CS 212 **DUE: Tuesday, March 19 at 11:59 PM**
Spring 2019
Assignment 3
3/11/2019 **10 points**

Introduction: This assignment will give you practice doing proofs by weak induction.

This is an individual effort homework assignment. You must write up your solutions in LaTeX. Use the `a3.tex` template that I provide and be sure to replace each "Put your answer for ___ here." with your answers but leave everything else alone. Your solutions must be written up in a clear, concise and rigorous manner.

When you are done, zip up your .TEX file and corresponding .PDF file. Upload your .ZIP file to the **a3** dropbox on d2l. After you have uploaded the file, double-check to ensure your file was uploaded correctly. It is your responsibility to ensure your submission was done correctly. Assignments that are not uploaded correctly are worth 0 points.

This assignment will focus exclusively on the following propositional function, which takes as its only input a positive integer n:

$$P(n) = \text{“ } 1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}\, n^2 = (-1)^{n-1} \frac{n(n+1)}{2} \text{ ”}$$

Notice that $P(n)$ does not evaluate to an integer, but rather, given a positive integer n as an input parameter, it evaluates to proposition.

In the rest of this assignment, you will prove that, for all positive integers n, $P(n)$ is true (that is, it evaluates to a proposition that is true), using weak mathematical induction.

1. (1 point) Explicitly verify that $P(10)$ is true. Make sure you evaluate both sides separately and then show that both sides evaluate to the same value.

We evaluate $P(n)$ at 10 by plugging in 10 for n like so:

$$\begin{aligned} 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2 - 8^2 + 9^2 - 10^2 &= (-1)^{10-1}\frac{10(10+1)}{2} \\ -55 &= (-1)^9\frac{110}{2} \\ -55 &= -55 \end{aligned}$$

Since both sides are equivalent, $P(10)$ is true.

2. (1 point) What is the base case?

The base case is $n = 1$. As stated above, the domain of the propositional function $P(n)$ is the set of all positive integers of which 1 is the first.

3. (2 points) Explicitly verify that the base case is true.

$$P(1) = 1^2 = (-1)^{1-1}\frac{1(1+1)}{2}$$
$$1 = 1$$

4. (2 points) What is the inductive hypothesis, expressed as a quantified statement?

$(\exists k \in \mathbb{N})P(k)$ where $\mathbb{N}$ is the set of all natural numbers (starting at 1)

5. (1 point) Assuming that the inductive hypothesis is true, then what do you need to prove in the inductive step?

$P(k+1)$

6. (3 points) Complete the inductive step, identifying very clearly where you use the inductive hypothesis. Show all the steps!

$$
\begin{aligned}
(-1)^{n-1}\frac{n(n+1)}{2}|_{n=k+1} &= (-1)^{k+1-1}\frac{(k+1)(k+1+1)}{2} \\
&= (-1)(-1)^{k-1}\frac{k^2+k+k+k+1+1}{2} \\
&= (-1)(-1)^{k-1}\left(\frac{k(k+1)}{2}+\frac{2k+2}{2}\right) \\
&= (-1)\left((-1)^{k-1}\frac{k(k+1)}{2}+(-1)^{k-1}(k+1)\right) \\
&\stackrel{I.H.}{=} (-1)\left(P(k)+(-1)^{k-1}(k+1)\right) \\
&= (-1)^k(k+1)-P(k)
\end{aligned}
$$

Now if we consider what would happen if we had $k+1$ terms, it would go something like this:

$$1^2-2^2+3^2-4^2+\cdots-(-1)^{k-1}k^2+(-1)^k(k+1)^2$$

If we summed the first k terms, then by the induction hypothesis, it would yield $P(k)$. Therefore, the above expression can also be written like so:

$$-P(k)+(-1)^k(k+1)^2$$

or

$$(-1)^k(k+1)^2-P(k)$$

As we can see, we arrive at the identical result as above when $k+1$ is plugged into the hypothesized formula. Therefore, $(\exists k\in\mathbb{N})P(k)$ where $\mathbb{N}$ is the set of all natural numbers (starting at 1). $\square$