

STAT 640 Section 1 - Homework 2

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Contents

19.	2
a.	2
b.	2
c.	3
20.	4
50.	6
a.	6
b.	6
c.	7
55.	8
a.	8
b.	10
c.	10
d.	10

19.

a.

First find the likelihood function and log likelihood functions:

$$L(\mu, \sigma^2) = \prod_{i=1}^n f(x_i) = \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}}$$

Find log-likelihood function:

$$\ell(\mu, \sigma^2) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2} \quad (1)$$

$$= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \sigma^2 - \frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2} \quad (2)$$

$$= -\frac{n}{2} \log(2\pi) - n \log \sigma - \frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2} \quad (3)$$

Since μ is known, we take the derivative with respect to σ :

$$\nabla \ell(\sigma) = \frac{\partial \ell(\mu, \sigma^2)}{\partial \sigma} \quad (4)$$

$$= -\frac{n}{\sigma} + \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^3} \quad (5)$$

Set gradient vector to 0 and solve for σ :

$$\nabla \ell(\sigma) = 0 \quad (6)$$

$$-\frac{n}{\sigma} + \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^3} = 0 \quad (7)$$

$$\hat{\sigma}_{mle} = \frac{\sqrt{\sum_{i=1}^n (x_i - \mu)^2}}{n} \quad (8)$$

b.

Using the same log-likelihood function, we take the derivative with respect to μ :

$$\nabla \ell(\mu) = \frac{\partial \ell(\mu, \sigma^2)}{\partial \mu} \quad (9)$$

$$= \frac{\sum_{i=1}^n (x_i - \mu)}{\sigma^2} \quad (10)$$

Set gradient vector to 0 and solve for μ :

$$\nabla \ell(\mu) = 0 \quad (11)$$

$$\frac{\sum_{i=1}^n (x_i - \mu)}{\sigma^2} = 0 \quad (12)$$

$$\hat{\mu}_{mle} = \frac{\sum_{i=1}^n x_i}{n} = \bar{x} \quad (13)$$

C.

Since an unbiased estimator that satisfies the Cramer-Rao lower bound is automatically the best unbiased estimator,

Find the Fisher information first:

$$I(\mu) = -E\left[\frac{\partial^2}{\partial \theta^2} \log f(X_1; \mu)\right]$$

Find the second derivative: $\theta = \mu$

$$\log f(X_1; \theta) = -\frac{1}{2} \log(2\pi\sigma^2) - \frac{(X_1 - \mu)^2}{2\sigma^2} \quad (14)$$

$$\frac{\partial}{\partial \theta} \log f(X_1; \theta) = 0 - (-1) \frac{2(X_1 - \mu)}{\sigma^2} = \frac{X_1 - \mu}{\sigma^2} \quad (15)$$

$$\frac{\partial^2}{\partial \theta^2} \log f(X_1; \theta) = -\frac{1}{\sigma^2} \quad (16)$$

$$\therefore I(\mu) = \frac{1}{\sigma^2} \quad (17)$$

The Cramer-Rao lower bound is:

$$\frac{1}{n \cdot I(\mu)} = \frac{\sigma^2}{n}$$

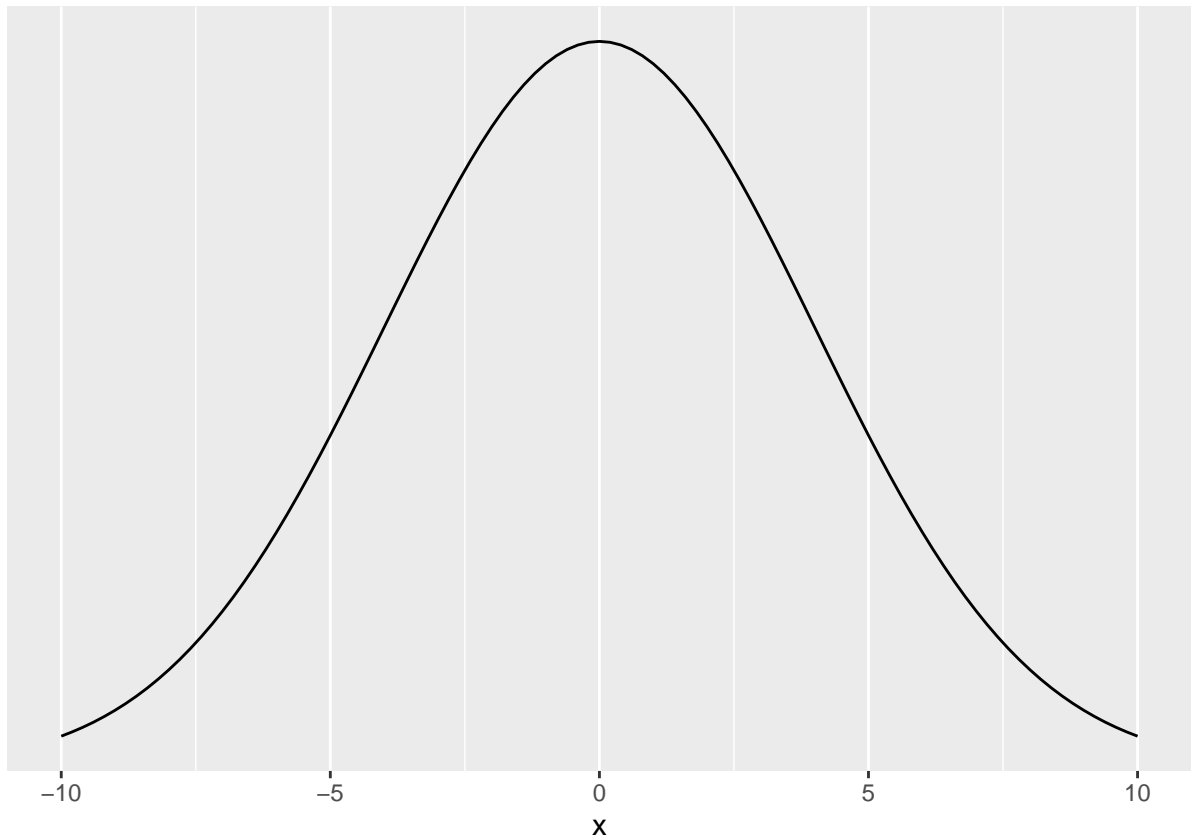
Since variance of \bar{X} is $\frac{\sigma^2}{n}$, MLE estimator of μ is the best unbiased estimator. That is. no other unbiased estimator for μ can have lower variance than MLE estimator.

20.

Get expected value and variance of \bar{X} :

- expected value: $E(\bar{X}) = E(\sum_{i=1}^{25} \frac{x_i}{n}) = \frac{1}{n} \sum_{i=1}^{25} x_i = 0$
- variance: $\text{Var}(\bar{X}) = \text{Var}(\frac{1}{25} \sum_{i=1}^{25} X_i) = \frac{1}{25^2} \text{Var} \sum_{i=1}^{25} (X_i) = \frac{1}{25^2} \cdot 25 \cdot 100 = 4$

```
library(ggplot2)
p1 <- ggplot(data = data.frame(x = c(-10, 10)), aes(x)) +
  stat_function(fun = dnorm, n = 101, args = list(mean = 0, sd = 4)) + ylab("") +
  scale_y_continuous(breaks = NULL)
p1
```



Since X follows a normal distribution, the variance follows chi-square distribution with $n - 1$ degrees of freedom.

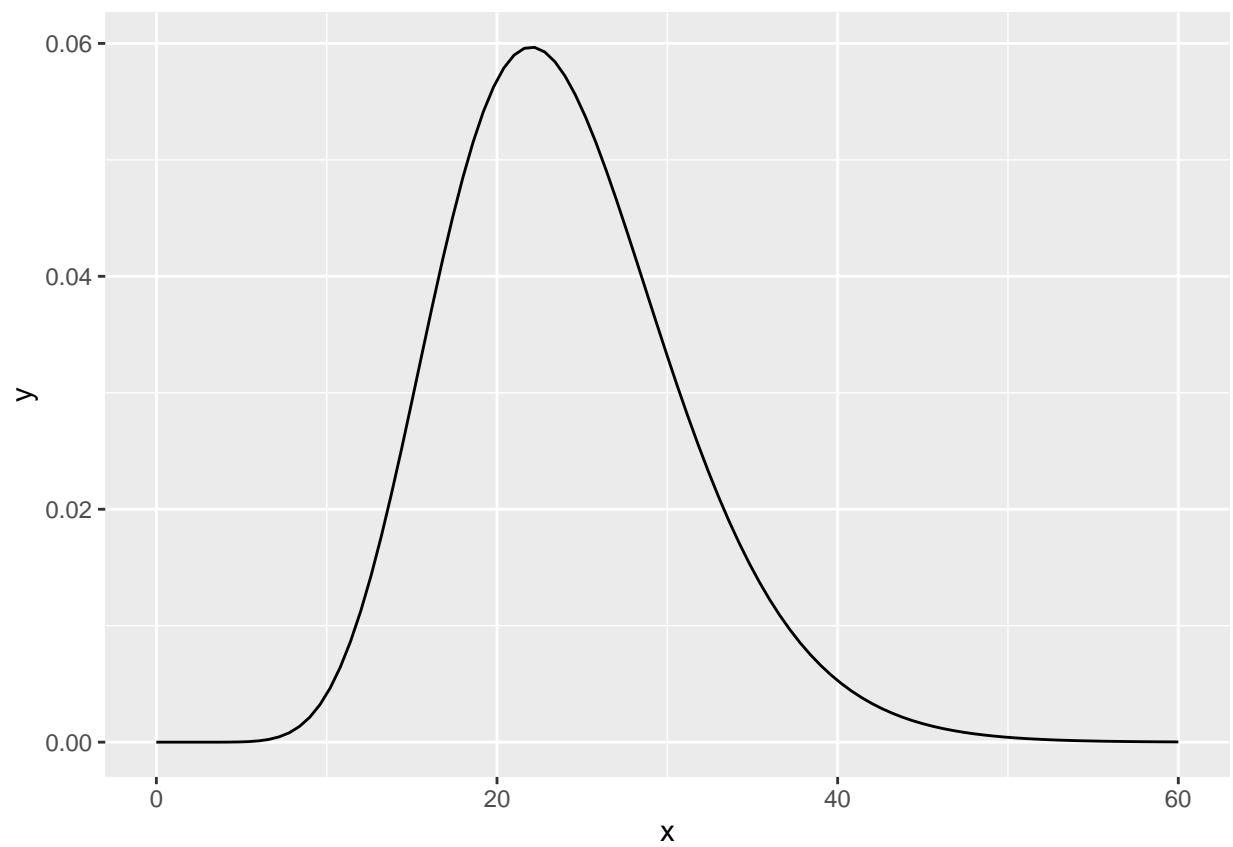
$$\hat{\sigma}^2 \sim \frac{\sigma^2}{n} \cdot \chi_{n-1}^2 \quad (18)$$

$$\sim \frac{100}{25} \cdot \chi_{24}^2 \quad (19)$$

$$\sim 4 \cdot \chi_{24}^2 \quad (20)$$

```
library(ggplot2)

ggplot(data.frame(x = c(0, 60)), aes(x = x)) +
  stat_function(fun = dchisq, args = list(df = 24))
```



50.

a.

Find the first theoretical population moment $k = 1$:

$$\mu_1 = E(X_1) = \int_0^\infty x \frac{x}{\theta^2} e^{-x^2/(2\theta^2)} dx$$

Variable substitution $v = \frac{x^2}{2\theta^2}$, $dv = \frac{x}{\theta^2} dx$, $x = \theta\sqrt{2v}$

$$E(X_1) = \int_0^\infty \theta\sqrt{2v} e^{-v} dv = \theta\sqrt{2} \int_0^\infty v^{\frac{1}{2}} e^{-v} dv$$

Using the Gamma trick, $\Gamma(x = \frac{3}{2})$:

$$E(X_1) = \theta\sqrt{2}\Gamma(\frac{3}{2}) = \theta\sqrt{2}\Gamma(\frac{1}{2} + 1) = \theta\frac{\sqrt{2}}{2}\Gamma(\frac{1}{2}) = \theta\frac{\sqrt{2\pi}}{2}$$

Equate to the first sample moment:

$$\theta\frac{\sqrt{2\pi}}{2} = \bar{X} \tag{21}$$

$$\hat{\theta}_{mom} = \bar{X} \frac{\sqrt{2\pi}}{\pi} \tag{22}$$

b.

Find the likelihood function:

$$L(\theta) = \prod_{i=1}^n f(x|\theta) \tag{23}$$

$$= \prod_{i=1}^n \frac{x_i}{\theta^2} e^{-x_i^2/(2\theta^2)} \tag{24}$$

$$= \frac{\prod_{i=1}^n x_i}{\theta^{2n}} e^{\sum_{i=1}^n -x_i^2/2\theta^2} \tag{25}$$

Find the log likelihood function:

$$\ell(\theta) = \sum_{i=1}^n \log x_i - 2n \log \theta - \sum_{i=1}^n \frac{x_i^2}{2\theta^2}$$

Take derivative:

$$\nabla \ell(\theta) = -2n \frac{1}{\theta} + \frac{1}{\theta^3} \sum_{i=1}^n x_i^2$$

Set to 0 and solve for θ :

$$\nabla \ell(\theta) = 0 \quad (26)$$

$$-2n \frac{1}{\theta} + \frac{1}{\theta^3} \sum_{i=1}^n x_i^2 = 0 \quad (27)$$

$$\frac{1}{\theta^3} \sum_{i=1}^n x_i^2 = 2n \frac{1}{\theta} \quad (28)$$

$$\frac{1}{\theta^2} = \frac{2n}{\sum_{i=1}^n x_i^2} \quad (29)$$

$$\hat{\theta}_{mle} = \sqrt{\frac{\sum_{i=1}^n x_i^2}{2n}} \quad (30)$$

c.

Find the Fisher information of the sample. Because X_1, \dots, X_n is an *i.i.d* sample,

$$I_n(\theta_0) \stackrel{i.i.d}{=} nE\left[\frac{\partial}{\partial \theta} \log(f(X, \theta))^2\right] \quad (31)$$

$$= -nE\left[\frac{\partial^2}{\partial \theta^2} \log\left[\frac{x}{\theta^2} e^{-x^2/(2\theta^2)}\right]\right] \quad (32)$$

$$= -nE\left[\frac{\partial^2}{\partial \theta^2} (\log x - \log \theta^2 - \frac{x^2}{2\theta^2})\right] \quad (33)$$

$$= -nE\left[\frac{\partial}{\partial \theta} \left(-\frac{2}{\theta} + x^2 \theta^{-3}\right)\right] \quad (34)$$

$$= -nE\left[\frac{2}{\theta^2} - 3\frac{x^2}{\theta^4}\right] \quad (35)$$

$$= -n \frac{2}{\theta^2} + n \frac{3}{\theta^4} E(x^2) \quad (36)$$

Find $E(X^2)$:

$$E(X^2) = \int_0^\infty x^2 \frac{x}{\theta^2} e^{-x^2/(2\theta^2)} dx$$

Variable substitution $v = \frac{x^2}{2\theta^2}$, $dv = \frac{x}{\theta^2} dx$, $x = \theta\sqrt{2v}$

$$E(X^2) = \int_0^\infty \theta^2 2v e^{-v} dv = \theta^2 2 \int_0^\infty v e^{-v} dv$$

Using the Gamma trick, $\Gamma(x=2)$:

$$E(X^2) = 2\theta^2 \Gamma(2) = 2\theta^2$$

Plug $E(X^2)$ back to $I(\theta)$:

$$I_n(\theta_0) = -n \frac{2}{\theta^2} + n \frac{3}{\theta^4} 2\theta^2 = \frac{4n}{\theta^2}$$

Therefore, the asymptotic variance of MLE:

$$\text{Var} \approx \frac{1}{I_n(\theta)} \approx \frac{\theta^2}{4n}$$

55.

a.

Get likelihood and log-likelihood function, and take first derivative:

$$L(\theta) = \prod f(\theta) \quad (37)$$

$$= [0.25(2 + \theta)]^{1997} \cdot [0.25(1 - \theta)]^{906} \cdot [0.25(1 - \theta)]^{904} \cdot [0.25\theta]^{32} \quad (38)$$

$$= (0.5 + 0.25\theta)^{1997} \cdot (0.25 - 0.25\theta)^{906} \cdot (0.25 - 0.25\theta)^{904} \cdot (0.25\theta)^{32} \quad (39)$$

$$= (0.5 + 0.25\theta)^{1997} \cdot (0.25 - 0.25\theta)^{1810} \cdot (0.25\theta)^{32} \quad (40)$$

$$\ell(\theta) = 1997 \log [0.5 + 0.25\theta] + 1810 \log [0.25 - 0.25\theta] + 32 \log [0.25\theta] \quad (41)$$

$$\nabla \ell(\theta) = \frac{1997}{2 + \theta} - \frac{1810}{1 - \theta} + \frac{32}{\theta} = \frac{-3839\theta^2 - 1655\theta + 64}{(2 + \theta)(1 - \theta)\theta} \quad (42)$$

Set the gradient to 0:

$$\nabla \ell(\theta) = 0 \quad (43)$$

$$\frac{-3839\theta^2 - 1655\theta + 64}{(2 + \theta)(1 - \theta)\theta} = 0 \quad (44)$$

$$-3839\theta^2 - 1655\theta + 64 = 0 \quad (45)$$

$$(46)$$

```
result <- function(a,b,c){
  if(delta(a,b,c) > 0){
    x_1 = (-b+sqrt(delta(a,b,c)))/(2*a)
    x_2 = (-b-sqrt(delta(a,b,c)))/(2*a)
    result = c(x_1,x_2)
  }
  else if(delta(a,b,c) == 0){
    x = -b/(2*a)
  }
  else {"try again."}
}

delta<-function(a,b,c){
  b^2-4*a*c
}

a <- result(-3839, -1655, 64);a
```

```
## [1] -0.4668142  0.0357123
```

From using quadratic formal, and keep the positive root, we get $\hat{\theta}_{mle} \approx 0.0357$.

Next, find the fisher information by taking second derivative of the log likelihood function:

$$\nabla^2 \ell(\theta) = -\frac{1997}{(2+\theta)^2} - \frac{1810}{(1-\theta)^2} - \frac{32}{\theta^2} \quad (47)$$

$$\text{Var}(\hat{\theta}_{mle}) = \frac{1}{I_n(\theta)} \quad (48)$$

$$= -\frac{1}{E\left[-\frac{1997}{(2+\theta)^2} - \frac{1810}{(1-\theta)^2} - \frac{32}{\theta^2}\right]} \quad (49)$$

$$= \frac{1}{\frac{1997}{(2+\theta)^2} + \frac{1810}{(1-\theta)^2} + \frac{32}{\theta^2}} \quad (50)$$

```
var <- 1/ (((1997/(2+0.0357)^2) + (1810/(1-0.0357)^2) + (32/0.0357^2) )) ;var
```

```
## [1] 3.631547e-05
```

```
sd <- var^{1/2}; sd
```

```
## [1] 0.006026232
```

Plug in the estimated θ , and we get asymptotic variance 0.00602.

b.

With 95% confidence interval, $\alpha = 0.05$, and

Using the asymptotic distribution of MLEs

$$(\hat{\theta}_n - z_{1-\alpha/2} \sqrt{\frac{1}{nI(\theta_0)}}, \hat{\theta}_n + z_{1-\alpha/2} \sqrt{\frac{1}{nI(\theta_0)}}) \quad (51)$$

$$(0.0357 - 1.96 \sqrt{3.6315 \times 10^{-5}}, 0.0357 + 1.96 \sqrt{3.6315 \times 10^{-5}}) \quad (52)$$

$$(0.0239, 0.0475) \quad (53)$$

```
0.0357 - 1.96 * 0.000036315^(1/2)
```

```
## [1] 0.02388866
```

```
0.0357 + 1.96 * 0.000036315^(1/2)
```

```
## [1] 0.04751134
```

c.

```
N <- 100000 # bootstrap samples
thetaMLE <- 0.0357 # initial MLE estimate
thetas <- array()

for (i in 1:N){
  data<- rmultinom(1, 3839 ,prob = c((2+thetaMLE)*.25, (1-thetaMLE)*.25, (1-thetaMLE)*.25, thetaMLE*.25)
  n<- sum(data) #sample size

  negative_loglik<- function(x){
    p<- c( (2+x)*.25, (1-x)*.25, (1-x)*.25, x*.25 ) #likelihood
    # log_likelihood
    ell<- -1 * sum(data*log(p))
    return(ell)}

  res<- optim(0.5, negative_loglik, hessian = TRUE, method = "Brent", lower=0, upper=1)
  thetas[i]<- res$par
}
sd(thetas)
```

```
## [1] 0.005845914
```

d.

```
quantile(thetas, c(0.025,0.975))
```

```
##          2.5%          97.5%
```

```
## 0.02471926 0.04750652
```