Project #2 12/03/2023

Part A:

$u_t - u_{xx} = f(x,t) \qquad (x,t) \in (0,1) \times (0,1)$

$u(x,0) = \sin(\pi x) \qquad u(0,t) = u(1,t) = 0$

$f(x,t) = (\pi^2 - 1)e^{-t}\sin(\pi x)$

- choose a test function, v, that satisfies the boundary conditions and integrate over the domain

$$\int_0^t \int_0^1 (u_t v - u_{xx} v)\, dx\, dt = \int_0^T \int_0^1 f(x,t)\, v\, dx\, dt$$

- integration by parts:

$$\int_0^t \int_0^1 u_t v\, dx\, dt + \int_0^t (u_x v)\Big|_0^1 dt - \int_0^t \int_0^1 u_x v_x\, dx\, dt - \int_0^T \int_0^1 f(x,t)\, v\, dx\, dt$$

note: u_x eventually disappears due to given dirichlet boundary conditions

- take u_t and apply the forward euler method : $u_t = \frac{(u(t+\Delta t) - u(t))}{\Delta t}$ where Δt is the stepsize

$$\int_0^1 \int_0^t \left(\frac{u(t+\Delta t) - u(t)}{\Delta t}\right) v\, dx\, dt - \int_0^t \int_0^1 u_x v_x\, dx\, dt - \int_0^T \int_0^1 f(x,t)\, v\, dx\, dt$$

- apply the Galerkin Method:

$$U_n(x,t) = \sum_{i=1}^{N} u_i(t)\, \phi_i(x)$$

$$V_n(x,t) = \sum_{j=1}^{N} v_j(t)\, \phi_j(x)$$

Final Equation:

$$B_i = \sum_{i=1}^{N} \int_0^t \int_0^1 \frac{\left(u(t+\Delta t) - u(t)\right)}{\Delta t} \left(\phi_i V_j\right) dx\, dt$$

$$- \sum_{i=1}^{N} \left(\int_0^t \int_0^1 u_{ix}\, V_{jx}\, dx\, dt - \sum_{i=1}^{N} \int_0^t \int_0^1 f(x,t)\, \phi_j\, dx\, dt \right.$$