2021 European Control Conference (ECC)
June 29 - July 2, 2021. Rotterdam, Netherlands

Switched nonlinear systems in the Koopman operator framework: Toward a Lie-algebraic condition for uniform stability

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Abstract— **In this paper, we exploit the Koopman operator framework to provide a novel Lie-algebraic sufficient condition of uniform stability of switched nonlinear systems. First, a constructive proof of the existence of a common Lyapunov function is presented, which is based on a geometric argument using the notion of invariant maximal flag. Then we exploit this result to infer uniform stability when the Koopman operators related to the subsystems have a common invariant subspace and are associated to solvable Lie algebras. In the general case (when no invariant subspace is identified), a numerical scheme is provided to estimate a stability region in which a given nonlinear switched system is uniformly stable.**

I. INTRODUCTION

Global stability analysis of nonlinear dynamical systems is a challenging problem, and it is even so in the context of switched systems. A switched system is a collection of subsystems that alternate according to a given commutation law and, in particular, it is said to be globally uniformly asymptotically stable (GUAS) if it is globally asymptotically stable under any commutation law. Uniform stability was studied in the linear case in [7], where it was shown that a switched linear system is GUAS if the matrices describing the subsystems are stable and generate a solvable Lie algebra. In [8], an open problem was proposed for the nonlinear case, suggesting that uniform stability could be characterized in terms of conditions on the Lie algebra generated by the subsystems vector fields. This problem was partially solved in [10], where it is proved that unifom stability holds for a commuting algebra of vector fields. A common Lyapunov function (CLF) for this case was constructed in [19] and [20]. In [18], it is shown that uniform stability is guaranteed for a pair of vector fields which generate a third-order nilpotent Lie algebra. And in [12], the authors proved uniform stability for particular r-order nilpotent Lie algebras. In particular, this result solves the problem for second-order nilpotent Lie algebras. However, it is noticeable that the conjectured connection between the more general solvability property of Lie algebras and uniform stability of switched systems has not been established in the nonlinear case.

In this paper, we provide preliminary steps toward a novel solution to the uniform stability problem, which is based on solvability properties. Our key idea is to exploit the Koopman operator approach describing a dynamical system through the evolution of an observable defined on the state space. In the continuous-time setting, the infinitesimal generator of the semigroup of Koopman operators leads to a linear (but infinite-dimensional) representation of the system. This approach provides a global linearization of the system that can be used to study global properties of nonlinear systems [5]. For example, as shown in [13], there is a strong connection between global stability properties of the nonlinear system and linear stability properties of the Koopman operator. We leverage this connection in the context of switched nonlinear systems. We first present a novel proof of the existence of a common Lyapunov function (CLF) for switched linear systems that have an invariant maximal flag. Secondly, we exploit this result in the Koopman operator framework, showing that switched nonlinear systems are GUAS if the associated Koopman operators have a common invariant finite-dimensional subspace. Finally, when no invariant subspace can be identified for the Koopman operators, the construction of the CLF allows us to derive a numerical method to approximate a stability region in which a given switched system is GUAS.

The rest of the paper is organized as follows. In Section II, we present some preliminary notions on uniform stability of switched nonlinear systems. Section III provides an introduction to the Koopman operator framework and to its use in the context of switched systems. Section IV contains a geometric constructive proof of the existence of a CLF based on the notion of invariant maximal flag. In Section V, we extend the previous result to switched nonlinear systems through the Koopman operator framework. Finally, concluding remarks and perspectives are given in Section VI.

II. PRELIMINARIES: STABILITY OF SWITCHED SYSTEMS

In this Section, we introduce some preliminary notions and results on stability theory of switched systems. We will focus on the uniform asymptotic stability property and the existence of a common Lyapunov function.

Definition 1 (Switched systems): A *switched system* is a (finite) set of subsystems

$$\{\dot{x} = F_i(x),\, x \in X \subset \mathbb{R}^n\}_{i=1}^{m} \tag{1}$$

associated with a commutation law $\sigma : \mathbb{R}^+ \to \{1, \cdots, m\}$ indicating which subsystem is activated at time t.
The vector fields F_i are assumed to be continuously differentiable and the commutation law σ is a piecewise constant function with a finite number of discontinuities on every bounded time interval [9].

A. Uniform stability

Suppose that all subsystems in the family (1) share a common asymptotically stable equilibrum point x^* (i.e.

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$F_i(x^*) = 0$ for all $i = 1, \ldots, m$). It is well known that even though all subsytems are individually stable, the whole switched system (1) can be unstable with a specific switching law. An important problem is to determine whether the equilibrium is stable under the action of the switched system for *any* commutation law σ, in which case the equilibrium is said to be uniformly stable.

Definition 2 (Uniform stability): Assume that $F_i(x^*) = 0$ for all $i = 1, \ldots, m$. The equilibrium x^* is

- *uniformly asymptotically stable* (UAS) if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$\|x(0) - x^*\| \leq \delta \Rightarrow \|x(t) - x^*\| \leq \epsilon \, \forall t > 0, \, \forall \sigma$$

$$\text{and} \quad \|x(0) - x^*\| \leq \delta \Rightarrow \lim_{t \to \infty} x(t) = x^* \; \forall \sigma;$$

- *globally uniformly asymptotically stable* (GUAS) on $D \subseteq \mathbb{R}^n$ if it is UAS and

$$x(0) \in D \Rightarrow \lim_{t \to \infty} x(t) = x^* \; \forall \sigma.$$

- *globally uniformly exponentially stable* (GUES) on $D \subseteq \mathbb{R}^n$ if $\exists \beta, \lambda > 0$ such that

$$x(0) \in D \Rightarrow \|x(t) - x^*\| \leq \beta \|x(0) - x^*\| e^{-\lambda t} \; \forall t > 0, \forall \sigma.$$

Uniform asymptotic stability can be studied through the notion of common Lyapunov function.

Definition 3 (Common Lyapunov function (CLF) [9]): A positive $\mathcal{C}^1$-function $V : \mathbb{R}^n \to \mathbb{R}$ is a *common Lyapunov function* on $D \subseteq \mathbb{R}^n$ for the family of subsystems (1) if

$$\nabla V \cdot F_i(x) < 0 \quad \forall x \in D \setminus \{x^*\}, \quad \forall i = 1, \ldots, m.$$

It follows from classic stability theory that the existence of a CLF implies uniform asymptotic stability. Moreover, for switched systems (1) with finite number of subsystems, a converse Lyapunov result holds ([9], [11]) so that the existence of a CLF is a necessary and sufficient condition for uniform asymptotic stability.

Theorem 1 ([11]): Suppose that $D \subseteq \mathbb{R}^n$ is compact and forward-invariant with respect to the flow induced by the subsystems (1). The switched system (1) is UAS if and only if all subsystems share a CLF on D.

B. Lie-algebraic conditions

In the case of switched linear systems $\{\dot{x} = A_i x\}_{i=1}^m$, Lie algebraic conditions can be used to guarantee the existence of a CLF (and therefore uniform asymptotic stability).

Definition 4 (Solvable Lie algebra): A Lie algebra $\mathfrak{g}$, with Lie bracket $[.,.]$, is said to be *solvable* if there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, where $\{\mathfrak{g}^j\}_{j \in \mathbb{N}}$ is a descendant sequence of ideals defined by $\mathfrak{g}^1 := \mathfrak{g}$, $\mathfrak{g}^{j+1} := \left[\mathfrak{g}^j, \mathfrak{g}^j\right]$.

Now denote by $\mathfrak{g} = Span\{A_i\}_{Lie}$ the Lie algebra generated by A_i and associated with the Lie bracket $[A_i, A_j] = A_i A_j - A_j A_i$. A general Lie-algebraic criteria of uniform asymptotic stability for switched linear systems is given in the following theorem.

Theorem 2 ([7]): If all matrices A_i, $i = 1, \cdots, m$, are stable and if the Lie algebra $\mathfrak{g}$ is solvable, then the switched linear system $\{\dot{x} = A_i x\}_{i=1}^m$ is GUES.

The proof of Theorem 2 relies on the simultaneous triangularization of the matrices A_i, a property which allows to build a CLF. This result is summarized in the celebrated Lie's theorem.

Theorem 3 (Lie's theorem [4]): Let X be a nonzero n-complex vector space, and $\mathfrak{g}$ be a solvable Lie subalgebra of the Lie algebra of $n \times n$ complex matrices. Then X has a basis $(v_1, \ldots, v_n)$ with respect to which every element of $\mathfrak{g}$ has an upper triangular form.

The property of simultaneous triangularization is in fact equivalent to the existence of a common invariant flag.

Definition 5 (Invariant flag): An *invariant maximal flag* for the matrices $\{A_i\}_{i=1}^m$ is a set of subspaces $\{S_j\}_{j=1}^n$ such that $A_i S_j \subset S_j$, $\dim(S_j) = j$ and $S_j \subsetneq S_{j+1}$.

The basis $(v_1, \cdots, v_n)$ which triangularizes simultaneously the matrices A_i in Lie's theorem allows to construct the invariant maximal flag by setting $S_j = \text{span}\{v_1, \cdots, v_j\}$.

In the context of switched nonlinear systems, it has been conjectured in [8] that conditions on the Lie algebra generated by the subsystems vector fields could be used to bring out uniform stability. This problem has been solved partially in [18] for third-order nilpotent Lie algebras and in [12] for particular r-order nilpotent Lie algebras. Here, we make a first step to obtain more general Lie-algebraic conditions based on solvability, leveraging the property of invariant flag within the linear framework of the Koopman operator.

III. Koopman approach to dynamical systems

We now present the Koopman operator framework that will allow us to extend the above results on uniform stability to nonlinear systems. We introduce the Koopman semigroup along with its infinitesimal generator, cast the framework in the context of Lie groups, and describe the finite-dimensional approximation of the operator.

A. Koopman operator

Consider the continuous-time dynamical system

$$\dot{x} = F(x), \qquad x \in X \subset \mathbb{R}^n, \quad F \in \mathcal{C}^1 \tag{2}$$

which generates a flow $\varphi^t : X \to X$, with $t \in \mathbb{R}^+$. The Koopman operator is defined on a (Banach) space $\mathcal{F}$ and acts on observables, i.e. functions $f : X \to \mathbb{R}$, $f \in \mathcal{F}$.

Definition 6 (Koopman semigroup ([14], chapter 1)): The *semigroup of Koopman operators* (in short, Koopman semigroup) is the family of operators $\left(U^t\right)_{t \geq 0}$ defined by

$$U^t : \mathcal{F} \to \mathcal{F} : U^t f = f \circ \varphi^t.$$

For appropriate spaces $\mathcal{F}$ (e.g. square integrable functions, continuous functions), the Koopman semigroup is strongly continuous, i.e. $\lim_{t \to 0^+} \|U^t f - f\|_{\mathcal{F}} = 0$. In this case, the limit $\lim_{t \to 0^+} (U^t f - f)/t$ exists for all $f \in \mathcal{D}$, where $\mathcal{D}$ is dense in $\mathcal{F}$, and allows to define the infinitesimal generator of the Koopman semigroup.

Definition 7 (Koopman generator): The *infinitesimal generator* of a strongly continuous Koopman semigroup

$\left(U^t\right)_{t\geq 0}$ (in short, Koopman generator) is the operator

$$L_F f := \lim_{t\to 0^+} \frac{U^t f - f}{t}, \quad \forall f \in \mathcal{D}. \tag{3}$$

When the Koopman semigroup is associated with a dynamical system of the form (2), the action of the Koopman generator on an observable is given by $L_F f = F \cdot \nabla f$ (see e.g. [6], chapter 7). The Koopman operator U^t and its associated infinitesimal generator L_F are both linear operators. This allows us to describe the dynamics of an observable f on $\mathcal{F}$ through the linear infinite-dimensional dynamical system

$$\dot{f} = L_F f. \tag{4}$$

In the case of a switched nonlinear system (1), this yields the switched linear infinite-dimensional system of the form

$$\left\{\dot{f} = L_{F_i} f,\, f \in \mathcal{D}\right\}_{i=1}^{m}. \tag{5}$$

It is noticeable that the switched nonlinear (finite-dimensional) dynamics (1) and the switched linear (infinite-dimensional) dynamics (5) are equivalent representations. In our study, we will focus on the latter.

B. Lie group framework and relationship between finite-dimensional and infinite-dimensional systems

We can exploit Lie group and Lie algebra theory in the context of the Koopman operator framework and push further the relationship between finite-dimensional and infinite-dimensional representations (2) and (4) (or (1) and (5)).

The flow $\varphi^t : X \subset \mathbb{R}^n \to X$ generated by the finite-dimensional system is a diffeomorphism that can be equivalently described in the Koopman framework by the automorphism $U^t = U_{\varphi^t} : \mathcal{C}^k(X) \to \mathcal{C}^k(X)$, $0 \leq k \leq \infty$ (see [1], chapter 2). According to infinite-dimensional Lie group theory, for a compact subspace $X \subset \mathbb{R}^n$, Frechet-Lie group structures can be defined on the set of diffeomorphisms $Diff(X)$ and the set of automorphisms $Aut\left(\mathcal{C}^\infty(X)\right)$. Using some technical tools (i.e. topology of local uniform convergence of all partial derivatives on $C^\infty(X)$), one can show that these two Frechet-Lie groups are isomorphic. Moreover, their associated Lie algebras are $Vect(X)$ (set of vector fields on X) and $Der\left(\mathcal{C}^\infty(X)\right)$ (set of derivations on $\mathcal{C}^\infty(X)$), respectively. See [17], [15] and [16] (chapters VI and XII) for more details. Also, the well-known Lie algebras isomorphism $Vect(X) \to Der\left(\mathcal{C}^\infty(X)\right)$ clearly indicates that a vector field F on X is equivalent to the derivation L_F acting on f according to $L_F f = F \cdot \nabla f$, which is precisely the Koopman generator. These equivalences are summarized in Fig. 1.

Note also that $Vect(X)$ is equipped with the Lie bracket

$$[F_i, F_j](x) = \frac{\partial F_j(x)}{\partial x} F_i(x) - \frac{\partial F_i(x)}{\partial x} F_j(x)$$

while $Der\left(\mathcal{C}^\infty(X)\right)$ is equipped with $\left[L_{F_i}, L_{F_j}\right] = L_{F_i} L_{F_j} - L_{F_j} L_{F_i}$. It therefore follows that $\left[L_{F_i}, L_{F_j}\right] = L_{[F_i, F_j]}$. This well-known relationship on Lie brackets can be exploited in the case of switched nonlinear systems (1). Let $\mathfrak{g} = \text{span}\left\{F_i, i = 1, \ldots, m\right\}_{\text{Lie}}$ be the Lie algebra spanned

$$\begin{array}{lccc}
\text{Lie groups :} & Diff(X) & \longrightarrow & Aut\left(\mathcal{C}^\infty(X)\right) \\
 & \varphi^t & & U_{\varphi^t} \\
 & \top\,\downarrow & & \top\,\downarrow \\
 & F & & L_F \\
\text{Lie algebras :} & Vect(X) & \longrightarrow & Der\left(\mathcal{C}^\infty(X)\right)
\end{array}$$

Fig. 1: Lie groups and Lie algebras correspondence.

by F_i and their Lie brackets. According to the previous description, a Lie-algebraic criteria of uniform stability in $\mathfrak{g}$ can be recast into a Lie-algebraic criteria in $\mathfrak{g}_K = \text{span}\left\{L_{F_i}, i = 1, \ldots, m\right\}_{\text{Lie}}$, that is, in terms of Koopman generators, where we may expect results for switched linear systems to remain valid. In particular, the solvability of $\mathfrak{g}$ is equivalent to the solvability of $\mathfrak{g}_K$, and we may exploit the latter to show the existence of a common Lyapunov function.

C. Finite-dimensional approximation of the Koopman operator

It is useful to approximate the infinite-dimensional operator with a finite-dimensional Koopman operator. Consider an N-dimensional subspace $\mathcal{F}_N$ of the space of observables $\mathcal{F}$ and a projection operator $\Pi : \mathcal{F} \to \mathcal{F}_N$. The finite-dimensional approximation of the Koopman generator is given by

$$\bar{L}_F := \left(\Pi \circ L_F\right)|_{\mathcal{F}_N} \to \mathcal{F}_N. \tag{6}$$

Let $\mathcal{F}_N = \text{span}\left\{\psi_1, \psi_2, \cdots, \psi_N\right\}$ and consider an observable $f = \sum_{i=1}^{N} c_i\, \psi_i \in \mathcal{F}_N$. The action of $\bar{L}_F$ on f can be described by the $(N \times N)$-matrix

$$A_L = \begin{pmatrix} | & | & \cdots & | \\ \left[\bar{L}_F \psi_1\right]^{\mathcal{B}} & \left[\bar{L}_F \psi_2\right]^{\mathcal{B}} & \cdots & \left[\bar{L}_F \psi_N\right]^{\mathcal{B}} \\ | & | & \cdots & | \end{pmatrix}^T,$$

where the column-vector $\left[\bar{L}_F \psi_j\right]^{\mathcal{B}}$ contains the components of $\bar{L}_F \psi_j$ in the basis $\mathcal{B} = \{\psi_1, \psi_2, \cdots, \psi_N\}$. More details can be found in [14], chapter 1.

If the finite-dimensional subspace $\mathcal{F}_N$ is invariant under the action of the Koopman operators U^t (and therefore also the infinitesimal generator L_F), the dynamics of the nonlinear system are *exactly* described by the lifted linear system

$$\dot{z} = A_L z, \quad z \in \mathbb{R}^N \tag{7}$$

where $z = (\psi_1(x), \ldots, \psi_N(x))$ is the "lifted state". This is the case with linear dynamics when the subspace $\mathcal{F}_n = \text{span}\left\{f_1, f_2, \cdots, f_n\right\}$ is considered, with the "identity observables" $f_i : X \subset \mathbb{R}^n \to \mathbb{R} : x \mapsto f_i(x) = x_i$ [3]. There also exist particular nonlinear systems for which this property holds (see Section V-A). If $\mathcal{F}_N$ is not invariant, the lifted dynamics (7) is an approximation, whose accuracy depends on the choice of the subspace $\mathcal{F}_N$.

IV. Uniform stability of switched systems: A geometric proof

In this Section, we present a geometric proof of Theorem 2, which is an alternative proof to the original one given in [7]. This proof is constructive in the sense that it allows to build the CLF. This result will be exploited in Section V, where it will be used to approximate the stability region in which switched nonlinear systems are GUAS. Note also that since the proof does not rely on linear algebra but directly exploits the concept of invariant flag, it is more amenable to the infinite-dimensional framework of the Koopman operator.

Proposition 1: Let

$$\left\{\dot{x} = A_i\, x,\ A_i \in \mathbb{C}^{n\times n},\, x \in \mathbb{C}^n\right\}_{i=1}^m \tag{8}$$

be a switched linear system. Suppose that

a) all matrices A_i are stable and
b) there exists an invariant maximal flag

$$\{0\} \subsetneq S_1 \subset \cdots \subset S_n = \mathbb{C}^n, \quad S_j = \mathrm{span}\{v_1, \ldots, v_j\}$$

for the matrices $\{A_i\}_{i=1}^m$.

Then there exist $\epsilon_j > 0$, $j = 1, \ldots, n$, such that in $\mathbb{C}^n$

$$V(x) = \sum_{k=1}^n \epsilon_k |v_k^* x|^2 \tag{9}$$

is a CLF for (8), with v_k^* the conjugate transpose of v_k.

Proof: Without loss of generality, we assume that $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of $\mathbb{C}^n$. It is clear that $V(0) = 0$ and $V(x) > 0$ for all $x \in S_n \setminus \{0\}$ since $\bigcap_{k=1}^n \{x \in \mathbb{C}^n : v_k^* x = 0\} = \{0\}$. Moreover we have

$$\dot{V}(x) = \sum_{k=1}^n \epsilon_k \left((v_k^* A_i x)\ (x^* v_k) + (v_k^* x)\ (x^* A_i^* v_k)\right)$$

and, for $x = \sum_{j=1}^n c_j v_j \in S_n$, it follows that

$$\dot{V}(x) = 2\sum_{j=1}^n \sum_{k=1}^n \epsilon_k \Re\left(c_j c_k^* \left(v_k^* A_i v_j\right)\right).$$

It is easy to see that

$$v_k^* A_i v_j = 0 \quad \text{for } j < k \tag{10}$$

since $A_i v_j \in S_j$ by invariance. Moreover, suppose that $v_j = v_{i,j} + w$, where $v_{i,j} \in S_j$ is an eigenvector of A_i associated with the eigenvalue $\Re\left(\lambda_{i,j}\right) < 0$ and where $w \in S_{j-1}$. We have that

$$v_j^* A_i v_j = v_j^* A_i (v_{i,j} + w) = \lambda_{i,j} v_j^* v_{i,j} = \lambda_{i,j} \tag{11}$$

where we used $A_i v_{i,j} \in S_{j-1}$ by invariance and $1 = v_j^* v_j = v_j^* v_{i,j}$. Using (10) and (11) and rearranging terms, we obtain

$$\begin{aligned}
\dot{V}(x) &= 2\sum_{j=1}^n \epsilon_j \left|c_j\right|^2 \Re\left(\lambda_{i,j}\right) + 2\sum_{j=2}^n \sum_{k=1}^{j-1} \epsilon_k \Re\left(c_j c_k^* \left(v_k^* A_i v_j\right)\right) \\
&= 2\sum_{j=2}^n \sum_{k=1}^{j-1} \frac{\epsilon_j \left|c_j\right|^2 \Re\left(\lambda_{i,j}\right) + \epsilon_k \left|c_k\right|^2 \Re\left(\lambda_{i,k}\right)}{n-1} \\
&\quad + 2\sum_{j=2}^n \sum_{k=1}^{j-1} \epsilon_k \Re\left(c_j c_k^* \left(v_k^* A_i v_j\right)\right).
\end{aligned}$$

Next, the value $\epsilon_1 > 0$ is chosen arbitrarily and we proceed recursively to select ϵ_j, $j = 2, \ldots, n$, so that

$$\frac{\epsilon_j \left|c_j\right|^2 \Re\left(\lambda_{i,j}\right) + \epsilon_k \left|c_k\right|^2 \Re\left(\lambda_{i,k}\right)}{n-1} + \epsilon_k \Re\left(c_j c_k^* \left(v_k^* A_i v_j\right)\right) < 0 \tag{12}$$

for all $i = 1, \ldots, m$, for all $k = 1, \ldots, j-1$, and for all $(c_j, c_k) \neq (0, 0)$. Note that, if $c_j = 0$, the inequality (12) holds for all ϵ_j. If $c_j \neq 0$, (12) is satisfied if

$$\begin{aligned}
\epsilon_j &> \epsilon_k \left(-\frac{\left|\Re\left(\lambda_{i,k}\right)\right|}{\left|\Re\left(\lambda_{i,j}\right)\right|} \frac{\left|c_k\right|^2}{\left|c_j\right|^2} + (n-1)\frac{\left|v_k^* A_i v_j\right|}{\left|\Re\left(\lambda_{i,j}\right)\right|} \frac{\left|c_k\right|}{\left|c_j\right|}\right) \\
&:= \epsilon_k F\left(\frac{\left|c_k\right|}{\left|c_j\right|}\right),
\end{aligned}$$

where we used the inequality $\left|c_j c_k^* \left(v_k^* A_i v_j\right)\right| \geq \left|\Re\left(c_j c_k^* \left(v_k^* A_i v_j\right)\right)\right|$. The maximal value of the quadratic function F is given by

$$\frac{(n-1)^2}{4} \frac{\left|v_k^* A_i v_j\right|^2}{\left|\Re\left(\lambda_{i,j}\right)\right| \left|\Re\left(\lambda_{i,k}\right)\right|}.$$

It follows that we can choose

$$\epsilon_j > \max_{\substack{i \in \{1,\ldots,m\} \\ k \in \{1,\ldots,j-1\}}} \epsilon_k \frac{(n-1)^2}{4} \frac{\left|v_k^* A_i v_j\right|^2}{\left|\Re\left(\lambda_{i,j}\right)\right| \left|\Re\left(\lambda_{i,k}\right)\right|} \tag{13}$$

so that (12) holds, and therefore $\dot{V}(x) < 0$ for all $x \in \mathbb{C}^n \setminus \{0\}$. This concludes the proof. ■

The above result can be used to construct the CLF. However, as a preliminary step, it is necessary to compute the invariant maximal flag for the matrices A_i. To do so, we can use the fact that the basis vector $v_k \in S_k$ is a common eigenvector of the matrices $(I - P_{k-1})A_i$, where I is the identity matrix and P_{k-1} is the orthogonal projection onto S_{k-1}. The numerical scheme used to obtain the invariant flag is summarized in Algorithm 1.

Remark 1: For a single matrix A_i, Algorithm 1 is a mere triangularization algorithm (i.e. Schur's decomposition).

V. Toward uniform stability of switched nonlinear systems

In this Section, we present uniform stability results for switched nonlinear systems through the Koopman operator framework. The Koopman operator is represented in a finite-dimensional subspace, so that the switched system is described by a switched linear system, whose uniform stability can be used through the results presented in the previous sections. We will distinguish the cases where the finite-dimensional representation is exact or approximate.

Input: Matrices A_i, $i = 1, \ldots, m$.
Output: Invariant maximal flag $S_1 \subset \cdots \subset S_n$.
Find a common eigenvector v_1 for the matrices A_i;
if *v_1 does not exist,* **then**
  there is no invariant maximal flag;
else
  set $S_1 = \text{span}\{v_1\}$ and compute the orthogonal projection onto S_1: $P_1 = v_1(v_1^* v_1)^{-1} v_1^*$;
end
for *k=2,$\cdots$ n* **do**
  find v_k a common eigenvector of the matrices $(I - P_{k-1})A_i$;
  if *v_k does not exist,* **then**
    there is no invariant maximal flag;
  else
    set $S_k = \text{span}\{v_1, v_2, \ldots, v_k\}$;
    if $k < n$ **then**
      compute the orthogonal projection onto S_k: $P_k = V(V^*V)^{-1}V^*$ with the matrix $V = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{pmatrix}$;
    end
  end
end

Algorithm 1: Invariant maximal flag

A. Exact finite-dimensional representation

For some specific dynamics, one can identify a subspace $\mathcal{F}_N \subset \mathcal{F}$ that is invariant under the action of the associated Koopman semigroup. In this case, the lifting dynamics (7) exactly describes the evolution of the observables $f \in \mathcal{F}_N$. Provided that there is an injective mapping from the state space x to the lifted space z, proving stability of the lifting dynamics is equivalent to proving stability of the state dynamics. In the case of switched systems, we have the following result.

Proposition 2: Consider a switched system of the form (1) such that x^* is a common equilibrium and suppose that its associated switched linear infinite-dimensional system (5) has an invariant subspace $\mathcal{F}_N \subset \mathcal{C}^0$, where it is represented by the lifted dynamics

$$\left\{\dot{z} = A_{L_i} z,\, z \in \mathbb{R}^N\right\}_{i=1}^{m}. \tag{14}$$

If the matrices A_{L_i} are stable, if the Lie algebra $\mathfrak{g} = \text{span}\{A_{L_i}\}_{\text{Lie}}$ is solvable and if $f(x) = 0\,\forall f \in \mathcal{F}_N \Rightarrow x = x^*$, then (1) is GUAS in $X \subset \mathbb{R}^n$.

Proof: Lie's theorem implies that there exists an invariant maximal flag in $\mathbb{C}^N$ for the matrices A_{L_i} (or equivalently an invariant flag in $\mathcal{F}_N$ for the Koopman generators L_{F_i}). Proposition 1 implies that there exists a CLF for the subsystems $\dot{z} = A_{L_i} z$ so that (14) is GUAS in $\mathbb{R}^N \subset \mathbb{C}^N$. By definition of the Koopman semigroup, it follows that $\lim_{t\to\infty} f(\varphi^t(x)) = 0$ (where φ^t is the flow generated by (1)) for all x and for all $f \in \mathcal{F}_N$. Since the observables f are continuous and all have zero value only at x^*, this implies that $\lim_{t\to\infty} \varphi^t(x) = x^*$, which concludes the proof. ■

Note that a CLF for (14) is given in Proposition 1 by (9) (where x is replaced by the lifted state z), so that a CLF for the switched nonlinear system is of the form

$$V(x) = \sum_{k=1}^{n} \epsilon_k \left|v_k^* \Psi(x)\right|^2 \tag{15}$$

with $\Psi(x) = (\psi_1(x), \ldots, \psi_N(x))^T$ (see also [14], chapter 2).

Example 1: Consider a switched nonlinear system with subsystems

$$\begin{cases} \dot{x}_1 = \mu_i x_1, \\ \dot{x}_2 = \beta_i \left(x_2 - x_1^2\right), \end{cases} \qquad \mu_i, \beta_i < 0, i = 1 \ldots, m, \tag{16}$$

which have a stable equilibrium at the origin. It is known that the subspace $\mathcal{F}_3 = \text{span}\left\{x_1, x_2, x_1^2\right\}$ is invariant under the action of the Koopman operator (see [2]). With the lifted state $z = (x_1, x_2, x_1^2)^T$, this yields the exact lifting dynamics

$$\dot{z} = A_{L_i} z \quad A_{L_i} = \begin{pmatrix} \mu_i & 0 & 0 \\ 0 & \beta_i & -\beta_i \\ 0 & 0 & 2\mu_i \end{pmatrix}. \tag{17}$$

The matrices A_i are stable and the Lie algebra $\mathfrak{g} = \text{span}\{A_{L_i}\}_{\text{Lie}}$ is solvable. Moreover, it is clear that $f(x) = 0$ $\forall f \in \mathcal{F}_3 \Rightarrow x = x^*$. Then it follows from Proposition 2 that the switched nonlinear system (16) is GUAS in $\mathbb{R}^2$.

If we have two subsystems with $\mu_1 = -1$, $\beta_1 = -2$, $\mu_2 = -3$, $\beta_2 = -4$, Algorithm 1 yields the following invariant maximal flag in $\mathcal{F}_3$: $S_j = \text{span}\{v_1, \cdots, v_j\}$ where v_j is the jth vector of the canonical basis of $\mathbb{R}^3$ and we can pair the eigenvalues according to

$$\begin{cases} \lambda_{1,1} = -1 \\ \lambda_{2,1} = -3, \end{cases} \quad \begin{cases} \lambda_{1,2} = -2 \\ \lambda_{2,2} = -4, \end{cases} \quad \begin{cases} \lambda_{1,3} = -2 \\ \lambda_{2,3} = -6 \end{cases}.$$

Using (13), one can choose $\epsilon_1 = \epsilon_2 = 1/9$ and $\epsilon_3 = 1/8$ and we obtain the CLF $V(x_1, x_2) = \dfrac{x_1^2}{9} + \dfrac{x_2^2}{9} + \dfrac{x_2^4}{8}$. ◇

B. Finite-dimensional approximation

Consider a switched nonlinear system of the form (1) and the associated finite-dimensional approximation of the form (14). Here, we suppose that the subspace $\mathcal{F}_N$ is not invariant under the action of the Koopman semigroups. However we assume that

1) all subsystems of (1) are globally asymptotically stable,
2) and $\mathfrak{g}_K = \text{span}\{A_{L_i}\}_{\text{Lie}}$ is solvable.

These assumptions imply that there exists a CLF for the switched lifting dynamics (14), so that we can use a candidate CLF of the form (15) to approximate a forward-invariant set D in which the switched system is GUAS. This is summarized in the following scheme:

- compute the region B such that
$$\dot{V}(x) = \max_{i\in\{1,\ldots,m\}} F_i(x) \cdot \nabla V(x) < 0 \quad \forall x \in B;$$
- D is the largest sublevel set of V contained in B.

The computation of the stability set D for a switched nonlinear system is illustrated with the following example.

Example 2: Consider the switched system described by the following subsystems:

$$\begin{cases}\dot{x}_1 = -2x_1 + x_1x_2 \\ \dot{x}_2 = -2x_2 - x_1^2\end{cases} \quad \text{and} \quad \begin{cases}\dot{x}_1 = -2x_1 + x_2^2 \\ \dot{x}_2 = -x_2 - x_1^2.\end{cases} \tag{18}$$

The subspace $\mathcal{F}_5 = \text{span}\left\{x_1, x_2, x_1^2, x_1x_2, x_2^2\right\}$ is not invariant by the Koopman operator. However, we can approximate the Koopman operator on $\mathcal{F}_5$ with a finite section method and, for the lifted state $z = (x_1, x_2, x_1^2, x_1x_2, x_2^2)^T$, we obtain the approximate dynamics

$$\dot{z}_i = A_{L_i} z, \text{ with} \tag{19}$$

$$A_{L_1} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & -4 & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}^T, A_{L_2} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & -4 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix}^T.$$

The Lie algebra $\mathfrak{g} = \text{span}\left\{A_{L_i}\right\}_{Lie}$ is solvable. It follows that there exists a CLF for the subsystems of (19) in $\mathcal{F}_5$. Algorithm 1 yields the following invariant maximal flag in $\mathcal{F}_5 : S_j = \text{span}\left\{v_1, \cdots, v_j\right\}$ where v_j is the jth vector of the canonical basis of $\mathbb{R}^5$ and we can pair the eigenvalues according to

$$\begin{cases}\lambda_{1,1} = \lambda_{1,2} = -2, \lambda_{1,3} = \lambda_{1,4} = \lambda_{1,5} = -4 \\ \lambda_{2,1} = -2,\ \lambda_{2,2} = -1, \lambda_{2,3} = -4,\ \lambda_{2,4} = -3,\ \lambda_{2,5} = -2.\end{cases}$$

Using (13), one can choose $\epsilon_1 = \epsilon_2 = 1/25$, $\epsilon_3 = 1/24$, $\epsilon_4 = 1/47$ and $\epsilon_5 = 1/46$ and we obtain the candidate CLF

$$V(x_1, x_2) = \frac{x_2^4}{25} + \frac{x_1^2x_2^2}{25} + \frac{x_1^4}{24} + \frac{x_2^2}{47} + \frac{x_1^2}{46}.$$

This allows to compute a stability set D in which we can guarantee that (18) is GUAS (see Fig. 2). ◇

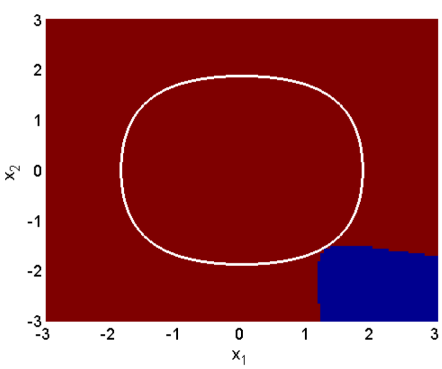

Fig. 2: The switched nonlinear system (18) is GUAS in the the stability region D (with white boundary). The candidate CLF is decreasing for all subsystems in the red region.

VI. Conclusion and perspectives

In this paper, we have considered the uniform stability problem for switched nonlinear systems, leveraging the equivalence between Lie algebras of vector fields and Lie algebras of Koopman generators. We have shown that uniform stability can be deduced from the solvability of Lie algebras of vector fields if the Koopman operators associated with the nonlinear subsystems possess a common invariant subspace. For the case where no invariant subspace can be identified, we have provided a numerical method to compute a stability region where the switched nonlinear systems are uniformly stable.

This paper lays the basis for a complete criteria of uniform stability of switched nonlinear systems in terms of solvability properties. In particular, we have proposed an alternative proof for the existence of a common Lyapunov function, which is based on the notion of invariant maximal flag. The extension of the proof to the infinite-dimensional setting of the Koopman operator will be presented in an upcoming paper. This will provide a final answer to the uniform stability problem and its connection to solvability properties.

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