4. Oblicz granicę

$$\lim_{n\to\infty} n\left(e-\left(1+\frac{1}{n}\right)^n\right).$$

$$\left(1+\frac{1}{x}\right)^x = e^{x\ln\left(1+\frac{1}{x}\right)}$$

$$\lim_{x\to\infty} x\left(e-\left(\underbrace{\left(1+\frac{1}{x}\right)^x}\right)\right) = \lim_{x\to\infty}\frac{e-\left(1+\frac{1}{x}\right)^x}{1/x} \overset{H}{=} \lim_{x\to\infty}\frac{-\left(1+\frac{1}{x}\right)^x\cdot\left(x\ln\left(1+\frac{1}{x}\right)\right)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\overset{e \uparrow}{\left(1+\frac{1}{x}\right)^x}\cdot\left(\ln\left(1+\frac{1}{x}\right)+x\frac{\left(-\frac{1}{x^2}\right)}{\left(1+\frac{1}{x}\right)}\right)}{1/x^2} =$$

$$x^2\,\frac{x\cdot\left(-\frac{1}{x^2}\right)}{\frac{x+1}{x}}$$

$$= e\lim_{x\to\infty}\left(x^2\ln\left(1+\frac{1}{x}\right)-\frac{x^2}{1+x}\right) = e\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)-\frac{1}{1+x}}{x^{-2}} \overset{H}{=}$$

$$= e\lim_{x\to\infty}\frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}+\frac{1}{(x+1)^2}}{-2x^{-3}} = -\frac{e}{2}\lim\left(\frac{x^3}{(x+1)^2}-\frac{x^2(x+1)}{(x+1)^2}\right) = \frac{e}{2}$$

$$= x^2\,\frac{-1}{(x+1)^2}$$

$$\left(\left(1+\frac{1}{x}\right)^x\right)' = \left(e^{x\ln\left(1+\frac{1}{x}\right)}\right)' =$$

$$= \underbrace{e^{x\cdot\ln\left(1+\frac{1}{x}\right)}}_{\left(1+\frac{1}{x}\right)^x}\left(x\ln\left(1+\frac{1}{x}\right)\right)'$$

$$f(x)^{g(x)} =$$

$$= e^{g(x)\ln f(x)}$$

4. Oblicz granicę

$$\lim_{n\to\infty} n\left(e-\left(1+\frac{1}{n}\right)^n\right).$$

$x = n = \frac{1}{y}$ $\quad y \to 0^+$

$$\lim_{y\to 0^+} \frac{e-(1+y)^{\frac{1}{y}}}{y}$$

$$(1+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} y^k$$

(f)

$$\lim_{x\to 2^+}(x-2)e^{\frac{1}{x-2}}, = \left[\begin{array}{l} \frac{1}{x-2} = y \\ x \to 2^+ \\ y \to \infty \end{array}\right] = \lim_{y\to\infty} \frac{e^y}{y} = \infty$$

6. Oszacuj błąd przybliżenia

$$\rightarrow \quad e^x \simeq 1 + x + x^2/2 + \ldots + x^n/(n!) \quad (x \in [0,1])$$

$$\sqrt{1+x} \simeq 1 + \frac{x}{2} - \frac{x^2}{8} \quad (x \in [0,1])$$

$f(x) = e^x$

$f^{(n)}(x) = e^x$

$f^{(n)}(0) = 1$

Wzór Taylora:

$$e^x = \sum_{k=0}^{n} 1 \cdot \frac{x^k}{k!} + R_{n+1}(x)$$

$$|R_{n+1}(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot x^{n+1} \right|$$

$$= \frac{e^{\xi} \cdot |x|^{n+1}}{(n+1)!} \leq \frac{e}{(n+1)!} \qquad 0 < \xi < x \leq 1$$

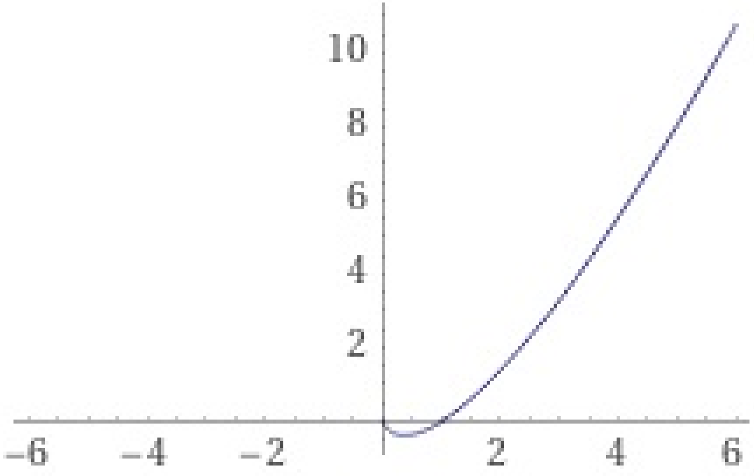
$$\left(1+\frac{a}{x}\right)^x \xrightarrow[x\to\infty]{} e^a$$

14. Czy funkcja $x \ln x$ określona na $(0, \infty)$ jest wypukła?

$f : (0, \infty) \to \mathbb{R}$ $\qquad f(x) = x \ln x$

$f'(x) = \ln x + 1$ $\qquad (f(x) = x^2)$

$f''(x) = \frac{1}{x} > 0$ $\quad f$ wypukła



(x from −6 to 6)

12. Niech

$$f(x) = \begin{cases} \frac{e^{x^2}-1}{x^2}, & \text{gdy } x \neq 0 \\ 1, & \text{gdy } x = 0. \end{cases}$$

Wyznaczyć szereg Taylora funkcji $f(x)$. w $x=0$.

$x \neq 0$ $\quad f(x) = \dfrac{e^{x^2}-1}{x^2} = \dfrac{\sum_{k=0}^{\infty} \frac{x^{2k}}{k!} - 1}{x^2} = \dfrac{\sum_{k=1}^{\infty} \frac{x^{2k}}{k!}}{x^2} = \sum_{k=1}^{\infty} \dfrac{x^{2k-2}}{k!} =$

$= \sum_{k=0}^{\infty} \dfrac{x^{2k}}{(k+1)!} =$

$= 1 + \dfrac{x^2}{2} + \dfrac{x^4}{3!} + \ldots$

dla $x=0$ też

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(k+1)!} \quad \text{dla } x \in \mathbb{R}$$

Wniosek 98. *Dla $x \in \mathbb{R}$ mamy:*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots .$$

Dla $|x| < 1$, $\alpha \in \mathbb{R}$ mamy:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots ,$$

$$\operatorname{arctg} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{2}{2n+1} \binom{2n}{n} \left(\frac{x}{2}\right)^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots .$$

8. Dana jest funkcja
$$f(x) = x\sin(2x^2).$$

(a) Znaleźć rozwinięcie w szereg Taylora (Maclaurina) wokół punktu $x = 0$ funkcji $f(x)$.

(b) Dla jakich x-ów szereg jest zbieżny?

(c) Wyznaczyć $f^{(2022)}(0)$ oraz $f^{(2023)}(0)$.

$\underbrace{f^{(2022)}(0)}_{= 0}$

$$\_\_ x^{2019} + \_\_ x^{2023}$$

$$\_\_ x^3 + \_\_ x^7 + \dots$$

$$(x^2)^{2k+1}$$

$$k = 505$$

$$x\sin(2x^2) = \underline{x} \sum_{k=0}^{\infty} \frac{(2x^2)^{2k+1}(-1)^k}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{2^{2k+1}(-1)^k}{(2k+1)!} \cdot x^{4k+3}$$

$$\frac{f^{(2023)}(0)}{(2023)!} = \text{wsp. przy } x^{2023} = \frac{2^{1011}\cdot(-1)}{(1011)!}$$

$$\frac{f^{(2022)}(0)}{(2022)!} = \text{—''— } x^{2022} = 0$$

$$\lim_{x\to 0} \frac{2\cos x + x^2 - 2}{x\sin x - x^2}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

$$\sin x = x - \frac{x^3}{6} + O(x^5)$$

$$= \lim_{x\to 0} \frac{\cancel{2} \cancel{- x^2} + \frac{x^4}{12} + O(x^6) \cancel{+ x^2} \cancel{- 2}}{\cancel{x^2} - \frac{x^4}{6} + O(x^6) \cancel{- x^2}} = \lim_{x\to 0} \frac{\frac{1}{12} + O(x^2)}{-\frac{1}{6} + O(x^2)} = -\frac{1}{2}$$

$$x\, O(x^5) = O(x^6)$$

10. Zbadać na odcinku $[0,1]$ zbieżność punktową i jednostajną ciągu funkcyjnego:

$$f_n(x) = \frac{x^{2n} - x^{3n}}{x^{2n}+1}.$$

$f \equiv 0$

$x = 0 \quad f_n(x) = 0 \qquad x = 1 \quad f_n(x) = 0 \qquad f_n \rightrightarrows f$ na $[0,1]$

$x \in (0,1) \qquad f_n(x) \xrightarrow[n]{} 0$

$\underline{x \in [0, 1-\delta]} \quad x^{2n} \leq (1-\delta)^{2n}$

$|f_n(x) - f| \leq (1-\delta)^{2n}$

$$f_n(x) = x^{2n} \frac{(1-x^n)}{x^{2n}+1}$$

$$x_n = 1 - \frac{1}{n} \qquad f_n(x_n) = \left(1-\frac{1}{n}\right)^{2n} \frac{\left(1-\left(1-\frac{1}{n}\right)^n\right)}{\left(1-\frac{1}{n}\right)^{2n}+1} \longrightarrow \frac{1}{e^2} \frac{1-\frac{1}{e}}{\frac{1}{e^2}+1}$$

$f_n \not\rightrightarrows 0$ na $[0,1]$ ale $f_n \rightrightarrows 0$ na $[0, 1-\delta]$ $(\delta > 0)$

$$f_n(x) = \frac{x^{2n} - x^{3n}}{x^{2n} + 1}.$$

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} f_n(x)$$

$$f_n(\underbrace{\sqrt[n]{\tfrac{2}{3}}}) = \frac{\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3}{\left(\frac{2}{3}\right)^2 + 1} = c \underset{n\to\infty}{\not\to} 0$$

$$g_n(x) = x^{2n} - x^{3n}$$

$$g_n'(x) = 2n\, x^{2n-1} - 3n\, x^{3n-1}$$

$$g'(x) = 0 \iff$$

$$2x^{2n-1} = 3x^{3n-1}$$

$$\frac{2}{3} = x^n$$

$$x_n = \sqrt[n]{\frac{2}{3}}$$

5. Rozwiń w szereg Taylora w punkcie $x = 0$ funkcje

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \qquad |x| < \delta$$

rozw. w $x=0$

(a) $x^3 \cos(x^2)$

(b) $\ln(1 + x^4)$

w $x = \frac{1}{2}$

$x = y + \frac{1}{2}$

$x - \frac{1}{2} = y$

$$\ln(1 + x^4) = \ln\left(1 + \left(y + \tfrac{1}{2}\right)^4\right) = \sum a_k y^k = \sum a_k \left(x - \tfrac{1}{2}\right)^k$$

$$\ln\left(1 \tfrac{1}{16} + \dots\right)$$

(c) $f(x) = \dfrac{2\cos x - 2}{x^2}$ $\quad (f(0) = 0)$

rozw w $x = \frac{1}{2}$

$$f(x) = \sum_{k=0}^{\infty} a_k \left(x - \tfrac{1}{2}\right)^k$$

$$\left|x - \tfrac{1}{2}\right| < \delta$$

10. Wykazać, że szereg funkcyjny

$$f(x) = \sum_{n=1}^{\infty} \underbrace{3^{-n}\cos(2^n x)}_{a_n(x)}$$

$f: \mathbb{R} \to \mathbb{R}$ ciągła

$\Uparrow$

$\underbrace{\sum a_n(x)}$ sz. zb. jednost.

(a) jest zbieżny jednostajnie na $\mathbb{R}$,

(b) zadaje funkcję różniczkowalną na $\mathbb{R}$.

a) $|a_n(x)| \leq 3^{-n}$ i $\sum_n 3^{-n} < \infty$ $\Rightarrow$ sz. zb. jednost.

b) $|a_n'(x)| = \left|-\left(\frac{2}{3}\right)^n \cdot \sin(2^n \cdot x)\right| \leq \left(\frac{2}{3}\right)^n$ $\quad \sum_n \left(\frac{2}{3}\right)^n < \infty \Rightarrow \sum a_n'(x)$ zb. jednost. na R

więc z tw o różnicz. sz. funkcyjnego

f jest różnicz. i $f'(x) = \sum_{n=1}^{\infty} -\left(\frac{2}{3}\right)^n \cdot \sin(2^n x)$