Chapter 10

Bilinear Transforms and IIR Digital Filter Design

10.1 Introduction

By now we have been exposed to a wealth of theory on continuous-time filter design. In this chapter we will continue to make use of the same continuous-time filter design techniques, and we will employ transform techniques to extend our filter designs to discrete-time.

In this chapter we introduce the method of *bilinear transformation* for designing infinite impulse response (IIR) digital filters. Bilinear transformation methods are particularly appropriate in cases where filter specifications are expressed in terms of passband and stopband frequency limits and attenuations. In this method the designer must *prewarp* the limiting frequencies, design a continuous time filter that satisfies the specifications in this warped frequency domain, and calculate the digital filter coefficients using the bilinear transformation. Since this process is quite tedious when done by hand, computer programs have been written and are an appropriate method for calculation in actual applications.

Although it is important to gain experience in using computer aided design software, we present graphical techniques in this chapter for IIR filter design in the hopes that the reader will gain insight and understanding of design techniques not attainable by using design software or by performing laborious calculations. Concepts of continuous time filter prototype design, relationships between pole/zero locations and frequency response, and the mapping between the s and z planes are reinforced by using graphical techniques for design.

10.2 Rationale Behind the Bilinear Transform Mapping

In order to easily use the powerful techniques developed over many years for continuous time filter design in the design of digital filters, an efficient method of mapping transfer functions from the s plane to the z plane is necessary. Desirable features of this mapping should include:

a. *Stable filter should map to stable filter*. For causal filters, this means the left half of the s plane should map to the interior of the unit circle in the z plane, the imaginary axis should map to the z-plane unit circle, and the right half of the s plane should map to the area outside the unit circle.

b. *The mapping should be one to one and should map bilinear transfer function (ratio of finite order polynomials in s or z) to bilinear transfer function.* This insures that the magnitude and phase of the frequency response in one domain exactly matches that in the other domain at the corresponding frequency.

Now we will examine what mappings might do a reasonable job of satisfying the requirements above. Our first intelligent choice might be the mapping $z=e^{sT}$ (where T corresponds to our sampling interval). This mapping satisfies requirement (a) above. Requirement (b), however, immediately disqualifies this mapping, since many points in the s plane map to the same point in the z plane, and transfer function does not map to transfer function.

To illustrate the "many-to-one" mapping of $z=e^{sT}$, consider the case where $s_n=j2n\pi/T$, n=0,1,..,N-1. This defines a family of N equally spaced points along the $j\omega$ axis in the s plane. In this case all N points map to $z=e^{j2\pi n}$ which corresponds to z=1 for any integer value n. Hence this mapping ($z=e^{sT}$) is a "many-to-one" mapping.

The one to one requirement, together with requirement (a) above means that the entire (infinite length) imaginary axis in the s plane must map to a finite length contour that goes exactly once around the unit circle in the z plane. Therefore nonlinear compression or "warping" in frequency is necessary to accomplish the mapping of an infinitely long contour (imaginary s plane axis) to the contour of the unit circle (of length 2π).

A candidate mapping from $j\omega$ axis of the s plane to the unit circle of the z plane might take the form of mapping s=0 to z=1, and the rest of the upper and lower $j\omega$ axis could map to upper and lower halves of the unit circle, respectively. One advantage of this type of mapping is that by mapping infinite frequency in the s plane to z = -1, zeros at $s = \infty$ (due to non-zero relative degree) map to zeros at one half the sampling frequency. Zeros at one half the sampling frequency ease the requirements on the front end analog anti-aliasing filter.

In selecting a candidate mapping (s = f(z)), *matching* continuous and discrete time frequency responses at DC implies s = 0 maps to z = 1, which together with (b) above implies a (z - 1) term in the numerator of f(z). Furthermore, $s = \infty$ mapping to z = -1 implies a (z + 1) term in the denominator, or

$$s = f(z) = K\left(\frac{z-1}{z+1}\right) \tag{10.1}$$

where K is some arbitrary constant. Similarly the inverse mapping is derived by solving for z from (10.1) such that

$$z = \left(\frac{K+s}{K-s}\right) \ . \tag{10.2}$$

If we evaluate equation 10.2 along the $s=j\omega$ axis,

$$z = \left(\frac{K+s}{K-s} \right) \Bigg|_{s=j\omega} = \left(\frac{K+j\omega}{K-j\omega} \right), \qquad (10.3)$$

and it is clear for any value ω that z has unit magnitude (and some associated angle). Said another way, the s plane jω axis (s=jω) maps to the z plane unit circle in this mapping. Further close examination of (10.2) shows that an s with a negative real part will map to a point in the interior of the unit circle, and an s with a positive real part will map to a point outside the unit circle. To see this, let s = σ + jω, where σ represents the real part and ω represents the imaginary part of the pole or zero in the s plane. From 10.3 we have

$$z = \left(\frac{K+s}{K-s} \right) = \left(\frac{K+(\sigma+j\omega)}{K-(\sigma+\omega)} \right), \qquad (10.4)$$

and

$$|z| = \left| \frac{K+(\sigma+j\omega)}{K-(\sigma+\omega)} \right| = \frac{\sqrt{(K+\sigma)^2+\omega^2}}{\sqrt{(K-\sigma)^2+\omega^2}}. \qquad (10.5)$$

Note from 10.5 that $|z|>1$ for $\sigma>0$, and $|z|<1$ for $\sigma<0$. Said another way, the right half of the s plane maps to the exterior of the z plane unit circle and the left half of the s plane maps to the interior of the z plane unit circle ($\sigma>0$). As a result we satisfy condition (a) from section 10.2 in that poles from a stable filter in the s plane (left half plane) will map to stable poles in the z plane (interior portion of the z plane unit circle).

Although any positive and real K will work in (10.2), we will be a bit more specific in our choice. Although K can be chosen such that continuous and discrete time transfer functions will match exactly at a desired reference frequency, here we will match frequency responses in the limit as frequency goes to zero (i.e., approaching DC). In that case,

$$z = e^{j\omega T}$$

$$\cong 1 + j\omega T \text{ for } \omega T << 1. \tag{10.6}$$

Substituting 10.6 into 10.1 we see that

$$s = K\left(\frac{z-1}{z+1}\right) \cong K\left(\frac{1 + j\omega T - 1}{1 + j\omega T + 1}\right)$$

$$\cong K\left(\frac{j\omega T}{2}\right). \tag{10.7}$$

If we select $K = \frac{2}{T}$ equation 10.7 reduces to $s = j\omega$, implying that we have in fact *matched* our frequency responses at the low frequency condition. Furthermore, in order to use the same mapping for any sampling frequency, we will arbitrarily set T=1 (and hence K=2). What this means is that all true frequencies in the discrete time domain will be expressed as fractions of sampling frequency. (For example, a lowpass filter transfer function designed for a filter cutoff of $0.2f_s$ will not change regardless of the sampling frequency. In other words this filter will have a cutoff of 1kHz at a sampling rate of 5kHz, and will have a cutoff at 2kHz at a sampling rate of 10kHz.) Our transformation then takes the form

$$z = \frac{2+s}{2-s} \tag{10.8}$$

If we consider mapping the s plane jω axis onto the unit circle in the z plane, equation 10.8 becomes

$$z = \frac{2 + j\omega_1}{2 - j\omega_1}$$

$$= e^{j\left[2\tan^{-1}\left(\omega_1/2\right)\right]}, \tag{10.9}$$

$$= e^{j\omega} \tag{10.10}$$

where ω_1 represents warped continuous time frequency, and ω represents the true discrete time frequency. Therefore the true discrete time frequency $\omega = 2\pi\left(\frac{f}{f_s}\right)$ (in terms of fraction of sampling frequency) is given in terms of the warped continuous time frequency $\omega_1 = 2\pi\left(\frac{f_1}{f_s}\right)$ by

$$\omega = 2\tan^{-1}\left(\frac{\omega_1}{2}\right) \tag{10.11}$$

and the inverse frequency mapping is

$$\omega_1 = 2\tan\left(\frac{\omega}{2}\right) \tag{10.12}$$

The computer-drawn bilinear transform chart shown in Figure 10.1 is the z plane with s plane polar coordinates mapped onto it. The chart which is based on the equations presented above can be used to map poles and zeros of transfer functions in either direction. The magnitude and phase of the discrete time system's frequency response at ω is exactly the same as the continuous time response at ω_1. This suggests that the first step of our design process will be to prewarp all passband and stopband limiting frequencies. If we prewarp our passband and stopband limiting frequencies in the beginning, in essence we can then design a continuous time filter which will meet these prewarped specifications. The final step will be to map those poles from our continuous time filter to the z plane using the bilinear transformation, which will "undo" our prewarping, and give us a discrete time filter which meets our original specifications!

Example 10.1: Frequency Warping

We want to design a digital low pass filter with a filter cutoff frequency (f_c) of 3kHz and a stopband frequency (f_r) of 4kHz. Assume the sampling frequency (f_s) is 10kHz. Calculate the appropriate filter cutoff and stopband frequencies for our continuous time prototype design (i.e. prewarped frequencies).

Solution:

$f_c=0.3f_s$, and $f_r=0.4f_s$. The corresponding prewarped continuous time frequencies for which we should design our continuous time prototype are given by (10.12),

$$\omega_1 = 2\tan(\omega/2)$$

so that

$$\omega_c' = 2\tan(2\pi*0.3/2) = 2.75276$$

and

$$\omega_r' = 2\tan(2\pi*0.4/2) = 6.15535 .$$

These numbers for ω_c' and ω_r' represent prewarped continuous time frequencies relative to fraction of sampling frequency; i.e., $\omega_r'=2\pi(f_r'/f_s)$. To obtain the actual prewarped frequencies we divide the numbers above by 2π and multiply by f_s to obtain actual prewarped continuous time frequencies of

$$f_c' = 4381.2\text{Hz}$$

$$f_r' = 9796.5\text{Hz} .$$

Instead of prewarping via these calculations, we could have directly used the Bilinear Transform Chart to "automatically" accomplish our prewarping. Observe that a frequency of $0.3f_s$ (labeled outside circle) corresponds to a warped radian frequency of roughly $2.75f_s$ rad/sec (labeled inside circle), and similarly $0.4f_s$ corresponds to a warped radian frequency of roughly $6.15f_s$ rad/sec.

In the sections that follow we will make use of the Bilinear Transform Chart to assist in design and analysis. In design, our general procedure to design a digital filter that will satisfy a set of required specifications will be as follows:

i. Just as we did in Example 10.1, we will use the chart to determine our prewarped frequencies for the continuous time design.

ii. We will then employ conventional continuous time techniques to determine the minimum order filter and the pole/zero locations of the filter to satisfy the required specifications in terms these warped (ω_1) frequencies.

iii. We will use the Bilinear Transform Chart to map these poles and zeros from s plane to z plane, and we will make use of the nomographs on this chart to simply read off our digital filter coefficients.

In filter or system analysis (Chapter 11), our general procedure will be as follows:

i. We will make use of the Bilinear Transform Chart to determine break frequencies and Q's of poles and zeros.

ii. We will then make use of conventional Bode plot techniques with a horizontal axis that is logarithmic in this warped frequency (but also labeled with true discrete time frequencies) to produce an accurate plot of the frequency response of a discrete time system.

10.3 The Bilinear Transform Chart

In this chapter we find that the Bilinear Transform Chart shown in Figure 10.1 will effectively bridge the gap between continuous time and discrete time for purposes of IIR digital filter design. We will use this chart to map continuous time poles and zeros to the equivalent discrete time poles and zeros (and vice versa), and we will find that the chart enables us to design infinite impulse response digital filters directly from a given set of filter specifications.

The Bilinear Transform Chart is essentially a map of the z plane unit circle which contains numerous arcs. The arcs labeled "15", "30", etc. correspond to the bilinear transform mappings of s plane vectors from the s plane origin of angles "15", "30", etc. with respect to the negative s axis. The other arcs in the chart (which appear at right angles to those arcs previously discussed) correspond to the bilinear transform mappings of s plane circles centered at s=0. These circles each have a radius in the s plane equal to some undamped natural frequency, ω_o, which represents some fraction of a desired discrete time sampling frequency, f_s. Around the outside of the bilinear transform chart are displayed fractions of f_s, and immediately inside the circle is each associated warped radian frequency equivalent.

To map poles (or zeros) from the s to the z plane, one enters the chart with the angle of the s-plane pole (or zero) in degrees from the negative s axis and its undamped natural frequency in terms of fractions of the sampling frequency. Once the poles and zeros have been plotted, the discrete time transfer function coefficient calculations are a straightforward extension.

For real poles (p_1, p_2, ..., p_n) and zeros (q_1, q_2, ... , q_m), the discrete time transfer function H(z) can be expressed as

$$H(z) = \frac{(z-q_1)(z-q_2)\ldots(z-q_n)}{(z-p_1)(z-p_2)\ldots(z-p_m)} \qquad (10.13)$$

where the p_i's and q_i's can be read directly off the Bilinear Transform Chart.

For a numerator or denominator in the quadratic form

$$F(z) = z^2 + az + b \qquad (10.14)$$

(with complex roots), where we assume a and b are real coefficients, F(z) can be written as

$$F(z) = (z-\rho)(z-\rho^*) \qquad (10.15)$$

where * denotes complex conjugate. F(z) from 10.15 can be further simplified to

$$\begin{aligned} F(z) &= z^2 - (\rho + \rho^*)z + \rho\rho^* \\ &= z^2 - 2\,\mathrm{Re}(\rho)z + |\rho|^2 \qquad (10.16) \end{aligned}$$

Observe that the coefficient "a" from 10.14 is -2 times the horizontal displacement from the centerline or imaginary axis to the z plane pole or zero, and b is the square of the radial distance from the center of the z plane unit circle to the pole or zero. Scales at the bottom of the Bilinear Transform Chart make use of these relationships, so one can directly measure the real part of a pole or zero and the radius from the center with a pair of dividers, and simply read off the coefficients for z^1 and z^0.

10.4 IIR Filter Design Strategy

In many cases, IIR filter design specifications will take the form of some set of required frequency domain magnitude response characteristics. Most often the required digital filter specifications will be expressed in terms of maximum allowed passband attenuation (A_{max}), minimum acceptable stopband attenuation (A_{min}), passband limits, stopband limits, system sampling frequency, and type of filter desired. In this case, our design strategy will be as follows:

1. Prewarp the required passband and stopband limits (which are usually expressed in terms of fractions of sampling frequency) according to the bilinear transform frequency warping characteristic given in equations 10.11 and 10.12. Recall that no calculation is required for this step since the Bilinear Transform Chart may be used to translate fraction of sampling frequency (labeled around the outside perimeter of the unit circle on the Bilinear Transform Chart) to corresponding warped radian frequencies (labeled just inside the unit circle).

2. From the requirements of Amax, Amin, pre-warped passband limits, pre-warped stopband limits, and type of filter desired, determine the minimum order of the continuous time low pass prototype filter (for low pass and band pass filters) or high pass prototype (for high pass and band stop filters) that will satisfy our specifications.

3. If necessary, accomplish a transformation from continuous time low pass prototype to continuous time bandpass, or continuous time high pass prototype to continuous time band stop characteristic via the bandpass mapping methods from Chapter 6 (Figure 10.3). Common continuous time design spectral transformations are listed in Table 10.1.

4. Determine the Q (or s plane angles) and ω_0 associated with each filter section to assist in plotting the poles and zeros onto the Bilinear Transform Chart.

5. After plotting poles and zeros, use a pair of dividers to measure the real part of each complex conjugate pole pair, and then move the dividers to the top scale of the Bilinear Transform Chart to read off the denominator coefficient of z. Similarly measure the radius of each complex conjugate pole pair, and then move the dividers to the center scale to read off the denominator constant coefficient. Simple real poles can be handled by inspection without the aid of these scales. This procedure will produce a filter transfer function H(z) which is a product of first and/or second order section transfer functions.

Table 10.1 Spectral Transformations for Continuous Time Designs

Assuming $H_c(s)$ is a lowpass filter with $\omega_c = 1$:

1. $H_c(s/\omega_c)$ is lowpass with cutoff ω_c.

2. $H_c(\omega_c/s)$ is highpass with cutoff ω_c.

3. $H_c\left\{\dfrac{s^2+\omega_1\omega_2}{s(\omega_2-\omega_1)}\right\}$ is bandpass with cutoffs ω_1 and ω_2.

4. $H_c\left\{\dfrac{s(\omega_2-\omega_1)}{s^2+\omega_1\omega_2}\right\}$ is bandstop with cutoffs ω_1 and ω_2.

10.5 IIR Filter Design Examples

The use of the Bilinear Transform Chart in filter design is best illustrated by examples.

Example 10.1. Butterworth Low Pass Filter

Design a digital Butterworth low pass filter to satisfy the following specifications: (f_s = sampling frequency)

Maximum passband attenuation (Amax)	3 dB
Minimum stopband attenuation (Amin)	25 dB
Passband limits	0-0.1 f_s
Stopband limits	0.2-0.5 f_s

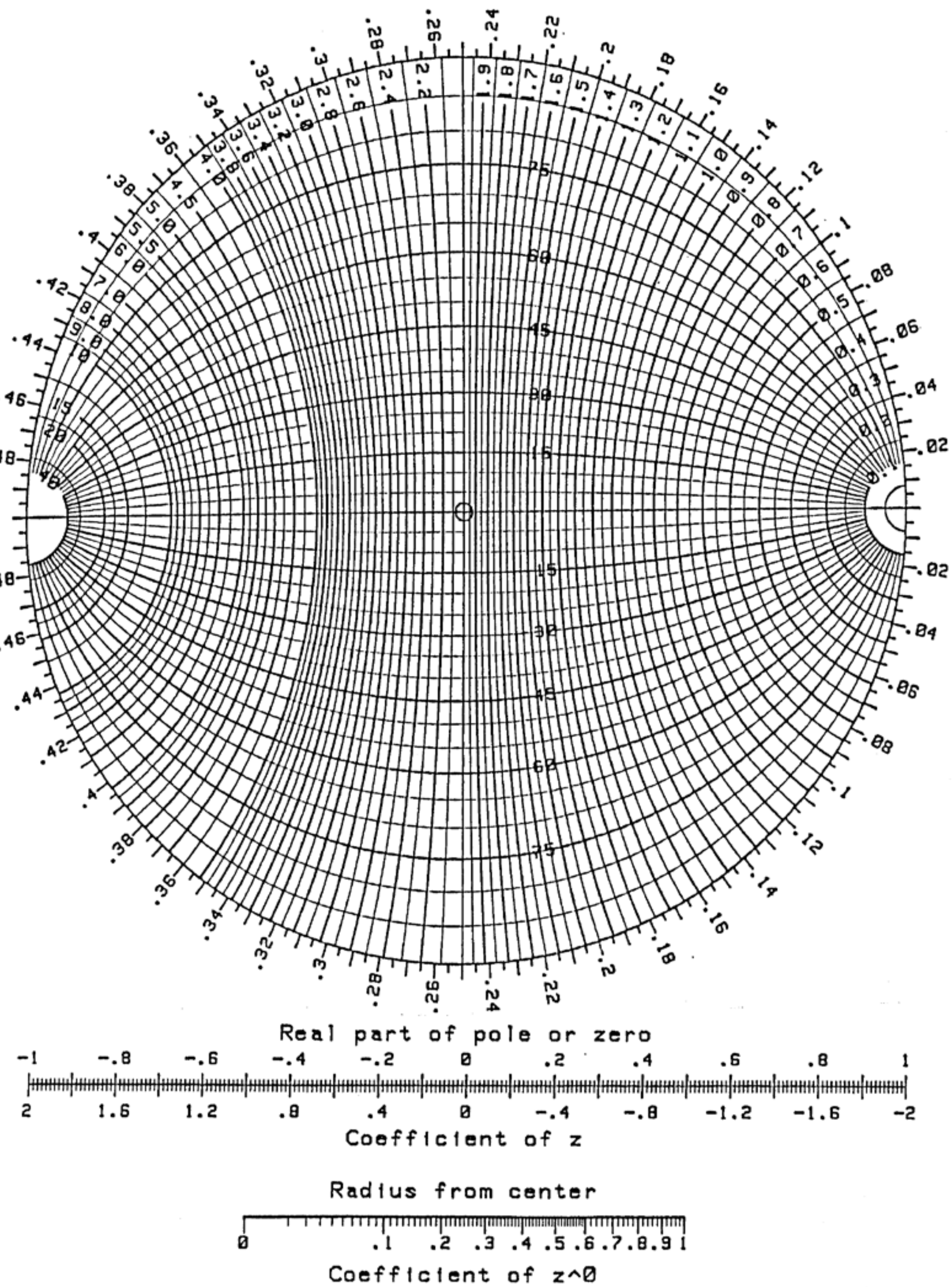


Figure 10.1 Bilinear Transform Chart

Solution:

According to our design strategy discussed in Section 10.1, we will design a continuous time prototype of this filter, and then map poles and zeros to the z plane via the Bilinear Transform Chart. Due to the warping characteristics of the bilinear transformation mapping from s to z planes, we must first "pre-warp" our passband and stopband limits before designing our continuous time prototype. Using the outer scale on the unit circle of the Bilinear Transform Chart (Figure 10.4), we look up our passband and stopband limits in terms of fractions of sampling frequency. Our frequency of 0.1 fs corresponds to a pre-warped frequency of 0.65 rad/sec as a pass band limit, and 0.2 fs corresponds to a pre-warped frequency of 1.45 rad/sec as a stop band limit. The minimum order required to satisfy the required specifications may be determined from a nomograph or by using standard calculation methods described in chapters 4 and 5. For a continuous time Butterworth filter, the minimum order n is determined from 4.11 as

$$n = \frac{\ln\left(\dfrac{10^{A\min/10}-1}{10^{A\max/10}-1}\right)}{2\ln\left(\omega_{stop}/\omega_{pass}\right)} \tag{4.11}$$

Since n in general will be non-integer from this calculation, we simply round up to the next highest integer order and use this value for n. Substituting the pre-warped stop band and pass band limits for ω_{stop} and ω_{pass} yields a minimum order of four. Our fourth order Butterworth low-pass filter has s plane poles at angles of +/-22.5 degrees and +/-67.5 degrees with respect to the negative s axis and has natural frequencies of 0.65 rad/sec (radius of poles from s plane origin). Observe that one can think of this Butterworth low pass transfer function as having these four poles, as well as having four zeros at $s = \infty$ (due to the non-zero relative degree). In plotting the poles and zeros (Figure 10.4), the four zeros at s = infinity map to four zeros at z =

-1. After plotting poles and zeros onto the Bilinear Transform Chart, we can use a pair of dividers to measure the real part of each complex conjugate pole pair, and to measure the radius of each complex conjugate pole pair. We can then read the denominator coefficients from the top two scales. Our discrete time transfer function is therefore

$$H(z) = K\left(\frac{z^2 + 2z + 1}{z^2 - 1.025z + 0.3}\right)\left(\frac{z^2 + 2z + 1}{z^2 - 1.305z + 0.63}\right)$$

The constant (K) can be evaluated two ways. If the complete continuous time transfer function (including the overall constant) is known, K in the expression for H(z) equals the continuous time transfer function evaluated at s = 2. (Refer to Appendix A for proof.) In many design problems it is of more interest to constrain the gain to a certain value at a reference frequency (0 dB at DC in this example). Evaluating the above at z = 1 yields H(1)=179.021, therefore K = 1/H(1) = 0.005586, thereby yielding a DC gain of 0 dB.

For purposes of comparison, one can use the same design specifications and design this filter via computer. Results of this design method for H(z) are as follows:

$$H(z) = 0.0048243\left(\frac{z^2 + 2z + 1}{z^2 - 1.04860z + 0.29614}\right)\left(\frac{z^2 + 2z + 1}{z^2 - 1.32091z + 0.63274}\right)$$

Example 10.2. Chebyshev Bandpass Filter

Design a Chebyshev bandpass digital filter to satisfy the following specifications:

Amax	0.5 dB
Amin	20 dB
Passband limits	0.2-0.3 f_s
Stopband limits	0.15 and 0.35 f_s

Solution:

First we need to prewarp the passband limits and stopband limits of our filter. From 10.12 we know

0.2f_s warps to: $\omega_{c_1}' = 2\tan(2\pi*0.2/2)=1.4531$

0.3f_s warps to: $\omega_{c_2}' = 2\tan(2\pi*0.3/2)=2.7528$

0.15f_s warps to: $\omega_{r_1}' = 2\tan(2\pi*0.15/2)=1.0191$

0.35f_s warps to: $\omega_{r_2}' = 2\tan(2\pi*0.35/2)=3.9252$

Recall that these numbers for ω_{c_1}', ω_{c_2}', ω_{r_1}', and ω_{r_2}' represent prewarped continuous time frequencies relative to fraction of sampling frequency; i.e., $\omega_{c_1}'=2\pi(f_{c1}'/f_s)$. If we needed to obtain the actual prewarped frequencies we would divide the numbers ω_{c_1}', ω_{c_2}', ω_{r_1}', and ω_{r_2}' above by 2π and multiply by f_s to obtain actual prewarped continuous time frequencies. (Note that we could accomplish the prewarping in one step by referring to the Bilinear Transform Chart (Figure 10.4). Now we need to verify that the filter passband and stopband limits are symmetric. That is, the center frequency of our filter as calculated from the passband limits is

$$CF_{pb} = \sqrt{(1.4531)(2.7528)} \cong 2.0$$

and the center frequency of the filter as calculated from the stopband limits is

$$CF_{sb} = \sqrt{(1.0191)(3.9252)} \cong 2.0$$

(Recognize if these had been different, we would have been forced to change one of the stopband limits or one of the passband limits to enforce symmetry. The choice of which to modify would have been determined by which modification would have produced the lower order filter.)

Using techniques described in Chapter 6 for bandpass filter design, we solve for the ratio of passband limits relative to the center frequency of the filter, and in this case we have

$$b = \frac{\Delta\omega_p}{\omega_o} = \frac{2.7528 - 1.4531}{2.0} = 0.64985 \cong 0.65$$

Our ratio of stopband limits to passband limits (which is required to calculate filter order) is

$$\frac{\Delta\omega_s}{\Delta\omega_p} = \frac{3.9252 - 1.0191}{2.7528 - 1.4531} = 2.236$$

The filter order is calculated from Chapter 6 as

$$n = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{A\min/10} - 1}{10^{A\max/10} - 1}}\right)}{\cosh^{-1}\left(\frac{\Delta\omega_s}{\Delta\omega_p}\right)} = 2.8049$$

which we round up to an integer order n=3. (Recall this is the order of our lowpass prototype filter.) At this point we know that the poles of our Chebyshev lowpass prototype will lie on an elliptical contour, and we need to find the minor axis of the ellipse. Using techniques described in Chapter 5, for n=3 and for Amax=0.5dB, we obtain a minor axis of

$$a = 0.6265$$

Entering the Chebyshev ellipse chart with minor axis $a = 0.6265$ and the Butterworth angles (for n=3) of 0 and $\pm$60 degrees we get for our lowpass prototype

$$\Omega_{o_{LP1}} = 1.0689 \text{ (relative to the lowpass prototype filter cutoff),}$$

and

$$Q_{LP1}=1.7062 \text{ (or a pole angle of approximately 73 degrees).}$$

Our second section (which is a first order section since n=3)

$$\Omega_{o_{LP2}} = 0.6265 \text{ (relative to the lowpass prototype filter cutoff),}$$

and

$$Q_{LP2}=0.5 \text{ (since it is a single real pole).}$$

At this point we map our lowpass prototype poles (n=3) to our bandpass poles (n=6) using our bandpass mapping techniques from Chapter 6. Recall we enter this mapping with the product of b and Ω_o, where b is approximately 0.65. These products of b and Ω_o are

$$b\Omega_{o_{LP1}} = (0.65)(1.0689) = 0.6949 \text{ (with LP prototype angle 73 degrees)}$$

$$b\Omega_{o_{LP2}} = (0.65)(0.6265) = 0.407 \text{ (with LP prototype angle 0 degrees)}$$

The mapping produces bandpass poles at:

$\pm$84.2 degrees (Q=5) at 1.4 times the center frequency (2.8 r/s)

$\pm$84.2 degrees (Q=5) at 1/1.4 times the center frequency (1.43 r/s)

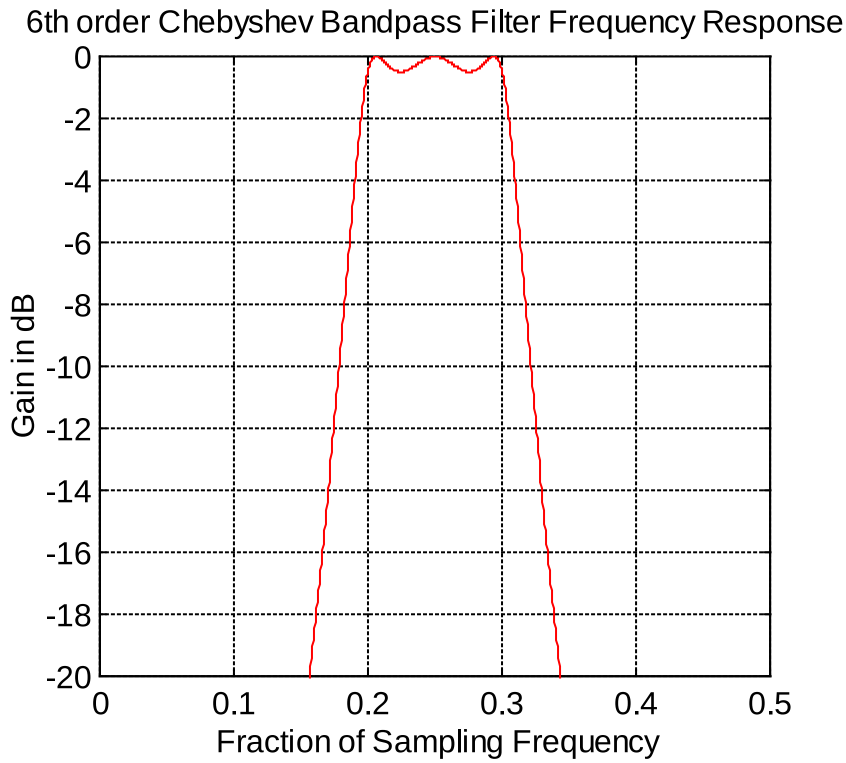
$\pm$78.5 degrees (Q=2.5) at the filter center frequency (2.0 r/s)

When considering the zero locations, note that the third order lowpass continuous time prototype contains three zeros at $s=\infty$, which map to three zeros at $s=0$ and three zeros at $s=\infty$ in the continuous time bandpass prototype.

Now we map poles and zeros into the z plane using the Bilinear Transform Chart. The three zeros at $s=\infty$ map to z=1, and the three zeros at $s=\infty$ map to z=-1. From the pole zero plot (or from computer analysis) we can obtain the coefficients of our transfer function as

$$H(z)=K\left(\frac{z^2-1}{z^2+0.5798z+0.8322}\right)\left(\frac{z^2-1}{z^2-0.5798z+0.8322}\right)\left(\frac{z^2-1}{z^2+0.6618}\right)$$

In order to set the center frequency gain to 0dB, we can simply evaluate the function H(z) at the center frequency $\left(z=e^{j\pi/2}=j\right)$. If we assume for a minute that the value of K=1, evaluating at z=j we obtain a center frequency gain equal to 64.9155, hence if we set $K=\left(\frac{1}{64.955}\right)=0.0154$ we would obtain a gain of 0dB at the center frequency of this filter.


6th order Chebyshev Bandpass Filter Frequency Response
Gain in dB
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
0
0.1
0.2
0.3
0.4
0.5
Fraction of Sampling Frequency

Example 10.3. Butterworth Notch Filter

Design a Butterworth band-stop digital filter to satisfy the following specifications:

Amax	2.0 dB
Amin	16 dB
Passband limits	0.1-0.25 fs
Stopband limits	0.15 and 0.2 fs

Solution:

First we need to prewarp the passband limits and stopband limits of our filter. From 10.12 we know

$0.1f_s$ warps to: $\omega_{c_1}' = 2\tan(2\pi*0.1/2)=0.65$

$0.25f_s$ warps to: $\omega_{c_2}' = 2\tan(2\pi*0.25/2)=2.0$

$0.15f_s$ warps to: $\omega_{r_1}' = 2\tan(2\pi*0.15/2)=1.02$

$0.2f_s$ warps to: $\omega_{r_2}' = 2\tan(2\pi*0.35/2)=1.45$

Observe that the center frequency based on passband limits is (0.65*2.0) or 1.14, and the center frequency based on stopband limits is (1.02*1.45) or 1.216. In effect, we have been given asymmetrical limits that must be satisfied by a symmetrical filter transfer function characteristic. More to the point, we cannot design to meet these specifications exactly, so we must change the specifications slightly in order to design a geometrically symmetric notch filter. One choice (choice #1) is to increase the lower passband limit of 0.65 rad/sec to 0.74 rad/sec so that (0.74*2.0)=(1.02*1.45). A second choice (choice #2) is to decrease the lower stopband limit from 1.02 rad/sec to .896 rad/sec so that (.65*2.0)=(.896*1.45).

Although this asymmetry of specifications is a subtle point, it is nevertheless worthwhile to consider such cases since they can easily occur! In either choice above, we are designing a filter that exactly meets the specifications on the upper band side of the notch, and performs better than our specifications on the low band side. We could have just as easily met specifications on the low side, and performed better than our requirements on the high band side. However, given that we want to limit ourselves to choices #1 or #2 above, which choice gives us the lower order filter?

One metric that is used in the case of our notch filter in the determination of required order is the ratio of stopband limits to passband limits. For choice #1, the ratio is $\frac{(2.0-0.74)}{(1.45-1.02)}=2.93$. For choice #2 the ratio is $\frac{(2.0-0.65)}{(1.45-0.896)}=2.44$. Using these ratios we determine from equation 10.6 that choice #1 would require a 2nd order prototype, whereas choice #2 would require a 3rd order prototype. We will choose the minimum order solution. (Remember, for a fractional order we must round to the next highest integer.) Recognizing from equation 10.11 that $f=\frac{2\tan^{-1}\left(\frac{\omega_1}{2}\right)}{2\pi}$, where f is expressed as a fraction of sampling frequency, our new lower passband specification from "choice #1" above can be written as ω_{c_1}' = 0.74 which corresponds to $f=0.1128f_s$.

Now our task is to design a continuous time high pass (since we are designing a notch filter instead of a bandpass filter) prototype in order to obtain a notch filter with center frequency of 1.216 rad/sec. From our previous work in analog filter design we know that this 2nd order Butterworth high pass prototype will have s-plane poles at angles of +/-45 degrees with respect to the negative s axis, and two zeros at the origin.

From our knowledge of Butterworth highpass response characteristics from Chapter 4, we know that the frequency of the highpass prototype poles is given as

$$\omega_o = \omega_p \left(10^{A\max/10} - 1\right)^{\frac{1}{2n}} \tag{10.7}$$

where in our case Amax=2, n=2, and ω_π is the frequency at which Amax has been specified. Normalizing to $\omega_p = 1$ we obtain

$$\omega_o = 1 \times \left(10^{A\max/10} - 1\right)^{\frac{1}{2n}} = 0.8745$$

such that our high pass prototype poles will be at +/-45 degrees (Q=0.707) with respect to the negative s axis at radii of 0.8745. As introduced in Chapter 6 and as used in Example 10.2 of this chapter, we use the Bandpass Mapping Chart (Figure 6.5) to map the high pass prototype poles and zeros to notch filter poles and zeros. Recall we enter this mapping with the product of b and Ω_o, where b is

$$b = \frac{\Delta\omega_p}{CF} = \frac{2.0 - 0.74}{1.2165} = 1.0357$$

and

Ω_o =0.8745.

Mapping these highpass prototype poles in the bandpass mapping we obtain a 4th order analog filter, with poles at

****

Section #1:

$$\omega_o = 0.71887 \times (CF) = 0.71887 \times 1.2165 = 0.8745$$

Q=1.65

Section #2:

$$\omega_o = 1.3911 \times (CF) = 1.3911 \times 1.2165 = 1.6923$$

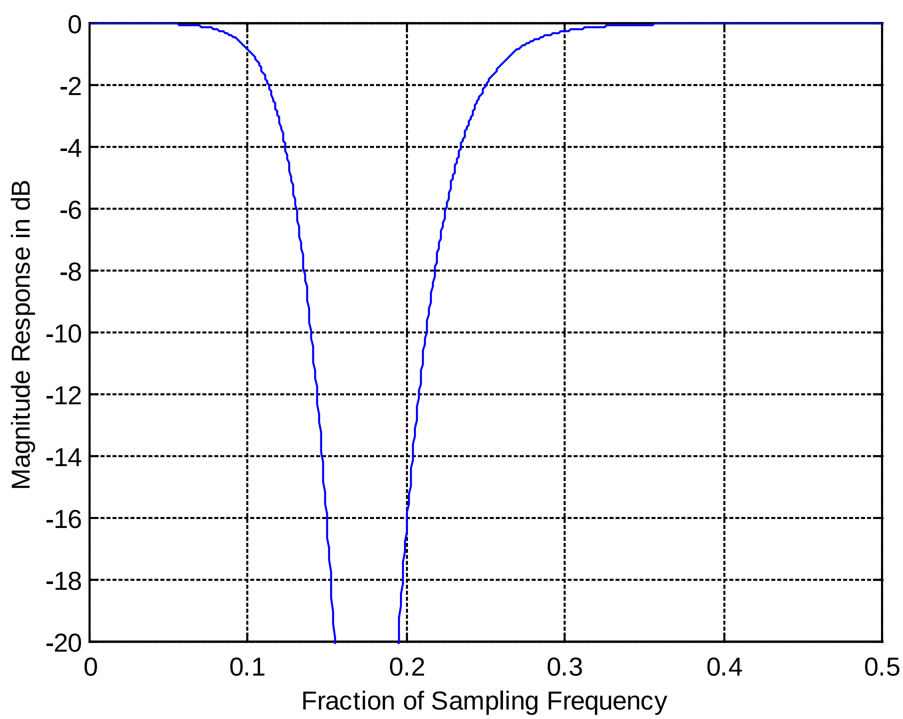
Q=1.65

Of course the two zeros at s=0 from our high pass prototype map to two zeros at $s = +j\omega_{cf}$ and two zeros at $s = -j\omega_{cf}$, where $\omega_{cf} = 1.2165$, or just the center frequency of the notch filter.

We then map these poles and zeros into the z-plane using the Bilinear Transform Chart or using the Bilinear Transformation from equation 10.8, namely $z = \dfrac{2+s}{2-s}$. From this mapping we obtain

$$H(z) = K\left(\frac{z^2 - 0.9197z + 1.0}{z^2 - 1.1105z + 0.6355}\right)\left(\frac{z^2 - 0.9197z + 1.0}{z^2 - 0.2548z + 0.5392}\right)$$

To set the passband gain (say at DC) to 0dB, we simply evaluate the transfer function at z=1. In doing so, we obtain H(1) = K(1.73), therefore setting K=0.5779 will give us a transfer function that has 0dB gain at DC. A plot of the frequency response for this notch filter is shown below. Observe from the frequency response that just as we had specified in our design process, we meet our high band response exactly, and we meet our lower stopband characteristic exactly as well. We improved on our requirement for having a maximum passband attenuation of 2dB at 0.1fs, and in fact we obtain roughly 1dB attenuation at $0.1f_s$.



EXAMPLE PROBLEMS for DIGITAL FILTER DESIGN:

Problem 10.1: Design a Chebyshev lowpass digital filter to satisfy the following specifications:

A_{max} = 1 dB	Passband limit = 800Hz
A_{min}= 30 dB	Stopband limit = 1600Hz
DC Gain = 0dB	Sampling Frequency = 8000Hz

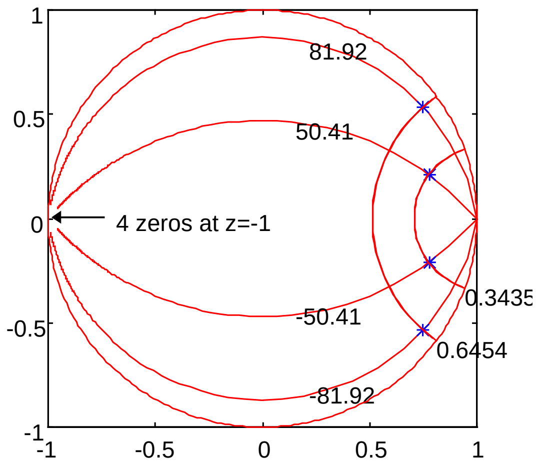
Solution:

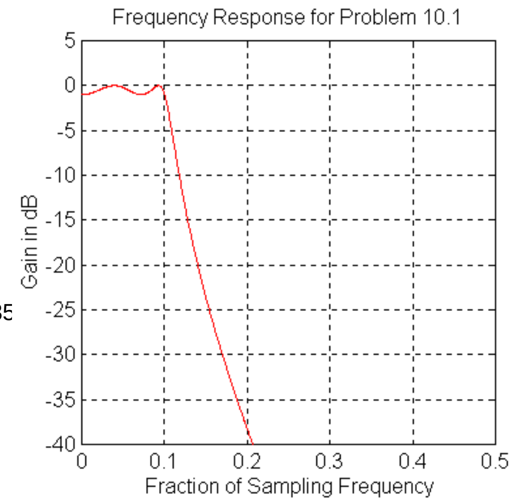
Passband limit of $0.1f_s$ is prewarped to 0.65 r/s, and the stopband limit frequency of $0.2f_s$ is prewarped to 1.45. The calculation for order yields n = 3.34 which we round to 4. The ellipse minor axis a=0.37 by charts (or a=0.3646 by computer). From Chebyshev chart Ω_o = 0.53 at 50° and 1.0 at 82° (recall these are normalized, relative to Ω_p…. by computer these normalized values are Ω_o = 0.5286 at 50.41° and 0.9932 at 81.92°.) After multiplying by Ω_p =0.65 we get our "real" analog prototype pole locations of Ω_{o1} = 0.3435 at +/-50.41° and Ω_{o2} = 0.6454 at +/-82°. (For a digital lowpass filter there are as many zeros at z=-1 as the order of the denominator of the filter, hence there are four zeros at z = -1.) Mapping our poles to the bilinear transform chart (0.3435 and 50°, and 0.65 and 82°), or using the bilinear transformation itself, we obtain the transfer function

$$H(z) = K\left(\frac{z^2 + 2z + 1}{z^2 - 1.4996z + 0.8482}\right)\left(\frac{z^2 + 2z + 1}{z^2 - 1.5548z + 0.6493}\right)$$

Evaluating H(z) (and assuming K=1 for a moment) we obtain a DC gain of 485.5497. Hence, if we wanted to obtain a DC gain of 0dB for H(z) we would set $K = \left(\frac{1}{485.597}\right)$.

Question: How would our design change if our specification had said "MAXIMUM gain of 0dB" instead of DC gain of 0dB? Recognize that we have a 4^{th} order Chebyshev filter, so if we used $K = \left(\frac{1}{485.597}\right)$, while the DC gain would be 0dB, the *maximum* gain would be +1dB and not 0dB. To decrease the filter magnitude response by 1dB, we would need to multiply our value of K by $10^{(-1/20)} = 0.8913$, such that our final value for K would be given by $K = \left(\frac{0.8913}{485.597}\right) = 1.8354 \times 10^{-3}$.





Our final transfer function for this Chebyshev lowpass filter is

$$H(z) = \left(1.8354 \times 10^{-3}\right)\left(\frac{z^2 + 2z + 1}{z^2 - 1.4996z + 0.8482}\right)\left(\frac{z^2 + 2z + 1}{z^2 - 1.5548z + 0.6493}\right)$$

Problem 10.2: Design a Butterworth high pass digital filter to satisfy the following specifications:

A_{max} = 2 dB	Stopband limit = 1600Hz
A_{min}= 30 dB	Passband limit = 2400Hz
Maximum Gain = 0dB	Sampling Frequency = 8000Hz

Solution:

Stopband limit of $0.2f_s$ is prewarped to 1.45 r/s, and the passband limit frequency of $0.3f_s$ is prewarped to 2.75. Solving for order we obtain a required order of 5.825 which we round to n=6. We obtain three second order sections as follows:

Section # 1
Wo = 2.632 rad/sec
Q = 1.932
(or angle = 75 degrees)

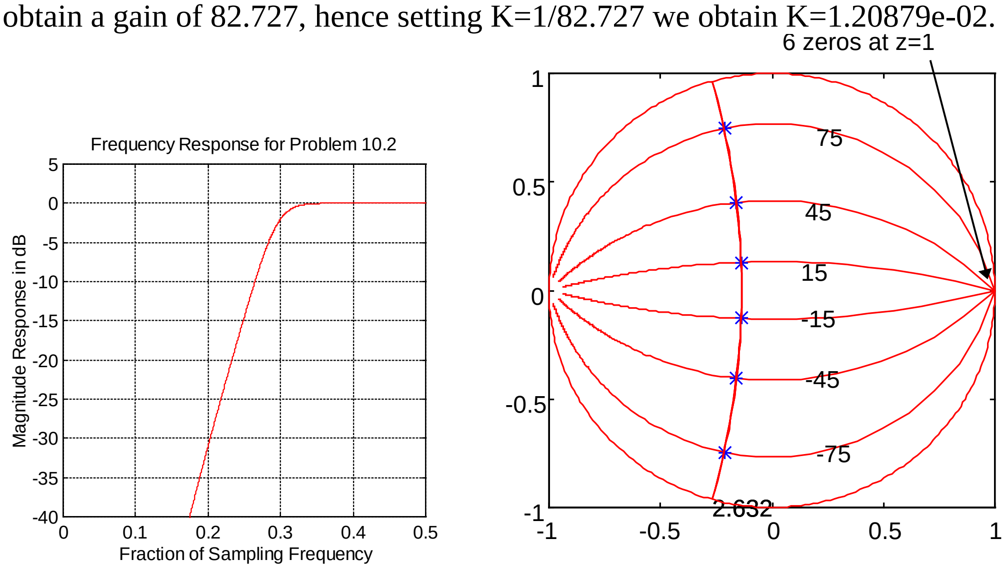
Section # 2
Wo = 2.632 rad/sec
Q = 0.7071
(or angle = 45 degrees)

Section # 3
Wo = 2.632 rad/sec
Q = 0.5176
(or angle = 15 degrees)

Mapping these poles to the bilinear transform chart (or using MATLAB to perform the mapping) we obtain

$$H(z) = (K)\left(\frac{z^2 - 2z + 1}{z^2 + 0.4291z + 0.6008}\right)\left(\frac{z^2 - 2z + 1}{z^2 + 0.31888z + 01896}\right)\left(\frac{z^2 - 2z + 1}{z^2 + 0.2777z + 0.03597}\right)$$

Assuming we want a maximum gain of 0dB, if we assume K=1 for just a second, we need to evaluate H(z) at z=-1 (since we have a highpass filter). Solving for H(-1) we obtain a gain of 82.727, hence setting K=1/82.727 we obtain K=1.20879e-02.



Problem 10.2 frequency response and pole/zero diagrams.

Problem 10.3: Design a Butterworth bandpass digital filter to satisfy the following specifications:

A_{max} = 2 dB	Stopband limits = 1kHz and 6kHz
A_{min}= 20 dB	Passband limits = 2kHz and 4kHz
Maximum Gain = 0dB	Sampling Frequency = 20kHz

Solution:

First we need to prewarp the passband limits and stopband limits of our filter. From 10.12 we know

$0.1f_s$ (2kHz) warps to: $\omega_{C_1}' = 2\tan(2\pi*0.1/2)=0.6498$

$0.2f_s$ (4kHz) warps to: $\omega_{C_2}' = 2\tan(2\pi*0.2/2)=1.4531$

$0.05f_s$ (1kHz) warps to: $\omega_{r_1}' = 2\tan(2\pi*0.05/2)=0.3168$

$0.3f_s$ (6kHz) warps to: $\omega_{r_2}' = 2\tan(2\pi*0.3/2)=2.7528$

Recall that these numbers for ω_{C_1}', ω_{C_2}', ω_{r_1}', and ω_{r_2}' represent prewarped continuous time frequencies relative to fraction of sampling frequency; i.e., $\omega_{C_1}'=2\pi(f_{c1}'/f_S)$. If we needed to obtain the actual prewarped frequencies we would divide the numbers ω_{C_1}', ω_{C_2}', ω_{r_1}', and ω_{r_2}' above by 2π and multiply by f_s to obtain actual prewarped continuous time frequencies. (Note that we could accomplish the prewarping in one step by referring to the Bilinear Transform Chart (Figure 10.4). Now we need to verify that the filter passband and stopband limits are symmetric. That is, the center frequency of our filter as calculated from the passband limits is

$$CF_{pb}=\sqrt{(1.4531)(0.6498)} \cong 0.9717$$

and the center frequency of the filter as calculated from the stopband limits is

$$CF_{sb}=\sqrt{(0.3168)(2.7528)} \cong 0.9339$$

Recognize these center frequencies are different, and we need to change one of the stopband limits or one of the passband limits to enforce symmetry. As discussed in Chapter 6, to minimize filter order in band pass filters, we will modify one of the stop band limits. Here, we will raise the lower stop band limit.

$$\text{New Lower Stop Band Limit} = \frac{(0.6498)(1.4531)}{2.7528} \cong 0.3430$$

Using techniques described in Chapter 6 for band pass filter design, we solve for the ratio of passband limits relative to the center frequency of the filter, and in this case we have

$$b = \frac{\Delta\omega_p}{\omega_o} = \frac{1.4531 - 0.6498}{0.9717} = 0.8267$$

Our ratio of stopband limits to passband limits (which is required to calculate filter order) is

$$\frac{\Delta\omega_s}{\Delta\omega_p} = \frac{2.7528 - 0.3430}{1.4531 - 0.6498} = 3.000$$

The Butterworth filter order is calculated from Chapter 6 as

$$n = \frac{\ln\left(\dfrac{10^{A\min/10} - 1}{10^{A\max/10} - 1}\right)}{2\ln\left(\dfrac{\Delta\omega_s}{\Delta\omega_p}\right)} = 2.3354$$

which we round up to an integer order n=3. (Recall this is the order of our low pass prototype filter.) At this point we know that the poles of our Butterworth low pass prototype will lie on a circular contour, and we can calculate that radius as

$$\Omega_{o_{LP}} = \frac{1}{\left(10^{A_{\max}/10} - 1\right)^{\frac{1}{2n}}} = 1.0935 \text{ rad/sec}$$

At this point we map our low pass prototype poles (n=3) to our band pass poles (n=6) using our band pass mapping techniques from Chapter 6. Recall we enter is mapping with the product of b and Ω_o, where b is approximately 0.8267, and Ω_o is approximately 1.0935. This product of of b and Ω_o is

$$b\Omega_{o_{LP}} = (0.8267)(1.0935) = 0.9040$$

The mapping produces band pass poles at the intersection of this $b\Omega_{o_{LP}} = 0.9040$ curve, and the angles of 0 and +/- 60 degrees. The actual band pass pole locations are at

ω_{o1} = 0.97174 rad/sec (or ω_{o1} = 1*(Center Frequency)), and Q = 1.1063 (angle = +/-63.13 degrees)

ω_{o2} = 0.65771 rad/sec (or ω_{o1} = 0.67684*(Center Frequency)), and Q = 2.3833 (angle = +/-77.89 degrees)

ω_{o3} = 1.4357 rad/sec (or ω_{o1} = 1.4775*(Center Frequency)), and Q = 2.3833 (angle = +/-77.89 degrees)

When considering the zero locations, note that the third order low pass continuous time prototype contains three zeros at $s = \infty$, which map to three zeros at $s = 0$ and three zeros at $s = \infty$ in the continuous time band pass prototype.

Now we map poles and zeros into the z plane using the Bilinear Transform Chart. The three zeros at $s = 0$ map to z=1, and the three zeros at $s = \infty$ map to z=-1. From the pole zero plot (or from computer analysis, with considerably more precision) we can obtain the coefficients of our transfer function as

$$H(z) = K\left(\frac{z^2 - 1}{z^2 - 0.91202z + 0.47569}\right)\left(\frac{z^2 - 1}{z^2 - 1.4314z + 0.77854}\right)\left(\frac{z^2 - 1}{z^2 - 0.53365z + 0.66838}\right)$$

In order to set the center frequency gain to 0dB, we can simply evaluate the function H(z) at the center frequency of the filter. Here, the center frequency W=0.97174rad/sec (warped frequency), so we could use the Bilinear Transformation Chart or the “unwarping equation” (below) to obtain the “discrete-time” frequency F’ (in terms of fraction of sampling frequency).

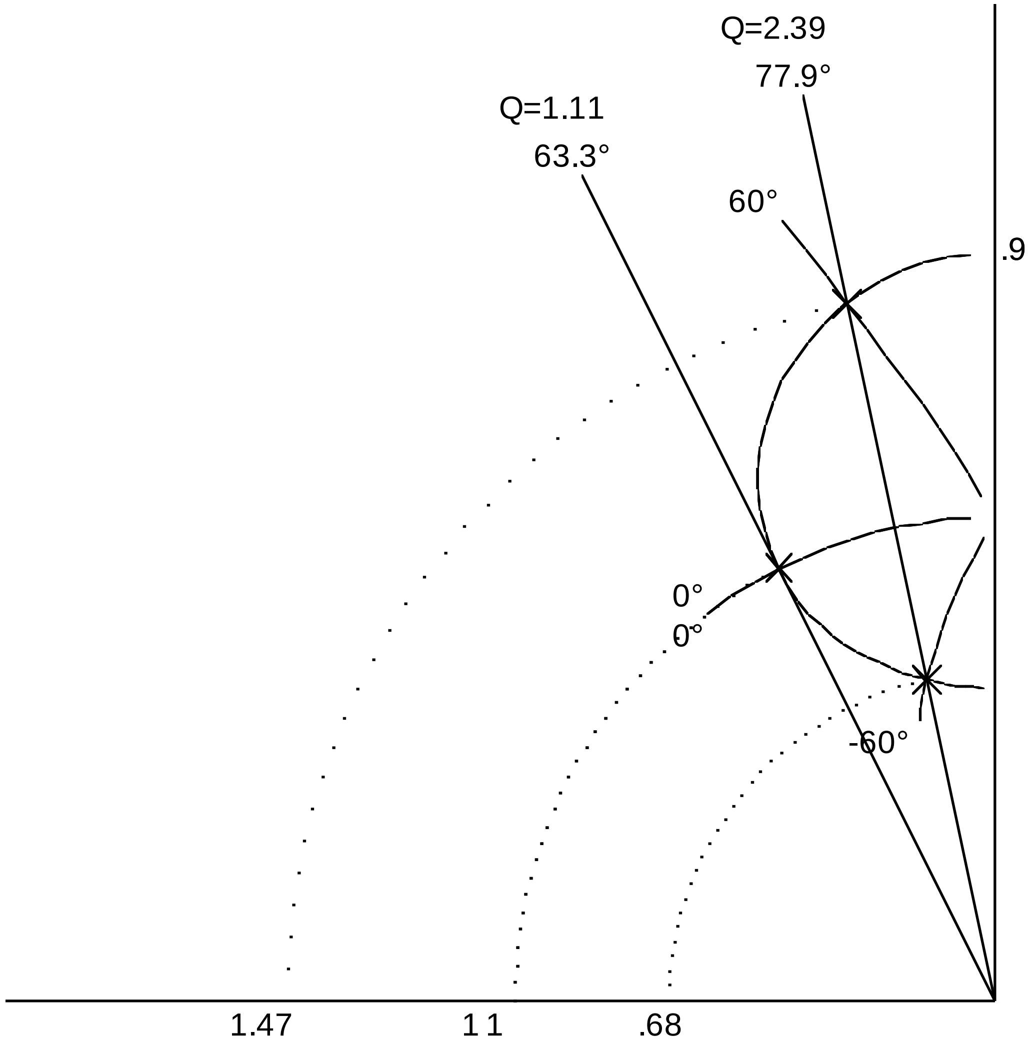
$$F' = \frac{2\tan^{-1}(W/2)}{2\pi} = 0.14396.$$

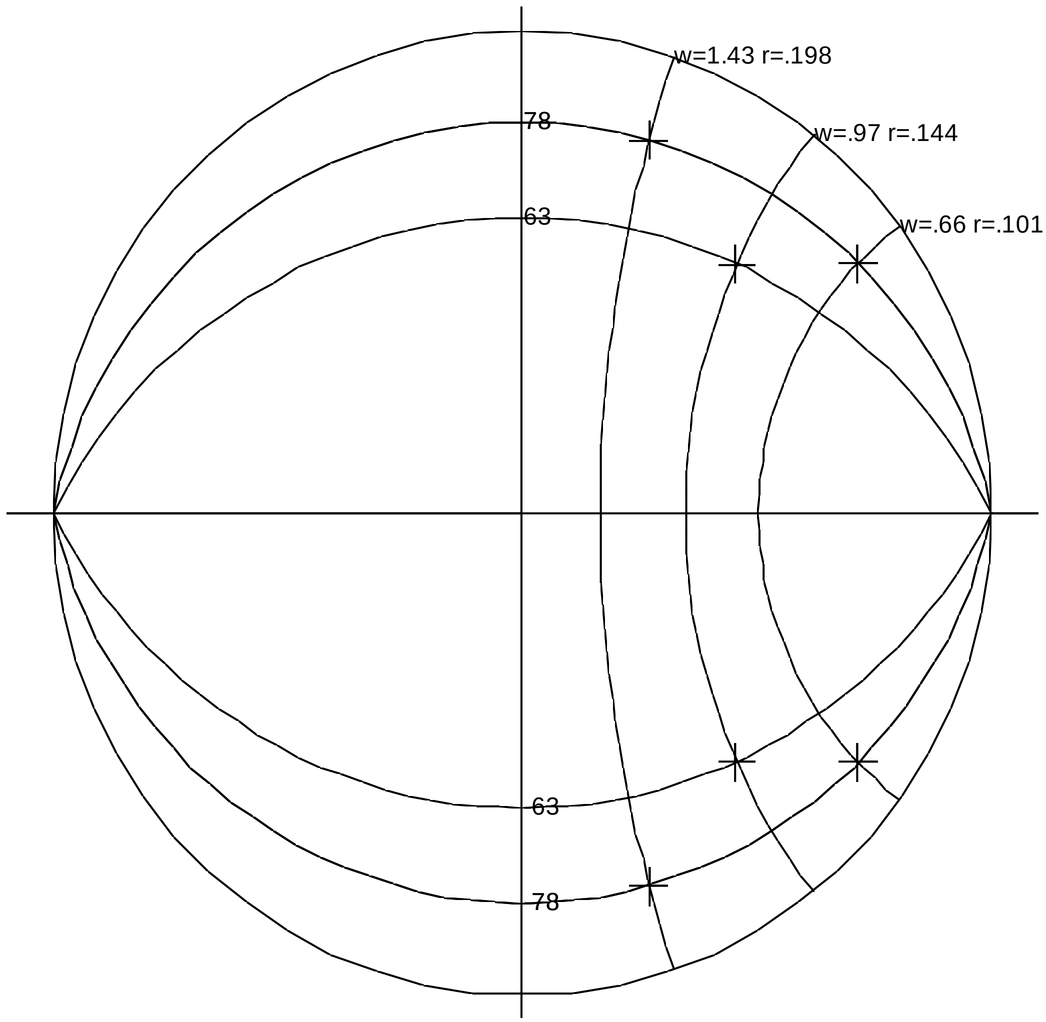
Since $F' = 0.5$ corresponds to half the sampling frequency (or angle π radians), the angle associated with this frequency is then calculated as

$$\theta = 2\pi F' = 2\pi(0.14396)$$

```
z=exp(j*2*pi*0.14396);
n1=z^2 -1;
n2=n1;
n3=n1;
d1=z^2-0.91202*z+0.47569;
d2=z^2-1.4314*z+0.77854;
d3=z^2-0.53365*z+0.66838;
Mag=20*log10(abs(n1*n2*n3/(d1*d2*d3)))
```

So if we evaluate H(z) at that frequency $\left(z = e^{j2\pi(0.14396)}\right)$, we get 33.02dB of gain at center frequency. THEREFORE we must have $K = 10^{-33.02/20}$ =0.02234 for 0dB center frequency gain.


Q=2.39
77.9°
Q=1.11
63.3°
60°
.9
0°
0°
-60°
1.47
1 1
.68



$$\frac{(z^2 - 1)}{(z^2 - .91z + .47)} \quad \frac{(z^2 - 1)}{(z^2 - .54z + .67)} \quad \frac{(z^2 - 1)}{(z^2 - 1.43z + .78)}$$