Chapter 2

Bode Plots

2.1 Introduction

In this chapter we consider the problem of plotting (by hand) the frequency responses (magnitude and phase) of continuous time transfer functions. As we will see it is possible to approximate the effects of real poles and zeros using straight line asymptotes on log magnitude (or dB) vs log frequency, and phase vs log frequency plots. These plots are referred to as Bode plots, in honor of Hendrik Wade Bode, who developed both the plotting techniques and stability results for closed loop systems based on these plots in the 1930's. In Chapter 11 we will introduce plotting techniques for discrete time Bode plots.

Certainly an accurate and expedient method for plotting a system's frequency response is through the use of a computer. Programs that accomplish this are easily written, and environments such as MATLAB® support the plot of frequency response by using simple one line commands. Frequency response plots may also be generated using a computer spreadsheet application such as Excel®. In fact a computer was used to produce most of the graphs in this chapter. Why then do we care about plotting frequency responses by hand? The short answer is that we are interested in the relationships between frequency response and the locations of the poles and zeros of the transfer function, and these can be illustrated via hand-drawn Bode plots. Furthermore, programs have been known to make errors, or mistakes could be made in entering transfer function coefficients. The best way to detect such errors is first to have an approximate idea of what the frequency response plot should look like. Bode analysis will do just that. Finally, someday we could find ourselves at a technical review, listening to engineering contractors presenting specifications for a proposed design, and we need to be able to estimate approximate magnitude and phase responses (without a computer), to assess the validity of design proposals!

Our strategy for this chapter will be to study the Bode plots of the simple "building block" transfer functions listed as items (a) through (d) below. For complicated transfer functions that are represented as a product of simpler transfer functions, we will analyze each simpler transfer function using Bode plot techniques, then combine the results to form a composite frequency response. More specifically, if we have some general transfer function H(s), we will express it in terms of a product of

a) a constant multiple C,
b) real poles or zeros of the form $(s + a)^N$,
c) poles or zeros at the origin s^N,
d) and complex poles or zeros of the form $(s^2 + \dfrac{\omega_o s}{Q} + \omega_o^2)^N$,

where N represents a non-zero integer in b-d above.

A more complicated system that can be expressed as the product of building blocks

$$H(s) = H_1(s)\, H_2(s)\, H_3(s) \;\ldots$$

has a frequency response that can be represented in terms of magnitude and phase as

$$|H(j\omega)| = |H_1(j\omega)|\, |H_2(j\omega)|\, |H_3(j\omega)| \;\ldots$$

and

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega) + \angle H_3(j\omega) + \ldots,$$

since continuous-time "frequency response" is defined as $H(s)\big|_{s=j\omega}$.

Since the Bode plot is really a complete magnitude (in dB) and phase (in radians or degrees) plot of $H(j\omega)$ as a function of log frequency, the composite magnitude and phase responses are simply additive, or

$$|H(j\omega)|_{dB} = |H_1(j\omega)|_{dB} + |H_2(j\omega)|_{dB} + |H_3(j\omega)|_{dB} + \ldots$$

and

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega) + \angle H_3(j\omega) + \ldots.$$

Therefore once we understand how to plot each of the "building block" transfer functions listed above, we will combine their effects in an additive fashion to produce both the magnitude (in dB) and phase of the more complex composite frequency response.

The first "building block" is a constant multiple C. Its magnitude as a function of frequency is constant, or

$$|H(j\omega)|_{dB} = 20\log_{10}(|C|).$$

Its phase is either 0 (if C is positive) or ± 180 degrees ($\pm\pi$ radians) if C is negative.

2.2 Real Poles and Zeros

Now we consider our second building block, namely real poles and zeros of the form $(s + a)^N$.

Example 2.1. Real Pole

Before considering the general case of real poles and zeros, we will first look at a very simple example, namely

$$H(s) = \frac{1}{s+1}. \tag{2.1}$$

Here we have a pole at s = -1. The frequency response is found by evaluating H(s) for s along the $j\omega$ axis, or

$$H(j\omega) = \frac{1}{j\omega + 1} \tag{2.2}$$

and the magnitude response is

$$H(j\omega) = \frac{1}{\sqrt{1+\omega^2}}. \tag{2.3}$$

For $\omega << 1$, $|H(j\omega)|$ is approximately 1. For $\omega >> 1$, $|H(j\omega)|$ is approximately $\frac{1}{\omega}$. Figures 2.1a and 2.1b illustrate these two cases graphically. The denominator vector **s + 1** (evaluated at $s = j\omega$) goes from the pole at s = -1 to s = jω. For $\omega << 1$ as in Figure 2.1a the denominator has magnitude of approximately 1. For $\omega >> 1$ as in Figure 2.1b the denominator has magnitude of approximately ω.

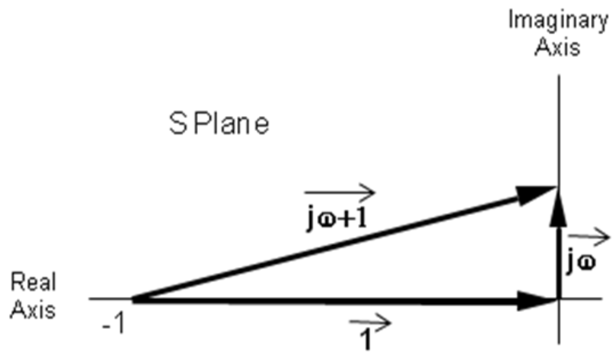

Figure 2.1a. **jω + 1** for $\omega << 1$.

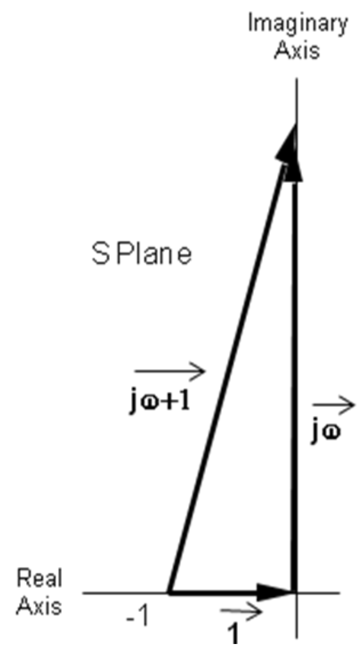

Figure 2.1b. **jω + 1** for $\omega >> 1$.

The example above illustrates the fundamental principle behind Bode plots for transfer functions with only real poles and zeros. More specifically, the magnitude (as a function of the radian frequency, ω) is approximately proportional to ω^k where k is an integer (that may vary with frequency) and can be positive, zero, or negative. More specifically,

$$|H(j\omega)| \approx |C|\,\omega^k . \tag{2.4}$$

In Example 2.1, C=1 and k = 0 for $\omega << 1$, and k = -1 for $\omega >> 1$. For examples of more complicated transfer functions, the magnitude response of an n^{th} order low-pass filter may be approximately constant (i.e., k = 0) at low frequencies, and proportional to ω^{-n} (i.e., k = -n) at high frequencies. The magnitude response of a second order band-pass filter will be proportional to ω (i.e., k = 1) at low frequencies and inversely proportional to ω (i.e., k = -1) at high frequencies.

If we take the logarithm of both sides of 2.4,

$$\log_{10}(|H(j\omega)|) \approx \log_{10}(|C|) + k\log_{10}(\omega). \tag{2.5}$$

Therefore a plot of $\log_{10}(|H(j\omega)|)$ vs $\log(\omega)$ can be approximated by straight lines of slope k. If 2.5 is expressed in decibels,

$$20\log_{10}(|H(j\omega)|) \approx 20\log_{10}(|C|) + 20\,k\log_{10}(\omega). \tag{2.6}$$

Returning to Example 2.1 for a moment, for $\omega << 1$, C=1 and k=0, and 2.6 becomes

$$20 \log_{10}(|H(j\omega)|) \approx 20 \log_{10}(1) = 0\text{dB for low frequencies.} \qquad (2.7)$$

For $\omega >> 1$, C=1 and k=-1, and 2.6 becomes

$$20 \log_{10}(|H(j\omega)|) \approx 20 \log_{10}(1) - 20 \log_{10}(\omega).$$

or

$$20 \log_{10}(|H(j\omega)|) \approx - 20 \log_{10}(\omega). \qquad (2.8)$$

This represents a -20dB/decade asymptote. (Recall that one decade is one factor of ten in frequency.) Figure 2.2 is a plot of gain in dB vs frequency on a logarithmic scale for example 2.1 above where $H(s) = \frac{1}{s+1}$. The plot shows both the actual transfer function and the straight line asymptotes from the approximation in 2.7 and 2.8.

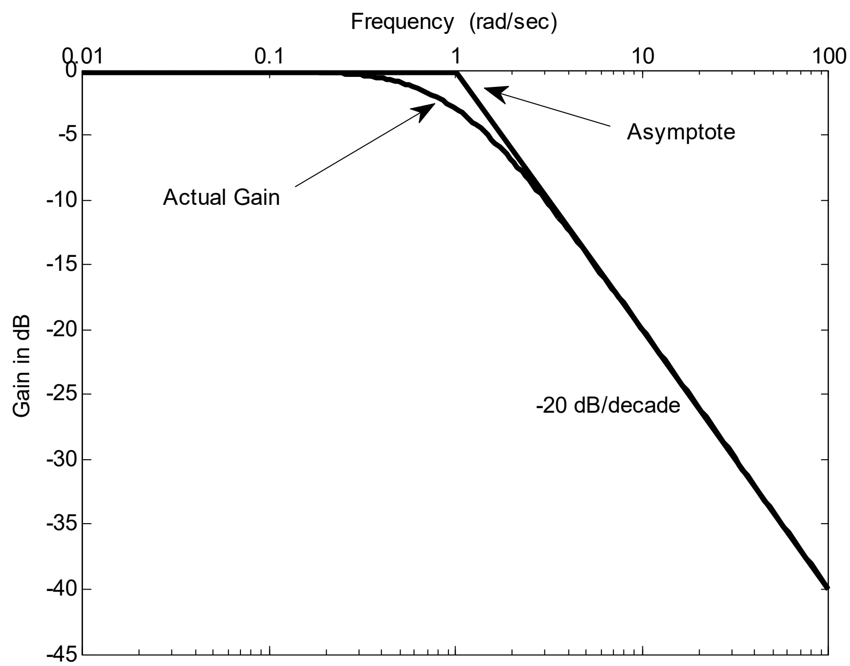

Figure 2.2. Magnitude response for $H(s) = \frac{1}{s+1}$.

As expected, the asymptotes match the actual plot very closely for ω either much larger than or much less than the pole frequency of one. The asymptotes intersect when the low and high frequency approximations are equal, or when

$$1 = \frac{1}{\omega}$$

or

$$\omega = 1. \qquad (2.9)$$

For a single real pole as in this example, the asymptotes differ from the actual plot by a factor of $\sqrt{2}$, equating to a 3 dB difference.

The phase response in Example 2.1 above,

$$H(j\omega) = \frac{1}{1 + j\omega} \tag{2.10}$$

is minus the denominator phase (since the numerator phase is zero), or

$$\phi(\omega) \; = \; - \arctan(\omega). \tag{2.11}$$

As can be seen from Figure 2.1 or 2.10, the angle with respect to the positive real axis (or phase) of the denominator vector goes from 0° to 90° as the frequency goes from zero to infinity. Since

$$\arctan(\omega) \; = \; 90° - \arctan(1/\omega) \tag{2.12}$$

observe that a plot of phase vs the log of frequency must be antisymmetric about the point of intersection of the pole frequency (ω=1) and phase = -45°. The phase can also be approximated by straight line segments. In this case, we approximate the phase by 0° for frequencies less than one tenth the pole frequency, by an asymptote with a slope of minus 45°/decade from one tenth to ten times the pole frequency, and by minus 90° beyond ten times the pole frequency. The actual phase and these straight line approximations are shown in Figure 2.3.

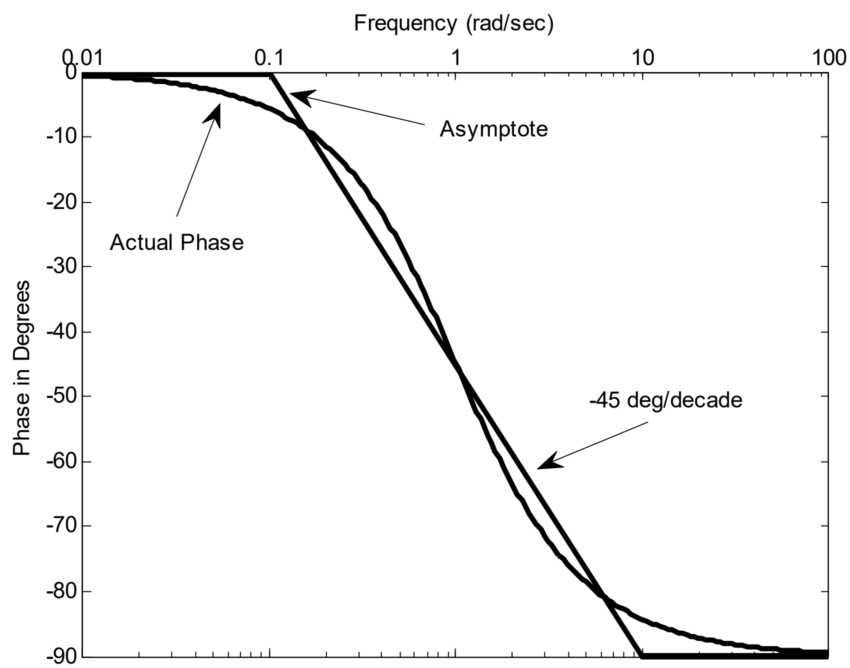

Figure 2.3. Phase response for $H(s) = \frac{1}{s+1}$.

In general, the Bode plots for general first order low-pass filters of the form

$$H(s) = \frac{b}{s+a} \tag{2.13}$$

can be summarized as follows:

a. The magnitude plot starts level at the DC gain.
{20 $\log_{10}$(H(0)) = 20 $\log_{10}$(|b/a|)}

b. It breaks downward with a slope of -20 dB/decade at the pole frequency, |a|.

c. Assuming a is positive, if b is also positive, the phase plots starts level at 0°, from one tenth the pole frequency to ten times the pole frequency it has a slope of minus 45°/decade, and it levels out at minus 90°. If b is negative, the phase plot looks exactly the same except that it starts level at 180°, and levels out at 90°.

Example 2.2 Real Zero

We will continue with a more complicated transfer function with a simple real pole (which we have already considered), and a simple real zero. When a transfer function contains zeros as in

$$H(s) = \frac{s+10}{s+1}, \tag{2.14}$$

the zeros are treated just as we treated poles except they have opposite signs for their asymptote slopes. Figure 2.4 shows graphically how this frequency response can be determined. At frequencies less than both the pole and zero frequencies (Figure 2.4a), the numerator and denominator are of lengths approximately 10 and one respectively. At frequencies between the pole and zero frequencies (Figure 2.4b), the numerator length is still approximately 10 and the denominator length is approximately ω. Above the zero frequency (Figure 2.4c) both numerator and denominator vectors are approximately of length ω.

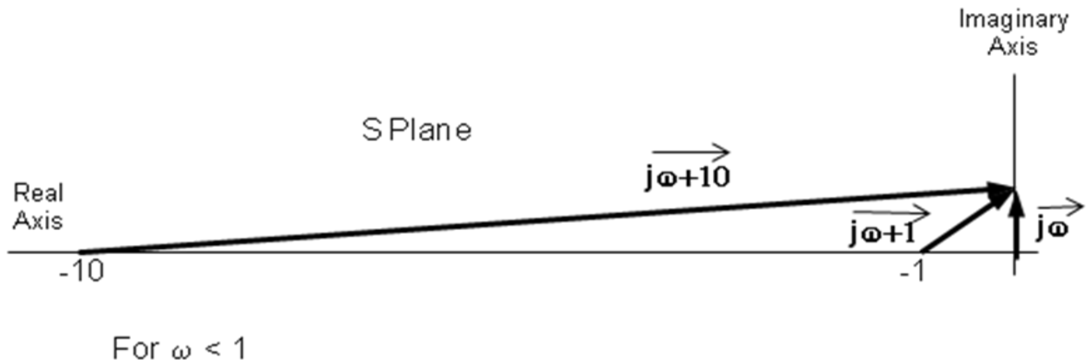

Figure 2.4a. Vector diagram for $H(j\omega) = \dfrac{j\omega + 10}{j\omega + 1}$ shown for ω < 1.

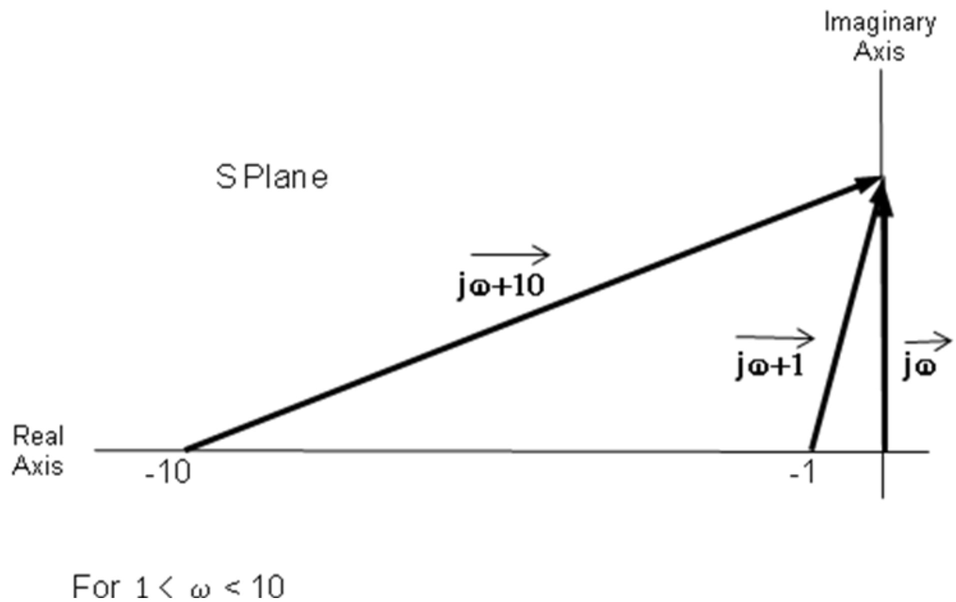

Figure 2.4b. Vector diagram for $H(j\omega) = \dfrac{j\omega + 10}{j\omega + 1}$ shown for 1 < ω < 10.

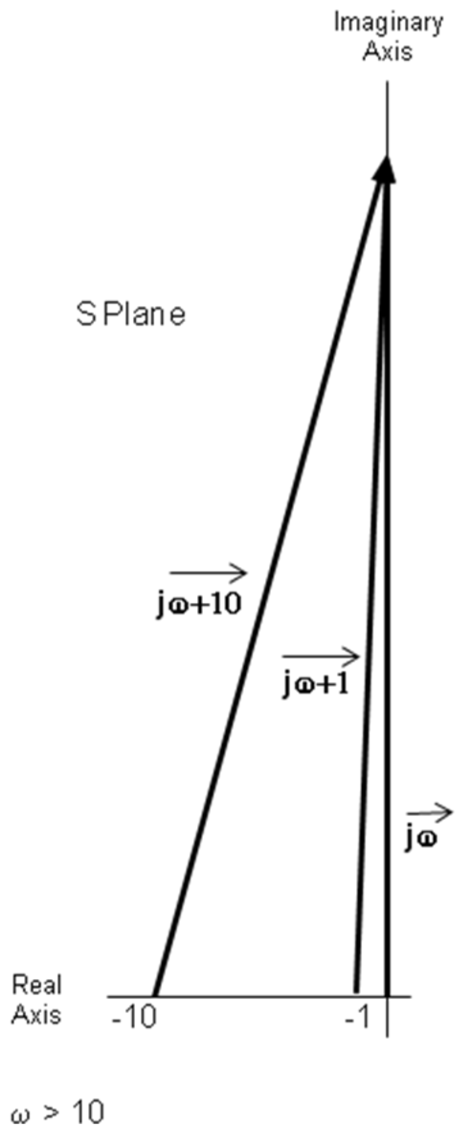

Figure 2.4c. Vector diagram for $H(j\omega) = \dfrac{j\omega + 10}{j\omega + 1}$ shown for ω > 10.

In summary,

$$|H(j\omega)| \approx 10 \qquad \text{for } \omega < 1,$$

$$\approx \frac{10}{\omega} \qquad \text{for } 1 < \omega < 10,$$

and

$$\approx 1 \qquad \text{for } \omega > 10.$$

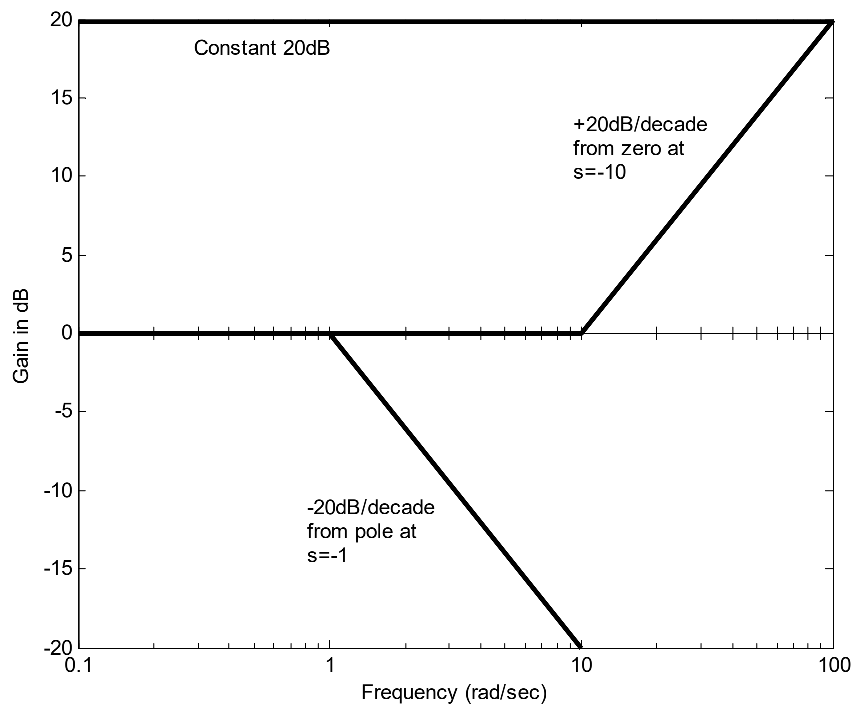

Figure 2.5a. Three elements of frequency response for Example 2.2.

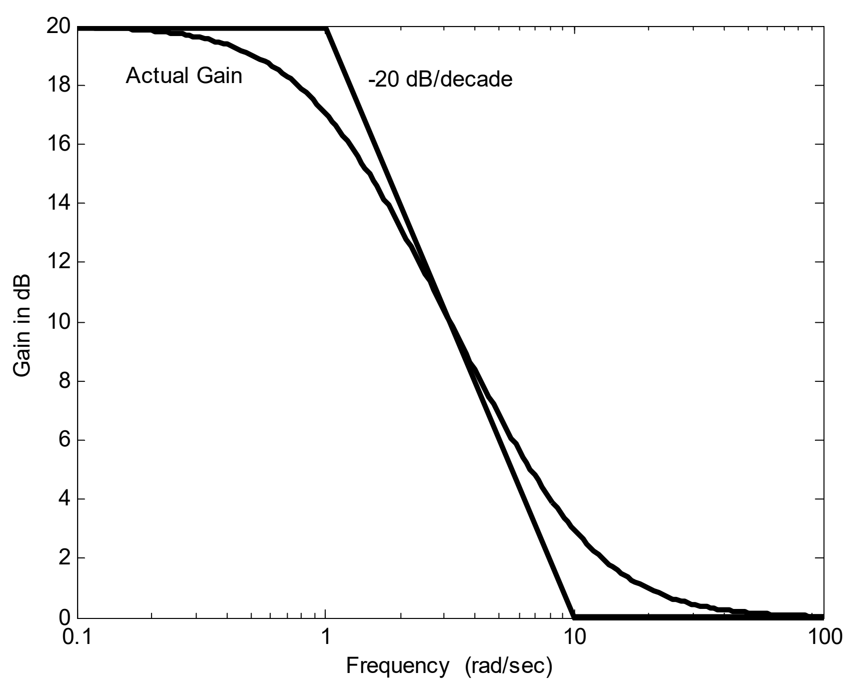

Figure 2.5b. Composite magnitude response for Example 2.2.

The actual gain and these asymptotes are shown in Figure 2.5b. The composite asymptotic response shown in Figure 2.5b is a result of three components as shown in Figure 2.5a.

The phase plot is the sum of an asymptote with a slope of -45°/decade from 0.1 to 10 due to the pole at -1 and a second asymptote with a slope of 45°/decade from 1 to 100 due to the zero at -10. These asymptotes are shown in Figure 2.6a and their sum is compared to the actual phase in Figure 2.6b.

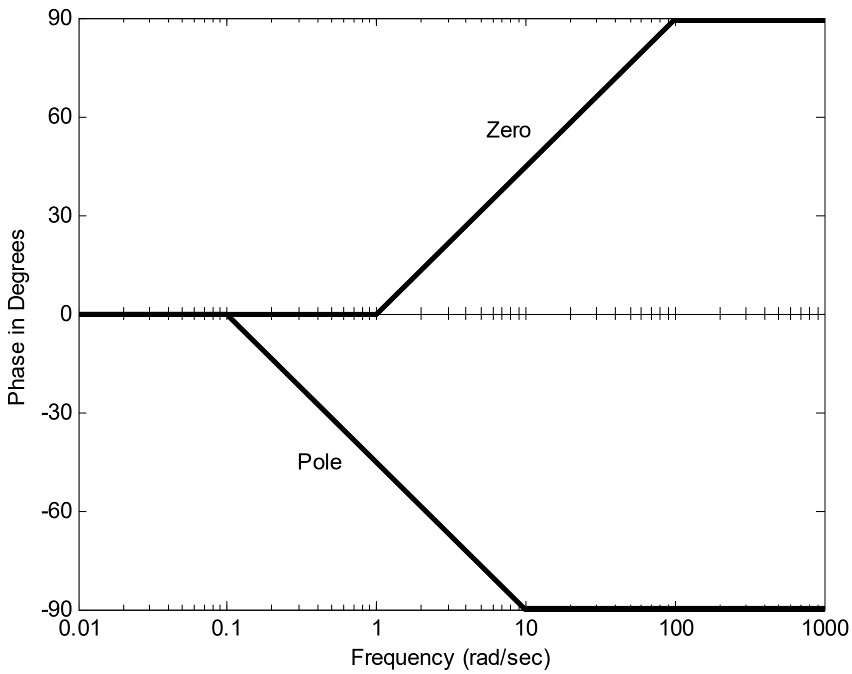

Figure 2.6a. The two elements of phase response for Example 2.2.

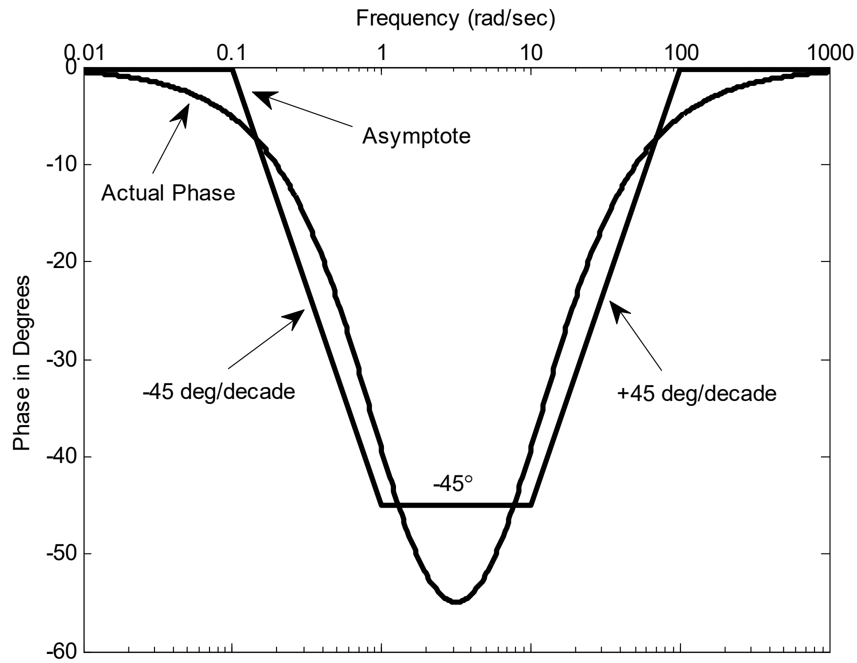

Figure 2.6b. Composite and actual phase responses for Example 2.2.

Example 2.3 Zero at Origin

For systems with poles or zeros at the origin of the s plane it is no longer possible to calculate the DC gain to establish the starting asymptote. The DC gain is infinite for a pole at the origin or zero for a zero at the origin, and is off scale. (Recall that zero gain corresponds to $-\infty$ in dB.) Consider the example

$$H(s) = \frac{100s}{(s+10)^2}.$$

For $\omega << 10$,

$$H(j\omega) \approx \frac{j100\omega}{100} = j\omega.$$

Therefore the magnitude plot starts with a slope of +20 dB/decade. The two poles at s = -10 result in a <u>change</u> in slope of - 40 dB/decade at $\omega = 10$ to a total slope of -20 dB/decade for frequencies greater than 10. Once we know the slopes we need to establish a starting point. A convenient method is to choose some reference ω (usually an integer power of 10) less than all non-zero "pole and zero frequencies," and to determine the <u>asymptotic</u> gain at that point. In this example $\omega = 1$ is a convenient reference. The <u>asymptotic</u> magnitude of the denominator is the same as the DC value (100) at $\omega = 1$ even if the actual magnitude is 101. Therefore our reference point is $\omega = 1$ and 0 dB (100/100). The <u>actual</u> gain is compared with these asymptotes in Figure 2.7. Observe that the <u>actual</u> gain is 6 dB (3 dB per pole) less than the intersection of the asymptotes at the break frequency of 10.

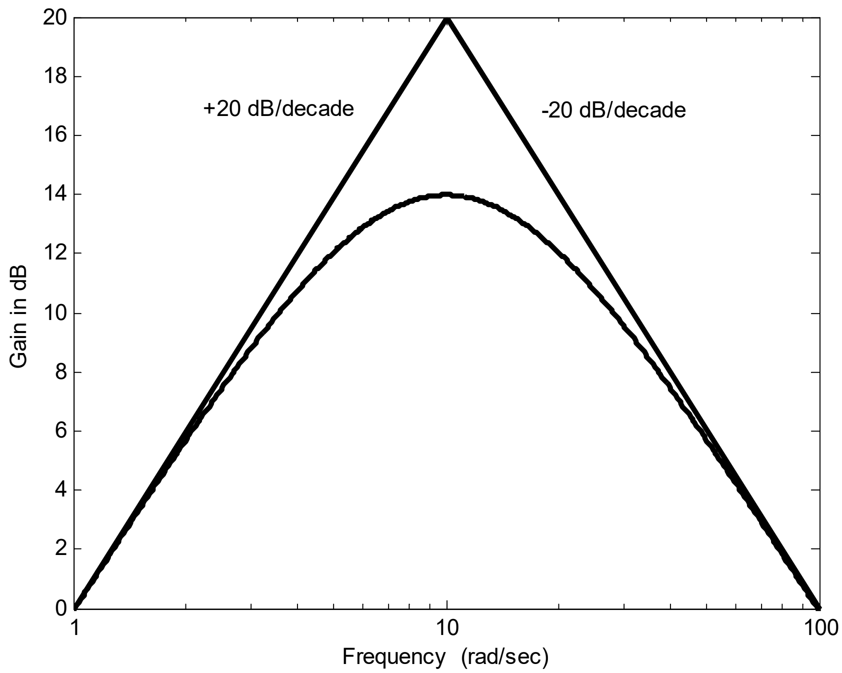

Figure 2.7. Asymptotic and actual frequency response for Example 2.3.

The phase plots for systems with poles or zeros at the origin will start at minus 90° per pole or +90° per zero (at the origin), assuming there is no overall negative constant. In this example the phase starts at +90° for low frequencies, has an asymptote with slope -90°/decade from $\omega = 1$ to 100 due to the double pole at $s = -10$, and ends at -90° for high frequencies. These asymptotes are compared to the actual phase response in Figure 2.8.

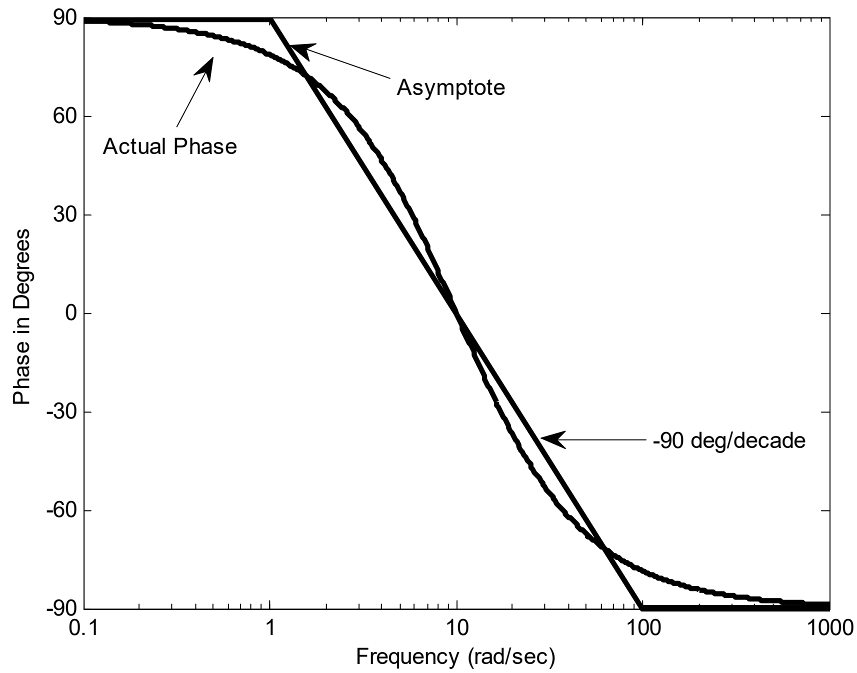

Figure 2.8. Asymptotic and actual phase response for Example 2.3.

Example 2.4 Pole at Origin

Transfer functions with a pole or poles at the origin of the s plane are treated the same as zeros at the origin except with opposite sign. For example, for

$$H(s) = \frac{10}{s(s+10)} .$$

For $\omega \ll 10$,

$$H(j\omega) \approx \frac{10}{j10\omega} = \frac{1}{j\omega} .$$

Therefore the magnitude plot starts at low frequencies with a slope of -20 dB/decade. The pole at s = -10 results in a <u>change</u> in slope of -20 dB/decade at $\omega = 10$ or a total slope of -40 dB/decade for frequencies greater than 10. Again we cannot use DC as our reference frequency due to the infinite gain at DC, so we choose a convenient starting point of $\omega = 1$ and 0 dB. The <u>actual</u> gain is compared to these asymptotes in Figure 2.9. The actual gain is 3 dB less than the point of intersection of the asymptotes at the break frequency of 10.

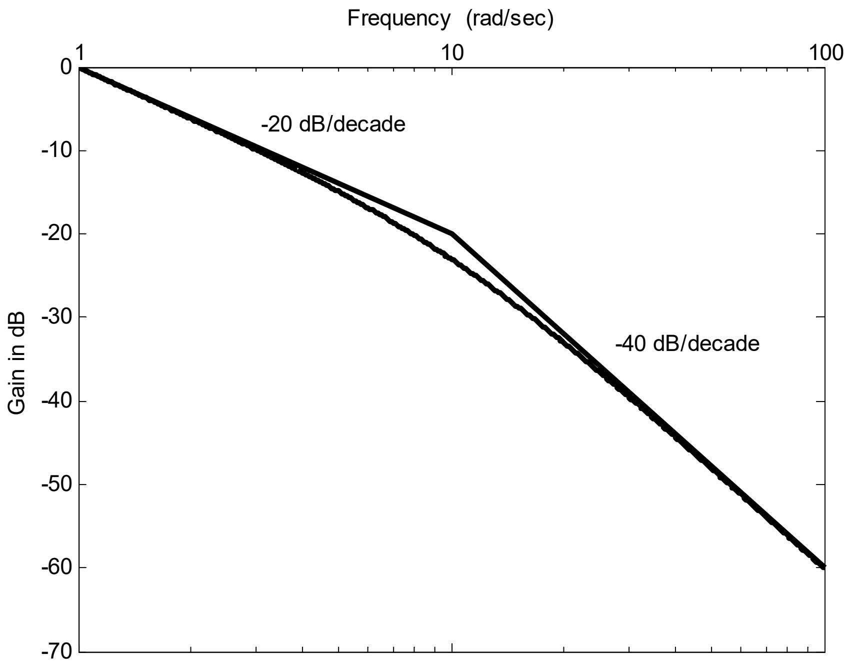

Figure 2.9. Asymptotic and actual frequency response for Example 2.4.

In this example the phase is -90° for low frequencies, an asymptote with slope -45°/decade from $\omega = 1$ to 100 due to the pole at $s = -10$, and -180° at high frequencies. These asymptotes are compared to the actual in Figure 2.10.

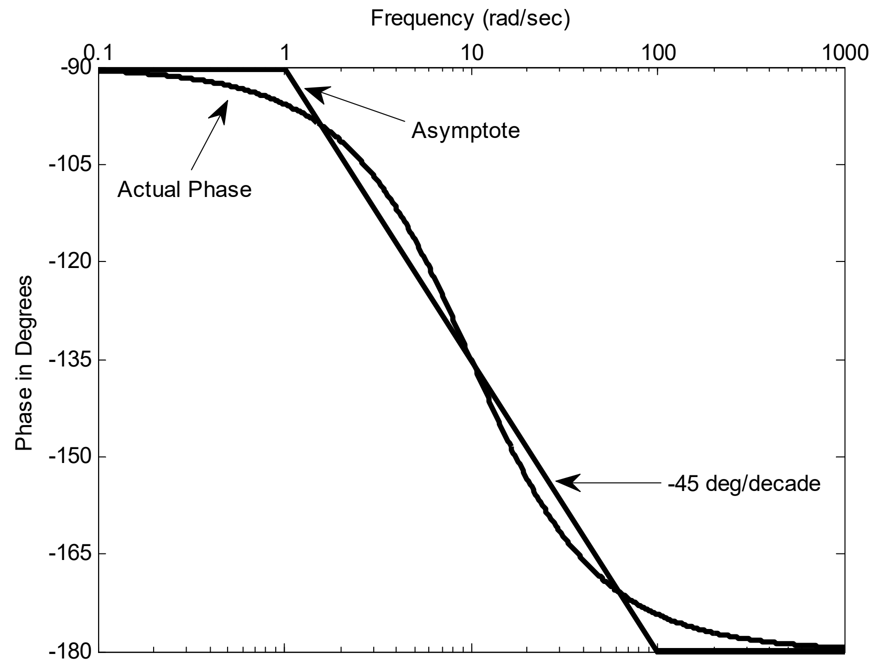

Figure 2.10. Asymptotic and actual phase response for Example 2.4.

Example 2.5 Zero in Right Half of s plane

Consider the transfer functions

$$H_1(s) = \frac{0.4(50 - s)}{(s + 2)}$$

and

$$H_2(s) = \frac{0.4(50 + s)}{(s + 2)}.$$

As illustrated in Figure 2.11, the phasor **50-jω** has the same magnitude but opposite phase (opposite sign of the angle with reference to the positive real axis) as the phasor **50+jω** in $H_2(s)$.

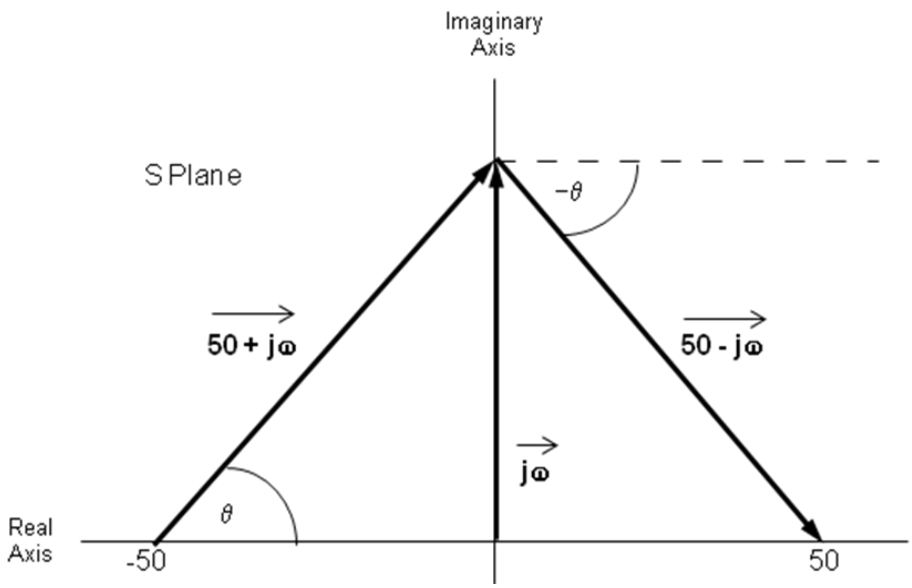

Figure 2.11. Phasors for **jω + 50** and **50 - jω**.

Since $H_1(0) = H_2(0) = 10$, both magnitude plots start level at +20 dB, break downward with a slope of -20 dB/decade at $\omega = 2$ due to the poles at $s = -2$, and then level out at $\omega = 50$ due to the zeros at $s = \pm 50$. In all the previous examples, we were lucky in that break frequencies were powers of 10 for simplicity. Certainly this won't always be the case. Using semilog plotting paper and exercising care in plotting our response is one solution. However, since understanding is our primary objective, a preferable method is to calculate the magnitude in dB at the break points. For example, how much does the gain change from $\omega = 2$ to $\omega = 50$ at a slope of -20 dB/decade? Since the frequency span between 2 and 50 is $\log_{10}(50/2) = 1.4$ decades on our logarithmic scale, this change is -28 dB. Alternatively, a slope of -20 dB/decade is a decrease in gain by a factor of 10 for each increase by a factor of 10 in frequency, or gain inversely proportional to frequency. Since the span from 2 to 50 in frequency is a factor of 25 increase, the gain must decrease by a factor of 25 over this range, which represents a change of -28 dB to a level of -8 dB. The magnitude plot for both transfer functions is shown in Figure 2.12.

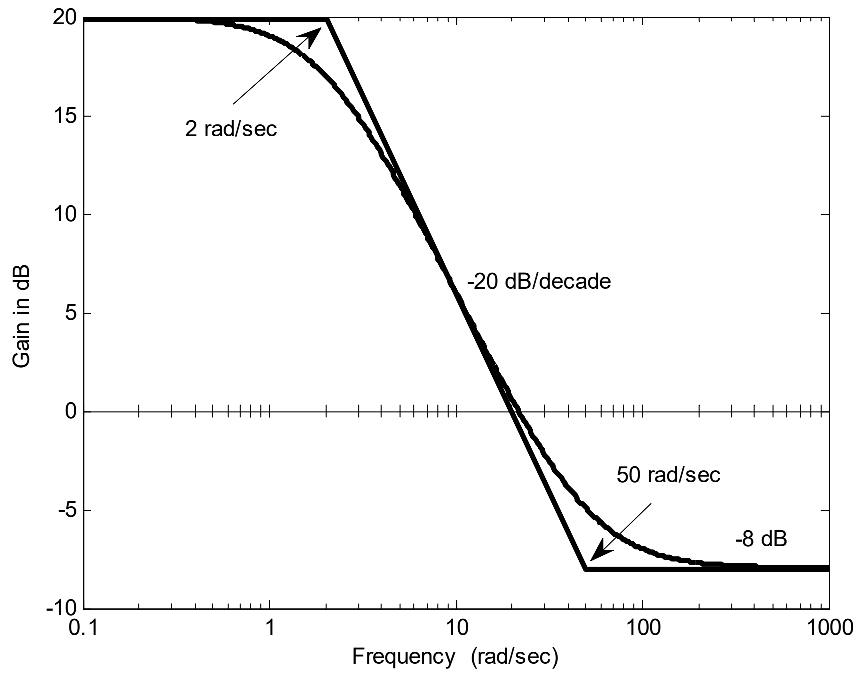

Figure 2.12. Magnitude responses for Example 2.5.

For the phase plots for both transfer functions, the poles at s = -2, contribute asymptotes with slopes of -45°/decade from $\omega = 0.2$ to $\omega = 20$. The zero at s = -50 in $H_2(s)$ contributes an asymptote with slope of +45°/decade from $\omega = 5$ to $\omega = 500$. The net result when these are added for $H_2(s)$ is level at 0° to $\omega = 0.2$, -45°/decade to $\omega = 5$, level from $\omega = 5$ to $\omega = 20$, +45°/decade from $\omega = 20$ to $\omega = 500$ and level for $\omega > 500$. Since 0.2 and 5 are separated by 1.4 decades the change in phase over this frequency range is

$$(1.4 \text{ decades}) \times (-45°/\text{decade}) = -63°.$$

The sum of these asymptotes is compared to the actual phase in Figure 2.13.

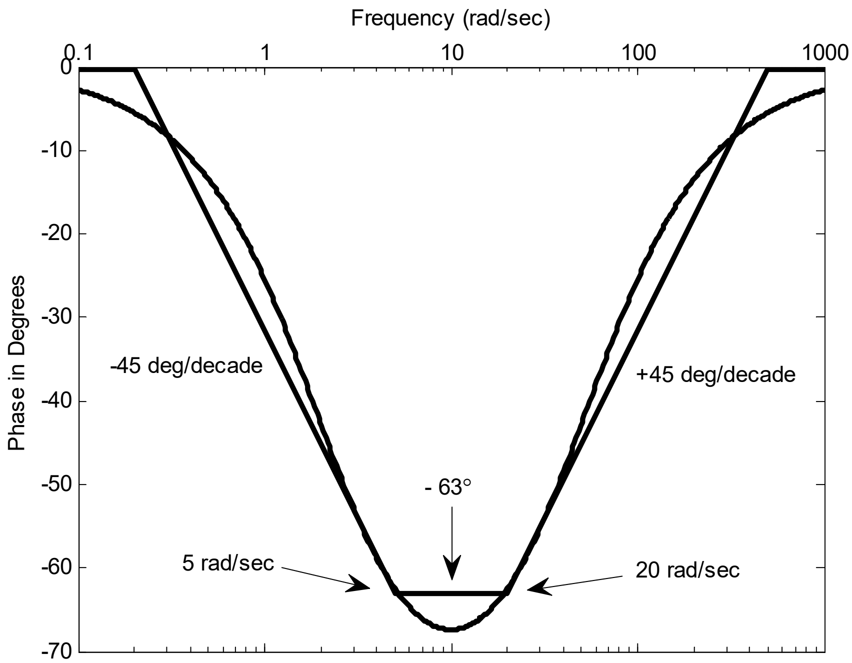

Figure 2.13. Asymptotic and actual phase responses for $H_2(s) = \dfrac{0.4(s+50)}{(s+2)}$.

In $H_1(s)$ the zero at s = +50 contributes an asymptote with slope of minus 45°/decade over the range $5 < \omega < 500$. Combined with the asymptote due to the pole at s = -2, the total phase starts level at 0°, slopes down at minus 45°/decade for $0.2 < \omega < 5$, changes to minus 90°/decade for $5 < \omega < 20$, then changes to minus 45°/decade for $20 < \omega < 500$, and becomes level for $\omega > 500$. Just as we calculated the phase response for $H_1(s)$, we find that the phase of $H_2(s)$ changes by -63° from 0.2 to 5 rad/sec. The phase of $H_2(s)$ changes by

$$\log_{10}(20/5) \text{ x } (-90°/\text{decade}) = (0.6 \text{ decades}) \text{ x } (-90°/\text{decade}) = -54°$$

from 5 to 20 rad/sec, and it changes by another minus 63° from 20 to 500 rad/sec. The asymptotic approximation and the actual phase are compared in Figure 2.14.

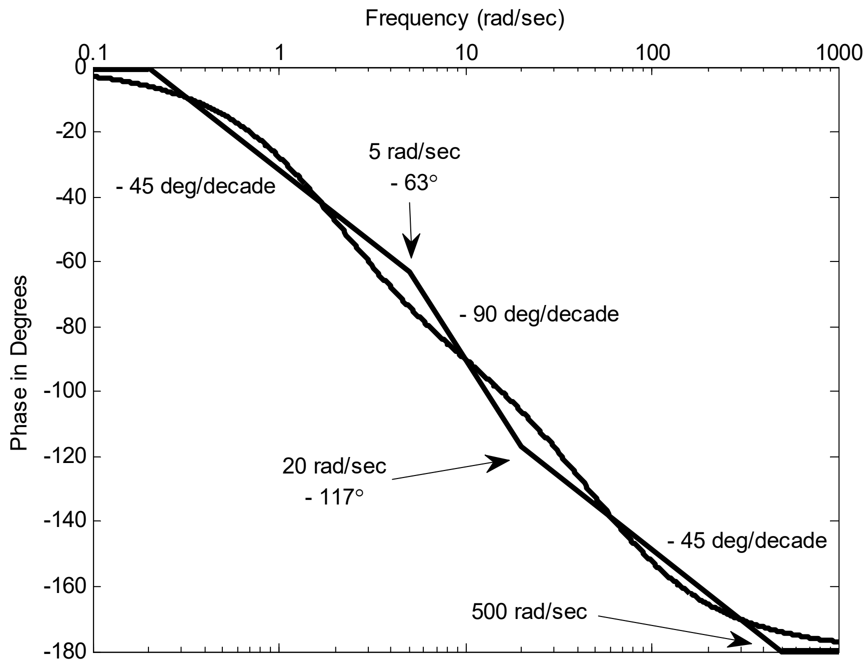

Figure 2.14. Asymptotic and actual phase responses for $H_1(s) = \dfrac{0.4(50-s)}{(s+2)}$.

Again, the purpose of studying the method of Bode asymptotic plots is that we want to be able to look at any transfer function, and have some immediate idea what to expect for its magnitude and phase response. Actual magnitude and phase response plots, however, are best generated with the assistance of a computer. Here we show the MATLAB® code used to generate the actual frequency response plots for $H_1(s) = \frac{0.4(50-s)}{(s+2)}$ shown in Figure 2.15.

```
*********************************************************
clear all; hold off;clf;
w=logspace(-1,3,100);            % generate 100 points equally spaced on a logarithmic axis
                                 % from 0.1 to 1000
b=[-0.4 20];                     %numerator   = -.4s + 20
a=[1 2];                         %denominator = 1s + 2
h=freqs(b,a,w);                  %evaluate frequency response at those frequencies in the w vector
mag=20*log10(abs(h));
phase=angle(h)*180/pi;
subplot(2,1,1);
semilogx(w,mag);
xlabel('Frequency in radians/sec');
ylabel('Gain in dB');grid;
subplot(2,1,2);
semilogx(w,phase);
xlabel('Frequency in radians/sec');
ylabel('Phase in Degrees');grid;

*********************************************************
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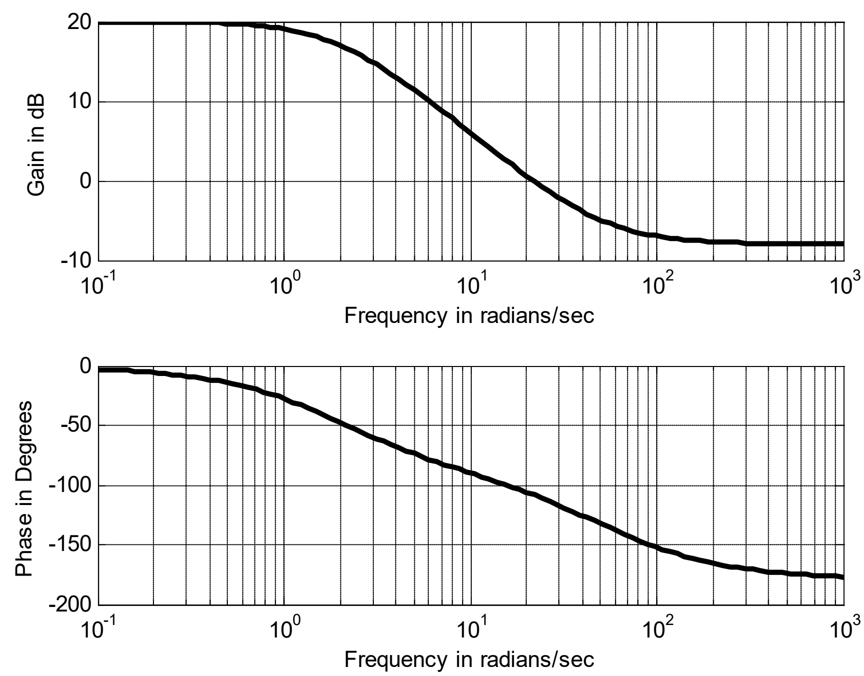


Figure 2.15. Frequency response for $H_1(s) = \dfrac{0.4(50-s)}{(s+2)}$ as generated from MATLAB®.

2.3 Complex Poles

Because all of the filters (of order 2 or greater) that we will design in this text will have complex poles, we must learn to draw Bode plots for transfer functions with such poles. Historically, passive filters with complex poles required inductors in addition to resistors and capacitors. We will begin by considering such circuits here. In later chapters we will introduce active filters employing operational amplifiers (op-amps), where complex poles can be realized without the expense and size of inductors.

Second Order Passive Band-pass Filter

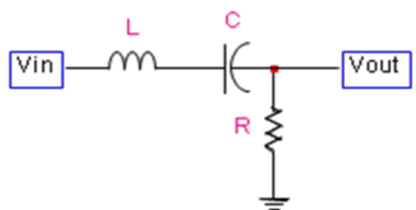


Figure 2.16. Passive band-pass filter.

For the circuit in Figure 2.16 the transfer function is derived using voltage division as

$$\frac{\text{Vout}}{\text{Vin}} = \frac{R}{sL + R + \dfrac{1}{sC}}$$

$$= \frac{\dfrac{R}{L}s}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}} . \qquad (2.15)$$

Setting the denominator equal to zero

$$= s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 ,$$

and using the quadratic formula, we can solve for the poles as

$$s = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} .$$

If

$$\frac{R^2}{4L^2} \geq \frac{1}{LC} ,$$

the poles are real and the Bode plot is drawn as discussed in the previous section. Of interest here will be the underdamped or complex pole case, where

$$\frac{R^2}{4L^2} < \frac{1}{LC}$$

and

$$s = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{2.16}$$

The magnitude or resonant frequency of each pole is

$$|s| = \sqrt{\frac{1}{LC}}\,, \tag{2.17}$$

and we define this as the parameter ω_o. Equation 2.15 can then be rewritten as

$$\frac{Vout}{Vin} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \omega_0^2}\,. \tag{2.18}$$

The Q or "quality factor" historically has been defined as the inductive reactance at resonance divided by the resistance in a series RLC circuit, or

$$Q = \frac{\omega_0 L}{R}\,. \tag{2.19}$$

Substituting for R/L in 2.18,

$$\frac{Vout}{Vin} = \frac{\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}\,. \tag{2.20}$$

In the context of our active filter design (without inductors), we will see that Q will serve a more general purpose as a measure of filter selectivity and as a measure of the angle of the poles with respect to the negative real s axis. For instance, in the case of a second order band-pass filter, a high Q filter is considered to be a more highly "selective" filter, whose poles are "close" to the $j\omega$ axis of the s-plane. Substituting in our expressions for Q and ω_o the expression for the pole locations in 2.16 becomes

$$s = -\omega_0\left(\frac{1}{2Q} \pm j\sqrt{1 - \frac{1}{4Q^2}}\right). \tag{2.21}$$

These poles are shown in Figure 2.17.

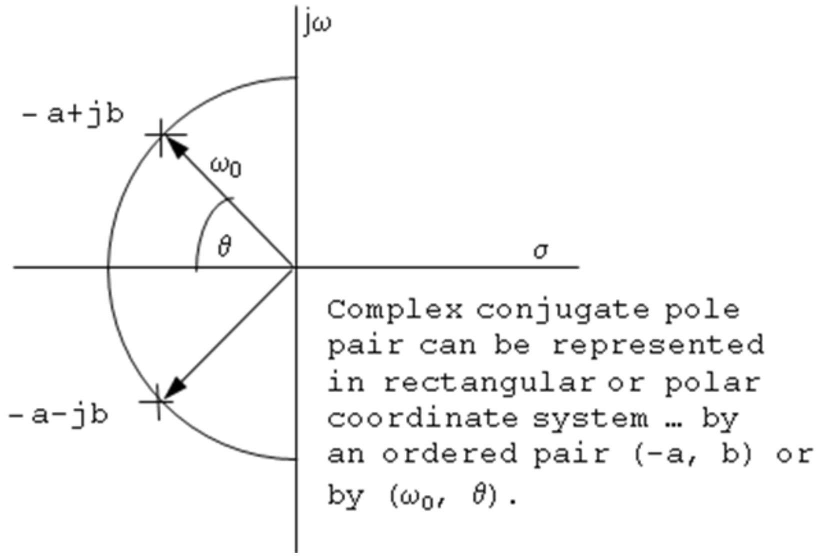


Figure 2.17. Complex conjugate poles showing different coordinate representations.

In Figure 2.17, the cosine of the angle between the pole and the negative real axis is

$$\cos(\theta) = \frac{-\text{Real(s)}}{|s|} = \frac{\left(\frac{\omega_0}{2Q}\right)}{\omega_0} = \frac{1}{2Q},$$

or

$$Q = \frac{1}{2\cos(\theta)}. \qquad (\text{Note}: \ \theta = \cos^{-1}(\frac{1}{2Q}) \ ) \qquad (2.22)$$

In other texts (particularly in Automatic Controls) you may have seen or will see quadratic factors of the form

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \qquad (2.23)$$

Before considering any examples, we should discuss further exactly what we mean by ω_o or ω_n, Q, and ζ. Refer to Figure 2.17 and observe that we could use either a rectangular or a polar coordinate system to describe a set of pole or zero locations. That is, -a+jb and (ω_o,θ) are both equally valid ways to specify the pole locations. The term ω_o in a filter design context (or ω_n in a control systems context) represents the resonant frequency of the filter or system. As shown in Figure 2.17 it is simply the distance of the pole from the s-plane origin. The term ζ in 2.23 is called the "damping coefficient" because we can often determine transient behavior or damping characteristics of each second order section from this parameter. The larger the damping coefficient, the more quickly the transient response decays over time. Equating coefficients of s in 2.23 and the denominator of 2.20, ζ is related to Q and $\cos(\theta)$ by

$$2\zeta = \frac{1}{Q}$$

and

$$\zeta = \cos(\theta)\,. \tag{2.24}$$

Before doing our Bode plot for the transfer function in 2.20, we should make some instructive observations about the magnitude and phase response of this transfer function at low and at high frequencies. Substituting $s = j\omega$ in 2.20,

$$H(j\omega) = \frac{Vout}{Vin}(j\omega) = \frac{j\dfrac{\omega\,\omega_0}{Q}}{-\omega^2 + j\dfrac{\omega\,\omega_0}{Q} + \omega_0^2}\,. \tag{2.25}$$

Observe that for low frequencies ($\omega \ll \omega_o$), $H(j\omega) \approx \dfrac{j\omega}{Q\omega_0}$, so that we should expect our Bode plot to begin with a slope of 20 dB/decade and 90° phase. For high frequencies ($\omega \gg \omega_o$), $H(j\omega) \approx \dfrac{j\omega_0}{-Q\omega}$, so that we should expect our Bode plot to end with a slope of -20 dB/decade and -90° phase. These two magnitude asymptotes intersect when

$$\frac{\omega}{Q\omega_0} = \frac{\omega_0}{Q\omega}$$

or when

$$\omega = \omega_o, \tag{2.26}$$

and they intersect at a magnitude of $\dfrac{1}{Q}$. This response is illustrated for $Q = 0.5$, 1, 2, and 4 in Figure 2.18.

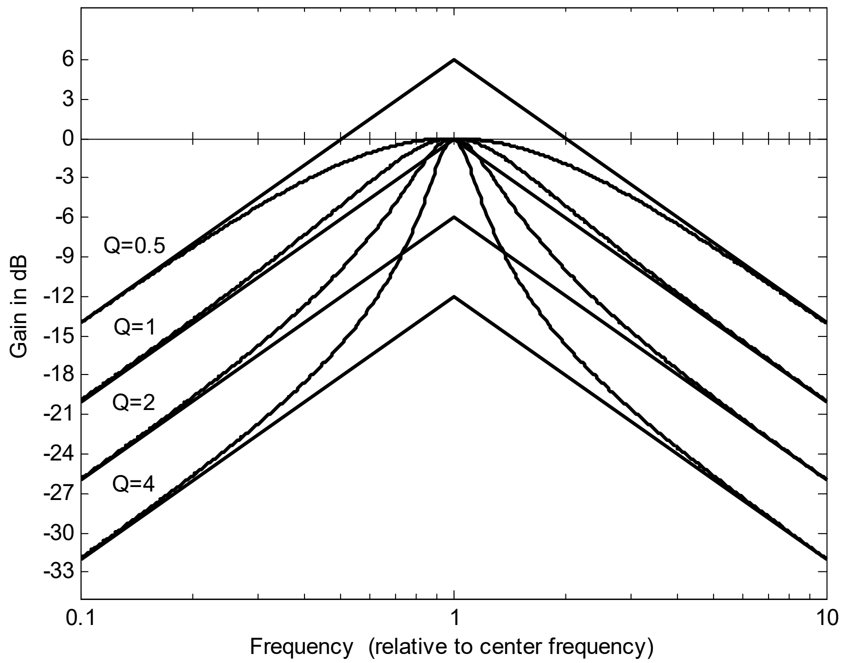


Figure 2.18. Magnitude response of passive band-pass filter.

At this frequency ($\omega = \omega_o$), $H(j\omega_o) = 1$, or a gain of Q relative to the gain at the intersection of the asymptotes. If we evaluate the gain as a function of frequency

$$|H(j\omega)| = \frac{\frac{\omega\omega_0}{Q}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \left(\frac{\omega_0\omega}{Q}\right)^2}}$$

$$= \frac{1}{\sqrt{Q^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + 1}} . \qquad (2.27)$$

Typically the bandwidth of a band-pass filter is defined as the difference between the two frequencies where the gain is $\frac{1}{\sqrt{2}}$ or - 3 dB relative to the center frequency gain. This gain is equal to $\frac{1}{\sqrt{2}}$ when

$$Q^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 = 1 ,$$

or equivalently when

$$Q\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)=\pm 1\,. \tag{2.28}$$

In solving for these half power frequencies or passband limits, we multiply both sides of 2.28 by ω and obtain the quadratic equation

$$\frac{Q}{\omega_0}\omega^2 \pm \omega - Q\omega_0 = 0\,. \tag{2.29}$$

Using the quadratic equation to solve for ω,

$$\omega = \omega_0\left(\pm\frac{1}{2Q}\pm\sqrt{\frac{1}{4Q^2}+1}\right)$$

$$= \pm\omega_0\left(\sqrt{\frac{1}{4Q^2}+1}+\frac{1}{2Q}\right) \tag{2.30}$$

and

$$= \pm\omega_0\left(\sqrt{\frac{1}{4Q^2}+1}-\frac{1}{2Q}\right). \tag{2.31}$$

As in all transfer functions with real coefficients, the magnitude response is symmetric or even about $\omega = 0$. We are interested in the absolute or positive values of the two passband limits, and these limits are

$$\omega_1 = \omega_0\left(\sqrt{\frac{1}{4Q^2}+1}-\frac{1}{2Q}\right) \tag{2.32}$$

and

$$\omega_2 = \omega_0\left(\sqrt{\frac{1}{4Q^2}+1}+\frac{1}{2Q}\right). \tag{2.33}$$

At this point we should make three important observations about these two frequencies ω_1 and ω_2 as defined in 2.32 and 2.33.

1. Since $\omega_1\omega_2 = \omega_o^2$, the frequencies ω_1 and ω_2 are symmetric about ω_o on a *logarithmic* frequency scale as we used for Bode plots.

2. From 2.32 and 2.33 we note that ω_1 and ω_2 are arithmetically symmetric about the frequency

$$\omega_0\sqrt{\frac{1}{4Q^2}+1} \approx \omega_0\left(\frac{1}{8Q^2}+1\right) \tag{2.34}$$

on a linear frequency scale.

3. Their difference $\omega_2 - \omega_1 = \left(\frac{\omega_0}{Q}\right)$ is called the half power or 3 dB bandwidth of the filter.

Our third observation suggests yet another interpretation of this parameter Q. It is the resonant frequency relative to the half power bandwidth in a second order band-pass filter.

Now we will examine the asymptotic phase response for our second order band-pass filter. The complete phase response as calculated from 2.25 is given as

$$\angle H(j\omega) = \angle\left(\frac{j\frac{\omega\,\omega_0}{Q}}{-\omega^2 + j\frac{\omega\,\omega_0}{Q} + \omega_0^2}\right) \tag{2.25}$$

$$= 90° - \angle\left(\omega_0^2 - \omega^2 + j\frac{\omega_0\,\omega}{Q}\right) \tag{2.35}$$

$$= 90° - \arctan\left(\frac{\frac{\omega_0\,\omega}{Q}}{\omega_0^2 - \omega^2}\right). \tag{2.36}$$

Observe that for $\omega \ll \omega_o$, 2.36 simplifies to

$$H(j\omega) \approx 90° - \arctan\left(\frac{\omega}{\omega_0 Q}\right) \approx 90°. \tag{2.37}$$

For $\omega \gg \omega_o$ 2.36 simplifies to

$$\angle H(j\omega) = 90° - \arctan\left(\frac{\frac{\omega_0\,\omega}{Q}}{\omega_0^2 - \omega^2}\right)$$

$$\approx 90° - \arctan\left(\frac{-\omega_0}{Q\omega}\right) \approx -90°. \tag{2.38}$$

At $\omega=\omega_o$ we observe from 2.35 that the phase is exactly 0°. Now we know the phase plot goes from 90° to -90° as we move from low to high frequencies, and passes through 0° at $\omega=\omega_o$. Unfortunately the <u>transition rate</u> or slope (degrees/decade) going through 0° is a function of Q, and we can no longer say that the slope is a constant as we did for the case of real poles or zeros.

Instead we recognize that if we had a double real pole we would have a Q=0.5 and a corresponding slope of -90°/decade over two decades. For the more general case including complex poles, we can approximate this slope as a function of Q by saying that the slope is negative Q x 180° per decade for $\frac{1}{Q}$ decades. Since the bandwidth relative to the center frequency goes as $\frac{1}{Q}$, the phase transition should occur at a rate proportional to Q, and should occur over a frequency span inversely proportional to Q. Figure 2.19 shows this asymptotic phase representation compared to the actual phase for values of Q= 0.5, 1, 2, and 4.

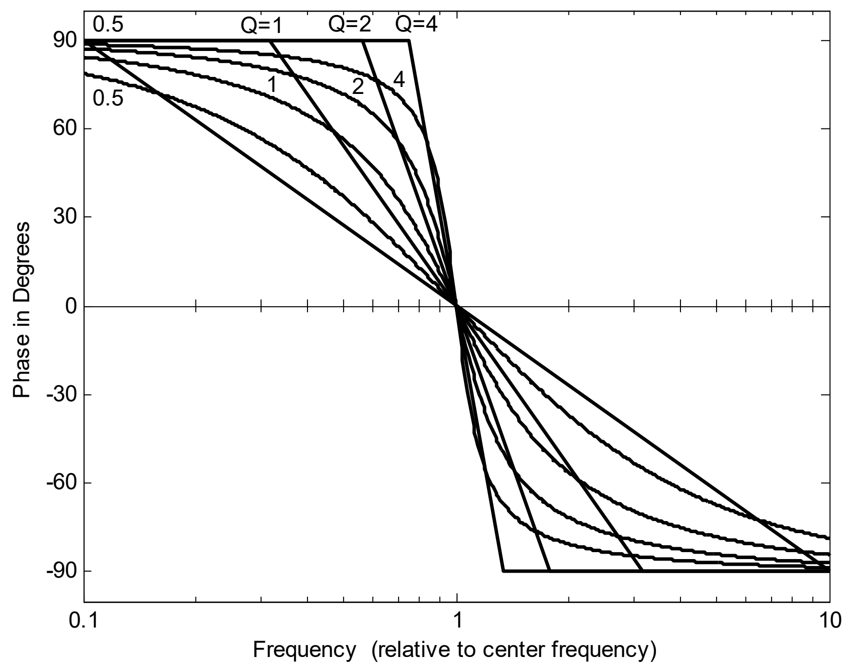


Figure 2.19. Asymptotic phase (in degrees) for second order band-pass filter.

Example 2.6 Second Order Band-pass Filter Bode Plot

To illustrate the concepts above we will consider the transfer function

$H(s) = \frac{500s}{s^2 + 5s + 100}$ as our example. Evaluating the parameters in 2.20 by inspection we observe that $\omega_O = \sqrt{100} = 10$ rad/sec, $\frac{w_O}{Q} = 5$, and Q = 2. The center frequency gain is $\frac{500}{5}$ or 40 dB. The intersection of the asymptotes occurs at 20 $\log_{10}$(Q) below 40 dB, or 6 dB lower at 34 dB. From 2.34, the two half power frequencies are arithmetically symmetric about the frequency

$\omega_0\left(\frac{1}{8Q^2}+1\right) = 10\left(\frac{1}{8(2)^2}+1\right)$ = 10.3 rad/sec, and are 5 rad/sec apart.

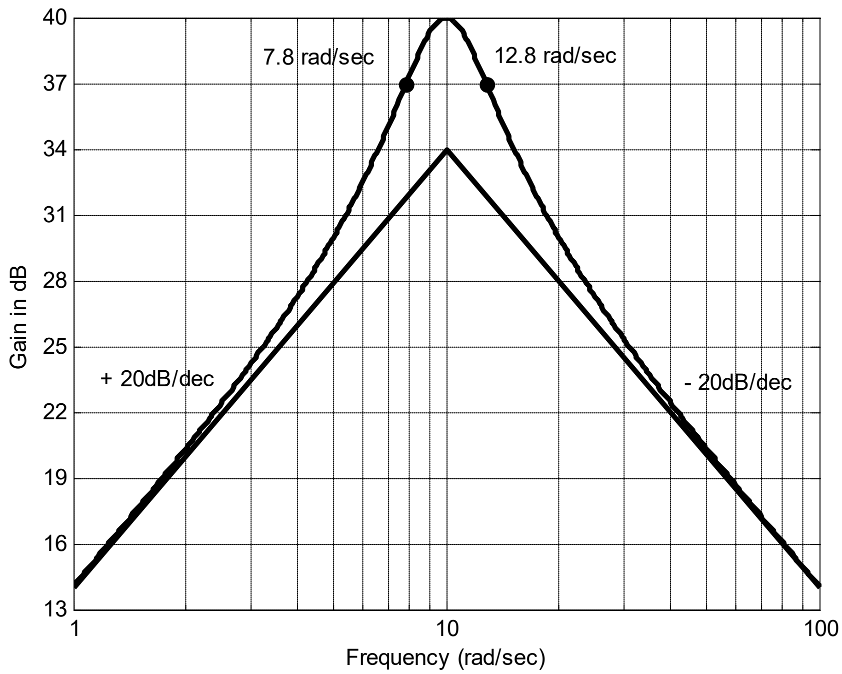


Figure 2.20. Magnitude response for Example 2.6.

The phase starts at +90°, for Q = 2 the asymptote is drawn at a slope of -360°/decade from 0.25 decades below 10rad/sec ($\omega_0 10^{\frac{-1}{2Q}} = 10 \bullet 10^{-0.25} = 10^{0.75} = 5.6$) to 0.25 decades above 10 ($\omega_0 10^{\frac{1}{2Q}} = 10^{1.25} = 17.8$). These asymptotes are compared to the actual phase in Figure 2.21.

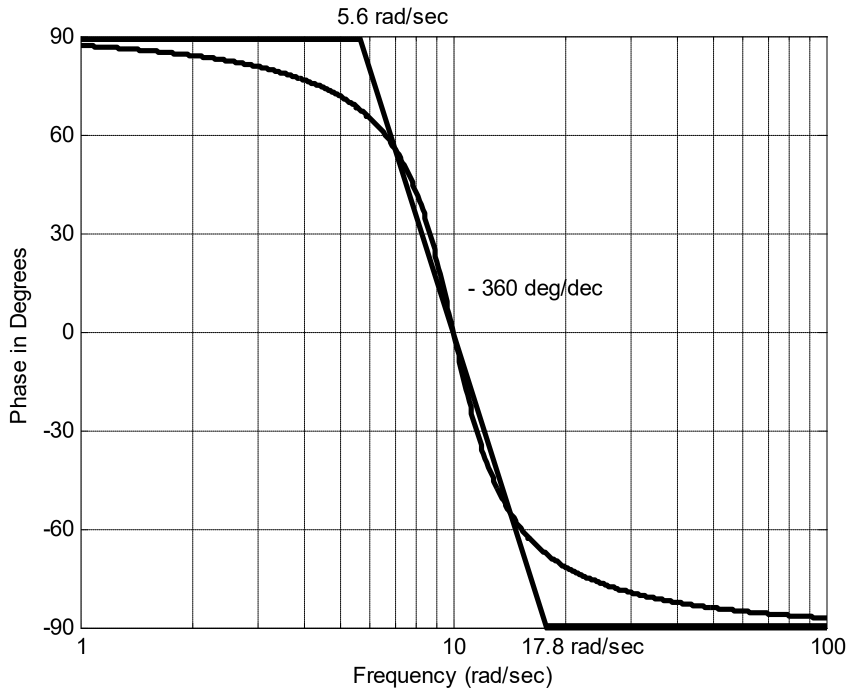
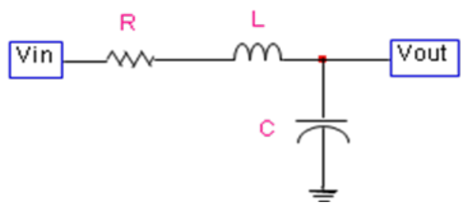
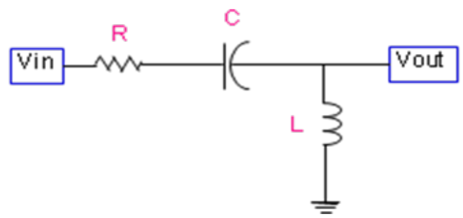


Figure 2.21. Phase response for Example 2.6.

Second Order Low-pass and High-pass Filters with Complex Poles



a. Passive low-pass filter



b. Passive high-pass filter

Figure 2.22. Second order LP and HP filters with complex poles.

Using voltage division the second order transfer functions for the circuits in Figure 2.22 become

$$H_{LP}(s) = \frac{Vout}{Vin} = \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}}$$

$$= \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}. \tag{2.39}$$

for the low-pass filter, and

$$H_{HP}(s) = \frac{Vout}{Vin} = \frac{sL}{sL + R + \frac{1}{sC}}$$

$$= \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}. \tag{2.40}$$

for the high-pass filter. Defining the parameters ω_o. and Q as we did in the band-pass case, these transfer functions can be rewritten as

$$H_{LP}(s) = \frac{\omega_0{}^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0{}^2} \tag{2.41}$$

and

$$H_{HP}(s) = \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0{}^2} \tag{2.42}$$

respectively. Substituting $j\omega$ for s, the frequency responses are

$$H_{LP}(j\omega) = \frac{\omega_0{}^2}{-\omega^2 + \dfrac{j\omega_0\,\omega}{Q} + \omega_0{}^2} \qquad (2.43)$$

and

$$H_{HP}(s) = \frac{-\omega^2}{-\omega^2 + \dfrac{j\omega_0\,\omega}{Q} + \omega_0{}^2}\,. \qquad (2.44)$$

At this point we note the following:

a. For $\omega \ll \omega_o$, the low-pass filter response in 2.43 has unity gain, zero phase.

b. For $\omega \gg \omega_o$, the high-pass filter response in 2.44 has unity gain, zero phase.

c. For $\omega \gg \omega_o$, the low-pass filter magnitude response in 2.43 equals $\frac{\omega_0{}^2}{\omega^2}$ or has an asymptote at -40 dB/decade that intersects the low frequency asymptote at $\omega = \omega_o$. The phase is -180°.

d. For $\omega \ll \omega_o$, the high-pass filter magnitude response in 2.44 equals $\frac{\omega^2}{\omega_0{}^2}$ or an asymptote at +40 dB/decade that intersects the high frequency asymptote at $\omega = \omega_o$. The phase is +180°.

e. For $\omega = \omega_o$, the gain of both filters equals Q. Note that for all second order filters (low-pass, high-pass, and band-pass) the gain at $\omega = \omega_o$ is 20 $\log_{10}(Q)$ relative to the gain at the intersection of the asymptotes. The phase is -90° for the low-pass case and 90° for the high-pass case.

f. The change in phase with frequency for all these second order filters is a function only of the denominator since the numerator phase is a constant 0°, 90° or 180°. Therefore, we will draw our phase plots exactly as we did in the band-pass case, except we will start at 90° x (number of zeros at s = 0).

Figures 2.23 a and b illustrate the magnitude response of low and high-pass filters for Q = 2 and Figure 2.24 illustrates the phase response for Q = 0.5, Q=1, and Q=2.

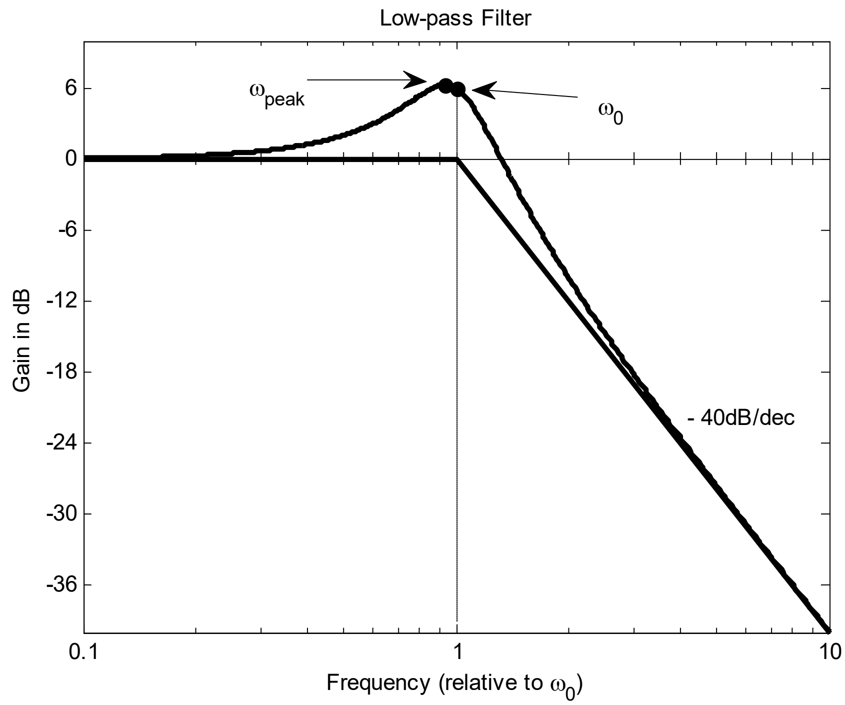


Figure 2.23a. Magnitude response of second order low-pass filter (Q=2).

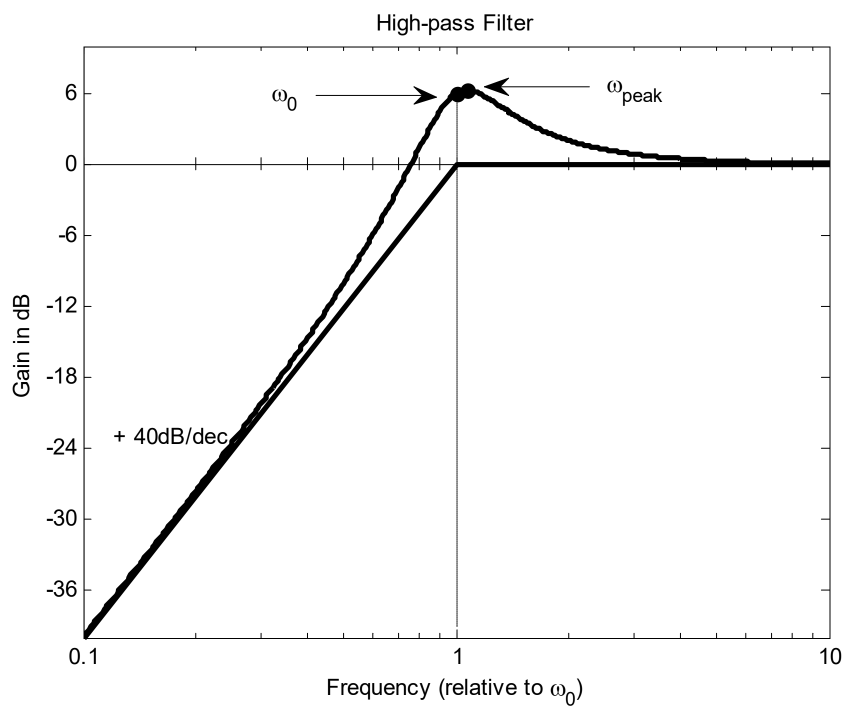


Figure 2.23b. Magnitude response of second order high-pass filter (Q=2).

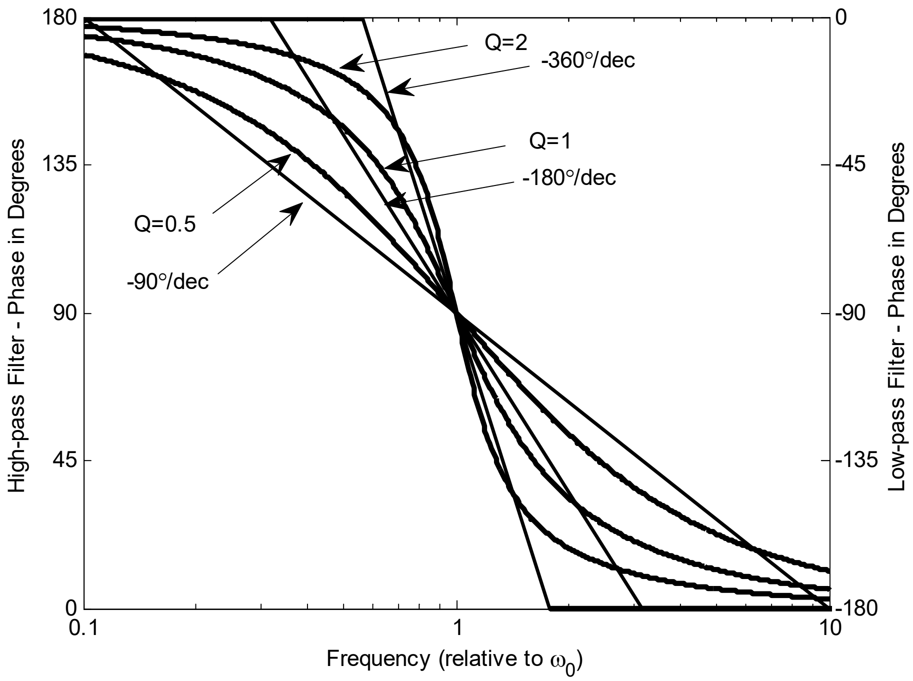


Figure 2.24. Phase response of second order low-pass and high-pass filters.

The magnitude response is 20 $\log_{10}(Q)$ = 20 $\log_{10}(2)$ = 6 dB at $\omega = \omega_o$ for both low-pass and high-pass cases as shown in Figure 2.23. Unlike the band-pass case the maximum gain does not occur exactly at $\omega = \omega_o$. In fact, the maximum occurs at a lower frequency for low-pass filters and at a higher frequency for high-pass filters. To evaluate this maximum gain we need to take the derivatives of the magnitudes of 2.43 and 2.44 with respect to ω and set the results equal to zero. It is left as a homework exercise to prove that if $Q > \frac{\sqrt{2}}{2}$, the actual low-pass peak magnitude occurs at

$$\omega_{peak} = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}\,, \tag{2.45}$$

and the high-pass peak magnitude occurs at

$$\omega_{peak} = \frac{\omega_0}{\sqrt{1 - \frac{1}{2Q^2}}}\,. \tag{2.46}$$

For both cases the value at that frequency ω_{peak} is

$$|H(j\omega)|_{dB_{peak}} = 20\log_{10}\left(\frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}\right).$$

$$= 20\log_{10}\left(\frac{2Q^2}{\sqrt{4Q^2-1}}\right). \qquad (2.47)$$

For high values of Q (Q>2.5), 2.45-2.47 may be simplified (for purposes of sketching our Bode plot) to

$\omega_{peak} \approx \omega_o$, and

$$|H(j\omega)|_{dB_{peak}} \approx 20\ \log_{10}(Q). \qquad (2.48)$$

Therefore our Bode magnitude plots will have the low and high frequency asymptotes as noted before, and to more accurately represent the true frequency response, we will include the response at the break point, which deviates by 20 $\log_{10}(Q)$ from the intersection of the asymptotes at ω_o.

Notch Filters

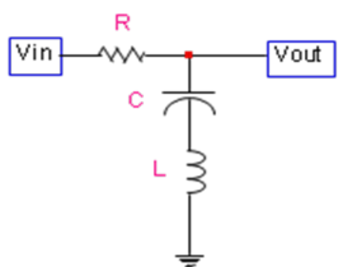


Figure 2.25. Passive notch filter.

For the circuit in Figure 2.25 we use voltage division once again to derive the transfer function. If we define ω_o and Q as before, the transfer function is

$$H(s) = \frac{sL + \frac{1}{sC}}{sL + R + \frac{1}{sC}} = \frac{s^2 + \omega_0{}^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0{}^2}, \qquad (2.49)$$

and substituting s=jω to obtain the frequency response we find

$$H(j\omega) = \frac{-\omega^2 + \omega_0{}^2}{-\omega^2 + \frac{j\omega_0\,\omega}{Q} + \omega_0{}^2}. \qquad (2.50)$$

We note the following about the frequency response:

a. For $\omega << \omega_o$ and for $\omega >> \omega_o$, the notch filter has unity gain, zero phase.

b. For $\omega = \omega_o$, the gain equals 0 or -infinity in dB.

c. At $\omega = \omega_o$, the numerator instantly changes sign and therefore changes phase from 0° to ±180°. The denominator phase at this frequency is exactly 90°, therefore the overall phase changes from -90° to 90° at the discontinuity. Except for this discontinuity the change with frequency is the same as for the previous second order filters and we draw the asymptotes accordingly.

d. Since the numerator is purely real and equal to the real part of the denominator, the gain is equal to $\frac{1}{\sqrt{2}}$ or - 3 dB when the imaginary part of the denominator equals plus or minus the real part. This is precisely the same as for the band-pass case and these half power frequencies in 2.34 are the same.

Example 2.7

Draw the Bode plot (magnitude and phase) for

$$H(s) = \frac{s^2 + 10^6}{s^2 + 800s + 10^6}$$

$$\omega_0 = \sqrt{10^6} = 10^3$$

$$Q = \frac{10^3}{800} = 1.25$$

The half power frequencies are (from 2.34) centered at $\omega_0\left(\frac{1}{8(1.25)^2} + 1\right) = 1080$ rad/sec. They are separated by 800 rad/sec, and therefore are at 680 and 1480 rad/sec. The magnitude response is illustrated in Figure 2.26.

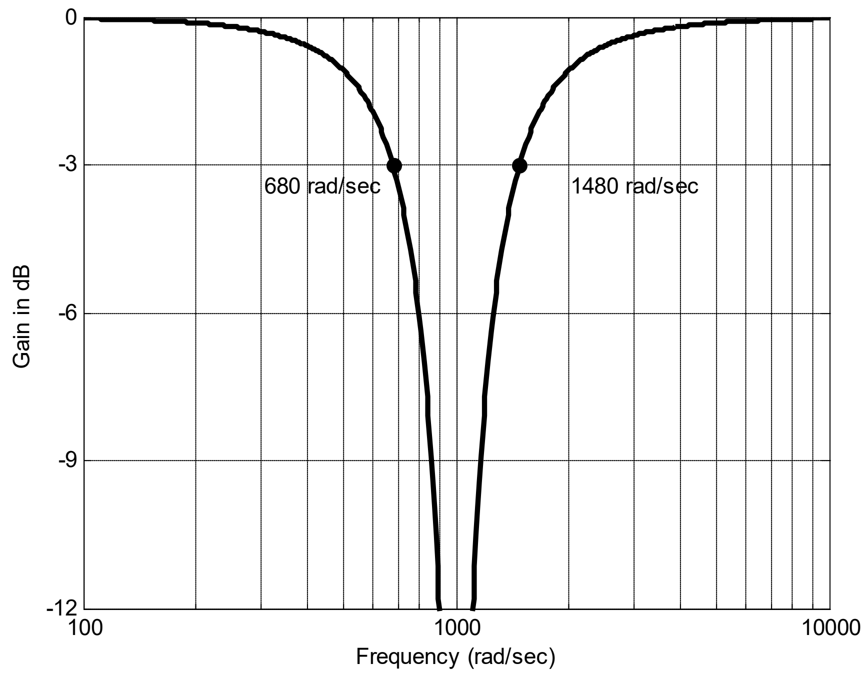


Figure 2.26. Magnitude response of notch filter.

The phase asymptotes have a slope of -180 x Q = -225°/decade and extend $\pm\frac{1}{2Q} = \pm 0.4$ decades from the center frequency or from 400 to 2500 rad/sec. The phase plot is shown in Figure 2.27.

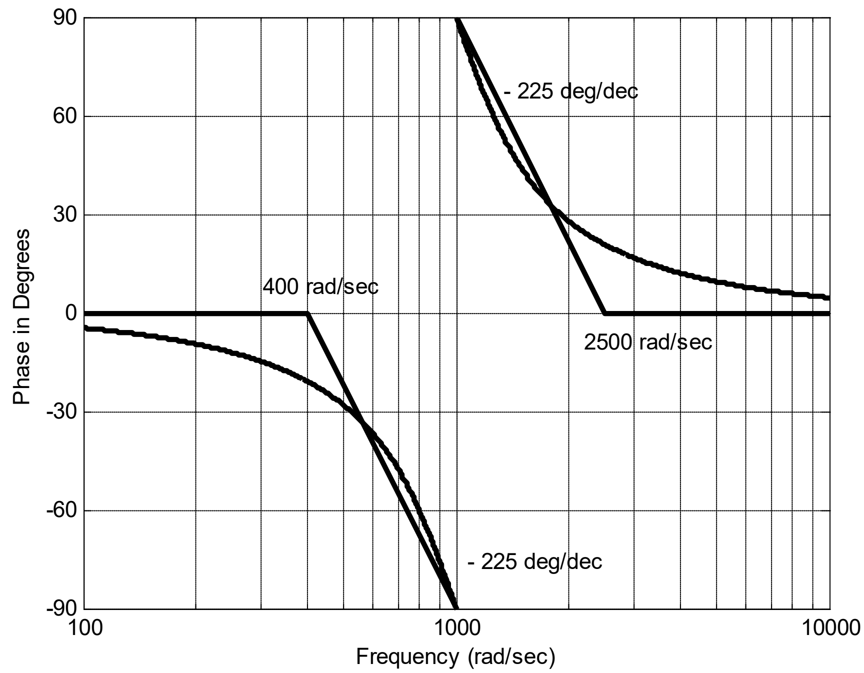
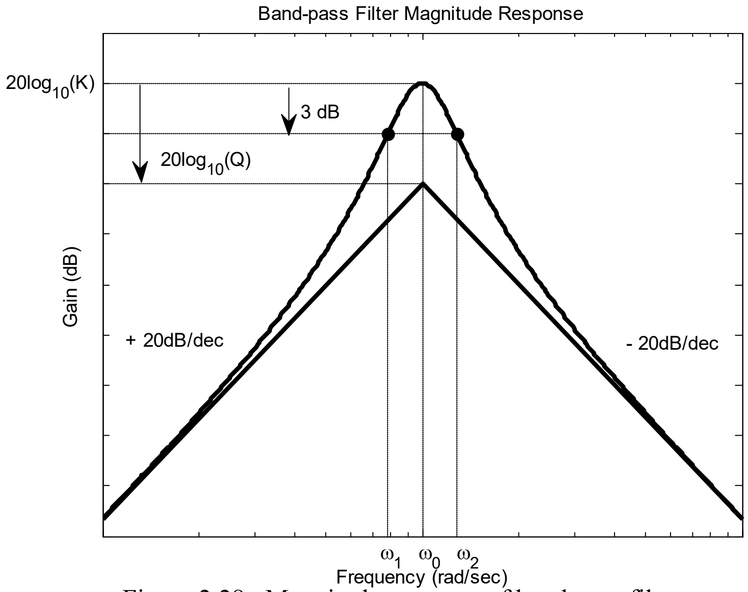


Figure 2.27. Phase response of notch filter.

Summary of Bode Plots for Complex Poles

2nd Order Band-pass Filter (with complex poles)

$$H(s) = \frac{K\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \qquad \text{(assuming K is a positive real number)}$$



$$\omega_1, \omega_2 = \omega_0\left(\frac{1}{8Q^2}+1\right) \pm \frac{\omega_0}{2Q}$$

Figure 2.28. Magnitude response of band-pass filter.

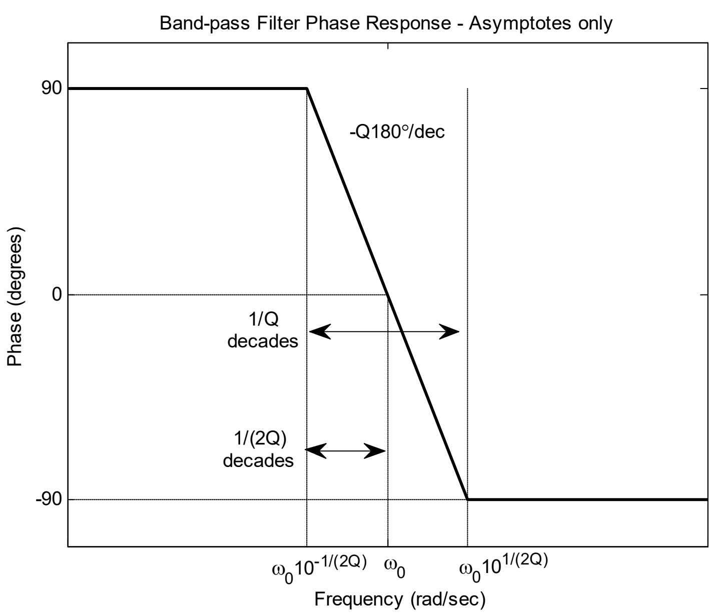
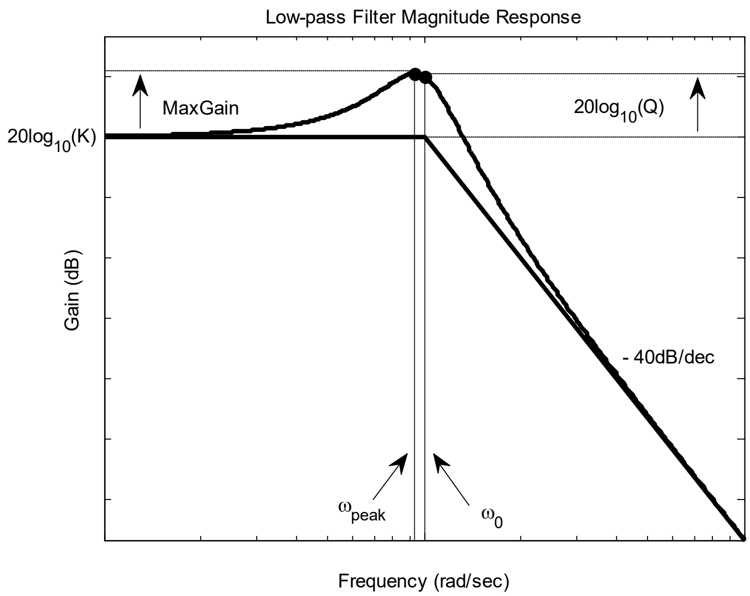


Figure 2.29. Phase response of band-pass filter.

2nd Order Low-pass Filter (with complex poles)

$$H(s) = \frac{K\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \qquad \text{(assuming K is a positive real number)}$$



$$\text{MaxGain} = 20\log_{10}\left(\frac{2Q^2}{\sqrt{4Q^2-1}}\right)$$

$$\omega_{peak} = \omega_0\sqrt{1-\frac{1}{2Q^2}}$$

Figure 2.30. Magnitude response of low-pass filter.

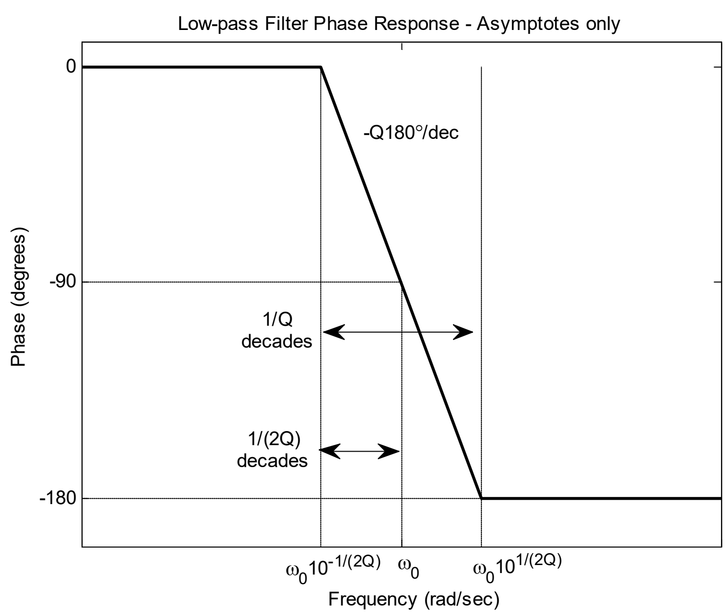
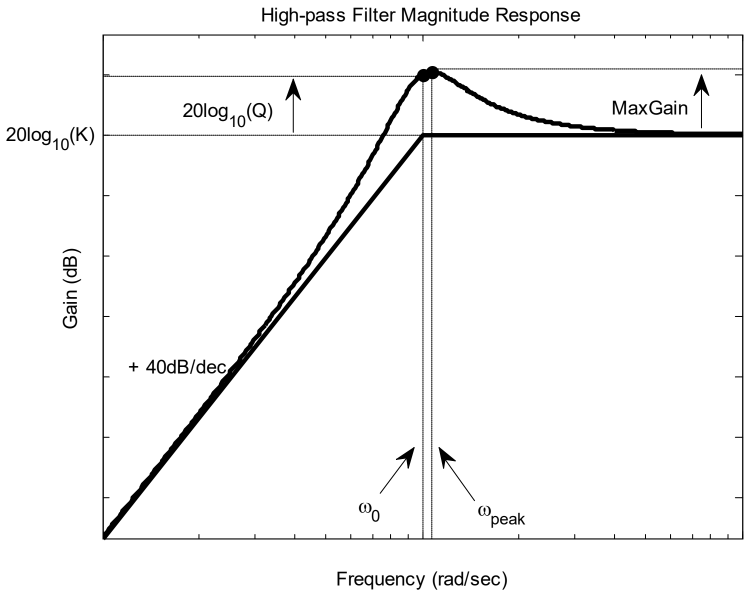


Figure 2.31. Phase response of low-pass filter.

2^{nd} Order High-pass Filter (with complex poles)

$$H(s) = \frac{Ks^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \qquad \text{(assuming K is a positive real number)}$$



$$\text{MaxGain} = 20\log_{10}\left(\frac{2Q^2}{\sqrt{4Q^2-1}}\right)$$

$$\omega_{peak} = \frac{\omega_0}{\sqrt{1-\frac{1}{2Q^2}}}$$

Figure 2.32. Magnitude response of high-pass filter.

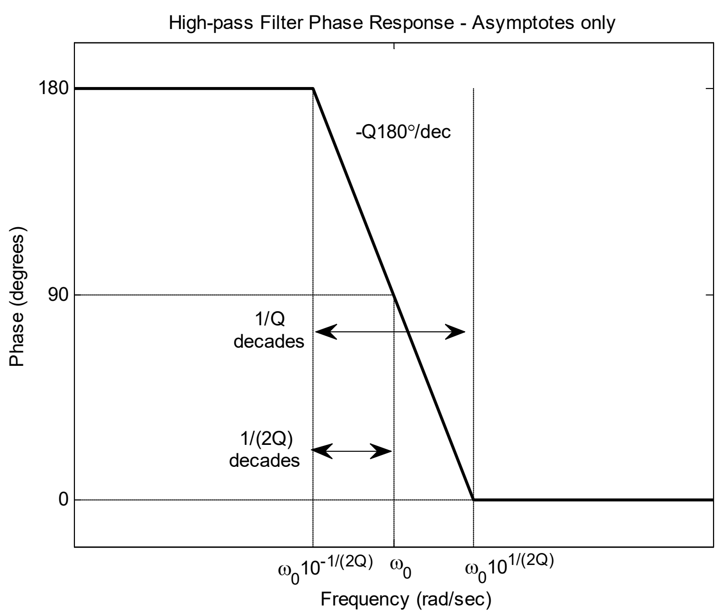
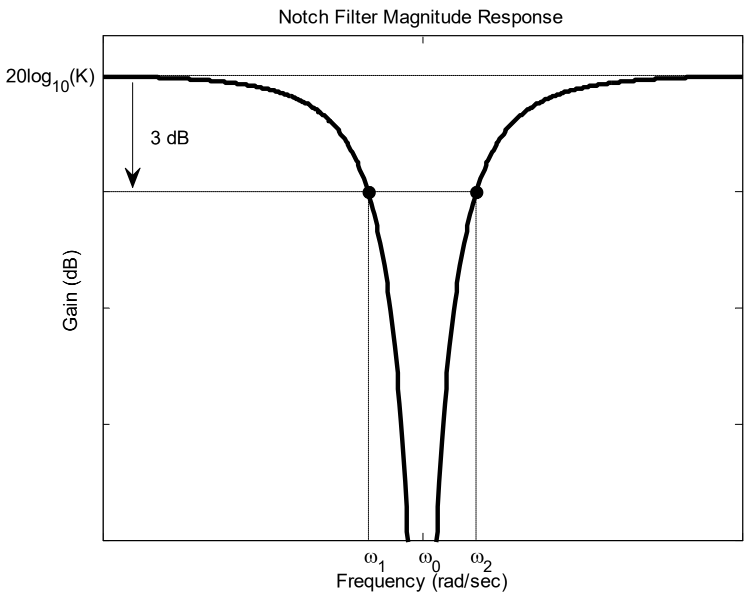


Figure 2.33. Phase response of high-pass filter.

2nd Order Notch Filter (with complex poles)

$$H(s) = \frac{K(s^2 + \omega_0^2)}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \qquad \text{(assuming K is a positive real number)}$$



$$\omega_1, \omega_2 = \omega_0\left(\frac{1}{8Q^2}+1\right) \pm \frac{\omega_0}{2Q}$$

Figure 2.34. Magnitude response of notch filter.

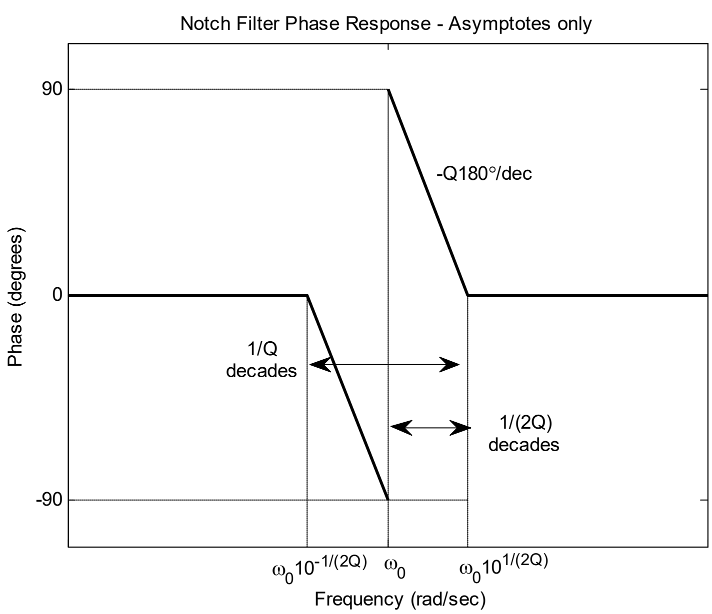


Figure 2.35. Phase response of notch filter.

2.4 Concluding Remarks

Our study of Bode plot techniques is important in that these techniques emphasize the relationship between poles and zeros of the system transfer function, and the inherent frequency response of the system. In this chapter we studied the Bode plots of simple "building block" transfer functions that can be combined in an additive fashion to represent the composite magnitude and phase responses of more complicated transfer functions. We were given a transfer function and we were asked to sketch the frequency response. We will soon make great use of these newly learned skills when we face a slightly different problem. As the title of the text suggests, we will soon be designing filters (both analog and digital) in order to meet some desired set of filter specifications. In most cases our designs will be a cascade of first and second order sections. Not only will Bode techniques form an integral part of the design process, but knowledge of these techniques will help us to verify quickly the theoretical frequency response for our designs.

Problems

2.1 Show that a filter roll-off characteristic of -60dB/decade is the same as a roll-off of -18dB/octave. (Recall a decade is a factor of 10 and an octave is a factor of 2 in frequency.)

2.2 Show that a Bode (asymptotic) magnitude plot for the transfer function

$$H(s) = \frac{a}{s+a}$$

deviates from the actual frequency response by approximately

a) 3dB at the break frequency,
b) 1dB at one octave above or below the break frequency, and
c) 0.3dB at two octaves above or below the break frequency.

2.3 a. Prove that for $H_{LP}(s) = \dfrac{\omega_0^{\;2}}{s^2 + \dfrac{\omega_0}{Q}s + \omega_0^{\;2}}$ and $H_{HP}(s) = \dfrac{s^2}{s^2 + \dfrac{\omega_0}{Q}s + \omega_0^{\;2}}$,

the peak magnitude of the frequency response occurs at

$$\omega_{peak} = \omega_0\sqrt{1 - \frac{1}{2Q^2}}\,.$$

and

$$\omega_{peak} = \frac{\omega_0}{\sqrt{1 - \dfrac{1}{2Q^2}}}\text{, respectively.}$$

b. Show that the magnitude of both transfer function at the ω_{peak} found in part (a) is

$$|H(j\omega)|_{dB_{peak}} = 20\log_{10}\left(\frac{2Q^2}{\sqrt{4Q^2 - 1}}\right).$$

2.4 Prove that for the transfer functions in Problem 2.3 at $\omega = \omega_o$ (i.e. at the break frequency), $|H(j\omega_o)|_{dB} = 20\log_{10}(Q)$.

2.5 Show that a linear, time invariant system characterized by a second order transfer function H(s) consisting of only two complex conjugate poles must have all real coefficients.

2.6 Sketch the Bode plot (both magnitude and phase) for the transfer functions below.

a) $H(s) = \frac{50}{s}$ b) $H(s) = 50s$ c) $H(s) = \frac{1}{s+6}$

d) $H(s) = s + 8$ e) $H(s) = s - 4$ f) $H(s) = \frac{10(s+2)}{(s+0.5)(s+20)}$

g) $H(s) = \frac{1}{s+30}$ h) $H(s) = s+300$ i) $H(s) = \frac{2(s+10)}{(s+1)(s+20)}$

2.7 Sketch the Bode plot (both magnitude and phase) for the transfer functions listed below.

a) $H(s) = \frac{2000}{(s+5)(s+200)}$ b) $H(s) = \frac{5s}{s+50}$

c) $H(s) = \frac{s-2}{s+2}$

2.8-11 Estimate the transfer functions illustrated by the magnitude and phase response plots that follow.

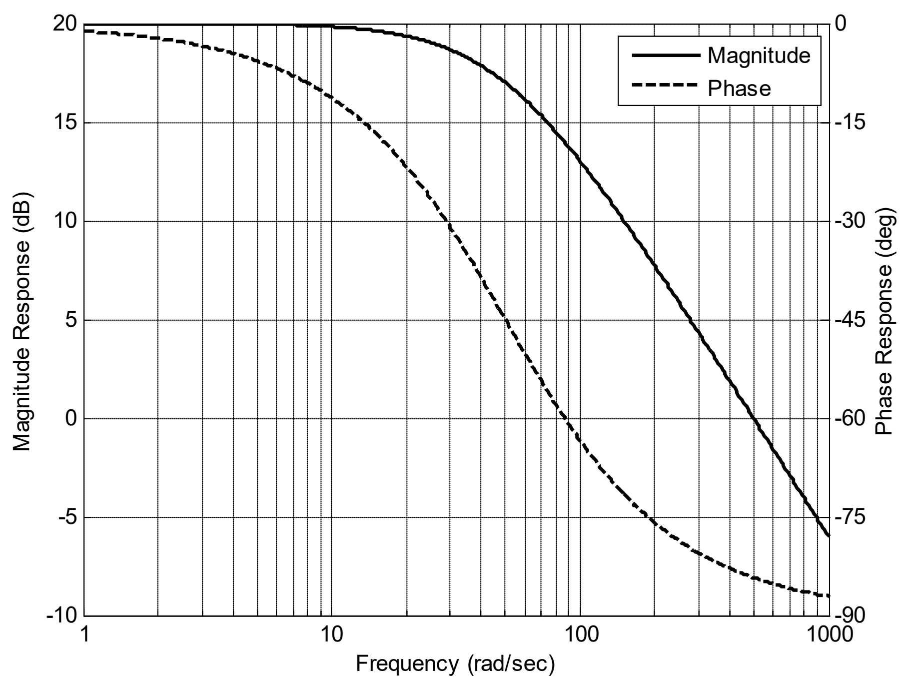


Figure P2.8. Magnitude and phase response for Problem 2.8.

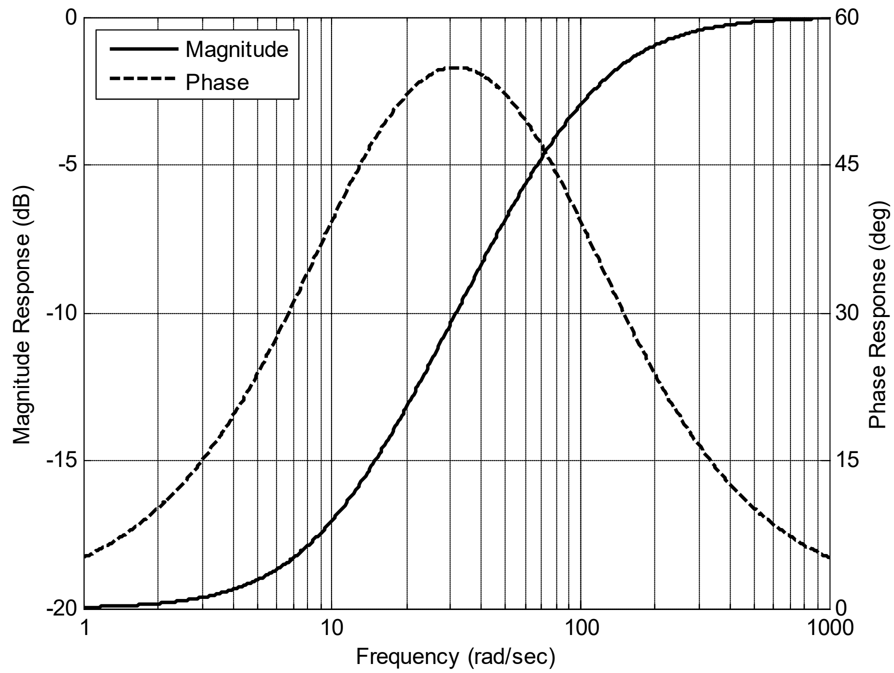


Figure P2.9. Magnitude and phase response for Problem 2.9.

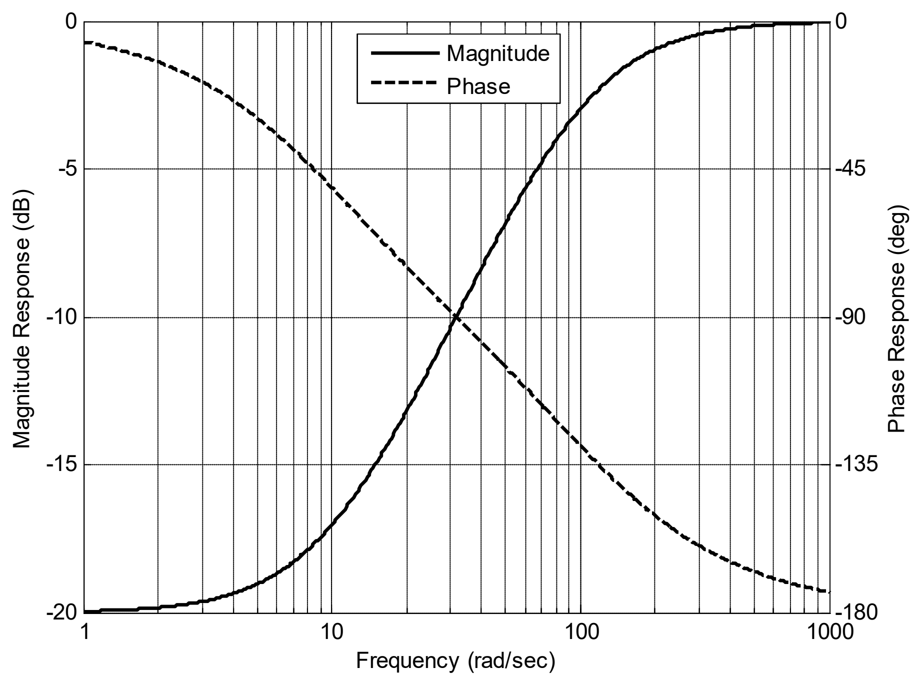


Figure P2.10. Magnitude and phase response for Problem 2.10.

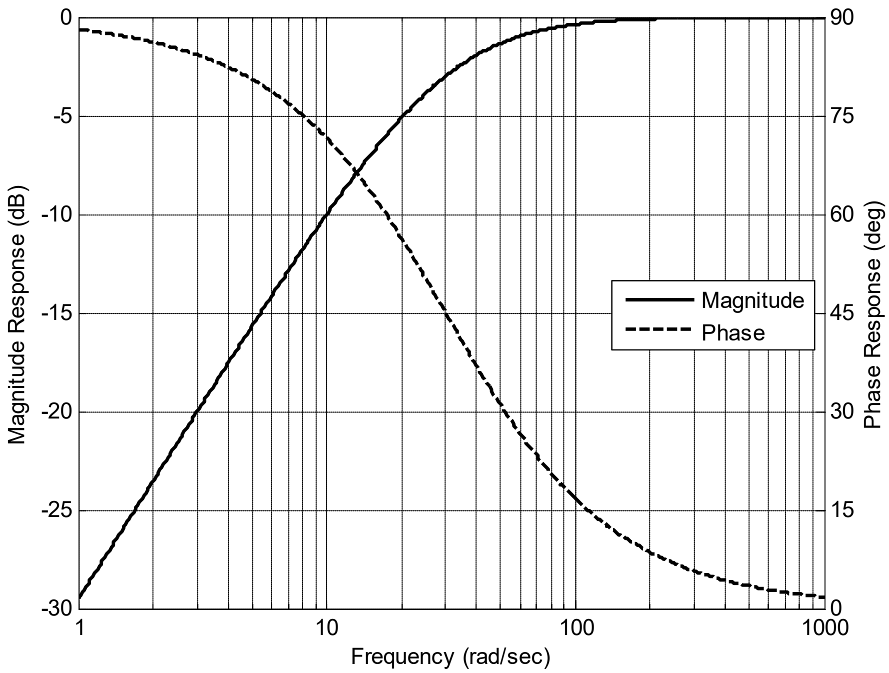


Figure P2.11. Magnitude and phase response for Problem 2.11.

2.12 Sketch the Bode plot (both magnitude and phase) and the pole/zero diagram for the transfer functions listed below.

a) $H(s) = \dfrac{10}{s^2 + s + 1}$

b) $H(s) = \dfrac{4s}{s^2 + 2s + 10}$

c) $H(s) = \dfrac{s^2}{s^2 + s + 1}$

d) $H(s) = \dfrac{s - 3}{s^2 + s + 1}$

e) $H(s) = \dfrac{s^2 + 4s + 12}{(s+3)(s+20)}$

f) $H(s)=\dfrac{50}{s^2+s+1}$

g) $H(s)=\dfrac{20s}{s^2+2s+10}$

h) $H(s)=\dfrac{2s^2}{s^2+s+1}$

i) $H(s)=\dfrac{s^2+64}{s^2+4s+64}$

j) $H(s)=\dfrac{20s^2}{s^2+2s+16}$

k) $H(s)=\dfrac{s^2+100}{s^2+5s+100}$

2.13-2.16 Estimate the transfer functions represented by the magnitude and phase response plots shown below. Sketch the pole and zero locations in the s plane.

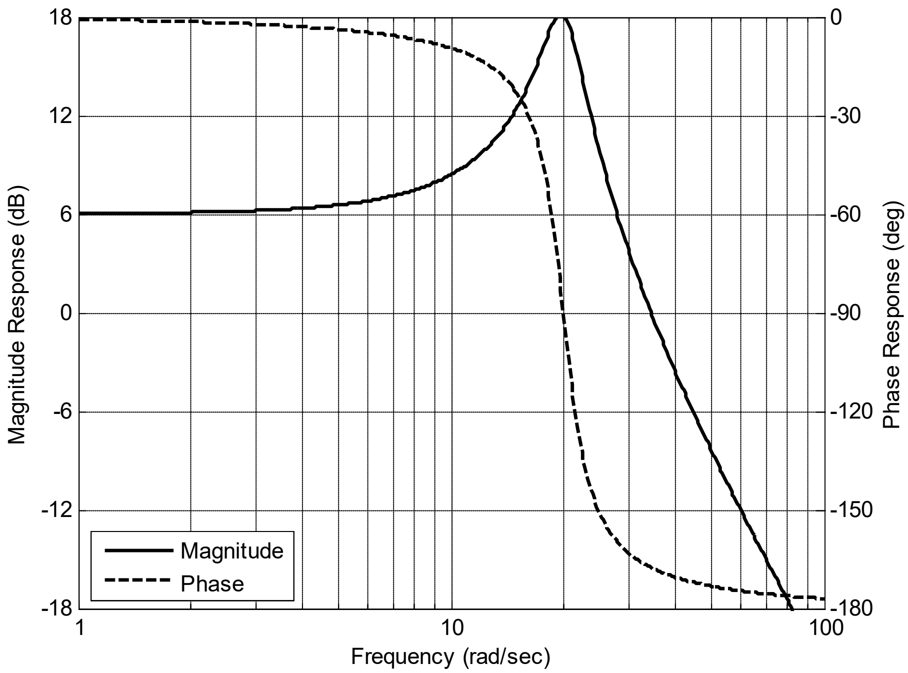


Figure P2.13. Magnitude and phase response for Problem 2.13.

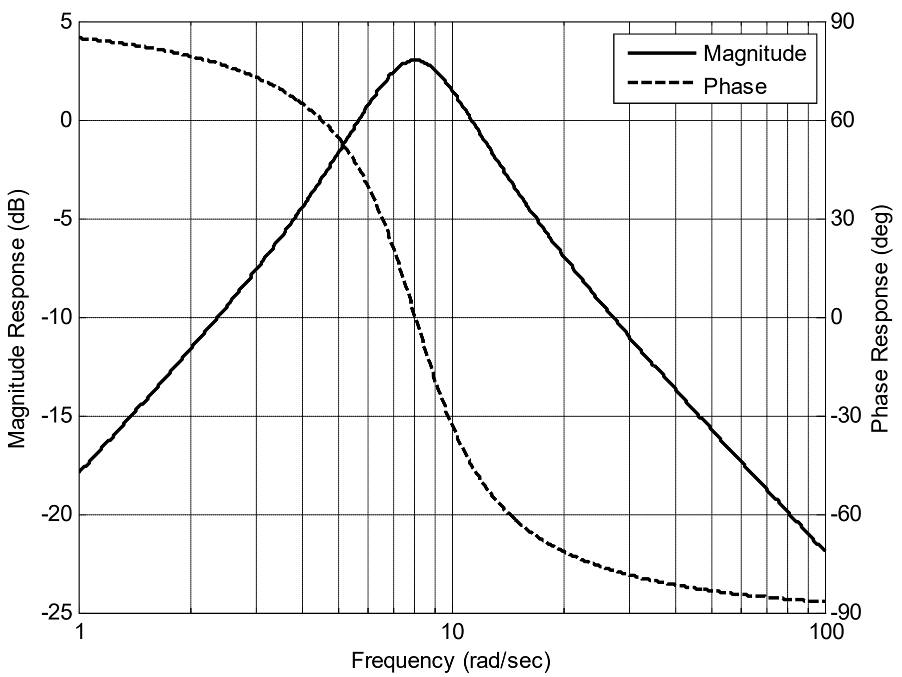


Figure P2.14. Magnitude and phase response for Problem 2.14.

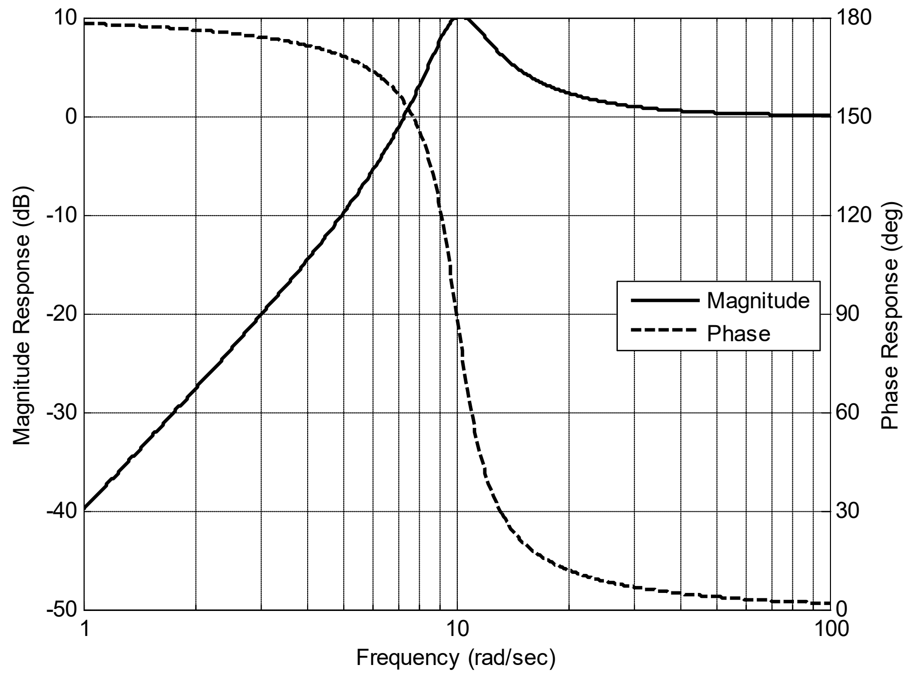


Figure P2.15. Magnitude and phase response for Problem 2.15.

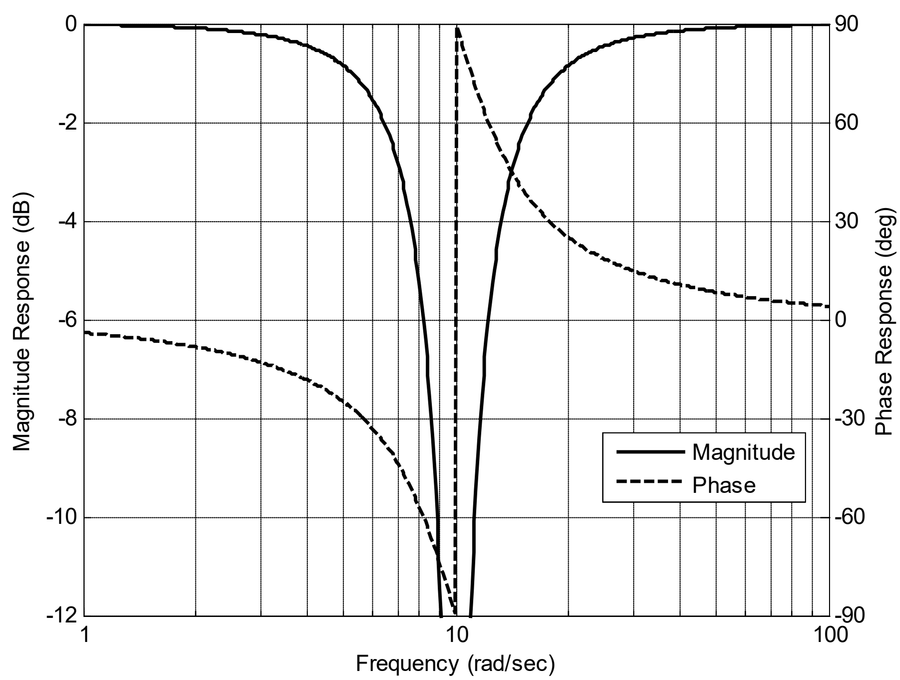


Figure P2.16. Magnitude and phase response for Problem 2.16.

2.17-20 Draw the Bode Plots (magnitude and phase) for the circuits below.

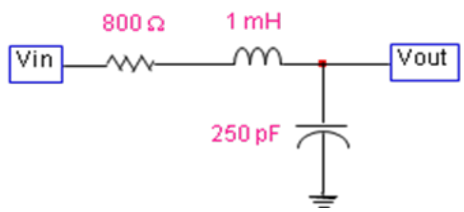


Figure P2.17.

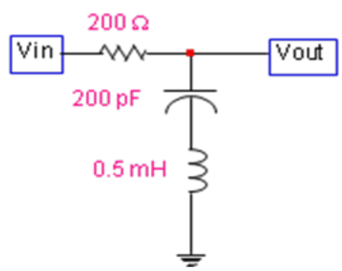


Figure P2.18

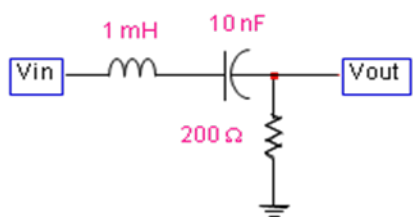


Figure P2.19

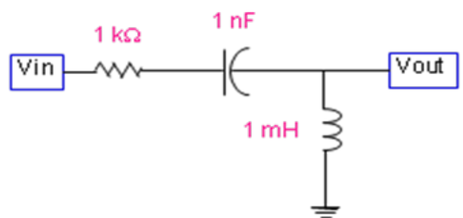


Figure P2.20