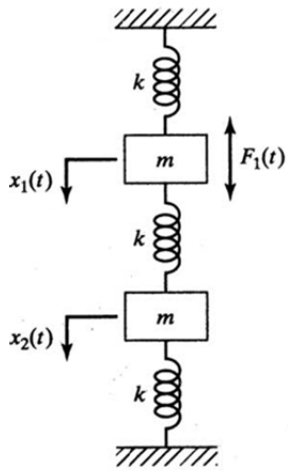
Tutorials: Advanced Dynamics and Applications 5

by Dr David García Cava

EXERCISES FOR LECTURES IN BLOCK I

Exercise 1.1 (Revision of Laplace Transform but applied. See Additional material)

Find the forced vibration response of the system below when $F_1(t)$ is the step function of magnitude $5N$ using the Laplace transform method. DATA: $x_1(0) = \dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$, $m = 1\ kg$, $k = 100\ N/m$.



Exercise 1.2

The 2DOF system below has the following properties $m_1 = m_2 = 1kg$, $k_1 = k_3 = 100N/m$ and $k_2 = 150N/m$ and the damping coefficients are $c_1 = c_2 = c_3 = 0$. It has been found to have

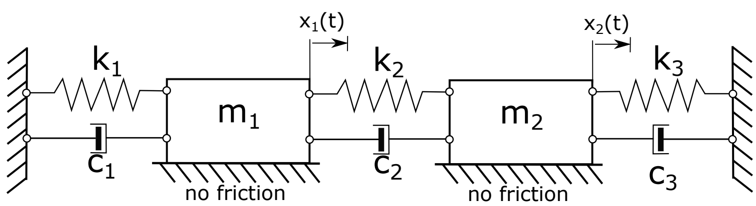
- poles of the system:

$$s_1 = \pm j\sqrt{100} = \pm j10$$
$$s_2 = \pm j\sqrt{400} = \pm j20$$

- and eigenvectors (mode shapes)

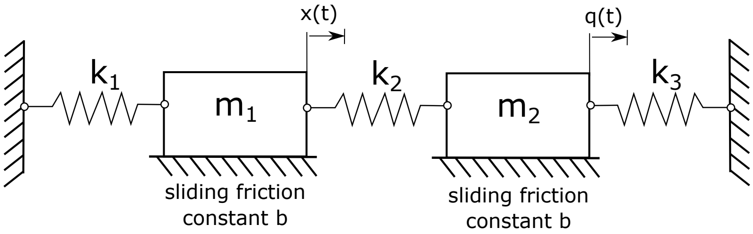
$$\boldsymbol{\phi}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$\boldsymbol{\phi}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Calculate the frequency response function for $H_{11}(j\omega)$ and $H_{12}(j\omega)$ for the 2DOF system below.



Exercise 1.3

Consider the case in figure below. Determine the state variable matrix differential equation for $m_1 = m_2 = m$ and $k_1 = k_3 = k$.

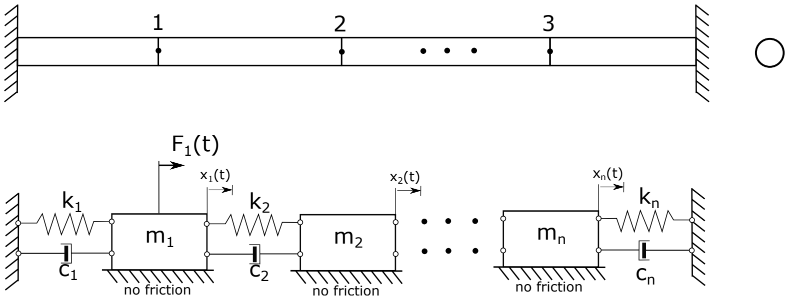


Exercise 1.4 (Code: Ex_CT1.4)

This exercise is solved in a Jupiter notebook Ex_CT1.4.

In this example we simulate and study the effect of the structural properties of a MDOF mass-spring structure in its dynamical parameters. The structure consists of a series of masses connected by springs, with a fixed-fixed boundary condition.

The structure has all the stiffness, mass and damping elements are the same. Thus, $k_1 = k_2 = \cdots = k_n = k$, $m_1 = m_2 = \cdots = m_n = m$ and $c_1 = c_2 = \cdots = c_n = c$. The code below calculates the natural frequencies and mode shapes for an aluminum beam with Young's modulus $E = 69GPa$, circular cross-section of diameter 40 mm, density $\rho = 2.7 \times 10^3\ kg/m^3$, and a total length of 4 m.



The Learning outcomes in this example:

- The modal parameters are obtained (i.e. natural frequencies, damping ratios and mode shapes).
- The FRF for each dof are obtained when excitation is entered in the first degree of freedom as unit.

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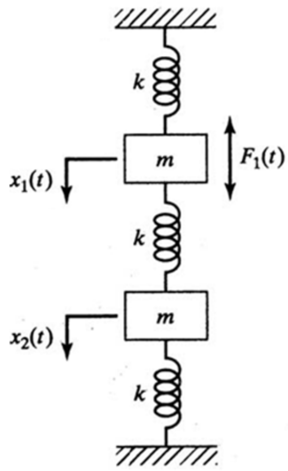
Exercise 1.1 (Revision of Laplace Transform but applied. See Additional material)

Find the forced vibration response of the system below when $F_1(t)$ is the step function of magnitude $5N$ using the Laplace transform method. DATA: $x_1(0) = \dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$, $m = 1\ kg$, $k = 100\ N/m$.



Solution:

The **equations of motion** of the system are:

$$
\begin{aligned}
m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_1(t) \\
m_2\ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = 0
\end{aligned}
\tag{1}
$$

Take the **Laplace Transform**

$$
\begin{aligned}
m_1\left[s^2X_1(s) - sx_1(0) - \dot{x}_1(0)\right] + (k_1 + k_2)X_1(s) - k_2X_2(s) = F_1(s) \\
m_2\left[s^2X_2(s) - sx_2(0) - \dot{x}_2(0)\right] - k_2X_1(s) + (k_2 + k_3)X_2(s) = 0
\end{aligned}
\tag{2}
$$

Substituting <u>initial conditions</u> and arranging:

$$
\begin{aligned}
\left(m_1s^2 + k_1 + k_2\right) X_1(s) - k_2X_2(s) = F_1(s) \\
\left(m_2s^2 + k_2 + k_3\right) X_2(s) - k_2X_1(s) = 0
\end{aligned}
\tag{3}
$$

As $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$

$$
\begin{aligned}
\left(ms^2 + 2k\right) X_1(s) - kX_2(s) = F_1(s) \\
\left(ms^2 + 2k\right) X_2(s) - kX_1(s) = 0
\end{aligned}
\tag{4}
$$

Substituting $X_1(s)$ from Eq. (4a) into Eq. (4b) and $X_2(s)$ from Eq. (4b) into Eq. (4a), it can be obtained the two following equations.

$$
\begin{aligned}
X_1(s)\left[(ms^2 + 2k)^2 - k^2\right] = F_1(s)(ms^2 + 2k) \\
X_2(s)\left[(ms^2 + 2k)^2 - k^2\right] = F_1(s)k
\end{aligned}
\tag{5}
$$

Also it is know that the force applied is a step function of $5N$.

$$F_1(t) = 5u(t) \tag{6}$$

The Laplace transform of the force is given by

$$F_1(s) = \frac{5}{s} \tag{7}$$

Then substituting the parameter values into equation (5)

$$\begin{aligned} X_1(s) &= \frac{F_1(s)(ms^2+2k)}{(ms^2+2k)^2-k^2} = \frac{5(s^2+200)}{s\left[(s^2+200)^2-10000\right]} = \frac{5(s^2+200)}{s\left[s^4+400s^2+30000\right]} \\ X_2(s) &= \frac{F_1(s)k}{(ms^2+2k)^2-k^2} = \frac{500}{s\left[(s^2+200)^2-10000\right]} = \frac{500}{s\left[s^4+400s^2+30000\right]} \end{aligned} \tag{8}$$

Factorisation of the denominator of the expressions in Eq.(8)
(Hint: Let's be $y = s^2$ then $y^2 + 400y + 30000$, calculate the roots and substitute)

$$\begin{aligned} X_1(s) &= \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} \\ X_2(s) &= \frac{500}{s(s^2+300)(s^2+100)} \end{aligned} \tag{9}$$

Now, take the **inverse of the Laplace Transform** and this can be done by partial fraction expansion and by the method second order polynomial

$$\begin{aligned} X_1(s) &= \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} = \frac{A_1}{s} + \frac{A_2s+A_3}{(s^2+300)} + \frac{A_4s+A_5}{(s^2+100)} \\ X_2(s) &= \frac{500}{s(s^2+300)(s^2+100)} = \frac{B_1}{s} + \frac{B_2s+B_3}{(s^2+300)} + \frac{B_4s+B_5}{(s^2+100)} \end{aligned} \tag{10}$$

After finding the values of the parameters $A_1 ... A_5$ and $B_1 ... B_5$, then equations (10) have the form:

$$\begin{aligned} X_1(s) &= \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} = \frac{1}{30} - \frac{s}{120(s^2+300)} - \frac{s}{40(s^2+100)} \\ X_2(s) &= \frac{500}{s(s^2+300)(s^2+100)} = \frac{1}{60} + \frac{s}{120(s^2+300)} - \frac{s}{40(s^2+100)} \end{aligned} \tag{11}$$

Therefore, the final inverse Laplace transform can be obtained by using the table of Laplace Transform $f(t) = cos\ \omega t \implies F(s) = \frac{s}{s^2+\omega^2}$

$$\begin{aligned} x_1(t) &= \frac{1}{30} - \frac{1}{40}cos10t - \frac{1}{120}cos10\sqrt{3}t \\ x_2(t) &= \frac{1}{60} - \frac{1}{40}cos10t + \frac{1}{120}cos10\sqrt{3}t \end{aligned} \tag{12}$$

Therefore, the **two natural frequencies** are 10, $10\sqrt{3}$ (rad/s).

NOTE: **How to prepare the equation (10) for the Inverse of Laplace transform by partial fraction expansion?**

$$X_1(s) = \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} = \frac{A_1}{s} + \frac{A_2s+A_3}{(s^2+300)} + \frac{A_4s+A_5}{(s^2+100)} \tag{13}$$

The quantities $A_1 ... A_5$ can be found from cross-multiplication.

$$\begin{aligned} X_1(s) &= \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} \\ &= \frac{A_1(s^2+300)(s^2+100)}{s(s^2+300)(s^2+100)} + \frac{(A_2s+A_3)s(s^2+100)}{s(s^2+300)(s^2+100)} + \frac{(A_4s+A_5)s(s^2+300)}{s(s^2+300)(s^2+100)} \end{aligned} \tag{14}$$

$$
\begin{aligned}
X_1(s) &= \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} \\
&= \frac{A_1(s^2+300)(s^2+100)}{s(s^2+300)(s^2+100)} + \frac{(A_2 s + A_3)s(s^2+100)}{s(s^2+300)(s^2+100)} + \frac{(A_4 s + A_5)s(s^2+300)}{s(s^2+300)(s^2+100)} \\
&= \frac{A_1(s^4+400s^2+30000) + A_2 s^4 + A_2 100S^2 + A_3 s^3 + A_3 100s + \ldots}{s(s^2+300)(s^2+100)} \\
&\frac{\cdots + A_4 s^4 + A_4 300 s^2 + A_5 s^3 + A_5 300 s}{s(s^2+300)(s^2+100)}
\end{aligned} \tag{15}
$$

Now, we equate the powers of s

$$
\begin{aligned}
s^4 &\longrightarrow 0 = A_1 + A_2 + A_4 \\
s^3 &\longrightarrow 0 = A_3 + A_5 \\
s^2 &\longrightarrow 5 = 400A_1 + 100A_2 + 300A_4 \\
s^1 &\longrightarrow 0 = 100A_3 + 300A_5 \\
s^0 &\longrightarrow 1000 = 30000A_1
\end{aligned} \tag{16}
$$

Then,

$$
\begin{aligned}
A_1 &= \frac{1}{30} \\
A_2 &= -\frac{1}{120} \\
A_3 &= 0 \\
A_4 &= -\frac{1}{40} \\
A_5 &= 0
\end{aligned} \tag{17}
$$

A similar approach than for equation (10a) can be applied for (10b)

$$
\begin{aligned}
X_2(s) &= \frac{500}{s(s^2+300)(s^2+100)} \\
&= \frac{B_1(s^4+400s^2+30000) + B_2 s^4 + B_2 100S^2 + B_3 s^3 + B_3 100s + \ldots}{s(s^2+300)(s^2+100)} \\
&\frac{\cdots + B_4 s^4 + B_4 300 s^2 + B_5 s^3 + B_5 300 s}{s(s^2+300)(s^2+100)}
\end{aligned} \tag{18}
$$

Now, we equate the powers of s

$$
\begin{aligned}
s^4 &\longrightarrow 0 = B_1 + B_2 + B_4 \\
s^3 &\longrightarrow 0 = B_3 + B_5 \\
s^2 &\longrightarrow 0 = 400B_1 + 100B_2 + 300B_4 \\
s^1 &\longrightarrow 0 = 100B_3 + 300B_5 \\
s^0 &\longrightarrow 500 = 30000B_1
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
B_1 &= \frac{1}{60} \\
B_2 &= \frac{1}{120} \\
B_3 &= 0 \\
B_4 &= -\frac{1}{40} \\
B_5 &= 0
\end{aligned} \tag{20}
$$

Then, the quantities $A_1 ... A_5$ and $B_1 ... B_5$ can be substituted in equation (10)

Exercise 1.2

The 2DOF system below has the following properties $m_1 = m_2 = 1kg$, $k_1 = k_3 = 100N/m$ and $k_2 = 150N/m$ and the damping coefficients are $c_1 = c_2 = c_3 = 0$. It has been found to have

- poles of the system:

$$s_1 = \pm j\sqrt{100} = \pm j10$$
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- and eigenvectors (mode shapes)

$$\boldsymbol{\phi}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
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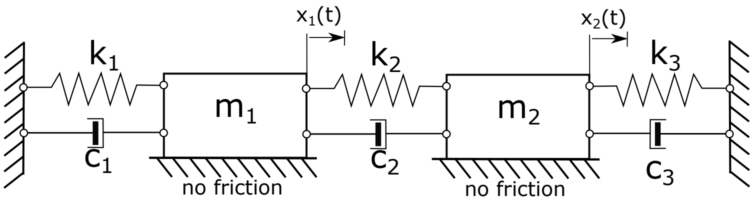
Calculate the frequency response function for $H_{11}(j\omega)$ and $H_{12}(j\omega)$ for the 2DOF system below.



Solution:

By using the equation $A_{pqr} = Q_r \phi_{pr} \phi_{qr}$ to calculate the residues per each FRF.

Then, for FRF H_{11}, we have

$$A_{111} = Q_1 \phi_{11} \phi_{11} \tag{21}$$

Then, the modal scaling constant can be calculated as follows (n.b. that for this exercise because $\zeta = 0$, then $\omega_d = \omega_n$).
For mode 1

$$Q_1 = \frac{1}{j2\omega_{d1} m_1} = \frac{1}{j2\omega_{n1} m_1} = \frac{1}{j20} \tag{22}$$

For mode 2

$$Q_2 = \frac{1}{j2\omega_{d2} m_2} = \frac{1}{j2\omega_{n2} m_2} = \frac{1}{j40} \tag{23}$$

and the residue for mode 1 when $p = 1, q = 1$ is then:

$$A_{111} = Q_1 \phi_{11} \phi_{11} = \frac{1}{j20} \times 1/\sqrt{2} \times 1/\sqrt{2} = \frac{1/2}{j20} = \frac{-j}{40} = -0.025j \tag{24}$$

similarly the residue for mode 2 when $p = 1, q = 1$ is:

$$A_{112} = Q_2 \phi_{12} \phi_{12} = \frac{1}{j40} \times 1/\sqrt{2} \times 1/\sqrt{2} = \frac{1/2}{j40} = \frac{-j}{80} = -0.0125j \tag{25}$$

Then, using equation 25 below we can calculate the H_{11} and H_{12}

$$H_{pq}(j\omega) = \sum_{r=1}^{n} \left(\frac{A_{pqr}}{j\omega - s_r} - \frac{A^*_{pqr}}{j\omega - s^*_r} \right) \tag{26}$$

For the frequency response function H_{11},

$$
\begin{aligned}
H_{11}(j\omega) =& \frac{A_{111}}{j\omega - s_1} - \frac{A^*_{111}}{j\omega - s^*_1} + \frac{A_{112}}{j\omega - s_2} - \frac{A^*_{112}}{j\omega - s^*_2} \\
=& \frac{-0.025j}{j\omega - s_1} + \frac{0.025j}{j\omega - s^*_1} - \frac{0.0125j}{j\omega - s_2} + \frac{0.0125}{j\omega - s^*_2}
\end{aligned} \tag{27}
$$

and similarly for H_{12}, we first calculate the corresponding residuals as above. The residue for mode 1 when $p = 1, q = 2$ is then:

$$
A_{121} = Q_1\phi_{11}\phi_{21} = \frac{1}{j20} \times 1/\sqrt{2} \times 1/\sqrt{2} = \frac{1/2}{j20} = \frac{-j}{40} = -0.025j \tag{28}
$$

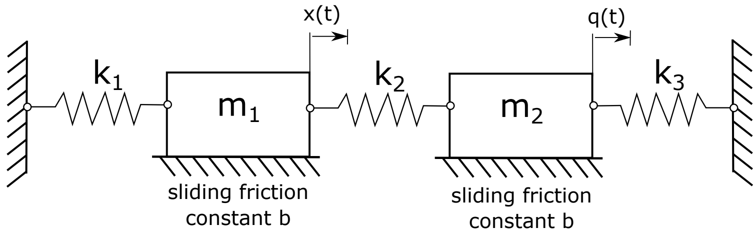
similarly the residue for mode 2 when $p = 1, q = 2$ is:

$$
A_{122} = Q_2\phi_{12}\phi_{22} = \frac{1}{j40} \times 1/\sqrt{2} \times -1/\sqrt{2} = \frac{-1/2}{j40} = \frac{j}{80} = 0.0125j \tag{29}
$$

$$
\begin{aligned}
H_{12}(j\omega) =& \frac{A_{121}}{j\omega - s_1} - \frac{A^*_{121}}{j\omega - s^*_2} + \frac{A_{122}}{j\omega - s_2} - \frac{A^*_{122}}{j\omega - s^*_2} \\
=& \frac{-0.025j}{j\omega - s_1} + \frac{0.025j}{j\omega - s^*_1} + \frac{0.0125j}{j\omega - s_2} - \frac{0.0125}{j\omega - s^*_2}
\end{aligned} \tag{30}
$$

Exercise 1.3

Consider the case in figure below. Determine the state variable matrix differential equation for $m_1 = m_2 = m$ and $k_1 = k_3 = k$.



Solution:

First let's calculate the equations of motion and for this less draw the Body diagram:

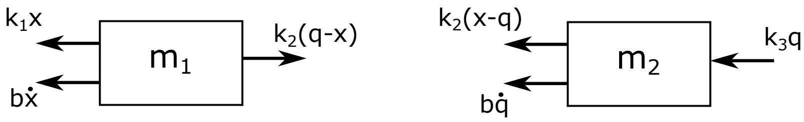


Figure 1: Body Diagram Mass-Spring system

Then for mass 1 and mass 2 the equation of motion are:

$$\begin{aligned} m_1\ddot{x} + b\dot{x} + k_1 x + k_2(x-q) &= 0 \\ m_2\ddot{q} + b\dot{q} + k_2(q-x) + k_3 q &= 0 \end{aligned} \tag{31}$$

As per $m_1 = m_2 = m$ and $k_1 = k_3 = k$, then:

$$\begin{aligned} m\ddot{x} + b\dot{x} + kx + k_2(x-q) &= 0 \\ m\ddot{q} + b\dot{q} + k_2(q-x) + kq &= 0 \end{aligned} \tag{32}$$

Now, we define the state vector as below

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x \\ q \\ \dot{x} \\ \dot{q} \end{bmatrix} \qquad \Rightarrow \qquad \dot{\mathbf{z}} = \begin{bmatrix} \dot{x} \\ \dot{q} \\ \ddot{x} \\ \ddot{q} \end{bmatrix} \tag{33}$$

$$\dot{\mathbf{z}} = \begin{bmatrix} \dot{x} \\ \dot{q} \\ \ddot{x} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} z_3 \\ z_4 \\ -\frac{(k+k_2)}{m} z_1 + \frac{k_2}{m} z_2 - \frac{b}{m} z_3 \\ \frac{k_2}{m} z_1 - \frac{k_2+k}{m} z_2 - \frac{b}{m} z_4 \end{bmatrix} \begin{bmatrix} x \\ q \\ \dot{x} \\ \dot{q} \end{bmatrix} \tag{34}$$

Then, the matrix $\mathbf{A}$

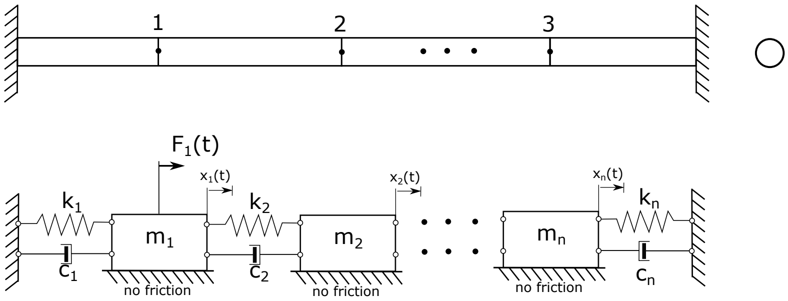
$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k+k_2}{m} & \frac{k_2}{m} & -\frac{b}{m} & 0 \\ \frac{k_2}{m} & -\frac{k_2+k}{m} & 0 & -\frac{b}{m} \end{bmatrix} \tag{35}$$

Exercise 1.4 (Code: Ex_CT1.4)

This exercise is solved in a Jupiter notebook Ex_CT1.4.

In this example we simulate and study the effect of the structural properties of a MDOF mass-spring structure in its dynamical parameters with motion in the longitudinal direction. The structure consists of a series of masses connected by springs, with a fixed-fixed boundary condition.

The structure has all the stiffness, mass and damping elements are the same. Thus, $k_1 = k_2 = \cdots = k_n = k$, $m_1 = m_2 = \cdots = m_n = m$ and $c_1 = c_2 = \cdots = c_n = c$. The code below calculates the natural frequencies and mode shapes for an aluminum beam with Young's modulus $E = 69GPa$, circular cross-section of diameter 40 mm, density $\rho = 2.7 \times 10^3\ kg/m^3$, and a total length of 4 m.



The Learning outcomes in this example:

- The modal parameters are obtained (i.e. natural frequencies, damping ratios and mode shapes).
- The FRF for each dof are obtained when excitation is entered in the first degree of freedom as unit.

Solution:
Solution on Jypter notebook Ex_CT1.4

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EXERCISES FOR LECTURES IN BLOCK II

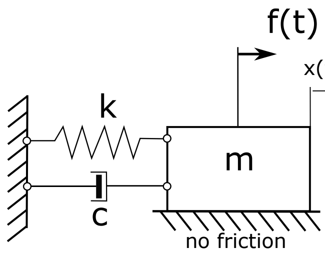
Exercise 2.1

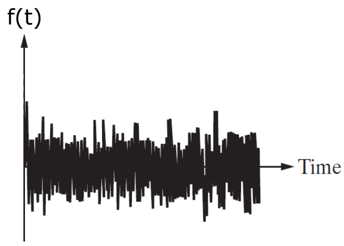
Consider a single-degree-of-freedom system below subject to a random (white noise) force input f(t).

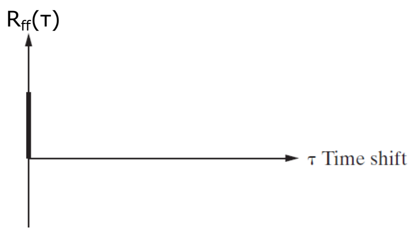
(a) Calculate the power spectral density of the response x(t) given that the PSD of the applied force is the constant value S_0 as illustrated in Figure below. Also sketch by hand the expected result.

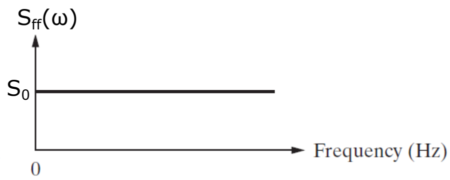
(b) The mean-square value is defined as $E[x^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_{ff}(\omega) d\omega$ and the form Eq. (1 has been tabulated by Newland, 1993 for constant S_{ff}. Calculate the mean-square value of the response of the x and explain (and proof) the result.

$$\int_{-\infty}^{\infty} \left| \frac{B_0 + j\omega B_1}{A_0 + j\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi(A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2} \tag{1}$$







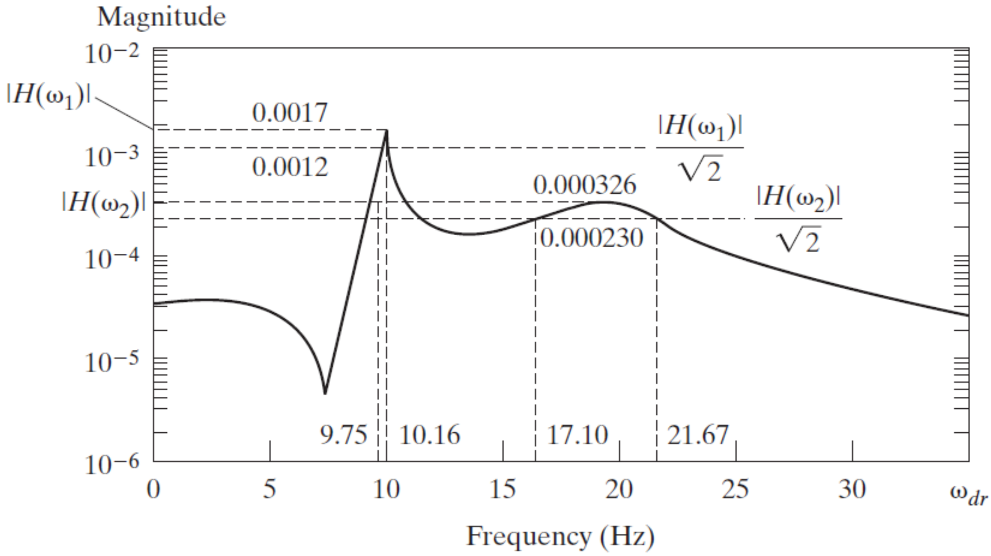


Exercise 2.2

Consider a plane moving over the runway before taking off. The situation can be modelled as 1DOF spring-damper system with values $m = 1000kg$, $c = 1500kg/s$ and $k = 35000N/m$. The effect of runaway has been considered to have a random stationary cross section producing a PSD of S_0. Calculate the PSD of the response and the mean-squared value.

Exercise 2.3

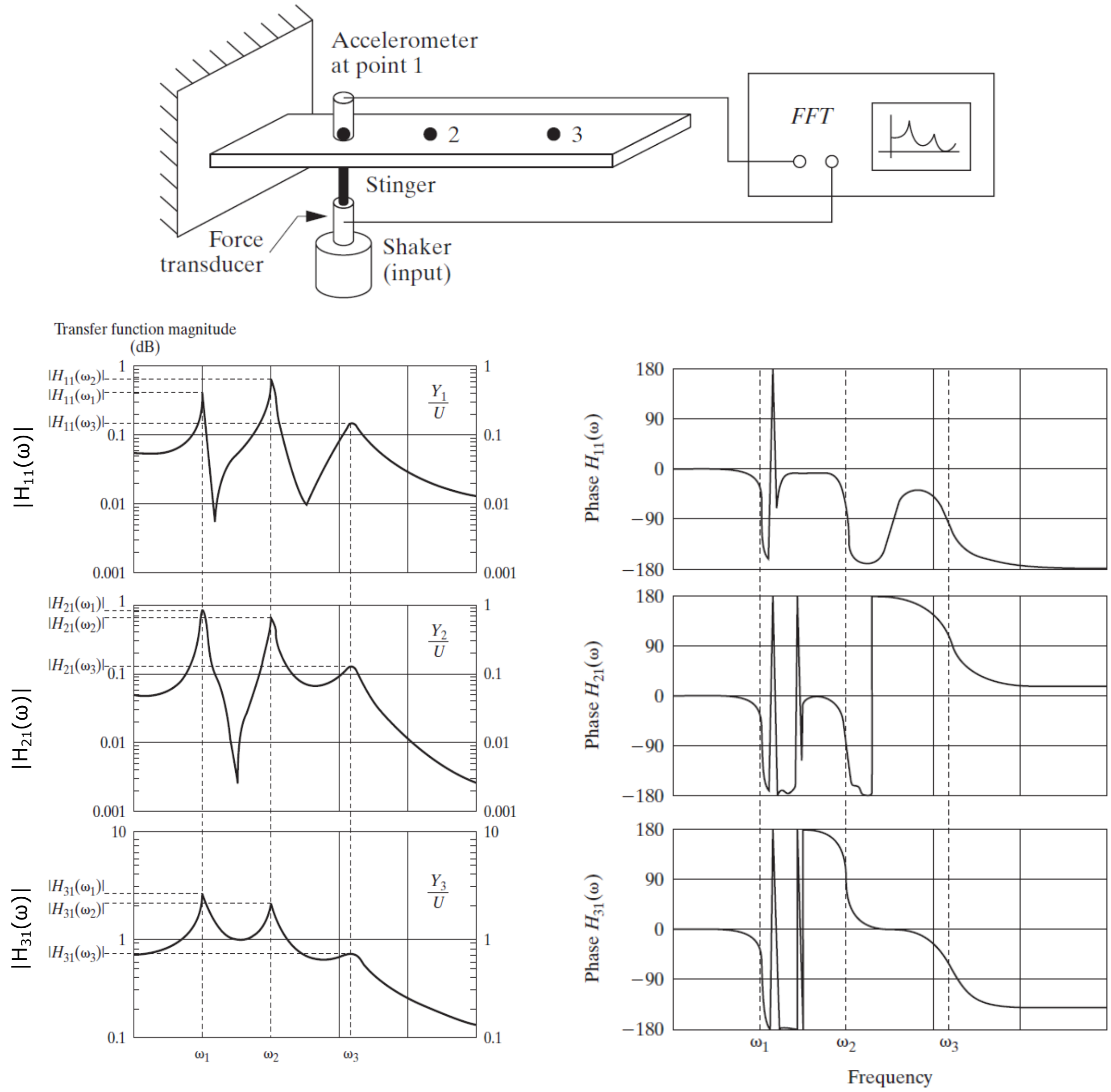
The compliance transfer function represent a dedicate experimental campaign. Calculate the number of degree of freedom, modal damping ratios and natural frequencies.



(Source: Inman. D, 2013. Engineering Vibrations)

Exercise 2.4

The experimental set up shown in the Figure below has been designed to dynamically characterise a cantilever beam. The set up is performed as follows. A modal shaker has been attached to point 1 with a load transducer to measure the input load at point 1. The vibration response has been measured in the points 1, 2 and 3 as marked in the Figure. Only one accelerometer was used and it was moved from point to point to obtain data at each measurement point. The acceleration in the time domain was transformed into the frequency domain via the Fast Fourier Transform (FFT) through a frequency analyzer. The Frequency Response Functions where calculated as Y/U being Y the acceleration response out and U the force load input. Each FRF was then obtained as H_{qp} where q is the output and p is the input. The obtained FRF and Phase diagrams are presented below.



(Source: Inman. D, 2013. Engineering Vibrations)

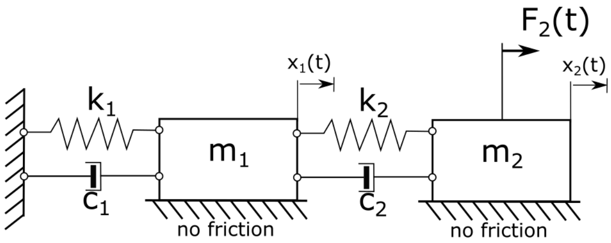
The natural frequencies and damping ratios have been estimated using peak-picking method and half-power spectrum method. The values are: $\{\omega_1, \zeta_1\} = 10\ rad/s, 0.01$, $\{\omega_2, \zeta_2\} = 20\ rad/s, 0.02$ and $\{\omega_3, \zeta_3\} = 32\ rad/s, 0.05$. The Magnitude values corresponding to the peaks has been found to be:

$$\begin{aligned}
&H_{11}(\omega_1) = 0.41 \quad H_{21}(\omega_1) = 0.98 \quad H_{31}(\omega_1) = 2.62 \\
&H_{11}(\omega_2) = 0.68 \quad H_{21}(\omega_2) = 0.65 \quad H_{31}(\omega_2) = 2.21 \\
&H_{11}(\omega_3) = 0.17 \quad H_{21}(\omega_3) = 0.12 \quad H_{31}(\omega_3) = 0.72
\end{aligned} \tag{2}$$

Given the information above, calculate all the mode shapes of the beam using Pick-peaking method

Exercise 2.5 (Code Ex_CT2.5)

Calculate and compute the Power Spectral Density (PSD), Cross Spectral Density (CSD), H1 & H2 indicator and Coherence at mass 1 on the simulated system below that correspond to Ex_C1.2.



The Learning outcomes in this example:

- How to compute PSD, CSD, H1, H2 and Coherence

Tutorials: Advanced Dynamics and Applications 5

by Dr David García Cava

EXERCISES FOR LECTURES IN BLOCK II

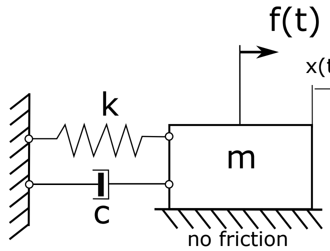
Exercise 2.1

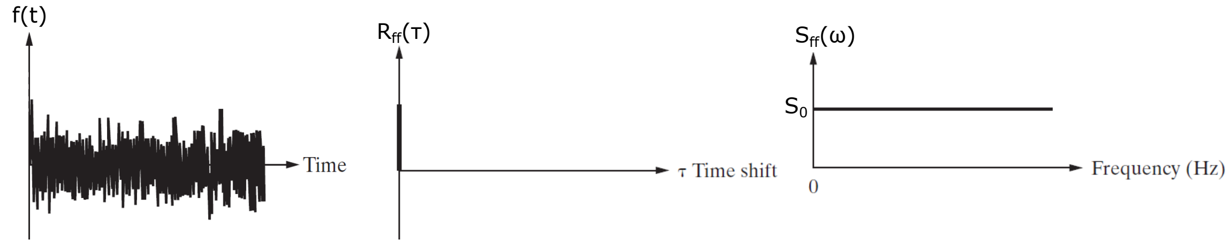
Consider a single-degree-of-freedom system below subject to a random (white noise) force input f(t).

(a) Calculate the power spectral density of the response x(t) given that the PSD of the applied force is the constant value S_0 as illustrated in Figure below. Also sketch by hand the expected result.

(b) The mean-square value is defined as $E[x^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_{ff}(\omega) d\omega$ and the form Eq. (1 has been tabulated by Newland, 1993 for constant S_{ff}. Calculate the mean-square value of the response of the x and explain (and proof) the result.

$$\int_{-\infty}^{\infty} \left| \frac{B_0 + j\omega B_1}{A_0 + j\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi(A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2} \tag{1}$$





Solution:

(a)
The equation of motion of the system can be defined as

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{2}$$

The frequency response function can be obtained doing the Laplace transform of Eq.2,

$$H(\omega) = \frac{1}{k - m\omega^2 + jc\omega} \tag{3}$$

Then, we know the that the FRF and PSD hold the following relationship

$$S_{xx}(\omega) = |H(\omega)|^2 S_{ff}(\omega) \tag{4}$$

So,

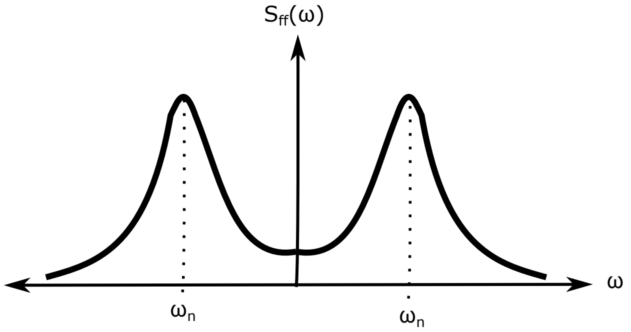


Figure 1: Sketch of S_{xx}

$$|H(\omega)|^2 = \left|\frac{1}{k - m\omega^2 + jc\omega}\right|^2 = \frac{1}{(k - m\omega^2) + jc\omega} \cdot \frac{1}{(k - m\omega^2) - jc\omega} = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} \tag{5}$$

Then, substituting into Eq.(4), the PSD response becomes

$$S_{xx}(\omega) = |H(\omega)|^2 S_{ff}(\omega) = \frac{S_0}{(k - m\omega^2)^2 + c^2\omega^2} \tag{6}$$

This states that if a single-degree-of-freedom system is excited by a stationary random force (of constant mean and rms value i.e. white noise) that has a constant PSD of value S_0, the response of the system will also be random with non-constant PSD as it depends on the frequency (see Eq. (6).

(b)

Then, since the PSD of the forcing function is the constant S_0, the expected mean-square vale can be calculated as:

$$E[x^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_{ff}(\omega) d\omega = \int_{-\infty}^{\infty} \left|\frac{1}{k - m\omega^2 + jc\omega}\right|^2 S_0 d\omega \tag{7}$$

if we compare with Eq. (1), we can see that $B_0 = 1, B_1 = 0, A_0 = k, A_1 = c$ and $A_2 = m$. Thus, we can obtain:

$$E[x^2] = S_0 \frac{\pi m}{kcm} = \frac{\pi S_0}{kc} \tag{8}$$

Explanation and reflections on solution of Eq. (8)

As observed, the mean-square value is independent of the mass m. The explanation can be considered by looking at the spectral peak in the sketch above (We focused on the right-had side only as remember the left hand-side is the mirror of the right-hand side).

As observed, the peak value occur, for small damping, when:

$$\omega \approx \omega_n = \sqrt{\frac{k}{m}} \quad ; \quad k = m\omega_n^2 \tag{9}$$

and then, its height is:

$$S_{xx}(\omega_n) = \frac{S_0}{(k - m\omega_n^2)^2 + c^2\omega_n^2} = \frac{S_0}{(m\omega_n^2 - m\omega_n^2)^2 + c^2\omega_n^2} = \frac{S_0}{c^2\omega_n^2} = \frac{S_0 m}{c^2 k} \tag{10}$$

Then, it is demonstrated that the peak of $S_{xx}(\omega)$ is proportional to m.

Now the peak width needs definition, We can arbitrarily define the difference in frequency $2\Delta\omega$ between two points on either side of ω_n with heights equals to half the height peak (a.k.a. as we know called "half-power" bandwidth/method). Then, we know that $2\zeta\omega = \frac{c}{m}$ but for small damping:

$$\Delta\omega \ll \omega_n \tag{11}$$

and so

$$2\Delta\omega \approx \frac{c}{m} \tag{12}$$

We can see that it is inversely proportional to m.

Therefore, we can see that increasing m increases the height of the spectra peak, but at the same time reduces width. Then, these two opposite effects exactly cancel out and the total area under the spectral density curve $S_x x(\omega)$ is independent of m as it is the mean-square value.

Exercise 2.2

Consider a plane moving over the runway before taking off. The situation can be modelled as 1DOF spring-damper system with values $m = 1000kg$, $c = 1500kg/s$ and $k = 35000N/m$. The effect of runaway has been considered to have a random stationary cross section producing a PSD of S_0. Calculate the PSD of the response and the mean-squared value.

Solution

Then, we know $S_{ff} = S_0$ and with the values m, c and k when obtain the damping ratio of the system.

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{1500}{2\sqrt{35000 \cdot 1000}} = 0.126 < 1 \quad \text{(underdamped)} \tag{13}$$

Therefore, the FRF of the systems can be defined as:

$$H(\omega) = \frac{1}{k - m\omega^2 + jc\omega} \tag{14}$$

$$\begin{aligned} |H(\omega)|^2 &= \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} \\ |H(\omega)|^2 &= \frac{1}{k^2 + m^2\omega^4 - 2kn\omega^2 + c^2\omega^2} \end{aligned} \tag{15}$$

Substituting the values of k, m and c gives

$$\begin{aligned} |H(\omega)|^2 &= \frac{1}{(3.5 \times 10^4)^2 + (1000)^2\omega^4 - 2(3.5 \times 10^4)(1000)\omega^2 + (1.5 \times 10^3)\omega^2} \\ |H(\omega)|^2 &= \frac{1}{1 \times 10^6\omega^4 - 6.77 \times \omega^2 + 1.22 \times 10^9} \end{aligned} \tag{16}$$

Then, the PSD is

$$S_{xx}(\omega) = |H(\omega)|^2 S_{ff}(\omega) = \frac{S_0}{1 \times 10^6\omega^4 - 6.77 \times \omega^2 + 1.22 \times 10^9} \tag{17}$$

Then, the mean-squared value is obtained as:

$$\begin{aligned} E[x^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_{ff}(\omega) d\omega &= \int_{-\infty}^{\infty} \left| \frac{1}{k - m\omega^2 + jc\omega} \right|^2 S_0 d\omega \\ &= \int_{-\infty}^{\infty} \left| \frac{1}{3.5 \times 10^4 - 1000\omega^2 + 1500j\omega} \right|^2 S_0 d\omega \end{aligned} \tag{18}$$

As we know, for S_{ff} constant.

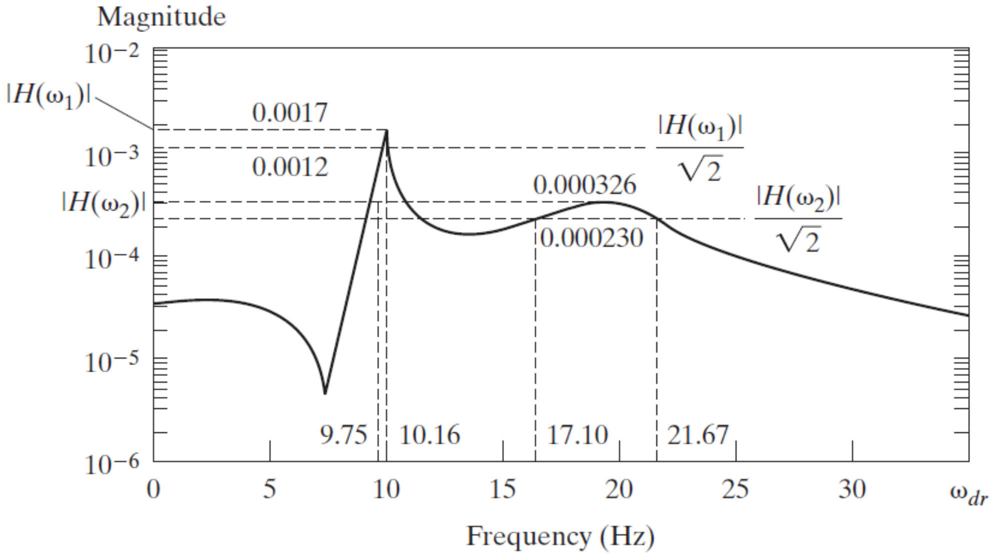
$$\int_{-\infty}^{\infty} \left| \frac{B_0 + j\omega B_1}{A_0 + j\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi(A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2} \tag{19}$$

and it can be simplified as:

$$E[x^2] = \frac{\pi S_0}{kc} = \frac{\pi S_0}{(3.5 \times 10^4)(1500)} = \frac{\pi S_0}{5.25 \times 10^7} \tag{20}$$

Exercise 2.3

The compliance transfer function represent a dedicate experimental campaign. Calculate the number of degree of freedom, modal damping ratios and natural frequencies.



(Source: Inman. D, 2013. Engineering Vibrations)

Solution

- Clearly two distinct peaks can be observed in the magnitude plot and for this reason it is assumed to have two degree of freedom. Bare in mind that the system might have more degree of freedom but they are outside the sampling frequency used to record the vibration response.
- The natural frequencies can be obtained by looking at the vertical line intersecting the horizontal axis where the peaks occur. In this case $\omega_1 = 10Hz$ and $\omega_2 = 20Hz$.
- The damping rations can be obtained using half-power method as follows:

For ζ_1:

The magnitude value at ω_1 is $|H(\omega_1)| = 0.0017$. From the maximum peak, the $3dB$ down are $|H(\omega_{1a})| = |H(\omega_{1b})| = 0.00017/\sqrt{2} = 0.0012$. Then, these plots leads to $\omega_{1a} = 9.95Hz$ and $\omega_{1b} = 10.16Hz$

$$\zeta_1 = \frac{\omega_{2a} - \omega_{2b}}{2\omega_1} = \frac{10.16 - 9.75}{2(10)} = 0.02 \tag{21}$$

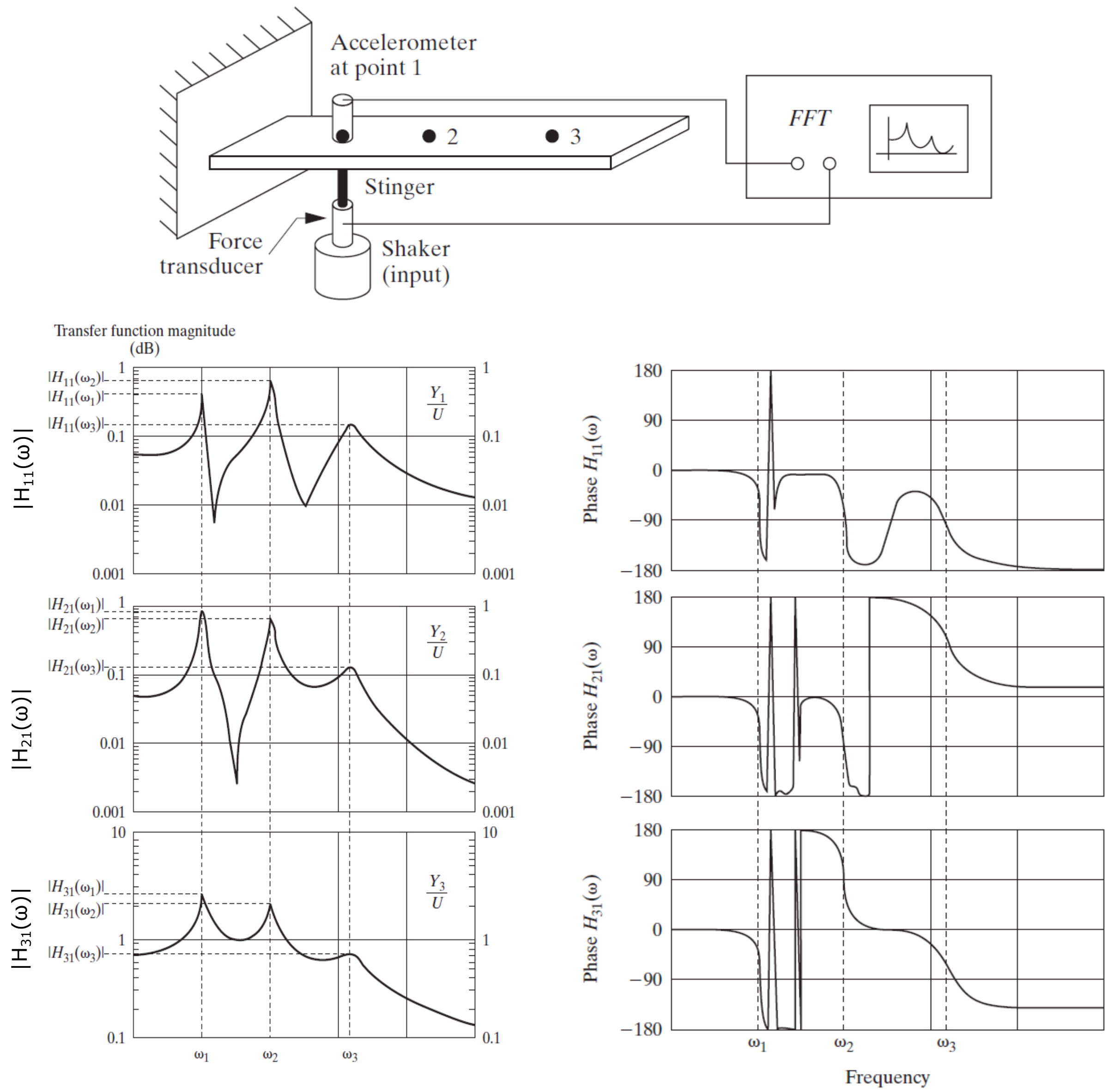
For ζ_2:

The magnitude value at ω_2 is $|H(\omega_2)| = 0.000326$. From the maximum peak, the $3dB$ down are $|H(\omega_{2a})| = |H(\omega_{2b})| = 0.00326/\sqrt{2} = 0.000230$. Then, these plots leads to $\omega_{2a} = 17.10Hz$ and $\omega_{2b} = 21.67Hz$

$$\zeta_1 = \frac{\omega_{2a} - \omega_{2b}}{2\omega_2} = \frac{21.67 - 17.10}{2(120)} = 0.11 \tag{22}$$

Exercise 2.4

The experimental set up shown in the Figure below has been designed to dynamically characterise a cantilever beam. The set up is performed as follows. A modal shaker has been attached to point 1 with a load transducer to measure the input load at point 1. The vibration response has been measured in the points 1, 2 and 3 as marked in the Figure. Only one accelerometer was used and it was moved from point to point to obtain data at each measurement point. The acceleration in the time domain was transformed into the frequency domain via the Fast Fourier Transform (FFT) through a frequency analyzer. The Frequency Response Functions where calculated as Y/U being Y the acceleration response out and U the force load input. Each FRF was then obtained as H_{qp} where q is the output and p is the input. The obtained FRF and Phase diagrams are presented below.



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The natural frequencies and damping ratios have been estimated using peak-picking method and half-power spectrum method. The values are: $\{\omega_1, \zeta_1\} = 10\ rad/s, 0.01$, $\{\omega_2, \zeta_2\} = 20\ rad/s, 0.02$ and $\{\omega_3, \zeta_3\} = 32\ rad/s, 0.05$. The Magnitude values corresponding to the peaks has been found to be:

$$\begin{aligned}
&H_{11}(\omega_1) = 0.41 \quad H_{21}(\omega_1) = 0.98 \quad H_{31}(\omega_1) = 2.62 \\
&H_{11}(\omega_2) = 0.68 \quad H_{21}(\omega_2) = 0.65 \quad H_{31}(\omega_2) = 2.21 \\
&H_{11}(\omega_3) = 0.17 \quad H_{21}(\omega_3) = 0.12 \quad H_{31}(\omega_3) = 0.72
\end{aligned} \tag{23}$$

Given the information above, calculate all the mode shapes of the beam using Pick-peaking method

Solution

As we know, one of the assumptions made is that around the resonance, the FRF is dominated by its main vibration mode and the contribution of other modes are (assumed) to be negligible, when we use Peak-peaking method.

Then, we can state that around the resonance ($\omega = \omega_r$)

$$A_r = |H(\omega_r)| 2\zeta_r \omega_r^2 \tag{24}$$

Now, we know that point 1 can be used as reference as it was the location of the excitation (input) p. So, we can start to reformulate the modal constant components at each mode r $A_{qpr} = \phi_{pr}\phi_{qr}$, where p is the excitation point and q the response point.

For ω_1,

$$\begin{aligned}
A_{111} &= \phi_{11}\phi_{11} = |H_{11}(\omega_1)| 2\zeta_1\omega_1^2 = (0.41)(2)(0.01)(10^2) = 0.82 \\
A_{211} &= \phi_{11}\phi_{21} = |H_{21}(\omega_1)| 2\zeta_1\omega_1^2 = (0.98)(2)(0.01)(10^2) = 1.96 \\
A_{311} &= \phi_{11}\phi_{31} = |H_{31}(\omega_1)| 2\zeta_1\omega_1^2 = (2.62)(2)(0.01)(10^2) = 5.24
\end{aligned} \tag{25}$$

Now, each component for mode $r = 1$ can be obtained as: First, we obtain the Point FRF

$$\phi_{11} = \sqrt{A_{111}} = \sqrt{0.82} = 0.906 \tag{26}$$

and the rest using transfer FRF

$$\begin{aligned}
\phi_{21} &= \sqrt{A_{211}} = \sqrt{1.96} = 2.164 \\
\phi_{31} &= \sqrt{A_{311}} = \sqrt{5.24} = 5.787
\end{aligned} \tag{27}$$

Now, looking at the phase plots, the phase at the resonance peaks is 90° and it can be + or - (i.e., $H(\omega_r)$ can be in phase or out of phase). So, to allocate signs, we look at the phase at each peak.

$$\boldsymbol{\phi}_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{bmatrix} = \begin{bmatrix} -0.906 \\ -2.164 \\ -5.787 \end{bmatrix} \tag{28}$$

Then, similarly the other mode shapes can be obtained:

For ω_2,

$$\begin{aligned}
A_{112} &= \phi_{12}\phi_{12} = |H_{11}(\omega_2)| 2\zeta_2\omega_2^2 = (0.68)(2)(0.02)(20^2) = 10.88 \\
A_{212} &= \phi_{12}\phi_{22} = |H_{21}(\omega_2)| 2\zeta_2\omega_2^2 = (0.65)(2)(0.02)(20^2) = 10.4 \\
A_{312} &= \phi_{12}\phi_{32} = |H_{31}(\omega_2)| 2\zeta_2\omega_2^2 = (2.21)(2)(0.02)(20^2) = 35.36
\end{aligned} \tag{29}$$

Now, each component for mode $r = 2$ can be obtained as: First, we obtain the Point FRF

$$\phi_{12} = \sqrt{A_{112}} = \sqrt{10.88} = 3.298 \tag{30}$$

and the rest using transfer FRF

$$\begin{aligned}
\phi_{22} &= \sqrt{A_{212}} = \sqrt{10.4} = 3.153 \\
\phi_{32} &= \sqrt{A_{312}} = \sqrt{35.36} = 10.720
\end{aligned} \tag{31}$$

Now, looking at the phase plots,

$$\boldsymbol{\phi}_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{bmatrix} = \begin{bmatrix} -3.298 \\ -3.153 \\ 10.720 \end{bmatrix} \tag{32}$$

And, for ω_3,

$$
\begin{aligned}
A_{113} &= \phi_{13}\phi_{13} = |H_{11}(\omega_3)|2\zeta_3\omega_3^2 = (0.17)(2)(0.05)(32^2) = 17.408 \\
A_{213} &= \phi_{13}\phi_{23} = |H_{21}(\omega_3)|2\zeta_3\omega_3^2 = (0.12)(2)(0.05)(32^2) = 12.288 \\
A_{313} &= \phi_{13}\phi_{33} = |H_{31}(\omega_3)|2\zeta_3\omega_3^2 = (0.72)(2)(0.05)(32^2) = 73.728
\end{aligned} \tag{33}
$$

Now, each component for mode $r = 3$ can be obtained as: First, we obtain the Point FRF

$$
\phi_{13} = \sqrt{A_{113}} = \sqrt{17.408} = 4.172 \tag{34}
$$

and the rest using transfer FRF

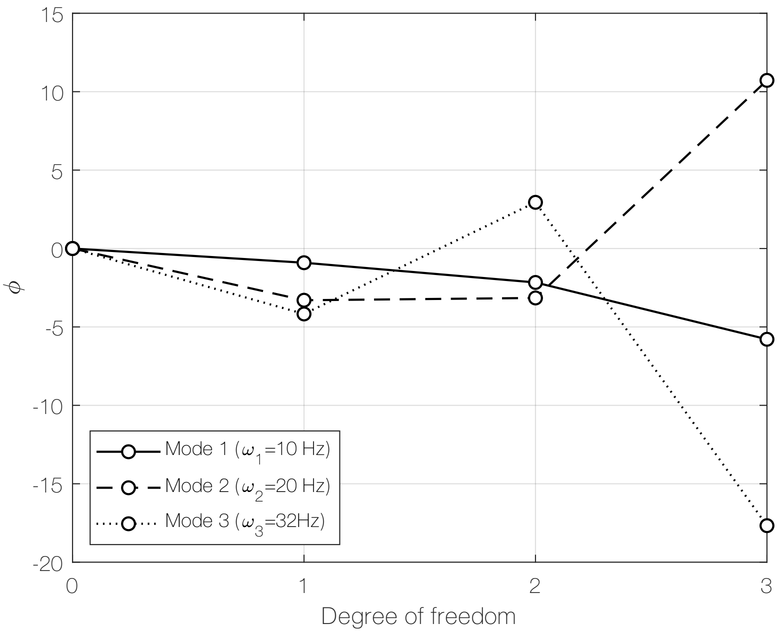
$$
\begin{aligned}
\phi_{23} &= \sqrt{A_{213}} = \sqrt{12.288} = 2.945 \\
\phi_{33} &= \sqrt{A_{313}} = \sqrt{73.728} = 17.671
\end{aligned} \tag{35}
$$

Now, looking at the phase plots,

$$
\boldsymbol{\phi}_3 = \begin{bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{bmatrix} = \begin{bmatrix} -4.172 \\ 2.945 \\ -17.671 \end{bmatrix} \tag{36}
$$

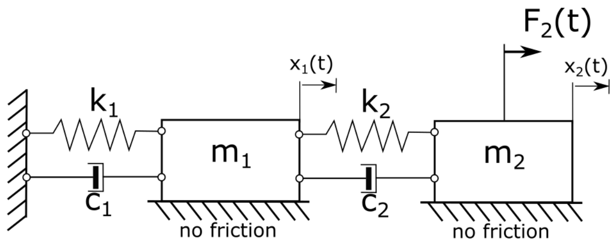
Then, the full mode shape matrix is

$$
\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \boldsymbol{\phi}_3 \end{bmatrix} = \begin{bmatrix} -0.906 & -3.298 & -4.172 \\ -2.164 & -3.153 & 2.945 \\ -5.787 & 10.720 & -17.671 \end{bmatrix} \tag{37}
$$



Exercise 2.5 (Code Ex_CT2.5)

Calculate and compute the Power Spectral Density (PSD), Cross Spectral Density (CSD), H1 & H2 indicator and Coherence at mass 1 on the simulated system below that correspond to Ex_C1.2.



The Learning outcomes in this example:

- How to compute PSD, CSD, H1, H2 and Coherence

Solution

The solution can be found in Jupyter notebook Ex_CT2.5

Tutorials: Advanced Dynamics and Applications 5

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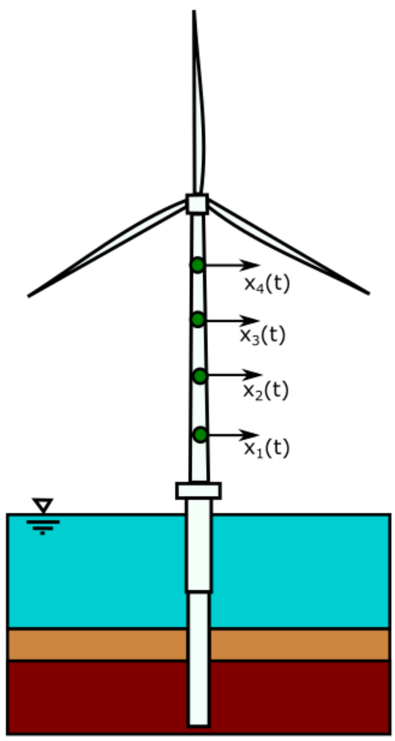
EXERCISES FOR LECTURES IN BLOCK III

Exercise 3.1 (Code Ex_CT3.1)

This exercise is solved in a Jupiter notebook Ex_CT3.1. This example requires to have the file 'utils.py' in the same folder of the Ex_CT3.1 to call the functions.

This example covers identification of a linear, 4-dof system of a simplified simulated wind turbine parked with non-flexible boundary conditions. The ssicov algorithm is used to estimate the state-space matrices.

Four accelerations responses were computed on the turbine tower. The wind turbine can be then simulated as 4-dof mass-spring-damper system where measurements were taken in the tower $m_1 = m_2 = m_3 = m_4 = 100kg$, the stiffness $k_1 = k_2 = k_3 = k_4 = 10000Nm^{-1}$ and the damping coefficients $c_1 = c_2 = c_3 = c_4 = 10N(ms)^{-1}$. It is also assumed that the turbine is parked and the boundary conditions bellow the sea level are not flexible, so it is simplified to be fixed below sea level.



The Learning outcomes in this example:

- Builds the state-space model of a simplified structure for 4-dof
- Estimates the **A** and **C** matrices using the exponential solution see in Lecture block 1 and additional material for further understanding.
- Obtain the theoretical modal parameters from the **A** and **A** to set the benchmark by using the function 'modalparams' in the file 'utils.py'.
- Calculates the time responses and PSD Welch of all the DoF for a simplified added noise for simulation operational excitation.
- Uses SSI-cov method to estimate **A** and **C** matrices using the function 'ssicov' in the file 'utils.py'.

- Compute the stabilisation diagram using 'plot_stabilization_diagram' in the file 'utils.py'.
- Obtain the operational modal parameters using a function 'modalparams' in the file 'utils.py'.

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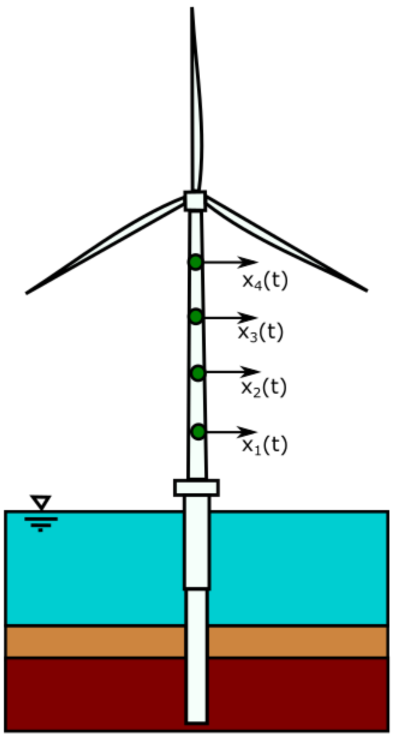

The Learning outcomes in this example:

- Builds the state-space model of a simplified structure for 4-dof
- Estimates the **A** and **C** matrices using the exponential solution see in Lecture block 1 and additional material for further understanding.
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- Compute the stabilisation diagram using 'plot_stabilization_diagram' in the file 'utils.py'.
- Obtain the operational modal parameters using a function 'modalparams' in the file 'utils.py'.

Solution:

The solutions can be found in Jupiter notebook Ex_CT3.1

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Dynamics 5

Exercise Sheet 1

Note

This Dynamics 5 exercise sheet covers the following topics:

- Low energy impact.
- Impulse and momentum.
- Coefficient of restitution.
- Energy balance.

Question 1.1

A medium size vehicle has a mass $M = 1250$ kg and is moving with a velocity $v_0 = 48$ km/h when it crashes into a rigid wall. If the duration of the impact is $\Delta t = 0.06$ s determine the average impulsive force acting on the vehicle, assuming the brakes are not applied.

Solution

The average impulsive force is $\bar{F} = 278$ kN.

Question 1.2

Two smooth discs A and B, with masses $m_A = 4$ kg and $m_B = 2$ kg, respectively, slide along a smooth surface and hit one another as shown in figure **??**. The coefficient of restitution is $e = 0.5$. If the initial velocities of the disks are $v_A = 3$ m/s and $v_B = 5$ m/s, determine the velocity and direction both disks after impact.

Solution

The final velocities of disks A and B are $v_{Af} = 1$ m/s (direction BA) and $v_{Bf} = 3$ m/s (direction AB), respectively.

Question 1.3

In case of an emergency, the explosive gas actuator (cylinder A) shown in figure 1 is used to move the 75 kg block B. Neglect the mass of the cylinder itself. Upon explosion, the displaced cylinder A moves B forward, giving it a velocity $v_B = 200$ mm/s in $\Delta t = 0.4$ s. If the coefficient of kinetic friction between B and the floor is $\mu_k = 0.5$ determine the impulse the actuator imparts to B.

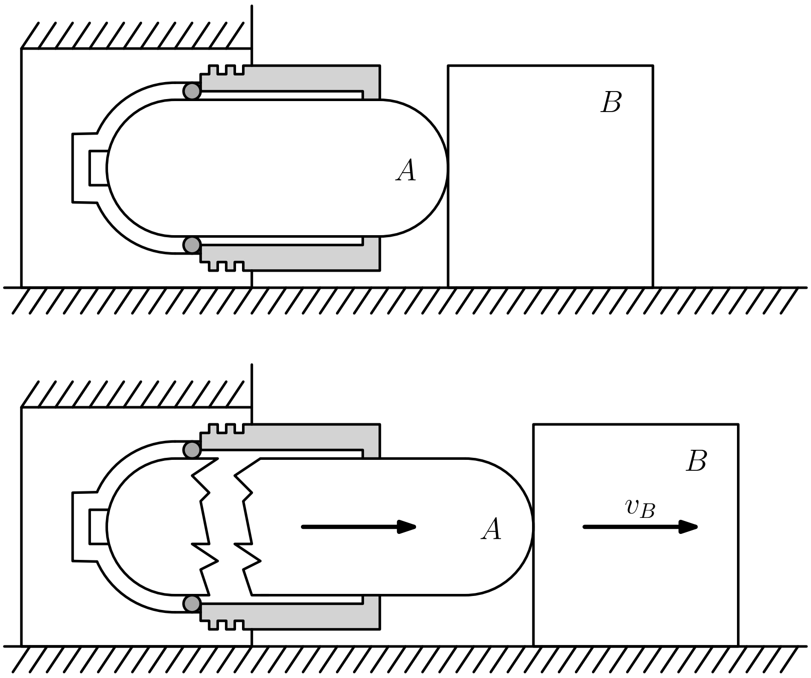

Figure 1: Explosive gas actuator for Question 1.5.

Solution

The impulse the actuator imparts on mass B is $I_B = 162.2$ Ns (the force is $P = 405.4$ N).

Question 1.4

A boxcar similar to the one shown in figure **??**, with mass $m_1 = 50,000$ kg is rolling at 2 m/s when it is hit by another boxcar of mass $100,000$ kg rolling in the same direction at 3 m/s. For a coefficient of restitution $e = 0.5$ find the velocity of each boxcar at separation and calculate the loss of kinetic energy in the collision.

Solution
The final velocities of the boxcars are $v_{1f} = 3$ m/s and $v_{2f} = 2.5$ m/s.

Question 1.5

Three elastic spherical balls B_1, B_2 and B_3 have their centres aligned. Balls B_2 and B_3 are stationary and slightly separated before ball B_1, travelling at speed v_0, collides against B_2.

a) Prove that the final velocity v_3' of ball B_3 is a maximum if ball B_2 has mass M_2 equal to the geometric mean of the masses of B_1 and B_3, that is, if $M_2 = \sqrt{M_1 M_3}$. Find the maximum value of the ratio v_3'/v_0 by letting the mass ratio $M_3/M_1 = \alpha^2$.

b) Let the number of balls be an arbitrary number n rather than 3. To obtain a maximum ratio of the final to the initial velocity v_n'/v_0, prove that the mass of each ball is related to that of its neighbours by

$$M_i = \sqrt{M_{i-1} M_{i+1}} \quad \text{where} \quad i = 1, \ldots, n$$

Find the maximum ratio v_n'/v_0 as a function of n.

THE UNIVERSITY of EDINBURGH
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Dynamics 5

Tutorial 3

Note

This Dynamics 5 exercise sheet covers the following topics:

- Stress waves.
- Transmission and reflection at interfaces.

Question 5.1

An impact generates a stress wave at the front face of a layered armour plate. The front plate is steel and the armour designer is trying different materials for the back layer.

a) Find the transmitted-to-incident and reflected-to-incident stress wave ratios at the interface between steel and the following materials: air, epoxy resin, PMMA, magnesium alloy AZ31B, aluminium alloy 6082-T6, titanium alloy Ti6Al4V, copper, steel, tungsten and tungsten carbide (WC).

b) Discuss the meaning of the sign of the stresses in each layer.

Solution To find these stress wave ratios we first need to calculate the mechanical impedance of each material,

$$Z_i = \rho_i c_i$$

The stress wave ratios can then be determined with the following two relations:

$$\frac{\sigma_\mathrm{t}}{\sigma_\mathrm{i}} = \frac{2R}{1+R} \qquad \text{and} \qquad \frac{\sigma_\mathrm{r}}{\sigma_\mathrm{i}} = -\frac{1-R}{1+R}$$

where $R = \dfrac{Z_2}{Z_1}$ The stress wave rations for all the pairs of materials are listed in the table below.

Material A	Material B	$Z_B \times 10^6$ [kg/m^{2}s]	$\sigma_\mathrm{t}/\sigma_\mathrm{i}$	$\sigma_\mathrm{r}/\sigma_\mathrm{i}$
Steel	Air	0.00	0.00	-1.00
Steel	Epoxy resin	2.41	0.11	-0.89
Steel	PMMA	2.69	0.12	-0.88
Steel	AZ31B	8.55	0.35	-0.65
Steel	6082-T6	14.09	0.52	-0.48
Steel	Ti6Al4V	22.40	0.71	-0.29
Steel	Copper	33.01	0.90	-0.10
Steel	Steel	40.58	1.00	-0.00
Steel	Tungsten	88.95	1.37	0.37
Steel	WC	94.11	1.40	0.40

Question 5.2

A 50 MPa elastic stress wave from an aluminium bar is transmitted into a steel bar. The steel bar has half the diameter of the aluminium bar.

a) Calculate the stress transmitted into the steel bar.

b) Calculate the reflected wave back into the aluminium bar.

Solution

As in the previous example, here the wave transmission area also changes at the interface. The diameter of the steel bar is half the diameter of the aluminium bar. Consequently, the area of the steel bar will be one-quarter of the aluminium bar, that is, $n = 0.25$. The equilibrium and compatibility equations are then

$$(\sigma_\mathrm{i} + \sigma_\mathrm{r}) = n\sigma_\mathrm{t} = 0.25\sigma_\mathrm{t}$$

and

$$\frac{\sigma_\mathrm{i}}{Z_1} - \frac{\sigma_\mathrm{r}}{Z_1} = \frac{\sigma_\mathrm{t}}{Z_2}$$

Solving these equations simultaneously gives

$$\frac{\sigma_\mathrm{t}}{\sigma_\mathrm{i}} = \frac{2A_1 Z_2}{A_1 Z_1 + A_2 Z_2} \qquad \text{and} \qquad \frac{\sigma_\mathrm{r}}{\sigma_\mathrm{i}} = \frac{A_2 Z_2 - A_1 Z_1}{A_1 Z_1 + A_2 Z_2}$$

Calculating the mechanical impedances of steel and aluminium and substituting in the above equations leads to

a) $\sigma_\mathrm{t} = +169.1$ MPa.

b) $\sigma_\mathrm{r} = -7.7$ MPa.

Wave propagation in solids

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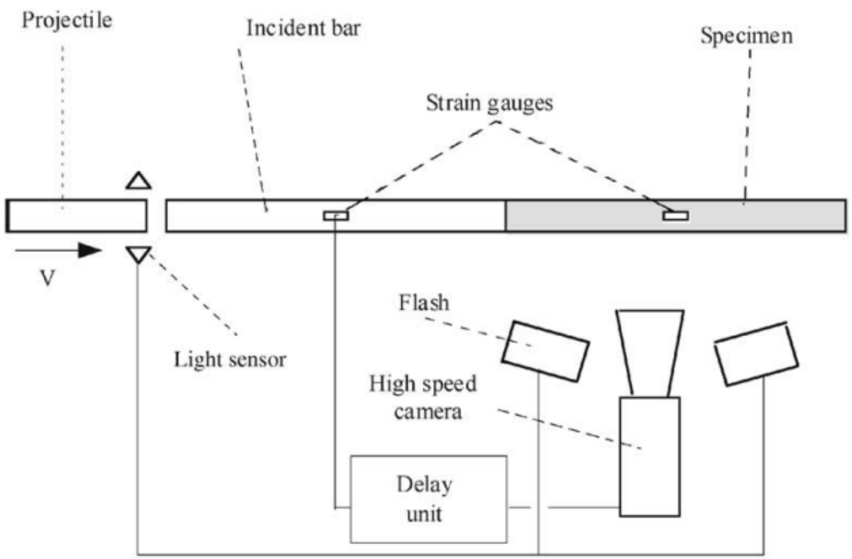
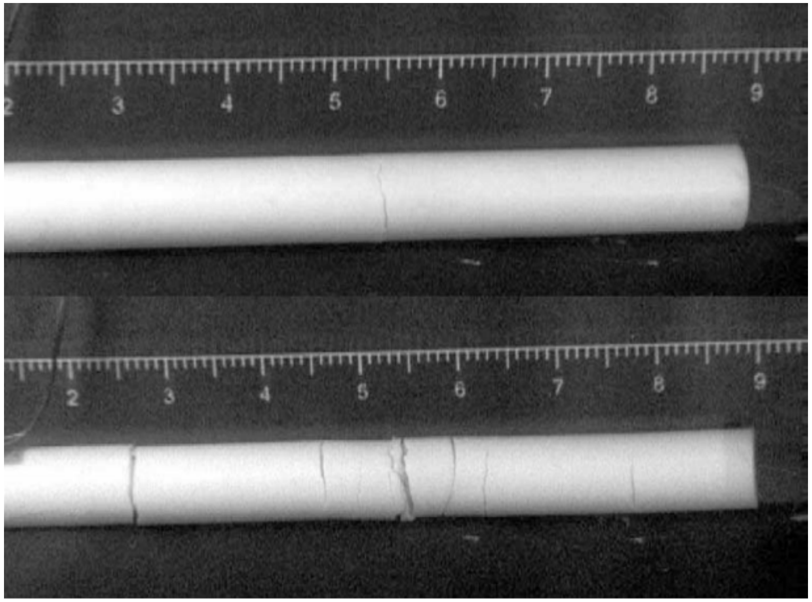
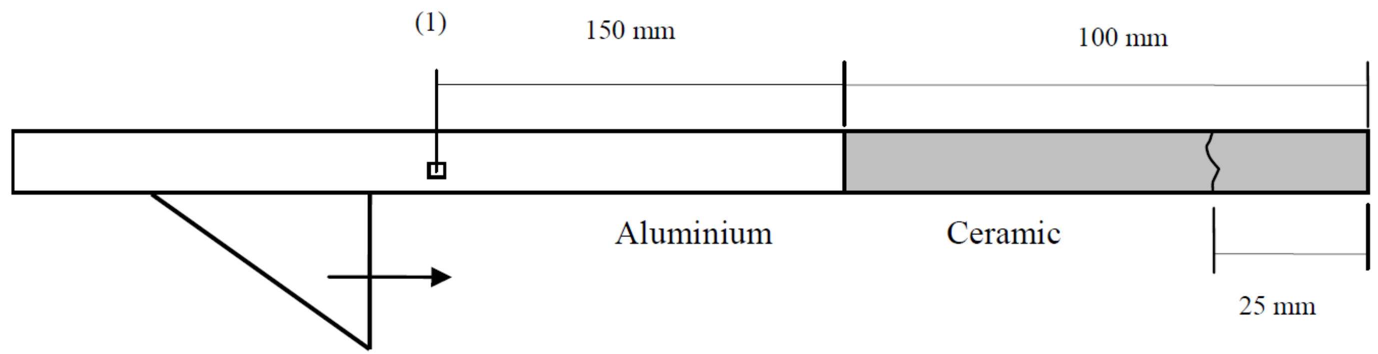




Spalling in Alumina

Exercise 1



A triangular compression wave, with a maximum value of 300MPa and 10μs of duration, travels on a aluminium bar as shown on the figure. The aluminium rod is in contact with a ceramic bar that behaves elastic until failure. This ceramic bar has no damage in compression but fails under tension when a stress value reaches its tensile strength. If the failure appears at 25 mm from the free end, obtain:

a) The time *t* that takes the wave to cross from the point (1) to the failure point. ($T_f = 42.5\ \mu s$)
b) The Lagrange diagram until time *t*
c) The tensile strength of the ceramic material ($\sigma_F \cong 219.15$ MPa)
d) Provide the expression of stress on the ceramic bar in function of time and position

Data:

Aluminium Bar	(E = 70 GPa, ρ = 2800 Kg/m3)
Ceramic Bar	(E = 380 GPa, ρ = 3800 Kg/m3)

Dynamic Properties of Materials

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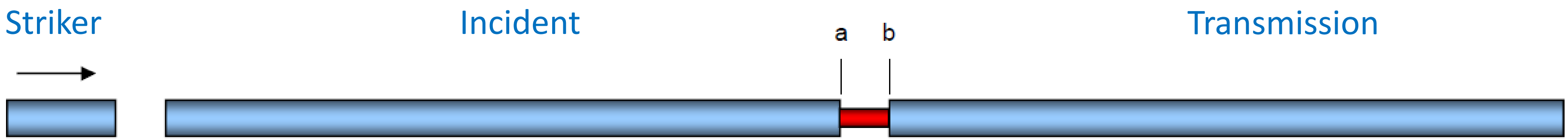
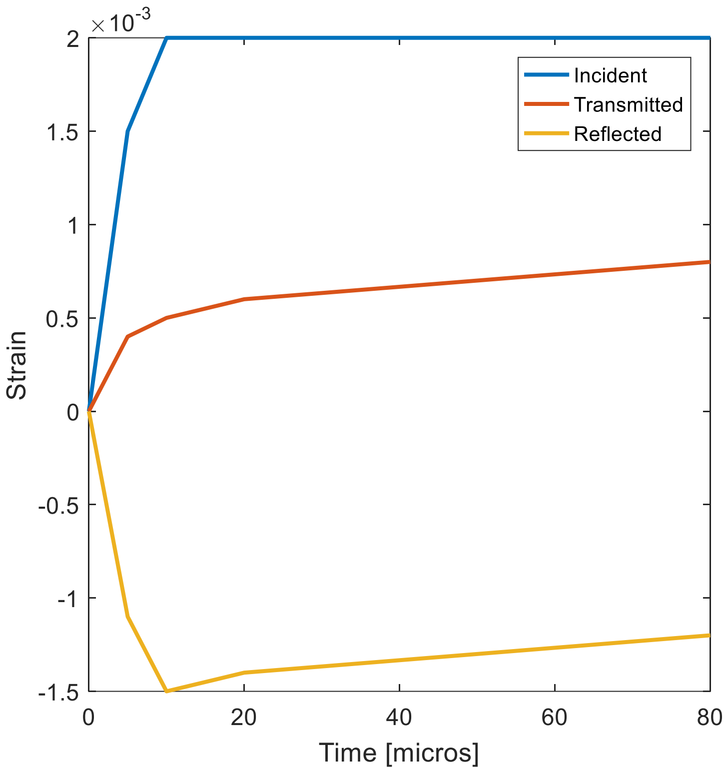
Exercise 1



To obtain the stress-strain curve of a material, a compression test on a Hopkinson bar experiment is performed. The equipment is composed of two steel bars of 20 mm diameter. Specimens have 10 mm in diameter and 10 mm in length. During the tests, incident, reflected and transmitted waves measured are plotted on the figure. Obtain:

a) The velocity of the bar ends in contact with the specimen as function of time
b) The forces on the bar ends
c) The history of the strain rate
d) The adiabatic stress – strain curve
e) The parameters k and n of the constitutive equation considering the expression: $\sigma = k\varepsilon^n$

Exercise 1



Data:

Steel Bar (ρ = 7850 Kg/m3) (E = 203 GPa)

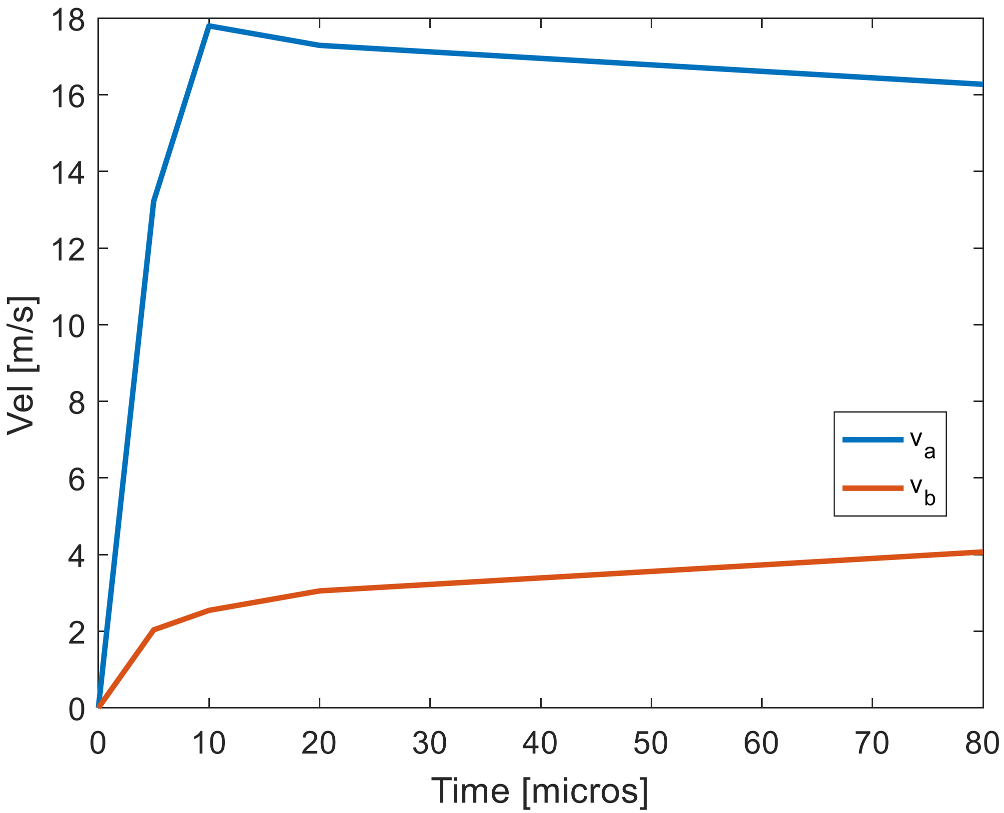
```
t  = [0,    5,   10,   20,   50,   80];          %[micros]
ei = [0,  1.5,    2,    2,    2,    2]*0.001;    %[strain]
et = [0,  0.4,  0.5,  0.6,  0.7,  0.8]*0.001;    %[strain]
er = [0, -1.1, -1.5, -1.4, -1.3, -1.2]*0.001;    %[strain]
```

Exercise 1. A. Velocities

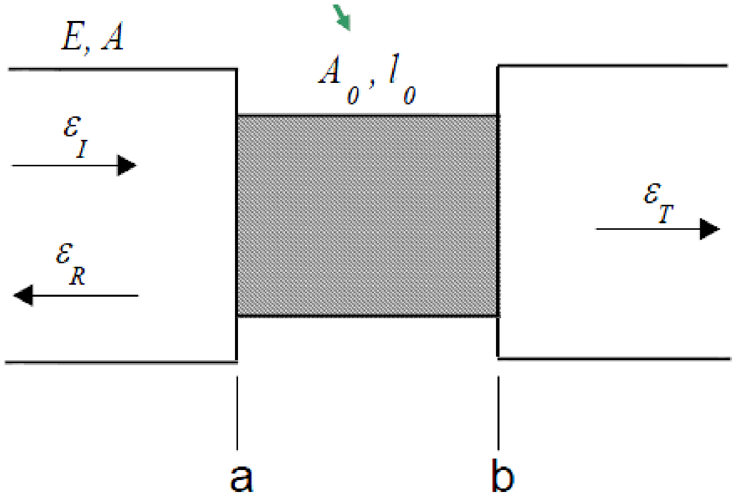
Bar velocities

$$v_a = c_b(\varepsilon_I - \varepsilon_R)$$

$$v_b = c_b \varepsilon_T$$

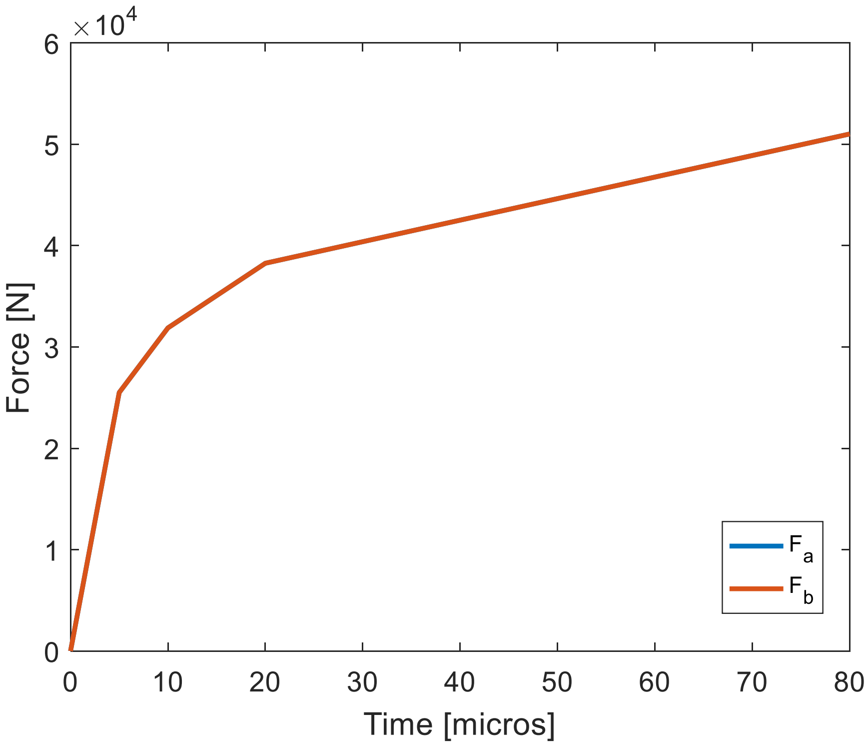


Exercise 1. B. Forces



$$F_a = E\,A(\varepsilon_I + \varepsilon_R)$$

$$F_b = E\,A\;\varepsilon_T$$



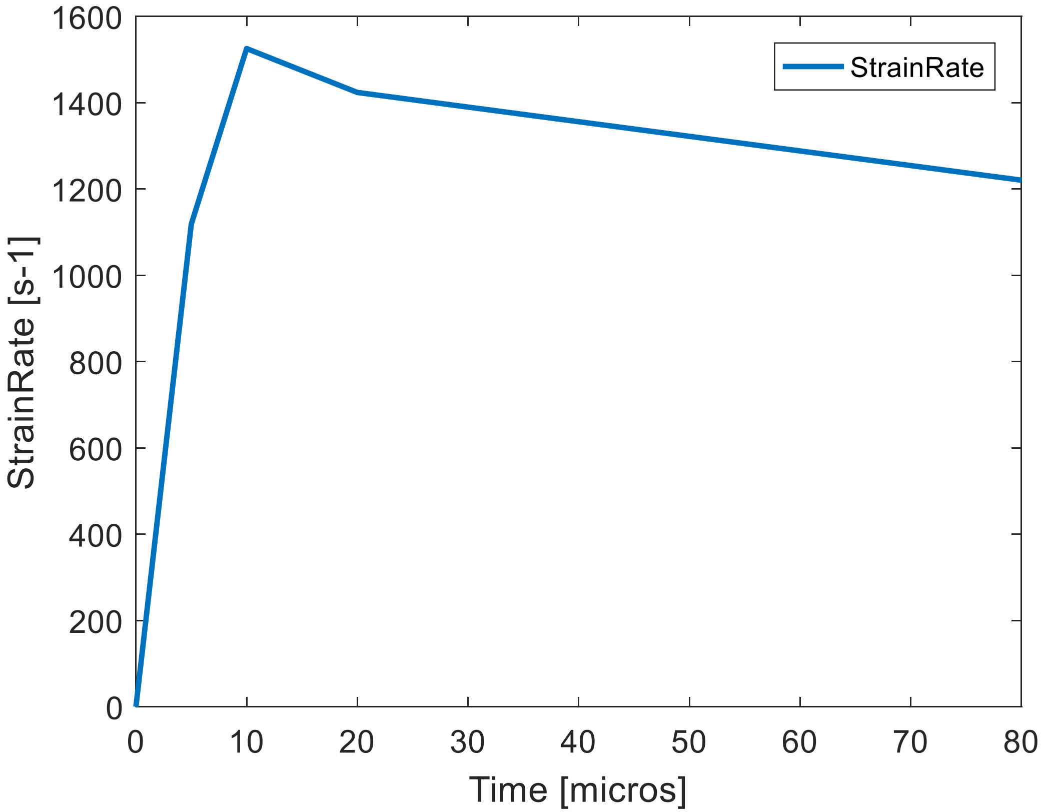
Dynamic force equilibrium in the bars

Exercise 1. C. Strain-rate

$$\dot{\varepsilon} = \frac{v_a - v_b}{l_0} = \frac{c_B}{l_0}(\varepsilon_I - \varepsilon_R - \varepsilon_T)$$

Under dynamic force equilibrium

$$\dot{\varepsilon} = -2\frac{c_B}{l_0}\varepsilon_R$$



Exercise 1. D. Stress-strain curve

Strain needs to be obtained by parts

$$\varepsilon = \int_0^t \dot{\varepsilon} dt = \frac{c_B}{l_0}\int_0^t (\varepsilon_I - \varepsilon_R - \varepsilon_T) dt = -2\frac{c_B}{l_0}\int_0^t \varepsilon_R(t) dt$$

$$\varepsilon_R(t_1, t_2) = kt + cte$$

Constants can be determined considering continuity of the strains

Stress as function of time can be determined by both equivalent equations

$$\sigma_a = \frac{A_B}{A_0} E_b(\varepsilon_I + \varepsilon_R)$$

$$\sigma_b = \frac{A_B}{A_0} E_b \varepsilon_T(t_1, t_2)$$

Exercise 1. D. Stress-strain curve

Strain vs time and stress vs time curves needs to be combined in a stress-strain unique curve

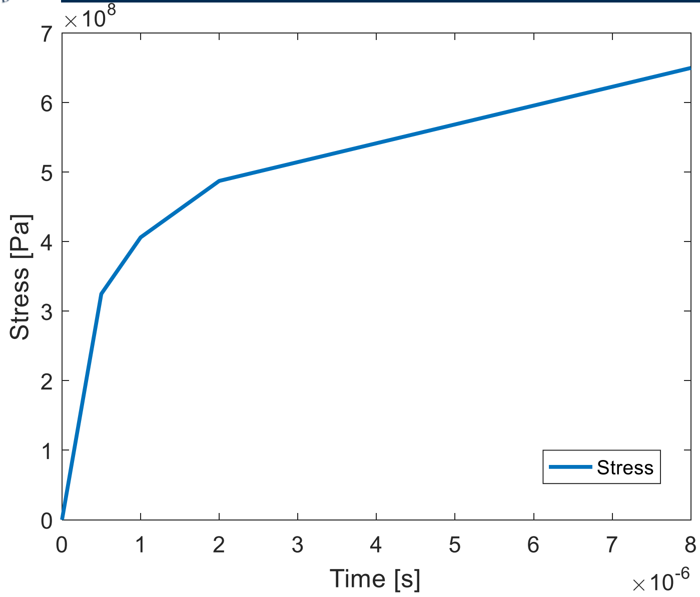
Using the expression strain vs time to rewrite the time as function of strain

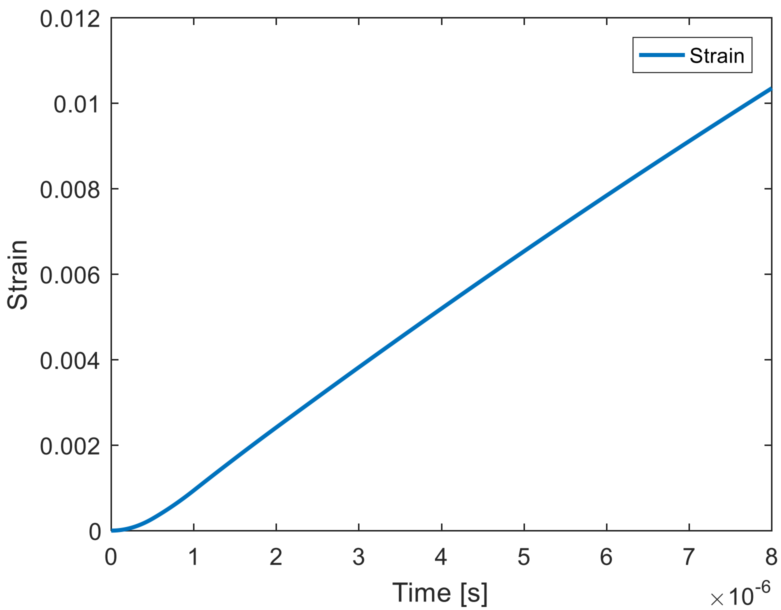
$$\varepsilon(t_1, t_2) \rightarrow t(\varepsilon)$$

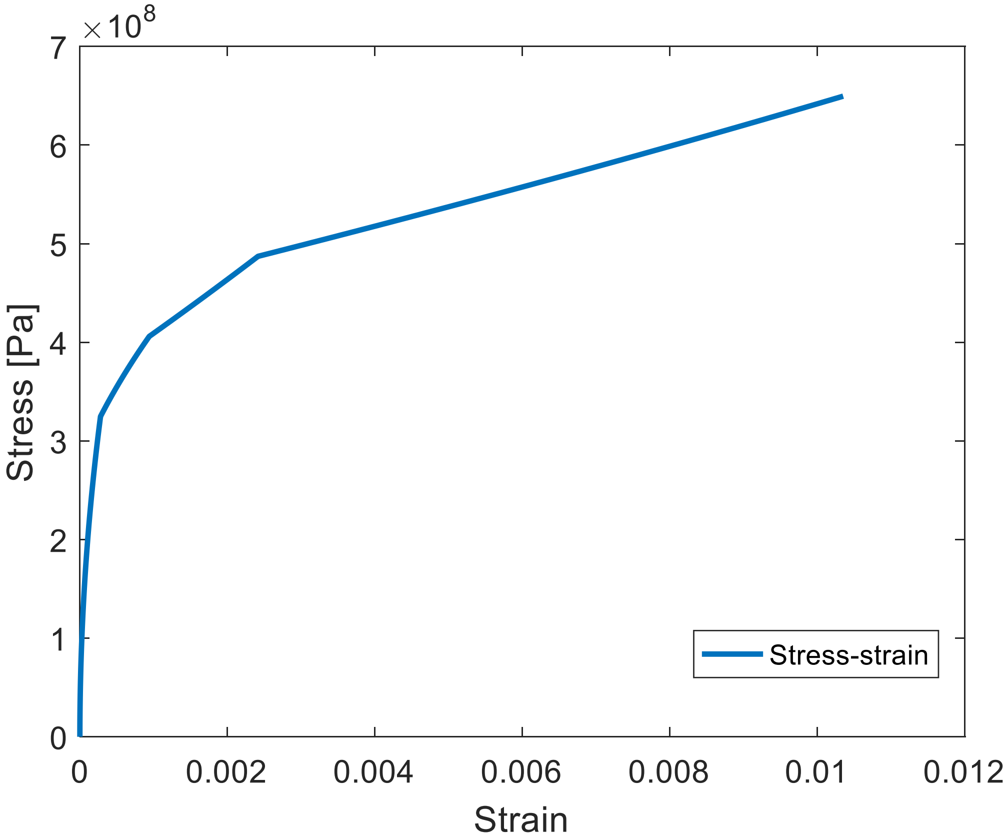
We can substitute in the expression stress vs time, to finally obtain stress vs strain

$$\sigma(t_1, t_2) \rightarrow \sigma(\varepsilon)$$

Exercise 1. D. Stress-strain curve







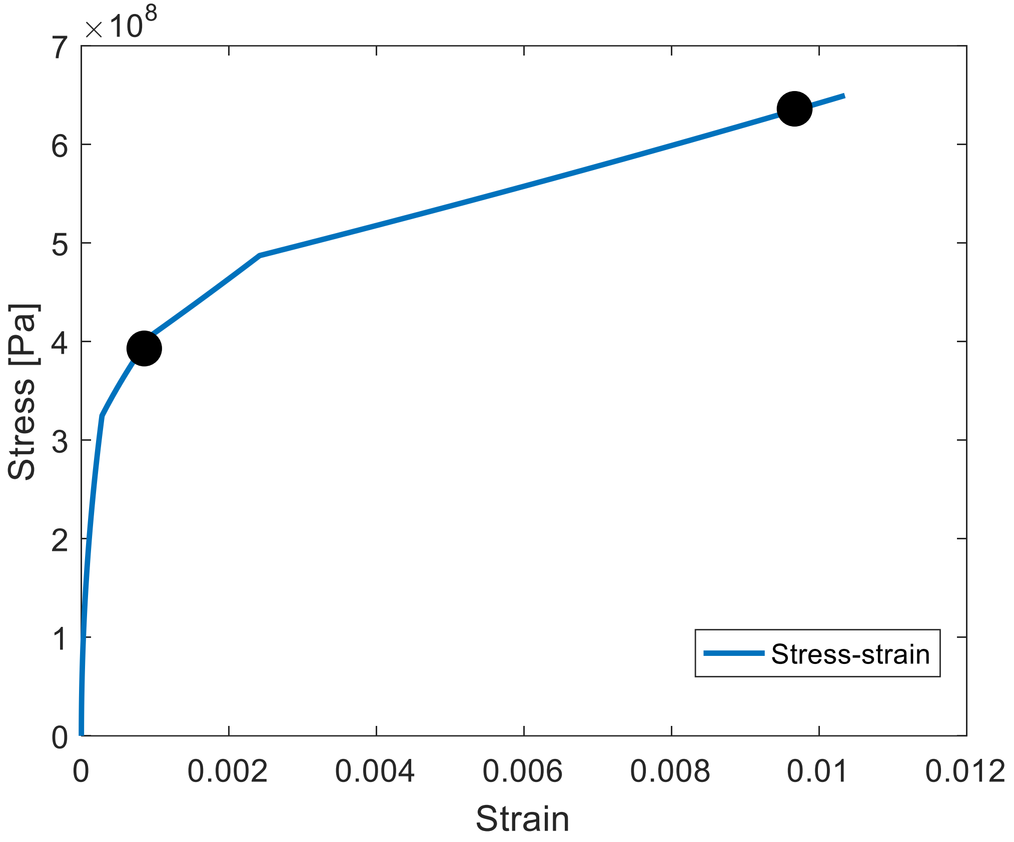
Exercise 1. E. Constitutive model

$$\sigma = k\varepsilon^n$$

Selecting two points for the fitting

$$n = \frac{\log\frac{\sigma(2)}{\sigma(6)}}{\log\frac{\varepsilon(2)}{\varepsilon(6)}}$$

$$k = \frac{\sigma(6)}{\varepsilon(6)^n}$$



Exercise 2

The previous material was also characterised under quasi-static loading conditions resulting in:

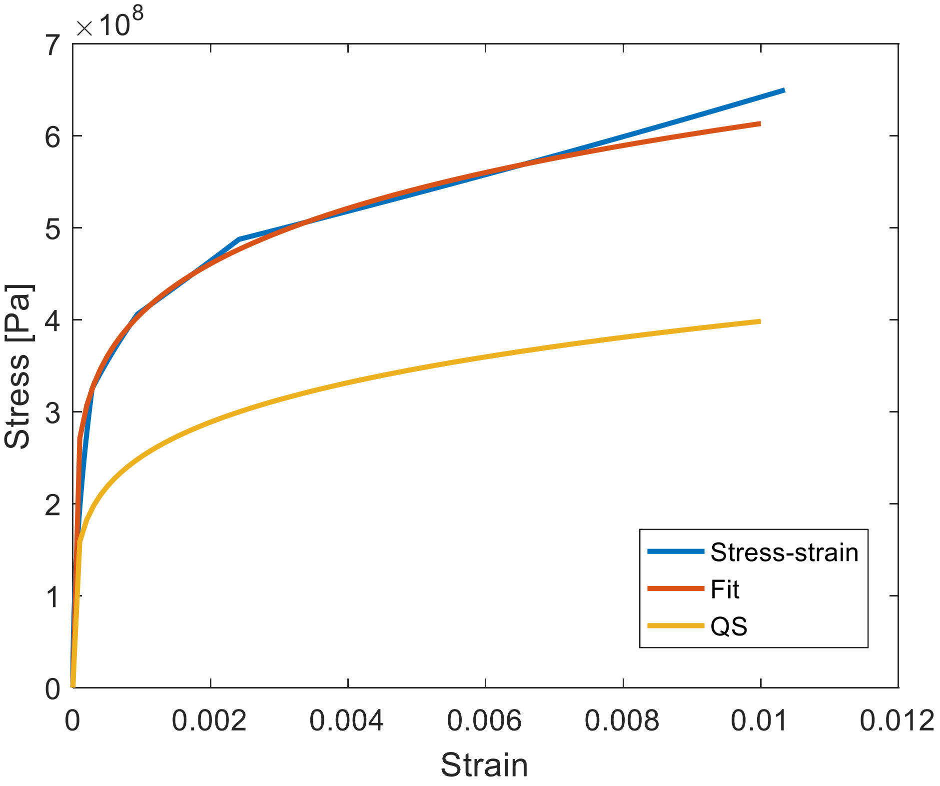
$$\sigma = k\varepsilon^n$$

k = 1 GPa
n = 0.2

Considering the strain rate formulation of the constitutive model is:

$$\sigma = k\varepsilon^n \dot{\varepsilon}^m$$

Find parameter m



Exercise 2

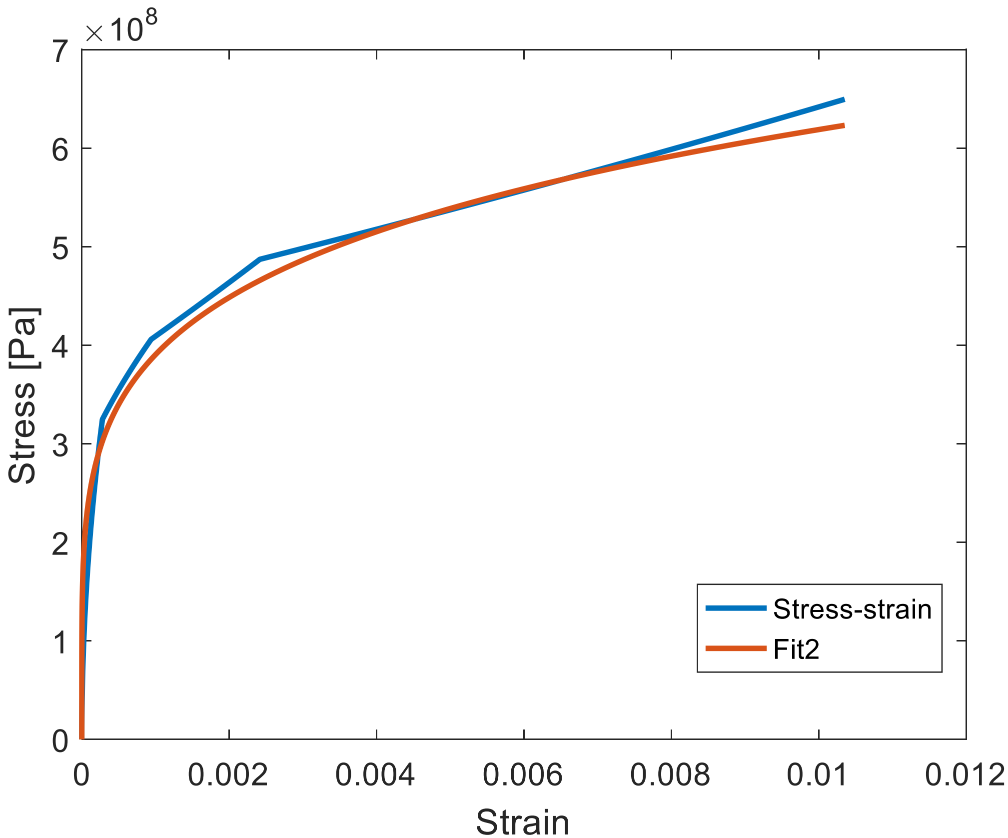
Considering:

$$\sigma_{Dyn} = k_{QS}\varepsilon^{n_{QS}}\dot{\varepsilon^{m}} = k_{Dyn}\varepsilon^{n_{Dyn}}$$

Solving:

$$\dot{\varepsilon^{m}} = \frac{k_{Dyn}}{k_{QS}}\varepsilon^{(n_{Dyn}-n_{QS})}$$

$$m = \frac{log\left(\frac{k_{Dyn}}{k_{QS}}\varepsilon^{(n_{Dyn}-n_{QS})}\right)}{log(\dot{\varepsilon})}$$



Parameter *m* can be study for each point of the curve, or, as an example, we can take the n^{th} point to obtain a good fitting