Homework 9

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Chapter 5

1. (25 points) A fair coin is tossed four times. The random variable X is the number of heads in the first three tosses and the random variable Y is the number of heads in the last three tosses.
a) What is the joint pmf of X and Y?

First, let's take a look at all possible outcomes of flipping a coin 4 times:

Outcome	(X,Y)
TTTT	(0,0)
TTTH	(0,1)
TTHT	(1,1)
TTHH	(1,2)
THTT	(1,1)
THTH	(1,2)
THHT	(2,2)
THHH	(2,3)
HTTT	(1,0)
HTTH	(1,1)
HTHT	(1,2)
HTHH	(2,2)
HHTT	(2,1)
HHTH	(2,2)
HHHT	(3,2)
HHHH	(3,3)

Now we can construct a pmf of X and Y in the form of a table by simply taking the number of times each ordered pair of X and Y pops up and dividing it by the total number of outcomes:

$Y \backslash X$	0	1	2	3	
0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{2}{16}$
1	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	0	$\frac{5}{16}$
2	0	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{7}{16}$
3	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{2}{16}$
	$\frac{2}{16}$	$\frac{7}{16}$	$\frac{5}{16}$	$\frac{2}{16}$	1

b) What are the marginal pmfs of X and Y?

The marginal pmf's of X and Y are listed in the bottom and right sides of the above table respectively. These represent the probabilities of a single variable taking a particular value regardless of the value of the other.

c) Are the random variables X and Y independent? Explain.

If the two variables are independent, then $P(X,Y) = P(X) \cdot P(Y)$ for all X and Y. Let's start by testing $P(X = 0)$ and $P(Y = 0)$:

$$P(X = 0) = \frac{2}{16}$$

$$P(Y = 0) = \frac{2}{16}$$

$$P(X = 0, Y = 0) = \frac{1}{16}$$

$$P(X = 0) \cdot P(Y = 0) = \frac{2}{16} \cdot \frac{2}{16} = \frac{1}{64}$$

As you can see, we have already disproven that X and Y are not independent for 1 case, therefore, they are not independent at all.

d) Find Cov(X,Y).

$$\mu_x = 0\left(\frac{2}{16}\right) + 1\left(\frac{7}{16}\right) + 2\left(\frac{5}{16}\right) + 3\left(\frac{2}{16}\right) = 1.4375$$
$$\mu_y = 0\left(\frac{2}{16}\right) + 1\left(\frac{5}{16}\right) + 2\left(\frac{7}{16}\right) + 3\left(\frac{2}{16}\right) = 1.5625$$

$$\begin{aligned}
Cov(X,Y) &= E(XY) - \mu_x\mu_y \\
&= 0\cdot 0\cdot\left(\frac{1}{16}\right) + 0\cdot 1\cdot\left(\frac{1}{16}\right) + 0\cdot 2\cdot(0) + 0\cdot 3\cdot(0) \\
&+ 1\cdot 0\cdot\left(\frac{1}{16}\right) + 1\cdot 1\cdot\left(\frac{3}{16}\right) + 1\cdot 2\cdot\left(\frac{1}{16}\right) + 1\cdot 3\cdot(0) \\
&+ 2\cdot 0\cdot(0) + 2\cdot 1\cdot\left(\frac{3}{16}\right) + 2\cdot 2\cdot\left(\frac{3}{16}\right) + 2\cdot 3\cdot\left(\frac{1}{16}\right) \\
&+ 3\cdot 0\cdot(0) + 3\cdot 1\cdot(0) + 3\cdot 2\cdot\left(\frac{1}{16}\right) + 3\cdot 3\cdot\left(\frac{1}{16}\right) \\
&- (1.4375)(1.5625) \\
&= \boxed{0.5039}
\end{aligned}$$

e) Find the correlation coefficient of X and Y.

$$\begin{aligned}
\sigma_x &= \sqrt{E[X^2] - \mu_x^2} \\
&= \sqrt{0^2\left(\frac{2}{16}\right) + 1^2\left(\frac{7}{16}\right) + 2^2\left(\frac{5}{16}\right) + 3^2\left(\frac{2}{16}\right) - (1.4375)^2} \\
&= 0.8638 \\
\sigma_y &= \sqrt{E[Y^2] - \mu_y^2} \\
&= \sqrt{0^2\left(\frac{2}{16}\right) + 1^2\left(\frac{5}{16}\right) + 2^2\left(\frac{7}{16}\right) + 3^2\left(\frac{2}{16}\right) - (1.5625)^2} \\
&= 0.8638
\end{aligned}$$

$$\rho_{(x,y)} = \frac{Cov(X,Y)}{\sigma_x\sigma_y} = \frac{0.5039}{(0.8638)(0.8638)} = \boxed{0.6753}$$

4. (30 points) Suppose X and Y are discrete random variables with joint pmf:

$$\begin{aligned} p(x,y) &= kxy & \quad x = 1, 2{,}3 \text{ and } y = 2,3,4 \\ &= 0 & \quad \text{otherwise} \end{aligned}$$

a) Find the value of k.

First, let's create a pmf for x and y:

$y \backslash x$	1	2	3	
2	$2k$	$4k$	$6k$	$12k$
3	$3k$	$6k$	$9k$	$18k$
4	$4k$	$8k$	$12k$	$24k$
	$9k$	$18k$	$27k$	1

Now we can take the marginal pmf of x or y, (let's do x), and set the sum of the terms equal to 1:

$$\begin{aligned} 12k + 18k + 24k &= 1 \\ 54k &= 1 \\ k &= \boxed{\frac{1}{54}} \end{aligned}$$

b) Find $P(X < Y)$.

Looking at our table, we can take the sum of all the cases where x is less than y and add the probabilities plugging in $\frac{1}{54}$ for k:

$$\frac{2}{54} + \frac{3}{54} + \frac{6}{54} + \frac{4}{54} + \frac{8}{54} + \frac{12}{54} = \boxed{\frac{35}{54}}$$

c) Find the marginal pmfs of X and Y.

Plugging in $\frac{1}{54}$ for k in our above table, we can get the following:

$y \backslash x$	1	2	3	
2	$\frac{2}{54}$	$\frac{4}{54}$	$\frac{6}{54}$	$\frac{12}{54}$
3	$\frac{3}{54}$	$\frac{6}{54}$	$\frac{9}{54}$	$\frac{18}{54}$
4	$\frac{4}{54}$	$\frac{8}{54}$	$\frac{12}{54}$	$\frac{24}{54}$
	$\frac{9}{54}$	$\frac{18}{54}$	$\frac{27}{54}$	1

d) Find Cov(X,Y).

$$\mu_x = 1\left(\frac{9}{54}\right) + 2\left(\frac{18}{54}\right) + 3\left(\frac{27}{54}\right) = 2.3333$$
$$\mu_y = 2\left(\frac{12}{54}\right) + 3\left(\frac{18}{54}\right) + 4\left(\frac{24}{54}\right) = 3.2222$$

$$\begin{aligned}
Cov(X,Y) &= E(XY) - \mu_x\mu_y \\
&= 1\cdot 2\cdot\left(\frac{2}{54}\right) + 1\cdot 3\cdot\left(\frac{3}{54}\right) + 1\cdot 4\cdot\left(\frac{4}{54}\right) \\
&+ 2\cdot 2\cdot\left(\frac{4}{54}\right) + 2\cdot 3\cdot\left(\frac{6}{54}\right) + 2\cdot 4\cdot\left(\frac{8}{54}\right) \\
&+ 3\cdot 2\cdot\left(\frac{6}{54}\right) + 3\cdot 3\cdot\left(\frac{9}{54}\right) + 3\cdot 4\cdot\left(\frac{12}{54}\right) \\
&- (2.3333)(3.2222) \\
&= \boxed{0.0001592}
\end{aligned}$$

e) Find the correlation coefficient of X and Y.

$$\begin{aligned}
\sigma_x &= \sqrt{E[x^2] - \mu_x^2} \\
&= \sqrt{1^2\left(\frac{9}{54}\right) + 2^2\left(\frac{18}{54}\right) + 3^2\left(\frac{27}{54}\right) - (2.3333)^2} \\
&= 0.7455 \\
\sigma_y &= \sqrt{E[y^2] - \mu_y^2} \\
&= \sqrt{2^2\left(\frac{12}{54}\right) + 3^2\left(\frac{18}{54}\right) + 4^2\left(\frac{24}{54}\right) - (3.2222)^2} \\
&= 0.7858
\end{aligned}$$

$$\rho_{(x,y)} = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{0.0001592}{(0.7455)(0.7858)} = \boxed{0.0002718}$$

f) Are X and Y independent? Explain.

Let's check this by verifying that $P(x,y) = P(x) \cdot P(y)$ for all x and y:

$$\begin{aligned}
P(1,2) &= \frac{2}{54} & P(1) \cdot P(2) &= \frac{2}{54} \\
P(1,3) &= \frac{3}{54} & P(1) \cdot P(3) &= \frac{3}{54} \\
P(1,4) &= \frac{4}{54} & P(1) \cdot P(4) &= \frac{4}{54} \\
P(2,2) &= \frac{4}{54} & P(2) \cdot P(2) &= \frac{4}{54} \\
P(2,3) &= \frac{6}{54} & P(2) \cdot P(3) &= \frac{6}{54} \\
P(2,4) &= \frac{8}{54} & P(2) \cdot P(4) &= \frac{8}{54} \\
P(3,2) &= \frac{6}{54} & P(3) \cdot P(2) &= \frac{6}{54} \\
P(3,3) &= \frac{9}{54} & P(3) \cdot P(3) &= \frac{9}{54} \\
P(3,4) &= \frac{12}{54} & P(3) \cdot P(4) &= \frac{12}{54}
\end{aligned}$$

$\boxed{\text{Yes}}$, X and Y are independent.