Variable aleatoria bidimensional continua

Función de densidad de probabilidad conjunta

$$f_{X,Y}(x,y)$$

$$f_{X,Y}(x,y) \times \Delta x \Delta y \approx P\left(x - \frac{\Delta x}{2} < X \leq x + \frac{\Delta x}{2}, y - \frac{\Delta y}{2} < Y \leq y + \frac{\Delta y}{2}\right)$$

$$f_{X,Y}(x,y) \geq 0$$

Variable aleatoria bidimensional continua

Función de densidad de probabilidad conjunta

$$f_{X,Y}(x,y) \times \Delta x \Delta y \approx P\left(x - \frac{\Delta x}{2} < X \leq x + \frac{\Delta x}{2}, y - \frac{\Delta y}{2} < Y \leq y + \frac{\Delta y}{2}\right)$$

$$P(a < X \leq b, c < Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,dx\,dy$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Variable aleatoria bidimensional continua

Función de probabilidad acumulada conjunta

$$F_{X,Y}(x,y)=P(X\le x, Y\le y)$$

$$f_{X,Y}(x,y)=\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

-

$$0\le F_{X,Y}(x,y)\le 1$$

- Monotónicamente no decreciente en cada variable
- Continua por derecha en cada variable

Variable aleatoria bidimensional continua

Función de densidad de marginal

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$$

Variable aleatoria bidimensional continua

Función de densidad condicional

$$f_{X \vee Y}(x \vee y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{Y \vee X}(y \vee x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Variable aleatoria bidimensional continua

Variables aleatorias independientes

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$f_{X \vee Y}(x \vee y) = f_X(x)$$

$$f_{Y \vee X}(y \vee x) = f_Y(y)$$

Variable aleatoria bidimensional continua

Valor esperado

$$E\left[g\left(X,Y\right)\right]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}g\left(x,y\right)f_{X,Y}\left(x,y\right)dxdy$$

Covarianza

$$\mathrm{Cov}\left(X,Y\right)=E\left[XY\right]-E\left[X\right]E\left[Y\right]$$

Variables independientes

Variable aleatoria bidimensional continua

Correlación

$$\varrho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$\varrho_{X,Y} \in [-1,+1]$$

Relación lineal con probabilidad 1

Variable aleatoria bidimensional continua

Experimento

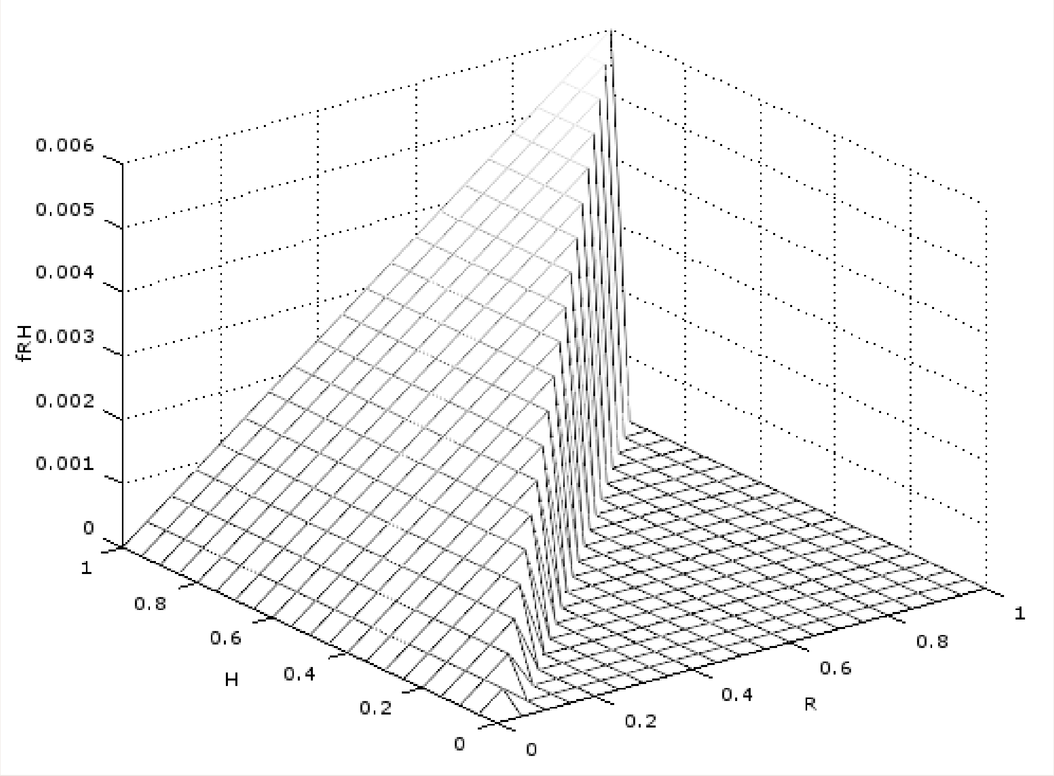
En cierto proceso industrial se fabrican piezas cilíndricas.

Se definen dos variables aleatorias continuas:

: el radio de la base (en cm) .
: la altura (en cm).

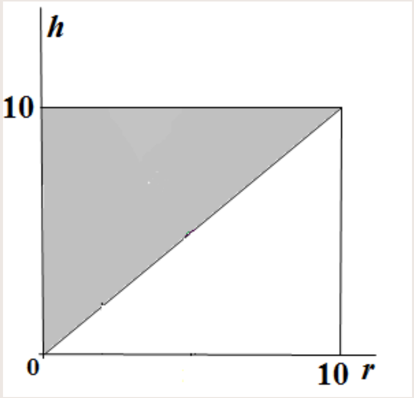
Variable aleatoria bidimensional continua

Densidad de probabilidad conjunta



Variable aleatoria bidimensional continua

¿Cuál es el valor de la constante ?



$$1=\int_{-\infty}^{+\infty}\int f_{R,H}(r,h)\,dh\,dr = \text{¿}\int_{0}^{10}\int_{r}^{10} 10kr\,dh\,dr\,\text{¿}$$

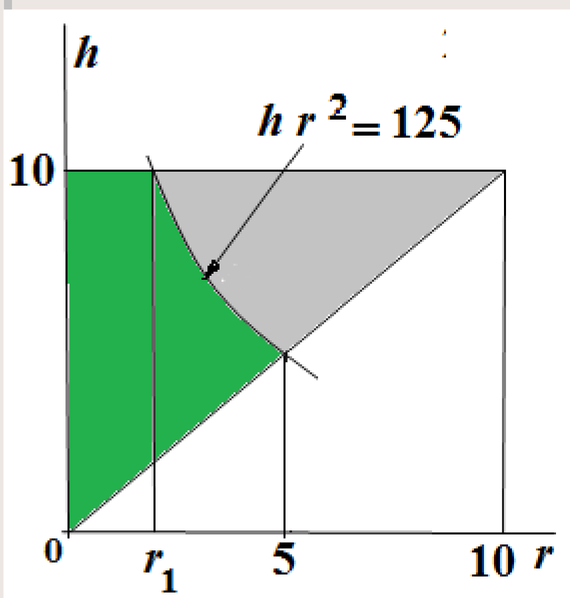
$$1=\int_{0}^{10} 10k(10-r)\,r\,dr = 10k\left(5r^2-\frac{r^3}{3}\right)\Bigg|_{0}^{10}$$

$$1=\frac{5000}{3}k$$

$$\Rightarrow k=\frac{3}{5000}=0.0006$$

Variable aleatoria bidimensional continua

¿Cuál es la probabilidad de que el volumen () sea menor que 125 cm^3?



$$r_1 = \sqrt{12.5}$$

Variable aleatoria bidimensional continua

$$\int_0^{r_1}\int_r^{10} 10k\,r\,dh\,dr=\int_0^{r_1} 10k(10-r)\,r\,dr=10k\left(5r^2-\frac{r^3}{3}\right)\Bigg|_0^{r_1}$$

$$¿10k\left(5r_1^2-\frac{r_1^3}{3}\right)$$

$$\int_{r_1}^{5}\int_r^{125/r^2} 10k\,r\,dh\,dr=\int_{r_1}^{5} 10k\left(\frac{125}{r}-r^2\right)dr=10k\left(125\ln(r)-\frac{r^3}{3}\right)\Bigg|_{r_1}^{5}$$

$$¿10k\left(125\ln\left(\frac{5}{r_1}\right)-\frac{125}{3}+\frac{r_1^3}{3}\right)$$

Variable aleatoria bidimensional continua

¿Cuál es la probabilidad de que el volumen () sea menor que 125 cm^3?

$$\mathrm{P}\left(V<125\pi\right)=10k\left(5r_1^2+125\ln\left(\frac{5}{r_1}\right)-\frac{125}{3}\right)$$

$$\approx 0.38493$$

Variable aleatoria bidimensional continua

Densidades de probabilidad marginales

$$f_R(r)=\int_{-\infty}^{+\infty} f_{R,H}(r,h)\,dh=\text{¿}\int_{r}^{10} 10kr\,dh=10kr(10-r)\text{¿}$$

$$f_R(r)=\begin{cases}\frac{3}{500}r(10-r) & r\in(0,10)\\ 0 & r\notin(0,10)\end{cases}$$

$$\mu_R=\mathrm{E}[R]=\int_0^{10} r\,\frac{3}{500}r(10-r)\,dr=\frac{3}{500}\left(\frac{10r^3}{3}-\frac{r^4}{4}\right)\Bigg|_0^{10}=5$$

$$\mathrm{E}\left[R^2\right]=\int_0^{10} r^2\,\frac{3}{500}r(10-r)\,dr=\frac{3}{500}\left(\frac{10r^4}{4}-\frac{r^5}{5}\right)\Bigg|_0^{10}=30$$

$$\sigma_R^2=30-5^2=5$$

Variable aleatoria bidimensional continua

Densidades de probabilidad marginales

$$f_H(h) = \int_{-\infty}^{+\infty} f_{R,H}(r,h)\, dr = ¿ \int_0^h 10 k r\, dr = 5 k h^2 ¿$$

$$f_H(h) = \begin{cases} \frac{3}{1000} h^2 & h \in (0,10) \\ 0 & h \notin (0,10) \end{cases}$$

$$\mu_H = \mathrm{E}[H] = \int_0^{10} h \frac{3}{1000} h^2\, dh = \frac{3}{4000} h^4 \Big|_0^{10} = 7.5$$

$$\mathrm{E}[H^2] = \int_0^{10} h^2 \frac{3}{1000} h^2\, dh = \frac{3}{5000} h^5 \Big|_0^{10} = 60$$

$$\sigma_H^2 = 60 - 7.5^2 = 3.75$$

Variable aleatoria bidimensional continua

Densidades de probabilidad marginales

$$f_R(r)=\begin{cases}\dfrac{3}{500}r(10-r) & r\in(0,10)\\ 0 & r\notin(0,10)\end{cases}$$

$$f_H(h)=\begin{cases}\dfrac{3}{1000}h^2 & h\in(0,10)\\ 0 & h\notin(0,10)\end{cases}$$

$$\mu_R=5$$

$$\mu_H=7.5$$

$$\sigma_R^2=5$$

$$\sigma_H^2=3.75$$

Variable aleatoria bidimensional continua

Densidades de probabilidad condicionales

$$f_R(r)=\begin{cases} \dfrac{3}{500} r(10-r) & r \in (0,10) \\ 0 & r \notin (0,10) \end{cases}$$

$$f_{H \vee R}(h|r)=\begin{cases} \dfrac{1}{10-r} & 0<r<10\,, r<h<10 \\ 0 & \text{en todo otro caso} \end{cases}$$

Variable aleatoria bidimensional continua

Densidades de probabilidad condicionales

$$f_H(h)=\begin{cases} \dfrac{3}{1000}h^2 & h \in (0,10) \\ 0 & h \notin (0,10) \end{cases}$$

$$f_{R\vee H}(r|h)=\begin{cases} \dfrac{2r}{h^2} & 0<r<10\,, r<h<10 \\ 0 & \text{en todo otro caso} \end{cases}$$

Variable aleatoria bidimensional continua

Covarianza

$$\mathrm{E}[RH] = \int_0^{10}\int_r^{10} rh\,10\,k\,r\,dh\,dr = \int_0^{10} 5\,k\,r^2\left(100 - r^2\right) dr = \text{¿}\,\square\,\text{¿}$$

$$\text{¿}\,5\,k\left(\frac{100\,r^3}{3} - \frac{r^5}{5}\right)\Bigg|_0^{10} = 40$$

$$\mathrm{Cov}(R, H) = 40 - 5\cdot 7.5 = 2.5$$

$$\varrho_{R,H} = \frac{2.5}{\sqrt{5\cdot 3.75}} \approx 0.57735$$

Variable aleatoria bidimensional continua

Experimento

Se sortea un número al azar en y a continuación se sortea otro número al azar pero en donde fue el valor observado de .

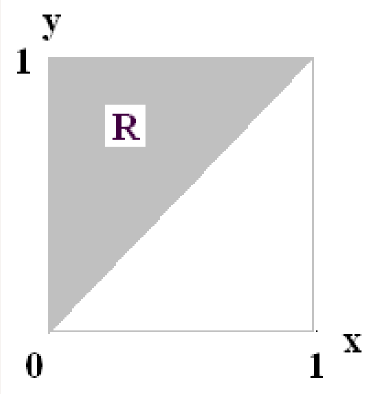
Variable aleatoria bidimensional continua

$$f_X(x)=\begin{cases}1 & x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

$$f_{Y\vee X}(y\vee x)=\begin{cases}\dfrac{1}{1-x} & y\in(x,1),x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

La función de densidad conjunta viene dada por :

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{1}{1-x} & y\in(x,1),x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$



Variable aleatoria bidimensional continua

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{1}{1-x} & y\in(x,1),x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

$$f_Y(y)=\int_0^1 f_{XY}(x,y)\,dy=\int_0^y \frac{dx}{1-x}=\ln\left(\frac{1}{1-y}\right)y\in(0,1)$$

$$f_Y(y)=\begin{cases}\ln\left(\dfrac{1}{1-y}\right) & y\in(0,1)\\ 0 & y\notin(0,1)\end{cases}$$

Variable aleatoria bidimensional continua

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{1}{1-x} & y\in(x,1),x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

$$f_X(x)=\begin{cases}1 & x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

$$f_Y(y)=\begin{cases}\ln\left(\dfrac{1}{1-y}\right) & y\in(0,1)\\ 0 & y\notin(0,1)\end{cases}$$

$$f_{X,Y}(0.5,0.1)=0\neq 1\cdot(-\ln(0.9))=f_X(0.5)\,f_Y(0.1)$$

No son independientes

Variable aleatoria bidimensional continua

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{1}{1-x} & y\in(x,1),x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

$$f_X(x)=\begin{cases}1 & x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

$$f_Y(y)=\begin{cases}\ln\left(\dfrac{1}{1-y}\right) & y\in(0,1)\\ 0 & y\notin(0,1)\end{cases}$$

Valores esperados

$$E[X]=0.5$$

$$E[Y]=\int_0^1\int_x^1\frac{y}{1-x}\,dy\,dx=\int_0^1\frac{dx}{2(1-x)}\left(1-x^2\right)=0.75$$

Variable aleatoria bidimensional continua

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{1}{1-x} & y\in(x,1), x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

$$f_X(x)=\begin{cases}1 & x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

$$f_Y(y)=\begin{cases}\ln\left(\dfrac{1}{1-y}\right) & y\in(0,1)\\ 0 & y\notin(0,1)\end{cases}$$

Covarianza

$$E[XY]=\int_0^1\int_x^1 \frac{xy}{1-x}\,dy\,dx=\int_0^1 \frac{xdx}{2}(1+x)=\frac{5}{12}$$

$$\mathbf{Cov}(¿X,Y)=E[XY]-E[X]E[Y]=\frac{5}{12}-\frac{1}{2}\frac{3}{4}=\frac{1}{24}¿$$

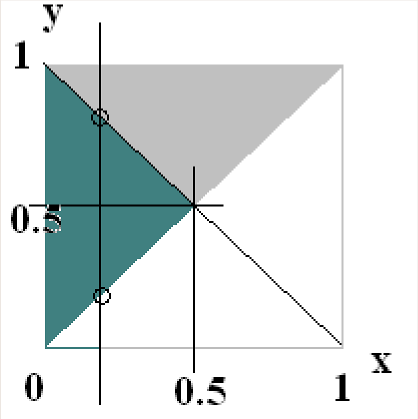
Variable aleatoria bidimensional continua

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{1}{1-x} & y\in(x,1),x\in(0,1)\\ 0 & x\notin(0,1)\end{cases}$$

Algunas probabilidades

$$P(X+Y\le 1)=\int_0^{0.5}\int_x^{1-x}\frac{1}{1-x}\,dy\,dx=\int_0^{0.5}\left(1-\frac{x}{1-x}\right)dx$$

$$P(X+Y\le 1)=1-\ln(\text{¿}2)\approx 0.3069\,\text{¿}$$



Variable aleatoria bidimensional continua

$$f_{X,Y}(x,y)=\begin{cases} \dfrac{1}{1-x} & y\in(x,1), x\in(0,1) \\ 0 & x\notin(0,1) \end{cases}$$

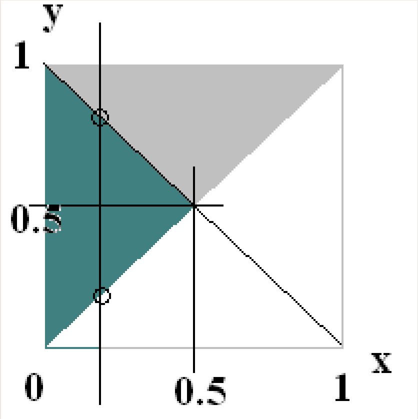
Algunas probabilidades

$$P(Y<0.5\vee X<0.25)=\frac{P(\{Y<0.5\}\cap\{X<0.25\})}{P(\{X<0.25\})}=\text{¿}$$

$$\text{¿}\frac{1}{0.25}\int_0^{0.25}\int_x^{0.5}\frac{1}{1-x}\,dy\,dx$$

$$\text{¿}\frac{1}{0.25}\int_0^{0.25}\left(1-\frac{0.5}{1-x}\right)dx$$

$$\text{¿}1+2\ln(\text{¿}0.75)\approx 0.4246\text{¿}$$

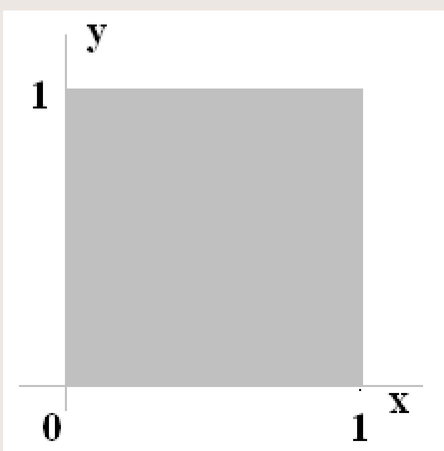


Variable aleatoria bidimensional continua

Experimento

Se sortean dos números e al azar en y en forma independiente

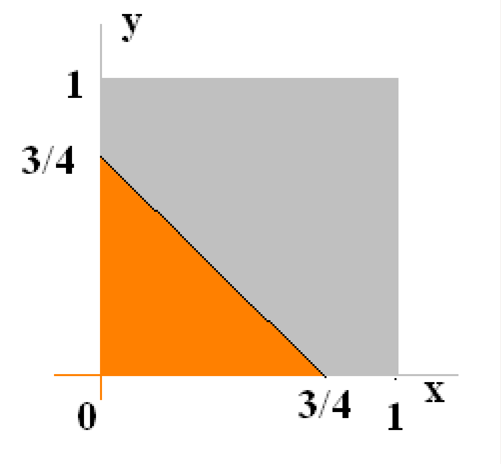
$$f_{XY}(x,y)=\begin{cases} \text{¿}1 & (x,y)\in(0,1)\times(0,1) \\ \text{¿}0 & (x,y)\notin(0,1)\times(0,1) \end{cases}$$



Como la función de densidad conjunta es igual a 1 en este cuadrado entonces el cálculo de probabilidades de sucesos en este dominio se reduce al cálculo de áreas de recintos.

Variable aleatoria bidimensional continua

Algunas probabilidades



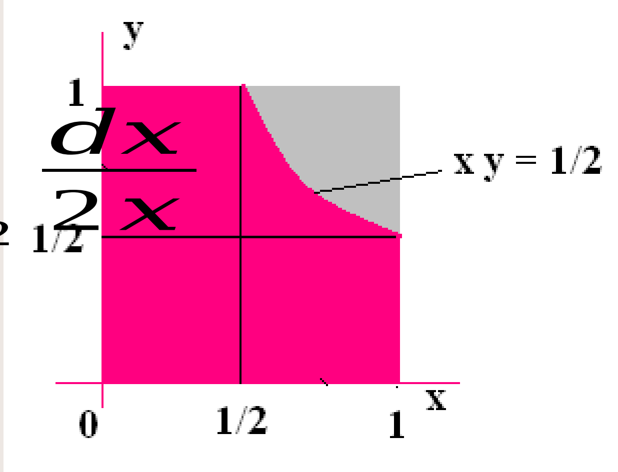
$$P\left(X+Y<3/4\right)=\frac{1}{2}\frac{3}{4}\frac{3}{4}=\frac{9}{32}$$

Variable aleatoria bidimensional continua

Algunas probabilidades

$$P(XY<1/2)=\frac{1}{2}+\int_{1/2}^{1}\frac{dx}{2x}$$

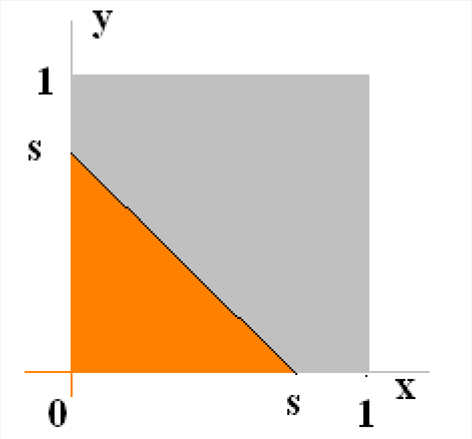
$$\text{¿}\,\frac{1}{2}+\frac{1}{2}\ln(\text{¿}\,2)\approx 0.8466\,\text{¿}$$



Variable aleatoria bidimensional continua

Algunas probabilidades

$$S = X + Y$$



$$P(S \leq s) = ¿$$

Variable aleatoria bidimensional continua

Algunas probabilidades

$$S = X + Y$$

$$F_S(s) = ¿$$

$$f_S(s) = \begin{cases} s & 0 < s < 1 \\ 2 - s & 1 < s < 2 \\ 0 & s \notin (0,1) \end{cases}$$