# Numerical Methods for Time Inconsistency, Private Information and Limited Commitment

Antonio Mele

University of Surrey

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# ONE-SIDED LACK OF COMMITMENT

$$
\begin{aligned}
&\max_{\{c_t\}_{t=0}^{\infty}} E_{-1} \sum_{t=0}^{\infty} \beta^t (y_t - c_t) \\
&s.t. \quad E_{-1} \sum_{t=0}^{\infty} \beta^t u(c_t) \geq U_0 \\
&\qquad u(c_t) + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} u\left(c_{t+j}\right) \geq u(y_t) + \beta U_{aut} \, \forall t
\end{aligned}
$$

Assume $y_t$ is i.i.d. and its support is $\mathbf{Y} \equiv \left\{\bar{y}_s\right\}_{s=1}^{S}$. Call its distribution function $\pi_s, s = 1, ..., S$.

# PROMISED UTILITY

Define the *agent's promised utility* as:

$$U_t \equiv E_{t-1} \sum_{j=0}^{\infty} \beta^j u\left(c_{t+j}\right) \tag{1}$$

This variable summarizes the promises that the principal makes to the agent in each period.

# Recursive contract

- Principal enters period $t$ with a promise $U_-$ made in $t-1$
- Complies with previous promise by choosing state-contingent consumption $c_s, s = 1, ..., S$ and a new promise $U_s, s = 1, ..., S$
- Let $P(U_-)$ be the value function of the principal:

$$P(U_-) = \max_{\{c_s, U_s\}_{s=1}^S} \sum_{s=1}^S \pi_s \left[\bar{y}_s - c_s + \beta P(U_s)\right] \tag{2}$$

$$s.t. \quad \sum_{s=1}^S \pi_s \left[u(c_s) + \beta U_s\right] \geq U_- \tag{3}$$

$$u(c_s) + \beta U_s \geq u(\bar{y}_s) + \beta U_{aut} \quad s = 1, ..., S \tag{4}$$

$$c_s \in C \tag{5}$$

$$U_s \in \mathscr{U} \tag{6}$$

# The set $\mathscr{U}$

The set $\mathscr{U}$ is obtained iterating on the APS operator $B$ defined as:

$$B(W) = \left\{ \begin{array}{l} U \in W : \sum_{s=1}^{S} \pi_s \left[ u(c_s) + \beta U_s \right] \geq U, \\ u(c_s) + \beta U_s \geq u(\bar{y}_s) + \beta U_{aut} \quad s = 1, ..., S \\ c_s \in C \quad s = 1, ..., S \end{array} \right\} \tag{7}$$

# Getting a solution

Apart from this set, the functional equation looks like a Bellman equation in which we have the agent's value function as a state variable. Therefore, we can characterize the optimal contracts in two steps:

1. Find the set $\mathscr{U}$ by repeatedly applying the operator $B$ defined in (7)
2. Solve the Bellman equation as usual, either with value function iteration or Howard's improvement algorithm.

# ENDOWMENT IS PRIVATE INFORMATION

$$\max_{\left\{\tau_t(y^t)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[-\tau_t\left(y^t\right)\right] \pi\left(y^t | y_{-1}\right) \tag{8}$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{y^t \in y^{t+1}} \beta^t u\left(c_t\left(y^t\right)\right) \pi\left(y^t | y_{-1}\right) \geq U_0 \tag{9}$$

$$DIC_t\left(y^{t-1}, \bar{y}_i; \bar{y}_{i-1}\right) \geq 0 \quad \forall y^{t-1}, \quad \forall \bar{y}_i, \quad i = 2, \dots, N \tag{10}$$

# RECURSIVE CONTRACT

$$P(U_-) = \max_{\{\tau_s, U_s\}_{s=1}^S} \sum_{s=1}^{S} \pi_s \left[-\tau_s + \beta P(U_s)\right] \tag{11}$$

$$s.t. \quad \sum_{s=1}^{S} \pi_s \left[u(\tau_s + \bar{y}_s) + \beta U_s\right] = U_- \tag{12}$$

$$u(\tau_s + \bar{y}_s) + \beta U_s \geq u(\tau_{s-1} + \bar{y}_s) + \beta U_{s-1} \quad s = 2, \ldots, S \tag{13}$$

$$c_s \in C \tag{14}$$

$$U_s \in \mathscr{U} \tag{15}$$

# THE SET $\mathscr{U}$

$\mathscr{U}$ is the fixed point of the operator:

$$B(W) = \left\{ \begin{array}{l} U \in W : \sum_{s=1}^{S} \pi_s \left[ u(\tau_s + \bar{y}_s) + \beta U_s \right] \geq U, \\ u(\tau_s + \bar{y}_s) + \beta U_s \geq u(\tau_{s-1} + \bar{y}_s) + \beta U_{s-1} \quad s = 2, ..., S \\ c_s \in C \quad s = 1, ..., S \end{array} \right\} \tag{16}$$

# HIDDEN EFFORT

$$\max_{\left\{\tau_t\left(y^t\right)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[-\tau_t\left(y^t\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right)$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right) \geq U_0$$

$$\left\{a_t\left(y^t\right)\right\}_{t=0}^{\infty} \in \arg \max_{\left\{a_t\left(y^t\right)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right)$$

# RECURSIVE CONTRACT

$$P\left(U_{-},\bar{y}_i\right)=\max_{\left\{c,\{U_s\}_{s=1}^{S},a^*\right\}}\left[\bar{y}_i-c+\beta\sum_{s=1}^{S}\pi_s\left(a^*\right)P\left(U_s,\bar{y}_s\right)\right] \tag{17}$$

$$s.t.\quad u(c)-\upsilon\left(a^*\right)+\beta\sum_{s=1}^{S}\pi_s\left(a^*\right)U_s=U_{-} \tag{18}$$

$$a^*=\arg\max_{a\in A}\left\{u(c)-\upsilon(a)+\beta\sum_{s=1}^{S}\pi_s(a)U_s\right\} \tag{19}$$

$$c\in C,a\in A \tag{20}$$

$$U_s\in\mathscr{U} \tag{21}$$

Notice different timing convention: effort affects the probability of the state tomorrow

# THE SET $\mathscr{U}$

$\mathscr{U}$ is the fixed point of the operator:

$$B(W) = \left\{ \begin{array}{l} U \in W : u(c) - \upsilon(a^*) + \beta \sum_{s=1}^{S} \pi_s(a^*) U_s = U, \\ a^* = \arg\max_{a \in A} \left\{ u(c) - \upsilon(a) + \beta \sum_{s=1}^{S} \pi_s(a) U_s \right\} \\ c \in C, a \in A \quad s = 1, ..., S \end{array} \right\} \tag{22}$$

# Optimal fiscal policy

$$\max_{\{c_t, b_t, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[ u(c_t) + v(l_t) \right]$$

$$s.t. \quad c_t - \beta \frac{E_t u'(c_{t+1})}{u'(c_t)} b_{t+1}^g = -(c_t + g_t) \frac{v'(c_t + g_t)}{u'(c_t)} - b_t^g$$

# OPTIMAL FISCAL POLICY (CONT.)

Small departure from APS:

- Instead of using the continuation value of the agent as a state variable, we are going to use the marginal utility of consumption
- We have already a state variable in this problem: debt holdings.
- We have to characterize a feasible set for continuation values for any value of debt holdings: i.e. a correspondence

# RECURSIVE CONTRACT

Let $m \equiv u'(c)$, govt expenditure can take $S$ values and is i.i.d. as in previous examples

$$P(m_-, \bar{g}_s, b_-) = \max_{\{c,b,l,\{m_s\}_{s=1}^S\}} u(c) + v(l) + \beta \sum_{s=1}^{S} P(m_s, \bar{g}_s, b) \tag{23}$$

$$s.t. \quad c - \beta \frac{\sum_{s=1}^S m_s}{m_-} b = -(c+\bar{g}_s)\frac{v'(c+\bar{g}_s)}{m_-} - b_- \tag{24}$$

$$c \in C, l \in L, b \in B \tag{25}$$

$$m_s \in \mathscr{U}(b_-) \tag{26}$$

# THE SET $\mathscr{U}(b_-)$

$\mathscr{U}(b_-)$ is the fixed point of the operator

$$B\left(W(b_-)\right) = \left\{ \begin{array}{c} m \in W(b_-) : c - \beta \frac{\sum_{s=1}^{S} m_s}{m_-} b = -(c + \overline{g}_s) \frac{v'\left(c + \overline{g}_s\right)}{m_-} - b_- \\ c \in C, l \in L, b \in B \end{array} \right\} \tag{27}$$

This set changes as $b_-$ changes

# $DP^2 = (\text{CURSE OF DIMENSIONALITY})^2$

- Any application can be solved by a two-step procedure:
    1. Characterize the set of continuation values
    2. Solve the Bellman equation
- First step causes many troubles if other ("natural") state variables or several agents (e.g., optimal fiscal policy)
- Set of admissible continuation values is a correspondence that maps from the set of natural states to the set of continuation values
- I don't know any paper that works with more than 2 natural states

# THE LAGRANGEAN APPROACH

# INTRODUCTION

- APS suggest to "recursify" the problem by using the continuation values as state variables
- Marcet and Marimon (2017): use duality theory (i.e., Lagrangean methods)
- Provide a recursive formulation in terms of a **saddle point functional equation**: auxiliary state variables are obtained from Lagrange multipliers
- Big advantage: does not suffer from "curse of dimensionality squared"
- Disadvantage: does not work in all situations

# NON-STANDARD DYNAMIC PROGRAMMING

Many dynamic problems we face in (macro)economics can be described with the following non-standard dynamic programming problem:

$$V(x,s) = \sup_{a_t} E_0 \sum_{t=0}^{\infty} \beta^t r(x_t, a_t, s_t) \qquad \textbf{PP}$$

$$s.t. \quad x_{t+1} = \ell(x_t, a_t, s_{t+1}), \qquad p(x_t, a_t, s_t) \geq 0$$

$$E_t \sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n}, a_{t+n}, s_{t+n}) + h_1^j(x_t, a_t, s_t) \geq 0 \tag{28}$$

$$j = 1, \ldots, l; \quad t \geq 0, \quad x_0 = x, s_0 = s,$$

$$N_j = \infty \quad for \quad j = 1, \ldots, k \quad N_j = 0 \quad for \quad j = k+1, \ldots, l$$

# GENERALIZATION OF THE PROBLEM

$$V_\mu(x,s) = \sup_{\{a_t\}} E\left[\sum_{j=0}^{l}\sum_{t=0}^{N_j} \beta^t \mu^j h_0^j(x_t,a_t,s_t) \mid s\right] \qquad \mathbf{PP}_\mu$$

$$s.t. \quad x_{t+1} = \ell(x_t,a_t,s_{t+1}), \qquad p(x_t,a_t,s_t) \geq 0 \tag{29}$$

$$E_t \sum_{n=1}^{N_j+1} \beta^n h_0^j(x_{t+n},a_{t+n},s_{t+n}) + h_1^j(x_t,a_t,s_t) \geq 0 \tag{30}$$

$$j = 0,\ldots,l; \quad t \geq 0, \quad x_0 = x, s_0 = s,$$

$$N_j = \infty \quad for \quad j = 1,\ldots,k \quad N_j = 0 \quad for \quad j = k+1,\ldots,l$$

# Generalization of the problem (cont.)

- $\mathbf{PP} = \mathbf{PP}_\mu$ when $\mu = (1, 0, 0, ..., 0)$, and $h_0^0(x, a, s) \equiv r(x, a, s)$, $h_0^1(x, a, s) \equiv r(x, a, s) - R$ where $R > 0$ is a large, never binding bound.
- The value function of this problem takes the form $V_\mu(x, s) = \mu \upsilon_\mu(x, s)$, i.e. it looks like a social welfare function with Pareto weights given by the vector $\mu$

# SADDLE-POINT PROBLEM

By attaching Lagrange multipliers $\gamma^j$ to the first-period constraints (28), you can transform ($\mathbf{PP}_\mu$) in the following saddle point problem ($\mathbf{SPP}_\mu$):

$$\inf_{\gamma \in R^l_+} \sup_{\{a_t\}} \sum_{j=0}^{l} \left( \mu^j h_0^j (x, a_0, s) + \gamma^j h_1^j (x, a_0, s) \right) + \qquad \mathbf{SPP}_\mu$$

$$+ \beta E \left[ \sum_{j=0}^{k} \left( \mu^j + \gamma^j \right) \sum_{n=1}^{\infty} \beta^n h_0^j (x_n, a_n, s_n) + \sum_{j=k+1}^{l} \gamma^j h_0^j (x_1, a_1, s_1) \mid s \right]$$

$$s.t. \quad x_{t+1} = \ell (x_t, a_t, s_{t+1}), \qquad p(x_t, a_t, s_t) \geq 0, \quad t \geq 0$$

$$E_t \sum_{n=1}^{N_j+1} \beta^n h_0^j (x_{t+n}, a_{t+n}, s_{t+n}) + h_1^j (x_t, a_t, s_t) \geq 0$$

$$j = 1, ..., l; \quad t \geq 1$$

# SADDLE-POINT PROBLEM (CONT.)

- We have incorporated the period-0 constraints (28) in the objective of this *inf sup* problem
- Crucial step: show that this problem has a recursive structure
- Define a law of motion $\varphi$ for the weights $\mu$ as follows: $\mu'^j = \varphi(\mu, \gamma, s) = \mu^j + \gamma^j$ if $N_j = \infty$, and $\mu'^j = \varphi(\mu, \gamma, s) = \gamma^j$ if $N_j = 0$
- The solution $\gamma^*$ of $\mathbf{SPP}_\mu$ in state $(x, s)$ generates a new problem $\mathbf{SPP}_{\varphi(\mu,\gamma^*,\mathbf{s})}$ in state $\left(x^{*\prime}, s'\right)$
- MM show that solutions of $\mathbf{SPP}_{\varphi(\mu,\gamma^*,\mathbf{s})}$ in state $\left(x^{*\prime}, s'\right)$ are one-period ahead solutions of $\mathbf{SPP}_\mu$ in state $(x, s)$, i.e. $\mathbf{SPP}_\mu$ is recursive

# SADDLE-POINT FUNCTIONAL EQUATION

This implies that the optimal solution of $\mathbf{SPP}_\mu$ satisfies the following functional equation:

$$W(x,\mu,s) = \inf_{\gamma\geq 0}\sup_{a}\left\{\mu h_0(x,a,s) + \gamma h_1(x,a,s) + \beta E\left[W\left(x',\mu',s'\right) \mid s\right]\right\}$$

**SPFE**

$$\begin{aligned} s.t. \quad & x' = \ell\left(x,a,s'\right), \qquad p(x,a,s) \geq 0 \\ & \mu' = \varphi(\mu,\gamma,s) \end{aligned} \tag{31}$$

# SADDLE-POINT FUNCTIONAL EQUATION (CONT.)

- This is a saddle-point functional equation which is the analog of Bellman equation for problems with constraints in the form of (28)
- Recursivity comes from the Lagrange multipliers: we use Lagrange multipliers as record-keeping tools, that incorporate the promises of the contract
  - For $N_j = \infty$, the relevant state is the sum of past Lagrange multipliers
  - For $N_j = 0$ we use the past Lagrange multiplier as a state variable

# SUMMARY

- MM show that problems like ($\mathbf{PP}_\mu$) can be solved recursively by using (cumulated) Lagrange multipliers as state variables
- Standard way to proceed: write down the Lagrangean of the problem at hand, and solve the functional equation (**SPFE**) associated with it
- Value function iteration or Howard's improvement algorithm work well for this methodology **if the policy correspondence is single-valued** (see Marimon, Messner and Pavoni (2009) for examples)
- The majority of applications solve Lagrangean's FOCs (even more problematic...)

# ASSUMPTIONS (ALL AT ONCE)

### ASSUMPTION 1

*$S$ is a countable set, and the process $\{s_t\}$ is Markov.*

### ASSUMPTION 2

*$X$ and $A$ are convex subsets of $R^n$ and $R^m$. Functions $p(\cdot,\cdot,\cdot)$ and $\ell(\cdot,\cdot,\cdot)$ are continuous. For any $(x,s) \in X \times S$ there exists $\widetilde{a} \in A$ such that $p(x,\widetilde{a},s) > 0$.*

### ASSUMPTION 3

*$\ell(\cdot,\cdot,s)$ is linear and $p(\cdot,\cdot,s)$ is concave.*

# ASSUMPTIONS (ALL AT ONCE)

## ASSUMPTION 4

*Given $(x,s) \in X \times S$ there exist constants $B > 0$ and $\varphi \in \left(0, \beta^{-1}\right)$ such that if $p(x,a,s) \geq 0$ and $x' = \ell\left(x,a,s'\right)$, then $\|a\| \leq B\|x\|$ and $\left\|x'\right\| \leq \varphi\|x\|$.*

## ASSUMPTION 5

*$h_i^j(\cdot,\cdot,s)$, $i = 0,1$, $j = 0,...,l$, are continuous and uniformly bounded, $h_0^j(\cdot,\cdot,s)$ is non-decreasing and $\beta \in (0,1)$.*

## ASSUMPTION 6

*$h_i^j(\cdot,\cdot,s)$, $i = 0,1$, $j = 0,...,l$, are continuously differentiable on $\left\{(x,a) : p(x,a,s) > 0\right\}$.*

# ASSUMPTIONS (ALL AT ONCE)

### ASSUMPTION 7

$h_i^j(\cdot,\cdot,s)$, $i=0,1$, $j=0,...,l$, *are concave.*

### ASSUMPTION 8

*Assumption 7 holds and moreover* $h_0^j(\cdot,\cdot,s)$, $j=0,...,l$, *are strictly concave.*

### ASSUMPTION 9

*For all* $(x,s)$, *there exists a plan* $\{\widetilde{a}\}_{n=0}^{\infty}$ *with initial conditions* $(x,s)$ *which satisfies the inequality constraints* (29) *and* (30) *with strict inequality.*

# ASSUMPTIONS (ALL AT ONCE)

- Assumptions 1-5 are quite standard for many models
- Assumption 4 allows for technologies that yield long-run growth (e.g. AK technology in Ramsey growth model)
- Under Assumptions 1 - 5 and 7, and given $\mu \in R_+^{l+1}$, there is a solution $\mathbf{a}^* \equiv \{a_t\}_{t=0}^{\infty}$ that solves $\mathbf{PP}_\mu$ with initial conditions $(x,s)$, achieving the value $V_\mu(x,s)$
- If Assumption 8 holds, the solution is unique.

# $PP_\mu \Rightarrow SPP_\mu$

The first theorem links the solution of $\mathbf{PP}_\mu$ to $\mathbf{SPP}_\mu$.

### THEOREM 1

$(\mathbf{PP}_\mu \Rightarrow \mathbf{SPP}_\mu)$ *Let Assumptions 1 - 5, 7 and 9 be satisfied. Given* $\mu \in R_+^{l+1}$, *let* $\mathbf{a}^*$ *be a solution of* $\mathbf{PP}_\mu$ *with initial conditions* $(x,s)$. *Then there exists* $\gamma^* \in R_+^l$ *such that* $(\mathbf{a}^*, \gamma^*)$ *is a solution to* $\mathbf{SPP}_\mu$ *in state* $(x,s)$, *and the value of the latter is* $V_\mu(x,s)$.

# $PP_\mu \Leftarrow SPP_\mu$

The second theorem proves the converse result, and it is worth mentioning that it is basically assumption-free:

THEOREM 2

$\left(\mathbf{SPP}_\mu \Rightarrow \mathbf{PP}_\mu\right)$ *Given* $\mu \in R_+^{l+1}$ *and initial conditions* $(x,s)$, *let* $(\mathbf{a}^*, \gamma^*)$ *be a solution to* $\mathbf{SPP}_\mu$. *Then* $\mathbf{a}^*$ *is a solution to* $\mathbf{PP}_\mu$ *in state* $(x,s)$.

# SADDLE-POINT POLICY CORRESPONDENCE

Define the saddle-point policy correspondence as:

$$\Psi(x,\mu,s) =$$

$$\left\{ (a^*,\gamma^*) : W(x,\mu,s) = \mu h_0(x,a^*,s) + \gamma^* h_1(x,a^*,s) + \beta E\left[ W\left(x^{*\prime},\mu^{*\prime},s'\right) \mid s \right] \right.$$

$$\left. s.t. \quad x^{*\prime} = \ell\left(x,a^*,s'\right), \qquad p(x,a^*,s) \geq 0, \qquad \mu^{*\prime} = \varphi(\mu,\gamma^*,s) \right\}$$

# $(\mathbf{SPP}_\mu \Rightarrow \mathbf{SPFE})$

We can therefore state the result:

THEOREM 3

$(\mathbf{SPP}_\mu \Rightarrow \mathbf{SPFE})$ *Let* $W(x,\mu,s) \equiv V_\mu(x,s)$ *be the value of* $\mathbf{SPP}_\mu$ *at* $(x,s)$, *for an arbitrary* $(x,\mu,s)$, *then the value function* $W$ *satisfies* **SPFE**. *In particular, if* $(\mathbf{a}^*,\gamma^*)$ *is a solution to* $\mathbf{SPP}_\mu$ *in state* $(x,s)$, *then* $(a_0^*,\gamma^*) \in \Psi(x,\mu,s)$.

# $\left(\mathbf{SPP}_\mu \Leftarrow \mathbf{SPFE}\right)$

The converse is also true under differentiability assumptions:

THEOREM 4

$\left(\mathbf{SPFE} \Rightarrow \mathbf{SPP}_\mu\right)$ *Let $W$ be continuous in $(x, \mu)$, concave in $z$, homogeneous of degree one in $\mu$, and satisfy* **SPFE**. *Let Assumption 6 hold. Then if $(\mathbf{a}^*, \gamma^*)$ is generated by the saddle-point policy function $\psi$ associated with $W$ from initial conditions $(x, \mu, s)$, and $p\left(x_t^*, a_t^*, s_t\right) > 0$ for any $t$ and any $s_t$, then $(\mathbf{a}^*, \gamma^*)$ is also a solution to* $\mathbf{SPP}_\mu$ *at $(x, s)$.*

# NOTICE THAT:

- Conditions under which Theorem 4 holds are quite standard
- Homogeneity, continuity and boundedness of $W$ are due to Assumptions 2-5
- Concavity and differentiability of $W$ are more restrictive, and due to Assumptions 7 and 6
- However, Theorem 4 can be proved also without Assumptions 7 and 6: in that case the only restrictive condition is that $(\mathbf{a}^*, \gamma^*)$ must be generated by a saddle-point policy function, i.e. must be uniquely determined.

# SPFE

- Next result: **SPFE** is a contraction mapping
- We can use iterative methods analogous to dynamic programming to solve for the optimal policy

# SPFE

- Next result: **SPFE** is a contraction mapping
- We can use iterative methods analogous to dynamic programming to solve for the optimal policy
- $W$ is homogeneous of degree one in $\mu$ and we can therefore rewrite it as $W = \mu\omega$. Define the set $M$ as:

$$M = \{\omega : X \times R_+^{l+1} \times S \to R^{l+1} \text{ s.t. for } j = 0, .., l$$

i. $\omega_j(\cdot,\cdot,s)$ is continuous, and $\omega_j(\cdot,\mu,s)$ is bounded if $\|\mu\| \leq 1$

ii. $\omega_j(\cdot,\mu,s)$ is concave, and

iii. $\omega_j(x,\cdot,s)$ is convex and homogeneous of degree zero $\}$

# DYNAMIC SADDLE-POINT PROBLEM

Define the Dynamic Saddle-Point Problem as:

$$\inf_{\gamma \geq 0} \sup_{a} \left\{ \mu h_0 (x,a,s) + \gamma h_1 (x,a,s) + \beta E \left[ \mu' \omega \left( x', \mu', s' \right) \mid s \right] \right\} \quad \mathbf{DSPP}_{(x,\mu,s)}$$

$$\begin{aligned} s.t. \quad & x' = \ell \left( x,a,s' \right), \qquad p(x,a,s) \geq 0 \\ & \mu' = \varphi (\mu, \gamma, s) \end{aligned}$$

# INTERIORITY ASSUMPTION

In order to have a well defined program in $\mathbf{DSPP}_{(x,\mu,s)}$, we need to make the following interiority assumption:

### ASSUMPTION 10

*For any* $(x,\mu,s) \in X \times R_+^{l+1} \times S$*, there exists* $\widetilde{a} \in A$ *satisfying Assumption 2 such that for any* $\mu' \in R_+^{l+1}$, $\|\mu'\| \leq 1$*, and* $j = 0,\ldots,l$ *the following holds:*
$h_1^j(x,\widetilde{a},s) + \beta E\left[\omega^j\left(\ell(x,\widetilde{a},s),\mu',s'\right) \mid s\right] > 0.$

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*For any $(x,\mu,s) \in X \times R_+^{l+1} \times S$, there exists $\widetilde{a} \in A$ satisfying Assumption 2 such that for any $\mu' \in R_+^{l+1}$, $\|\mu'\| \leq 1$, and $j = 0,...,l$ the following holds:* $h_1^j(x,\widetilde{a},s) + \beta E\left[\omega^j\left(\ell(x,\widetilde{a},s),\mu',s'\right) \mid s\right] > 0.$

This assumption is satisfied whenever Assumption 9 is and $\mu'\omega\left(\ell(x,\widetilde{a},s),\mu',s'\right)$ is the value function of $\mathbf{SPP}_{\left(\ell(x,\widetilde{a},s),\mu',s'\right)}$.

# $\mathbf{DSPP}_{(x,\mu,s)}$ HAS A SOLUTION

### PROPOSITION 1

*Let $\omega \in M$ and assume Assumptions 1-5, 7 and 10 hold. Then there exists $(a^*, \gamma^*)$ that solves $\mathbf{DSPP}_{(x,\mu,s)}$. If Assumption 8 is satisfied, then $a^*(x, \mu, s)$ is unique.*

# $\mathbf{DSPP}_{(x,\mu,s)}$ HAS A SOLUTION

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Assume from now on that:

### ASSUMPTION 11

$\mathbf{DSPP}_{(x,\mu,s)}$ *has a unique solution*

# The Saddle-Point operator

We can write **SPFE** as

$$\mu\omega(x,\mu,s) = \mu h_0(x,a^*,s) + \gamma^* h_1(x,a^*,s) + \\ + \beta E\left[\varphi(\mu,\gamma^*,s)\,\omega\left(x^{*\prime},\varphi(\mu,\gamma^*,s),s'\right) \mid s\right]$$

# THE SADDLE-POINT OPERATOR

We can write **SPFE** as

$$\mu\omega(x,\mu,s) = \mu h_0(x,a^*,s) + \gamma^* h_1(x,a^*,s) + \\ + \beta E\left[\varphi(\mu,\gamma^*,s)\,\omega\left(x^{*\prime},\varphi(\mu,\gamma^*,s),s'\right) \mid s\right]$$

and notice that at the optimum:

$$\gamma^{*j}\left\{h_1^j(x,a^*,s) + \beta E\left[\omega_j\left(x^{*\prime},\varphi(\mu,\gamma^*,s),s'\right) \mid s\right]\right\} = 0$$

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We can use these two facts to define **SPFE** operator $T : M \to M$

$$\omega_j(x,\mu,s) = h_0^j(x,a^*(x,\mu,s),s) + \beta E\left[\omega_j\left(x'(x,\mu,s),\mu'(x,\mu,s),s'\right) \mid s\right] \quad (32)$$

$$\text{if } j = 0,\ldots,k, \text{ and}$$

$$\omega_j(x,\mu,s) = h_0^j(x,a^*(x,\mu,s),s) \quad \text{if } j = k+1,\ldots,l \quad (33)$$

# THE SADDLE-POINT OPERATOR IS A CONTRACTION MAPPING

- Let $M^j$ be the $j$-th projection of $M$, $j = 0, ..., l$.
- Let $T^j : M^j \to M^j$ be the "individual" operator defined by (32)-(33) under Assumption 11.

# THE SADDLE-POINT OPERATOR IS A CONTRACTION MAPPING

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- Therefore it is possible to show:

**LEMMA 5**

*Let Assumptions 1-5, 7, 8, 10 and 11 hold. Then $T^j : M^j \to M^j$ is a contraction mapping.*

# $\mathbf{DSPP}_{(x,\mu,s)} \Rightarrow \mathbf{PP}_\mu(x,s)$

By using Lemma 5, and Theorems 2 and 4, we obtain the final result:

### THEOREM 6

$\left(\mathbf{DSPP}_{(x,\mu,s)} \Rightarrow \mathbf{PP}_\mu(x,s)\right)$ *Let Assumptions 1-5, 7, 8, 10 and 11 hold. Then* $T: M \to M$ *has a unique solution* $\omega$*, which defines a value function* $W(x,\mu,s) = \mu\omega(x,\mu,s)$ *and a saddle-point policy function* $\psi$*, such that if* $(\mathbf{a}^*, \gamma^*)$ *is generated by* $\psi$ *from* $(x,\mu,s)$*, then* $\mathbf{a}^*$ *is the unique solution to* $\mathbf{PP}_\mu$ *at* $(x,s)$.

# A THEOREM WITHOUT CONCAVITY ASSUMPTIONS

We can have unique solutions also without concavity. Define $\widetilde{M}$ as the set $M$ without the concavity assumption:

### COROLLARY 7

***(Bounded returns)*** *Let Assumptions 1-5 and that for all $(x,\mu,s)$ Assumption 11 hold. Then $T:\widetilde{M}\to\widetilde{M}$ has a unique solution $\omega$, which defines a value function $W(x,\mu,s)=\mu\omega(x,\mu,s)$ and a saddle-point policy function $\psi$, such that if $(\mathbf{a}^*,\gamma^*)$ is generated by $\psi$ from $(x,\mu,s)$, then $\mathbf{a}^*$ is the unique solution to $\mathbf{PP}_\mu$ at $(x,s)$.*

# NUMERICAL APPROACHES

- We showed that the operator defined by the functional equation is a contraction mapping
- We can solve it with an iterative procedure in the same spirit of value function iteration or Howard's improvement algorithm, adapted to the min max operator

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- In almost all applications: solve first-order conditions of the Lagrangean associated with the dynamic optimization
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- Another approach is Parameterized Expectations

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## Setup

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$$g\left(E_t\left[\phi\left(z_{t+1}, z_t\right) \mid x_t\right], z_t, z_{t-1}, u_t\right) = 0 \tag{34}$$

where $x_t$ is a subset of $\left(z_{t-1}, u_t\right)$ and a vector of state variables.

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where $x_t$ is a subset of $(z_{t-1}, u_t)$ and a vector of state variables.

- Conditional expectations in (34) is recursive

$$E_t\left[\phi\left(z_{t+1}, z_t\right) \mid x_t\right] = \mathscr{E}\left(x_t\right) \tag{35}$$

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## PEA CONCEPT

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## PEA concept

- Approximate the conditional expectations in (35) by a polynomial or a similar functional form (splines or finite methods are ok too).
- Call this approximation $\psi(\beta;x)$, where $\beta$ is a vector of coefficient for the polynomial approximation.
- We can rewrite (34) as:

$$g\left(\psi\left(\beta;x_t\left(\beta\right)\right),z_t\left(\beta\right),z_{t-1}\left(\beta\right),u_t\right)=0 \tag{36}$$

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- The task is to find a good approximation

# Parameterized Expectations Algorithm

1. Write the system of equations (34) such that it is invertible in its second argument. Find a set of state variables that satisfy (35). Replace the true conditional expectation by the parameterized function $\psi(\beta;\cdot)$ and get (36). Fix initial conditions $(z_0, u_0)$. Generate a series $\{u_t\}_{t=0}^{T}$ for T large.

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3. Find a new vector of parameters $G(\beta)$ that solves the following non-linear least squares problem:

$$G(\beta) = \arg\min_{\xi} \frac{1}{T}\sum_{t=0}^{T} \left\| \phi\left(z_{t+1}(\beta), z_t(\beta)\right) - \psi\left(\xi; x_t(\beta)\right) \right\| \tag{37}$$

which is easy to solve if we perform a NLLS regression of $\phi\left(z_{t+1}(\beta), z_t(\beta)\right)$ on $\psi\left(\xi; x_t(\beta)\right)$.

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4. Iterate on steps 2-3 until you get the fixed point of $G(\cdot)$, i.e. until $G\left(\beta_f\right) \approx \beta_f$ in numerical terms.

# Optimal policy

The problem was:

$$\max_{\{c_t, b_t, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[ u(c_t) + v(l_t) \right]$$

$$s.t. \quad c_t - \beta \frac{E_t u'(c_{t+1})}{u'(c_t)} b_{t+1}^g = -(c_t + g_t) \frac{v'(c_t + g_t)}{u'(c_t)} - b_t^g$$

# Optimal policy

Define:

$$y_t \equiv \frac{-(c_t+g_t)\, v'(c_t+g_t) + b_t u'(c_t) - c_t u'(c_t)}{b_{t+1}} \tag{38}$$

Notice that we can transform it in a $\mathbf{PP}_\mu$ problem in the following way. Let $l \equiv 1$, $N_0 = \infty$ and $N_1 = 0$, $s \equiv g$, $x \equiv b$, $a \equiv (c, l)$, $p(x,a,s) \equiv l - (c+g)$, $h_0^0(x,a,s) \equiv u(c) + v(c+g)$, $h_0^1(x,a,s) \equiv u'(c)$, $h_1^0(x,a,s) \equiv h_0^0(x,a,s) - R$, $R$ big, $h_1^1(x,a,s) \equiv -y$, and $\ell(x,a,s) \equiv \frac{-(c+g)v'(c+g)+bu'(c)-cu'(c)}{y}$.

# OPTIMAL POLICY

$$\max_{\{c_t,b_t\}_{t=0}^{\infty}} E\left[\left\{\sum_{t=0}^{\infty}\beta^t\mu^0\left[u(c_t)+v(c_t+g_t)\right]\right\}+\mu^1\left[u'(c_0)\right]|s\right] \quad (\mathbf{PP}_\mu \text{ Optimal Policy})$$

$$s.t. \quad b_{t+1} = \frac{-(c_t+g_t)v'(c_t+g_t)+b_tu'(c_t)-c_tu'(c_t)}{y_t}$$

$$E_t\sum_{n=1}^{\infty}\beta^n\left[u(c_n)+v(c_n+g_n)\right]+\left[u(c_t)+v(c_t+g_t)-R\right]\geq 0 \qquad t\geq 0 \quad (39)$$

$$\beta E_t\left[u(c_{t+1})\right]-y_t\geq 0 \qquad t\geq 0 \quad (40)$$

# Optimal policy

- Write down the Lagrangean of (($\mathbf{PP}_{\mu}$ Optimal Policy)) (Lagrange multipliers $\gamma_t^j$)
- Use constraints (40) multiplied by $b_{t+1}$
- Constraint (39) is always not binding because of our interiority assumptions $\rightarrow \gamma_t^0 = 0$ for any $t$
- We can consider the relaxed program where there is no the sequence of constraints (39).

# The Lagrangean

We can write the Lagrangean as:

$$\mathscr{L}\left(\{c_t, g_t, b_t, \mu_t, \gamma_t\}_{t=0}^{\infty}\right) =$$

$$= E\left[\sum_{t=0}^{\infty} \beta^t \left\{\mu^0 \left[u\left(c_t\right) + v\left(c_t + g_t\right)\right] + \mu_t^1 \left[u'\left(c_t\right)\right] b_t - \right.\right.$$

$$\left.\left. -\gamma_t^1 \left[-\left(c_t + g_t\right) v'\left(c_t + g_t\right) + b_t u'\left(c_t\right) - c_t u'\left(c_t\right)\right]\right\} | s\right]$$

where $\mu_{t+1}^1 = \gamma_t^1$, with $\mu_0^1 = \mu^1 = 0$, and $\mu^0 = 1$.

# The Lagrangean FOCs

$$/c_t: \qquad u'(c_t) + v'(c_t + g_t) + \mu_t^1 u''(c_t) b_t - \gamma_t^1 \left[ -v''(c_t + g_t) - \right.$$
$$\left. -(c_t + g_t) v''(c_t + g_t) + b_t u''(c_t) - u'(c_t) - c_t u''(c_t) \right] = 0 \quad (41)$$

$$/b_{t+1}: \qquad E_t \left[ \left( \mu_{t+1}^1 - \gamma_{t+1}^1 \right) u'(c_{t+1}) \right] = 0 \quad (42)$$

plus all the constraints of the original problem

# Collocation

Given that we can apply MM, we know that the policy functions will be:

$$\left(c_t, b_{t+1}, \gamma_t^1\right) = \psi\left(b_t, g_t, \mu_t^1\right)$$

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We can approximate $\psi$ with Chebichev polynomials, substitute this approximation in the first order conditions (41)-(42) and all the constraints of the original problem, and iterate until the approximated policy function solves them.

# OPTIMAL POLICY

**Exercise 1 Optimal policy**

Modify the code for collocation over the first order conditions of the stochastic growth model to solve the Ramsey taxation problem.

# LACK OF COMMITMENT

The problem was:

$$\max_{\{c_t\}_{t=0}^{\infty}} E_{-1} \sum_{t=0}^{\infty} \beta^t (y_t - c_t)$$

$$s.t. \quad E_{-1} \sum_{t=0}^{\infty} \beta^t u(c_t) \geq U_0 \tag{43}$$

$$u(c_t) + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} u\left(c_{t+j}\right) \geq u(y_t) + \beta U_{aut}, \quad t = 0, 1, \ldots \tag{44}$$

# REWRITE THE PROBLEM

We can get rid of (43): this constraint only requires the principal to give a weight $\alpha$ (which depends on the minimum value $U_0$) to agent's utility in the principal's maximization problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} E_{-1} \sum_{t=0}^{\infty} \beta^t \left(y_t - c_t + \alpha u\left(c_t\right)\right)$$

$$s.t. \quad u\left(c_t\right) + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} u\left(c_{t+j}\right) \geq u\left(y_t\right) + \beta U_{aut}, \quad t = 0, 1, \ldots$$

# REWRITE THE PROBLEM

We can see this problem is already in the form $\mathbf{PP}_{\mu}$, by letting $l \equiv 1$, $N_0 = \infty$ and $N_1 = \infty$, $s \equiv y$, $a \equiv c$, $h_0^0(x,a,s) \equiv y - c + \alpha u(c)$, $h_0^1(x,a,s) \equiv u(c)$, $h_1^0(x,a,s) \equiv h_0^0(x,a,s) - R$, $h_1^1(x,a,s) \equiv h_0^1(x,a,s) - u(y) - \beta U_{aut}$, and using $\mu_0 = (1,0)$. We can write it as:

$$\max_{\{c_t\}_{t=0}^{\infty}} E_{-1} \sum_{t=0}^{\infty} \beta^t \left[ \mu^0 \left( y_t - c_t + \alpha u(c_t) \right) + \mu^1 u(c_t) \right]$$

$$s.t. \quad u(c_t) + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} u(c_{t+j}) \geq u(y_t) + \beta U_{aut}, \quad t = 0, 1, \ldots$$

$$E_t \sum_{n=0}^{\infty} \beta^t \left[ y_{t+n} - c_{t+n} + \alpha u(c_{t+n}) \right] - R \geq 0 \tag{45}$$

# The Lagrangean

The Lagrangean is:

$$\mathscr{L}\left(\{c_t, y_t, \mu_t, \gamma_t\}_{t=0}^{\infty}\right) = E\sum_{t=0}^{\infty}\beta^t\Big[\mu^0\left(y_t - c_t +\right) + \left(\mu^0\alpha + \mu_t^1\right)u\left(c_t\right) + \\ + \gamma_t^1\left(u\left(c_t\right) - u\left(y_t\right) - \beta U_{aut}\right)\Big]$$

with $\mu^0 = 1$, $\mu^1 = 0$ and $\mu_{t+1}^j = \mu_t^j + \gamma_t^j$ for $j = 0, 1$.

# The Lagrangean FOCs

Take first order conditions with respect to consumption:

$$-1+\left(\alpha+\mu_{t+1}^{1}\right)u'\left(c_t\right)=0$$

Rearranging, we get:

$$\frac{1}{u'\left(c_t\right)}=\left(\alpha+\mu_{t+1}^{1}\right) \tag{46}$$

# Collocation

Participation constraint is not always binding, therefore we must solve for the two possible cases: the case of binding constraint, and the case with no binding constraint:

1. We solve the problem without taking into account the participation constraint. We calculate the RHS of (44) consistent with this solution.
2. For those gridpoints such that $RHS > u(y_t) + \beta U_{aut}$, our solution is good. For those gridpoints such that $RHS < u(y_t) + \beta U_{aut}$, assume the participation constraint is binding, and re-solve the problem.

# Endowment is private information

For this problem, we use the methodology of Sleet and Yeltekin (2009). Remember our maximization problem:

$$\max_{\left\{\tau_t\left(y^t\right)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[-\tau_t\left(y^t\right)\right] \pi\left(y^t | y_{-1}\right)$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{y^t \in y^{t+1}} \beta^t u\left(c_t\left(y^t\right)\right) \pi\left(y^t | y_{-1}\right) \geq U_0 \tag{47}$$

$$\beta^t \left[u\left(c_t\left(y^{t-1}, \bar{y}_i\right)\right) + \beta U_{t+1}\left(y^{t-1}, \bar{y}_i\right) - u\left(c_t\left(y^{t-1}, \bar{y}_{i-1}\right)\right) - \beta U_{t+1}\left(y^{t-1}, \bar{y}_{i-1}\right)\right] \geq 0 \tag{48}$$

$$\forall y^{t-1}, \quad \forall \bar{y}_i, \quad i = 2, \ldots, N$$

where

$$U_{t+1}\left(y^{t-1}, \bar{y}_i\right) \equiv \sum_{j=1}^{\infty} \sum_{y^{t+j} \in Y^{t+j+1}} \beta^{j-1} u\left(c_{t+j}\left(y^{t-1}, \bar{y}_i, y^{t+j}\right)\right) \pi\left(y^{t+j} | y_{-1}\right)$$

# Rewrite the problem

Eliminate the constraint (47):

$$\max_{\left\{c_t\left(y^t\right)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[y_t - c_t\left(y^t\right) + \alpha u\left(c_t\left(y^t\right)\right)\right] \pi\left(y^t | y_{-1}\right)$$

$$\beta^t \left[u\left(c_t\left(y^{t-1}, \bar{y}_i\right)\right) + \beta U_{t+1}\left(y^{t-1}, \bar{y}_i\right) - u\left(c_t\left(y^{t-1}, \bar{y}_{i-1}\right)\right) - \beta U_{t+1}\left(y^{t-1}, \bar{y}_{i-1}\right)\right] \geq 0$$

$$\forall y^{t-1}, \quad \forall \bar{y}_i, \quad i = 2, \ldots, N$$

# Rewrite the problem

Assume shocks are i.i.d., and define:

$$p_{ik} \equiv \frac{\pi(\bar{y}_k)}{\pi(\bar{y}_i)}$$

$$\varsigma_t^j(y^{t-1}, \bar{y}_i) \equiv \gamma_t^j(y^{t-1}, \bar{y}_i)$$

$$\varsigma_t^j(y^{t-1}, \bar{y}_k) \equiv -\gamma_t^j(y^{t-1}, \bar{y}_i) p_{ik}$$

where $\gamma_t^j(y^{t-1}, \bar{y}_i)$ is the Lagrange multiplier attached to constraint $j = 0, 1$ as usual, and in particular $\gamma_t^1(y^{t-1}, \bar{y}_i)$ is attached to $DIC_t\left(y^{t-1}, \bar{y}_i; \bar{y}_{i-1}\right)$.

# THE LAGRANGEAN

We can due our standard algebra, and by defining $\mu_t^j(y^{t-1},\overline{y}_i) \equiv \mu_{t-1}^j(y^{t-1}) + \varsigma_t^j(y^{t-1},\overline{y}_i)$ for any $\overline{y}_i \in Y$ with $\mu_{-1}^0(y_{-1}) = \mu_{-1}^0 = 1$, and $\mu_{-1}^1(y_{-1}) = \mu_{-1}^1 = 0$, we can write the Lagrangean as:

$$\mathscr{L}(c^\infty, \gamma^\infty; \alpha) = \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[\mu^0 \left(y_t - c_t\left(y^t\right)\right) + \left(\mu^0 \alpha + \mu_t^1\left(y^t\right)\right) u\left(c_t\left(y^t\right)\right)\right] \pi^t\left(y^t\right)$$

# HIDDEN EFFORT

The problem of the planner is:

$$\max_{\left\{c_t\left(y^t\right)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[y_t - c_t\left(y^t\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right)$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right) \geq U_0$$

$$\left\{a_t\left(y^t\right)\right\}_{t=0}^{\infty} \in \arg \max_{\left\{a_t\left(y^t\right)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right) \tag{49}$$

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# A (BIG) CAVEAT

- We need to restrict the class of models we can solve
- We assume we can use a first-order approach: instead of using (49), we use the first-order conditions of that optimization problem with respect to effort
- In order to guarantee that we get the same solution as in the original problem, we assume that two conditions are satisfied. They are known as **Rogerson conditions**.

# Rogerson's conditions

**Condition 1 (Monotone Likelihood-Ratio Condition (MLRC))**

$\widehat{a} \leq \widehat{\widehat{a}} \Longrightarrow \frac{\pi\left(\overline{y}_s|\widehat{a}\right)}{\pi\left(\overline{y}_s|\widehat{\widehat{a}}\right)}$ *is nonincreasing in s.*

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- If $\pi(\cdot)$ is differentiable, then MLRC is equivalent to $\frac{\pi_a\left(\overline{y}_s|a\right)}{\pi\left(\overline{y}_s|a\right)}$ being nondecreasing in $s$ for any $a$

# Rogerson's conditions (cont)

### Condition 2 (Convexity of the Distribution Function Condition (CDFC))

*$F''(\bar{y}_s \mid a)$ is nonnegative for any $s$ and every $a$.*

CDFC is a sort of decreasing marginal returns for effort in stochastic terms.

The two conditions make sure the agent's problem is strictly concave

# Rewrite the problem

$$W^{SWF}(s_0) = \max_{\left\{a_t(y^t), c_t(y^t)\right\}_{t=0}^{\infty}} \mu_0^0 \sum_{t=0}^{\infty} \sum_{y^t} \beta^t \left[ y(y_t) - c_t\left(y^t\right) \right] \pi\left(y^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right) +$$

$$+ \mu_0^1 \sum_{t=0}^{\infty} \sum_{y^t} \beta^t \left[ u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right) \right] \pi\left(y^t \mid s_0, a^{t-1}\left(s^{t-1}\right)\right)$$

$$s.t. \quad \upsilon'\left(a_t\left(y^t\right)\right) = \sum_{j=1}^{\infty} \beta^j \sum_{y^{t+j}|y^t} \frac{\pi_a\left(y_{t+1} \mid y_t, a_t\left(y^t\right)\right)}{\pi\left(y_{t+1} \mid y_t, a_t\left(y^t\right)\right)} \times$$

$$\times \left[ u\left(c_{t+j}\left(y^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(y^{t+j}\right)\right) \right] \pi\left(y^{t+j} \mid y^t, a^{t+j-1}\left(y^{t+j-1} \mid y^t\right)\right)$$

# The Lagrangean

$$
\begin{aligned}
L\left(y_0, c^{\infty}, a^{\infty}, \lambda^{\infty}\right) = & \sum_{t=0}^{\infty} \sum_{y^t} \beta^t \left\{ \mu_0^0 \left( y_t - c_t\left(y^t\right)\right) + \mu_0^1 \left( u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right)\right\} \pi\left(y^t \mid s_0, a^{t-1}\left(y^{t-1}\right)\right) + \\
& - \sum_{t=0}^{\infty} \sum_{y^t} \beta^t \lambda_t\left(y^t\right) \left\{ \upsilon'\left(a_t\left(y^t\right)\right) - \sum_{j=1}^{\infty} \beta^j \sum_{y^{t+j} \mid y^t} \frac{\pi_a\left(y_{t+1} \mid y_t, a_t\left(y^t\right)\right)}{\pi\left(y_{t+1} \mid y_t, a_t\left(y^t\right)\right)} \times \right. \\
& \left. \times \left[ u\left(c_{t+j}\left(y^{t+j}\right)\right) - \upsilon\left(a_{t+j}\left(y^{t+j}\right)\right)\right] \pi\left(y^{t+j} \mid y^t, a^{t+j-1}\left(y^{t+j-1} \mid y^t\right)\right)\right\} \pi\left(y^t \mid y_0, a^{t-1}\left(y^{t-1}\right)\right)
\end{aligned}
$$

# The Lagrangean (cont.)

$$L\left(y_0, c^\infty, a^\infty, \lambda^\infty\right) = \sum_{t=0}^{\infty} \sum_{y^t} \beta^t \left\{ \mu_0^0 \left( y_t - c_t\left(y^t\right)\right) + \mu_t^1\left(y^t\right) \left[ u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right] + \right.$$

$$\left. -\lambda_t\left(y^t\right) \upsilon'\left(a_t\left(y^t\right)\right)\right\} \pi\left(y^t \mid s_0, a^{t-1}\left(y^{t-1}\right)\right)$$

where

$$\mu_t^1\left(y^{t-1}, y_t\right) = \mu_0^1 + \sum_{i=0}^{t-1} \lambda_i\left(y^i\right) \frac{\pi_a\left(y_{i+1} \mid y_i, a_i\left(y^i\right)\right)}{\pi\left(y_{i+1} \mid y_i, a_i\left(y^i\right)\right)}$$

# The law of motion for $\mu_t^1\left(y^t\right)$

$$\mu_{t+1}^1\left(y^t,\widehat{y}_s\right) = \mu_t^1\left(y^t\right) + \lambda_t\left(y^t\right)\frac{\pi_a\left(y_{t+1}=\widehat{y}_s \mid y_t, a_t\left(y^t\right)\right)}{\pi\left(y_{t+1}=\widehat{y}_s \mid y_t, a_t\left(y^t\right)\right)} \quad \forall \widehat{y}_s \in Y$$

$$\mu_0^1\left(y^0\right) = \mu_0^1$$

# The saddle-point functional equation

$$J\left(y^{i},\mu\right)=\max_{a,c,}\min_{\lambda}\mu^{0}\left(y^{i}-c\right)+\mu^{1}\left[u\left(c\right)-\upsilon\left(a\right)\right]-\lambda\upsilon'\left(a\right)+$$
$$+\beta\sum_{s}J\left(y^{s},\mu'_{s}\right)\pi\left(y^{s}\mid a\right)$$
$$s.t.\quad \mu_{s}^{h\prime}=\mu^{h}+\lambda\frac{\pi_{a}\left(y_{s}\mid a_{i}\right)}{\pi\left(y_{s}\mid a_{i}\right)}$$

# How to solve it numerically

- Collocation algorithm over the first-order conditions of the Lagrangean

# HOW TO SOLVE IT NUMERICALLY

- Collocation algorithm over the first-order conditions of the Lagrangean
- $\varsigma \equiv$ vector of allocations
- $\chi \equiv$ vector of Lagrange multipliers
- $x \equiv$ vector of natural states
- $\theta \equiv$ vector of costates
- $R(y, \varsigma, \chi, x, \theta) \equiv$ objective function in the Lagrangean
- $r(y, \varsigma, \chi, x, \theta) \equiv$ instantaneous utility function for the agent

# The algorithm

1. Fix $\mu_0^1$ and define a discrete grid $G \subset X \times \Theta$ for natural states and costates.
2. Approximate policy functions for allocations $\varsigma$ and Lagrange multipliers $\chi$, the value function of the principal $J$ and the continuation value of the agent $U$ using cubic splines (or Chebychev polynomials, depending on the application), and set initial conditions for the approximation parameters
3. For any $(y, x, \theta) \in G$, use a nonlinear solver to solve for the Lagrangean first order conditions and the following two equations for the continuation value $U$ and the value function $J$:

$$U(y,x,\theta) = r(y,\varsigma,\chi,x,\theta) + \beta\left[\pi(a)\,U\left(y^H,x'^H,\theta'^H\right) + (1-\pi(a))\,U\left(y^L,x'^L,\theta'^L\right)\right] \tag{50}$$

$$J(y,x,\theta) = R(y,\varsigma,\chi,x,\theta) + \beta\left[\pi(a)\,J\left(y^H,x'^H,\theta'^H\right) + (1-\pi(a))\,J\left(y^L,x'^L,\theta'^L\right)\right] \tag{51}$$

# MORE DETAILS

$$c_t : \quad u'(c_t) = \frac{1}{\mu_t^1} \tag{52}$$

$$\begin{aligned}
a_t : \quad 0 = & -\lambda_t \upsilon''(a_t) - \mu_t^1 \upsilon'(a_t) + \\
& + \pi_a(a_t) \beta E_{t+1}^a \left\{ \sum_{j=1}^{\infty} \beta^{j-1} \left\{ \left( y_{t+j} - c_{t+j} \right) - \lambda_{t+j} \upsilon'\left(a_{t+j}\right) + \mu_{t+j}^1 \left[ u\left(c_{t+j}\right) - \upsilon\left(a_{t+j}\right) \right] \right\} \mid y_{t+1} = y^H \right\} + \\
& - \pi_a(a_t) \beta E_{t+1}^a \left\{ \sum_{j=1}^{\infty} \beta^{j-1} \left\{ \left( y_{t+j} - c_{t+j} \right) - \lambda_{t+j} \upsilon'\left(a_{t+j}\right) + \mu_{t+j}^1 \left[ u\left(c_{t+j}\right) - \upsilon\left(a_{t+j}\right) \right] \right\} \mid y_{t+1} = y^L \right\} + \\
& + \beta \lambda_t \pi(a_t) \frac{\partial \left( \frac{\pi_a(a_t)}{\pi(a_t)} \right)}{\partial a_t} \left[ u(c_{t+1}) - \upsilon(a_{t+1}) \mid y_{t+1} = y^H \right] + \\
& + \beta \lambda_t (1 - \pi(a_t)) \frac{\partial \left( \frac{-\pi_a(a_t)}{1-\pi(a_t)} \right)}{\partial a_t} \left[ u(c_{t+1}) - \upsilon(a_{t+1}) \mid y_{t+1} = y^L \right]
\end{aligned} \tag{53}$$

# MORE DETAILS

$$\lambda_t : \quad 0 = -\upsilon'(a_t) + \pi_a(a_t)\beta E^a_{t+1}\left\{\sum_{j=1}^{\infty}\beta^{j-1}\left[u\left(c_{t+j}\right) - \upsilon\left(a_{t+j}\right)|y_{t+1} = y^H\right]\right\} + \\ -\pi_a(a_t)\beta E^a_{t+1}\left\{\sum_{j=1}^{\infty}\beta^{j-1}\left[u\left(c_{t+j}\right) - \upsilon\left(a_{t+j}\right)|y_{t+1} = y^L\right]\right\} \tag{54}$$

# More details

Notice that

$$J\left(y^i,\mu_{t+1}^{1,i}\right)=$$
$$=E_{t+1}^a\left\{\sum_{j=1}^{\infty}\beta^{j-1}\left\{\left(y_{t+j}-c_{t+j}\right)-\lambda_{t+j}\upsilon'\left(a_{t+j}\right)+\mu_{t+j}^1\left[u\left(c_{t+j}\right)-\upsilon\left(a_{t+j}\right)\right]\right\}\mid y_{t+1}=y^i\right\}$$
$$i=H,L$$

and

$$U\left(y^i,\mu_{t+1}^{1,i}\right)=E_{t+1}^a\left\{\sum_{j=1}^{\infty}\beta^{j-1}\left[u\left(c_{t+j}\right)-\upsilon\left(a_{t+j}\right)|y_{t+1}=y^i\right]\right\}$$
$$i=H,L$$

# Conclusion

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- APS complicated and easily untractable when state space is large

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- MM easy and no big problems with many state variables

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- APS complicated and easily untractable when state space is large
- MM easy and no big problems with many state variables
- MM + collocation on Lagrangean FOCs: easy and FAST


KEEP
CALM
AND
CODE
ON