CS 381 - A8

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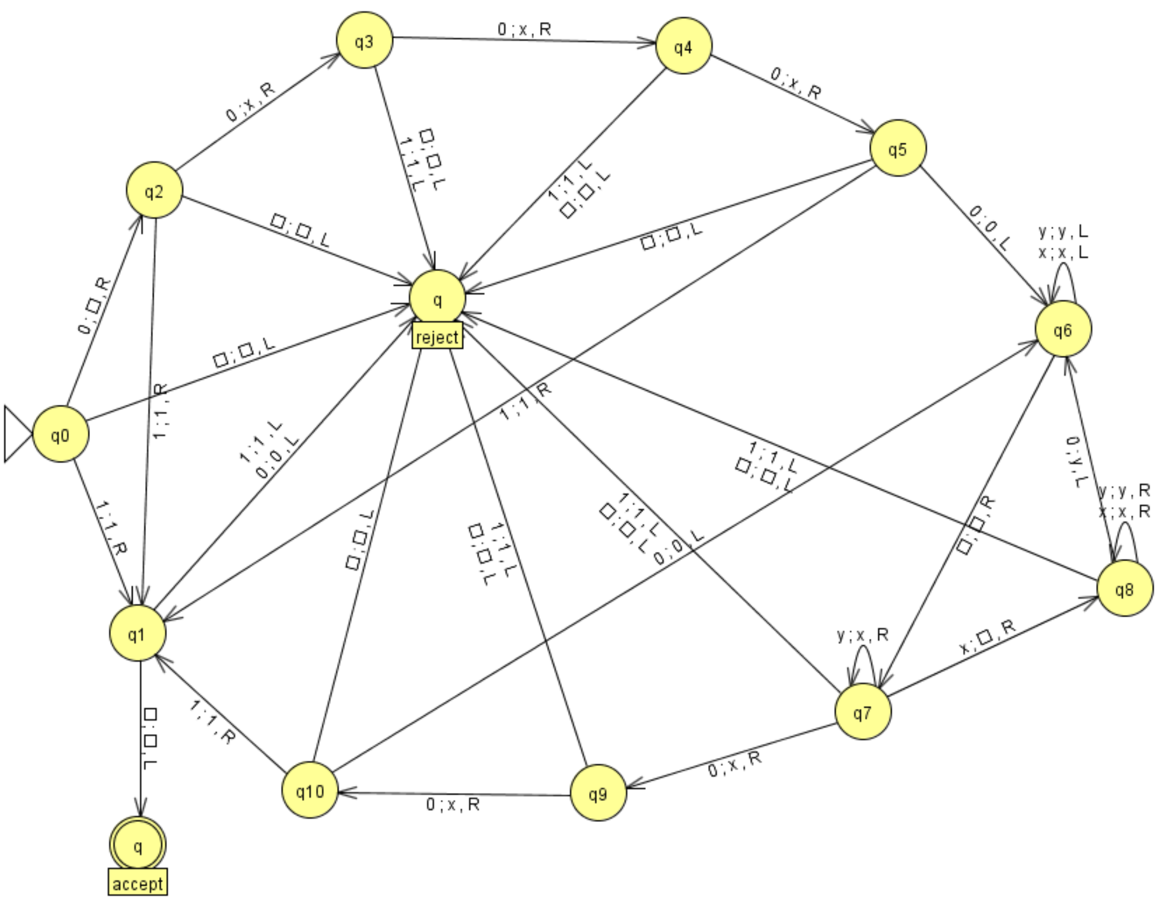
1 Problem 1

Design a standard Turing machine that decides the language $A = \{0^{n^2}1 \mid n \geq 0\}$ in place.

1.1 Formal Description

- $Q = \{q_0, q_1, ..., q_{10}, q_{accept}, q_{reject}\}$
- $\Sigma = \{0, 1\}$
- $\Gamma = \{\sqcup, 0, 1, x, y\}$
- δ is described by the state transition diagram.
- The start, accept, and reject states are q_0, q_{accept}, and q_{reject} respectively.



1.2 High-level Description

1. Reject if the Turing machine's head starts on a blank symbol.
2. Accept if the input is just a single 1.
3. Replace the first 0 with a blank symbol and move right.
4. Reject if there is a blank symbol.
5. Accept if only a single 1 is at the end of the input, but reject if other symbols follow the 1.
6. Replace the next three 0's with x's. Reject if the next three symbols are not 0's.
7. Reject if the next symbol is a blank symbol.
8. Repeat the following steps:
   (a) Accept if only a single 1 is at the end of the input, but reject if other symbols follow the 1.
   (b) Replace the next several 0's with the entire preceding string of x's, replace the old x's with blank symbols, and replace the next two 0's with x's. If this process is interrupted due to a lack of symbols or a symbol that is not 0, reject.

1.3 Justification

This Turing machine utilizes a special property of square numbers: the square of a natural number n is equal to the sum of the first n odd natural numbers. For example:

$$
\begin{aligned}
1^2 &= 1 \\
2^2 &= 1 + 3 \\
3^2 &= 1 + 3 + 5 \\
&\dots
\end{aligned}
$$

When the Turing machine copies over the preceding string of x's and adds two additional x's to the end, it's really adding the next odd natural number to the current sum. Each time the machine does this, it must check that there are enough 0's to do this with, and that only 0's exist in the next $n + 2$ spaces. Strings that interrupt this copy and add 2 section of the algorithm are rejected. At the end, the machine must simply check if the very last symbol is a 1. Any strings that do not fit this final pattern are immediately rejected.

2 Problem 2

Assuming A is an arbitrary language, prove that A and the $\overline{A}$ are Turing-recognizable only if A is decidable.

2.1 Proof

A language A is decidable if there exists a decider M that accepts all $w \in A$. This means that M must be able to definitively tell of a string is or is not in A. Such a machine would be able to recognize A and $\overline{A}$ because it halts on both inputs in A and inputs not in A.