Chapter 5
Chebyshev Filters

5.1 Introduction

In the previous chapter we required the response of Butterworth low pass filters to be maximally flat in the pass band. In this chapter we will relax that constraint and allow ripple within the pass band. We will allow the attenuation to vary between 0 and A_{max} dB within the pass band, but we will still require that it to go monotonically to infinity in dB (or the gain to a magnitude of zero) as the frequency approaches infinity. This filter, called a Chebyshev (type 1) filter, will have a transfer function numerator equal to a constant (just as in the case of Butterworth low pass filters), and the overall magnitude response will roll off at the maximum rate of 20n dB/decade, where n is the filter order.

The major advantage of allowing ripple within the pass band will be that we can satisfy the identical specifications (A_{max}, A_{min} and ratio of stop band to pass band limiting frequencies) with a lower order filter when compared to Butterworth filter order. Unfortunately, the calculations required to determine the poles of that minimum order Chebyshev filter will be a bit more tedious than those for Butterworth filters.

We will start out by providing the reader some necessary mathematical background, and we will solve for exact Chebyshev pole locations. As we present the design method, we will introduce design charts that will allow us to quickly plot these poles for filters of order six or less. If we exercise reasonable care in our use of these charts, we can obtain filter parameters to an accuracy of approximately 1%, and this will prove sufficient if our final design is to be implemented with 5% tolerance resistors and capacitors. If higher order or more precision is required, we can always calculate exact pole locations (and component values) via calculator or computer.

5.2 The Chebyshev Response

Just as in the case for Butterworth filters (or for all transfer functions with real-valued impulse responses) the squared magnitude response is a ratio of polynomials in ω containing only even powers. In addition, the requirement for the gain to go monotonically to zero in the stop band means there will be no zeros. We will write the squared magnitude response of an n^{th} order Chebyshev filter as

$$H(j\omega)H(-j\omega) = \frac{1}{1+\varepsilon^2 {C_n}^2(\omega)} \qquad (5.1)$$

where $C_n^2(\omega)$ is a polynomial of order 2n containing only even powers of ω. Also, to simplify our algebra, we will define the pass band limit ω_p to be equal to 1. Therefore all pole frequencies we determine later will be interpreted as frequencies relative to the pass band limit. (Said another way, to get back to the *true* ω_o or radius for each pole, we will have to remember to multiply by ω_p.) $C_n^2(\omega)$ will be constrained to vary between 0 and 1 within the pass band, and $C_n^2(1) = 1$. (By pass band we mean for ω between -1 and +1, because we have normalized to $\omega_p = 1$.) As a result, the squared magnitude pass band gain will vary between 1 and $1/(1 + \varepsilon^2)$. From (5.1), A_{max} and ε are related by

$$A_{\max} = 10\log_{10}\left(1+\varepsilon^2\right)$$

or

$$\varepsilon = \sqrt{10^{\left(A_{\max}/10\right)} - 1} \tag{5.2}$$

In searching for candidate forms for $C_n(\omega)$, we will start by considering what $C_n(\omega)$ looks like for first and second order filters, and then extend this response to general order. For first order filters (i.e. a single real pole), there is no difference between the Butterworth and Chebyshev responses. Since the magnitude squared gain at the pass band limit ($\omega_p = 1$) is

$$H(j)H(-j) = \frac{1}{1+\varepsilon^2} \tag{5.3}$$

this implies that

$$C_1^2(\omega) = \omega^2 \ . \tag{5.4}$$

Next the magnitude responses of second order Butterworth and Chebyshev low pass filters with A_{max} = 3 dB are compared in Figure 5.1. Because the pass band gain is allowed to increase with frequency for the Chebyshev filter, it can have a larger Q and a steeper slope outside the pass band. Note the response starts at $-A_{max}$, goes to 0 dB at ω approximately equal to 0.7 and returns to $-A_{max}$ at the pass band limit of one.

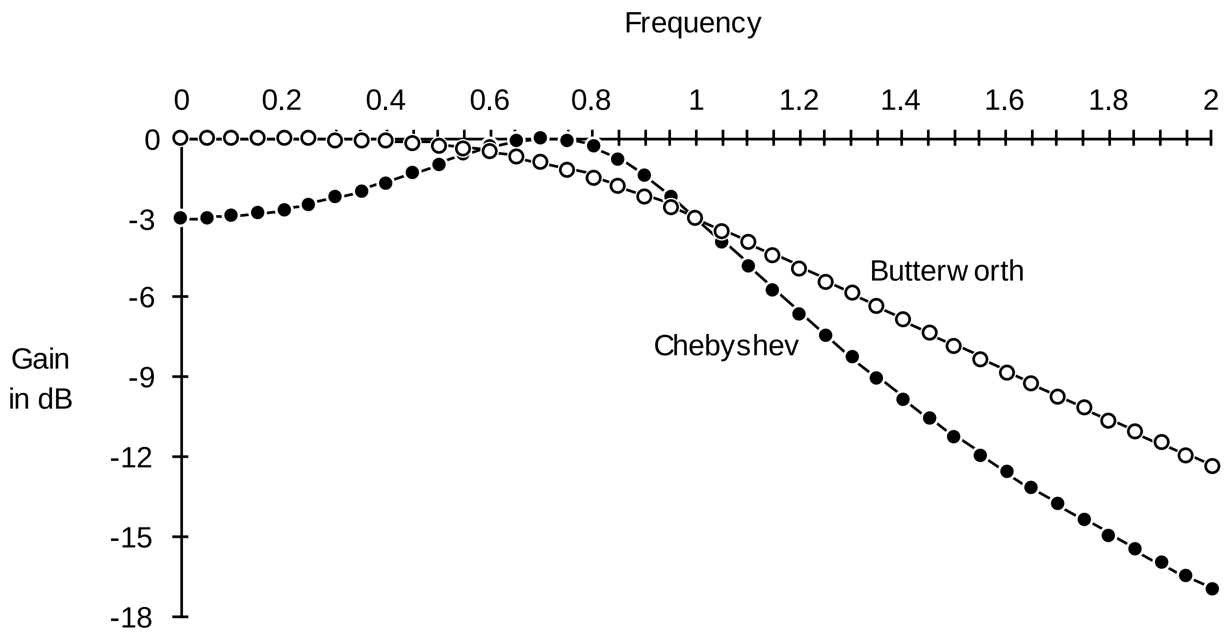

Figure 5.1. Magnitude responses for 2nd order Butterworth and Chebyshev filters.

The Chebyshev polynomial for n = 2 is in general

$$C_2^2(\omega) = c_o + c_2\,\omega^2 + c_4\,\omega^4 \;. \tag{5.5}$$

Referring to Figure 5.1, since the transfer function magnitude response at DC is a minimum, from (5.1) this implies $C_2^2(0) = 1$ and therefore $c_o = 1$. Since $C_2^2(1)$ must also equal 1 (since the response returns to $-A_{max}$ at the pass band limit of one),

$$c_2 + c_4 = 0 \tag{5.6}$$

and 5. 5 can be rewritten

$$C_2^2(\omega) = 1 - c_4\,\omega^2 + c_4\,\omega^4 \,. \tag{5.7}$$

$C_2^2(\omega)$ is a minimum when the Chebyshev response in Figure 5.1 achieves its maximum. To find the frequency at which this occurs (and thereby solve for c_4 in equation 5.7), we take the first derivative of (5.7) with respect to ω and set this equal to zero. This implies

$$\frac{dC_2^{\,2}(\omega)}{d(\omega)} = 0 = -2c_4\omega + 4c_4\omega^3 \tag{5.8}$$

which occurs at

$$\omega = \frac{1}{\sqrt{2}} \quad . \tag{5.9}$$

(The other root ($\omega = 0$) is a maximum.) At $\omega = \frac{1}{\sqrt{2}}$ the response should be 0 dB (unity gain) so that $C_2^{\,2}\left(\frac{1}{\sqrt{2}}\right)$=0, and from (5.8) we know

$$C_2^{\,2}\left(\frac{1}{\sqrt{2}}\right) = 1 - \frac{c_4}{2} + \frac{c_4}{4} = 0 \tag{5.10}$$

Therefore $c_4 = 4$, $c_2 = -4$, so that

$$C_2^{\,2}(\omega) = 1 - 4\,\omega^2 + 4\,\omega^4 \tag{5.11}$$

or

$$C_2(\omega) = \pm\left(2\omega^2 - 1\right) \quad . \tag{5.12}$$

While deriving the 2nd order Chebyshev polynomial was instructive, it is more useful to obtain a more general expression for $C_n(\omega)$. Unfortunately, using similar techniques to obtain a general expression for the Chebyshev polynomial $C_n(\omega)$ is considerably more tedious. We do know, however, that sinusoids are the most familiar functions that vary between -1 and 1. While in general sinusoids cannot be expressed as finite polynomials of their arguments, cosine functions of integer multiples of some angle can be expressed as finite polynomial functions of the cosine of that angle. For example

$$\cos 2\alpha = 2\cos^2(\alpha) - 1 \; . \tag{5.13}$$

Therefore if

$$\alpha = \cos^{-1}(\omega) \tag{5.14}$$

then

$$\begin{aligned} \cos(2\cos^{-1}(\omega)) &= 2\,\omega^2 - 1 \\ &= C_2(\omega) \; . \end{aligned} \tag{5.15}$$

In general we can write

$$C_n(\omega) = \cos(n \cos^{-1}(\omega)) . \tag{5.16}$$

Because $\cos^{-1}(\omega)$ varies from $\pi/2$ to 0 as ω varies from 0 to 1, $C_n^2(\omega)$ will vary from $\cos^2(n\pi/2)$ to $\cos^2(0) = 1$ over the same frequency range. The resulting magnitude squared response in the pass band for filters of order 2, 3, 4, 5, and 6 is shown in Figure 5.2.

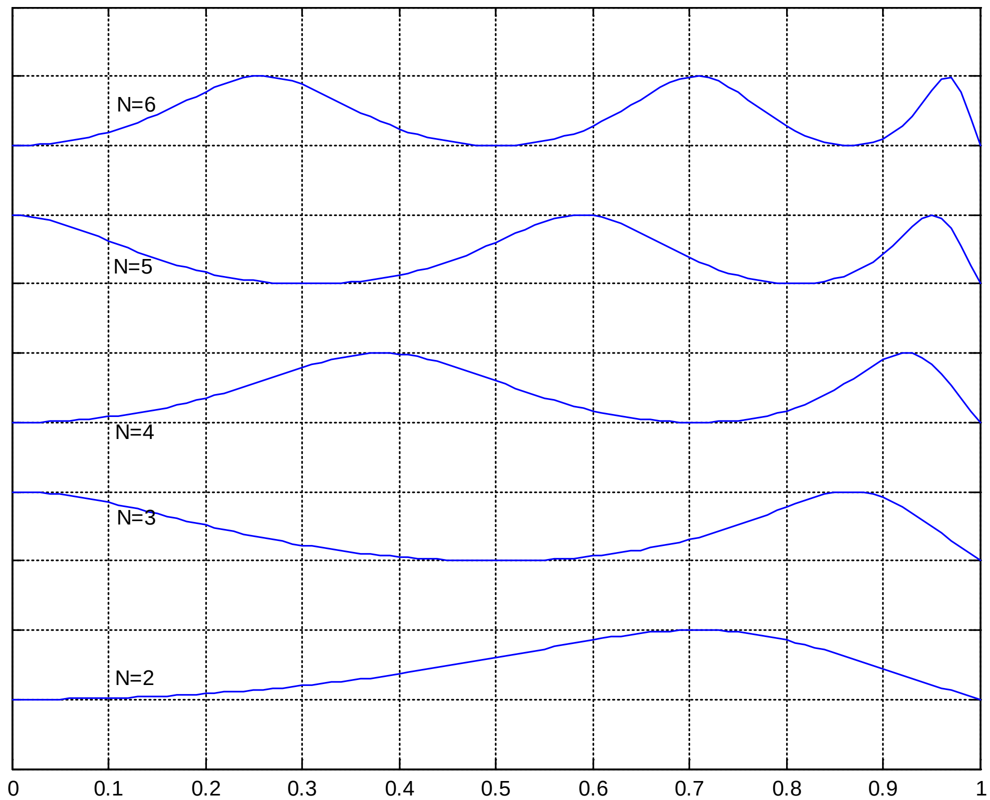

Figure 5.2. Passband magnitude squared responses for Chebyshev lowpass filters of order 2 through 6.

Also note that for n even, $C_n^2(0)$ is 1, and $C_n^2(\omega)$ will equal 0 exactly n/2 times in the pass band. In terms of the magnitude squared response, this implies the DC gain and the gain at the pass band limit are both

$$\begin{aligned} H(0)^2 &= H(j)\, H(-j) \\ &= \frac{1}{1+\varepsilon^2}, \end{aligned} \tag{5.17}$$

and the gain will go a positive peak (of one), n/2 times (for n even).

For n odd $C_n^2(0)$ is 0 and $C_n^2(\omega)$ goes to 0 exactly (n - 1)/2 additional times in the pass band. The frequency response for odd order filters is thus a positive maximum at DC with (n - 1)/2 more positive peaks, and the response is equal to that of even order filters at the pass band limit.

In general (for n even or odd), the maxima in $C_n^2(\omega)$ (or minima in gain) occur when

$$2n \cos^{-1}(\omega) = 0, 2\pi, 4\pi, ...2m\pi, \qquad m = \text{integer}$$

or when

$$\omega = \cos\left(\frac{m\pi}{n}\right) \tag{5.18}$$

Minima in attenuation (or maxima in gain) occur when

$$2n \cos^{-1}(\omega) = \pi, 3\pi, 5\pi ....(2m + 1)\pi$$

or when

$$\omega = \cos\left(\frac{(2m+1)\pi}{2n}\right) \tag{5.19}$$

These maxima and minima for n = 2 to 5 are summarized below in Table 5.1.

Order, n	Maxima in gain	Minima in gain
2	0.707	0
3	0, 0.833	0.5
4	0.383, 0.923	0, 0.707
5	0, 0.588, 0.951	0.309, 0.809

Table 5.1. Table of maxima and minima for Chebyshev passbands.

We can recursively generate the polynomials using the trigonometric identity

$$\cos n\alpha = 2 \cos (n - 1)\alpha \cos \alpha - \cos(n - 2)\alpha . \tag{5.20}$$

Therefore

$$C_n(\omega) = 2\omega\, C_{n-1}(\omega) - C_{n-2}(\omega) . \tag{5.21}$$

Table 5.2 shows these functions for n = 0 to 5.

Order, n	$C_n(\omega)$	$C_n^2(\omega)$
0	1	1
1	ω	ω^2
2	$2\omega^2 - 1$	$4\omega^4 - 4\omega^2 + 1$
3	$4\omega^3 - 3\omega$	$16\omega^6 - 24\omega^4 + 9\omega^2$
4	$8\omega^4 - 8\omega^2 + 1$	$64\omega^8 - 128\omega^6 + 80\omega^4 - 16\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$	$256\omega^{10} - 640\omega^8 + 560\omega^6 - 200\omega^4 + 25\omega^2$

Table 5.2 Chebyshev polynomials for order 0 through 5.

As evidenced by Table 5.2 and Figure 5.2, $C_n^2(\omega)$ has the following features:

a. It is equal 1 for $\omega = 1$.

b. At $\omega = 0$, it is 1 for n even and 0 for n odd.

c. Between $\omega = 0$ and 1, it equals 0 exactly n/2 times for n even, and exactly $\frac{(n-1)}{2}$ times for n odd.

d. For ω larger than 1, $C_n^2(\omega)$ goes monotonically to infinity.

Features (a), (b) and (c) are quite consistent with our definition of $C_n^2(\omega)$ in terms of cosines (5.16) and with the features we noted above using that definition. Unfortunately, feature (d), while consistent with our requirement to have the frequency response tend to zero as the frequency tends to infinity, is inconsistent with what we normally interpret a sinusoid to be. Said another way, what is $\cos^{-1}(\omega)$ for ω larger than 1? Stated in other terms …what angle has a cosine larger than 1?! Using Euler's identity,

$$\omega = \cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$$

what value can we substitute for α and get ω larger than 1? Certainly a purely imaginary angle (α = ja, where a is real) will work, and substituting α=ja we obtain

$$\omega = \cos\alpha = \frac{e^{-a} + e^{a}}{2}$$
$$= \cosh(a) . \qquad (5.22)$$

Certainly ω is greater than or equal to l for all real a. Now $C_n(\omega)$ is given by

$C_n(\omega) = \cos[n \cos^{-1}(\omega)] = \cos n\alpha = \cos jna$

$= \cos[jn \cosh^{-1}(\omega)]$

$= \cosh[n \cosh^{-1}(\omega)]$. (5.23)

The magnitude squared response <u>in the stop band</u> can then be expressed as

$$\boxed{H(j\omega)H(-j\omega) = \frac{1}{1+\varepsilon^2 \cosh^2\left(n \cdot \cosh^{-1}(\omega)\right)}}, \tag{5.24}$$

and as before, the magnitude squared response in the pass band (where $|\omega| \leq 1$) is expressed as

$$\boxed{H(j\omega)H(-j\omega) = \frac{1}{1+\varepsilon^2 \cos^2\left(n \cdot \cos^{-1}(\omega)\right)}}.$$

If the attenuation at the stop band limit ω_s is A_{min}

$$A_{\min} = 10 \cdot \log_{10}\left[1+\varepsilon^2 \cosh^2\left(n \cdot \cosh^{-1}(\omega_s)\right)\right]. \tag{5.25}$$

Solving for n using 5.25 and the relationship between A_{max} and ε (equation 5.2),

$$10^{\left(A_{\min}/10\right)} - 1 = \varepsilon^2 \cosh^2\left(n \cdot \cosh^{-1}(\omega_s)\right), \tag{5.26}$$

so

$$\cosh^{-1}\left(\frac{\sqrt{10^{\left(A_{\min}/10\right)} - 1}}{\varepsilon}\right) = n \cdot \cosh^{-1}(\omega_s) \tag{5.27}$$

and

$$n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{\left(A_{\min}/10\right)} - 1}}{\varepsilon}\right)}{\cosh^{-1}(\omega_s)} \tag{5.28}$$

or

$$n = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{\left(A_{\min}/10\right)} - 1}{10^{\left(A_{\max}/10\right)} - 1}}\right)}{\cosh^{-1}(\omega_s)} .$$

Recall that all frequencies (including ω_s) are normalized frequencies "relative to ω_p," so that $\omega_s = \omega_{s_{actual}}/\omega_{p_{actual}}$, so

$$n=\boxed{\frac{\cosh^{-1}\left(\sqrt{\dfrac{10^{\left(A_{\min}/10\right)}-1}{10^{\left(A_{\max}/10\right)}-1}}\right)}{\cosh^{-1}\left(\omega_{s_{actual}}/\omega_{p_{actual}}\right)}} \tag{5.29}$$

Just as in the case of Butterworth filters, the order n is rounded up to the next largest integer. Remember also that all of our calculations assumed a pass band limit of 1. Thus the argument in the denominator of 5.29 became the ratio of the stop band to pass band limit. Also, just as in the Butterworth case, the minimum order is a function of two parameters, namely the ratio of stop band to pass band limit, and the term

$$\frac{10^{\left(A_{\min}/10\right)}-1}{10^{\left(A_{\max}/10\right)}-1}.$$

Figure 5.3 is a plot of minimum order as a function of these two parameters, for both Butterworth and Chebyshev responses, showing that by allowing ripple within the pass band we can meet identical specifications with lower order Chebyshev filters. In Figure 5.3 the two scales on the left may be used for the graphical calculation of the parameter

$$\frac{10^{\left(A_{\min}/10\right)}-1}{10^{\left(A_{\max}/10\right)}-1}$$

by simply extending a straight line through A_{max} and A_{min} to the $\left(\dfrac{10^{\left(A_{\min}/10\right)}-1}{10^{\left(A_{\max}/10\right)}-1}\right)$ scale. From the point of intersection on that scale, draw a horizontal line from that point across to the right until it intersects the vertical line drawn upward from the axis corresponding to the ratio of stop band to pass band frequency (in the case of low pass filters). Draw a dot at the point of intersection of these lines, and read the next higher order filter from the chart.

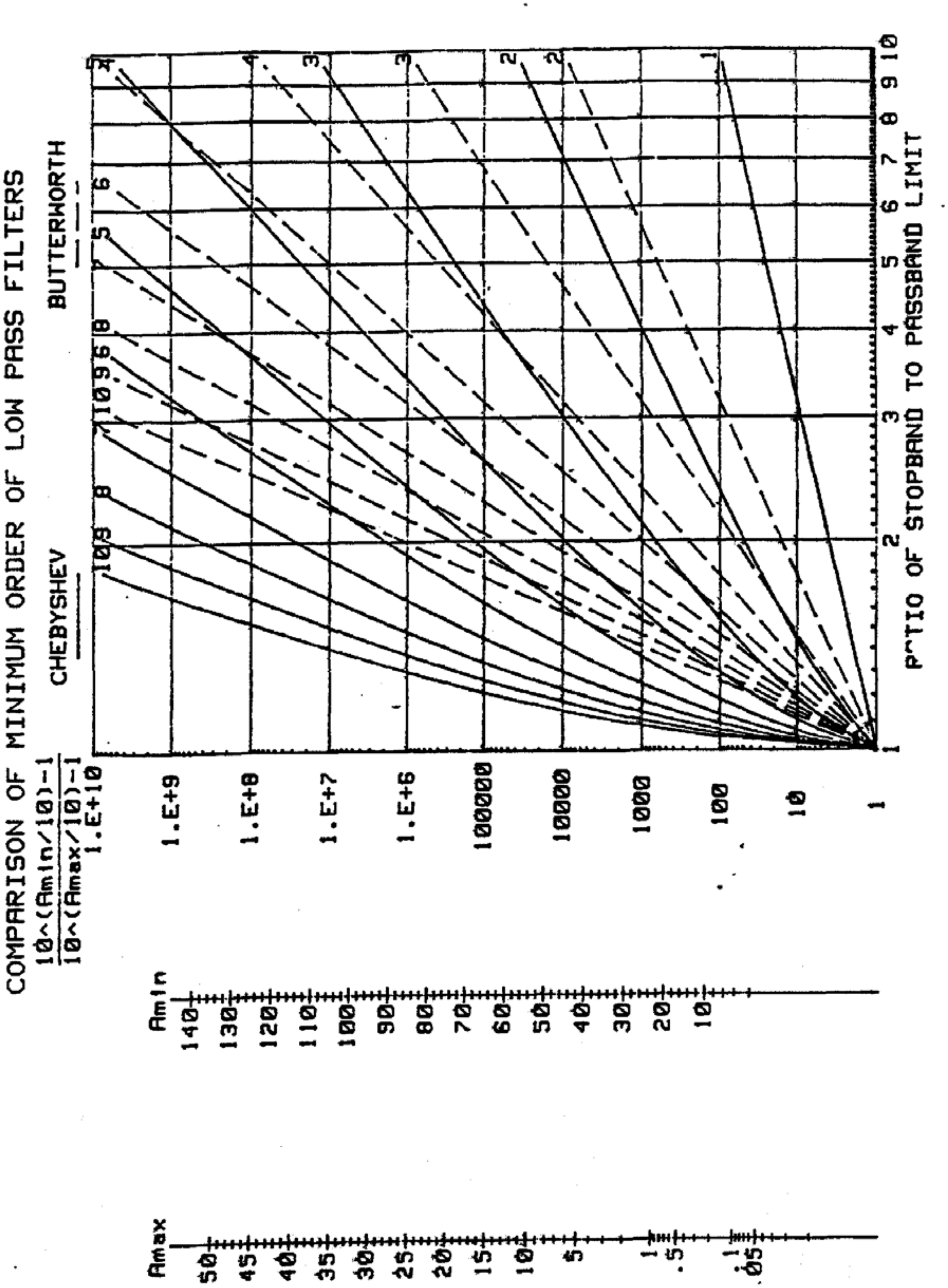

Figure 5.3. Nomograph for calculation of Butterworth and Chebyshev filter orders.

Example 5.1

Find the minimum order Butterworth and Chebyshev filters to satisfy the following specifications:

A_{max} = 1 dB

A_{min} = 40 dB

Pass band limit = 1 kHz

Stop band limit = 1.85 kHz

Calculating:

$$\frac{10^{\left(A_{min}/10\right)}-1}{10^{\left(A_{max}/10\right)}-1}=\frac{10^4-1}{10^{0.1}-1}=3.86\times10^4$$

The minimum order for a Butterworth filter from equation 4.11 is

$$n=\frac{\ln\left(\frac{10^{A_{min}/10}-1}{10^{A_{max}/10}-1}\right)}{2\cdot\ln\left(\frac{\omega_s}{\omega_p}\right)}=\frac{\ln\left(3.86\times10^4\right)}{2\cdot\ln\left(1.85\right)}$$

n=8.58 (which we round to 9).

For a Chebyshev filter, from equation 5.29 we know that

$$n=\frac{\cosh^{-1}\left(\sqrt{\frac{10^{\left(A_{min}/10\right)}-1}{10^{\left(A_{max}/10\right)}-1}}\right)}{\cosh^{-1}\left(\omega_{s_{actual}}/\omega_{p_{actual}}\right)}$$ =4.87 (which we round to n=5).

Figure 5.3 shows how these values can be determined graphically. The intersection of the lines drawn horizontally from 3.86×10^4 and vertically from 1.85 is between the 8^{th} and 9^{th} order Butterworth curves and just below the 5^{th} order Chebyshev curve.

If your calculator or computer does not have the inverse cosh function, the following identity may prove useful:

$$\cosh^{-1}\left(x\right)=\ln\left(x+\sqrt{x^2-1}\right) \tag{5.30}$$

$$\cong \ln(2x) \quad \text{for } x>>1.$$

This approximation is less than 1% in error for x larger than 4. In addition, this approximation can be applied to the numerator in equation 5.29 to obtain the following approximation for n when $A_{\min}$ is reasonably large (typically larger than 15 dB):

$$n = \frac{0.693 - \ln(\varepsilon) + 0.115 A_{\min}}{\cosh^{-1}\left(\omega_{s_{actual}} / \omega_{p_{actual}} \right)} \tag{5.31}$$

5.3 Chebyshev Poles

As we saw from Chapter 4 in Butterworth filter design, solving for pole locations to meet a desired set of specifications was very important. In Chebyshev design, those pole locations are equally as important. Here we will solve for the Chebyshev pole locations by substituting s/j for ω in equation 5.24 to obtain

$$H(s)H(-s) = \frac{1}{1 + \varepsilon^2 \cdot \cosh^2\left[n \cdot \cosh^{-1}\left(s/j \right) \right]}. \tag{5.32}$$

Pole locations are determined by setting the denominator of (5.32) to zero, and we obtain

$$\cosh^2\left[n \cdot \cosh^{-1}\left(s/j \right) \right] = \frac{-1}{\varepsilon^2} \tag{5.33}$$

Taking the square root and then the inverse cosh of each side results in

$$n \cdot \cosh^{-1}\left(s/j \right) = \cosh^{-1}\left(\pm j/\varepsilon \right) \tag{5.34}$$

Dividing each side by n and taking the cosh we have

$$s = j \cdot \cosh\left(\frac{\cosh^{-1}\left(\pm j/\varepsilon \right)}{n} \right) \tag{5.35}$$

While we now have a closed form expression for the pole locations, the evaluation of this expression is not all that simple! Let us first consider the evaluation of $\cosh^{-1}\left(\pm j/\varepsilon \right)$.

Let $x = \cosh^{-1}\left(\frac{\pm j}{\varepsilon}\right)$. (5.36)

Therefore

$$\pm\frac{j}{\varepsilon} = \cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{5.37}$$

Note that if x were either purely real or purely imaginary the <u>right hand side</u> of equation 5.37 would be real. Since the left side of 5.37 is purely imaginary, we know x must be some complex number (i.e., x = α + jβ). This implies

$$\pm\frac{j}{\varepsilon} = \frac{e^{\alpha + j\beta} + e^{-\alpha - j\beta}}{2} = \cosh(\alpha + j\beta)$$

or

$$\pm\frac{j}{\varepsilon} = \frac{e^{\alpha}(\cos\beta + j\sin\beta) + e^{-\alpha}(\cos\beta - j\sin\beta)}{2} \tag{5.38}$$

Because the right hand side of equation 5.38 must be purely imaginary, cos β must be zero, which implies that $\boxed{\beta = \pm\frac{(2m+1)\pi}{2}}$ where m is an integer. This forces sin β to be ± 1. Equation 5.38 now becomes

$$\pm\frac{j}{\varepsilon} = \pm\frac{j}{2}\left(e^{\alpha} - e^{-\alpha}\right) = \pm j\cdot\sinh(\alpha) \tag{5.39}$$

or

$$\boxed{\alpha = \pm\sinh^{-1}\left(1/\varepsilon\right)}.$$

Since

$$\pm\frac{j}{\varepsilon} = \frac{e^{\alpha + j\beta} + e^{-\alpha - j\beta}}{2} = \cosh(\alpha + j\beta),$$

we know

$$\cosh^{-1}\left(\pm\frac{j}{\varepsilon}\right) = \alpha + j\beta$$

$$= \pm\sinh^{-1}\left(\frac{1}{\varepsilon}\right) \pm j\frac{(2m+1)\pi}{2} \tag{5.40}$$

Using 5.40 in equation 5.35 to solve for the pole locations

$$s = j\cdot\cosh\left(\frac{\cosh^{-1}\left(\pm j/\varepsilon\right)}{n}\right)$$

$$= j \cdot \cosh\left(\frac{\pm \sinh^{-1}\left(1/\varepsilon \right) \pm \dfrac{j\left(2m+1\right)\pi}{2}}{n} \right)$$

$$= \left(\frac{j}{2}\right) \cdot \left\{ \exp\left(\frac{\pm \sinh^{-1}\left(1/\varepsilon \right) \pm \dfrac{j\left(2m+1\right)\pi}{2}}{n} \right) + \exp\left(-\frac{\pm \sinh^{-1}\left(1/\varepsilon \right) \pm \dfrac{j\left(2m+1\right)\pi}{2}}{n} \right) \right\} \tag{5.41}$$

Now we employ the identity for inverse sinh,

$$\sinh^{-1}\left(1/\varepsilon \right) = \ln\left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right) \tag{5.42}$$

which implies

$$= \exp\left(\frac{\sinh^{-1}\left(1/\varepsilon \right)}{n} \right) = \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{1/n} \tag{5.43}$$

Observe from (5.41) that the angles $\dfrac{\left(2m+1\right)\pi}{2n}$ are the angles between the imaginary axis in the s plane and the <u>Butterworth</u> poles for filters of order n, or the <u>complements</u> of the Butterworth angles as defined in the previous chapter. For example for n = 5, these angles are π/10 (18^o), 3π/10 (54^o), and 5π/10 (90^o), corresponding to Butterworth poles at 72^o, 36^o, and 0^o with respect to the negative real s-plane axis.

The expression for the Chebyshev pole locations (5.41) then becomes

$$\begin{aligned} s = \left(\pm\frac{j}{2}\right) \cdot \Bigg\{ & \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{1/n} \left[\cos\left(\frac{\left(2m+1\right)\pi}{2n} \right) + j\sin\left(\frac{\left(2m+1\right)\pi}{2n} \right) \right] \\ & + \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{-1/n} \left[\cos\left(\frac{\left(2m+1\right)\pi}{2n} \right) - j\sin\left(\frac{\left(2m+1\right)\pi}{2n} \right) \right] \Bigg\} \end{aligned} \tag{5.44}$$

which means that

$$REAL\{s\} = \left(\pm\frac{1}{2}\right) \cdot \left\{ \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{1/n} - \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{-1/n} \right\} \left[\sin\left(\frac{\left(2m+1\right)\pi}{2n} \right) \right]$$

and

$$IMAG\{s\} = \left(\pm\frac{1}{2}\right) \cdot \left\{ \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{1/n} + \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{-1/n} \right\} \left[\cos\left(\frac{\left(2m+1\right)\pi}{2n} \right) \right] \tag{5.45}$$

At this point it is convenient to define new variables a and b,

$$a = \left(\frac{1}{2}\right) \cdot \left\{ \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{1/n} - \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{-1/n} \right\} \tag{5.46}$$

$$b = \left(\frac{1}{2}\right) \cdot \left\{ \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{1/n} + \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{-1/n} \right\} \tag{5.47}$$

such that the pole locations for H(s)H(-s) can be expressed as

$$s = \pm a \cdot \sin\left[\frac{(2m+1)\pi}{2n} \right] \pm jb \cdot \cos\left[\frac{(2m+1)\pi}{2n} \right]. \tag{5.48}$$

Note from equation (5.48) that the pole locations of H(s)H(-s) lie on an elliptical contour in the s-plane, where “a” is the minor axis of that ellipse, and “b” is the major axis of that ellipse. Figure 5.4 is a plot of the variable "a" as a function of A_{max} and the filter order for filters of order 2 to 6.

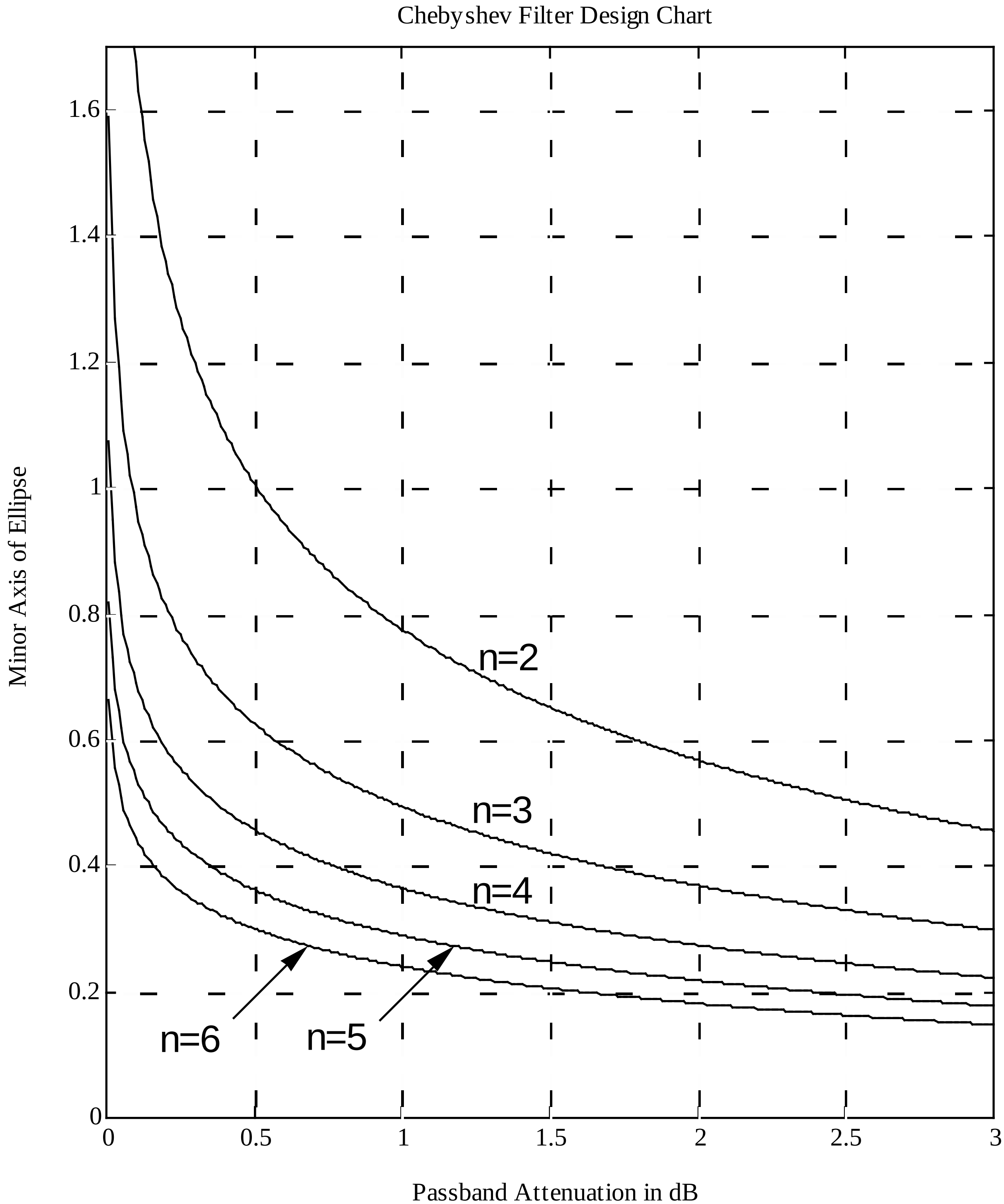

Figure 5.4. Chebyshev filter design chart. The minor axis of the Chebyshev ellipse is “a” from (5.47)

Recall that the angles $\frac{(2m+1)\pi}{2n}$ in equation 5.48 were the angles between the imaginary axis in the s plane and the Butterworth poles for filters of order n, or the complements of the Butterworth angles as defined in the previous chapter. The poles of H(s)H(-s) can then be described as

$$s_{mn} = \pm a \cdot \cos\theta_{mn} \pm jb \cdot \sin\theta_{mn}, \qquad (5.49)$$

where the θ_{mn} are the angles relative to the negative real axis in the s plane of the poles of the n^{th} order Butterworth filters presented in the previous chapter, and are merely the complements of the angles in equation 5.48.

To express the poles in terms of undamped natural frequency (ω_o) and Q we first note that

$$b^2 - a^2 = 1 \quad , \qquad (5.50)$$

so that

$$\begin{aligned} \omega_{o_{mn}} &= \sqrt{a^2\cos^2(\theta_{mn}) + b^2\sin^2(\theta_{mn})} \\ &= \sqrt{a^2 + \sin^2(\theta_{mn})} \end{aligned} \qquad (5.51)$$

and for those stable poles,

$$Q_{mn} = \frac{\omega_o}{2\cdot\left|\mathrm{Re}\{s_{mn}\}\right|} = \frac{1}{2\cdot\left|\cos\theta_{mn}\right|} \qquad (5.52)$$

Remember that we have normalized the pass band limit to unity so the result of applying equation 5.51 is the pole frequency relative to the pass band limit. To obtain the "actual" pole frequency values from equation 5.51, we remind the student to multiply the normalized value for ω_o obtained from equation 5.51 by the actual value for ω_p.

Figure 5.5 is a plot of the poles of H(s) H(-s) for a fourth order filter and minor axis (a) equal to 0.58. In this example the angles are $\pi/8$ or 22.5^o and $3\pi/8$ or 67.5^o. The real components of the poles are the real values at the intersections of radials at the Butterworth angles and the small circle of radius a. Similarly, the imaginary components are the imaginary values at the intersections of these radials and the large circle of radius b. Just as in the Butterworth case we desire a stable filter and therefore assign the poles in the left half plane to H(s) and the poles in the right half plane to H(-s). The poles can be determined graphically as illustrated by Figure 5.5. The procedure is as follows:

1. Determine the minimum order using either Equation 5.29 or Figure 5.3.

2. Determine the variables a and b using equations 5.46, 5.47, and recalling that $\varepsilon = \sqrt{10^{(A_{\max}/10)} - 1}$ from equation (5.2). Alternatively you may use Figure 5.4.

3. Draw two circles of radii a and b. Draw radials at the Butterworth angles.

4. Draw vertical lines from the intersections of the radials and the smaller circle, and horizontal lines from the intersections of the radials and the larger circle. The poles are at the intersections of these vertical and horizontal lines.

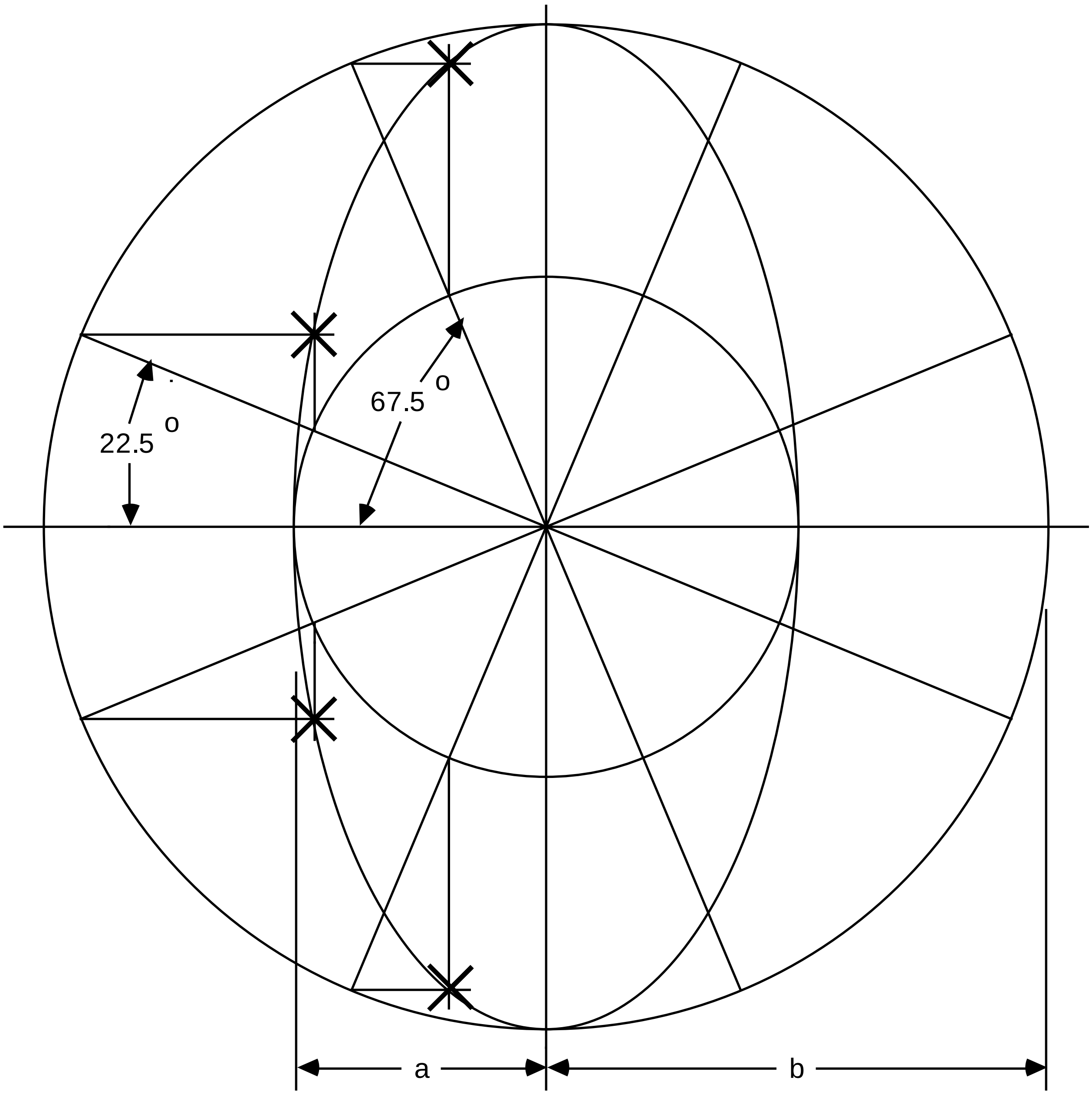

Figure 5.5. Graphical method for determining pole locations (4th order Chebyshev lowpass filter.)

If s is expressed as $\sigma + j\omega$ it follows that

$$\frac{\sigma^2}{a^2} + \frac{\omega^2}{b^2} = \sin^2\theta_{mn} + \cos^2\theta_{mn}$$

$$= 1 ,$$

or our Chebyshev poles are located on an ellipse of minor axis a and major axis b. Since

$$b^2 - a^2 = 1 , \tag{5.53}$$

the foci of the ellipse are at $\pm$ j. Figure 5.6 is a plot of these ellipses for a from 0.1 to 1 for poles with positive imaginary parts. The remaining poles are their complex conjugates. Also plotted are curves labeled by their Butterworth angles, showing the pole locations for filters of order 2 to 6. Now the graphical procedure for determining Chebyshev pole locations for a design problem becomes very simple:

a. Determine the minimum order to meet desired specifications.
b. Determine the minor axis of the Chebyshev ellipse.
c. Plot the poles at intersections of that ellipse and the curves corresponding to the Butterworth angles for the order found in (a).

The angles or Q's can be read by using a straight-edge, lining up the s-plane origin with the pole, and reading Q or angle from the outer scale. Each pole's resonant frequency (relative to the pass band limit) is simply the radius of the pole from the origin. This can be measured by using a set of dividers, and noting the length as measured from the scale on either axis. Note that because the poles for a Chebyshev filter lie on an elliptical contour, the resonant frequency of each pole (or section) will be different. Recall from Chapter 4 that Butterworth poles lie on a circular contour, hence the resonant frequency of each of those poles was the same.

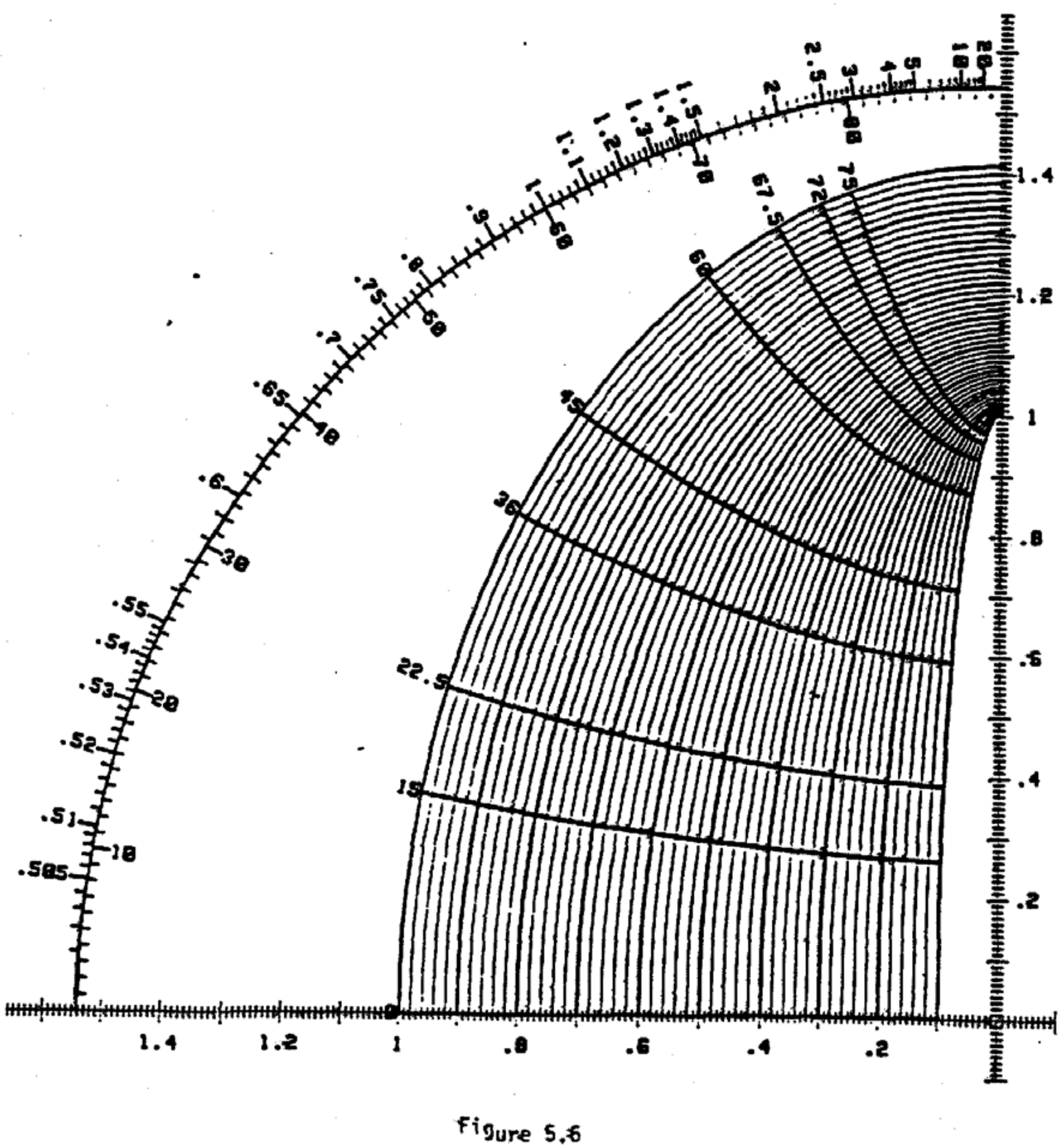

Figure 5.6

Figure 5.6.

5.4 Design Examples

Example 5.2

Design a Chebyshev lowpass filter to satisfy the specifications given in Example 5.1, namely:

A_{max} = 1 dB

A_{min} = 40 dB

Passband limit = 1 kHz

Stopband limit = 1.85 kHz

As we calculated in Example 5.1 the minimum order is 5. The variable "a" (the minor axis of the ellipse) can be determined from Figure 5.4, or may be calculated as follows:

$$\varepsilon = \sqrt{10^{(A_{\max}/10)} - 1} = \sqrt{10^{0.1} - 1} = 0.509 \,.$$

From equation (5.46) we know the minor axis of the ellipse is calculated as

$$a = \left(\frac{1}{2}\right) \cdot \left\{ \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{1/n} - \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{-1/n} \right\} \tag{5.46}$$

$$= \left(\frac{1}{2}\right) \cdot \left\{ \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{1/5} - \left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1} \right)^{-1/5} \right\}$$

$$= 0.289 \,.$$

A plot of the poles is shown in Figure 5.7.

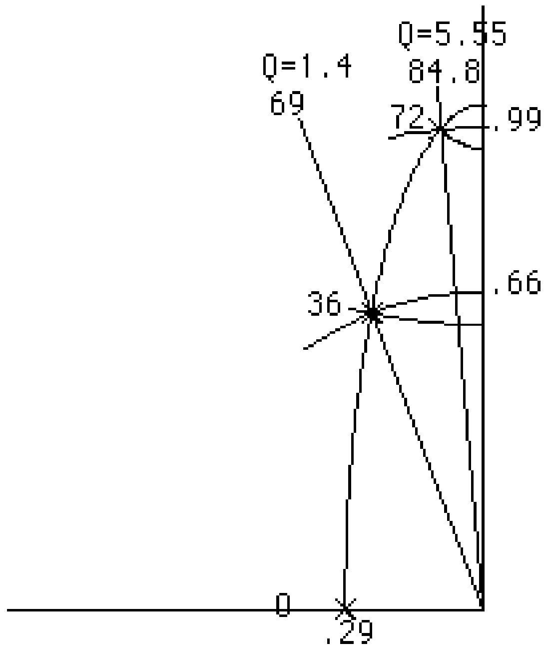

Figure 5.7. Pole locations for 5th order Chebyshev filter from Example 5.2.

The poles are at the intersections of the ellipse whose minor axis equals 0.289 (or 0.29) and the curves labeled 0^o, 36^o and 72^o. The resonant frequencies (relative to ω_p) are determined by measuring the pole radii. The actual resonant frequencies in this case, given that $\omega_p = 2\pi * 1$ kHz = 6283 rad/sec, are

$$\omega_{o1} = 0.289\ \omega_p = 1820 \text{ rad/sec}$$
$$\omega_{o2} = 0.66\ \omega_p = 4150 \text{ rad/sec}$$
$$\omega_{o3} = 0.99\ \omega_p = 6220 \text{ rad/sec} \quad .$$

The Q values are read from the outer scale and are 1.4 and 5.5 for the two second order sections respectively. (Of course the single first order pole has a Q=0.5.)

The circuit design proceeds exactly as in the previous chapter on Butterworth design, except that the resonant frequencies of each section are not the same, and therefore the equivalent capacitance value for each section will be different. The circuit is shown in Figure 5.8.

If we let each resistor be 10 kΩ,

$$C_{eq1} = \frac{1}{\omega_{o1} \times 10k\Omega} = \frac{1}{1820r/s \times 10k\Omega} = 0.055\mu F$$

Similarly,

$$C_{eq2} = \frac{1}{\omega_{o2} \times 10k\Omega} = \frac{1}{4150r/s \times 10k\Omega} = 0.024\mu F$$

$$C_{21} = \frac{C_{eq2}}{2Q_2} = 8.57nF$$

$$C_{22} = C_{eq2} \times 2Q_2 = 0.067\mu F$$

And

$$C_{eq3} = \frac{1}{\omega_{o3} \times 10k\Omega} = \frac{1}{6220r/s \times 10k\Omega} = 0.016\mu F$$

$$C_{31} = \frac{C_{eq3}}{2Q_3} = 1.45nF$$

$$C_{32} = C_{eq3} \times 2Q_3 = 0.176\mu F$$

Because the filter is odd order, the DC gain is the maximum gain (see Figure 5.2), which is 0 dB in this example (since we used unity DC Gain Sallen-Key sections, plus a unity DC gain first order section).

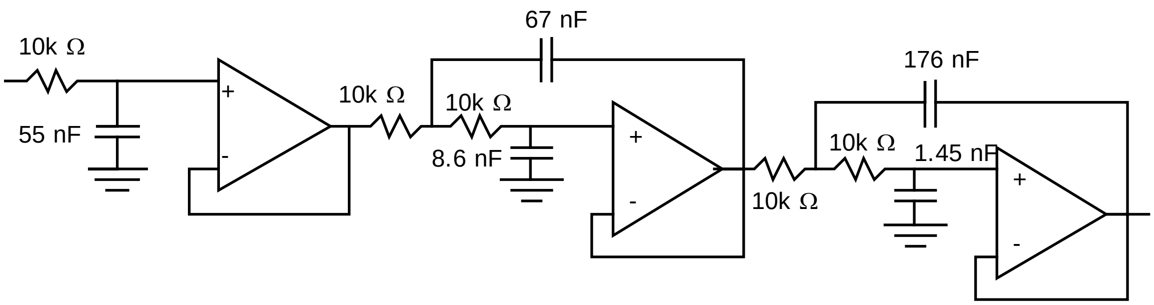

Figure 5.8. Unity gain Sallen-Key cascade design for Example 5.2.

Example 5.3

Design a Chebyshev low pass filter to satisfy the following specifications:

A_{max} = 0.5 dB

A_{min} = 30 dB

Pass band limit = 1 kHz

Stop band limit = 2 kHz

Minimum pass band attenuation (or equivalently, maximum pass band gain) = 0 dB

Using either Figure 5.3 or equation 5.29 the minimum order is 4. This implies that the corresponding Butterworth angles would be 22.5^{o} and 67.5^{o}. Using either Figure 5.4 or equations 5.2 and 5.46, the variable "a" (ellipse minor axis) is 0.46. Plotting the poles (Figure 5.9) we see the two second order sections have resonant frequencies of approximately 0.6 and 1.03 relative to the pass band limit, and Q's of approximately 0.7 and 2.9 respectively.

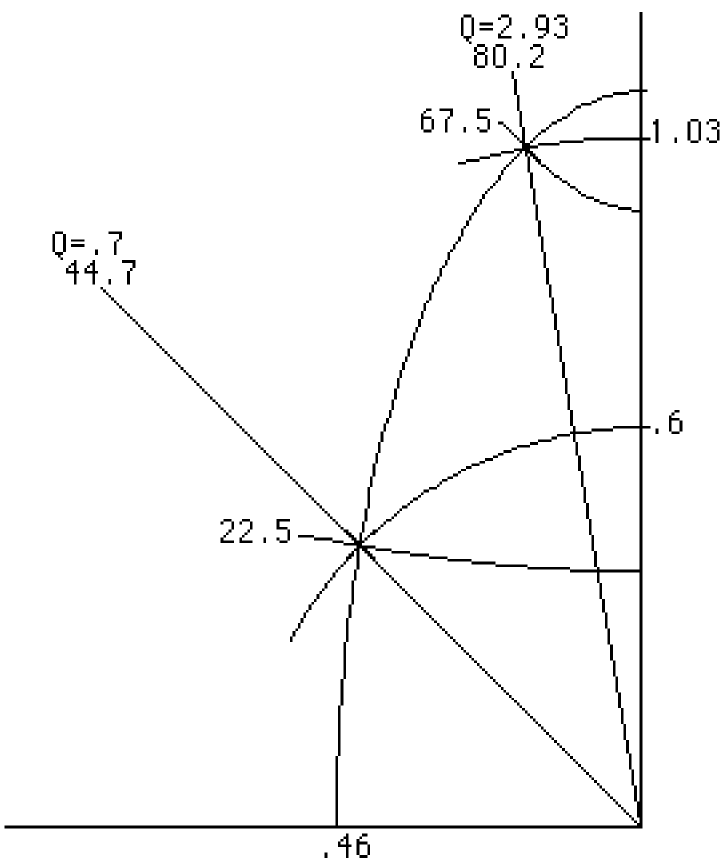

Figure 5.9. Pole locations for 4th order Chebyshev low pass design in Example 5.3.

Proceeding with the design as in Example 5.2, given that $\omega_p = 2\pi * 1$ kHz = 6283 rad/sec

$\omega_{o1} = 0.6\,\omega_p = 3770$ rad/sec
$\omega_{o2} = 1.03\,\omega_p = 6470$ rad/sec .

If we let each resistor be 10 kΩ,

$$C_{eq1} = \frac{1}{\omega_{o1} \times 10k\Omega} = \frac{1}{3770 r/s \times 10k\Omega} = 0.0265\mu F$$

$$C_{11} = \frac{C_{eq1}}{2Q_1} = 0.0189\mu F$$

$$C_{12} = C_{eq1} \times 2Q_1 = 0.0371\mu F$$

And

$$C_{eq2} = \frac{1}{\omega_{o2} \times 10k\Omega} = \frac{1}{6470 r/s \times 10k\Omega} = 0.0155\mu F$$

$$C_{21} = \frac{C_{eq2}}{2Q_2} = 2.66nF$$

$$C_{22} = C_{eq2} \times 2Q_2 = 0.09\mu F$$

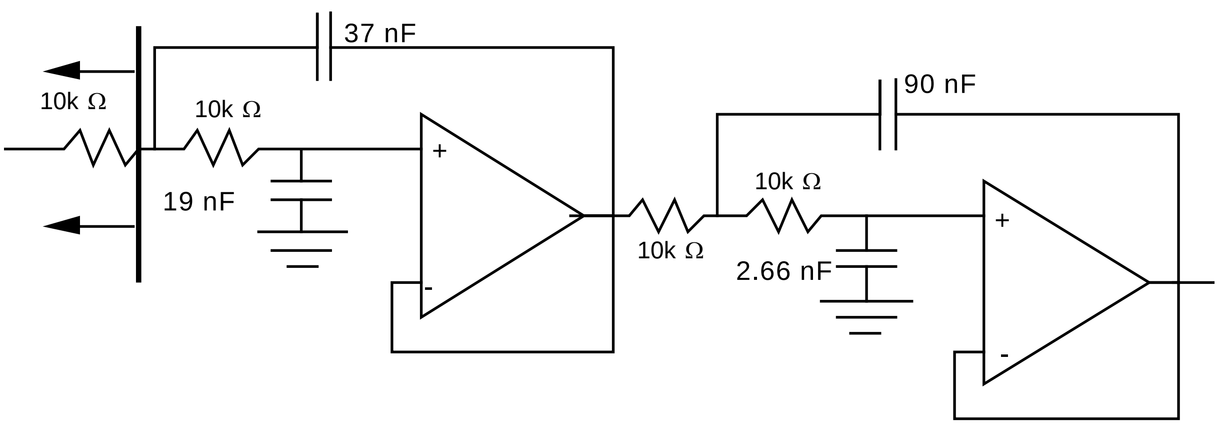

Figure 5.10a. Initial 4th order Chebyshev low pass filter design. (The vertical black "bar" with arrows indicates that we want our circuit to "look outward in that direction and see 10kΩ resistance," however the final circuit will not actually have a single 10kΩ resistor to perform that function.

This circuit is shown in Figure 5.10a. We have now designed a filter that meets the all specifications *except the one requiring the minimum attenuation to be 0 dB, (or equivalently, a maximum pass band gain of 0dB).* The circuit we have used has a *DC gain of 0 dB*, but because this is an even order Chebyshev low pass filter (Figure 5.2), $\omega = 0$ is a passband minimum in gain. The gain at the two maxima is +0.5 dB, or too high for the specifications. There are a number of ways to adjust the gain downward by 0.5 dB. The simplest way is to add a single resistor to ground at the junction of the two resistors in either section. Conceptually, this can best be understood in the context of a Thevenin equivalent circuit for everything to the left of the bold line with arrows in Figure 5.10a. If the Thevenin equivalent of that portion of the circuit were an ideal source of 0.944 V_{in} in series with 10KΩ, the overall gain would be correct. This voltage divider circuit is shown in Figure 5.10b. The two equations needed to determine R_1 and R_2 are just those for voltage division and parallel resistance, namely

$$\frac{R_2}{R_1 + R_2} = 10^{\left(-0.5/20\right)} = 0.9441$$

and

$$\frac{R_1 R_2}{R_1 + R_2} = 10k\Omega$$

Dividing the second equation by the first, we obtain

$$R_1 = \frac{10k\Omega}{0.9441} = 10.6k\Omega\ ,$$

and then solving the first equation for R_2, we obtain

$$R_2 = 0.944\,(R_1 + R_2)$$
$$R_2 = 16.9\,R_1 = 179\ k\Omega.$$

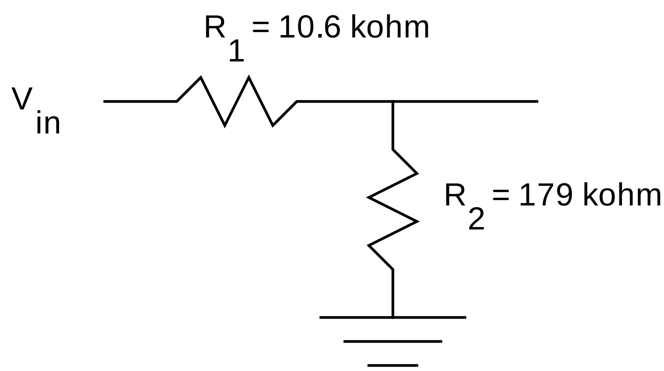

Figure 5.10b. Voltage divider which replaces the single 10kΩ resistor to the left of the vertical bar in Figure 5.10a.

Example 5.4

Design a Chebyshev <u>high pass filter</u> to satisfy the following specifications:

A_{max} = 0.5 dB

A_{min} = 30 dB

Pass band limit = 2 kHz

Stop band limit = 1 kHz

Minimum attenuation (or maximum gain) = 0 dB

This example is virtually identical to the previous one except it is high pass versus low pass design. The modification to the low pass circuit described in Chapter 4 is used. The minimum order is the same as in the previous example. The determination of the minor axis of the ellipse, the plotting of the poles, and the determination of the Q's are also the same. Now, however, the actual resonant frequencies relative to the pass band limit are at the reciprocals of the radii measured on the chart (since our high pass response is just our previous low pass response rotated about ω_p on a logarithmic frequency axis). This implies that

$\omega_p = 2\pi * 2$ kHz = 12,566 rad/sec ,

$\omega_{o1} = \omega_p/0.6 = 2.09 \times 10^4$ rad/sec,

and

$\omega_{o2} = \omega_p/1.03 = 1.22 \times 10^4$ rad/sec .

(Observe in the previous example, which was a <u>low</u> pass design, $\omega_{o1} = \omega_p \times 0.6$ and $\omega_{o2} = \omega_p \times 1.03$.)

Now we will fix the value for the capacitors at some convenient value (say C=0.01 μF) and calculate the resistors:

$$R_{eq1} = \frac{1}{\omega_{o1} \times 0.01\mu F}$$

$$= \frac{1}{2.09 \times 10^4\, rad/\sec \times 0.01\mu F} = 4.78k\Omega$$

$$R_{11} = 2Q_1 \times R_{eq1} = 6.7k\Omega$$

$$R_{12} = \frac{R_{eq1}}{2Q_1} = 3.42k\Omega$$

And

$$R_{eq2} = \frac{1}{\omega_{o2} \times 0.01\mu F}$$

$$= \frac{1}{1.22 \times 10^4\, rad/\sec \times 0.01\mu F} = 8.2k\Omega$$

$$R_{21} = 2Q_2 \times R_{eq2} = 47.5k\Omega$$

$$R_{22} = \frac{R_{eq2}}{2Q_2} = 1.41k\Omega$$

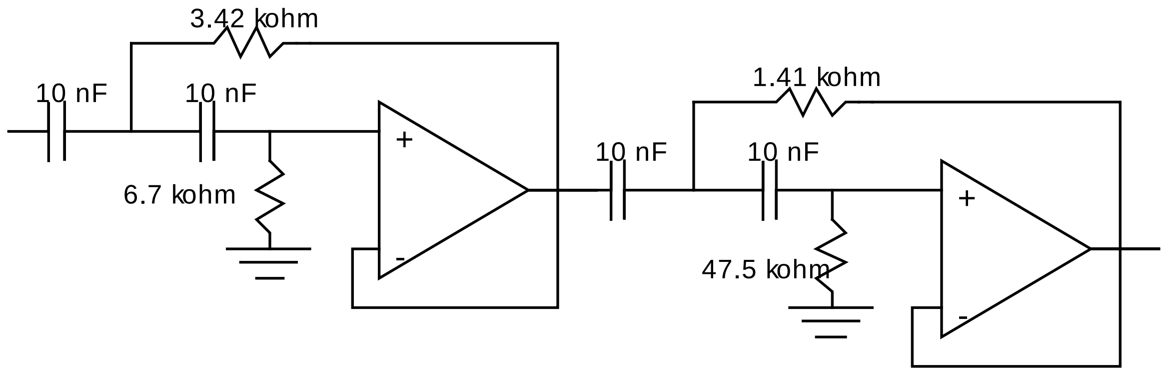

Figure 5.11a. Initial 4th order Chebyshev high pass filter design. The first 10nF capacitor must be changed to a voltage divider configuration to satisfy the 0dB maximum gain constraint.

Analogous to the previous example, the high frequency gain of the circuit in Figure 5.11a is 0 dB and is 0.5 dB lower than the maximum gain. Therefore to satisfy the specifications we need to lower the gain by 0.5 dB. The solution is virtually the same as the previous example except now the Thevenin equivalent should be 0.944 V_{in} in series with 0.01 µF. This circuit is shown in Figure 5.11b and the equations for solving for C_1 and C_2 are:

$$C_1 + C_2 = 0.01\ \mu F$$

and

$$\frac{1/C_2}{1/C_1 + 1/C_2} = 10^{-0.5/20} = 0.9441$$

or

$$\frac{C_1}{C_1 + C_2} = \frac{C_1}{0.01\mu F}\ .$$

This implies C_1 = 9.44 nF and C_2 = 560 pF.

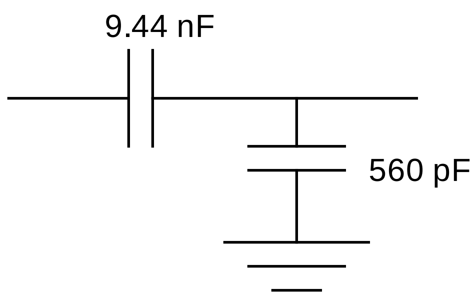

Figure 5.11b. Capacitive voltage divider circuit which replaces the left-most 10pF capacitor in Figure 5.11a.

5.5 Summary

In this chapter we learned about a different class of low and high pass filters that were no longer "maximally flat" in the pass band. By relaxing this constraint and letting the gain vary (ripple) between two specified limits within the pass band, we saw that we could meet our filter specifications with a lower order filter, called a Chebyshev filter. This reduction in required order did not come without drawbacks, and we witnessed that there were more complicated calculations required to determine the minimum order and the pole locations for the Chebyshev filter. Although the equations were more complicated, the general design strategy was the same as for the Butterworth filter, except that the undamped natural frequency (ω_o) was no longer the same for each cascaded filter section as it was in the Butterworth case.

To accomplish the design calculations there were several alternatives. We learned some graphical techniques that are appropriate for low order filters when accuracy of approximately 1% is sufficient. This 1% accuracy represented an insignificant limitation, especially since our designs often use resistors, capacitors, and op-amps with greater than 1% component tolerance. If better accuracy is required, a second method is to write short computer routines to perform the calculations presented in this chapter. Writing simple computer routines (problems 5.2, 5.4, and 5.5) is quite efficient and is less prone to mistakes than using a hand-held calculator. Of course the final method is to perform the calculations with the assistance of a hand-held calculator.

As a preview for Chapter 10 (and as a motivation for further study in this chapter), we should advise the reader that we will use all the methods presented in this chapter as first steps toward the design of *digital* Chebyshev filters.

Problems

5.1 For the following sets of specifications, find the minimum order Butterworth and Chebyshev lowpass filters, using the method of Figure 5.3 and the method of exact calculations:

	A_{max}	A_{min}	ω_p	ω_s
a.	1 dB	20 dB	1000 rad/sec	2000 rad/sec
b.	0.5dB	30 dB	1000 rad/sec	2500 rad/sec

5.2 Write an interactive computer program or function that will calculate the minimum order Butterworth or Chebyshev filter after entering the low pass specifications.

5.3 Determine the pole locations of the minimum order Chebyshev filters for the specifications given in Problem 5.1 by the following methods:

a. By calculation, using Equations 5.2 and 5.46-5.49

b. Graphically, as illustrated in Figure 5.5

c. Graphically, using Figure 5.6

5.4 Write a computer subroutine, procedure, or function, that will return Q and ω_o for the Chebyshev poles when passed ε and the order n.

5.5 Combine Problems 5.2 and 5.4 into a single program.

5.6 Design minimum order Chebyshev lowpass filter circuits to satisfy the specifications given in Problem 5.1. Use reasonable values for circuit elements. Design your circuits such that the maximum gain is unity.

Problems 5.7-5.11 are intended as practice problems in the design of Chebyshev low pass and high pass filters. In each case, determine the minimum order and the pole locations, and then design the circuit. Use reasonable values for circuit elements. Design your circuits such that the maximum gain is unity.

5.7 Design a Chebyshev low pass filter to meet the following set of specifications:

A_{max} = 1dB
A_{min} = 30dB
ω_p = 5000 rad/sec
ω_s = 15000 rad/sec

5.8 Design a Chebyshev low pass filter to meet the following set of specifications:

A_{max} = 0.1dB
A_{min} = 20dB
ω_p = 2000 rad/sec
ω_s = 3200 rad/sec

5.9 Design a Chebyshev low pass filter to meet the following set of specifications:

A_{max} = 0.4dB
A_{min} = 50dB
ω_p = 40,000 rad/sec
ω_s = 80,000 rad/sec

5.10 Design a Chebyshev high pass filter to meet the following set of specifications:

A_{max}=1.0 dB
A_{min}=20 dB
ω_p =1000 rad/sec
ω_s = 250 rad/sec

5.11 Design a Chebyshev high pass filter to meet the following set of specifications:

A_{max}=0.5 dB

A_{min}=25 dB

ω_p =10,000 rad/sec

ω_s = 3500 rad/sec