Robotic Navigation and Exploration

Week 2: Kinematic Model & Path Tracking Control

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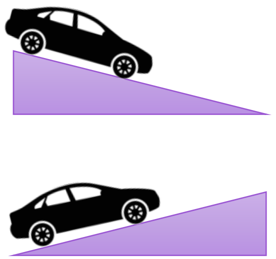
Outline

- Basics of Control System for Automobile
- PID Control
- Kinematic Model
- Differential Drive
- Pure-Pursuit Control
- Bicycle Model
  - Pure Pursuit Control
  - Stanley Control (Path Coordinate and Control Stabilization)
  - Linear Quadratic Regulator (LQR)

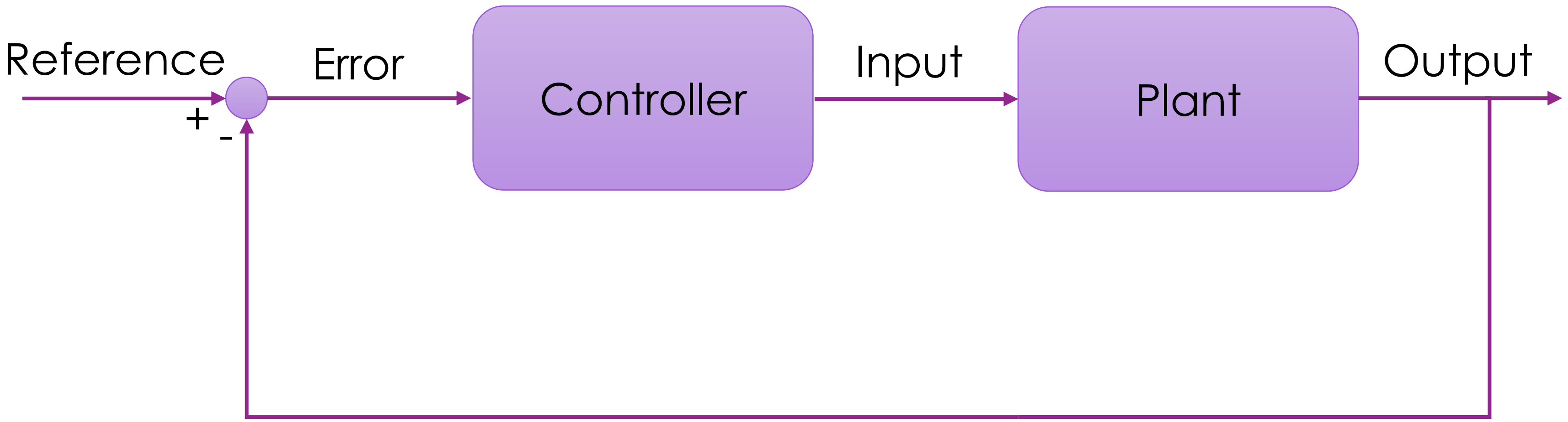
Control Theory: Open Loop Control

- Control System: the mechanism that affects the future state of a system
- Control Theory: a strategy to change input to desired output

Open Loop Control

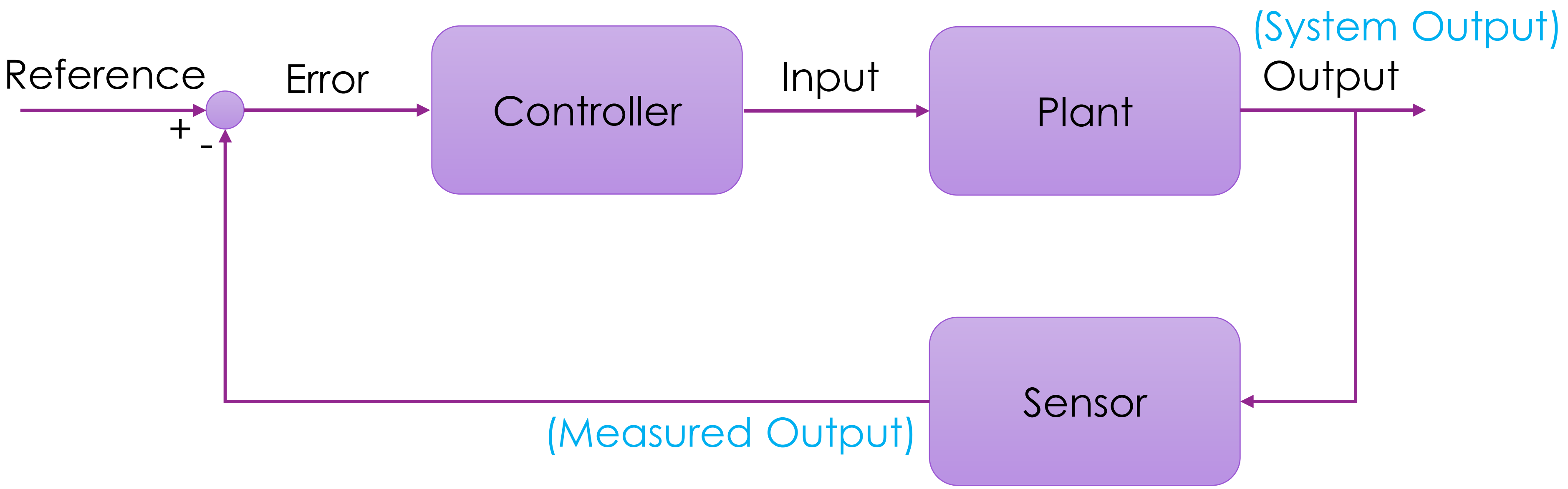


Control Theory: Close Loop Control



Close Loop Control
(Feedback Control)

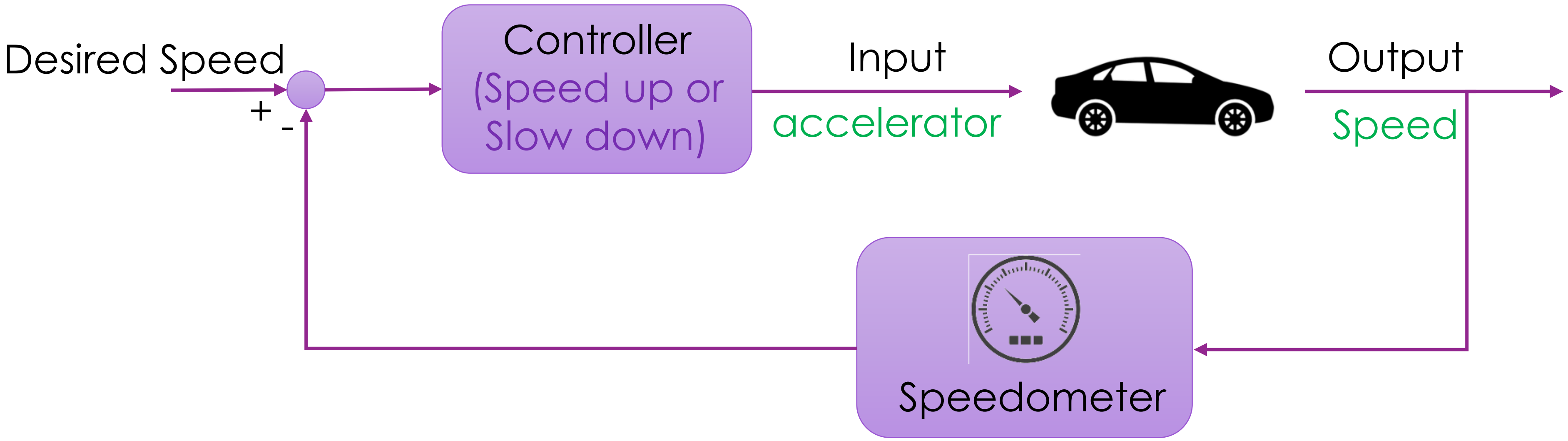
Control Theory: Close Loop Control



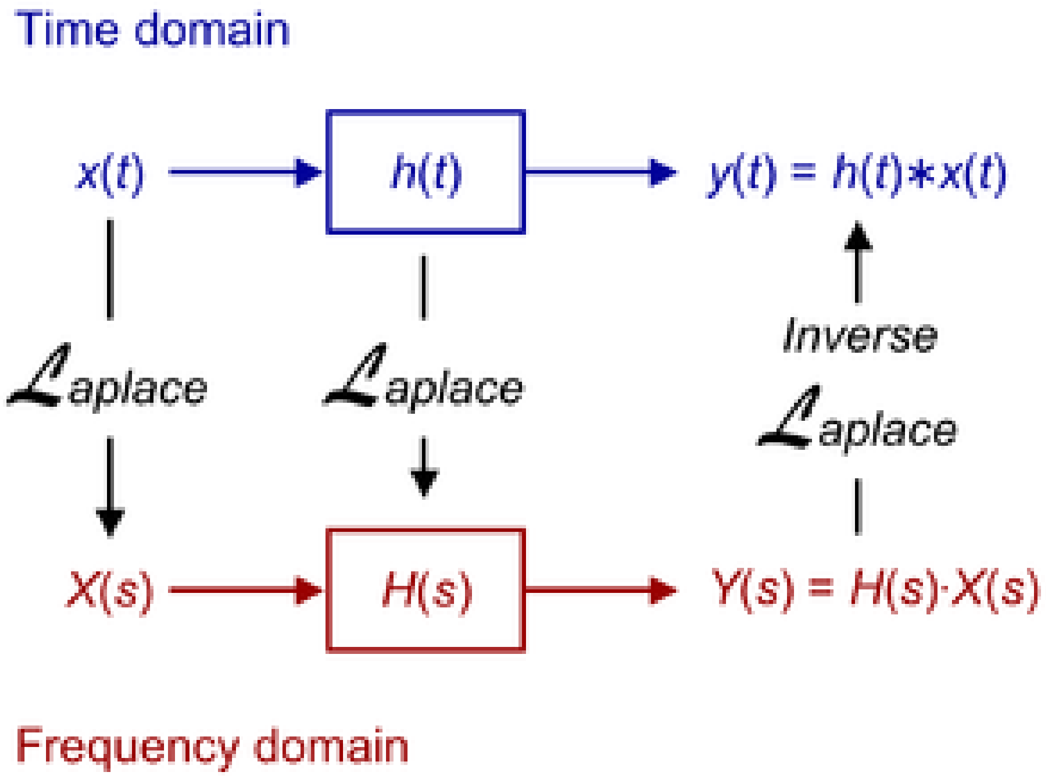
Close Loop Control

(Feedback Control)

Control Theory : Car Example



Linear Time Invariant System



Laplace transform

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_{0^-}^{\infty} e^{-st} f(t)\, dt \\
&= \left[\frac{f(t)e^{-st}}{-s}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{e^{-st}}{-s} f'(t)\, dt \quad \text{(by parts)} \\
&= \left[-\frac{f(0^+)}{-s}\right] + \frac{1}{s}\mathcal{L}\{f'(t)\},
\end{aligned}
$$

Basic Laplace Transform Pairs

Signal or Function	f(t)	F(s)
Impulse	$\delta(t)$	1
Step	$u(t)=1, \quad t \ge 0$	$\frac{1}{s}$
Ramp	$r(t)=t, \quad t \ge 0$	$\frac{1}{s^2}$
Exponential	$e^{-\alpha t} \quad e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
Damped Ramp	$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$
Sine	$\sin(\beta t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$\cos(\beta t)$	$\frac{s}{s^2+\beta^2}$
Damped Sine	$e^{-\alpha t}\sin(\beta t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped Cosine	$e^{-\alpha t}\cos(\beta t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$
Simple Complex Pole	see next pg	see next pg

Linear Time Invariant System

$$e = r - yH$$

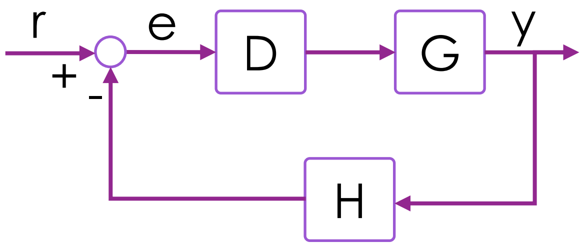
$$y = e \cdot D \cdot G$$

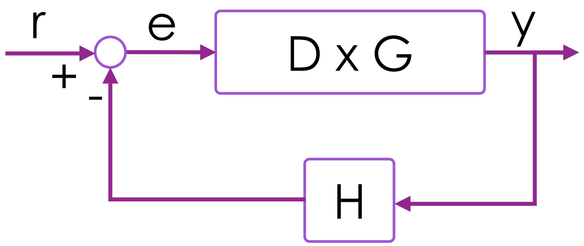
$$e = \frac{y}{DG}$$

$$r - yH = \frac{y}{DG}$$

$$(DG)(r - yH) = y$$

$$DGr = y(1+DGH) \quad \text{or} \quad y = \frac{DGr}{1+DGH}$$





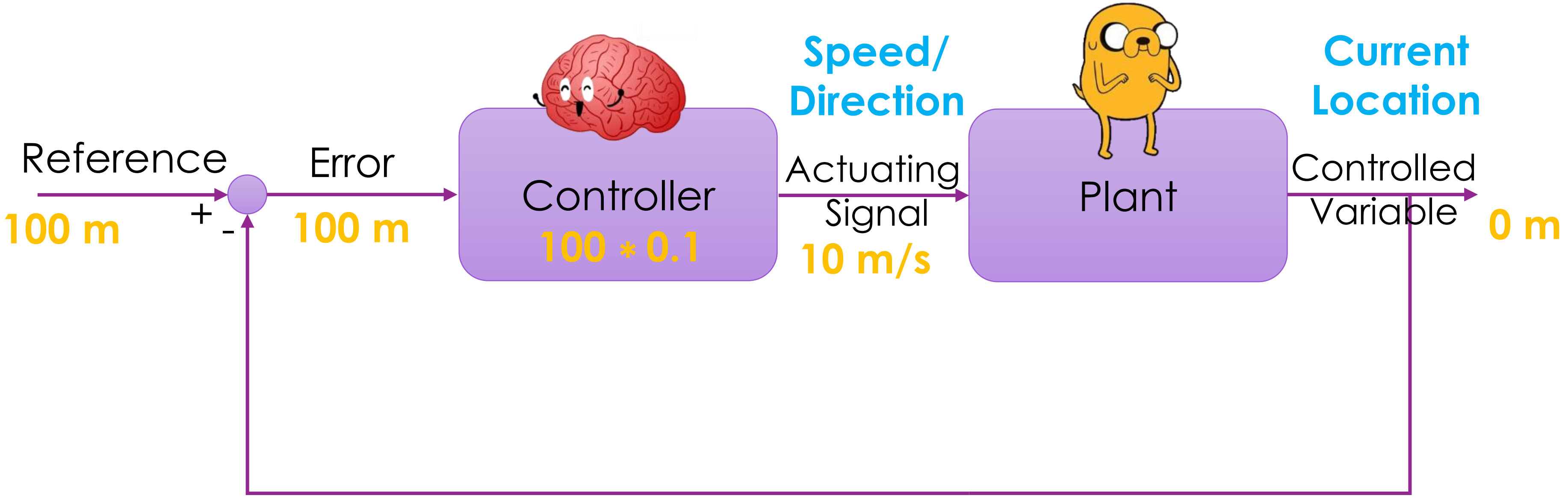
$\Longleftrightarrow$

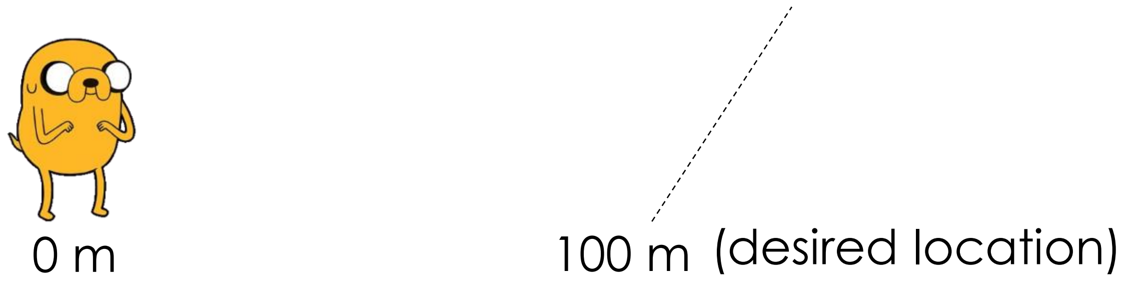
Equivalent

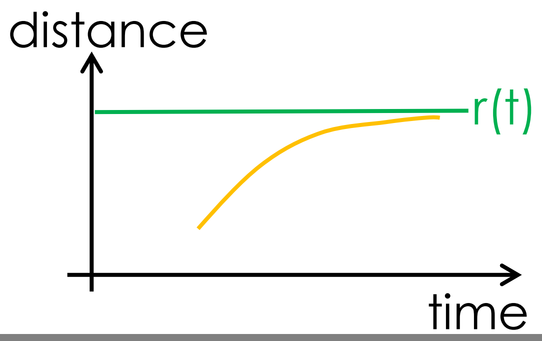
Plant

Open Loop

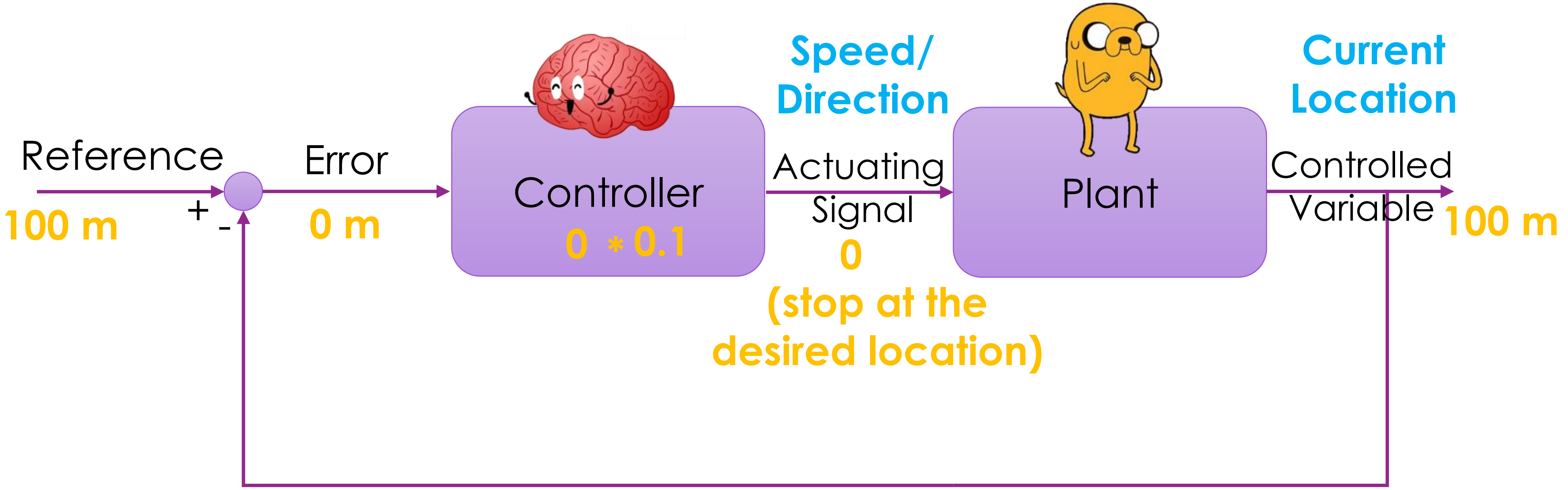
PID Control : Proportional Gain

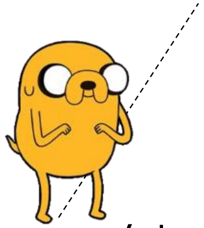






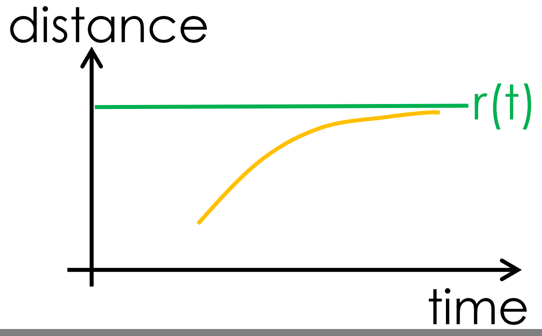
PID Control : Proportional Gain

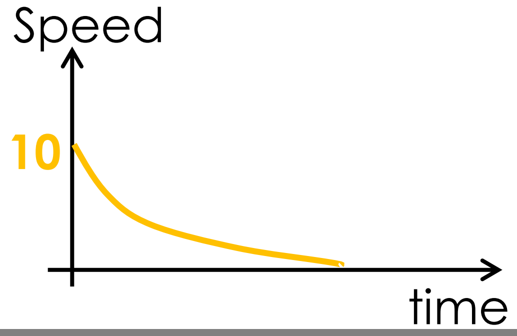




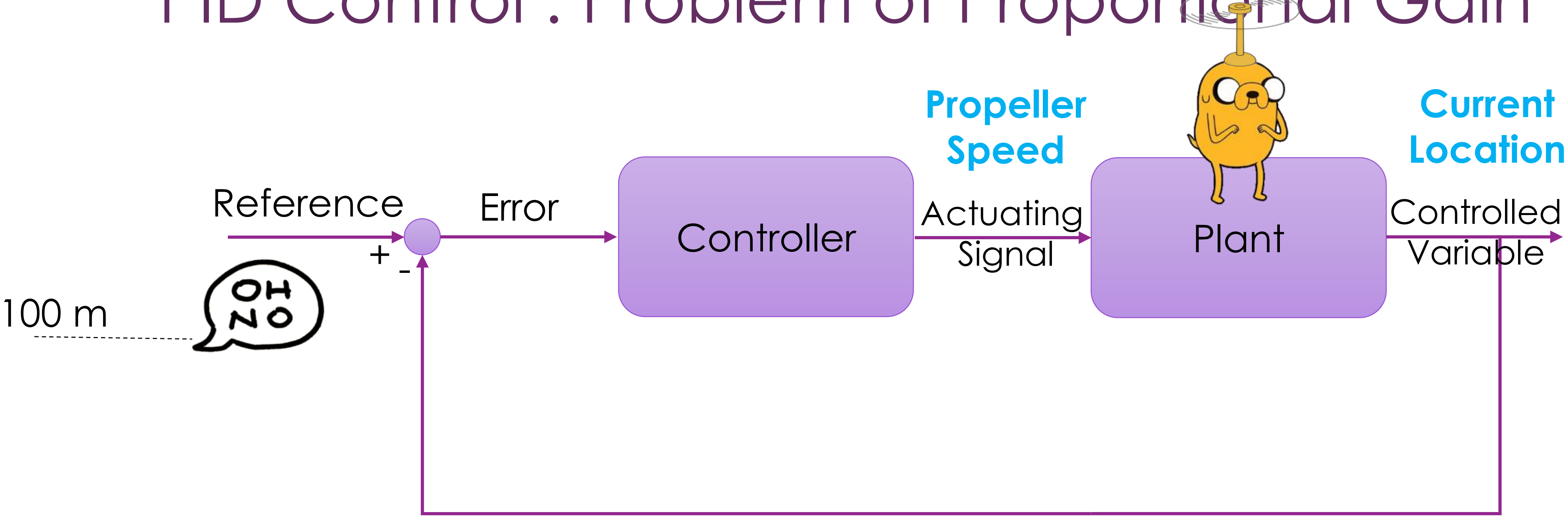
0 m

100 m (desired location)





PID Control : Problem of Proportional Gain



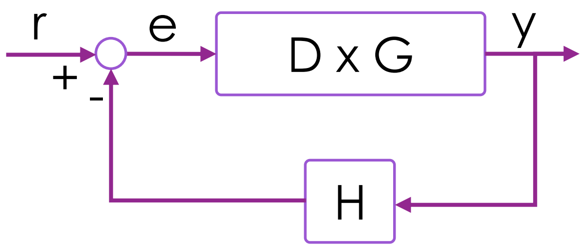
Error * Gain = Propeller Speed

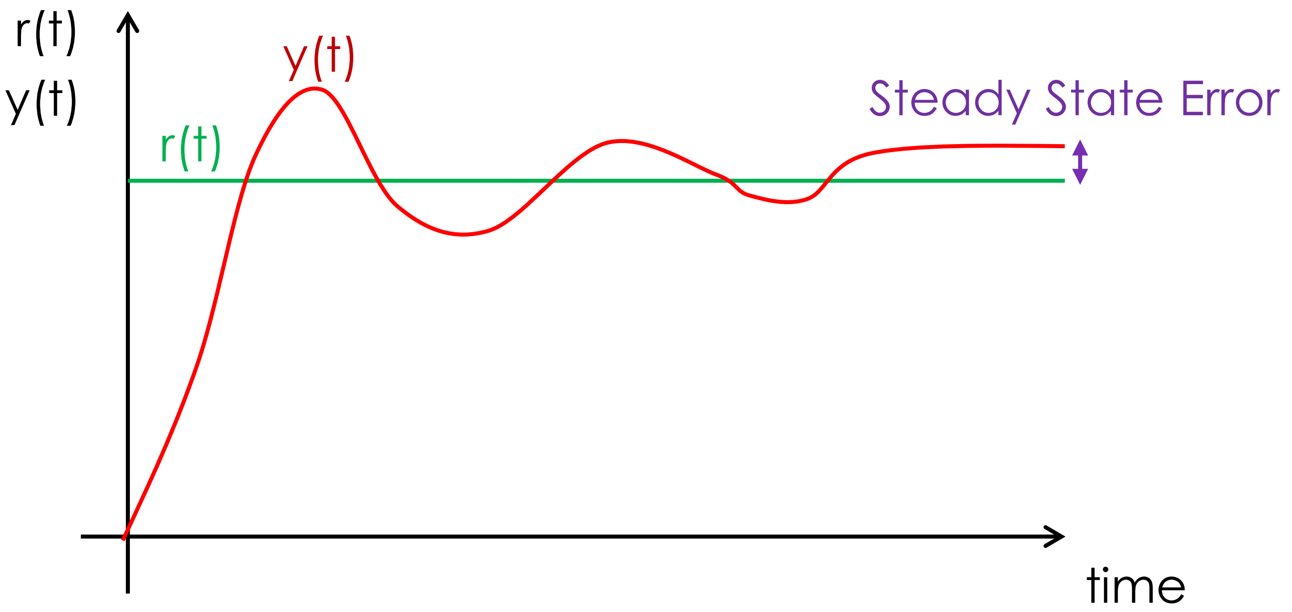
Steady state error

100	*	2	= 200 rpm
40	*	5	= 200 rpm
20	*	10	= 200 rpm
2	*	100	= 200 rpm

Idea: Consider past information !

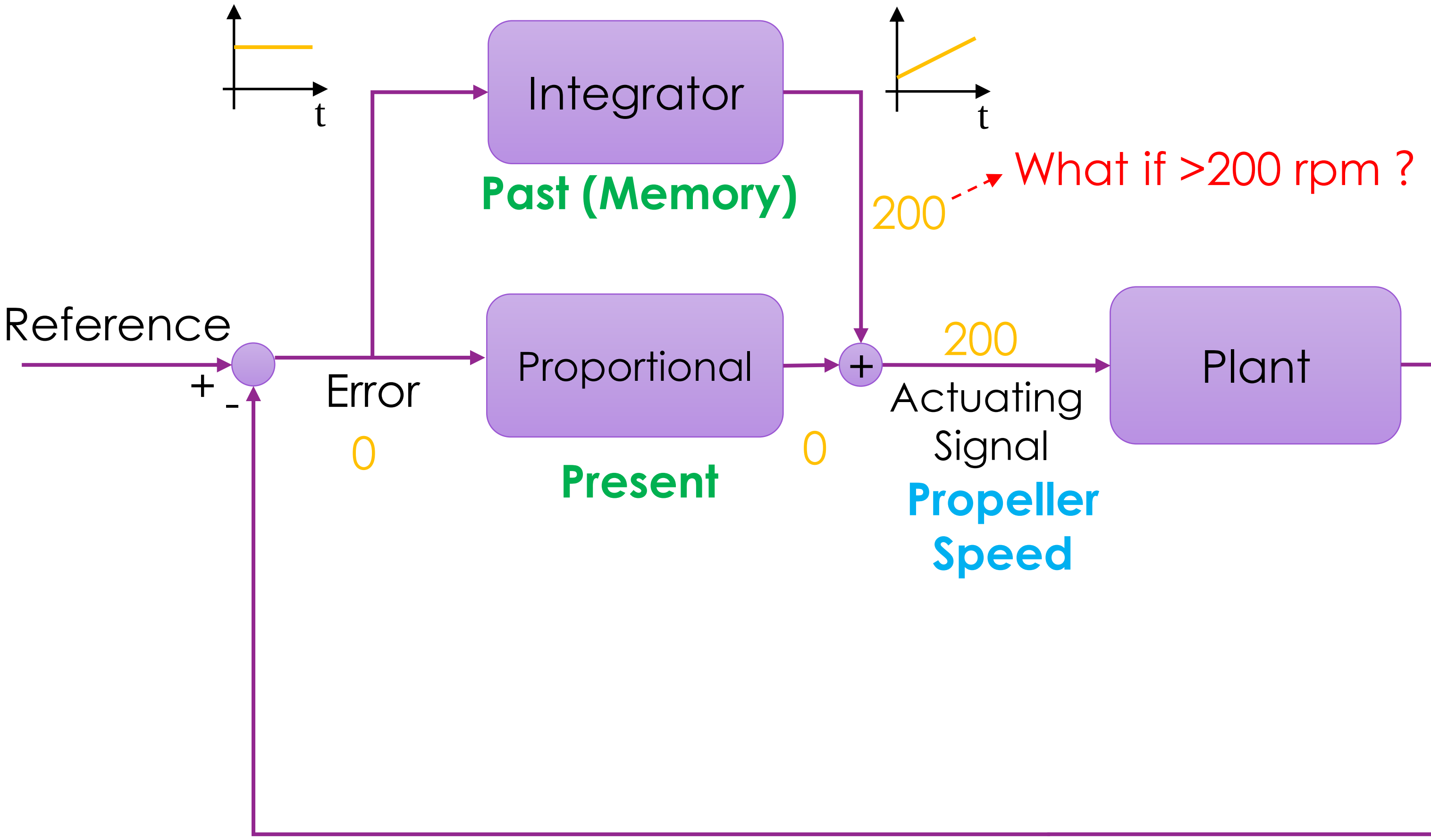
Steady State Error



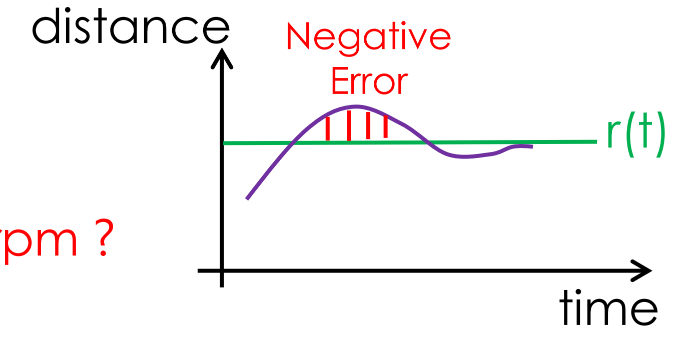


PID Control : Integral Gain

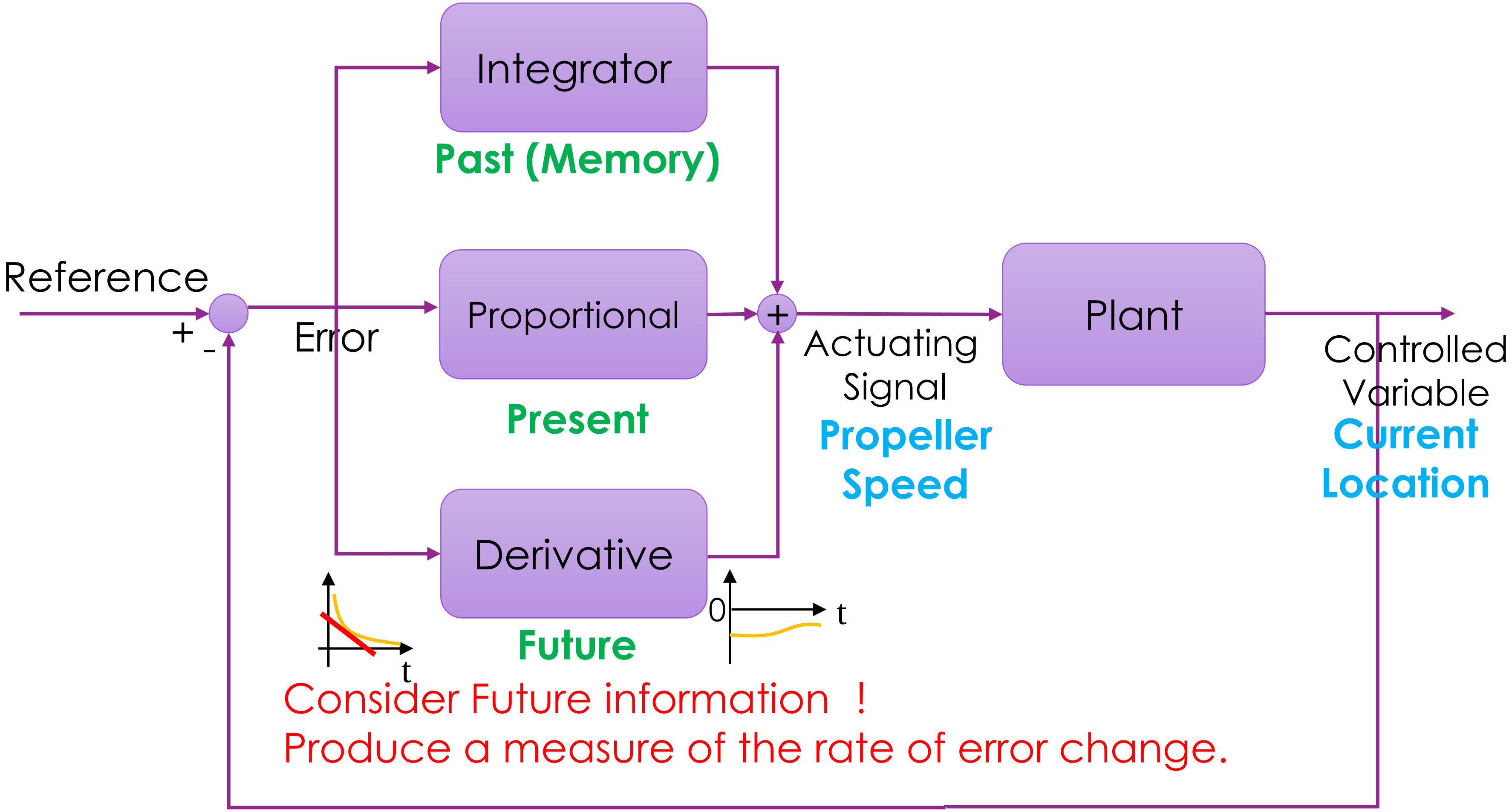
Consider past information !
Sum up non-zero steady state error over time



Overshooting!



PID Control : Differential Gain



Consider Future information !
Produce a measure of the rate of error change.

PID Control

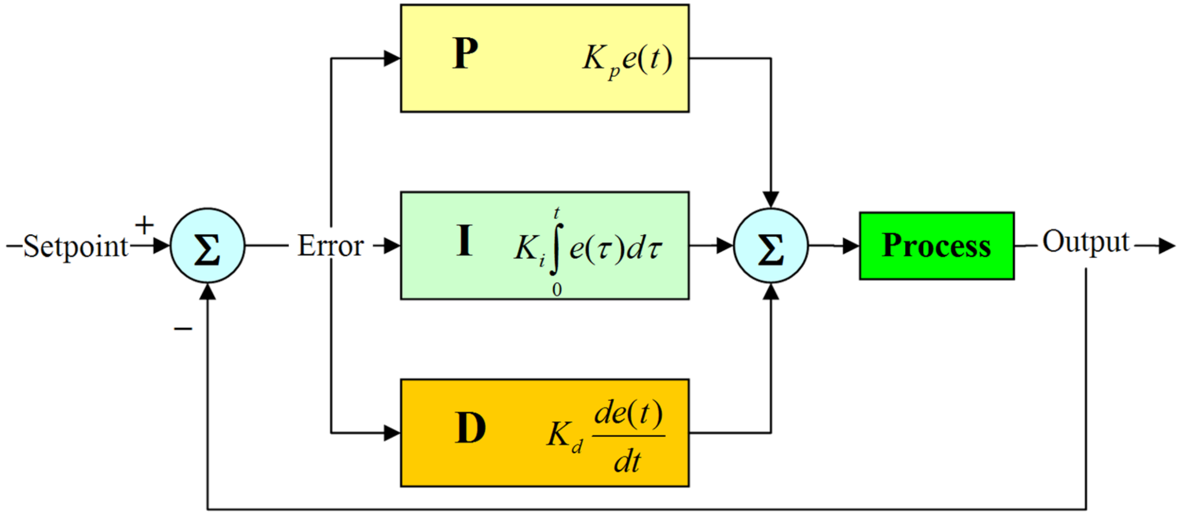
- **P**roportional / **I**ntegral / **D**ifferential Control

Continuous Form :

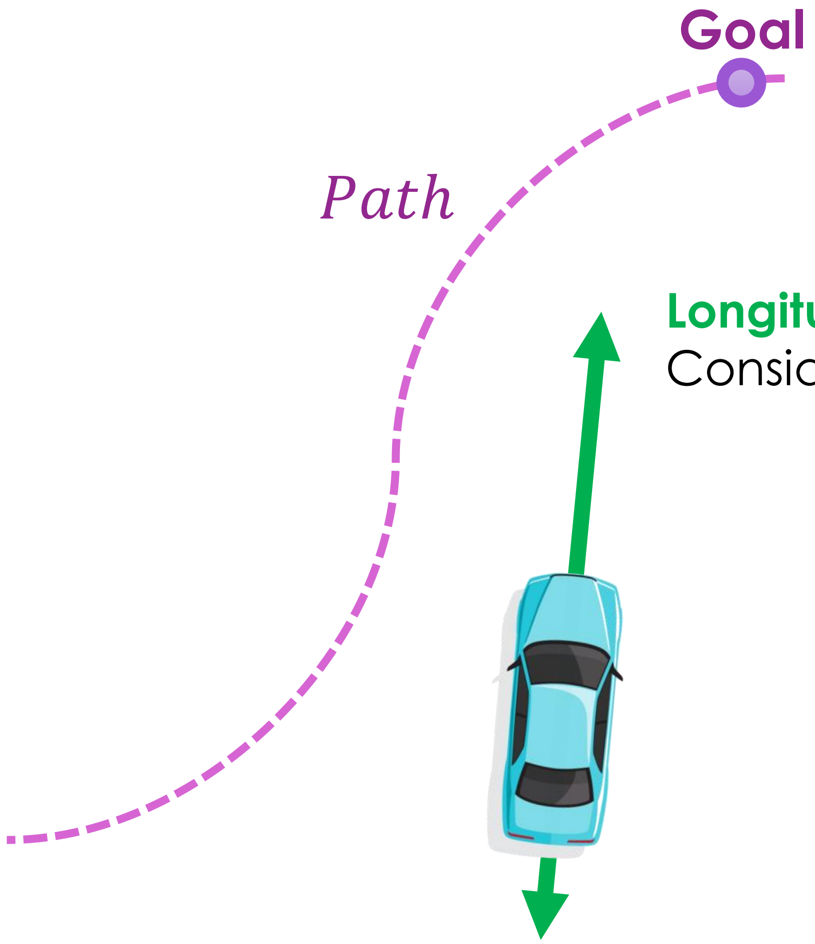
$$Output = K_p e(t) + K_i \int_0^t e_t \, dt + K_d \frac{de(t)}{dt}$$

Discrete Form :

$$Output = K_p e(t) + K_i \sum_0^t e_t + K_d (e(t) - e(t-1))$$

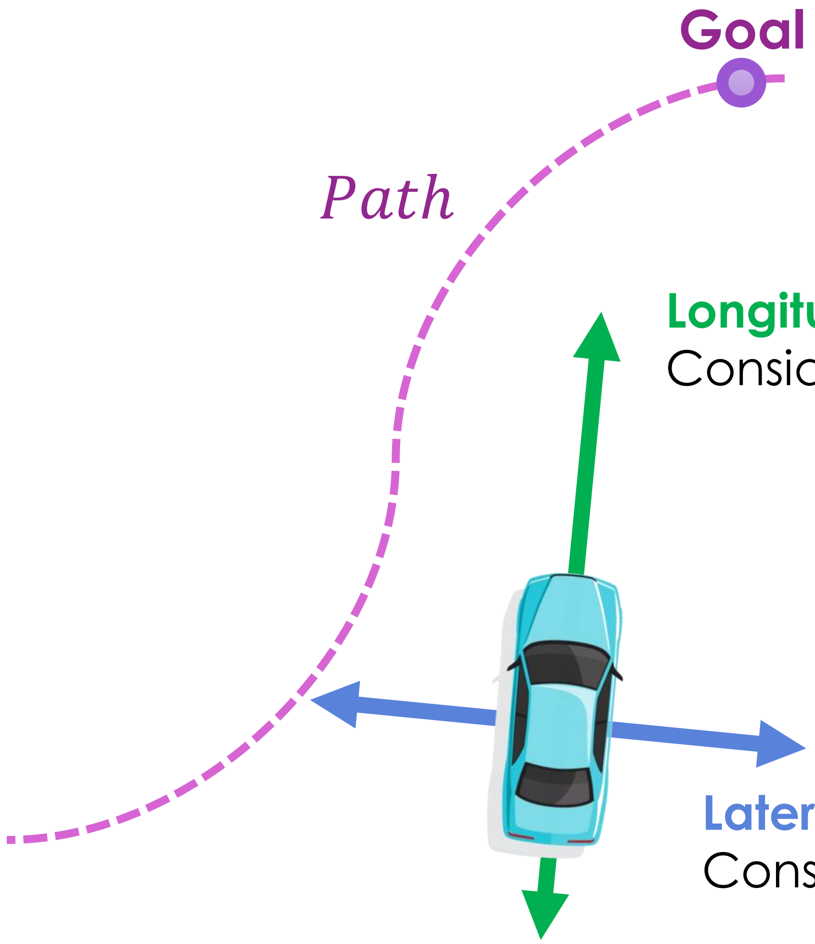


Path Tracking Problem



Longitude Control:
Consider the distance to the goal.

Path Tracking Problem



Longitude Control:
Consider the distance to the goal.

Lateral Control:
Consider the error distance to the path.

Basic Kinematic Model

State:

$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Rotation Matrix:

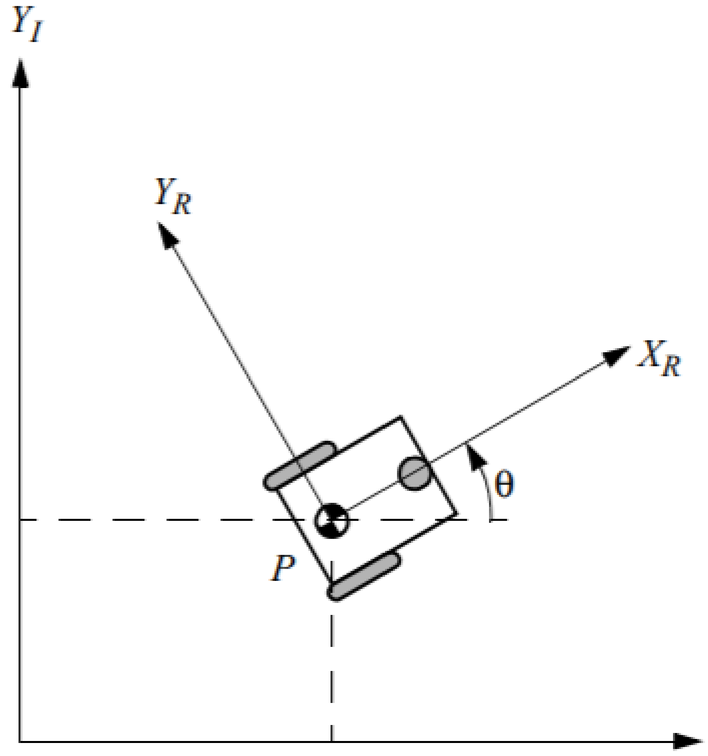
$$R(\theta) = \begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Kinematic Model:

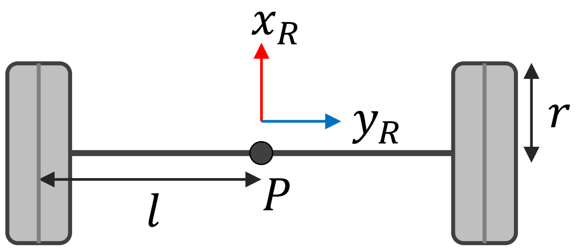
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \dot{x_R} \\ \dot{y_R} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ 0 \\ \omega \end{bmatrix}$$

$$= \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ \omega \end{bmatrix}$$

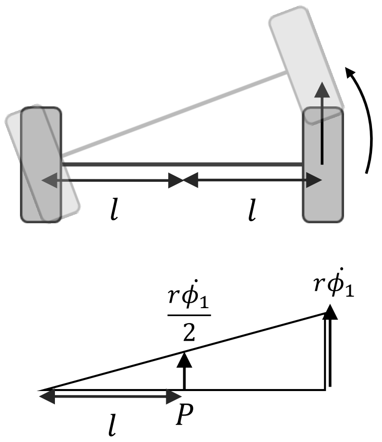


Differential Drive Vehicle (cont.)



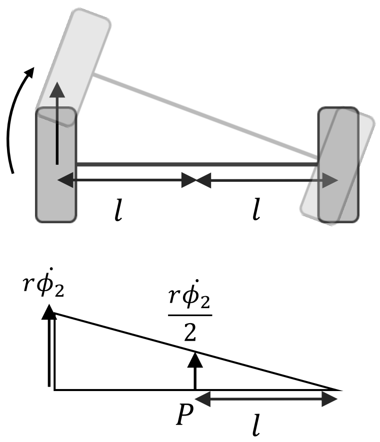
Right Wheel:

$$\dot{x}_{R1} = \frac{r\dot{\phi}_1}{2}$$
$$\omega_1 = \frac{r\dot{\phi}_1}{2l}$$



Left Wheel:

$$\dot{x}_{R2} = \frac{r\dot{\phi}_2}{2}$$
$$\omega_2 = \frac{-r\dot{\phi}_2}{2l}$$



Kinematic model for differential drive:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} - \frac{r\dot{\phi}_2}{2l} \end{bmatrix}$$

Differential Drive Vehicle

- Given target velocity $\boldsymbol{v}$ and angular velocity $\boldsymbol{\omega}$

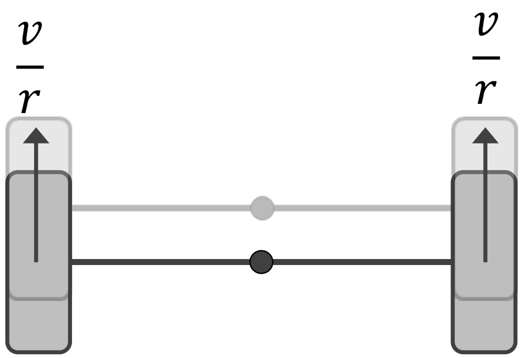
$$\begin{cases} v = \dfrac{r\dot{\phi}_1}{2} + \dfrac{r\dot{\phi}_2}{2} \\ \omega = \dfrac{r\dot{\phi}_1}{2l} - \dfrac{r\dot{\phi}_2}{2l} \end{cases}$$

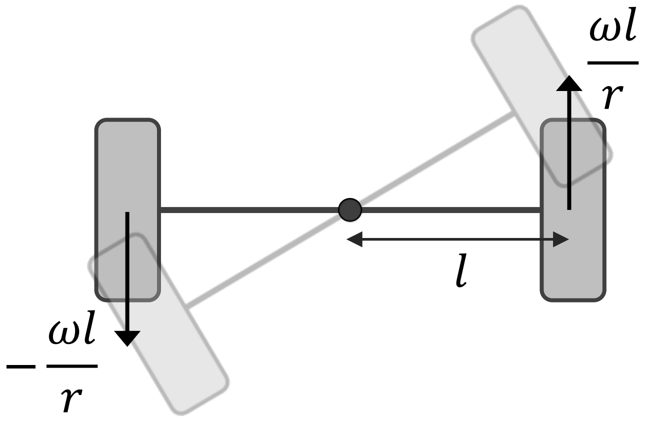
$$\dot{\phi}_2 = \left(v - \frac{r\dot{\phi}_1}{2}\right)\frac{2}{r} = \frac{2v}{r} - \dot{\phi}_1$$

$$\omega = \frac{r\dot{\phi}_1}{2l} - \frac{r\left(\frac{2v}{r} - \dot{\phi}_1\right)}{2l} = \frac{r\dot{\phi}_1 - v}{l}$$

$$\dot{\phi}_1 = \frac{v}{r} + \frac{\omega l}{r}$$

$$\dot{\phi}_2 = \frac{v}{r} - \frac{\omega l}{r}$$





Pure Pursuit Control

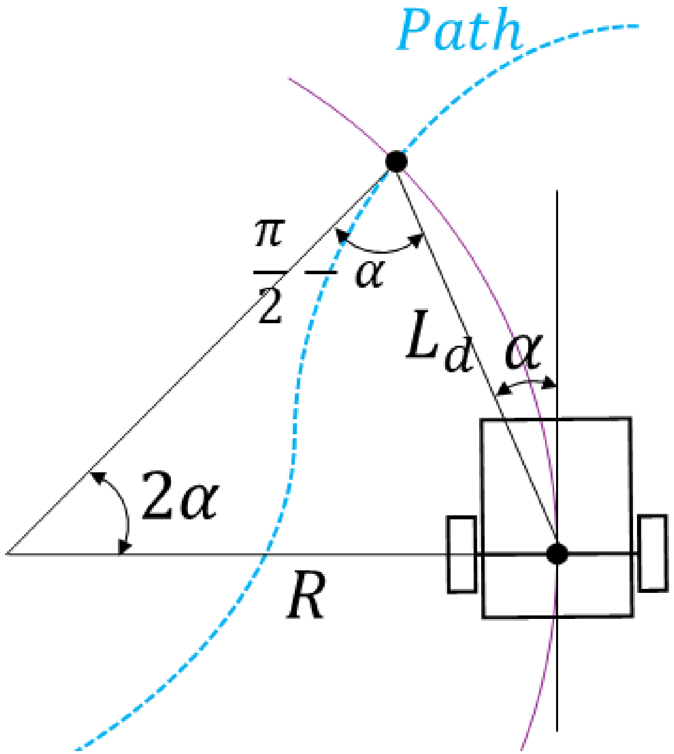
- Concept:
  - Modify the angular velocity to let the center achieve a point on path

$$\alpha = \arctan\left(\frac{y - y_g}{x - x_g}\right) - \theta$$

$$\frac{L_d}{\sin(2\alpha)} = \frac{R}{sin\left(\frac{\pi}{2} - \alpha\right)}$$

$$R = \frac{L_d \sin\left(\frac{\pi}{2} - \alpha\right)}{\sin(2\alpha)} = \frac{L_d \cos(\alpha)}{2\sin(\alpha)\cos(\alpha)} = \frac{L_d}{2\sin(\alpha)}$$

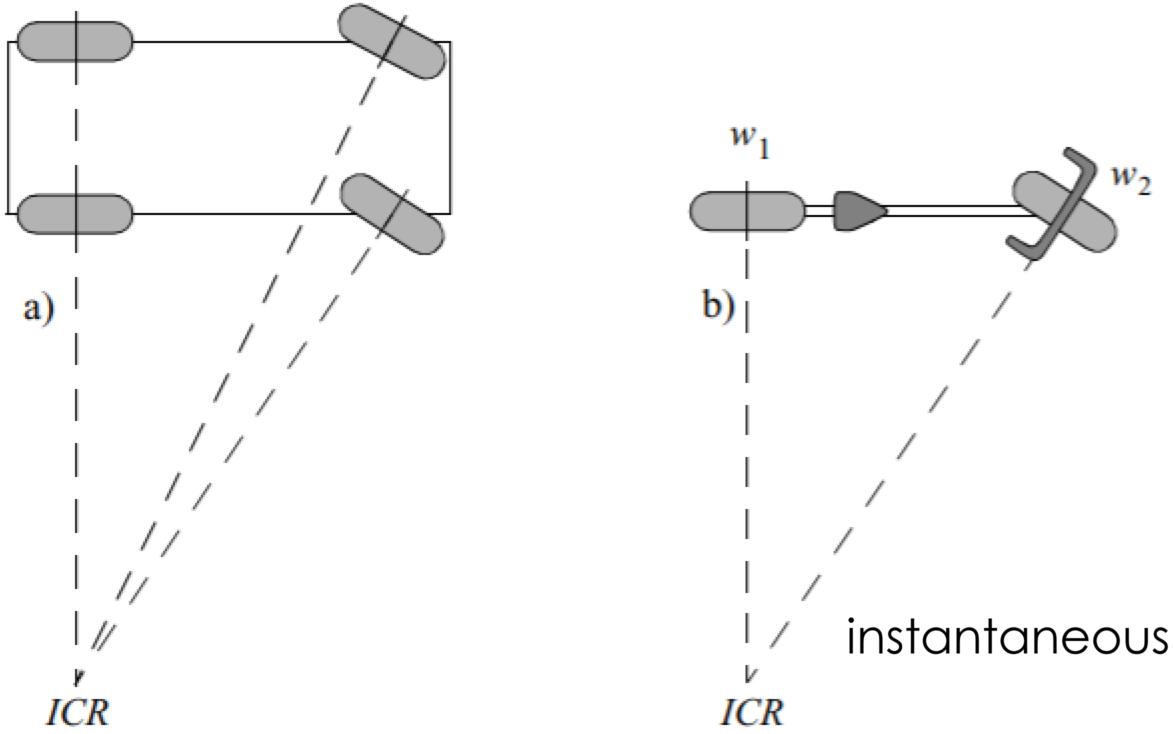
$$\omega = \frac{v}{R} = \frac{2v\sin(\alpha)}{L_d}$$



L_d usually set to $(kv + L_{fc})$, where k, L_{fc} are parameters.

Kinematic Bicycle Model

- Speed and Steering Control



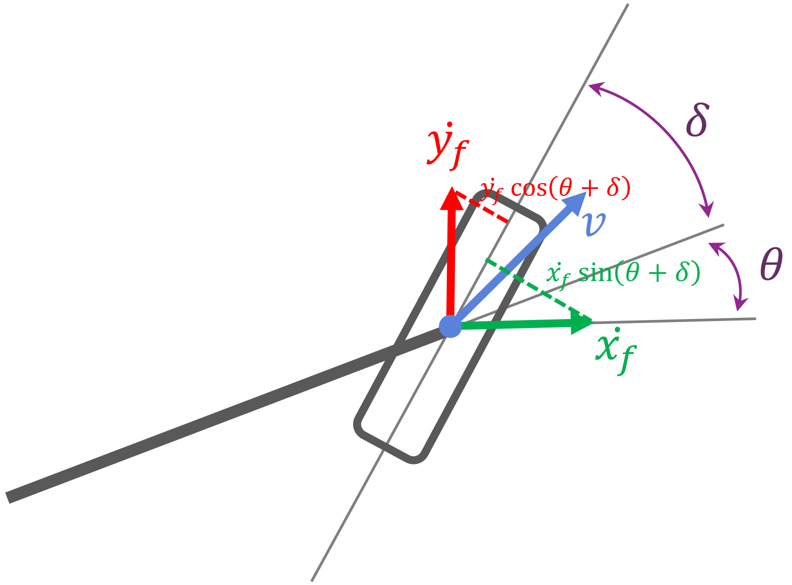
instantaneous center of rotation (ICR)

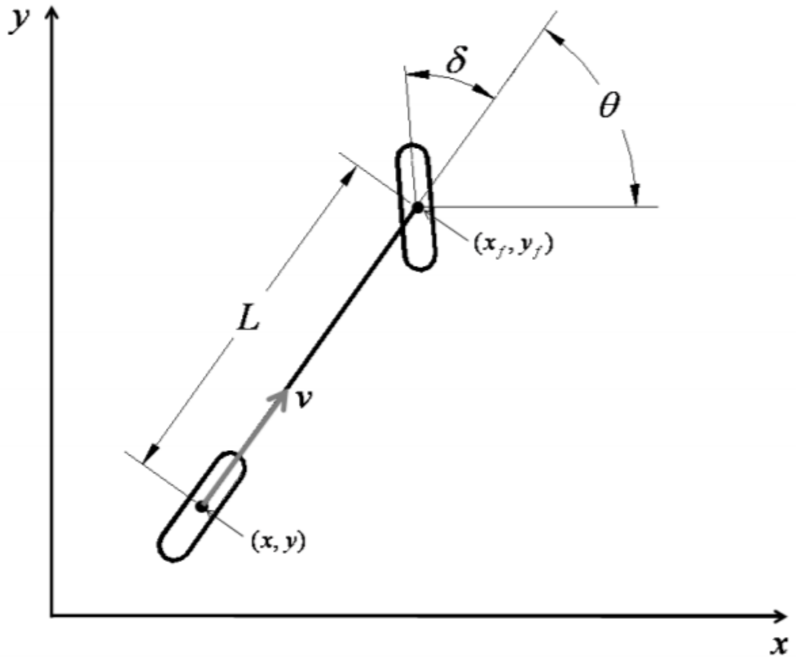
Figure 3.12
(a) Four-wheel with car-like Ackerman steering. (b) bicycle.

Kinematic Bicycle Model

- nonholonomic constraint equations

$$\dot{x}_f \sin(\theta + \delta) - \dot{y}_f \cos(\theta + \delta) = 0 \qquad (1) \quad \text{Front Wheel}$$
$$\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = 0 \qquad (2) \quad \text{Rear Wheel}$$



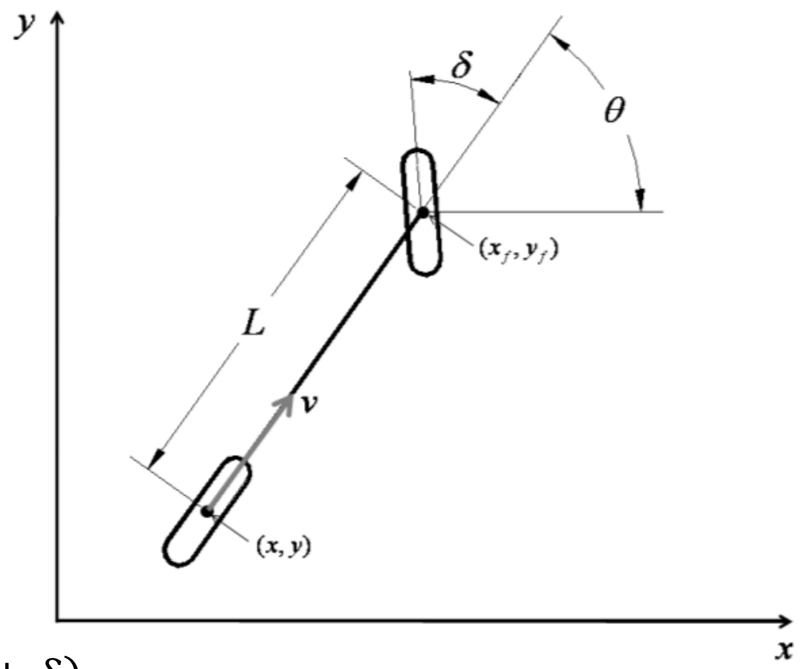


Kinematic Bicycle Model

- nonholonomic constraint equations

$$\dot{x}_f \sin(\theta+\delta) - \dot{y}_f \cos(\theta+\delta) = 0 \qquad (1)$$ Front Wheel

$$\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = 0 \qquad (2)$$ Rear Wheel



- Front Wheel Position

$$x_f = x + L\cos(\theta)$$
$$y_f = y + L\sin(\theta)$$

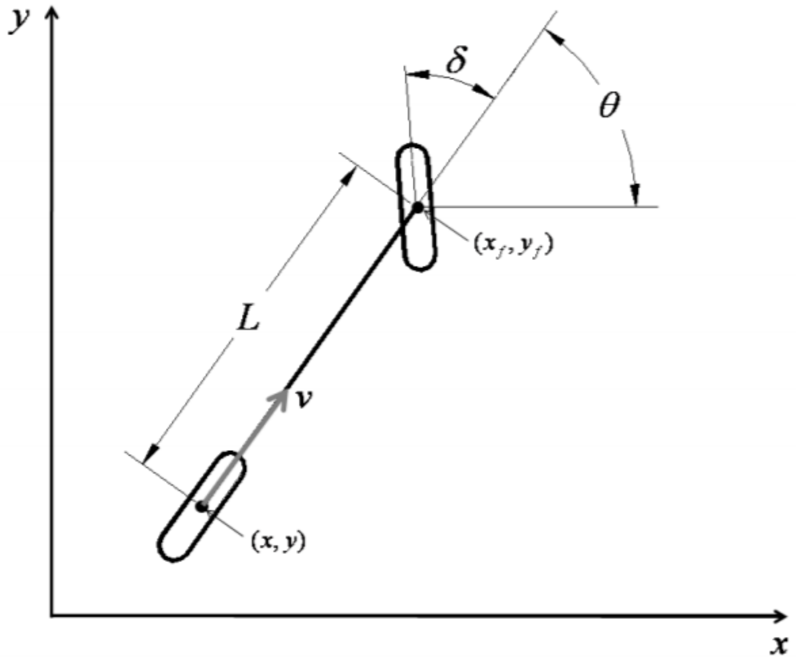
- Eliminating front wheel position from (1)

$$\begin{aligned} 0 &= \left(\dot{x} - \dot{\theta}L\sin(\theta)\right)\sin(\theta+\delta) - \left(\dot{y} + \dot{\theta}L\cos(\theta)\right)\cos(\theta+\delta) \\ &= \dot{x}\sin(\theta+\delta) - \dot{y}\cos(\theta+\delta) - \dot{\theta}L\sin(\theta)(\sin(\theta)\cos(\delta) + \cancel{\cos(\theta)\sin(\delta)}) \\ &\qquad - \dot{\theta}L\cos(\theta)(\cos(\theta)\cos(\delta) + \cancel{\sin(\theta)\sin(\delta)}) \\ &= \dot{x}\sin(\theta+\delta) - \dot{y}\cos(\theta+\delta) - \dot{\theta}L\cos(\delta) \qquad (3) \end{aligned}$$

Kinematic Bicycle Model

- nonholonomic constraint equations

$$\dot{x}\sin(\theta+\delta)-\dot{y}\cos(\theta+\delta)-\dot{\theta}L\cos(\delta)=0 \quad (3)$$
$$\dot{x}\sin(\theta)-\dot{y}\cos(\theta)=0 \quad (2)$$



- Rear wheel satisfied the constrain (2) when

$$\dot{x}=v\cos(\theta) \quad (4)$$
$$\dot{y}=v\sin(\theta) \quad (5)$$

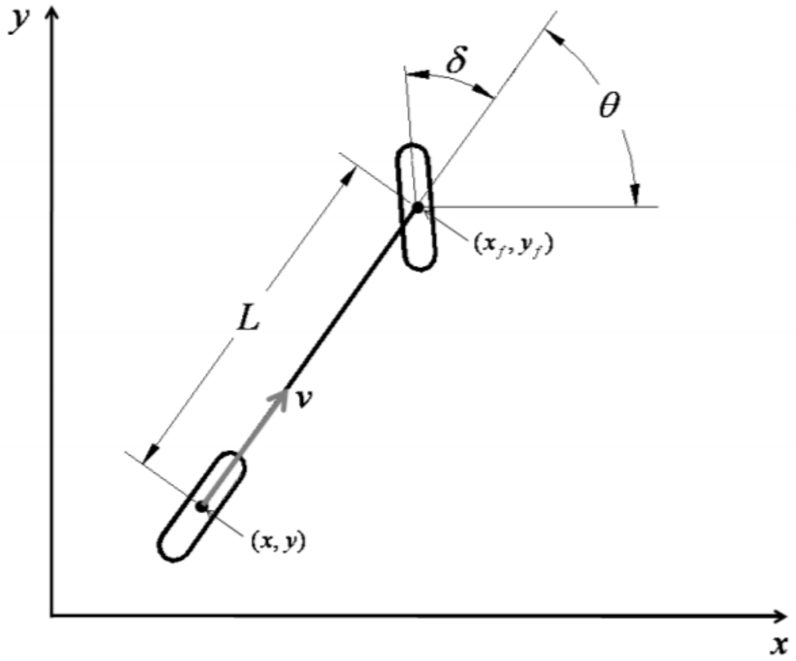
- Applying (4)(5) to (3)

$$\dot{\theta}=\frac{\dot{x}\sin(\theta+\delta)-\dot{y}\cos(\theta+\delta)}{L\cos(\delta)}$$
$$=\frac{v\cos(\theta)(\cancel{\sin(\theta)\cos(\delta)}+\cos(\theta)\sin(\delta))-v\sin(\theta)(\cancel{\cos(\theta)\cos(\delta)}+\sin(\theta)\sin(\delta))}{L\cos(\delta)}$$
$$=\frac{v(\cos^2(\theta)+\sin^2(\theta))\sin(\delta)}{L\cos(\delta)}=\frac{v\tan(\delta)}{L}$$

Kinematic Bicycle Model

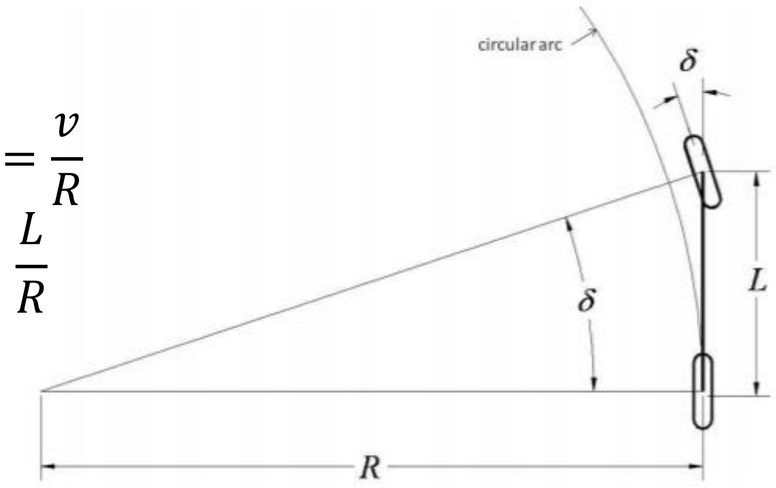
- Kinematic Model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \dfrac{\tan(\delta)}{L} \end{bmatrix} v$$



- Some Property

$$R\dot{\theta} = v$$
$$\frac{v\tan(\delta)}{L} = \frac{v}{R}$$
$$\tan(\delta) = \frac{L}{R}$$



Pure Pursuit Control for Bicycle Model

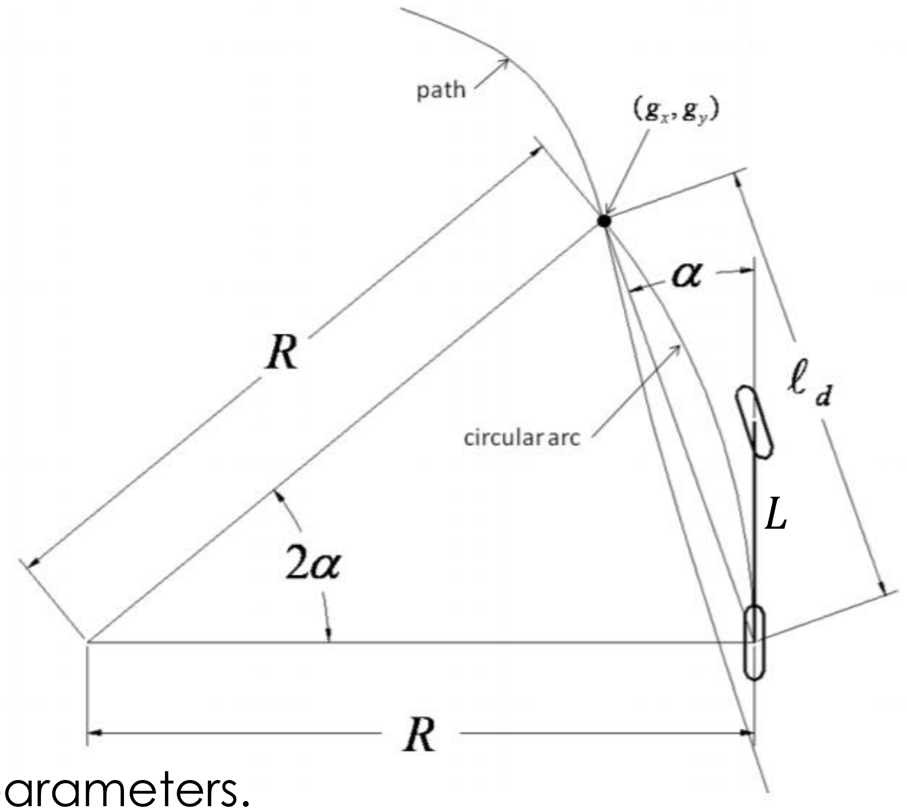
- Concept:
  - Control the steer to let the rear wheel achieve a point on the path.

$$\alpha = \arctan\left(\frac{y - y_g}{x - x_g}\right) - \theta$$

$$R = \frac{L_d \sin\left(\frac{\pi}{2} - \alpha\right)}{\sin(2\alpha)} = \frac{L_d \cos(\alpha)}{2\sin(\alpha)\cos(\alpha)} = \frac{L_d}{2\sin(\alpha)}$$

$$\tan(\delta) = \frac{L}{R}$$

$$\delta = \arctan\left(\frac{L}{R}\right) = \arctan(\frac{2Lsin(\alpha)}{L_d})$$



L_d usually set to $(kv + L_{fc})$, where k, L_{fc} are parameters.

Stanley Control

- Concept:
  - Exponential stability for front wheel feedback
- Differential of error distance

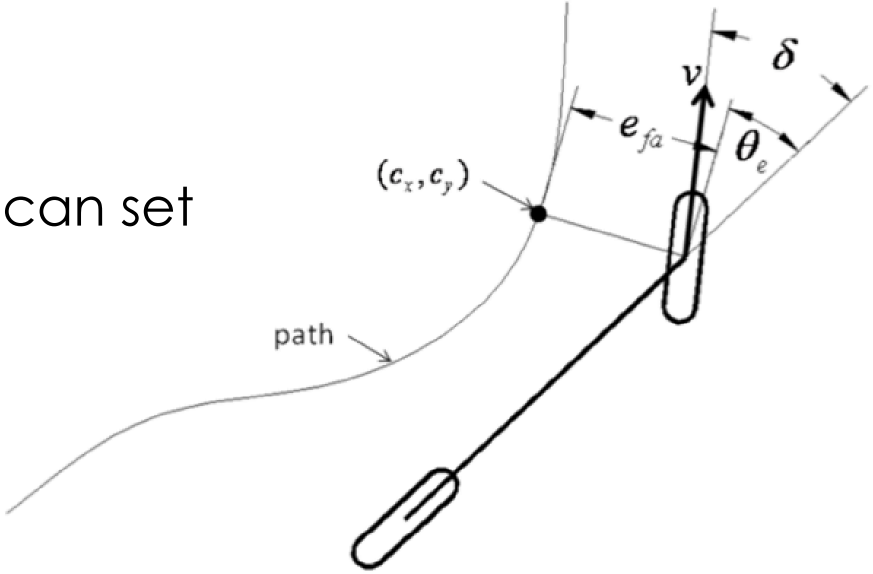
$$\dot{e} = v_f sin(\delta - \theta_e)$$

- To achieve exponential stability to path, we can set

$\dot{e} = -ke$, where k>0

$$-ke = v_f sin(\delta - \theta_e)$$

$$\delta = \arcsin\left(-\frac{ke}{v_f}\right) + \theta_e$$



- It is not defined when **|–ke/vf|>1**. We can modify the control law to

$\delta = \arctan\left(-\frac{ke}{v_f}\right) + \theta_e$, which satisfy the **local exponential stability (LES).**

LQR Control

- If we use the motion model with more complex form (e.g. dynamic model), it is hard to directly analyze the error function.
- **L**inear **Q**uadratic **R**egulator (LQR) introduce the concept of **cost function**, and try to solve the optimization problem when the motion model is **linear form** and the cost function is **quadratic form**.
- The formulation of LQR problem:
  - Define state **x** and control **u,** the motion model is $\dot{x} = Ax + Bu.$
  - The cost function is setting to the quadratic form $c = \underbrace{x^T Q x}_{\text{State Error}} + \underbrace{u^T R u}_{\text{Minimum Control}}$

    , in which **Q** is the state weighting matrix and **R** is the control weighting matrix.
  - The total objective function of an episode $J = \int_0^T [x(t)^T Q x(t) + u(t)^T R u(t)] dt + x^T(T) S x(T)$

LQR Control (cont.)

- The goal is to find the optimal control **u*** which minimize the total object function: $\min_u J = \min_{\mathrm{u}} \int_0^T x(t)^T Q x(t) + u(t)^T R u(t)\, dt + x(T)^T S x(T)$

- To solve this problem, we first introduce the concept of optimal principle. If we have a optimal control sequence $[u_t^*, u_{t+1}^*, u_{t+2}^*, \dots, u_T^*]$, then the subsequence $[u_{t+1}^*, u_{t+2}^*, \dots, u_T^*]$ is also an optimal control sequence.

- Follow the concept, we can apply **dynamic programming** to recursively solve the optimal control from terminal state to current time.

LQR Control (cont.)

- However, we do not know the terminal time or even the terminal time is infinite in most time. In this case, we can solve the LQR using the recursive relation of value function.

- Introduce the value function **V(x)**, which is the summing of the future cost. We can write down the recursive form of the discrete time value function:

$$V(x_t) = \min_{u}\{ x_t^T Q x_t + u_t R u_t + V(x_{t+1})\}$$

- We can guess the value function to be quadratic form $V(x_t) = x_t^T P_t x_t$ (which P is symmetric positive-definite), and apply the linear motion model $Ax_t + Bu_t$ to value function:

$$V(x_t) = \min_{u}\{ x_t^T Q x_t + u_t R u_t + x_{t+1}^T P_{t+1} x_{t+1}\}$$
$$= \min_{u}\{ x_t^T Q x_t + u_t R u_t + (Ax_t + Bu_t)^T P_{t+1}(Ax_t + Bu_t)\}$$
$$= \min_{u}\{x_t^T (Q + A^T P_{t+1} A) x_t + 2x^T A^T P B u + u_t^T (R + B^T P_{t+1} B) u_t\}$$

LQR Control (cont.)

- Solve the minimum equation

$$V(x_t) = x_t^T P_t x_t = \min_{\mathrm{u}}\{x_t^T (Q + A^T P_{t+1} A) x_t + 2x^T A^T P B u + u_t^T (R + B^T P_{t+1} B) u_t\}$$

$$\frac{\partial}{\partial u}\left[x_t^T (Q + A^T P_{t+1} A) x_t + 2x^T A^T P B u_t^* + u_t^{*T} (R + B^T P_{t+1} B) u_t^*\right] = 0$$

$$2(x^T A^T P_{t+1} B)^T + 2(R + B^T P_{t+1} B) u_t^* = 0$$

$$u_t^* = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A x_t$$

- Apply u* to the value function, and get the equation of P

$$x_t^T P_t x_t = x_t^T (Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A) x_t$$

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

Discrete **A**lgebra **R**iccati **E**quation (DARE)

Remark: In continuous case, $\dot{P} = -PA - A^T P + PBR^{-1}P - Q$ is the **C**ontinuous **A**lgebra **R**iccati **E**quation (CARE)

LQR Control (cont.)

- Given discrete Riccati algebra equation

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

- Suppose the value function is time-invariant, then

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$

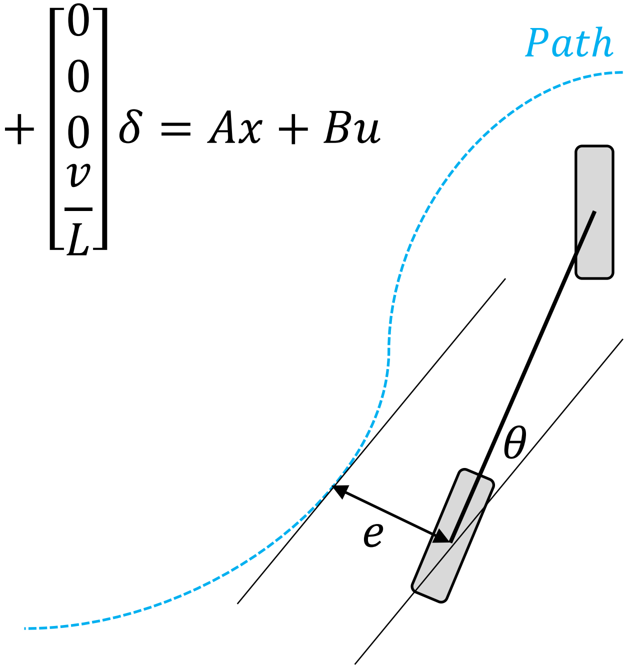
- In practice, we can first initialize $P^{(0)} = Q$, then iteratively apply the Riccati equation on until converge :

```
INITIALIZE: P ← Q
REPEAT
   P_next ← Q + A^T PA − A^T PB(R + B^T PB)^{-1} B^T PA
   ϵ ← ||P_next − P||
   IF ϵ < threshold THEN
      return P_next
   ENDIF
   P ← P_next
END
```

LQR Control for Kinematic Model

- Take an example to solve the LQR optimal control of the kinematic model.
- Define State: $x = [e, \dot{e}, \theta, \dot{\theta}]$, and set the matrix Q and R
- the linear approximate of kinematic motion model:

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{v\tan(\delta)}{L} \end{bmatrix} \approx \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{v}{L} \end{bmatrix}\delta = Ax + Bu$$



- Solve the DARE to get the P matrix

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$

- Finally, we can get the optimal control

$$u_t^* = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A x_t$$

Review of Control Algorithms

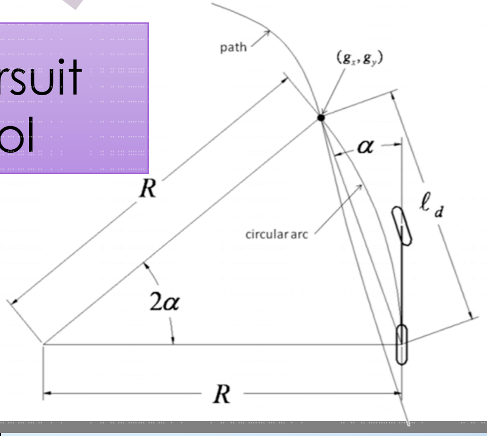
$$\delta = K_p e(t) + K_i \sum_{0}^{t} e_t + K_d(e(t) - e(t-1))$$

PID Control

Apply the kinematic property.

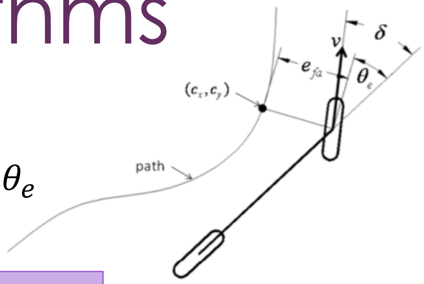
Pure-Pursuit Control

$$\delta = \arctan\left(\frac{2L\sin(\alpha)}{L_d}\right)$$



Consider the progressive stability.

$$\delta = \arctan\left(-\frac{ke}{v_f}\right) + \theta_e$$



Stanley Control

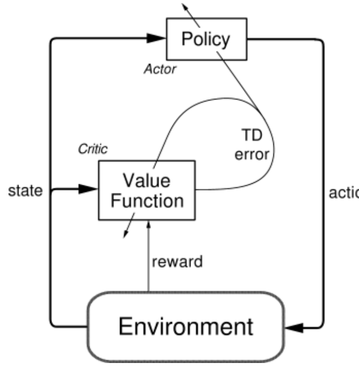
More complex motion model.

LQR Control

DARE:

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

Don't need model.
Non-linear case.



Model-free Reinforcement Learning

Next Week

- Path Planning