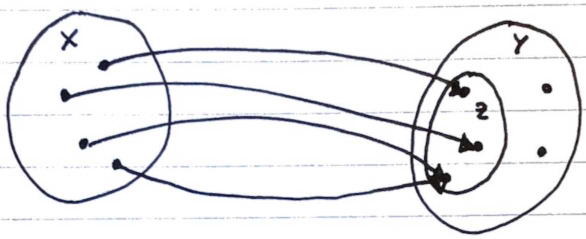
CS130 - Functions Summary

Definition

A function $f: X \rightarrow Y$ is a relation $R \subseteq X \times Y$ where for every $x \in X$ there is a unique $y \in Y$ such that xRy, i.e. every item in X unambiguously maps to an item in Y.

Values which could come out



X : domain

Y : co-domain

Z : range — Values which do come out

($Z \subseteq Y$ for it to be a function)

Nomenclature

$f(x)$ is called the "image of x under f", or the value of f at x, and is the unique y that is mapped to from an x by f.

Image must be unique for f to be a function

$$f(x) = y \iff x \xmapsto{f} y$$

We can also consider the "pre-image" of a value, all the values in the domain that map to it, an element in the co-domain. We write this set as $f^{-1}(y)$

Values where $f^{-1}(y) = \emptyset$ are in the co-domain, but not the range

$$f^{-1}(y) = \{x \in X : f(x) = y\}$$

If only one x maps to y, $f^{-1}(y)$ is a singleton. For a well defined inverse function, there must be no multi-element pre-images.

Failure Conditions

A relation is not a function if:

- Elements in the domain don't map into the co-domain
- Elements in the domain map to multiple values in the co-domain.

Restricting the domain

Sometimes, we can simplify a problem by ignoring irrelevant cases in the domain. We can define a new function as a relation based off the old one.

$$f : X \to Y \quad \& \quad \underbrace{X'}_{\text{restricted domain}} \subseteq X$$

$$f\big|_{X'} = \underbrace{\{(x, f(x)) : x \in X'\}}_{\text{Relation definition}}$$

The codomain can only be restricted if no values in the domain map to the elements to be removed.

Composition of functions

We can compose multiple functions together, applying one's output to another's input in sequence:

$$R \subseteq A \times B \quad \& \quad Q \subseteq B \times C$$

Range & domain need to be equal

$$R \circ Q = Q(R(x)) \subseteq A \times C.$$

By the previous definitions of functions:

$$|A| \geqslant |C|$$

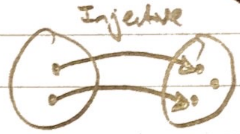
is necessary for composition, but other requirements on the sets A, B & C exist.

Properties of functions

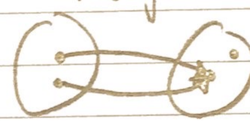
Given a function $f : X \to Y$

- Injectivity (one-to-one) is when every value in the domain maps to only one value in the codomain, i.e. $\forall x_1, x_2 : x_1 \neq x_2 \to f(x_1) \neq f(x_2)$, or $|f^{-1}(y)| \leq 1 \ \forall y \in Y$.

Injective

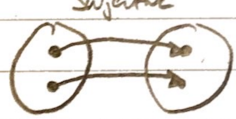


Not injective

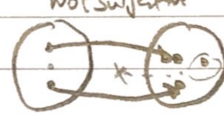


- Surjectivity (onto) is when all the elements of the codomain are in the range, i.e. $\forall y \in Y \ \exists x \in X : f(x) = y$

Surjective



Not surjective



- Bijectivity (one-to-one correspondence) is when a function is both injective <u>and</u> surjective.

$f : X \to Y$ is bijective iff its inverse relation $f^{-1} \subseteq Y \times X$ is also a function.

Equinumerous Sets

The sets ~~[illegible]~~ A and B are equinumerous if there exists a bijective function between them. This is denoted by:

$$A \cong B$$

this property can be used to define categories of sets based on their cardinality

- finite sets are equinumerous with one ~~arbitrary~~ set $X \in 2^{\mathbb{N}}$

- countably infinite sets are equinumerous with the natural numbers, $\mathbb{N}$

- countable sets are equinumerous with either a finite set or countably infinite sets.

- uncountably infinite sets are not countable, i.e. not equinumerous with any countable set

we can also prove this by writing a pseudo-code algorithm that maps one set onto another