frac{(2000-500)\times 2\pi}{(1250-800)\times 2\pi} = 3.3333$$

Substituting this ratio to calculate the order for our HIGH PASS prototype filter (or using the nomograph from Figure 5.3), we obtain

$$n = \frac{\ln\left(\frac{10^{A_{min}/10}-1}{10^{A_{max}/10}-1}\right)}{2\ln\left(\frac{\Delta\omega_p}{\Delta\omega_s}\right)} = 1.9822$$, which we round to n=2.

The pole frequency relative to Ω_p (the pass band limit) of our <u>high</u> pass prototype is given by

$$\Omega_o = \left(10^{A_{max}/10} - 1\right)^{\frac{1}{2N_{HP}}} = 0.7133.$$

The difference of pass band limits relative to the center frequency of the notch filter is

$$b = \frac{2000-500}{1000} = 1.5$$

At this point we enter Figure 6.5a with $b\Omega_o = (1.5)(0.7133) = 1.07$ and $\pm 45^\circ$. Figure 6.21 shows the resulting poles and zeros of our notch filter.

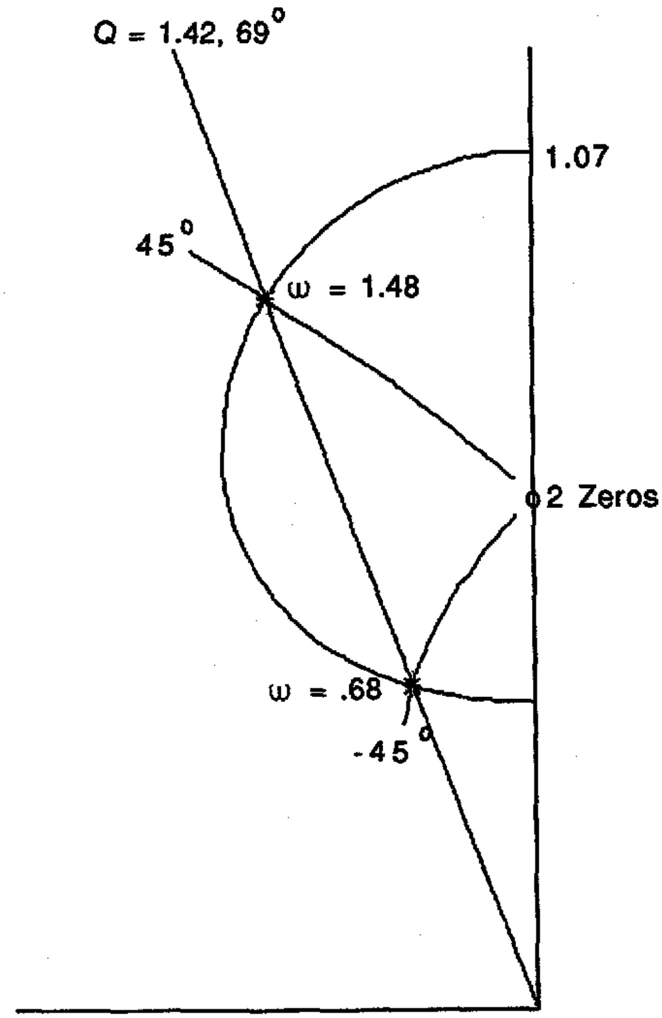

Figure 6.21. Pole/zero diagram for notch filter Example 6.8.

The poles therefore have Q's of approximately 1.4255 and frequencies of

$$\omega_o = 1.4825 \times 2\pi \times 1000 = 9314.7 r/s$$

and

$$\omega_o = \left(\frac{1}{1.4825}\right) \times 2\pi \times 1000 = 4238.3 r/s$$

The two remaining poles are below the real s-plane axis, and are just the complex conjugates of the ones shown in Figure 6.21. Finally, there are four zeros, two each at $s = j \times (2\pi) \times 1000$ r/s and $s = -j \times (2\pi) \times 1000$ r/s (i.e. at the center frequency of the notch filter).

Implementation of notch filters is a bit more complicated since there are now three entering parameters: Q, ω_o, and ω_z. There are several choices of single op-amp sections for notch filter implementation in the literature. We will consider only two versions of one basic circuit here for purposes of analysis. The first version is shown in Figure 6.22, for $\omega_o > \omega_z$. A slightly different version (for $\omega_o < \omega_z$) will be analyzed later. Finally, for the case where $\omega_o = \omega_z$ (as will be the case for a second order section originating from a high pass prototype real pole), the capacitor C_1 in Figure 6.22 is removed.

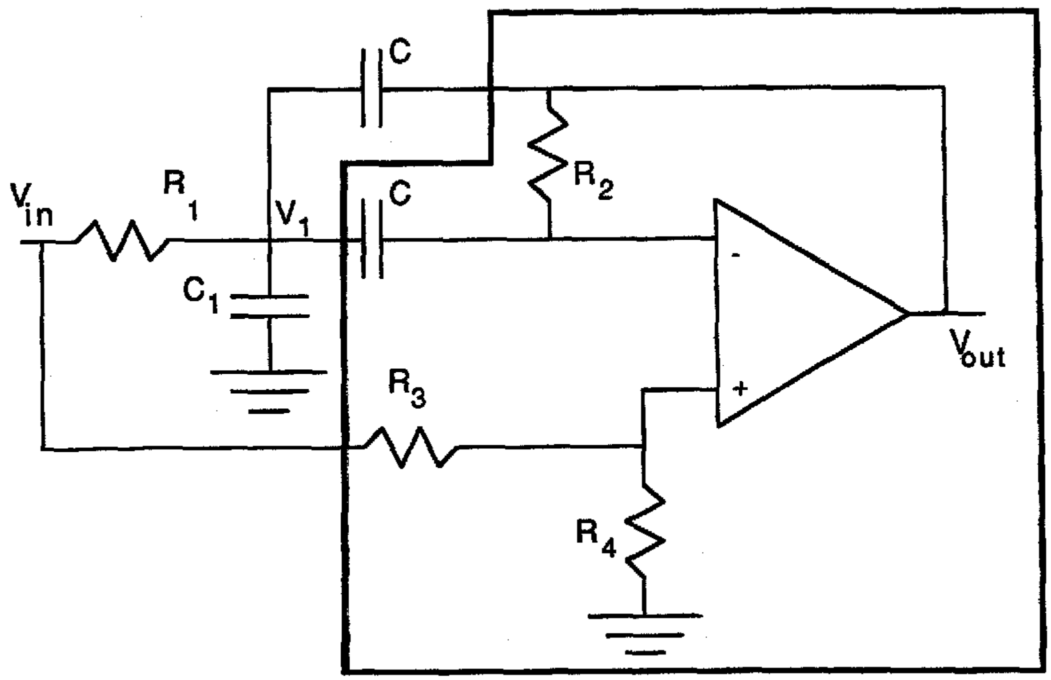

Figure 6.22. Notch filter for $\omega_o > \omega_z$.

We first recognize from analysis techniques introduced in Chapter 3 that the portion inside the bold line in Figure 6.22 represents an inverting differentiator from V_1 to V_{out}, and unity gain plus a non-inverting differentiator from V_+ to V_{out}. More specifically,

$$V_{out} = -R_2CsV_1 + (1+R_2Cs)V_+ \quad . \tag{6.68}$$

Defining the parameter "k" by

$$k = \frac{R_4}{R_4 + R_3}, \text{ this says}$$

$$V_+ = kV_{in}$$

and

$$V_{out} = -R_2CsV_1 + \left(1 + R_2Cs\right)kV_{in} \tag{6.69}$$

or

$$V_1 = \frac{kV_{in} - V_{out}}{R_2Cs} + kV_{in} \tag{6.70}$$

The node equation at V_1 is

$$\frac{V_1 - V_{in}}{R_1} + sC_1V_1 + sC\left(V_1 - V_{out}\right) + sC\left(V_1 - kV_{in}\right) = 0\,. \tag{6.71}$$

After collecting terms, 6.71 becomes

$$\left(\frac{1}{R_1} + 2sC + sC_1\right)V_1 - \left(\frac{1}{R_1}\right)V_{in} - sC\left(V_{out} + kV_{in}\right) = 0\,. \tag{6.72}$$

Using 6.70 to substitute for V_1,

$$\left(\frac{1}{R_1} + 2sC + sC_1\right)\left(\frac{kV_{in} - V_{out}}{R_2Cs} + kV_{in}\right) - \frac{V_{in}}{R_1} - sC\left(V_{out} + kV_{in}\right) = 0\,. \tag{6.73}$$

Grouping terms in V_{in} and V_{out} we obtain

$$\left[\left(\frac{1}{R_1} + 2sC + sC_1\right)\left(\frac{1}{R_2Cs} + 1\right) - \frac{1}{kR_1} - sC\right]kV_{in} =$$

$$\left(\frac{1}{R_1R_2sC} + \frac{2}{R_2} + \frac{C_1}{R_2C} + \frac{1}{R_1} + sC + sC_1 - \frac{1}{kR_1}\right)kV_{in} = \left(\frac{1}{R_1R_2sC} + \frac{2}{R_2} + \frac{C_1}{R_2C} + sC\right)V_{out} \tag{6.74}$$

Multiplying both sides by $\left(\dfrac{s}{C}\right)$ and defining ω_o as

$$\omega_o = \frac{1}{C\sqrt{R_1R_2}} = \frac{1}{CR_{eq}},$$

the transfer function simplifies to

$$\frac{V_{out}}{V_{in}} = \frac{k\left[\left(1 + \frac{C_1}{C}\right)s^2 + \left(\frac{2}{CR_2} + \frac{C_1}{R_2C^2} + \frac{1}{CR_1} - \frac{1}{kCR_1}\right)s + \omega_o^2\right]}{s^2 + \left(\frac{2}{CR_2} + \frac{C_1}{R_2C^2}\right)s + \omega_o^2}\,. \tag{6.75}$$

Recall that we would like 6.75 to be expressed in the general form for a notch filter (taken from 6.67) as

$$\frac{V_{out}}{V_{in}} = \frac{A\left(s^2 + \omega_z^2\right)}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2} \quad . \tag{6.76}$$

Recognize that if we were to divide numerator and denominator of 6.75 by $\left(1 + \frac{C_1}{C}\right)$ (the coefficient of s^2 in the numerator), we would see that the constant coefficient in the numerator becomes $\omega_z^2 = \frac{\omega_o^2}{\left(1 + \frac{C_1}{C}\right)}$,

thus we choose C_1 such that

$$C_1 = \left(\frac{\omega_o^2}{\omega_z^2} - 1\right)C .$$

Note that when $\omega_z = \omega_o$ (as will be the case when mapping a real pole of a high pass prototype), the capacitor C_1 is eliminated. (Also note that the case of $\omega_o < \omega_z$ is not possible with this circuit!)

Setting the coefficients of s in the denominators of 6.76 and 6.75 we obtain

$$\frac{\omega_o}{Q} = \frac{1}{\sqrt{R_1 R_2}CQ} = \frac{2}{CR_2} + \frac{C_1}{R_2 C^2} . \tag{6.77}$$

Multiplying both sides of 6.77 by R_2C we realize that

$$\sqrt{\frac{R_2}{R_1}}\frac{1}{Q} = 2 + \frac{C_1}{C} = 1 + \frac{\omega_o^2}{\omega_z^2} ,$$

thus R_1 and R_2 are selected by

$$\boxed{R_2 = Q\left(1 + \frac{\omega_o^2}{\omega_z^2}\right)R_{eq}} \tag{6.78}$$

and

$$\boxed{R_1 = \frac{R_{eq}}{Q\left(1 + \frac{\omega_o^2}{\omega_z^2}\right)}} \quad . \tag{6.79}$$

Finally, in order to obtain a notch filter, we must set the coefficient of s in the numerator of 6.75 equal to zero, which means

$$\left(\frac{2}{CR_2}+\frac{C_1}{R_2C^2}+\frac{1}{CR_1}-\frac{1}{kCR_1}\right)=0\,. \tag{6.80}$$

Multiplying both sides by CR_1 we obtain

$$\frac{1}{k}=\frac{2R_1}{R_2}+\frac{C_1R_1}{R_2C}+1$$

or

$$=\frac{R_1}{R_2}\left(2+\frac{C_1}{C}\right)+1$$

$$=\frac{1+\frac{\omega_o^2}{\omega_z^2}}{Q^2\left(1+\frac{\omega_o^2}{\omega_z^2}\right)^2}+1$$

so

$$\frac{1}{k}=\frac{1}{Q^2\left(1+\frac{\omega_o^2}{\omega_z^2}\right)}+1\ . \tag{6.81}$$

From 6.68 we know $k=\dfrac{R_4}{R_3+R_4}$, so $\dfrac{1}{k}=1+\dfrac{R_3}{R_4}$, therefore from 6.81 we choose R_3 and R_4 by

$$\boxed{\frac{R_4}{R_3}=Q^2\left(1+\frac{\omega_o^2}{\omega_z^2}\right)} \tag{6.82}$$

For the case where we need to design a circuit such that $\omega_o<\omega_z$, the capacitor C_1 is eliminated, and a resistor R_5 is added between the inverting input and ground, as shown in Figure 6.23. Note that this resistor does not change the transfer function from V_1 to V_{out}, but adds a DC term $\left(\dfrac{R_2}{R_5}\right)$ to the transfer function from V_+ to V_{out}, or

$$V_{out}=-R_2CsV_1+\left(1+\frac{R_2}{R_5}+R_2Cs\right)V_+$$

Defining parameter "k" as we did in 6.68 $\left(k=\dfrac{R_4}{R_3+R_4}\right)$, and recognizing that $V_+=kV_{in}$, we have that

$$V_{out} = -R_2CsV_1 + \left(1 + \frac{R_2}{R_5} + R_2Cs\right)kV_{in} \tag{6.83}$$

or

$$V_1 = \frac{-V_{out}}{R_2Cs} + \left(\frac{1}{R_2Cs} + \frac{1}{R_5Cs} + 1\right)kV_{in} \tag{6.84}$$

The node equation at V_1 is:

$$\frac{V_1 - V_{in}}{R_1} + sC\left(V_1 - V_{out}\right) + sC\left(V_1 - kV_{in}\right) = 0\,. \tag{6.85}$$

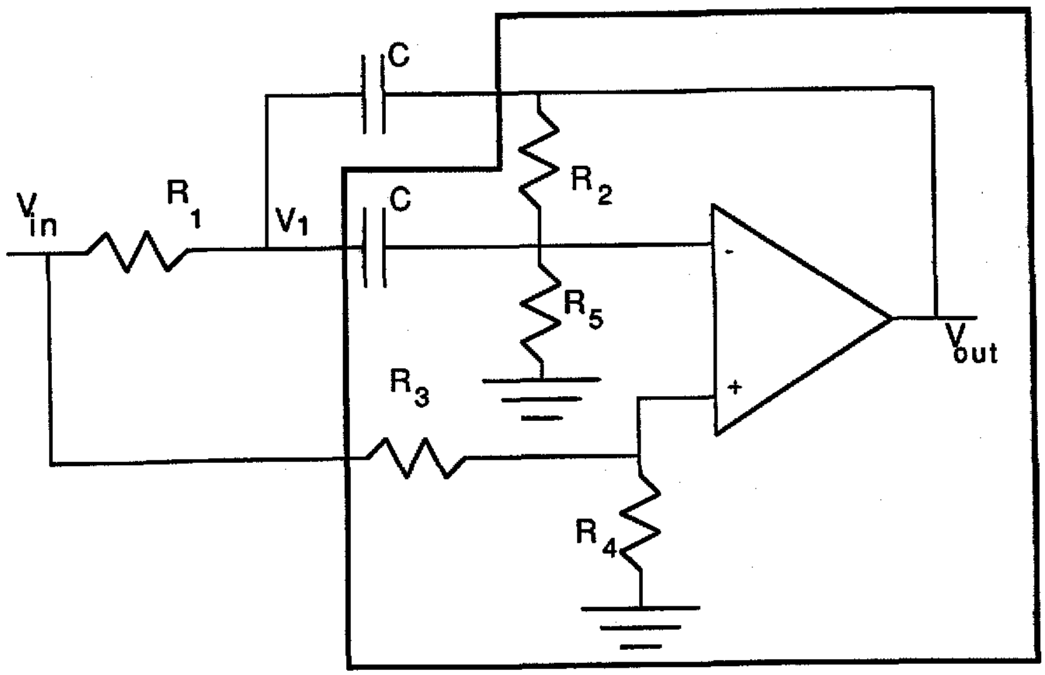

Figure 6.23. Notch filter circuit for $\omega_o < \omega_z$.

Note that this resistor does not change the transfer function from

Collecting terms from 6.85,

$$\left(\frac{1}{R_1} + 2sC\right)V_1 - \left(\frac{1}{R_1}\right)V_{in} - sC\left(V_{out} + kV_{in}\right) = 0 \tag{6.86}$$

Using 6.84 to substitute for V_1 we obtain

$$\left(\frac{1}{R_1}+2sC\right)\left[\frac{-V_{out}}{R_2Cs}+\left(\frac{1}{R_2Cs}+\frac{1}{R_5Cs}+1\right)kV_{in}\right]-\frac{1}{R_1}V_{in}-sC\left(V_{out}+kV_{in}\right)=0 \tag{6.87}$$

and collecting terms results in

$$\left[\left(\frac{1}{R_1}+2sC\right)\left(\frac{1}{R_2Cs}+\frac{1}{R_5Cs}+1\right)-\frac{1}{kR_1}-sC\right]kV_{in}=\left(\frac{1}{R_1R_2Cs}+\frac{2}{R_2}+sC\right)V_{out}\ . \tag{6.88}$$

At this point we multiply both sides by $\left(\frac{s}{C}\right)$, and define

$$\omega_o=\frac{1}{\sqrt{R_1R_2}C}=\frac{1}{R_{eq}C} \tag{6.89}$$

and we derive the transfer function from 6.88 as

$$\frac{V_{out}}{V_{in}}=\frac{k\left[s^2+\left(\frac{2}{CR_2}+\frac{2}{CR_5}+\frac{1}{CR_1}-\frac{1}{kCR_1}\right)s+\left(1+\frac{R_2}{R_5}\right)\omega_o^2\right]}{s^2+\left(\frac{2}{CR_2}\right)s+\omega_o^2}\quad . \tag{6.90}$$

We would like 6.90 to be of the form given in 6.76, namely

$$\frac{V_{out}}{V_{in}}=\frac{A\left(s^2+\omega_z^2\right)}{s^2+\frac{\omega_o}{Q}s+\omega_o^2} \tag{6.76}$$

so we begin our design by selecting $R_{eq}=\frac{1}{\sqrt{R_1R_2}}$ and C such that

$$R_{eq}C=\frac{1}{\omega_o} \tag{6.91}$$

Setting the coefficients of s in the denominators of 6.76 and 6.90 equal we obtain

$$\frac{\omega_o}{Q}=\frac{1}{\sqrt{R_1R_2}CQ}=\frac{2}{CR_2}, \tag{6.92}$$

or

$$\sqrt{\frac{R_2}{R_1}}\left(\frac{1}{Q}\right)=2 \tag{6.93}$$

so that R_1 and R_2 are selected by

$$\boxed{R_2 = (2Q)R_{eq}} \tag{6.94}$$

and

$$\boxed{R_1 = \left(\frac{1}{2Q}\right)R_{eq}} \ . \tag{6.95}$$

Since

$$\omega_z^2 = \omega_o^2\left(1+\frac{R_2}{R_5}\right) \tag{6.96}$$

R_5 is then given by

$$\boxed{R_5 = \frac{R_2}{\left(\frac{\omega_z^2}{\omega_o^2}-1\right)}} \ . \tag{6.97}$$

Finally, in order to obtain a notch filter, we must set the coefficient of s in the numerator of 6.90 equal to zero, which means

$$\left(\frac{2}{CR_2}+\frac{2}{CR_5}+\frac{1}{CR_1}-\frac{1}{kCR_1}\right)=0\,. \tag{6.98}$$

Multiplying both sides by CR_1 we obtain

$$\frac{1}{k}=\frac{2R_1}{R_2}+\frac{2R_1}{R_5}+1$$

or

$$\frac{1}{k}=\frac{2R_1}{R_2}\left(1+\frac{R_2}{R_5}\right)+1=\frac{\omega_z^2}{2Q^2\omega_o^2}+1 \ . \tag{6.99}$$

Since

$$k=\frac{R_4}{R_3+R_4}, \tag{6.100}$$

or

$$\frac{1}{k}=1+\frac{R_3}{R_4}, \tag{6.101}$$

we know that R_3 and R_4 are selected by

$$\boxed{\frac{R_4}{R_3}=2Q^2\left(\frac{\omega_o^2}{\omega_z^2}\right)} \tag{6.102}$$

Example 6.9. Design a circuit to satisfy the specifications given in Example 6.8.

Solution: Recall from Example 6.8 our given specifications were:

A_{max}=1dB; A_{min}=15dB
Maximum gain = 0dB
Pass band limits = 500Hz and 2kHz
Stop band limits = 800Hz to 1250Hz

Recall also from Example 6.8 that the calculated poles had Q's of 1.4255 and frequencies of

$$\omega_o = 1.4825 \times 2\pi \times 1000 = 9314.7 r/s \text{ and } \omega_o = \left(\frac{1}{1.4825}\right) \times 2\pi \times 1000 = 4238.3 r/s$$

Recall also there are four zeros, two each at $s = j \times (2\pi) \times 1000$ r/s and $s = -j \times (2\pi) \times 1000$ r/s (i.e. at the center frequency of the notch filter).

We start our design by arbitrarily selecting C=10nF in each section. The design equations as well as the specifics of this design are summarized in Table 6.3.

Parameter	High Pass Section (Figure 6.22)	Parameter	Low Pass Section (Figure 6.23)
$\omega_o (rad/s)$	9.29×10^3	$\omega_o (rad/s)$	4.24×10^3
Q	1.42	Q	1.42
$\omega_z (rad/s)$	6.28×10^3 r/s	$\omega_z (rad/s)$	6.28×10^3 r/s
$\frac{\omega_o^2}{\omega_z^2}$	2.19	$\frac{\omega_o^2}{\omega_z^2}$	$\frac{1}{2.19}$
$R_{eq} = \frac{1}{\omega_o C}$	10.8kΩ	$R_{eq} = \frac{1}{\omega_o C}$	23.6kΩ
R_1	$\frac{R_{EQ}}{Q\left(1+\frac{\omega_o^2}{\omega_z^2}\right)} = 2.38k\Omega$	$\frac{R_{eq}}{2Q}$	$\frac{R_{eq}}{2Q} = 8.3k\Omega$
R_2	$Q\left(1+\frac{\omega_o^2}{\omega_z^2}\right)R_{EQ} = 49k\Omega$	$R_{eq} \times 2Q$	$R_{eq} \times 2Q = 67k\Omega$
C_1	$\left[\left(\frac{\omega_o^2}{\omega_z^2}\right)-1\right] \times C$	R_5	$R_5 = \frac{R_2}{\left(\frac{\omega_z^2}{\omega_o^2}-1\right)} = 56k\Omega$
$\frac{R_4}{R_3}$	$Q^2\left[1+\left(\frac{\omega_o^2}{\omega_z^2}\right)\right] = 6.4$	$2Q^2\left(\frac{\omega_o^2}{\omega_z^2}\right)$	$2Q^2\left(\frac{\omega_o^2}{\omega_z^2}\right) = 1.84$
DC Gain	$k = \frac{R_4}{R_4+R_3} = 0.865$	DC Gain	$\left(\frac{R_4}{R_4+R_3}\right)\left(1+\frac{R_2}{R_5}\right) = 1.42$

Table 6.3. Design summary for Example 6.9.

The circuit is shown in Figure 6.24. If the overall DC gain (1.23) is not correct, it could be reduced as before with voltage division at the input, or increased by using an additional stage.

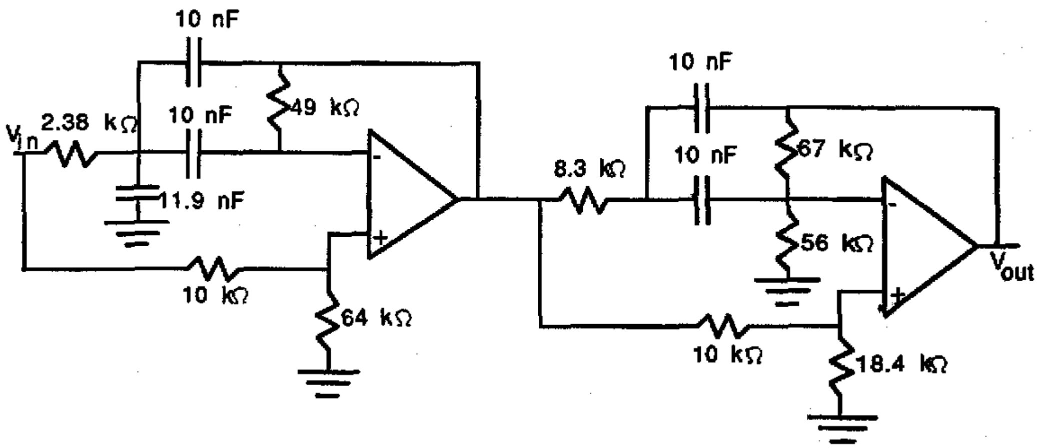

Figure 6.24 Notch Filter Circuit for Example 6.9

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GO TO SECTION 6.7 of 06-Bandpass and Notch Filters on Desire2Learn (older chapter 6), which can be found on page 6-52.