Chapter 6
Bandpass and Notch Filters

6.1 Introduction

In this chapter we consider the design of both bandpass and notch filters. Bandpass filters by definition select some range of frequencies and reject frequencies above and below that range. Notch filters reject a range of frequencies and select frequencies above and below that range. In either case we may specify that the filter be a Butterworth or Chebyshev type, depending on whether we require the response to be maximally flat (Butterworth) or whether we can tolerate some ripple (Chebyshev.)

Our strategy in design of bandpass and notch filters makes use of the lowpass and highpass filter design techniques discussed in Chapters 4 and 5. More specifically, we will generate the bandpass response by mapping a lowpass magnitude response (centered about DC) to a bandpass response centered at the geometric mean of the passband limits. Likewise for notch filters we will start with a highpass prototype, and map its response to a notch response as shown in Figure 6.1. This mapping of the frequency axis will be extended to mapping the entire lowpass (or highpass) prototype complex (S) plane to the the bandpass (or notch) filter complex (s) plane. The mapping will be one to two in that an n^{th} order lowpass prototype will map to a $2n^{th}$ order bandpass filter, or that a single lowpass pole will map to two bandpass poles.

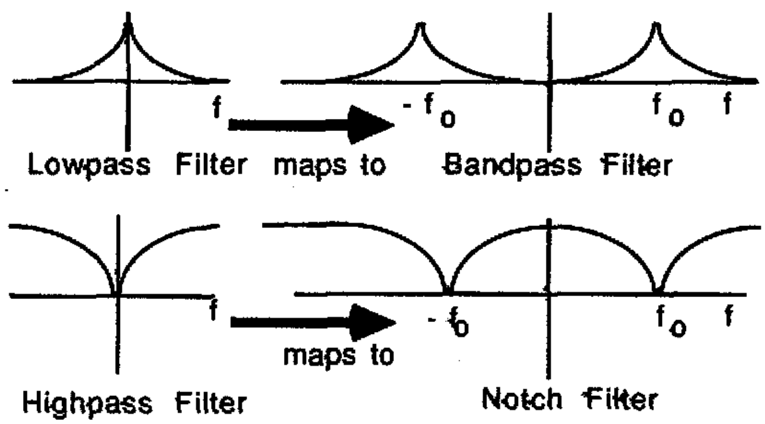

Figure 6.1. Frequency Response Mappings.

The big problem in this mapping is calculating the bandpass or notch filter pole locations from the lowpass (or highpass) pole locations. Two closed form solutions to the problem are presented in this chapter. Both are rather tedious for a handheld calculator and are best done with the aid of a short computer program. As a less tedious alternative we propose a method that uses a computer generated chart of this mapping, where the lowpass S plane has been mapped onto the bandpass s plane.

Later in the chapter we present circuits to synthesize second order filter sections for a given Q and resonant frequency, and we consider some specific design examples. As in Chapter 4 we also analyze the effects finite op-amp gain bandwidth product. Here we observe similar effects as in the lowpass or highpass case, when poorer gain-bandwidth product op-amps are used for these circuits. However, since the Q's of bandpass filters are generally higher, these effects (pole movements) will be more pronounced at lower frequencies than in the lowpass case.

Finally we consider an innovative alternative for producing a 4th order circuit which uses the cascade of two identical 2nd order sections together with negative feedback. This structure has been implemented in an integrated circuit (MF8) by National Semiconductor, making the design of bandpass filters (to 20 kHz) quite simple.

6.2 Bandpass Response

In Chapter 2 we learned a general expression for a second order bandpass filter transfer function, namely

$$H_{BP}(s) = \frac{s/Q}{s^2 + s/Q + 1} \quad (6.1)$$

As we will soon see, when we design bandpass filters graphically, we will determine the resonant frequency of each second order section by measuring the radius from the origin of a plot of the poles. This radius will be interpreted as the resonant frequency relative to the center frequency. Therefore, for all calculations in this chapter, the center frequency will be normalized to unity and other frequencies will be interpreted relative to this center frequency. The magnitude squared response of 6.1 is

$$H_{BP}(j\omega)H_{BP}(-j\omega) = \frac{\left(\frac{\omega}{Q}\right)^2}{(1-\omega^2)^2 + \left(\frac{\omega}{Q}\right)^2}$$

$$= \frac{1}{Q^2\left(\frac{1}{\omega} - \omega\right)^2 + 1}, \quad (6.2)$$

and the attenuation in dB is

$$A(\omega) = 10\log_{10}\left(1 + Q^2\left(\frac{1}{\omega} - \omega\right)^2\right). \quad (6.3)$$

Recall from Chapter 2 that the magnitude response is symmetric on a logarithmic scale about the center frequency. For example for a center frequency of unity (as above), the response at ω is the same as at that at $1/\omega$. If we define $\Delta\omega$ as the difference between these frequencies, namely

$$\Delta\omega = \frac{1}{\omega} - \omega \quad , \tag{6.4}$$

then from 6.3 we have

$$A(\Delta\omega) = 10 \log_{10}\left(1 + Q^2(\Delta\omega)^2\right) \tag{6.5}$$

as the attenuation in dB at each of these two frequencies separated by $\Delta\omega$. Note the symmetry about $\omega = 1$ on a logarithmic scale. Recall from Chapter 2 that Q was defined as the half power (or 3 dB) bandwidth relative to the center frequency. In 6.5, $A(\Delta\omega)$ is clearly 3 dB when $\Delta\omega$ equals $1/Q$. We will define this 3 dB bandwidth relative to the center frequency as $\Delta\omega_o$, and substituting into 6.5 we obtain

$$A(\Delta\omega) = 10 \log_{10}\left(1 + \left(\frac{\Delta\omega}{\Delta\omega_o}\right)^2\right) . \tag{6.6}$$

Now compare 6.6 to the attenuation of a first order lowpass filter with a half power bandwidth of Ω_o. (In an effort to avoid as much confusion as possible we will use uppercase variables (S, Ω) for lowpass prototypes and lowercase (s, ω) for bandpass filters.) More specifically, the transfer function of a first order lowpass filter with half power bandwidth of Ω_o is

$$H_{LP}(S) = \frac{\Omega_o}{S + \Omega_o} , \tag{6.7}$$

and

$$A(\Omega) = 10 \log_{10}\left(1 + \left(\frac{\Omega}{\Omega_o}\right)^2\right) . \tag{6.8}$$

Note that the magnitude response of a second order bandpass filter is exactly the same as that of first order lowpass filter when we substitute $\Delta\omega$ for Ω and $\Delta\omega_o$ for Ω_o! This could be extended to higher order filters as well. For example we could write the attenuation of an n^{th} order Butterworth bandpass filter as

$$A(\Delta\omega) = 10 \log_{10}\left(1 + \left(\frac{\Delta\omega}{\Delta\omega_o}\right)^n\right) \tag{6.9}$$

and similarly for Chebyshev bandpass filters. One important point is that the magnitude response will always be symmetric about the center frequency on a logarithmic frequency scale. For example, the response at twice the center frequency is the same magnitude as the response at half the center frequency, etc.

Rather than attempting to substitute s/j for ω in equations such as 6.9, and solve for the poles as we did in the previous two chapters, we will develop expressions that will relate poles (and zeros) of lowpass prototypes to those of bandpass filters. Setting 6.1 and 6.7 equal

$$H_{LP}(S) = H_{BP}(s)$$

or

$$\frac{\Omega_o}{S + \Omega_o} = \frac{s/Q}{s^2 + s/Q + 1} \quad . \tag{6.10}$$

Solving 6.10 for S

$$S + \Omega_o = \frac{s^2 + s/Q + 1}{s/Q} \, \Omega_o \tag{6.11}$$

$$S = \left(\frac{s^2 + s/Q + 1}{s/Q} - 1\right)\Omega_o$$

$$= \left(\frac{s^2 + 1}{s}\right) Q \, \Omega_o \quad . \tag{6.12}$$

Recall that Q is the center frequency relative to the 3 dB bandwidth. Just as in the case of Chebyshev lowpass filters in Chapter 5 we will set (normalize) the lowpass prototype passband frequency limit (defined by A_{max}) equal to unity. Therefore Ω_o is the 3 dB bandwidth relative to the passband width as defined by A_{max}. Their product is the center frequency relative to the passband width, or

$$\frac{\text{center freq}}{\text{3dB BW}} \times \frac{\text{3dB BW}}{(\text{BW defined by } A_{max})} = Q \, \Omega_o$$

$$= \frac{\text{center freq}}{(\text{BW defined by } A_{max})}$$

$$= \frac{1}{\Delta\omega_p} \quad . \tag{6.13}$$

Substituting into 6.12,

$$S = \frac{1}{\Delta\omega_p} \left(\frac{s^2 + 1}{s}\right)$$

or

$$= \frac{1}{b} \left(\frac{s^2 + 1}{s}\right) \tag{6.14}$$

where b is defined as the passband width (defined by A_{max}) relative to the center frequency. Although our derivation thus far has used an example of mapping a first order lowpass prototype to a second order lowpass filter, the mathematical mapping in 6.14 is valid for filters of arbitrary order. Substituting $j\Omega$ for S and $j\omega$ for s in 6.14 results in a frequency mapping of

$$\Omega = \frac{1}{b}\left(\omega - \frac{1}{\omega}\right) \quad . \tag{6.15}$$

Since b is the passband width, if ω_p (>1) is the upper passband limit

$$b = \omega_p - \frac{1}{\omega_p} \tag{6.16}$$

From 6.15 and 6.16 it follows that $\omega = \omega_p$ and $\omega = 1/\omega_p$ map to $\Omega = +1$ and -1 respectively. These mappings in frequency are illustrated in Figure 6.2.

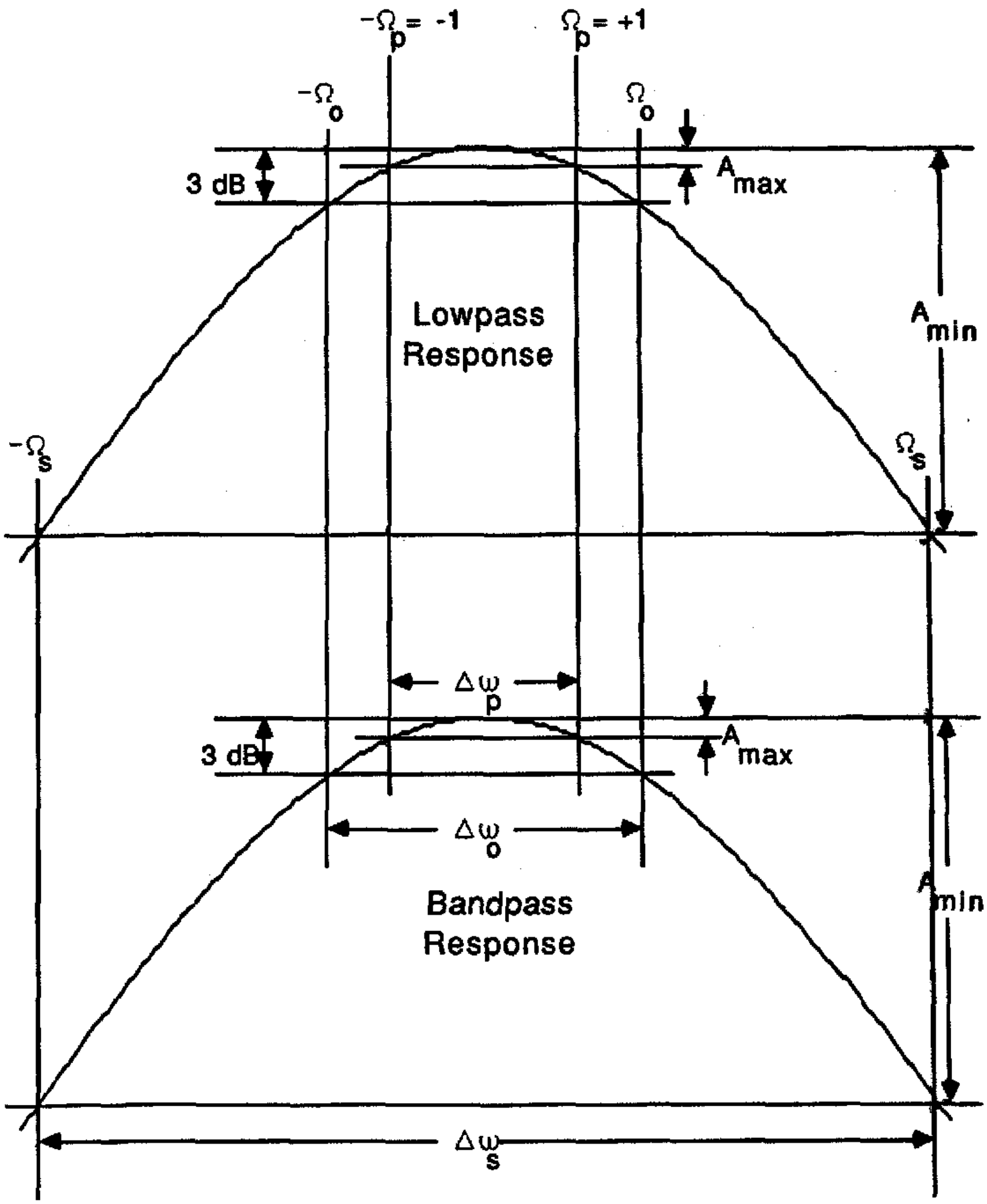

Figure 6.2. Comparison of Lowpass and Bandpass Frequency Response.

The determination of the minimum order to satisfy a given set of specifications follows directly from the relationships illustrated in Figure 6.2. The ratio of the stopband width to the passband width in the bandpass filter is the same as the ratio of the stopband limit to the passband limit (normalized tol) of the lowpass prototype or

$$\frac{\Delta\omega_s}{\Delta\omega_p} = \frac{\Omega_s}{\Omega_p}$$

$$= \Omega_s \ . \qquad (6.17)$$

Therefore the minimum order of the lowpass prototype or the number of <u>second</u> order sections in the bandpass filter is determined as in Chapters 4 and 5 for Butterworth and Chebyshev bandpass filters respectively, with $\Delta\omega_s/\Delta\omega_p$ substituted for ω_s/ω_p. Likewise when using the design chart presented in Chapter 5 (Figure 5.6) the entering argument for the horizontal axis is $\Delta\omega_s/\Delta\omega_p$ instead of ω_s/ω_p.

Example 6.1

Determine the minimum order Butterworth and Chebyshev bandpass filters to satisfy the following specifications.

A_{max} = 1 dB
A_{min} = 30 dB
Passband limits = 1 kHz to 2 kHz
Stopband limits = 500 Hz and 4 kHz

Note that the passband and stopband limits are symmetric about the same center frequency (1414 Hz) and the ratio of stopband width to passband width is

$$\frac{4\ \text{kHz} - 500\ \text{Hz}}{2\ \text{kHz} - 1\ \text{kHz}} = 3.5 \qquad (6.18)$$

Calculating for minimum order for a lowpass prototype Butterworth filter,

$$\frac{10^{A\,min/10} - 1}{10^{A\,max/10} - 1} = \frac{10^3 - 1}{10^{.1} - 1} = 3.86 \times 10^3, \qquad (6.19)$$

and from equation (4.11) we have

$$N_{LP} = \frac{\log(3.86 \times 10^3)}{2 \log(3.5)} = 3.29 \text{ or } 4 \qquad (6.20)$$

In a similar fashion from (5.29) for a Chebyshev filter we obtain

$$N_{LP} = \frac{\cosh^{-1}[\sqrt{3.86 \times 10^3}]}{\cosh^{-1}(3.5)} = 2.32 \text{ or } 3 \quad . \tag{6.21}$$

Therefore our Butterworth bandpass filter would be 8^{th} order (4 second order sections) and our Chebyshev bandpass filter would be 6^{th} order.

Example 6.2

Determine the minimum order Butterworth bandpass filter to satisfy the following specifications.

A_{max} = 0.5 dB
A_{min} = 40 dB
Passband limits = 800 Hz to 1.25 kHz
Stopband limits = 400 Hz and 5 kHz

First we calculate

$$\frac{10^{A_{min}/10} - 1}{10^{A_{max}/10} - 1} = \frac{10^4 - 1}{10^{.05} - 1} = 8.20 \times 10^4 \tag{6.22}$$

Now observe the passband and stopband limits are not symmetric about the same center frequency. However we know that the magnitude frequency response must be symmetric about some center frequency. Therefore we have two choices. We can rewrite the specifications to force the response to be symmetric about the geometric mean of the passband limits (1 kHz), or to force the response to be symmetric about the geometric mean of the stopband limits (1.414 kHz). That is,

Passband limits = 800 Hz to 1.25 kHz
Stopband limits = 400 Hz and 2.5 kHz

or

Passband limits = 800 Hz to 2.5 kHz
Stopband limits = 400 Hz and 5 kHz

In the first case the ratio of stopband width to passband width is

$$\frac{2.5 \text{ kHz} - 400 \text{ Hz}}{1.25 \text{ kHz} - 800 \text{ Hz}} = 4.67 \tag{6.23}$$

and

$$N_{LP} = \frac{\log(8.20 \times 10^4)}{2 \log(4.67)} = 3.67 \text{ or } 4 \quad , \tag{6.24}$$

which implies an 8^{th} order bandpass filter is required. In the second case

$$\frac{5\ \text{kHz} - 400\ \text{Hz}}{2.5\ \text{kHz} - 800\ \text{Hz}} = 2.71$$

and

$$N_{LP} = \frac{\log(8.20 \times 10^4)}{2\log(2.71)} = 5.67 \text{ or } 6\ , \qquad (6.25)$$

which implies a12th order bandpass filter is required.

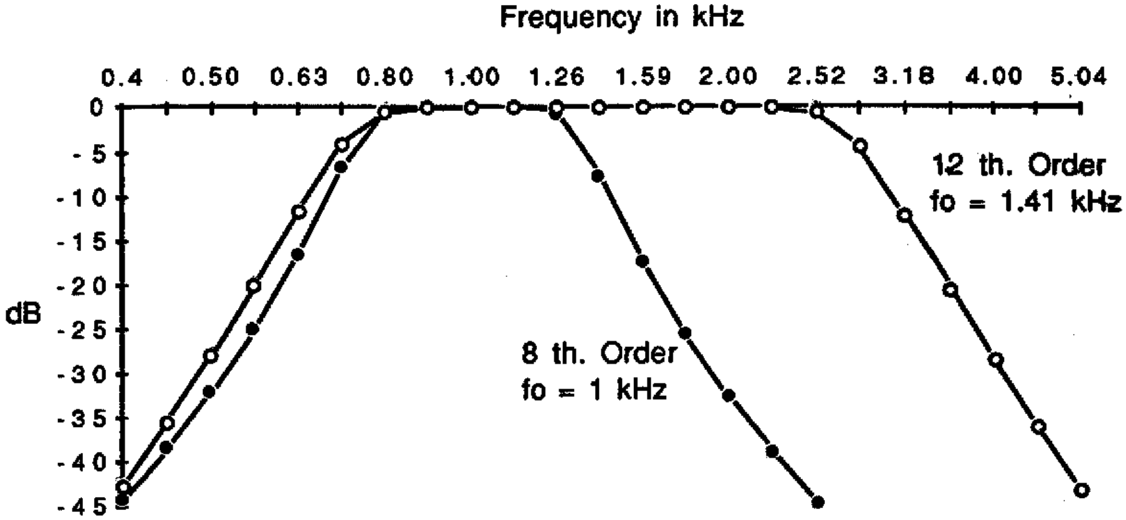

Figure 6.4. Comparison of 8th and 12th order Bandpass Responses.

Observe from Figure 6.3 that the minimum required order decreases as the ratio of stopband width to passband width increases. This suggests that we can always satisfy the specifications with a lower order filter if we force the center frequency to be the geometric mean of the passband limits, and our example above supports this idea. A somewhat more physical explanation is that by choosing the first case, the bandwidth relative to the center frequency is smaller, which implies our filter will have higher Q's and steeper rolloff characteristics between passband and stopband. The magnitude responses of these two filters are compared in Figure 6.4.

6.3 Bandpass Poles

Thus far we have considered the problem of mapping lowpass frequency response to bandpass response, and the problem of determining the minimum order for a bandpass filter design. In order to complete the design process for bandpass (and notch) filters, we will also need to determine the complete transfer function, and more specifically, the Q's and resonant frequencies of each second order section. The next step from the first order example in section 6.2 is a second order lowpass prototype.

Example 6.3

The mapping defined by equation 6.14 can be used directly to design higher order bandpass filters. To illustrate this we will determine the pole locations of a fourth order Butterworth bandpass filter with a 3 dB bandwidth of $\sqrt{2}$ relative to the center frequency. Based on our work in Chapter 4 we know that the frequency of the lowpass prototype poles is given as

$$\Omega_0 = \frac{\Omega_p}{(10^{A_{max}/10} - 1)^{1/2n}} ,$$

where in our case A_{max}=3dB, n=2, and Ω_p is the frequency at which A_{max} has been specified, or unity. This implies the lowpass prototype is a second order Butterworth filter with a resonant frequency of Ω_0=1, with pole angles of +/- 45^o (Q=0.707) relative to the negative real S-plane axis. Therefore

$$H_{LP}(S) = \frac{1}{S^2 + \sqrt{2}\ S + 1} \quad \text{, and} \tag{6.26}$$

$$H_{BP}(s) = H_{LP}\left(S = \frac{1}{\sqrt{2}}\left(\frac{s^2+1}{s}\right)\right)$$

$$= \frac{1}{\frac{1}{2}\left(\frac{s^2+1}{s}\right)^2 + \left(\frac{s^2+1}{s}\right) + 1} \tag{6.27}$$

Multiplying both numerator and denominator by 2 s^2,

$$H_{BP}(s) = \frac{2s^2}{s^4 + 2s^3 + 4s^2 + 2s + 1} \quad . \tag{6.28}$$

The magnitude squared of the frequency response is

$$|H(\omega)|^2 = H_{BP}(j\omega)H_{BP}(-j\omega) = \frac{4}{\left(\frac{1}{\omega^2} - 4 + \omega^2\right)^2 + \left(\frac{2}{\omega} - 2\omega\right)^2}$$

$$= \frac{4}{\omega^4 - 4\omega^2 + 10 - \frac{4}{\omega^2} + \frac{1}{\omega^4}}$$

$$= \frac{1}{1 + \left(\frac{\omega - 1/\omega}{\sqrt{2}}\right)^4} \quad ,$$

or precisely what we should expect for a 4^{th}order Butterworth bandpass filter.

At this point recall that in our final design we will need to know the resonant frequency and Q of each second order section in order to construct the filter as a cascade of two second order stages. Although equation 6.28 is a valid result, it really does not tell us much about the resonant frequencies or Q's of each second order section! To obtain this information we will need to find the roots of this fourth order denominator. Our solution strategy for this example will present three different methods:

1.) First we will use the quadratic equation to solve equation 6.14 for s. Then by substituting the lowpass prototype poles for S the result will be the bandpass poles. This method is quite straightforward for real lowpass poles, but not so simple for complex poles.

2.) Next we will present a second closed form solution for solving for the Q's and resonant frequencies of the poles. The calculations are somewhat tedious and are best done with the aid of a computer program.

3.) Finally we will use a graph (generated using solution method 2 above) that maps the lowpass S plane to the bandpass s plane.

Method 1

Solving 6.14 for the bandpass poles:

$$S = \frac{1}{b}\left(\frac{s^2 + 1}{s}\right) \tag{6.14}$$

$$bS\,s = s^2 + 1 \tag{6.29}$$

or

$$s^2 - bSs + 1 = 0 \tag{6.30}$$

Therefore our bandpass poles appear at

$$s = \frac{bS}{2} \pm \sqrt{\frac{(bS)^2}{4} - 1}\ . \tag{6.31}$$

The solution for the poles in Example 6.3 using Equation 6. 31 proceeds as follows, with each intermediate step illustrated in Figure 6.5:

$$S = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}} \quad \text{(our first lowpass prototype pole)}$$

and

$$b = \sqrt{2}$$

$$\frac{bS}{2} = -\frac{1}{2} + \frac{j}{2}$$

$$\frac{(bS)^2}{4} = -\frac{j}{2}$$

$$\sqrt{\frac{(bS)^2}{4} - 1} = \sqrt{-\frac{j}{2} - 1}$$

$$= \sqrt{\sqrt{1.25}\, \exp\left(j \arctan\left(\frac{-1/2}{-1}\right)\right)}$$

$$= 1.057 \exp(j\ 1.803)$$

$$= -0.243 + j\ 1.029$$

Finally our two pole locations are

$$s = \frac{bS}{2} \pm \sqrt{\frac{(bS)^2}{4} - 1}$$

$$= -0.5 + j0.5 \pm (-0.243 + j\ 1.029)$$

$$= (-0.743 + j\ 1.529) \quad \text{and} \quad (-0.257 - j\ 0.529)$$

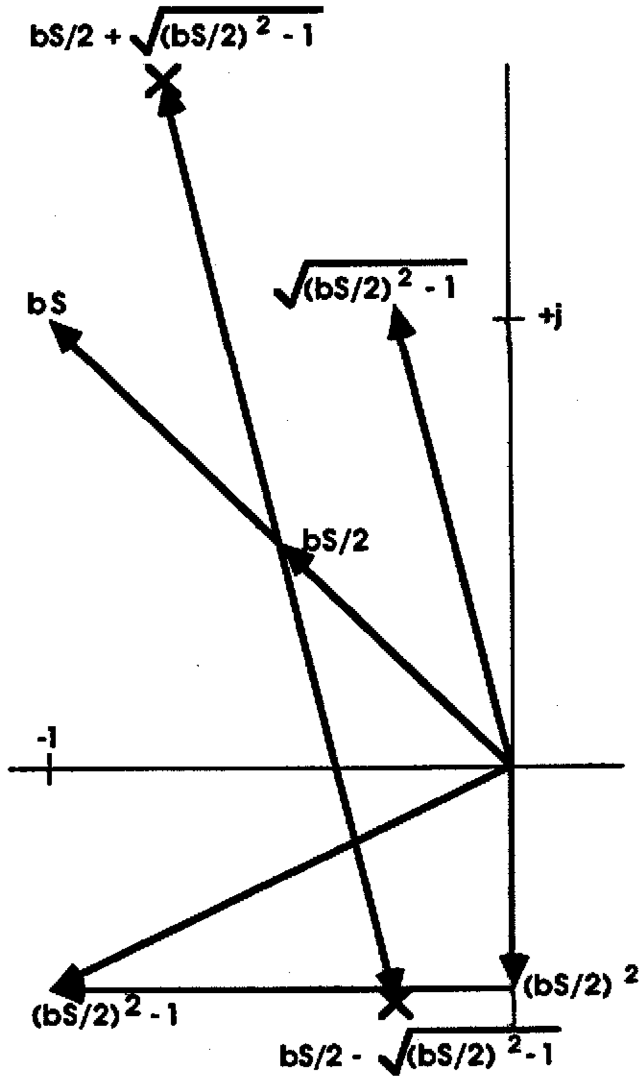

Figure 6.5 Graphical Determination of Pole Locations for Example 6.3.

Had we started instead with the other lowpass prototype pole (namely $S = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$), we would have obtained the complex conjugates of the two poles above.

There are two features of the poles above worth noting. First, each pair of bandpass poles is at angles of ±64.1° with respect to the negative real axis, or at a Q of 1.144. Their resonant frequencies (1.70 and 0.588) are symmetric on a logarithmic scale about the center frequency of 1 (i.e., 1.70 * 0.588 = 1). As we will see this will always be true for any two pairs of bandpass poles resulting from a second order lowpass prototype. Therefore, once we have determined one of the four poles we know where the other three are.

This example was chosen to illustrate a general solution method on a problem with simple calculations. In general the solution for bandpass poles from Equation 6.31 is rather tedious on a hand held calculator, but it is quite simple to handle on a computer. If a language that supports complex numbers such as FORTRAN is used, the solution is quite simple. If not, the code is longer, but still not difficult. An example of a BASIC program is shown below. In reviewing this program it is helpful to recognize the following relationships while referring to Figure 6.5:

1) $\text{real}\{S\} = -\omega_0 \cos\alpha = -\dfrac{\omega_0}{2Q_{LP}}$, where α is the angle of the pole relative to the negative real axis. This implies the program variable RbS2 (or real{bS/2}) equals $-\dfrac{b\omega_0}{4Q_{LP}}$.

2) $\text{imag}\{S\} = \omega_0 \sin\alpha = \omega_0\sqrt{1 - \dfrac{1}{4{Q_{LP}}^2}}$, where α is the angle of the pole relative to the negative real axis. This implies the program variable IbS2 is

$$\text{IbS2} = \text{imag}\left\{bS/2\right\} = b\omega_0 \frac{\sqrt{1 - \dfrac{1}{4{Q_{LP}}^2}}}{2} .$$

3) $R = \text{real}\left\{(bS/2)^2 - 1\right\} = \left(\text{real}\{bS/2\}\right)^2 - \left(\text{imag}\{bS/2\}\right)^2 - 1$

4) $I = \text{imag}\left\{(bS/2)^2 - 1\right\} = 2\,\text{real}\left\{bS/2\right\}\ \text{imag}\left\{bS/2\right\}$

5) $M = \text{magnitude}\left\{\sqrt{(bS/2)^2 - 1}\right\}$

6) $A = \text{angle}\left\{\sqrt{(bS/2)^2 - 1}\right\}$ with respect to positive real axis

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```
print "Q and Wo of lowpass prototype"
input Qlp,Wo
print "Bandwidth relative to center frequency"
input b
let RbS2 = -b*Wo/(4*Qlp)
let IbS2 = b*Wo*sqr(1-1/(4*Qlp^2))/2
let R = RbS2^2 - IbS2^2 - 1
let I = 2*RbS2*IbS2
let M = (R^2 + I^2)^.25
let A = 0.5*angle(R,I)
let Rp = RbS2 + M*cos(A)
let Ip = IbS2 + M*sin(A)
let Qbp = -1/(2*cos(angle(Rp,Ip)))
let wbp = sqr(Rp^2 +Ip^2)
print "Pole locations"
print Rp; "+/- j";abs(Ip)
print Rp/wbp^2;"+/- j";abs(Ip)/wbp^2
print "Q = ";Qbp;"w's = ";wbp;1/wbp
```

end

An alternative solution for the bandpass poles can be obtained by using equation 6.14 to substitute for S in the transfer function for a general second order lowpass prototype, and then set the coefficients of each power of s in the resulting fourth order characteristic equation equal to the coefficients obtained by the product of two general second order polynomials in s. Substituting in our expression for $H_{LP}(S)$,

$$H_{LP}(S) = \frac{\Omega_0^2}{S^2 + \frac{\Omega_0}{Q_{LP}}S + \Omega_0^2}, \tag{6.32}$$

and

$$S = \frac{1}{b}\left(\frac{s^2+1}{s}\right), \text{ so} \tag{6.14}$$

$$H_{BP}(s) = \frac{\Omega_0^2}{\frac{1}{b^2}\left(\frac{(s^2+1)^2}{s^2}\right) + \frac{\Omega_0}{Q_{LP}}\frac{1}{b}\left(\frac{s^2+1}{s}\right) + \Omega_0^2}. \tag{6.33}$$

Multiplying both numerator and denominator in equation 6.33 by b^2s^2 in order that the denominator be a monic polynomial,

$$H_{BP}(s) = \frac{b^2\,\Omega_0^2\,s^2}{(s^2+1)^2 + \frac{b\,\Omega_0}{Q_{LP}}(s^2+1)\,s + b^2\,\Omega_0^2\,s^2}$$

$$= \frac{b^2\,\Omega_0^2\,s^2}{s^4 + \frac{b\,\Omega_0}{Q_{LP}}s^3 + (b^2\,\Omega_0^2 + 2)\,s^2 + \frac{b\,\Omega_0}{Q_{LP}}s + 1}. \tag{6.34}$$

This denominator of $H_{BP}(s)$, denoted $D_{BP}(s)$, must also be the product of two general second order sections, and therefore must be equal to

$$D_{BP}(s) = (s^2 + \frac{\omega_1}{Q_1}s + \omega_1^2)\;\;(s^2 + \frac{\omega_2}{Q_2}s + \omega_2^2)$$

$$= s^4 + \left(\frac{\omega_1}{Q_1} + \frac{\omega_2}{Q_2}\right)s^3 + \left(\omega_1^2 + \omega_2^2 + \frac{\omega_1\omega_2}{Q_1Q_2}\right)s^2 + \left(\frac{\omega_2^2\omega_1}{Q_1} + \frac{\omega_1^2\omega_2}{Q_2}\right)s + \omega_1^2\omega_2^2 \tag{6.35}$$

Equating coefficients in equations 6.34 and 6.35 for like powers of s, we first observe that the product $\omega_1^2\omega_2^2$ must be unity. Substituting $1/\omega_1$ for ω_2 in 6.35,

$$s^4 + \left(\frac{\omega_1}{Q_1} + \frac{1}{\omega_1 Q_2}\right)s^3 + \left(\omega_1{}^2 + \frac{1}{\omega_1{}^2} + \frac{1}{Q_1Q_2}\right)s^2 + \left(\frac{1}{\omega_1 Q_1} + \frac{\omega_1}{Q_2}\right)s + 1 = 0 \qquad (6.36)$$

Since the coefficients of s^3 and s are equal in Equation 6.34 they must also be equal in 6.36. This implies Q_1 equals Q_2 (equals Q). Therefore we simplify $D_{BP}(s)$ further to obtain

$$s^4 + \frac{1}{Q}\left(\omega_1 + \frac{1}{\omega_1}\right)s^3 + \left(\omega_1{}^2 + \frac{1}{\omega_1{}^2} + \frac{1}{Q^2}\right)s^2 + \frac{1}{Q}\left(\omega_1 + \frac{1}{\omega_1}\right)s + 1 = 0 \qquad (6.37)$$

There are now two equations (6.37 and 6.34) to solve for two unknowns (ω_1 and Q). Equating coefficients of s or s^3 we find that

$$\frac{1}{Q}\left(\omega_1 + \frac{1}{\omega_1}\right) = \frac{b\,\Omega_o}{Q_{LP}}, \qquad (6.38)$$

and equating coefficients of s^2 we obtain

$$\left(\omega_1{}^2 + \frac{1}{\omega_1{}^2} + \frac{1}{Q^2}\right) = b^2\,\Omega_o{}^2 + 2 \qquad . \qquad (6.39)$$

Equation 6.38 can be rewritten as

$$\omega_1{}^2 - \frac{Q\,b\,\Omega_o}{Q_{LP}}\omega_1 + 1 = 0 \qquad , \qquad (6.40)$$

and solving for ω_1 in terms of Q by using the quadratic equation,

$$\omega_1 = \frac{Q\,b\,\Omega_o}{2Q_{LP}} \pm \sqrt{\left(\frac{Q^2b^2\Omega_o{}^2}{4Q_{LP}{}^2}\right) - 1} \qquad . \qquad (6.41)$$

Note that the product of the two solutions to equation 6.41 is unity! We know that the product of ω_1 and ω_2 is unity, and therefore 6.41 must really be an expression for ω_1 <u>and</u> ω_2, depending on the sign. Using equation 6.41 to substitute for $\omega_1{}^2 + 1/\omega_1{}^2$ in equation 6.39,

$$\omega_1{}^2 + \frac{1}{\omega_1{}^2} = \left(\frac{Qb\Omega_o}{2Q_{LP}} + \sqrt{\left(\frac{Q^2b^2\Omega_o{}^2}{4Q_{LP}{}^2}\right) - 1}\right)^2 + \left(\frac{Qb\Omega_o}{2Q_{LP}} - \sqrt{\left(\frac{Q^2b^2\Omega_o{}^2}{4Q_{LP}{}^2}\right) - 1}\right)^2$$

$$= \frac{Q^2b^2\Omega_o{}^2}{Q_{LP}{}^2} - 2 \qquad ,$$

therefore equation 6.39 becomes

$$\frac{Q^2 b^2 \Omega_o^2}{Q_{LP}^2} - 2 + \frac{1}{Q^2} = b^2 \Omega_o^2 + 2 \tag{6.42}$$

or

$$\frac{Q^4 b^2 \Omega_o^2}{Q_{LP}^2} - (b^2 \Omega_o^2 + 4) Q^2 + 1 = 0 \quad . \tag{6.43}$$

Using the quadratic equation to solve for Q^2,

$$Q^2 = \frac{2 + \frac{b^2\Omega_o^2}{2} + \sqrt{\left(2 + \frac{b^2\Omega_o^2}{2}\right)^2 - \frac{b^2\Omega_o^2}{Q_{LP}^2}}}{\frac{b^2\Omega_o^2}{Q_{LP}^2}} \tag{6.44}$$

or

$$Q = \frac{Q_{LP}}{b\,\Omega_o}\sqrt{2 + \frac{b^2\Omega_o^2}{2} + \sqrt{\left(2 + \frac{b^2\Omega_o^2}{2}\right)^2 - \frac{b^2\Omega_o^2}{Q_{LP}^2}}} \quad . \tag{6.45}$$

Our reasons for chosing only the positive square root in 6.44 will become clear in the example below.

Now we have closed form expressions to solve for Q and ω_1. Unless you are much better using your calculator than we are, you will probably make several errors if you try to calculate Q and ω_1 using equations 6.41 and 6.45 without writing down several intermediate results. The calculations are best done with a short computer program. Equations 6.41 and 6.45 are less formidable if written in algorithmic form. For example if we let

$$A = b\Omega_o$$

$$B = 2 + \frac{A^2}{2}$$

and

$$C = \frac{A}{Q_{LP}}$$

then

$$Q = \frac{1}{C}\sqrt{B + \sqrt{B^2 - C^2}} \quad ,$$

and if

$$x = \frac{Q\, b\, \Omega_o}{2Q_{LP}} = \frac{QC}{2}$$

then

$$\omega_1 = x \pm \sqrt{x^2 - 1} \quad .$$

As an additional caution if using these two expressions with a hand held calculator note that for narrowband filters both ω_1 and the parameter x become close to one. If we examine the sensitivity of ω_1 to this parameter,

$$\frac{d\omega_1}{dx} = 1 \pm \frac{2x}{\sqrt{x^2 - 1}}$$

we note that it tends to inifinity for narrowband filters. What this all means is that although we may not care what Q is to an accuracy of better than a few per cent, when we use equation 6.45, we can't round off any intermediate results or our calculation of ω_1 will be in error. For computer programs single precision will suffice for filters we can implement with op-amps.

Example 6.3 (method #2)

Determine the transfer function of a fourth order Butterworth bandpass filter with a 3 dB bandwidth of $\sqrt{2}$ relative to the center frequency.

Solution: Repeating Example 6.3 from before using equations 6.41 and 6.45, we know $b = \sqrt{2}$, $Q_{LP} = \frac{1}{\sqrt{2}}$, and $\Omega_0 = 1$. This says that $b^2\Omega_0^2 = 2$, and from 6.45 we have that

$$Q = \frac{1}{2}\sqrt{2 + 1 + \sqrt{(2 + 1)^2 - 4}}$$
$$= 1.144 \; .$$

Using equation 6.41 for ω_1,

$$\omega_1 = \frac{Q\,b\,\Omega_0}{2Q_{LP}} \pm \sqrt{\left(\frac{Q^2 b^2 \Omega_0^2}{4Q_{LP}^2}\right) - 1}$$

$$= \frac{1.144\sqrt{2}}{\sqrt{2}} \pm \sqrt{\frac{2(1.144)^2}{2} - 1}$$

$$= 1.144 \pm \sqrt{0.308}$$

$$= 1.70 \text{ and } 0.588.$$

Therefore our bandpass filter will have two second order sections, each with a Q of 1.144, one with a resonant frequency of 1.70, and the other with a resonant frequency of 0.588. These are the same results as obtained before.

To illustrate why we chose the positive square root for our solution for Q^2 in equation 6.44 above, consider using the negative square root in the previous numerical example. Now

$$Q = \frac{1}{2}\sqrt{3 - \sqrt{5}}$$
$$= 0.437 \quad .$$

Q's of less than 0.5 generally indicate real and distinct roots. In this case however when we solve for ω_1:

$$\omega_1 = \frac{0.437\sqrt{2}}{\sqrt{2}} \pm \sqrt{\frac{2(0.437)^2}{2} - 1}$$

$$= 0.437 \pm \sqrt{-0.809}$$

$$= 0.437 \pm j0.899$$

or

$$\omega_1 = 0.437 + j0.899$$

and

$$\omega_2 = 0.437 - j0.899.$$

When we solve for the pole locations using

$$s^2 + \frac{\omega_1}{Q}s + \omega_1{}^2 = 0 \quad ,$$

$$s = -\omega_1\left(\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - 1}\right)$$

$$= -(0.437 + j0.899)(1.144 \pm 0.556)$$

$$= -0.743 + j\,1.529 \quad \text{and} \quad -0.257 + j\,0.529$$

or the two 2nd quadrant poles. The 3rd quadrant poles would result from using ω_2. Therefore while Q = 0.437 is not necessarily wrong, it results from grouping the poles in equation 6.35 by quadrant and not by complex conjugate pairs. This results in each second order section having complex coefficients even if their 4th order product has only real coefficients. Since we can implement only transfer functions with real coefficients, this solution (although interesting) is of no practical use.

6.4 Graphical Technique

In order to ease the burden of calculating bandpass poles from lowpass prototype poles and also to further illustrate the nature of the mapping from the lowpass prototype S plane to the bandpass s plane, we have developed computer generated charts of the mapping. Note that in both closed form expressions for the bandpass poles (Equation 6.31 and Equations 6.41 and 6.45), only the product of the bandwidth relative to the center frequency and the lowpass pole enter into the expressions and not these variables separately. Therefore the charts are entered with this product in the form of a magnitude ($b\ \Omega_o$) and the angle of the lowpass prototype pole relative to the negative real S axis. We have produced the chart in two versions, Figure 6.6a for narrowband filters ($b\ \Omega_o \leq 1.2$) and a second (Figure 6.6b) for wider band filters ($b\ \Omega_o \leq 3.4$). Very wide band filters ($b\ \Omega_o > 3.4$) can be viewed as the cascade of low and high pass filters and designed accordingly.

Using these charts to design filters proceeds as follows:

1. Determine the minimum order of the lowpass prototype as described in Section 6.2.

2. Determine the angle (or Q) and the frequency (Ω_o) of each lowpass prototype pole. Again, the passband (defined by A_{max}) limit is normalized to unity. For Chebyshev filters this is merely the radius measured the chart for plotting Chebyshev lowpass poles (Figure 5.6). For Butterworth filters it is merely equation 4.6 with the passband limit set equal to unity

$$\Omega_o = \frac{1}{\{10^{A_{max}/10} - 1\}^{1/(2N_{LP})}} \qquad (6.46)$$

where N_{LP} is the order of the lowpass prototype.

3. Plot the bandpass poles at the intersections of the curves for the lowpass prototype angles and the curves for $b\Omega_o$. The frequency of each pole relative to the center frequency is merely its radius, and its angle and Q are determined by extending a line from the origin through the pole to the outer two scales. Typically for second order lowpass sections, it is only necessary to plot the bandpass pole of higher frequency. The other pair has the same Q and reciprocal frequency.

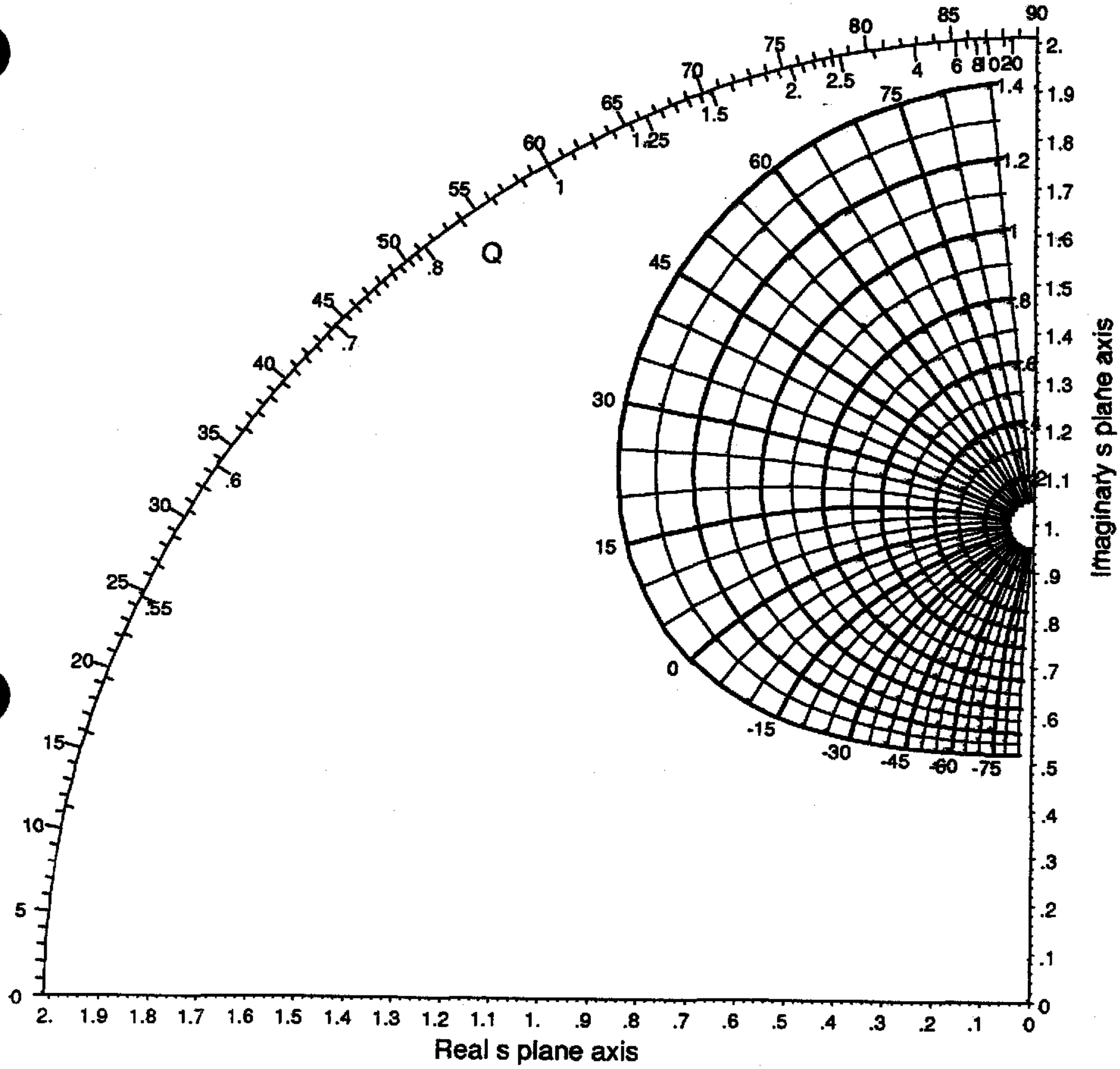

Figure 6.5. Lowpass to Bandpass Mapping (Narrowband)

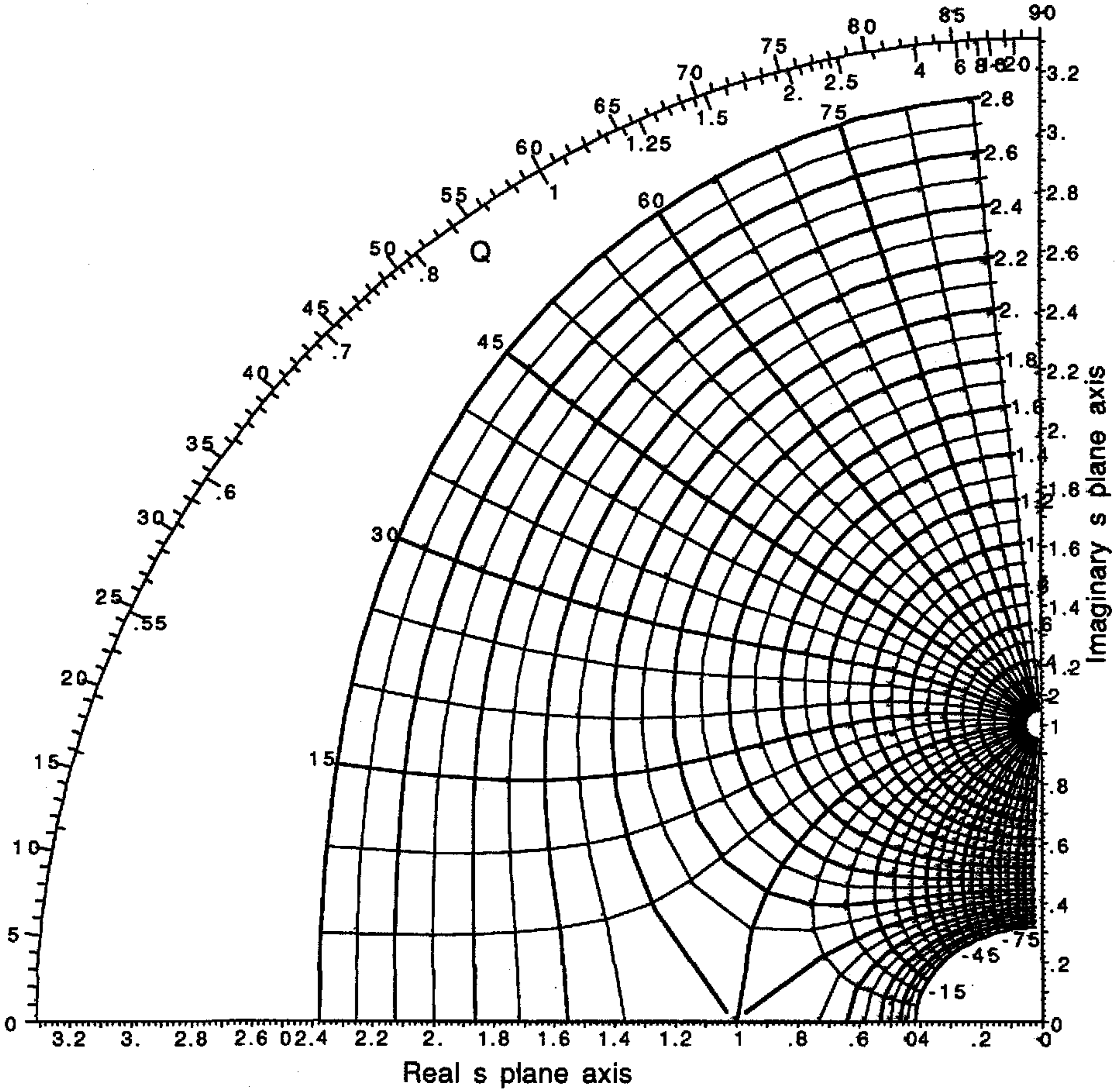

Figure 6.6b. Lowpass to Bandpass Mapping
(Wideband)

To illustrate the use of the charts we will consider some examples.

Example 6.4

Graphically determine the poles in Example 6.3 above, a fourth order Butterworth bandpass filter with a 3 dB bandwidth of $\sqrt{2}$ relative to the center frequency. The lowpass prototype was a second order Butterworth lowpass filter with a resonant frequency of one. Therefore the lowpass prototype poles are at $\pm 45^{o}$ and $b\Omega_o = \sqrt{2}$. Figure 6.7 shows a plot of the (second quadrant) poles with results identical to those found in Example 6.3.

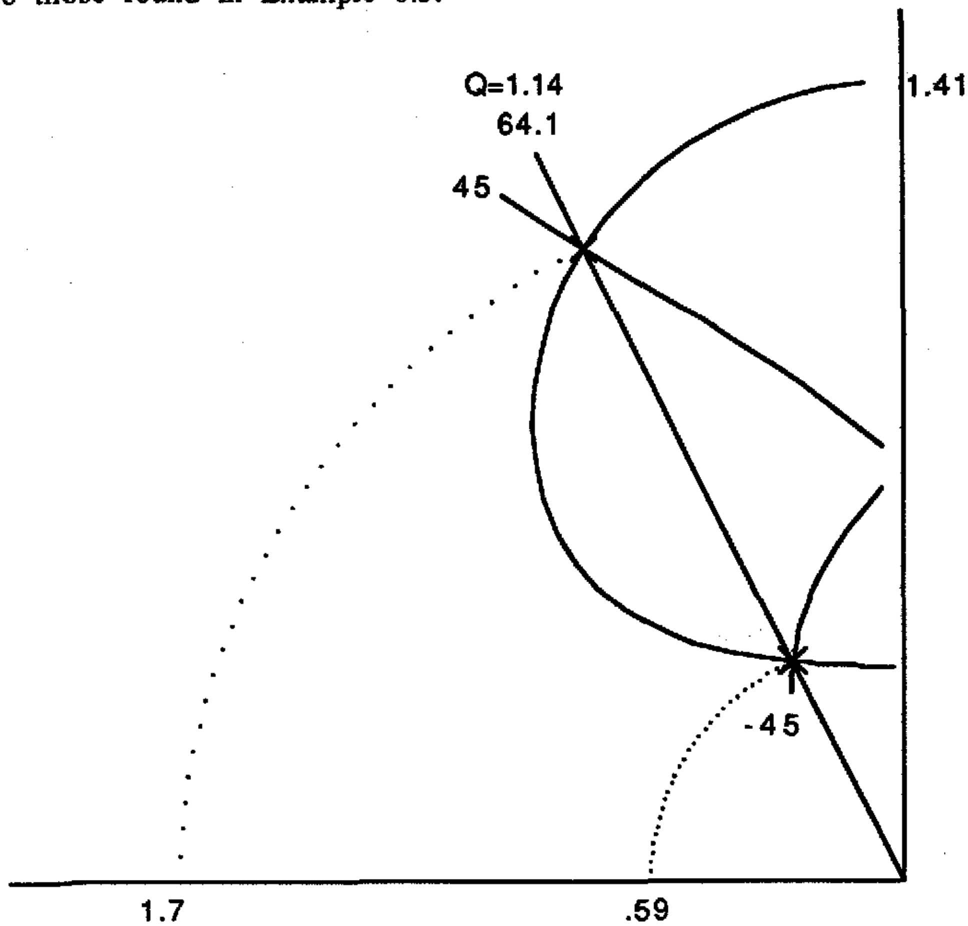

Figure 6.7 Plot of Poles for Example 6.4

Example 6.5

Graphically determine the poles of the Butterworth and Chebyshev bandpass filters in Example 6.1 above. Here

A_{max} = 1 dB

Amin = 30 dB
Passband limits = 1 kHz to 2 kHz
Stopband limits = 500 Hz and 4 kHz

For the Butterworth case $N_{LP} = 4$ and

$$\Omega_o = \frac{1}{(10^{A\,max/10} - 1)^{1/2N_{LP}}}$$

$$= \frac{1}{(10^{.1} - 1)^{1/8}}$$

$$= 1.18 \quad .$$

The bandwidth reltive to the center frequency is

$$b = \frac{2\ \text{kHz} - 1\ \text{kHz}}{1.414\ \text{kHz}}$$

$$= 0.707 \quad .$$

The poles are plotted at the intersections of the $b\Omega_o = 0.837$ curve and the $\pm 22.5°$ and $\pm 67.5°$ curves. These poles are shown in Figure 6.8. These poles are seen to be at $\pm 67.5°$ (Q = 1.3) and frequencies of 1.19 and 1/1.19 = 0.84 relative to the center frequency (of 2π x 1.414 kHz = 8.88 x 10^3 rad/sec.) and at $\pm 81°$ (Q = 3.3) and frequencies of 1.46 and 1/1.46 = 0.68 relative to the center frequency. For the four second order sections we have the following specifications:

Second Order Section #1

ω_{o1} = 1.19 * 8.88 x 10^3 rad/sec
= 1.06 x 10^4 rad/sec

Q_{o1} = 1.3

Second Order Section #2

ω_{o2} = 0.84 * 8.88 x 10^3 rad/sec
= 7.46 x 10^3 rad/sec

Q_{o2} = 1.3

Second Order Section #3

ω_{o3} = 1.46 * 8.88 x 10^3 rad/sec
= 1.30 x 10^4 rad/sec

Q_{o3} = 3.3

Second Order Section #4

ω_{o4} = 0.68 * 8.88 x 10^3 rad/sec

$= 6.04 \times 10^3$ rad/sec

$Q_{o4} = 3.3$

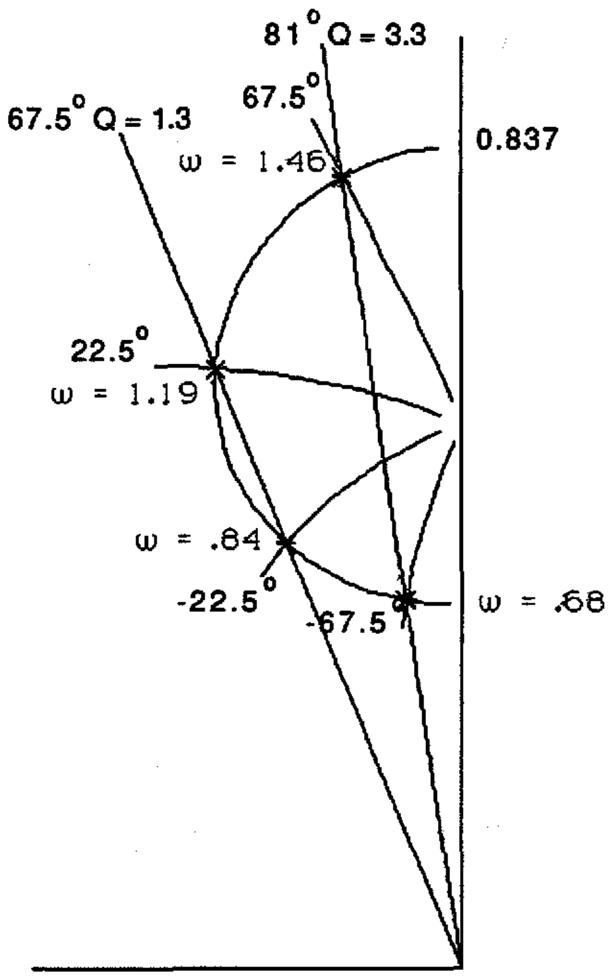

Figure 6.8 Butterworth Filter Poles for Example 6.5

For the Chebyshev filter N_{LP} is 3 and from either equation 5.46 or Figure 5.4, the lowpass prototype poles are on an ellipse with minor axis a = 0.49. A plot of these poles is shown in Figure 6.9. There is one real pole at S = -0.49 and complex conjugate pair at $\pm 76^{o}$ and $\Omega_o = 1$.

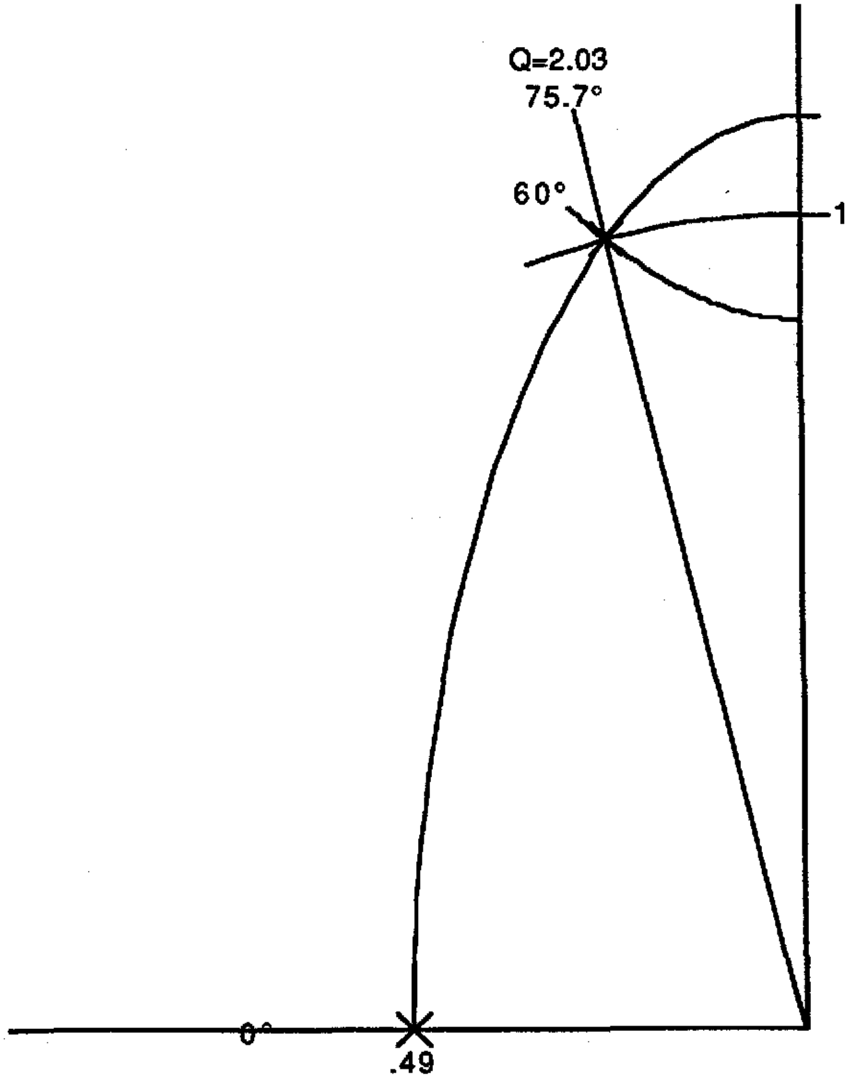

Figure 6.9 Lowpass Prototype Poles for Chebyshev Filter in Example 6.5

The bandpass poles (shown in Figure 6.10) are therefore at the intersection of the 0° curve and the (0.707)(0.49) = 0.35 curve and the intersections of the 0.7 curve and the $\pm 76^{\circ}$ curves. The bandpass filter then consists of a single second order section at $\pm 80^{\circ}$ (Q = 2.9) and at the center frequency (8.88×10^3 rad/sec), and a pair of second order sections at $\pm 85.5^{\circ}$ (Q = ~~6.2~~) and at frequencies of 1.4 and .72 relative to the center frequency (1.24×10^4 and 6.3×10^3 rad/sec.)

6.05 = Q

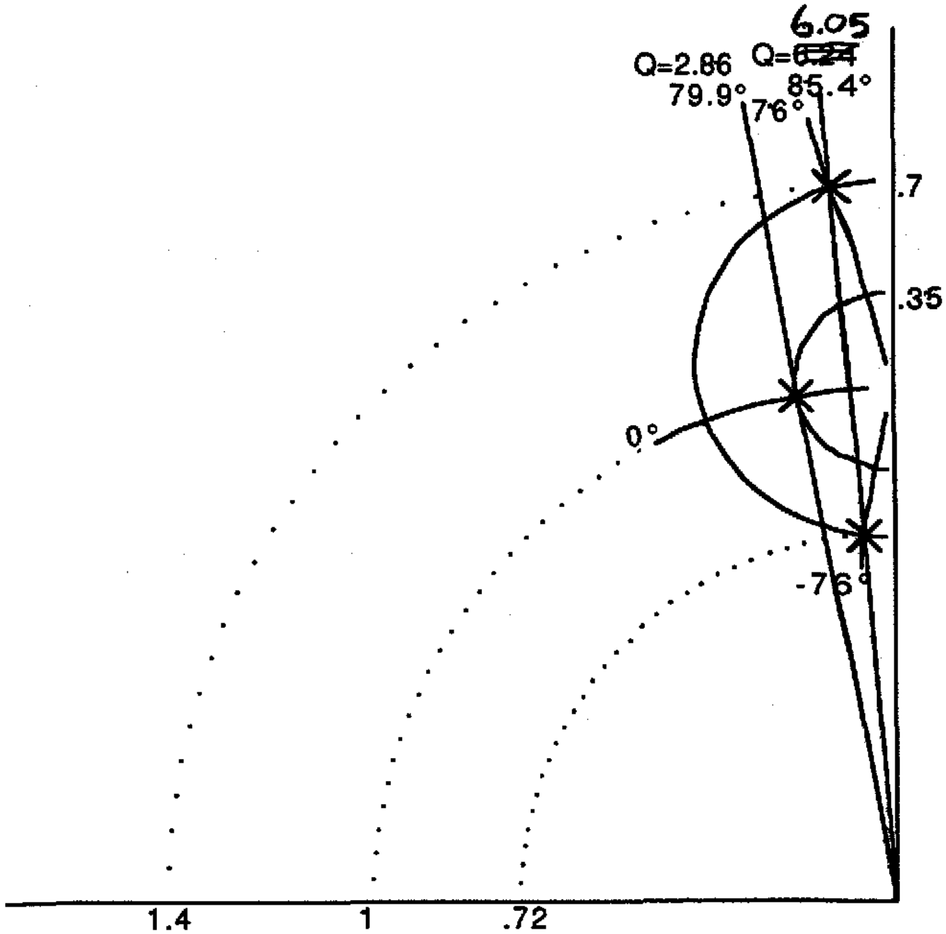

Figure 6.10 Chebyshev Filter Poles for Example 6.5

6.5 Bandpass Circuits

We will consider two circuits for implementation of bandpass filters. The first is similar to the Sallen Key lowpass circuit introduced in Chapter 4, and is shown in Figure 6.11.

Case 1

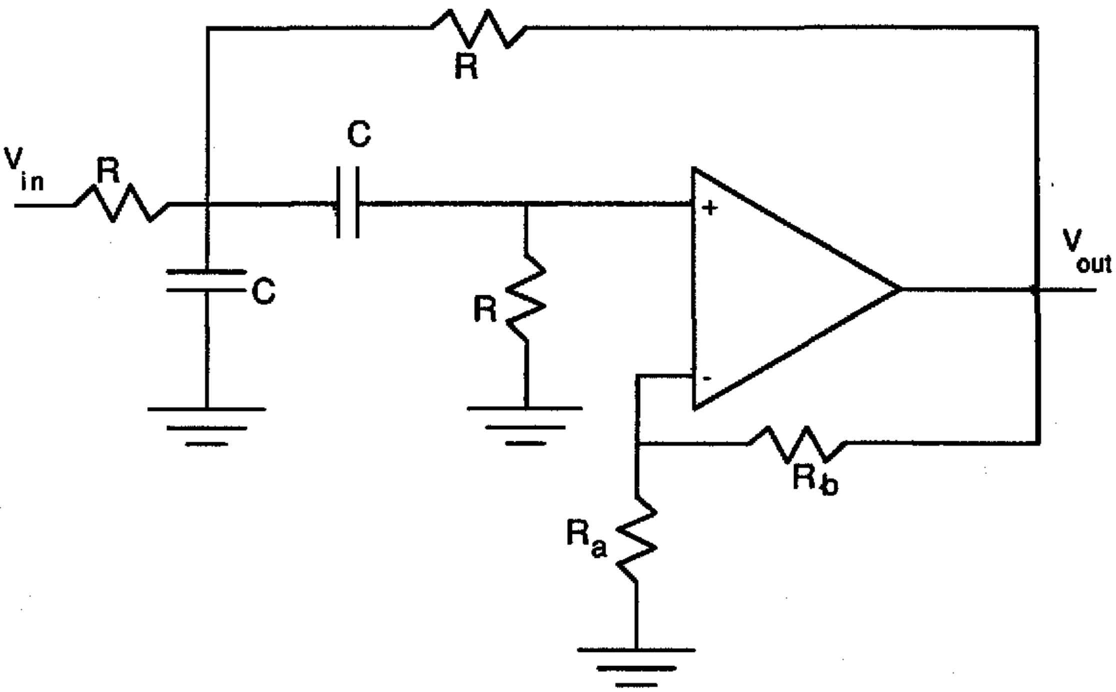

Figure 6.11. Second order bandpass filter section.

Just as in our analysis of the Sallen Key circuit, here we will replace the combination of the op-amp and negative feedback with a non-inverting amplifier labeled A(s) in Figure 6.12. For design purposes in this section we will assume ideal op-amp characteristics, and A(s) will be considered to have constant gain (= 1 + R_b/R_a) as a function of frequency. Later we will model A(s) as a first order lowpass device and analyze the effects of finite gain bandwidth product on our desired filter characteristics.

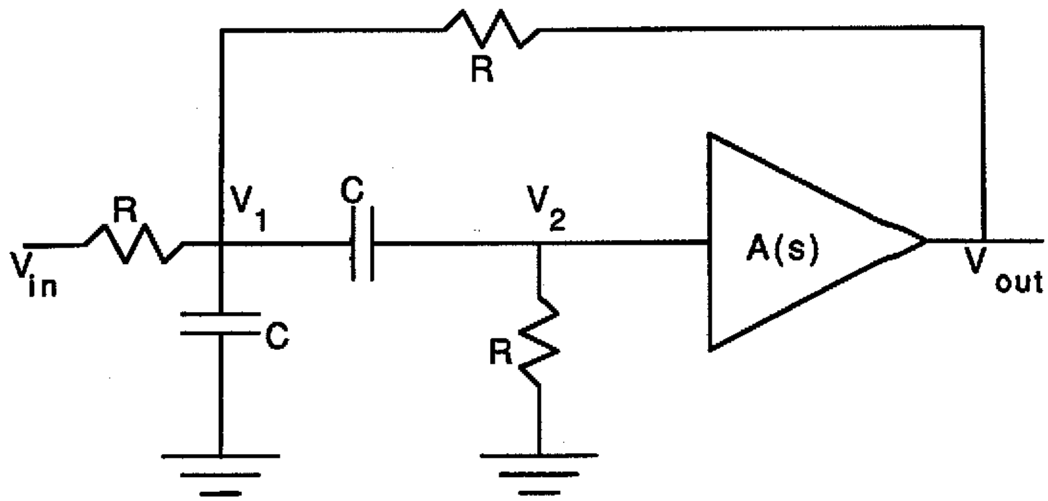

Figure 6.12. Simplified model of second order bandpass filter section.

To analyze the circuit in Figure 6.12 we will first write the node voltage equation at V_1 and the voltage divider equation relating V_2 and V_1.

$$\frac{V_1 - V_{in}}{R} + (V_1 - V_2)\, sC + \frac{V_1 - V_{out}}{R} + sC\, V_1 = 0 \tag{6.47}$$

and

$$V_2 = \frac{R\,C\,s}{R\,C\,s+1}\, V_1 . \tag{6.48}$$

Solving for V_1 in terms of V_2,

$$V_1 = \left(1 + \frac{1}{R\,C\,s}\right) V_2 . \tag{6.49}$$

Collecting terms in equation 6.47, we have

$$\left[\frac{2}{R} + 2sC\right] V_1 - sC\, V_2 = \frac{1}{R} V_{in} + \frac{1}{R} V_{out} . \tag{6.50}$$

Substituting for V_1 in equation 6.50, we find

$$\left[\frac{2}{R} + 2sC\right]\left(1 + \frac{1}{RCs}\right) V_2 - sC\, V_2 = \frac{1}{R} V_{in} + \frac{1}{R} V_{out} , \tag{6.51}$$

and after collecting terms,

$$\left(\frac{4}{R} + sC + \frac{2}{R^2Cs}\right)V_2 = \frac{1}{R}V_{in} + \frac{1}{R}V_{out} \,. \tag{6.52}$$

Noting that

$$V_{out} = A(s)\,V_2 \tag{6.53}$$

or

$$V_2 = \frac{V_{out}}{A(s)}, \tag{6.54}$$

we can using 6.54 in 6.52 to eliminate V_2, and

$$\left(\frac{4}{R} + sC + \frac{2}{R^2Cs}\right)\frac{V_{out}}{A(s)} = \frac{1}{R}V_{in} + \frac{1}{R}V_{out} \quad . \tag{6.55}$$

At this point we will multiply both sides of 6.55 by sA(s)/C to obtain

$$\left(s^2 + \left(\frac{4 - A(s)}{RC}\right)s + \frac{2}{R^2C^2}\right)V_{out} = \frac{A(s)s}{RC}V_{in} \,. \tag{6.56}$$

Now we will define

$$\omega_o = \frac{\sqrt{2}}{RC} \quad , \tag{6.57}$$

so that 6.56 becomes

$$\left(s^2 + \frac{1}{\sqrt{2}}\big(4 - A(s)\big)\omega_o s + \omega_o^2\right)V_{out} = \frac{A(s)}{\sqrt{2}}\omega_o s V_{in} \tag{6.58}$$

or

$$\frac{V_{out}}{V_{in}} = \frac{\frac{A(s)}{\sqrt{2}}\omega_o s}{s^2 + \frac{1}{\sqrt{2}}\big(4 - A(s)\big)\omega_o s + \omega_o^2} \quad . \tag{6.59}$$

If A(s) is a constant A, 6.59 is a second order bandpass filter with

$$Q = \frac{\sqrt{2}}{4 - A} \tag{6.60}$$

or

$$A = 4 - \frac{\sqrt{2}}{Q}, \tag{6.61}$$

and center frequency gain of

$$\frac{A}{4\ -\ A} = \frac{4 - \frac{\sqrt{2}}{Q}}{\frac{\sqrt{2}}{Q}} = 2\sqrt{2}\,Q - 1 \tag{6.62}$$

From this point the design procedure is relatively straightforward. Once we have determined the center frequency and Q of each section in order to meet the desired filter specifications, we need only refer to two equations to determine component values. Equation 6.57 is used to select R and C such that

$$RC = \frac{\sqrt{2}}{\omega_o} \quad , \tag{6.63}$$

and equation 6.61 is used to select R_a and R_b such that

$$\frac{R_b}{R_a} = 3 - \frac{\sqrt{2}}{Q} \quad . \tag{6.64}$$

Example 6.6

Design a Butterworth bandpass filter to satisfy the specifications of Example 6.1 with the additional requirement that the center frequency (or minimum) attenuation be 0 dB. These specifications are

A_{max} = 1 dB
A_{min} = 30 dB
Minimum attenuation = 0 dB
Passband limits = 1 kHz to 2 kHz
Stopband limits = 500 Hz and 4 kHz

As we saw earlier in Example 6.1, an 8th order bandpass filter is required. The Q's and resonant frequencies of each section were determined as part of Example 6.5 and are repeated in Table 6.1. If we choose some reasonable value of capacitance, say 10 nF for each capacitor, the resistors are given by equation 6.63 or

$$R = \frac{\sqrt{2}}{C\omega_o}$$

$$= \frac{\sqrt{2}\ \text{x}\ 10^8}{\omega_o}\ \text{ohms} \quad .$$

The ratio $\frac{R_b}{R_a}$ is given by equation 6.64. The results of the calculations are summarized in Table 6.1.

Section #	ω_0(rad/sec)	R(kΩ)	Q	$R_b/R_a=3-\frac{\sqrt{2}}{Q}$
1	1.06×10^4	13.3	1.3	1.91
2	7.46×10^3	19.0	1.3	1.91
3	1.30×10^4	10.9	3.3	2.57
4	6.04×10^3	23.4	3.3	2.57

Table 6.1

Now we must consider center frequency gain. In order to design for a center frequency gain of 0 dB we first need to know the gain of each second order section at the center frequency of the entire 8th order bandpass filter. (One might initially be tempted to say that the overall gain is simply the product of the center frequency gains of each of the four stages. Unfortunately this is incorrect since these individual center frequency gains occur at different frequencies!) Since sections 1 and 2 above have the same Q and are equidistant from the center frequency (on a logarithmic scale) they have the same gain at the overall center frequency. The same is true for sections 3 and 4. Therefore if we know the gain of one of these pairs at the center frequency, we know the other. In equation 6.3 we expressed the attenuation of a second order bandpass filter with a center frequency of unity and a center frequency gain of 0 dB as

$$A(\omega) = 10\log_{10}\left(1 + Q^2\left(\frac{1}{\omega} - \omega\right)^2\right) . \tag{6.3}$$

By combining (6.3) with (6.62) we can express the <u>gain</u> of a single second order section at the overall center frequency as

$$G(\omega) = -10\log_{10}\left(1 + Q^2\left(\frac{1}{\omega} - \omega\right)^2\right) + 20\log_{10}(2\sqrt{2}\,Q - 1) , \tag{6.65}$$

where ω represents the ratio of the overall center frequency to the section resonant frequency. For this example, 6.65 yields

$$-10\log_{10}\left(1 + (1.3)^2\left(\frac{1}{1.19} - 1.19\right)^2\right) + 20\log_{10}(2\sqrt{2}\,(1.3) - 1) = 7.74 \text{ dB}$$

for sections 1 and 2, and

$$-10\ \log_{10}\left(1 + (3.3)^2\left(\frac{1}{1.46} - 1.46\right)^2\right) + 20\ \log_{10}(2\sqrt{2}\ (3.3) - 1) = 9.64\ \text{dB}$$

for sections 3 and 4. Therefore our 8th order filter would have a center frequency gain of 34.8 dB if our design were not somehow modified. Using a similar technique as we used for Chebyshev lowpass filters in Chapter 5, we replace the resistor at the input to each section with a voltage divider. The parallel combination of the resistors will be the value listed in Table 6.1 above. The gain of each section will be adjusted to give 0 dB gain at the overall center frequency. For example, for the first section we require

$$\frac{R_1R_2}{R_1 + R_2} = 13.3\ \text{k}\Omega$$

and

$$\frac{R_2}{R_1 + R_2} = 10^{(-7.74/20)}$$

$$= 0.41\ .$$

This voltage divider network is shown in Figure 6.13 and the final circuit is shown in Figure 6.14. The frequency response of each individual section is shown in Figure 6.15.

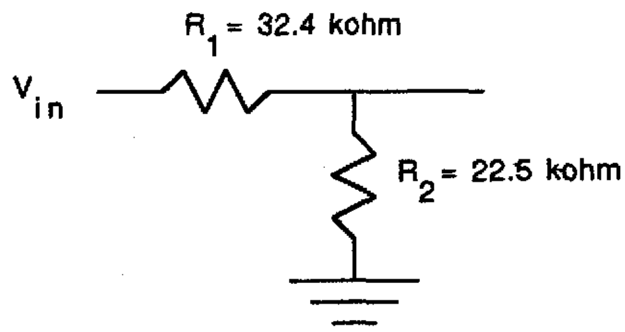

Figure 6.13. Voltage divider network for Example 6.6.

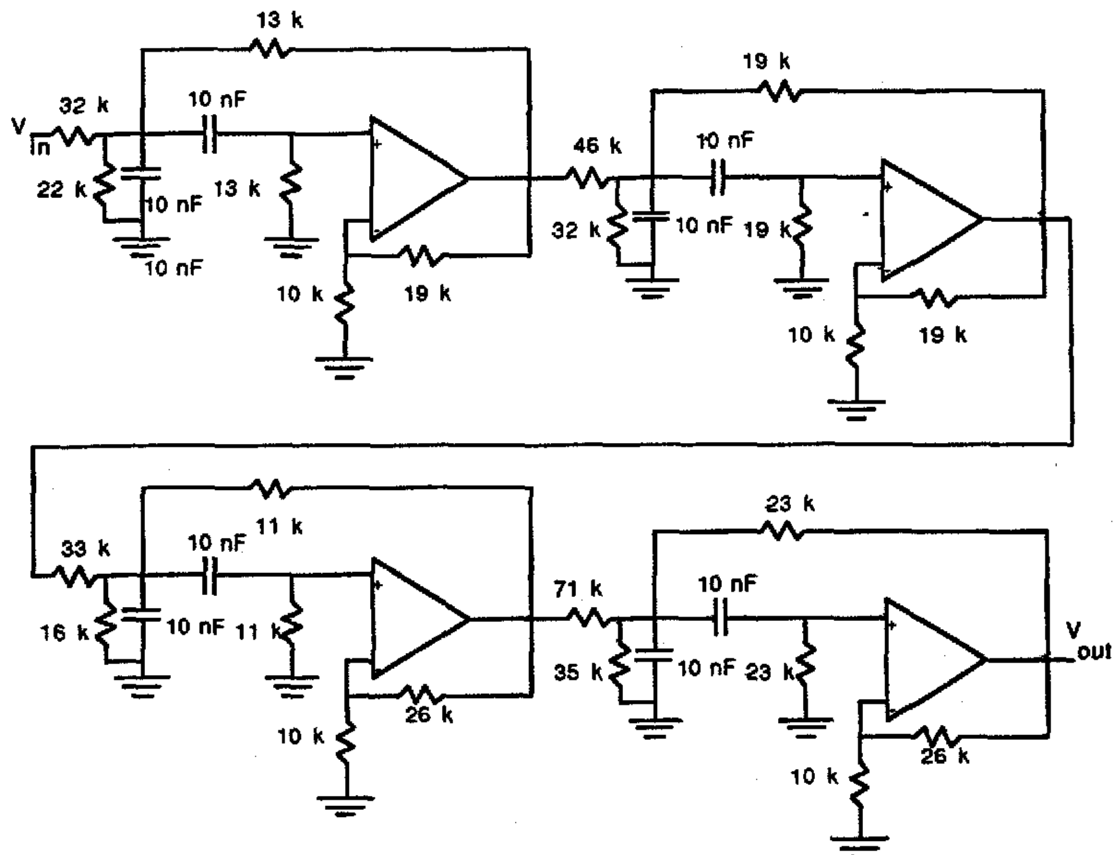

Figure 6.14. Completed circuit design for Example 6.6.

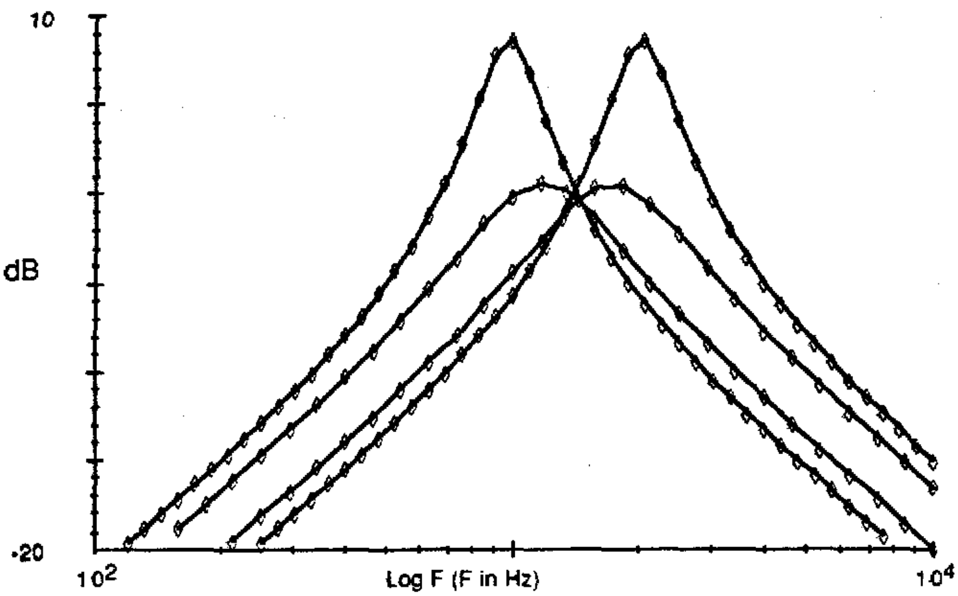

Figure 6.15. Frequency response of individual second order sections for Example 6.6.

At this point one might question why we should "spread the attenuation" over all four stages. Certainly we could have elected to reduce the gain by 34.8 dB with a single voltage divider. The reason for spreading the attenuation across the four sections is to preserve as much dynamic range as possible. If we were to attenuate some incoming signal by 34.8 dB before entering the first section, our signals would be down at the millivolt level, and not appreciably far above the noise! If we were to attenuate by 34.8 dB in the last section, the earlier stages would be driven into saturation at small signal levels.

Case 2

A second bandpass circuit is shown in Figure 6.16.

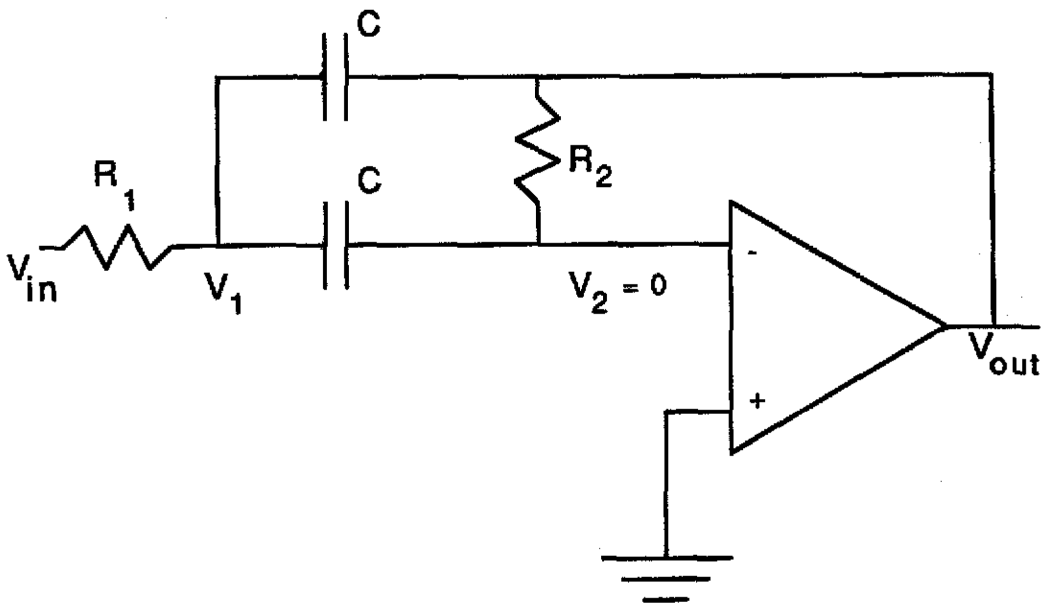

Figure 6.16. Second order bandpass filter section.

To analyze the circuit we will assume an ideal op-amp and that the inverting terminal is a virtual ground (as first presented in Chapter 3). Writing the node voltage equation at V_1,

$$\frac{V_1 - V_{in}}{R_1} + V_1\,sC + (V_1 - V_{out})\,sC = 0 \qquad (6.66)$$

After collecting terms in 6.66

$$\left(\frac{1}{R_1} + 2sC\right)V_1 - V_{out}\,sC = \frac{V_{in}}{R_1} \qquad (6.67)$$

From Chapter 3, you should recognize the response from V_1 to V_{out} as a differentiator,

$$\frac{V_{out}}{V_1} = -sCR_2 \tag{6.68}$$

or

$$V_1 = \frac{-V_{out}}{sCR_2} \quad . \tag{6.69}$$

Using 6.69 in 6.67

$$-\left(\left(\frac{1}{R_1} + 2sC\right) \frac{1}{sCR_2} + sC\right)V_{out} = -\left(\frac{1}{sR_1R_2C} + \frac{2}{R_2} + sC\right)V_{out}$$

$$= \frac{V_{in}}{R_1} \quad . \tag{6.70}$$

Multiplying both sides of 6.70 by s/C we see that

$$-\left(s^2 + \frac{2s}{R_2C} + \frac{1}{R_1R_2C^2}\right)V_{out} = \frac{sV_{in}}{R_1C} \quad . \tag{6.71}$$

If ω_o is defined by

$$\omega_o = \frac{1}{C\sqrt{R_1R_2}}$$

$$= \frac{1}{R_{eq}C} \tag{6.72}$$

the transfer function becomes

$$\frac{V_{out}}{V_{in}} = -\frac{\sqrt{\frac{R_2}{R_1}}\,\omega_o s}{s^2 + 2\sqrt{\frac{R_1}{R_2}}\,\omega_o s + \omega_o^2} \quad . \tag{6.73}$$

Therefore

$$Q = \frac{1}{2}\sqrt{\frac{R_2}{R_1}} \quad , \tag{6.74}$$

or for design purposes,

$$R_2 = 2Q\,R_{eq} \tag{6.75}$$

and

$$R_1 = \frac{R_{eq}}{2Q} \quad . \tag{6.76}$$

From 6.73 the center frequency gain is

$$\frac{R_2}{2R_1} = 2Q^2 \qquad (6.77)$$

Example 6.7

Design a Chebyshev bandpass filter to satisfy the specifications of Example 6.1 with the additional requirement that the center frequency (or minimum) attenuation be 0 dB. These specifications are

A_{max} = 1 dB
A_{min} = 30 dB
Minimum attenuation = 0 dB
Passband limits = 1 kHz to 2 kHz
Stopband limits = 500 Hz and 4 kHz

As we saw from Example 6.1, a 3rd order LP prototype is required, which implies that we need a 6th order bandpass filter in order to meet the specifications above. The Q's and resonant frequencies of each section were determined as part of Example 6.5 and are repeated for convenience in Table 6.2. If we choose 10 nF for each capacitor, the resistors are given by equations 6.72, 6.75 and 6.76.

Section Number	ω_o(rad/sec)	R_{eq}(kΩ)	Q	R_1(kΩ)	R_2(kΩ)
1	8.88×10^3	11.3	2.9	1.95	66
2	6.3×10^3	15.8	6.2	1.27	196
3	1.24×10^4	8.06	6.2	0.65	100

Table 6.2

Just as we did in the case of the Butterworth bandpass filter in Example 6.6, in order to force the maximum gain to be 0 dB, we need to know the gain of each section at the overall filter center frequency. Since the lowpass prototype was third order, its DC gain was a maximum and therefore the center of our filter corresponds to a point of maximum gain. (Recall for Chebyshev filters this is not always the case due to passband ripple!) Similar to the case of equation 6.65 in the previous example, if we combine equations 6.3 and 6.77, the gain in dB of an individual section at the overall filter center frequency is

$$-10\log_{10}\left(1 + Q^2\left(\frac{1}{\omega} - \omega\right)^2\right) + 20\log_{10}(2Q^2) , \qquad (6.78)$$

or for this example,

$$-10\ \log_{10}\left(1 + (6.2)^2\left(\frac{1}{1.4} - 1.4\right)^2\right) + 20\ \log_{10}(2(6.2)^2) = 24.9\ \text{dB} \qquad (6.79)$$

for sections 2 and 3 above. In addition, section 1 has a gain of

$$20\ \log_{10}(2(2.9)^2) = 24.5\ \text{dB} \qquad (6.80)$$

at the filter center frequency. Again we will use a voltage divider at the input to each section so that the gain of each section at the overall filter center frequency is 0 dB. The circuit itself is shown in Figure 6.17, while the frequency response of each section and the overall frequency response are shown in Figures 6.18 and 6.19 respectively.

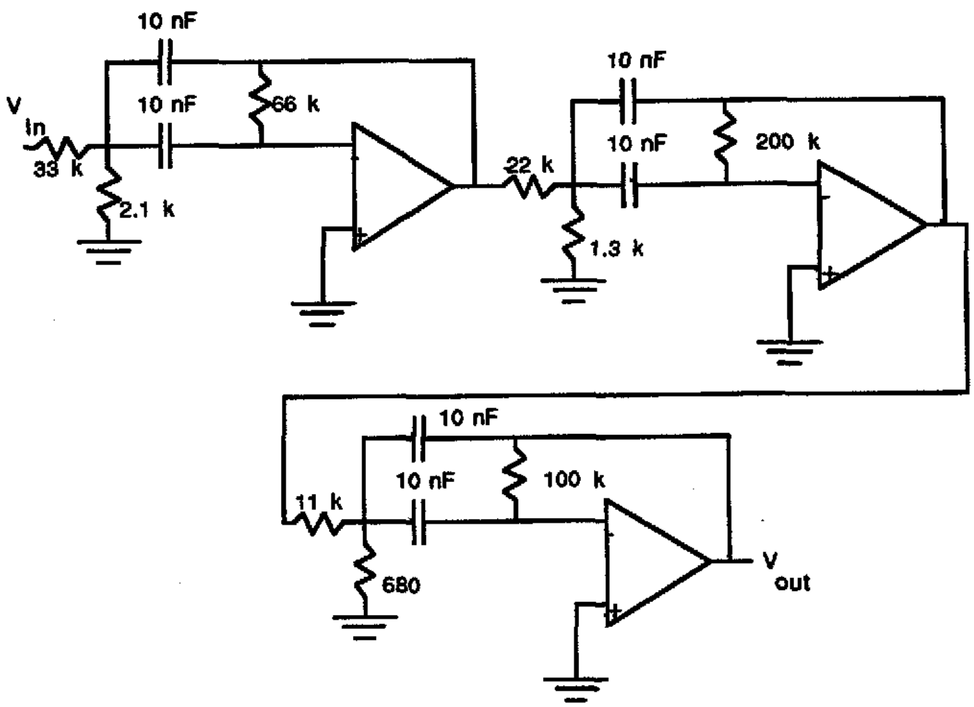

Figure 6.17. Final design for Example 6.7.

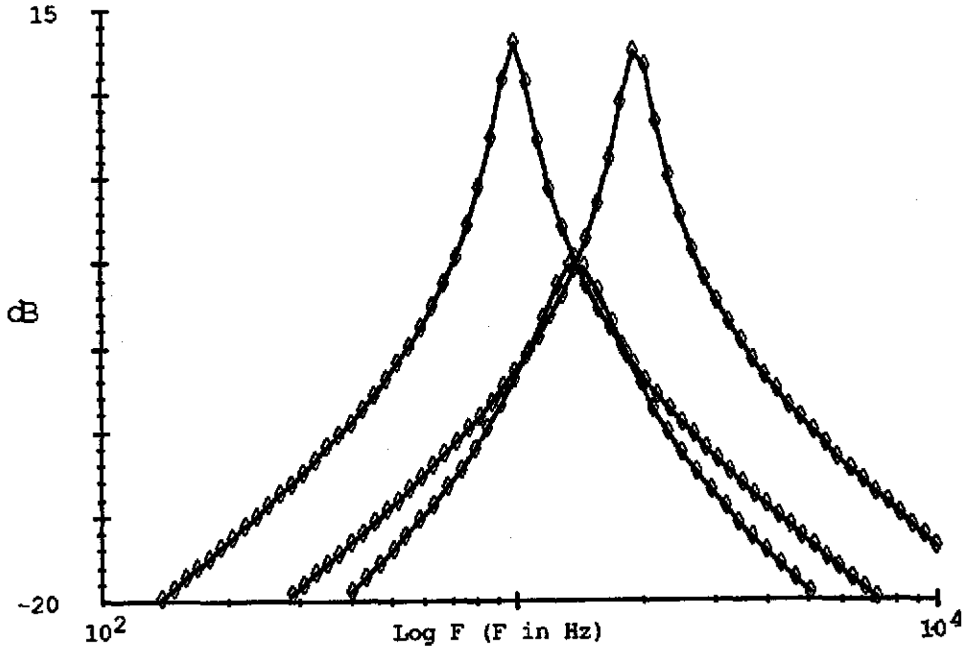

Figure 6.18. Frequency response of individual second order sections of Example 6.7.

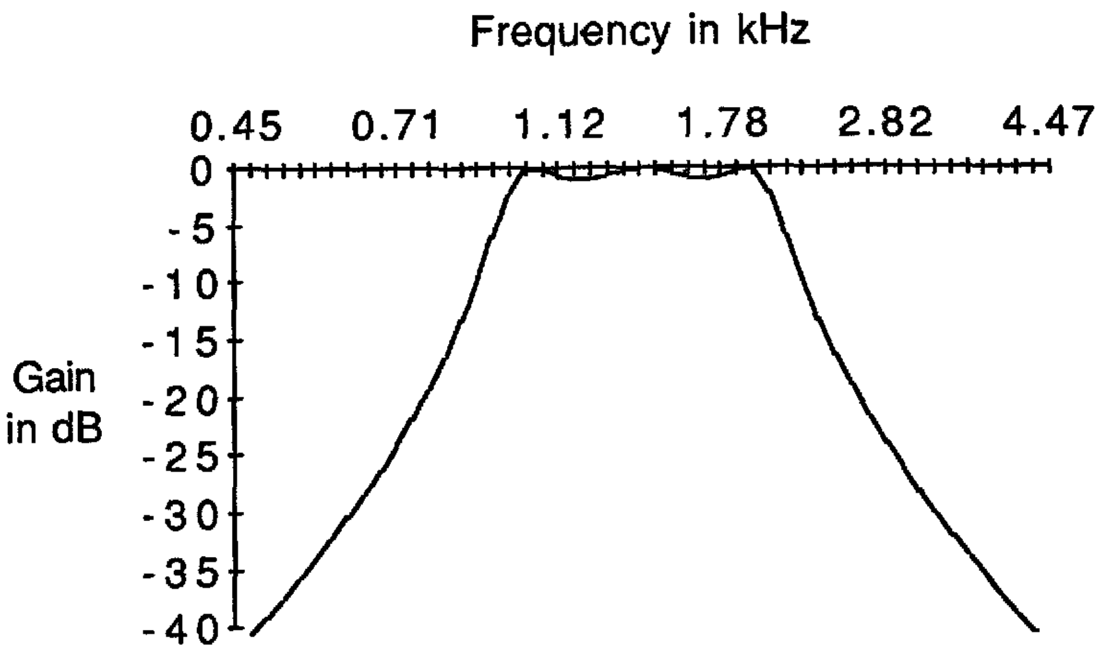

Figure 6.19. Composite frequency response for filter design in Example 6.7.

6.6 Notch Filters

The relationship between notch filters and highpass prototypes is exactly the same as that between bandpass filters and lowpass prototypes. This relationship is illustrated in Figure 6.20.

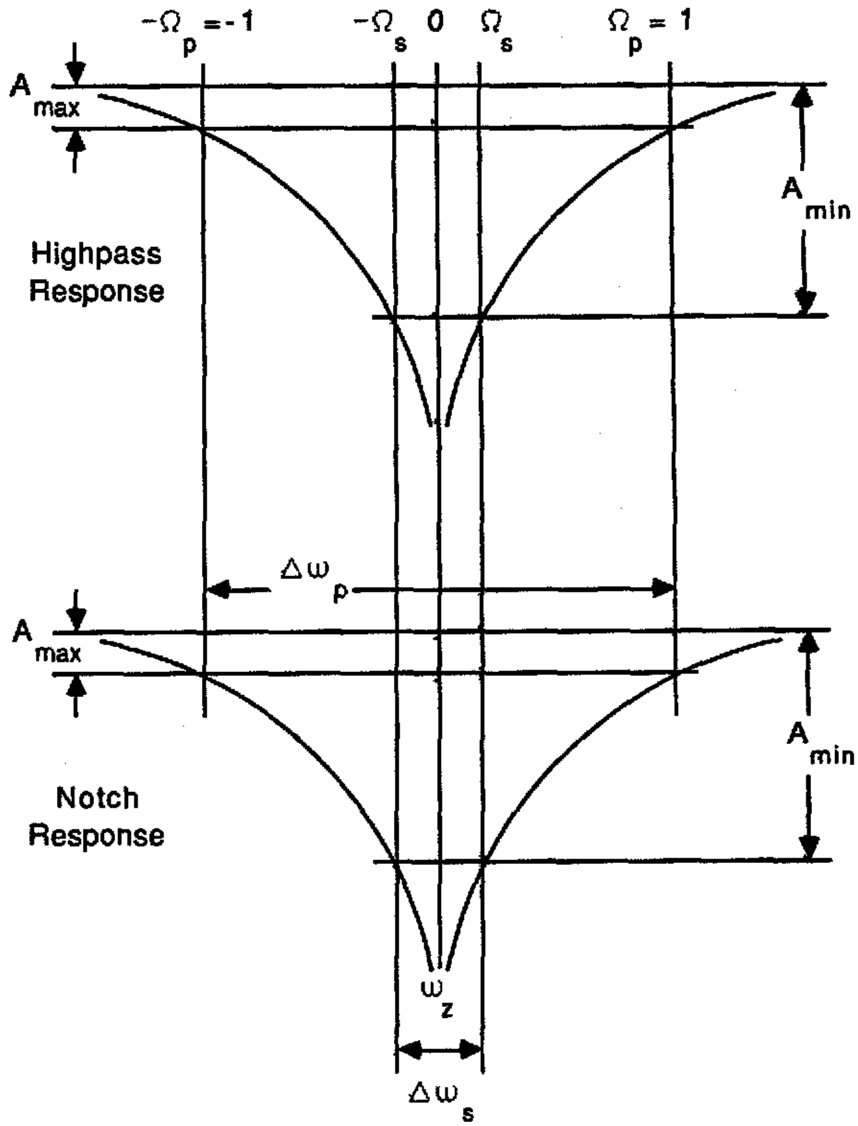

Figure 6.20. Frequency response relationships between highpass prototype and notch filter response.

Again we will set the passband limit of our (highpass) prototype equal to unity. The poles will map from our highpass S plane to the notch s plane exactly as poles mapped from the lowpass S plane to the bandpass s plane. In addition, using 6.31 (or Figure 6.5) the zeros at S = 0 in our highpass prototypes will map to

$$s = \frac{bS}{2} \pm \sqrt{\frac{(bS)^2}{4} - 1} \qquad (6.31)$$

$$= \pm\sqrt{-1}$$

$$= \pm\, j \text{ (normalized to } \omega_z = 1).$$

Since the zero at s=0 in our highpass prototype maps to a zero at $\pm$ j in our notch filter, it means we need to implement filters with zeros on the $j\omega$ axis and with complex conjugate poles. This implies the general form of the notch filter transfer function will be

$$\frac{V_{out}}{V_{in}} = \frac{A(s^2 + \omega_z^2)}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2} \quad . \qquad (6.81)$$

Example 6.8

Determine the minimum order and the pole/zero locations for a Butterworth notch filter to satisfy the following specifications:

A_{max} = 1 dB
A_{min} = 15 dB
Minimum attenuation = 0 dB
Passband limits = 500 Hz and 2 kHz
Stopband limits = 800 to 1250 Hz

The passband and stopband limits are both geometrically symmetric about the same frequency (1 kHz). Their ratio (3.33), A_{max}, and A_{min} result in a second order highpass prototype using either equation 4.47 or Figure 5.3. The pole frequency relative to the passband limit of our highpass prototype is given by

$$\Omega_0 = (10^{A_{max}/10} - 1)^{1/2N_{HP}} = (10^{\cdot 1} - 1)^{1/4} = 0.713$$

The difference between passband limits relative to the center frequency is

$$b = \frac{2\text{ kHz} - 500\text{ Hz}}{1\text{ kHz}} = 1.5 \quad .$$

At this point we enter Figure 6.6a with $b\Omega_0$ = (1.5)(0.713) = 1.07 and $\pm$ 45°. Figure 6.21 shows the resulting poles and zeros of our notch filter.

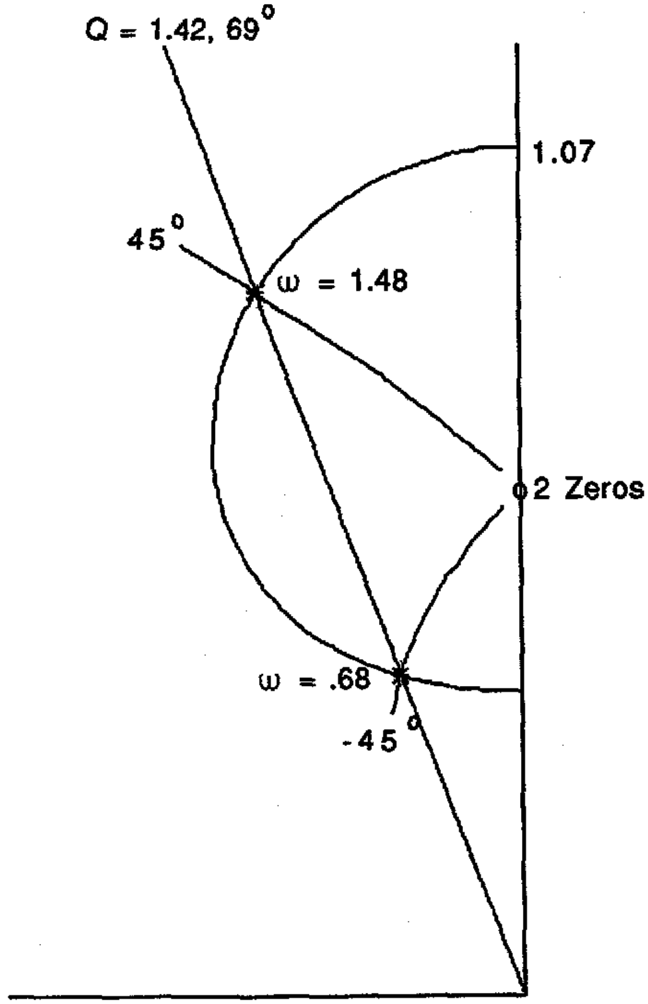

Figure 6.21. Pole/zero diagram for notch filter Example 6.8.

The poles therefore have Q's of 1.43 and frequencies of

$$\omega_{o1} = (1.48) \times 2\pi \times 1 \text{ kHz}$$
$$= 9.29 \times 10^3 \text{ rad/sec} \qquad (9314.7 \text{ r/s})$$

and

$$\omega_{o1} = (1/1.48) \times 2\pi \times 1 \text{ kHz}$$
$$= 4.24 \times 10^3 \text{ rad/sec} \quad . \qquad (4238.3 \text{ r/s})$$

using computer program

The two remaining poles are below the real s-plane axis and are just the complex conjugates of the ones shown in Figure 6.21. There are four zeros, two each at $j(2\pi \times 1 \text{ kHz})$ and $-j(2\pi \times 1 \text{ kHz})$.

Implementation of notch filters is a bit more complicated since there are now three entering parameters, namely Q, ω_o, and ω_z. There are several choices of single op-amp sections for notch filter implementation in the literature. We will consider only two versions of one basic circuit here for purposes of analysis. The first version (Figure 6.22) is for $\omega_o > \omega_z$. A slightly different version for $\omega_o < \omega_z$. will be analyzed later. For the case where $\omega_o = \omega_z$ (as will be the case for the single section resulting from the highpass prototype real pole), the capacitor C_1 in Figure 6.22 is removed.

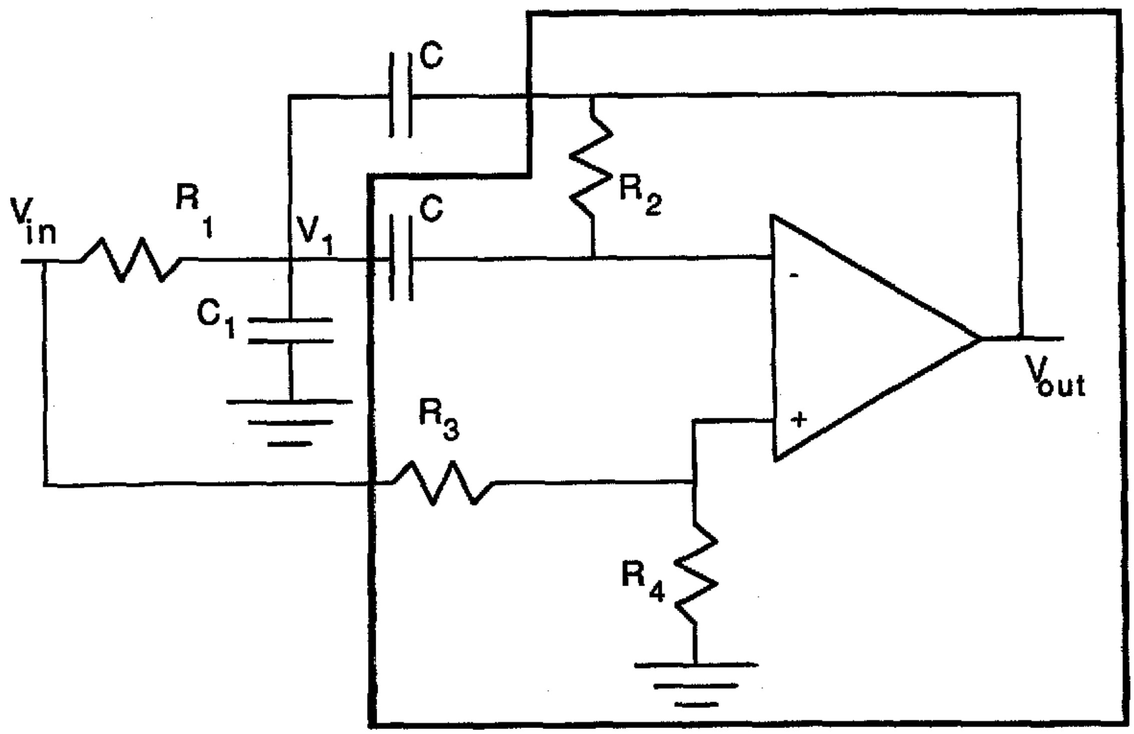

Figure 6.22. Notch filter circuit for $\omega_o > \omega_z$.

We first recognize from analysis techniques introduced in Chapter 3, the portion inside the bold line in Figure 6.22 represents an inverting differentiator from V_1 to V_{out}, and unity gain plus a non-inverting differentiator from V_+ to V_{out}. More specifically,

$$V_{out} = -R_2Cs\, V_1 + (1 + R_2Cs)V_+ \quad . \tag{6.82}$$

Defining the parameter k by

$$k = \frac{R_4}{R_4 + R_3} \text{ , this says}$$

$$V_+ = k\, V_{in}$$

and

$$V_{out} = -R_2Cs\, V_1 + (1 + R_2Cs)\, kV_{in} \tag{6.83}$$

or

$$V_1 = \frac{kV_{in} - V_{out}}{R_2Cs} + kV_{in}. \tag{6.84}$$

The node equation at V_1 is

$$\frac{V_1 - V_{in}}{R_1} + sC_1V_1 + sC(V_1 - V_{out}) + sC(V_1 - kV_{in}) = 0. \tag{6.85}$$

After collecting terms 6.85 becomes

$$\left(\frac{1}{R_1} + 2sC + sC_1\right)V_1 - \frac{1}{R_1}V_{in} - sC(V_{out} + kV_{in}) = 0. \tag{6.86}$$

Using 6.84 to substitute for V_1,

$$\left(\frac{1}{R_1} + 2sC + sC_1\right)\left(\frac{kV_{in} - V_{out}}{R_2Cs} + kV_{in}\right) - \frac{1}{R_1}V_{in} - sC(V_{out} + kV_{in}) = 0.$$

Grouping terms in V_{in} and V_{out} we obtain

$$\left[\left(\frac{1}{R_1} + 2sC + sC_1\right)\left(\frac{1}{R_2Cs} + 1\right) - \frac{1}{kR_1} - sC\right]kV_{in} =$$

$$\left(\frac{1}{R_1R_2Cs} + \frac{2}{R_2} + \frac{C_1}{R_2C} + \frac{1}{R_1} + sC + sC_1 - \frac{1}{kR_1}\right)kV_{in} =$$

$$\left(\frac{1}{R_1R_2Cs} + \frac{2}{R_2} + \frac{C_1}{R_2C} + sC\right)V_{out}. \tag{6.87}$$

Multiplying both sides by s/C and defining ω_o as

$$\omega_o = \frac{1}{\sqrt{R_1R_2}\,C}$$

$$= \frac{1}{R_{eq}\,C},$$

the transfer function simplifies to

$$\frac{V_{out}}{V_{in}} = \frac{k\left[\left(1 + \frac{C_1}{C}\right)s^2 + \left(\frac{2}{CR_2} + \frac{C_1}{R_2C^2} + \frac{1}{CR_1} - \frac{1}{kCR_1}\right)s + \omega_o^2\right]}{s^2 + \left(\frac{2}{CR_2} + \frac{C_1}{R_2C^2}\right)s + \omega_o^2} \tag{6.88}$$

Recall that we would like 6.88 to be in the general form for a notch filter as first presented in 6.81, or

$$\frac{V_{out}}{V_{in}} = \frac{A(s^2 + \omega_z^2)}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2} \quad . \tag{6.81}$$

We start by selecting $R_{eq} = \frac{1}{\sqrt{R_1 R_2}}$ and C such that

$$R_{eq}C = \frac{1}{\omega_o} \quad .$$

Since

$$\omega_z^2 = \frac{\omega_o^2}{1 + \frac{C_1}{C}} \quad , \tag{6.89}$$

we next select C_1 such that

$$C_1 = \left(\frac{\omega_o^2}{\omega_z^2} - 1\right)C \quad . \tag{6.90}$$

Note that when $\omega_z = \omega_o$ as will be the case when mapping the real pole of the highpass prototype, the capacitor C_1 is eliminated. (Also note that the case of $\omega_o < \omega_z$ would not be possible.)

Setting the coefficients of s in the denominators of 6.87 and 6.88 equal

$$\frac{\omega_o}{Q} = \frac{1}{\sqrt{R_1 R_2}\,CQ}$$

$$= \frac{2}{CR_2} + \frac{C_1}{R_2 C^2} \, , \tag{6.91}$$

or

$$\sqrt{\frac{R_2}{R_1}}\frac{1}{Q} = 2 + \frac{C_1}{C} = 1 + \frac{\omega_o^2}{\omega_z^2} \quad . \tag{6.92}$$

Therefore R_1 and R_2 are selected by

$$R_2 = Q\left(1 + \frac{\omega_o^2}{\omega_z^2}\right)R_{eq} \, , \tag{6.93}$$

and

$$R_1 = \frac{R_{eq}}{Q\left(1 + \frac{\omega_o^2}{\omega_z^2}\right)} \quad . \tag{6.94}$$

Finally, to obtain a notch we set the coefficient of s in the numerator of 6.88 to zero, or

$$\frac{2}{CR_2} + \frac{C_1}{R_2C^2} + \frac{1}{CR_1} - \frac{1}{kCR_1} = 0 \; . \tag{6.95}$$

This says

$$\frac{1}{k} = \frac{2R_1}{R_2} + \frac{C_1R_1}{R_2C} + 1$$

$$= \frac{R_1}{R_2}\left(2 + \frac{C_1}{C}\right) + 1$$

$$= \frac{1 + \frac{\omega_o^2}{\omega_z^2}}{Q^2\left(1 + \frac{\omega_o^2}{\omega_z^2}\right)^2} + 1$$

$$= \frac{1}{Q^2\left(1 + \frac{\omega_o^2}{\omega_z^2}\right)} + 1 \; , \tag{6.96}$$

Since

$$k = \frac{R_4}{R_4 + R_3}$$

or

$$\frac{1}{k} = 1 + \frac{R_3}{R_4} \tag{6.97}$$

Therefore R_3 and R_4 are selected by

$$\frac{R_4}{R_3} = Q^2\left(1 + \frac{\omega_o^2}{\omega_z^2}\right) \quad . \tag{6.98}$$

For $\omega_o < \omega_z$ the capacitor C_1 is eliminated and a resistor R_5 is added between the inverting input and ground as shown in Figure 6.23.

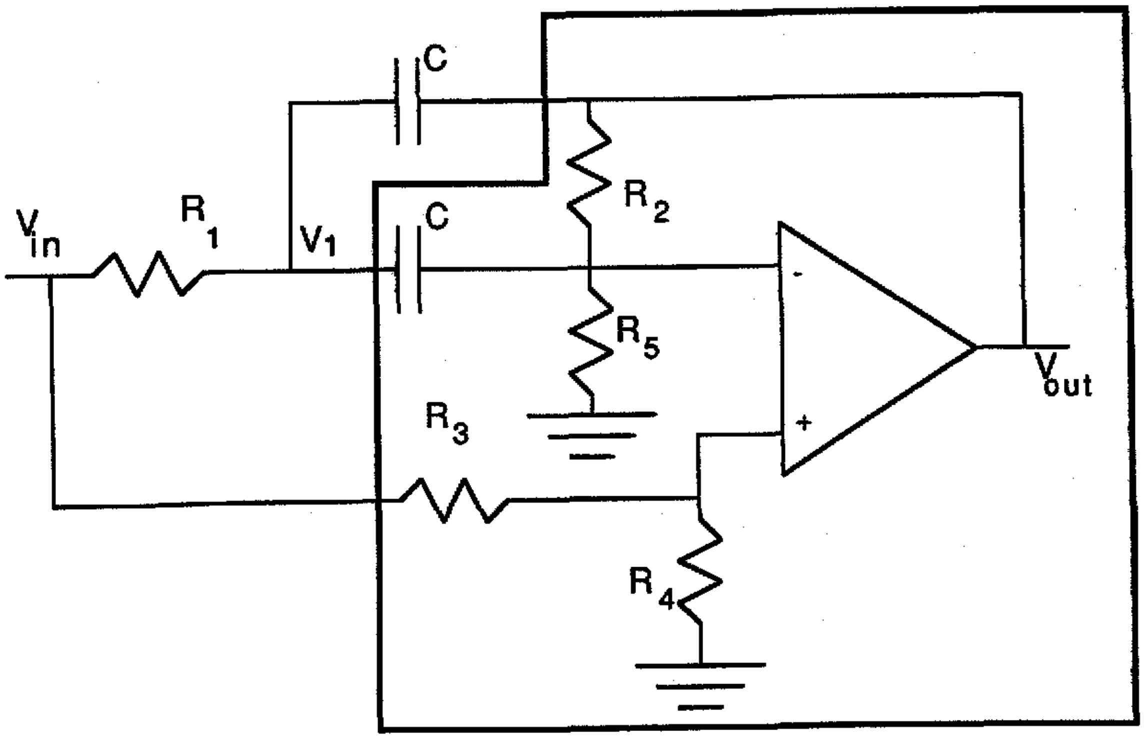

Figure 6.23 Notch filter circuit for $\omega_0 < \omega_z$.

This resistor does not change the transfer function from V_1 to V_{out} but adds a DC term (R_2/R_5) to the transfer function from V_+ to V_{out} or

$$V_{out} = -R_2Cs\,V_1 + \left(1 + \frac{R_2}{R_5} + R_2Cs\right)V_+ \tag{6.99}$$

Defining the parameter k by

$$k = \frac{R_4}{R_4 + R_3} \tag{6.100}$$

$$V_+ = k\,V_{in} \tag{6.101}$$

and

$$V_{out} = -R_2Cs\,V_1 + \left(1 + \frac{R_2}{R_5} + R_2Cs\right)kV_{in} \tag{6.102}$$

or

$$V_1 = \frac{-V_{out}}{R_2Cs} + \left(\frac{1}{R_2Cs} + \frac{1}{R_5Cs} + 1\right)kV_{in} \tag{6.103}$$

The node equation at V_1 is

$$\frac{V_1 - V_{in}}{R_1} + sC(V_1 - V_{out}) + sC(V_1 - kV_{in}) = 0 \tag{6.104}$$

Collecting terms

$$\left(\frac{1}{R_1} + 2sC\right)V_1 - \frac{1}{R_1}V_{in} - sC(V_{out} + kV_{in}) = 0 \tag{6.105}$$

Using 6.103 to substitute for V_1

$$\left(\frac{1}{R_1} + 2sC\right)\left(\frac{-V_{out}}{R_2Cs} + \left(\frac{1}{R_2Cs} + \frac{1}{R_5Cs} + 1\right)kV_{in}\right) - \frac{1}{R_1}V_{in} - sC(V_{out} + kV_{in}) = 0 \tag{6.106}$$

Collecting terms results in

$$\left(\left(\frac{1}{R_1} + 2sC\right)\left(\frac{1}{R_2Cs} + \frac{1}{R_5Cs} + 1\right) - \frac{1}{kR_1} - sC\right)kV_{in} = \left(\frac{1}{R_1R_2Cs} + \frac{2}{R_2} + sC\right)V_{out} \tag{6.107}$$

Multiplying both sides by s/C and defining ω_0 by

$$\omega_0 = \frac{1}{\sqrt{R_1R_2}\,C}$$

$$= \frac{1}{R_{eq}\,C} \tag{6.108}$$

the transfer function (6.107) becomes

$$\frac{V_{out}}{V_{in}} = \frac{k\left(s^2 + \left(\frac{2}{CR_2} + \frac{2}{CR_5} + \frac{1}{CR_1} - \frac{1}{kCR_1}\right)s + \left(1 + \frac{R_2}{R_5}\right)\omega_0^2\right)}{s^2 + \frac{2}{CR_2}s + \omega_0^2} \tag{6.109}$$

We would like 6.109 to be of the form

$$\frac{V_{out}}{V_{in}} = \frac{A(s^2 + \omega_z^2)}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \tag{6.81}$$

To design using we start by selecting R_{eq} and C such that

$$R_{eq} C = \frac{1}{\omega_o} \quad . \tag{6.110}$$

Setting the coefficients of s in the denominators of 6.81 and 6.109 equal

$$\frac{\omega_o}{Q} = \frac{1}{\sqrt{R_1 R_2}\, CQ}$$

$$= \frac{2}{CR_2} \tag{6.111}$$

or

$$\sqrt{\frac{R_2}{R_1}}\frac{1}{Q} = 2 \tag{6.112}$$

Therefore R_1 and R_2 are selected by

$$R_2 = 2Q\, R_{eq}$$

and

$$R_1 = \frac{R_{eq}}{2Q} \quad . \tag{6.113}$$

Since

$$\omega_z^2 = \omega_o^2\left(1 + \frac{R_2}{R_5}\right) \quad , \tag{6.114}$$

R_5 is given by

$$R_5 = \frac{R_2}{\left(\frac{\omega_z^2}{\omega_o^2} - 1\right)} \tag{6.115}$$

Finally to obtain a notch we set the coefficient of s in the numerator of 6.109 to zero

$$\frac{2}{CR_2} + \frac{2}{CR_5} + \frac{1}{CR_1} - \frac{1}{kCR_1} = 0 \tag{6.116}$$

or

$$\frac{1}{k} = \frac{2R_1}{R_2} + \frac{2R_1}{R_5} + 1$$

$$= \frac{2R_1}{R_2}\left(1 + \frac{R_2}{R_5}\right) + 1 = \frac{\omega_z^2}{2Q^2\omega_o^2} + 1 \tag{6.117}$$

Since

$$k = \frac{R_4}{R_4 + R_3} \tag{6.118}$$

or

$$\frac{1}{k} = 1 + \frac{R_3}{R_4} \tag{6.119}$$

Therefore R_3 and R_4 are selected by

$$\frac{R_4}{R_3} = 2Q^2 \frac{\omega_o^2}{\omega_z^2} \tag{6.120}$$

Example 6.9

Design a circuit to satisfy the specifications given in Example 6.8 above. The poles therefore have Q's of 1.42 and frequencies of

$$\omega_{o1} = 9.29 \times 10^3 \text{ rad/sec}$$

and

$$\omega_{o2} = 4.24 \times 10^3 \text{ rad/sec} \quad .$$

There are four zeros, two each at $j(2\pi * 1 \text{ kHz})$ and $-j(2\pi * 1 \text{ kHz})$.

We start by arbitrarily selecting C =10 nF in each section. The design equations as well as the specifics of this design are summarized in Table 6.3

	Highpass Section (Figure 6.22)	Lowpass Section (Figure 6.23)
ω_o (rad/sec)	9.29×10^3	4.24×10^3
Q	1.42	1.42
ω_z (rad/sec)	6.28×10^3	6.28×10^3
$\frac{\omega_o^2}{\omega_z^2}$	2.19	$\frac{1}{2.19}$
$R_{eq} = \frac{1}{\omega_o C}$	10.8 kΩ	23.6 kΩ
R_1	$\frac{R_{eq}}{Q\left(1 + \frac{\omega_o^2}{\omega_z^2}\right)} = 2.38\ \text{k}\Omega$	$\frac{R_{eq}}{2Q} = 8.3\ \text{k}\Omega$
R_2	$Q\left(1 + \frac{\omega_o^2}{\omega_z^2}\right)R_{eq} = 49\ \text{k}\Omega$	$2Q\ R_{eq} = 67\ \text{k}\Omega$
	$C_1 = \left(\frac{\omega_o^2}{\omega_z^2} - 1\right)C = 11.9\ \text{nF}$	$R_5 = \frac{R_2}{\left(\frac{\omega_z^2}{\omega_o^2} - 1\right)} = 56\ \text{k}\Omega$
$\frac{R_4}{R_3}$	$Q^2\left(1 + \frac{\omega_o^2}{\omega_z^2}\right) = 6.4$	$2Q^2\frac{\omega_o^2}{\omega_z^2} = 1.84$
DC gain	$k = \frac{R_4}{R_4+R_3} = 0.865$	$\frac{R_4}{R_4+R_3}\left(1 + \frac{R_2}{R_5}\right) = 1.42$

Table 6.3

The circuit is shown in Figure 6.24. If the overall DC gain (1.23) is not correct it could be reduced as before with voltage division at the input or increased with another stage.

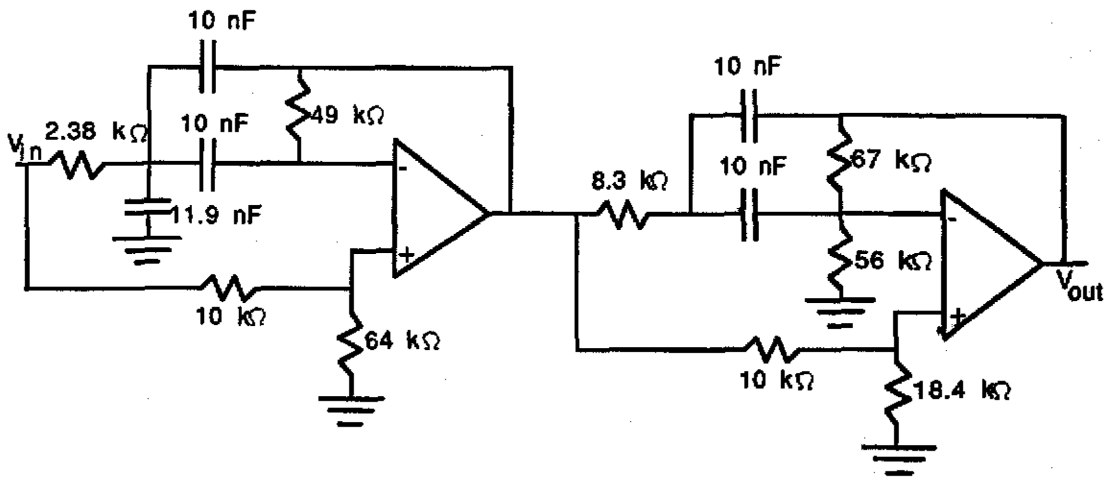

Figure 6.24 Notch Filter Circuit for Example 6.9

6.7 Effects of Finite Gain Bandwidth Product in Bandpass Filter Circuits

In Chapter 4 we saw that using an op-amp with a finite gain bandwidth product in our design of lowpass filters would result in poles being displaced from our desired pole locations. Constructing bandpass filters with real op-amps produces similar results, but the effects of finite gain bandwidth product may be more pronounced in the bandpass case for two reasons:

1). Generally speaking, bandpass filters will be designed with higher Q sections, and as we saw in Chapter 4, higher Q sections have poles which deviate more from their desired location.

2). A high Q bandpass filter inherently suggests we may need to control the resonant frequency to a much closer tolerance. (For example, in a low pass filter we might not care that the actual cutoff frequency is 5% too low, but we probably cannot accept such errors in a very narrow bandpass filter.)

To illustrate these effects, we will first go through the analysis of both circuits presented earlier in Figures 6.11 and 6.16. Then we will consider a specific example where we will use an op-amp whose gain bandwidth product is 70-100 times the desired center frequencies of the sections of our high Q bandpass filter. Although one might suspect that such a device could be accurately modeled as having an infinite gain bandwidth product relative to desired center frequency in this design, we will find that this is not the case. To design a reasonably symmetric, high Q bandpass filter at our desired center frequency, we must account for this finite gain bandwidth product, or suffer a deviation of over 10% in our desired filter center frequency, and also suffer a serious degradation of filter shape.

Case 1. Circuit from Figures 6.11-12.

Earlier in Section 6.5 we derived the transfer function for the bandpass circuit in Figure 6.11 as

$$\frac{V_{out}}{V_{in}} = \frac{\frac{A(s)}{\sqrt{2}}\omega_o s}{s^2 + \frac{1}{\sqrt{2}}\left(4 - A(s)\right)\omega_o s + \omega_o^2} \tag{6.59}$$

Again we will model A(s) (as in Chapter 3) as a lowpass filter with DC gain A_o and bandwidth $\frac{G}{A_o}$. If we normalize ω_o to unity and interpret G as the gain bandwidth product relative to ω_o then we can say

$$A(s) = \frac{G}{s + \frac{G}{A_o}}, \tag{6.121}$$

and equation 6.59 can be rewritten as

$$\frac{V_{out}}{V_{in}} = \frac{\frac{1}{\sqrt{2}} \frac{Gs}{s + \frac{G}{A_o}}}{s^2 + \frac{1}{\sqrt{2}}\left(4 - \frac{G}{s + \frac{G}{A_o}}\right)s + 1}$$

$$= \frac{\frac{Gs}{\sqrt{2}}}{s^2\left(s + \frac{G}{A_o}\right) + \frac{1}{\sqrt{2}}\left(4\left(s + \frac{G}{A_o}\right) - G\right)s + s + \frac{G}{A_o}}$$

$$= \frac{\frac{Gs}{\sqrt{2}}}{s^3 + \frac{4}{\sqrt{2}}s^2 + s + \frac{G}{A_o}\left(s^2 + \frac{1}{\sqrt{2}}\left(4 - A_o\right)s + 1\right)}. \tag{6.122}$$

If we define Q as in equation 6.60 then

$$Q = \frac{\sqrt{2}}{4 - A_o},$$

and

$$\frac{V_{out}}{V_{in}} = \frac{\frac{Gs}{\sqrt{2}}}{s^3 + \frac{4}{\sqrt{2}}s^2 + s + \frac{G}{A_o}\left(s^2 + \frac{s}{Q} + 1\right)}. \tag{6.123}$$

Observe that equation 6.123 is very similar to equation 4.59 for the Sallen Key lowpass circuit except the second order term in the denominator is 2.828 vice 3. The resulting plot of pole locations versus the parameter G (Figure 6.25), is therefore quite similar to Figure 4.21.

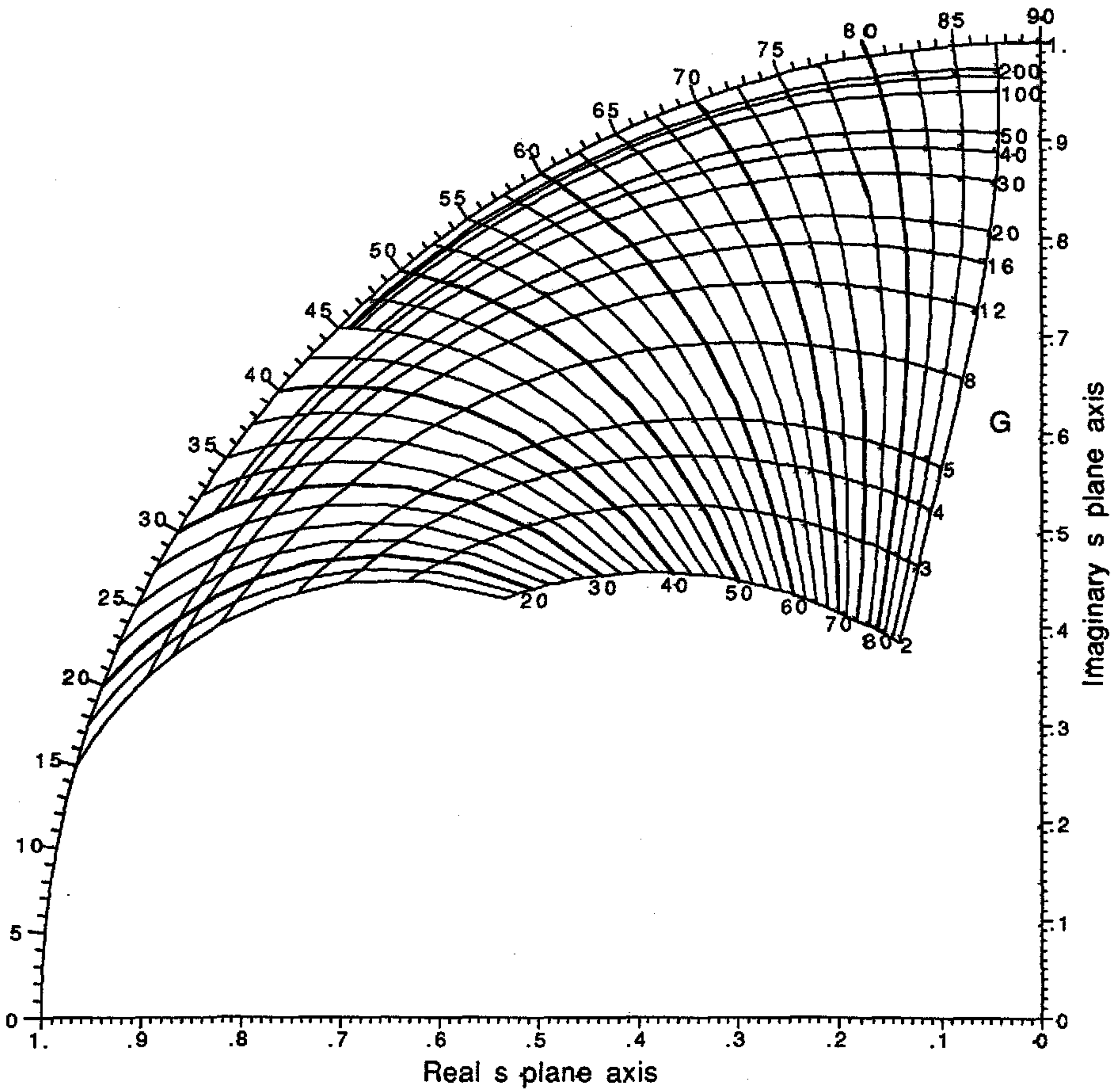

Figure 6.25. Roots of $s^3 + 2.828s^2 + s + (G/A_0)(s^2 + s/Q + 1) = 0$.

Case 2. Figure 6.26

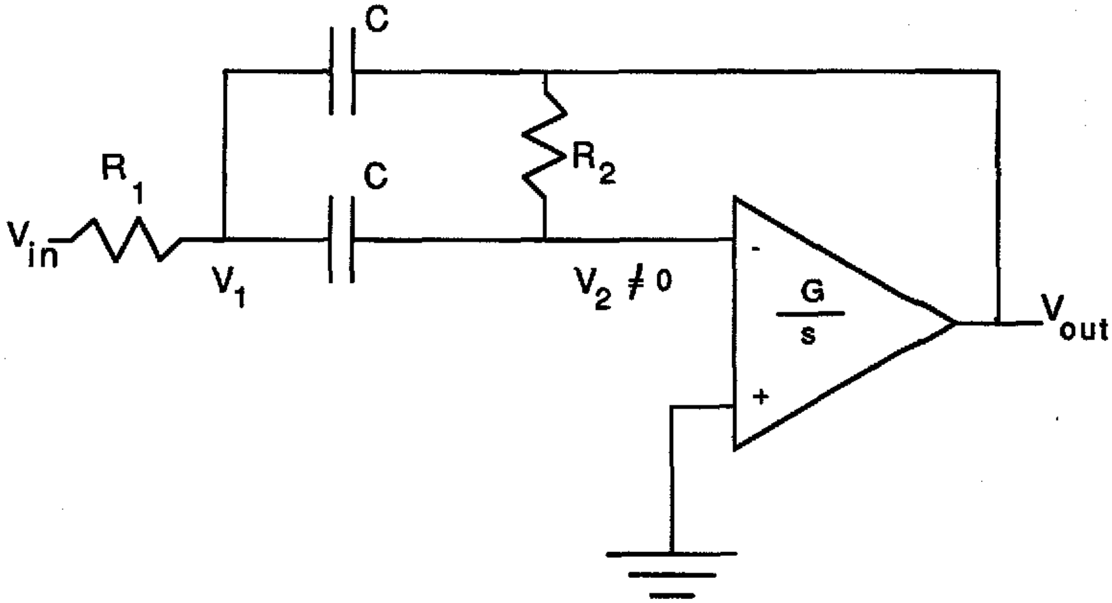

Figure 6.26. Second Order Bandpass Filter Section.

The analysis the affects of finite gain bandwidth for the second bandpass filter circuit presented is more somewhat more tedious because the previous assumption that the inverting input (V_2) was a virtual ground is no longer valid. What is interesting is that the results are identical to the second lowpass case, even though the circuit is very different. The node equations at V_1 and V_2 are now

$$\frac{V_1 - V_{in}}{R_1} + (V_1 - V_2)\,sC + (V_1 - V_{out})\,sC = 0 \qquad , \qquad (6.124)$$

and

$$\frac{V_2 - V_{out}}{R_2} + (V_2 - V_1)\,sC = 0 \qquad . \qquad (6.125)$$

Solving equation 6.125 for V_1:

$$V_1 = \frac{V_2 - V_{out}}{sCR_2} + V_2 \quad . \qquad (6.126)$$

and using this result in 6.124

$$\frac{V_1 - V_{in}}{R_1} + \frac{V_2 - V_{out}}{R_2} + (V_1 - V_{out})\,sC = 0 \quad ,$$

$$\left(\frac{1}{R_1} + sC\right) V_1 + \frac{V_2}{R_2} = \frac{V_{in}}{R_1} + \left(sC + \frac{1}{R_2}\right) V_{out} \quad ,$$

$$\left(\frac{1}{R_1} + sC\right)\left(\frac{V_2 - V_{out}}{sCR_2} + V_2\right) + \frac{V_2}{R_2} = \frac{V_{in}}{R_1} + \left(sC + \frac{1}{R_2}\right) V_{out} \quad ,$$

$$\left(\frac{1}{R_1} + sC\right)\left(\frac{V_2}{sCR_2} + V_2\right) + \frac{V_2}{R_2} = \frac{V_{in}}{R_1} + \left(sC + \frac{2}{R_2} + \frac{1}{sCR_1R_2}\right) V_{out} \quad ,$$

$$\left(\frac{1}{R_1} + \frac{2}{R_2} + sC + \frac{1}{sCR_1R_2}\right) V_2 = \frac{V_{in}}{R_1} + \left(sC + \frac{2}{R_2} + \frac{1}{sCR_1R_2}\right) V_{out} \quad . \tag{6.127}$$

If the transfer function of the op-amp is A(s) or:

$$V_{out} = -A(s)\, V_2$$

$$-\left(\frac{1}{R_1} + \frac{2}{R_2} + sC + \frac{1}{sCR_1R_2}\right)\frac{V_{out}}{A(s)} = \frac{V_{in}}{R_1} + \left(sC + \frac{2}{R_2} + \frac{1}{sCR_1R_2}\right) V_{out} \,, \tag{6.128}$$

We will model the op-amp by:

$$A(s) = \frac{G}{s} \quad , \tag{6.129}$$

where as before we will normalize ω_o to unity and interpret G as the gain bandwidth product of the op-amp relative to ω_o. Using the definitions of Q and ω_o above:

$$\omega_o = \frac{1}{C\sqrt{R_1R_2}} \quad , \tag{6.72}$$

and

$$Q = \frac{1}{2}\sqrt{\frac{R_2}{R_1}} \tag{6.74}$$

it follows that:

$$\frac{1}{R_1C} = 2Q\omega_o$$

Equation 6.128 now becomes:

$$-\left(s^2 + \left(\frac{1}{Q} + 2Q\right)s + 1\right)\frac{s\,V_{out}}{G} = 2Q\,s\,V_{in} + \left(s^2 + \frac{s}{Q} + 1\right)V_{out} \,, \tag{6.130}$$

and the transfer function becomes:

$$\frac{V_{out}}{V_{in}} = - \frac{2QG\,s}{s^3 + \left(\frac{1}{Q} + 2Q\right)s^2 + s + G\left(s^2 + \frac{s}{Q} + 1\right)} \quad . \qquad (6.131)$$

The denominator of 6.131 is exactly that of 4.61 or the lowpass second order section with the op-amp connected as a voltage follower. Therefore the poles are exactly the same as plotted in Figure 4.22 (repeated below as Figure 6.27).

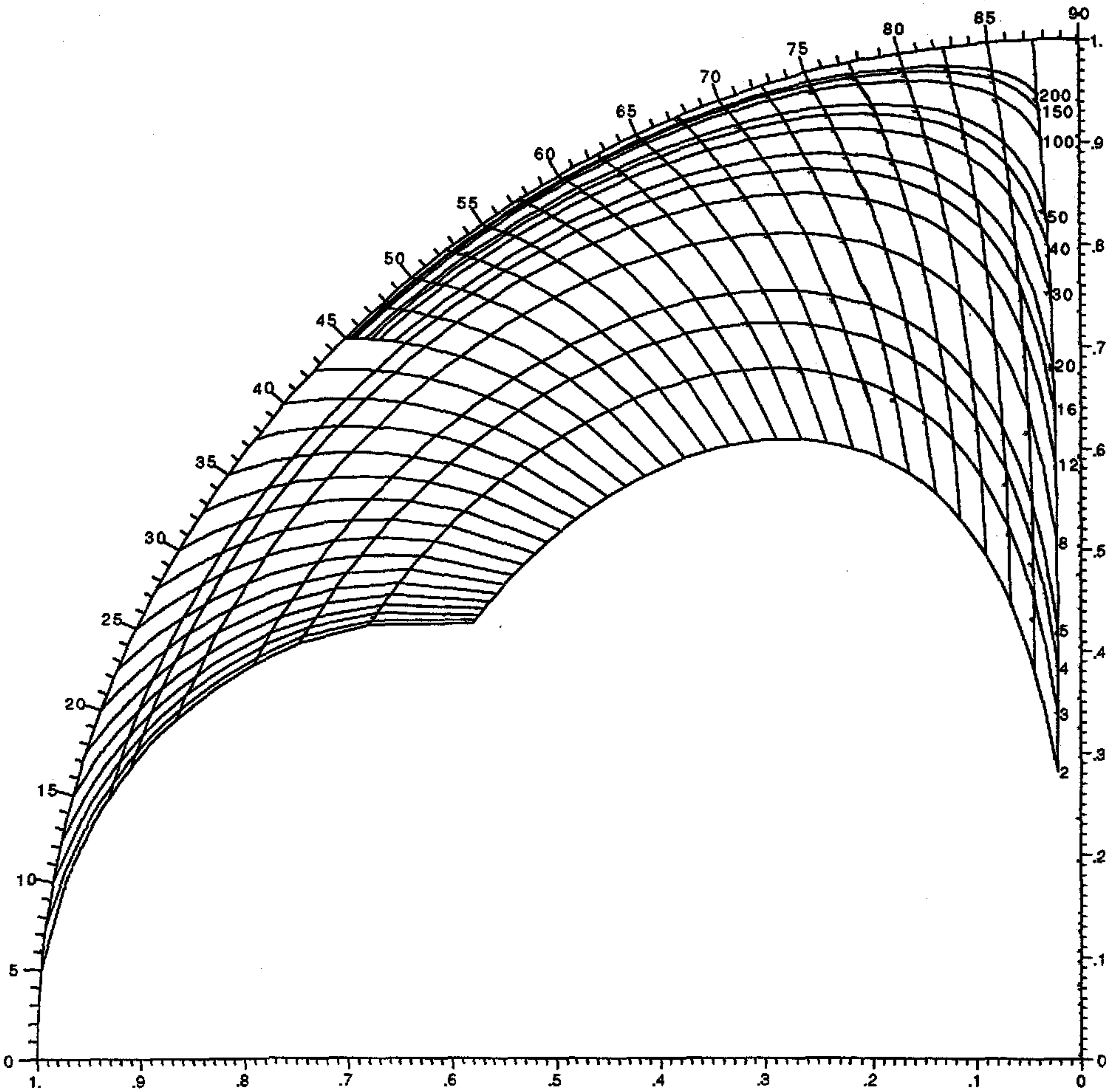

Figure 6.27 Roots of $s^3 + (1/Q + 2Q)s^2 + s + G(s^2 + s/Q + 1) = 0$

Example 6.10 Omega Receiver Pre-Amp/Pre-Selector

The Omega Navigation System consists of eight stations (Norway, Liberia, Hawaii, North Dakaota, La Reunion Island, Argentina, Austrailia and Japan) and provides wordwide electronic navigation. Each station transmits one unique and four common frequencies within the 10.2-13.6 kHz band. The phase differences from station pairs on the common frequencies are measured resulting in hyperbolic lines of position and a position.

Figure 6.28 shows the spectrum from 5 to 25 kHz and from a 35 foot whip antenna. Due to the much longer propagation path length, the Omega signals are seen to be much weaker than the frequency shift keyed (FSK) signals from Naval communications stations in the 15.1 to 24 kHz band. We would like to design an amplifier/filter to amplify the Omega signals and attenuate the interfering signals. We will use a bandpass vice lowpass filter design so as to eliminate 60 Hz or harmonics from the power distribution system. Our design will meet the following specifications:

Filter type	Chebyshev
Passband	10.2 to 13.6 kHz
Upper Stopband Limit	15.1 kHz
A_{max}	1 dB
A_{min}	18 dB

The lower stopband limit is $\dfrac{13.6 \times 10.2}{15.1} = 9.2$ kHz.

Using the charts, the minimum order of the lowpass prototype Chebyshev filter is 3, the lowpass prototype poles are on an ellipse with minor axis a = 0.49, radii of 0.49 and 1.00, and at 0° and 74° with respect to the negative real axis. The bandwidth realtive to the center frequency is $\dfrac{13.6 - 10.2}{\sqrt{13.6 \times 10.2}} = \dfrac{3.4}{11.8} = 0.29$. A plot of the pole locations is shown in Figure 6.29.

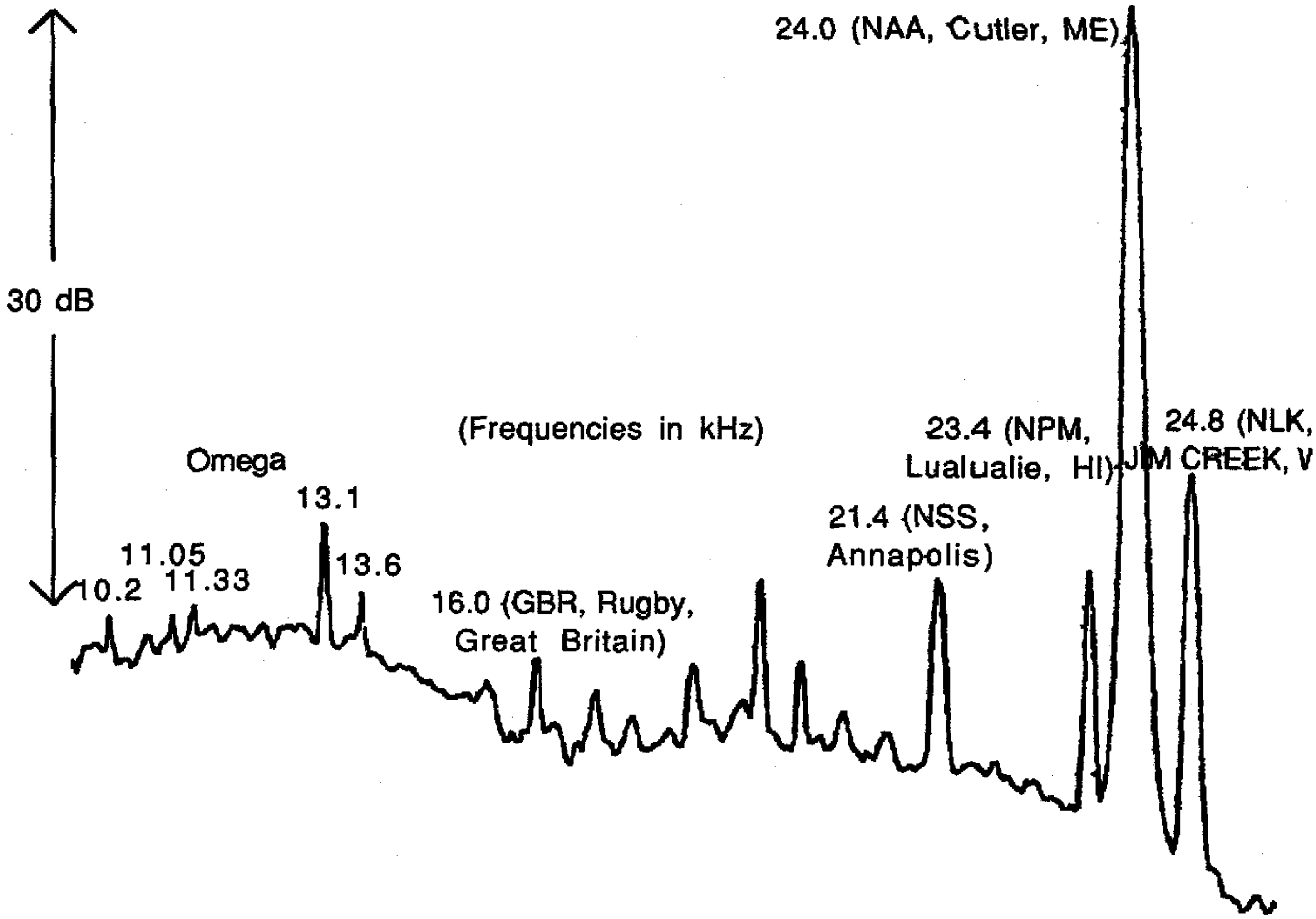

Figure 6.28. VLF Spectrum

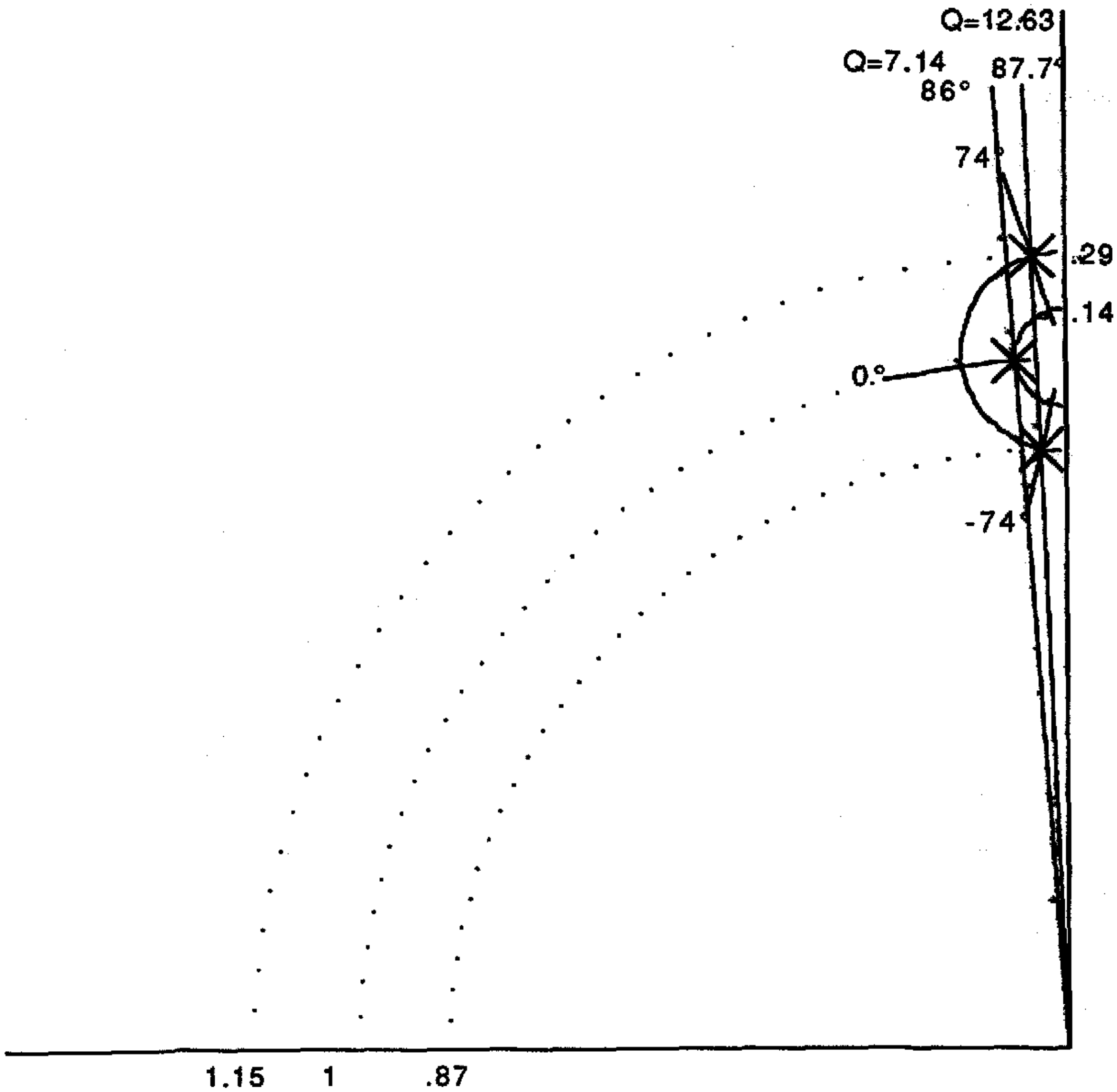

Figure 6.29. Plot of Pole Locations for Omega Filter.

Table 6.2 summarizes the design calculations using the circuit in Figure 6.16 and1000 pF for all capacitors. The circuit design is shown in Figure 6.30.

	Section1	Section 2	Section 3
Q	7.14	12.6	12.6
ω_o	$2\pi \times 11.8$ kHz = 7.41×10^4	$0.87 \times 7.41 \times 10^4$ = 6.45×10^4	$1.15 \times 7.41 \times 10^4$ = 8.52×10^4
$R_{eq} = \dfrac{1}{\omega_0 C}$	13.5 kΩ	15.5 kΩ	11.7 kΩ
$R_1 = \dfrac{R_{eq}}{2Q}$	945 Ω	615 Ω	464 Ω
$R_2 = 2Q\,R_{eq}$	188kΩ	390kΩ	295kΩ

Table 6.4

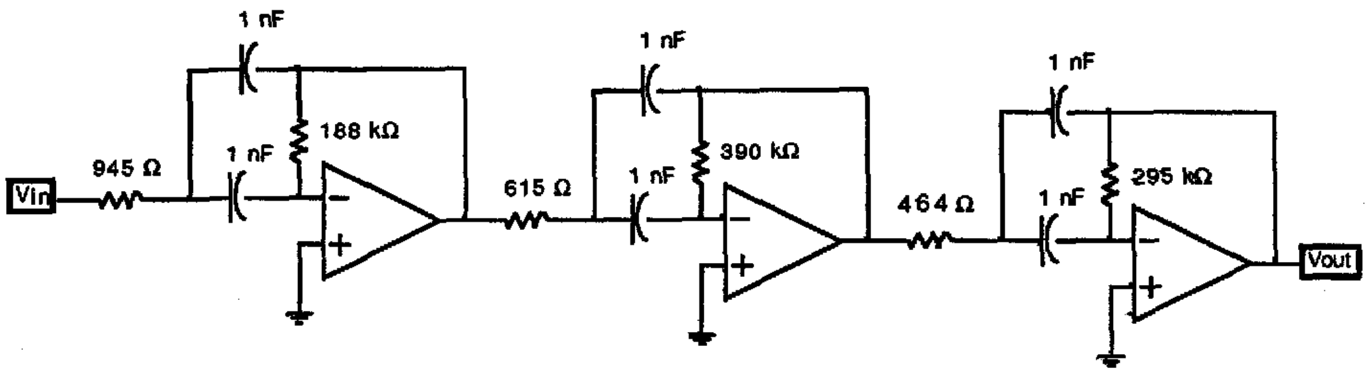

Figure 6.30. Omega Amplifier/Filter.

When we analyze the circuit using a linear circuit analysis program assuming 1 MHz gain bandwidth product op-amps, the frequency response (Figure 6.31) is not even close to what we designed for.

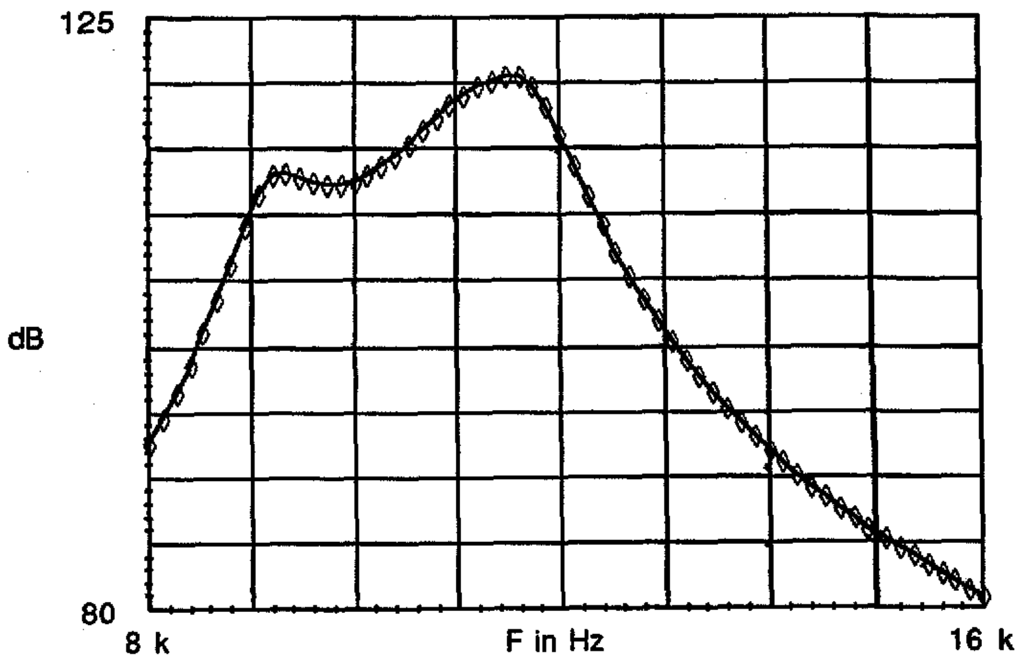

Figure 6.31. Frequency Response of Circuit in Figure 6.30.

The reason becomes obvious when we look at the frequency response of each section (Figure 6.32) and the pole location versus gain bandwidth chart (Figure 6.27). Even though the gain bandwidth product is 85, 98 and 74 times the respective resonant frequencies the poles are still signficantly moved.

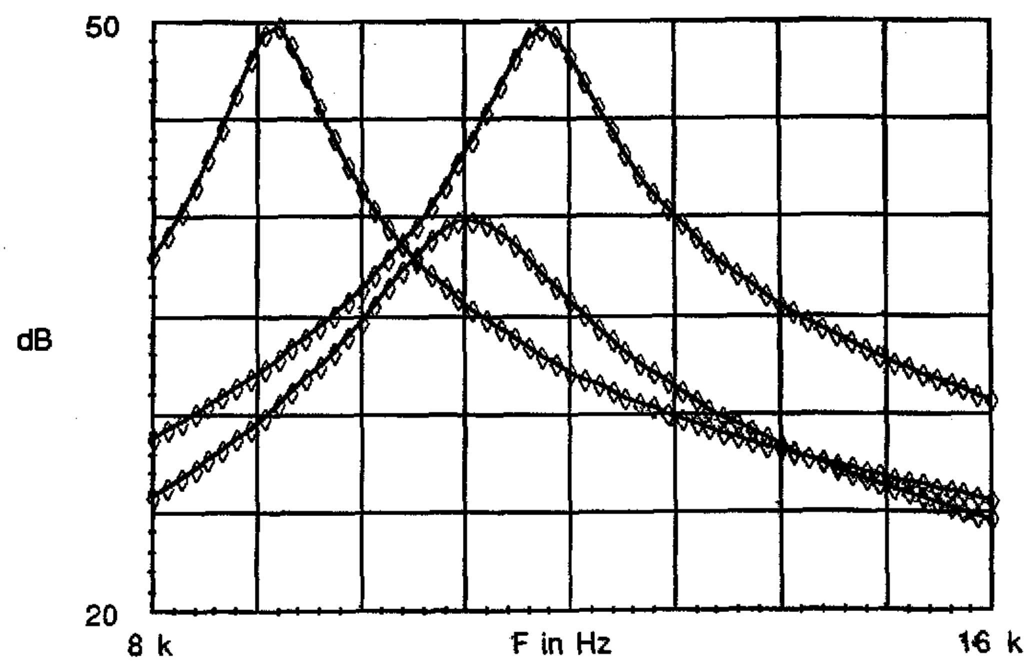

Figure 632. Frequency Response in Individual Sections of Circuit in Figure 6.30.

All of the pole frequencies have been shifted lower, but the higher Q poles have been shifted by a larger percentage. The Q's of each section are almost what we designed for, therefore the only modification necessary is to reduce $R_{eq}C$ and therefore increase the resonant frequency of each section. The present resonant frequencies are seen to be at 11.1, 9.2 and 11.8 kHz respectively or at 94%, 90%, and 87% of what the were designed to be. Because as we increase $\frac{1}{R_{eq}C}$ we are going to decrease the parameter G we need to adjust by somewhat more than these percentages, and by an even greater amount for sections with higher Q's and ω_0's. The modified circuit and its frequency response are shown in Figures 6.33 and 6.34. Even though we may want large gain due to weak signals we are trying to detect, 113 dB is too much and we would have to reduce the gain using techniques introduced earlier.

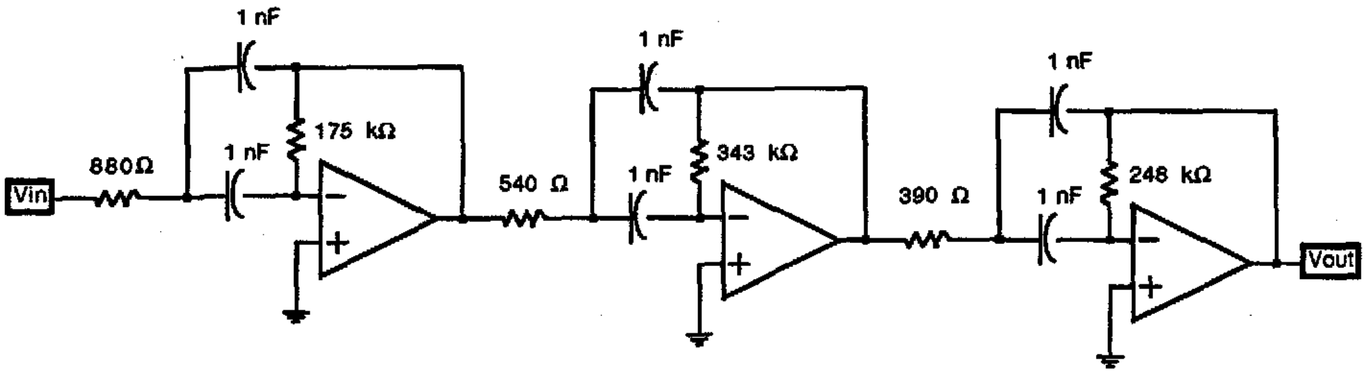

Figure 6.33. Modified Circuit

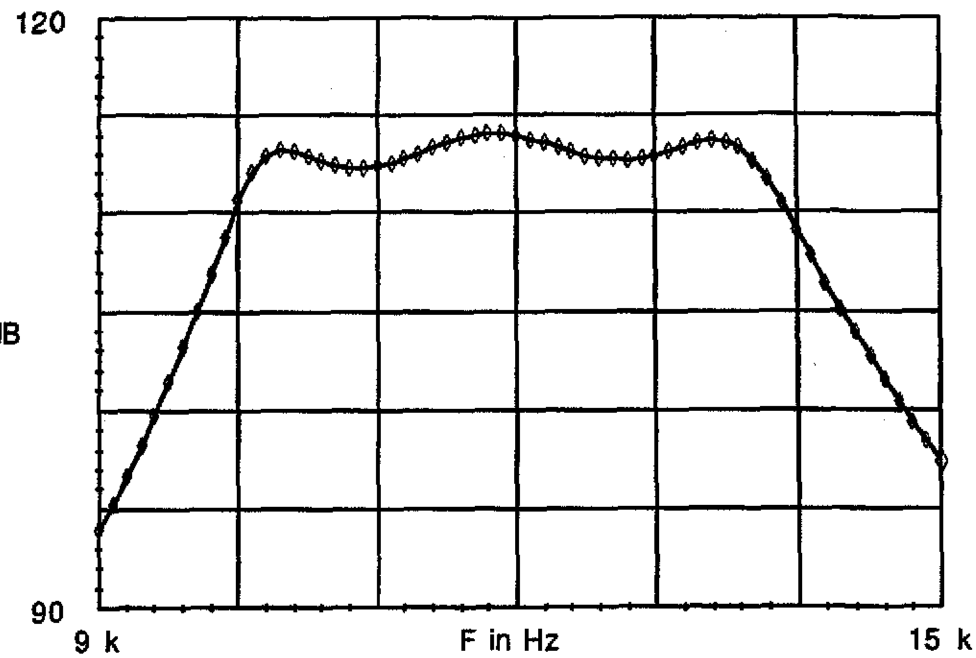

Figure 6.34. Frequency Response of Modified Circuit.

6.8 Bandpass Filter Design Using the National Semiconductor MF8

To this point we have designed higher order bandpass filters by cascading second order sections. A slightly different approach (for audio frequencies to 20 kHz) is followed in design using the National Semiconductor MF8 Switched Capacitor Bandpass Filter. Here two identical (same Q, same center frequency) second order sections are cascaded, and then with a separate op-amp, negative feedback is applied. The design and programming of the circuit are quite simple as the center frequency is controlled with a clock signal and the Q's are set with a five bit digital word. The two sections and the op-amp are integrated into a single inexpensive integrated circuit.

The study of switched capacitor filters per se is beyond the scope of this book. While in reality they are discrete time systems with a sampling frequency equal to the clock frequency (either 50 or 100 times the center frequency), for our purposes here, they will be a continuous time transfer function in a block diagram. Consistent with our algebra to this point we will normalize the center frequency to unity. The gain at the center frequency of each section is 2 (or 6 dB). A block diagram is shown in Figure 6.35.

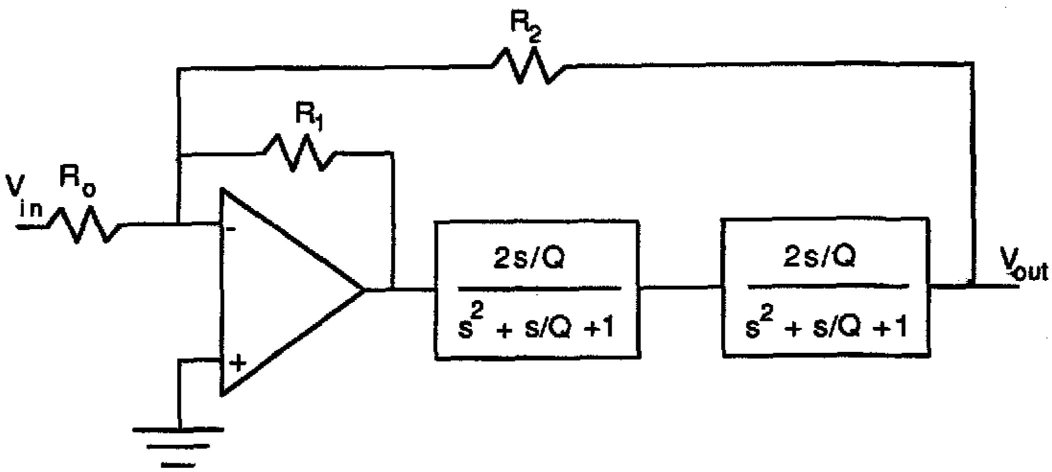

Figure 6.35. Block Diagram of MF8 Bandpass Filter.

This block diagram is redrawn in Figure 6.36 taking into account the summing of the op-amp.

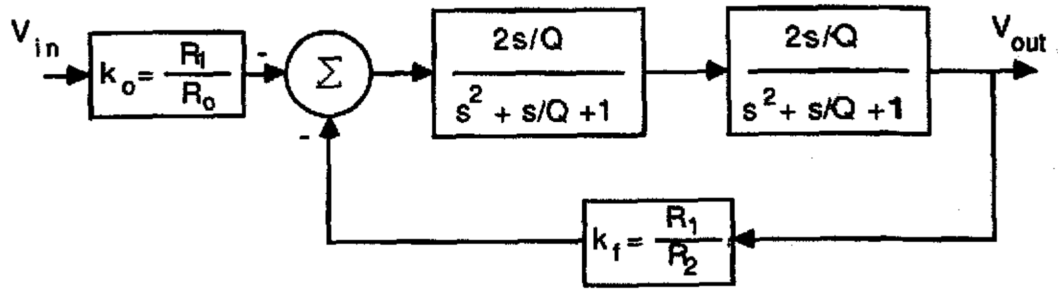

Figure 6.36. Redrawn block diagram of MF8 Bandpass Filter.

This system is of the form illustrated in Figure 6.37 which using standard block diagram algebra has the transfer function

$$\frac{V_{out}}{V_{in}} = \frac{-G_1G_2}{1 + G_2H} \tag{6.132}$$

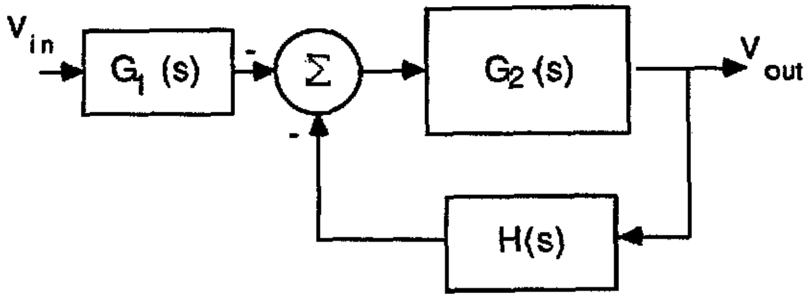

Figure 6.37. General form of block diagram.

Therefore the overall transfer function is

$$\frac{V_{out}}{V_{in}} = -k_o \frac{\left(\frac{2s/Q}{s^2 + s/Q + 1}\right)^2}{1 + k_f\left(\frac{2s/Q}{s^2 + s/Q + 1}\right)^2}$$

$$= -k_o \frac{4s^2/Q^2}{s^4 + \frac{2}{Q}s^3 + \left(2 + \frac{1 + 4k_f}{Q^2}\right)s^2 + \frac{2}{Q}s + 1} \tag{6.133}$$

If 6.133 is compared to equation 6.34

$$\frac{V_{out}}{V_{in}} = \frac{b^2\,\Omega_o^2\,s^2}{s^4 + \frac{b\,\Omega_o}{Q_{LP}}s^3 + (b^2\,\Omega_o^2 + 2)\,s^2 + \frac{b\,\Omega_o}{Q_{LP}}s + 1} \qquad (6.34)$$

we see that it has precisely the form we want, namely the coefficients of s^3 and s are equal and by adjusting Q and k_f we can make it term for term the same as 6.34.

$$\frac{2}{Q} = \frac{b\,\Omega_o}{Q_{LP}}$$

or

$$Q = \frac{2Q_{LP}}{b\,\Omega_o} \qquad (6.134)$$

and

$$2 + \frac{1 + 4k_f}{Q^2} = b^2\,\Omega_o^2 + 2$$

$$1 + 4k_f = b^2\,\Omega_o^2\,Q^2$$

$$= 4Q_{LP}^2$$

or

$$k_f = Q_{LP}^2 - \frac{1}{4} \qquad (6.135)$$

Since the center frequency gain of each section is 2, the overall gain at the center frequency is

$$\frac{4\,k_o}{1 + 4k_f} \qquad (6.136)$$

Example 6.11

Using the MF8, design a fourth order Butterworth bandpass filter with A_{min} as large as possible to satisfy the following specifications

A_{max} = 1 dB
Minimum attenuation = 0 dB
Passband limits = 4 kHz to 5 kHz
Stopband limits = 3.64 kHz and 5.5 kHz

The passband and stopband limits are symmetric about the same center frequency (4.47 kHz). The bandwidth relative to the center frequency is

$$b = \frac{5\text{ kHz} - 4\text{kHz}}{4.47\text{ kHz}} = 0.224$$

The lowpass prototype is a second order Butterworth filter with Q_{LP} = .707

and

$$\Omega_o = \frac{1}{(10^{A_{max}/10} - 1)^{1/2N_{LP}}} \tag{6.46}$$

$$= \frac{1}{(10^{.1} - 1)^{1/4}} = 1.4$$

Therefore

$$Q = \frac{2 * 0.707}{0.224 * 1.4} = 4.5$$

and

$$k_f = (0.707)^2 - 0.25 = 0.25$$

To set the maximum (and center frequency) gain at 0 dB we set

$$\frac{4\ k_o}{1 + 4k_f} = 1$$

or

$$k_o = 0.5 \quad .$$

Therefore setting $R_1 = 10\ k\Omega$, $R_o = 20\ k\Omega$, and $R_2 = 40\ k\Omega$ will result in a Butterworth filter with a center frequency gain of 0 dB.

Because Q is controlled with a 5 bit binary number, 32 choices of Q are available. The nearest to 4.5 are 4 and 5. Which is the logical choice? Because A_{max} and the passband limits are specified but A_{min} was not we must error on the side of making the passband wider and not narrower than specified. Therefore we must choose the lower value of Q which results in a slightly wider passband. Had A_{min} been specified and our design objective was to make A_{max} as small as possible we would have chosen the higher value for Q.

Problems

6.1-4 Determine the minimum order Butterworth and Chebyshev bandpass filters to satisfy the following specifications:

6.1 A_{max} = 2dB
A_{min} = 30dB
Passband Limits = 3kHz to 4kHz
Stopband Limits = 1.5kHz to 8kHz

6.2 A_{max} = 1dB
A_{min} = 36dB
Passband Limits = 3kHz to 4kHz
Stopband Limits = 1.5kHz to 8kHz

6.3 $A_{max} = 0.5dB$
$A_{min} = 30dB$
Passband Limits = 3kHz to 5kHz
Stopband Limits = 1.5kHz to 8kHz
(Note the asymmetrical limits.)

6.4 $A_{max} = 1dB$
$A_{min} = 30dB$
Passband Limits = 5kHz to 10kHz
Stopband Limits = 3kHz to 12kHz
(Note the asymmetrical limits.)

6.5-8 Determine the pole locations of the filters in problems 6.1 to 6.4 by

a) Method 1, (Eq. 6.31)
b) Method 2, (Eq. 6.41 and 6.45)
c) Graphically

6.9 Design a Butterworth bandpass filter using the circuit of Figure 6.11 to satisfy the following specifications:

$A_{max} = 2dB$
$A_{min} = 30dB$
Passband Limits = 15kHz to 20kHz
Stopband Limits = 7.5kHz to 40kHz
Center Frequency Gain = 0dB

6.10 Write a short program (Basic, Pascal or Fortran) that will calculate the minimum required order for a Butterworth or Chebyshev bandpass or notch filter, given a set of filter specifications. Inputs and outputs for your routine are as follows:

<u>Inputs</u>

Notch or Bandpass-
Buterworth or Chebyshev-
Passband limits-
Stopband limits-

<u>Outputs</u>

Required filter order-
<u>Actual</u> attenuation at passband and stopband limits-

Your routine should be able to handle the case of asymmetrical limits.

6.11 Design a Butterworth bandpass filter using the circuit of Figure 6.11 to satisfy the following specifications:

A_{max} = 2dB
A_{min} = 30dB
Passband Limits = 15kHz to 20kHz
Stopband Limits = 7.5kHz to 40kHz
Center Frequency Gain = 0dB

6.12 One of the most important applications of analog filters is for prefiltering or anti-alias filtering of an analog signal prior to a sampling/quantization operation. Suppose we are interested in receiving and studying a Loran-C navigation signal, whose center frequency is known to be 100kHz and whose bandwidth is approximately 20kHz. We are interested in viewing a passband of 20kHz. Our measurement system consists of a sample and hold and 8 bit A/D circuitry, sampling at 140kHz. This means that our sampling operation shifts the bandpass Loran-C spectrum down to a center frequency of 40kHz. (More specifically, we will interpret 50kHz as 90kHz, 40kHz as 100kHz, 30kHz as 110kHz, etc.) However, if we sample at 140kHz, note that we shift any signal components that happen to be at 170kHz down to 30kHz, which we would incorrectly interpret as part of the 110kHz component of Loran-C. Likewise, any signal components we receive at 50kHz would show up after sampling at 50kHz and would be incorrectly interpreted as part of the 90kHz component. Hence we need anti-aliasing protection, and our goal is to get the aliasing terms down to the same level as the quantization error. If we assume that there could be other signal sources as large as the 100kHz Loran-C signal at frequencies of 50kHz and 170kHz, and if we want to get the aliasing terms down to the same level as the quantization error, this implies that we need at least 48dB stopband rejection for an 8 bit A/D converter (6dB/bit since each bit represents a factor of 2). You are to design a minimum order Butterworth anti-aliasing bandpass filter with a 1dB passband of 90-110kHz, stopband limits of 50kHz and 170kHz, and stopband rejection of 48dB for this measurement system.

6.13 Design a Chebyshev notch filter to satisfy the following specifications

A_{max} = 1dB
A_{min} = 18dB
Passband Limits = 1 kHz and 20kHz
Stopband Limits = 2.5 kHz to 8 kHz
Maximum Gain = 0dB