射影几何与相机模型

Projective Geometry and Camera model

Class 2

points, lines, planes
conics and quadrics
transformations
camera model

点，线，面，二次曲线和二次曲面变换
相机模型

Chapter 1, 2 and 5 in Hartley and Zisserman

对应课本的第1，2，5章

Homogeneous coordinates

Homogeneous representation of 2D points and lines

$$ax + by + c = 0 \qquad\qquad (a,b,c)^{\mathsf{T}}(x,y,1) = 0$$

The point x lies on the line l if and only if

$$\mathrm{l}^{\mathsf{T}}\mathrm{x} = 0$$

Note that scale is unimportant for incidence relation

$$(a,b,c)^{\mathsf{T}} \sim k(a,b,c)^{\mathsf{T}}, \forall k \neq 0 \qquad (x,y,1)^{\mathsf{T}} \sim k(x,y,1)^{\mathsf{T}}, \forall k \neq 0$$

equivalence class of vectors, any vector is representative

Set of all equivalence classes in $\mathbf{R}^3$–$(0,0,0)^{\mathsf{T}}$ forms $\mathbf{P}^2$

Homogeneous coordinates $(x_1, x_2, x_3)^{\mathsf{T}}$ but only 2DOF

Inhomogeneous coordinates $(x, y)^{\mathsf{T}}$

Points from lines and vice-versa

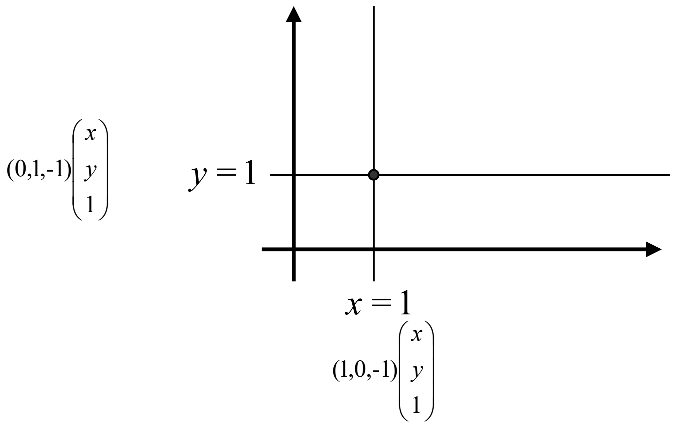
Intersections of lines

The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

Line joining two points

The line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

Example



Note:

$$\mathbf{x} \times \mathbf{x}' = [\mathbf{x}]_\times \mathbf{x}'$$

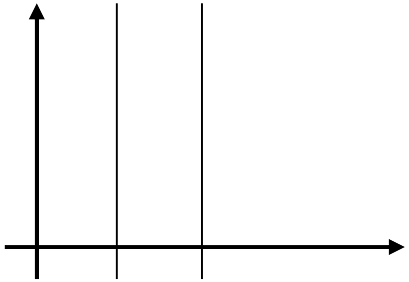
with $$[\mathbf{x}]_\times = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}$$

Ideal points and the line at infinity

Intersections of parallel lines

$\mathbf{l} = (a, b, c)^{\mathsf{T}}$ and $\mathbf{l}' = (a, b, c')^{\mathsf{T}}$ $\quad\quad$ $\mathbf{l} \times \mathbf{l}' = (b, -a, 0)^{\mathsf{T}}$

Example



$(b, \text{-}a)$ tangent vector

(a, b) normal direction

Ideal points $\quad (x_1, x_2, 0)^{\mathsf{T}}$

Line at infinity $\quad \mathbf{l}_\infty = (0, 0, 1)^{\mathsf{T}}$

$$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathbf{l}_\infty$$

Note that in **P**^{2} there is no distinction between ideal points and others

3D points and planes

Homogeneous representation of 3D points and planes

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

The point X lies on the plane π if and only if

$$\pi^{\mathsf{T}}\mathrm{X} = 0$$

The plane π goes through the point X if and only if

$$\pi^{\mathsf{T}}\mathrm{X} = 0$$

Planes from points

Solve π from $X_1^\mathsf{T}\pi = 0$, $X_2^\mathsf{T}\pi = 0$ and $X_3^\mathsf{T}\pi = 0$

$$\begin{bmatrix} X_1^\mathsf{T} \\ X_2^\mathsf{T} \\ X_3^\mathsf{T} \end{bmatrix}\pi = 0 \quad \text{(solve } \pi \text{ as right nullspace of } \begin{bmatrix} X_1^\mathsf{T} \\ X_2^\mathsf{T} \\ X_3^\mathsf{T} \end{bmatrix} \text{)}$$

Points from planes

Solve X from $\pi_1^{\mathsf{T}}\mathrm{X}=0$, $\pi_2^{\mathsf{T}}\mathrm{X}=0$ and $\pi_3^{\mathsf{T}}\mathrm{X}=0$

$$\begin{bmatrix}\pi_1^{\mathsf{T}}\\ \pi_2^{\mathsf{T}}\\ \pi_3^{\mathsf{T}}\end{bmatrix}\mathrm{X}=0$$

(solve X as right nullspace of $\begin{bmatrix}\pi_1^{\mathsf{T}}\\ \pi_2^{\mathsf{T}}\\ \pi_3^{\mathsf{T}}\end{bmatrix}$)

Representing a plane by its span

$$\mathrm{X}=\mathbf{M}\,\mathrm{x} \qquad \mathbf{M}=\left[\mathrm{X}_1\mathrm{X}_2\mathrm{X}_3\right]$$

$$\pi^{\mathsf{T}}\,\mathbf{M}=0$$

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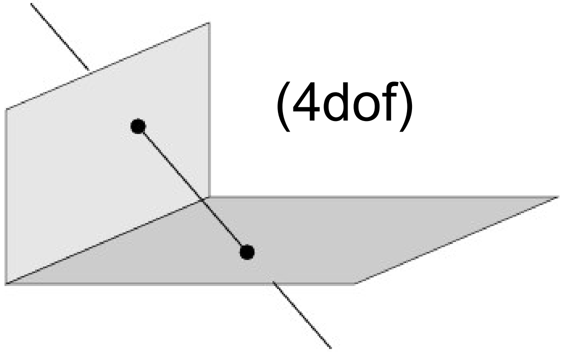
Lines



Representing a line by its span

$$W = \begin{bmatrix} A^\mathsf{T} \\ B^\mathsf{T} \end{bmatrix} \qquad \lambda A + \mu B$$

Dual representation

$$W^* = \begin{bmatrix} P^\mathsf{T} \\ Q^\mathsf{T} \end{bmatrix} \qquad \lambda P + \mu Q$$

$$W^* W^\mathsf{T} = W W^{*\mathsf{T}} = 0_{2\times 2}$$

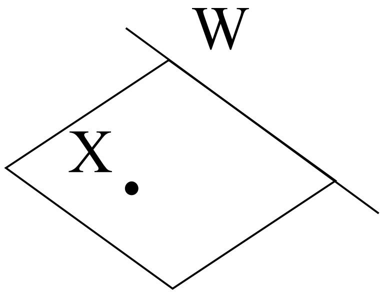
Example: X-axis

$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad W^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

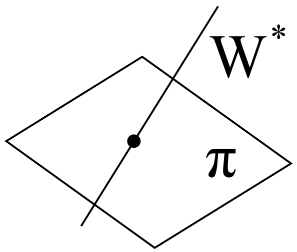
(Alternative: Plücker representation, details see e.g. H&Z)

Points, lines and planes

$$\mathbf{M} = \begin{bmatrix} W \\ X^T \end{bmatrix} \qquad \mathbf{M}\pi = 0$$



$$\mathbf{M} = \begin{bmatrix} W^* \\ \pi^T \end{bmatrix} \qquad \mathbf{M}X = 0$$



Plücker coordinates

Elegant representation for 3D lines

$$l_{ij} = A_i B_j \quad B_i A_j$$ (with A and B points)

$$\mathsf{L} = \left[l_{12}, l_{13}, l_{14}, l_{23}, l_{42}, l_{34} \right]^{\mathsf{T}} \in \mathbf{P}^5$$

$$\left(\mathsf{L} \mid \hat{\mathsf{L}} \right) = l_{12}\hat{l}_{34} + l_{13}\hat{l}_{42} + l_{14}\hat{l}_{23} + l_{23}\hat{l}_{14} + l_{42}\hat{l}_{13} + l_{34}\hat{l}_{12}$$

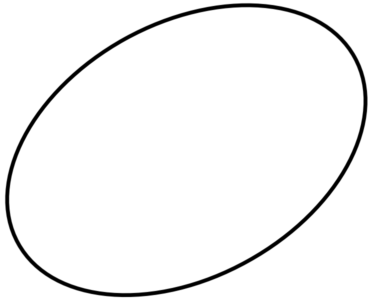
$$\left(\mathsf{L} \mid \mathsf{L} \right) = 0$$ (Plücker internal constraint)

$$\left(\mathsf{L} \mid \hat{\mathsf{L}} \right) = \det\left[\mathrm{A}, \mathrm{B}, \hat{\mathrm{A}}, \hat{\mathrm{B}} \right] = 0$$ (two lines intersect)

(for more details see e.g. H&Z)

Conics

Curve described by 2nd-degree equation in the plane



$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^{\ 2} + bx_1x_2 + cx_2^{\ 2} + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a : b : c : d : e : f\}$

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_i y_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

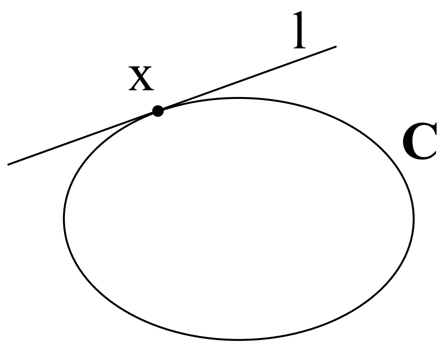
$$\left(x_i^2, x_i y_i, y_i^2, x_i, y_i, 1\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^\mathsf{T}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Tangent lines to conics

The line l tangent to $\mathbf{C}$ at point x on $\mathbf{C}$ is given by $\mathrm{l}=\mathbf{C}\mathrm{x}$

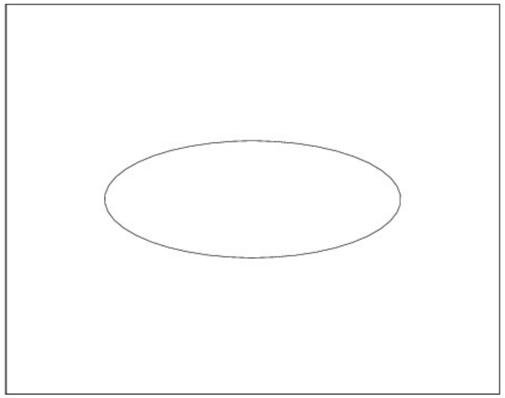


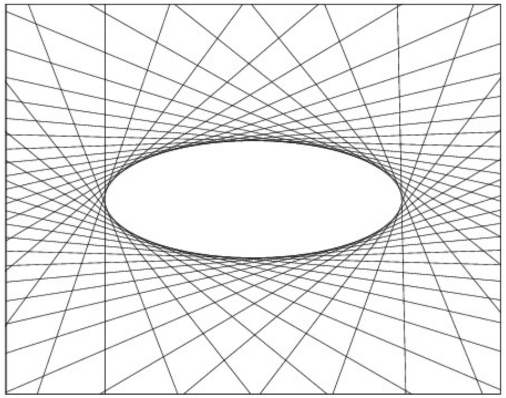
Dual conics

A line tangent to the conic $\mathbf{C}$ satisfies $\mathbf{l}^{\mathsf{T}}\, \mathbf{C}^{*}\, \mathbf{l} = 0$

In general ($\mathbf{C}$ full rank): $\mathbf{C}^{*} = \mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes

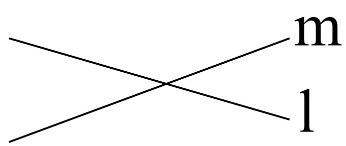




Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

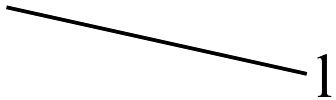
e.g. two lines (rank 2)



$$\mathbf{C} = \mathrm{l}\mathrm{m}^{\mathsf{T}} + \mathrm{m}\mathrm{l}^{\mathsf{T}}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathrm{l}\mathrm{l}^{\mathsf{T}}$$



Degenerate line conics: 2 points (rank 2), double point (rank1)

Note that for degenerate conics $\left(\mathbf{C}^{*}\right)^{*} \neq \mathbf{C}$

Quadrics and dual quadrics

$X^T Q X = 0$ (Q : 4x4 symmetric matrix)

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

- 9 d.o.f.
- in general 9 points define quadric
- det Q=0 $\boxed{\leftrightarrow}$ degenerate quadric
- tangent plane $\pi = QX$

$\pi^T Q^* \pi = 0$

- relation to quadric $Q^* = Q^{-1}$ (non-degenerate)

2D projective transformations

<u>Definition:</u>

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

<u>Theorem:</u>

A mapping $h: P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix $\mathbf{H}$ such that for any point in P^2 reprented by a vector x it is true that $h(x) = \mathbf{H}x$

<u>Definition:</u> Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad x' = \mathbf{H}\, x$$

8DOF

projectivity=collineation=projective transformation=homography

Transformation of 2D points, lines and conics

For a point transformation

$$\mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

Transformation for lines

$$\mathbf{l}' = \mathbf{H}^{-T}\,\mathbf{l}$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

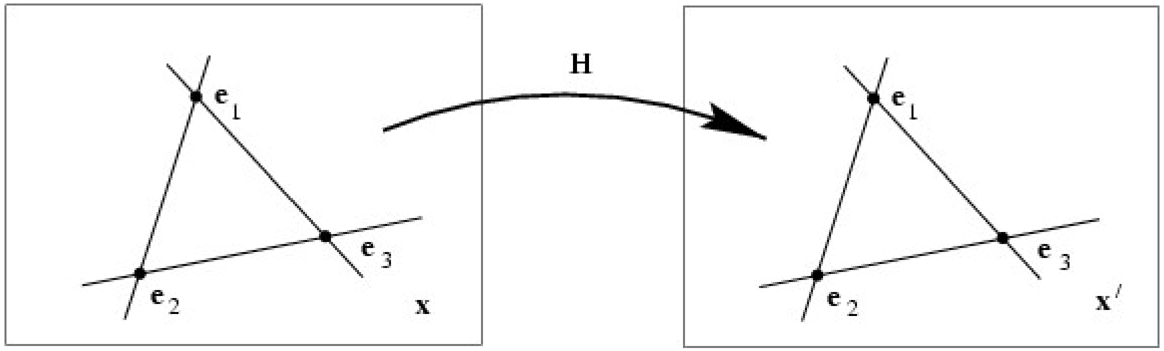
$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{T}$$

Fixed points and lines

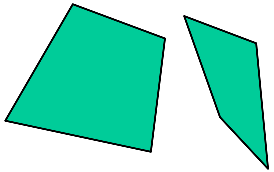
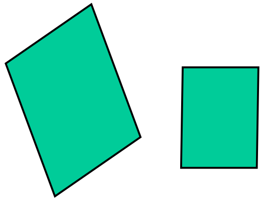
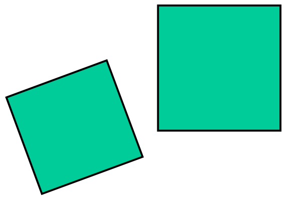
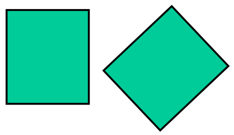
$\mathbf{H}\,\mathrm{e} = \lambda\,\mathrm{e}$ (eigenvectors $\mathbf{H}$ =fixed points)

(λ_1=λ_2 $\Rightarrow$ pointwise fixed line)

$\mathbf{H}^{\mathsf{T}}\,\mathrm{l} = \lambda\,\mathrm{l}$ (eigenvectors $\mathbf{H}^{-\mathsf{T}}$ =fixed lines)



Hierarchy of 2D transformations

		transformed squares	invariants
Projective 8dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio
Affine 6dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids). **The line at infinity l_∞**
Similarity 4dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratios of lengths, angles. **The circular points I,J**
Euclidean 3dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		lengths, areas.

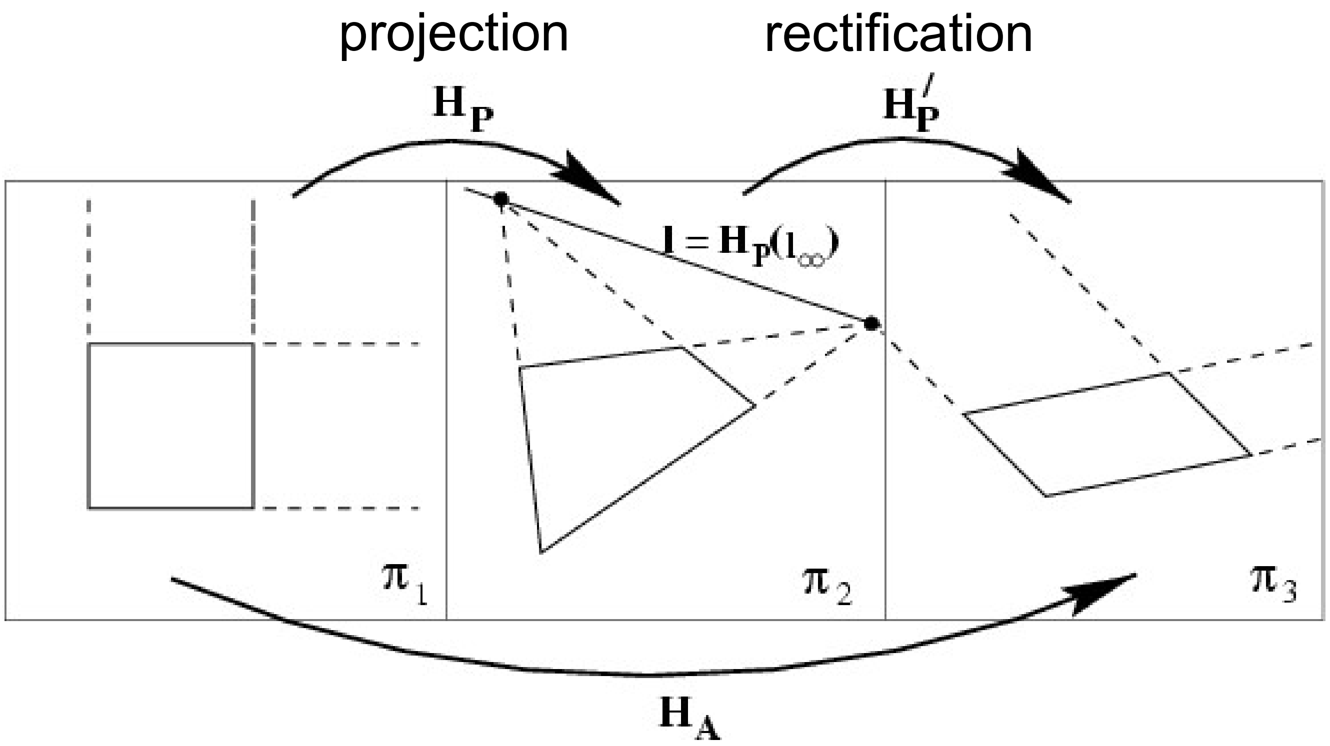
The line at infinity

$$\mathbf{l}'_\infty = \mathbf{H}_A^{-\mathsf{T}}\,\mathbf{l}_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{t}^{\mathsf{T}}\,\mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty$$

The line at infinity l_∞ is a fixed line under a projective transformation H if and only if H is an affinity

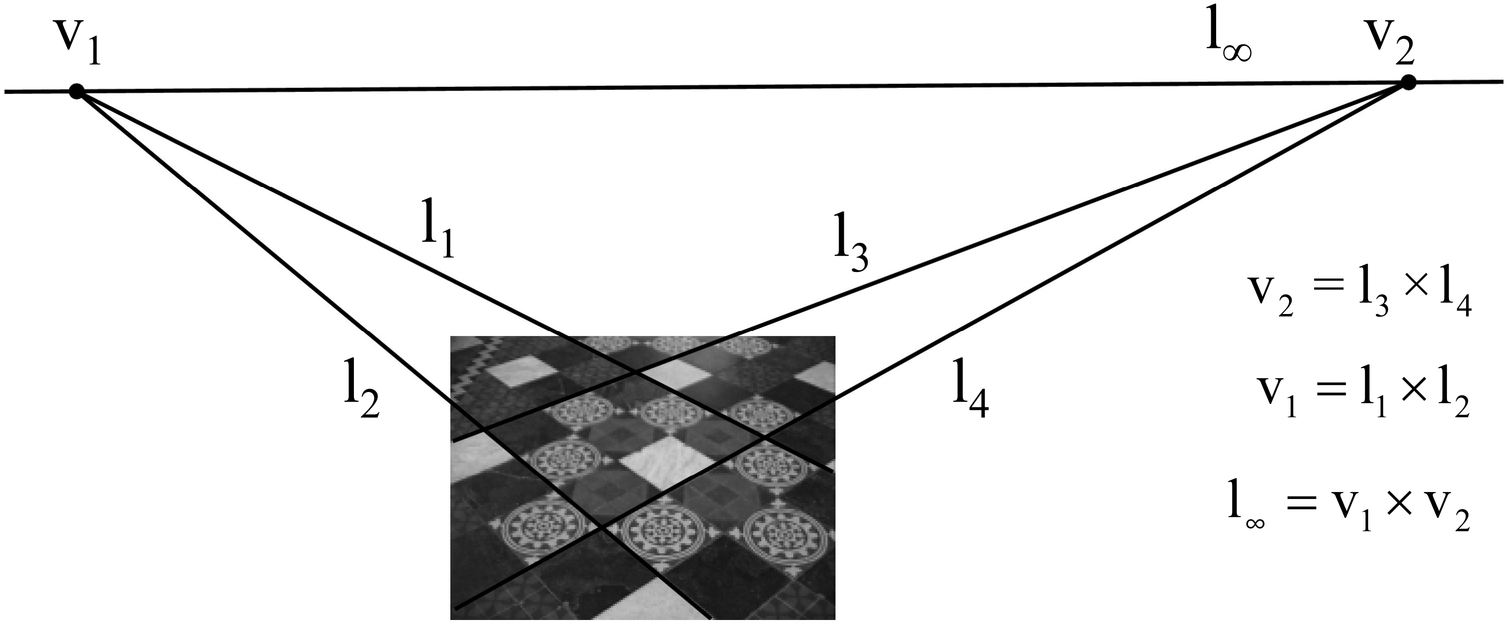
Note: not fixed pointwise

Affine properties from images



$$\mathbf{H}_{PA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} \mathbf{H}_A \qquad \mathbf{l}_\infty = [l_1 \quad l_2 \quad l_3]^\mathsf{T}, l_3 \neq 0$$

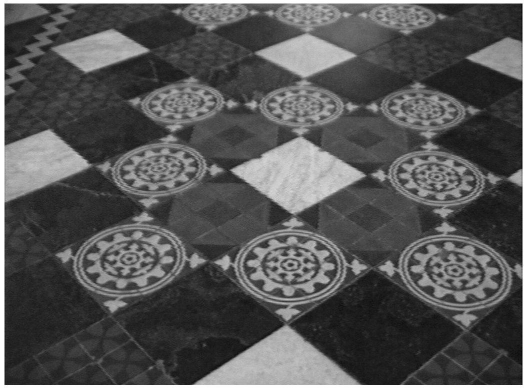
Affine rectification

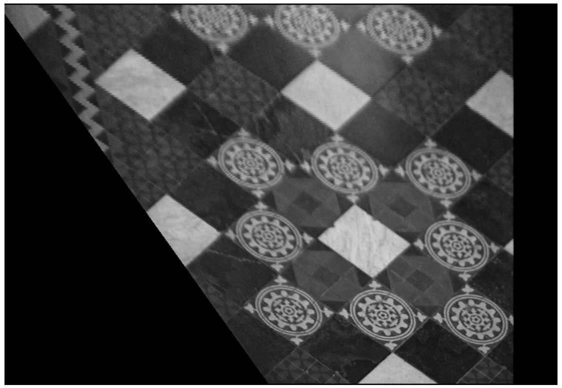


$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

$$l_\infty = v_1 \times v_2$$





The circular points

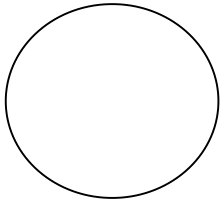
$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathrm{I}' = \mathbf{H}_S\,\mathrm{I} = \begin{bmatrix} s\cos\theta & s\sin\theta & t_x \\ -s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

The circular points I, J are fixed points under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

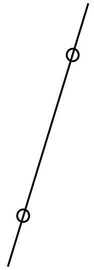
The circular points

"circular points"



$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

$$x_3 = 0$$



l_∞

$$x_1^2 + x_2^2 = 0$$

$$\mathrm{I} = (1, i, 0)^\mathsf{T}$$

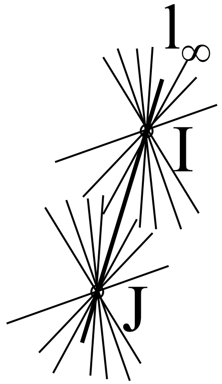
$$\mathrm{J} = (1, -i, 0)^\mathsf{T}$$

Algebraically, encodes orthogonal directions

$$\mathrm{I} = (1,0,0)^\mathsf{T} + i(0,1,0)^\mathsf{T}$$

Conic dual to the circular points

$$\mathbf{C}^*_\infty = \mathrm{IJ}^\mathsf{T} + \mathrm{JI}^\mathsf{T} \qquad \mathbf{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$\mathbf{C}^*_\infty = \mathbf{H}_S \mathbf{C}^*_\infty \mathbf{H}_S^\mathsf{T}$$

The dual conic $\mathbf{C}^*_\infty$ is fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

Note: $\mathbf{C}^*_\infty$ has 4DOF

l_∞ is the nullvector

Angles

Euclidean: $\mathrm{l} = (l_1, l_2, l_3)^\mathsf{T} \qquad \mathrm{m} = (m_1, m_2, m_3)^\mathsf{T}$

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

Projective:
$$\cos\theta = \frac{\mathrm{l}^\mathsf{T}\,\mathbf{C}_\infty^*\,\mathrm{m}}{\sqrt{(\mathrm{l}^\mathsf{T}\,\mathbf{C}_\infty^*\,\mathrm{l})(\mathrm{m}^\mathsf{T}\,\mathbf{C}_\infty^*\,\mathrm{m})}}$$

$\mathrm{l}^\mathsf{T}\,\mathbf{C}_\infty^*\,\mathrm{m} = 0$ (orthogonal)

Transformation of 3D points, planes and quadrics

For a point transformation (cfr. 2D equivalent)

$$X' = \mathbf{H}\,X \qquad (x' = \mathbf{H}\,x)$$

Transformation for lines

$$\pi' = \mathbf{H}^{-T}\,\pi \qquad (l' = \mathbf{H}^{-T}\,l)$$

Transformation for conics

$$Q' = H^{-T} Q H^{-1} \qquad (\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1})$$

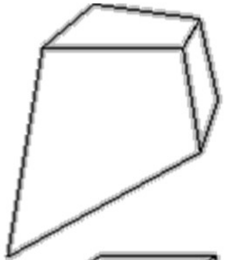
Transformation for dual conics

$$Q'^{*} = H Q^{*} H^{T} \qquad (\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{T})$$

Hierarchy of 3D transformations

Projective
15dof

$$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$$



Intersection and tangency

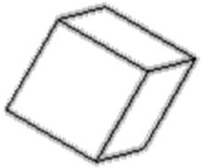
Affine
12dof

$$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$$

Parallellism of planes,
Volume ratios, centroids,
The plane at infinity π_∞

Similarity
7dof

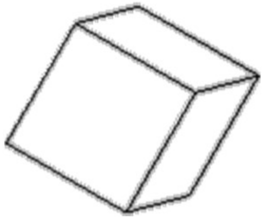
$$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$$



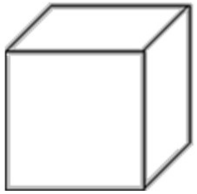
Angles, ratios of length
The absolute conic Ω_∞

Euclidean
6dof

$$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$$



Volume



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The plane at infinity

$$\pi'_\infty = \mathbf{H}_A^{-\mathsf{T}} \pi_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathrm{t}^{\mathsf{T}} \mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

1. canonical position $\pi_\infty = (0,0,0,1)^{\mathsf{T}}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^{\mathsf{T}}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_∞
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_∞

The absolute conic

The absolute conic Ω_∞ is a (point) conic on π_∞.

In a metric frame:

$$\left.\begin{array}{c} X_1^2+X_2^2+X_3^2 \\ X_4 \end{array}\right\} = 0$$

or conic for directions: $(X_1, X_2, X_3)\mathrm{I}(X_1, X_2, X_3)^\mathsf{T}$

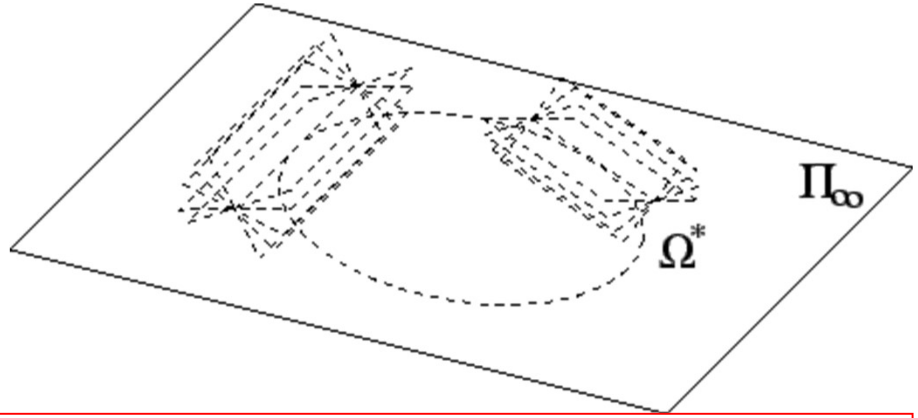
(with no real points)

> The absolute conic Ω_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. Ω_∞ is only fixed as a set
2. Circle intersect Ω_∞ in two circular points
3. Spheres intersect π_∞ in Ω_∞

The absolute dual quadric

$$\Omega^*_\infty = \begin{bmatrix} \mathrm{I} & 0 \\ 0^\mathsf{T} & 0 \end{bmatrix}$$



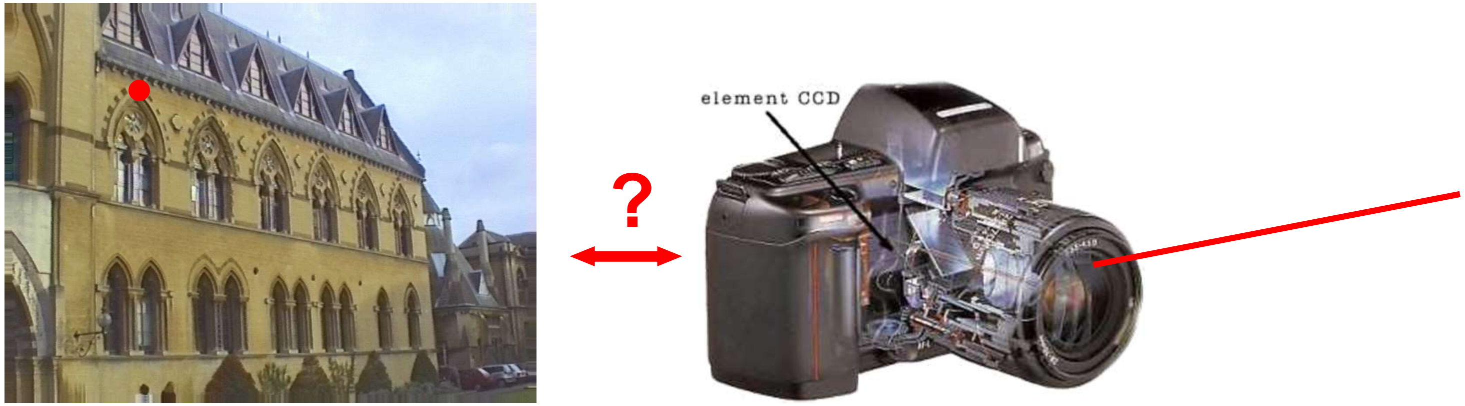
The absolute dual quadric Ω^*_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. 8 dof
2. plane at infinity π_∞ is the nullvector of Ω_∞
3. Angles:

$$\cos\theta = \frac{\pi_1^\mathsf{T}\Omega^*_\infty\pi_2}{\sqrt{\left(\pi_1^\mathsf{T}\Omega^*_\infty\pi_1\right)\left(\pi_2^\mathsf{T}\Omega^*_\infty\pi_2\right)}}$$

Camera model

Relation between pixels and rays in space



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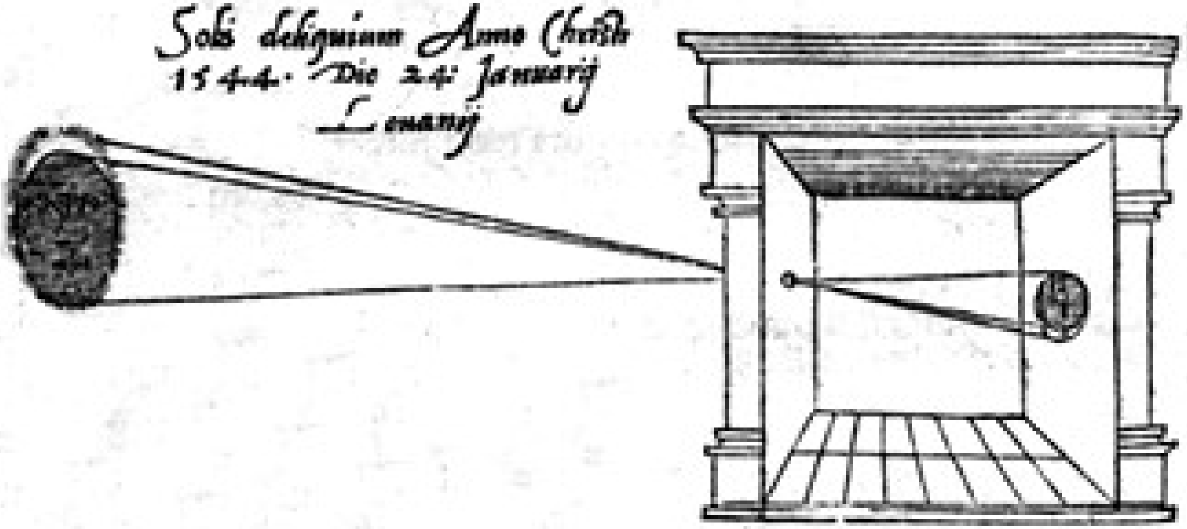
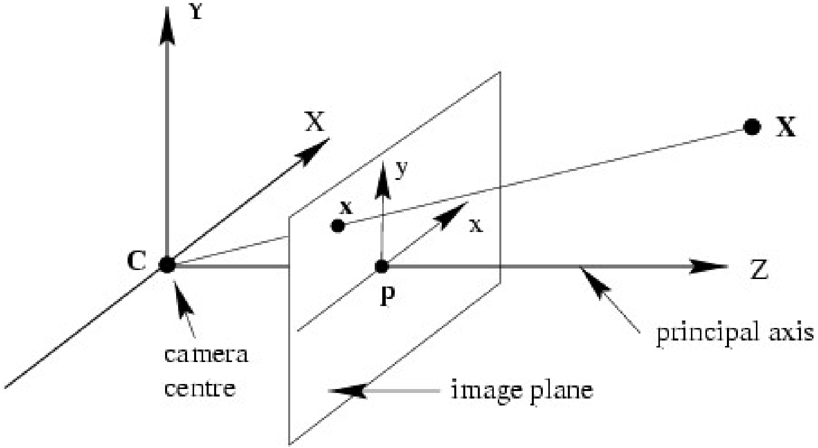
Pinhole camera

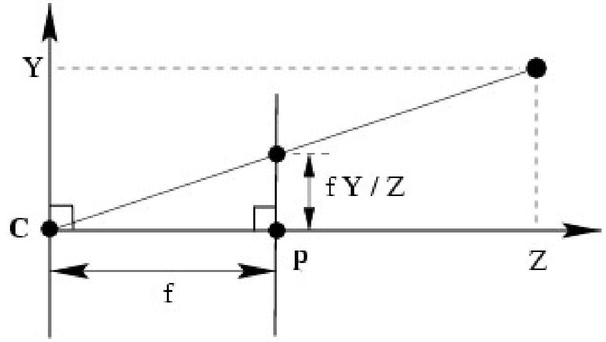


illum in tabula per radios Solis, quàm in cœlo contin-
git: hoc eſt, ſi in cœlo ſuperior pars deliquiũ patiatur, in
radiis apparebit inferior deficere, vt ratio exigit optica.

Sic nos exactè Anno .1544. Louanii eclipſim Solis
obſeruauimus, inuenimusq; deficere paulò plus q̃ dex-

Pinhole camera model



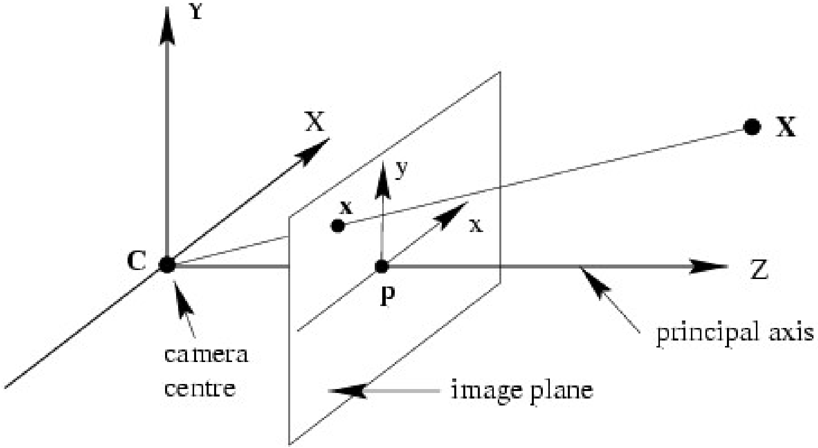


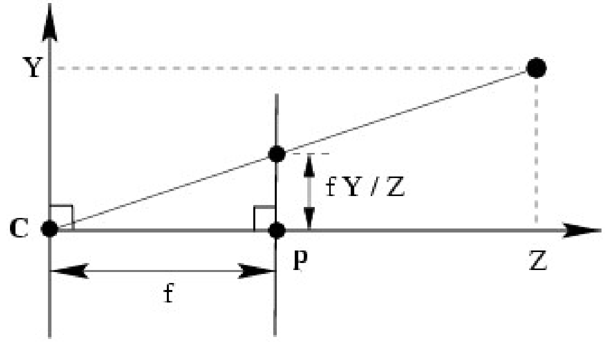
$$(X, Y, Z)^T \mapsto (fX/Z, fY/Z)^T$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

linear projection in homogeneous coordinates!

Pinhole camera model





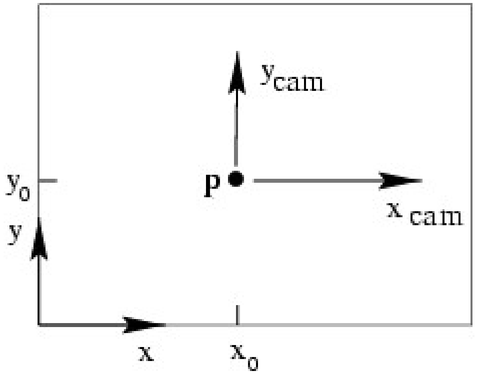
$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{x} = \mathrm{PX}$$

$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{P} = \mathrm{diag}(f, f, 1)\left[\mathrm{I} \mid 0\right]$$

Principal point offset

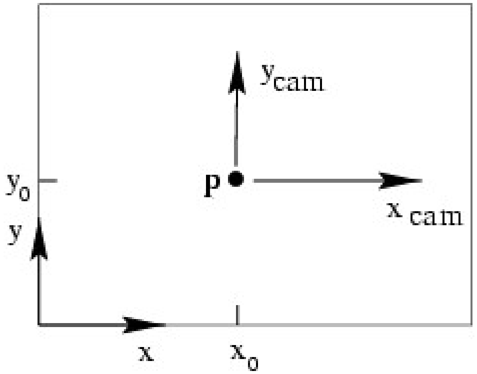


$$(X,Y,Z)^T \mapsto (fX/Z+p_x, fY/Z+p_y)^T$$

$(p_x, p_y)^T$ principal point

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX+Zp_x \\ fY+Zp_x \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Principal point offset

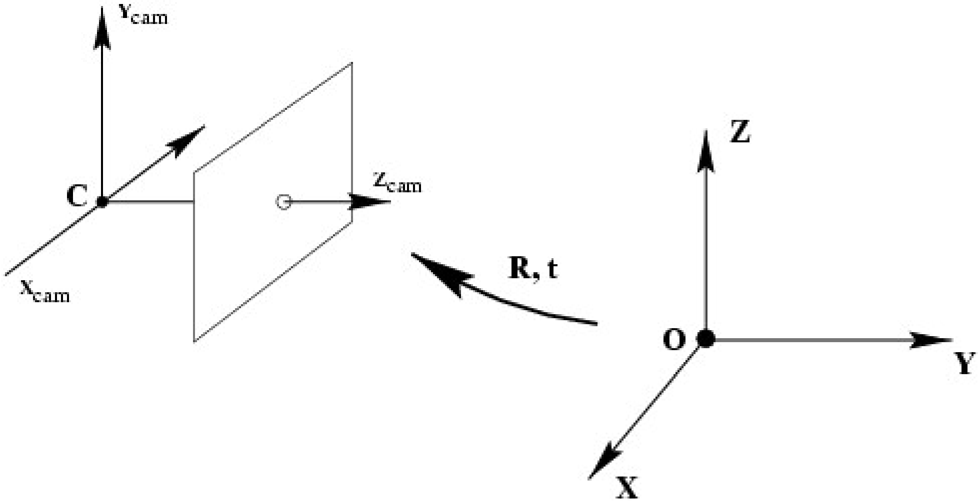


$$\begin{pmatrix} fX + Zp_x \\ fY + Zp_x \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{x} = \mathrm{K}[\mathrm{I} \mid 0]\mathrm{X}_{cam}$$

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$ calibration matrix

Camera rotation and translation



$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

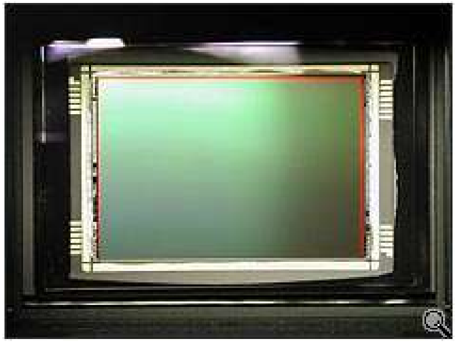
$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} X$$

$$\mathbf{x} = \mathbf{K}[\mathrm{I}\,|\,0]X_{cam}$$

$$\mathbf{x} = \mathbf{K}R[\mathrm{I}\,|\,-\tilde{C}]X$$

$$x = PX \qquad P = K[R\,|\,t] \qquad t = -R\tilde{C}$$

CCD camera



$$K = \begin{bmatrix} m_x & & \\ & m_y & \\ & & 1 \end{bmatrix}\begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} \alpha_x & & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

General projective camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$

$$P = \underbrace{KR}_{\text{non-singular}}\left[I \mid \quad \tilde{C}\right]$$

11 dof (5+3+3)

$$P = K\left[R \mid t\right]$$

intrinsic camera parameters
extrinsic camera parameters

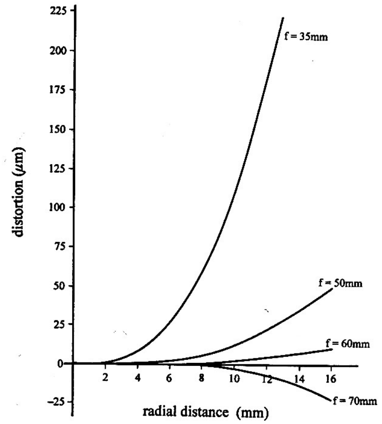
Radial distortion

- Due to spherical lenses (cheap)
- Model:

$$\begin{bmatrix} x\\ y\\ 1\end{bmatrix} \sim \begin{bmatrix} f_x & s & c_x\\ 0 & f_y & c_y\\ 0 & 0 & 1\end{bmatrix} \mathcal{R}\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}\mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t}\\ 0_3^\top & 1\end{bmatrix}\begin{bmatrix}X\\ Y\\ Z\\ 1\end{bmatrix}\right)$$

$$\mathcal{R} \qquad (x,y) = (1+K_1(x^2+y^2)+K_2(x^4+y^4)+\ldots)\begin{bmatrix}x\\ y\end{bmatrix}$$

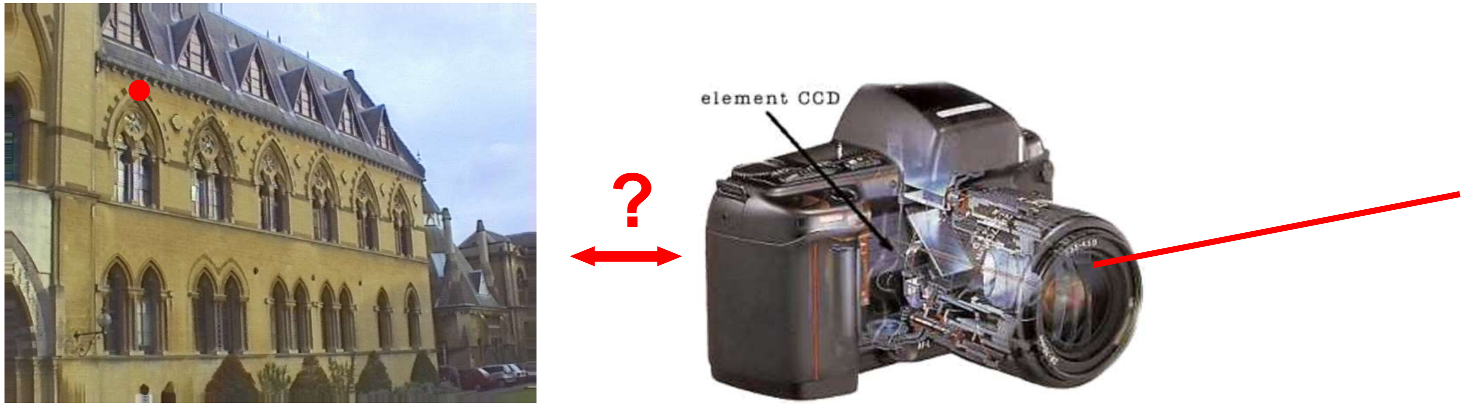
straight lines are not straight anymore



http://foto.hut.fi/opetus/260/luennot/11/atkinson_6-11_radial_distortion_zoom_lenses.jpg

Camera model

Relation between pixels and rays in space



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Projector model

Relation between pixels and rays in space (dual of camera)

(main geometric difference is vertical principal point offset to reduce keystone effect)

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Meydenbauer camera

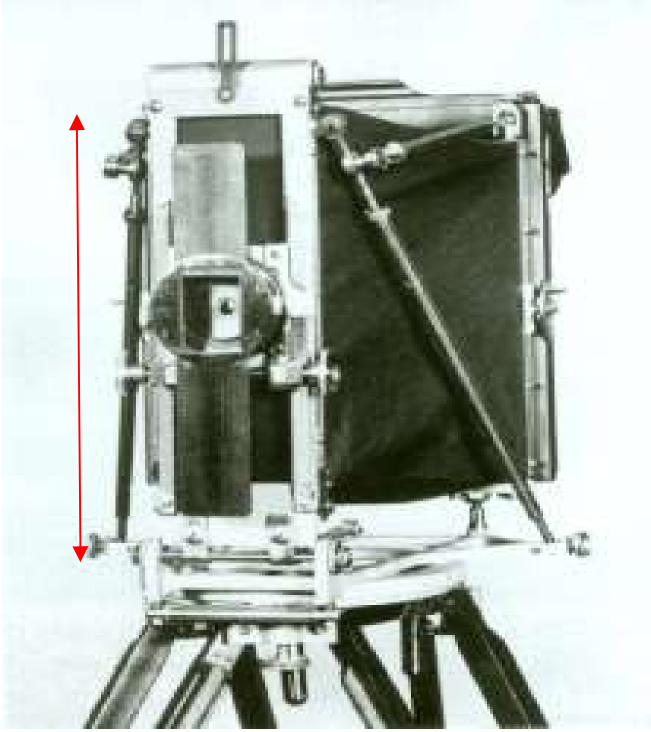

Fig. 5: The principle of »Plane-Table Photogrammetry«
(after an instructional poster of Meydenbauer's institute)

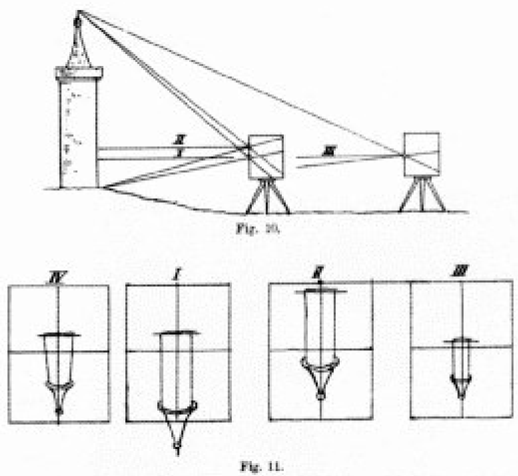

Fig. 6: The effect of a vertical shift of the camera lens;
the position II makes the best use of the image format
(after Meydenbauer's textbook from 1912)

vertical lens shift
to allow direct
ortho-photographs

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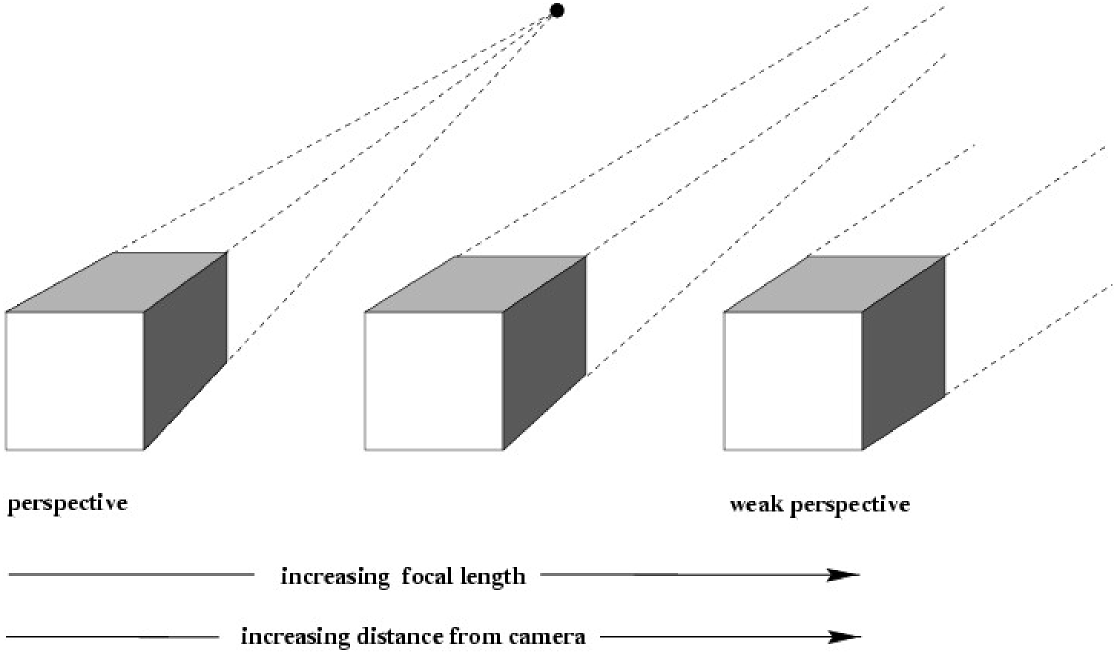
Affine cameras



Action of projective camera on points and lines

projection of point

$$x = PX$$

forward projection of line

$$X(\mu) = P(A + \mu B) = PA + \mu PB = a + \mu b$$

back-projection of line

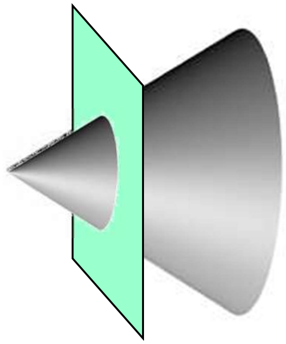
$$\Pi = P^T l$$

$$\Pi^T X = l^T PX \qquad \left(l^T x = 0;\ x = PX\right)$$

Action of projective camera on conics and quadrics

back-projection to cone

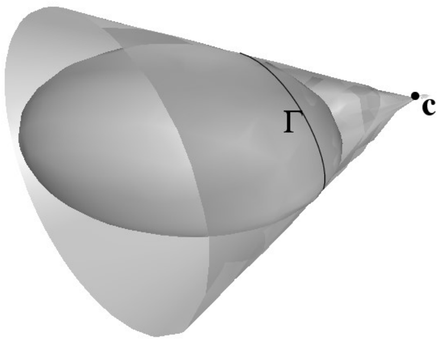
$$Q_{co} = P^T C P \qquad x^T C x = X^T P^T C P X = 0$$



$$(x = PX)$$

projection of quadric

$$C^* = P Q^* P^T \qquad \Pi^T Q^* \Pi = l^T P Q^* P^T l = 0$$



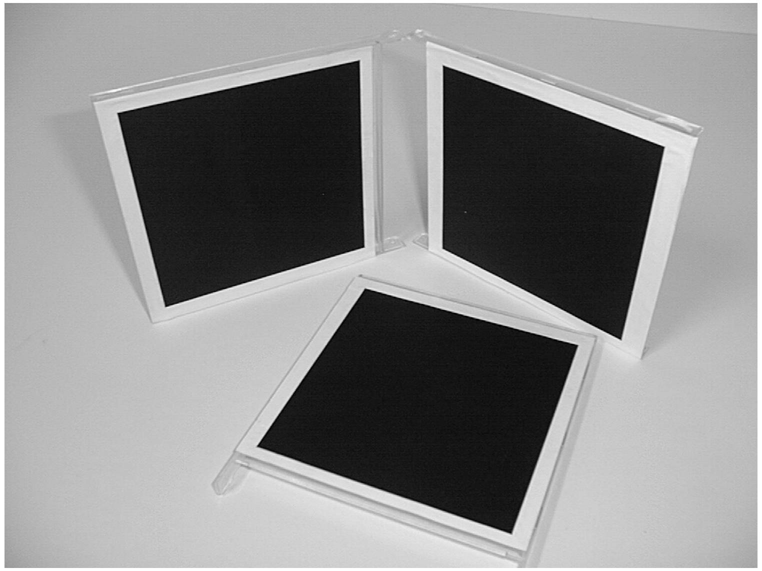
$$(\Pi = P^T l)$$

Image of absolute conic

$$
\begin{aligned}
\omega^* &= \mathbf{P}\Omega^*\mathbf{P}^\top \\
&= \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathtt{t} \end{bmatrix}\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{R}^\top \\ \mathtt{t}^\top \end{bmatrix}\mathbf{K}^\top \\
&= \mathbf{K}\mathbf{K}^\top
\end{aligned}
$$

$$
\omega = \mathbf{K}^{-1}\mathbf{K}^{-\top}
$$

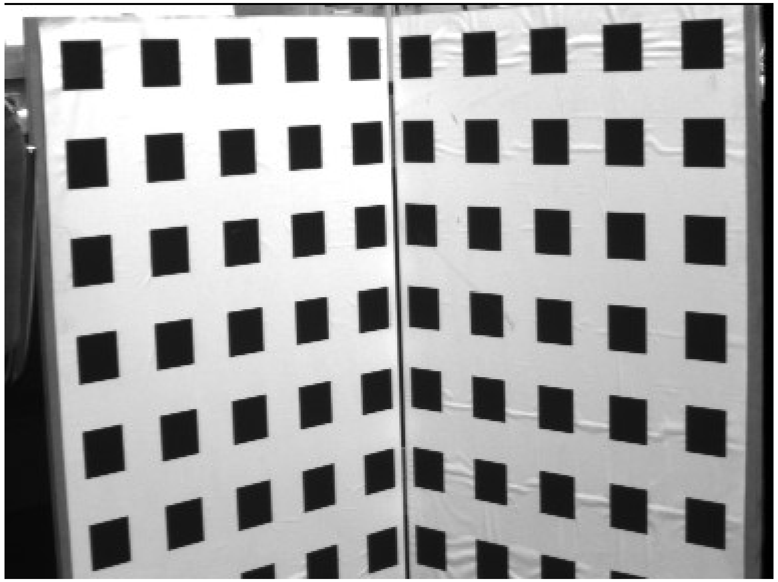
A simple calibration device



(i) compute H for each square (corners @ (0,0),(1,0),(0,1),(1,1))
(ii) compute the imaged circular points $H(1,\pm i,0)^T$
(iii) fit a conic to 6 circular points
(iv) compute K from ω through cholesky factorization

(≈ Zhang's calibration method)

Exercises: Camera calibration



Next class:
Single View Metrology

Antonio Criminisi ETH