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TIME-VARYING DELAYED FEEDBACK CONTROL FOR AN INTERNET CONGESTION CONTROL MODEL

Shu Zhang and Jian Xu*

School of Aerospace Engineering and Applied Mechanics
Tongji University, Shanghai, 200092, China

Abstract. A proportionally-fair controller with time delay is considered to control Internet congestion. The time delay is chosen to be a controllable parameter. To represent the relation between the delay and congestion analytically, the method of multiple scales is employed to obtain the periodic solution arising from the Hopf bifurcation in the congestion control model. A new control method is proposed by perturbing the delay periodically. The strength of the perturbation is predicted analytically in order that the oscillation may disappear gradually. It implies that the proved control scheme may decrease the possibility of the congestion derived from the oscillation. The proposed control scheme is verified by the numerical simulation.

1. **Introduction.** Since Jacobson put forward the concepts of congestion avoidance and control of computer networks [13], much attention was paid to the research on the Internet congestion control [7, 15, 16, 17, 20]. Floyd and Jacobson [7] employed a kind of Active Queue Management (AQM) called Random Early Detection (RED) to implement TCP congestion control for good network services. Kelly et al. [16] presented a framework to investigate the congestion control and analyzed the stability and fairness of rate control algorithm. In [15], a single proportionally-fair congestion controller with time delay was proposed and the stability condition was obtained. The other stability analysises were also available in [27, 22, 1].

In recent years, there was an increasing interest in the study of nonlinear dynamical behavior of congestion control model. In [19] and [33], the stability of the equilibrium of the model given by Kelly [15] was analyzed and center manifold reduction and normal form computation were employed to study the Hopf bifurcation qualitatively. Based on the analytical technique of nonlinear dynamics, in [3] and [29], some control methods for the Hopf bifurcation in the model from [19, 33] were proposed by introducing feedback controllers. The authors of [23, 6] investigated the Hopf bifurcation in dual models of congestion control. The other authors [32, 24, 10, 11, 9] also studied the similar problem for different versions of congestion control algorithm such as TCP-RED, TCP-REM [32, 24, 10, 11] and discrete models of congestion control [14, 21]. In [28] the stability and direction of the bifurcating periodic solution were determined by the perturbation procedure. All of these

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*Corresponding author.

researches focused on a qualitative analysis. Few papers contributed to compute periodic solutions derived from the Hopf bifurcation quantitatively [31, 30, 5, 8].

In the present paper, a proportionally-fair controller with a single source and a single link is considered as [15]

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = k\,(w\, - y(t-\tau)\,p(y(t-\tau))), \tag{1}$$

where y represents the sending rate of the source, k a positive gain parameter, w a target, and τ the time delay or round trip time (RTT) which represents the sum of propagation and queueing delays. p is an increasing function in its variable and non-negative and can be considered as the probability marked in a data packet. It should be noted that $p(.)$ is usually embedded in the router and thus controllable. Therefore, Eq. (1) is a closed loop control system with negative feedback.

The time delay may induce the oscillation. It is necessary to consider effects of the RTT in Eq. (1) on the congestion. It is easily understood that the oscillation of the system will increase the risk of congestion or deteriorate the situation of the exited congestion. Two questions arise: (1) whether the delay can induce the oscillation and (2) whether the oscillation may be controlled by utilizing the delay existed in the congestion model. This two questions provide our motivation in the present research. To answer the first question, the method of multiple scales is employed to obtain the delay-induced periodic solution analytically when the time delay is chosen to be a controllable parameter. Thus, the relation between the delay and the oscillation can be seen clearly. For the second problem, a new control method is proposed by perturbing the delay periodically. The strength of the perturbation is predicted analytically in order that the oscillation may disappear gradually. It implies that the proved control scheme may decrease the possibility of the congestion derived from the oscillation. The proposed control scheme is verified by numerical simulations.

The rest of this paper is organized as follows. In the next section, we study the equilibrium as well as its stability and the existence of the Hopf bifurcation. In Section 3, we present the procedure of multiple scales analysis. The periodic solution of Eq. (1) arising from the Hopf bifurcation will be obtained by the method of multiple scales in Section 4. In Section 5, the oscillation of the congestion control model is destroyed by means of introducing time-varying delay and further discussions on the perturbation parameters to time delay are presented. Finally, there is a conclusion in Section 6. Except for summarizing the results of the paper, we also discuss the possibility of realizing the time-varying delayed feedback control in the conclusion.

2. **Equilibrium and stability.**

In this paper, the function p in Eq. (1) has the same form as that in [19] and [26], i.e. $p(y) = \frac{\theta\sigma^2\,y}{\theta\sigma^2\,y+2\,(C-y)}$ where σ^2 denotes the variability of the traffic at the packet level and C is the capacity of the link. We assume $\theta\,\sigma^2 = 0.5$ and $C = 5$ thus $p(y) = \frac{y}{20-3\,y}$. If there is a unique equilibrium, say, y^*, then $w = y^*\,p(y^*)$ yields $y^* = \frac{1}{2}\,(-3\,w + \sqrt{w\,(80+9w)})$. Without loss of generality, letting $x = y\, - y^*$, for $w \neq 0$ and $w \neq -\frac{80}{9}$. Eq. (1) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2\,k\,x_\tau\,(b\, + x_\tau)}{3\,b\, - a\, + 6\,x_\tau}, \tag{2}$$

where $x_\tau = x(t-\tau)$, $a = 40 + 9\,w$ and $b = \sqrt{w\,(80 + 9\,w)}$. For $b \neq 0$ and $3\,b - a \neq 0$, to avoid the error caused by Taylor expansion, rewrite Eq. (2) as follows

$$\dot{x} = f(x_\tau, \dot{x}), \tag{3}$$

where

$$f(x_\tau, \dot{x}) = -\frac{2\,k\,b\,x_\tau + 2\,k\,{x_\tau}^2 - 6\,x_\tau\,\dot{x}}{a - 3\,b}. \tag{4}$$

The characteristic equation of Eq. (3) at the trivial equilibrium is expressed as

$$\lambda - C - D\,\mathrm{e}^{-\lambda\,\tau} = 0, \tag{5}$$

where $C = \frac{\partial f}{\partial x}|_{x=0}$, $D = \frac{\partial f}{\partial x_\tau}|_{x_\tau=0}$. Then one may easily obtain that $C = 0$ and $D = -\frac{2\,k\,b}{a - 3\,b}$.

The roots of the characteristic equation (5) are commonly called the eigenvalues of the equilibrium of Eq. (4). The stability of the trivial equilibrium changes when the eigenvalues of Eq. (3) are zero or purely imaginary pairs [18]. The former may lead to a static bifurcation of equilibrium points such that the number of equilibrium points changes when the bifurcation parameters vary. The latter deals with a Hopf bifurcation such that dynamical behavior of the system changes from a static stable state to a periodic motion or vice versa. Especially, if there is only one pair of purely imaginary eigenvalues given by $\pm i\,\omega$ at τ_c (or the values of other parameter) and $\lambda = 0$ is not a root of Eq. (5), then a Hopf bifurcation may occur in Eq. (3). We will concentrate on this case in the subsequent discussion. To this end, letting $\alpha = \frac{2\,b}{a-3\,b}$ and substituting $\lambda = i\,\omega$ where $\omega > 0$ into Eq. (5) yield the critical boundary of the stability zones in terms of τ and k as

$$\alpha\,k\,\cos(\omega\,\tau) + i\,(\omega - \alpha\,k\,\sin(\omega\,\tau)) = 0. \tag{6}$$

From (6), the critical boundary is determined by the sets of points (τ^j, k), where

$$\tau^j = \frac{\pi}{2\,\omega} + \frac{2\,j\,\pi}{\omega}, j = 0, 1, 2, \cdots, \omega = k\,\alpha. \tag{7}$$

The sets of points (τ^j, k), for all j describes several curves in parameters space. Denote these curves by S_j. When a pair of parameters (τ^j, k) crosses a curve S_j, the sign of the real part of an eigenvalue changes and there will be a switch of stability of the stationary state. The sign of the quantities

$$d_\tau^j = \frac{\partial Re(\lambda)}{\partial \tau}|_{(\tau^j,k)\in S_j}, d_k^j = \frac{\partial Re(\lambda)}{\partial k}|_{(\tau^j,k)\in S_j},$$

indicates the direction of change of the real parts of the eigenvalue on the imaginary axis. If $d_\tau^j \neq 0$ and $d_k^j \neq 0$, a Hopf bifurcation may occur at S_j, provided that all other eigenvalues of (5) have nonzero real parts. We point out that only $j = 0$ corresponds to the case for the Hopf bifurcation. Choosing τ to be the unique bifurcation parameter and noticing that $C = 0$, one can obtain from (5) that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = -\frac{D\,\lambda}{\mathrm{e}^{\lambda\,\tau} + D\,\tau}.$$

Let $\lambda = m + n\,i$ $(n > 0)$. Note that for $\tau = \tau^j$, one has $m = 0$ and $n = \omega$. Provided $D < 0$, we obtain

$$\begin{aligned} Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})|_{\tau^j} &= -\frac{D\,m\,(\mathrm{e}^{m\,\tau}\,\cos(n\,\tau) + D\,\tau) + D\,n\,(\mathrm{e}^{m\,\tau}\,\sin(n\,\tau))}{(\mathrm{e}^{m\,\tau}\,\cos(n\,\tau) + D\,\tau)^2 + (\mathrm{e}^{m\,\tau}\,\sin(n\,\tau))^2}|_{\tau^j} \\ &= -\frac{D\,n}{(D\,\tau)^2 + 1} > 0. \end{aligned}$$

For $\tau < \tau^0$, from the theory of functional differential equations [12], it is clear that all the eigenvalues of (3) have negative real parts. For $\tau = \tau^0$, $Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}) > 0$ implies that the equilibrium loses the stability for $\tau > \tau^0$. Thus, according to the theory of Hopf bifurcation, it occurs at $\tau = \tau^0$. Furthermore, noticing that some of the eigenvalues have positive real parts for $\tau > \tau^0$ and that $Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}) > 0$ is still valid for $j \neq 0$, we conclude that the bifurcation will not occur for $\tau = \tau^j$ where $j \neq 0$. Thus, from (7) we have

$$\tau_c = \tau^0 = \frac{\pi}{2\,\omega}, \omega = k\,\alpha, \tag{8}$$

where τ_c represents the critical value for the Hopf bifurcation.

Following this understanding one may easily obtain that system (3) undergoes a Hopf bifurcation at S_0 determined by the set of points (τ_c, k), as shown in Fig. 1, where the shadow shows the region in which the trivial equilibrium of (3) is stable.

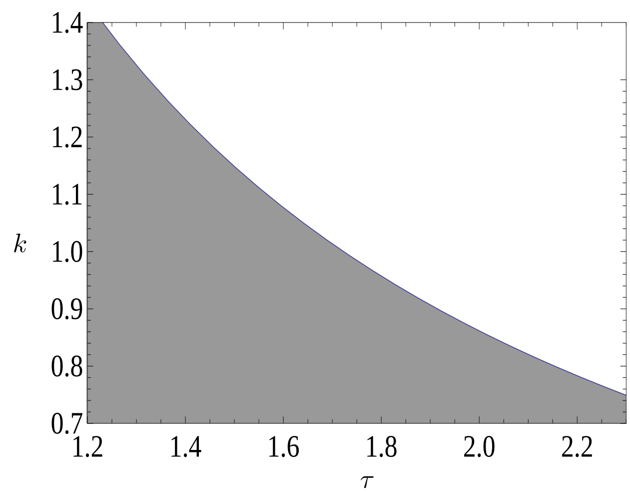

FIGURE 1. The boundary of the Hopf bifurcation of (3) on parameters space in (τ, k), where the shadow is the stable region of the trivial equilibrium of (3)

3. **Method of multiple scales for delayed differential equation.** A class of scalar differential equations with time delay can be written as

$$\dot{z}(t) = c\,z(t) + d\,z(\,t - \tau) + f(\,z(t),\,\dot{z}(t)), \tag{9}$$

where $f(.)$ represents a function with quadratic nonlinearity. As to consider the problem of the Hopf bifurcation, a perturbation to the critical value of the bifurcation parameter of Eq. (9), namely $\tau = \tau_c + \epsilon\,\tau_\epsilon$, yields

$$\dot{z}(t) = c\,z(t) + d\,z(\,t - \tau_c) + f(\,z(t),\, z(t - \tau_c - \epsilon\,\tau_\epsilon),\, z(t - \tau_c),\, \dot{z}(t),\, \epsilon),$$

where ϵ takes a small positive value. Then following the method of multiple scales for delayed differential system proposed in [5], one can expand the solution of Eq. (9) in powers of ϵ, i.e.

$$\begin{aligned} z(t) &= Z(T_0,\, T_1,\, T_2,\, \cdots) \\ &= \epsilon\, Z_1(T_0,\, T_1,\, T_2,\, \cdots) + \epsilon^2\, Z_2(T_0,\, T_1,\, T_2,\, \cdots) + \cdots, \end{aligned} \tag{10}$$

where $T_i = \epsilon^i t$, $i = 0, 1, 2, \cdots$. Substituting (10) in Eq. (9) and expanding $Z_i(t - \tau_c - \epsilon\,\tau_\epsilon, \epsilon\,(t - \tau_c - \epsilon\,\tau_\epsilon), \cdots)$ at $(T_0 - \tau_c, T_1, \cdots)$ in powers of ϵ and considering the lowest order of ϵ, we obtain

$$D_0\, Z_1(T_0, T_1, T_2, \cdots) + \omega\, Z_1(T_0 - \tau_c, T_1, T_2, \cdots) = 0, \tag{11}$$

where $D_0 = \frac{\partial}{\partial T_0}$ and ω is the frequency of the system while τ exactly locates at the bifurcation point. The solution of Eq. (11) has the form

$$Z_1(T_0, T_1, T_2, \cdots) = A_{11}\, \sin(\omega\, T_0) + B_{11}\, \cos(\omega\, T_0), \tag{12}$$

where $A_{11} = A_{11}(T_1, T_2, \cdots)$ and $B_{11} = B_{11}(T_1, T_2, \cdots)$. Henceforth we will write $A_{11}(T_1, T_2, \cdots)$ and $B_{11}(T_1, T_2, \cdots)$ as A_{11} and B_{11} for brevity, respectively. Substituting Eqs. (12) and (11) in Eq. (10), the following equation is obtained at the second order of ϵ

$$\begin{aligned} &D_0\, Z_2(T_0, T_1, T_2, \cdots) + \omega\, Z_2(T_0 - \tau_c, T_1, T_2, \cdots) + F_2 \\ &+ P_{21}\, \sin(\omega\, T_0) + Q_{21}\, \cos(\omega\, T_0) + P_{22}\, \sin(2\,\omega\, T_0) + Q_{22}\, \cos(2\,\omega\, T_0) = 0, \end{aligned} \tag{13}$$

where $P_{2i} = P_{2i}(D_1A_{11}, D_1B_{11}, A_{11}, B_{11})$, $Q_{2i} = Q_{2i}(D_1A_{11}, D_1B_{11}, A_{11}, B_{11})$, $D_1 = \frac{\partial}{\partial T_1}$, $i = 1, 2$, and $F_2 = F_2(A_{11}, B_{11})$. Note that F_2 appears due to the existence of quadratic nonlinearity in $f(.)$. To avoid the occurrence of the secular terms in the solution of Eq. (9), the harmonic terms of first order are assumed to vanish. According to this consideration a system of equations of D_1A_{11} and D_1B_{11} is given. Solving the equations one can obtain

$$D_1A_{11} = M_1(A_{11}, B_{11}), D_1B_{11} = N_1(A_{11}, B_{11}).$$

Then, for Eq. (13), it is seen that the solution has the form

$$Z_2(T_0, T_1, T_2, \cdots) = C_2 + A_{22}\, \sin(2\,\omega\, T_0) + B_{22}\, \cos(2\,\omega\, T_0). \tag{14}$$

Substituting (14) in Eq. (13) we obtain

$$A_{22} = A_{22}(A_{11}, B_{11}), B_{22} = B_{22}(A_{11}, B_{11}), C_2 = C_2(A_{11}, B_{11}). \tag{15}$$

The above procedure can be performed similarly to higher orders to obtain D_iA_{11} and D_iB_{11}, $D_i = \frac{\partial}{\partial T_i}$, $i = 1, 2, \cdots$, in the expressions of A and B. Finally, we have

$$\frac{\mathrm{d}A_{11}}{\mathrm{d}t} = \epsilon\, D_1A_{11} + \epsilon^2\, D_2A_{11} + \cdots, \quad \frac{\mathrm{d}B_{11}}{\mathrm{d}t} = \epsilon\, D_1B_{11} + \epsilon^2\, D_2B_{11} + \cdots. \tag{16}$$

By using the following transformations

$$A_{11} = R(t)\, \cos(\varphi(t)), B_{11} = R(t)\, \sin(\varphi(t)). \tag{17}$$

Eq. (16) in polar coordinates is given by

$$\begin{aligned} &\dot{R}(t) = r_1(\epsilon, \tau_\epsilon)\, R(t) + r_3(\epsilon, \tau_\epsilon)\, R(t)^3 + r_5(\epsilon, \tau_\epsilon)\, R(t)^5 + \cdots, \\ &\dot{\varphi}(t) = f_0(\epsilon, \tau_\epsilon) + f_2(\epsilon, \tau_\epsilon)\, R(t)^2 + f_4(\epsilon, \tau_\epsilon)\, R(t)^4 + \cdots. \end{aligned} \tag{18}$$

4. **Hopf bifurcation and periodic oscillation.** Following the case studied in [19], we assume that k and w are fixed in unit. Thus it is easily computed that $\alpha = 0.911581$. For the occurrence of the Hopf bifurcation, it follows from Eq. (8) that $\tau_c = \frac{\pi}{2\,\alpha} = 1.72315$ and $\omega = 0.911581$. Employing the method of multiple scales which is described above, one can assume that the periodic solution has the form

$$\begin{aligned} x(t) =& \epsilon^2\, C_2 + \epsilon\, (A_{11}\, \sin(\omega\, t) + B_{11}\, \cos(\omega\, t)) \\ &+ \epsilon^2\, (A_{22}\, \sin(2\,\omega\, t) + B_{22}\, \cos(2\,\omega\, t)) + \cdots, \end{aligned}$$

where each term has the same meaning as in Section 3. Using (15), (16), and (17) and letting $\tau = \tau_c + \epsilon\,\tau_\epsilon$, the normal form equations (or the amplitude-frequency response) are given by

$$\begin{aligned} \dot R(t) &= r_1\,R(t) + r_3\,R(t)^3 + r_5\,R(t)^5, \\ \dot\varphi(t) &= f_0\, + f_2\,R(t)^2 + f_4\,R(t)^4, \end{aligned} \tag{19}$$

where

$$\begin{aligned} r_1 &= 0.239655\,\epsilon\tau_\epsilon\, - 0.205538\,\epsilon^2\,\tau_\epsilon^2\, + 0.144392\,\epsilon^3\,\tau_\epsilon^3\, - 0.0945551\,\epsilon^4\,\tau_\epsilon^4, \\ r_3 &= -0.0274087\,\epsilon^2\, + 0.0335403\,\epsilon^3\,\tau_\epsilon\, - 0.0260517\,\epsilon^4\,\tau_\epsilon^2, \\ r_5 &= 0.0000237954\,\epsilon^4, \end{aligned} \tag{20}$$

and

$$\begin{aligned} f_0 &= -0.376449\,\epsilon\,\tau_\epsilon\, + 0.150621\,\epsilon^2\,\tau_\epsilon^2\, - 0.0522287\,\epsilon^3\,\tau_\epsilon^3\, + 0.0107349\,\epsilon^4\,\tau_\epsilon^4, \\ f_2 &= -0.0356391\,\epsilon^2\, + 0.0641506\,\epsilon^3\,\tau_\epsilon\, - 0.0640275\,\epsilon^4\,\tau_\epsilon^2, \\ f_4 &= -0.00152543\,\epsilon^4. \end{aligned} \tag{21}$$

When a periodic solution bifurcates from the bifurcation point one can obtain a stationary solution of $R(t)$ by letting $\dot R(t) = 0$, which yields

$$R_{st} = 0.5\,\sqrt{2303.7\, - 2819.06\,\tau_\Delta\, + 2189.65\,\tau_\Delta^2 - 2204.12\,\Delta)}, \tag{22}$$

where $\tau_\Delta = \epsilon\,\tau_\epsilon$, $\Delta = \sqrt{(0.933131\, - 1.33365\,\tau_\Delta\, + \tau_\Delta^2)\,(1.17069 - 1.22754\,\tau_\Delta\, + \tau_\Delta^2)}$. Substituting (22) into (19) we have

$$\begin{aligned} \varphi(t) =& \varphi_0\, + (-1032.47 + 987.835\,\Delta\, + \tau_\Delta\,(2553.69 - 1220.14\,\Delta) \\ &+ \tau_\Delta^2\,(-3553.65 + 955.542\,\Delta) + 2443.47\,\tau_\Delta^3 - 955.32\,\tau_\Delta^4)\,t, \end{aligned}$$

where φ_0 is the initial phase. Thus an approximate solution of Eq. (1) to second order is given by

$$\begin{aligned} x(t) =& \epsilon\,R_{st}\,\sin(\omega\,t\, + \varphi(t)) \\ &+ \epsilon^2\,(\rho_0\, + 0.0396\,R_{st}^2\,(\cos(2\,\omega\,t + 2\,\varphi(t)) - 2\,\sin(2\,\omega\,t + 2\,\varphi(t)))), \end{aligned}$$

where $\rho_0 = -0.198\,R_{st}^2$. Noting that $y_{\substack{max\\min}} = 3.21699 + \epsilon^2\,\rho_0\, \pm \epsilon\,R_{st}$, one obtains the bifurcation diagram in which the values of $y_{\substack{max\\min}}$ are plotted while τ varies near the critical value τ_c. Fig. 2 shows graphical comparisons of time history, phase trajectories and bifurcation curves between solutions obtained by MMS and numerical simulation, respectively. From Fig. 2 it is seen that the analytical results are in good agree with the numerical simulations.

5. **Controlling oscillation by perturbing time delay periodically.** It is seen that the delay existed in (1) can induce the oscillation as mentioned in above section when $\tau = \tau_c + \epsilon\tau_\epsilon$, where $\epsilon\tau_\epsilon > 0$. In this section, we try to provide a control strategy to control such oscillation by a time-varying perturbation to the delay [25]. To this end, we note $\tau_s = \tau_c + \epsilon\tau_\epsilon$ for $\epsilon\tau_\epsilon > 0$. Perturbing τ_s by a periodic excitation yields that the delay can be expressed as

$$\tau(t) = \tau_s\, + B\,\sin(\Omega\,t), \tag{23}$$

where Ω is a frequency of the perturbation, as shown in Fig. 3. For any values of τ_s, we try to predict an variable range of B analytically so that the oscillation can be controlled.

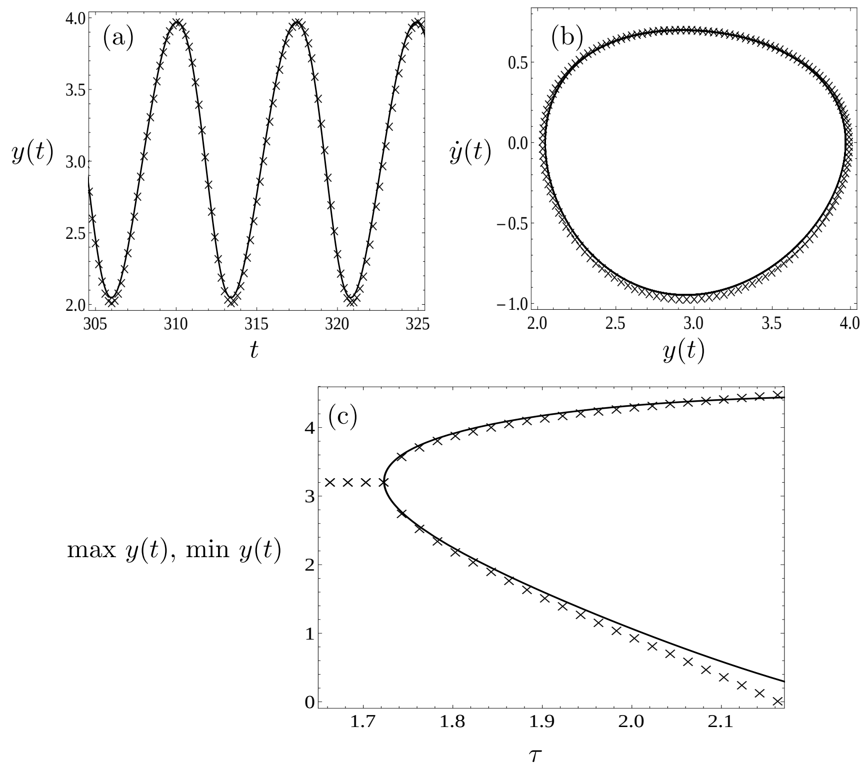

FIGURE 2. Comparisons between the results obtained by MMS and numerical simulation in (a) time history; (b) phase trajectory; and (c) bifurcation curve where (a) and (b) are plotted for $\epsilon\,\tau_\epsilon = 0.1$. Solid lines for analytical method, $\times$ for numerical simulation

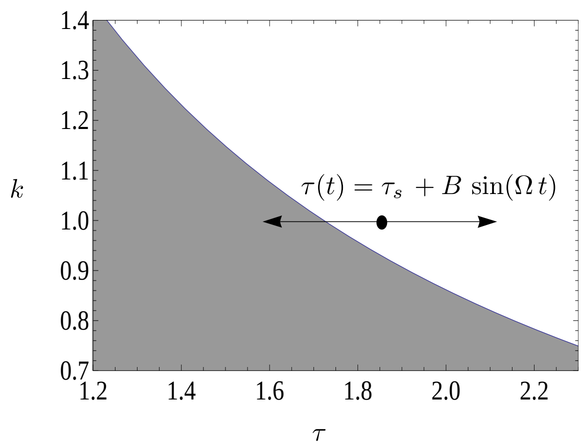

FIGURE 3. Traversing the boundary of the stable and unstable regions of the equilibrium by perturbing τ_s periodically

It follows from Eq. (23) that Eq. (1) becomes

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = k\,(w - y(t-\tau(t))\,p(y(t-\tau(t)))). \tag{24}$$

For time-varying delayed differential system, some qualitative studies are available [2, 4]. The present paper will follow the idea proposed in [4]. Since $B\sin(\Omega t)$ can be considered as a perturbation, the MMS can be also applied to obtain the amplitude-frequency response, given by

$$\begin{aligned}\dot{R}(t) &= r_1(t)\,R(t) + r_3(t)\,R(t)^3 + r_5(t)\,R(t)^5,\\ \dot{\varphi}(t) &= f_0(t) + f_2(t)\,R(t)^2 + f_4(t)\,R(t)^4.\end{aligned} \tag{25}$$

Notice that one has to rescale $\epsilon\,R(t) \to R(t)$ to obtain Eq. (25). We will first give an analytical expression of the solution of Eq. (25). The dynamical behavior of the system for different values of τ_s and B are discussed in Subsections 2 and 3.

5.1. **Analytical solution of time-varying delayed system (25).** Assuming that all the physical parameters and the function p are the same as in Section 4 and considering $B\sin(\Omega t)$ as a perturbation to τ_s, from the method of multiple scales, we have

$$\begin{aligned} r_1(t) =& g_0 + g_1\sin(\Omega\,t) + g_2\cos(\Omega\,t) + g_3\sin(2\,\Omega\,t) + g_4\cos(2\,\Omega\,t)\\ &+ g_5\sin(3\,\Omega\,t) + g_6\cos(3\,\Omega\,t) + g_7\sin(4\,\Omega\,t) + g_8\cos(4\,\Omega\,t),\\ r_3(t) =& h_0 + h_1\sin(\Omega\,t) + h_2\cos(\Omega\,t) + h_3\sin(2\,\Omega\,t) + h_4\cos(2\,\Omega\,t),\\ r_5(t) =& i_0,\end{aligned} \tag{26}$$

and

$$\begin{aligned} f_0(t) =& m_0 + m_1\sin(\Omega\,t) + m_2\cos(\Omega\,t) + m_3\sin(2\,\Omega\,t) + m_4\cos(2\,\Omega\,t)\\ &+ m_5\sin(3\,\Omega\,t) + m_6\cos(3\,\Omega\,t) + m_7\sin(4\,\Omega\,t) + m_8\cos(4\,\Omega\,t),\\ f_2(t) =& n_0 + h_1\sin(\Omega\,t) + n_2\cos(\Omega\,t) + n_3\sin(2\,\Omega\,t) + n_4\cos(2\,\Omega\,t),\\ f_4(t) =& l_0,\end{aligned} \tag{27}$$

where

$$\begin{aligned} g_0 =& -0.1B^2 - 0.035B^4 + 0.24(\tau_s-\tau_c) + 0.22B^2(\tau_s-\tau_c) - 0.21(\tau_s-\tau_c)^2\\ &- 0.28B^2(\tau_s-\tau_c)^2 + 0.14(\tau_s-\tau_c)^3 - 0.1(\tau_s-\tau_c)^4 + 0.07B^2\Omega^2,\end{aligned} \tag{28}$$

and

$$
\begin{aligned}
g_1 =& 0.24B + 0.11B^3 - 0.4B(\tau_s - \tau_c) - 0.28B^3(\tau_s - \tau_c) + 0.43B(\tau_s - \tau_c)^2 \\
& - 0.38B(\tau_s - \tau_c)^3 + 0.17B\Omega^2 + 0.17B(\tau_s - \tau_c)\Omega^2, \\
g_2 =& 0.29B\Omega + 0.0001B^3\Omega + 0.13B(\tau_s - \tau_c)\Omega + 0.0005B(\tau_s - \tau_c)^2\Omega \\
& - 0.07B\Omega^3, \\
g_3 =& 0.06B^2\Omega + 0.0005B^2(\tau_s - \tau_c)\Omega, \\
g_4 =& 0.1B^2 + 0.05B^4 - 0.22B^2(\tau_s - \tau_c) + 0.28B^2(\tau_s - \tau_c)^2 - 0.1B^2\Omega^2, \\
g_5 =& -0.04B^3 + 0.1B^3(\tau_s - \tau_c), \\
g_6 =& -0.0001B^3\Omega, \\
g_7 =& 0, \\
g_8 =& -0.012B^4, \\
h_0 =& -0.027 - 0.046B^2 + 0.068(\tau_s - \tau_c) - 0.092(\tau_s - \tau_c)^2, \\
h_1 =& 0.068B - 0.184B(\tau_s - \tau_c), \\
h_2 =& 0.005B\Omega, \\
h_3 =& 0, \\
h_4 =& 0.046B^2, \\
i_0 =& -0.004,
\end{aligned}
$$

and

$$
\begin{aligned}
m_0 =& 0.08B^2 + 0.004B^4 - 0.38(\tau_s - \tau_c) - 0.08B^2(\tau_s - \tau_c) + 0.15(\tau_s - \tau_c)^2 \\
& + 0.03B^2(\tau_s - \tau_c)^2 - 0.05(\tau_s - \tau_c)^3 + 0.01(\tau_s - \tau_c)^4 - 0.05B^2\Omega^2, \\
m_1 =& -0.376B + 0.3B(\tau_s - \tau_c) + 0.03B^3(\tau_s - \tau_c) - 0.16B(\tau_s - \tau_c)^2 \\
& - 0.039B^3 + 0.04B(\tau_s - \tau_c)^3 - 0.08B\Omega^2 - 0.11B(\tau_s - \tau_c)\Omega^2, \\
m_2 =& -0.14B\Omega - 0.0005B^3\Omega - 0.11B(\tau_s - \tau_c)\Omega - 0.002B(\tau_s - \tau_c)^2\Omega \\
& + 0.03B\Omega^3, \\
m_3 =& -0.06B^2\Omega - 0.002B^2(\tau_s - \tau_c)\Omega, \\
m_4 =& B^2(-0.075 - 0.005B^2 + 0.08(\tau_s - \tau_c) - 0.03(\tau_s - \tau_c)^2 + 0.06\Omega^2), \\
m_5 =& 0.01B^3 - 0.01B^3(\tau_s - \tau_c), \\
m_6 =& 0.0005B^3\Omega, \\
m_7 =& 0, \\
m_8 =& 0.001B^4, \\
n_0 =& -0.036 - 0.057B^2 + 0.086(\tau_s - \tau_c) - 0.113(\tau_s - \tau_c)^2, \\
n_1 =& 0.086B - 0.227B(\tau_s - \tau_c), \\
n_2 =& -0.005B\Omega, \\
n_3 =& 0, \\
n_4 =& 0.057B^2, \\
l_0 =& -0.004.
\end{aligned}
$$

Notice that the first equation of Eq. (25) is self-consistent and the solution of the second equation of Eq. (25) depends on the solution of the first one. Therefore we are concerned only with the first equation of (25), namely,

$$\dot{R}(t) = r_1(t)\,R(t) + r_3(t)\,R(t)^3 + r_5(t)\,R(t)^5 \tag{29}$$

It can be proved that there is no obvious difference for the solution of Eq. (29) whether we consider the term $R(t)^5$ or not as long as the value of the coefficient of $R(t)^5$ is very small. For simplicity, we neglect the term $R(t)^5$ in the right hand side of Eq. (25) for $r_5(t)$ is close to zero. Then Eq. (29) is reduced to

$$\dot{R}(t) = r_1(t)\,R(t) + r_3(t)\,R(t)^3 \tag{30}$$

After that, we have the following theorem:

Theorem 5.1. *The analytical solution of non-autonomous equation (30) can be solved explicitly, i.e.,*

$$R(t) = C\,\mathrm{e}^{\int_0^t r_1(s)\,ds} \Big/ \sqrt{1 - 2\,C^2 \int_0^t r_3(s_2)\,\mathrm{e}^{2\int_0^{s_2} r_1(s_1)\,ds_1}\,ds_2} \tag{31}$$

where C appears to meet the initial condition.

Proof of Theorem 5.1. Notice that Eq. (30) is a Bernoulli equation. □

5.2. **Discussion on the oscillation control for system (24).** In this part, one sufficient condition for the decay of the periodic solution of Eq. (30) is obtained. Note that it is in correspondence with the decay of bursting-like oscillation of the original time-varying delayed system (24). We have one theorem as follows:

Theorem 5.2. *A sufficient condition for the decay of the periodic solution of Eq. (30) is that $g_0 < 0$. The meaning of g_0 is referred to Subsection 5.1, Eq. (28).*

Proof of Theorem 5.2. If the initial condition is set to be positive, namely, $C > 0$ in Eq. (31), then this equation can be rewritten as

$$R(t) = 1 \Big/ \sqrt{C^{-2}\,\mathrm{e}^{-2\int_0^t r_1(s)\,ds} - 2\,\mathrm{e}^{-2\int_0^t r_1(s)\,ds} \int_0^t r_3(s_2)\,\mathrm{e}^{2\int_0^{s_2} r_1(s_1)\,ds_1}\,ds_2}$$

Let $\widetilde{C} = C^{-2}$. By Fourier series expansions and Eq. (26), it is obtained that

$$\widetilde{C}\,\mathrm{e}^{-2\int_0^t r_1(s)\,ds} - 2\,\mathrm{e}^{-2\int_0^t r_1(s)\,ds} \int_0^t r_3(s_2)\,\mathrm{e}^{2\int_0^{s_2} r_1(s_1)\,ds_1}\,ds_2 = \widetilde{C}\,\mathrm{e}^{-2\,g_0\,t}\,G(t) + H(t) \tag{32}$$

where $G(t)$ and $H(t)$ are periodic functions with frequency Ω. Clearly that $\widetilde{C}$ is positive and the sign of g_0 determines the long time behavior of $u(t) = \widetilde{C}\,\mathrm{e}^{-2\,g_0\,t}\,G(t) + H(t)$. If $g_0 > 0$, $\widetilde{C}\,\mathrm{e}^{-2\,g_0\,t}\,G(t)$ will die out exponentially fast then the long time behavior of $u(t)$ is determined by $H(t)$ and thus $R(t)$ performs a periodic motion. Clearly that the result remains the same for the case $g_0 = 0$. If $g_0 < 0$, $u(t)$ increases exponentially then $R(t) = 1/\sqrt{u(t)}$ will move towards zero at an exponential speed. That finishes the proof. □

It follows from Theorem 5.2 that the oscillation vanishes in the region bounded by $g_0 = g_0(B, (\tau_s - \tau_c), \Omega) < 0$ on $(B, (\tau_s - \tau_c))$ for a fixed value of Ω (say 0.02), as shown in Fig. 4. If a critical boundary of $(B, (\tau_s - \tau_c))$ is determined by solving $g_0 = 0$, then one may compare the analytical prediction with the numerical simulation.

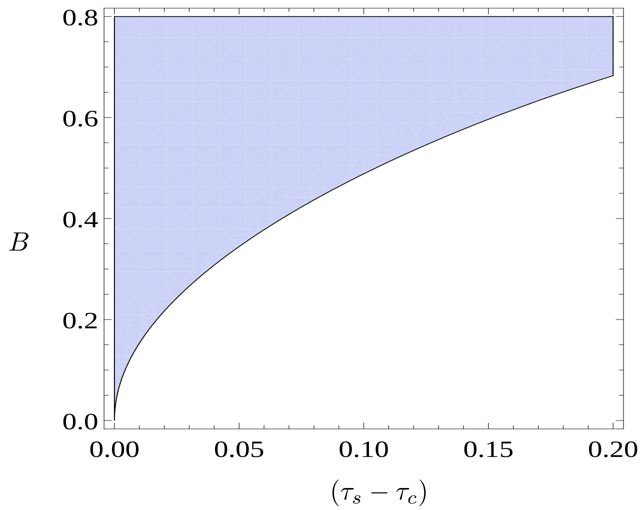

FIGURE 4. The region for the oscillation disappearing on the plane of $(B, (\tau_s - \tau_c))$ marked in shadow, where $g_0(B, (\tau_s - \tau_c), 0.02) < 0$

For example, for $(\tau_s - \tau_c) = 0.05$, the critical value of B is solved to be 0.3444 analytically in term of $g_0((\tau_s - \tau_c), B, 0.02) = 0$. To verify the analytical prediction, we give the numerical results respectively for $B = 0.325$ and $B = 0.365$, as shown in Figs. 5(a) and (b). It is seen from Fig. 5(a) that the oscillation exists in Eq. (24) but from Fig. 5(b) it vanishes. This suggests that the analytical predication is valid. We may verify the other cases in a similar way, as shown in Figs. 6 and 7, where the critical values of B are solved to be 0.4885 and 0.5966 for $(\tau_s - \tau_c) = 0.1$ and $(\tau_s - \tau_c) = 0.15$, respectively.

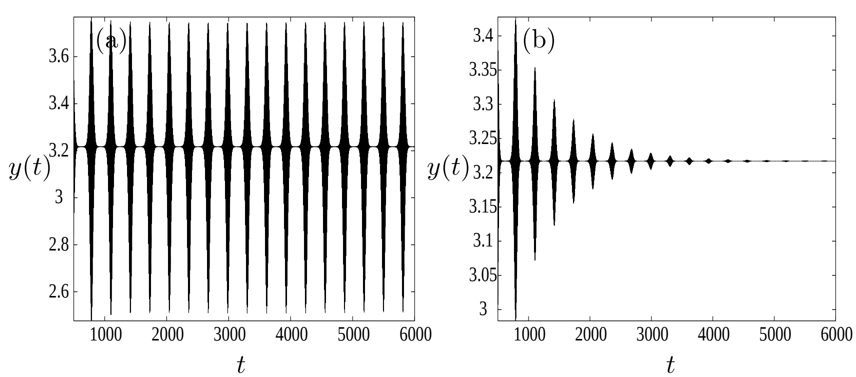

FIGURE 5. Time history for $(\tau_s - \tau_c) = 0.05$, $\Omega = 0.02$ and (a) $B = 0.325$, (b) $B = 0.365$

Now one can conclude that it is possible to destroy the oscillation of congestion control system (1) by the periodic perturbation to the time delay if this operation is compatible with the working circumstances of the real network system.

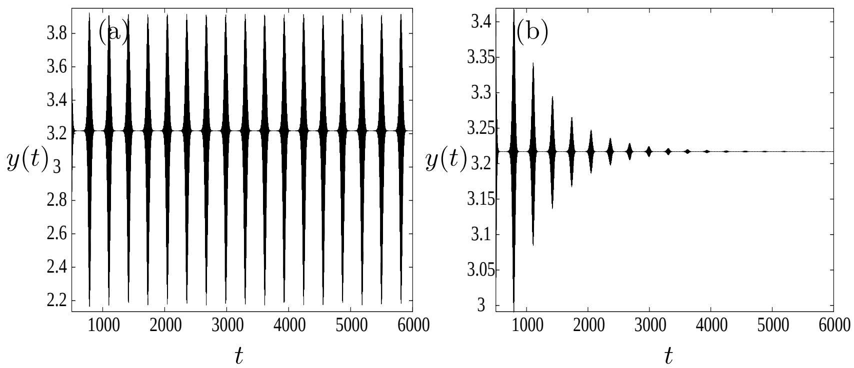

FIGURE 6. Time history for $(\tau_s - \tau_c) = 0.1$, $\Omega = 0.02$ and (a) $B = 0.47$, (b) $B = 0.51$.

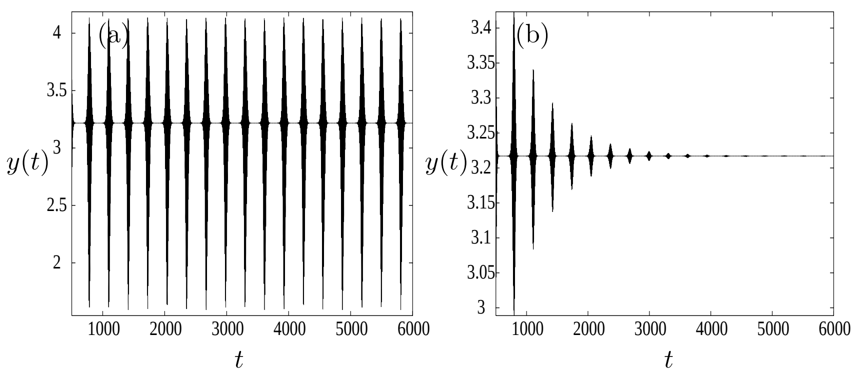

FIGURE 7. Time history for $(\tau_s - \tau_c) = 0.15$, $\Omega = 0.02$ and (a) $B = 0.575$, (b) $B = 0.625$

5.3. **Effects of perturbation parameters on the speed of oscillation decay.** It is seen from Figs. 5(b), 6(b) and 7(b) that the speed of the oscillation decay in Eq. (24) is related to perturbation parameters $\tau_s - \tau_c$ and B. In this subsection, we will observe such effect analytically with $\tau_s - \tau_c$ and B varying. It follows from Eqs. (31) and (32), one has

$$R(t) = \frac{1}{\sqrt{\widetilde{C}\,\mathrm{e}^{-2\,g_0\,t}\,G(t) + H(t)}}, \tag{33}$$

where g_0 is a function of $\tau_s - \tau_c$ and B, given by Eq. (28). It is seen from Eq. (33) that the oscillation decays for $g_0 < 0$ and the speed of the decay is determined by g_0. Thus, for $\widetilde{C} > 0$, g_0 can be used to estimate the exponential index of the amplitude decay of the oscillation, as shown in Fig. 8, where $\Omega = 0.05$. For the small values of B and $(\tau_s - \tau_c)$, one has

$$\begin{aligned}\frac{\mathrm{d}g_0}{\mathrm{d}B} =& -0.205538B - 0.141833B^3 + 0.142461B\Omega^2 + 0.433175B(\tau_s - \tau_c) \\ & - 0.567331B(\tau_s - \tau_c)^2 < 0\end{aligned} \tag{34}$$

and

$$\begin{aligned}\frac{\mathrm{d}g_0}{\mathrm{d}(\tau_s-\tau_c)} =& 0.239655 + 0.216587B^2 - 0.4111(\tau_s-\tau_c) + 0.433175(\tau_s-\tau_c)^2 \\ & - 0.567331B^2(\tau_s-\tau_c) - 0.378221(\tau_s-\tau_c)^3 > 0\end{aligned} \tag{35}$$

It follows from Eqs. (34) and (35) that g_0 decreases with B increasing and $(\tau_s - \tau_c)$ decreasing. Such analytical prediction is verified by the result from numerical simulation as shown in Figs. 9 and 10.

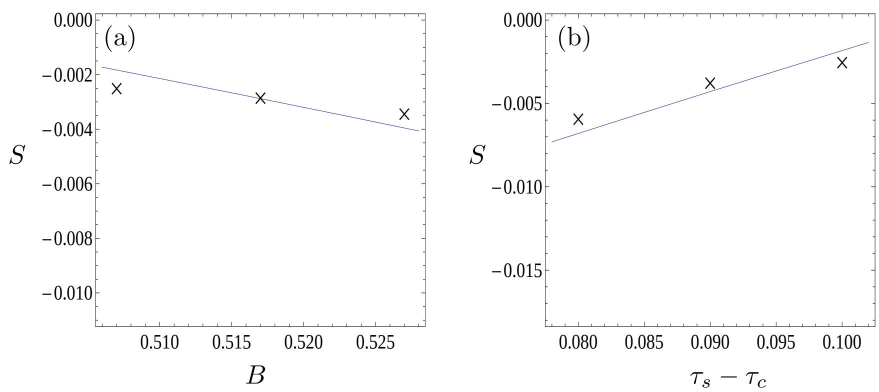

FIGURE 8. Comparisons in the exponential index of oscillation decay (denoted by S) between numerical simulations and its analytical predictions, for (a) $(\tau_s - \tau_c) = 0.1$ and (b) $B = 0.507$. The analytical predictions marked in solid lines and the numerical simulations in $\times$

6. **Conclusions.** A proportionally-fair controller with time delay is considered to control Internet congestion. The time delay is chosen to be a controllable parameter. To represent the relation between the delay and congestion analytically, the method of multiple scales is employed to obtain the periodic solution arising from the Hopf bifurcation in the congestion control model. A new control method is proposed by perturbing the delay periodically. For this case, the MMS is employed to predict the relation between the strength of the perturbation and the speed of the oscillation decay. It is seen that the proved control scheme may control not only the oscillation but also the speed of the oscillation decay. These results are shown both in analytical prediction and numerical simulation. This paper provides a method to decrease the risk of the congestion.

For Internet network, the data sending rate may be oscillatory. This is often induced by RTT. Based on our results, such phenomenon may be controlled by changing the RTT periodically. Usually, RTT is composed of two parts, namely, propagation delay for data transmission in the link and queueing delay for data processing in the router. Changing the RTT periodically implies changing the queueing delay in the router. Thus, the control strategy provided in this paper can be realized by coding the algorithm in the router.

It should be also noted that the provided control strategy will work well if the congestion occurs in local area network (LAN) since the propagation delay is relatively small and the factor of the queueing delay in RTT is large. For example, the

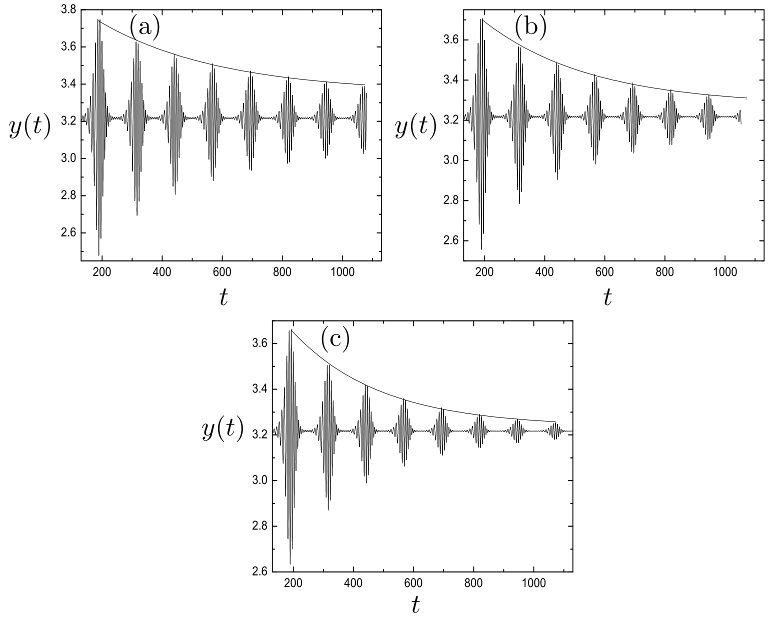

FIGURE 9. Time history and the fitting curve of the maximum value in each burst of system (24) for $(\tau_s - \tau_c) = 0.1$ and (a) $B = 0.507$; (b) $B = 0.517$; (c) $B = 0.527$, where each fitting curve is approximately (a) $3.35 + 0.6154\,\mathrm{e}^{-t/406.134}$; (b) $3.275 + 0.7285\,\mathrm{e}^{-t/356.242}$; (c) $3.2365 + 0.8133\,\mathrm{e}^{-t/294.74}$

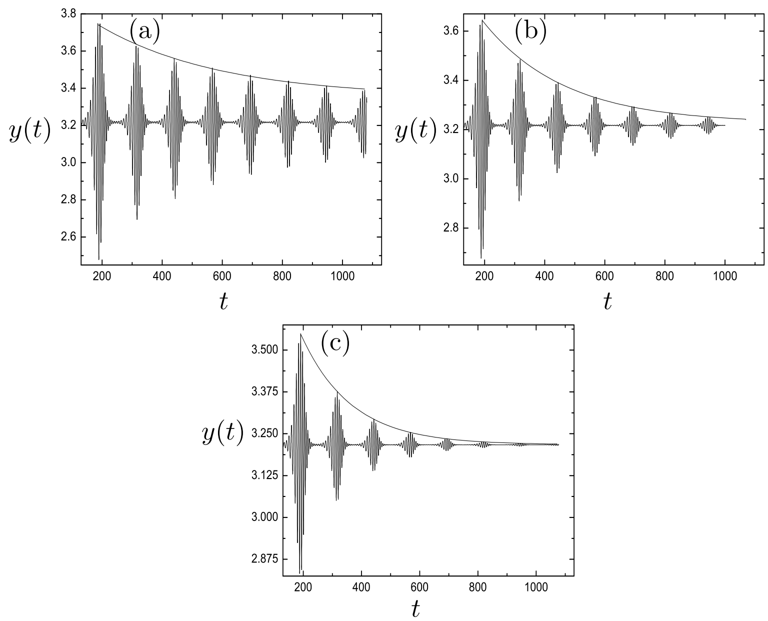

FIGURE 10. Time history and the fitting curve of the maximum value in each burst of system (24) for $B = 0.507$ and (a) $(\tau_s - \tau_c) = 0.1$; (b) $(\tau_s - \tau_c) = 0.09$; (c) $(\tau_s - \tau_c) = 0.08$, where each fitting curve is approximately (a) $3.35 + 0.6154\,\mathrm{e}^{-t/406.134}$; (b) $3.2258 + 0.8475\,\mathrm{e}^{-t/271.1158}$; (c) $3.2174 + 1.0089\,\mathrm{e}^{-t/171.0625}$

factor of the queueing delay will increase several times over the common level if the congestion has already occurred in a key router or there are always excessive data packets waiting to be processed in the buffer of the router.

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E-mail address: zhsh886@yahoo.com.cn
E-mail address: xujian@tongji.edu.cn