Numerical Methods for Time Inconsistency, Private Information and Limited Commitment

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MODELS WITH TIME INCONSISTENCY, PRIVATE INFORMATION AND LIMITED COMMITMENT

When standard DP does not work

Till now, we have seen problems of the form:

$$\begin{aligned}
\max_{\{u_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \\
s.t. \quad & x_{t+1} = h(x_t, u_t), \quad t = 0, 1, \ldots \\
& x_0 \in X \text{ given}
\end{aligned}$$

WHEN STANDARD DP DOES NOT WORK

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However there are many situations in (macro)economics that cannot be represented in the previous form.

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- Three categories:
  - Problems of optimal policy choice by a benevolent government which suffer from time inconsistency
  - Models of limited commitment
  - Private information environments

CAN WE SOLVE THE PROBLEM? YES!

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- We can use a generalization of the Bellman theory
- Dynamic programming squared:
  - We transform the problem in order to introduce a new auxiliary state variable (sometimes more than one)
  - We can show that this new formulation of the problem has a recursive structure

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- Abreu, Pearce and Stacchetti (1990): auxiliary state is continuation value for each agent
  - Very popular for private info and limited commitment
- Marcet and Marimon (2017): auxiliary state is derived from Lagrange multipliers
  - Very popular in optimal policy and limited commitment case
  - More recently, extended to private information environments (Sleet and Yeltekin in a series of papers) and to repeated moral hazard framework (Mele (2014))

Optimal fiscal policy: HH problem

OPTIMAL FISCAL POLICY: HH PROBLEM

- Representative agent, benevolent government
- Government must finance exogenous random expenditure by means of distorsive taxation and debt.
- The agent solves his own maximization problem:

$$\max_{\{c_t, b_t, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(l_t) \right]$$

$$s.t. \quad c_t + p_t^b b_{t+1} = l_t (1 - \tau_t) + b_t \tag{1}$$

FOCS HH PROBLEM

FOCs HH problem

Substitute the budget constraint in the utility function, take FOCs:

$$/b_{t+1}: \qquad p_t^b u'(c_t) = \beta E_t u'(c_{t+1}) \tag{2}$$

$$/l_t: \qquad -\frac{v'(l_t)}{u'(c_t)} = (1-\tau_t) \tag{3}$$

Government's problem

Government's problem

The government wants to maximize the utility of representative agent subject to its own budget constraint, resource constraint, and **taking as given the optimal choices of the agent** summarized by equations (2)-(3) and the individual budget constraint (1):

$$\max_{\{c_t,b_t,l_t\}_{t=0}^{\infty}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(l_t)\right]$$

$$s.t. \quad (1),(2),(3)$$

$$c_t + g_t = l_t \tag{4}$$

$$g_t + p_t^b b_{t+1}^g = b_t^g + \tau_t l_t \tag{5}$$

$$b_t^g + b_t = 0 \tag{6}$$

PLAY WITH ALGEBRA AND GET:

Play with algebra and get:

$$\max_{\{c_t, b_t, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) + v\left(l_t\right) \right]$$

$$s.t. \quad c_t - \beta \frac{E_t u'\left(c_{t+1}\right)}{u'\left(c_t\right)} b_{t+1}^g = -\left(c_t + g_t\right) \frac{v'\left(c_t + g_t\right)}{u'\left(c_t\right)} - b_t^g$$

PLAY WITH ALGEBRA AND GET:

$$\max_{\{c_t,b_t,l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(l_t)\right]$$

$$s.t. \quad c_t - \beta \frac{E_t u'(c_{t+1})}{u'(c_t)} b^g_{t+1} = -(c_t + g_t) \frac{v'(c_t + g_t)}{u'(c_t)} - b^g_t$$

The optimal choice of c_t depends on the value of c_{t+1}

ONE-SIDED LACK OF COMMITMENT

One-sided lack of commitment

- Risk neutral Principal, risk averse Agent
- Agent receive y_t each period
- Principal borrows and lends at interest rate $R = \beta^{-1}$, agent cannot save and has no access to credit market
- Principal and agent want to share the risk associated with endowment
- write a contract at time 0, that defines the sharing rule for each period t and each possible realization of the endowment process

ONE-SIDED LACK OF COMMITMENT (CONT.)

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- Principal is a nice guy and can commit forever to stay in the risk-sharing contract
- Agent is not a loyal guy, and can leave the arrangement at any time, being at that point on his own and having to consume just his individual endowment
- write a contract such that the risk-sharing rule gives always the agent a sufficient amount to stay in the agreement

ONE-SIDED LACK OF COMMITMENT (CONT.)

One-sided lack of commitment (cont.)

Participation constraint makes sure the value of staying in the contract for the agent is always larger than the value of the contract in autarky (i.e., consuming only his own endowment):

$$\max_{\{c_t\}_{t=0}^{\infty}} E_{-1} \sum_{t=0}^{\infty} \beta^t (y_t - c_t)$$

$$s.t. \quad u(c_t) + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} u(c_{t+j}) \geq$$

$$u(y_t) + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} u(y_{t+j}), \quad t = 0, 1, \ldots \tag{7}$$

Endowment is private information

ENDOWMENT IS PRIVATE INFORMATION

- Assume full commitment but endowment is agent's private information
- Revelation principle $\Rightarrow$ truthful revelation mechanisms
- The principal chooses transfers $\tau_t\left(y^t\right) \equiv c_t\left(y^t\right) - y_t$ trying to minimize the expected discounted cost of implementing an incentive compatible allocation.
- Let $y \in \left\{\overline{y}_i\right\}_{i=1}^{N}$ such that $\overline{y}_i < \overline{y}_{i+1}$ for any i.

INCENTIVE COMPATIBILITY

INCENTIVE COMPATIBILITY

Define the local downward incentive-compatibility constraints as

$$DIC_t\left(y^{t-1},\bar{y}_i;\bar{y}_{i-1}\right)=\left\{\beta^t\left[u\left(c_t\left(y^{t-1},\bar{y}_i\right)\right)+\beta U_{t+1}\left(y^{t-1},\bar{y}_i\right)-\right.\right.$$

$$\left.\left.-u\left(c_t\left(y^{t-1},\bar{y}_{i-1}\right)\right)-\beta U_{t+1}\left(y^{t-1},\bar{y}_{i-1}\right)\right]\right\}$$

$$\forall y^{t-1},\quad \forall \bar{y}_i,\quad i=2,\ldots,N$$

where

$$U_{t+1}\left(y^{t-1},\bar{y}_i\right)\equiv\sum_{j=1}^{\infty}\sum_{y^{t+j}\in Y^{t+j+1}}\beta^{j-1}u\left(c_{t+j}\left(y^{t-1},\bar{y}_i,y^{t+j}\right)\right)\pi\left(y^{t+j}|y_{-1}\right)$$

OPTIMAL CONTRACT

OPTIMAL CONTRACT

I can solve a relaxed Pareto-constrained problem in which I only impose the local downward incentive-compatibility constraints (I need to impose some strong concavity conditions)

$$\max_{\left\{\tau_t(y^t)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[-\tau_t\left(y^t\right)\right] \pi\left(y^t | y_{-1}\right)$$

$$s.t. \qquad \sum_{t=0}^{\infty} \sum_{y^t \in y^{t+1}} \beta^t u\left(c_t\left(y^t\right)\right) \pi\left(y^t | y_{-1}\right) \geq U_0$$

$$DIC_t\left(y^{t-1}, \bar{y}_i; \bar{y}_{i-1}\right) \geq 0 \quad \forall y^{t-1}, \quad \forall \bar{y}_i, \quad i = 2, \ldots, N$$

HIDDEN EFFORT

Hidden effort

- Endowment is observable and i.i.d.
- Its future distribution is affected by an unobservable agent's action a_t through the transition function $\pi\left(y_{t+1} \mid a_t\right)$
- a_t is costly for the agent:

$$u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)$$

Optimal contract

OPTIMAL CONTRACT

We can characterize the optimal contract by solving:

$$\max_{\left\{\tau_t(y^t)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[-\tau_t\left(y^t\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right)$$

$$s.t. \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right) \geq U_0$$

$$\left\{a_t\left(y^t\right)\right\}_{t=0}^{\infty} \in \arg\max_{\left\{a_t(y^t)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right] \times$$
$$\pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right)$$

THE PROMISED UTILITIES APPROACH

THE APPROACH OF ABREU PEARCE AND STACCHETTI

- Main idea in DP: we make the problem recursive by summarizing all the history in a state variable
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THE APPROACH OF ABREU PEARCE AND STACCHETTI

- Main idea in DP: we make the problem recursive by summarizing all the history in a state variable
- In non-DP problems: not possible, therefore decisions today depend on the whole past history
- Look for a variable that summarizes history
- APS says: use continuation values of the individuals (costate variable)
- Continuation value summarizes the promises that the planner made to the agent in the past

THE APPROACH OF ABREU PEARCE AND STACCHETTI

- Given continuation value promised yesterday, planner chooses future continuation value
- Choosing a future continuation value means choosing a promise to the agent
- New continuation values must be consistent with promises made in the past, and with all the remaining constraints of the problem

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- Strategy profile: σ_i generates a sequence of action profiles

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- Discounted payoff:

$$v_i(\sigma) = \sum_{t=0}^{\infty} \beta^t u_i\left(a^t\right)$$

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$$v_i(\sigma) = \sum_{t=0}^{\infty} \beta^t u_i\left(a^t\right)$$

- profile of continuation strategies for the subgame starting at h^t: $\sigma\mid_{h^t}$

$$v_i\left(\sigma\mid_{h^t}\right) = \sum_{j=t}^{\infty} \beta^{j-t} u_i\left(a^j\right)$$

Subgame perfect equilibria

Definition 1

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DEFINITION 2

Let $W \subset \mathbb{R}^N$. Then (a, v) is admissible with respect to W if $a \in A$, $v \in W$ and

$$u_i(a) + \beta v_i(a) \geq u_i(\widehat{a}_i, a_{-i}) + \beta v_i(\widehat{a}_i, a_{-i}) \quad \forall \widehat{a}_i \quad \forall i$$

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Let $B(W)$ be the set of all values of admissible pairs for W

SELF-GENERATION AND FACTORIZATION

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The set of equilibrium values is a fixed point of the set-valued operator B

CALCULATING V

THEOREM 5

Let $W_0 \subseteq \mathbb{R}^{\mathbb{N}}$ be compact and such that $V \subseteq B(W_0) \subseteq W_0$. For $n = 1, 2, ...,$ let $W_n = B(W_{n-1})$. Then $\{W_n\}$ is a decreasing sequence and $V = \lim_{n\to\infty} W_n$.

Calculating V

Theorem 5

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We can follow this procedure:

1. Start with a set $W_0 \subseteq \mathbb{R}^{\mathbb{N}}$ such that $V \subseteq B(W_0) \subseteq W_0$.
2. For any $n = 1, 2, \ldots$, calculate $W_n = B(W_{n-1})$
3. Iterate until convergence.

GENERAL APPLICABILITY

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General applicability

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- We can generate the set of all payoffs that satisfy certain constraints by applying Theorem 5
- We are typically interested in efficient equilibria
- Standard application of APS has two stages: first, get the set of continuation values that satisfy some constraints, and then look for the efficient ones in that set

CHARACTERIZATION OF THE FEASIBLE SET

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- Conklin, Judd and Yeltekin (2003)

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- Player i's payoff in the stage game will be $\Pi_i : A_i \to \mathbf{R}$.
- Player i responds optimally to other players' actions, getting a payoff given by

$$\Pi_i^*(a_{-i}) \equiv \max_{a_i \in a_i} \Pi_i(a_i, a_{-i})$$

CHARACTERIZATION OF THE FEASIBLE SET

THE B OPERATOR

Define $B(W)$ as

$$B(W) = \bigcup_{(a,w)\in A\times W} \left\{(1-\delta)\Pi(a) + \delta w \mid \forall i(IR_i \geq 0)\right\}$$

where

$$IR_i \equiv [(1-\delta)\Pi(a) + \delta w_i] - [(1-\delta)\Pi^*(a_{-i}) + \delta \underline{w}_i]$$

is the individual rationality constraint for player i, and

$$\underline{w}_i \equiv \inf_{w\in W} w_i$$

is player i's worst possible continuation value in W .

Characterization of the feasible set

Outer and inner approximation

Characterization of the feasible set

Outer and inner approximation

- **Outer Approximation.** W is the convex combination of a finite number of half-spaces which can be represented as a collection of linear inequalities. In this case, $W = \bigcap_{\ell=1}^{L} \left\{ z \in \mathbf{R}^N \mid h_\ell z \leq c_\ell \right\}$ where $h_\ell \in \mathbf{R}^N$ is the gradient orthogonal to the face ℓ of W, and $c_\ell \in \mathbf{R}$ is a scalar which we call a level.

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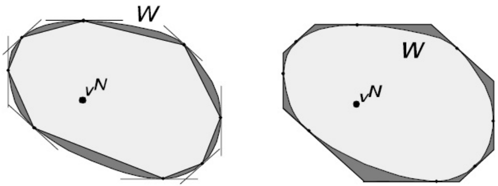

(a) Inner Approx. (b) Outer Approx.

Characterization of the feasible set

Outer approximation

CHARACTERIZATION OF THE FEASIBLE SET

OUTER APPROXIMATION

- Suppose we know N points on the boundary of W, call them $Z = \{z_1, z_2, ..., z_N\}$, and the corresponding subgradients $G = \{g_1, g_2, ..., g_N\}$. In other words, the hyperplane $z_i \cdot g_i = z \cdot g_i$ is tangent to W at z_i, and the gradients are such that $g_i \cdot w \leqq g_i \cdot z_i$ for $w \in W$.

CHARACTERIZATION OF THE FEASIBLE SET

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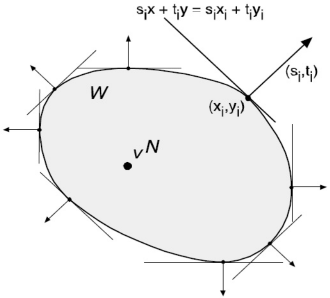

In the graph, $z_i = (x_i, y_i)$, $g_i = (s_i, t_i)$. Each tangent hyperplane generates two half-spaces. The one containing W is called the interior half-space. The outer approximation is therefore the intersection of all the interior half-spaces generated by all z_i's, and it is in fact the smallest convex set containing W, given our choice of Z and G.

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RAY AND EXTREMAL PROCEDURES

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Ray and extremal procedures

- Ray procedure: (for inner approx.)

- Extremal procedure: (for both)

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  - $\theta =$ polar coordinate angle from v^N

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    - Also a linear programming problem
    - Repeating the same maximization for several subgradients, we can generate a set of pairs (Z, G). In fact, when Z is available, we can use it for calculating the inner approximation as the convex hull of Z.

CHARACTERIZATION OF THE FEASIBLE SET

OUTER MONOTONE APPROXIMATION OF $B(W)$

Algorithm 1: Outer Monotone Approximation of $B(W)$

1. Define L search subgradients $H = \{h_1, h_2, \ldots, h_L\} \subset R^N$
2. Define W with M approximation subgradients $G = \{g_1, g_2, \ldots, g_M\} \subset R^N$ and levels $C = \{c_m \mid m = 1, \ldots, M\} \subset R$ such that $W \equiv \bigcap_{\ell=1}^{M} \{z \mid g_m \cdot z \leqq c_m\}$

CHARACTERIZATION OF THE FEASIBLE SET

OUTER MONOTONE APPROXIMATION OF $B(W)$

3. For each $h_\ell \in H$:
    1. For each $a \in A$, find optimal feasible equilibrium value in the h_ℓ direction, by assuming that action a is the current action profile, by solving

$$\begin{aligned} c_\ell(a) = \max_w \, & h_\ell \cdot [(1-\delta)\Pi(a) + \delta w] \\ & w \in W \\ & (1-\delta)\Pi^i(a) + \delta w_i \geq (1-\delta)\Pi_i^*(a_{-i}) + \delta \underline{w}_i \\ & \qquad i = 1, \ldots, N \end{aligned} \tag{8}$$

       and set $c_\ell(a) = -\infty$ if there is no w that satisfies the constraint (8)
    2. Choose the best action profile $a \in A$ by computing $c_\ell^+ = \max\{c_\ell(a) \mid a \in A\}$
4. Get $B^O(W;H) = W^+$, where levels are in $C^+ = \{c_1^+, \ldots c_L^+\}$, the approximation gradients are in H and $W^+ = \bigcap_{\ell=1}^L \{z \mid g_\ell \cdot z \leqq c_\ell^+\}$

CHARACTERIZATION OF THE FEASIBLE SET

OUTER HYPERPLANE ALGORITHM FOR APPROXIMATING V

Algorithm 2: Outer Hyperplane Algorithm for Approximating V

1. Set initial guess $W^0 \supset \mathscr{W}$ and the elements of the approximation:
   1. Set L search subgradients $H = \{h_1, h_2, ..., h_L\} \subset R^N$
   2. Select boundary points $Z^0 = \{z_1^0, ..., z_L^0\} \subset R^N$
   3. Compute hyperplane levels $c_\ell^0 = g_\ell^0 \cdot z_\ell^0$, $\ell = 1, ..., L$ and collect them in C^0
   4. Let $W^0 = \bigcap_{\ell=1}^{L} \{z \mid g_\ell \cdot z \leqq c_\ell^0\}$
2. Generate $W^{k+1} = B^O(W^k; H)$, where $C^{k+1} = \{c_1^{k+1}, ..., c_L^{k+1}\}$ define $W^{k+1} = \bigcap_{\ell=1}^{L} \{z \mid g_\ell \cdot z \leqq c_\ell^{k+1}\}$
3. Stop if W^{k+1} is close to W^k, i.e. if $\max_\ell \left| c_\ell^{k+1} - c_\ell^k \right| < \varepsilon$.

CHARACTERIZATION OF THE FEASIBLE SET

MONOTONE INNER HYPERPLANE APPROXIMATION OF $B(W)$

Algorithm 3: Monotone Inner Hyperplane Approximation of $B(W)$

1. Define L search subgradients $H = \{h_1, h_2, ..., h_L\} \subset R^N$
2. Define W with M approximation subgradients $G = \{g_1, g_2, ..., g_M\} \subset R^N$ and levels $C = \{c_m \mid m = 1, ..., M\} \subset R$ such that $W \equiv \bigcap_{\ell=1}^{M} \{z \mid g_m \cdot z \leqq c_m\}$

Characterization of the feasible set

Monotone Inner Hyperplane Approximation of $B(W)$

3. For each $h_\ell \in H$:
   1. For each $a \in A$, find optimal feasible equilibrium value in the h_ℓ direction by solving

$$
\begin{aligned}
c_\ell(a) = \max_w h_\ell \cdot [(1-\delta)\Pi(a) + \delta w] \\
w \in W \\
(1-\delta)\Pi^i(a) + \delta w_i \geq (1-\delta)\Pi_i^*(a_{-i}) + \delta \underline{w}_i \\
i = 1, \ldots, N
\end{aligned}
\tag{9}
$$

   and set $c_\ell(a) = -\infty$ if there is no w that satisfies the constraint (9)
   2. Choose the best action profile $a \in A$ and the corresponding continuation value:

$$a_\ell^* = \arg\ \max\{c_\ell(a) \mid a \in A\}$$

$$z_\ell' = (1-\delta)\Pi^i(a_\ell^*) + \delta w_\ell(a_\ell^*)$$

   3. Collect set of vertices of convex hull $Z' = \{z_\ell' \mid \ell = 1, \ldots, L\}$ and define $W^+ = co(Z')$

CHARACTERIZATION OF THE FEASIBLE SET

MONOTONE INNER HYPERPLANE APPROXIMATION OF $B(W)$

4. Compute $Z^+ = \{z' \in Z' \mid z' \in \partial W^+\}$, and find subgradients $G^+ = \{g_1^+, g_2^+, ..., g_M^+\} \subset R^N$ and levels $C^+ = \{c_m^+ \mid m = 1, ..., M\} \subset R$ such that $co(Z^+) \equiv \bigcap_{\ell=1}^{M} \{z \mid g_m^+ \cdot z \leq c_m^+\} = W^+$
5. Get $B^I(W;H) = W^+$, where levels are in $C^+ = \{c_1^+, ... c_L^+\}$, the approximation subgradients are in G^+ and vertices are in Z^+

CHARACTERIZATION OF THE FEASIBLE SET

CONCEPT OF DISTANCE USED BY CJY

To define convergence for the inner approximation iterations, CJY use the following distance:

$$d(Z_1, Z_2) = \max\left\{ \max_{z_1 \in Z_1} \min_{z_2 \in Z_2} \|z_1 - z_2\|, \max_{z_2 \in Z_2} \min_{z_1 \in Z_1} \|z_1 - z_2\| \right\} \qquad (10)$$

Characterization of the feasible set

Inner Hyperplane Algorithm for Approximating V

Algorithm 4: Inner Hyperplane Algorithm for Approximating V

1. Set initial elements of the approximation:
   1. Set L search subgradients $H = \{h_1, h_2, ..., h_L\} \subset R^N$
   2. Select vertices $Z^0 = \{z_1^0, ..., z_M^0\} \subset R^M$ for initial guess $W^0 = co(Z^0)$
   3. Define the gradients $G = \{g_1^0, ..., g_M^0\} \subset R^M$ and levels $C^0 = \{c_1^0, ..., c_M^0\} \subset R^M$ which define $W^0 = \bigcap_{m=1}^{M} \{z \mid g_m^0 \cdot z_m^0 \leq c_m^0\}$
2. Generate $W^{k+1} = B^I(W^k; H)$, with vertices $Z^{k+1} = \{z_1^{k+1}, ..., z_M^{k+1}\}$
3. Stop if $d(Z^{k+1}, Z^k) < \varepsilon$

CHARACTERIZATION OF THE FEASIBLE SET

REPEATED PRISONER'S DILEMMA

Discount factor $\delta = 0.8$

Payoffs of the prisoner's dilemma:

		player 2	
	Actions	C	D
player 1	C	(4,4)	(0,6)
	D	(6,0)	(2,2)

ONE-SIDED LACK OF COMMITMENT

$$\max_{\{c_t\}_{t=0}^{\infty}} \quad E_{-1}\sum_{t=0}^{\infty}\beta^t(y_t - c_t)$$

$$s.t. \quad E_{-1}\sum_{t=0}^{\infty}\beta^t u(c_t) \geq U_0$$

$$u(c_t) + \beta E_t \sum_{j=1}^{\infty}\beta^{j-1}u\left(c_{t+j}\right) \geq u(y_t) + \beta U_{aut}\,\forall t \qquad (11)$$

Assume y_t is i.i.d. and its support is $\mathbf{Y} \equiv \left\{\bar{y}_s\right\}_{s=1}^{S}$. Call its distribution function $\pi_s, s = 1, ..., S$.

PROMISED UTILITY

Define the *agent's promised utility* as:

$$U_t \equiv E_{t-1} \sum_{j=0}^{\infty} \beta^j u\left(c_{t+j}\right) \tag{12}$$

This variable summarizes the promises that the principal makes to the agent in each period.

RECURSIVE CONTRACT

- Principal enters period t with a promise U_{-} made in $t-1$
- Complies with previous promise by choosing state-contingent consumption $c_s, s = 1, ..., S$ and a new promise $U_s, s = 1, ..., S$
- Let $P(U_{-})$ be the value function of the principal:

$$P(U_{-}) = \max_{\{c_s, U_s\}_{s=1}^{S}} \sum_{s=1}^{S} \pi_s \left[\bar{y}_s - c_s + \beta P(U_s)\right] \tag{13}$$

$$s.t. \quad \sum_{s=1}^{S} \pi_s \left[u(c_s) + \beta U_s\right] \geq U_{-} \tag{14}$$

$$u(c_s) + \beta U_s \geq u(\bar{y}_s) + \beta U_{aut} \quad s = 1, ..., S \tag{15}$$

$$c_s \in C \tag{16}$$

$$U_s \in \mathscr{U} \tag{17}$$

The set $\mathscr{U}$

The set $\mathscr{U}$ is obtained iterating on the APS operator B defined as:

$$B(W) = \left\{ \begin{array}{l} U \in W : \sum_{s=1}^{S} \pi_s \left[u(c_s) + \beta U_s \right] \geq U, \\ u(c_s) + \beta U_s \geq u(\bar{y}_s) + \beta U_{aut} \quad s = 1, \ldots, S \\ c_s \in C \quad s = 1, \ldots, S \end{array} \right\} \tag{18}$$

GETTING A SOLUTION

Apart from this set, the functional equation looks like a Bellman equation in which we have the agent's value function as a state variable. Therefore, we can characterize the optimal contracts in two steps:

1. Find the set $\mathscr{U}$ by repeatedly applying the operator B defined in (18)
2. Solve the Bellman equation as usual, either with value function iteration or Howard's improvement algorithm.

ENDOWMENT IS PRIVATE INFORMATION

$$\max_{\{\tau_t(y^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[-\tau_t\left(y^t\right)\right] \pi\left(y^t | y_{-1}\right)$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{y^t \in y^{t+1}} \beta^t u\left(c_t\left(y^t\right)\right) \pi\left(y^t | y_{-1}\right) \geq U_0$$

$$DIC_t\left(y^{t-1}, \bar{y}_i; \bar{y}_{i-1}\right) \geq 0 \quad \forall y^{t-1}, \quad \forall \bar{y}_i, \quad i = 2, \ldots, N$$

RECURSIVE CONTRACT

$$
\begin{aligned}
P(U_-) = \quad & \max_{\{\tau_s, U_s\}_{s=1}^{S}} \sum_{s=1}^{S} \pi_s \left[-\tau_s + \beta P(U_s)\right] && (19) \\
s.t. \quad & \sum_{s=1}^{S} \pi_s \left[u\left(\tau_s + \bar{y}_s\right) + \beta U_s\right] = U_- && (20) \\
& u\left(\tau_s + \bar{y}_s\right) + \beta U_s \geq u\left(\tau_{s-1} + \bar{y}_s\right) + \beta U_{s-1} \quad s = 2, \ldots, S && (21) \\
& c_s \in C && (22) \\
& U_s \in \mathscr{U} && (23)
\end{aligned}
$$

The set $\mathscr{U}$

$\mathscr{U}$ is the fixed point of the operator:

$$B(W) = \left\{ \begin{array}{l} U \in W : \sum_{s=1}^{S} \pi_s \left[u(\tau_s + \bar{y}_s) + \beta U_s \right] \geq U, \\ u(\tau_s + \bar{y}_s) + \beta U_s \geq u(\tau_{s-1} + \bar{y}_s) + \beta U_{s-1} \quad s = 2, \ldots, S \\ c_s \in C \quad s = 1, \ldots, S \end{array} \right\} \tag{24}$$

HIDDEN EFFORT

$$\max_{\left\{\tau_t\left(y^t\right)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[-\tau_t\left(y^t\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right)$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right) \geq U_0$$

$$\left\{a_t\left(y^t\right)\right\}_{t=0}^{\infty} \in \arg \max_{\left\{a_t\left(y^t\right)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{y^t \in Y^{t+1}} \beta^t \left[u\left(c_t\left(y^t\right)\right) - \upsilon\left(a_t\left(y^t\right)\right)\right] \pi\left(y^t \mid a^{t-1}\left(y^{t-1}\right)\right)$$

RECURSIVE CONTRACT

$$P\left(U_{-},\bar{y}_i\right) = \max_{\left\{c,\{U_s\}_{s=1}^{S},a^*\right\}} \left[\bar{y}_i - c + \beta \sum_{s=1}^{S} \pi_s\left(a^*\right) P\left(U_s,\bar{y}_s\right)\right] \tag{25}$$

$$s.t. \quad u(c) - \upsilon\left(a^*\right) + \beta \sum_{s=1}^{S} \pi_s\left(a^*\right) U_s = U_{-} \tag{26}$$

$$a^* = \arg\max_{a \in A} \left\{u(c) - \upsilon(a) + \beta \sum_{s=1}^{S} \pi_s(a) U_s\right\} \tag{27}$$

$$c \in C, a \in A \tag{28}$$

$$U_s \in \mathscr{U} \tag{29}$$

Notice different timing convention: effort affects the probability of the state tomorrow

THE SET $\mathscr{U}$

$\mathscr{U}$ is the fixed point of the operator:

$$B(W) = \left\{ \begin{array}{l} U \in W : u(c) - \upsilon(a^*) + \beta \sum_{s=1}^{S} \pi_s(a^*) U_s = U, \\ a^* = \arg\max_{a \in A} \left\{ u(c) - \upsilon(a) + \beta \sum_{s=1}^{S} \pi_s(a) U_s \right\} \\ c \in C, a \in A \quad s = 1, ..., S \end{array} \right\} \tag{30}$$

OPTIMAL FISCAL POLICY

$$\max_{\{c_t,b_t,l_t\}_{t=0}^{\infty}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(l_t)\right]$$

$$s.t. \quad c_t - \beta \frac{E_t u'(c_{t+1})}{u'(c_t)} b^g_{t+1} = -(c_t + g_t) \frac{v'(c_t + g_t)}{u'(c_t)} - b^g_t$$

OPTIMAL FISCAL POLICY (CONT.)

Small departure from APS:

- Instead of using the continuation value of the agent as a state variable, we are going to use the marginal utility of consumption
- We have already a state variable in this problem: debt holdings.
- We have to characterize a feasible set for continuation values for any value of debt holdings: i.e. a correspondence

Recursive contract

Let $m \equiv u'(c)$, govt expenditure can take S values and is i.i.d. as in previous examples

$$P(m_-, \overline{g}_s, b_-) = \max_{\{c,b,l,\{m_s\}_{s=1}^S\}} u(c) + v(l) + \beta \sum_{s=1}^{S} P(m_s, \overline{g}_s, b) \tag{31}$$

$$s.t. \quad c - \beta \frac{\sum_{s=1}^{S} m_s}{m_-} b = -(c + \overline{g}_s) \frac{v'(c + \overline{g}_s)}{m_-} - b_- \tag{32}$$

$$c \in C, l \in L, b \in B \tag{33}$$

$$m_s \in \mathscr{U}(b_-) \tag{34}$$

THE SET $\mathscr{U}(b_-)$

$\mathscr{U}(b_-)$ is the fixed point of the operator

$$B\left(W(b_-)\right) = \left\{ \begin{array}{c} m \in W(b_-) : c - \beta \frac{\sum_{s=1}^{S} m_s}{m_-} b = -(c + \overline{g}_s) \frac{v'\left(c + \overline{g}_s\right)}{m_-} - b_- \\ c \in C, l \in L, b \in B \end{array} \right\} \tag{35}$$

This set changes as b_- changes

$DP^2 = (\text{CURSE OF DIMENSIONALITY})^2$

- Any application can be solved by a two-step procedure:
  1. Characterize the set of continuation values
  2. Solve the Bellman equation
- First step causes many troubles if other ("natural") state variables or several agents (e.g., optimal fiscal policy)
- Set of admissible continuation values is a correspondence that maps from the set of natural states to the set of continuation values
- I don't know any paper that works with more than 2 natural states