$F^{\omega}_{..}$ Types are Strongly Normalizing

Cameron Wong

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1 Semantics of $F^{\omega}_{..}$

The presentation here is taken, slightly modified, from that of Sandro et. al.

We omit the rules concerning terms for brevity.

$$
\begin{array}{rll}
X, Y, Z\ldots & & \textbf{Type Variable} \\
A, B, \tau ::= & X \mid \top \mid \bot \mid A \to B \mid \forall(X : K).A \mid \lambda(X : K).A \mid A\ B & \textbf{Type} \\
J, K, L ::= & * \mid A..B \mid \Pi(X : J).K & \textbf{Kind} \\
\Gamma ::= & \varnothing \mid \Gamma, x : A \mid \Gamma, X : K & \textbf{Context}
\end{array}
$$

Figure 1: Syntax of $F^{\omega}_{..}$

$$
\frac{}{\varnothing\ \texttt{ctx}}\ \textsc{C-empty}
\qquad
\frac{\Gamma\ \texttt{ctx} \qquad \Gamma \vdash K\ \texttt{kd}}{\Gamma, X : K\ \texttt{ctx}}\ \textsc{C-KdBind}
$$

$$
\frac{\Gamma\ \texttt{ctx} \qquad \Gamma \vdash A : *}{\Gamma, x : A\ \texttt{ctx}}\ \textsc{C-TpBind}
$$

Figure 2: Context formation

$$
\frac{}{\Gamma \vdash *\ \texttt{kd}}\ \textsc{Wf-Type}
\qquad
\frac{\Gamma \vdash A : * \qquad \Gamma \vdash B : *}{\Gamma \vdash A..B\ \texttt{kd}}\ \textsc{Wf-Intv}
$$

$$
\frac{\Gamma \vdash J\ \texttt{kd} \qquad \Gamma, X : J \vdash K\ \texttt{kd}}{\Gamma \vdash \Pi(X : J).K\ \texttt{kd}}\ \textsc{Wf-DArr}
$$

Figure 3: Kind formation

As per Wang and Rompf, we use a runtime environment lookup-based evaluation strategy rather than a substitution-based one, detailed in figure 6.

$$\frac{\Gamma\ \texttt{ctx} \qquad \Gamma(X) = K}{\Gamma \vdash X : K}\ \text{K-Var} \qquad \frac{}{\Gamma \vdash \top : *}\ \text{K-Top} \qquad \frac{}{\Gamma \vdash \bot : *}\ \text{K-Bot}$$

$$\frac{\Gamma \vdash A : * \qquad \Gamma \vdash B : *}{\Gamma \vdash A \to B : *}\ \text{K-Arr} \qquad \frac{\Gamma \vdash K\ \texttt{kd} \qquad \Gamma, X : K \vdash A : *}{\Gamma \vdash \forall(X : K).A : *}\ \text{K-All}$$

$$\frac{\Gamma \vdash J\ \texttt{kd} \qquad \Gamma, X : J \vdash A : K \qquad \Gamma, X : J \vdash K\ \texttt{kd}}{\Gamma \vdash \lambda(X : J).A : \Pi(X : J).K}\ \text{K-Abs}$$

$$\frac{\Gamma \vdash A : \Pi(X : J).K \qquad \Gamma \vdash B : J \qquad \Gamma, X : J \vdash K\ \texttt{kd} \qquad \Gamma \vdash K[B/X]\ \texttt{kd}}{\Gamma \vdash A\ B : K[B/X]}\ \text{K-App}$$

$$\frac{\Gamma \vdash A : B..C}{\Gamma \vdash A : A..A}\ \text{K-Sing} \qquad \frac{\Gamma \vdash A : J \qquad \Gamma \vdash J \leq K}{\Gamma \vdash A : K}\ \text{K-Sub}$$

Figure 4: Kind assignment

$$\frac{\Gamma \vdash A_2 \leq A_1 : * \qquad \Gamma \vdash B_1 \leq B_2 : *}{\Gamma \vdash A_1..B_1 \leq A_2..B_2}\ \text{SK-Intv}$$

$$\frac{\Gamma \vdash \Pi(X : J_1).K_1\ \texttt{kd} \qquad \Gamma \vdash J_2 \leq J_1 \qquad \Gamma, X : J_2 \vdash K_1 \leq K_2}{\Gamma \vdash \Pi(X : J_1).K_1 \leq \Pi(X : J_2).K_2}\ \text{SK-DArr}$$

Figure 5: Subkinding

We use the judgment $H \vdash A \Downarrow^n \tau$ to say that A steps to a type value τ in n steps.

2 Analysis

We follow the method of Girard and Tait. For kinds K, we define its denotation $[\![K]\!]_\Gamma$ as the set of type values inhabiting K, given some type environment Γ. Type variables can appear in kinds in the bounds of some interval $A..B$, so this context is necessary to enforce that the correct subtyping relations are satisfied.

Definition 1 (Kind-Type Relation)**.**

$$\begin{aligned}
[\![*]\!]_\Gamma &= \{\tau\} \\
[\![A..B]\!]_\Gamma &= \{\langle H, \tau\rangle \mid \Gamma \vdash A \leq B : *\} \\
[\![\Pi(X : J).K]\!]_\Gamma &= \{\langle H, \lambda(X : J).A\rangle \mid \forall \tau_x \in [\![J]\!]_\Gamma . \langle H(X \mapsto \tau_x), \tau\rangle \in \mathscr{E}[\![K]\!]_{\Gamma(X \mapsto \tau_x)}\} \\
\mathscr{E}[\![K]\!]_\Gamma &= \{\langle H, A\rangle \mid \exists n, \tau . H \vdash A \Downarrow n\tau \wedge \tau \in [\![K]\!]_\Gamma\}
\end{aligned}$$

$$H ::= \emptyset \mid H, X = \tau \qquad\qquad \textbf{Runtime environment}$$

$$\frac{}{\emptyset\ \texttt{env}}\ \textsc{H-Emp} \qquad\qquad \frac{H \vdash \tau\ \texttt{val}}{H, X = \tau\ \texttt{env}}\ \textsc{H-Bind}$$

Figure 6: Runtime structures

$$\frac{}{H \vdash \top\ \texttt{val}} \qquad \frac{}{H \vdash \bot\ \texttt{val}} \qquad \frac{H \vdash A\ \texttt{val} \qquad H \vdash B\ \texttt{val}}{H \vdash A \to B\ \texttt{val}}$$

$$\frac{}{H \vdash \forall(X : K).A\ \texttt{val}} \qquad \frac{}{H \vdash \lambda(X : K).A\ \texttt{val}}$$

$$\frac{H(X) = \tau}{H \vdash X \Downarrow^0 \tau} \qquad \frac{H \vdash A \Downarrow^n \tau_A \qquad H \vdash B \Downarrow^m \tau_B}{H \vdash A \to B \Downarrow^{n+m} \tau_A \to \tau_B}$$

$$\frac{H \vdash A \Downarrow^a \lambda(X : K).A' \qquad H \vdash B \Downarrow^b \tau \qquad H, X = \tau \vdash A' \Downarrow^n \tau'}{H \vdash A\ B \Downarrow^{a+b+n} \tau'}$$

Figure 7: Type reduction

2.1 Inversion of Function Kinds

Lemma 1 (Dependent Function canonical form and denotation inversion)**.**

$$\frac{T \in [\![\Pi(X : J).K]\!]_\Gamma}{T = \langle H, \lambda(X : J).A\rangle \qquad \forall \tau_x \in [\![J]\!]_\Gamma . \langle H(X \to \tau_x), A\rangle \in \mathscr{E}[\![K[\tau_x/X]]\!]_\Gamma}$$

2.2 The main proof

Theorem 1 (Strong Normalization of $F^\omega_{\because}$ types)**.**

$$\frac{\Gamma \vdash A : K \qquad \Gamma \models H}{\langle H, A\rangle \in \mathscr{E}[\![K]\!]_\Gamma}$$

Proof. By induction on the derivation $\Gamma \vdash A : K$.

- Case K-Top, K-Bot, K-All, K-Abs: Immediate.
- Case K-Var: By the definition of consistent environments.
- Case K-Arr, K-Sing: By the inductive hypotheses.
- Case K-App:

□