Modern Robotics

Generated by Zettlr

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1 4.1.1: Product of Exponentials Formula in the Space Frame

1.1 Terms:

Forward Kinematics

- Refers to the use of the kineamtic equations of a robot to compute the position of the **end-effector** from specified values for the joint parameters.

End Effector

- Device at the end of a robotic arm designed to interact with the environment.
  - End effectors are typically the **grippers** of a robot

Denavit-Hartenberg Parameters

- Referred to as D-H Parameters
- Commonly used convention for selecting frames of reference to the links of a spatial kinematic chain

Product of Exponentials

- Referred to as PoE
- Adjacent frames do not need to be considered.
- Only 2 frames are needed: Space Frame {s} and Tool Frame {b}
- 6n numbers are needed to describe n screw axes

1.2 Forward Kinematics

- Forward kinematics refers to the use of the kinematic equations of a robot to compute the position of the end-effector from specified values for the joint parameters.

1.3 Problem

Define a frame {s} (often fixed at the base of a robot) and a frame {b} at the end effector of the robot arm.

Calculate the Forward Kinematics of the robot: I.e., $Find$ $T(\theta)$

1.3.1 Representing Forward Kinematics in the {s} Frame

Procedure:

1. Let M be the transformation matrix of the end-effector frame {b} when θ = 0

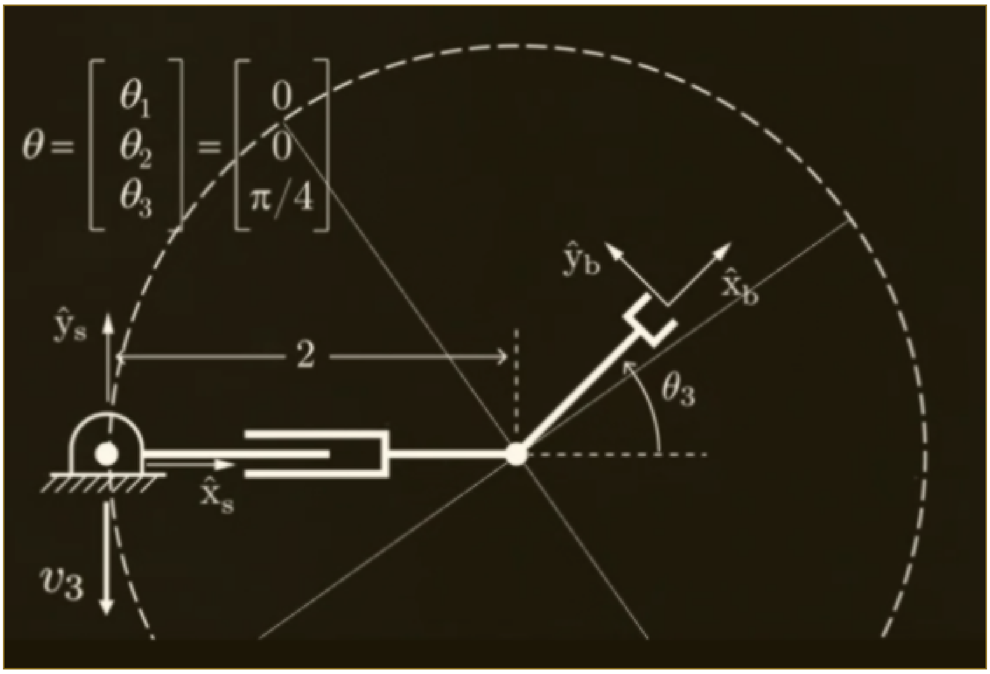
   In other words, M is the position and orientation of the end effector when all joint angles are set to zero:

$$
\begin{array}{cc}
 & \begin{array}{cccc} x_b & y_b & z_b & \phantom{\sum L_{x_s}} \end{array} \\
M = & \begin{bmatrix} \hat{x}_s & \hat{x}_s & \hat{x}_s & \sum L_{x_s} \\ \hat{y}_s & \hat{y}_s & \hat{y}_s & \sum L_{y_s} \\ \hat{z}_s & \hat{z}_s & \hat{z}_s & \sum L_{z_s} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

   - The first three 3-vectors (x,y,z) denote which axis in the {s} frame corresponds to a value in the {b} frame. For example, a value of $[0, 1, 0]^T$ in the first column means that the x axis of the {b} frame is aligned with the positive y axis of the {s} frame
   - The bottom row is added to simplify matrix operations
   - The $\sum L$ represents the sum of all of the links from the space frame {s} to the end effector.

2. Find the {s} frame Screw Axes $S_1, ..., S_n$ for each of the n joint axes when $\theta = 0$

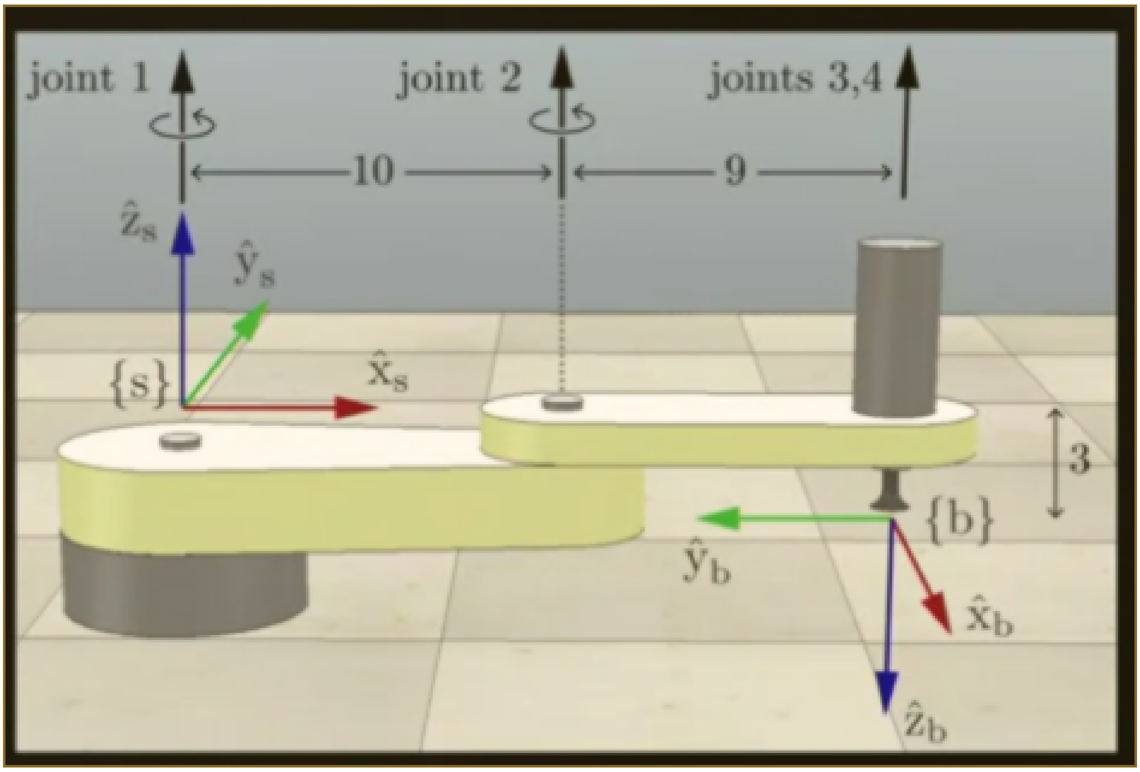
$$
S_n = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{x}_s \\ \hat{y}_s \\ \hat{z}_s \\ \\ \dot{x}_s \\ \dot{y}_s \\ \dot{z}_s \end{bmatrix}
$$

- The first 3-vector (angular velocity) represents which axis the space frame {s} is rotating about. For example, a value of $[0, 0, 1]^T$ means that this joint in the {s} frame is rotating about the positive z-axis
- The linear velocity v can be found by identifying which axis is tangental to the turntable created at the center of the joint n and then multiplying that number with the distance that the joint is from the origin of the {s} frame.
- 

3. Given θ, calculate the product of exponentials (PoE) formula in the space frame:

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} .. e^{[s_n]\theta_n} M$$

Example:
Given a 4 joint RRRP Robot Arm, find the Forward Kinematics:



1st, Find M:

$$M = \begin{bmatrix} 0 & -1 & 0 & 19 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2nd, Find all Screw Axes:

$$S_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \\ 0 \\ 0 \\ 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \\ 0 \\ -10 \\ 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \\ 0 \\ -19 \\ 0 \end{bmatrix}, S_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Note, for S_4, the angular velocity is 0 because joint 4 is a prismatic joint which can only slide

3rd, that's it!

We now have the elements necessary to calculate the forward kinematic of the robot T(θ) for any given theta

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1.3.2 Representing Forward Kinematics in the {b} Frame

Procedure:

1. **Unchanged:** Let M be the transformation matrix of the end-effector frame {b} when θ = 0

$$M = \begin{array}{c} \begin{array}{ccc} x_b & y_b & z_b \end{array} \\ \begin{bmatrix} \hat{x}_s & \hat{x}_s & \hat{x}_s & \sum L_{x_s} \\ \hat{y}_s & \hat{y}_s & \hat{y}_s & \sum L_{y_s} \\ \hat{z}_s & \hat{z}_s & \hat{z}_s & \sum L_{z_s} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

2. Find the {b} frame Screw Axes β_{1}, ..., β_n for each of the n joint axes when θ = 0

$$\beta_n = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{x}_b \\ \hat{y}_b \\ \hat{z}_b \\ \\ \dot{x}_b \\ \dot{y}_b \\ \dot{z}_b \end{bmatrix}$$

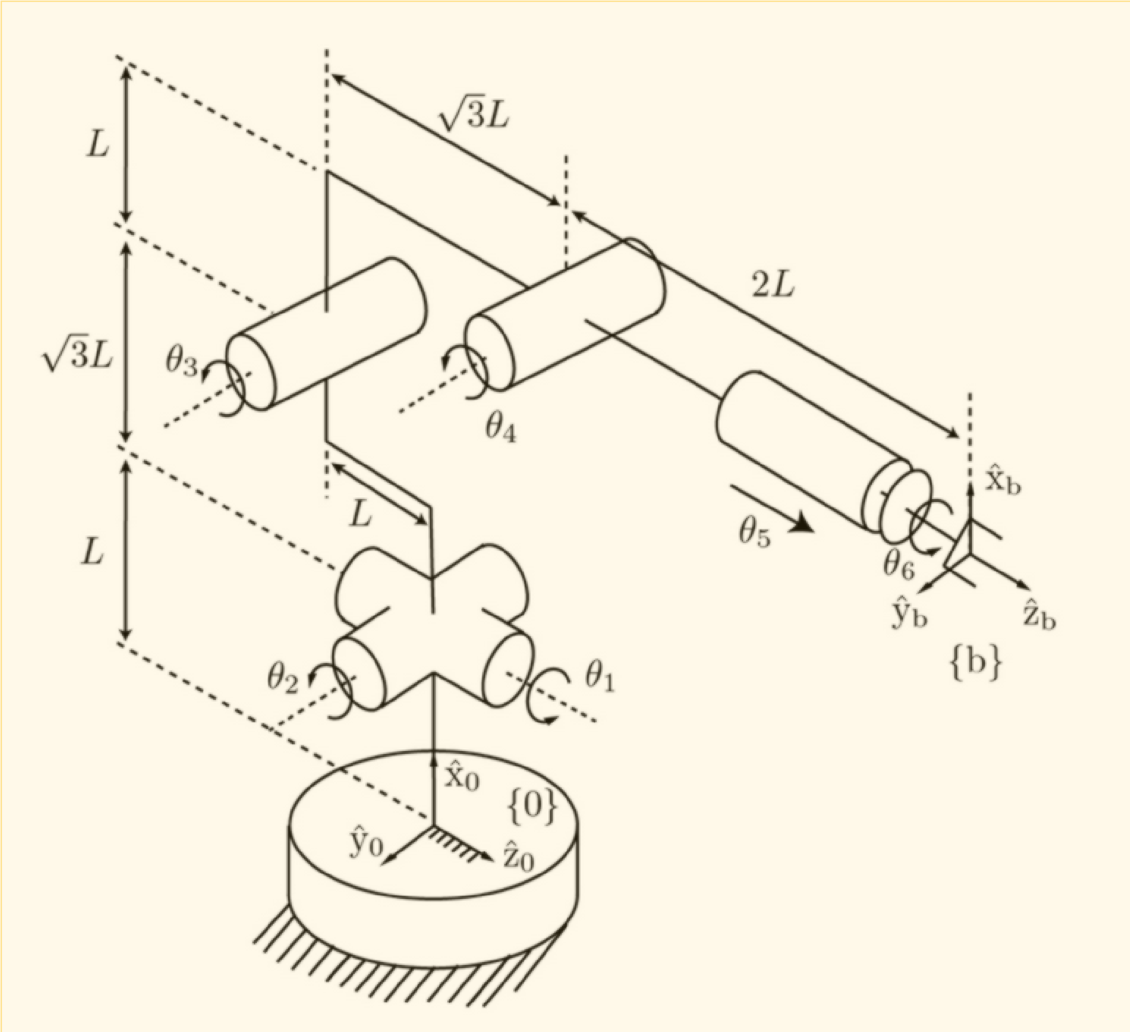
   - The first 3-vector (angular velocity) represents which axis the space frame {b} is rotating about. For example, a value of [0,0,1]^T means that this joint in the {b} frame is rotating about the positive z-axis
   - The linear velocity v can be found by identifying which axis is tangental to the turntable created at the center of the joint n and then multiplying that number with the distance that the joint is from the origin of the {b} frame.

3. Given θ, calculate the product of exponentials (PoE) formula in the space frame:

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} .. e^{[s_n]\theta_n} M$$

Example:

Given a URRPR spatial open chain robot, determin the screw axis β_i in {b} when the robot is in its zero position. **L = 1.**



1st, Find M:

$$M = \begin{bmatrix} 1 & 0 & 0 & 3.73 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2.73 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2nd, Find all Screw Axes:

$$\beta_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \\ 0 \\ 2.73 \\ 0 \end{bmatrix}, \beta_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \\ 2.73 \\ 0 \\ -2.73 \end{bmatrix}, \beta_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \\ 3.73 \\ 0 \\ -1 \end{bmatrix}, \beta_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \\ 2 \\ 0 \\ 0 \end{bmatrix} \beta_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ 1 \end{bmatrix} \beta_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Calculating the linear velocity looks complicated but is still quite simple. Take a look at θ_1. If you were to map that rotation onto the body frame {b}, you would notice that axis y is tangental to the rotation. It is also **2.73 units away from the {b} frame** which is why it gets the value assigned.

Next, take a look at θ_2. It's rotation is tangental to both the x and z frames, so those will be assigned values. It is translated along the z_b axis by a unit of 2.73 and along the x_b axis by a unit of 2.73. The sign is produced from applying the multiplication formula: $v_i = -\omega_i \times q_i$. Here, -$\omega_i$ is (0,-1,0), which will invert the sign of the z axis.

Alternatively, you could just apply the formula for each vector to determine the linear velocity. Less error prone even though it takes more time.

3rd, that's it!

We now have the elements necessary to calculate the forward kinematic of the robot $T(\theta)$ for any given theta

2 Chapter 5: Velocity, Kinematics, and Statics

2.1 Terms

Jacobian

- A determinate which describes the transformation factor of a matrix
  - I.e., after the determinate is applied, the area of the original matrix will change by a factor of what?
- Represented by the time derivative of the matrix

Singularity

- Robot configurations where $J_1(\theta)$ and $J_2(\theta)$ will be co-linear, resulting in a singular matrix for the Jacobian $J(\theta)$

Space Jacobian

- Jacobian in fixed (space) frame coordinate

Body Jacobian

- Jacobian in the end-effector (or body) frame coordinates

End-Effector Velocity

- Also referred to as endpoint velocity (v_{tip})
- Since the end effector can have any x, y, and theta value depending on its configuration, the time derivative of this vector will yield a Jacobian matrix that can be used to calculate the end-effector's v_{tip}

2.2 Velocity Kinematics

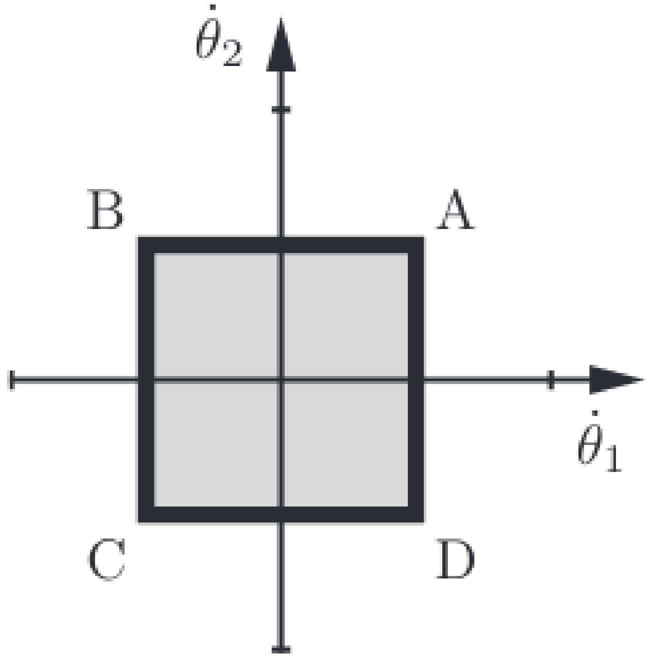
- The Jacobian matrix provides the relation between joint velocities and end-effector velocitities of a robot manipultor.
- Since the joints move with certain velocities, it is useful to know the velocity of the end-effector velocity: v_{tip}
- Here, we are calculating the velocity with Twists.
- Since the Jacobian is a matrix, we need to understand what all of the values mean:

$$J(\theta) = \begin{bmatrix} \dot{x}_{j1} & \dot{x}_{j2} & \dots & \dot{x}_{jn} \\ \dot{y}_{j1} & \dot{y}_{j2} & \dots & \dot{y}_{jn} \\ \dot{z}_{j1} & \dot{z}_{j2} & \dots & \dot{z}_{jn} \end{bmatrix}$$

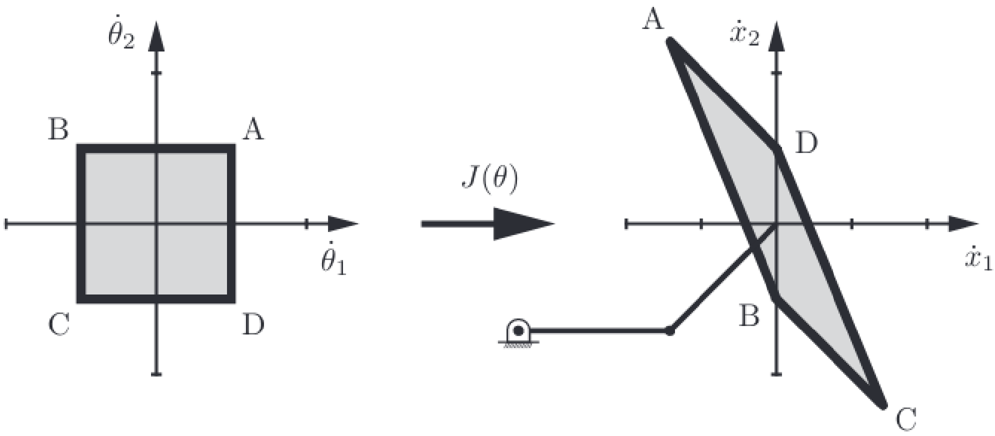
- Each column represents the effect on v_{tip} due to variation in each joint velocity

2.3 Manipulability Ellipsoids - Joint Velocities

The joint velocities are typically graphed in a 2D grid, where the possible joint velocities are represented as a square in the $\dot{\theta}_1 - \dot{\theta}_2$ space.

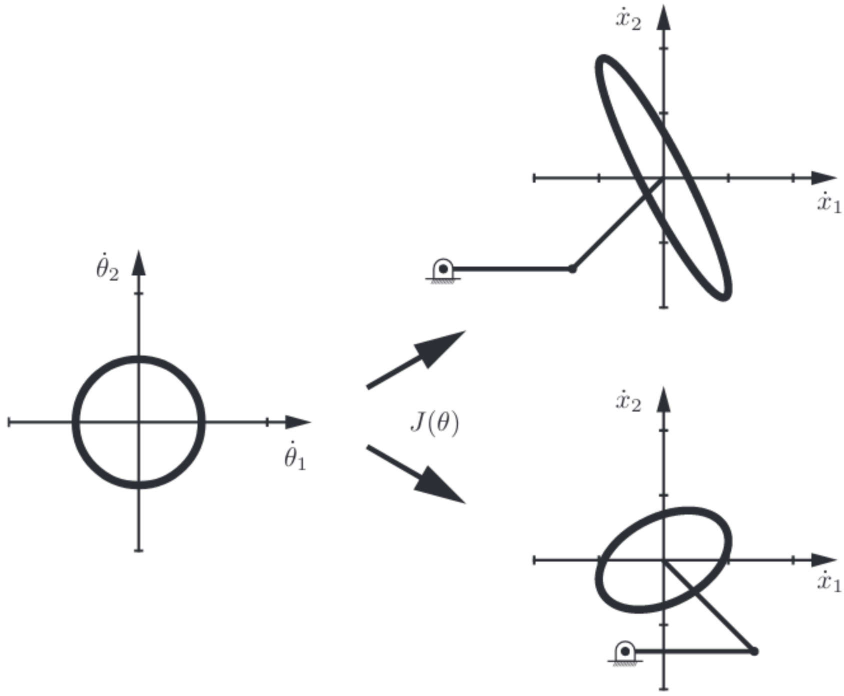


This set of values can then be passed to the Jacobian to return the parallelogram of **possible end-effector velocities.** :



Note: Typically, the joint velocities are shown as a sphere. When they are, an ellipsoid appears to show the end-effector velocities instead of a parallelogram.

- These ellipses are called **Manipulability Ellipsoids**



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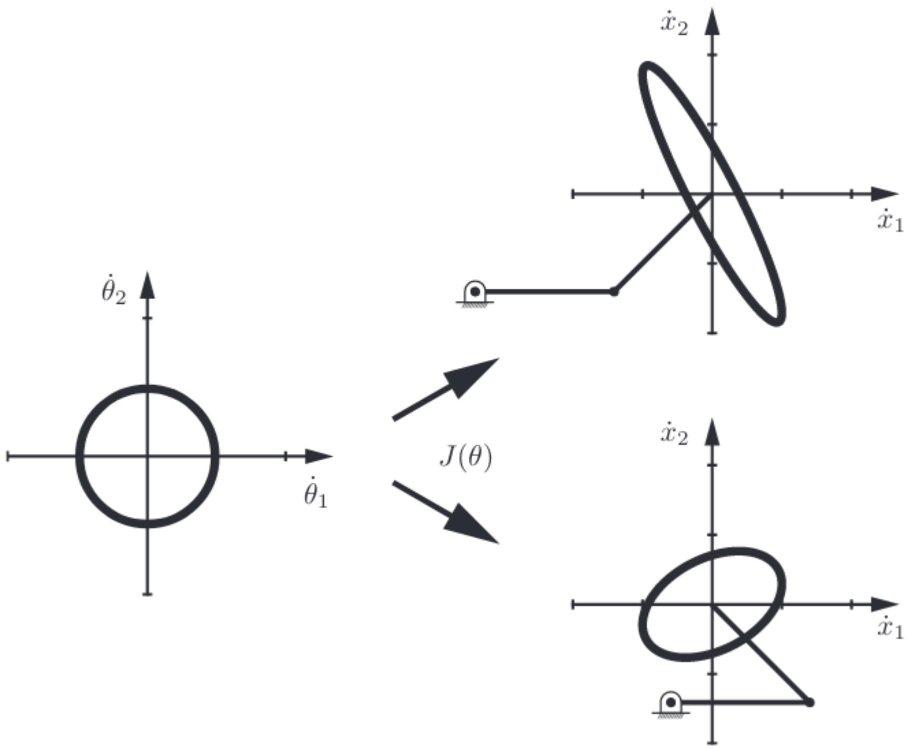
2.4 Force Ellipsoids - Joint Torques

You can also map the joint torques onto a 2D grid. When passed through the Jacobian, this will return the limits of the end-effector forces.



- The resulting ellipsoid is called the **force ellipsoid.**

Here's how to find Tau (τ)

- Let τ = vector of joint torques (forces)

$$power = \dot{\theta}^T \times \tau = v_{tip}^T \times f_{tip}$$

$$= \dot{\theta} \times \tau = (J(\theta)\dot{\theta})^T \times f_{tip} = \dot{\theta} \times \tau = \dot{\theta} \times J^T(\theta) \times f_{tip}\tau = J^T(\theta) \times f_{tip}$$

- Tau is useful for **force control:** If we want the robot to generate the force f_{tip} as its end-effector, the motors must generate joint torques and forces equal to $J^T(\theta) \times f_{tip}$
- If $J^T(\theta)$ is invertible: $J^{-T}(\theta)\tau = f_{tip}$
  - This formula is used to produce the force ellipsoids

2.5 Space Jacobian

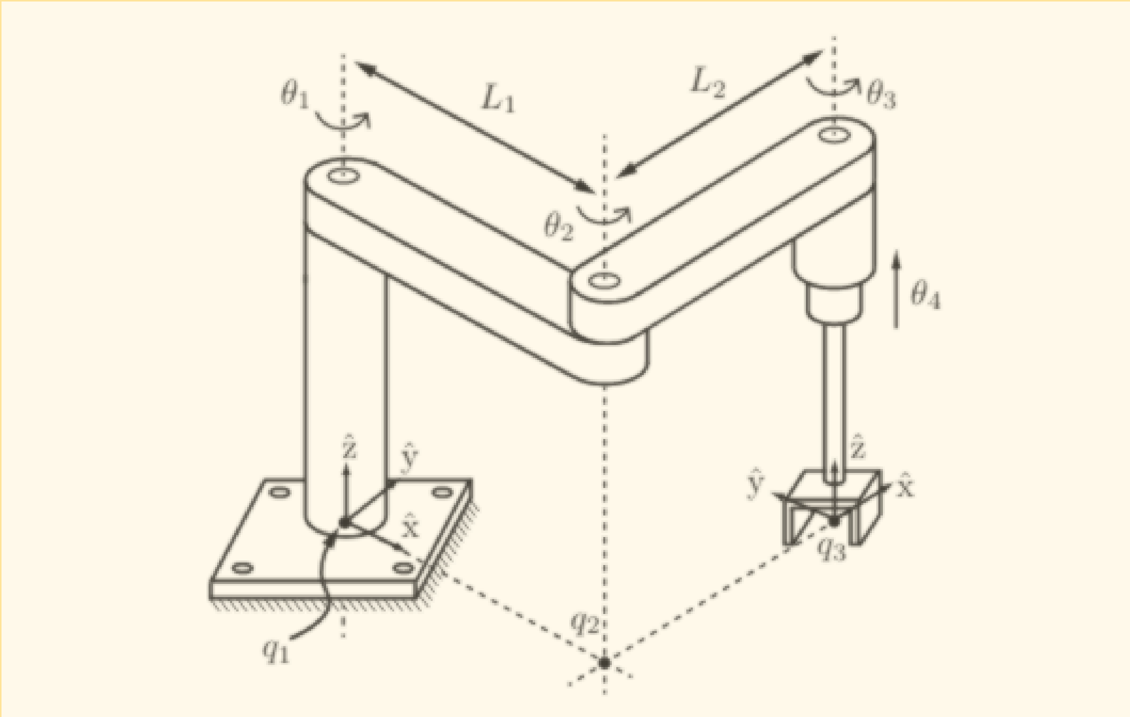
Other than representing the end-effector velocity as v_{tip}, we can represent it by the twist V_s

$$V_s = J_s(\theta)\dot{\theta} = [J_{s1}(\theta)J_{s2}(\theta)...J_{sn}(\theta)]\dot{\theta}$$

Generalized the **Space Jacobian** is defined as:

$$J_s(\theta) \in^{6\times n}, n = 2$$

Since the space jacobian is dependend on joints and angle rotations, we can derive the following example for a RRRP robot:



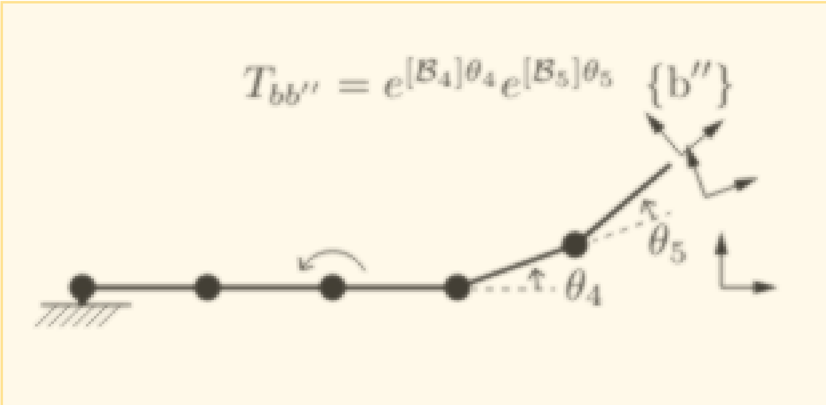
Since the Jacobian is a matrix, we will need to fill in every ith column:

- Denote the ith column of $J_S(\theta)$ by $J_{si} = (\omega_{si}, v_{si})$
- Observe that ω_{s1} is constant and in the $\hat{z}_s$ direction: $\omega_{s1} = (0, 0, 1)$. Choosing q1 as the origin, $v_{s1} = (0, 0, 0)$
- ω_{s2} is also constant in the $\hat{z}_s$ direction, so $\omega_{s2} = (0, 0, 1)$. Choose q2 as the point $(L_1c_1, L_1s_1, 0)$, where $c_1 = cos\theta_1, s_1 = sin\theta_1$. Then $v_{s2} = -\omega_2 \times q_2 = (L_1s_1, -L_1c_1, 0)$.
- The direction of ω_{s3} is always fixed in the $\hat{z}_s$ direction regardless of the values of θ_1 and θ_2, so $\omega_{s3} = (0, 0, 1)$. Choosing $q_3 = (L_1c_1 + L_2c_{12}, L_1s_1 + L_2s_{12}, 0)$, where $c_{12} = cos(\theta_1 + \theta_2), s_{12} = sin(\theta_1 + \theta_2)$, it followsthat $v_{s3} = (L_1s_1 + L_2s_{12}, -L_1c_1 - L_2c_{12}, 0)$.
- Since the finl joint is prismatic, $\omega_{s4} = (0, 0, 0)$, and the joint-axis direction is given by $v_{s4} = (0, 0, 1)$, the Space Jaobian is therefore:

- $$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & L_1 s_1 & L_1 s_1 + L_2 s_{12} & 0 \\ 0 & -L_1 c_1 & -L_1 c_1 - L_2 c_{12} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2.6 Body Jacobian

The Body Jacobian transforms joint veocities into the body twist:



Here is a 5R Robot. The end effector frame is denoted as b'' since the frame is shifted twice due to the angle changes from the rotations of joints 4 and 5.

To find J_{b3}, find $T_{bb''}$

$$J_{b3} = [Ad_{T_{b''b}}]\beta_3 = [Ad_{T^{-1}_{bb''}}]\beta_3 = [Ad_{e^{-[\beta_5]\theta_5} e^{-[\beta_4]\theta_4}}]\beta_3$$

Generalized: the Body Jacobian $J_b(\theta)$ is defined by

$V_b = J_b(\theta)\dot{\theta}$

where $J_b(\theta) = [J_{b1}(\theta) ... J_{b(n-1)}(\theta) J_{bn}] \in^{6\times n}$

with $J_{bn} = B_n$

and $J_{bi}(\theta) = [Ad_{e^{-[B_n]\theta_n} ... e^{-[B_{i+1}]\theta_{i+1}}}]$

2.6.1 Relationship between Space Jacobian and Body Jacobian

$J_b(\theta) = [Ad_{T_{bs}}] J_s(\theta)$

$J_s(\theta) = [Ad_{T_{sb}}] J_b(\theta)$

2.7 Statics of Open Chains

$\tau_{motion}(t) : Inverse\ Dynamics$

To resist a wrench $-F_*$ applied tothe end-effector at configuration θ, the joint torques/forces must be at:

$\tau = J_*^T(\theta)F_*$

where * = b indicates th Jacobian and the wrench represented in {b} and * = s indicates the Jacobian and wrench represented in {s}. This can be used in force control of the robot.

Remember, the Jacbian shows the relationships for te end-effector velocities. These relationships can be used to determine the **corrective force** needed to maintain stability when an opposing force is applied.

2.8 Kinematic Singularities

Since the Jacobian is a matrix, we can determine its rank:

2.8.1 Ranking Jacobians

The rank of a matrix is the number of independent rows contained in the matrix. A row is independent if it matches all of the following criteria:

- The row must contain at least one non-zero value
- The row must be unique
- The row must not be a multiple of another
- The row must not be a linear combination of another row

Remember that:

- $J(\theta) \in^{6\times n}$

Therefore:

- $rank\ J(\theta) <= min(6, n)$

if $J(\theta) = min(6, n)$, then the Jacobian is at **full rank**.
If $J(\theta) < max\ rank\ J(\theta)$, then the Jacobian is in **singularity at** θ

2.8.2 Classifing Jacobians

For $J(\theta) \in^{6\times n}$:

- When n < 6: The Jacobian is **tall** and **kinematically deficient**.
- When n = 6: The Jacobian is **square** (capable of general 6-dimenional body motions)
- When n > 6: The Jacobian is **fat** (it has more columns than rows like a 7R robot) and is considered **redundant**.

2.9 Manipulability

A robot configuration is either singular, or it is not. However, it is useful to describe "how close" the robot is to being singular.

We can assign a measure ofjust how close the robot is to being singular according to how close the ellipsoid is to collapsing.

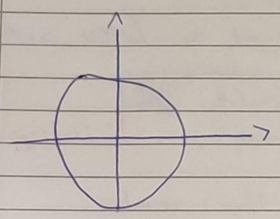
Eigenvector:

- A vector when after a transformation is applied to it only scaled instead of rotating

Eigenvalue:

- The amount that the vector scaled

$v_{tip} = J(\theta)\dot{\theta}$, $v_{tip} \in \mathbb{R}^n$, $\dot{\theta} \in \mathbb{R}^n$, $J(\theta) \in \mathbb{R}^{m \times n}$



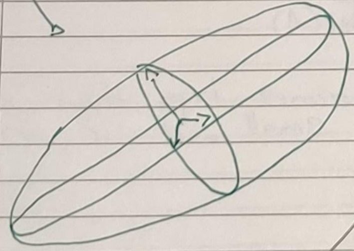
$$\dot{\theta}^T \dot{\theta} = 1$$
$$(J^{-1} v_{tip})^T (J^{-1} v_{tip}) = 1$$
$$\boxed{\begin{array}{c} v_{tip}^T A^{-1} v_{tip} = 1 \\ A = J J^T \in \mathbb{R}^{m \times m} \end{array}}$$

Replacing v_{tip} with x...

$$x^T A^{-1} x = 1$$

$A \in \mathbb{R}^{m \times m}$ (symmetric, positive definite)

eigenvalues (e-vals) of $A = \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m$
eigenvectors (e-vecs) of $A = v_1, v_2, v_3, \dots, v_m$



ellipsoid in x space

if $A = JJ^T$, then $x = v_{tip}$ and... ellipsoid is called manipulability ellipsoid.

if $A = (JJ^T)^{-1}$, then $x = f_{tip}$ and... ellipsoid is called force ellipsoid

Both ellipsoid have the same principal axis.

We can then use a single number to represent how close the ellipsoid is to singularity: **This is called Manipulabiliy**

We can then use a single number to represent how close the ellipsoid is to singularity: (Called Manipulability)

Manipulability Measurement 1: Ratio of longest to shortest axes

$$\mu_1(A) = \frac{\sqrt{\lambda_{max}(A)}}{\sqrt{\lambda_{min}(A)}} = \sqrt{\frac{\lambda_{max}(A)}{\lambda_{min}(A)}} \geq 1 \quad (=1 : \text{isotropic})$$

↳ As the ellipsoid approaches singularity, this number grows large.

Manipulability Measurement 2: Condition number of A

$$\mu_2(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)} \geq 1$$

Manipulability Measurement 3: Proportional to volume of ellipsoid

$$\mu_3(A) = \sqrt{\lambda_1 \lambda_2 \cdots} = \sqrt{\det(A)}$$

↳ If the manipulability ellipsoid becomes large, then the force ellipsoid volume becomes small and vice versa.

$J_{bw}(\theta) \in \mathbb{R}^{3\times n}$ $\quad J_{bv}(\theta) \in \mathbb{R}^{3\times n}$

It's more useful to visualize the manipulability ellipsoid using the body Jacobian than the space Jacobian, since the body Jacobian measures linear velocities at the origin of the end-effector frame, which has a more intuitive meaning than the linear velocity at the origin of the space frame. If the robot has nnn joints, then the body Jacobian J_b is 6 x n. We can break J_b into two sub-Jacobians, the angular and linear Jacobians:

$$J_b = \begin{bmatrix} J_{bw} \\ J_{bv} \end{bmatrix}$$

The dimension of $J_{bv} J_{bv}^T$, which is used to generate the linear component of the manipulability ellipsoid is:

- 3x3

As a Full Rank Jacobian approaches singular configuration, the following occurs to the **manipulability ellipsoid**:

- The length of one principal axis approaches zero
- The interior volume of the ellipsoid approaches zero

As a Full Rank Jacobian approaches singular configuration, the following occurs to the **force ellipsoid**

- The length of one principal axis approaches infinity.
- The interior volume of the ellipsoid approaches infinity.