

Hopfield network

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This is a demo for illustrating the concept of associative memory using the Hopfield network. The Hopfield network stores a number of unique patterns in a recurrent neuronal network. It also illustrates the concept of content addressable memory, and was extremely influential as a model for episodic memory, where each stored pattern forms the representation of an event.

The idea is the following: A particular event can be defined by the unique combination sensory inputs available for the subject: the description of a visual scene, the noises present along with potential tactile and olfactory cues. These sensory stimuli elicit a neuronal activity pattern which is associated with the stimulus: it represents the event. If we want to memorise the event, we need to store this neuronal activity pattern somehow in the brain in a way that the brain should be able to re-create the same activity pattern in the absence of the sensory stimuli. When the brain internally re-creates the stored neuronal activity pattern is called memory recall. Here we will see an example neuronal network that is able to store and recall activity patterns.

Important concepts:

- storage and recall
- attractor
- convergence
- energy function

Storing patterns in a recurrent network

We will use a standard binary Hopfield network: each neuron has two state, 0 and 1. The sparseness of the patterns defines the proportion of 1s in the data, which is defined by the parameter `f`. We will have $32 \cdot 32 = 1024$ neurons, and will store M patterns.

```
f <- 0.5 # the sparseness of the representation: percent of 1s
N <- 1024 # number of neurons

M <- 100 # number of patterns stored try with 10, 20, 50, 100, 150, 200 and 300
X <- matrix(rbinom(N*M, 1, f), M, N)*2-1 # the patterns to be stored
```

The first 2 patterns are meaningful images - these patterns do not differ from the other patterns in any other way, just their pixels are ordered in a way that seems meaningful for us. We read these patterns from png imagefiles in the directory `Demos/Fig32s/`.

```
library(png)
ims <- array(NA, dim=c(2, 32, 32))
imnames <- list.files('./Figs32/')
par(mfcol=c(2,4)) # two rows, 4 columns
par(mar=c(1,1,1,1)) # small margins
for (i.image in 1:2){
  imname <- paste('./Figs32/', imnames[i.image], sep='')
  im1 <- readPNG(imname)
  for (i in 1:32){
    for (j in 1:32){
      ims[i.image, i,j] <- round(1-im1[33-j, i,1]) * 2 - 1
    }
  }
}
```

```

}
image(ims[i.image,,], col=grey(seq(0,1, by=0.1)), axes=F)
X[i.image,] <- as.vector(ims[i.image,,])
}

```



Figure 1: The first 8 stored patterns

```
print(round(apply(ims, 1, sum) / 32/32, 2))
```

```
## [1] 0.00 -0.06
```

Training in a neuronal network means to change the strength of the synaptic weights in order to improve the network's performance.

There are several ways to train a Hopfield network. We will use the simple, offline covariance rule, i.e., the recurrent synaptic weight are set to be the covariance of the training data. This is an offline rule, because to set the weight we need to know first all of the stored patterns. The diagonal is set to zero.

```

mu <- 1/M
W <- mu * t(X) %*% X
diag(W) <- 0

```

To test the network's performance, we will initialise the network with a noisy version of one of the originally stored patterns. We add noise in way to preserve the sparseness of the network.

```

x <- (X[1,]+1)/2
px <- rep(1-f/3, N); px[!x] <- f/3
nx <- rbinom(N, 1, px)*2-1 # the noisy version of the original

par(mfcol=c(1,2)) # two rows, 4 columns
par(mar=c(1,1,1,1)) # small margos
image(matrix(x, 32), col=grey(seq(0,1, by=0.1)), axes=F)
image(matrix(nx, 32), col=grey(seq(0,1, by=0.1)), axes=F)

```

Next we will test the network with the first 50 patterns. We start the network from the noisy recall cue and then update the neurons simultaneously according to the following rule:

$$\mathbf{x} = \text{sign}(\mathbf{W}\mathbf{x})$$

where the function sign returns -1 or 1 depending on the sign of the argument.

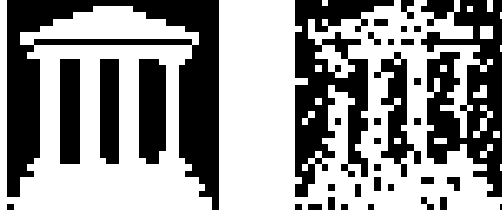


Figure 2: Left: stored image, Right: recall cue

We will check the following measures for each pattern:

- recalled vs target: the difference between the stored and the recalled versions
- recalled vs best: the difference between the stored and the recalled versions
- cue vs. target: the difference between the stored and the original cue
- i.recalled: identity of the recalled pattern

The network works if the correlation with the original, stored pattern increases during recall.

Note, that testing all patterns in a Hopfield network can be quite slow!

```
MM <- min(10, M)
stats.recall <- rep(NA, 9)
kk <- 0
mems <- matrix(NA, 4, MM)
rownames(mems) <- c("recalled vs target", "recalled vs best", "cue vs. target", "i.recalled")

noiseLevel <- f/2 # 0: no noise, f/1=0.5: no signal

cors.orig.t <- matrix(NA, 10, 1001)

par(mfcol=c(3,7))
par(mar=c(1,1,1,1))

for (m in 1:MM){ # for each pattern
  x <- (X[m,]+1)/2 # we start from the stored pattern

  px <- rep(1-noiseLevel, N); px[!x] <- noiseLevel
  x.t <- rbinom(N, 1, px)*2-1 # the noisy version of the original
  nx <- x.t
  converged <- F
  k <- 0

  while (converged == F){
    cors.orig.t[m, k+1] <- cor(x.t, x)
    I <- (W %*% x.t) # dynamics after MacKay - Eqs. 42.2-42.3
    x.I <- rep(F, N)
    x.I [I>0] <- T
    change.x <- which(xor(x.t==1, x.I))
    if (length(change.x) == 0){ # convergence
      converged <- T
    } else {
      ch.x <- sample(change.x, 1)
      x.t[ch.x] <- (-1) * x.t[ch.x]
    }
  }
}
```

```

    k <- k+1
    if (k>1000) break
  }

  if (m < 8){
    image(matrix(x, 32), col=grey(seq(0,1, by=0.1)), axes=F)
    image(matrix(nx, 32), col=grey(seq(0,1, by=0.1)), axes=F)
    image(matrix(x.t, 32), col=grey(seq(0,1, by=0.1)), axes=F)
  }

  # plot (X %% x)
  # points (X %% nx, col=2, pch=21, bg=2, cex=0.7)
  mem <- X %% x.t
  # points (mem, col=3, t="h")
  i.recalled <- which.max(mem)
  x.recalled <- X[i.recalled,]
  x <- X[m,]
  mems[1,m] <- 2 * x %% x.t / (x.t %% x.t + x %% x)
  mems[2,m] <- 2 * x.recalled %% x.t / (x.t %% x.t + x.recalled %% x.recalled)
  mems[3,m] <- 2 * nx %% x / (x %% x + nx %% nx)
  mems[4,m] <- i.recalled
  if (mems[2,m] < mems[3,m]) mems[4,m] <- NA
}

```

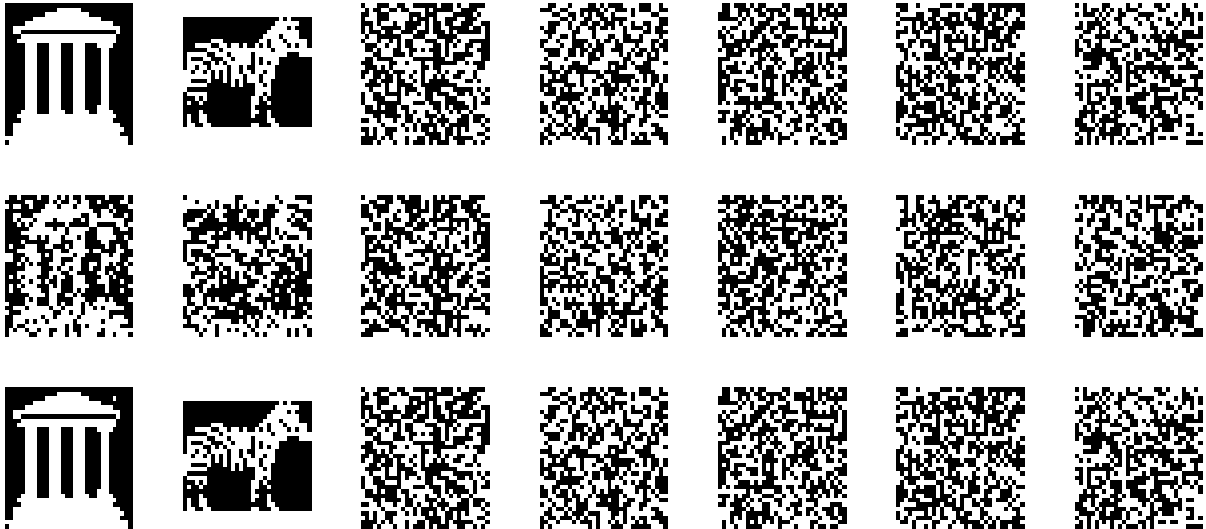


Figure 3: stored pattern, recall cue and recalled pattern

Let's evaluate the network:

- The average correlation of the recalled with the stored: 0.9994141
- The average correlation of the *recall cue* with the stored: 0.4988281
- Percent of correctly recalled patterns: 100%

We can also see how the patterns become similar to the originally stored patterns during recall. We quantify this by the correlation between the current and the stored pattern and plot it in the function of time:

```

matplot(t(cors.orig.t[,1:1000]), t='l', lty=1, col=rainbow(12), axes=F, xlab='timestep', ylab='correlation')
axis(1); axis(2, las=2)

```

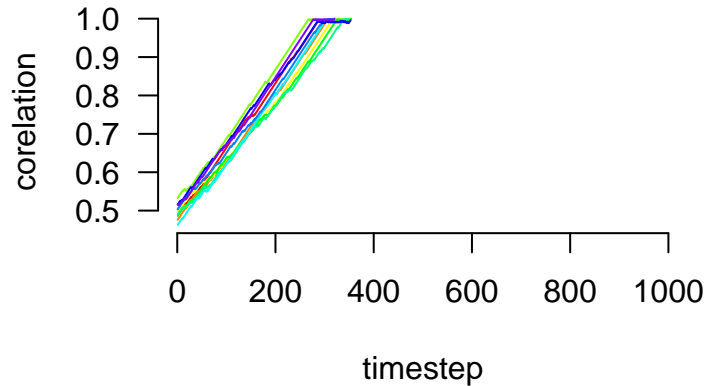


Figure 4: correlation between the stored pattern and the activity in the network

Homework

Homework 1 - theory

Show that the function

$$E = -\frac{1}{2} \sum_{ij} w_{ij} x_i x_j$$

is an energy function, i.e., it is guaranteed that the function $E(x)$ will not increase if the units are updated sequentially using the simplified update rule. [3 points]

What are the implications of having an energy function for a network? [2 points]

Show, that if a pattern y is an attractor of the network then $1 - y$ is also a stable pattern, stored in the network! [2 points]

Homework 2 - programming

Demonstrate, that if y is an attractor of the network, then $1 - y$ is also a stable pattern, stored in the network! [2 points]

Change the noise level of the system. Observe how the recall quality changes when the recall cue becomes more noisy! [2 points]

How would you measure the capacity of the network? Try to store more patterns in the Hopfield network (try e.g., 50-200-1000-5000). Check what happens to the first 50 stored patterns! What is the capacity of the network? [5 points]

Homework 3 - open questions

How would you improve the Hopfield network to

- make it more efficient - i.e., storing more patterns without interference? [5 points]
- make it more biologically realistic? Identify at least 3 different features of the Hopfield net that are probably not similar to what we see in the nervous system. How would you change it to be more similar to the nervous system? [3 points]