$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{(x^2+a^2)(x^2+b^2)}\,dx = 2\,\mathrm{Im}\left(\int_0^{\infty} \frac{xe^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx\right), \quad I = \int_C \frac{ze^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz, \quad \mathrm{Re}(a)>0 \wedge \mathrm{Re}(b)>0,$$

poles: $z = \pm ib$, $z = \pm ia$

$$\oint_C = \int_\Gamma + \int_{-R}^{R}$$

$$\oint_C = 2\pi i \sum \mathrm{Res}(f)$$

$$= 2\pi i \left[\lim_{z\to ib} (z-ib)\frac{ze^{iz}}{(z^2+a^2)(z-ib)(z+ib)} + \lim_{z\to ia} (z-ia)\frac{ze^{iz}}{(z-ia)(z+ia)(z^2+b^2)}\right]$$

$$= 2\pi i \left[\frac{ibe^{-b}}{(-b^2+a^2)2ib} + \frac{iae^{-a}}{2ia(-a^2+b^2)}\right] = 2\pi i \frac{e^{-b}}{(-b^2+a^2)2} + 2\pi i \frac{e^{-a}}{2(-a^2+b^2)}$$

$$= \frac{\pi i e^{-b}}{a^2-b^2} + \frac{\pi i e^{-a}}{b^2-a^2} = -\frac{i\pi(e^{-a}-e^{-b})}{a^2-b^2}$$

$$-\frac{i\pi(e^{-a}-e^{-b})}{a^2-b^2} = \int_\Gamma + \int_{-R}^{R}$$

$$\left|\int_\Gamma\right| \le \int_0^{\pi} \left|\frac{Re^{i\phi}\cdot e^{iRe^{i\phi}}}{(R^2e^{2i\phi}+a^2)(R^2e^{2i\phi}+b^2)}\, iRe^{i\phi}\right| d\phi = R^2\int_0^{\pi} \frac{e^{-R\sin(\phi)}}{\left|a^2b^2+a^2R^2+b^2R^2+R^4\right|}\,d\phi$$

$$= R^2\int_0^{\pi} \frac{e^{-R\sin(\phi)}}{\left|(a^2+R^2)(b^2+R^2)\right|}\,d\phi = \frac{R^2}{R^4}\int_0^{\pi} \frac{e^{-R\sin(\phi)}}{\left|\frac{a^2b^2}{R^4}+\frac{a^2+b^2}{R^2}+1\right|}\,d\phi$$

$$= \frac{1}{R^2}\int_0^{\pi} \frac{e^{-R\sin(\phi)}}{\left|\frac{a^2b^2}{R^4}+\frac{a^2+b^2}{R^2}+1\right|}\,d\phi \xrightarrow{R\to\infty} 0$$

$$\therefore \frac{-i\pi(e^{-a}-e^{-b})}{a^2-b^2} = 2\,\mathrm{Im}[I]$$

$$\Rightarrow \frac{-\pi(e^{-a}-e^{-b})}{a^2-b^2} = \int_{-\infty}^{+\infty} \frac{x\sin(x)}{(x^2+a^2)(x^2+b^2)}\,dx$$