國立清華大學

碩士論文

廣義相對論下的奇異等溫球
之重力塌縮與震波解

The Relativistic Shockwave Solution in the Collapse of Singular Isothermal Sphere

所別：天文所

學號：946904

研究生：陳建廷

指導教授：蔡駿博士

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Chen, Chien-Ting
Institute of Astronomy, National Tsing-Hua University

Supervisor: Dr. Mike J. Cai

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摘要

本篇論文中，我們在廣義相對論的範疇之內，研究奇異等溫球(一團球對稱的等溫氣體，具有自相似的性質，中心的密度為無限大，對應了一個奇異點)的重力塌縮現象。針對這個問題，我們試著以震波的存在，來延伸原本由中心向外塌縮的數學解。與僅考慮牛頓力學的情形不同，廣義相對論中，震波發生的原因，乃是奇異等溫球內部的重力塌縮而產生的時空變化；一旦氣體球的中心開始塌縮形成黑洞，時空的改變會導致塌縮區域之外的重力降低，因此產生一個由中心向外、超音速的震波。此一相對論性的震波與牛頓力學下的震波不同，並不需要額外的能量來維持震波的存在，而是時空本身變化的結果。我們認為此一震波是中心的黑洞釋放重力位能的一個現象。本篇論文中，震波的存在受到等溫氣體聲速的限制，我們計算的結果顯示，只有聲速的平方在 17.32% ~ 45.83% 光速的範圍之內，可以找到對應的震波解。在低聲速的情況下，此一問題的數學形式會逐漸趨近於牛頓力學的結果，由於我們所計算的震波僅存在廣義相對論的範疇之內，故在聲速過低的情形下，無法找到對應的震波解；而聲速的上限則可能與奇異等溫球本身的物理特性有關。對應到不同的聲速，我們所計算出的能量釋放比率約為中心黑洞靜止質量的 3% ~ 15%，若假設中心的黑洞重十個太陽質量，所釋放的重力位能可達10^{54}爾格。此一龐大的能量若全以電磁波的形式釋放出來，與最強烈的伽瑪射線爆發所需的能量相當接近。

關鍵辭：相對論、震波、重力塌縮

Abstract

We studied the collapse solution to a singular isothermal sphere with consideration of relativistic collapse solution, and extended the collapse solutions with critical points into the shock wave solution. Apart from its Newtonian counterpart, the general relativistic shock wave is generated by the change of space-time after the collapse of the sphere begins. The shock wave propagate outward at a super sonic speed, accompanied by the collapse of the sphere inside and the formation of a central black hole. In our calculation, the shock wave solutions are capped by the isothermal sound speed. The solution only exists for the sound speed ranged at 17.32%   45.83% of speed of light. We postulate this shock wave can release the gravitational binding energy of the central black hole by the ratio of 3%   15% of its rest mass. A simple estimation shows that the released energy is about $10^{54} erg$ for a black hole with mass $M_{BH} \sim 10 M_{\odot}$, which correspond to the energy scale of those most energetic gamma-ray bursts in the sky.

Key words: relativity, gravitation, shock wave, gamma-ray burst

Contents

Chapter 1

Introduction

Based on observation, black holes can be categorized into three classes. The observation of high mass x-ray binaries showed the evidence of the existence of stellar mass black holes (McClintock & Remillard, 2003.) There is also evidence for the existence of super massive black holes(SMBHs) which reside in the center of most galaxies (Begelman et al. 1984; Rees 1984; Kormendy & Richstone 1995.) Besides SMBHs and stellar mass black holes, there is one more class of black holes reside in the center of globular clusters with mass $M_{BH} \sim 10^3 M_\odot$ which are called "intermediate mass black holes"(IMBHs).

The stellar mass black holes have long been known as a product of the death of massive stars, although the details of the formation is yet to be studied in more detail due to the complexity of core-collapse. However, it can still be understood via an Oppenheimer-Snyder collapse (Oppen heimer & Snyder,1939) in a fundamental level. More precisely, the stellar mass black holes can be formed in two ways. The first way is to directly collapse into a black hole without supernova (Mirabel & Rodrigues 2003.) Alternatively, a supernova occurred and end up with a neutron star, if the supernova explosion is not energetic enough to unbind the whole stellar envelope, it would fall back on the neutron star to form a black hole (see review of Heger et al. 2003.) In both scenarios, a black hole is formed with a release of gravitational binding energy, through a supernova or the long-term gamma-ray burst.

However, there is still lack of consensus on the formation of SMBHs and IMBHs. Observations suggested that the mass of SMBHs and IMBHs are related to the velocity dispersion (σ) of their host galaxy or star cluster by a correlation of $M_{BH} \propto \sigma^4$ (Gebhardt et al. 2000; Ferrarese & Merritt 2000; Tremaine et al. 2002). To understand this unexpected relationship between the massive region and its vast surroundings, a viable process is to study the gravitational collapse of the inner part of a preexisting unstable gas sphere. The collapse of singular isothermal sphere (SIS) is one of the most tractable problem in Newtonian regime. In the paper of Shu (1977), the collapse of SIS was described through an inside-out fashion named "expansion wave collapse solution". If the center of the preexisting unstable gas sphere is perturbed slightly, then the unstable equilibrium between gravity and pressure gradient brakes down to trigger the collapse of the sphere. The unbalance between forces would propagate outward at speed of sound, leads to the expanding of collapsing region. The solution has been extended into the general relativistic regime by Cai & Shu(2005), with the formation of a massive black hole in the center. In this article we extend the study of the collapse of general relativistic singular isothermal sphere(SIS) of Cai & Shu(2005), and try to augment the model with the presence of a shock wave.

Tsai & Hsu(1995) expanded the solutions of Shu(1977) and found the shock wave solution in the Newtonian regime, they postulated that if a central star formed during the collapse, the gas might possess external energy to form a shock wave. However, the story could be told in a totally different way with consideration of general relativity. In Newtonian gravity, a spherical mass distribution could be treated as a mass point at its center of mass, this is known as the iron sphere theorem. Essentially, the iron sphere theorem is due to the linearity of mass and Gauss theorem, neither is valid for general relativity. In particular, if a spherical object becomes more compact, it's gravitational influence on the surrounding becomes weaker. Consider a gaseous cloud with unstable equilibrium initial state, if its central part collapses into a black hole, then the gravitational field due to this part becomes weaker. However, the equilibrium state is achieved through balancing gravity with pressure gradient, therefore the outside cloud elements will feel an unbalanced force and starts to move out, then the shock formed result in the outgoing wind hits the static region.

In this work, we search for the mathematical description through general relativistic frame work. we introduce the background knowledge and some solutions to the problem of singular isothermal sphere in chapter 2, including Newtonian mechanics and general relativity. Chapter 3 describes the techniques and process to find a shock wave solution. In chapter 4 we discuss the physical insight of our shock wave solution, then we conclude in chapter 5.

Chapter 2

Collapse of Singular Isothermal Sphere

In this chapter, we review some of the studies of the collapse of singular isothermal sphere. We start on the problem in the Newtonian regime, which provides a basic physical picture of the problem, then we extend the discussion into General Relativity. In section 2.1, most equations are quoted from Shu(1977), and we quote most equations from Cai & Shu(2005) in section 2.2.

2.1 Newtonian Collapse

2.1.1 Preliminary

The study of singular isothermal sphere (SIS) was motivated by the problem of star formation. The collapse of a pre-existing gas sphere is due to the gravity overcoming the pressure gradient. So the calculation can be started by studying Jean's instability. Consider a static gaseous configuration:

$$\bigtriangledown \mathbf{P} = -\rho \bigtriangledown \Phi, \tag{2.1}$$

where $\mathbf{P}$ stands for the gas pressure, ρ is the density of gas and Φ represents gravitational potential. The problem can be simplified by considering spherical symmetry and isothermal equation of state, $P = {c_s}^2\rho$ where ${c_s}^2 = constant$ =(sound speed)^2. Let r be the radius, then equation(2.1) reduces to

$$\frac{c_s^2}{\rho}\frac{d\rho}{dr} = -\frac{d\Phi}{dr}. \tag{2.2}$$

Equation (2.1) would now have a solution

$$\rho = \rho_0 exp(\Phi/{c_s}^2), \tag{2.3}$$

where ρ_0 is the integration constant which can be determined by an initial condition.

The gravitational potential Φ should satisfy the Poisson's equation,

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\Phi}{dr}) = 4\pi G\rho \tag{2.4}$$

where G is the gravitational constant. This nonlinear ordinary differential equation is known as the *Lane-Emden equation* for an isothermal sphere.

Bodenheimer & Sweigart(1968) , have studied this configuration, they argued that the self-gravitating isothermal gas can evolve into basically the same conclusion before collapse. For a boundary condition that $\Phi = 0, \quad d\Phi/dr = 0$ at $r = 0$, if the size of the region of interest is much smaller than the outer boundary, the system can evolve secularly to an **singular isothermal sphere** (SIS)), with the density profile

$$\rho = \frac{a^2}{2\pi G r^2} \tag{2.5}$$

(see Chandrasekhar, 1957). The term singular is from the formal divergent of density when $r = 0$. In reality, this can be treated as a tiny hydrostatic protostar (or an incipient black hole in the general relativistic case) to prevent from singularity (see Shu, 1992, p.248). Once this state is reached, it becomes unstable to dynamical collapse.

2.1.2 Basic Equations

The governing equations for a spherically symmetric system are the equation of continuity (written in mass shell form) and the Euler equation,

$$\begin{aligned}\frac{\partial M}{\partial t} &= -u\frac{\partial M}{\partial r}, \quad \frac{\partial M}{\partial r} = 4\pi r^2 \rho \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} &= -\frac{a^2}{\rho}\frac{\partial \rho}{\partial r} - \frac{GM}{r^2},\end{aligned} \tag{2.6}$$

where M is the mass of the gas sphere at radius r, u is the fluid velocity and ρ is the density.

For the density profile mentioned in equation (2.5), there are no physical quantities to scale the system properly except the gravitational constant G and the speed of sound c_s. In this case, a self-similar variable is introduced to make the equations dimensionless,

$$x \equiv r/at \tag{2.7}$$

The governing equations can be rewritten into dimensionless form by writing every physical quantities into functions of the self-similar variable,

$$\rho(r,t) = \frac{\alpha(x)}{4\pi G t^2}, \quad M(r,t) = \frac{a^3 t}{G} m(x), \quad u(r,t) = a v(x). \tag{2.8}$$

With these substitutions, the partial differential equations can now be written into ordinary ones,

$$\begin{aligned}\frac{dv}{dx} &= \frac{[(x-v)\alpha - \frac{2}{x}](x-v)}{(x-v)^2 - 1} \\ \frac{1}{\alpha}\frac{d\alpha}{dx} &= \frac{[\alpha - \frac{2}{x}(x-v)](x-v)}{(x-v)^2 - 1}.\end{aligned} \tag{2.9}$$

The derivatives will diverge when the denominator on the right hand side becomes 0, that means there will be a set of critical points when

$$(x - v) = \pm 1$$

The equation of lower sign is actually mathematically extraneous for it cannot satisfy the fluid equation.

In order to ensure the continuity for the gas to accelerate smoothly through the critical line, the numerator must vanishing simultaneously. If this condition is not satisfied, there could only be a shock wave solution otherwise the derivative would diverge.

2.1.3 Solutions

The equation (2.9) has an analytic static solution,

$$v = 0, \quad \alpha = \frac{2}{x^2}, \quad m = 2x \tag{2.10}$$

Shu(1977), has found several sets of solutions to equation 2.9, here we list 2 sets of solutions in figure 2.1, the **expansion wave solution**(EWS) and the **minus solution with critical points**($MSCP$). The physical picture of EWS is, for a static SIS with initial condition given by equation (2.10), if the pressure of the central point is reduced slightly through some process, the unstable equilibrium will break and the gas sphere will start to collapse from the very center of the sphere. Once the bottom part of the gas sphere collapses, the gas right above the collapsed region would find that the gas supporting itself with gas pressure vanished, and begins to collapse as well. This solution gives an inside-out collapse scenario, the gas outside of the collapsing region will remain unchanged until the expanding front of collapse region reached. The front of collapse expands in the speed of sound, this solution is then called the *Expansion wave solution*.

Ultimately, in this paper we hope to find a shock wave solution to the problem in general relativity, thus we review the shock wave solution found by Tsai &

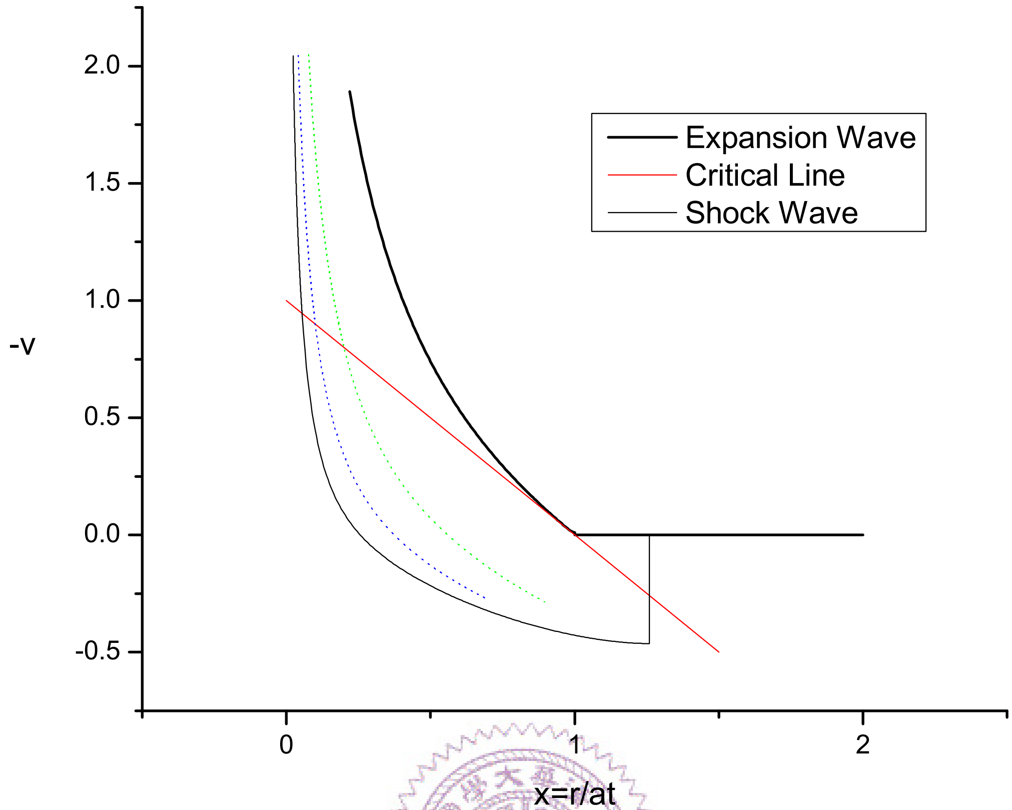

Figure 2.1: Newtonian Solutions:The thick solid curve is the expansion wave solution and the thin solid curve is the shock wave solution. The red straight line represents the critical line, while the dotted lines represent the MSCPs which cannot satisfy the shock jump condition.

Hsu(1995). Their solution extended the MSCP in the discussion of Shu (1977) by considering the existence of a shock front. The MSCP can be integrated toward the direction of x^+ which would intersect with the critical line again. However, the density has been fixed by requiring the numerator and denominator in equation (2.9) vanishing simultaneously, a shock jump condition is needed to connect the MSCP with the static solution,

$$v_2 = \frac{{v^2}_{shock} - a^2}{v_{shock}}, \quad \frac{\rho_2}{\rho_1} = (\frac{v_{shock}}{a})^2, \tag{2.11}$$

where the footnotes 1,2 denote the physical quantities in the preshock and postshock region, v_{shock} is the propagation speed of shock front and c_s is the speed of sound. Region 1, the preshock region is static so $v_1 = 0$. The isothermal jump conditions are basically the conservation of particle number and momentum across the shock front.

Their shock wave solution represents a mode of collapse, which allows some material being accreted onto the central core, but interacting with the external energy source at the same time. This extra energy brought in can make the outer part of the gas unbound to become a shock wave.

2.2 Relativistic Collapse

2.2.1 Basic Equations

In the following discussions, except when stated explicitly, we'll adopt the geometric unit where $c = G = 1$.

Schwarzschild Coordinate

In general relativity, a SIS can be described by the Einstein equations and the equation of conservation, i.e. divergence free of the stress-energy tensor:

$$G^{\mu\nu} = 8\pi T^{\mu\nu}, \quad T^{\mu\nu}{}_{;\mu} = 0. \tag{2.12}$$

The very first step is to choose the appropriate coordinate. The Schwarzschild coordinate is commonly used, for its simplicity and intuitive physical interpretation. (For simplicity, we use "SSS" to denote the Self-Similar Schwarzschild coordinate) In the discussion of Cai & Shu(2005), the Schwarzschild metric can be written in a self-similar form which the metric coefficients depend only on a self-similar variable,

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$
$$\alpha = \alpha(\zeta), a = a(\zeta), \zeta = \frac{r}{t}, \tag{2.13}$$

where the notation r,t,θ, ϕ are the radius, time and angles used in a spherical coordinate, ζ is defined to be the self-similar variable. This metric is equivalent to a tetrad basis:

$$e_{(0)}{}^{\mu} = (\alpha^{-1}, 0, 0, 0) \tag{2.14}$$

$$e_{(1)}{}^{\mu} = (0, a^{-1}, 0, 0,) \tag{2.15}$$

$$e_{(2)}{}^{\mu} = (0, 0, r^{-1}, 0) \tag{2.16}$$

$$e_{(3)}{}^{\mu} = (0, 0, 0, r^{-1} sin^{-1}\theta). \tag{2.17}$$

With these basis, the four-velocity is defined by

$$u^{(a)} = (\frac{1}{\sqrt{1-v^2}}, \frac{v}{\sqrt{1-v^2}}, 0, 0), \tag{2.18}$$

for v is the three-velocity measured by the observer at r equals to a constant. The scaled energy density and the velocity function are defined here for later calculational convenience, by

$$\varepsilon = 4\pi r^2 (1+\gamma)\rho a^2, \quad \beta = -\frac{2v}{1-v^2}, \tag{2.19}$$

where ρ and γ are the rest energy density and the square of the sound speed ($\gamma = {c_s}^2$). The isothermal equation of state gives a simple relation between the pressure and the density, $P = \gamma\rho$.

With these substitutions and the isothermal equation of state, the Einstein equation can be written as

$$(\ln a'' + \ln a \ln x')(1 - x^2) - \ln x'' + (2 + 2\ln a' - \ln x')(1 - \ln x') \tag{2.20}$$

$$= \varepsilon(\frac{\beta^2}{\sqrt{\beta^2+1} - 1} - \Gamma) \tag{2.21}$$

$$\ln a'' = -\frac{\beta\varepsilon}{x} \tag{2.22}$$

$$\ln x' = 1 - \varepsilon\sqrt{\beta^2+1} - \frac{\beta\varepsilon}{x} \tag{2.23}$$

$$-1 - \varepsilon\sqrt{\beta^2+1} - \frac{\beta\varepsilon}{x} + a^2 = \Gamma\varepsilon \tag{2.24}$$

($' \equiv \frac{d}{d\ln\zeta}$, for notational economy)

After some significant algebra, the Einstein equation and the conservation equation can be reduced into the following ordinary differential equations,

$$x' = x - x\varepsilon\sqrt{\beta^2+1} - \beta\varepsilon \tag{2.25}$$

$$\varepsilon' = -\varepsilon^2\frac{\beta}{x}(2 - \frac{\Gamma}{D}) - \frac{\beta'}{D}(\frac{1}{x} + \frac{\beta}{\sqrt{\beta^2+1}}) \tag{2.26}$$

$$\beta' = -\frac{\varepsilon\beta\Gamma[\beta(x + \frac{1}{x}) + 2\sqrt{\beta^2+1}] + 2x(\varepsilon - 1 + \Gamma)D}{(x^2-1)/\sqrt{\beta^2+1} \ + \ (2\beta x/\sqrt{\beta^2+1} \ + \ x^2+1)\Gamma}. \tag{2.27}$$

In the equations (2.20 $\sim$ 2.26) several substitutions are used to simplify the form of the equations, the first is x, which can be understood as the "proper self-similar variable",

$$x^2 = \frac{a^2}{\alpha^2}\zeta^2 = -\frac{g_{rr}r^2}{g_{tt}t^2}, \tag{2.28}$$

and

$$D \equiv \frac{\beta}{x} + \sqrt{\beta^2+1} + \Gamma, \Gamma \equiv \frac{1-\gamma}{1+\gamma}. \tag{2.29}$$

Comoving Coordinates

In order to study the dynamics of the growing event horizon and the massive singularity, where the SSS is not applicable, another coordinate must be introduced. Choose the comoving coordinate here, the metric can be written in this form,

$$ds^2 = -e^{2\Phi} dT^2 + e^{2\Lambda} dR^2 + e^{2\omega} R^2 (d\theta^2 + \sin^2\theta d\Phi^2), \tag{2.30}$$

where $r = e^{\omega} R$ is the circumferential radius as before. $\xi = R/T$ is the self-similar variable in the comoving frame. All the unknown metric coefficients Φ, Λ, and ω will be functions of ξ if self-similarity is enforced.

In this coordinate, the four-velocity is now a unit vector pointing in the time direction,

$$u^{(a)} = (1, 0, 0, 0). \tag{2.31}$$

In this coordinate the stress-energy tensor will take a quite simple form

$$T^{(0)(0)} = \rho, \quad T^{(i)(j)} = \gamma\rho\delta^{(i)(j)}. \tag{2.32}$$

The equation $T^{(a)(b)}{}_{;(b)} = 0$ has 2 components:

$$\ln\bar{\varepsilon} = \frac{2\Gamma\Lambda - 4\omega}{1+\Gamma} + C_1 \tag{2.33}$$

$$\ln\bar{\varepsilon} = \frac{2\ln y - 2\Gamma\Lambda - 2\Gamma\ln\xi}{1-\Gamma} + C_2, \tag{2.34}$$

where

$$\bar{\varepsilon} = 4\pi R^2 (1+\gamma)\rho e^{2\Lambda}. \tag{2.35}$$

$\bar{\varepsilon}$ is the energy function and $y = e^{\Lambda-\Phi}\xi$ is the "proper self-similar variable" in the comoving coordination, (we use SSC to denote the Self-Similar Comoving frame in the following discussion) the notation $'$ denotes $\frac{d}{d\ln\xi}$. The constants C_1

and C_2 are integration constants that correspond to the freedom of rescaling the radial coordinate. They can be fixed by the boundary condition given by the equilibrium solution outside the collapsing region.

Equation (2.33),(2.34) can be reduced into

$$\ln y = (1-\gamma)\Lambda + \Gamma \ln \xi - 2\gamma\omega + \frac{1-\Gamma}{2}(C_1 - C_2) \tag{2.36}$$

Again, one can get a set of ordinary equations from calculating Einstein equation and the equation of conservation with the substitution above,

$$\Lambda' = \frac{\epsilon - 1 + \Gamma - 2\Omega\gamma}{\gamma - y^2} \tag{2.37}$$

$$\bar{\varepsilon}' = \bar{\varepsilon}(-2(\gamma+1)\Omega + (1-\gamma)\Lambda') \tag{2.38}$$

$$y' = y(\Lambda'(1-\gamma) + \Gamma - 2\Omega\gamma) \tag{2.39}$$

$$\Omega' = \Omega^2(2\gamma - 1) + \Lambda'[\Omega(1+\gamma) + 1] - \Gamma\Omega, \tag{2.40}$$

where Ω is defined by $\Omega \equiv \omega'$.
Note that Λ and ξ are absent in the equation (2.19), so there are only two degrees of freedom.

2.2.2 Transformation between coordinates

A coordinate transformation can be written in the following form

$$\begin{aligned} g^{rr} &= \frac{\partial r}{\partial R}\frac{\partial r}{\partial R}g^{RR} + \frac{\partial r}{\partial T}\frac{\partial r}{\partial T}g^{TT} \\ g^{tt} &= \frac{\partial t}{\partial R}\frac{\partial t}{\partial R}g^{RR} + \frac{\partial t}{\partial T}\frac{\partial t}{\partial T}g^{TT} \\ g^{rt} &= \frac{\partial r}{\partial R}\frac{\partial t}{\partial R}g^{RR} + \frac{\partial r}{\partial T}\frac{\partial t}{\partial T}g^{TT}. \end{aligned} \tag{2.41}$$

In details, for $r = e^w R, \quad t = e^\tau T, \quad \zeta = \xi e^{w-\tau}$,

$$\frac{\partial r}{\partial R} = e^\omega(\Omega + 1), \quad \frac{\partial t}{\partial R} = \frac{\tau' e^\tau}{\xi}$$
$$\frac{\partial r}{\partial T} = -\xi \Omega e^\omega, \quad \frac{\partial t}{\partial T} = e^\tau(1 - \tau').$$

Then, the metric coefficient can now be written as

$$\begin{aligned} g^{rr} &= e^{2\omega-2\Lambda}(1+\Omega)^2 + e^{2\omega-2\Phi}\xi^2\Omega^2 \\ g^{tt} &= e^{2\tau-2\Lambda}\xi^{-2}\tau'^2 + e^{2\tau-2\Phi}(1-\tau')^2 \\ g^{rt} &= e^{\omega+\tau-2\Lambda}\xi^{-1}\tau'(1+\Omega) + e^{\omega+\tau-2\Phi}\xi\Omega(1-\tau') = 0. \end{aligned} \tag{2.42}$$

Note that $g^{rt} = 0$ for the coordinate r and t are demanded to be orthogonal basis, τ' can than be evaluated

$$\tau' = -\frac{y^2\Omega}{(1+\Omega) - y^2\Omega}. \tag{2.43}$$

In addition, the transformations of four-velocity are similar,

$$\begin{aligned} u^r &= \frac{\partial r}{\partial T}u^T = -\xi\Omega e^{\omega-\Phi} \\ u^t &= \frac{\partial t}{\partial T}u^T = e^{\tau-\Phi}(1-\tau') \end{aligned} \tag{2.44}$$

The energy function should be a scalar, so the relation between $\varepsilon, \bar{\varepsilon}$ would be

$$\begin{aligned} \bar{\varepsilon} &= 4\pi R^2(1+\gamma)\rho e^{2\Lambda} \\ \varepsilon &= 4\pi r^2(1+\gamma)\rho a^2 \\ \rightarrow \bar{\varepsilon} &= e^{2\Lambda-2\omega}\frac{\varepsilon}{a^2} \end{aligned} \tag{2.45}$$

Now, rewrite the equations with the following substitution:

$$\begin{aligned} &\xi = e^{\Phi-\Lambda}y, \quad \zeta = e^{\Phi-\Lambda+\omega-\tau} \\ &e^{2\omega-2\Lambda} = \frac{\varepsilon}{\bar{\varepsilon}a^2} \end{aligned} \tag{2.46}$$

and rearrange some notations, the transformations become

$$
\begin{aligned}
u^r &: -\sqrt{\frac{\varepsilon}{\bar{\varepsilon}}}y\Omega = \frac{v}{\sqrt{1-v^2}} \\
u^t &: \frac{\varepsilon}{\bar{\varepsilon}}(1-\tau')^2 y^2 = \frac{x^2}{\sqrt{1-v^2}} \\
g^{rr} &: \frac{1}{a^2} = \frac{\varepsilon}{\bar{\varepsilon}a^2}[(1+\Omega)^2 - y^2\Omega^2] \\
g^{tt} &: x^2 = y^2\frac{\varepsilon}{\bar{\varepsilon}}(\frac{\tau'}{y^2} - (1-\tau')^2)
\end{aligned}
\tag{2.47}
$$

These equations are written into the functions of quantities that actually being used in the numerical calculation.

2.2.3 Critical Points

In the equation (2.27) and (2.37), one can easily discover that there are critical points in both coordinates. For SSS, when the denominator in equation 2.27 vanishes

$$\frac{x^2-1}{\sqrt{\beta^2+1}} + \Gamma(\frac{2\beta x}{\sqrt{\beta^2+1}} + x^2 + 1) = 0,$$

β' becomes singular. In order for gas to accelerate smoothly across these critical curves, the numerator is required vanishing at the same time,

$$\varepsilon\beta\Gamma[\beta(x + \frac{1}{x} + 2\sqrt{\beta^2+1})] + 2x(\varepsilon - 1 + \Gamma)D = 0$$

Solving these 2 equations, one can obtain the condition for crossing critical curves without a shock:

$$x = \frac{-\beta\Gamma \pm \sqrt{1-\Gamma^2}}{1+\Gamma\sqrt{\beta^2+1}}, \varepsilon = 1 - \Gamma \mp \beta\Gamma\sqrt{\frac{1-\Gamma}{1+\Gamma}} \tag{2.48}$$

The lower sign correspond to negative x, which is physically extraneous. To evaluate the value of the derivatives on the critical curve, the l'Hopital's rule is applied here for the expression β'. Again, after some algebra, the value of β' on the critical curve can be expressed as

$$\beta' = \frac{-m \pm \sqrt{m^2 - 4ln}}{2l} \tag{2.49}$$

l, m and n are notations to make this equation shorter,

$$l \equiv \frac{\beta - \beta x^2 + 2x\Gamma}{(\beta^2+1)^{3/2}}$$

$$m \equiv 2(x - \varepsilon(\beta + x\sqrt{\beta^2+1}))\frac{\sqrt{1-\Gamma^2}}{\sqrt{\beta^2+1}} + 2x\Gamma D\sqrt{\frac{1-\Gamma}{1+\Gamma}} - \frac{\varepsilon\beta(x^2-1)}{x\sqrt{\beta^2+1}}$$

$$- 2(1 + \frac{x\beta}{\sqrt{\beta^2+1}})(\beta\Gamma\sqrt{\frac{1-\Gamma}{1+\Gamma}} + 1 - \Gamma)$$

$$n \equiv \Gamma(1 - \varepsilon(D\Gamma))[\frac{\varepsilon\beta^2}{x}(x^2-1) - 2\varepsilon\beta(2D-\Gamma)(1-\Gamma)]$$

That means, one can arbitrarily pick a value β_c on the critical curve and then obtain the value of the derivatives of β with equation (2.49) hence the derivatives of ε and x by equation (2.25),(2.26). This initial condition on the critical curve makes the ordinary differential equations (eq 2.25∼2.27) numerically solvable via a fourth-order Runge-Kutta method.

As for SSC, same thing happens on the equation of Λ' (eq. 2.37), in order to avoid the singularity here, the denominator and the numerator are demanded to be vanishing simultaneously

$$\bar{\varepsilon} = 1 - \Gamma + 2\Omega\gamma, \quad y^2 = \gamma \tag{2.50}$$

Then Λ' can be evaluated using l'Hopital's rule:

$$\Lambda' = 4\gamma\Omega + 1 - 2\Gamma \pm \sqrt{(1-2\Gamma)^2 + 8[\gamma + \frac{\Gamma^2}{1+\Gamma}]\Omega + 4\Omega^2(3\gamma+1)} \times [2(1-\gamma)]^{-1} \tag{2.51}$$

With equation (2.50) and (2.51) , the value of Λ' and $\bar{\varepsilon}$ on the critical curve in SSC can be obtained by choosing an arbitrary Ω_c. The derivatives of $y, \bar{\varepsilon}, \Omega$ on the critical curve can also be determined by equation (2.38),(2.39),(2.40). Now the ODEs in SSC can be solved numerically with fourth-order Runge-Kutta method.

2.2.4 Solutions

Equilibrium Solutions

An analytic solution to the above equations in SSS is given by

$$\beta = 0, \quad \varepsilon = 1 - \gamma, \quad a^2 = 2 - \Gamma^2, \quad x = (C\zeta)^{\Gamma}, \tag{2.52}$$

which represents the hydrostatic equilibrium. C is an arbitrary integration constant, since equations (2.25)∼(2.27) are invariant to a rescaling $\zeta \to C\zeta$, it can be set to $C = 1$.

A counterpart can be found in SSC, too

$$\bar{\varepsilon_0} = 1 - \Gamma, \quad \omega_0 = \Omega_0 = 0, \quad e^{2\Lambda_0} = 2 - \Gamma^2, \quad y_0 = \xi^{\Gamma}. \tag{2.53}$$

The integration constants in equation (2.36) can be determined with this equilibrium solution,

$$C_1 = \ln \bar{\varepsilon}_0 - (1-\gamma)\Lambda_0, C_2 = \ln \bar{\varepsilon}_0 + (\gamma^{-1} - 1)\Lambda_0. \tag{2.54}$$

General Solutions Cai & Shu(2005), have found a few sets of solutions to equation (2.25)∼ (2.27). Like its Newtonian counter part, the expansion wave solution in General Relativity proceeds basically the same scenario, but this time it is accompanied by a formation of massive black hole in the center. The expansion wave of collapse in general relativity also travels at speed of sound.

We suspect the shock wave solution also exists in general relativity, so there's another set of solution we're interested in. That is, the collapse solution with critical point(CSWCP). As shown in fig 2.2, the CSWCP hits the line $\beta = 0$ and tends to hit the critical curve again at $\beta < 0$, which corresponds to a wind solution. This makes them potential candidates of shock wave solution.

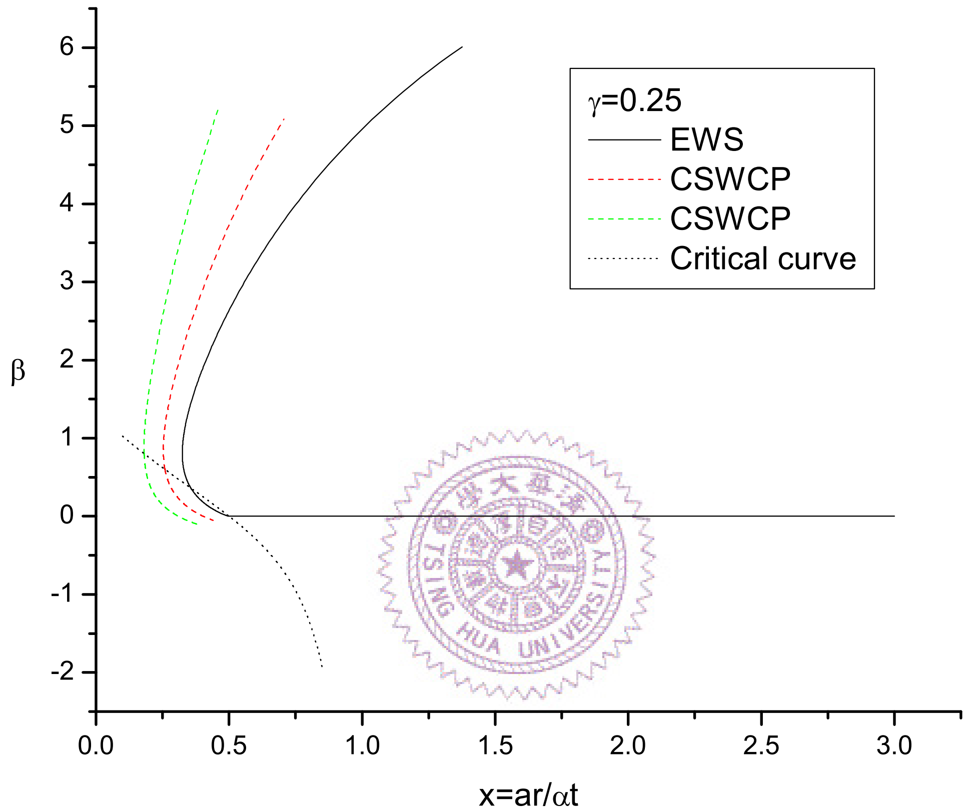

Figure 2.2: General Relativistic Solutions:$\gamma = 0.25$, the dotted curve is the critical curve, the solid black curve the expansion wave solution and the dashed curve are the collapse solution with critical points, different color stands for different initial conditions.

Chapter 3

Relativistic Shockwave solution

In the reviews in the previous chapter, the **collapse solution with critical points** could be the possible candidate of shock wave solution. If we integrate them toward positive x, we'll find them rendezvous with the critical curve again. At the initial point on the critical curve, the singularity was avoid by requiring the numerator vanished simultaneously(eq 2.48). Physically, if we want the CSWCPs inside to connect with the static solutions outside, a shock wave must introduced at the connecting point. A **shock jump condition** can achieve our goal.

3.1 Shockwave Jump Condition

As mentioned in Landau & Lifshitz(1959), in a frame that comoving with the shock front, the jump conditions for the general relativistic fluid moving only in radial direction through a discontinuous surface can be written as

$$n_1 v_1^r = n_2 v_2^r \tag{3.1}$$

$$T_1^{tr} = T_2^{tr} \tag{3.2}$$

$$T_1^{rr} = T_2^{rr} \tag{3.3}$$

The first condition is the conservation of number flux, the baryon number flows from one region to another should be a conserved quantity. The second is the energy flux conservation and the last one is the momentum conservation. In the case of singular isothermal sphere, we assume that the radiating efficiency is so good that any heat or energy produced on the shock front will be radiated away immediately, then the conservation of energy should not be valid on the shock front. There are now only two jump conditions here, written in components,

$$\rho_1 v_1 \gamma_1 = \rho_2 v_2 \gamma_2 \tag{3.4}$$

$$(\rho_1 + P_1){v_1}^2{\gamma_1}^2 + P_1 = (\rho_2 + P_2){v_2}^2{\gamma_2}^2 + P_2 \tag{3.5}$$

where $\gamma_1 = \frac{1}{\sqrt{1-v_1^2}}, \quad \gamma_2 = \frac{1}{\sqrt{1-v_2^2}}$.

Here we replace the number density by rest energy density divided by rest mass of a particle ($n = \frac{\rho}{\mu}$), the footnote 1, 2 represent the physical quantities in the postshock and preshock region, the notations ρ, P and v are the internal energy, pressure and the velocity, γ_1,γ_2 are the Lorenz factor in region 1 and 2 with respect to shock rest fame. (Reminder: γ is differed from γ_1 and γ_2, it's the square of isothermal sound speed.)

The Landau-Lifshitz jump conditions are written in a flat space-time, with Cartesian coordinate basis comoving with the shock front. In our calculations, we use a set of tetrad basis as our coordinate, which is by definition an orthonormal basis. The principle of relativity also ensures the tetrad basis local flatness near the shock front, thus the Landau-Lifshitz jump condition can be directly applied to our calculation. A local Lorenz transformation can be made between the shock rest frame (SRF) and SSS.

That is,

$$T^{\alpha\beta}{}_{sss} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta T^{\mu\nu}{}_{srf}$$

The transformation matrix is

$$\Lambda = \begin{pmatrix} \gamma_s & -\gamma_s v_s & 0 & 0 \\ -\gamma_s v_s & \gamma_s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

while

$\gamma_s = \frac{1}{\sqrt{1-{v_s}^2}}$,

v_s is the speed of shock front and v is the post-shock fluid velocity in SSS. In practice, we can substitute v_s with x, the proper self-similar variable in SSS. Since $x \equiv \frac{\sqrt{g_{rr}}r}{\sqrt{-g_{tt}}t}$, the value of x on the location of the shock front in the $x-\beta$ diagram will be the proper expansion speed of the shock wave, that is, $v_s = x_s$.

We now have the jump conditions in SSS:

Number flux conservation

$$\rho = \rho_0 \frac{-x\sqrt{1-v^2}}{v-x} \tag{3.6}$$

Momentum flux conservation:

$$\rho[x^2{\gamma_v}^2(1+\gamma) - x^2\gamma + v^2{\gamma_v}^2(1+\gamma) + \gamma - 2xv{\gamma_v}^2(1+\gamma)] - x^2\rho_0 = \gamma\rho_0 \tag{3.7}$$

$\gamma_v = \frac{1}{\sqrt{1-v^2}}$,

the footnote 0 represents the quantities in the static region and the quantities without any footnote are in the postshock region.

To evaluate the jump conditions in practice, the following substitutions are applied on the equation(3.2) and (3.3):

$$\beta = -\frac{2v}{1-v^2} \rightarrow v = \frac{1 \pm \sqrt{1-\beta^2}}{\beta},$$

minus sign is preferred here, since we're looking for a wind solution where $\beta < 0$ and $v > 0$. As for the density ρ, it can be replaced by the energy function ε

$$\frac{\varepsilon}{4\pi r^2(1+\gamma)a^2} = \rho.$$

Newtonian Limit

To search the jump conditions in Newtonian limit, the metric coefficients are rewritten into the form

$$g_{tt} = -1 - 2\Phi + O(\epsilon^4), \quad g_{ij} = \delta_{ij} + O(\epsilon^2),$$

where ϵ is the order of smallness. The velocity v and speed of sound $\sqrt{\gamma}$ are typically of the order of $O(\epsilon)$, and the gravitational potential Φ is $O(\epsilon^2)$. In the nonrelativistic limit, the self-similar variable $\zeta = r/t$ is on the order of sound speed, then the following dimensionless variables can be defined,

$$\eta = \frac{\zeta}{\sqrt{\gamma}}, \quad u(\eta) = \frac{v}{\sqrt{\gamma}}, \quad \lambda(\eta) = 4\pi t^2 \rho.$$

These variables are the counterpart of the self-similar variable, velocity variable and density variable in the Newtonian limit, which are identical to equation 2.8, except the notational difference.

Keep the auccuracy to the second order of ϵ, the fluid variables and the metric coefficients now reduce to

$$\beta = -2\sqrt{\gamma}u, \quad \varepsilon = \lambda\eta^2$$
$$a^2 = 1 + \epsilon^2 q, \quad \alpha^2 = 1 + 2\Phi, \quad x = \sqrt{\gamma}\eta(1 - \Phi + \frac{\epsilon^2}{2q})$$

Substitute these new variables into the jump equation (eq. 3.6). Neglect the terms higher the $O(\epsilon^2)$ and make use of the Newtonain static solution (equation 2.10), the jump condition for number flux can be written as

$$\lambda = \frac{-2}{\eta(u-\eta)}. \tag{3.8}$$

Similarly, rewrite the jump condition for momentum to the accuracy of $O(\epsilon^2)$ and make use of equation 3.8, 2.10,

$$u = \frac{-1+\eta^2+2\gamma\eta^2}{(1+\gamma)\eta}. \tag{3.9}$$

Combine this equation with equation 3.8, and take the limit further to the first order of $O(\epsilon)$, equation 3.8, 3.9 now reduce to

$$\begin{aligned} &\lambda = 2 \\ &u = \eta_s - \frac{1}{\eta_s} \end{aligned} \tag{3.10}$$

Apart from the notational difference, these jump conditions in Newtonian limit are identical to the jump conditions in Tsai & Hsu(1995). We use this results to make sure there is no calculational mistake in the relativistic jump conditions.

3.2 Numerical Results

3.2.1 Shockwaves in Schwarzschild Coordinate

In section 2, the Einstein equation has been reduced to ordinary differential equations, which are now manageable via the classic fourth-order Runge-Kutta method.

At first, we arbitrarily choose the value of β on the critical line and use equation (2.48, 2.49) to construct the necessary initial conditions. Since ζ is absent in the ODEs, it's only an arbitrary number.

While integrating the differential equations, we can get the value of x, β and ε for every steps, substitute them into the left-hand-side of equation (3.6) and (3.7), we can simultaneously check if the jump conditions are satisfied during the integration. We used different β as our initial value of integration and found that

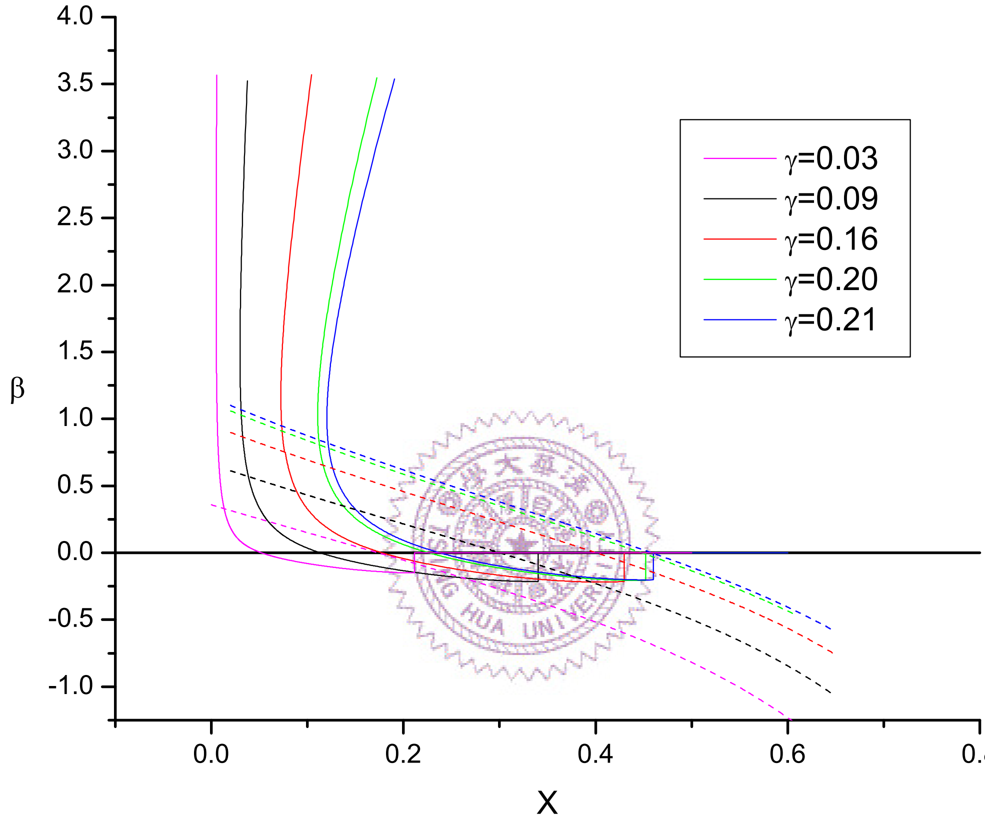

Figure 3.1: Shock Solutions : Different color correspond to different sound speeds, the solid lines are the solutions while the dash lines are the corresponding critical curves.

there is only one unique initial value for each γ that the jump conditions are both satisfied. If the jump conditions are satisfied before the solution curve hits the critical curve and $x_s > \sqrt{\gamma}$, we claim that a shock solution is located.

Several results are listed in Table.(1) and shown in Fig (3.1). The results show that shock solutions do exist in general relativistic case, and we also find an intriguing fact that the shock solutions are capped by the sound speed of fluid. In Table.(1), the Mach number of the shock fronts decrease with $\gamma = (c_s)^2$ increases, the Mach number decreases to 1 roughly at $\gamma = 0.22$. It is not surprising for the existance for an upper limit, but this limit is differed from the ultra-relativistic limit for sound speed, $\gamma = 0.33$. Currently we do not have a good explanation to this value of upper limit, we speculate that the problem lies in one of the assumption we made to simplify the problem, for the SIS is not a physically realistic configuration. Besides an upper limit, there is also a lower limit of sound speed for the shock solution. In our calculation, we cannot find a shock solution for $\gamma < 0.02$. This lower limit might correspond to the case of Newtonian limit, where the relativistic version of shock solution does not exist.

Table 1: Shock wave solutions

γ	x_{shock}	M(Mach number)	x_0
0.03	0.2116	1.2192	0.052264
0.05	0.2655	1.1874	0.072645
0.09	0.3400	1.1344	0.113447
0.16	0.4287	1.1025	0.176686
0.20	0.4520	1.0213	0.226811
0.21	0.4603	1.0088	0.238191
0.22	N/A	N/A	0.251464

γ is the square of sound speed, x_{shock} is the position of shock front in SSS, which correspond to the propagating speed of shock front. x_0 denotes the position of the temporarily static front, the region $x > x_0$ is the wind solution and the region $x < x_0$ correspond to the collapse solution.

3.2.2 Comoving Frame

One of the biggest differences between the Newtonian and general relativistic SIS is the mass of the central object formed from collapse. The study of the expansion wave of collapse for general relativity leads to the formation of a black hole in the center. The shock wave solution, as shown in the previous section, is also accompanied by the formation of a massive singularity in the center. The Schwarzschild coordinate is insufficient for the physics on the event horizon, we shall again adopt the comoving coordinate. Although we have found the jump conditions in the Schwarzschild coordinate, but it is not as simple in the comoving coordinate. Since the velocity in the comoving frame is by definition zero, the number flux on the discontinuous surface would be zero. In this case, there won't be a simple continuity equation connecting the 2 sides of the shock front. The coordinate itself is now discontinuous for the both sides of the shock front. That means the Lorentz transformations for both side are different. A discontinuous coordinate is quite complicate, besides it is not necessary for us to re-calculate the equations in the SSC through the whole space. The shock solution has been found with SSS, the most important thing concerned in SSC is how the solution behave near the event horizon and the central singularity. Cai & Shu (2005), discussed the propagation speed of different surfaces, the expansion speed of event horizon is slower than the speed of expansion wave front, i.e. speed of sound, hence the shock propagating speed. Since the shock front is always located at the outer region away from the event horizon, the physics near the shock front can than be throughly explained with the Schwarzschild coordinate.

The shock wave solution in SSS is not completed yet, the behavior of the shock wave solution around the event horizon should be evaluated in the comoving frame. We start the evaluation by using the initial condition on critical curve (equation 2.50, 2.51.) This time the boundary is not arbitrarily chosen, the solutions in SSC should correspond to the physically identical shock wave solution. In order to make sure the solutions in different coordinates correspond to the same shock wave, the initial condition for SSC should be assigned by the initial condition for SSS through the coordinate transformation equation (eq 2.47). In equation 2.50, $y = \sqrt{\gamma}$ is the definition of critical curve which can be determined without coordinate transformation. Written in components for the both coordinates, the coordinate transformations for the rest variables (equation (2.47)) become

$$
\begin{aligned}
&-\sqrt{\frac{\varepsilon_c}{\bar{\varepsilon}}}y\Omega = \frac{v_c}{\sqrt{1-{v_c}^2}} \\
&\frac{\varepsilon_c}{\bar{\varepsilon}a_c^2}((1+\Omega)^2 - y^2\Omega^2) = \frac{1}{{a_c}^2} \\
&\frac{\gamma\varepsilon_c\left(1-\frac{\gamma\Omega}{1+\Omega-\gamma\Omega}\right)^2}{1-\frac{1-\gamma}{1+\gamma}+2\gamma\Omega} = \frac{x_c^2}{1-v_c^2} \\
&\frac{\gamma\varepsilon_c\left(\frac{\gamma\Omega^2}{(1+\Omega-\gamma\Omega)^2} - \left(1-\frac{\gamma\Omega}{1+\Omega-\gamma\Omega}\right)^2\right)}{1-\frac{1-\gamma}{1+\gamma}+2\gamma\Omega} = -{x_c}^2,
\end{aligned}
\tag{3.11}
$$

where the footnote c for x_c, v_c, a_c represents these quantities are from the initial condition on the critical curve for the shock wave solution in SSS. y is known once γ is chosen and $\bar{\varepsilon}$ can be substitute by equation 2.50.

After all, these coordinate transformation equations all acquire the same solution for Ω, giving no informations for ξ, Λ and ω. Additional equations is needed, luckily the starting point on critical point is not the only connection between both coordinates. Recall that all the shock solutions (or CSWCPs) have similar behavior on the x-β diagram, they pass the critical curve smoothly at the 1st quadrant ($\beta > 0$), which correspond to a collapse solution inside, and intersect with the line $\beta = 0$ than approach the critical curve at the point $\beta < 0$, which correspond to a wind solution. The line $\beta = 0$ is a temporarily static region separating the collapse and wind region The transformation relations of velocity is $u^r = -\xi\Omega e^{\omega-\Phi}$, applied on the intersection of shock wave solution and $\beta = 0$:

$$u^r = 0 \rightarrow \Omega = 0 \tag{3.12}$$

The fluid velocity in SSC was defined to be $(1, 0, 0, 0)$, the fact that $u^r = 0$ in SSS satisfies the comoving definition, then we can conclude that at this particular event in space-time, the observer in Schwarzschild coordinate is **comoving** with the fluid itself, i.e. the Schwarzschild coordinate and the comoving frame are identical at the intersection point.

Now we have additional equations to determine ξ, ω ,Λ and the integral coefficients:

$$g_{rr} = g_{RR} \rightarrow a^2 = e^{\Lambda}$$
$$\Lambda = \frac{1}{2}\ln(1 + \varepsilon D)$$
$$\varepsilon = \bar{\varepsilon} \rightarrow \omega = 0$$

here we use equation 2.24 to solve the energy function,

$$\varepsilon = \frac{a^2 - 1}{D},$$

and make use of equation 2.44 as well: $e^{2\omega-2\Lambda} = \frac{\varepsilon}{\bar{\varepsilon}a^2}$

The self-similar variable ξ in SSC can then be obtained using eq 2.36:

$$\xi = \frac{exp(\ln y - (1-\gamma)\Lambda + 2\gamma\omega - \frac{1-\Gamma}{2}(C_1 - C_2))}{\Gamma}. \tag{3.13}$$

The integral coefficients $C_1 - C_2$ is the same as those obtained from the equilibrium solution. Since they're embedded in the metric coefficients which must be continuous everywhere(except the central singularity). The discontinuities through the shock front are presented in the form of jump conditions, which are the discontinuities for derivatives and second derivatives on metric coefficient.

Now we have everything determined.

The self-similar variable ξ and Λ was absent in the equation 2.37∼2.40, thus one can find equation 2.37∼2.40 invariant to a rescaling $\xi \to C\xi$ and $\Lambda \to C\Lambda$. But they can be fixed by the relations in equation 310. After the rescaling, some physical quantities on comoving curve are shown in figure 3.2, we can see the solution terminates at ξ_* where y, g_{RR} and $-u^r$ diverges, which corresponds to a massive singularity. The behavior of the system in SSC near the event horizon are basically similar to the expansion wave solution, which implies the black hole solution.

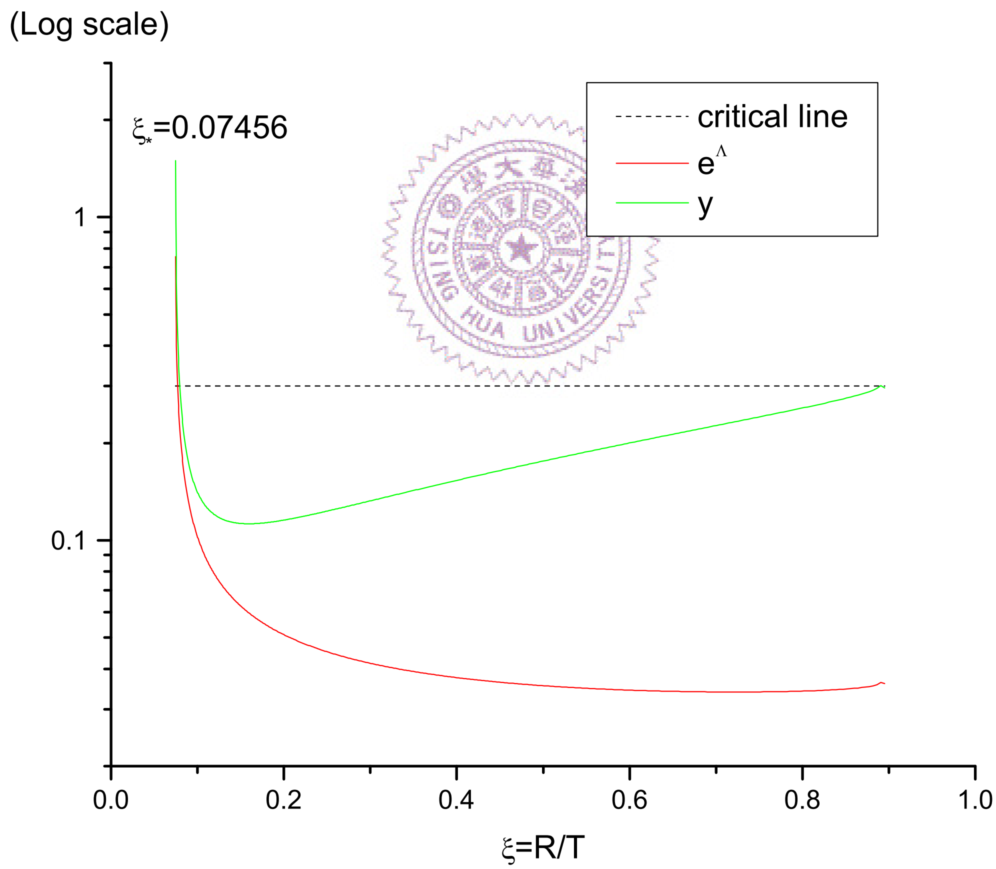

Figure 3.2: Metric coefficients in comoving frame

Chapter 4

Physics in Shockwave Solution

Mass accretion rate

The mass accretion rate can be obtained using the same way derived by Cai & Shu(2005), by evaluating the rest mass-energy enclosed in a radius R in comoving coordinate

$$\begin{aligned} M &= \int_0^R \rho\sqrt{-g}d^3x \\ &= \frac{\bar{\varepsilon}_0}{1+\gamma}\frac{T}{2-\Gamma}\xi^{2-\Gamma}. \end{aligned}$$

Since there is no mass shell crossing during the collapse, the total baryon number inside a radius R is conserved, we can use R to label each mass shell. In the comoving coordinate, the singularity is given by $R = \xi_* T$, then the mass collapsed into the singularity would be

$$M_* = \frac{\bar{\varepsilon}_0}{1+\gamma}\frac{T_*}{2-\Gamma}\xi_*^{2-\Gamma}.$$

Energy of Shock wave

To estimate the energy released in the shock front, we start by the evaluation of the equations of energy flux,

$$[T^{0x}] = T_1^{0x} - T^{0x}{}_2,$$

where the footnotes 1,2 correspond to the postshock and preshock region.
It is one of the jump condition derived by Landau & Lifshitz(1959), stands for *Conservation of energy* which we spared for considering isothermal equation of state. Written in components at the SRF(shock rest frame),

$$\begin{aligned}\triangle T^{0x} &= (1+\gamma)\rho_1 v_1 \gamma_1^2 - (1+\gamma)\rho_2 v_2 \gamma_2^2 \\ &= \frac{\varepsilon_1 v_1 \gamma_1^2}{4\pi r^2 a_1^2} - \frac{\varepsilon_2 v_2 \gamma_2^2}{4\pi r^2 a_2^2}, \quad \gamma_1 = \frac{1}{1-\sqrt{v_1^2}}, \gamma_2 = \frac{1}{1-\sqrt{v_2^2}}.\end{aligned} \tag{4.1}$$

Then we integrate over the shell of the shock front,

$$\begin{aligned}&\int \triangle T^{0x}\sqrt{-g}d^2x \\ &= \frac{1}{a_0}(\varepsilon_1 v_1 \gamma_1^2 - \varepsilon_2 v_2 \gamma_2).\end{aligned} \tag{4.2}$$

Since the equation is valid in SRF only, it is important for us to adopt the Jacobian $\sqrt{-g}$ in the flat space-time, $\sqrt{-g} = r^2 \sin\theta$. The preshock region corresponds to the static region, thus ε_2 is identical to the energy function in static region, which can substituted by equation (2.52).

The integration gives a result of the energy output rate at the shock front, compare to the baryonic mass enclosed in the shock front,

$$\begin{aligned}M &= \int_0^{r_s} \rho\sqrt{-g}d^3x, \quad (\sqrt{-g} = \alpha a r^2 \sin\theta) \\ &= \frac{\varepsilon}{1+\gamma}\frac{t_s}{2-\Gamma}\zeta_s^{2-\Gamma}.\end{aligned} \tag{4.3}$$

For a given radius r_s, there is a corresponding time t_s, which implies the time traveled by the shock front from the origin to $r = r_s$. The total energy released can be denoted by $(\int \triangle T^{0x}\sqrt{-g}d^2x) \times t_s$. Comparing the energy released and the total baryonic mass enclosed in the shock front, we can acquire an estimated energy releasing ratio. Several results are listed in Table 2.
We can see that the energy extraction ratio is about 10% of the enclosed rest mass, which is a significant number, considering the maximum energy extraction ratio of a Schwarzschild black hole to be 4.7%(The energy extracted by an object collapsing from infinite to the black hole's innermost stable circular orbit, see

Misner, Thorne & Wheeler, 1973.) The energy extraction ratio increases with increasing sound speed, which might explain why the sound speed is capped, for the total energy extraction ratio is limited. We can also find that the extracted energy for the lower limit of sound speed is only 3.38%. In the limit of small sound speed, the extracted energy might not be sufficient enough to trigger a shock wave, therefore the relativistic shock wave could only exist in the case of $\gamma > 0.03$.

Table 2. Energy of Shock Wave

γ	x_{shock}	Energy extraction ratio
0.03	0.21158	3.38%
0.05	0.26553	5.42%
0.09	0.34062	8.91%
0.16	0.42872	13.60%
0.20	0.45196	14.30%
0.21	0.46931	15.08%

Chapter 5

Conclusion and Discussion

In the study of Shu(1977), singular isothermal sphere can collapse through an inside-out fashion. Perturbation at the center of the SIS in precarious equilibrium induces an imbalance between gravity and gas pressure gradient. The central region of SIS then starts to collapse. The gas layer above the collapsed region would lose the support from gas pressure then starts to collapse as well, which triggers the collapse of next layer of gas. This front of collapse propagates outward at speed of sound, which is called "expansion wave of collapse". Cai & Shu(2005) extended the solution to the general relativistic regime, found the similar scenario happening, but the gas collapsed to form a black hole rather the to form a star.

The formation of black hole gives rise to an interesting problem. Since the Birkhoff theorem, the general relativistic version of the iron sphere theorem, is not valid in the non-vacuum dynamical space-time, the cloud outside the collapsing region would find a reduction of gravity when a black hole is formed in the center. However, the pressure gradient remains unchanged, the imbalanced force then triggered the static gas into an outgoing wind. If the wind travels at a super sonic speed, there could be a shock wave. In general, the gas can be accelerated from subsonic to supersonic smoothly as long as one can adopt a suitable density profile. However, this condition cannot be used again on another sonic point, which means the gas cannot be decelerated from supersonic to subsonic smoothly unless a shock wave is introduced. One of the sets of solutions in Cai & Shu(2005), the collapse solution with critical point(CSWCP), tend to hit the critical curve again at $\beta < 0$. In order to find a shockwave solution, we introduce the shock jump condition to the CSWCPs. With the numerical fourth order Runge-Kutta method, we evaluate the solution and the functions of jump conditions at the same time on every data point. If the jump conditions(eq 3.6,3.7) are satisfied in the supersonic region, there will be a shockwave solution.

We have considered no external energy input, for the shockwave is an intrinsic phenomenon embedded in the collapse of general relativistic SIS. Once the shock front carrying the information of gravity reduction reaches the outside static region , the static gas would become a wind. Before long, the wind will be slowed down by the growing gravity in the center and eventually collapses into the central black hole. We remind the readers that the concept of gravity reduced due to the difference of gravitational energy is not perfect. In fact, there is no "gravitational force" in General Relativity. The shock wave solution is the product of complex dynamical process of space-time. The explanation of gravity can help us understand the physical insight of this problem.

In this article, we have use the assumption of isothermal equation of state to simplify the mathematical challenge. Physically, isothermal equation of state implies the heat conducting rate is so well such that any energy produced on the shock front would be radiated away immediately. The "proper" energy in SIS is undetermined, since the dynamical space-time gives rise to the ambiguous definition of energy. There is neither time-like Killing vector nor well-defined outer boundary in SIS for one to really evaluate the energy. Nevertheless, a simplified method can still be used to evaluate the energy release ratio, by comparing the energy flowing out of the shock front to the baryonic mass enclosed. This gives us a flavor about the scale of the energy released. The range of isothermal sound of speed corresponds to the result that 5%-15% of the rest mass can be turned into radiation. For a stellar mass black hole with mass $M_{BH} \sim 10M_{\odot}$, the energy released is about $10^{54}erg$. These event are so energetic that can only be interpreted by a gamma-ray burst, for the typical energy for an extremely high energy GRB is about $10^{54}erg$. However, the case of spherically symmetric collapse is unrealistic, the existence of magnetic field and accretion disk could lead to a strong beaming effect which might even exaggerate the energy of gamma-ray photons observed.

We have made several assumptions to simplify the model, the SIS is assumed to be infinitely large and the space time will not be asymptotically flat, the solution must truncate at some finite radius then connect to a realistic boundary condition. The spherical symmetry also limited the possibility of gravitational radiation which carries the information of central gravitational influence. However, the shockwave solutions we have constructed still provides a basic idea to those most energetic events in the sky.

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