Catering The Food - Solution

1 Using DP

- $V(C, V, S)$ = number of valid ways to allocate all flavors
- $V_C(C, V, S)$ = number of valid ways to allocate all flavors given that the first flavor is Chocolate
- $V_V(C, V, S)$ = number of valid ways to allocate all flavors given that the first flavor is Vanilla
- $V_S(C, V, S)$ = number of valid ways to allocate all flavors given that the first flavor is Strawberry

Base cases

- $V_C(1, 0, 0) = 1$, $V_V(0, 1, 0) = 1$, $V_S(0, 0, 1) = 1$
- $V_w(x, y, z) = 0$ if $x < 0$ or $y < 0$ or $z < 0$

Recursive cases

- $V_C(C, V, S) = V_V(C - 1, V, S) + V_S(C - 1, V, S)$
- $V_V(C, V, S) = V_C(C, V - 1, S) + V_S(C, V - 1, S)$
- $V_S(C, V, S) = V_C(C, V, S - 1) + V_V(C, V, S - 1)$
- $V(C, V, S) = V_C(C, V, S) + V_V(C, V, S) + V_S(C, V, S)$

Complexity

- This solution runs in $\mathcal{O}(C \cdot V \cdot S)$
- It is too slow when the number of scoops is close to 200,000

Valid Arrangement

- WLOG, assume $C \geq V \geq S$
- For any valid arrangement, chocolate scoops partition the sequence into $C + 1$ bins
- Every bin must contain some vanilla or strawberry flavors
  - Except for the first and the last bins, which may be empty
- Depending on whether the first and/or last bins are empty, we have four cases:
  1. **Both first and last bins are non-empty:** All $C + 1$ bins contain at least one flavor (either vanilla or strawberry).
  2. **Only the last bin is empty:** The first C bins contain flavor, but the last bin is empty, so only C bins are used.

3. **Only the first bin is empty:** The last C bins contain flavor, but the first bin is empty, again resulting in C bins.
4. **Both first and last bins are empty:** Only the middle $C-1$ bins contain flavor, meaning $C-1$ bins are used.

Counting Valid Arrangements

- Denote $\text{count}_{V,S}(k)$ as the number of ways to arrange V vanilla and S strawberry flavors into k bins such that adjacent flavors are different.
- **Theorem 1:** The number of valid arrangements of C Chocolate, V vanilla, and S strawberry flavors is:
$$\text{count}_{V,S}(C-1) + 2 \cdot \text{count}_{V,S}(C) + \text{count}_{V,S}(C+1)$$
- **Theorem 2:** Assume $V < S$. The number of ways to distribute vanilla and strawberry flavors into k bins is $\text{count}_{V,S}(k) =$
$$\sum_{t_1=0}^{V} \left\{ \binom{k}{t_1, t_2, t_3} \binom{V+t_2-1}{V-t_1-t_3} 2^{t_3} \,\middle|\, t_2 = S - V + t_1,\, t_3 = k - t_1 - t_2 \right\}$$

Can be proved by basic PnC

- By Theorems 1 and 2, we have the following algorithm:

Algorithm `ValidArrangement`(C, V, S)
Input: Assume $C > V > S$
Return: $\text{count}_{V,S}(C-1) + 2\text{count}_{V,S}(C) + \text{count}_{V,S}(C+1)$

Algorithm $\texttt{Count}_{V,S}(k)$
Return:
$$\sum_{t_1=0}^{V} \left\{ \binom{k}{t_1, t_2, t_3} \binom{V+t_2-1}{V-t_1-t_3} 2^{t_3} \,\middle|\, t_2 = S - V + t_1,\, t_3 = k - t_1 - t_2 \right\}$$