CS130 - Relations Summary

Relations

↳ Links between objects, defined as:

Given two sets A & B, a relation is a subset of $A \times B$

↳ $R \subseteq A \times B$

↳ The first term in each tuple is an element in the co-domain, and the second an element in the range.

↳ If $A = B$, the relation is said to be "on" A

↳ Examples:

~~$A = B = \mathbb{Z}$~~

1. $R_1 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \, . \, x = y\}$
   ↳ the "equality" relation.
2. $R_2 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \, . \, x \leq y\}$
   ↳ one of the "inequality" relations
3. $R_3 = \{(x, y) \in \mathbb{Z} \times \{0, 1\} \, . \, x$ has the remainder y modulo $2\}$
   $= \{\ldots, (1,1), (2,0), (3,1), \ldots\}$

↳ The inverse of a relation $R \subseteq A \times B$ is:

$$R^{-1} = \{(b, a) \in B \times A \, . \, (a, b) \in R\}$$

↳ Relations are in essence sets, so normal operations (e.g. $R_1 \setminus R_2$, $R_1 \cup R_2$) can be used, and the can contain more concrete or more abstract objects.

↳ aRb is used to denote $(a, b) \in R$.

↳ Multiple relations can be composed together
↳ e.g. if there are relations between A&B and B&C, you can form one between A&C.

↳ $R \subseteq A \times B$, $Q \subseteq B \times C$

The composition of R & Q is:

$$R \circ Q = \{ (a,c) \in A \times C. \text{ there is } b \in B \text{ such that } (a,b) \in R \text{ and } (b,c) \in Q \}.$$

↳ Relations can be said to have various properties
A relation R on set S is:

- Reflexive if aRa for all $a \in S$
  ↳ every element relates to itself
- Symmetric if aRb implies bRa for all $a, b \in S$
- Anti-Symmetric if aRb and bRa implies that $a = b$
- Transitive if aRb an bRc together imply aRc

↳ Having combinations of these properties are named types of relations:

- Equivalence relations, which are reflexive, symmetric and transitive
- Partial order relations which are reflexive, anti symmetric and transitive.

Equivalence relations & classes

↳ Equivalence classes are defined as:

Given a set S on which there is an equivalence relation R

For every $a \in S$, $[a]$ denotes the equivalence class containing a with respect to the relation R

$$[a] = \{x \in S \,.\, aRx\}$$

↳ Equivalence classes a sets of items in a set with a relation to a given item.

↳ Examples

① $R_1 = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \,.\, x = y\}$ "equality"

for R_1 over $\mathbb{Z}$, $\forall n \in \mathbb{Z} \,.\, [n] = \{n\}$

i.e. all equality equivalence classes are singletons

② $R_2 = \{(x,y) \in \mathbb{Z} \times \{0,1\} \,.\, y = x \text{ MOD } 2\}$

for R_2 over $\mathbb{Z}$:

$[5]_{R_2} = \{\ldots 5, 3, 1, -1, \ldots\}$

Sometimes used to denote which relation is being considered

↳ Since equivalence relations are by definition reflexive, for $[n] = A$, $n \in A$.

↳ $\forall b \in [a]$, $[b] = [a]$

↳ every element in an equivalence ~~relation~~ class has that same equivalence class

↳ $\forall a, b \in S$, either $[a] \cap [b] = \emptyset$
or $[a] = [b]$

↳ every equivalence class either wholly coincides or doesn't overlap with every other equivalence class

↳ Set partitions are defined as:

↳ Initial nomenclature:

↳ we can consider a group of sets by index by having a set of all of their indices.

For each $i \in I$, A_i is a set

e.g. $I = \{1, 2\} \Rightarrow A_1, A_2$

$I = \mathbb{N} \Rightarrow A_1, A_2, A_3, \ldots$

The sets $(A_i)_{i \in I}$ form a partition of the set B if:

cover the entire set ← $\bigcup_{i \in I} A_i = B$ and

Are all disjoint from each other ← $A_i \cap A_j = \emptyset \quad \forall i, j \in I : i \neq j$

↳ ~~Further nomenclature:~~ Set quotients.

$$S/R = \{ [a]_R : a \in S \}$$

'the quotient of S with respect to R', meaning the set of all the equivalence classes in S

↳ Examples

1. $E = \{(x,x) . x \in \mathbb{Z}\}$ "equality"

Since every element maps only to itself, it is the set of singletons of every integer

$$\mathbb{Z}/E = \{\{x\} : x \in \mathbb{Z}\}$$

2. $T = \{(x,y) : x, y \in \mathbb{Z}\}$

Every element maps to every other element, so it is the set of every element only.

$$\mathbb{Z}/T = \{\mathbb{Z}\} \quad (\mathbb{Z} = [0]_T)$$

3. $R_m = \{(x,y) : x-y \text{ is divisible by } m\}$

"congruence modulo m"

Finite cardinality

wraps around, due to modulo

$[m] = [0]$
$[m+1] = [1]$ etc.

$$\mathbb{Z}/R_m = \{[0]_{R_m}, [1]_{R_m}, \dots, [m-1]_{R_m}\}$$