

Linearization of Dynamics

1. Linearization of Arbitrary Link's Energy

For i th link, its kinematic energy follows:

$$\begin{aligned}
 T_i &= \frac{1}{2} \int \vec{v}^\top \vec{v} dm \\
 &= \frac{1}{2} \int (\vec{v}_i + [\vec{\omega}_i] \vec{r})^\top (\vec{v}_i + [\vec{\omega}_i] \vec{r}) dm \\
 &= \frac{1}{2} \int \vec{v}_i^\top \vec{v}_i dm + \int \vec{v}_i^\top [\vec{\omega}_i] \vec{r} dm + \frac{1}{2} \int \vec{r}^\top [\vec{\omega}_i]^\top [\vec{\omega}_i] \vec{r} dm \\
 &= \frac{1}{2} \vec{v}_i^\top \vec{v}_i \int dm + \vec{v}_i^\top [\vec{\omega}_i] \int dm \cdot \vec{r}_{C_i} \\
 &\quad + \frac{1}{2} \int (y^2 + z^2) dm \cdot (\omega_x^i)^2 + \frac{1}{2} \int (x^2 + z^2) dm \cdot (\omega_y^i)^2 \\
 &\quad + \frac{1}{2} \int (x^2 + y^2) dm \cdot (\omega_z^i)^2 - \int xy dm (\omega_x^i \omega_y^i) \\
 &\quad - \int xz dm (\omega_x^i \omega_z^i) - \int yz dm (\omega_y^i \omega_z^i) \\
 &= \frac{1}{2} \vec{v}_i^\top \vec{v}_i m_i + \vec{v}_i^\top [\vec{\omega}_i] m_i \vec{r}_{C_i} + \frac{1}{2} \vec{\omega}_i^\top \begin{bmatrix} xx_i & -xy_i & -xz_i \\ -xy_i & yy_i & -yz_i \\ -xz_i & -yz_i & zz_i \end{bmatrix} \vec{\omega}_i \\
 &= \frac{1}{2} \vec{\omega}_i^\top J_{C_i}^{\text{Former Joint}} \vec{\omega}_i + \vec{v}_i^\top [\vec{\omega}_i] m_i \vec{r}_{C_i} + \frac{1}{2} \vec{v}_i^\top \vec{v}_i m_i,
 \end{aligned}$$

where $m_i \vec{r}_{C_i} = R_i^0 \begin{pmatrix} mx_i \\ my_i \\ mz_i \end{pmatrix}$, elements in $J_{C_i}^{\text{Former Joint}} = \begin{bmatrix} xx_i & -xy_i & -xz_i \\ -xy_i & yy_i & -yz_i \\ -xz_i & -yz_i & zz_i \end{bmatrix}$ are

moment of inertia intergrated from former joint along the link, which are different from those in $J_{C_i}^{\text{CoM}}$ matrixes.

We have

$$\begin{aligned}
xx_i &= \iiint (y^2 + z^2) \rho dx dy dz \\
&= \iiint [(y - y_{C_i})^2 + (z - z_{C_i})^2] \rho dx dy dz \\
&\quad + 2 \iiint [y_{C_i} y + z_{C_i} z] \rho dx dy dz \\
&\quad - \iiint [(y_{C_i})^2 + (z_{C_i})^2] \rho dx dy dz \\
&= J_{xx_i} + m_i [(y_{C_i})^2 + (z_{C_i})^2], \\
xy_i &= \iiint xy \rho dx dy dz \\
&= \iiint [(x - x_{C_i})(y - y_{C_i})] \rho dx dy dz \\
&\quad + \iiint (x_{C_i} y + y_{C_i} x) \rho dx dy dz \\
&\quad - \iiint x_{C_i} y_{C_i} \rho dx dy dz \\
&= J_{xy_i} + m_i x_{C_i} y_{C_i}.
\end{aligned}$$

Similarly, it's easy to verify that

$$\begin{aligned}
yy_i &= J_{yy_i} + m_i [(x_{C_i})^2 + (z_{C_i})^2], & zz_i &= J_{zz_i} + m_i [(x_{C_i})^2 + (y_{C_i})^2], \\
xz_i &= J_{xz_i} + m_i x_{C_i} z_{C_i}, & yz_i &= J_{yz_i} + m_i y_{C_i} z_{C_i}.
\end{aligned}$$

Denote a vector

$$\begin{aligned}
\vec{p}^i &\triangleq [xx_i, xy_i, xz_i, yy_i, yz_i, zz_i, mx_i, my_i, mz_i, m_i]^\top \\
&\triangleq [p_1^i, p_2^i, \dots, p_{10}^i]^\top,
\end{aligned}$$

to collect inertial parameters. It's easy to verify that

$$\begin{aligned}
J_{C_i}^{\text{Former Joint}} \vec{\omega}_i &= \begin{bmatrix} xx_i & -xy_i & -xz_i \\ -xy_i & yy_i & -yz_i \\ -xz_i & -yz_i & zz_i \end{bmatrix} \vec{\omega}_i \\
&= \begin{bmatrix} \omega_x^i & -\omega_y^i & -\omega_z^i & 0 & 0 & 0 \\ 0 & -\omega_x^i & 0 & \omega_y^i & -\omega_z^i & 0 \\ 0 & 0 & -\omega_x^i & 0 & -\omega_y^i & \omega_z^i \end{bmatrix} \begin{pmatrix} xx_i \\ xy_i \\ xz_i \\ yy_i \\ yz_i \\ zz_i \end{pmatrix} \\
&\triangleq K(\vec{\omega}_i) \begin{pmatrix} xx_i \\ xy_i \\ xz_i \\ yy_i \\ yz_i \\ zz_i \end{pmatrix},
\end{aligned}$$

Then the kinematic energy can be rewritten as

$$\begin{aligned}
T_i &= \frac{1}{2} \vec{\omega}_i^\top J_{C_i}^{\text{Former Joint}} \vec{\omega}_i + \vec{v}_i^\top [\vec{\omega}_i] m_i \vec{r}_{C_i} + \frac{1}{2} \vec{v}_i^\top \vec{v}_i m_i \\
&= \left[\frac{1}{2} \vec{\omega}_i^\top K(\vec{\omega}_i), \vec{v}_i^\top [\vec{\omega}_i] R_i^0, \frac{1}{2} \vec{v}_i^\top \vec{v}_i \right] \vec{p}^i \\
&\triangleq \left(\tilde{T}^i \right)^\top \vec{p}^i,
\end{aligned}$$

The potential energy follows

$$\begin{aligned}
V_i &= -m_i \vec{g} \cdot (\vec{r}_i + \vec{r}_{C_i}) \\
&= -\vec{g}^\top (\vec{r}_i m_i + R_i^0 m_i \vec{r}_{C_i}) \\
&= \left[0_{1 \times 6}, -\vec{g}^\top R_i^0, -\vec{g}^\top \vec{r}_i \right] \vec{p}^i \\
&\triangleq \left(\tilde{V}^i \right)^\top \vec{p}^i.
\end{aligned}$$

2. Linearized Dynamics

Then total kinematic energy T and potential energy V follow

$$\begin{aligned}
T &= \sum_{i=1}^n \left(\tilde{T}^i \right)^\top \vec{p}^i \\
&= \left[\left(\tilde{T}^1 \right)^\top, \dots, \left(\tilde{T}^n \right)^\top \right] \vec{p} \\
&\triangleq \tilde{T} \vec{p}, \\
V &= \sum_{i=1}^n \left(\tilde{V}^i \right)^\top \vec{p}^i \\
&= \left[\left(\tilde{V}^1 \right)^\top, \dots, \left(\tilde{V}^n \right)^\top \right] \vec{p} \\
&\triangleq \tilde{V} \vec{p},
\end{aligned}$$

where $\vec{p} \triangleq \left[\left(\vec{p}^1 \right)^\top, \dots, \left(\vec{p}^n \right)^\top \right]^\top$ contains all dynamical parameters. Noting that it's highly possible that \vec{p} has zero-valued or linearly correlated elements.

With the 2nd Lagrangian equation, it can be obtained that

$$\begin{aligned}
\vec{\tau} &= \frac{d}{dt} \frac{\partial T}{\partial \dot{\vec{q}}} - \frac{\partial T}{\partial \vec{q}} + \frac{\partial V}{\partial \vec{q}} \\
&= \left[\frac{d}{dt} \frac{\partial \tilde{T}}{\partial \dot{\vec{q}}} \right] \vec{p} - \frac{\partial \tilde{T}}{\partial \vec{q}} \vec{p} + \frac{\partial \tilde{V}}{\partial \vec{q}} \vec{p} \\
&= \left[\frac{d}{dt} \frac{\partial \tilde{T}}{\partial \dot{\vec{q}}} - \frac{\partial \tilde{T}}{\partial \vec{q}} + \frac{\partial \tilde{V}}{\partial \vec{q}} \right] \vec{p} \\
&\triangleq Y_{n \times 10n} \vec{p}_{10n \times 1}
\end{aligned}$$

The above equation gives a linearized form of manipulator's dynamics. $Y = \frac{d}{dt} \frac{\partial \tilde{T}}{\partial \dot{\vec{q}}} - \frac{\partial \tilde{T}}{\partial \vec{q}} + \frac{\partial \tilde{V}}{\partial \vec{q}}$ is the parameter regression matrix and \vec{p} is dynamical parameter vector.

3. Simplification of Linearized Dynamics

Although we've already obtained

$$\begin{aligned}\vec{\tau} &= Y\vec{p}, \\ Y &= \frac{d}{dt} \frac{\partial \tilde{T}}{\partial \dot{\vec{q}}} - \frac{\partial \tilde{T}}{\partial \vec{q}} + \frac{\partial \tilde{V}}{\partial \vec{q}} \\ \vec{p} &= \left[(\vec{p}^i)^\top, \dots, (\vec{p}^n)^\top \right]^\top,\end{aligned}$$

zero-valued columns and linearly correlated columns in Y may lower correctness when performing online dynamical parameter estimation. Therefore, further simplification should be done.

1. If the i th column of Y , denoted as Y_i , follows

$$Y_i \equiv \vec{0}$$

Then the corresponding i th element in \vec{p} has no influence on dynamics. Therefore, the i th column of Y and i th element of \vec{p} can be eliminated when $Y_i = \vec{0}$ (or $p_i = 0$).

2. If a certain column of Y is a linear correlation of other columns:

$$Y_i \equiv \beta_{i_1} Y_{i_1} + \dots + \beta_{i_m} Y_{i_m},$$

where $\beta_{i_1}, \dots, \beta_{i_m}$ are constants. Then it's obvious that

$$Y_i p_i \equiv Y_{i_1} (\beta_{i_1} p_{i_1}) + \dots + Y_{i_m} (\beta_{i_m} p_{i_m})$$

Then we may set

$$p_{i_j} \rightarrow p_{i_j} + \beta_{i_j} p_i,$$

and $p_i \rightarrow 0$.