$F^{\omega}_{..}$ Types are Strongly Normalizing

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1 Semantics of $F^{\omega}_{..}$

The presentation here is taken, slightly modified, from that of Stucki et. al.

We omit the rules concerning terms for brevity.

$$\begin{array}{rll} X, Y, Z... & & \textbf{Type Variable} \\ A, B, \tau ::= & X \mid \top \mid \bot \mid A \to B \mid \forall(X:K).A \mid \lambda(X:K).A \mid A\ B & \textbf{Type} \\ J, K, L ::= & * \mid A..B \mid \Pi(X:J).K & \textbf{Kind} \\ \Gamma ::= & \varnothing \mid \Gamma, x : A \mid \Gamma, X : K & \textbf{Context} \end{array}$$

Figure 1: Syntax of $F^{\omega}_{..}$

$$\frac{}{\varnothing\ \texttt{ctx}}\ \textsc{C-empty} \qquad \frac{\Gamma\ \texttt{ctx} \qquad \Gamma \vdash K\ \texttt{kd}}{\Gamma, X : K\ \texttt{ctx}}\ \textsc{C-KdBind}$$

$$\frac{\Gamma\ \texttt{ctx} \qquad \Gamma \vdash A : *}{\Gamma, x : A\ \texttt{ctx}}\ \textsc{C-TpBind}$$

Figure 2: Context formation

$$\frac{}{\Gamma \vdash *\ \texttt{kd}}\ \textsc{Wf-Type} \qquad \frac{\Gamma \vdash A : * \qquad \Gamma \vdash B : *}{\Gamma \vdash A..B\ \texttt{kd}}\ \textsc{Wf-Intv}$$

$$\frac{\Gamma \vdash J\ \texttt{kd} \qquad \Gamma, X : J \vdash K\ \texttt{kd}}{\Gamma \vdash \Pi(X : J).K\ \texttt{kd}}\ \textsc{Wf-DArr}$$

Figure 3: Kind formation

As per Wang and Rompf, we use a runtime environment lookup-based evaluation strategy rather than a substitution-based one, detailed in figure 6.

$$\frac{\Gamma\ \texttt{ctx} \qquad \Gamma(X) = K}{\Gamma \vdash X : K}\ \textsc{K-Var} \qquad \frac{}{\Gamma \vdash \top : *}\ \textsc{K-Top} \qquad \frac{}{\Gamma \vdash \bot : *}\ \textsc{K-Bot}$$

$$\frac{\Gamma \vdash A : * \qquad \Gamma \vdash B : *}{\Gamma \vdash A \to B : *}\ \textsc{K-Arr} \qquad \frac{\Gamma \vdash K\ \texttt{kd} \qquad \Gamma, X : K \vdash A : *}{\Gamma \vdash \forall(X : K).A : *}\ \textsc{K-All}$$

$$\frac{\Gamma \vdash J\ \texttt{kd} \qquad \Gamma, X : J \vdash A : K \qquad \Gamma, X : J \vdash K\ \texttt{kd}}{\Gamma \vdash \lambda(X : J).A : \Pi(X : J).K}\ \textsc{K-Abs}$$

$$\frac{\Gamma \vdash A : \Pi(X : J).K \qquad \Gamma \vdash B : J \qquad \Gamma, X : J \vdash K\ \texttt{kd} \qquad \Gamma \vdash K[B/X]\ \texttt{kd}}{\Gamma \vdash A\ B : K[B/X]}\ \textsc{K-App}$$

$$\frac{\Gamma \vdash A : B..C}{\Gamma \vdash A : A..A}\ \textsc{K-Sing} \qquad \frac{\Gamma \vdash A : J \qquad \Gamma \vdash J \leq K}{\Gamma \vdash A : K}\ \textsc{K-Sub}$$

Figure 4: Kind assignment

$$\frac{\Gamma \vdash A_2 \leq A_1 : * \qquad \Gamma \vdash B_1 \leq B_2 : *}{\Gamma \vdash A_1..B_1 \leq A_2..B_2}\ \textsc{SK-Intv}$$

$$\frac{\Gamma \vdash \Pi(X : J_1).K_1\ \texttt{kd} \qquad \Gamma \vdash J_2 \leq J_1 \qquad \Gamma, X : J_2 \vdash K_1 \leq K_2}{\Gamma \vdash \Pi(X : J_1).K_1 \leq \Pi(X : J_2).K_2}\ \textsc{SK-DArr}$$

Figure 5: Subkinding

We use the judgment $H \vdash A \Downarrow^n \tau$ to say that A steps to a type value τ in n steps. We also use $H \vdash A \Downarrow \tau$ as shorthand for $\exists n. A \Downarrow^n \tau$.

2 Analysis

We follow the method of Girard and Tait. For kinds K, we define its denotation $[\![K]\!]_\Gamma$ as the set of type values inhabiting K, given some type environment Γ. Type variables can appear in kinds in the bounds of some interval $A..B$, so this context is necessary to enforce that the correct subtyping relations are satisfied.

Unlike Wang and Rompf, we do not need to track bounds in our denotation. In System $D_{<:}$, interval types are themselves types. This meant that the $D_{<:}$ interpretation $[\![\text{Type } T_1..T_2]\!]_\rho$ could not enforce any restrictions on its Type T members while remaining well-founded, leading to a loss of information when performing semantic widening. In System $F^\omega_{..}$, however, interval *kinds* are of a different syntactic sort than the types they restrict, permitting the simpler definition of $[\![\cdot]\!]_\Gamma$.

$$H ::= \emptyset \mid H, X = \tau \qquad\qquad \textbf{Runtime environment}$$

$$\frac{}{\emptyset\ \texttt{env}}\ \textsc{H-Emp} \qquad\qquad \frac{H \vdash \tau\ \texttt{val}}{H, X = \tau\ \texttt{env}}\ \textsc{H-Bind}$$

Figure 6: Runtime structures

$$\frac{}{H \vdash \top\ \texttt{val}} \qquad \frac{}{H \vdash \bot\ \texttt{val}} \qquad \frac{}{H \vdash \forall(X:K).A\ \texttt{val}} \qquad \frac{}{H \vdash \lambda(X:K).A\ \texttt{val}}$$

$$\frac{H \vdash A\ \texttt{val} \qquad H \vdash B\ \texttt{val}}{H \vdash A \to B\ \texttt{val}} \qquad \frac{H \vdash A\ \texttt{val} \qquad H \vdash B\ \texttt{val}}{H \vdash A..B\ \texttt{val}}$$

$$\frac{H(X) = \tau}{H \vdash X \Downarrow^0 \tau}\ \textsc{Eval-var} \qquad \frac{H \vdash A \Downarrow^n \tau_A \qquad H \vdash B \Downarrow^m \tau_B}{H \vdash A \to B \Downarrow^{n+m} \tau_A \to \tau_B}\ \textsc{Eval-arr}$$

$$\frac{H \vdash A \Downarrow^a \lambda(X:K).A' \qquad H \vdash B \Downarrow^b \tau \qquad H, X = \tau \vdash A' \Downarrow^n \tau'}{H \vdash A\ B \Downarrow^{a+b+n} \tau'}\ \textsc{Eval-app}$$

$$\frac{H \vdash A \Downarrow^a \tau_A \qquad H \vdash B \Downarrow^b \tau_B}{H \vdash A..B \Downarrow^{a+b} \tau_A..\tau_B}\ \textsc{Eval-intv}$$

Figure 7: Type reduction

2.1 Semantic Subtyping

Lemma 1 (Semantic Widening)**.**

$$\frac{\Gamma \vdash K_1 \leq K_2}{[\![K_1]\!]_\Gamma \subseteq [\![K_2]\!]_\Gamma}$$

Proof. By induction on the subkinding relation.

- Case SK-Intv:

  By transitivity of subtyping.

- Case SK-DArr:

  By the inductive hypotheses.

□

As bounds can no longer be nested, we do not need to enforce that they are "good" – while we can still *construct* the ill-bounded kind $\top..\bot$, there is no

Definition 1 (Kind-Type Relation)**.**

$$\begin{aligned}
[\![*]\!]_\Gamma &= \{\langle H, \tau\rangle\} \\
[\![A..B]\!]_\Gamma &= \{\langle H, \tau\rangle \mid \Gamma \vdash A \leq \tau : *, \Gamma \vdash \tau \leq B : *, \} \\
[\![\Pi(X : J).K]\!]_\Gamma &= \{\langle H, \lambda(X : J).A\rangle \mid \forall \tau_x \in [\![J]\!]_\Gamma . \langle H(X \mapsto \tau_x), A\rangle \in \mathscr{E}[\![K[\tau_x/X]]\!]_\Gamma\} \\
\mathscr{E}[\![K]\!]_\Gamma &= \{\langle H, A\rangle \mid \exists \tau . H \vdash A \Downarrow \tau \wedge \langle H, \tau\rangle \in [\![K]\!]_\Gamma\}
\end{aligned}$$

Definition 2 (Consistent environments)**.**

$$\frac{}{\varnothing \models \emptyset} \qquad \frac{\Gamma \models H \qquad \Gamma \vdash \tau : K \qquad H, X = \tau \ \mathit{env}}{\Gamma, X : K \models H, X = \tau}$$

type-level `null` to provide a spurious witness.

2.2 Inversion of Function Kinds

Lemma 2 (Inversion of Dependent Functions)**.**

$$\frac{\langle H, \tau\rangle \in [\![\Pi(X : J).K]\!]_\Gamma \qquad \Gamma \models H}{\tau = \lambda(X : J).A \qquad \forall \tau_x \in [\![J]\!]_\Gamma . \langle H(X \mapsto \tau_x), A\rangle \in \mathscr{E}[\![K[\tau_x/X]]\!]_\Gamma}$$

Proof. By definition of $[\![\cdot]\!]_\Gamma$. □

2.3 The main proof

Theorem 1 (Strong Normalization of types)**.**

$$\frac{\Gamma \vdash A : K \qquad \Gamma \models H}{\langle H, A\rangle \in \mathscr{E}[\![K]\!]_\Gamma}$$

Proof. By induction on the derivation $\Gamma \vdash A : K$.

- Case K-Top, K-Bot, K-All: Immediate.
- Case K-Var: By the definition of consistent environments.
- Case K-Arr, K-Sing: By the inductive hypotheses.
- Case K-Abs: By the inductive hypothesis.
- Case K-App:

  By the inductive hypothesis on $\Gamma \vdash A : \Pi(X : J).X$, we have $\langle H, A\rangle \in \mathscr{E}[\![\Pi(X : J).K]\!]_\Gamma$, so $H \vdash A \Downarrow^n \tau$ for some n and $\langle H, \tau\rangle \in [\![\Pi(X : J).K]\!]_\Gamma$. Then, use lemma 2 and the other inductive hypothesis to recover the premises to Eval-app.

  Note that τ' is a type value and is therefore closed, so it is valid to strengthen $\Gamma, X : \tau_B$ back to Γ

- Case K-Sub: By the inductive hypothesis and semantic widening.

□

A Structural Lemmas

Lemma 3 (Substitution commutes with normalization under denotation)**.**

$$\frac{\Gamma \vdash K \ \boldsymbol{\mathit{kd}} \qquad X \notin \Gamma \qquad \Gamma \models H \qquad H \vdash A \Downarrow^n \tau}{[\![K[A/X]]\!]_\Gamma = [\![K[\tau_x/X]]\!]_\Gamma}$$

Proof. By induction on the judgment $H \vdash A \Downarrow \tau$. □

Lemma 4 (Big-step evaluation results in a value)**.**

$$\frac{\Gamma \models H \qquad H \vdash A \Downarrow^n \tau}{\Gamma \vdash \tau \ \boldsymbol{\mathit{val}}}$$

Proof. By induction on the stepping judgment. □

Lemma 5 (Type values are closed)**.**

$$\frac{H \vdash \tau \ \boldsymbol{\mathit{val}}}{\emptyset \vdash \tau \ \boldsymbol{\mathit{val}}}$$

Proof. By induction on the judgment $H \vdash \tau$ `val`. □

Lemma 6 (Weakening/Strengthening)**.** *The type-kind relation is invariant under extending and shrinking the context.*

$$\frac{X \notin FV(K)}{[\![K]\!]_\Gamma = [\![K]\!]_{\Gamma(X \mapsto K')}}$$

Proof. By induction on K. □