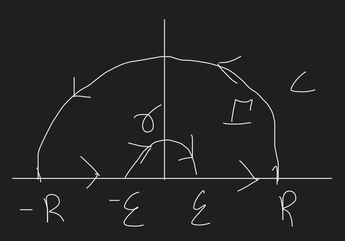
$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{(x^2+a^2)(x^2+b^2)}\,dx = 2\,\mathrm{Im}\left(\int_0^{\infty} \frac{xe^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx\right), \quad I = \int_0^{\infty} \frac{ze^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz, \quad \text{Pole: } z=\pm ib,\ z=\pm ia$$



$$\oint_C = \int_\Gamma + \int_\gamma + \int_{-R}^{R} + \int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}$$

$$\oint_C = 2\pi i \sum \mathrm{Res}(f)$$

$$= 2\pi i \left[\lim_{z\to ib} (z-ib)\frac{ze^{iz}}{(z^2+a^2)(z-ib)(z+ib)} + \lim_{z\to ia} (z-ia)\frac{ze^{iz}}{(z-ia)(z+ia)(z^2+b^2)}\right]$$

$$= 2\pi i \left[\frac{ibe^{-b}}{(-b^2+a^2)2ib} + \frac{ia e^{-a}}{2ia(-a^2+b^2)}\right]$$

$$= \frac{\pi i e^{-b}}{a^2-b^2} + \frac{\pi i e^{-a}}{b^2-a^2} = \frac{\pi i e^{-b}}{-(-a^2+b^2)} + \frac{\pi i e^{-a}}{b^2-a^2} = -\frac{\pi i e^{-b}}{b^2-a^2} + \frac{\pi i e^{-a}}{b^2-a^2} = \frac{\pi i e^{-a}}{b^2-a^2} - \frac{\pi i e^{-b}}{b^2-a^2} = \frac{\pi i(e^{-a}-e^{-b})}{b^2-a^2}$$

$$\frac{\pi i(e^{-a}-e^{-b})}{b^2-a^2} = \int_\Gamma + \int_\gamma + \int_{-\infty}^{-\varepsilon} + \int_{-\infty}^{0} + \int_0^{\infty}$$

$$\left|\int_\Gamma\right| \le \int_0^{\pi} \left|\frac{Re^{i\phi}\cdot e^{iRe^{i\phi}}}{(R^2e^{2i\phi}+a^2)(R^2e^{2i\phi}+b^2)} iRe^{i\phi}\right| d\phi = R^2\int_0^{\pi} \frac{e^{-R\sin(\phi)}}{|a^2b^2+a^2R^2+b^2R^2+R^4|}\,d\phi$$

$$= R^2\int_0^{\pi} \frac{e^{-R\sin(\phi)}}{|(a^2+R^2)(b^2+R^2)|}\,d\phi = \frac{R^2}{R^4}\int_0^{\pi} \frac{e^{-R\sin(\phi)}}{\left|\frac{a^2b^2}{R^4}+\frac{a^2+b^2}{R^2}+1\right|}\,d\phi$$

$$= \frac{1}{R^2}\int_0^{\pi} \frac{e^{-R\sin(\phi)}}{\left|\frac{a^2b^2}{R^4}+\frac{a^2+b^2}{R^2}+1\right|}\,d\phi \xrightarrow{R\to\infty} 0$$

$$\int_\gamma = \int_\pi^0 \frac{\varepsilon e^{i\phi}\cdot e^{i\varepsilon e^{i\phi}}}{(a^2+\varepsilon^2e^{2i\phi})(b^2+\varepsilon^2e^{2i\phi})} i\varepsilon e^{i\phi}\,d\phi = -\int_0^{\pi} \frac{\varepsilon e^{i\phi}\cdot e^{i\varepsilon e^{i\phi}}}{(a^2+\varepsilon^2e^{2i\phi})(b^2+\varepsilon^2e^{2i\phi})} i\varepsilon e^{i\phi}$$

$$= -i\int_0^{\pi} \frac{\varepsilon e^{i\phi}\cdot e^{i\varepsilon e^{i\phi}}}{(a^2+\varepsilon^2e^{2i\phi})(b^2+\varepsilon^2e^{2i\phi})}\,d\phi \xrightarrow{\varepsilon\to 0} 0$$

$$\frac{\pi i(e^{-a}-e^{-b})}{b^2-a^2} = \int_{-\infty}^{\infty} + \int_{-\infty}^{0} + \int_0^{\infty}$$

$$= \int_{-\infty}^{\infty} + \int_{-\infty}^{\infty} = 2I + 2I$$

$$\therefore \frac{\pi i(e^{-a}-e^{-b})}{b^2-a^2} = 4I$$

$$\Rightarrow \underline{I} = \frac{1}{4}\cdot\frac{\pi i(e^{-a}-e^{-b})}{b^2-a^2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x\sin(x)}{(x^2+a^2)(x^2+b^2)}\,dx = 2\,\mathrm{Im}\left(\frac{1}{4}\cdot\frac{\pi i(e^{-a}-e^{-b})}{b^2-a^2}\right) = \frac{1}{2}\cdot\frac{\pi(e^{-a}-e^{-b})}{b^2-a^2}$$