UNIVERSITY OF CAPE TOWN

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# Shooting Method Project 3

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# 1 Introduction

In Project 3 we are tasked with finding the eigenvalues and eigenfunctions of the given boundary equation:

$$\frac{d^2y}{dx^2} = (x - \lambda)y \tag{1}$$

with the given boundary conditions, y(0) = y(1) = 0. We are to solve this numerically using the shooting method.

For the second part of this project we are tasked with finding another eigenvalue and eigenfunction. This ODE has multiple eigenvalues which produce eigenfunctions that satisfy the boundary conditions. Using a similar approach used for the first part of this project find another eigenvalue of (1)

# 2 Background

## 2.1 Initial Value Problems and Boundary Value Problems

So far the ordinary differential equations (ODE) we have solved numerically have been initial value problems (IVP). That is equations of the form:

$$\frac{d^ny}{dx^n} = f(x, y, y', y'', ...., y(n-1))$$

where we need the initial values y(x0), y′(x0), y′′(x0) ,..., y(n−1)(x0) to be known to solve the equation.

Now, let's consider a different kind of ODE. Instead of defining y and it's derivatives for a particular x, we define values at multiple boundary points(boundary values).

Consider a second order ODE of the form:

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

For a second order IVP, we need two initial conditions to solve them:y(x0)and y′(x0). Similarly, for a BVP we need 2 boundary values: either y(x0) or y′(x0), and either y(x1) or y′(x1).

### 2.2 Shooting Method

In the shooting method, we start off by approaching our boundary value problem (1) as though it is an initial value problem. The first initial value is just the boundary condition y(0) = 0. The second is any non-zero value for y′. A value of zero will result in a trivial solution (y(x) = 0 everywhere), and the value chosen for y′ will only affect the scale of the solution, not it's functional form.

Now, we need to find a value for $\lambda$ that produces a solution that satisfies the boundary condition y(1) = 0. This can be achieved by using root finding techniques, where we search for the $\lambda$ such that f($\lambda$) = y$\lambda$ (1) = 0 (where y$\lambda$(x) is the solution to (1) for the given value of ). Though the proof of this is beyond the scope of the course, you can take it in faith that this f($\lambda$) is continuous within our region of interest.[1]

## 3 Method

We will start off by approaching (1) as if an IVP with y(0) = 0 and y′(0) = 1.Use $\lambda = 8$ and $\lambda = 12$, and plot these on 2 separate axis.This is achieved by using the RK4 approach of the higher order differential equation.

Now plot the function f($\lambda$) on the interval [8,12]. There should be a clear distinct difference between these two graphs. We then need to find  and its corresponding solution for y in this interval this is done by using root finding techniques in this particular case we made use of the bisection method.This solves the boundary value problem (1).[2]

By using the same method above we then need to find another Eigenvalue and Eigenfunction which satisfies the boundary conditions.

# 4 Results and Discussion

For the first part we are plotting an solving (1) as if in IVP problem, with the boundary and initial conditions given above in the method section of the part.

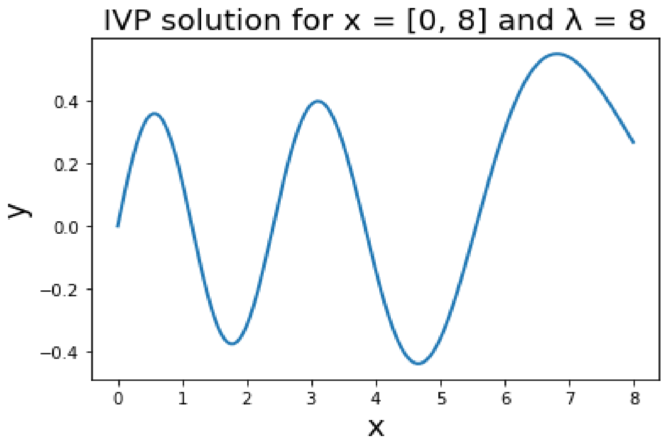


Figure 1: IVP Plot for x = [0,9] and $\lambda = 8$

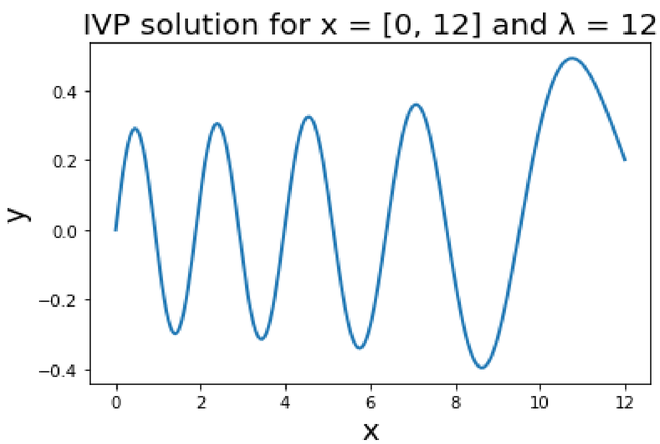


Figure 2: IVP Plot for x = [0,9] and $\lambda = 12$

Figures 1 and 2 are the independent plots of each boundary point [8,12]. We plotted them separately to do a comparison and have a look at how the boundary equation behaves and the shape its takes across a different boundary value.

Figures 1 and 2 behaves similar to that of the sin wave. Starting off with a positive slope and moving sinusoidally until it reaches its boundary condition which brings the boundary equation to a halt of course. Notice how as we progress through the function, the size of the waves increase, similarly the crest and trough are increasing as well. This tells me that x and y are directly proportional to each other.

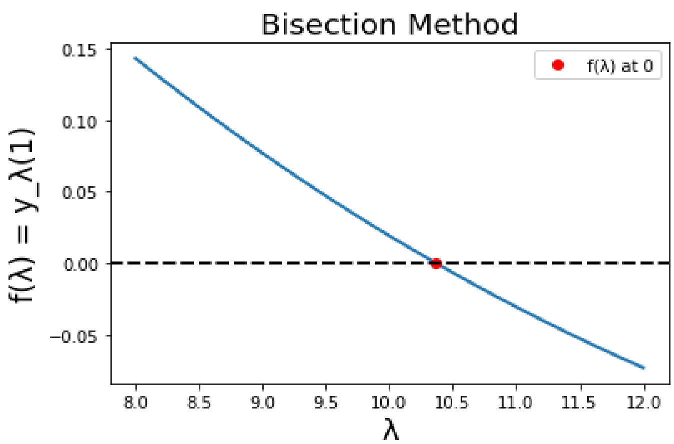


Figure 3: Bisection method used to plot f($\lambda$)

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We will no plot the boundary equation as a BVP using the shooting method outlined in the method section. Figure 3 is a plot of the fucntion vs lambda in the boundary [8,12]. The bisection method is a root finding technique, used to find the roots in the boundary equation.

Since Figure 3 is approached as a BVP as suppose to a IVP, the first thing we notice is a clear distinct change in the plot. Figure 3 is not a sinusoidal wave its looks like a straight line but if you look closely its clear that its not linear. The reason for this I would assume is because we are limiting the range of the boundary equation between [8,12] so we don't have a large range of data values to work with and this makes it difficult for function to take its natural shape.

The straight horizontal, f($\lambda$) at 0, intersects the graph at the following point.

$$f(\lambda) = 0, \lambda = 10.37$$

This tells us that the eigenfunction = 0 at an eigenvalue value of 10.37.Note that there is no error recorded here because the method that we are doing has discrepancies of its own, so the error that was given during the bisection method was not the real error since the shooting method has its own errors as well.

For the second part of this project we investigate finding another Eigenvalue and Eigenfunction. The ODE which we has been given has multiple eigenvalues which produce eigenfunctions that satisfies the boundary conditions. We used the exact same approach as above to solve this.

By changing the interval boundary points of the lambda values in conjunction with the range of values selected for the bisection method we are able to find the next eigenvalue and eigenfunction.

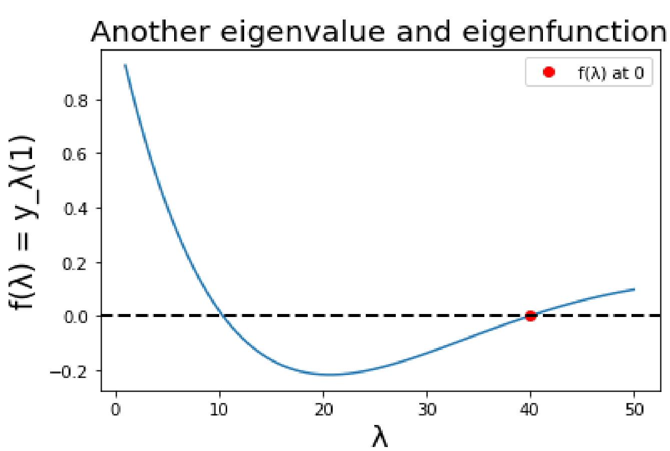


Figure 4: Another Eigenvalue and Eigenfunction

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$$f(\lambda) = 0, \lambda = 39.98$$

Notice how figure 4 behaves. It has a negative slope as it approaches the negative axis, this is also where its intersects the horizontal line first. At roughly -0.2 it reaches its minimum and starts gradually increasing until its intersects the horizontal line for a second time, this intersection points is also where the eigenvalue and eigenfunction is situated.

# References


[1] Unknown.Shooting method for two boundary problems. Available:https://www.math.usm.edu/lambers/mat461/spr10/lecture25.pdf [2020, July 5]

[2] Unknown.Bisection Method, root finding technique. Available:https://www.math.ubc.ca/pwalls/math-python/roots-optimization/bisection [2020, July 5]