Chapter 4

Butterworth Filters

4.1 Introduction

In Chapter 2 we concerned ourselves with the analysis of the frequency response of transfer functions. In previous courses you may have determined the frequency response of a given circuit. In this chapter we will deal with synthesizing a filter that yields a desired frequency response. More specifically we will address the question, "How do we go about designing a circuit to satisfy a given set of specifications?" Here we consider the Butterworth low-pass and high-pass responses. In later chapters we will study more complex responses. We will discover that the design problem is more interesting and actually is easier than the analysis problem.

The design of analog filters usually proceeds as follows:

a) Determine specifications and filter type.
b) Determine minimum order that satisfies specifications.
c) Solve for the poles and zeros of the transfer function.
d) Design the circuit to synthesize the transfer function assuming ideal operational amplifiers.
e) Analyze the circuit using a more accurate op-amp model, and if necessary, modify circuit elements to satisfy specifications.

In this and in successive chapters we will focus mainly on steps (b), (c), and (d). Step (e) is most easily accomplished with assistance from a circuit analysis program, however we will do some examples of step (e) in this and in later chapters to illustrate exactly how use of a non-ideal op-amp moves pole locations from desired locations. Step (a) is probably the most difficult (and important) step, but it is subjective in most cases and generally outside the scope of this text. As technology progresses and discrete time systems become more important, a major use of analog filters will be as anti-aliasing filters before analog to digital (A/D) converters or as reconstruction filters after digital to analog (D/A) converters (discussed in Chapter 8). In these cases, establishing specifications for the analog filter is more obvious and we will include step (a) in some examples of this type in later chapters on discrete time systems.

4.2 Butterworth Response

The magnitude of the frequency response of Butterworth filters is characterized by two distinctive features. First, the response is maximally flat within the pass band. In a low-pass filter this means the gain must be a maximum at DC and monotonically decrease with increasing frequency. (This is not a requirement for all low-pass responses. In Chapter 5 we will consider the Chebyshev response where the magnitude need only remain within upper and lower bounds, and ripple within the pass band is permissible.) The term maximally flat also means that the pass band gain should decrease "as slowly as possible" with frequency. As we will soon see, maximum flatness is a function of the order of the filter. A second feature is that the Butterworth response in the stop band should go monotonically to zero (or to negative infinity in dB) as frequency increases. This requirement will eliminate our using zeros on the imaginary axis in the s plane to put notches in the stop band.

In order to satisfy a given set of specifications with the minimum order filter, another requirement is that the gain should fall off as quickly as possible for a given order. Because the asymptotic magnitude response falls off at $20n_R$ dB/decade where n_R is the relative degree (or the difference in the order of the denominator and the numerator of the transfer function), the quickest possible roll-off is when there are no zeros and the relative degree is just the order of the denominator.

Finally, note that the magnitude response of any transfer function with real coefficients is an even function in frequency. This results in a magnitude squared response which is a function of only even powers of ω and can be written in general terms as

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega)$$

$$= \frac{1}{1+\sum_{i=1}^{n} a_i \omega^{2i}} \ . \tag{4.1}$$

So what are the a_i's ? The coefficient a_n is certainly non-zero or the transfer function would not be n^{th} order! If the a_i's for i from 1 to n-1 were greater than zero the frequency response would roll off faster than necessary in the pass band due to the dominance of the lower powers of ω at lower frequencies. If any coefficient were negative, the magnitude response would increase with frequency at some point due to the decrease in the magnitude of the denominator. We can gain added insight if we consider the maximally flat condition to require that as many derivatives of the magnitude response with respect to frequency as possible be zero when evaluated at DC. This occurs when the a_i's for i from 1 to n-1 are equal to zero and the first non-zero derivative at ω equal to zero is the $2n^{th}$. This is in fact the case for the Butterworth low-pass response. Therefore we write the Butterworth response as

$$\left|H(j\omega)\right|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_o}\right)^{2n}} , \tag{4.2}$$

where ω_o is the half power or -3 dB frequency. As we shall observe shortly it is also the undamped natural frequency of each section of the filter. Figure 4.1 shows the Butterworth low-pass magnitude response.

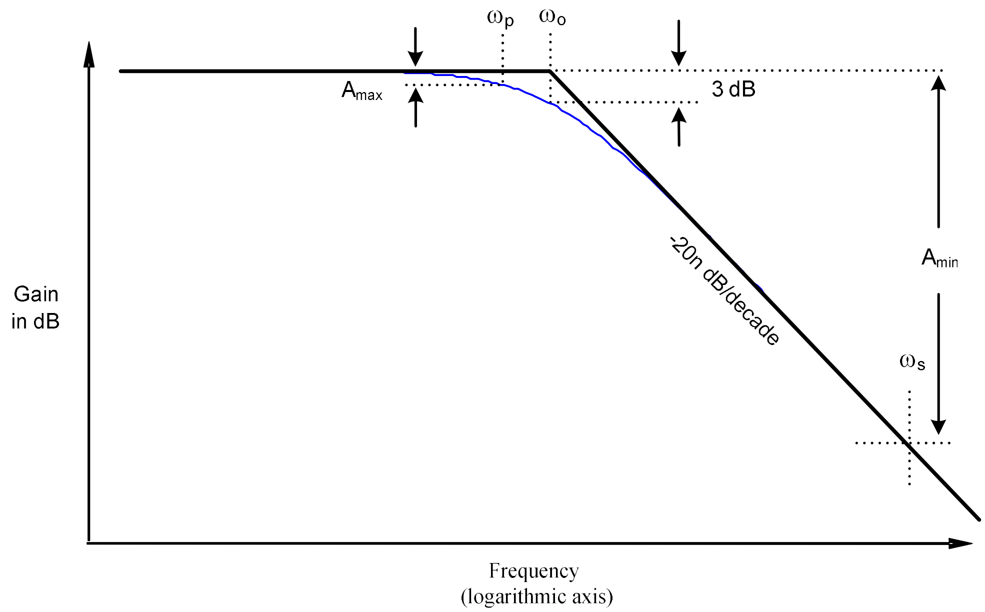

Figure 4.1. Butterworth low-pass magnitude response (n^{th} order).

As discussed in Chapter 1, A_{max} is the maximum allowable pass band attenuation in dB, A_{min} is the minimum acceptable stop band attenuation, ω_p is the pass band limit, and ω_s is the stop band limit. For our purposes we assume these specifications are given. Our tasks are to determine the minimum filter order necessary, to determine the poles of the filter's transfer function, and to design a circuit to satisfy these specifications.

Using equation 4.2, the *attenuation* in dB as a function of ω is

$$A(\omega) = -10\log_{10}\left(\left|H(j\omega)\right|^2\right)$$

$$= 10\log_{10}\left[1+\left(\omega/\omega_o\right)^{2n}\right] \tag{4.3}$$

Using the definitions of A_{max}, A_{min}, ω_p, and ω_s in equation 4.3 and Figure 4.1,

$$A_{max} = 10\log_{10}\left[1+\left(\omega_p/\omega_o\right)^{2n}\right] \tag{4.4}$$

and

$$A_{min} = 10\log_{10}\left[1+\left(\omega_s/\omega_o\right)^{2n}\right] \tag{4.5}$$

Solving 4.4 for ω_o ,

$$10^{A_{max}/10} - 1 = \left(\omega_p \big/ \omega_o \right)^{2n}$$

and

$$\boxed{\omega_o = \frac{\omega_p}{\left(10^{A_{max}/10} - 1\right)^{\frac{1}{2n}}}} \; . \tag{4.6}$$

Similarly for equation 4.5,

$$\boxed{\omega_o = \frac{\omega_s}{\left(10^{A_{min}/10} - 1\right)^{\frac{1}{2n}}}} \tag{4.7}$$

In solving for n we first set ω_o in equations 4.6 and 4.7 equal, so

$$\frac{\omega_p}{\left(10^{A_{max}/10} - 1\right)^{\frac{1}{2n}}} = \frac{\omega_s}{\left(10^{A_{min}/10} - 1\right)^{\frac{1}{2n}}} \; , \tag{4.8}$$

so

$$\frac{\omega_s}{\omega_p} = \frac{\left(10^{A_{min}/10} - 1\right)^{\frac{1}{2n}}}{\left(10^{A_{max}/10} - 1\right)^{\frac{1}{2n}}} \; . \tag{4.9}$$

Taking the logarithm of both sides,

$$\ln\left(\frac{\omega_s}{\omega_p}\right) = \left(\frac{1}{2n}\right) \ln\left(\frac{10^{A_{min}/10} - 1}{10^{A_{max}/10} - 1}\right) \tag{4.10}$$

or

$$\boxed{n = \frac{\ln\left(\dfrac{10^{A_{min}/10} - 1}{10^{A_{max}/10} - 1}\right)}{2\ln\left(\dfrac{\omega_s}{\omega_p}\right)}} \; . \tag{4.11}$$

Since n must be an integer number in solving for the minimum order, we must round up to the next highest integer from the value obtained in 4.11.

After determining the minimum order using equation 4.11, the next step is to use either equation 4.6 or 4.7 to find ω_o. As we will see in the next section, ω_o will be the undamped natural frequency of each section of our filter. When we round n up to the next highest integer and substitute back into these equations, each equation will give us a different answer for ω_o. A logical question is 'Which answer is correct?' The answer is that they both are, and so is any value in between! If we use 4.6, we force the attenuation at ω_p to be exactly A_{max} but the attenuation at ω_s will be slightly greater (better) than A_{min}. Conversely if we use 4.7, we force the attenuation at ω_s to be exactly A_{min} but the attenuation at ω_s will be slightly less (better) than A_{max}. If we choose a value in between we will exceed (do better than) the specifications by a slight amount at each limit. In other words, if we increase n up to the next integer our design is going to be better than our specifications somewhere. We can choose exactly where we want to match and where we want to "do better than" our specifications.

4.3 Butterworth Poles

Once we have found the minimum order and ω_o the next step is to find the poles of the transfer function. To solve for the Butterworth poles in the s plane we first write equation 4.2 in terms of s:

$$H(s)\,H(-s) = H(j\omega)\,H(-j\omega)\big|_{\omega=s/j}$$

$$= \frac{1}{1+\left(\dfrac{s}{j\omega_o}\right)^{2n}}$$

$$= \frac{1}{1+\left(\dfrac{1}{j}\right)^{2n}\left(\dfrac{s}{\omega_o}\right)^{2n}}$$

$$= \frac{1}{1+(-1)^n\left(\dfrac{s}{\omega_o}\right)^{2n}} \qquad (4.12)$$

We next set the denominator of 4.12 to zero and solve for s:

$$s^{2n} = \omega_o^{2n} \qquad \text{for n odd} \qquad (4.13)$$

and

$$s^{2n} = -\omega_o^{2n} \qquad \text{for n even.} \qquad (4.14)$$

In each case there are 2n solutions to the equation. For n an odd integer s equal to $\pm\omega_o$ are two solutions (and the complete solution for n equal to 1.) To determine the other roots, recall the following concept regarding complex numbers:

$$\left(Ae^{j\theta}\right)^k = A^k e^{j\theta k}$$

or equivalently

$$\left(Ae^{j\theta}\right)^{\frac{1}{k}} = A^{\frac{1}{k}} e^{\frac{j\theta}{k}} \ . \qquad (4.15)$$

Applying this to 4.13 for odd n and realizing that ω_o^{2n} is also $\omega_o^{2n}e^{\pm j2\pi}$, $\omega_o^{2n}e^{\pm j4\pi}$, ... , $\omega_o^{2n}e^{\pm j2k\pi}$ (since $e^{\pm j2k\pi} = 1$ for all integer k),

$$s = \left[\omega_o^{2n} e^{\pm j2k\pi}\right]^{\frac{1}{2n}} \qquad 0 \le k \le n \qquad (4.16)$$

$$= \omega_o e^{\pm jk\pi/n} \qquad 0 \le k \le n\,. \qquad (4.17)$$

Note the limits on k are chosen to give us 2n unique solutions. Other values are valid but would give redundant solutions.

From equation 4.17 we can see that the poles are evenly spaced at intervals of π/n radians on a circle of radius ω_o. Figures 4.2 and 4.3 are plots of these poles for n equal to 3 and 5 respectively.

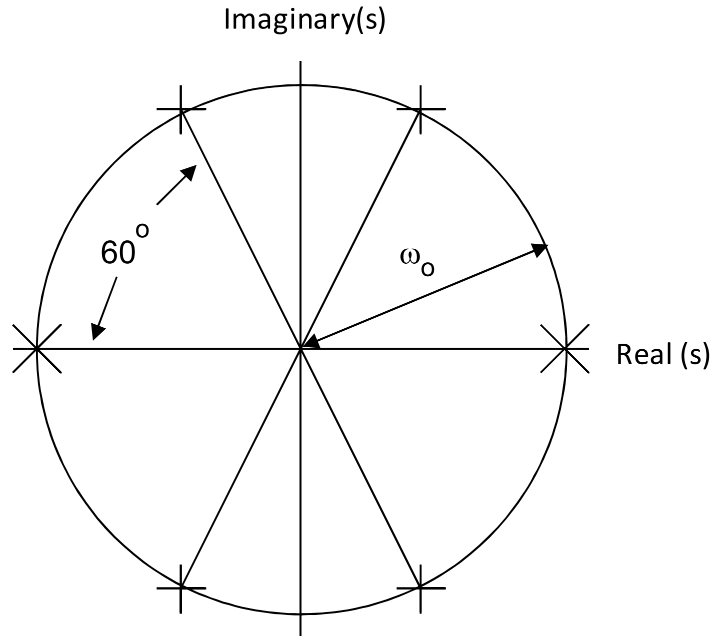

Figure 4.2. Butterworth poles of H(s)H(-s) for n=3.

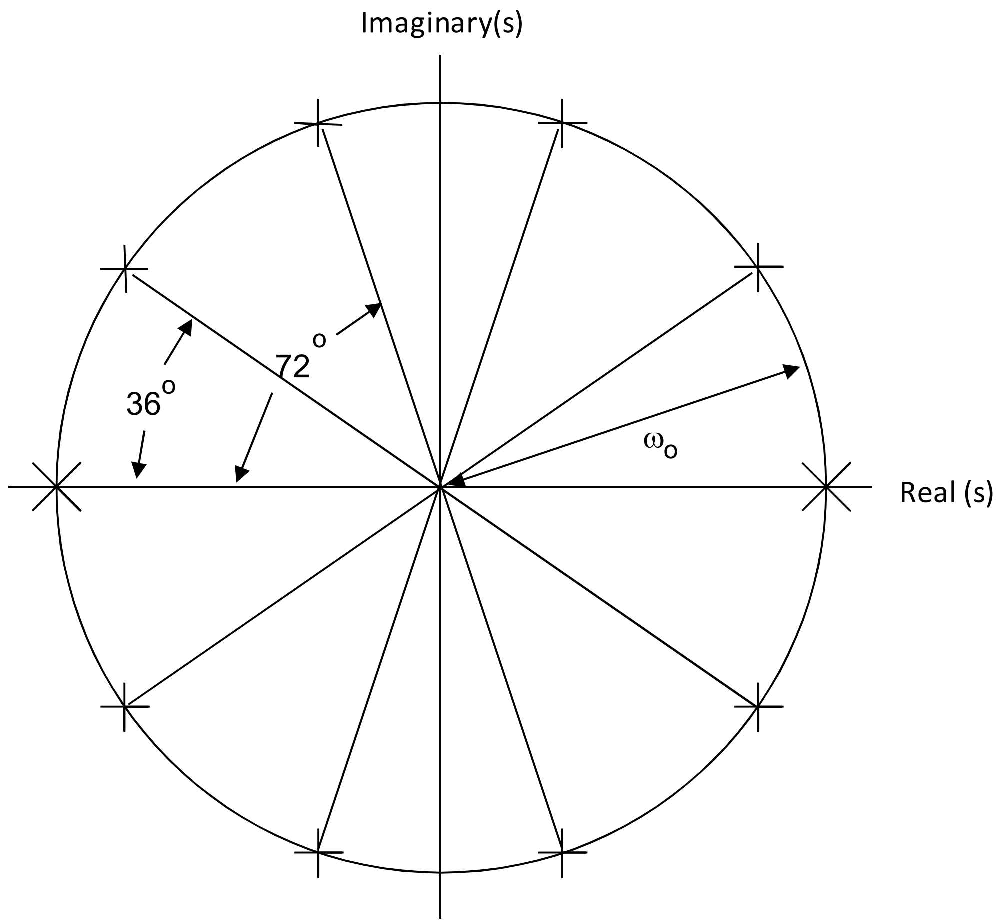

Figure 4.3. Butterworth poles of H(s)H(-s) for n=5.

In both cases there are poles at $\pm\ \omega_o$ and the remainder are spaced at $\pi/3$ radians or 60^o for n equal to 3 and $\pi/5$ or 36^o for n equal to 5.

Realize that the 2n poles in each case are the poles of H(s) H(-s). In order to obtain a stable filter we choose the stable poles (in the left half of the s plane) for the poles of H(s) and the right-half-plane poles for H(-s).

Using similar analysis for n even,

$$
\begin{aligned}
s^{2n} &= -\omega_o^{2n} \\
&= -\omega_o^{2n} e^{\pm j2k\pi} \\
&= \omega_o^{2n} e^{\pm j(2k+1)\pi}
\end{aligned}
$$

and

$$s = \omega_o e^{\pm j(2k+1)\pi/(2n)} \,. \qquad (4.18)$$

Just as for odd n, the poles are on circle of radius ω_o and are spaced at π/n radians. In this case however there are no real poles at $\pm\omega_o$. The poles closest to the positive and negative real axes in the s plane are at $\pm\ \pi/(2n)$ radians with respect to these axes. Figures 4.4 and 4.5 are plots of these poles for second and fourth order filters respectively.

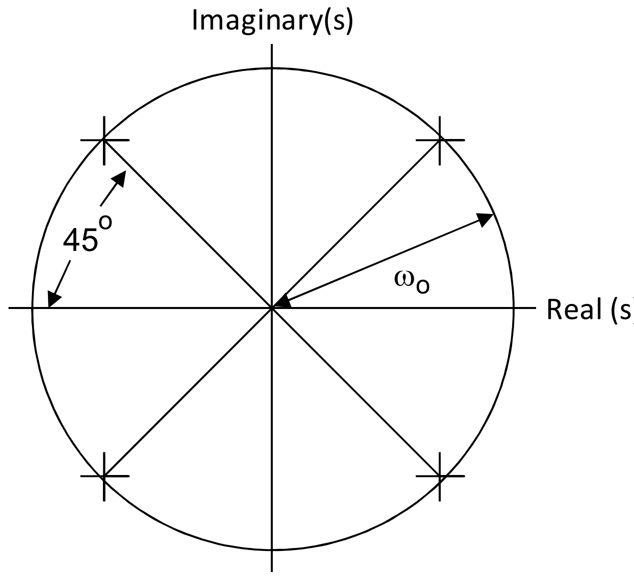

Figure 4.4. Butterworth poles of H(s)H(-s) for n=2.

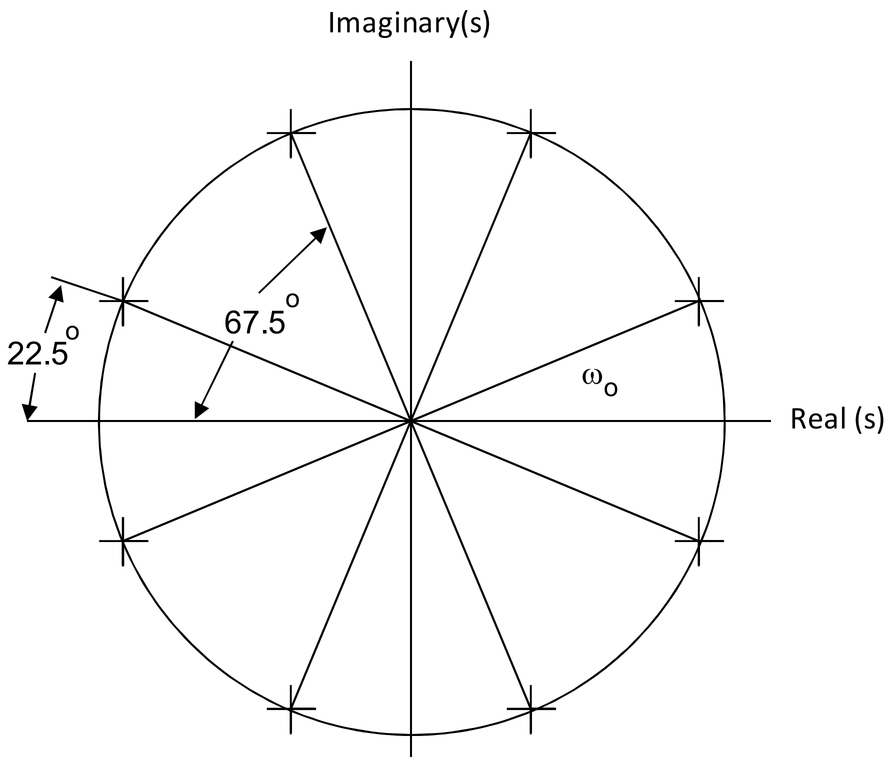

Figure 4.5. Butterworth poles of H(s)H(-s) for n=4.

Just as before the stable poles are those corresponding to H(s). For n equal to 2 the poles are separated by 90° and are at ±45° with respect to the negative real axis. For n equal to 4 the poles are separated by 45° and are at ±22.5° and ±67.5° with respect to the negative real axis.

In our designs it will be convenient to use the Q of each second order section as an entering argument to the design process. Recall from Chapter 2 the relationship between the Q of a second order transfer function and the angle of the poles (α) with respect to the negative real axis, namely

$$Q = \frac{1}{2\cos(\alpha)}. \tag{4.19}$$

Using 4.19 and the pole locations derived in (4.17) and (4.18) angles we can easily generate a table of Q's to use in our designs.

Table 4.1. Angles and Q's of Butterworth poles.

Order	Angles	Q	Angles	Q	Angles	Q	Angles	Q
1	0°	0.5	****	****	****	****	****	****
2	±45°	0.707	****	****	****	****	****	****
3	0°	0.5	±60°	1.0	****	****	****	****
4	±22.5°	0.541	±67.5°	1.306	****	****	****	****
5	0°	0.5	±36°	0.618	±72°	1.618	****	****
6	±15°	0.518	±45°	0.707	±75°	1.932	****	****
7	0°	0.5	±25.7°	0.555	±51.4°	0.802	±77.1°	2.247
8	±11.25°	0.510	±33.75°	0.601	±56.25°	0.900	±78.75°	2.563

At this stage we have presented methods to determine the minimum order and the pole locations for Butterworth low-pass filters. In the following section we will discuss how to implement these filters using op-amps and passive components. Later we will consider Butterworth high-pass, band-pass, and notch filters.

4.4 Sallen-Key Circuit

Although we did promise this to be a chapter on design and not analysis, we will need to do a little analysis to understand the circuits we will use in design. In general, our circuits will be designed as the cascade of second order sections (with an additional first order section for odd order filters.) Each second order section will be implemented with a single op-amp and will look exactly the same except for different values of resistance and capacitance. Therefore, all we need do is analyze one generic second order section to understand and design filters of arbitrary order and specifications. In this chapter we will consider low and high pass sections. In Chapter 6 we will discuss band-pass and notch sections.

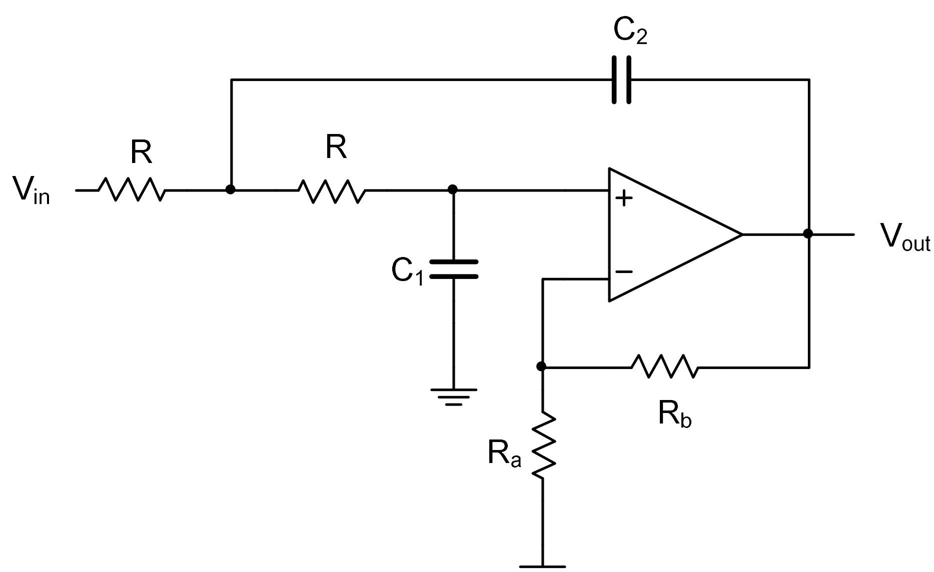

Figure 4.6. Second order low-pass filter section.

For the circuit in Figure 4.6, we first recall from Chapter 3 that from the non-inverting terminal to the output is merely a non-inverting amplifier with a DC gain of $1 + R_b/R_a$, represented by the block labeled A(s) in Figure 4.7 For design purposes we will assume an ideal op-amp and therefore will treat A(s) as a constant gain. Later we will take into account the finite gain bandwidth product of the op-amp and treat A(s) as a first order low-pass filter. In that case our second order transfer function will become a third order overall transfer function. In all cases we will assume infinite input and zero output impedance for the op-amp. Accounting for finite input and non-zero output impedances can easily be accomplished for specific cases by using circuit analysis programs, so we will not analyze these situations for general circuits. Besides, in almost all cases the finite gain bandwidth product is by far the most important (linear) deviation from the ideal.

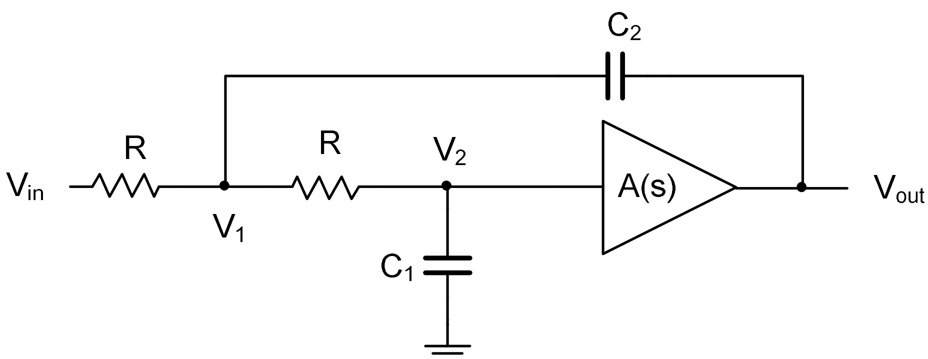

Figure 4.7. Simplified second order low-pass filter section.

To analyze the circuit in Figure 4.7 we will first write the node voltage equation at V_1 and the voltage divider equation relating V_2 and V_1.

$$\frac{V_1 - V_{in}}{R} + \frac{V_1 - V_2}{R} + (V_1 - V_{out})sC_2 = 0 \qquad (4.20)$$

and

$$V_2 = \left(\frac{1}{RC_1 s + 1}\right)V_1 \qquad (4.21)$$

or

$$V_1 = (RC_1 s + 1)V_2 \qquad (4.22)$$

Collecting terms in equation (4.20),

$$\left(\frac{2}{R} + sC_2\right)V_1 - \frac{V_2}{R} = \frac{V_{in}}{R} + sC_2 V_{out} \qquad (4.23)$$

Eliminating V_1 (by substituting 4.22 in 4.23),

$$\left[s^2 RC_1C_2 + (2C_1 + C_2)s + \frac{1}{R}\right]V_2 = \frac{V_{in}}{R} + sC_2 V_{out} \qquad (4.24)$$

Noting that

$$V_{out} = A(s)V_2 \tag{4.25}$$

or

$$V_2 = \frac{V_{out}}{A(s)}, \tag{4.26}$$

we use 4.26 in 4.24 to eliminate V_2 so that

$$\left[s^2RC_1C_2 + (2C_1 + C_2)s + \frac{1}{R}\right]\left(\frac{V_{out}}{A(s)}\right) - sC_2V_{out} = \frac{V_{in}}{R} \tag{4.27}$$

Multiplying both sides by A(s) and dividing each by RC_1C_2, (4.27) becomes

$$\left[s^2 + \left(\frac{2}{RC_2} + \frac{1}{RC_1} - \frac{A(s)}{RC_1}\right)s + \frac{1}{R^2C_1C_2}\right]V_{out} = A(s)\frac{V_{in}}{R^2C_1C_2}\ . \tag{4.28}$$

Now we define ω_o as

$$\begin{aligned}\omega_o &= \frac{1}{\sqrt{R^2C_1C_2}} \\ &= \frac{1}{RC_{eq}}\end{aligned} \tag{4.29}$$

where C_{eq} is the geometric mean of C_1 and C_2. Substituting this into equation 4.28 the transfer function becomes

$$\frac{V_{out}}{V_{in}} = \frac{A(s)\omega_o^2}{s^2 + \left[\frac{2}{RC_2} + \frac{1}{RC_1} - \frac{A(s)}{RC_1}\right]s + \omega_o^2}\ . \tag{4.30}$$

<u>Case 1.</u>

First consider the case when C_1 and C_2 are equal (to C) and A(s) is a constant (equal to $1 + R_b/R_a = A$). In this case

$$\frac{V_{out}}{V_{in}} = \frac{A\omega_o^2}{s^2 + (3 - A)\omega_o s + \omega_o^2} \tag{4.31}$$

This represents a general second order low pass filter of the form

$$\frac{V_{out}}{V_{in}} = \frac{A\omega_o^2}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2}$$

with a DC gain of A (>1) and

$$Q = \frac{1}{3 - A} \ ,$$

and since

$$A = 1 + \frac{R_b}{R_a} \ ,$$

$$Q = \frac{1}{2 - \frac{R_b}{R_a}} \ . \tag{4.32}$$

Once one has determined Q and ω_o for each section, one simply selects R and C such that

$$RC = \frac{1}{\omega_o} \ , \tag{4.33}$$

and R_b and R_a such that

$$\frac{R_b}{R_a} = 2 - \frac{1}{Q} \ . \tag{4.34}$$

<u>Case 2.</u>

If the op-amp is connected as a voltage follower vice a non-inverting amplifier as in Figure 4.8, A is now equal to unity (in equation 4.30) and the transfer function becomes

$$\frac{V_{out}}{V_{in}} = \frac{\omega_o^2}{s^2 + \left(\frac{2}{RC_2}\right)s + \omega_o^2} \ , \tag{4.35}$$

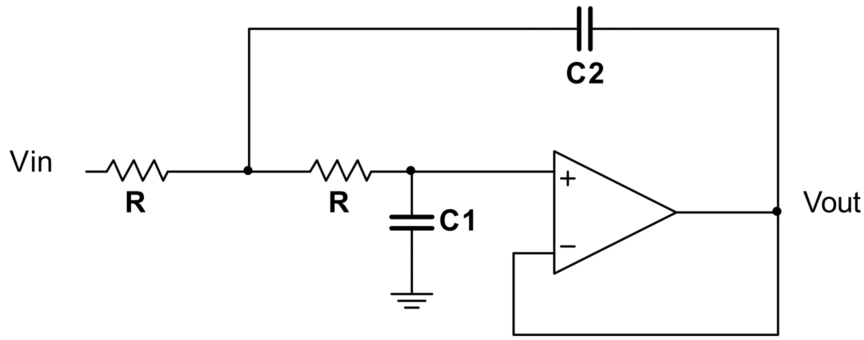

Figure 4.8. Unity gain voltage follower configuration.

or a second order low-pass filter with a DC gain of unity. Equating coefficients of s in the denominators of 4.35 and the general low-pass filter,

$$\frac{2}{RC_2} = \frac{\omega_o}{Q} \tag{4.36}$$

or

$$Q = \frac{\omega_o RC_2}{2} \ .$$

Furthermore since

$$\omega_o = \frac{1}{\sqrt{R^2 C_1 C_2}}$$

$$Q = \frac{RC_2}{2R\sqrt{C_1 C_2}}$$

$$= \frac{\sqrt{C_2 / C_1}}{2} \ , \tag{4.37}$$

and

$$\frac{C_2}{C_1} = 4Q^2 \tag{4.38}$$

To design using this circuit after Q and ω_o of each section have been determined one simply selects R and C_{eq} such that

$$RC_{eq} = \frac{1}{\omega_o} \ . \tag{4.39}$$

Since C_2 and C_1 have a ratio of $4Q^2$ and a geometric mean of C_{eq} they are given by

$$C_2 = 2QC_{eq} \tag{4.40}$$

and

$$C_1 = \frac{C_{eq}}{2Q} \; . \tag{4.41}$$

Given a choice of these two forms of second order low-pass filters, which is an appropriate choice for a given application? As we will see later in this chapter, the frequency response of the circuit in Figure 4.8 is less sensitive to variations in passive component values and to the finite gain bandwidth of the op-amps, particularly for sections with large Q. It also has two fewer resistors per stage. It has a DC gain of 0 dB while the circuit in Figure 4.6 has a DC gain of greater than 0 dB. Obviously, if one desires an overall DC gain of 0 dB, the circuit in Figure 4.8 is much simpler to design. If gain in addition to filtering is desired, the circuit in Figure 4.6 may be appropriate. One advantage of the circuit in Figure 4.6 with both capacitors the same size, we may find it easier to find the passive components to implement the filter. Generally standard resistors are available with smaller steps between adjacent values compared to standard capacitors. Using the circuit in 4.6 allows us to pick a standard capacitance value for both capacitors, calculate the resistances, and choose the standard values closest to our calculations.

4.5 Low-pass Design Examples

Example 4.1

Design a Butterworth low-pass filter to satisfy the following specifications.

DC gain	=	0 dB
A_{max}	=	2 dB
A_{min}	=	20 dB
f_p	=	5 kHz
f_s	=	10 kHz

Using equation 4.11 the minimum order is 4. Therefore the poles are at angles of ±22.5^o and ±67.5^o with respect to the negative real axis in the s plane (as in Figure 4.5) or at Q's of 0.541 and 1.306 (see Table 4.1). Using equation 4.6 to calculate the undamped natural frequency of each section as a function of A_{max} and ω_p

$$\omega_o = \frac{2\pi * 5000}{(10^{2/10} - 1)^{1/8}}$$

$$= 3.36 \text{ x } 10^4 \text{ rad/sec .}$$

In this case we have forced the attenuation at 5 kHz to be exactly 2 dB but the attenuation at 10 kHz (or $2\pi * 10^4$ rad/sec) using equation 4.3 is

$$A(2\pi * 10^4) = 10\log_{10}\left[1 + \left(\frac{2\pi * 10^4}{3.36 * 10^4}\right)^8\right]$$

$$= 21.8 \text{ dB ,}$$

or slightly in excess of A_{min}.

Had we instead calculated ω_o as a function of A_{min} and ω_s (equation 4.7):

$$\omega_o = \frac{2\pi * 10000}{(10^{20/10} - 1)^{1/8}}$$

$$= 3.54 \text{ x } 10^4 \text{ rad/sec}$$

Now the actual attenuation at 10 kHz is exactly A_{min} and the attenuation at 5 kHz is less (or better) than A_{max}.

$$A(2\pi * 5000) = 10\log_{10}\left[1 + \left(\frac{2\pi * 5000}{3.54 * 10^4}\right)^8\right]$$

$$= 1.41 \text{ dB .}$$

As you can see we can meet the specifications at either limiting frequency, and perform better than specifications at the other, or we could choose any ω_o between 3.36 x 10^4 rad/sec and 3.54 x 10^4 rad/sec and exceed (perform better than) the specifications at both limiting frequencies. Figure 4.9 illustrates the effects of choosing either of these two frequencies.

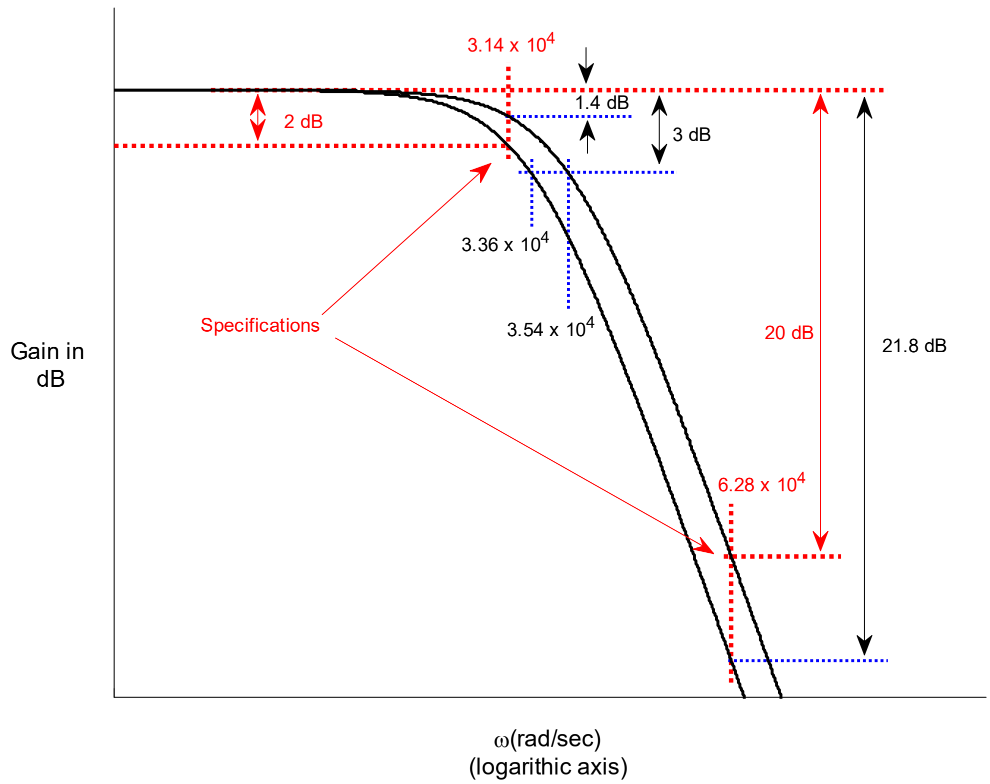

Figure 4.9. Example 4.1 diagram.

The circuit in Figure 4.8 will be the simplest solution. If we choose ω_o equal to 3.36 x 10^4 rad/sec and 1 kΩ for each resistor in our circuit, from equation 4.39 we calculate the equivalent capacitance of each section as

$$C_{eq} = \frac{1}{3.36 \times 10^4 * 1 \times 10^3} \text{ F}$$
$$= 29.8 \text{ nF} .$$

The individual capacitors are determined using equations 4.40 and 4.41. In the first second order section (Note: C_{mn} indicates the n^{th} capacitor in the m^{th} section.),

$$C_{11} = \frac{C_{eq}}{2Q_1}$$
$$= 27.5 \text{ nF}$$

and

$$C_{12} = 2C_{eq}Q_1 = 32.2 \text{ nF} .$$

In the second section,

$$C_{21} = \frac{C_{eq}}{2Q_2} = 11.5 \text{ nF}$$

and

$$C_{22} = 2C_{eq}Q_2 = 77.5 \text{ nF} .$$

The circuit is shown in Figure 4.10.

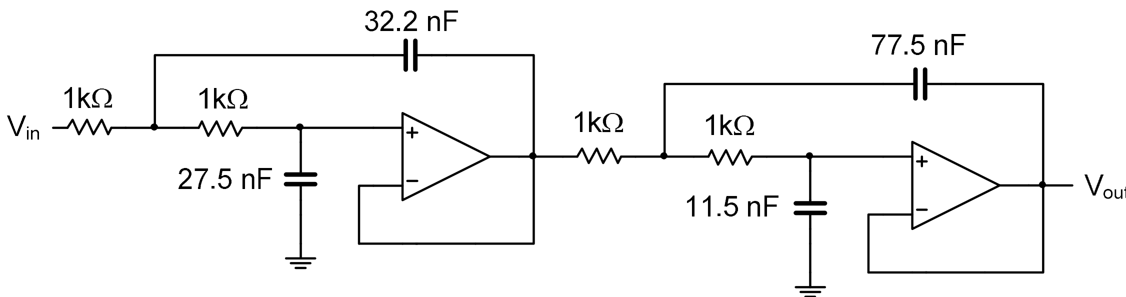

Figure 4.10. Final design for Example 4.1.

Example 4.2

Design a Butterworth low-pass filter to satisfy the following specifications.

DC gain	=	20 dB
A_{max}	=	1 dB
A_{min}	=	30 dB
f_p	=	2 kHz
f_s	=	10 kHz

Using equation 4.11 the minimum order is 3. Therefore the poles are at angles of 0^o and $\pm 60^o$ (see Figure 4.2) with respect to the negative real axis in the s plane or at Q's of 0.5 and 1 (see Table 4.1). Using equation 4.6 the undamped natural frequency of each section is

$$\omega_o = \frac{2\pi * 2000}{(10^{1/10} - 1)^{1/6}}$$
$$= 1.57 \text{ x } 10^4 \text{ rad/sec} .$$

Since we need some overall gain higher than 0 dB from the circuit we will choose the circuit in Figure 4.6. Now we will choose all our capacitors to be of some convenient value, for example 10 nF, and calculate all resistors. For the second order section from equation 4.33,

$$R = \frac{1}{1.57 \text{ x } 10^4 \text{ rad/sec} * 10\,\text{nF}}$$
$$= 6.37 \text{ k}\Omega .$$

For a Q of 1 using equation 4.34, the second order section should have

$$\frac{R_b}{R_a} = 2 - \frac{1}{Q}$$
$$= 1$$

and a DC gain of

$$A = 1 + \frac{R_b}{R_a}$$
$$= 2 .$$

To obtain an overall gain of 20 dB (a linear gain of 10) the first order section should therefore have a DC gain of 5. The circuit is shown in Figure 4.11.

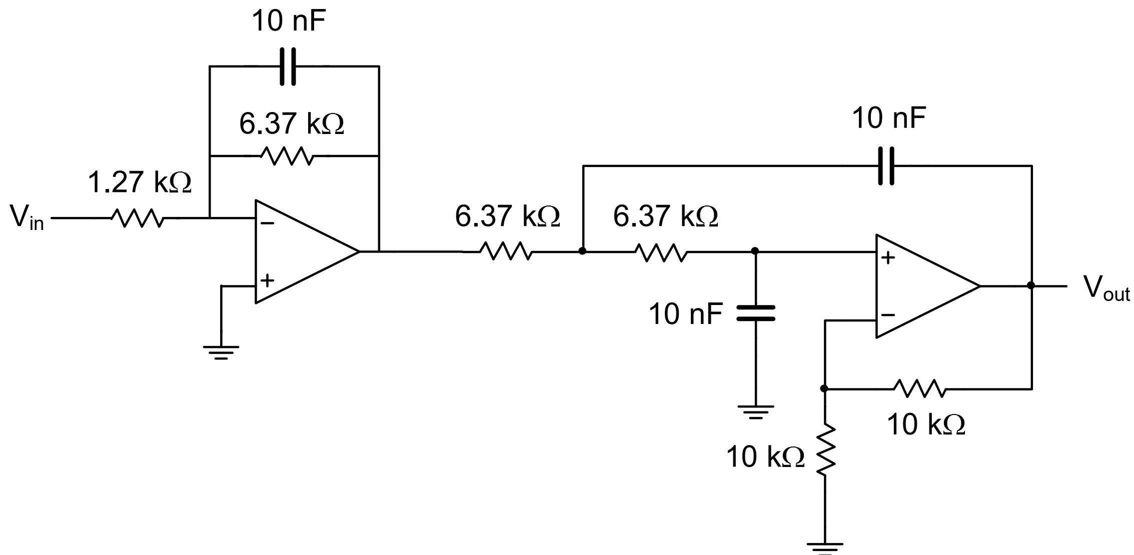

Figure 4.11. Final design for Example 4.2.

4.6 Butterworth High-pass Filters

The easiest way to consider the Butterworth high-pass response is to take the low-pass response plotted in Figure 4.1 and flip the response horizontally on the logarithmic frequency scale. Figure 4.12 illustrates the Butterworth high-pass response that results from this rotation.

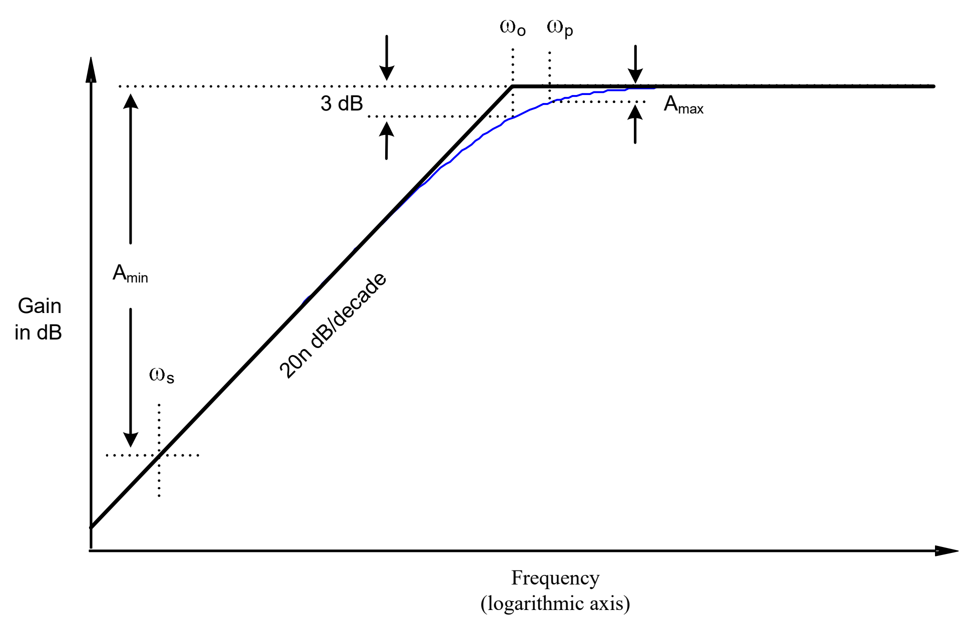

Figure 4.12. Illustration of high-pass Butterworth response (n^{th} order).

While rotation about any frequency would have produced a high-pass response, we will choose to rotate about ω_o so that exactly as in the low-pass case, ω_o remains the half power (-3 dB) frequency and the pole frequency. To accomplish this rotation, in the Butterworth low-pass magnitude response presented in section 4.2,

$$|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_o}\right)^{2n}}\ , \tag{4.2}$$

we substitute ω_o/ω for ω/ω_o, and the resulting magnitude response becomes

$$|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega_o}{\omega}\right)^{2n}} \tag{4.42}$$

$$= \frac{\left(\frac{\omega}{\omega_o}\right)^{2n}}{1+\left(\frac{\omega}{\omega_o}\right)^{2n}}\ . \tag{4.43}$$

Note that the low-pass magnitude response at $2\omega_o$ becomes the high-pass response at $\omega_o/2$, etc.

Since the denominator of the high-pass response in equation 4.43 is identical to 4.2 the high-pass poles are the same as the low-pass poles. By substituting s/j for ω in the numerator we find there are n zeros at the origin in the s plane. Using equation 4.42, the attenuation in dB as a function of ω is

$$A(\omega) \ = \ = 10\log_{10}\left[1+\left({\omega_o}/{\omega}\right)^{2n}\right] \tag{4.44}$$

Using the definitions of A_{max}, A_{min}, ω_p, and ω_s in equation 4.44 and following identical steps as in equations 4.4 to 4.11 in the low-pass case we observe

$$\boxed{\omega_o = \omega_p\left(10^{A_{max}/10}-1\right)^{\frac{1}{2n}}} \tag{4.45}$$

or

$$\boxed{\omega_o = \omega_s\left(10^{A_{min}/10}-1\right)^{\frac{1}{2n}}} \tag{4.46}$$

and

$$\boxed{n = \frac{\ln\left(\frac{10^{A_{min}/10}-1}{10^{A_{max}/10}-1}\right)}{2\ln\left(\frac{\omega_p}{\omega_s}\right)}}\ . \tag{4.47}$$

To implement the high-pass transfer function, we can use circuits very similar to those shown in Figures 4.6 and 4.8. With the exception of R_a and R_b in the feedback path we simply replace resistors with capacitors and vice versa as shown in Figure 4.13.

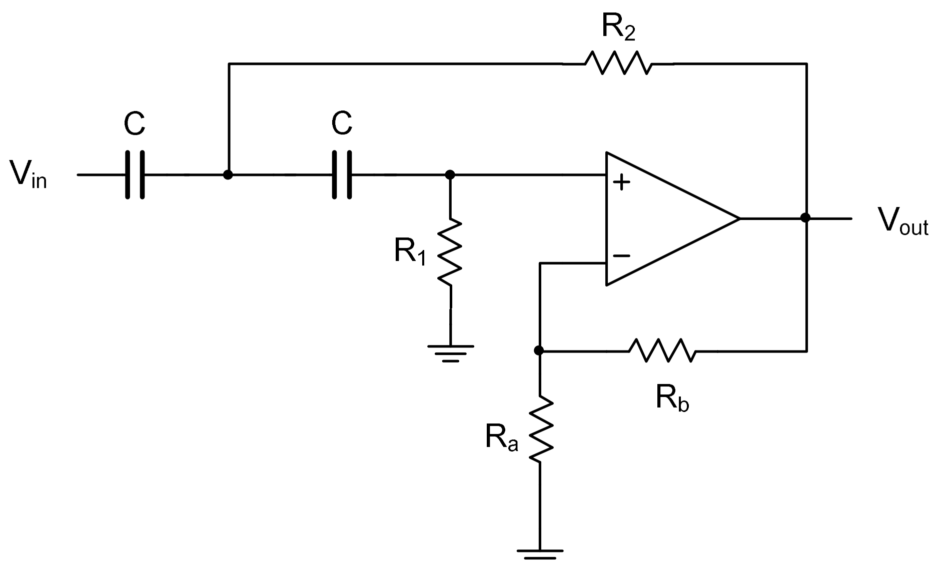

Figure 4.13. High-pass Sallen-Key circuit.

For analysis purposes we will redraw (Figure 4.14) the circuit shown in 4.13 as we did in the low-pass case by treating the non-inverting amplifier with a DC gain of $1 + R_b/R_a$ as a block labeled A(s).

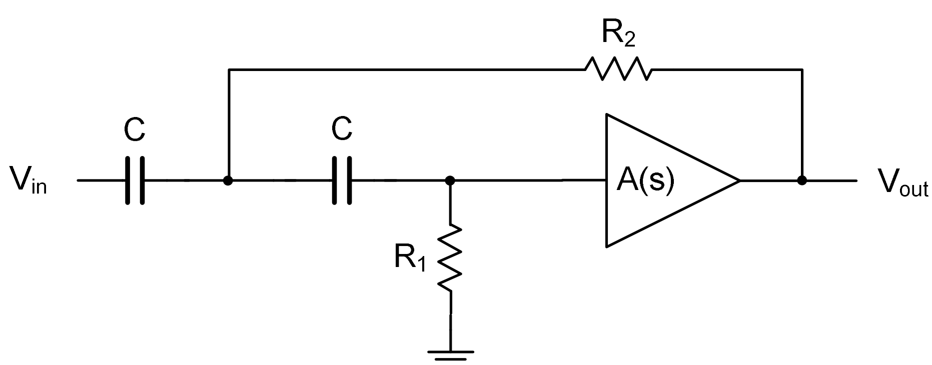

Figure 4.14. Simplified version of Figure 4.13.

The analysis of the circuit in Figure 4.14 is exactly the same as that of the circuit in Figure 4.7 with the following impedance substitutions:

<u>Low-pass</u> becomes <u>High-pass</u>

$$R \rightarrow \frac{1}{sC}$$

$$\frac{1}{sC_1} \rightarrow R_1$$

and

$$\frac{1}{sC_2} \rightarrow R_2 .$$

Making these substitutions in equation 4.27 (repeated below for convenience),

$$\left[s^2RC_1C_2 + (2C_1 + C_2)s + \frac{1}{R}\right]\left(\frac{V_{out}}{A(s)}\right) - sC_2V_{out} = \frac{V_{in}}{R} \quad . \tag{4.27}$$

becomes

$$\left[\frac{1}{sCR_1R_2} + \frac{2}{R_1} + \frac{1}{R_2} + sC\right]\left(\frac{V_{out}}{A(s)}\right) - \frac{1}{R_2}V_{out} = sCV_{in} \quad . \tag{4.48}$$

In the same manner as for the low-pass case where we defined ω_o in equation 4.29, here we define

$$\omega_o = \frac{1}{\sqrt{C^2R_1R_2}}$$

or

$$\omega_o = \frac{1}{CR_{eq}} \quad , \tag{4.49}$$

where R_{eq} is the geometric mean of R_1 and R_2. Substituting this into equation 4.48 and multiplying both sides of the equation by s A(s)/C, the transfer function is

$$\frac{V_{out}}{V_{in}} = \frac{A(s)s^2}{s^2 + \left[\frac{2}{CR_1} + \frac{1}{CR_2} - \frac{A(s)}{CR_2}\right]s + \omega_o^2} \quad . \tag{4.50}$$

High-pass Case 1.

First consider the case when R_1 and R_2 are equal (to R) and A(s) is a constant (equal to 1 + R_b/R_a = A). Our transfer function is

$$\frac{V_{out}}{V_{in}} = \frac{As^2}{s^2 + (3-A)\omega_o s + \omega_o^2} \tag{4.51}$$

or a general second order high-pass filter of the form

$$\frac{V_{out}}{V_{in}} = \frac{As^2}{s^2 + \frac{\omega_o}{Q} s + \omega_o^2}$$

with a high frequency gain of A (>1) . The equations for Q are the same as for the low-pass case in equation 4.32, namely

$$Q = \frac{1}{3 - A}$$

$$= \frac{1}{2 - \frac{R_b}{R_a}} \ . \tag{4.52}$$

To design filters using this circuit once one has determined Q and ω_o for each section, one simply selects R and C (from equation 4.49) such that

$$RC = \frac{1}{\omega_o} \ , \tag{4.53}$$

and R_b and R_a such that

$$\frac{R_b}{R_a} = 2 - \frac{1}{Q} \ . \tag{4.54}$$

High-pass Case 2.

If the op-amp is instead connected as a voltage follower as shown in Figure 4.15, A is now equal to unity and the transfer function from equation 4.50 simplifies to

$$\frac{V_{out}}{V_{in}} = \frac{s^2}{s^2 + \left(\frac{2}{CR_1}\right)s + \omega_o^2} \ . \tag{4.55}$$

The design proceeds as follows:

1. Choose C and R_{eq} such that $(C\,R_{eq}) = \frac{1}{\omega_o}$.
2. Since

$$Q = \frac{1}{2}\sqrt{\frac{R_1}{R_2}} \text{ , this implies}$$

$$\frac{R_1}{R_2} = 4Q^2 \ ,$$

or

$$R_1 = 2QR_{eq}$$

and

$$R_2 = \frac{R_{eq}}{2Q} \ .$$

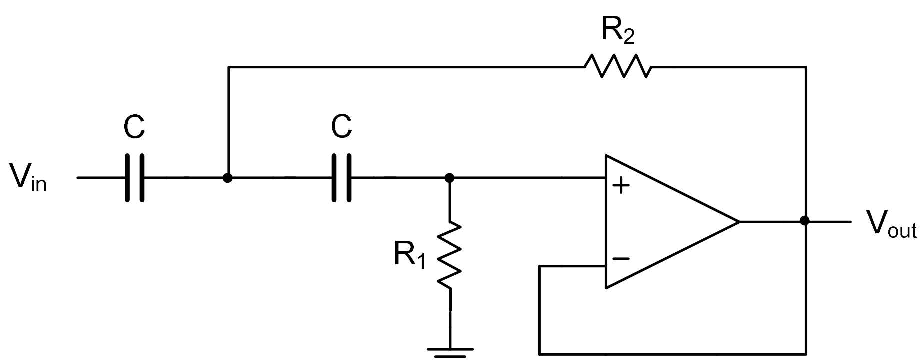

Figure 4.15. Unity gain high-pass circuit.

Just as in the low-pass case we have a choice of circuits to use in our design. Virtually the same advantages as in the low-pass case apply except that now in both circuits all capacitors have the same value so that availability of a particular value is not an issue.

Example 4.3

Design a Butterworth high-pass filter to satisfy the following specifications:

High frequency gain	=	0 dB
A_{max}	=	0.5 dB
A_{min}	=	20 dB
f_p	=	3 kHz
f_s	=	1 kHz

Using equation 4.47 the minimum order is 4. Therefore the poles are at angles of ±22.5° and ±67.5° with respect to the negative real axis in the s plane or at Q's of 0.541 and 1.306. Using equation 4.45 to calculate the undamped natural frequency of each section as a function of A_{max} and ω_p,

$$\begin{aligned}\omega_o &= 2\pi * 3000(10^{0.5/10} - 1)^{1/8} \\ &= 1.45 \text{ x } 10^4 \text{ rad/sec .}\end{aligned}$$

If we choose 10 nF for each capacitor in the two sections of the circuit of Figure 4.15,

$$\begin{aligned}R_{eq} &= \frac{1}{1.45 \text{ x } 10^4 \text{ rad/sec} * 10 \text{ nF}} \\ &= 6.9 \text{ k}\Omega \text{ .}\end{aligned}$$

As in previous low-pass examples, we use the notation R_{ij} to indicate the j^{th} resistor in the i^{th} stage. For the first section with Q_1 equal to 0.541,

$$\begin{aligned}R_{11} &= 2Q_1 R_{eq} \\ &= 7.45 \text{ k}\Omega\end{aligned}$$

and

$$\begin{aligned}R_{12} &= \frac{R_{eq}}{2Q_1} \\ &= 6.39 \text{ k}\Omega\end{aligned}$$

For the second section (Q_2 equal to 1.306),

$$\begin{aligned}R_{21} &= 2Q_2 R_{eq} \\ &= 18.0 \text{ k}\Omega\end{aligned}$$

and

$$\begin{aligned}R_{22} &= \frac{R_{eq}}{2Q_2} \\ &= 2.64 \text{ k}\Omega \text{ .}\end{aligned}$$

Figure 4.16 shows the complete circuit for this example.

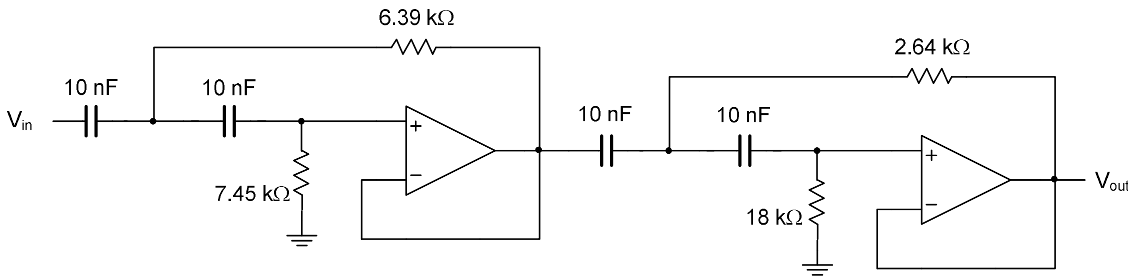

Figure 4.16. Final design for Example 4.3.

4.7 Effects of Using Real Components

In our designs to this point we have assumed the op-amps used are ideal, and that the components are exactly the values we specified. As we saw in Chapter 3, the gain of typical op-amps falls off inversely with frequency, or the product of gain and frequency is a constant. This constant is often referred to as the gain bandwidth product. Intuitively this suggests that if the gain bandwidth product of the op-amp is much larger than any frequencies of interest in the design, the assumption of large (although not infinite) gain is valid and our design will come close to specifications. While this is certainly true, precisely how large a gain bandwidth product is required for a particular design? For a low-pass filter, is 10 times the pass band limit enough? 100 times? Are there ways (other than specifying a wider bandwidth op-amp) to modify the design so that it meets specifications? Which of the two general forms for second order low-pass filters is less susceptible to the effects of finite gain bandwidth product op-amps?

To illustrate these effects we will first do a design example and then analyze the effects of a finite gain bandwidth product. Later we will also consider slew rate limitations and consider what happens if resistors and capacitors are not at the exact values we specify.

Example 4.4

Design a Butterworth low-pass filter to satisfy the following specifications.

DC gain	=	0 dB
A_{max}	=	1 dB
A_{min}	=	10 dB
f_p	=	400 kHz
f_s	=	800 kHz

Following the same procedures as in the examples above, the minimum order is 3. The poles are at 0^o and $\pm 60^o$ or Q's of 0.5 and 1.

$$\omega_o = \frac{2\pi * 4x10^5}{(10^{1/10} - 1)^{1/6}}$$
$$= 3.15 \text{ x } 10^6 \text{ rad/sec} .$$

In order to compare the two general forms for low-pass filter circuits, we will design using both circuits. In both cases we will choose 1 kΩ for resistor values. Therefore

$$C_{eq} = \frac{1}{R\omega_o}$$

$$= \frac{1}{1\text{k}\Omega * 3.15\,\text{x}\,10^6\,\text{rad/sec}}$$

$$= 318\ \text{pF}\ .$$

The two candidate designs are shown in Figures 4.17 and 4.18.

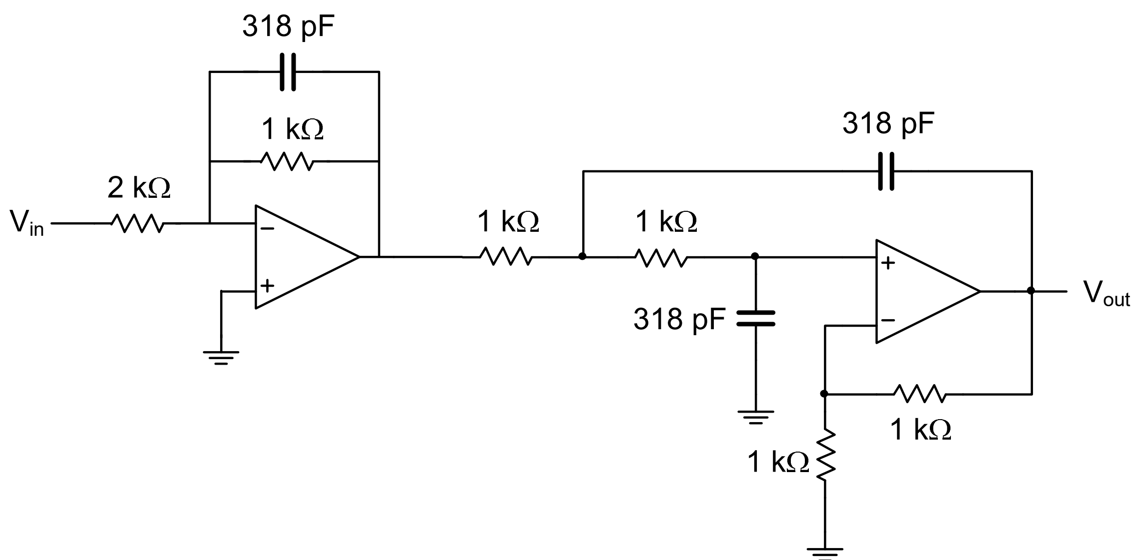

Figure 4.17. Candidate design 1 for Example 4.4.

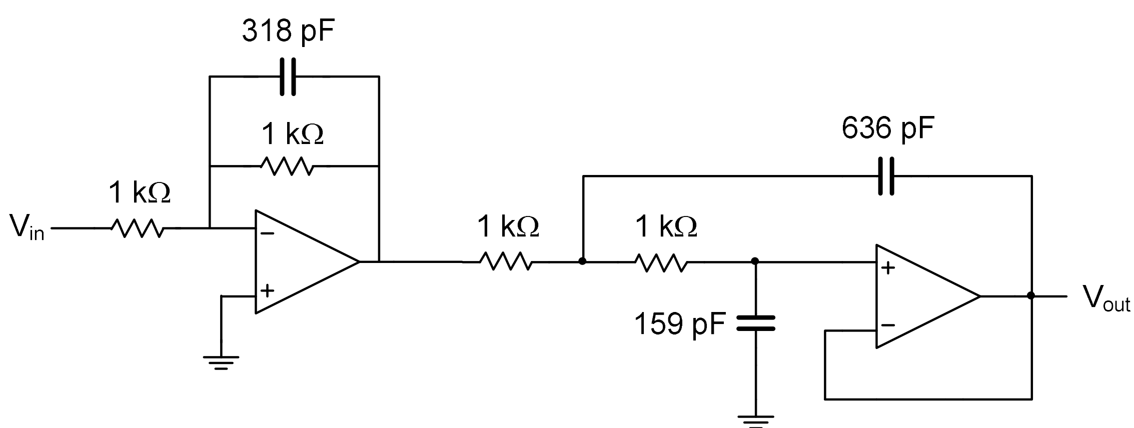

Figure 4.18. Candidate design 2 for Example 4.4.

The simplest method to analyze the effects of finite gain bandwidth product is to use an AC circuit analysis program. Figure 4.19 shows the result of analyzing the circuit in Figure 4.17 with op-amps of 1 MHz, 3 MHz and 15 MHz gain bandwidth products. At 1 MHz the gain at 400 kHz is less than -9 dB and is not even close to meeting specifications. With 3 MHz the gain is closer to the desired specification at 400 kHz, but it peaks at +0.8 dB and therefore is no longer a true Butterworth (maximally flat) response. At 15 MHz the response is still not quite perfect but would be considered close enough in most cases. Note that with a gain bandwidth product of 15 MHz the gain of the op-amp is 30 at the resonant frequency (500 kHz) of the second order section.

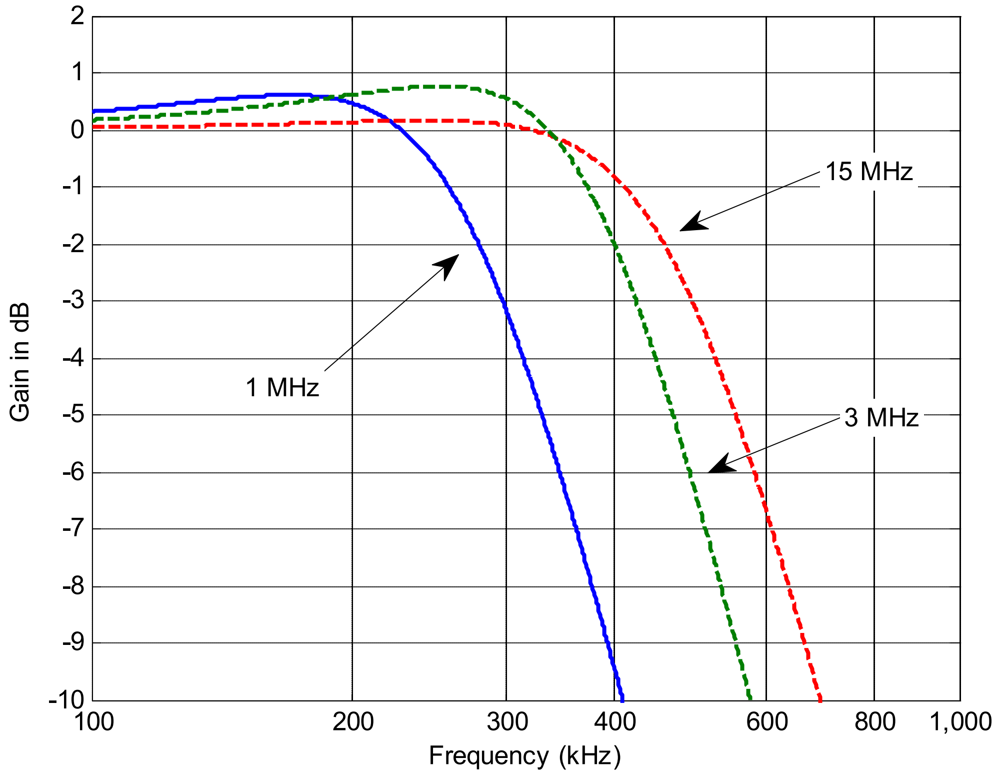

Figure 4.19. Magnitude response for design 1 (circuit from Figure 4.17) with op-amps of 1 MHz, 3 MHz and 15 MHz gain bandwidth products.

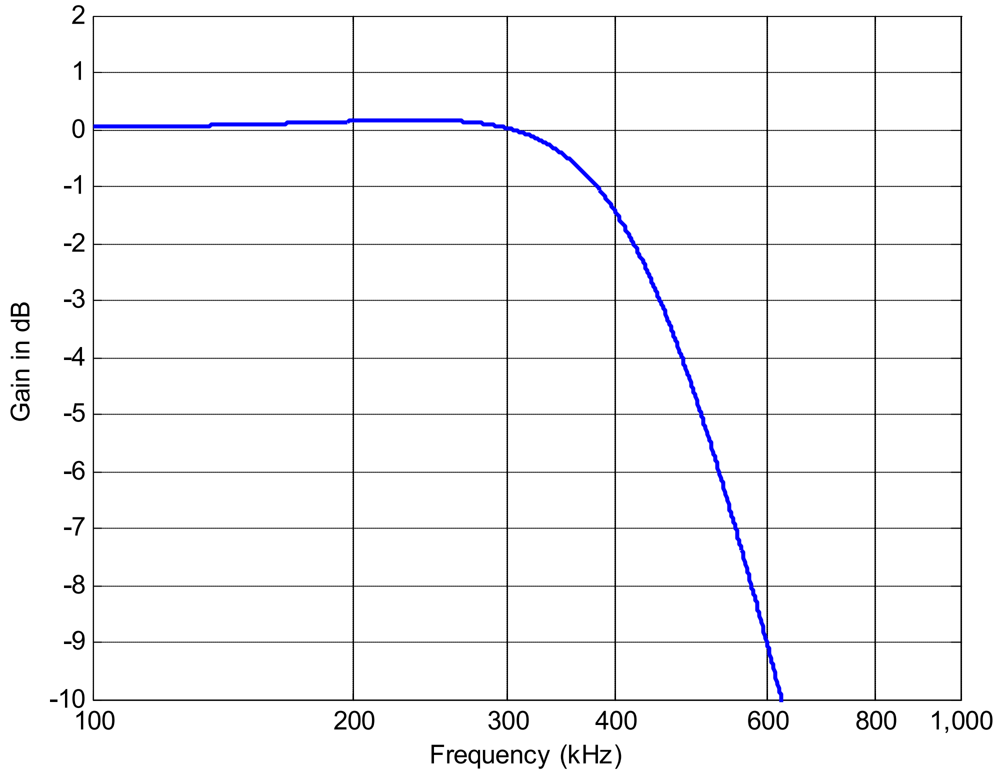

Figure 4.20. Magnitude response for design 2 (circuit from Figure 4.18) with 3MHz gain bandwidth product op-amp

In comparison, Figure 4.20 shows the frequency response of the circuit in Figure 4.18 when the op-amps have a gain bandwidth product of 3 MHz. While not perfect, it certainly is much closer to specifications than the circuit in Figure 4.17 using the same op-amps. To explain the reasons why these responses look like they do and why one circuit has better high frequency response, we will return to the analysis of the Sallen Key circuit in section 4.4. Now A(s) will be a first order low-pass filter vice a constant gain. In order to keep the algebra as simple as possible, we will normalize the resonant frequency (ω_o) to unity and define the variable (G) as the gain bandwidth product of the op-amp <u>relative to the designed</u> resonant frequency (ω_o). As we will see shortly, the <u>actual</u> resonant frequency and Q achieved will vary from those for which we have designed due to the finite gain bandwidth product of the op-amp.

Starting with equation 4.30 derived earlier,

$$\frac{V_{out}}{V_{in}} = \frac{A(s)\omega_o^2}{s^2 + \left[\frac{2}{RC_2} + \frac{1}{RC_1} - \frac{A(s)}{RC_1}\right]s + \omega_o^2} \quad . \tag{4.30}$$

Case 1.

In this case $C_1 = C_2 = C$
and let A_o be the closed-loop DC gain, or

$$A_o = 1 + \frac{R_b}{R_a} \tag{4.56}$$

$$= 3 - \frac{1}{Q} \ . \tag{4.57}$$

As shown in Chapter 3, A(s) is a low-pass filter with a DC gain of A_o and a bandwidth (relative to ω_o) of G/A_o or

$$A(s) = \frac{G}{s + \frac{G}{A_o}} \ . \tag{4.58}$$

Substituting 4.58 into 4.30 and letting ω_o equal one,

$$\frac{V_{out}}{V_{in}} = \frac{\dfrac{G}{s + \frac{G}{A_o}}}{s^2 + \left(3 - \dfrac{G}{s + \frac{G}{A_o}}\right)s + 1}$$

$$= \frac{G}{s^2\left(s + \frac{G}{A_o}\right) + 3s\left(s + \frac{G}{A_o}\right) - Gs + s + \frac{G}{A_o}}$$

$$= \frac{G}{s^3 + 3s^2 + s + \frac{G}{A_o}\left(s^2 + (3 - A_o)s + 1\right)}$$

$$= \frac{G}{s^3 + 3s^2 + s + \frac{G}{A_o}\left(s^2 + \frac{s}{Q} + 1\right)} \ . \tag{4.59}$$

By taking into account the first order dynamics of the op-amp, our original second order transfer function becomes a third order low-pass filter transfer function. Analysis of the frequency response of 4.59 can be accomplished by considering pole locations. The three poles consist of a complex conjugate pair (for all practical circuits) and a single real pole. Figure 4.21 shows a plot of the complex poles (in the second quadrant) as a function of the gain bandwidth product relative to ω_o (G) and the angle for which the poles were designed. Since the product of the three poles is the negative of the coefficient of

s^o, or G/A_o, the real pole is on the negative real axis at

$$s = -\frac{G}{A_o r^2}$$

where r is the radius of the complex conjugate pair. As expected as G tends to infinity the poles are positioned exactly as they were designed. As G decreases, the poles move to lower frequencies and higher Q's. In the example above the second order section was designed to have a resonant frequency of 500 kHz and the poles were to be at $\pm 60^o$. The actual poles are therefore at the intersection of the 60^o curve and the G equal to 2, 6, and 30 curves for gain bandwidth products of 1 MHz, 3 MHz, and 15 MHz respectively. Table 4.2 summarizes the locations of these poles.

Table 4.2. Summary of actual pole locations

Gain Bandwidth Product	G	Angle	Actual Q	Frequency Relative to Design Frequency
1 MHz	2	63^o	1.1	0.53
3 MHz	6	64^o	1.17	0.75
15 MHz	30	62^o	1.05	0.93

The reasons behind the frequency responses plotted in Figure 4.19 should now be apparent. The peaks in gain over the 0 dB DC value are caused by the actual Q's being greater than the design value of 1 and the lowered cutoff frequencies by the decrease in resonant frequencies. In addition for the 1 MHz case the extra real poles in both the first and second order sections cause extra attenuation in the pass band. For the 3 MHz and 15 MHz cases these extra poles also exist but are much larger than the pass band limit and are of no significant effect in the pass band.

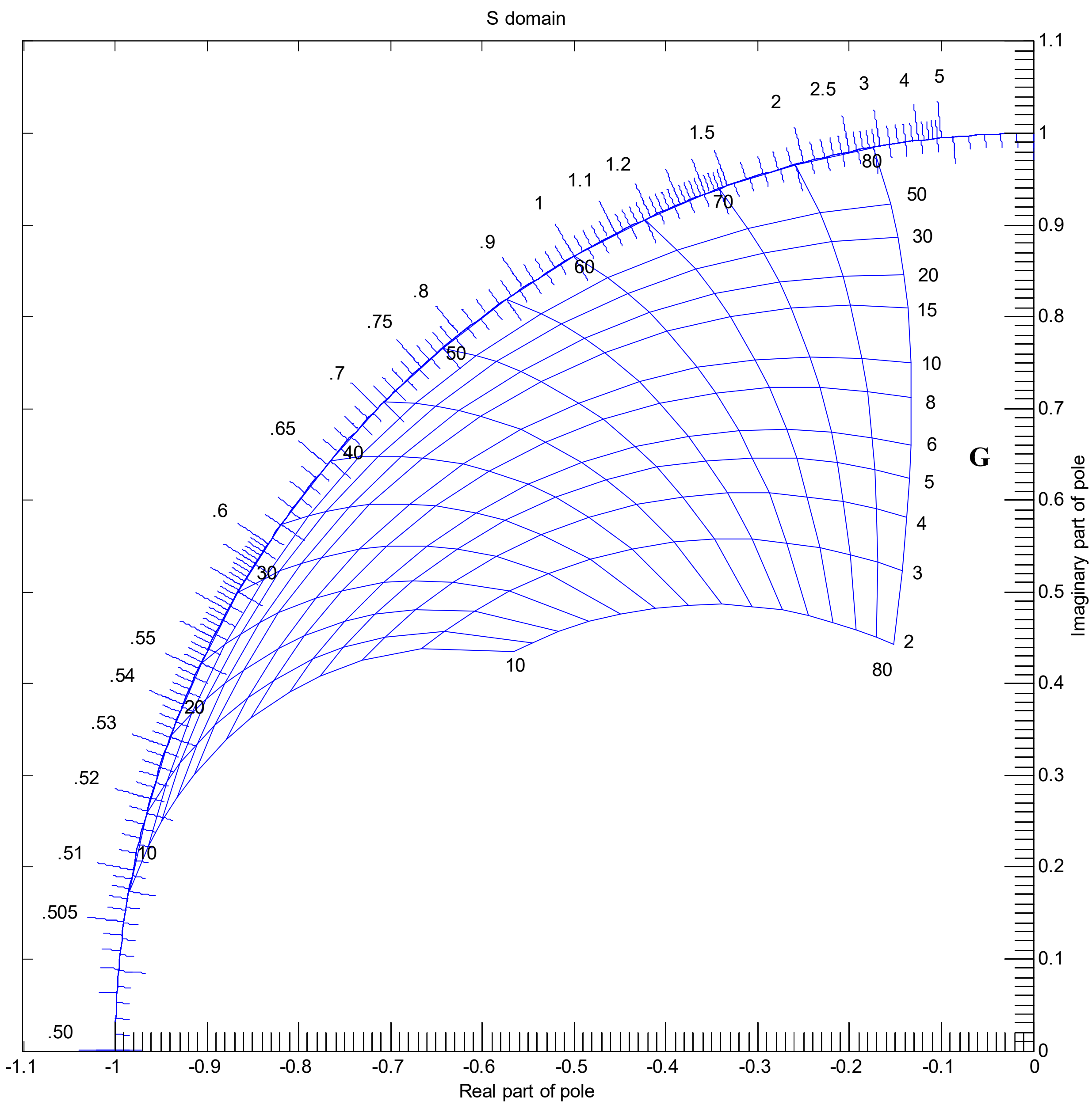

Figure 4.21. Complex poles of equation 4.59 (Case 1).

Case 2.

For the circuit in Figure 4.18 the analysis of the second order section is quite similar. In equation 4.30, A(s) is now given by

$$A(s) = \frac{G}{s+G} \quad . \tag{4.60}$$

For Case 2

$$C_2 = 2QC_{eq}$$

and

$$C_1 = \frac{C_{eq}}{2Q} \quad .$$

Substituting in equation 4.30 and again normalizing to ω_o= 1, we have

$$\frac{V_{out}}{V_{in}} = \frac{\dfrac{G}{s+G}}{s^2 + \left(\dfrac{1}{Q} + 2Q - \dfrac{2QG}{s+G}\right)s + 1}$$

$$\frac{V_{out}}{V_{in}} = \frac{G}{s^3 + \left(\dfrac{1}{Q} + 2Q\right)s^2 + s + G\left(s^2 + \dfrac{s}{Q} + 1\right)} \quad . \tag{4.61}$$

A plot of the complex poles as a function of G and the designed angle is shown in Figure 4.22. Again the poles tend to their design locations as G tends to infinity and move to larger Q's and lower resonant frequencies as G decreases. Note however that the movement of the pole locations in the s plane for comparable G is not as much as for the first circuit (Figure 4.21), which explains why the frequency response in Figure 4.20 is better than that of the 3 MHz curve in Figure 4.19. The reason for this is that the 3 dB bandwidth of the op-amp/negative feedback combination [A(s)] is the gain bandwidth product when connected as a voltage follower and the gain bandwidth product divided by the DC gain (A_o) when connected as a non-inverting amplifier.

The easiest way to determine if an op-amp has sufficient bandwidth for a particular design (and also to verify the correctness of the design), is to analyze the circuit with an AC circuit analysis program. However, the pole location plots do lend insight and also suggest what can be done to modify our original design to help meet specifications (other than finding a wider bandwidth op-amp). Intuitively if we use resistors and capacitors that would have resulted in higher frequencies and lower Q's with ideal op-amps we can compensate for the finite gain bandwidth product and come closer to meeting specifications with real op-amps.

As an example, consider modifying the circuit in Figure 4.17 when the op-amp gain bandwidth is 3 MHz. Note that we will need to increase the design resonant frequency (defined by 1/(RC)) by more than the 1/0.75 suggested by Table 4.2 because this increase results in a G of somewhat less than 6. Decreasing the capacitors by a factor of 1.4 is approximately correct. In addition, we will move the poles 4^o closer to the real axis by designing for a Q of 0.9 (i.e., designing for an angle of 56^o instead of 60^o). Similar analysis would show the finite gain bandwidth product of the op-amps also affects the first order section (but to a much lesser degree) and we can compensate with a small change in capacitance value.

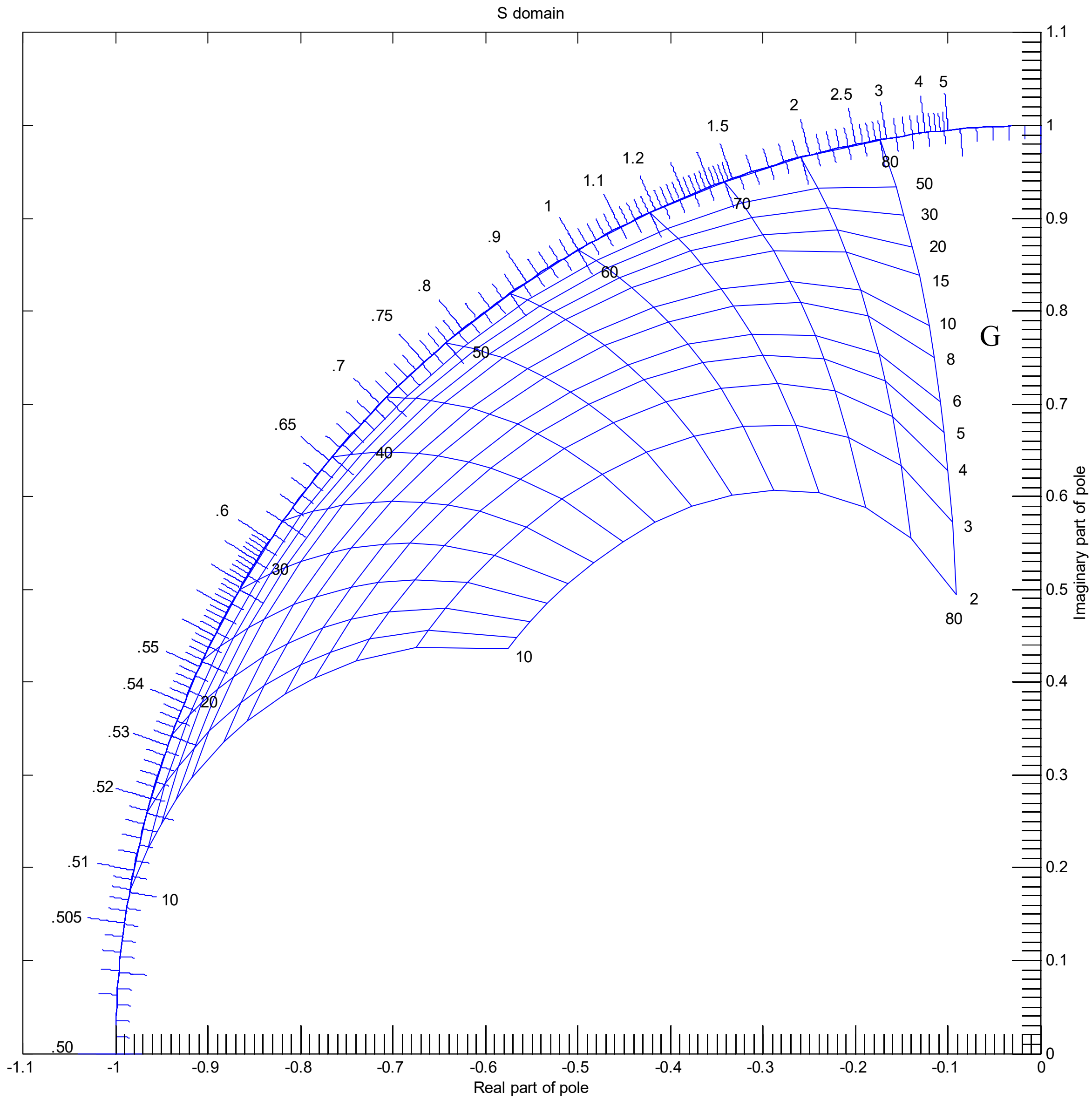

Figure 4.22. Complex poles of equation 4.61 (Case 2).

Our modified circuit is shown in Figure 4.23, and its corresponding frequency response is shown in Figure 4.24.

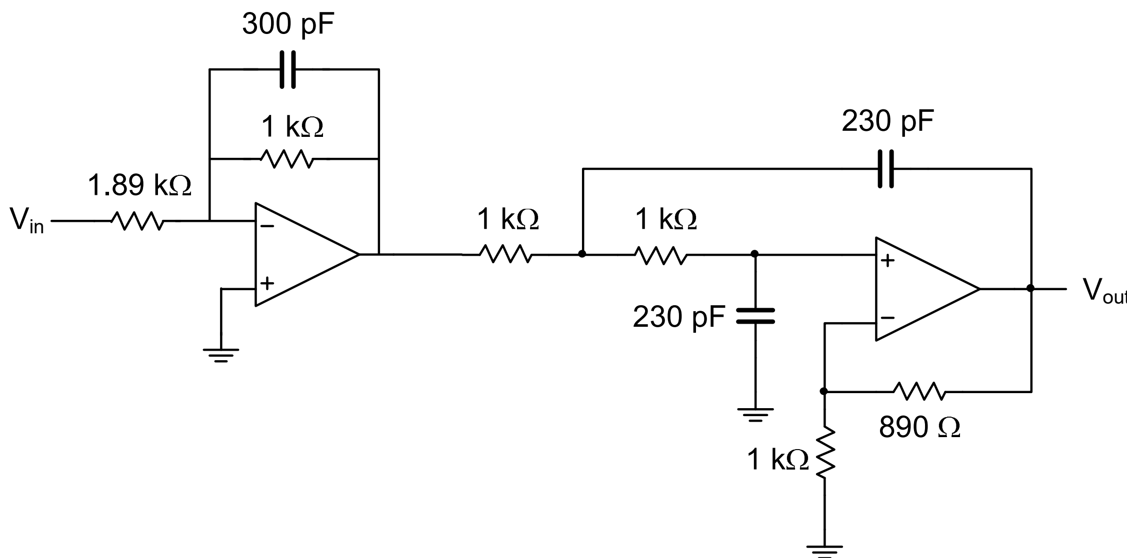

Figure 4.23. Modified circuit (from Figure 4.17) to account for use of 3MHz gain bandwidth product op-amps.

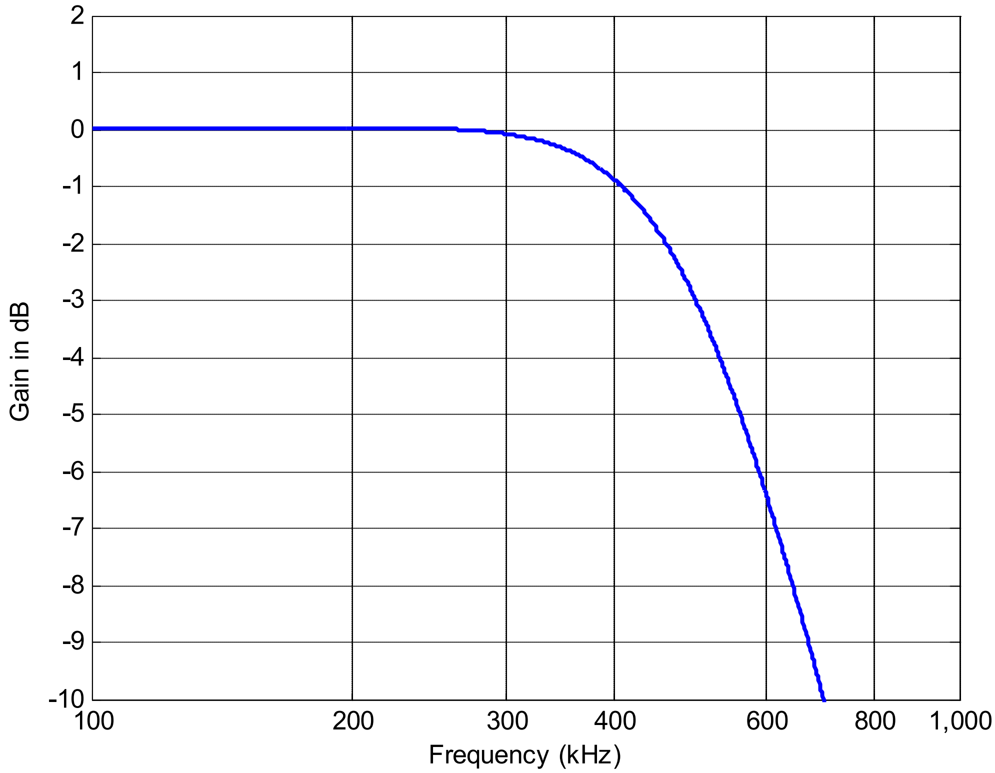

Figure 4.24. Magnitude response of modified circuit in Figure 4.23.

The finite gain bandwidth product of real op-amps is a linear effect that we have modeled with transfer functions and analyzed using linear circuit analysis programs. In addition, the finite slew rate of the op-amp is a non-linear effect that also limits its response at high frequencies. For example, we could have used a 1 MHz op-amp and modified our design to compensate. (In fact we could have reduced our design to a single op-amp because the extra real pole would have been close enough to satisfy the specifications.) However the common 741 type op-amp has a 1 MHz gain bandwidth product and a maximum slew rate of 0.5 volts/μsec. At the pass band limit of 400 kHz, a sinusoid of amplitude A volts has a maximum slope of

$$2\pi * A * 400 \text{ kHz} = 2.5 \times 10^6 \text{ A volts/sec},$$

hence any signal amplitude at the op-amp output greater than 0.2V would violate the maximum device slew rate of 0.5V/μsec for a 741 type op-amp. Therefore, even if we could make the linear frequency response correct, we would need a much faster op-amp in order to avoid problems due to slew rate limitations.

If we compare the general high-pass filter transfer function (equation 4.50) to the equivalent low-pass transfer function (equation 4.30) it should be clear that the plots of actual pole locations in Figures 4.21 and 4.22 apply to high pass filters as well. The extra real pole in the high-pass case is in the pass band of the filter and our filter will always look like a first order low-pass filter at high frequencies.

The selection of op-amps is now strictly a function of how high in frequency we wish our filter to look like a high pass filter. This frequency is merely the gain bandwidth product divided by A_O.

In future chapters we will look at op-amp implementations of other types of low-pass filters and at band-pass and notch filters. In general these filters will have larger Q's than the Butterworth low-pass and high-pass filters considered here. From Figures 4.21 and 4.22 we can see that higher Q filters are more affected by finite gain bandwidth product.

An additional consideration in our design is how close to our calculated values are the actual resistances and capacitances and what are the effects of any variations. A complete study of this topic (sensitivity) is beyond the scope of this book, but to illustrate some basic concepts we will consider one simple example.

Example 4.5

Figures 4.25 and 4.26 are two possible second order low-pass sections for Q equal to 2.5 and ω_o equal to 10,000 rad/sec.

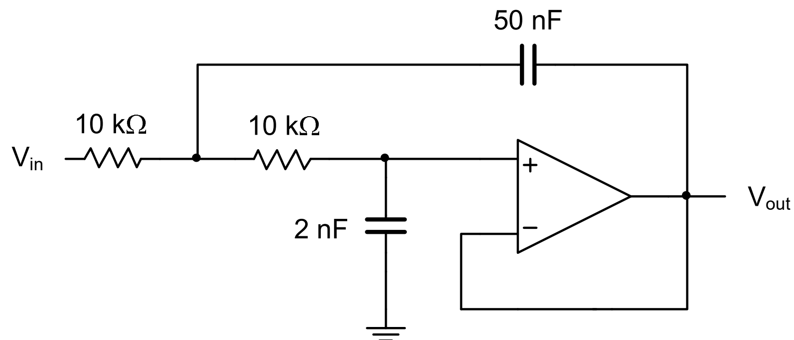

Figure 4.25. Candidate design 1 for Example 4.5.

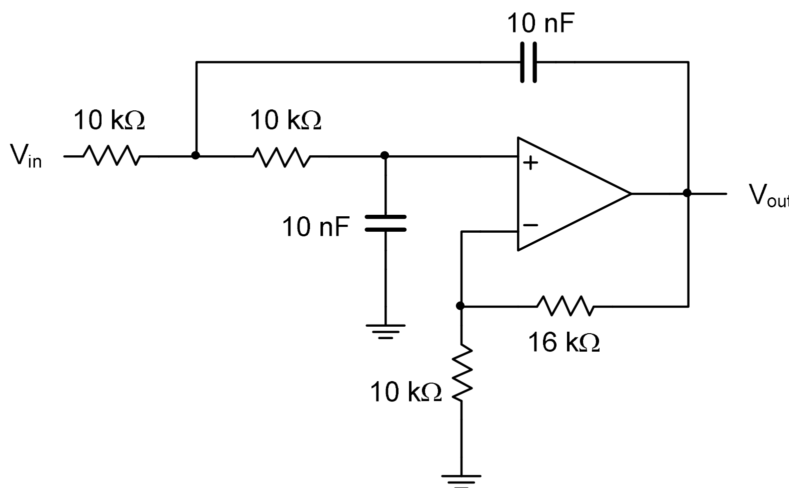

Figure 4.26. Candidate design 2 for Example 4.5.

Let us assume the passive components in each circuit have tolerances of 10%. In the circuit in Figure 4.25 variations of 10% in any one of the components will cause a variation of no more than 10% in the parameters of the circuit. In particular Q is a function of the square root of the ratio of the capacitors and therefore varies only 5% when one capacitor varies by 10% and varies by as much as 10% if both capacitors vary. In the circuit in Figure 4.26 however Q is given by

$$Q = \frac{1}{2 - \frac{R_b}{R_a}} \quad .$$

If R_a were 9 kΩ versus 10 kΩ, Q would increase from 2.5 to 5.5 or by much greater than 10%. If R_b were to increase 10% to 17.6 kΩ as well, Q would increase to 22.5. In general there is a very real danger of instability when designing high Q circuits of this type. Again this will become more of a concern when we consider higher Q circuits in later chapters. At low Q's the two circuits are so similar that their sensitivities to component variations are similar as well.

4.8 Concluding Remarks

In this chapter we presented the Butterworth frequency response for low-pass and high-pass filters. This response was characterized by a maximally flat pass band, and roll-off in the stop band of 20n dB/decade, where n is the filter order. For low pass filters we saw a response of the form

$$|H(j\omega)|^2 = \quad = \frac{1}{1+a_n\omega^{2n}}$$

$$= \frac{1}{1+\left(\frac{\omega}{\omega_o}\right)^{2n}} \ , \tag{4.2}$$

where ω_o is the half power (or -3 dB) frequency and was also the undamped natural frequency of each section of the filter. Using equation 4.2 and the definitions of A_{max} (the maximum pass band attenuation in dB), A_{min} (the minimum stop band attenuation), ω_p (the pass band limit), and ω_s (the stop band limit) the minimum order that satisfied these specifications was

$$\boxed{n = \frac{\ln\left(\frac{10^{A_{min}/10}-1}{10^{A_{max}/10}-1}\right)}{2\ln\left(\frac{\omega_s}{\omega_p}\right)}} \ . \tag{4.11}$$

The poles for Butterworth low-pass and high-pass filters were evenly spaced around a circle of radius ω_o in the s plane and at intervals of π/n radians. Odd order filters have a pole on the negative real axis with the remaining poles at intervals of π/n. Even order filters have a complex conjugate pair of poles at $\pm\ \pi/2n$ radians from the negative real axis.

We examined two versions of a second order circuit using a single op-amp, resistors, and capacitors. The design of filters for either version was quite straightforward. One need only select resistance and capacitance such that

$$RC = \frac{1}{\omega_o} \ .$$

Q was a function of a resistance ratio for one version and a capacitance ratio for the other.

We presented high-pass filters by rotating the low-pass frequency response about ω_o on a logarithmic frequency scale. The circuit design was similar to the low-pass case with resistors substituted for capacitors and capacitors for resistors.

Finally the effect of using op-amps with finite gain bandwidth products was analyzed. The poles moved toward lower frequencies and higher Q's as the gain bandwidth product decreased. Methods to modify the circuit to compensate for this movement were discussed.

Problems

For problems 4.1 to 4.8 find:

a. The minimum order and the Q's of each section.
b. ω_o and A_{min} if the attenuation is forced to be exactly A_{max} at ω_p.
c. ω_o and A_{max} if the attenuation is forced to be exactly A_{min} at ω_s.

	A_{max}	A_{min}	ω_p	ω_s
4.1	1 dB	20 dB	1000 rad/sec	3000 rad/sec
4.2	0.5dB	30 dB	1000 rad/sec	2500 rad/sec
4.3	2 dB	20 dB	2000 rad/sec	9000 rad/sec
4.4	0.5dB	40 dB	3000 rad/sec	15000 rad/sec

	A_{max}	A_{min}	f_p	f_s
4.5	1 dB	20 dB	2000 Hz	6000 Hz
4.6	0.5dB	30 dB	2000 Hz	5000 Hz
4.7	2 dB	20 dB	1000 Hz	4500 Hz
4.8	0.5dB	40 dB	2000 Hz	10000 Hz

For problems 4.9 to 4.16 design low-pass Butterworth filters to satisfy the specifications given. Use reasonable values for the resistors and capacitors in your circuits.

	DC Gain	A_{max}	A_{min}	ω_p	ω_s
4.9	0 dB	1 dB	30 dB	1000 rad/sec	3000 rad/sec
4.10	20 dB	0.5dB	30 dB	2000 rad/sec	5000 rad/sec
4.11	0 dB	2 dB	25 dB	2000 rad/sec	12000 rad/sec
4.12	6 dB	0.5dB	40 dB	4000 rad/sec	14000 rad/sec

	DC Gain	A_{max}	A_{min}	f_p	f_s
4.13	0 dB	1 dB	30 dB	2000 Hz	6000 Hz
4.14	20 dB	0.5dB	30 dB	1000 Hz	2500 Hz
4.15	0 dB	2 dB	25 dB	1000 Hz	6000 Hz
4.16	6 dB	0.5dB	40 dB	2000 Hz	7000 Hz

For problems 4.17 to 4.22 design high-pass Butterworth filters to satisfy the specifications given. Use reasonable values for the resistors and capacitors in your circuits:

	High Frequency Gain	A_{max}	A_{min}	ω_p	ω_s
4.17	0 dB	0.5 dB	30 dB	10000 rad/sec	3000 rad/sec
4.18	20 dB	0.2 dB	20 dB	11000 rad/sec	5000 rad/sec
4.19	0 dB	1 dB	25 dB	7000 rad/sec	2000 rad/sec

	High Frequency Gain	A_{max}	A_{min}	f_p	f_s
4.20	0 dB	0.5 dB	30 dB	5000 Hz	1500 Hz
4.21	20 dB	0.2 dB	20 dB	5500 Hz	2500 Hz
4.22	0 dB	1 dB	25 dB	3500 Hz	1000 Hz

4.23.

a. Assuming you are using ideal op-amps, design a low-pass Butterworth filter to satisfy the following specifications:

DC Gain	A_{max}	A_{min}	ω_p	ω_s
6 dB	0.5 dB	15 dB	1×10^6 rad/sec	3×10^6 rad/sec

b. Analyze your circuit using a linear circuit analysis package using 1 MHz for the gain bandwidth product of your op-amps. Where are your poles now compared to the desired locations?

c. Modify the passive components in your circuit to match the specifications more closely.

4.24. You have been tasked to design a Butterworth analog low pass filter which serves as part of a sampled-data system being used to analyze signals originating in the human brain. The sampling rate for this system is 44kHz. Your analog filter is allowed a maximum pass band deviation of 2dB, and since there are numerous interfering signals outside the band of interest (0-11kHz), you are required to have a minimum of 30dB attenuation for frequency components above 22kHz.

a. Determine appropriate filter specifications for this application (A_{max}, A_{min}, ω_p, ω_s).

b. Calculate the minimum required filter order, and specify ω_o and Q for each first or second order section.

c. Design your filter using ideal op-amps and ideal passive components. (Note: Assume 0dB DC gain is needed.)