# Robotic Navigation and Exploration Lab 3

Fast-SLAM

Min-Chun Hu <u>anitahu@cs.nthu.edu.tw</u> CS, NTHU

# Requirement

- -Python 3.X
- Numpy
- Opency-Python

## Useful NumPy Operation

Matrix Multiply.

```
- np.matmul(A,B) / A.dot(B) / A @ B
```

- Matrix Transpose.
  - np.transpose(A) / A.T
- Identity Matrix.
  - np.eye(dimension)
- Sample from normal distribution with mean mu and variance var.

```
- mu + np.sqrt(var)*np.random.randn()
```

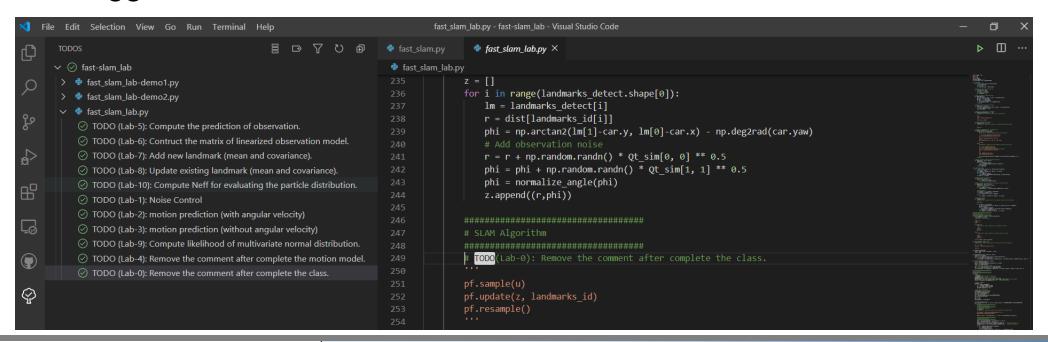
Matrix Inverse.

```
- np.linalg.inv(A)
```

- Matrix Determinant.
  - np.linalg.det(A)

#### TODO List

- In this lab, you need to complete each "TODO" comment in the codes:
  - # TODO(Lab-<ID>): <Explanation>
- The "Lab ID" implies the implementation order of the codes (except Lab-0).
- Suggest to install the "Todo Tree" extension in VSCode.



#### Main Function Workflow

- Set Parameters
- Create Landmarks
- Initialize Environment
- Repeat
  - Simulate Controlling
  - Simulate Observation
  - SLAM Algorithm

```
>pf.sample(u)
>pf.update(z, landmarks_id)
>pf.resample()
```

- Render Canvas
- Keyboard Events

```
u = (v, \omega, \Delta t)
z = [(r_{id1}, \theta_{id1}), (r_{id2}, \theta_{id2}), \dots]
landmarks_id = [id1, id2, \dots]
```

## Parameters Setting

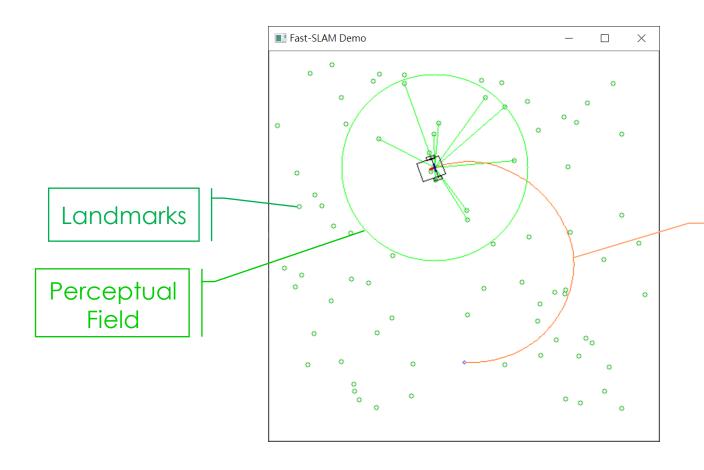
```
# Parameters
N_PARTICLES = 40  # Number of Particles
N_LANDMARKS = 80  # Number of Landmarks
PERCEPTUAL_RANGE = 120  # Landmark Detection Range
MOTION_NOISE = np.array([1e-5, 1e-2, 1e-5, 1e-2])  # Motion Noise
Qt_sim = np.diag([4, np.deg2rad(4)]) ** 2  # Observation Noise
```

Initialize Pose: (250,100,0)

Initialize Velocity: 24

Initialize Angular Velocity: 10

## Run the Code



Ground Truth Path

### Fast-SLAM

- Steps of Fast-SLAM
- 1. Predict the next pose  $x_t^{(i)}$  by motion model.

$$x_t^{(i)} \sim p(x_t^{(i)} | x_{t-1}^{(i)}, u_{t-1})$$

2. Update the distribution of each landmark  $(\mu_{j,t}^{(i)}, \Sigma_{j,t}^{(i)})$  via measurement  $z_t$ .

$$Q = H\Sigma_{j,t-1}^{(i)}H^{T} + Q_{t}, K_{t} = \Sigma_{j,t-1}^{(i)}H^{T}Q^{-1}$$

$$\mu_{j,t}^{(i)} = \mu_{j,t-1}^{(i)} + K_{t}\left(z_{t} - h(\mu_{j,t-1}^{(i)}, x_{t}^{(i)})\right)$$

$$\Sigma_{j,t}^{(i)} = (I - K_{t}H)\Sigma_{j,t-1}^{(i)}$$

3. Update the importance weight of particles.

$$w^{(i)} \sim |2\pi Q|^{-\frac{1}{2}} \exp\{-\frac{1}{2} \left(z_k - h\left(\mu_{j,t-1}^{(i)}, x_t^{(i)}\right)\right)^T Q^{-1} \left(z_k - h(\mu_{j,t-1}^{(i)}, x_t^{(i)})\right)\}$$

4. Resampling.

## Class Design

#### Particle

- init\_pos
- deepcopy: Copy the memory of whole particle.
- sample (TODO): Sample next pose from motion model.
- observation\_model (TODO): Predict the observation of landmark.
- compute\_H (TODO): Construct the matrix of linearized observation model.
- update\_landmark (TODO): Update one landmark given the observation.
- update: Update observed landmarks and get likelihood.

#### Particle Filter

- sample: Sample next pose of particles.
- update: Update the map and weight of particles given the observation.
- resample (TODO): Compute Neff for evaluating the particle distribution.

## Utility Functions

- multi\_normal() (TODO)
  - Compute the probability of multivariate normal distribution.
- normalize\_angle()
  - Normalize the angle to the range – $\pi$ ~  $\pi$ .

## Fast-SLAM

- Steps of Fast-SLAM
- 1. Predict the next pose  $x_t^{(i)}$  by motion model.

$$x_t^{(i)} \sim p(x_t^{(i)} | x_{t-1}^{(i)}, u_{t-1})$$

2. Update the distribution of each landmark  $(\mu_{j,t}^{(i)}, \Sigma_{j,t}^{(i)})$  via measurement  $z_k$ .

$$Q = H\Sigma_{j,t-1}^{(i)}H^{T} + Q_{t}, K_{t} = \Sigma_{j,t-1}^{(i)}H^{T}Q^{-1}$$

$$\mu_{j,t}^{(i)} = \mu_{j,t-1}^{(i)} + K_{k}\left(z_{k} - h(\mu_{j,t-1}^{(i)}, x_{t}^{(i)})\right)$$

$$\Sigma_{j,t}^{(i)} = (I - K_{t}H)\Sigma_{j,t-1}^{(i)}$$

3. Update the importance weight of particles.

$$w^{(i)} \sim |2\pi Q|^{-\frac{1}{2}} \exp\{-\frac{1}{2} \left(z_k - h\left(\mu_{j,t-1}^{(i)}, x_t^{(i)}\right)\right)^T Q^{-1} \left(z_k - h(\mu_{j,t-1}^{(i)}, x_t^{(i)})\right)\}$$

4. Resampling.

## Sample from Velocity Motion Model

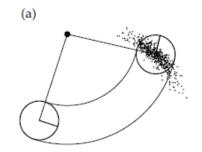
With Angular Velocity:

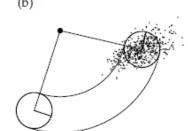
$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} -\frac{\hat{v}}{\widehat{\omega}}\sin(\theta) + \frac{\hat{v}}{\widehat{\omega}}\sin(\theta + \widehat{\omega}\Delta t) \\ \frac{\hat{v}}{\widehat{\omega}}\cos(\theta) - \frac{\hat{v}}{\widehat{\omega}}\cos(\theta + \widehat{\omega}\Delta t) \\ \widehat{\omega}\Delta t + \widehat{\gamma}\Delta t \end{bmatrix}$$

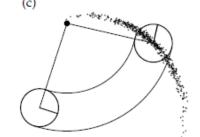
Without Angular Velocity:

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} \hat{v}\cos(\theta) * \Delta t \\ \hat{v}\sin(\theta) * \Delta t \\ \hat{\gamma}\Delta t \end{bmatrix}$$

1: Algorithm sample\_motion\_model\_velocity(
$$u_t, x_{t-1}$$
):
$$\mu \qquad \sigma^2$$
2:  $\hat{v} = v + \text{sample}(\alpha_1 v^2 + \alpha_2 \omega^2)$ 
3:  $\hat{\omega} = \omega + \text{sample}(\alpha_3 v^2 + \alpha_4 \omega^2)$ 
4:  $\hat{\gamma} = \text{sample}(\alpha_5 v^2 + \alpha_6 \omega^2)$ 
5:  $x' = x - \frac{\hat{v}}{\hat{\omega}} \sin \theta + \frac{\hat{v}}{\hat{\omega}} \sin(\theta + \hat{\omega} \Delta t)$ 
6:  $y' = y + \frac{\hat{v}}{\hat{\omega}} \cos \theta - \frac{\hat{v}}{\hat{\omega}} \cos(\theta + \hat{\omega} \Delta t)$ 
7:  $\theta' = \theta + \hat{\omega} \Delta t + \hat{\gamma} \Delta t$ 
8:  $\text{return } x_t = (x', y', \theta')^T$ 







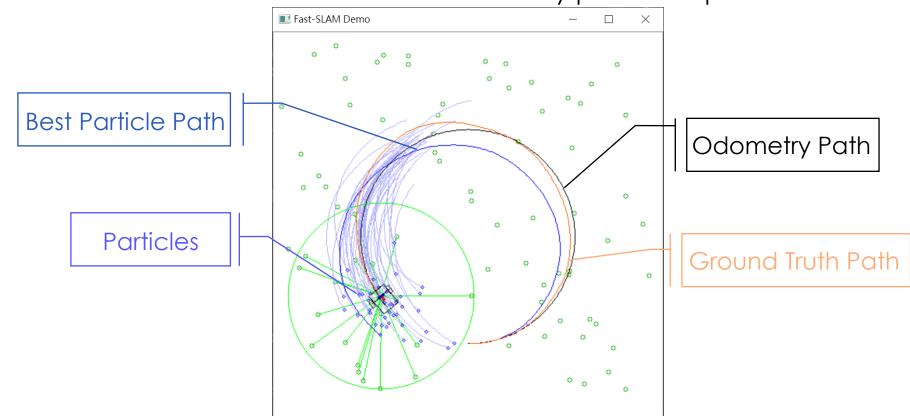
## Sample from Velocity Motion Model

• Lab-1~Lab-3: Complete the velocity motion model.

```
def motion_model(pos, control, motion_noise=[0]*6):
    x, y, yaw = pos
    v, w, delta_t = control
    # TODO(Lab-1): Noise Control
    v hat = v +
    w_hat = w +
    g_hat =
    if w_hat != 0:
        # TODO(Lab-2): motion prediction (with angular velocity)
        x next = x +
        y_next = y +
        yaw_next = yaw +
    else:
        # TODO(Lab-3): motion prediction (without angular velocity)
        x next = x +
        y_next = y +
        yaw_next = yaw +
    return [x_next, y_next, yaw_next]
```

## Run the code

- Lab-0: Remove the comment of "pf.sample(u)".
- Lab-4: Remove the comment of odometry path computation.



#### Fast-SLAM

- Steps of Fast-SLAM
- 1. Predict the next pose  $x_t^{(i)}$  by motion model.

$$x_t^{(i)} \sim p(x_t^{(i)} | x_{t-1}^{(i)}, u_{t-1})$$

2. Update the distribution of each landmark  $(\mu_{j,t}^{(i)}, \Sigma_{j,t}^{(i)})$  via measurement  $z_k$ .

$$Q = H\Sigma_{j,t-1}^{(i)}H^{T} + Q_{t}, K_{t} = \Sigma_{j,t-1}^{(i)}H^{T}Q^{-1}$$

$$\mu_{j,t}^{(i)} = \mu_{j,t-1}^{(i)} + K_{k}\left(z_{k} - h(\mu_{j,t-1}^{(i)}, x_{t}^{(i)})\right)$$

$$\Sigma_{j,t}^{(i)} = (I - K_{t}H)\Sigma_{j,t-1}^{(i)}$$

3. Update the importance weight of particles.

$$w^{(i)} \sim |2\pi Q|^{-\frac{1}{2}} \exp\{-\frac{1}{2} \left(z_k - h\left(\mu_{j,t-1}^{(i)}, x_t^{(i)}\right)\right)^T Q^{-1} \left(z_k - h(\mu_{j,t-1}^{(i)}, x_t^{(i)})\right)\}$$

4. Resampling.

#### Observation Model

Given observation model

$$z_{i} = \begin{bmatrix} \sqrt{q} \\ atan2(\delta_{x}, \delta_{y}) - \theta \end{bmatrix}, \delta = \begin{bmatrix} m_{i,x} - x \\ m_{i,y} - y \end{bmatrix}, q = \delta^{T} \delta$$

Linearized the observation model :

$$H^{i} = \frac{\partial z_{i}}{\partial(x,y,\theta,m_{i,x},m_{i,y})} = \begin{bmatrix} \frac{\partial\sqrt{q}}{\partial x} & \frac{\partial\sqrt{q}}{\partial y} & \cdots \\ \frac{\partial \tan 2(\delta_{x},\delta_{y})}{\partial x} & \frac{\partial \tan 2(\delta_{x},\delta_{y})}{\partial y} & \cdots \end{bmatrix}$$

$$= \frac{1}{q} \begin{bmatrix} -\sqrt{q}\delta_{x} & -\sqrt{q}\delta_{y} & 0 \\ \delta_{y} & -\delta_{x} & -q \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$

$$\frac{\partial\sqrt{q}}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{q}}2\delta_{x}(-1) = \frac{1}{q}(-\sqrt{q}\delta_{x})$$

$$\frac{\partial}{\partial x} \arctan(\frac{y}{x}) = -\frac{y}{x^{2} + y^{2}},$$

$$\frac{\partial}{\partial y} \arctan(y, x) = \frac{\partial}{\partial y} \arctan(\frac{y}{x}) = \frac{x}{x^{2} + y^{2}}.$$

$$\frac{\partial}{\partial y} \arctan(y, x) = \frac{\partial}{\partial y} \arctan(\frac{y}{x}) = \frac{x}{x^{2} + y^{2}}.$$

Only Consider the Landmarks

$$H = \begin{bmatrix} \delta_x / \sqrt{q} & \delta_y / \sqrt{q} \\ -\delta_y / q & \delta_x / q \end{bmatrix}$$

$$\frac{\partial \sqrt{q}}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{q}} 2\delta_x(-1) = \frac{1}{q} (-\sqrt{q} \delta_x)$$

$$rac{\partial}{\partial x} \operatorname{atan2}(y, x) = rac{\partial}{\partial x} \arctan\left(rac{y}{x}
ight) = -rac{y}{x^2 + y^2},$$

$$rac{\partial}{\partial y} \operatorname{atan2}(y, x) = rac{\partial}{\partial y} \arctan\left(rac{y}{x}
ight) = rac{x}{x^2 + y^2}.$$

#### Observation Model

- Lab-5: Compute the prediction of observation.
- Lab-6: Construct the matrix of linearized observation model.

```
# Predict the observation of landmark.
def observation model(self, lx, ly):
    x, y, yaw = self.pos
    # TODO(Lab-5): Compute the prediction of observation.
    # [Hint 1] The parameter "lx,ly" is the location of landmark.
   z th = normalize angle(z th)
   return (z_r, z_th)
# Linearized Observation Matrix
def compute_H(self, lx, ly):
   x, y, yaw = self.pos
    # TODO(Lab-6): Contruct the matrix of linearized observation model.
    # [Hint 1] The parameter "lx,ly" is the location of landmark.
   return H
```

#### Add New Landmark

• Obtain the relative measurement of landmarks:  $z_i = (r_i, \phi_i)^T$ 

$$h^{-1}(\mathbf{z}, \mathbf{x}) = \begin{bmatrix} \mu_{i,x} \\ \mu_{i,y} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} r_i \cos(\phi_i + \theta) \\ r_i \sin(\phi_i + \theta) \end{bmatrix}$$

• Given observation noise  $Q_t$ , the covariance of the observation will be scaled by the inverse of linearized observation matrix  $H^{-1}Q_t(H^{-1})^T$ .

$$H^{-1} = \frac{\partial h^{-1}(z,x)}{\partial x} = \begin{bmatrix} \cos(\phi_i + \theta) & -r_i \sin(\phi_i + \theta) \\ \sin(\phi_i + \theta) & r_i \cos(\phi_i + \theta) \end{bmatrix}$$

$$\mu_{j,t}^{[k]} = h^{-1}(z_t, x_t^{[k]})$$

$$H = h'(x_t^{[k]}, \mu_{j,t}^{[k]})$$

$$\Sigma_{j,t}^{[k]} = H^{-1} Q_t (H^{-1})^T$$

$$w^{[k]} = p_0$$

Landmark  $Y_R$   $r_i$   $\phi_i$   $X_R$ 

Initialized probability of new landmark, set to 1

#### Add New Landmark

Lab-7: Add new landmark, compute mean and covariance.

```
if lid not in self.landmarks:
    # TODO(Lab-7): Add new landmark (mean and covariance).
    # [Hint 1] The parameter "z" is a list of one landmark [r, phi].
    # [Hint 2] The observation noise is "self.Qt (numpy array)".
    # [Hint 3] The mean of landmark "mu" is a numpy array with shape (2,1).
    c = np.cos(yaw+z[1])
    s = np.sin(yaw+z[1])
    mu =
    H = self.compute_H(mu[0,0], mu[1,0])
    Hinv = np.linalg.inv(H)
    sig =
    self.landmarks[lid] = {"mu":mu, "sig":sig}
    p = 1.0
```

## Update Existing Landmark

Extended Kalman Filter

$$x_t^{pre} = f(x_{t-1}^{est}, u_t) \rightarrow \mu_{t-1}$$
 (Landmarks do not move)

$$P_t^{pre} = F_t P_{t-1}^{pre} F_t^T + Q \rightarrow \Sigma_{t-1}$$
 (Landmarks do not move)

$$K_{t} = P_{t}^{pre}H^{T}(\underbrace{HP_{t}^{pre}H^{T} + R})^{-1}$$

$$Q_{t}(\text{Observation Noise})$$

$$x_{t}^{est} = x_{t}^{pre} + K_{t}(z_{t} - \underbrace{Hx_{t}^{pre}})$$

$$\hat{z} \text{ (Predict Observation)}$$

$$P_{t}^{est} = (I - K_{t}H)P_{t}^{pre}$$

$$\hat{z} \text{ (Predict Observation)}$$

$$\hat{z} \text{ (Predict Observation)}$$

$$\hat{z} = h(\mu_{j,t-1}^{[k]}, x_t^{[k]}) 
H = h'(x_t^{[k]}, \mu_{j,t-1}^{[k]}) 
Q = H \sum_{j,t-1}^{[k]} H^T + Q_t 
K = \sum_{j,t-1}^{[k]} H^T Q^{-1} 
\mu_{j,t}^{[k]} = \mu_{j,t-1}^{[k]} + K(z_t - \hat{z}) 
\sum_{j,t}^{[k]} = (I - K H) \sum_{j,t-1}^{[k]} 
w^{[k]} = |2\pi Q|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(z_t - \hat{z}_n)^T Q^{-1}(z_t - \hat{z}_n)\right\}$$

## Update Existing Landmark

Lab-8: Update existing landmark, compute mean and covariance.

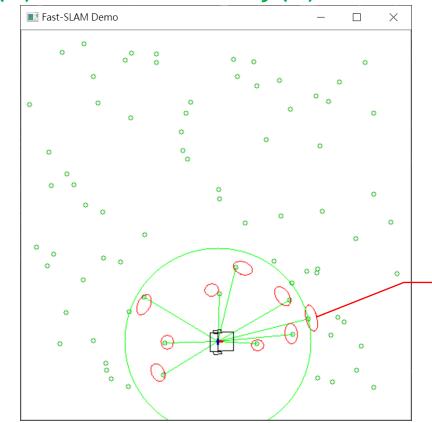
```
else:
    # TODO(Lab-8): Update existing landmark (mean and covariance).
    mu = self.landmarks[lid]["mu"]
    sig = self.landmarks[lid]["sig"]
    z_hat = self.observation_model(mu[0,0], mu[1,0])
   H = self.compute_H(mu[0,0], mu[1,0])
    0 =
    K =
    e = np.array(z) - np.array(z_hat)
    e[1] = normalize_angle(e[1])
    self.landmarks[lid]["mu"] =
    self.landmarks[lid]["sig"] =
    p = multi_normal(np.array(z).reshape(2,1),np.array(z_hat).reshape(2,1), Q)
```

## Run the Code

• Lab-0: Remove the comment of "pf.update(z, landmarks\_id)".

• Set "cv2.waitkey(1)" to "cv2.waitkey(0)" to observe the distribution of

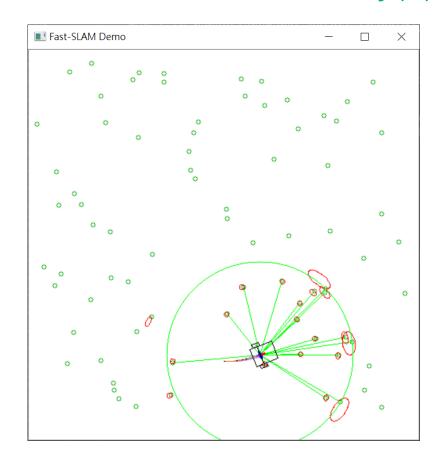
new landmarks.

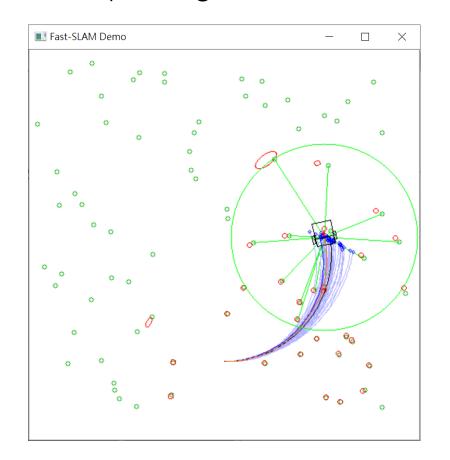


Landmark of best particle

## Run the Code

• Set back to "cv2.waitkey(1)" to observe the updating of landmarks.





#### Fast-SLAM

- Steps of Fast-SLAM
- 1. Predict the next pose  $x_t^{(i)}$  by motion model.

$$x_t^{(i)} \sim p(x_t^{(i)} | x_{t-1}^{(i)}, u_{t-1})$$

2. Update the distribution of each landmark  $(\mu_{j,t}^{(i)}, \Sigma_{j,t}^{(i)})$  via measurement  $z_k$ .

$$Q = H\Sigma_{j,t-1}^{(i)}H^{T} + Q_{t}, K_{t} = \Sigma_{j,t-1}^{(i)}H^{T}Q^{-1}$$

$$\mu_{j,t}^{(i)} = \mu_{j,t-1}^{(i)} + K_{k}\left(z_{k} - h(\mu_{j,t-1}^{(i)}, x_{t}^{(i)})\right)$$

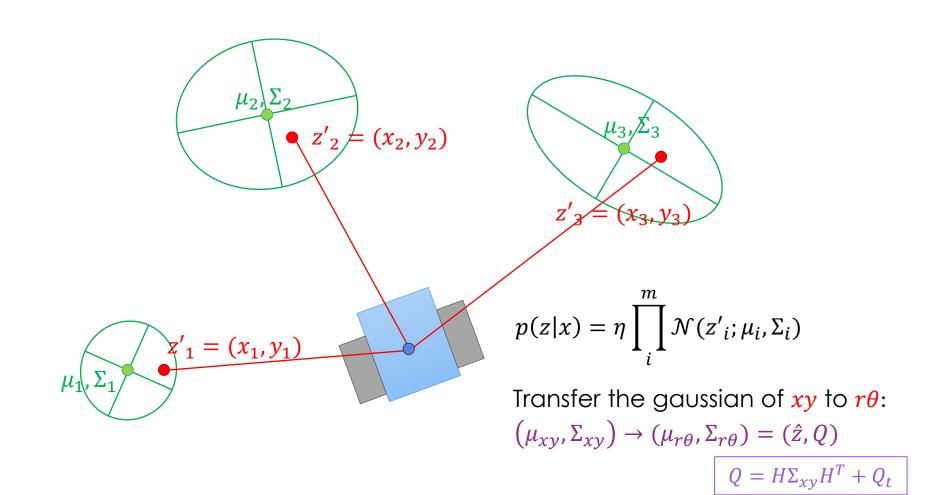
$$\Sigma_{j,t}^{(i)} = (I - K_{t}H)\Sigma_{j,t-1}^{(i)}$$

3. Update the importance weight of particles.

$$w^{(i)} \sim |2\pi Q|^{-\frac{1}{2}} \exp\{-\frac{1}{2} \left(z_k - h\left(\mu_{j,t-1}^{(i)}, x_t^{(i)}\right)\right)^T Q^{-1} \left(z_k - h(\mu_{j,t-1}^{(i)}, x_t^{(i)})\right)\}$$

4. Resampling.

## Likelihood of Measurement



#### Likelihood

Lab-9: Compute likelihood of multivariate normal distribution.

```
def multi_normal(x, mean, cov):
    # TODO(Lab-9): Compute likelihood of multivariate normal distribution.
    err = x - mean
    err[1,0] = normalize_angle(err[1,0])
    return 1
```

$$\mathcal{N}_{k}(\mu, \Sigma) = \frac{\exp\left\{-\frac{1}{2}(x-\mu)^{T}\Sigma^{-1}(x-\mu)\right\}}{\sqrt{(2\pi)^{k}|\Sigma|}} \qquad w^{(i)} \sim \frac{\exp\left\{-\frac{1}{2}(x-\mu)^{T}Q^{-1}(x-\mu)\right\}}{2\pi\sqrt{|Q|}}$$
Determinant

## Fast-SLAM

- Steps of Fast-SLAM
- 1. Predict the next pose  $x_t^{(i)}$  by motion model.

$$x_t^{(i)} \sim p(x_t^{(i)} | x_{t-1}^{(i)}, u_{t-1})$$

2. Update the distribution of each landmark  $(\mu_{j,t}^{(i)}, \Sigma_{j,t}^{(i)})$  via measurement  $z_k$ .

$$Q = H\Sigma_{j,t-1}^{(i)}H^{T} + Q_{t}, K_{t} = \Sigma_{j,t-1}^{(i)}H^{T}Q^{-1}$$

$$\mu_{j,t}^{(i)} = \mu_{j,t-1}^{(i)} + K_{k}\left(z_{k} - h(\mu_{j,t-1}^{(i)}, x_{t}^{(i)})\right)$$

$$\Sigma_{j,t}^{(i)} = (I - K_{t}H)\Sigma_{j,t-1}^{(i)}$$

3. Update the importance weight of particles.

$$w^{(i)} \sim |2\pi Q|^{-\frac{1}{2}} \exp\{-\frac{1}{2} \left(z_k - h\left(\mu_{j,t-1}^{(i)}, x_t^{(i)}\right)\right)^T Q^{-1} \left(z_k - h(\mu_{j,t-1}^{(i)}, x_t^{(i)})\right)\}$$

4. Resampling.

# Evaluating the Particle Weights

 Measure of how well the target distribution is approximated by samples drawn from the proposal.

 $N_{eff} = \frac{1}{\sum_{i} \left( w_t^{(i)} \right)^2}$ 

•  $N_{eff}$  denotes the inverse variance of the normalized particle weights. For equal weights, the results is the number of the particles. And the sample approximation is close to the target.

$$N_{eff}^* = \frac{1}{\sum_i \frac{1}{N^2}} = \frac{1}{N \frac{1}{N^2}} = N$$

• If  $N_{eff}$  drops below a given threshold (usually set to half of the particles), we will resample the particle.

$$N_{eff} < \frac{N}{2}$$

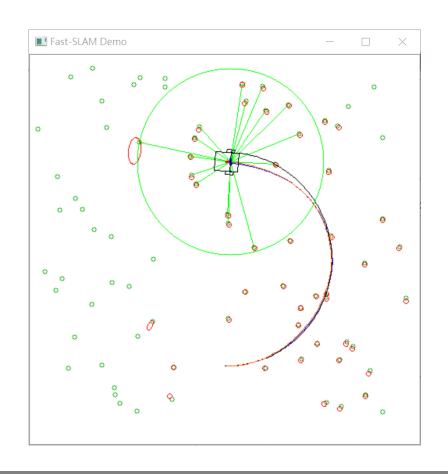
## Evaluating the Particle Weights

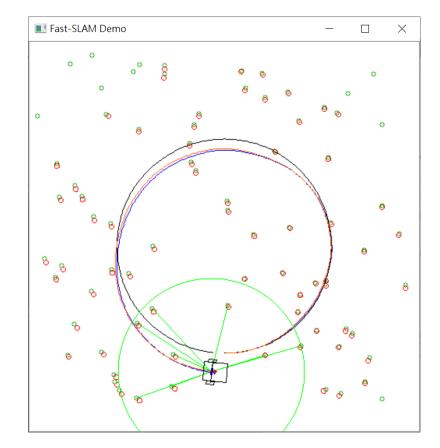
Lab-10: Compute Neff

```
# Resampling Process
def resample(self):
    # TODO(Lab-10): Compute Neff for evaluating the particle distribution.
    self.Neff =
        if self.Neff < self.psize/2:
        re_id = np.random.choice(self.psize, self.psize, p=list(self.weights))
        new_particles = []
        for i in range(self.psize):
             new_particles.append(self.particles[re_id[i]].deepcopy())
        self.particles = new_particles
        self.weights = np.ones(self.psize) / self.psize</pre>
```

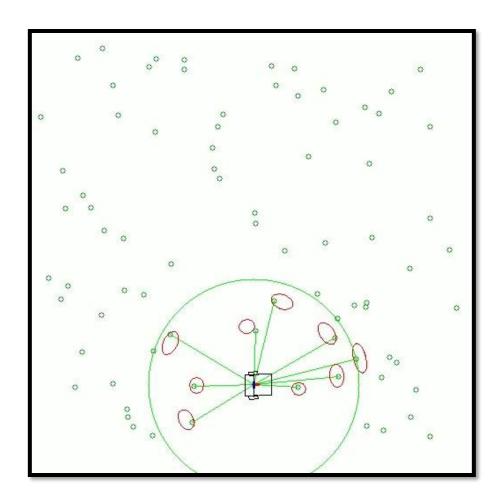
## Run the Code

• Lab-0: Remove the comment of "pf.resample()".





## Demo



#### Hint:

You can press "R" to reset the environment and control the car by keyboard (WSAD).

https://youtu.be/eFjTG5mVpJI