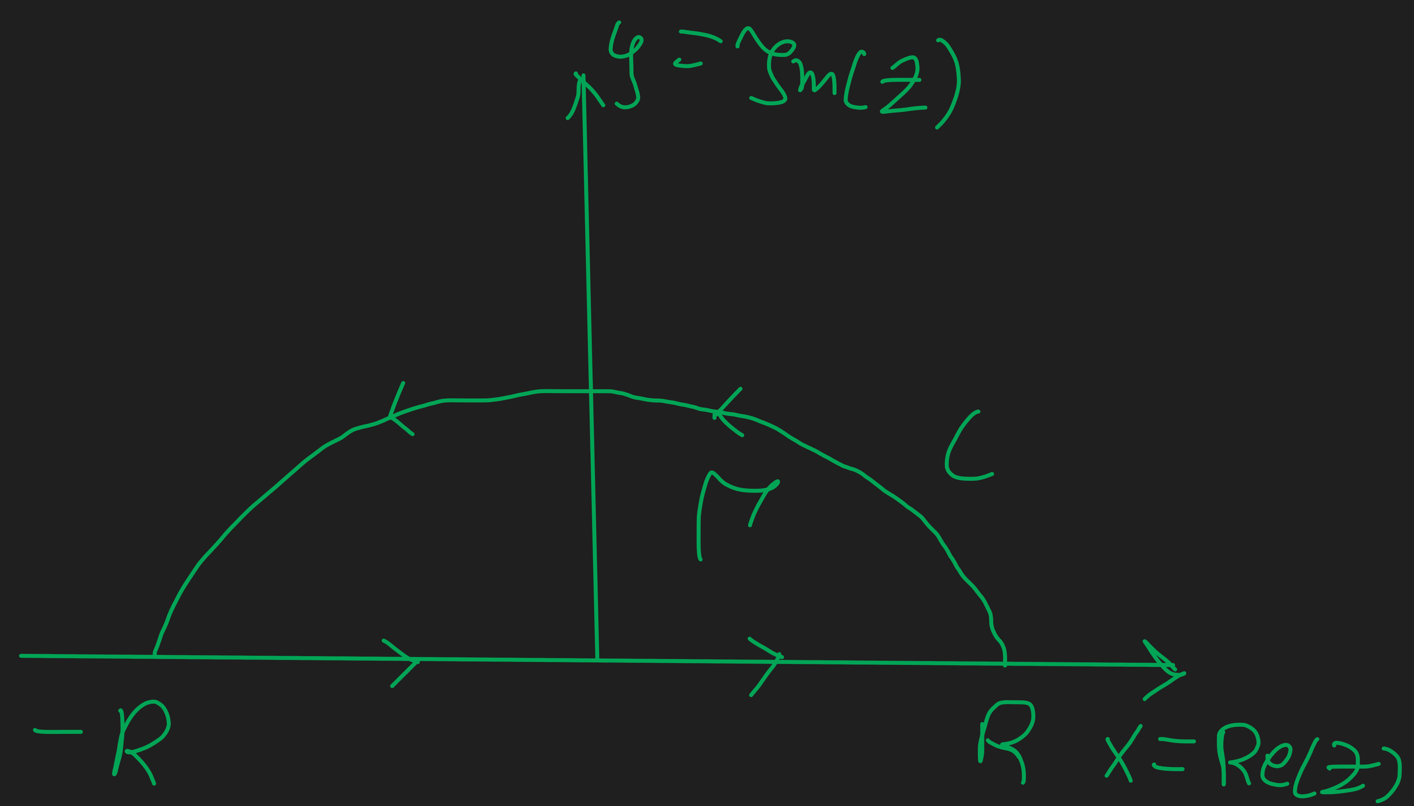
$$I = \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx$$

$$f(z) = \frac{e^{iz}}{z^2+1}, \text{ poles}: z^2+1=0 \Rightarrow z = \pm i$$

$$\oint_C = \int_\Gamma + \int_{-R}^{R}$$



$$\oint_C \frac{e^{iz}}{z^2+1}\,dz = 2\pi i \cdot \lim_{z\to i}(z-i)\cdot\frac{e^{iz}}{z^2+1}$$

$$= 2\pi i \cdot \lim_{z\to i} \cancel{(z-i)} \cdot \frac{e^{iz}}{\cancel{(z-i)}(z+i)}$$

$$= \cancel{2}\pi \cancel{i} \cdot \frac{e^{iz}}{\cancel{2i}} = \frac{\pi}{e}$$

$$\frac{\pi}{e} = \int_\Gamma + \int_{-R}^{R}$$

$$\left|\int_\Gamma\right| \leq \int_0^\pi \left|\frac{e^{iRe^{i\phi}}}{R^2e^{2i\phi}+1}\, iRe^{i\phi}\right| d\phi = \int_0^\pi \left|\frac{Re^{iRe^{i\phi}}}{R^2e^{2i\phi}+1}\right| d\phi$$

$$= \int_0^\pi \left|\frac{Re^{iR[\cos(\phi)+i\sin(\phi)]}}{R^2e^{2i\phi}+1}\right| d\phi = \int_0^\pi \left|\frac{Re^{iR\cos(\phi)}e^{-\sin(\phi)}}{R^2e^{2i\phi}+1}\right| d\phi$$

$$= \int_0^\pi \left|\frac{Re^{-\sin(\phi)}}{R^2e^{2i\phi}+1}\right| d\phi = \int_0^\pi \frac{\cancel{R}e^{-\sin(\phi)}}{R^{\cancel{2}}\left|e^{2i\phi}+\frac{1}{R^2}\right|}\,d\phi \Rightarrow \lim_{R\to\infty}\left|\int_\Gamma\right| = 0 \Rightarrow \int_\Gamma = 0$$

$$\therefore \frac{\pi}{e} = \int_{-R}^{R} \Rightarrow \mathrm{Re}\,\frac{\pi}{e} = \mathrm{Re}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x^2+1}\,dx \Rightarrow \int_{-\infty}^{+\infty}\frac{\cos(x)}{x^2+1}\,dx = \frac{\pi}{e}$$