Robotic Navigation and Exploration Lab 3

Fast-SLAM

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Requirement

- Python 3.X
- Numpy
- Opencv-Python

Useful NumPy Operation

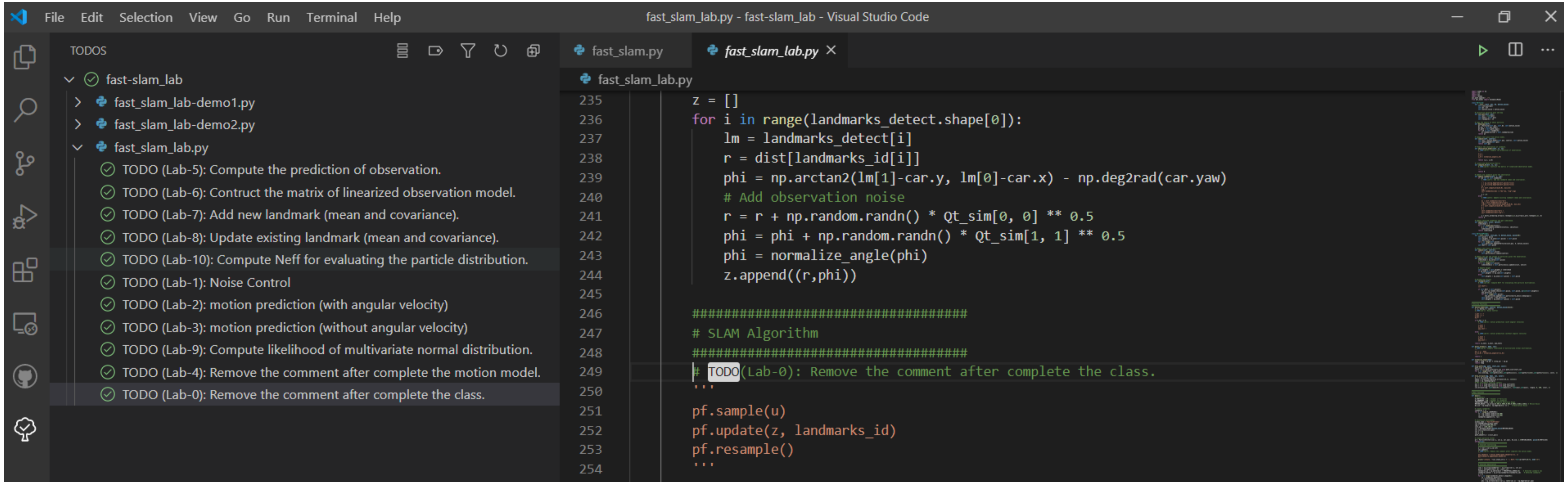
- Matrix Multiply.
  - `np.matmul(A,B) / A.dot(B) / A @ B`
- Matrix Transpose.
  - `np.transpose(A) / A.T`
- Identity Matrix.
  - `np.eye(dimension)`
- Sample from normal distribution with mean **mu** and variance **var**.
  - `mu + np.sqrt(var)*np.random.randn()`
- Matrix Inverse.
  - `np.linalg.inv(A)`
- Matrix Determinant.
  - `np.linalg.det(A)`

TODO List

- In this lab, you need to complete each “TODO” comment in the codes:

```
# TODO(Lab-<ID>): <Explanation>
```

- The “Lab ID” implies the implementation order of the codes (except Lab-0).
- Suggest to install the “Todo Tree” extension in VSCode.



Main Function Workflow

- Set Parameters
- Create Landmarks
- Initialize Environment
- Repeat
  - Simulate Controlling
  - Simulate Observation
  - **SLAM Algorithm**
    - `pf.sample(u)`
    - `pf.update(z, landmarks_id)`
    - `pf.resample()`
  - Render Canvas
  - Keyboard Events

$$u = (v, \omega, \Delta t)$$

$$z = [(r_{id1}, \theta_{id1}), (r_{id2}, \theta_{id2}), \dots]$$

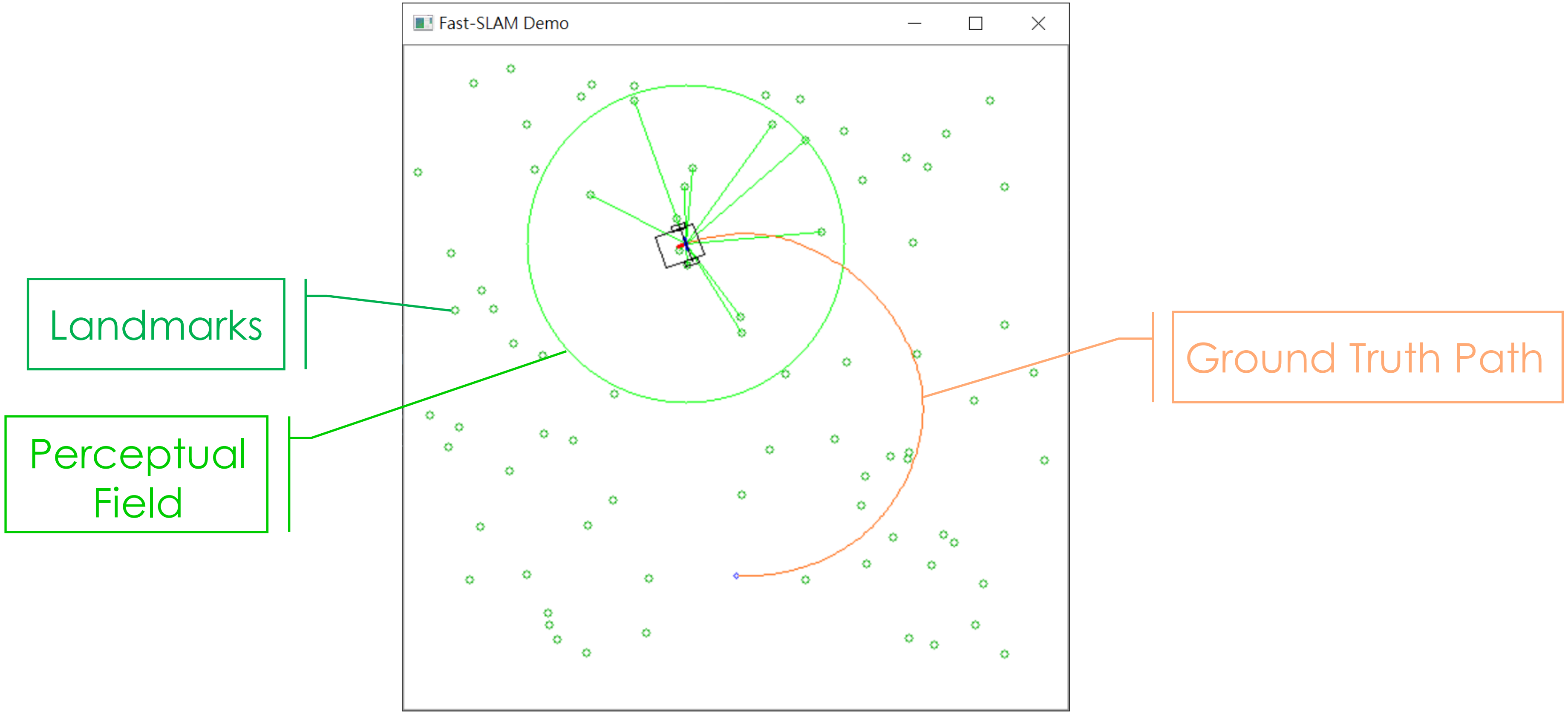
$$\text{landmarks\_id} = [id1, id2, \dots]$$

Parameters Setting

```
# Parameters
N_PARTICLES = 40  # Number of Particles
N_LANDMARKS = 80  # Number of Landmarks
PERCEPTUAL_RANGE = 120 # Landmark Detection Range
MOTION_NOISE = np.array([1e-5, 1e-2, 1e-5, 1e-2, 1e-5, 1e-2]) # Motion Noise
Qt_sim = np.diag([4, np.deg2rad(4)]) ** 2    # Observation Noise
```

Initialize Pose: **(250,100,0)**
Initialize Velocity: **24**
Initialize Angular Velocity: **10**

Run the Code


Fast-SLAM Demo
Landmarks
Perceptual Field
Ground Truth Path

Fast-SLAM

- Steps of Fast-SLAM

1. Predict the next pose $x_t^{(i)}$ by motion model.

$$x_t^{(i)} \sim p(x_t^{(i)} | x_{t-1}^{(i)}, u_{t-1})$$

2. Update the distribution of each landmark $(\mu_{j,t}^{(i)}, \Sigma_{j,t}^{(i)})$ via measurement z_t.

$$Q = H\Sigma_{j,t-1}^{(i)} H^T + Q_t, \qquad K_t = \Sigma_{j,t-1}^{(i)} H^T Q^{-1}$$

$$\mu_{j,t}^{(i)} = \mu_{j,t-1}^{(i)} + K_t \left(z_t - h(\mu_{j,t-1}^{(i)}, x_t^{(i)}) \right)$$

$$\Sigma_{j,t}^{(i)} = (I - K_t H)\Sigma_{j,t-1}^{(i)}$$

3. Update the importance weight of particles.

$$w^{(i)} \sim |2\pi Q|^{-\frac{1}{2}} \exp\{-\frac{1}{2}\left(z_k - h\left(\mu_{j,t-1}^{(i)}, x_t^{(i)}\right)\right)^T Q^{-1} \left(z_k - h(\mu_{j,t-1}^{(i)}, x_t^{(i)})\right)\}$$

4. Resampling.

Class Design

- Particle
  - init_pos
  - deepcopy: Copy the memory of whole particle.
  - sample **(TODO)**: Sample next pose from motion model.
  - observation_model **(TODO)**: Predict the observation of landmark.
  - compute_H **(TODO)**: Construct the matrix of linearized observation model.
  - update_landmark **(TODO)**: Update one landmark given the observation.
  - update: Update observed landmarks and get likelihood.
- Particle Filter
  - sample: Sample next pose of particles.
  - update: Update the map and weight of particles given the observation.
  - resample **(TODO)**: Compute Neff for evaluating the particle distribution.

Utility Functions

- multi_normal() **(TODO)**
  - Compute the probability of multivariate normal distribution.
- normalize_angle()
  - Normalize the angle to the range –π~ π.

Fast-SLAM

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3. Update the importance weight of particles.

$$w^{(i)} \sim |2\pi Q|^{-\frac{1}{2}} \exp\{-\frac{1}{2} \left(z_k - h\left(\mu_{j,t-1}^{(i)}, x_t^{(i)}\right) \right)^T Q^{-1} \left(z_k - h(\mu_{j,t-1}^{(i)}, x_t^{(i)}) \right)\}$$

4. Resampling.

Sample from Velocity Motion Model

With Angular Velocity:

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} -\frac{\hat{v}}{\hat{\omega}}\sin(\theta) + \frac{\hat{v}}{\hat{\omega}}\sin(\theta + \hat{\omega}\Delta t) \\ \frac{\hat{v}}{\hat{\omega}}\cos(\theta) - \frac{\hat{v}}{\hat{\omega}}\cos(\theta + \hat{\omega}\Delta t) \\ \hat{\omega}\Delta t + \hat{\gamma}\Delta t \end{bmatrix}$$

Without Angular Velocity:

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} \hat{v}\cos(\theta) * \Delta t \\ \hat{v}\sin(\theta) * \Delta t \\ \hat{\gamma}\Delta t \end{bmatrix}$$

Algorithm sample_motion_model_velocity(u_t, x_{t-1}):

$\hat{v} = \overset{\mu}{\boxed{v}} + \text{sample}(\overset{\sigma^2}{\boxed{\alpha_1 v^2 + \alpha_2 \omega^2}})$

$\hat{\omega} = \omega + \text{sample}(\alpha_3 v^2 + \alpha_4 \omega^2)$

$\hat{\gamma} = \text{sample}(\alpha_5 v^2 + \alpha_6 \omega^2)$

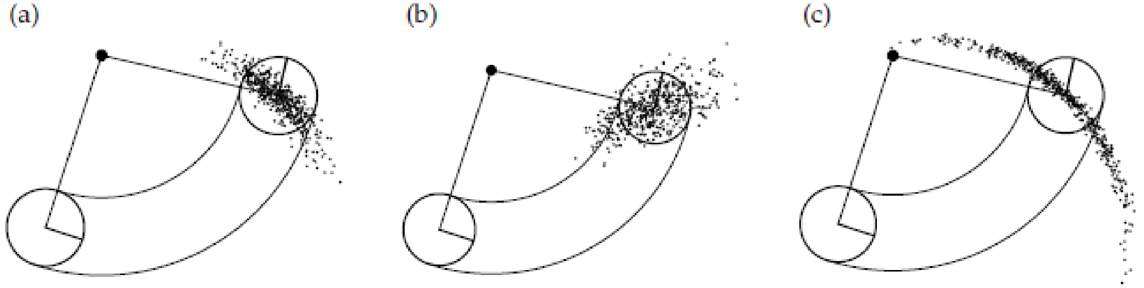
$x' = x - \frac{\hat{v}}{\hat{\omega}}\sin\theta + \frac{\hat{v}}{\hat{\omega}}\, sin(\theta + \hat{\omega}\Delta t)$

$y' = y + \frac{\hat{v}}{\hat{\omega}}\cos\theta - \frac{\hat{v}}{\hat{\omega}}\, cos(\theta + \hat{\omega}\Delta t)$

$\theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$

return $x_t = (x', y', \theta')^T$

$$\hat{v} \sim \mathcal{N}(v, \sigma_v^2) \qquad \sigma_v^2 = \alpha_1 v^2 + \alpha_2 \omega^2$$
$$\hat{\omega} \sim \mathcal{N}(\omega, \sigma_\omega^2) \qquad \sigma_\omega^2 = \alpha_3 v^2 + \alpha_4 \omega^2$$
$$\hat{\gamma} \sim \mathcal{N}(\gamma, \sigma_\gamma^2) \qquad \sigma_\gamma^2 = \alpha_5 v^2 + \alpha_6 \omega^2$$



Sample from Velocity Motion Model

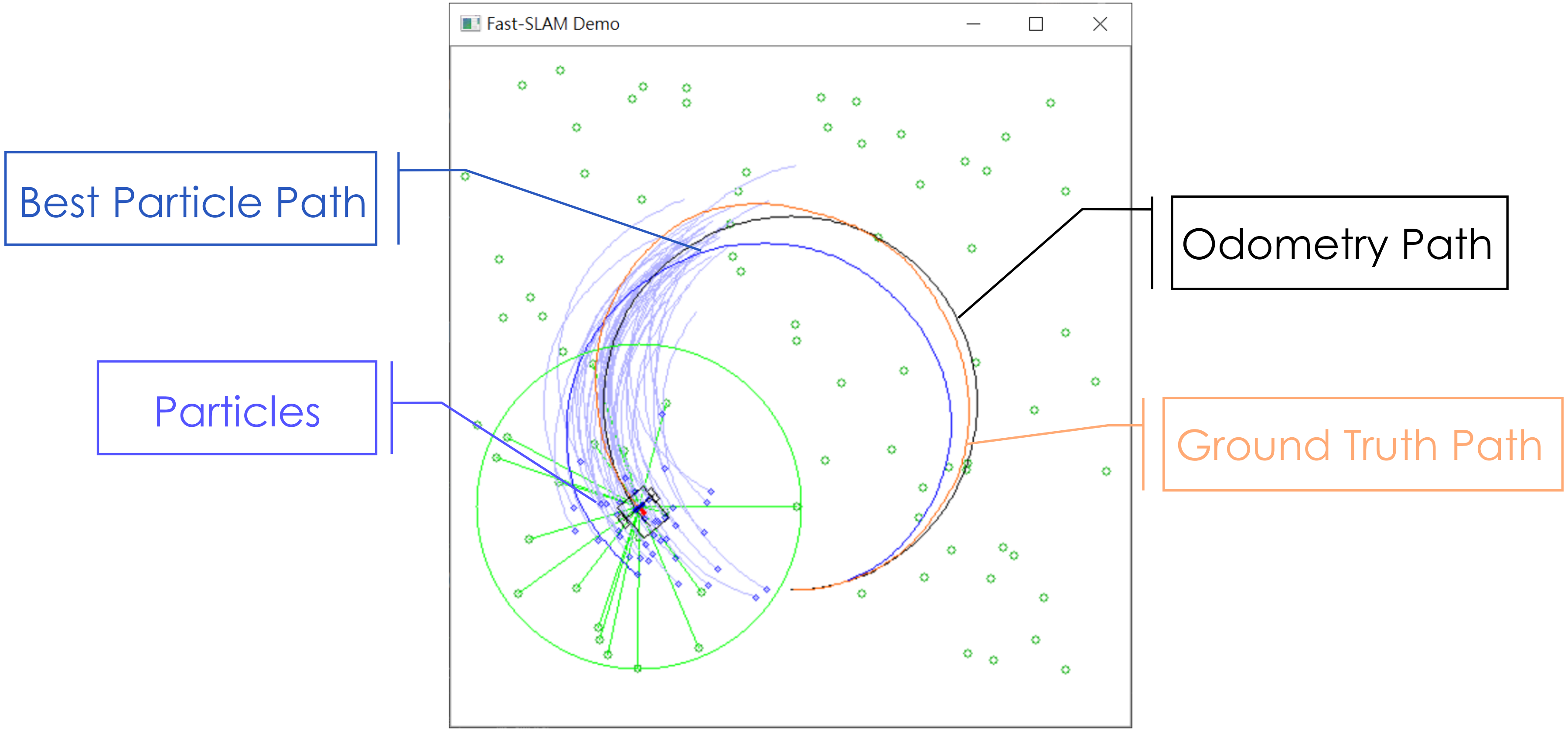
- Lab-1~Lab-3: Complete the velocity motion model.

```python
def motion_model(pos, control, motion_noise=[0]*6):
    x, y, yaw = pos
    v, w, delta_t = control

    # TODO(Lab-1): Noise Control
    '''
    v_hat = v +
    w_hat = w +
    g_hat =
    '''
    if w_hat != 0:
        # TODO(Lab-2): motion prediction (with angular velocity)
        '''
        x_next = x +
        y_next = y +
        yaw_next = yaw +
        '''
    else:
        # TODO(Lab-3): motion prediction (without angular velocity)
        '''
        x_next = x +
        y_next = y +
        yaw_next = yaw +
        '''
    return [x_next, y_next, yaw_next]
```

Run the code

- Lab-0: Remove the comment of “`pf.sample(u)`”.
- Lab-4: Remove the comment of odometry path computation.



Fast-SLAM

- Steps of Fast-SLAM

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3. Update the importance weight of particles.

$$w^{(i)} \sim |2\pi Q|^{-\frac{1}{2}} \exp\{-\frac{1}{2}\left(z_k - h\left(\mu_{j,t-1}^{(i)}, x_t^{(i)}\right)\right)^T Q^{-1}\left(z_k - h(\mu_{j,t-1}^{(i)}, x_t^{(i)})\right)\}$$

4. Resampling.

Observation Model

- Given observation model

$$z_i = \begin{bmatrix} \sqrt{q} \\ atan2(\delta_x, \delta_y) - \theta \end{bmatrix}, \delta = \begin{bmatrix} m_{i,x} - x \\ m_{i,y} - y \end{bmatrix}, q = \delta^T \delta$$

- Linearized the observation model :

  - $H^i = \frac{\partial z_i}{\partial(x, y, \theta, m_{i,x}, m_{i,y})} = \begin{bmatrix} \frac{\partial\sqrt{q}}{\partial x} & \frac{\partial\sqrt{q}}{\partial y} & \cdots \\ \frac{\partial atan2(\delta_x,\delta_y)}{\partial x} & \frac{\partial atan2(\delta_x,\delta_y)}{\partial y} & \cdots \end{bmatrix}$

$$= \frac{1}{q}\begin{array}{c} \begin{bmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & \sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{bmatrix} \\ \begin{array}{ccccc} \partial/\partial x & \partial/\partial y & \partial/\partial\theta & \partial/\partial r & \partial/\partial\phi \end{array} \end{array}$$

$$\frac{\partial\sqrt{q}}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{q}}2\delta_x(-1) = \frac{1}{q}(-\sqrt{q}\delta_x)$$

$$\frac{\partial}{\partial x}\operatorname{atan2}(y, x) = \frac{\partial}{\partial x}\arctan\left(\frac{y}{x}\right) = -\frac{y}{x^2+y^2},$$
$$\frac{\partial}{\partial y}\operatorname{atan2}(y, x) = \frac{\partial}{\partial y}\arctan\left(\frac{y}{x}\right) = \frac{x}{x^2+y^2}.$$

- Only Consider the Landmarks

$$H = \begin{bmatrix} \delta_x/\sqrt{q} & \delta_y/\sqrt{q} \\ -\delta_y/q & \delta_x/q \end{bmatrix}$$

Observation Model

- Lab-5: Compute the prediction of observation.
- Lab-6: Construct the matrix of linearized observation model.

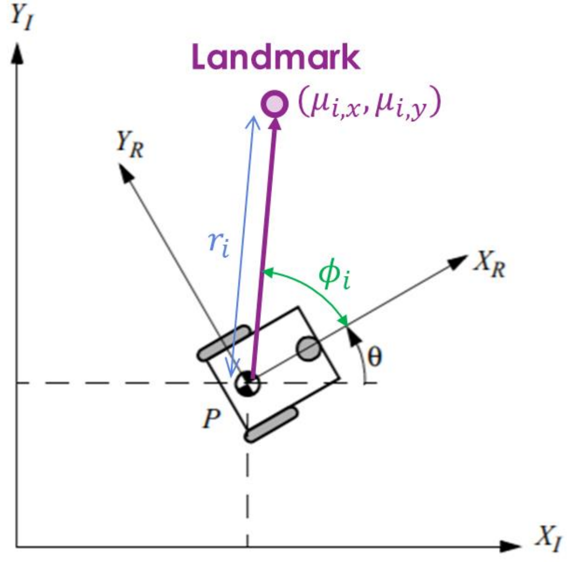
```
# Predict the observation of landmark.
def observation_model(self, lx, ly):
    x, y, yaw = self.pos
    # TODO(Lab-5): Compute the prediction of observation.
    # [Hint 1] The parameter "lx,ly" is the location of landmark.
    '''
    z_r =
    z_th =
    z_th = normalize_angle(z_th)
    '''
    return (z_r, z_th)

# Linearized Observation Matrix
def compute_H(self, lx, ly):
    x, y, yaw = self.pos
    # TODO(Lab-6): Contruct the matrix of linearized observation model.
    # [Hint 1] The parameter "lx,ly" is the location of landmark.
    '''
    '''
    return H
```

Add New Landmark

- Obtain the relative measurement of landmarks: $z_i = (r_i, \phi_i)^T$
  - $h^{-1}(\mathrm{z}, \mathrm{x}) = \begin{bmatrix} \mu_{i,x} \\ \mu_{i,y} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} r_i \cos(\phi_i + \theta) \\ r_i \sin(\phi_i + \theta) \end{bmatrix}$
- Given observation noise Q_t, the covariance of the observation will be scaled by the inverse of linearized observation matrix $H^{-1} Q_t (H^{-1})^T$.
  - $H^{-1} = \frac{\partial h^{-1}(\mathrm{z},\mathrm{x})}{\partial \mathrm{x}} = \begin{bmatrix} \cos(\phi_i + \theta) & -r_i \sin(\phi_i + \theta) \\ \sin(\phi_i + \theta) & r_i \cos(\phi_i + \theta) \end{bmatrix}$

$$\boxed{\begin{aligned} \mu_{j,t}^{[k]} &= h^{-1}(z_t, x_t^{[k]}) \\ H &= h'(x_t^{[k]}, \mu_{j,t}^{[k]}) \\ \Sigma_{j,t}^{[k]} &= H^{-1}\, Q_t\, (H^{-1})^T \\ w^{[k]} &= \underline{p_0} \end{aligned}}$$

Initialized probability of new landmark, set to 1



Add New Landmark

- Lab-7: Add new landmark, compute mean and covariance.

```python
if lid not in self.landmarks:
    # TODO(Lab-7): Add new landmark (mean and covariance).
    # [Hint 1] The parameter "z" is a list of one landmark [r, phi].
    # [Hint 2] The observation noise is "self.Qt (numpy array)".
    # [Hint 3] The mean of landmark "mu" is a numpy array with shape (2,1).
    '''
    c = np.cos(yaw+z[1])
    s = np.sin(yaw+z[1])
    mu =
    H = self.compute_H(mu[0,0], mu[1,0])
    Hinv = np.linalg.inv(H)
    sig =
    self.landmarks[lid] = {"mu":mu, "sig":sig}
    '''
    p = 1.0
```

Update Existing Landmark

- Extended Kalman Filter

$x_t^{pre} = f(x_{t-1}^{est}, u_t)$ $\rightarrow \mu_{t-1}$ (Landmarks do not move)

$P_t^{pre} = F_t P_{t-1}^{pre} F_t^T + Q$ $\rightarrow \Sigma_{t-1}$ (Landmarks do not move)

$$K_t = P_t^{pre} H^T \big(\overbrace{\boxed{H P_t^{pre} H^T + \underline{R}}}^{Q}\big)^{-1}$$

Q_t (Observation Noise)

$$x_t^{est} = x_t^{pre} + K_t\big(z_t - \underline{H x_t^{pre}}\big)$$

$\hat{z}$ (Predict Observation)

$$P_t^{est} = (I - K_t H) P_t^{pre}$$

$$\hat{z} = h(\mu_{j,t-1}^{[k]}, x_t^{[k]})$$
$$H = h'(x_t^{[k]}, \mu_{j,t-1}^{[k]})$$
$$Q = H\ \Sigma_{j,t-1}^{[k]}\ H^T + Q_t$$
$$K = \Sigma_{j,t-1}^{[k]}\ H^T\ Q^{-1}$$
$$\mu_{j,t}^{[k]} = \mu_{j,t-1}^{[k]} + K(z_t - \hat{z})$$
$$\Sigma_{j,t}^{[k]} = (I - K\ H)\Sigma_{j,t-1}^{[k]}$$
$$w^{[k]} = |2\pi Q|^{-\frac{1}{2}}\ \exp\left\{-\frac{1}{2}(z_t - \hat{z}_n)^T Q^{-1}\ (z_t - \hat{z}_n)\right\}$$

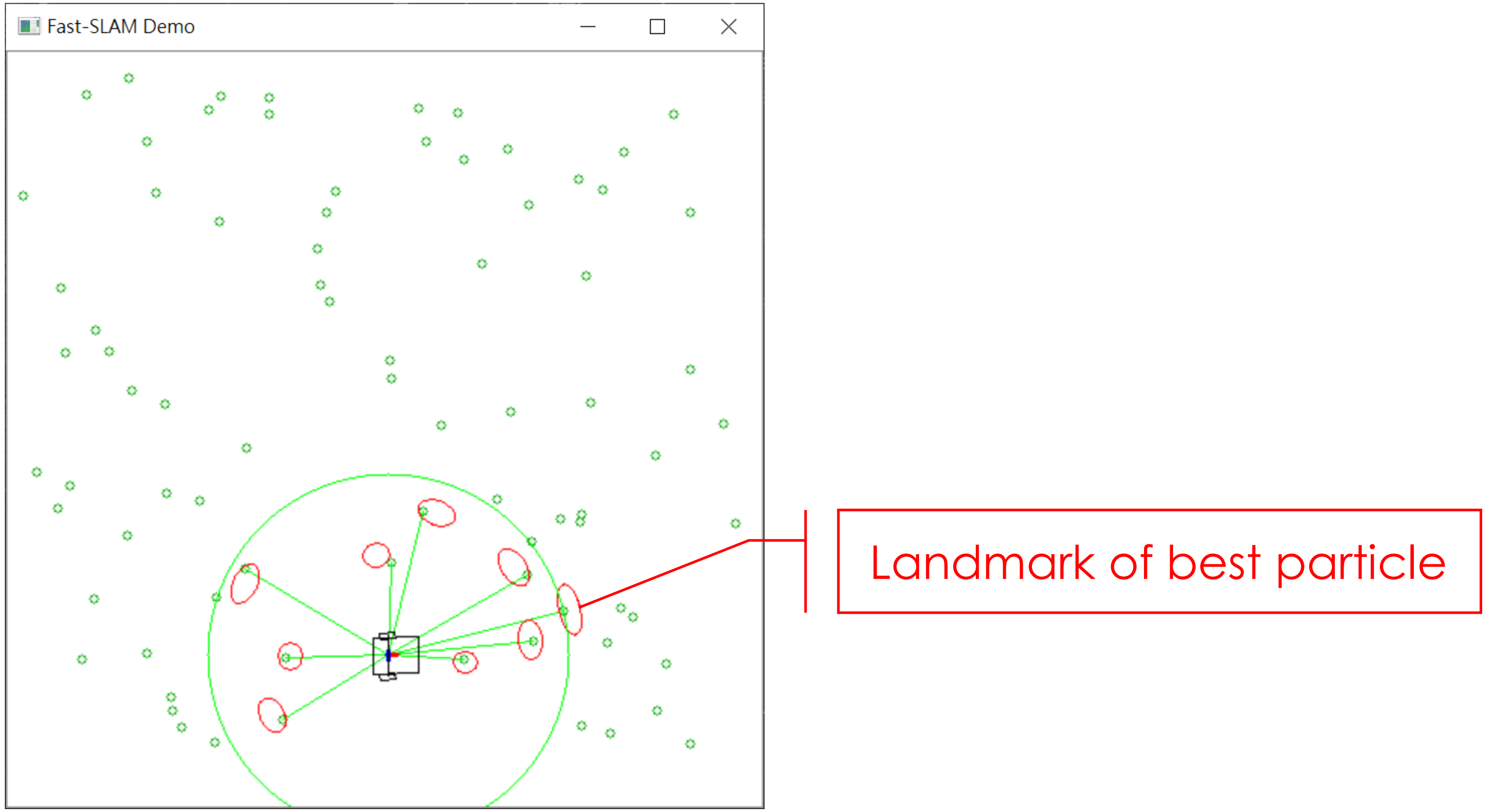
Update Existing Landmark

- Lab-8: Update existing landmark, compute mean and covariance.

```
else:
    # TODO(Lab-8): Update existing landmark (mean and covariance).
    '''
    mu = self.landmarks[lid]["mu"]
    sig = self.landmarks[lid]["sig"]
    z_hat = self.observation_model(mu[0,0], mu[1,0])
    H = self.compute_H(mu[0,0], mu[1,0])
    Q =
    K =
    e = np.array(z) - np.array(z_hat)
    e[1] = normalize_angle(e[1])
    self.landmarks[lid]["mu"] =
    self.landmarks[lid]["sig"] =
    '''
    p = multi_normal(np.array(z).reshape(2,1),np.array(z_hat).reshape(2,1), Q)
```

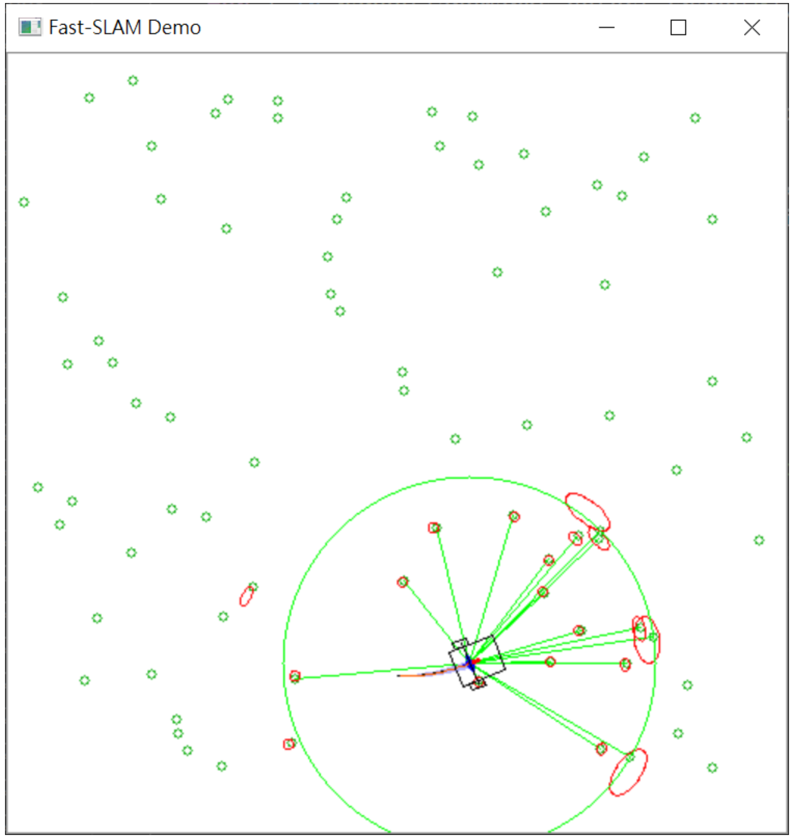
Run the Code

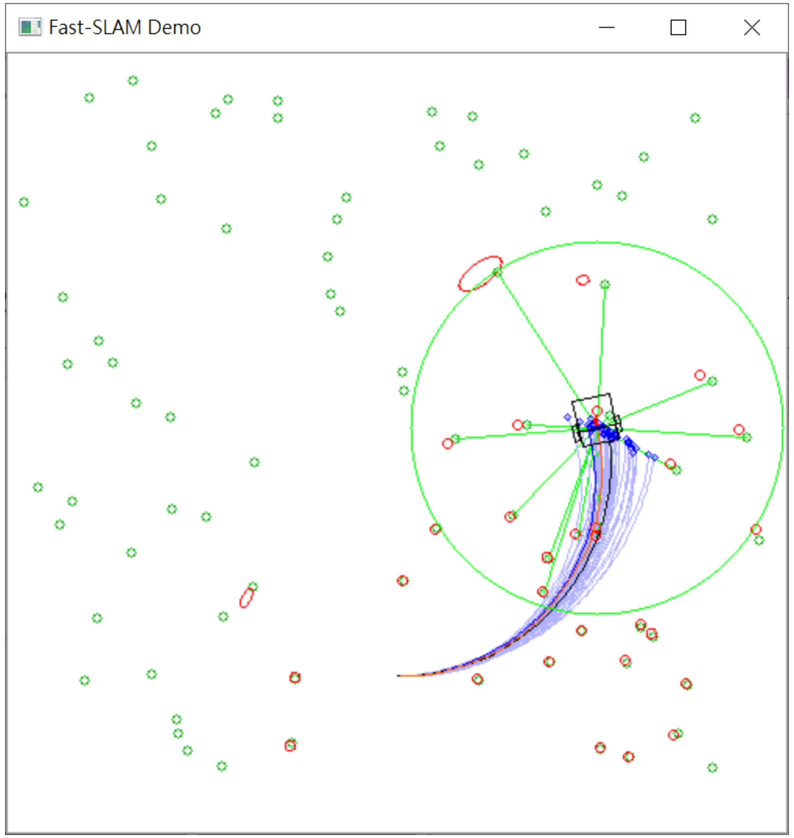
- Lab-0: Remove the comment of “`pf.update(z, landmarks_id)`”.
- Set “`cv2.waitkey(1)`” to “`cv2.waitkey(0)`” to observe the distribution of new landmarks.



Run the Code

- Set back to "`cv2.waitkey(1)`" to observe the updating of landmarks.





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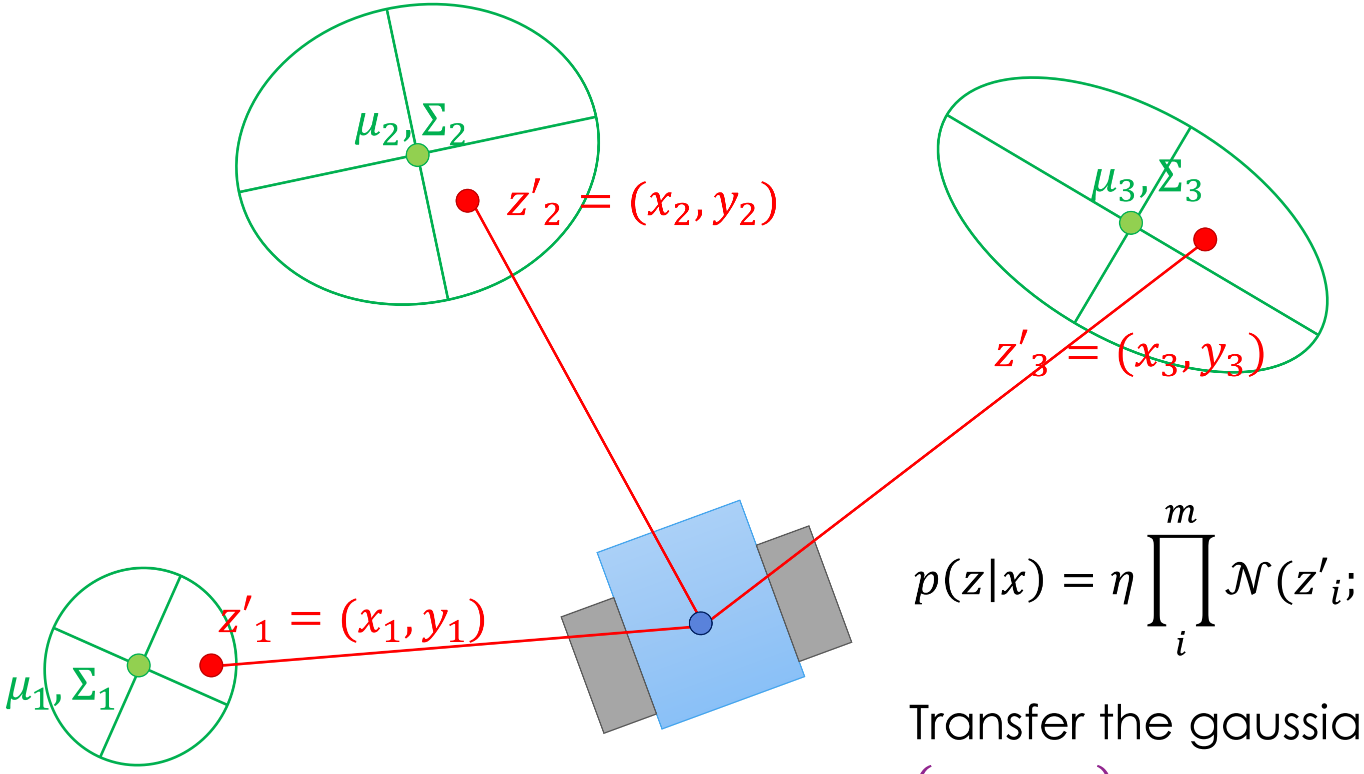
$$\Sigma_{j,t}^{(i)} = (I - K_t H)\Sigma_{j,t-1}^{(i)}$$

3. Update the importance weight of particles.

$$w^{(i)} \sim |2\pi Q|^{-\frac{1}{2}} \exp\{-\frac{1}{2}\left(z_k - h\left(\mu_{j,t-1}^{(i)}, x_t^{(i)}\right)\right)^T Q^{-1}\left(z_k - h(\mu_{j,t-1}^{(i)}, x_t^{(i)})\right)\}$$

4. Resampling.

Likelihood of Measurement



$$p(z|x) = \eta \prod_i^m \mathcal{N}(z'_i; \mu_i, \Sigma_i)$$

Transfer the gaussian of xy to $r\theta$:

$$(\mu_{xy}, \Sigma_{xy}) \rightarrow (\mu_{r\theta}, \Sigma_{r\theta}) = (\hat{z}, Q)$$

$$\boxed{Q = H\Sigma_{xy}H^T + Q_t}$$

Likelihood

- Lab-9: Compute likelihood of multivariate normal distribution.

```
def multi_normal(x, mean, cov):
    # TODO(Lab-9): Compute likelihood of multivariate normal distribution.
    '''
    err = x - mean
    err[1,0] = normalize_angle(err[1,0])
    '''
    return 1
```

$$\mathcal{N}_k(\mu,\Sigma) = \frac{\exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}}{\sqrt{(2\pi)^k\underline{|\Sigma|}}} \quad \Rightarrow \quad w^{(i)} \sim \frac{\exp\left\{-\frac{1}{2}(x-\mu)^T Q^{-1}(x-\mu)\right\}}{2\pi\sqrt{|Q|}}$$

Determinant

Fast-SLAM

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3. Update the importance weight of particles.

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4. Resampling.

Evaluating the Particle Weights

- Measure of how well the target distribution is approximated by samples drawn from the proposal.

$$N_{eff} = \frac{1}{\sum_i \left(w_t^{(i)}\right)^2}$$

- N_{eff} denotes the inverse variance of the normalized particle weights. For equal weights, the results is the number of the particles. And the sample approximation is close to the target.

$$N_{eff}^* = \frac{1}{\sum_i \frac{1}{N^2}} = \frac{1}{N \frac{1}{N^2}} = N$$

- If N_{eff} drops below a given threshold (usually set to half of the particles), we will resample the particle.

$$N_{eff} < \frac{N}{2}$$

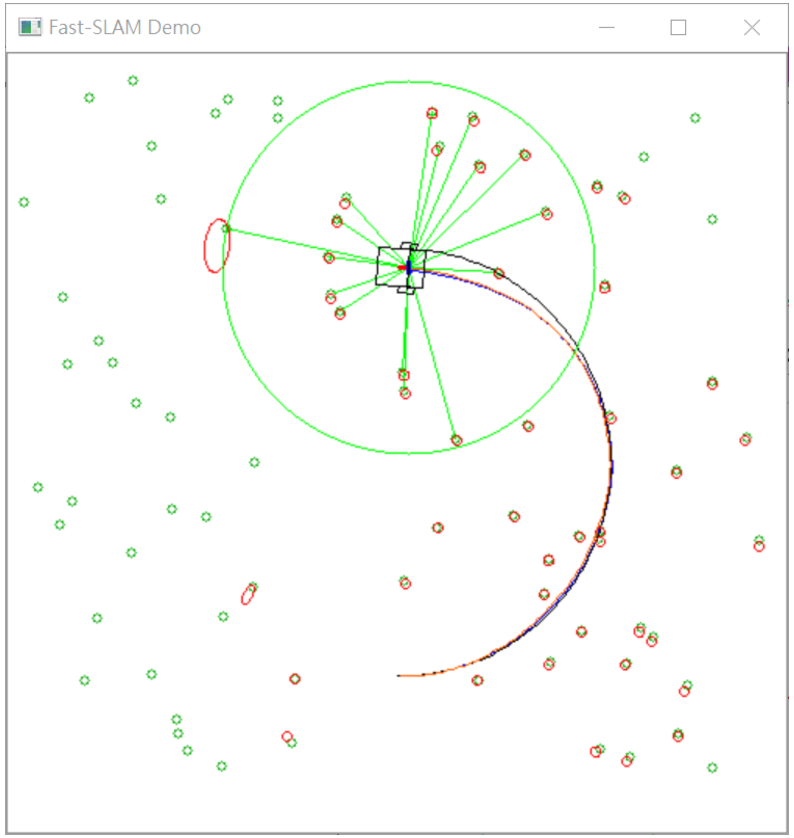
Evaluating the Particle Weights

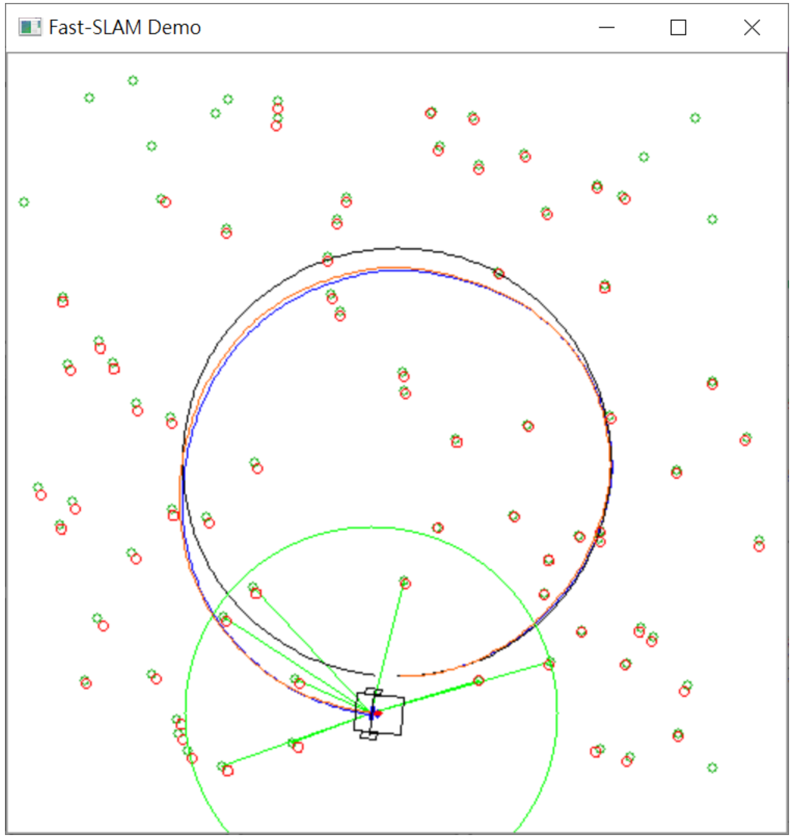
- Lab-10: Compute Neff

```python
# Resampling Process
def resample(self):
    # TODO(Lab-10): Compute Neff for evaluating the particle distribution.
    '''
    self.Neff =
    '''
    if self.Neff < self.psize/2:
        re_id = np.random.choice(self.psize, self.psize, p=list(self.weights))
        new_particles = []
        for i in range(self.psize):
            new_particles.append(self.particles[re_id[i]].deepcopy())
        self.particles = new_particles
        self.weights = np.ones(self.psize) / self.psize
```

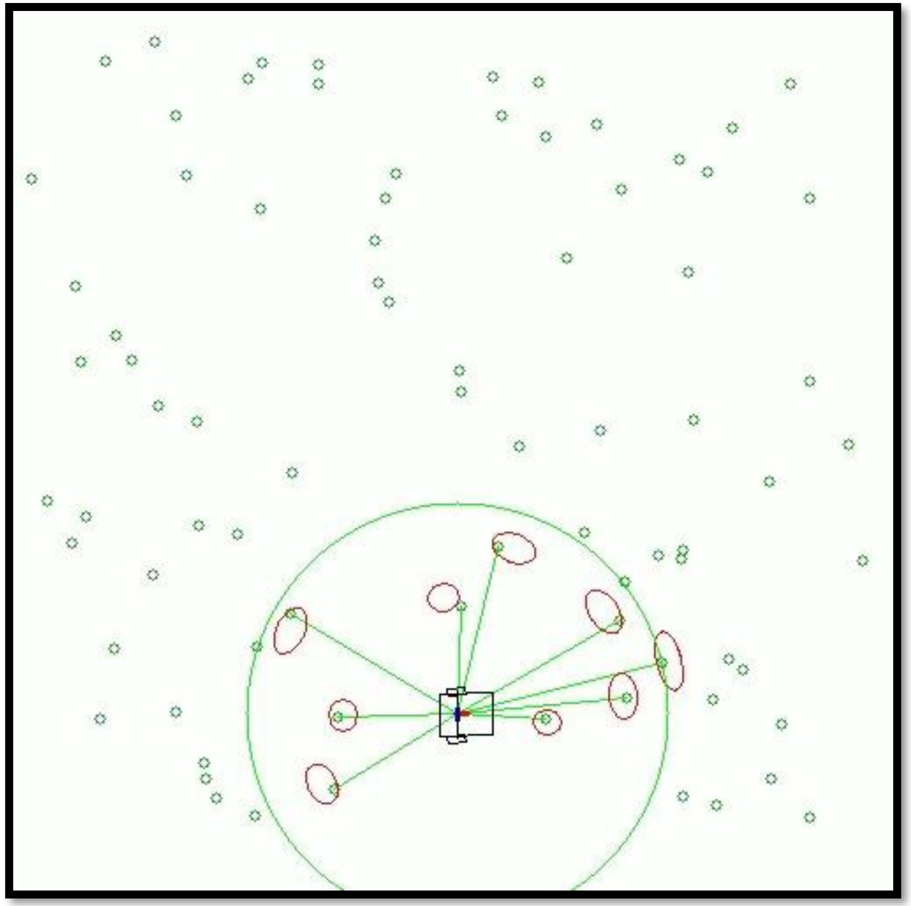
Run the Code

- Lab-0: Remove the comment of "`pf.resample()`".





Demo



Hint:
You can press "R" to reset the environment and control the car by keyboard (WSAD).

https://youtu.be/eFjTG5mVpJI