Linearization of Dynamics

1. Linearization of Arbitrary Link's Energy

For ith link, its kinematic energy follows:

$$
\begin{aligned}
T_i = & \frac{1}{2}\int \vec{v}^\top \vec{v} dm \\
= & \frac{1}{2}\int \left(\vec{v}_i + [\vec{\omega}_i]\vec{r}\right)^\top \left(\vec{v}_i + [\vec{\omega}_i]\vec{r}\right) dm \\
= & \frac{1}{2}\int \vec{v}_i^\top \vec{v}_i dm + \int \vec{v}_i^\top [\vec{\omega}_i]\vec{r} dm + \frac{1}{2}\int \vec{r}^\top [\vec{\omega}_i]^\top [\vec{\omega}_i]\vec{r} dm \\
= & \frac{1}{2}\vec{v}_i^\top \vec{v}_i \int dm + \vec{v}_i^\top [\vec{\omega}_i] \int dm \cdot \vec{r}_{C_i} \\
& + \frac{1}{2}\int \left(y^2+z^2\right) dm \cdot \left(\omega_x^i\right)^2 + \frac{1}{2}\int \left(x^2+z^2\right) dm \cdot \left(\omega_y^i\right)^2 \\
& + \frac{1}{2}\int \left(x^2+y^2\right) dm \cdot \left(\omega_z^i\right)^2 - \int xydm \left(\omega_x^i \omega_y^i\right) \\
& - \int xzdm \left(\omega_x^i \omega_z^i\right) - \int yzdm \left(\omega_y^i \omega_z^i\right) \\
= & \frac{1}{2}\vec{v}_i^\top \vec{v}_i m_i + \vec{v}_i^\top [\vec{\omega}_i] m_i \vec{r}_{C_i} + \frac{1}{2}\vec{\omega}_i^\top \begin{bmatrix} xx_i & xy_i & xz_i \\ xy_i & yy_i & yz_i \\ xz_i & yz_i & zz_i \end{bmatrix} \vec{\omega}_i \\
= & \frac{1}{2}\vec{\omega}_i^\top J_{C_i}^{\text{Former Joint}} \vec{\omega}_i + \vec{v}_i^\top [\vec{\omega}_i] m_i \vec{r}_{C_i} + \frac{1}{2}\vec{v}_i^\top \vec{v}_i m_i,
\end{aligned}
$$

where $m_i\vec{r}_{ci} = R_i^0 \begin{pmatrix} mx_i \\ my_i \\ mz_i \end{pmatrix}$, elements in $J_{C_i}^{\text{Former Joint}} = \begin{bmatrix} xx_i & xy_i & xz_i \\ xy_i & yy_i & yz_i \\ xz_i & yz_i & zz_i \end{bmatrix}$ are moment of inertia intergrated from former joint along the link, which are different from those in $J_{C_i}^{\text{CoM}}$ matrixes.

We have

$$
\begin{aligned}
xx_i = & \iiint \left(y^2+z^2\right) \rho dxdydz \\
= & \iiint \left[(y-y_{C_i})^2+(z-z_{C_i})^2\right] \rho dxdydz \\
& + 2\iiint \left[y_{C_i} y + z_{C_i} z\right] \rho dxdydz \\
& - \iiint \left[\left(y_{C_i}\right)^2+\left(z_{C_i}\right)^2\right] \rho dxdydz \\
= & J_{xx_i} + m_i \left[\left(y_{C_i}\right)^2+\left(z_{C_i}\right)^2\right], \\
xy_i = & -\iiint xy\rho dxdydz \\
= & -\iiint \left[\left(x-x_{C_i}\right)\left(y-y_{C_i}\right)\right] \rho dxdydz \\
& - \iiint \left(x_{C_i} y + y_{C_i} x\right) \rho dxdydz \\
& + \iiint x_{C_i} y_{C_i} \rho dxdydz \\
= & -\left(J_{xy_i} + m_i x_{C_i} y_{C_i}\right).
\end{aligned}
$$

Similarly, it's easy to verify that

$$
\begin{aligned}
yy_i = J_{yy_i} + m_i \left[\left(x_{C_i}\right)^2+\left(z_{C_i}\right)^2\right], \qquad & zz_i = J_{zz_i} + m_i \left[\left(x_{C_i}\right)^2+\left(y_{C_i}\right)^2\right], \\
xz_i = -\left(J_{xz_i} + m_i x_{C_i} z_{C_i}\right), \qquad & yz_i = -\left(J_{yz_i} + m_i y_{C_i} z_{C_i}\right).
\end{aligned}
$$

Denote a vector

$$
\begin{aligned}
\vec{p}^{\,i} & \triangleq \left[xx_i, xy_i, xz_i, yy_i, yz_i, zz_i, mx_i, my_i, mz_i, m_i\right]^\top \\
& \triangleq \left[p_1^i, p_2^i, \cdots, p_{10}^i\right]^\top,
\end{aligned}
$$

to collect inertial parameters. It's easy to verify that

$$
\begin{aligned}
J_{C_i}^{\text{Former Joint}} \vec{\omega}_i &= \begin{bmatrix} xx_i & xy_i & xz_i \\ xy_i & yy_i & yz_i \\ xz_i & yz_i & zz_i \end{bmatrix} \vec{\omega}_i \\
&= \begin{bmatrix} \omega_x^i & \omega_y^i & \omega_z^i & 0 & 0 & 0 \\ 0 & \omega_x^i & 0 & \omega_y^i & \omega_z^i & 0 \\ 0 & 0 & \omega_x^i & 0 & \omega_y^i & \omega_z^i \end{bmatrix} \begin{pmatrix} xx_i \\ xy_i \\ xz_i \\ yy_i \\ yz_i \\ zz_i \end{pmatrix} \\
&\triangleq K(\vec{\omega}_i) \begin{pmatrix} xx_i \\ xy_i \\ xz_i \\ yy_i \\ yz_i \\ zz_i \end{pmatrix},
\end{aligned}
$$

Then the kinematic energy can be rewritten as

$$
\begin{aligned}
T_i &= \frac{1}{2} \vec{\omega}_i^\top J_{C_i}^{\text{Former Joint}} \vec{\omega}_i + \vec{v}_i^\top [\vec{\omega}_i] m_i \vec{r}_{C_i} + \frac{1}{2} \vec{v}_i^\top \vec{v}_i m_i \\
&= \left[\tfrac{1}{2} \vec{\omega}_i^\top K(\vec{\omega}_i), \vec{v}_i^\top [\vec{\omega}_i] R_i^0, \tfrac{1}{2} \vec{v}_i^\top \vec{v}_i \right] \vec{p}^i \\
&\triangleq \left(\tilde{T}^i \right)^\top \vec{p}^i,
\end{aligned}
$$

The potential energy follows

$$
\begin{aligned}
V_i &= -m_i \vec{g} \cdot (\vec{r}_i + \vec{r}_{Ci}) \\
&= -\vec{g}^\top \left(\vec{r}_i m_i + R_i^0 m_i \vec{r}_{Ci} \right) \\
&= \left[\vec{0}_{1\times 6}, -\vec{g}^\top R_i^0, -\vec{g}^\top \vec{r}_i \right] \vec{p}^i \\
&\triangleq \left(\tilde{V}^i \right)^\top \vec{p}^i.
\end{aligned}
$$

2. Linearized Dynamics

Then total kinematic energy T and potential energy V follow

$$
\begin{aligned}
T &= \sum_{i=1}^{n} \left(\tilde{T}^{i}\right)^{\top} \vec{p}^{i} \\
&= \left[\left(\tilde{T}^{1}\right)^{\top}, \cdots, \left(\tilde{T}^{n}\right)^{\top}\right] \vec{p} \\
&\triangleq \tilde{T}^{\top} \vec{p}, \\
V &= \sum_{i=1}^{n} \left(\tilde{V}^{i}\right)^{\top} \vec{p}^{i} \\
&= \left[\left(\tilde{V}^{1}\right)^{\top}, \cdots, \left(\tilde{V}^{n}\right)^{\top}\right] \vec{p} \\
&\triangleq \tilde{V}^{\top} \vec{p},
\end{aligned}
$$

where $\vec{p} \triangleq \left[\left(\vec{p}^{i}\right)^{\top}, \cdots, \left(\vec{p}^{n}\right)^{\top}\right]^{\top}$ contains all dynamical parameters. Noting that it's highly possible that $\vec{p}$ has zero-valued or linearly correlated elements.

With the 2nd Lagrangian equation, it can be obtained that

$$
\begin{aligned}
\vec{\tau} &= \frac{d}{dt} \frac{\partial T}{\partial \dot{\vec{q}}} - \frac{\partial T}{\partial \vec{q}} + \frac{\partial V}{\partial \vec{q}} \\
&= \left[\frac{d}{dt} \frac{\partial \tilde{T}^{\top}}{\partial \dot{\vec{q}}}\right] \vec{p} - \frac{\partial \tilde{T}^{\top}}{\partial \vec{q}} \vec{p} + \frac{\partial \tilde{V}^{\top}}{\partial \vec{q}} \vec{p} \\
&= \left[\frac{d}{dt} \frac{\partial \tilde{T}^{\top}}{\partial \dot{\vec{q}}} - \frac{\partial \tilde{T}^{\top}}{\partial \vec{q}} + \frac{\partial \tilde{V}^{\top}}{\partial \vec{q}}\right] \vec{p} \\
&\triangleq Y_{n \times 10n} \vec{p}_{10n \times 1}
\end{aligned}
$$

The above equation gives a linearized form of manipulator's dynamics. $Y = \frac{d}{dt} \frac{\partial \tilde{T}^{\top}}{\partial \dot{\vec{q}}} - \frac{\partial \tilde{T}^{\top}}{\partial \vec{q}} + \frac{\partial \tilde{V}^{\top}}{\partial \vec{q}}$ is the parameter regression matrix and $\vec{p}$ is dynamical parameter vector.

3. Simplification of Linearized Dynamics

Although we've already obtained

$$
\begin{aligned}
\vec{\tau} &= Y\vec{p}, \\
Y &= \frac{d}{dt}\frac{\partial \tilde{T}^{\top}}{\partial \dot{\vec{q}}} - \frac{\partial \tilde{T}^{\top}}{\partial \vec{q}} + \frac{\partial \tilde{V}^{\top}}{\partial \vec{q}} \\
\vec{p} &= \left[\left(\vec{p}^{i}\right)^{\top}, \cdots, \left(\vec{p}^{n}\right)^{\top}\right]^{\top},
\end{aligned}
$$

zero-valued columns and linearly correlated columns in Y may lower correctness when performing online dynamical parameter estimation. Therefore, further simplification should be done.

1. If the ith column of Y, denoted as Y_i, follows

$$Y_i \equiv \vec{0}$$

Then the corresponding ith element in $\vec{p}$ has no influence on dynamics. Therefore, the ith column of Y and ith element of $\vec{p}$ can be eliminated when $Y_i = \vec{0}$ (or $p_i = 0$).

2. If a certain column of Y is a linear correlation of other columns:

$$Y_i \equiv \beta_{i_1} Y_{i_1} + \cdots + \beta_{i_m} Y_{i_m},$$

where $\beta_{i_1}, \cdots, \beta_{i_m}$ are constants. Then it's obvious that

$$Y_i p_i \equiv Y_{i_1}\left(\beta_{i_1} p_{i_1}\right) + \cdots + Y_{i_m}\left(\beta_{i_m} p_{i_m}\right)$$

Then we may set

$$p_{i_j} \rightarrow p_{i_j} + \beta_{i_j} p_i,$$

and $p_i \rightarrow 0$.