Homework 10

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Due: April 19^{th}, 2019

Chapter 5

7. (15 points) There are three traffic lights on my way to work. Let X_1 be the number of lights at which I must stop, and suppose the probability distribution of X_1 as follows:

X_1	0	1	2	3
$p(X_1)$	.2	.3	.3	.2

Let X_2 be the number of lights at which I must stop on the way home; X_2 is independent of X_1. Assume that X_2 has the same probability distribution as X_1, so that X_1 and X_2 is a random sample of size $n = 2$.

a) Let $T = X_1 + X_2$. Determine the probability distribution of T.

Since X_1 and X_2 are independent, we can just multiply the probabilities together to get the probability of X_1 and X_2 each taking on a particular value. Doing that, we can create the following table:

X_2 \ X_1	0	1	2	3	
0	.04	.06	.06	.04	.20
1	.06	.09	.09	.06	.30
2	.06	.09	.09	.06	.30
3	.04	.06	.06	.04	.20
	.20	.30	.30	.20	1

Using that table, we can determine the probability that the sum of X_1 and X_2 will be a particular number by adding the probabilities together. Doing this, we can create the following probability distribution for T:

T	0	1	2	3	4	5	6
$p(T)$	.04	.12	.21	.26	.21	.12	.04

b) Find $E(T)$. How does it relate to the population mean μ?

$$\begin{aligned} E(T) &= 0(.04) + 1(.12) + 2(.21) + 3(.26) + 4(.21) + 5(.12) + 6(.04) \\ &= \boxed{3} \end{aligned}$$

$$\begin{aligned} E(T) &= n\mu \\ \mu &= \frac{E(T)}{n} \\ &= \frac{3}{2} \\ &= \boxed{1.5} \end{aligned}$$

c) Find $V(T)$. How does it relate to the population variance σ^2?

$$\begin{aligned} V(T) &= 0^2(.04) + 1^2(.12) + 2^2(.21) + 3^2(.26) + 4^2(.21) + 5^2(.12) + 6^2(.04) \\ &= \boxed{11.1} \end{aligned}$$

Since both X_1 and X_2 have the same distribution:

$$\begin{aligned} V(T) &= n\sigma^2 \\ \sigma^2 &= \frac{V(T)}{n} \\ &= \frac{11.1}{2} \\ &= \boxed{5.55} \end{aligned}$$

8. (15 points) A random sample of size 2 is taken from a population with the following pmf:

X	0	1	2	3
$p(X)$	.1	.2	.4	.3

a) Find the probability distribution of the sample mean $\bar{X}$.

$$P(X = x) = p(x_1)p(x_2)$$

$X_2 \backslash X_1$	0	1	2	3	
0	.01	.02	.04	.03	.10
1	.02	.04	.08	.06	.20
2	.04	.08	.16	.12	.40
3	.03	.06	.12	.09	.30
	.10	.20	.40	.30	1

$$\bar{X} = \frac{X_1 + X_2}{2}$$

$\bar{X}$	0.0	0.5	1.0	1.5	2.0	2.5	3.0
$p(\bar{x})$	.01	.04	.12	.22	.28	.24	.09

b) Find $E(\bar{X})$. How does it relate to the population mean?

$$\begin{aligned} E(\bar{X}) &= 0.0(.01) + 0.5(.04) + 1.0(.12) + 1.5(.22) \\ &\quad + 2.0(.28) + 2.5(.24) + 3.0(.09) \\ &= \boxed{2.08} \end{aligned}$$

$$\boxed{E(\bar{X}) = \mu_{\bar{X}} = \mu}$$

c) Find $V(\bar{X})$. How does it relate to the population variance?

$$\begin{aligned} V(\bar{X}) &= 0.0^2(.01) + 0.5^2(.04) + 1.0^2(.12) + 1.5^2(.22) \\ &\quad + 2.0^2(.28) + 2.5^2(.24) + 3.0^2(.09) \\ &= \boxed{4.055} \end{aligned}$$

$$\begin{aligned} V(\bar{X}) &= \frac{\sigma^2}{n} \\ \sigma^2 &= V(\bar{X})n \\ &= 4.055 \cdot 2 \\ &= \boxed{8.11} \end{aligned}$$

12. (10 points) The life of a bread-making machine is a random variable with a mean of 7 years and a standard deviation of 1 year. A random sample of 36 machines is chosen. Find the probability that the...

a) ...average life of the sample is less than 7.45 years.

Since n is greater than 30, we can use the Central Limit Theorem:

$$\text{mean} = \mu$$

$$\text{variance} = \frac{\sigma^2}{n} = \frac{1^2}{36} = \frac{1}{36}$$

$$\text{standard deviation} = \sqrt{\text{variance}} = \sqrt{\frac{1}{36}} = \frac{1}{6}$$

$$\begin{aligned} \phi\left(\frac{x-\mu}{\sigma}\right) &= \phi\left(\frac{7.45-7}{\frac{1}{6}}\right) \\ &= \phi(2.70) \\ &= \boxed{.9965} \end{aligned}$$

b) ...average life of the sample is between 6.65 and 7.45 years.

$$\phi\left(\frac{7.45-7}{\frac{1}{6}}\right)-\phi\left(\frac{6.65-7}{\frac{1}{6}}\right)=\phi(2.70)-\phi(-2.10)$$
$$=0.9965-.0179$$
$$=\boxed{.9786}$$

14. (5 points) The time until recharge for a battery in a laptop computer is normally distributed with mean of 260 minutes and standard deviation of 50 minutes. If an office has 4 laptop computers, find the probability that the total time the batteries lasted were between 868 and 1189 minutes.

Even though n is less than or equal to 30, we are told that we are allowed to use a normal distribution. Because there are 4 different laptops, we need to find the probability of the sample total being between 1189 and 868. This means the average laptop should last a quarter of these times:

$$\frac{1189}{4}=297.25$$

$$\frac{868}{4}=217$$

$$\phi\left(\frac{x_2-\mu}{\sigma}\right)-\phi\left(\frac{x_1-\mu}{\sigma}\right)=\phi\left(\frac{297.25-260}{50}\right)-\phi\left(\frac{217-260}{50}\right)$$
$$=\phi(0.75)-\phi(-0.86)$$
$$=.7734-.1949$$
$$=\boxed{.5785}$$

16. (5 points) Components of type A have heights that are independently distributed as a normal distribution with a mean 190 and a standard deviation of 10. Components of type B have heights that are independently distributed as a normal distribution with a mean 150 and a standard deviation of 8. What is the probability that a stack of four components of type A placed one on top of the other will be taller than a stack of five components of type B placed one on top of the other?

This problem is essentially asking for the probability that the average height of five B's is less than the average height of four component A's:

$$4 \cdot (\text{mean height of component } A) = 760$$

Dividing this number by 5 will give us the threshold that the average of the 5 B's cannot go over:

$$760/5 = 152$$

Now we find the probability that the average height of five component B's is less than or equal to this number:

$$\begin{aligned}\phi\left(\frac{x-\mu}{\sigma}\right) &= \phi\left(\frac{152-150}{8}\right)\\ &= \phi(0.25)\\ &= \boxed{.5987}\end{aligned}$$

18. (10 points) Let X_1, X_2, X_3, and X_4 be four independent, normal random variables with means 14, 12, 18, and 26 and standard deviations 4, 5, 2, and 3, respectively.

a) Find $P(46 < X_1 + X_2 + X_3 + X_4 < 88)$.

$$\mu = 14 + 12 + 18 + 26 = 70$$

$$\sigma^2 = 4^2 + 5^2 + 2^2 + 3^2 = 54$$

$$\sigma = \sqrt{54} = 7.35$$

$$
\begin{aligned}
\phi\left(\frac{88-70}{7.35}\right)-\phi\left(\frac{46-70}{7.35}\right) &= \phi(2.45)-\phi(-3.27) \\
&= .9929-.0005 \\
&= \boxed{.9924}
\end{aligned}
$$

b) Find $P(22 < 3X_1 - 2X_2 + 4X_3 - X_4 < 120)$**.**

$$\mu = 3(14) - 2(12) + 4(18) - 1(26) = 64$$

$$\sigma^2 = 3(4^2) - 2(5^2) + 4(2^2) - 1(3^2) = 5$$

$$\sigma = \sqrt{5} = 2.24$$

$$
\begin{aligned}
\phi\left(\frac{120-64}{2.24}\right)-\phi\left(\frac{22-64}{2.24}\right) &= \phi(25)-\phi(-18.75) \\
&= 1-0 \\
&= \boxed{1}
\end{aligned}
$$