Advanced Dynamics and Applications 5

Dr Francisca Martínez-Hergueta

Francisca.mhergueta@ed.ac.uk

William Rankine Building, WR 1.19

Dr David Garcia Cava

david.garcia@ed.ac.uk

William Rankine Building, WR 2.20

This presentation includes:

1. General information about the course
2. Learning outcomes
3. Course material and resources
4. Assessment
5. Feedback
6. Planning

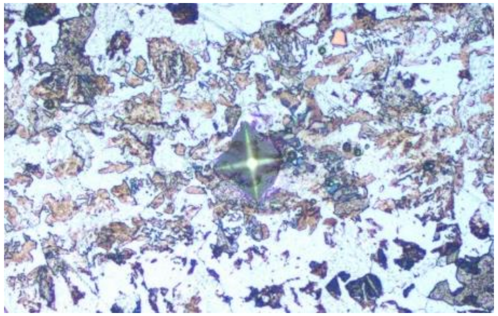
Welcome!

Dr. Francisca Martínez Hergueta
Impulsive Dynamics

Francisca.mhergueta@ed.ac.uk

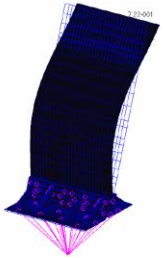
WR 1.19 William Rankine Building

2008
Institute for Regional Development



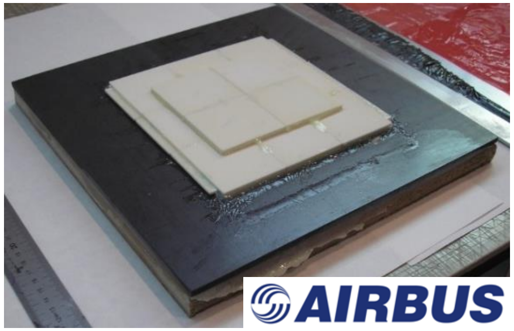
Mechanical characterization of microalloyed steels

2009-2010
National Institute for Aerospace Technology



Design of composite structures for aerospace

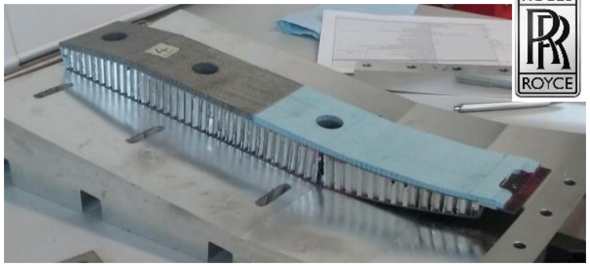
2011-2016
IMDEA Materials Institute



AIRBUS

Advanced composite materials for impact

2016-2018
Impact Laboratory University of Oxford



Certification of composites for aerospace

Welcome!

Dr David Garcia Cava

David.garcia@ed.ac.uk
WR 2.20 William Rankine Building

4 yrs in industry
5 yrs as researcher
7 yrs as academic

Expertise: →

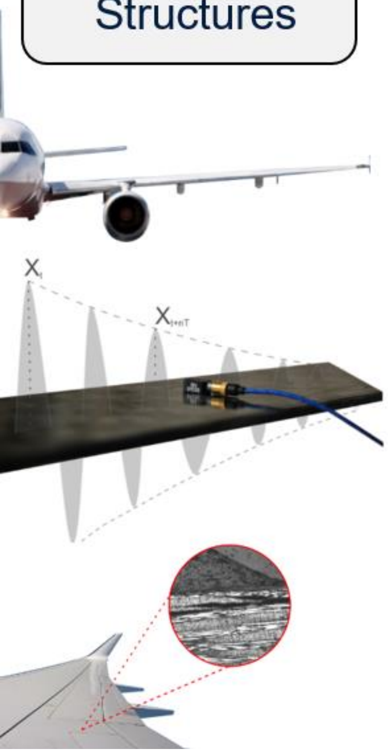
- Structural dynamics
- Engineering vibrations
- Structural health monitoring
- Uncertainty quantification

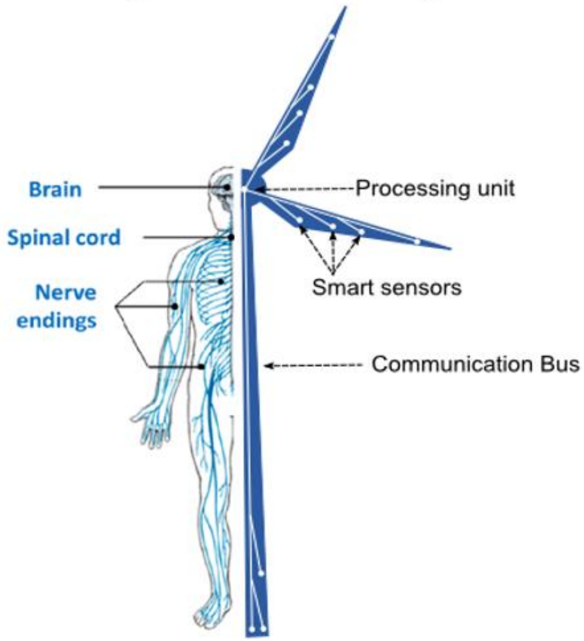
Composite Structures

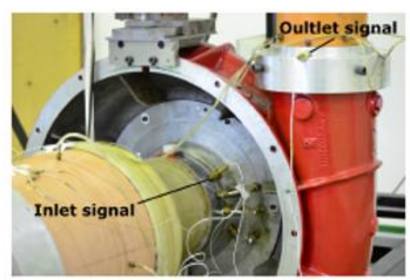
Wind Energy

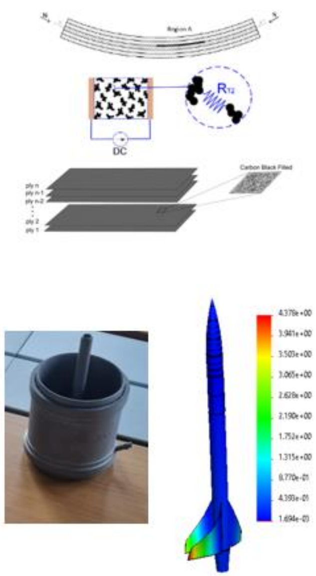
Rotating Machinery

Advanced Materials









Surgery/Office hours

Dr Francisca Martínez Hergueta

Lecturer

Surgery/Office hour: Thursday 12 am

Please write an email to book an appointment

Francisca.mhergueta@ed.ac.uk

Dr David Garcia Cava

Senior Lecturer

Surgery/Office hour: Thursday 12 am

Please write an email to book an appointment

david.garcia@ed.ac.uk

Course Information

Lectures: ASH_Lecture Theatre 3, Ashworth Labs

Monday 12-1 pm

Wednesday 12-1 pm

Tutorials: Alrick Building, Classroom 14

Friday 12-1 pm

Starting from week 2!!!!

Learning Outcomes

On completion of this course, you will be able to:

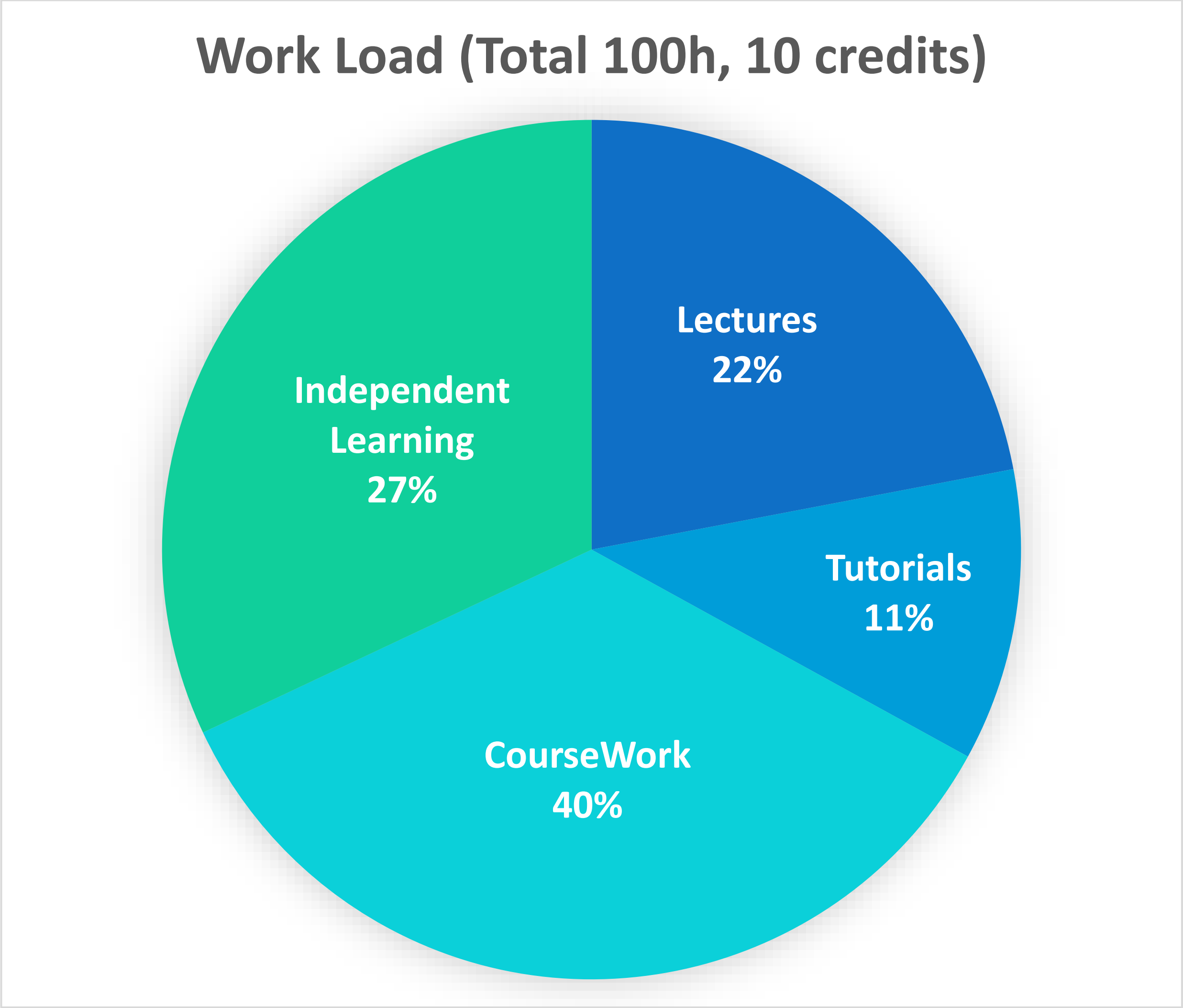
1. Knowledge of vibration analysis of mechanical systems in time and frequency domain, and their applications.
2. Knowledge of the principles of modal analysis as well as how to implement them for different engineering applications.
3. Understand wave propagation and interaction, and its effect on the behaviour of solid materials.
4. Select appropriate materials for impact applications.
5. Analyse simple structures subjected to impulsive loads.

Course Material

Everything available in Learn:

1. Lecture notes & Slides.
2. Demonstrations & Worked examples.
3. Tutorials and solutions.
4. Reading list / Chapter books.

Work Load


Work Load (Total 100h, 10 credits)
Lectures
22%
Tutorials
11%
CourseWork
40%
Independent
Learning
27%

Assessment

60% exam

- 3 questions (2 hours)
- Open book (bring anything you like)

40% Course Work

- Groups of 3 students, 40 hours per student
- Poster presentation (Voluntary). Week 5 (16th of February)
- See information on Coursework for more details:
    - Part I - Technical review report (20%)
    - Part II - Technical simulation report (20%)

COURSEWORK DEADLINE: Tuesday, 19th March 2024 at 16:00

Feedback

1. Self-assessment exercises
2. Tutorials
3. Report feedback
4. WooClap Questionnaire
5. Revision week
6. **Surgery/Office hour**

Planning

Week	Lesson
1	Introduction & key concepts / Modelling - Frequency Response Function
2	State-space form representation / Experimental spectral analysis
3	Principles for modal testing & techniques / Operational Modal Analysis
4	Stochastic Subspace Identification / Engineering applications
5	Design considerations for impact loads
6	Elastic wave propagation
7	Plastic waves
8	Shock waves
9	Terminal ballistics & Perforation mechanics
10	Soft body armour
11	Revision Week

Week 1 to 4 → Advanced Dynamics

Week 5 to 10 → Impulsive Dynamics

Tutorials will be running from week 2 to 11

Feedback... to us!

Check the material uploaded, and send suggestions NOW!!

Week 1 - 4 to david.garcia@ed.ac.uk

Week 5 - 10 to Francisca.mhergueta@ed.ac.uk

If you have a schedule of adjustments, please get in touch!

Advanced Dynamics and Applications 5

Dr Francisca Martínez-Hergueta

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William Rankine Building, 1.19

&

Dr David Garcia Cava

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Introduction to Part I of Advanced Dynamics for ADA5

MECE11014 - Advanced Dynamics and Applications 5

Dr David García Cava

- General Overview of the Part I - Advanced Dynamics for ADA5
- General Overview of the coursework topics

Contact details

David García Cava

- Office: William Rankine Building WR 2.20
- Email: david.garcia@ed.ac.uk
- Surgery hours/office hour: Thursdays at 12:00 mid-day

Why this course is so fascinating!

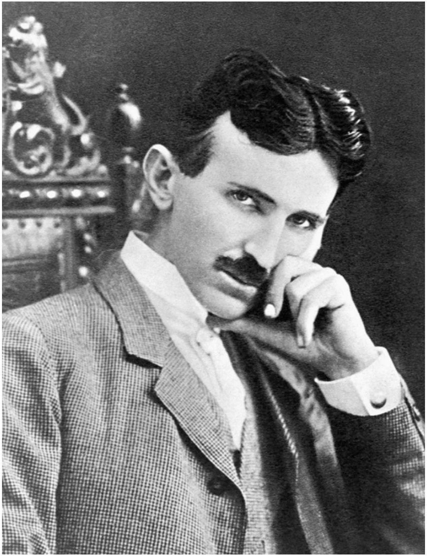


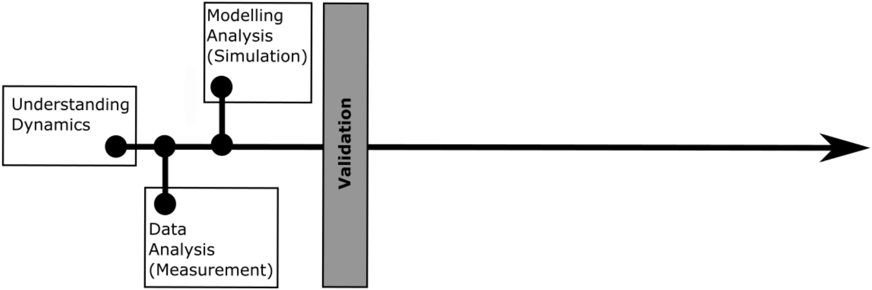
Figure 1: Nikola Testa (1856-1943)

"If you want to find the secrets of the universe, think in terms of energy, frequency and vibration"

Block I - Timetable

Month	Mo	Tu	We	Th	Fr	Sa	Su	Wk	Block	Lecture I	Lecture II	Tutorial
Jan-24	8	9	10	11	12	13	14	Welcome week				
	15	16	17	18	19	20	21	1 - Block 1 starts	Block I	Intro + Coursework	Lecture I (Freq. + Time)	--
	22	23	24	25	26	27	28	2	Block I	Lecture I (Freq. + Time)	Lecture II - A (Spectral)	Tutorial LI
	29	30	31	1	2	3	4	3	Block I	Lecture II - B (EMA)	Lecture III - A (SS - Modal)	Tutorial LII
Feb-24	5	6	7	8	9	10	11	4	Block I	Lecture III - B (SSI)	Lecture IV - A (Applications)	Tutorial LIII
	12	13	14	15	16	17	18	5 - Block 1 ends	Block II			Feedback
	19	20	21	22	23	24	25	Flexible week				
	26	27	28	29	1	2	3	6	Block II			
Mar-24	4	5	6	7	8	9	10	7	Block II			
	11	12	13	14	15	16	17	8	Block II			
	18	19	20	21	22	23	24	9	Block II	**Submission at 16:00**		
	25	26	27	28	29	30	31	10	Block II			
Apr-24	1	2	3	4	5	6	7	11	Revision			
	8	9	10	11	12	13	14	Vacation (12)				
	15	16	17	18	19	20	21	Vacation (13)				
	22	23	24	25	26	27	28	Revision (14)				
	29	30	1	2	3	4	5	Exams starts (15)				

WHY? - The Big Picture



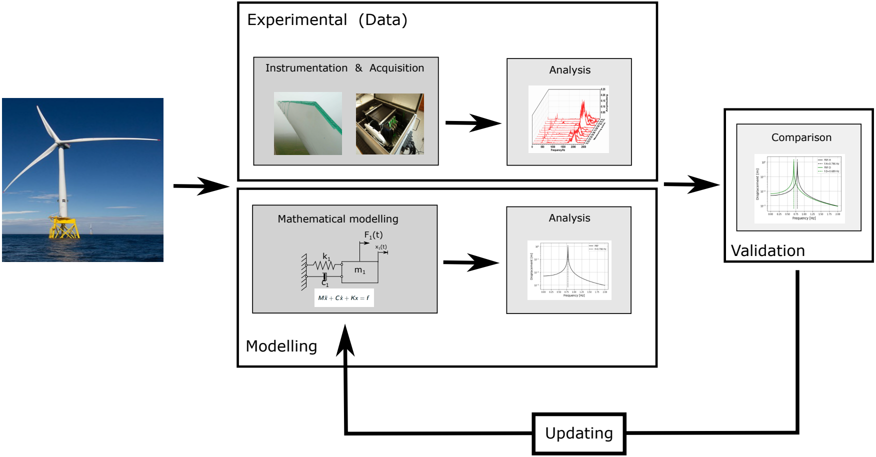
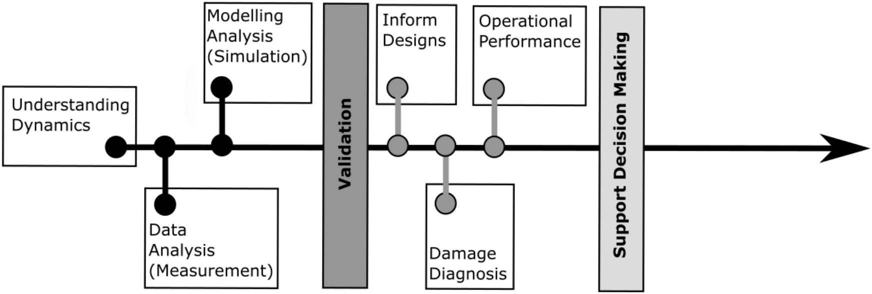


Figure 2: Graphical representation of structural dynamics-modelling approach

WHY? - The Big Picture



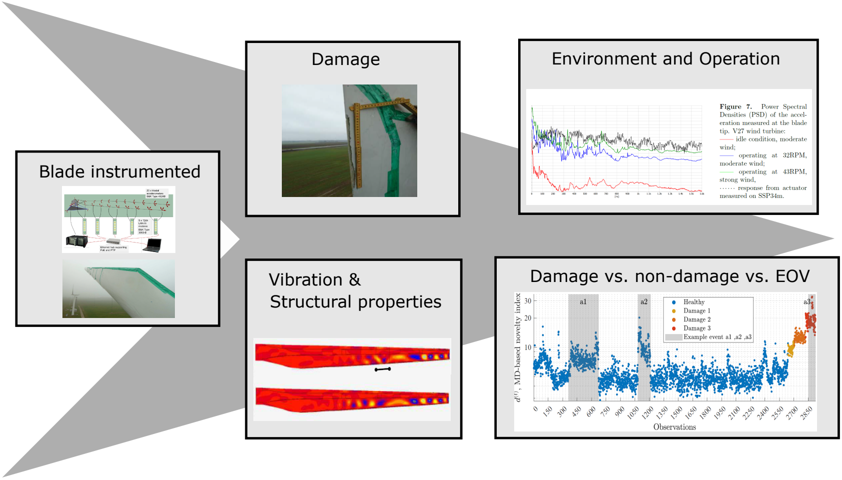
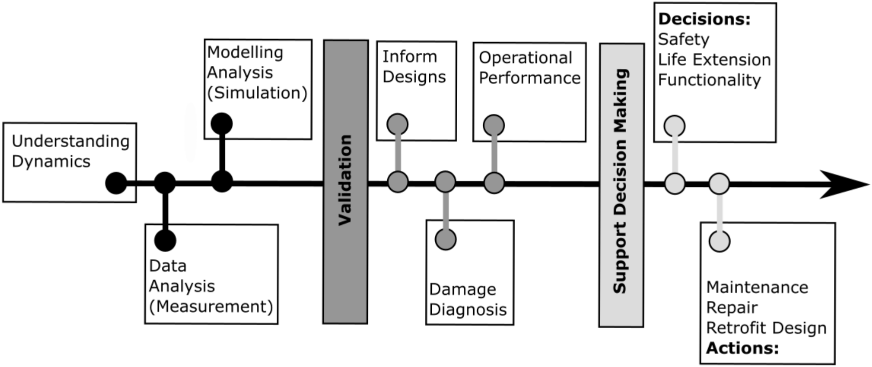


Figure 3: VSHM principle

WHY? - The Big Picture



WHY? - Decision support

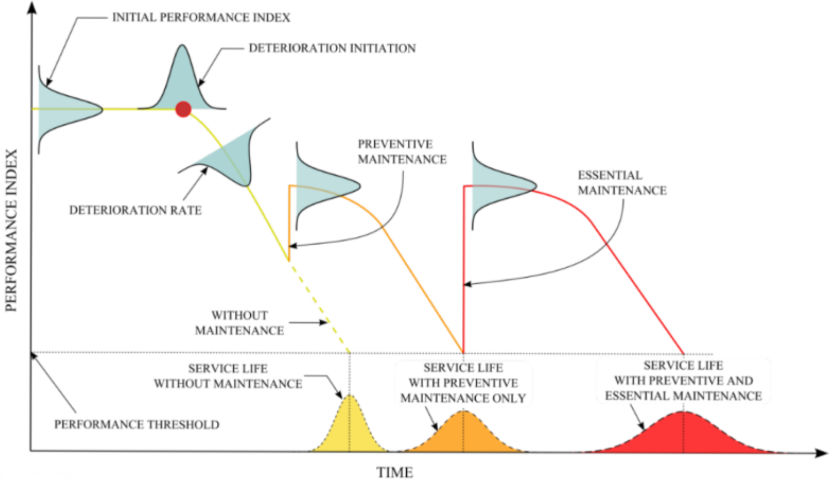
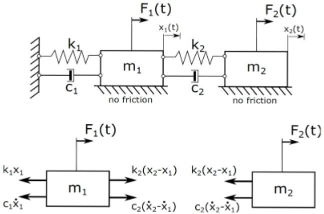
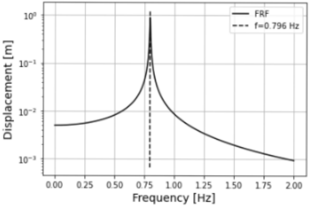


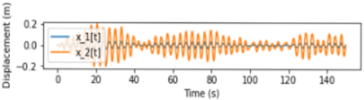
Figure 4: Decision Support. Source: ETH Zurich

BLOCK I
Modelling in Time and Frequency domain

- How to model dynamics using Laplace Transform modelling (Frequency domain)
- State-Space form representation (Time domain)

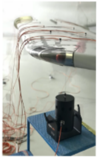


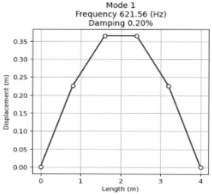


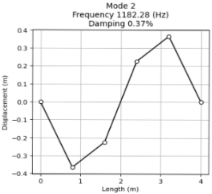


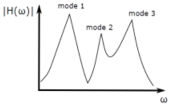
BLOCK II
Experimental spectral analysis. Modal testing

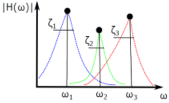
- Spectral types and analysis methods
- Modal identification: Experimental modal analysis

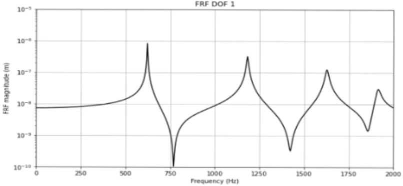










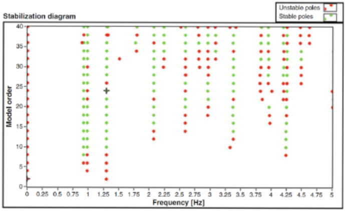


BLOCK III
Operational Modal Analysis. Stochastic Subspace Identification

- Fundamentals of Operational Modal Analysis (OMA)
- Output-only system identification. Stochastic process
- Stochastic Subspace Identification (SSI)

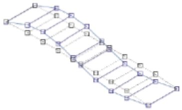
$$\boldsymbol{z}_{k+1} = \boldsymbol{A}\boldsymbol{z}_k + \boldsymbol{w}_k$$

$$\boldsymbol{x}_k = \boldsymbol{C}\boldsymbol{z}_k + \boldsymbol{v}_k$$

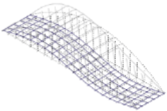


BLOCK IV
Engineering applications and Challenges

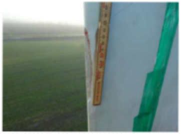
- Model updating
- Damage identification (Structural Health Monitoring (SHM))
- Environmental and Operational Variability (EOV)

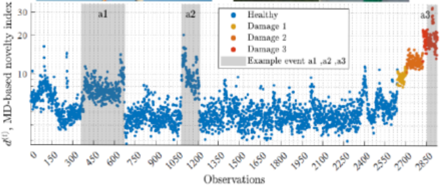


Mode 1
$f_1^{FDD} = 3.564$ Hz



Mode 1 (**V1**)
$f_1^{\mathrm{bm}} = 3.785$ Hz





The approach

- Setting up the theoretical aspects
- Tutorial exercises to set-up fundamentals
- Coding the content
- Discussing the applications and beyond

Delivery

- Lectures
- Tutorials
- Surgery hours
- Working in groups (Peer-support)
- Learn forums

DEADLINE
Tuesday, 19th March 2024 at 16:00.

Read carefully **ADA5 - Coursework for Academic year 2023-2024** in Learn.

The coursework counts for 40% of the total grade of the course and consists of two parts:

- Part I - Technical review report (20%)
- Part II - Technical simulation report (20%)

You should work in groups of 3 people.

Note: Each student group should select a topic sending an email with all the group components to David García Cava via eamil david.garcia@ed.ac.uk and Francisca Martínez Hergueta via email francisca.mhergueta@ed.ac.uk by the end of week 2 - **Friday, 26th January 2024**. Topics will be assigned in a first comes first serves basis, and the list of taken projects will be published in the board. The material for the report will be provided (1 scientific publication and book chapter per group).

Coursework - Part I: Topics Overview

For **Part II** just read the ADA5 - Coursework for Academic year 2023-2024 in Learn and use the theory as well as the codes provided during the course. The codes will serve as a starting point to answer the questions.

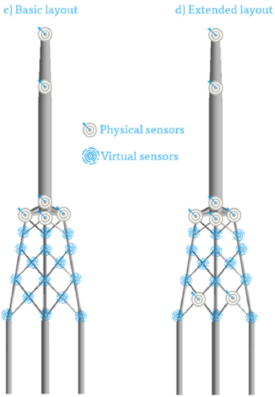
For **Part I**. For all topics:

- Require theoretical understating of concepts covered in the course
- New innovations will be presented, it requires the team to document themselves beyond the course content
- The groups will receive a Journal paper and a book chapter
- More or less all topics are related through the content of the course. The book chapter is to support the journal paper.
- A book chapter will be also provided to support the Topic on the Environmental and operational variabilites (EOV). If your topic will not directly address the EOV, you should reflect on how a EOV mitigation technique could be applied in the context of your journal paper.
- A *voluntary* feedback session will be hold during the tutorial of week 5. Each group could make a poster about the Part I (only) of their topics addressing the requirements for the coursework. Formative feedback will be provided.

A short summary of the topics is as follows

Topic 1 - Modal Analysis for virtual sensing in offshore wind jacket substructures

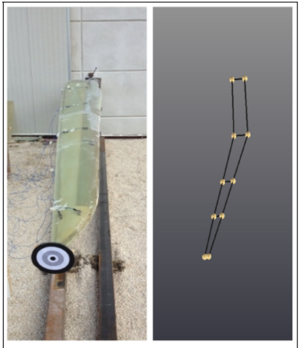
Citation: Augustyn, Dawid, et al. "Feasibility of modal expansion for virtual sensing in offshore wind jacket substructures." Marine Structures 79 (2021): 103019.



The idea:
How Operational Modal Analysis could help to support virtual sensing technique in offshore structures

Topic 2 - Damage detection using OMA

Citation: Lorenzo, Emilio Di, et al. "Damage detection in wind turbine blades by using operational modal analysis." Structural Health Monitoring 15.3 (2016): 289-301.



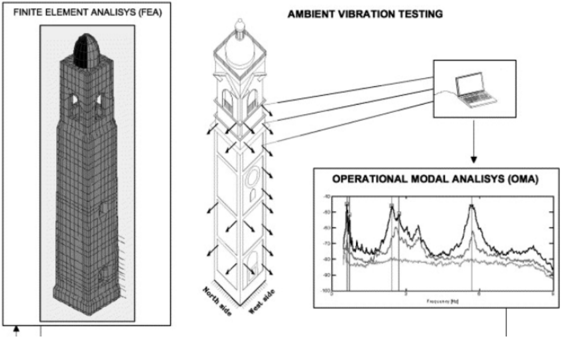
(a) (b)

The idea:

Can Operational Modal Analysis serve as good technique to detect damage in wind turbine blades?

Topic 3 - System identification and damage detection of historical buildings

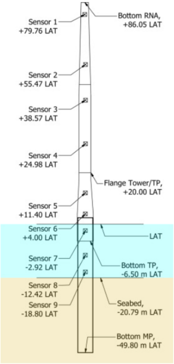
Citation: Gentile, Carmelo, and Antonella Saisi. "Ambient vibration testing of historic masonry towers for structural identification and damage assessment." Construction and building materials 21.6 (2007): 1311-1321.



The idea:
Can Operational Modal Analysis serve as good technique to detect damage in historical buildings?

Topic 4 - System identification in operational wind offshore wind turbines

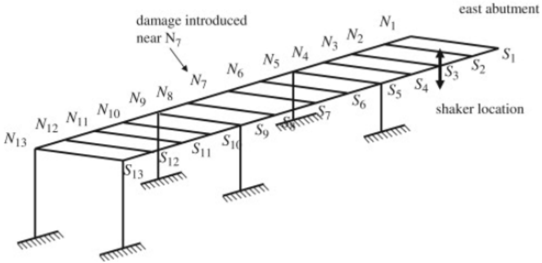
Citation: Kjeld, Jonas Gad, et al. "Towards minimal empirical uncertainty bounds of damping estimates of an offshore wind turbine in idling conditions." Mechanical Systems and Signal Processing 191 (2023): 110180.



The idea:
Can an automated Operational Modal Analysis be build to estimate modal parameters (i.e. damping) in offshore wind turbines for long-term monitoring?

Topic 5 - Frequency response function under EOVs mitigation in a Bridge application I-40

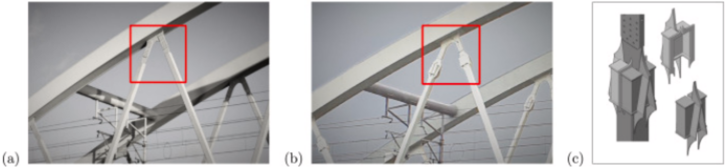
Citation: Limongelli, M. P. "Frequency response function interpolation for damage detection under changing environment." Mechanical Systems and Signal Processing 24.8 (2010): 2898-2913.



The idea:
How mitigate the effect of Environmental and operational variabilities in the estimated Frequency response function for Bridge applications?

Topic 6 - Structural health monitoring on retrofitted railway bridge KW51 with Environmental and operational variabilities mitigation

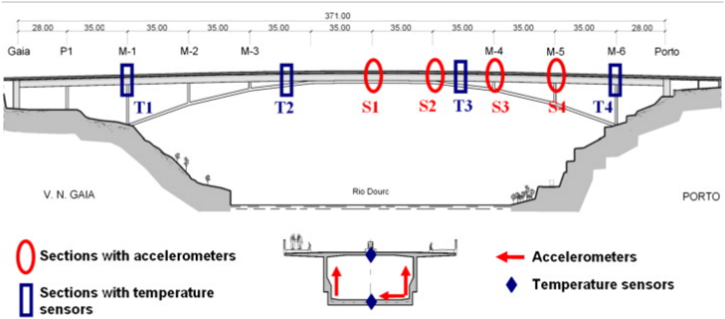
Citation: Maes, K., et al. "Validation of vibration-based structural health monitoring on retrofitted railway bridge KW51." Mechanical Systems and Signal Processing 165 (2022): 108380.



The idea:
How mitigate the effect of Environmental and operational variabilities in a Structural health monitoring can be performed in a bridge that has been retrofitted?

Topic 7 - An automated Operational modal analysis for Bridge monitoring

Citation: Magalhães, Filipe, Álvaro Cunha, and Elsa Caetano. "Vibration based structural health monitoring of an arch bridge: From automated OMA to damage detection." Mechanical Systems and signal processing 28 (2012): 212-228.



The idea:
How automated operational modal analysis can be used for-long term monitoring of arch bridges?

Topic 8 - Second-Order Blind Identification for Environmental and operational variability mitigation in bridges

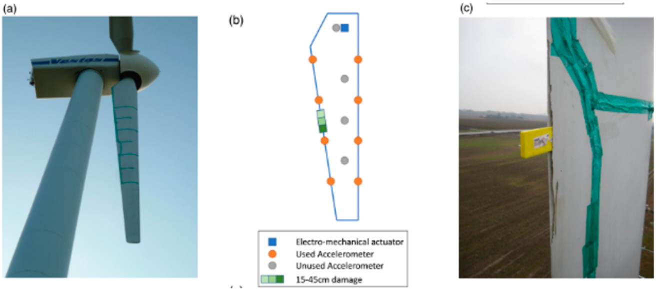
Citation: Rainieri, Carlo, et al. "Predicting the variability of natural frequencies and its causes by Second-Order Blind Identification." Structural Health Monitoring 18.2 (2019): 486-507.

(a) (b)

The idea:
How to automatically identify modal parameters to mitigate environmental and operational variability in civil structures?

Topic 9 - **Mitigation of environmental and operational variability via non-linear regression in Wind Turbine blades**

Citation: Roberts, Callum, David Garcia Cava, and Luis D. Avendaño-Valencia. "Addressing practicalities in multivariate nonlinear regression for mitigating environmental and operational variations." Structural Health Monitoring 22.2 (2023): 1237-1255



The idea:
How to mitigate multiple environmental and operational parameters for robust Structural health monitoring applications.

Coursework - Part I: Topics Overview

Topic 10 - Damage Identification by Vibration Monitoring Using Optical Fiber Strain Sensors

Citation: Reynders, Edwin, et al. "Damage identification on the Tilff Bridge by vibration monitoring using optical fiber strain sensors." Journal of engineering mechanics 133.2 (2007): 185-193.

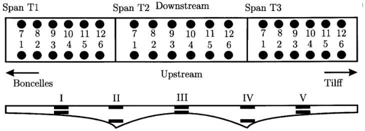


Fig. 2. Bridge of Tilff: top view with location of accelerometers (circles) and side view with location of optical fibers (rectangles)

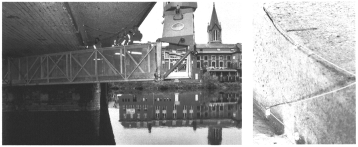


Fig. 3. Two optical fiber strain sensors are attached to bridge at each of five sections

The idea:

Can Operational modal analysis and optical fiber strain sensors be used to detect damage in bridges?

Impulsive Dynamics Summary

Dr Francisca Martínez-Hergueta

Francisca.mhergueta@ed.ac.uk

William Rankine Building, 1.19

This presentation contains:

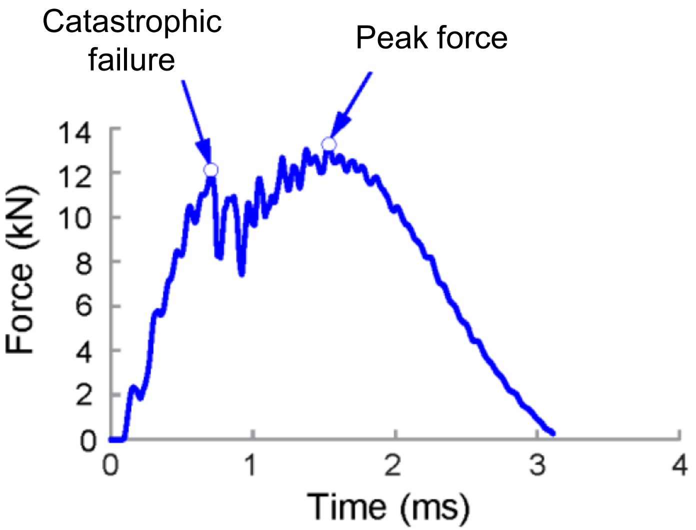
1. Definition of impulsive load
2. Weekly summary

Impulsive Load Definition

Impulse definition

$$I = \int_{t_1}^{t_2} \vec{F} \cdot dt$$

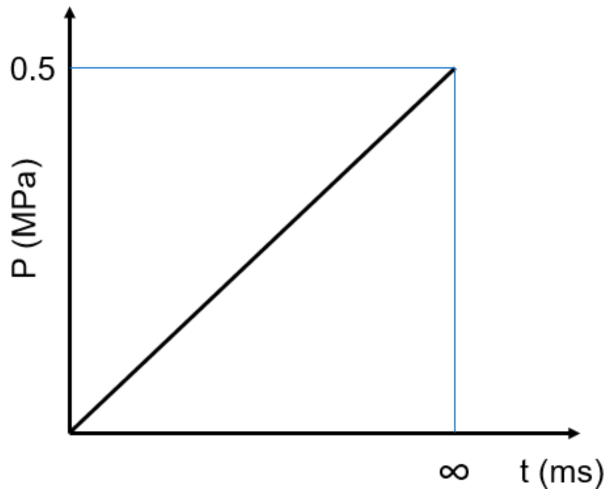
$$I = F\Delta t = m\Delta v$$

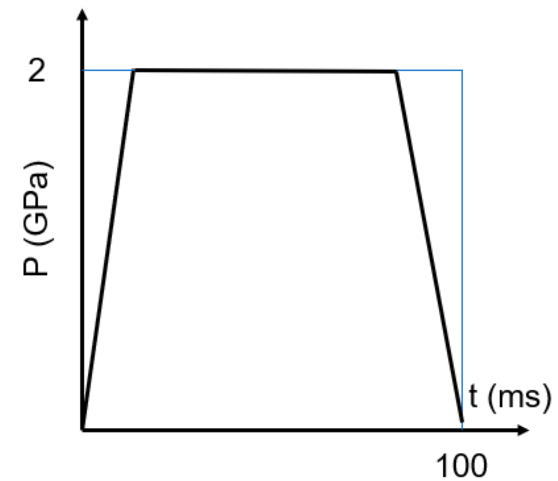


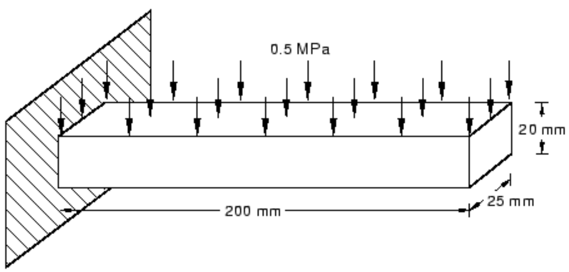
We call impulsive forces certain very large forces of very short duration. Includes impact and explosions.

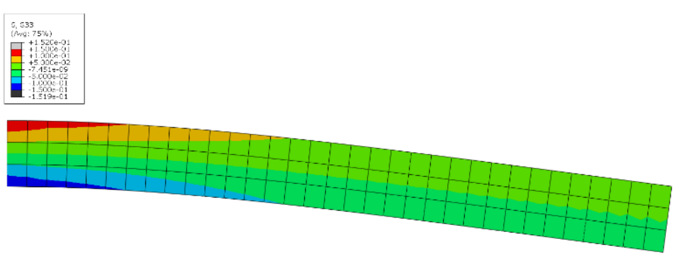
Quasi-static vs Dynamic load

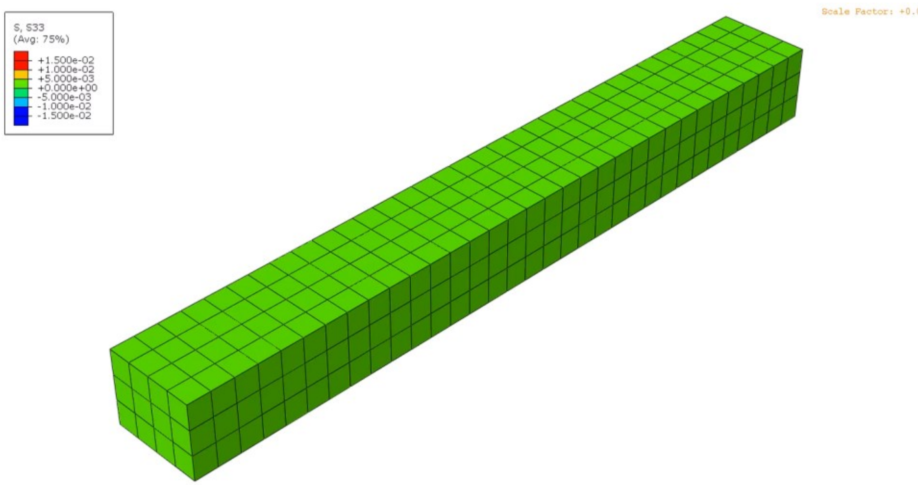
Quasi-static load *vs* Impulsive Load



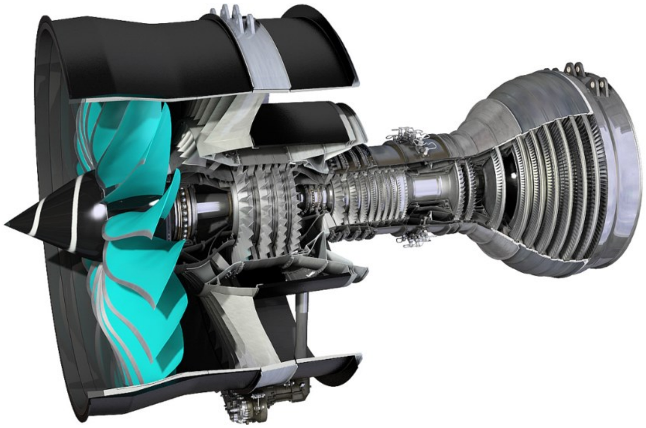






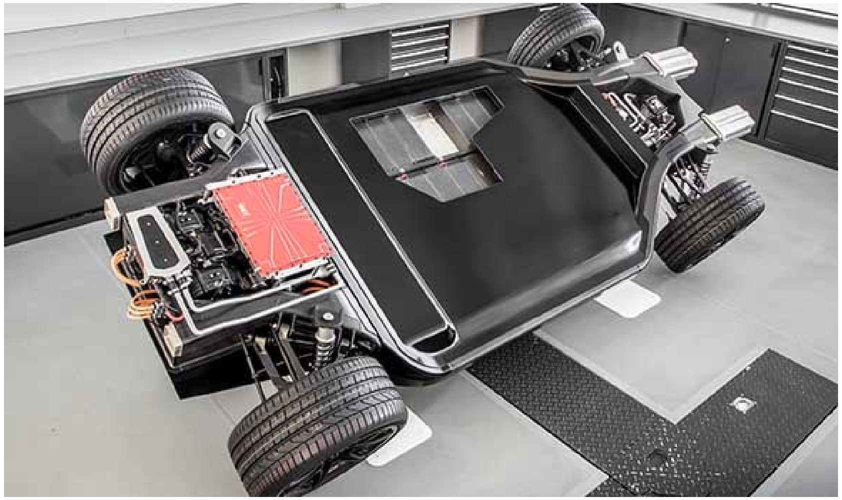


Week 5. Design criteria for different industry applications



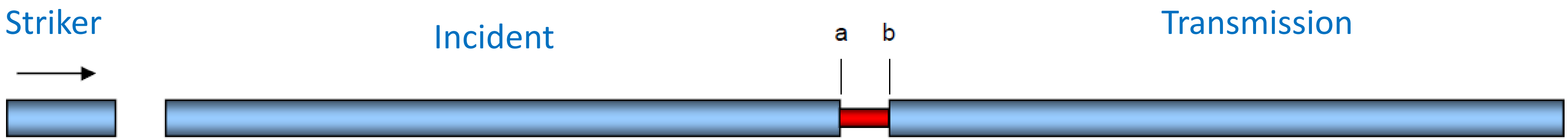
Aeronautics

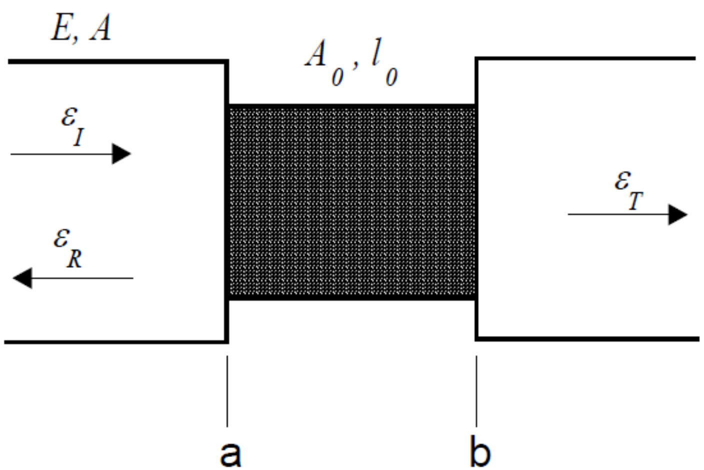
Space



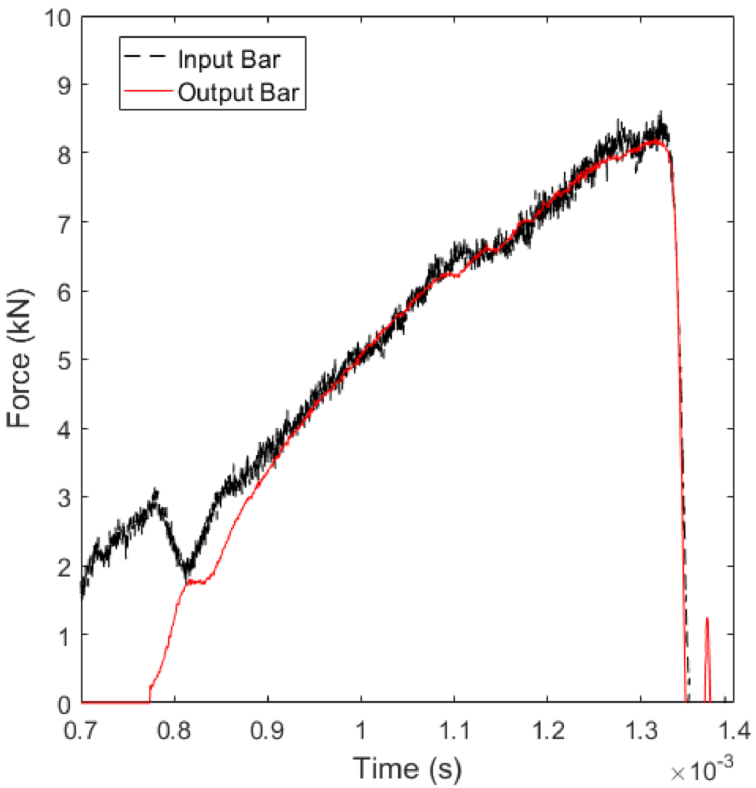
Automotive

Civil

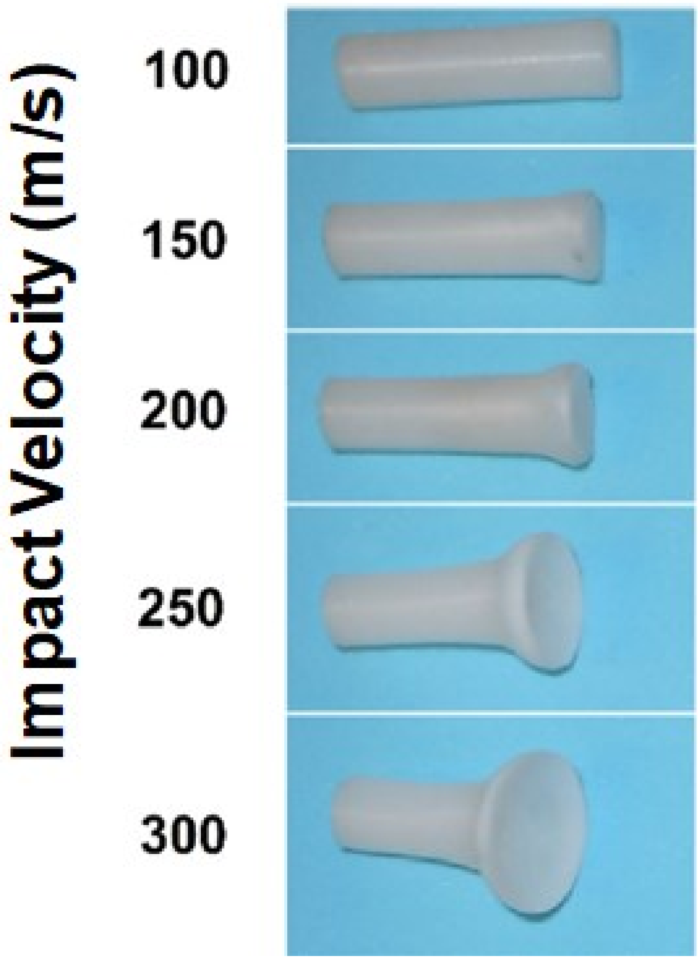




- Wave propagation velocity, *c*
- Relationship between impact velocity and stresses
- Material impedance
- Boundary problems, transmitted and reflected waves
- Split Hopkinson bar experiment



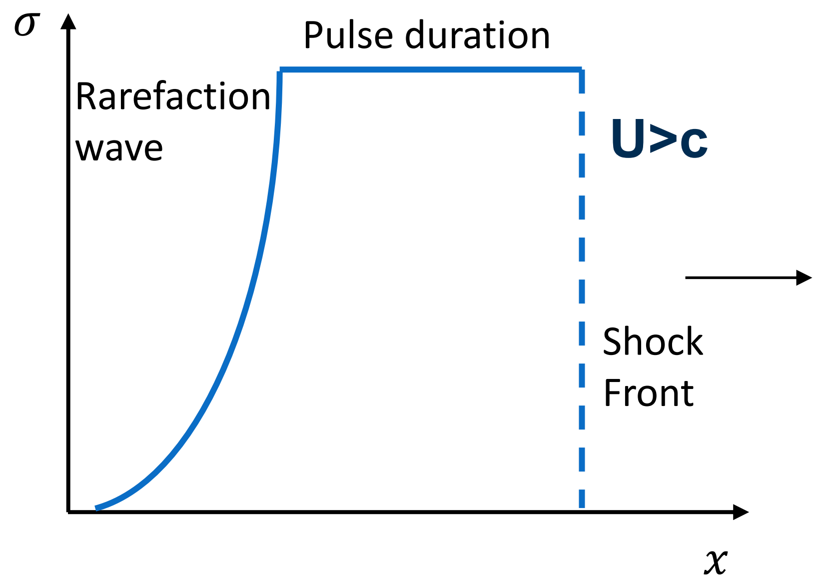
Week 7. Plastic wave propagation



- Plastic hardening and plastic wave propagation velocity
- Plastic/elastic wave decomposition
- Plastic loading front vs elastic unloading front
- Taylor cylinder experiment

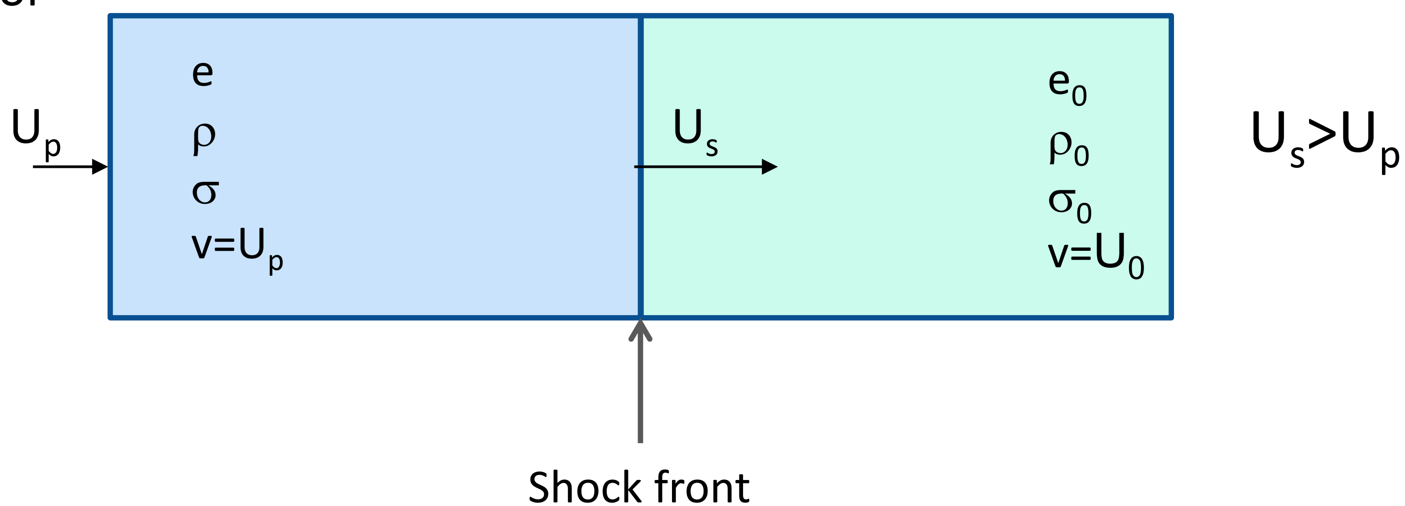
Discontinuous (supersonic) wave front

In detonations, the imposed particle velocity *U* is higher than the wave propagation velocity *c*, so there is a **discontinuity** between the shocked and unshocked material.



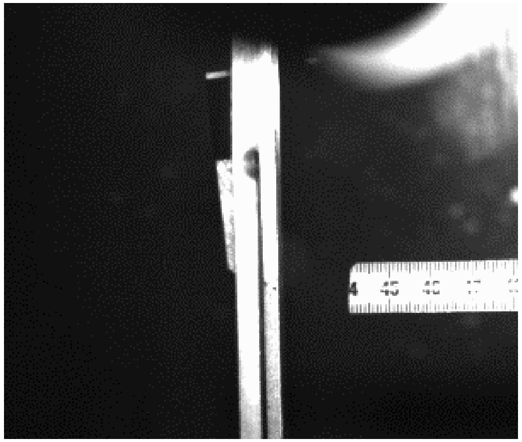
Conservation equations

Volume control

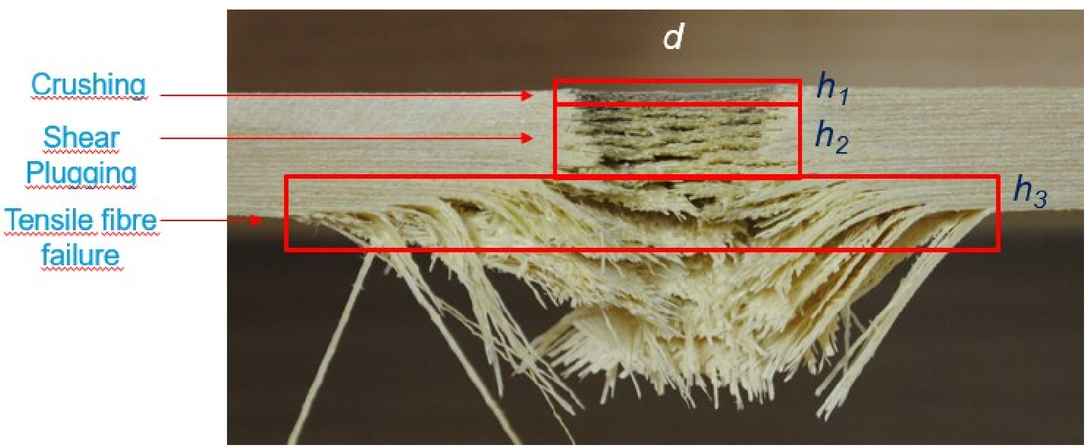


Absorbed energy during impact

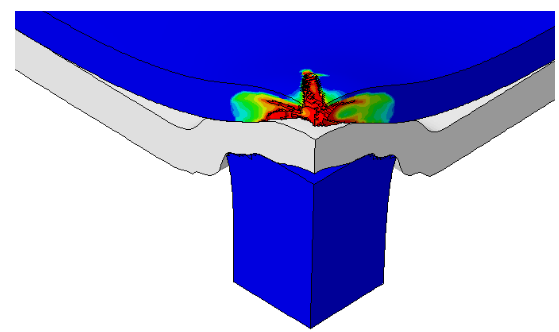
$$E_{abs} = E_{impact} - E_{res} = KE_{target} + \boxed{PE_{target} + PE_{projectile}}$$



Failure modes and absorbed energy

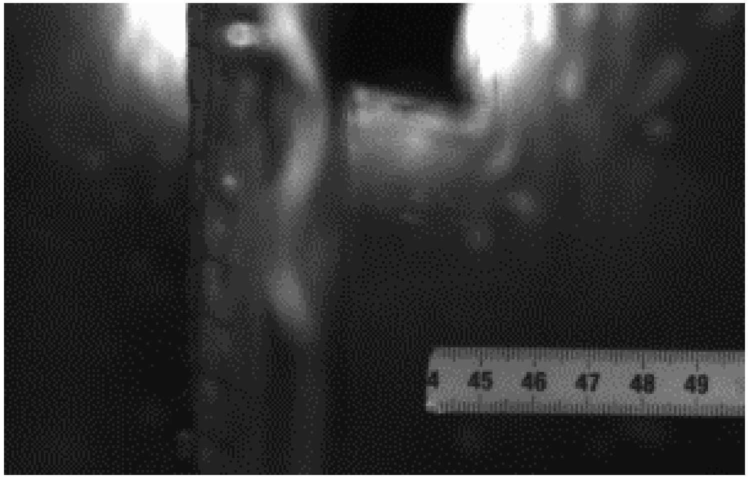


Hybrid shields



Absorbed energy during impact

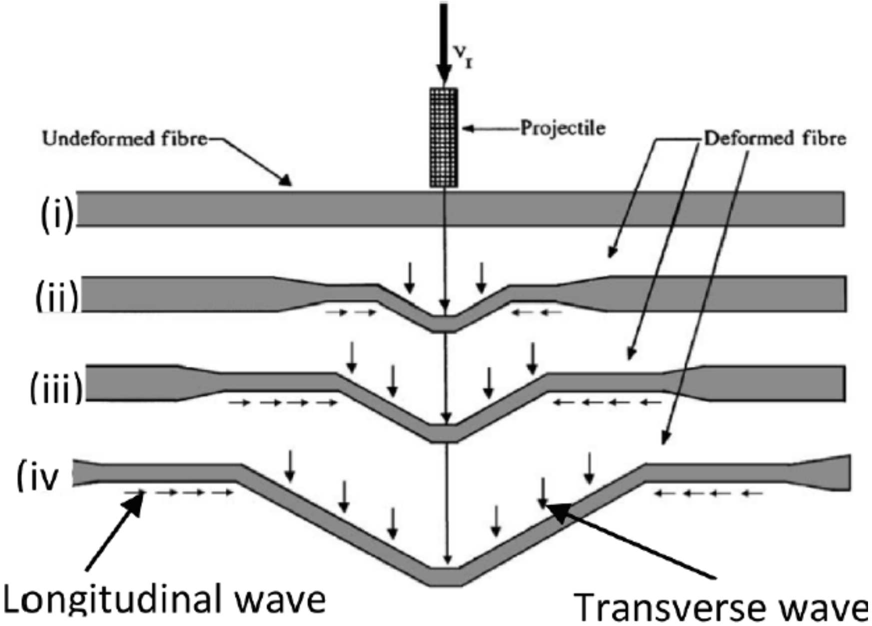
$$E_{abs} = E_{impact} - E_{res} = \boxed{KE_{target} + PE_{target}}$$



2 layers of Fraglight. V_{ini} = 327m/s

High **deflection** of the whole target

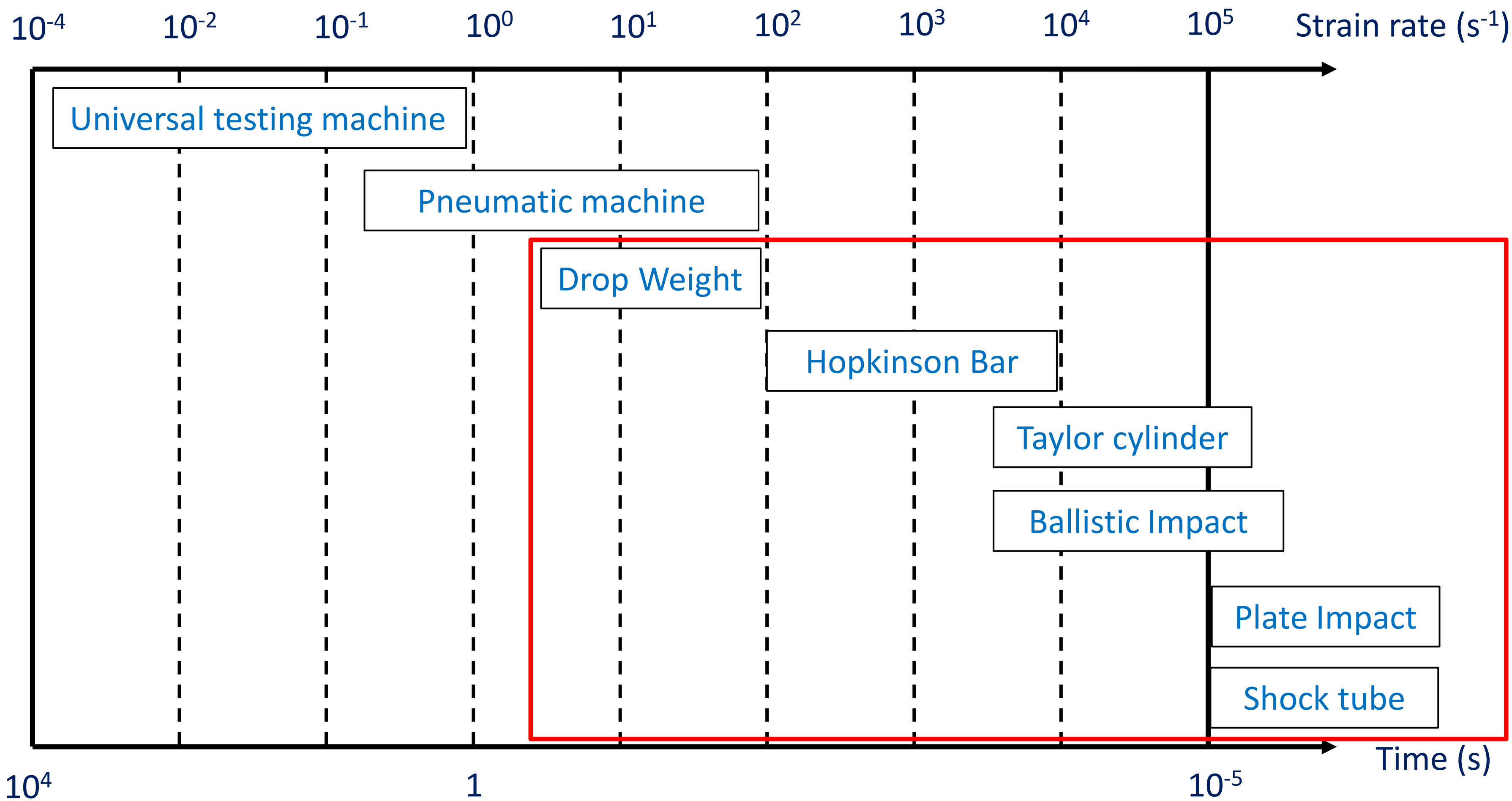
Civilian bulletproof vest



$$SE(t) = 2E\varepsilon^2 d_{proj} hct$$

$$KE(t) = \rho_{areal} l(t)^2 V_{proj}^2(t)$$

Dynamic testing regimes



Impulsive Dynamics

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Modelling - Frequency Response Function and State-space form representation

MECE11014 - Advanced Dynamics and Applications 5

Dr David García Cava

- Calculation of Frequency Response Functions (FRFs)
- Simulation of the time response of dynamical mechanical systems

A dynamical mechanical system - in brief

A dynamical mechanical system with n-DOF is defined by the following differential equation, which defines its equation of motion.

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f} \tag{1}$$

where

- $\boldsymbol{M}$ is the Mass matrix $(n \times n)$. Units (kg)
- $\boldsymbol{C}$ is the Damping matrix $(n \times n)$. Units (Ns/m)
- $\boldsymbol{K}$ is the Stiffness matrix $(n \times n)$. Units (N/m)
- $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ are the displacement $\{x(t)\}$, velocity $\{\dot{x}(t)\}$ and acceleration $\{\ddot{x}(t)\}$ vectors respectively at each DOF $(n \times 1)$. Units (m, m/s, m/s^2)
- $\boldsymbol{f}$ is the force vector $(n \times 1)$. Units (N)

The n-DOF system can be transformed into modal coordinates vectors $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{q}}$, so that each coordinate $\boldsymbol{q}_r$, $\dot{\boldsymbol{q}}_r$ and $\ddot{\boldsymbol{q}}_r$will represent an independent or decoupled form in which the dynamical mechanical system can move.

$$\boldsymbol{M}_r\ddot{\boldsymbol{q}} + \boldsymbol{C}_r\dot{\boldsymbol{q}} + \boldsymbol{K}_r\boldsymbol{q} = \boldsymbol{p} \tag{2}$$

where

- $\boldsymbol{M}_r$ is the diagonal modal mass matrix ($n \times n$).
- $\boldsymbol{C}_r$ is the diagonal modal damping matrix ($n \times n$).
- $\boldsymbol{K}_r$ is the diagonal modal stiffness matrix ($n \times n$).
- $\boldsymbol{p}$ is the modal force vector ($n \times 1$).

A dynamical mechanical system - in brief

The transformation from the physical (original) coordinates to the modal coordinates is obtained thanks to the modal shape matrix $\mathbf{\Phi}$ that are used to obtain the generalised or modal matrices.
In particular, the modal mass matrix $\boldsymbol{M}_r$, modal damping matrix $\boldsymbol{C}_r$ and modal stiffness matrix $\boldsymbol{k}_r$

$$\begin{aligned} \boldsymbol{M}_r &= \mathbf{\Phi}^T \boldsymbol{M} \mathbf{\Phi} \\ \boldsymbol{C}_r &= \mathbf{\Phi}^T \boldsymbol{C} \mathbf{\Phi} \\ \boldsymbol{K}_r &= \mathbf{\Phi}^T \boldsymbol{K} \mathbf{\Phi} \end{aligned} \tag{3}$$

The transformation from physical to modal coordinate vectors

$$\begin{aligned} \boldsymbol{q} &= \mathbf{\Phi}^T \boldsymbol{x} \\ \dot{\boldsymbol{q}} &= \mathbf{\Phi}^T \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{q}} &= \mathbf{\Phi}^T \ddot{\boldsymbol{x}} \end{aligned} \tag{4}$$

and the modal force excitation vector

$$\boldsymbol{p} = \mathbf{\Phi}^T \boldsymbol{f} \tag{5}$$

FREQUENCY RESPONSE FUNCTIONS

Frequency Response Functions

The Frequency Response Function (FRF) of a system is defined as the ratio of the response to the excitation in frequency domain.

So, if the input of the system is represented by $X(j\omega)$ and its response by $Y(j\omega)$, then the FRF is defined as:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \tag{6}$$

An MDOF system may have several excitation points and measured responses. So, let's assume that the excitation is applied on DOF q and the response measured on DOF p. The FRF then is as:

$$H_{pq}(j\omega) = \frac{Y_p(j\omega)}{X_q(j\omega)} \tag{7}$$

Frequency Response Functions

In general, we can calculate FRFs from any DOF to any DOF.

Thus, we can build an FRF matrix, containing the relation from the input DOFs to output DOFs:

$$\boldsymbol{H}(j\omega) = \begin{bmatrix} H_{1a}(j\omega) & H_{1b}(j\omega) \\ H_{2a}(j\omega) & H_{2b}(j\omega) \\ H_{3a}(j\omega) & H_{3b}(j\omega) \end{bmatrix} \tag{8}$$

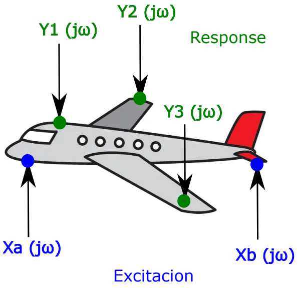


Figure 1: Excitation/response schematic representation

Types of FRFs in Mechanical systems

Mechanical systems may be characterized with different types of FRFs, depending on the selection of input and output variables.

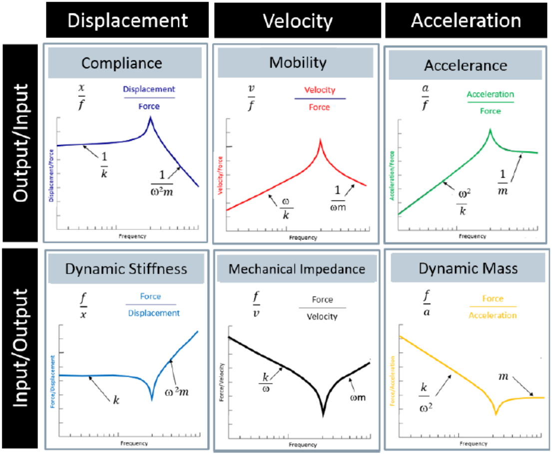


Figure 2: See more in: https://community.sw.siemens.com/s/article/dynamic-stiffness-compliance-mobility-and-more

Calculating FRFs from $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ matrices

NOTE: See additional to refresh Laplace transform

By applying the Laplace transform to the original MDOF equations, we obtain:

$$(\boldsymbol{M}s^2 + \boldsymbol{C}s + \boldsymbol{K})\boldsymbol{x}(s) = \boldsymbol{f}(s) \tag{9}$$

$$\boldsymbol{Z}(s)\boldsymbol{x}(s) = \boldsymbol{f}(s) \tag{10}$$

The matrix $\boldsymbol{Z}(s)$ is referred to as mechanical impedance or dynamic stiffness.
The dynamic flexibility is the inverse of the mechanical impedance, so that:

$$\boldsymbol{H}(s) = [\boldsymbol{Z}(s)]^{-1} \tag{11}$$

An FRF is obtained by evaluating the dynamic flexibility on $s = j\omega = \pm\sqrt{\lambda}$ being λ the eigenvalues of $\boldsymbol{A} - \lambda\boldsymbol{I}$ for $\boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{K}$ and $j = \sqrt{-1}$, which means

Frequency Response Function (FRF)

$$\boldsymbol{H}(j\omega) = [\boldsymbol{Z}(j\omega)]^{-1} = (\boldsymbol{K} - \boldsymbol{M}\omega^2 + j\boldsymbol{C}\omega)^{-1} \tag{12}$$

Therefore, the FRF calculation then implies the calculation of the matrix for each frequency of interest $\omega = 2\pi f$ and then calculation of the matrix inverse $[\boldsymbol{Z}(j\omega)]^{-1}$.

Calculating FRFs from modal parameters

Now, If we replace $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ matrices by their modal matrices $\boldsymbol{M}_r$, $\boldsymbol{C}_r$ and $\boldsymbol{K}_r$ matrices.
Including the modal matrices from Eq.(3) into Eq. (9) gives:

$$(\boldsymbol{\Phi} \boldsymbol{M}_r \boldsymbol{\Phi}^T s^2 + \boldsymbol{\Phi} \boldsymbol{C}_r \boldsymbol{\Phi}^T s + \boldsymbol{\Phi} \boldsymbol{K}_r \boldsymbol{\Phi}^T)\boldsymbol{x}(s) = \boldsymbol{f}(s) \tag{13}$$

Now arranging:

$$\boldsymbol{\Phi}(\boldsymbol{M}_r s^2 + \boldsymbol{C}_r s + \boldsymbol{K}_r)\boldsymbol{\Phi}^T \boldsymbol{x}(s) = \boldsymbol{f}(s) \tag{14}$$

The FRF is calculated as:

Frequency Response Function (FRF) from modal parameters

$$\boldsymbol{H}(j\omega) = \boldsymbol{\Phi}(\boldsymbol{K}_r - \boldsymbol{M}_r \omega^2 + j\boldsymbol{C}_r \omega)^{-1} \boldsymbol{\Phi}^T \tag{15}$$

or simplified as

$$\boldsymbol{H}(j\omega) = \boldsymbol{\Phi}[\boldsymbol{Z}_r(j\omega)]^{-1} \boldsymbol{\Phi}^T \tag{16}$$

where $\boldsymbol{Z}_r(j\omega) = \boldsymbol{K}_r - \boldsymbol{M}_r \omega^2 + j\boldsymbol{C}_r \omega$ is the modal stiffness matrix. This matrix is diagonal.

Calculating FRFs from modal parameters

FOR COMPLETENESS, LET'S FINALISE THE FORMULATION.

$$\boldsymbol{Z}_r(s) = \begin{bmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{bmatrix} s^2 + \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} s + \begin{bmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{bmatrix} \tag{17}$$

$$\boldsymbol{Z}_r(s) = \begin{bmatrix} m_1 s^2 + c_1 s + k_1 & & 0 \\ & \ddots & \\ 0 & & m_n s^2 + c_n s + k_n \end{bmatrix} \tag{18}$$

$$\boldsymbol{Z}_r(s) = \begin{bmatrix} m_1 (s - s_1)(s - s_1^*) & & 0 \\ & \ddots & \\ 0 & & m_n (s - s_n)(s - s_n^*) \end{bmatrix} \tag{19}$$

Where s_r is the pole corresponding to mode $r = 1, \ldots, n$

Calculating FRFs from modal parameters

Based on the stiffness matrix $\boldsymbol{Z}_r(s)$, we can calculate the FRF.

$$\boldsymbol{H}(s) = \boldsymbol{\Phi}[\boldsymbol{Z}_r(s)]^{-1}\boldsymbol{\Phi}^T \tag{20}$$

NOTE: The inverse of a diagonal matrix is nothing but he reciprocal of each value of the diagonal. Therefore, the Eq. (20) can be expanded as:

$$\boldsymbol{H}(s) = \begin{bmatrix} \phi_{11} & & \phi_{1n} \\ \vdots & \dots & \vdots \\ \phi_{n1} & & \phi_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{m_1(s-s_1)(s-s_1^*)} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{m_n(s-s_n)(s-s_n^*)} \end{bmatrix} \begin{bmatrix} \phi_{11} & & \phi_{n1} \\ \vdots & \dots & \vdots \\ \phi_{1n} & & \phi_{nn} \end{bmatrix} \tag{21}$$

Then, after some algebra, it turns out that each entry of the FRF matrix $H_{pq} = \frac{X_p}{F_q}$ can be computed simply as:

$$H_{pq}(s) = \sum_{r=1}^{n} \frac{\phi_{pr}\phi_{qr}}{m_r(s-s_r)(s-s_r^*)} \tag{22}$$

where ϕ_{pr} and ϕ_{qr} is the mode shape coefficient in point p and q respectively of the mode r.

or substituting $s = j\omega$

$$H_{pq}(j\omega) = \sum_{r=1}^{n} \frac{\phi_{pr}\phi_{qr}}{m_r(j\omega - s_r)(j\omega - s_r^*)} \tag{23}$$

A more generic representation of the transfer function can be obtained after partial fraction expansion and by substituting $s = j\omega$.

$$H_{pq}(j\omega) = \sum_{r=1}^{n} \left(\frac{A_{pqr}}{j\omega - s_r} + \frac{A_{pqr}^*}{j\omega - s_r^*} \right) \tag{24}$$

where: $s_r, s_r^* = -\omega_r\zeta_r \pm j\omega_r\sqrt{1-\zeta^2}$ and with residues defined as

$$A_{pqr} = Q_r\phi_{pr}\phi_{qr} = \frac{1}{j2\omega_{dr}m_r}\phi_{pr}\phi_{qr} \tag{25}$$

composed of the *modal scaling constant* Q_r for each mode r and mode shapes coefficients in the point p and q.

And $H_{pq}(j\omega) = H_{qp}(j\omega)$ called *Maxwell's reciprocity*. So, that the FRF is the same if we excite in p and measured in q than if the force and response is reversed.

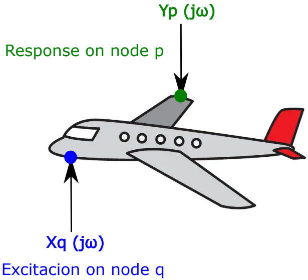
Calculating FRFs from modal parameters

Then, the total FRF matrix can be calculated as:

$$\boldsymbol{H}(j\omega) = \sum_{r=1}^{n} \frac{\boldsymbol{A}_{pqr}}{m_r(s - s_r)(s - s_r^*)} \tag{26}$$

with the matrix residues defined as $\boldsymbol{A}_r = \frac{1}{j2\omega_{dr} m_r} \phi_r \phi_r^T$ so that,

$$\boldsymbol{A}_r = \frac{1}{j2\omega_{dr} m_r} \begin{bmatrix} \phi_{1r}\phi_{1r} & & \phi_{1r}\phi_{nr} \\ \vdots & \dots & \vdots \\ \phi_{nr}\phi_{1r} & & \phi_{nr}\phi_{nr} \end{bmatrix} \tag{27}$$



Then, It is important to observe:

- Eq. 23 is called superposition equation and it relates frequency responses with modal parameters (i.e. poles and mode shapes).
- It also shows that a MDOF systems have frequency response functions that consists of sums of SDOF frequency response functions

Let's open to the exercise Ex. 1.1

- This exercise will calculate the FRFs of a single DOF system with damping.
- System:
  $x(t)$ displacement, $f(t)$ is the excitation force and $m = 10kg$, $k = 250N/m$ and damping is defined as damping ration $\zeta = 5\%$.

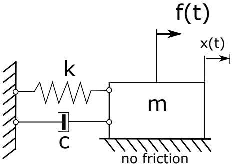


Figure 3: 1 DoF schematic representation

SIMULATION OF THE TIME RESPONSE

State-space form

- Modern dynamical systems deals with more complex system which can have multiple inputs and multiple outputs, time varying or nonlinear.

- **State variable models** are a time domain method. (compare it with transfer function which was in s-domain)

  The time domain is the mathematical domain that incorporates the response and description of system in term of time, t.

- It can be used for nonlinear, time-varying and multivariable system (e.g. the mass of a space rocket varies as a function of time as the fuel is consumed during a flight.).

  In a time-varying system, one or more parameters of the system change with time.

The general form of the state-space equation

The Eq.(28) **state differential equation**:

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \tag{28}$$

and the Eq.(29) is the **output equation**

$$\boldsymbol{x} = \boldsymbol{C}\boldsymbol{z} + \boldsymbol{D}\boldsymbol{u} \tag{29}$$

where:

$\boldsymbol{z}$ is the state vector with dimension $[n \times 1]$
$\boldsymbol{x}$ is the output vector with dimension $[r \times 1]$
$\boldsymbol{u}$ is the input (control) vector with dimension $[m \times 1]$
$\boldsymbol{A}$ is the system matrix with dimension $[n \times n]$
$\boldsymbol{B}$ is the input matrix with dimension $[n \times m]$
$\boldsymbol{C}$ is the output matrix with dimension $[r \times n]$
$\boldsymbol{D}$ is the feedthrough matrix with dimension $[r \times m]$

For many physical systems the matrix $\boldsymbol{D}$ is the null matrix, and the output equation reduces to a simple weighted combination of the state variables:

$$\boldsymbol{x} = \boldsymbol{C}\boldsymbol{z} \tag{30}$$

If D is not zero, it is because the presence of $\boldsymbol{D}$ means that there is a component of the output that changes instantaneously when the input changes.

State equation based Modelling procedure

Bringing the State-Space representation of the state equations again:

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \tag{31}$$

$$\boldsymbol{x} = \boldsymbol{C}\boldsymbol{z} + \boldsymbol{D}\boldsymbol{u} \tag{32}$$

It can mentioned the following:

- The matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ are the properties of the system and are determined or defined by the system structure and elements.
- The matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ are determined by the particular selection of the output variables.

Procedure of State equation Based modelling:

1. Determine of the system order n and select the set of state variables.
2. Generate a set of state equations and the system matrices $\boldsymbol{A}$ and $\boldsymbol{B}$
3. Determine a suitable set of output equations and derive an appropriate $\boldsymbol{C}$ and $\boldsymbol{D}$ matrices.

State equation based Modelling procedure

If we remember from Eq.(1), the Equations of motion of dynamical mechanical system is defined as:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{u} \tag{33}$$

and its corresponding state-space form can re-written in the compress from as follows: The **state transition equation**:

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{u} \tag{34}$$

The **measurement equation**:

$$\boldsymbol{x} = \begin{bmatrix} 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{bmatrix} \tag{35}$$

where $\boldsymbol{z} = \begin{bmatrix} \boldsymbol{x} & \dot{\boldsymbol{x}} \end{bmatrix}^T$ is the state vector, $\boldsymbol{B} = \begin{bmatrix} 0 & \boldsymbol{I} \end{bmatrix}^T$ is the input coupling matrix, $\boldsymbol{C} = \begin{bmatrix} 0 & \boldsymbol{I} \end{bmatrix}$ is the output (measurement) matrix and:

$$\boldsymbol{A} = \begin{bmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \tag{36}$$

is the state (system matrix).

Response of a state space system

Then, the solution of the response of a state space system can be demonstrated to be of the form:

$$\boldsymbol{x}(t) = \underbrace{e^{\boldsymbol{A}t}\boldsymbol{x}_0}_{x_h(t)} + \underbrace{\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau}_{x_p(t)} \tag{37}$$

given the initial conditions $\boldsymbol{x}_0$ and excitation $\boldsymbol{u}(t)$. Then:

- $\boldsymbol{x}_h(t)$ is the homogeneous solution that describes the response to an arbitrary set of initial conditions $\boldsymbol{x}(0)$
- $\boldsymbol{x}_p(t)$ is the particular solution that satisfies the state equations for the given input $\boldsymbol{u}(t)$

NOTE: See additional file to demonstrate the state-space solution

Let's move to the exercise Ex. 1.2

This exercise will calculate the time response of two 2 DOF system with mass, stiffness and damping. Define the state-space form representation.

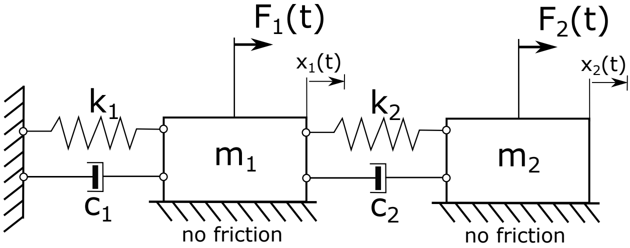


Figure 4: 2 DoF schematic representation

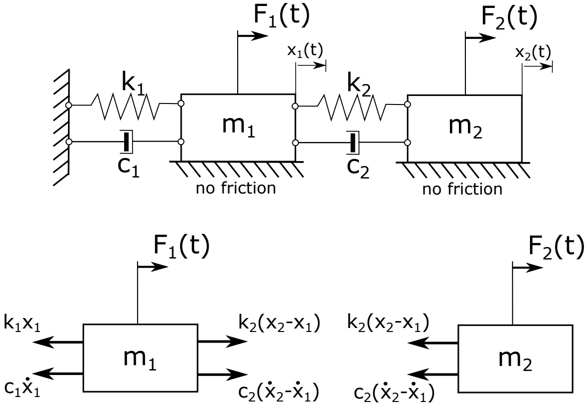


Figure 5: 2DoF system simulation and Free body diagram

Ex. Two degree of freedom system - State Space representation

The equations of motion can be written as

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) = F_1 \\ m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = F_2 \end{aligned} \tag{38}$$

$$\mathrm{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \qquad \Rightarrow \qquad \dot{\mathrm{z}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} \tag{39}$$

$$\dot{\mathrm{z}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_4 \\ -\frac{(k_1+k_2)}{m_1}z_1 + \frac{k_2}{m_1}z_2 - \frac{(c_1+c_2)}{m_1}z_3 + \frac{c_2}{m_1}z_4 \\ \frac{k_2}{m_2}z_1 - \frac{k_2}{m_2}z_2 + \frac{c_2}{m_2}z_3 - \frac{c_2}{m_2}z_4 \end{bmatrix} \tag{40}$$

Then, the matrix A and B

$$\mathrm{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \quad ; \quad \mathrm{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \tag{41}$$

Ex. Two degree of freedom system - State Space representation

Then, the State Equation is:

$$\dot{z} = Az + Bu \tag{42}$$

$$\dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{43}$$

The Output Equation is:

$$x = Cz + Du \tag{44}$$

Displacement at m_1

$$x = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{45}$$

Ex. Two degree of freedom system - State Space representation

Then, the State Equation is:

$$\dot{z} = Az + Bu \tag{46}$$

$$\dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{47}$$

The Output Equation is:

$$x = Cz + Du \tag{48}$$

Velocity at m_2

$$x = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{49}$$

Ex. Two degree of freedom system - State Space representation

Then, the State Equation is:

$$\dot{\mathrm{z}} = \mathrm{Az} + \mathrm{Bu} \tag{50}$$

$$\dot{\mathrm{z}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{51}$$

The Output Equation is:

$$\mathrm{x} = \mathrm{Cz} + \mathrm{Du} \tag{52}$$

Displacement at m_1 and **Acceleration at** m_2

$$\mathrm{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{53}$$

Exercise 1.2

Let's open to the exercise Ex. 1.3

- This exercise will calculate the time response using Runge-kutta method of the state-space form of a 2DOF system as in the Figure.
- System:
  $x_1(t), x_2(t)$ displacemenst, $F_2(t)$ is the excitation force and $m_1 = m_2 = 10kg$, $k_1 = 250N/m, k_2 = 50N/m$ and $c_1 = c_2 = 0.5Ns^2/m$

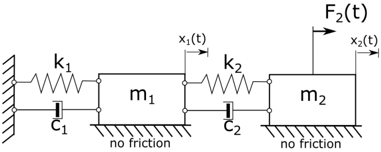


Figure 6: 2 DoF schematic representation

Additional - In detail Solution of the State-space form

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The state-space equation

The Eq.(1) **state differential equation**:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \tag{1}$$

and the Eq.(2) is the **output equation**

$$\mathbf{x} = \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u} \tag{2}$$

where:

- $\mathbf{z}$ is the state vector with dimension $[n \times 1]$
- $\mathbf{x}$ is the output vector with dimension $[r \times 1]$
- $\mathbf{u}$ is the input (control) vector with dimension $[m \times 1]$
- $\mathbf{A}$ is the system matrix with dimension $[n \times n]$
- $\mathbf{B}$ is the input matrix with dimension $[n \times m]$
- $\mathbf{C}$ is the output matrix with dimension $[r \times n]$
- $\mathbf{D}$ is the feedthrough matrix with dimension $[r \times m]$

For many physical systems the matrix $\mathbf{D}$ is the null matrix, and the output equation reduces to a simple weighted combination of the state variables:

$$\mathbf{x} = \mathbf{C}\mathbf{z} \tag{3}$$

If D is not zero, it is because the presence of $\mathbf{D}$ means that there is a component of the output that changes instantaneously when the input changes.

State equation Based Modelling procedure

Bringing the State-Space representation of the state equations again:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \tag{4}$$

$$\mathbf{x} = \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u} \tag{5}$$

It can mentioned the following:

- The matrix **A** and **B** are the properties of the system and are determined or defined by the system structure and elements.
- The matrices **C** and **D** are determined by the particular selection of the output variables.

Procedure of State equation Based modelling:

1. Determine of the system order n and select the set of state variables.
2. Generate a set of state equations and the system matrices **A** and **B**
3. Determine a suitable set of output equations and derive an appropriate **C** and **D** matrices.

Solution for a State differential equation

The solution of the state differential equation Eq.(4) can be obtained as a first order differential equation:

$$\dot{z} = az + bu \tag{6}$$

where $x(t)$ and $u(t)$ are scalar functions of time.

Taking the Laplace Transform of Eq.(6)

$$sZ(s) - z(0) = aZ(s) + bU(s) \tag{7}$$

therefore,

$$Z(s) = \frac{z(0)}{s-a} + \frac{b}{s-a}U(s) \tag{8}$$

The inverse Laplace transform

$$z(t) = e^{at}z(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \tag{9}$$

Solution for a State differential equation

It is expected to find a similar solution for state differential equation Eq.(9) and to be exponential form. The **matrix exponential function** is defined as:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^k t^k}{k!} + \dots \tag{10}$$

which converges for all finite t and any $\mathbf{A}$. Then the solution of Eq.(9) is

$$\mathbf{z}(t) = e^{\mathbf{A}t}\mathbf{z}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{11}$$

This can be then verified by taking the Laplace transform of Eq.(4) which is in the form of Eq.(7) and re-arranging it:

$$\mathbf{Z}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{z}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) \tag{12}$$

Then, it can be defined $[s\mathbf{I} - \mathbf{A}]^{-1} = \mathbf{\Phi}(s)$ is the Laplace Transform of $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$.

Then, if the Inverse of the Laplace Transform of Eq.(12) is taken, Eq.(11) will be obtained.

Solution for a State differential equation

The matrix exponential function describes the unforced response of the system and is called the **fundamental or state transition matrix $\boldsymbol{\Phi}(t)$**.

The Eq.(11) can be written as:

$$\mathbf{z}(t) = \boldsymbol{\Phi}(t)\mathbf{z}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau \tag{13}$$

The solution for unforced system ($\mathbf{u} = 0$):

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) & \dots & \phi_{1n}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \dots & \phi_{2n}(t) \\ \vdots & & \ddots & \vdots \\ \phi_{n1}(t) & \phi_{n2}(t) & \dots & \phi_{nn}(t) \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ \vdots \\ z_n(0) \end{bmatrix} \tag{14}$$

Therefore to determine the state transition matrix:

- All initial conditions are set to zero
- Except for one state variable, and the output of each state variable is evaluated.

Additional - Refreshing Laplace Transform Knowledge

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Laplace transformation involves a domain change so that the problem can be converted from one using differential equations into one which is algebraic.
This defines the process of **Laplace Transformation:**

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \mathscr{L}\{f(t)\} \tag{1}$$

Here are some useful Laplace Transforms:

$f(t)$	$F(s)$
Step function, $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

Figure 1: Selected - Important Laplace Transform Pairs. More in table attached to lectures (source: Dorf and Bishop)

Laplace Transform of the **step function** $u(t)$

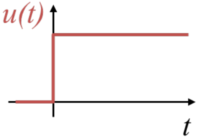

Figure 2: Step function

$$\mathscr{L}\left\{u(t)\right\} = \int_0^\infty e^{-st}dt = \left[\frac{-e^{-st}}{s}\right]_0^\infty = \frac{1}{s} \tag{2}$$

Laplace Transform of the **exponential decay** e^{-at}

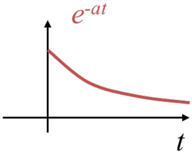

Figure 3: Exponential decay

$$\begin{aligned} \mathscr{L}\left\{e^{-at}\right\} &= \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt \\ \mathscr{L}\left\{e^{-at}\right\} &= \left[\frac{-e^{-(s+a)t}}{(s+a)}\right]_0^\infty = \frac{1}{(s+a)} \end{aligned} \tag{3}$$

The Inverse Laplace Transform

Once the **Laplace Transformation** of the problem has been done, and the algebraic problem that then arises has been solved, the result has to be **'inverse transformed'** back into the original domain. We can directly use the pairs in the table opposite, as long as they are in recognisable form. Partial fractions are frequently used to achieve that in practice.

The **inverse Laplace transform** is given by the following expression:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds \tag{4}$$

where σ is some real number representing the numerical limit of the function. We rarely need to use this explicitly for reasons alreay given.

EXAMPLE:

$$\mathscr{L}^{-1}\left\{ \frac{1}{(s+a)} \right\} = e^{-at} \tag{5}$$

Laplace Transforms of Derivatives

The **Laplace Transformation of the derivative** (taking the time domain as the based domain from which we transform to the s-domain, but it could equally be another physically real domain) is given by:

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = s\mathscr{L}\left\{f(t)\right\} - f(0) \tag{6}$$

The Laplace variable s can be considered to be the differential operator: $s \equiv \frac{d}{dt}$

$$\mathscr{L}\left\{\frac{d^k f(t)}{dt^k}\right\} = s^k F(s) - s^{k-1} f(0^+) - s^{k-2} f'(0) - \ \ldots \ - f^{k-1}(0) \tag{7}$$

For example for $k = 2$

$$\mathscr{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0) - \frac{df(0)}{dt} \tag{8}$$

Example: Mass-Spring-Damper system

As presented earlier the second order linear differential equation for a SMD system is given by:

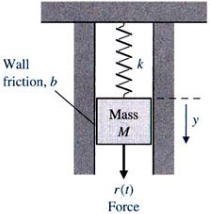

Figure 4: Spring - Mass - Damper system.
(source: Dorf and Bishop)

$$M\ddot{y} + b\dot{y} + ky = r(t) \tag{9}$$

The Laplace transform of the equation (9) is given as:

$$M\left(s^2Y(S) - sy(0) - \frac{dy(0)}{dt}\right) + b\left(sY(s) - y(0)\right) + kY(s) = R(s) \tag{10}$$

Now, for a free vibration context $r(t) = 0$
The initial conditions are: $y(0) = y_0$ and $\frac{dy(0)}{dt} = \dot{y}_0$ (often zero)

Example: Mass-Spring-Damper system

Then, Equation (10) simplifies as:

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0 \tag{11}$$

Now, it can be simplified for $Y(s)$ to get:

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)} \tag{12}$$

Note:

- When $q(s) = 0$, it is called the characterization equation.
- The roots of $q(s)$ are called **POLES**
- The roots of $p(s)$ are called **ZEROS**
- The Poles and Zeros are critical frequencies

EXAMPLE: Take $k/M = 2$ and $b/M = 3$ and try to show that:

$$\begin{aligned} Y(s) &= \frac{(Ms + 3M)y_0}{Ms^2 + 3Ms + 2M} \\ Y(s) &= \frac{(s + 3)y_0}{(s^2 + 3s + 2)} \\ Y(s) &= \frac{(s + 3)y_0}{(s + 1)(s + 2)} = \frac{p(s)}{q(s)} \end{aligned} \tag{13}$$

Damping ratio and natural frequency

For the Mass-Spring-damper (SMD)

$$Y(s) = \frac{(s + \frac{b}{M})y_0}{s^2 + \frac{b}{M}s + \frac{k}{M}} = \frac{(s + 2\zeta\omega_n)y_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{14}$$

where $\omega_n = \sqrt{k/M}$ and $\zeta = b/(2\sqrt{kM})$

$$s_{1,2} = -\zeta\omega \pm \omega\sqrt{\zeta^2 - 1} \tag{15}$$

$$\textbf{Cases for damped vibration} \begin{cases} 0 < \zeta < 1 & \text{underdamped} \\ \zeta = 1 & \text{critically damped} \\ \zeta > 1 & \text{overdamped} \end{cases} \tag{16}$$

Operational Modal Analysis. Stochastic Subspace Identification

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- Fundamentals of Operational Modal Analysis (OMA)
- Output-only system identification
- Stochastic Subspace Identification (SSI)

Operational Modal Analysis (OMA): Fundamentals

- The experimental set-up comprises the structure in **its normal operating environment**.This means changing loading and environmental conditions and operational regimes.
- While the operating environment cannot be controlled, it may be possible to select a most favorable period for analysis.
- In most cases the excitation cannot be measured, and only response signals are available.
- As excitation cannot be controlled, often some modes are not excited, or the excitation dynamics may be confounded with dynamics of the structure.

Why EMA and/or OMA?

- **For troubleshooting**: Understanding a vibration problem
- **For validation**: Do the actual dynamics of the structure correspond with design specifications (from an analytical model/FEM)?
- **For improved modelling**: Structural parameters such as damping are often unknown/guessed. Experimental data may provide actual values to be used in more accurate modelling.
- **For Structural Health Monitoring (SHM)**: Continuous tracking of the modal properties of a structure may provide information on its condition (e.g. deterioration or fatigue accumulation) and the potential emergence of damage.

Modal Analysis from the state-space form

Let's refresh ourselves from Lecture 1 state-space form formulation.
If we remember, the Equations of motion of dynamical mechanical system is defined as:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{u} \tag{1}$$

and its corresponding state-space form can re-written in the compress from as follows:
The **state transition equation**:

$$\underbrace{\begin{bmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{bmatrix}}_{\dot{\boldsymbol{z}}} = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{bmatrix}}_{\boldsymbol{z}} + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{B}} \boldsymbol{u} \tag{2}$$

The **measurement equation**:

$$\boldsymbol{x} = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{C}} \underbrace{\begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{bmatrix}}_{\boldsymbol{z}} \tag{3}$$

where $\boldsymbol{z}$ is the state vector, $\boldsymbol{B}$ is the input matrix, $\boldsymbol{C}$ is the output (measurement) matrix and $\boldsymbol{A}$ is the state (system matrix).

Modal Analysis from the state-space form

Modal Analysis relies on **Eigenvalue problem** to define the dynamical parameters. This can be done by doing the an eigen-analysis of the state matrix $\boldsymbol{A}$ by solving the equation:

$$(\boldsymbol{A} - \lambda_n \boldsymbol{I})\, \boldsymbol{v}_n = 0 \tag{4}$$

where λ_n and $\boldsymbol{v}_n$ are the n^{th}-eigenvalues and eigenvectors.They can be arranged in matrices as $\boldsymbol{\Lambda} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n]$ for the eigenvalues and eigenvectors (in columns) respectively.

Then, the eigenvalues λ_n correspond to the system (structural) modes, while the quantities:

$$\boldsymbol{\Phi}_n = \boldsymbol{C}\boldsymbol{v}_n \tag{5}$$

compromise the corresponding mode shapes.

Modal Analysis from the state-space form

Looking at the eigenvalue equation again

$$\boldsymbol{A}\boldsymbol{v}_1 - \lambda_1 \boldsymbol{v}_1 = 0 \tag{6}$$

$$\boldsymbol{A}\boldsymbol{v}_2 - \lambda_2 \boldsymbol{v}_2 = 0 \tag{7}$$

$$\vdots \tag{8}$$

$$\boldsymbol{A}\boldsymbol{v}_n - \lambda_n \boldsymbol{v}_n = 0 \tag{9}$$

Then, $\boldsymbol{AV} = \boldsymbol{V}\boldsymbol{\Lambda}$ if and only if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{A}$ and each $\boldsymbol{v}_n$ is an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue λ_n

Then, the Equation (11) demonstrates how the eigenvalues and eigenvectors reconstruct the state matrix.

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1} \tag{10}$$

and the Equation (12) how the eigenvectors diagonalise the state matrix.

$$\boldsymbol{\Lambda} = \boldsymbol{V}^{-1}\boldsymbol{AV} \tag{11}$$

where $\boldsymbol{A}$ is the state matrix (invertible square matrix), $\boldsymbol{\Lambda}$ is a diagonal matrix with the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{V}$ is the eigenvectors in columns of $\boldsymbol{A}$

Therefore, using the diagonalisation property of the eigenvalue decomposition, the state equation can be re-written.

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \longrightarrow \dot{\boldsymbol{z}} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \tag{12}$$

The eigenvectors are orthogonal, therefore $\boldsymbol{V}^{-1} = \boldsymbol{V}^T$, and by multiplying by $\boldsymbol{V}^T$ at both sides of the equal, we can have

$$\underbrace{\boldsymbol{V}^T\dot{\boldsymbol{z}}}_{\dot{\boldsymbol{q}}} = \underbrace{\boldsymbol{V}^T\boldsymbol{V}}_{\boldsymbol{I}}\boldsymbol{\Lambda}\underbrace{\boldsymbol{V}^T\boldsymbol{z}}_{\boldsymbol{q}} + \underbrace{\boldsymbol{V}^T\boldsymbol{B}}_{\tilde{\boldsymbol{B}}}\boldsymbol{u} \longrightarrow \dot{\boldsymbol{q}} = \boldsymbol{\Lambda}\boldsymbol{q} + \tilde{\boldsymbol{B}}\boldsymbol{u} \tag{13}$$

Now the system is diagonalised so that $\boldsymbol{\Lambda}$ is a diagonal matrix.
Then, we have the transformation to modal coordinates: $\boldsymbol{q} = \boldsymbol{V}^T\boldsymbol{z}$
and the Reconstruction of physical coordinates $\boldsymbol{z} = \boldsymbol{V}\boldsymbol{q}$

The system in state space is now represented in terms of the modal coordinates q, each representing individual dynamic components in the state response

Discretisation in modal domain

Then, the solution of **the response of a state space system** can be demonstrated to be of the form:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \tag{14}$$

given the initial conditions $\boldsymbol{x}_0$ and excitation $\boldsymbol{u}(t)$.
This can be written in the modal coordinates, as follows:

$$\boldsymbol{q}(t) = e^{\boldsymbol{A}t}\boldsymbol{q}_0 + \int_0^t e^{\boldsymbol{A}(t-\tau)}\tilde{\boldsymbol{B}}\boldsymbol{u}(\tau)d\tau \tag{15}$$

Descretisation of the response:
Let's assume that the system is observed at regularly sampled times (nT_s), where (T_s) is the sampling period. Then, the homogeneous response of the state space system may be discretised as follows:

$$\boldsymbol{q}(nT_s) = e^{\boldsymbol{A}nT_s}\boldsymbol{q}_0 = e^{(\boldsymbol{A}T_s)^n}\boldsymbol{q}_0 = \boldsymbol{A}_d^n\boldsymbol{q}_0 \tag{16}$$

The matrix $\boldsymbol{A}_d = e^{\boldsymbol{A}T_s}$ is the state matrix of the discretized system.

Discretisation in modal domain

Consider the eigenvalue decomposition of the state matrix $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$, where $\boldsymbol{\Lambda} = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ with the eigenvalues in the diagonal and $\boldsymbol{V} = [\mathbf{v}_1 \dots \mathbf{v}_2]$. Based on the properties of the exponential matrix, we have that the continuous-time modes can be associated with discrete-time modes ($\boldsymbol{A}_d$), as follows:

$$\boldsymbol{A}_d = e^{\boldsymbol{A}T_s} = e^{\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}T_s} = \boldsymbol{V}e^{\boldsymbol{\Lambda}T_s}\boldsymbol{V}^{-1} \tag{17}$$

Again due to the eigenvectors being orthogonal, therefore $\boldsymbol{V}^{-1} = \boldsymbol{V}^T$, we can have

$$\boldsymbol{V}e^{\boldsymbol{\Lambda}T_s}\boldsymbol{V}^T = \begin{bmatrix} \boldsymbol{v}_1 & \dots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 T_s} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n T_s} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^T \\ \vdots \\ \boldsymbol{v}_n^T \end{bmatrix} \tag{18}$$

Then, the continuous eigenvalues (λ_n) in the diagonal of $\boldsymbol{\Lambda}$ can be found as

$$\mu_n = e^{\lambda_n T_s} \iff \lambda_n = \frac{\ln(\mu_n)}{T_s} \tag{19}$$

Modal Analysis from the state-space form

Modal parameters extraction

Natural frequency (Hz):

$$f_n = \frac{|\lambda_n|}{2\pi} = \frac{\sqrt{Re(\lambda_n)^2 + Im(\lambda_n)^2}}{2\pi} \tag{20}$$

Damped Natural frequency (Hz):

$$f_{dn} = \frac{Im(\lambda_n)}{2\pi} \tag{21}$$

Damping ratio:

$$\zeta_n = -\frac{Re(\lambda_n)}{|\lambda_n|} = -\frac{Re(\lambda_n)}{\sqrt{Re(\lambda_n)^2 + Im(\lambda_n)^2}} \tag{22}$$

Mode Shape by the matrix $\boldsymbol{C}$:

$$\boldsymbol{\Phi} = Re(\boldsymbol{C}\boldsymbol{v}) \tag{23}$$

An useful tool: Singular Value Decomposition (SVD)

Singular Value Decomposition (SVD) is one of the most important matrix factorisation methods of this era. It can be used for many applications with the main aim of matrix decomposition. It can be considered as an extension of the EVD but for rectangular matrices. Then ,it can be defined as:

$$\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T \tag{24}$$

for a real-valued matrix $\boldsymbol{X}$ of dimensions $L \times M$, with $L \geq M$ and $r(\boldsymbol{A}) \leq M$. Then, the Eq.(24) contains:

- $\boldsymbol{\Sigma}$ of dimension $L \times M$ is rectangular diagonal matrix containing the singular values arranged in decreasing order. r singular values are positive and rest zero.
- $\boldsymbol{U}$ contains in its columns the left singular vectors with dimension $L \times L$
- $\boldsymbol{V}$ contains in its columns the right singular vectors with dimension $M \times M$

Relationship between SVD (Eq.(24)) and EVD (Eq.(10)) is below:

$$\boldsymbol{A} = \boldsymbol{X}^T\boldsymbol{X} \longrightarrow \boldsymbol{A} = (\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T)^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{U}^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^T \tag{25}$$

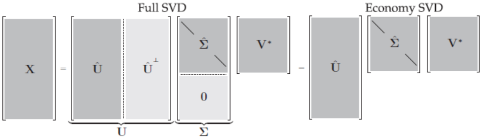


Figure 1: Schematic SVD. Source: Brunton 2022

Stochastic process model: Definition

BUILDING THE FUNDAMENTALS - Defining an stochastic model

The following stochastic dynamical model is defined for a discrete-time based on state-space model where $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ can be related to the process noise due to disturbances and measurement noise due to sensor inaccuracies, respectively over a over a finite discrete-time interval $k = 0, 1, \ldots, N-1$

$$\boldsymbol{z}_{k+1} = \boldsymbol{A}\boldsymbol{z}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{w}_k \tag{26}$$

$$\boldsymbol{x}_k = \boldsymbol{C}\boldsymbol{z}_k + \boldsymbol{D}\boldsymbol{u}_k + \boldsymbol{v}_k \tag{27}$$

As the input (excitation) is not available the response is only generated by the two stochastic processes $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ $\longrightarrow$ zero mean Gaussian white noise process.

$$\boldsymbol{z}_{k+1} = \boldsymbol{A}\boldsymbol{z}_k + \boldsymbol{w}_k \tag{28}$$

$$\boldsymbol{x}_k = \boldsymbol{C}\boldsymbol{z}_k + \boldsymbol{v}_k \tag{29}$$

The process noise and the measurement noise are zero mean, stationary Gaussian white noise process with covariance matrices as follows:

$$\mathbb{E}\left\{\begin{bmatrix}\boldsymbol{w}_p \\ \boldsymbol{v}_p\end{bmatrix}\begin{bmatrix}\boldsymbol{w}_q^T & \boldsymbol{v}_q^T\end{bmatrix}\right\} = \begin{bmatrix}\boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R}\end{bmatrix}; p = q \tag{30}$$

where p and q are two arbitrary time instants.

Stochastic process model: Framework for OMA

In agreement with the **stochastic framework of OMA**, the system response in the state-space model is represented by a zero mean Gaussian process.

The **output covariance matrix** is (carries all the information of the process):

$$\boldsymbol{R}_i = \mathbb{E}\{\boldsymbol{x}_{k+i}\boldsymbol{x}_k^T\} \tag{31}$$

The **current state covariance matrix** (independent of any time instance k) is

$$\boldsymbol{\Sigma} = \mathbb{E}\{\boldsymbol{z}_k\boldsymbol{z}_k^T\} \tag{32}$$

the **next state-output covariance matrix** is defined as:

$$\boldsymbol{G} = \mathbb{E}\{\boldsymbol{z}_{k+1}\boldsymbol{x}_k^T\} \tag{33}$$

and it is uncorrelated with the process of noise

$$\mathbb{E}\{\boldsymbol{z}_k\boldsymbol{w}_k^T\} = \boldsymbol{0} \quad \& \quad \mathbb{E}\{\boldsymbol{z}_k\boldsymbol{v}_k^T\} = \boldsymbol{0} \tag{34}$$

Fundamental equations: Using the state-space and the assumptions

$$\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^T + \boldsymbol{Q} \tag{35}$$

$$\boldsymbol{R}_0 = \boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}^T + \boldsymbol{R} \tag{36}$$

$$\boldsymbol{G} = \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{C}^T + \boldsymbol{S} \tag{37}$$

Stochastic Subspace Identification (SSI) - Introduction

There are mainly two different approaches the **Covariance-driven SSI (Cov-SSI)** and the **Data-driven SSI (DD-SSI)**. We will cover only Cov-SSI in this course. To see details on DD-SSI see Ranieri C. (2014) in Chapter 4.5.3.2.

Controllability and Observability matrices: Definition

- A **state of a system is controllable** if it can be reached from any initial state of the system in a finite time interval by some control actions.
- A **state of the system is observable** if the knowledge of input and output over a finite time interval completely determines the state.

This will then define the Controllability $\mathcal{O}$ and Observability $\mathcal{C}$ matrices

A system of order n is observable/controllable if and only if the observability/ controllability matrix is of rank n.

↓

This very complex task. A suitable approach is to overestimate the model order and this leads to nonphysical poles, next to physical poles.

Stochastic Subspace Identification (SSI) - Estimating Output Correlations

The method starts by computing the **output correlations** (i.e. the vibration responses of the structure) as:

$$\boldsymbol{R}_i = \frac{1}{N-i}\boldsymbol{X}_{(1:N-i)}\boldsymbol{X}_{(i:N)}^T \tag{38}$$

where:

- The matrix $\boldsymbol{X}$ are constructed by l dynamical responses of length N measured from l sensors
- The counter i defines the number of time lags in a finite number of data
- $\boldsymbol{X}_{(1:N-i)}$ is obtained from the matrix $\boldsymbol{X}$ with dimension $[l \times N]$ by removing the last i samples
- $\boldsymbol{X}_{(i:N)}$ is obtained from the matrix $\boldsymbol{X}$ with dimension $[l \times N]$ by removing the first i samples
- $\boldsymbol{R}_i$ correlation matrix at time lag i

Then, the correlations matrices estimated a different time lags can be used to construct the block Toeplitz as Eq. (40) or Hankel as Eq.(41) matrix.

Toeplitz matrix

$$\boldsymbol{T}_i = \begin{bmatrix} \boldsymbol{R}_i & \boldsymbol{R}_{i-1} & \dots & \boldsymbol{R}_1 \\ \boldsymbol{R}_{i+1} & \boldsymbol{R}_i & \dots & \boldsymbol{R}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{R}_{2i-1} & \boldsymbol{R}_{2i-2} & \dots & \boldsymbol{R}_i \end{bmatrix} \tag{39}$$

Hankel matrix

$$\boldsymbol{H}_i = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{R}_2 & \dots & \boldsymbol{R}_i \\ \boldsymbol{R}_2 & \boldsymbol{R}_3 & \dots & \boldsymbol{R}_{i+1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{R}_i & \boldsymbol{R}_{i+1} & \dots & \boldsymbol{R}_{2i-1} \end{bmatrix} \tag{40}$$

As an example lets assume a time lag $i = 3$, then, they will be:

Toeplitz matrix

$$\boldsymbol{T}_3 = \begin{bmatrix} \boldsymbol{R}_3 & \boldsymbol{R}_2 & \boldsymbol{R}_1 \\ \boldsymbol{R}_4 & \boldsymbol{R}_3 & \boldsymbol{R}_2 \\ \boldsymbol{R}_5 & \boldsymbol{R}_4 & \boldsymbol{R}_3 \end{bmatrix} \tag{41}$$

Hankel matrix

$$\boldsymbol{H}_3 = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{R}_2 & \boldsymbol{R}_3 \\ \boldsymbol{R}_2 & \boldsymbol{R}_3 & \boldsymbol{R}_4 \\ \boldsymbol{R}_3 & \boldsymbol{R}_4 & \boldsymbol{R}_5 \end{bmatrix} \tag{42}$$

Stochastic Subspace Identification (SSI) - Consideration in practice

Important considerations on the Block matrices

- Each individual correlation matrix (i.e. $\boldsymbol{R}_i$) has a dimension $l \times l$
- The Block matrix has a dimension $li \times li$

For the identification of the system of order n

$$li \geq n$$

NOTE: In practical applications the actual order of the system is obviously unknown. And it is generally difficult to identify.

IN PRACTICE:

- Assuming the model order n has been estimated.
- The number of outputs l is constant (i.e. responses from sensors).

$$\Downarrow$$

$$i \geq \frac{n}{l} \tag{43}$$

Stochastic Subspace Identification (SSI) - Factorisation

If we focus now in the Toeplitz matrix on Eq. (40), It can be factorised as:

$$\boldsymbol{T}_i = \underbrace{\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^{i-1} \end{bmatrix}}_{\text{Observability}} \underbrace{\begin{bmatrix} \boldsymbol{A}^{i-1}\boldsymbol{G} & \dots & \boldsymbol{AG} & \boldsymbol{G} \end{bmatrix}}_{\text{Reversed Controllability}} = \mathcal{O}_i \mathcal{C}_i \tag{44}$$

Then, if Eq.(44) fulfills and the system is 'Observable' and 'Controllable' the rank of $\boldsymbol{T}_i$ is rank n

Stochastic Subspace Identification (SSI) - SVD

Apply Singular Value Decomposition (SVD) to the Toeplitz matrix as follows:

$$\boldsymbol{T}_i = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1^T \\ \boldsymbol{V}_2^T \end{bmatrix} \tag{45}$$

As it can be seen in the decomposition the zero singular values and their corresponding singular vectors are divided. And if they are omitted, then:

$$\boldsymbol{T}_i = \boldsymbol{\mathcal{O}}_i \boldsymbol{\mathcal{C}}_i = \boldsymbol{U}_1 \boldsymbol{\Sigma}_1 \boldsymbol{V}_1^T \tag{46}$$

Finally, by splitting the SVD, it can be obtained:

Observability matrix

$$\boldsymbol{\mathcal{O}}_i = \boldsymbol{U}_1 \boldsymbol{\Sigma}_1^{1/2} \tag{47}$$

Controllability matrix

$$\boldsymbol{\mathcal{C}}_i = \boldsymbol{\Sigma}_1^{1/2} \boldsymbol{V}_1^T \tag{48}$$

NOTE: In fact, it is decomposed like $\boldsymbol{\mathcal{O}}_i = \boldsymbol{U}_1 \boldsymbol{\Sigma}_1^{1/2} \boldsymbol{P}$ and $\boldsymbol{\mathcal{C}}_i = \boldsymbol{P}^{-1} \boldsymbol{\Sigma}_1^{1/2} \boldsymbol{V}_1^T$ but it can be set to $\boldsymbol{P} = \boldsymbol{I}$ (identity matrix).

Stochastic Subspace Identification (SSI)

Then, the matrices $\boldsymbol{C}$, $\boldsymbol{G}$ and $\boldsymbol{A}$ can be obtained as:

The $\boldsymbol{C}$ the first l rows of $\boldsymbol{\mathcal{O}}_i$

$$\boldsymbol{C} = \boldsymbol{\mathcal{O}}_{(1:l,:)} \tag{49}$$

The $\boldsymbol{G}$ the last l columns of $\boldsymbol{\mathcal{C}}_i$

$$\boldsymbol{G} = \boldsymbol{\mathcal{C}}_{(:,N-l+1:N)} \tag{50}$$

And $\boldsymbol{A}$ as

$$\boldsymbol{A} = \boldsymbol{\mathcal{O}}_i^{\uparrow +}\boldsymbol{\mathcal{O}}_i^{\downarrow} \tag{51}$$

where $[\bullet]^+$ denominate the pseudo-inverse. Then, $\mathcal{O}_i^{\uparrow}$ and $\mathcal{O}_i^{\downarrow}$ are obtained from $\mathcal{O}_i$ by removal the last and the first l rows, respectively.

SSI: Steps in short

1. Define l number of dynamical responses (e.g. one vibration response per sensors) and i number of lags.
   - In Practice: Assume a model order n, and then as l is generally fixed, define i as per Eq.(43).
2. Compute the output correlation functions per Eq.(38)
3. Build a Toeplitz $\boldsymbol{T}$(Eq. (39)) or Hankel $\boldsymbol{H}$ (Eq. (40)) matrix.
   - Each individual matrix should have a dimension $l \times l$.
   - The "big" block matrix has a dimension $li \times li$.
4. Compute Singular Value Decomposition (SVD) to the $\boldsymbol{H}$ or $\boldsymbol{T}$ matrix.
5. Split the SVD to obtain the observabiliy $\boldsymbol{\mathcal{O}}$ (Eq.(47)) and controlability $\boldsymbol{\mathcal{C}}$ (Eq.(48)) matrices.
6. Estimate the $\boldsymbol{A}$, $\boldsymbol{C}$ and $\boldsymbol{G}$ matrices as per Eq. (49), (51) and (50) respectively.

Modal parameter estimation: Steps in short

1. Solve the eigenvalue problem for $\boldsymbol{A}$ as per Eq.(4).
2. Natural frequencies in (Hz) are calculated from the eigenvalues as per Eq.(20).
3. Damped natural frequencies (Hz) using only the imaginary part of the eigenvalue as per Eq.(21).
3. Damping ratio is obtained from the eigenvalues as per Eq. (22).
4. Mode shape by the matrix $\boldsymbol{C}$ and eigenvectors $\boldsymbol{v}$ as per Eq.(23).

Why?

- For parameter estimation in OMA, it is required to have previous knowledge of the model order but even if Eq. (43) somehow constraints this in SSI, noise and mathematical inaccuracies do not allow this. (i.e. It is difficult to identify the a gap in the SVD for correct model order selection).
- A conservative approach will be to select a big model order n. BUT... SPURIOUS MODES might appear:
  - Noise modes (e.g. poles from the excitation system).
  - Mathematical modes (e.g. poles created by the model to ensure the mathematical description).

To separate physical poles from spurious poles

Stabilization diagram

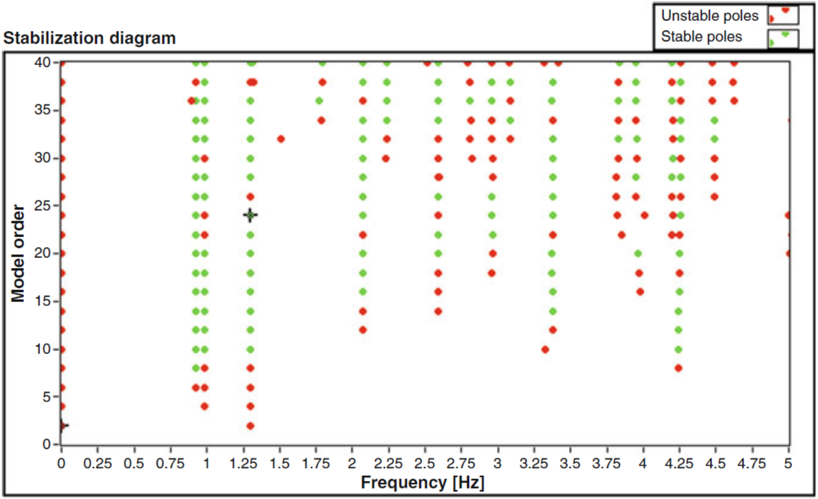


Figure 2: Illustration of a stabilisation diagram. Source: Ranieri, C., OMA for Civil Engineering

How to build it?

- Comparing poles at one model order to the ones in an model order below.
- Fulfill criteria typically criteria around frequency and damping (but also Mode shapes based on Modal Assurance Criterion (MAC)

$$\begin{aligned} \frac{f(n) - f(n+1)}{f(n)} &< 0.01 \quad \text{frequency scatter less than 1\%} \\ \frac{\zeta(n) - \zeta(n+1)}{\zeta(n)} &< 0.05 \quad \text{Damping scatter less than 5\%} \end{aligned} \tag{52}$$

if Mode shapes are available:

$$1 - MAC(\phi(n), \phi(n+1)) < 0.02 \quad \text{mode shape scatter less than 2\%} \tag{53}$$

Spectral analysis. Principles for modal testing

MECE11014 - Advanced Dynamics and Applications 5

Dr David García Cava

- Spectral types and analysis methods
- Modal identification: Experimental modal analysis

Experimental spectral analysis

The response vector from a Frequency Response Function (FRF) and excitation is defended as:

$$\ddot{\boldsymbol{X}}(f) = \boldsymbol{H}(f)\boldsymbol{F}(f) \tag{1}$$

where

$\ddot{\boldsymbol{X}}(f)$ is the frequency domain of the acceleration vector response $\ddot{\boldsymbol{x}}(t)$
$\boldsymbol{F}(f)$ is the frequency domain of the force vector input $\boldsymbol{f}(t)$
$\boldsymbol{H}(f)$ is the FRF matrix

(Note: It is in upper case because it is in the Frequency domain)

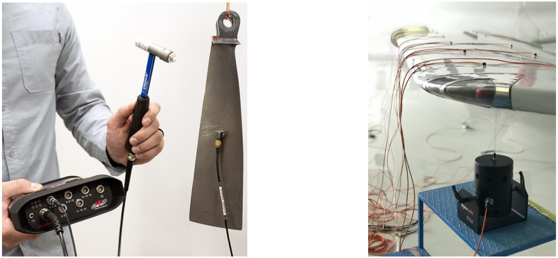


Figure 1: Illustration of experimental vibration testing. Source: Vibration research (left) and Dynalabs (right)

What is the Challenge?

Estimate the spectral content of the vibration response of a structure.

Equipment & Set-up

- Measuring: Sensors - Where?, Quantity? Which type?
- Data Acquisition: Sampling rate, How long?

Analysis for the estimation

- Parametric or Non-parametric methods
- Single Input Single Output (SISO) or Multiple Input Multiple Output (MIMO)
- Output only or Input-Output

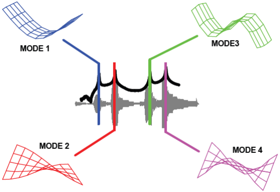


Figure 2: Illustration of experimental Spectral analysis. Source: Peter Avitabile.

Type of spectra (in an ideal world)

Fourier Series:

$$X_k = \int_0^T x(t) e^{-j2\pi k f_0 t} dt \quad (2)$$

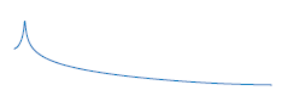
Periodic signal (period $T = 1/f_0$)

Lines at frequencies $n \cdot f_0$

Fourier transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad (3)$$

Transient signal (finite energy)



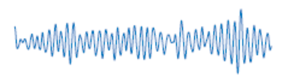
Continuous spectra in energy units

Power Spectral Density:

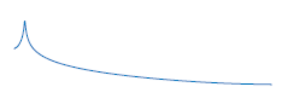
$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f \tau} d\tau \tag{4}$$

$R_{xx}(\tau)$: Autocorrelation function defined as

$$R_{xx}(\tau) = \lim_{T \to \infty} \int_{0}^{T} x(t)x(t+\tau)dt$$



Random signal (finite power)



Continuous spectra in power units

What are the Practical Challenges?

- Difficult to define a single period in a "periodic signals"
- Signals are sampled and quantized (i.e. continuous-to-discretised)
- Signals have finite length
- Noise is embedded in the signals
- Signals are non-stationary

Discrete Fourier Transform

The non-parametric spectral estimators are based on some form of the Discrete Fourier Transform (DFT).

The DFT of a discrete-time signal $x(n) = x(nT)$, with a sampling period $T = 1/f_s$ being f_s the sampling frequency can be defined as follows:

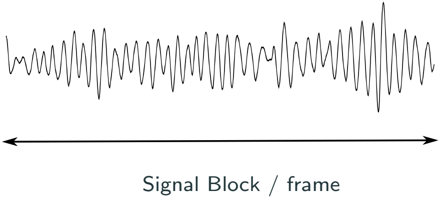
Discrete Fourier Transform

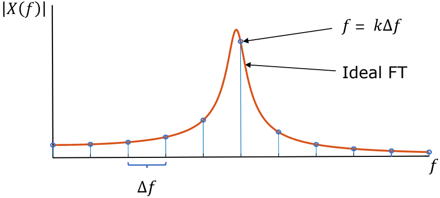
$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi}{N}kn} \tag{5}$$

Inverse Discrete Fourier Transform

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi}{N}kn} \tag{6}$$

for $k = 0, 1, \ldots, N-1$ and $n = 0, 1, \ldots, N-1$





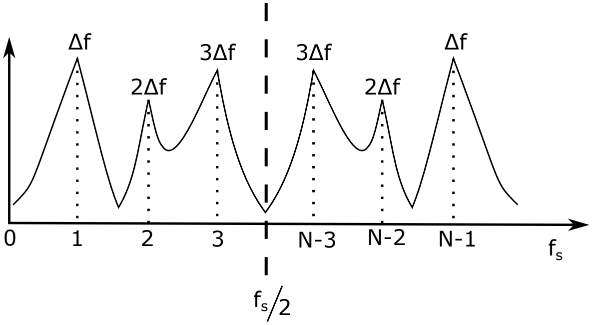
NOTES:

N : Blocksize

$N \cdot T$: Length of the block (general units seconds)

Each $X(k)$ is referred as a frequency bin or frequency line

Frequency resolution of the DFT: The minimum frequency is $\Delta_f = {}^{f_s}/_{N}$



NOTES:
N : Only one half of the spectrum is useful.
$X^*_{N/2} = X_{N/2}$ so that the maximum frequency that we can represent is $N\frac{\Delta f}{2} = f_s/2$

Fast Fourier exercise - toy example

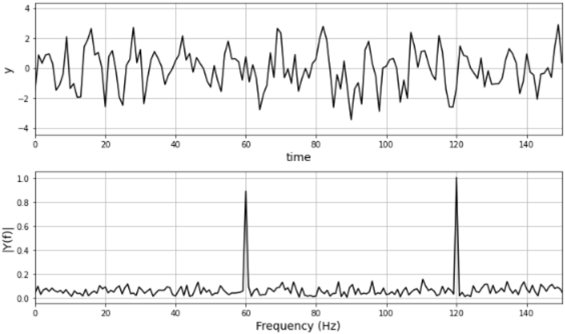


Figure 3: Frequency domain using FFT

Leakage

- Leakage is the effect on calculating the DFT of a signal whose frequency does not match the grid of frequencies determined by the DFT.
- This effect is noticed as ripples around the target frequency content value, or as if the frequency content was "leaking" from the target frequency.

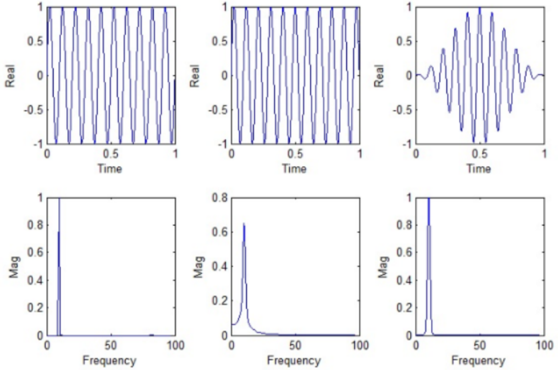


Figure 4: Illustration of Leakage. Left: pure periodic sine non-windowing. Middle: non-periodic sine and non-windowing. Right: non-periodic sine and Hanning windowing. Credit: Crystal instruments

Leakage

Some type of windows to be applied.

- Rectangular: Easy to apply but it might produce spectral leakage.
- Hanning: Designed to reduce the spectral leakage caused by the rectangular window.
- Flattop: Designed to minimize amplitude distortion when accurately measuring the amplitudes of multiple sinusoidal components in a signal.

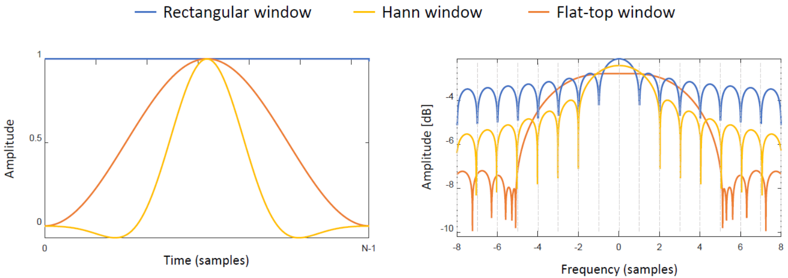


Figure 5: Illustration of Leakage. Left: periodic sine non-windowing. Middle: non-periodic sine and non-windowing. Right: non-periodic sine and Hanning windowing. Credit: Crystal instruments

Leakage exercise - toy example

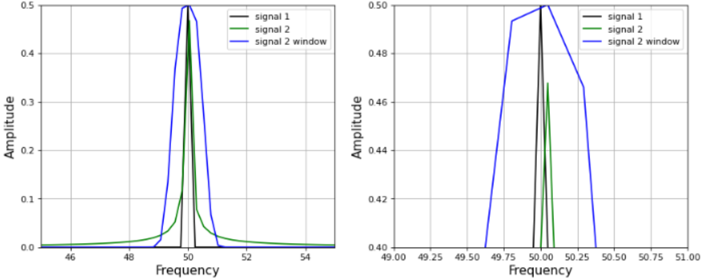


Figure 6: Effect of leakage

Spectral Estimation

- The issue of spectral estimation relates to the calculation of the power of a signal at specific values of frequency (frequency bins).
- While this maybe achieved by different methods, non-parametric DFT-based methods are by far the most popular, due to their simplicity and flexibility.
- As a rule of thumb, non-parametric PSD analysis should be the first step before moving forward to more advanced signal processing methods.

Spectral Estimation - Definitions

Power Spectral Density (PSD):
$S_{xx}(f)$ is defined as the FT of the autocorrelation function $R_{xx}(\tau) = E\{x(t)x(t-\tau)\}$

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f \tau} d\tau \tag{7}$$

with the Wiener-Kintchine relations

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f) e^{j2\pi f \tau} df \tag{8}$$

Properties

- PSD of a real valued process is symmetric $S_{xx}(-f) = S_{xx}(f)$

Cross-Power Spectral Density (CSD):
$S_{yx}(f)$ s defined as the FT of the cross-correlation function $R_{yx}(\tau) = E\{y(t)x(t-\tau)\}$

$$S_{yx}(f) = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-j2\pi f \tau} d\tau \tag{9}$$

with the Wiener-Kintchine relations

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} S_{yx}(f) e^{j2\pi f \tau} df \tag{10}$$

Properties

- CSD is in general a complex function, satisfying $S_{yx}(-f) = S^*_{yx}(f) = S_{xy}(f)$

Welch's PSD estimator:
Calculated on an average of PSD estimates at different signal windows.

$$S_{yy}(k) = \sum_{m=1}^{M} \frac{1}{MN} |Y^m(k)|^2 \tag{11}$$

where $Y^m(k)$ is the DFT of the signal window $y^m(n)$

$$Y^m(k) = \sum_{n=0}^{N-1} w(n) y^m(n) e^{-\frac{j2\pi}{N}kn} \tag{12}$$

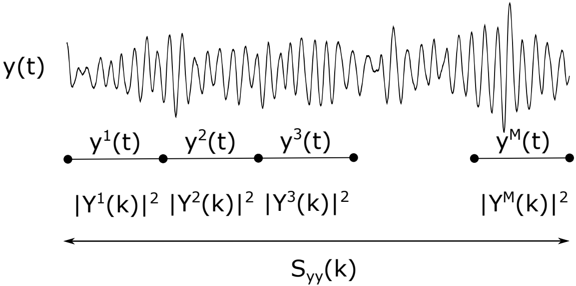


Figure 7: Graphical sketch of Welch's PSD estimator

Spectral Estimation - Single-Input Single-Output

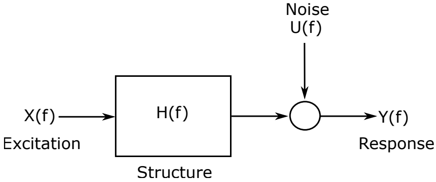
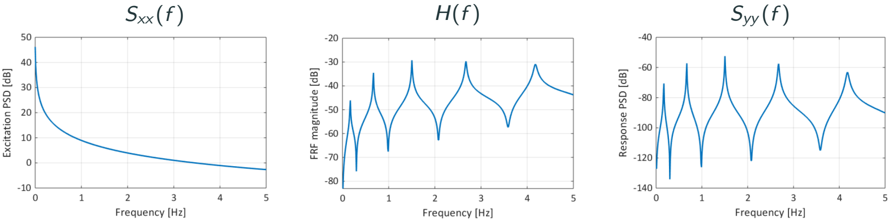


Figure 8: Graphical representation of a Single-Input Single-Output (SISO) Linear Time Invariant (LTI) system with noise

The **Frequency-domain input-output** characterization is:



Relationship of FRF and PSDs

Output PSD

$$S_{yy}(f) = |H(f)|^2 S_{xx}(f) + S_{uu}(f) \quad (13)$$

Output-Input CSD

$$S_{yx}(f) = H(f) S_{xx}(f) \quad (14)$$

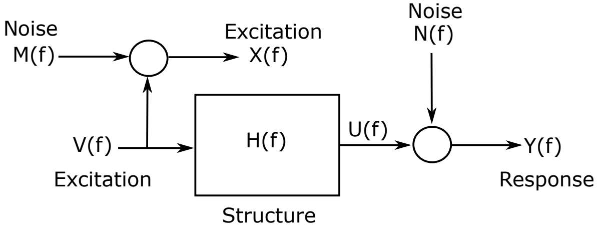


Figure 9: Graphical representation of a Single-Input Single-Output (SISO) Linear Time Invariant (LTI) system with noise in Input and Output

The **Frequency-domain input-output** characterization is

PSD estimated in the output

$$G_{yy}(f) = |H(f)|^2 G_{vv}(f) + G_{nn}(f) \tag{15}$$

PSD estimated in the input

$$G_{yx}(f) = H(f) G_{mm}(f) \tag{16}$$

Then:

Can we separate the effect of noise from the FRF?

No, but we can understand how much it affects.

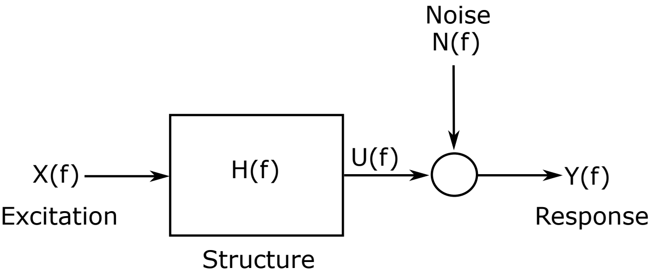


Figure 10: Graphical representation of a Single-Input Single-Output (SISO) Linear Time Invariant (LTI) system with noise in the output.

The assumptions are:

- The input is noise-free
- There is noise in the output

Spectral Estimation - H_1 estimator

The input-output relation in frequency domain is

$$Y(f) = X(f)H(f) + N(f) \tag{17}$$

If we multiply by the conjugate of the input

$$X^*(f)Y(f) = X^*(f)X(f)H(f) + X^*(f)N(f) \tag{18}$$

we can calculate the expected value to obtain PSD and CSD

$$G_{yx}(f) = G_{xx}(f)H(f) + G_{nx}(f) \tag{19}$$

If we assume to have uncorrelated input with noise

$$G_{nx}(f) = 0 \tag{20}$$

H_1 estimator

$$H_1(f) = \frac{G_{yx}(f)}{G_{xx}(f)} \tag{21}$$

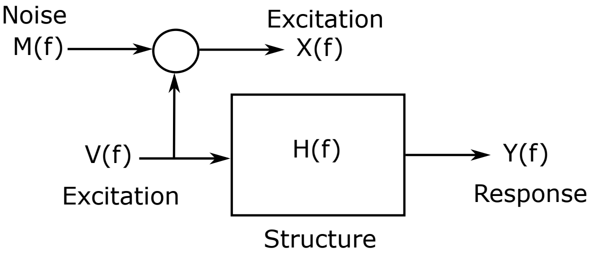


Figure 11: Graphical representation of a Single-Input Single-Output (SISO) Linear Time Invariant (LTI) system with noise in the input.

The assumptions are:

- The output is noise-free
- There is noise in the input

Spectral Estimation - H_2 estimator

The input-output relation in frequency domain is

$$Y(f) = [X(f) - M(f)]\, H(f) \tag{22}$$

If we multiply by the conjugate of the output

$$Y^*(f)Y(f) = [Y^*(f)X(f) - Y^*(f)M(f)]\, H(f) \tag{23}$$

we can calculate the expected value to obtain PSD and CSD

$$G_{yy}(f) = G_{xy}(f)H(f) + G_{my}(f) \tag{24}$$

If we assume to have uncorrelated input with noise

$$G_{my}(f) = 0 \tag{25}$$

H_2 estimator

$$H_2(f) = \frac{G_{yy}(f)}{G_{xy}(f)} \tag{26}$$

Coherence function

Then, we have $\boldsymbol{H_1}$ estimator

$$H_1(f) = \frac{G_{yx}(f)}{G_{xx}(f)} \tag{27}$$

and $\boldsymbol{H_2}$ estimator

$$H_2(f) = \frac{G_{yy}(f)}{G_{xy}(f)} \tag{28}$$

Both estimators are based on the assumption of no noise in the output or input respectively, but it can happen that both assumptions are not possible in practice. Let's assume Measured, Actual and Noise.
Then for H_1,

$$G_{xx}(f) = G_{vv}(f) + G_{mm}(f) \tag{29}$$

$$H_1(f) = \frac{G_{yx}(f)}{G_{vv} + G_{mm}(f)} \neq H(f) \tag{30}$$

Then for H_2,

$$G_{yy}(f) = G_{uu}(f) + G_{nn}(f) \tag{31}$$

$$H_2(f) = \frac{G_{uu}(f) + G_{nn}(f)}{G_{xy}} \neq H(f) \tag{32}$$

Coherence function

In general, the FRF magnitude $|H(f)|$ is bounded by H_1 and H_2 as shown in Eq.(33).

$$|H_1(f)| \leq |H(f)| \leq |H_2(f)| \tag{33}$$

Coherence function:
Defined as the ratio between H_1 and H_2 estimators:

$$\gamma^2_{yx}(f) = \frac{H_1(f)}{H_2(f)} = \frac{|G_{yx}(f)|^2}{G_{xx}(f)G_{yy}(f)} \tag{34}$$

With limits: $0 \leq \gamma^2_{yx}(f) \leq 1$.
Then, $\gamma^2_{yx}(f) \equiv 1$ if $H_1(f) \equiv H_2(f)$

Potential sources of error in the FRF estimation

Bias error:

- Poor frequency resolution will add bias in the PSD estimation. (Specially for low-damping cases).
- Delay between input and output will also generate some bias.
- The best solution is to increase the blocksize for the above cases.

Random error:

- Noise in the input or output will generate random error.
- Increasing number of averages when estimating PSD will mitigate the potential random error.

In a similar manner, the procedure described in SISO can be expanded for Multiple-Input Multiple-Output (MIMO).

Structural dynamics modelling-validation idea

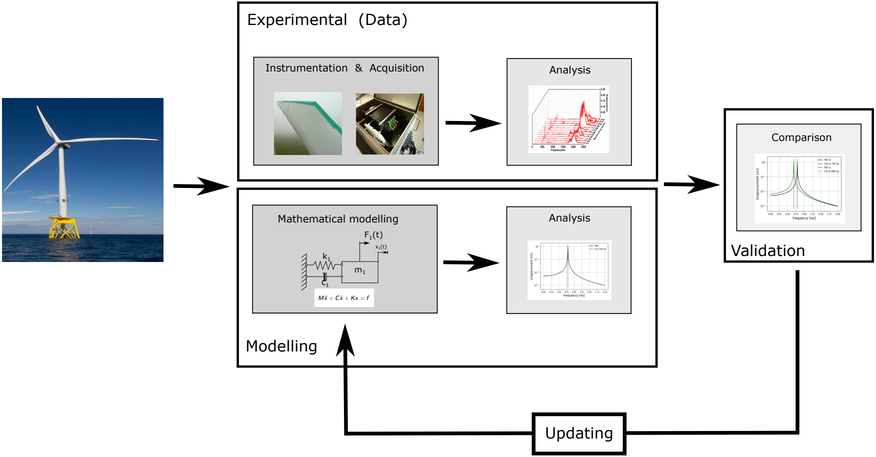


Figure 12: Graphical representation of structural dynamics-modelling approach

Definition:

Is the process of determining and describe the **dynamic characteristics** of a system (e.g. mechanical, civil...) by means of *natural frequencies, damping factors and mode shapes*.
And eventually **using them** to formulate a mathematical model for understanding their behaviour, design, structural integrity, monitoring.

Observations:

- The **vibration response** of a linear time-invariant (LTI) dynamic system can be defined as the linear combination of simple harmonic called **natural modes of vibration**. (a.k.a. *Fourier transform coverts a waveform in sum of sine and cosines*)
- The can be classified based on number of inputs and outputs: SISO, SIMO, MISO and MIMO.
- They can be used to identify properties:

Indirect Methods (Modal) - Natural frequency, Damping ratio, Mode shape.
Direct Methods (Mechanical) - Stiffness, Mass, Damping coefficient.

Parametric vs. non-parametric identification methods

Non-parametric:
FFT-based, PSD or FRF

- Simple and quick
- Basic user expertise
- Less accurate

Parametric:
Auto-Regressive model family

- A more intricate identification process
- Require significant input from user
- Potentially high accuracy
- Perhaps simple for simulation and dynamic analysis

Experimental Modal Analysis (EMA) and Operational Modal Analysis (OMA) are techniques used to extract the modal information of a structure based on measured vibration responses.

In Short:

- Excitation/responses are measured from the structure
- A mathematical model is built for the measured data (i.e. vibration responses)
- The mathematical model is then estimated
- Modal parameters are extracted from such model

There are many methods that will rely on the **data available**, excitation and/or response, and based on the **extraction's domain** from the frequency or time .

Generalities: Experimental Modal Analysis (EMA)

- An experimental set-up is prepared out of its normal operating environment
- The excitation is controlled: An impulse hammer or a shaker is used to excite the structure. The location of the excitation and its dynamics can be defined a-priori.
- Excitation and response signals are captured.

For EMA there are two main ways of exciting structures:

- Impact hammer excitation
- Shaker excitation

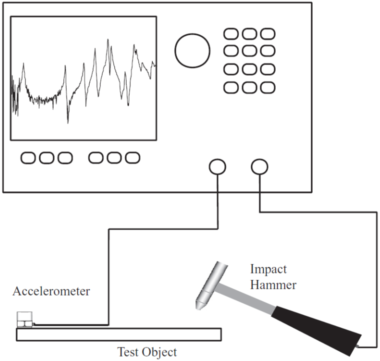
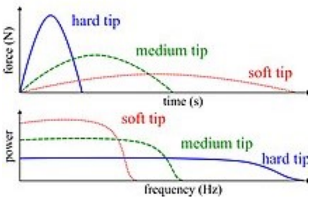


Figure 13: Graphical representation of a Impact hammer excitation test. Source: Brandt 2011



accelerometer

Modal Analysis with hammer can be implemented by:

- **Roving hammer**: Measuring the vibration response at one point (accelerometer fixed location) and exciting at different locations (impact hammer moving).
- **Roving accelerometer**: Measuring the vibration response at different points (accelerometer location moving) and exciting at fixed location (impact hammer fixed).
  Note: It is also possible to have many accelerometers fixed and hit at one location, so that all the readings from the accelerotmers come at once.

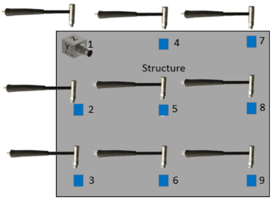


Figure 14: Roving hammer

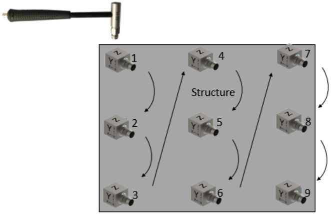


Figure 15: Roving accelerometer

More info:
https://community.sw.siemens.com/s/article/modal-tips-roving-hammer-versus-roving-accelerometer

EMA - Shaker excitation

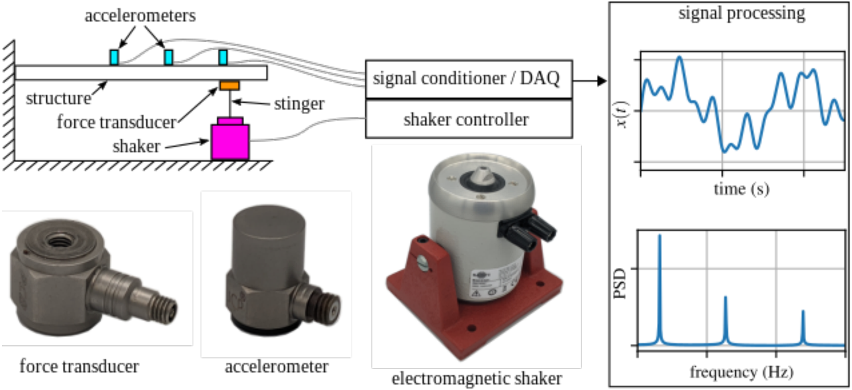


Figure 16: Graphical representation of a Shaker excitation test. Source: wikipedia

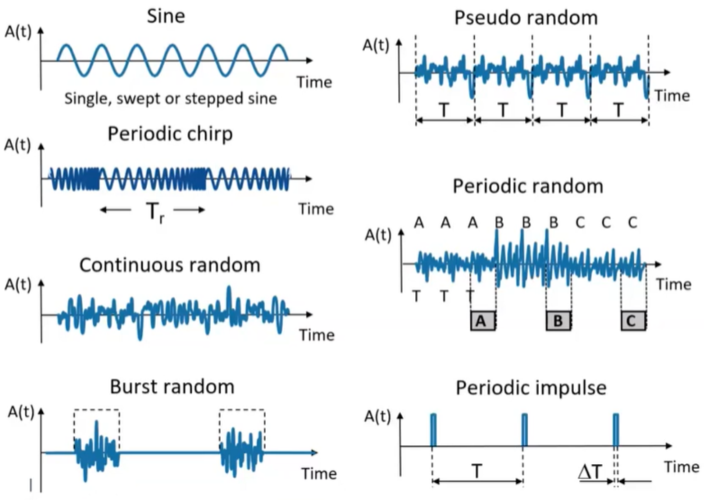


Figure 17: Shaker excitation signals type

More info:

Modal Analysis using Hammer or Shaker Excitation by HBK

Impact hammer excitation

ADVANTAGES

- Fast, No fixture, Relatively easy to implement, Portable and not expensive
- Easy to excite "high" frequencies (it depends on the object to test)

DISADVANTAGE

- Signal-to-Noise Ratio (SNR) is low in comparison with continuous excitation
- Risk on exciting nonlinearities is high
- High peak force needed for large structure (i.e. it might cause damage)
- It doesn't allow multiple-input force

Why may be preferred compared with Shaker excitation?

- Measuring quickly, precision less important
- Lightly damped structures and difficult to excite by shakers at their natural freq.
- Investigation of suitable excitation locations for shaker excitation

Shaker excitation

ADVANTAGES

- Broad rage of excitation signals
- Parameter optimisation possible (e.g. SNR, Leakage...)
- Good SNR compared with impact and allows multiple-input

DISADVANTAGE

- Fixture of structure (i.e. shaker, force transducer...)
- Potential dynamic loading from shaker and stringer
- Potential coupling between shakers (MINO)
- More expensive and less user friendly

Modal parameters estimation - Basic fundamentals

By the definition of the FRF (Lecture 1 - Eq. 24, here now Eq.(35)), the FRF can be re-written in partial expansion form.

$$H_{pq}(j\omega) = \sum_{r=1}^{n} \left(\frac{A_{pqr}}{j\omega - s_r} + \frac{A^*_{pqr}}{j\omega - s^*_r} \right) \tag{35}$$

with residues defined as $A_{pqr} = Q_r \phi_{pr} \phi_{qr} = \frac{1}{j2\omega_{dr} m_r} \phi_{pr} \phi_{qr}$

$\boldsymbol{A}_r$ has a rank one, meaning that they can be decomposed as:

$$\boldsymbol{A}_r = Q_r \boldsymbol{\Phi}_r \boldsymbol{\Phi}_r^T = \begin{bmatrix} \phi_{1r} \\ \vdots \\ \phi_{Nr} \end{bmatrix} [\phi_{1r} \dots \phi_{Nr}] \tag{36}$$

being $\boldsymbol{\Phi}_r$ the eigenvector of mode r.

Then, the FRF of a lineal MDOF system with N SDOF modes can be obtained as the sum of the N SDOF FRFs, this is called **modal superposition** (see Eq.(35). The transfer function matrix is completely characterised by the modal parameters.

- At (or near) the **natural frequency** of a mode, the **vibration response** is nominated by their corresponding **mode shape at the resonance**.
- In reality there is always a small contribution of the other modes
- Exciting harmonically a one DOF r at the damped frequency of the r^{th}-mode, results in a displacement vector (approx.) proportionally to the mode shape vector of the r^{th}-mode.

Modal parameters estimation methods

Frequency domain-based methods

- Peak-Picking method
- Circle fit-method
- Frequency domain decomposition
- Inverse FRF method
- Least-square method
- Dobson's method
- Rational fraction polynomials

Time domain-based methods

- Ibrahim time domain (ITD) method
- Stochastic subspace identification (SSI) method
- Random decrement (RD) method
- Auto-regressive family methods (ARMA, ARMAX)
- Least-squares time domain method
- Least-squares complex exponential (LSCE) method

Peak-piking: Assumption

Around the resonance, the FRF is dominated by its main vibration mode. The contribution of other modes are (assumed) to be negligible.

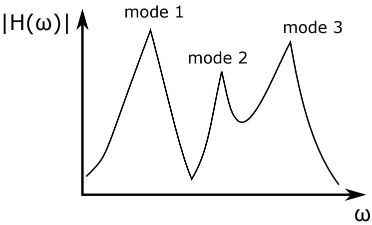


Figure 18: FRF of a dynamical systems

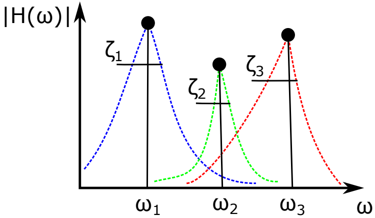


Figure 19: SDOF interpretation of the FRF

The modal frequencies can be estimated from the frequency response data by observing:

- The **magnitude** of the frequency response is a <u>maximum</u>.
- The **imaginary part** of the frequency response is a <u>maximum or minimum</u>.
- The **real part** of the frequency response is <u>zero</u>.
- The **phase** of the frequency response is <u>90°</u> and its sign will indicate the orientation.

Identification method: Peak Picking - Half Power method

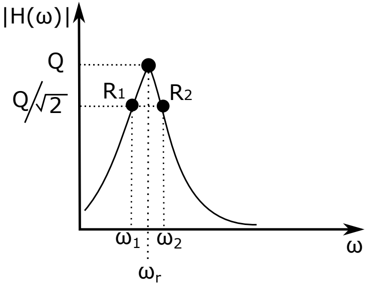


Figure 20: Half Power method

Modal identification per peak

- Natural frequency selection: $\omega_r = \omega_{peak}$
- Damping ratio: $\zeta_r = \frac{\omega_2 - \omega_1}{2\omega_r}$
- Modal constant (related to the mode shape) can be proven to be: $A_r = 2Q_r\zeta_r\omega_r^2$

The modal constant is derived from the transfer function of SDOF written for a $r^{th}-mode$ knowing that $2\zeta_r\omega_r = \frac{c_r}{m_r}$ and $\omega_r^2 = \frac{k_r}{m_r}$.

$$H(\omega) = \frac{A_r}{-\omega^2 + j\omega\frac{c_r}{m_r} + \frac{k_r}{m_r}} = \frac{A_r}{-\omega^2 + 2j\zeta_r\omega\omega_r + \omega_r^2} \tag{37}$$

Then, for $\omega = \omega_r$ (around resonance)

$$|H(\omega_r)| = \frac{A_r}{2\zeta_r\omega_r^2} \tag{38}$$

So, that:

$$A_r = |H(\omega_r)| 2\zeta_r\omega_r^2 \tag{39}$$

Mode shape Extraction

We know the modal constant components are per mode r as

$$A_{pqr} = Q_r \phi_{pr} \phi_{qr} \tag{40}$$

Therefore when looking at the FRF point component H_{pp} , for excitation at p and response at p, we have that for each mode $r = 1 \dots R$

$$A_{ppr} = Q_r \phi_{pr} \phi_{pr} \tag{41}$$

NOTE:

- Q_r is a scaling factor and does not affect the estimate of the mode.
- A_{ppr} is obtained before for the mode r using $|H_{pp}(\omega)|$

Then, the q^{th} component of the r mode can be obtained either from:

Point FRF $H_{qq}(\omega)$

$$\phi_{qr} = \sqrt{A_{qqr}} \tag{42}$$

Transfer FRF $H_{pq}(\omega)$

$$\phi_{qr} = \frac{A_{pqr}}{\phi_{pr}} \tag{43}$$

Identification method: Peak Picking - Quadrature approach

An alternative approach: **Quadrature approach**. General Assumptions:

- Assumption that the coupling between the modes is light.
- Mechanical structures are often very lightly damped ($< 1\%$).
- When there is little modal coupling between the modes, the structural response at a modal frequency is completely controlled by that mode.
- For Single-Degree-of-Freedom systems, the frequency response function (accelerance) at resonances is purely imaginary.
- The value of the imaginary part of the frequency response function at resonance, for structures with lightly coupled modes, is proportional to the modal displacement.
- The magnitude of the imaginary part of the frequency response function at a number of points on the structure, the relative modal displacement at each point can be found.
- Making an excitation and response measurement at the same point and in the same direction, the mode shape can be scaled in absolute units.

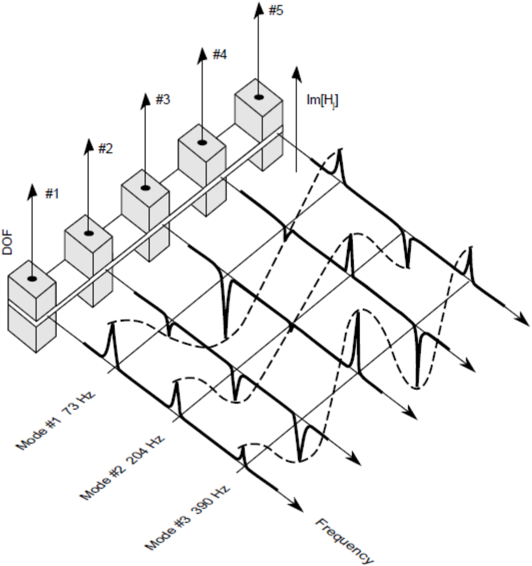


Figure 21: Graphical representation of Quadrature approach. Source: B&K

Peak Piking: Disadvantages

- This method relies on the peak FRF values, which are difficult to measure accurately
- Therefore, it struggles of producing accurate modal data
- Damping is estimated from the half power points only. No other FRF data points are used. The half power points have to be interpolated, as it is unlikely that they are two of the measured data points.
- Requires knowledge (recorded data) of the input signal (excitation).
- Cannot handle noise efficiently.
- The assumption of SDOF behavior in the vicinity of resonance is not accurate; in practice the nearby modes will also contribute.

$\Downarrow$

However, despite all these disadvantages, it is still used in practice.

Engineering applications

Advanced Dynamics and Applications 5

Dr David García Cava

- Model updating
- Damage identification (Structural Health Monitoring (SHM))
- Environmental and Operational Variability (EOV)

Why and how? - In general words

- Understanding how structures (things) vibrate is basically helping us to describe how they performance
- Identifying the real (i.e. experimental, operational) modal parameters help us to **update** models to add information to the design.
- Tracking modal parameters or other features coming from the vibration responses, can support short- and long-term performance of the structures/mechanical systems such as **damage diagnosis**.
- **Structural health monitoring** is heavily based on the vibration response of the structures and mechanical systems to dynamical loading
- There is a big challenge to address: the **Environmental and operational variability**(EOV) affect to the vibration response and therefore modal identification and its applications might face complications.

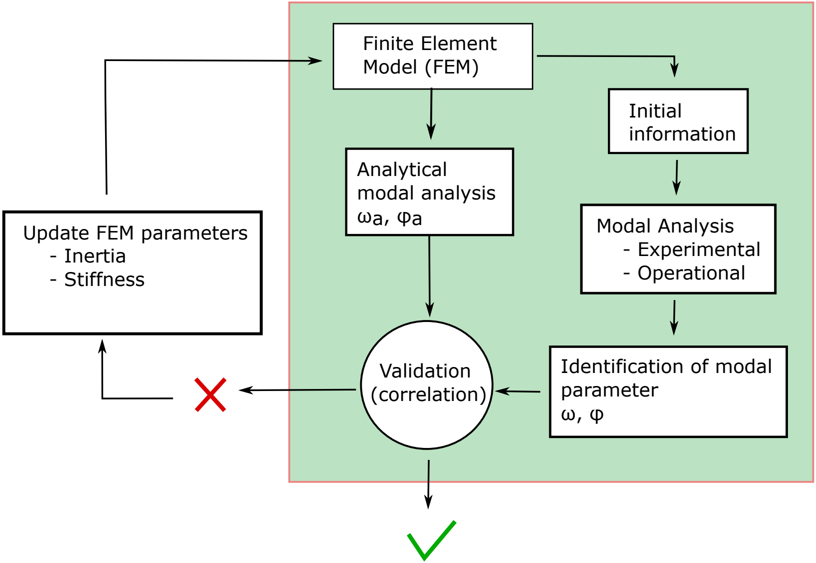


Figure 1: Graphical representation of a generic model updating strategy

How we can compare modal parameters from models and experimental?

- We can compare time series, frequency responses correlations
- We look at the natural frequencies
- We can look at the mode shapes
- IN SHORT: Anything that can be compared based on vibration responses from models and experimental

One metric for model shapes is **Modal Assurance Criterion (MAC)** and its different forms.

$$\mathrm{MAC}(\phi_{ir}^T, \phi_{jk}) = \frac{|\phi_{ir}^T \phi_{jk}|^2}{(\phi_{ir}^T \phi_{ir})(\phi_{jk}^T \phi_{jk})} \tag{1}$$

where ϕ_{jk} is the test modal vector j for mode k and ϕ_{ir} is the compatible analytical/model modal vector i for mode r.

- MAC values are preferred as a measure of similarity between vectors, regardless of their magnitude or phase.
- MAC values can be transformed to a distance using the transformation $d = 1 - \mathrm{MAC}$

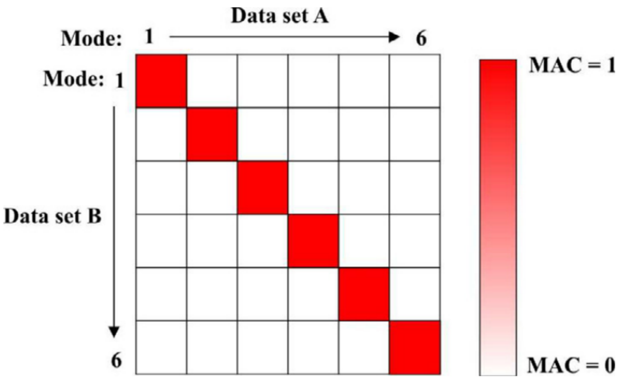


Figure 2: MAC representation. Source [1]

More information:

- [1] Pastor, Miroslav, Michal Binda, and Tomáš Harčarik. Modal assurance criterion. Procedia Engineering 48 (2012): 543-548.
- [2] Allemang, Randall J. The modal assurance criterion–twenty years of use and abuse. Sound and vibration 37.8 (2003): 14-23.

Model Updating - Example: Description

Let's evaluate this example: Ferrari, Rosalba, et al. Model updating of a historic concrete bridge by sensitivity-and global optimization-based Latin Hypercube Sampling. Engineering Structures 179 (2019): 139-160.

Figure 3: Picture of the Bridge

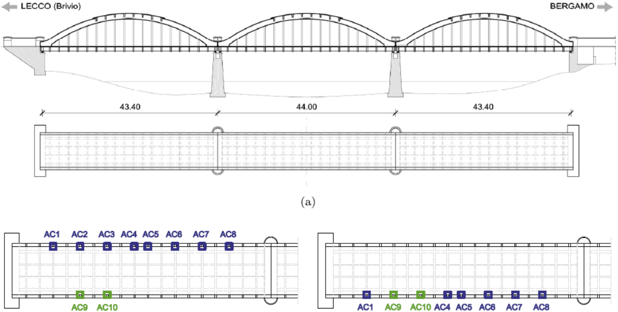


Figure 4: Geometry and accelerometers location

Model Updating - Example: Modes identification

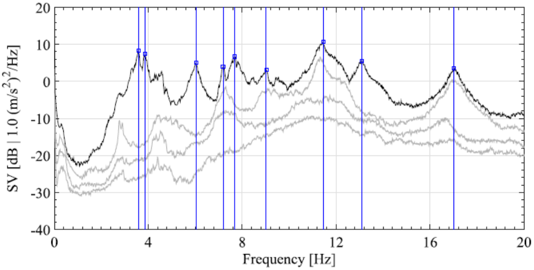


Figure 5: Singular Value curves resulting from classical Frequency Domain Decomposition

Table 1
Experimentally identified modal frequencies (FDD and SSI methods) and damping ratios (SSI method), and mutual MAC indexes.

Mode	1	2	3	4	5	6	7	8	9
f^{FDD} [Hz]	3.564	3.857	6.018	7.178	7.690	9.009	11.38	13.09	17.02
f^{SSI} [Hz]	3.449	3.887	5.968	7.146	7.592	8.928	11.39	13.04	16.99
Δf^{exp} [%]	−3.23	0.78	−0.83	−0.45	−1.27	−0.90	0.11	−0.35	−0.16
ζ^{SSI} [%]	4.60	4.09	3.17	1.51	2.82	1.67	1.28	2.01	1.44
MAC	0.997	0.991	0.998	0.989	0.991	0.990	0.938	0.987	0.935

Figure 6: Comparison FDD an SSI method estimation

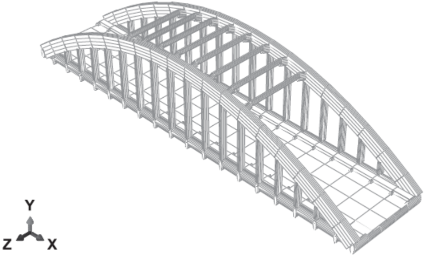


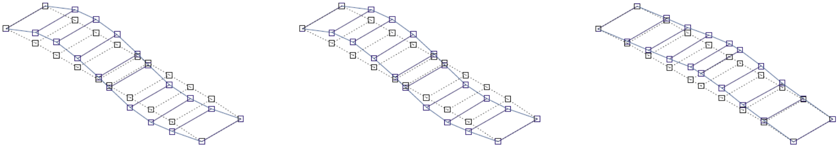
Figure 7: FEM Bridge

Table 3
Comparison between experimental (identified) and numerical (FEM base model) modal properties.

Mode	Description		Identified freq. (FDD) [Hz]	FEM base model freq. [Hz]	Δf^{bm} [%]	Mutual MAC
V1	1^{st}	bending	3.564	3.785	6.20	0.9986
T1	1^{st}	*torsion*	*3.857*	*4.040*	*4.74*	*0.0424*
V2	2^{nd}	bending	6.018	5.914	−1.73	0.9921
T2	2^{nd}	torsion	7.178	6.523	−9.13	0.9913
V3	3^{rd}	bending	7.690	7.856	2.16	0.9951
T3	3^{rd}	torsion	9.009	8.904	−1.17	0.9340
T4	4^{th}	torsion	11.38	10.66	−6.31	0.6859
V4	4^{th}	bending	13.09	12.33	−5.84	0.9721
T5	5^{th}	torsion	17.02	15.74	−7.51	0.9627

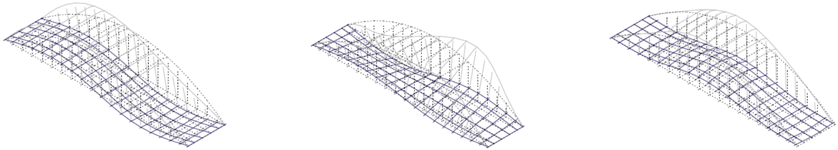
Figure 8: FEM Bridge natural frequencies

Model Updating - Example: Mode Shapes



Mode 1 $f_1^{FDD} = 3.564\,\mathrm{Hz}$ | Mode 2 $f_2^{FDD} = 3.857\,\mathrm{Hz}$ | Mode 3 $f_3^{FDD} = 6.018\,\mathrm{Hz}$

Figure 9: First 3 Mode Shapes Bridge Experimental



Mode 1 **(V1)** $f_1^{\mathrm{bm}} = 3.785\,\mathrm{Hz}$ | Mode 2 **(T1)** $f_2^{\mathrm{bm}} = 4.040\,\mathrm{Hz}$ | Mode 3 **(V2)** $f_3^{\mathrm{bm}} = 5.914\,\mathrm{Hz}$

Figure 10: First 3Mode Shapes Bridge FEM

Model Updating - Example: MAC

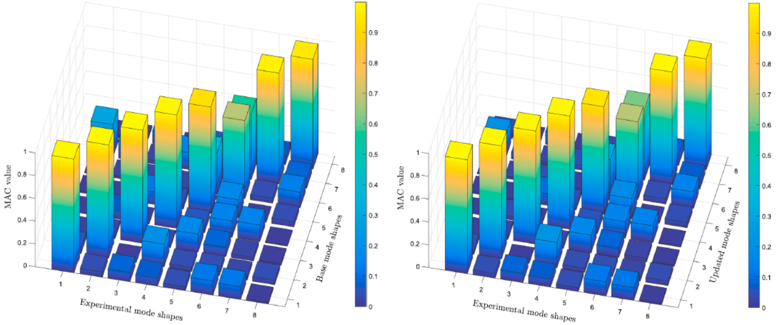


Figure 11: MAC Exp vs. FEM before update(left) and MAC Exp vs. FEM after update (right)

MAIN PHASES OF CASE STUDY ON BRIVIO BRIDGE (1917)

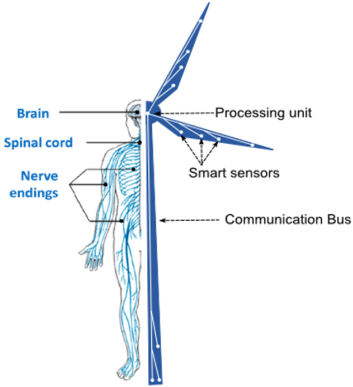
- Experimental campaign with vibrational measurements on spans of the bridge
- OMA modal dynamic identification based on experimental data (FDD + SSI)
- Assembly of linear elastic FEM base model by manual tuning
- LHS Sensitivity Analysis to select structural parameters to be updated
- LHS Global optimization based on modal properties
- Selection of updated parameters and final model-updated FEM model

Figure 12: Updating strategy

Structural Health Monitoring (SHM)

Definition: The process of implementing a damage identification strategy for aerospace, civil and mechanical engineering infrastructure.

(Farrar, C.R., and Keith W. Phi. Trans. R. Soc. A: Math., Phy. Eng. Sci. 365.1851 (2007): 303-315.)



SHM has the different levels?

- **Damage detection**:
  Is damage present?
- **Damage location**:
  Where is damage located?
- **Damage quantification**:
  How big is damage located?
- **Damage Prognosis**:
  How long can the structure operate safely?

Structural Health Monitoring (SHM)

A patter recognition approach

- Operational evaluation
- Data acquisition and cleansing (also normalisation)
- Feature selection and normalisation
- Statistical model development for feature discrimination

Vibration-based SHM

- Refers to SHM based on vibration signals
- Vibration signals include accelerations, displacements, strains, loads, torques...

The VSHM principle

- Structure or mechanical system is instrumented
- Damage introduces changes in local properties (mass, stiffness...)
- Structural properties determine the way structures vibrate.
- Environmental and Operational parameters will affect structural properties and therefore how structures vibrate
- Damage can be "potentially" capture in the features extracted by the vibration responses

VSHM - Principle

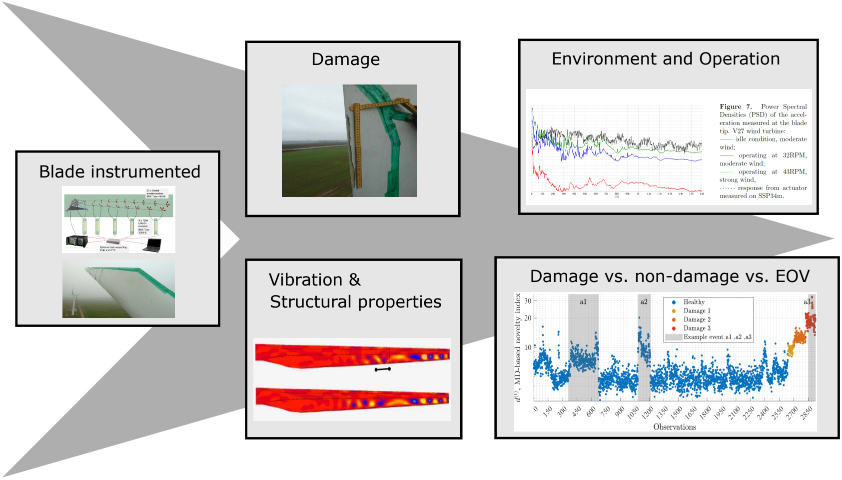


Figure 13: VSHM principle

VSHM - Why?

- **Verification of designs**: Making the structure complies with the designs and, if not, update the design models.
- **Damage diagnosis**: Preventing structures to failure while minimising Operational and Maintenance (O& M) costs.
- **Life extension**: Maximising the operational life span of structure.

Benefits

- Early detection of damages
- Reduction of the O&M Costs and better planning of maintenance interventions
- Reduction of the probability of catastrophic failures
- Updating and informing deigns

Pitfalls

- Increase costs in hardware
- Complex data analysis
- Difficulties in data storage

Definition of damage

Let's observe differences in the dynamics of a healthy and damaged structure.
Looking at FRF.

$$\begin{aligned} \boldsymbol{H}_o(j\omega) &= (\boldsymbol{K}_o - \boldsymbol{M}_o\omega^2 + j\boldsymbol{C}_o\omega)^{-1} \Longrightarrow \text{Healthy FRF} \\ \boldsymbol{H}_d(j\omega) &= (\boldsymbol{K}_d - \boldsymbol{M}_d\omega^2 + j\boldsymbol{C}_d\omega)^{-1} \Longrightarrow \text{Damaged FRF} \end{aligned} \tag{2}$$

The difference between FRFs will be used to detect damage.

$$\Delta\boldsymbol{H}(j\omega) = \boldsymbol{H}_o(j\omega) - \boldsymbol{H}_d(j\omega) \tag{3}$$

In theory (because of EOVs), if $\Delta\boldsymbol{H}(j\omega) = 0$ no damage.
Looking at eigenvalue problem.

$$(\boldsymbol{K} - \lambda_n\boldsymbol{M})\,\phi_n = \boldsymbol{0} \tag{4}$$

Assume a perturbation $\Delta\boldsymbol{K}_d$ (alteration of stiffness is more common) for the damage case.

$$\left([\boldsymbol{K} - \Delta\boldsymbol{K}_d] - \tilde{\lambda}_n\boldsymbol{M}\right)\tilde{\phi}_n = \boldsymbol{0} \tag{5}$$

Then, damage alters the eigenvalues

$$D = \left(\boldsymbol{K} - \tilde{\lambda}_n\boldsymbol{M}\right)\tilde{\phi}_n = \Delta\boldsymbol{K}_d\tilde{\phi}_n \tag{6}$$

Note:
In general $k \downarrow \Rightarrow \omega \downarrow$. However, in early stages of damage changes modal parameters might not be sensitive to damage.

Damage sensitive features (DSF)

- Mathematically, we define a DSF as the column vector $\boldsymbol{x}_i \in \mathbb{R}^n$ where n indicates the dimension of the DSF.
- In general, we will try to compact the DSF as much as possible while carrying as much information on the structure as possible.

Many DSF can be used

- Modal parameters (i.e. ω, ζ, ϕ) and/or PSD, FRF...
- Simple DSF from signal statistics. For a univariate signal y_n bein $n = 1, \ldots N$

Feature	Definition
Peak amplitude	$y_{peak} = \max_{n=1,\ldots,N} \lvert y_n \rvert$
Mean	$\bar{y} = \frac{1}{N}\sum_{n=1}^{N} y_n$
Root-Mean-Squared (RMS) value	$y_{rms} = \sqrt{\frac{1}{N}\sum_{n=1}^{N} y_n^2}$
Variance	$\sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(y_n - \bar{y})^2$

Feature	Definition
Standard deviation	$\sigma = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(y_n - \bar{y})^2}$
Skewness	$\gamma = \frac{1}{n\sigma^3}\sum_{n=1}^{N}(y_n - \bar{y})^3$
Kurtosis	$\kappa = \frac{1}{n\sigma^4}\sum_{n=1}^{N}(y_n - \bar{y})^4$
Crest factor	$y_{CF} = y_{peak}/y_{rms}$

Figure 14: Basic statistical DSF

Mitigating Environmental and Operational Variations (EOV)

Challenge: Structures might be exposed to Environmental and Operational Variations (EOV) that will affect the their dynamics.

Those EOVs can be more significant that the effect of damage on the structural dynamics. Therefore, this might mask damage sensitivity in the DSF and therefore damage diagnosis ad prognosis could be badly affected.

Figure 15: Wind farm. Source: www.offshorewind.biz. November 29, 2013. Siemens

How EOVs can affect DSF?

Variations imposed by Operational conditions of the structure

Examples:
Rotational speed turbine,
Playload of vehicle

→

DSFs exhibit changes of regime:
DSFs shift to a different type of distribution

Variations imposed by transient events

Examples:
Extreme meteorological conditions, earthquakes, blasts. . . ,
Transition periods between operational regimes

→

DSFs show inconsistent values:
DSFs have an sudden of irregular values – outliers

Variations imposed by Sensitivity of physical properties to environmental factors

Examples:
Dynamics that are dependent on environmental conditions (temp., aeroelasticity, hydroelasticity. . .)

→

DSFs show a drifting behavior:
DSFs appear to have time-dependent characteristics (non-stationary)

Examples of EVOs affecting dynamics

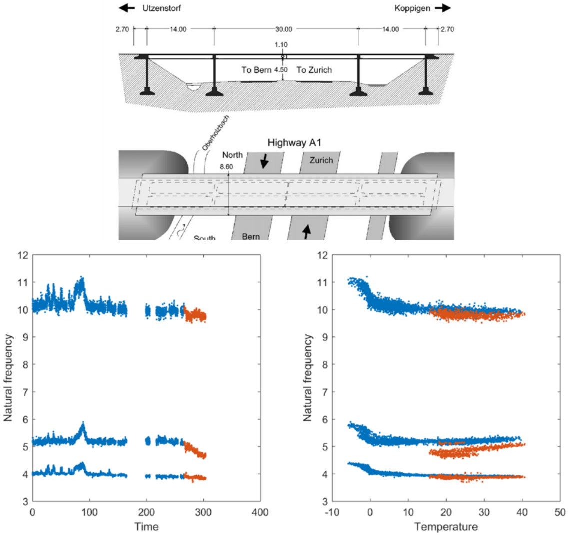


Figure 16: Z24 Bridge. Natural Freq. over time (left), Natural Freq. over Temp (right).

Ref: Maeck and de Roeck, "Description of Z24 Benchmark", Mechanical Systems and Signal Processing, 17(1), 127-131, 2003

Examples of EVOs affecting dynamics

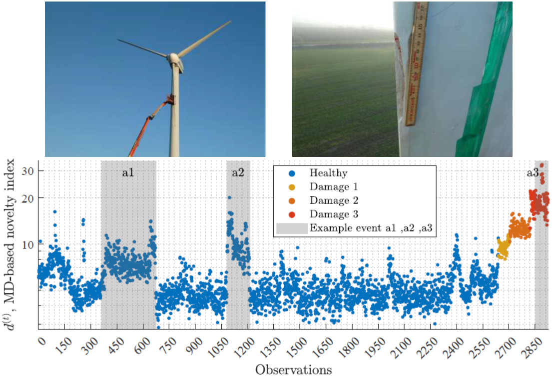
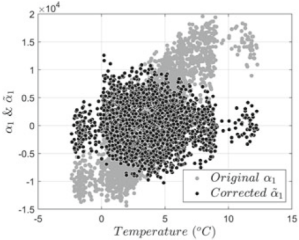


Figure 17: V27 Wind Turbine



Ref: Movsessian, Artur. "Data-driven frameworks for robust and interpretable damage detection in wind turbine blades." (2022).

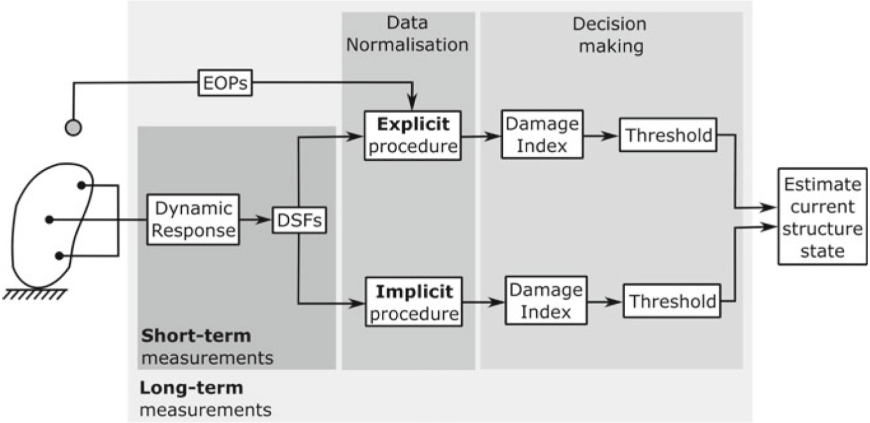


Figure 18: Explicit vs. Implicit procedure

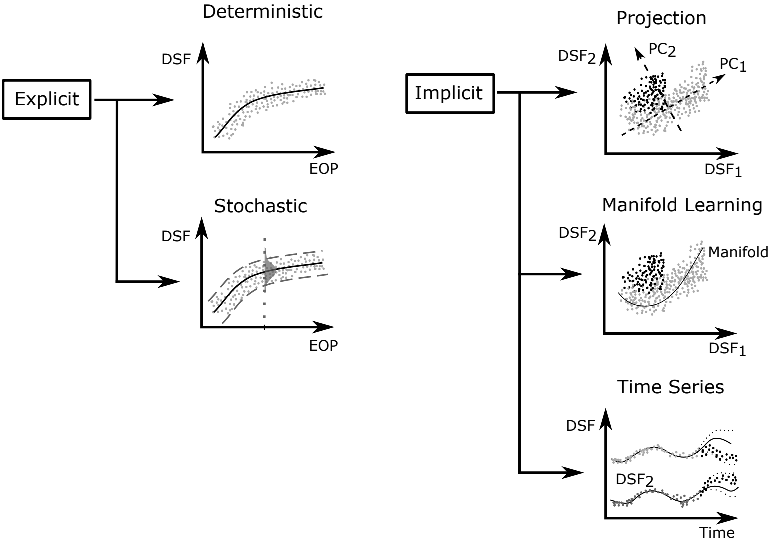


Figure 19: Types Explicit vs. Implicit procedures

EOVs mitigation example

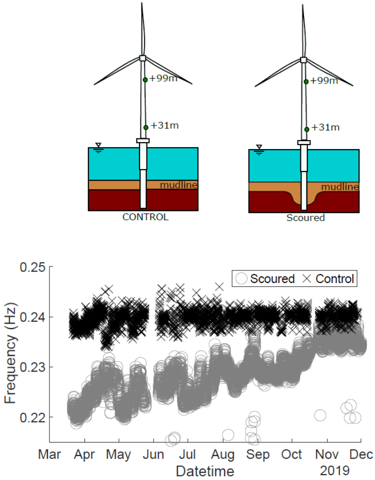


Figure 20: Explicit example

Ref.: Qu, K., Cava, D. G., Killbourn, S., & Logan, A. (2022, June). Operational modal analysis for scour detection in mono-pile offshore wind turbines. In European Workshop on Structural Health Monitoring (pp. 668-678). Cham: Springer International Publishing.

Impulsive Dynamics. Initial concepts and Applications

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William Rankine Building, 1.19

This presentation contains:

1. Definition of impulsive load
2. Differences between quasi-static and dynamics

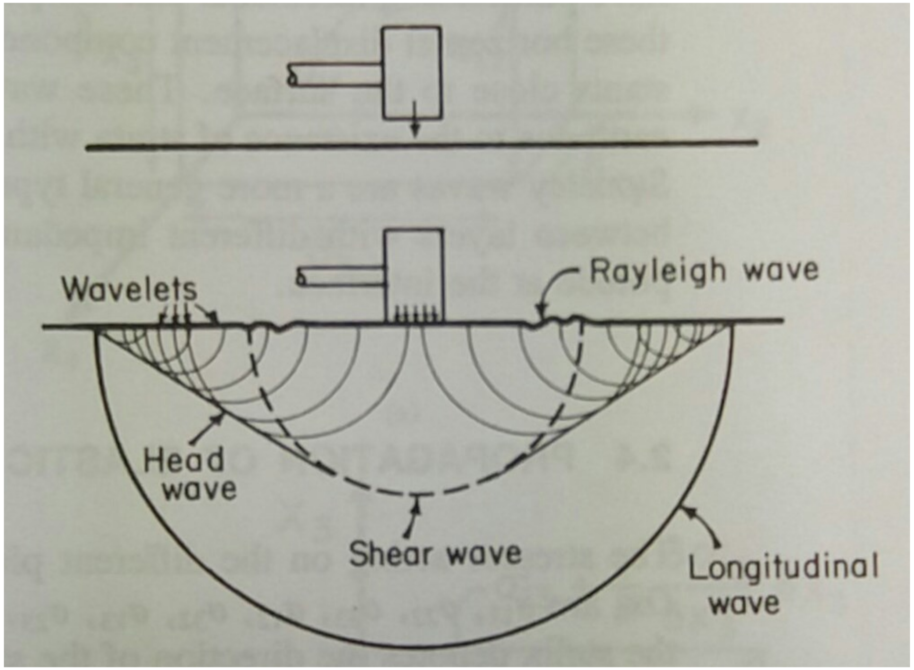
Initial definitions

Mechanical waves

A mechanical wave is the result of discrete particle oscillation. The summation of these oscillations results in a wave that travels across the medium. They transfer mechanical (strain) and kinetic energy, which induces the strains and stresses in our structures.

In this Youtube example, you can imaging each person as a material particle, and the resulting disturbance, as the mechanical wave. https://youtu.be/kZF7QC3tsJY.

Mechanical waves



Tensile and compression waves
Impacts and explosions produce tensile waves which propagates through the structures.

These waves propagates at very high velocity, so the failure of the structures might occur far away from the impact point.

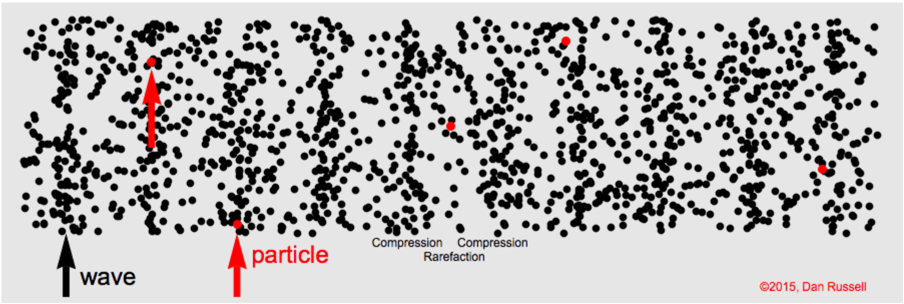
Understanding the wave propagation phenomenon is crucial for the design an evaluation of components subjected to impact

Wave propagation

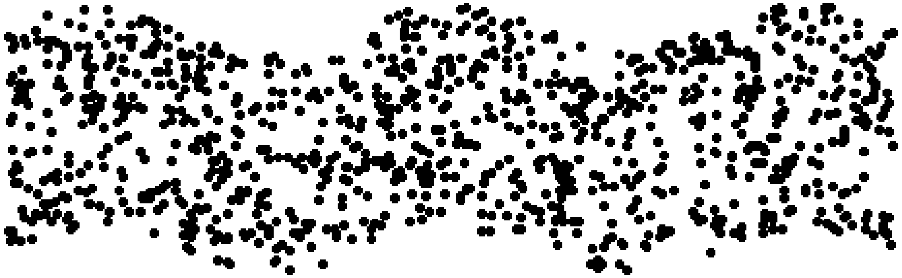
Wave propagation velocity (c) is defined as the distance travelled per unit of time by a sound wave as it propagates through an elastic medium.

Wave propagation velocity in air is about 343 m/s

Types of mechanical waves



Longitudinal waves
Particle movement **parallel** to the direction of the wave propagation.

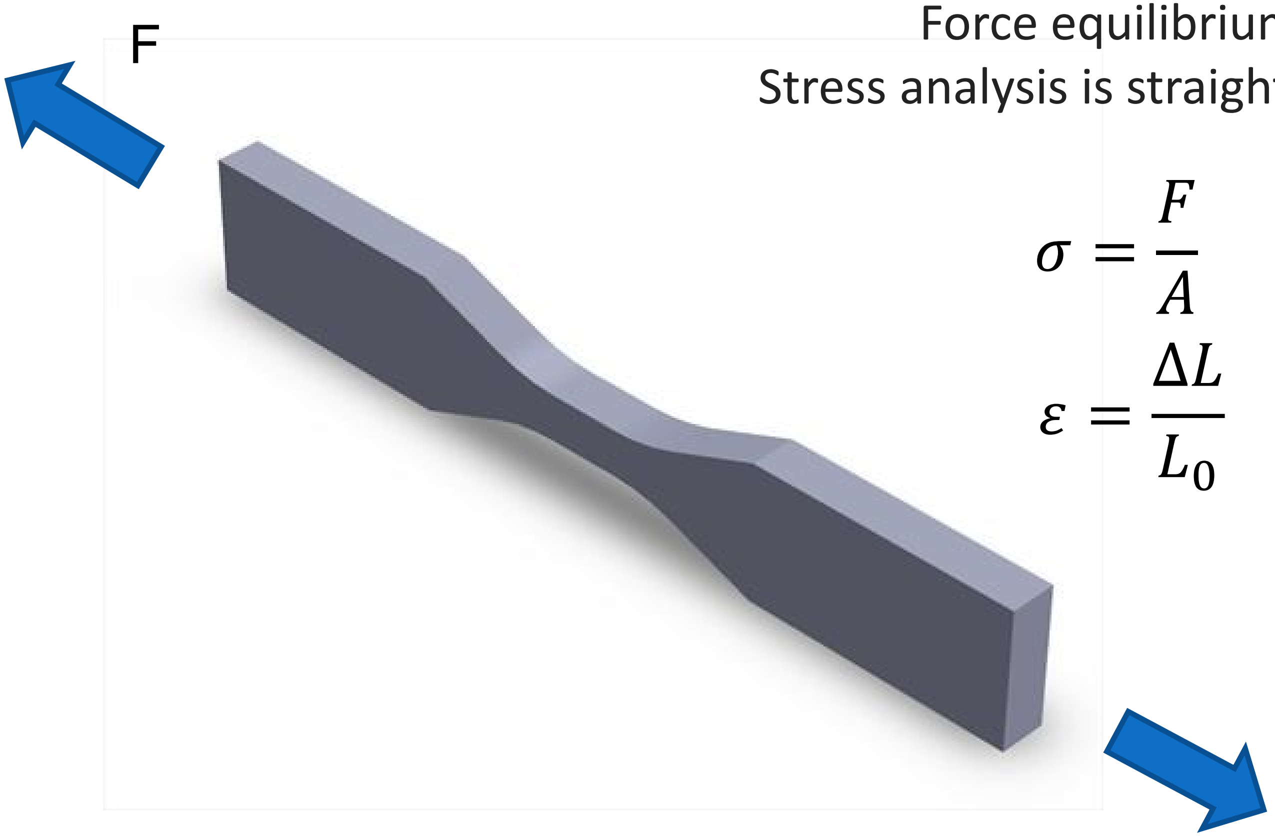


Transverse waves
Particle movement **perpendicular** to the direction of the wave propagation.

Combined
Seismic waves: Rayleigh surface waves, love waves, etc.

Wave propagation in dynamic testing

Force equilibrium.
Stress analysis is straight forward



$$\sigma = \frac{F}{A}$$

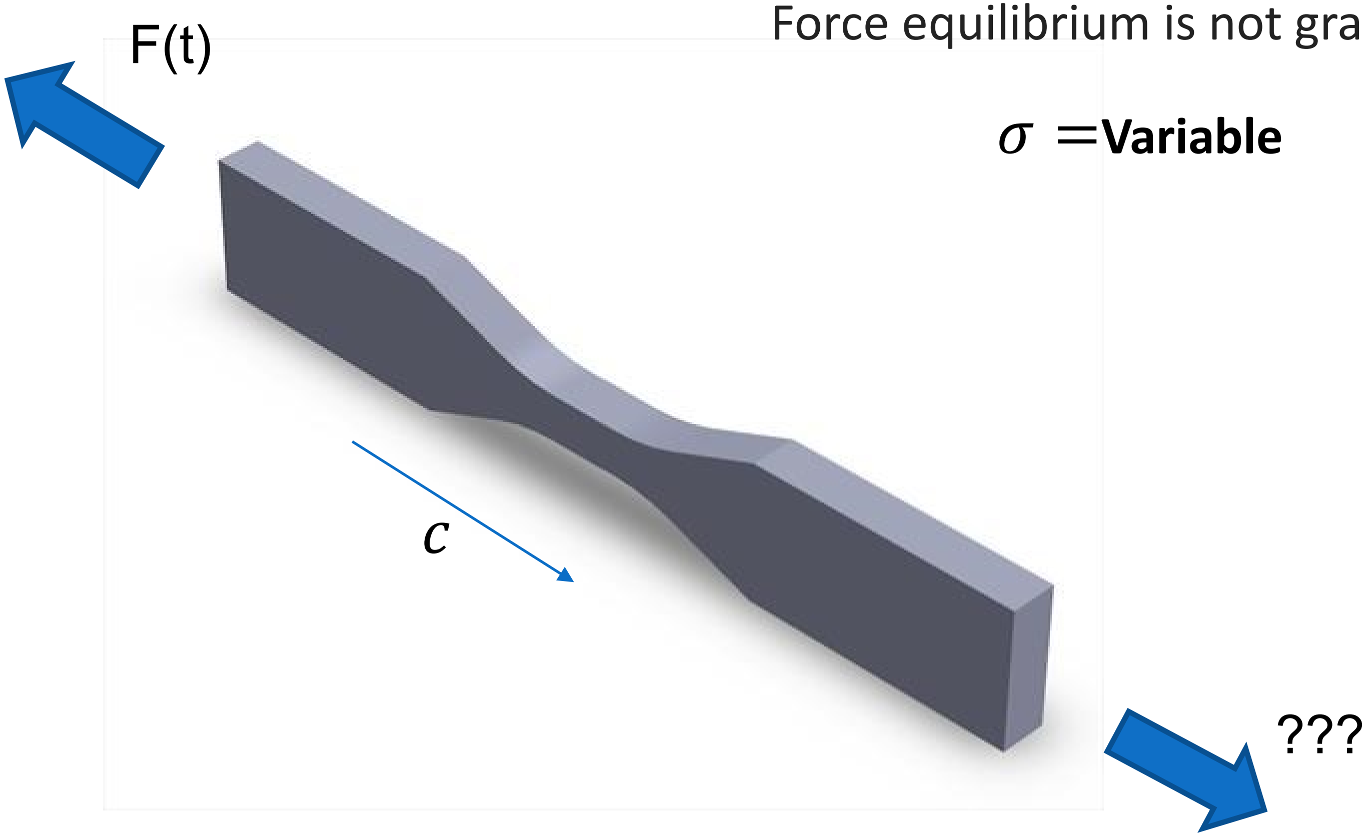
$$\varepsilon = \frac{\Delta L}{L_0}$$

Tensile tests under **quasi-static** loading
https://youtu.be/D8U4G5kcpcM

Wave propagation in dynamic testing

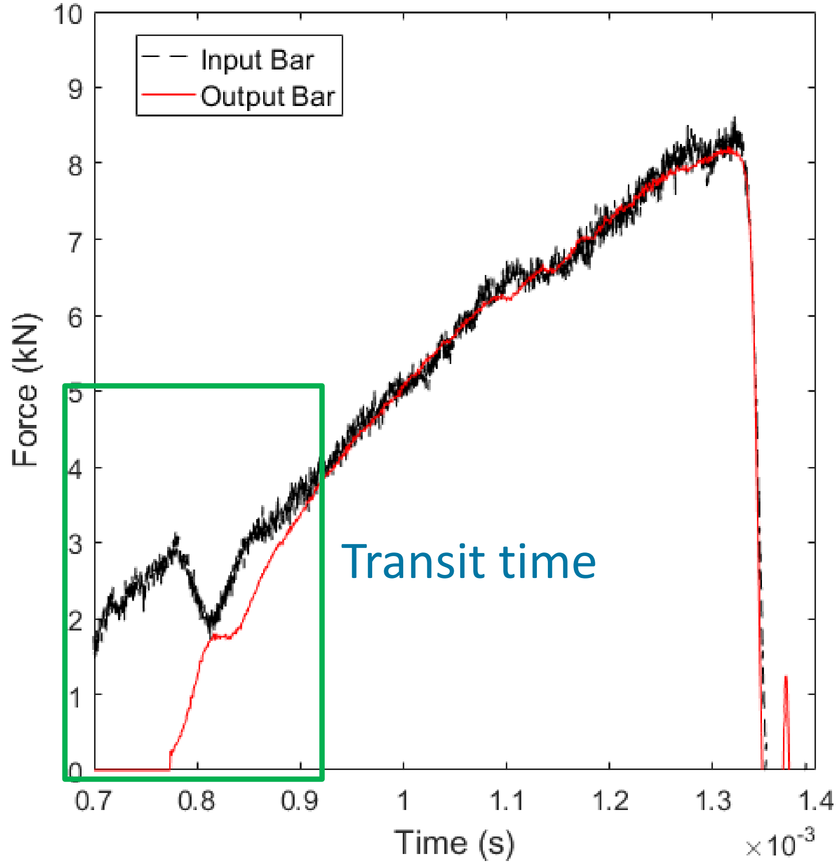
Force equilibrium is not granted

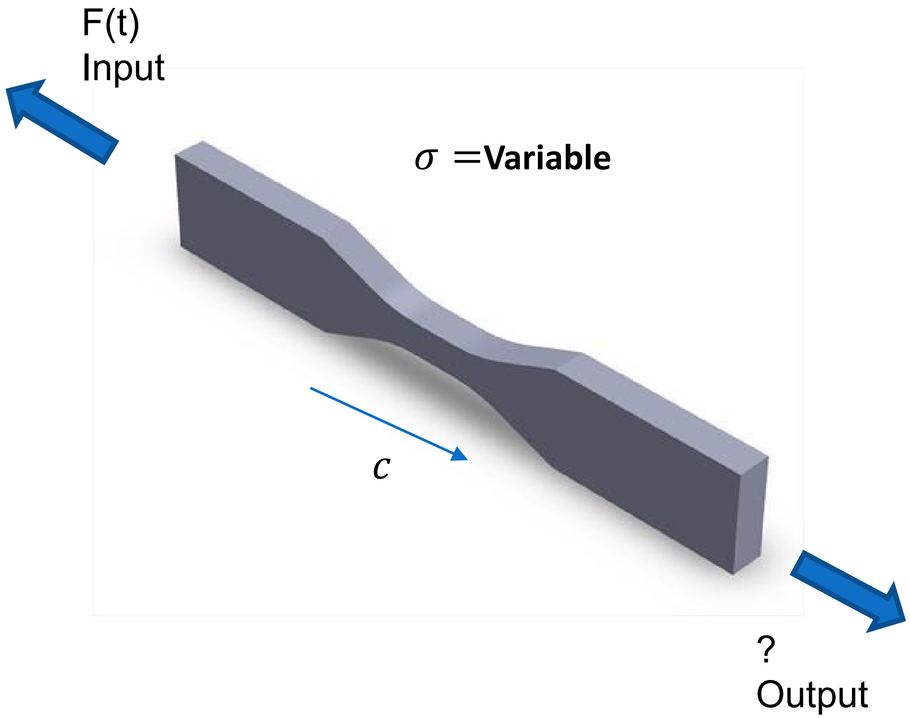
σ =**Variable**



Tensile tests under **dynamic** loading

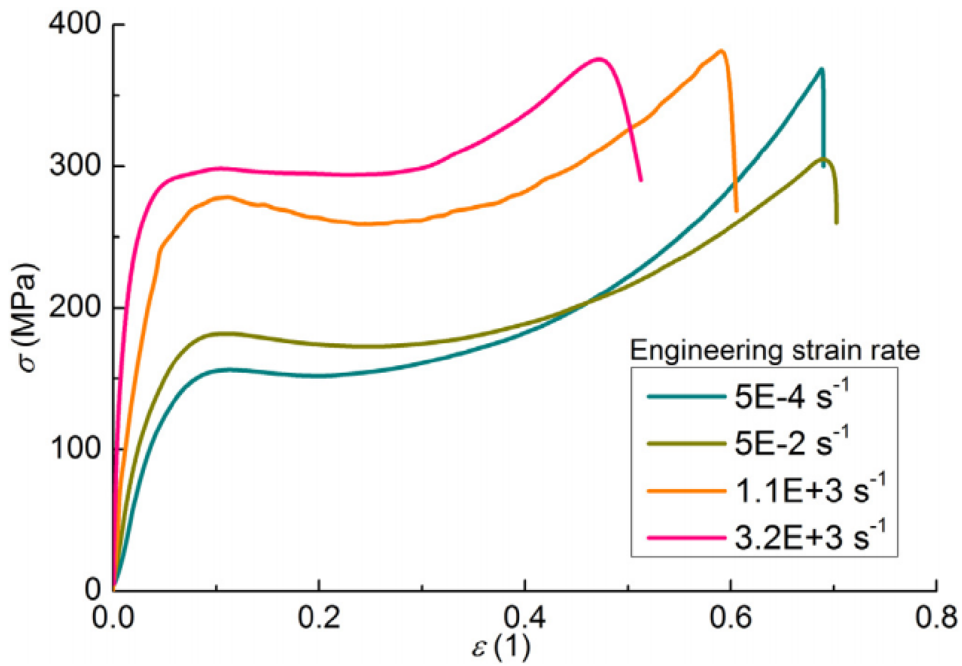
Wave propagation in mechanical testing





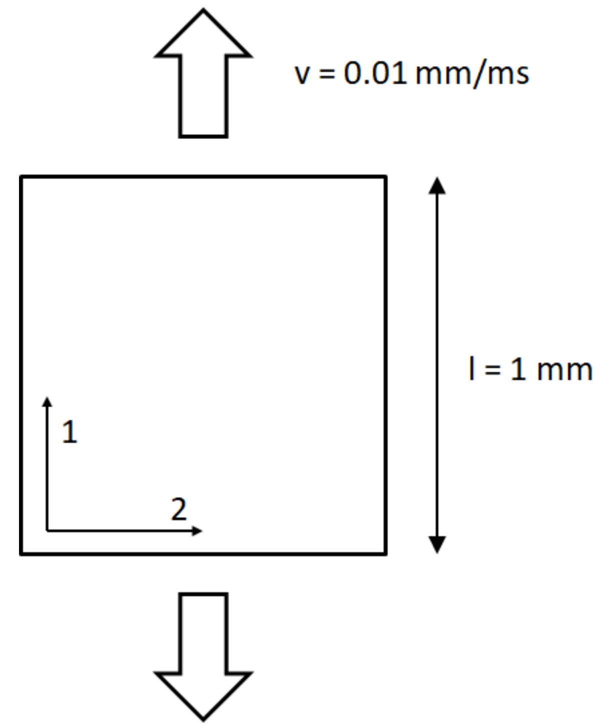
Under **dynamic** loading the force equilibrium is not granted

Strain rate dependency



Epoxy resin 8552
Cui et al, 2016

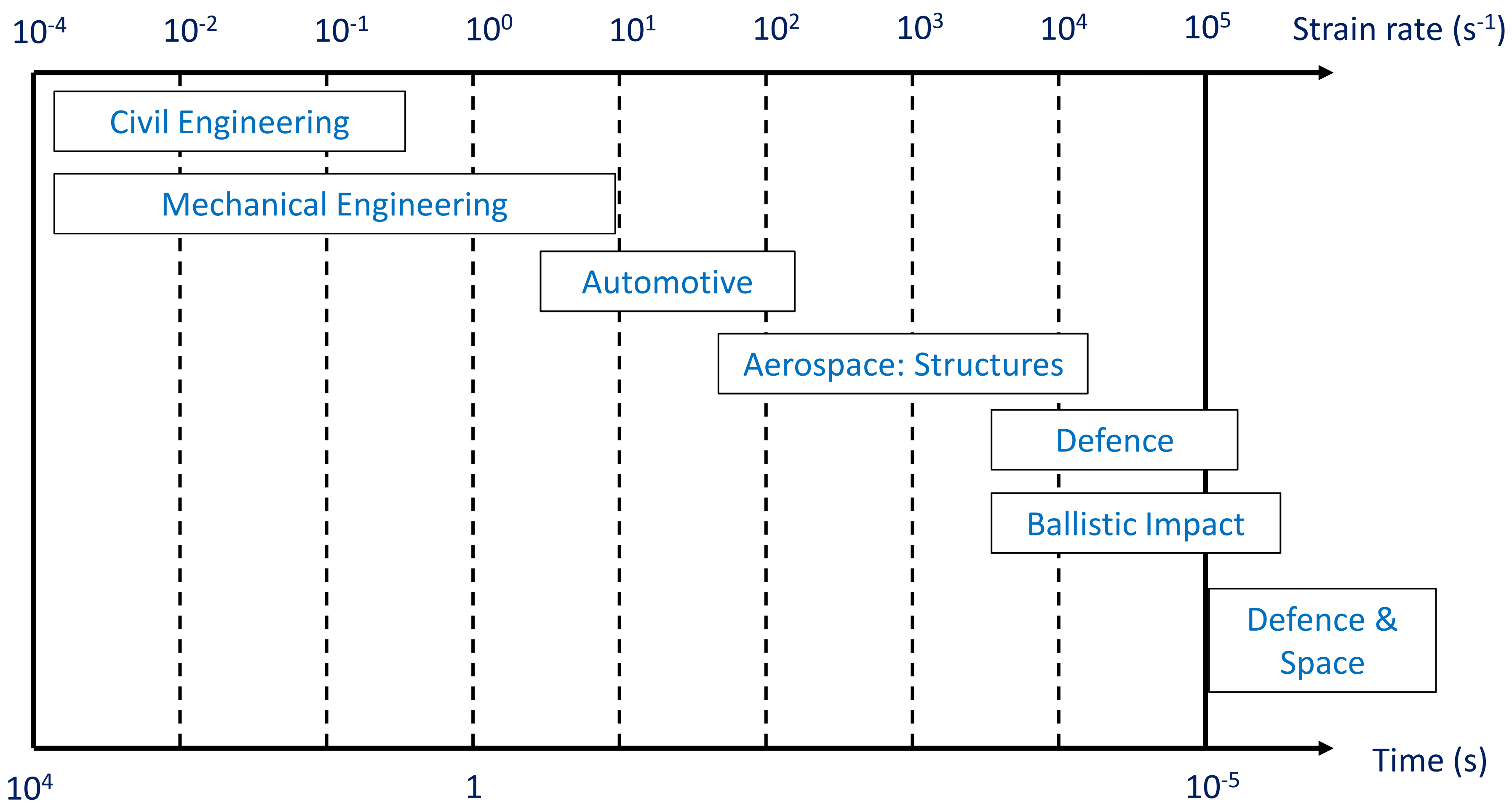
Most materials tend to increase the strength and stiffness under high strain rates



$$\varepsilon = \frac{\Delta l}{L} \qquad \dot{\varepsilon} = \frac{\Delta \varepsilon}{\Delta t} = \frac{\Delta l}{\Delta t L}$$

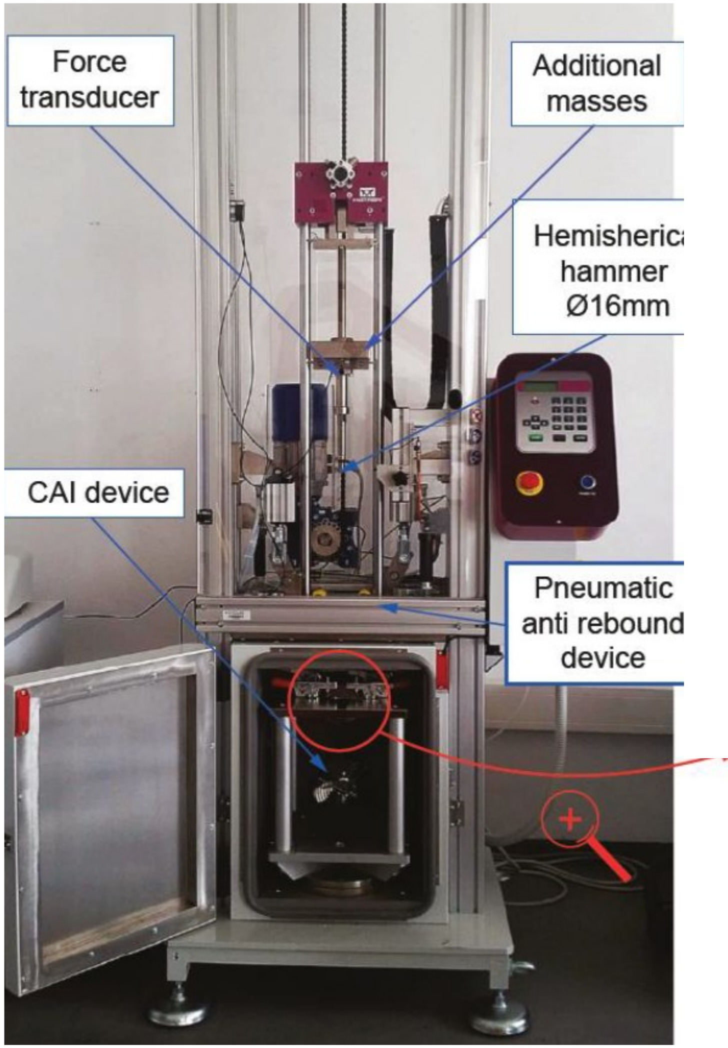
$$\dot{\varepsilon} = \frac{0.01mm}{1ms}\frac{1}{1mm} = 0.01ms^{-1}$$

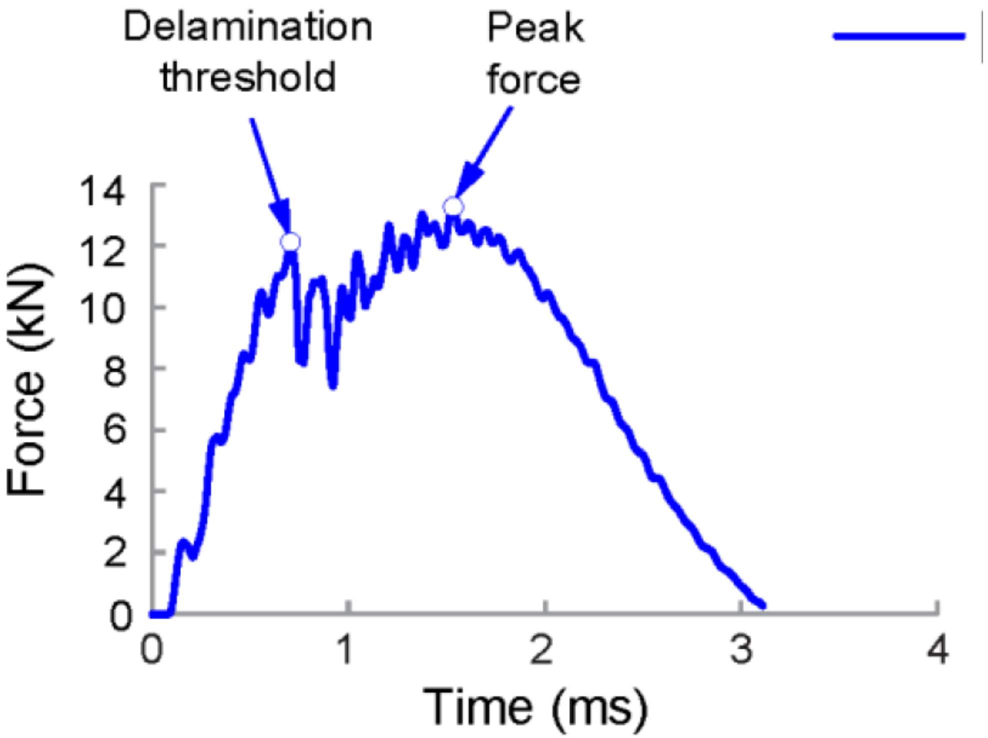
Time scales, quasi-static vs dynamic



Why an impulsive load produces more damage than a quasi-static load?

Reaction forces during impact. E.g., drop weight tower





García-Moreno et al. Polymers. 2019

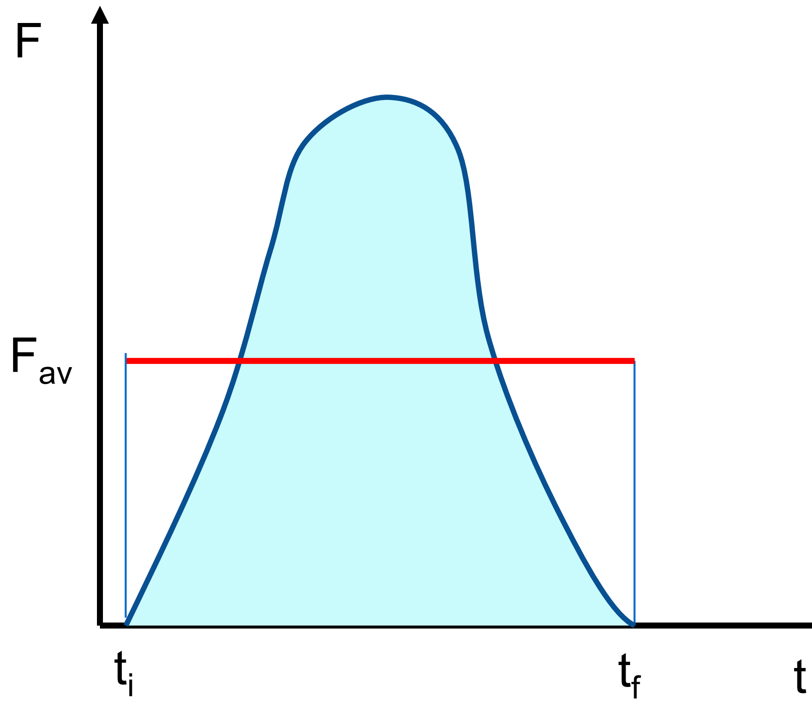
Action-reaction contact forces (accelerations to change the impact velocities). Rest of forces, such as gravity, are negligible.

Impulse and Average Force

Impulse of a force $\vec{I}$ is defined as the increment of momentum $\vec{p}$:

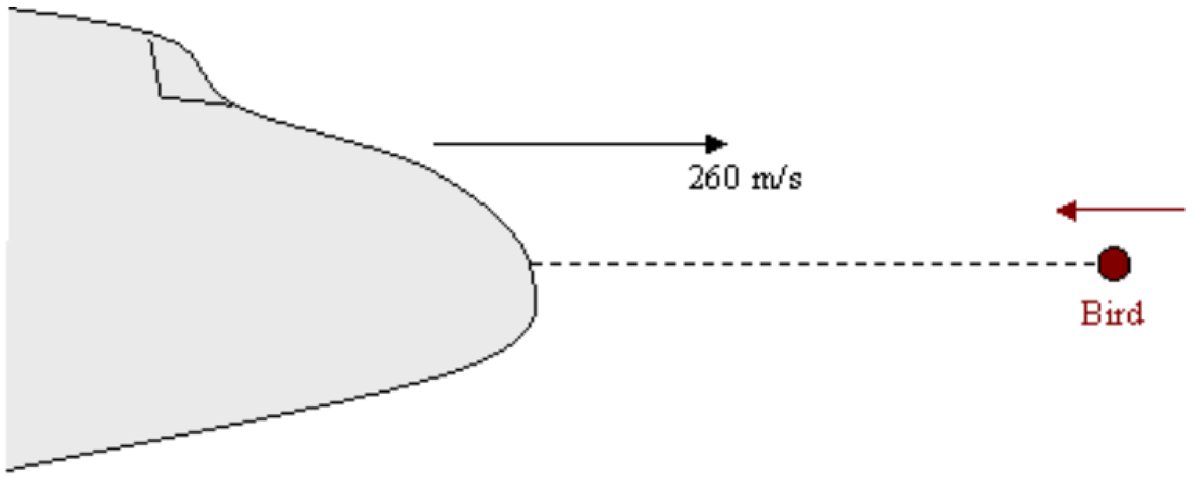
$$\vec{I} = \int_{t_i}^{t_f} \vec{F} \cdot dt = \int_{t_i}^{t_f} \frac{d\vec{p}}{dt} dt = \Delta\vec{p}$$

This force varies on time, so average values are usually used to simplify the resolution of the problem F_{av}



Example:

Estimate the average impact force between an airliner traveling at 260 m/s and a 0.3 kg bird whose length is 20 cm?



The change in velocity of the bird is estimated to be 260 m/s.
Main assumptions:

- Head on collision
- Bird is riding with the airliner after the collision
- Bird's velocity is negligible compared to that of the airliner

Example:

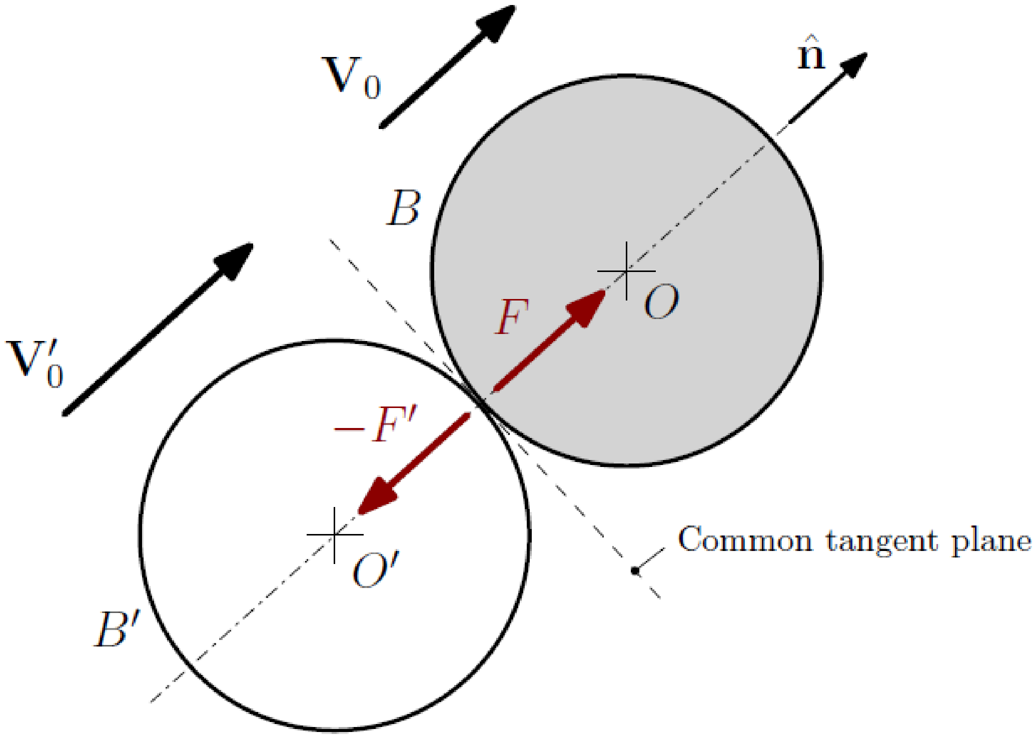
Estimate the average impact force between an airliner traveling at 260 m/s and a 0.3 kg bird whose length is 20 cm?

$$v_i = 0\, m/s \qquad v_f = 260\, m/s$$

$$\overrightarrow{F_{av}} = \frac{I}{\Delta t} = \frac{m(v_f - v_i)}{\frac{\delta}{\frac{1}{2}v_f}} = \frac{mv_f^2}{2\delta} = 50.7kN$$

Linear momentum, kinetic energy, coefficient of restitution and energy absorbed

Rigid Body Theory for Collinear Impact



Two bodies **collide** when they come together with an initial difference in velocity.

They first touch at a coincident point that will be termed the ***contact point****.* If one of the bodies has a surface that is topologically smooth, there is a ***tangent plane*** of this surface. The direction of the normal to the tangent plane is specified by a unit vector **n,** denominated the ***common normal direction***.

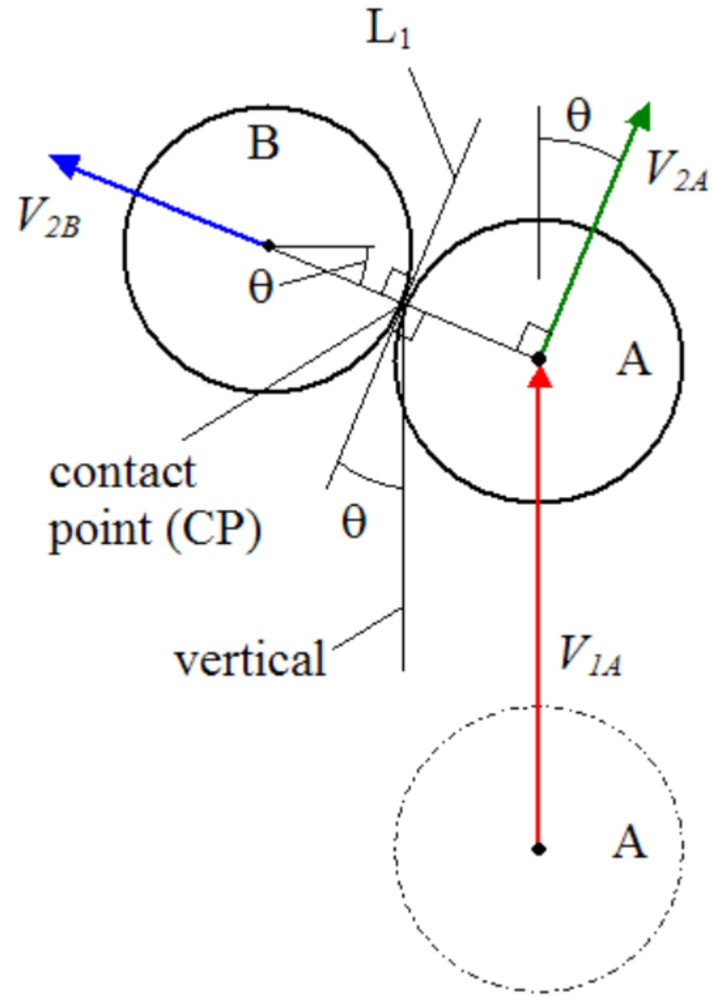
Equation of Relative Motion for Direct Impact

During contact there are equal but opposite compressive **reaction forces** which develop at the contact point. If the colliding bodies are hard, all rest of body forces will be negligible.

That is why for, example, in impact problems we do not consider gravitational forces.

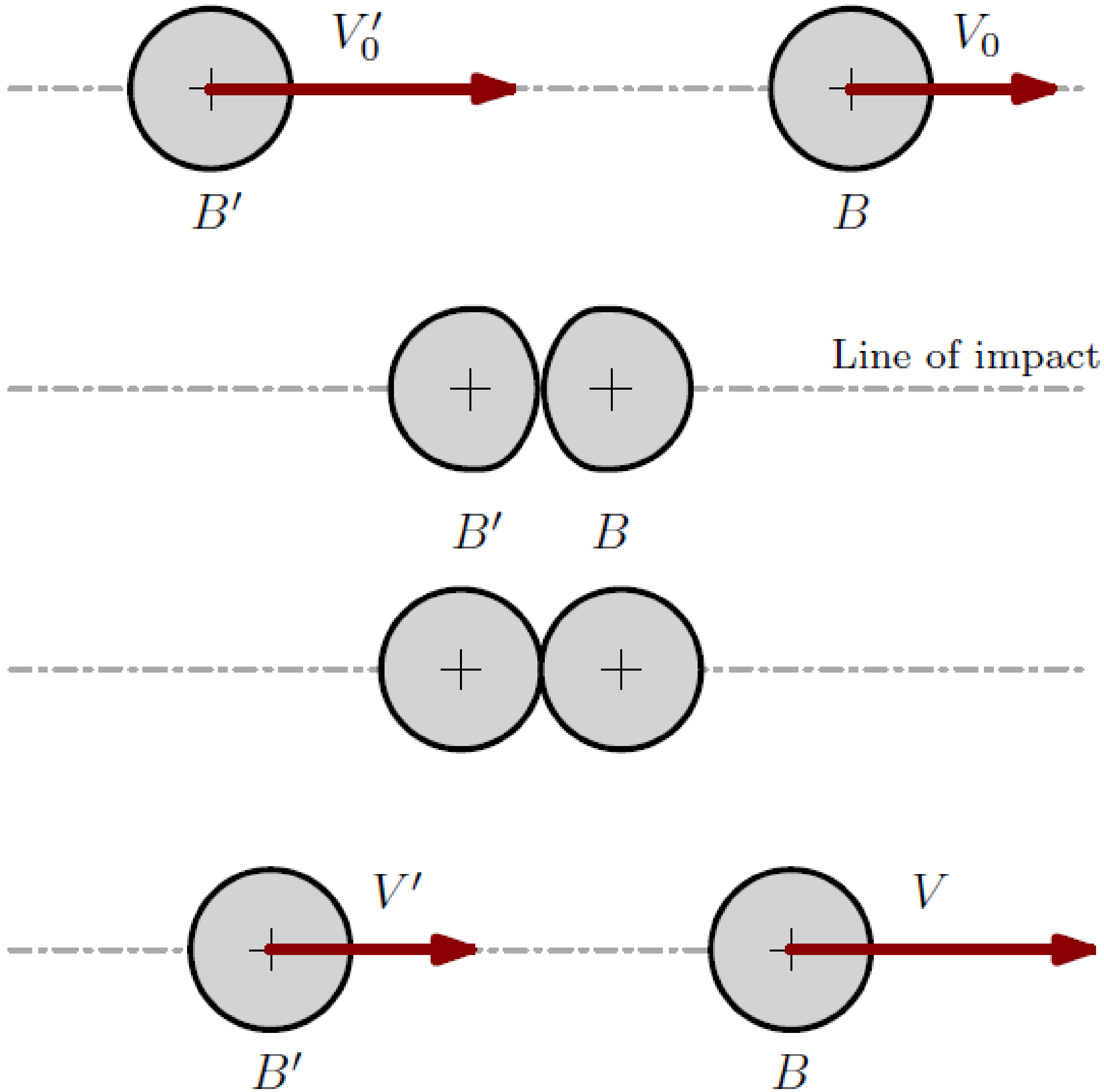
In the case of direct impact, the relative velocity between the contact points remains parallel to the common normal direction throughout the contact period.

- The initial velocity vector is given for a certain problem.
- The final velocity depends on the contact point and the parallel plane.



Compression and Restitution

- Approach
- Compression
- Restitution
- Separation



Equation of Relative Motion for Direct Impact

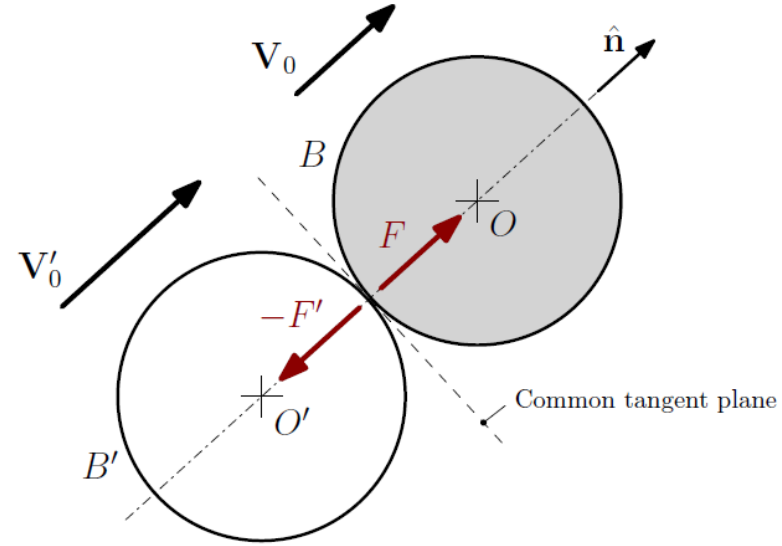
If deformation of the colliding bodies is considered *negligible*, it is possible to analyse changes in velocity as a function of **impulse**.

Impulse

$$I = \int_{t_1}^{t_2} \vec{F} \cdot dt$$

$$dP = Fdt$$

$$dP' = F'dt$$



Conservation of Linear Momentum

$$dP = mdV$$

$$dP' = m'dV$$

$$mv_{initial} + m'v'_{initial} = mv_{final} + m'v'_{final}$$

Kinetic Energy and Coefficient of Restitution

In an **elastic** event, kinetic energy is conserved.

$$E_{kin}^{initial} = E_{kin}^{final}$$

In a perfectly **inelastic** collision, the particles stick together after the collision (kinetic energy is not conserved).

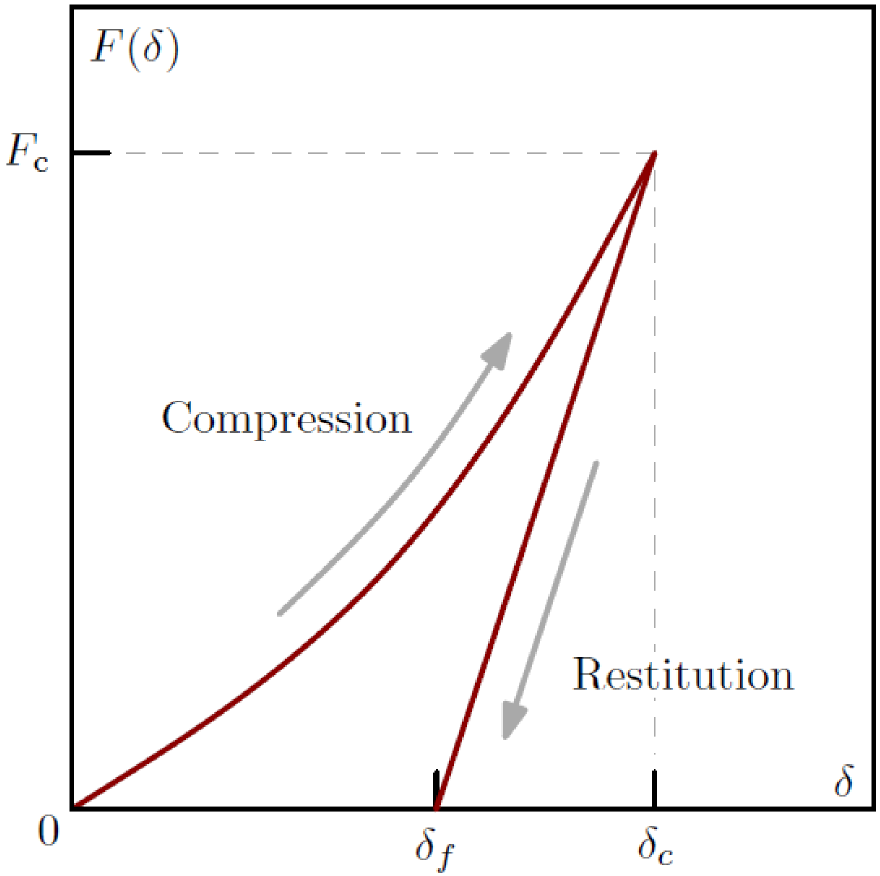
$$v_1^{final} = v_2^{final}$$

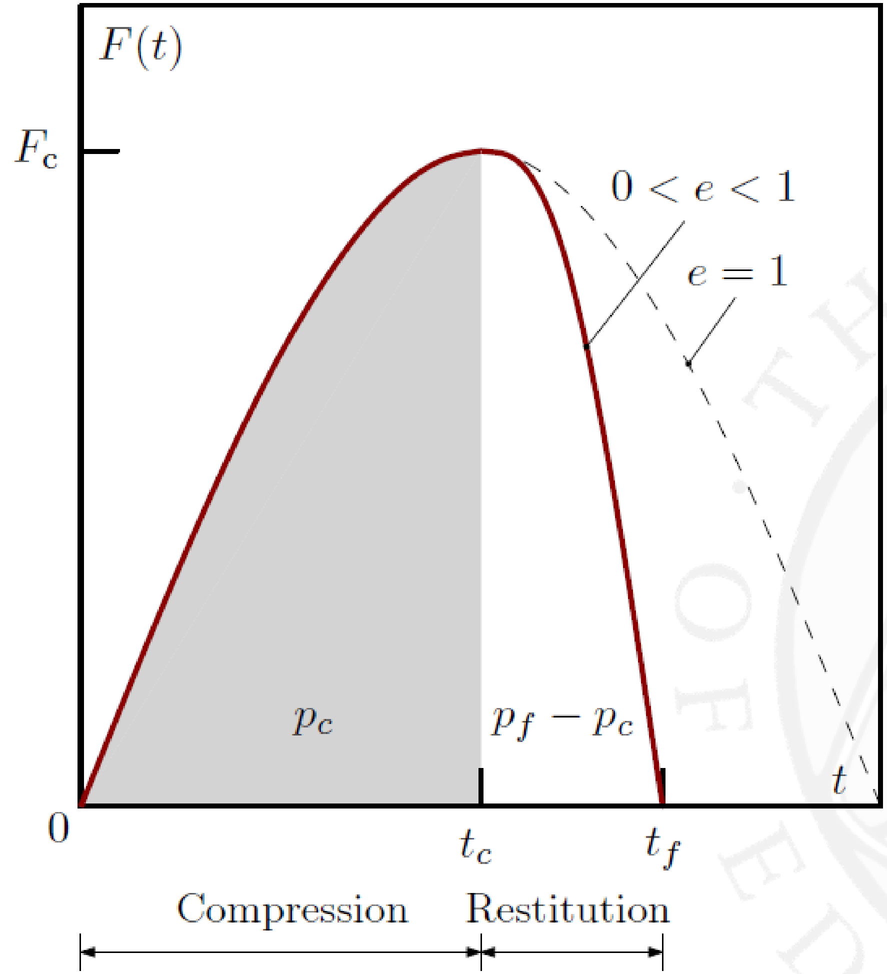
The **Coefficient of Restitution** is a measure of the elasticity of a collision.

For an elastic collision, **e = 1**.
For a perfectly inelastic collision, **e=0**.

$$e = -\frac{v_2^f - v_1^f}{v_2^i - v_1^i}$$

Compression and Restitution





Internal potential energy

Although kinetic energy of the system can differ overtime, the total energy of the system is always ***preserved***.

$$E_T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 {v'}_1^2 + \frac{1}{2}m_2 {v'}_2^2 + \boxed{E_{abs}}$$

Dissipated energy during impact

This internal energy can stand for:

- Plastic permanent deformation (ductile materials)
- Damage (brittle materials)
- Friction
- Heat

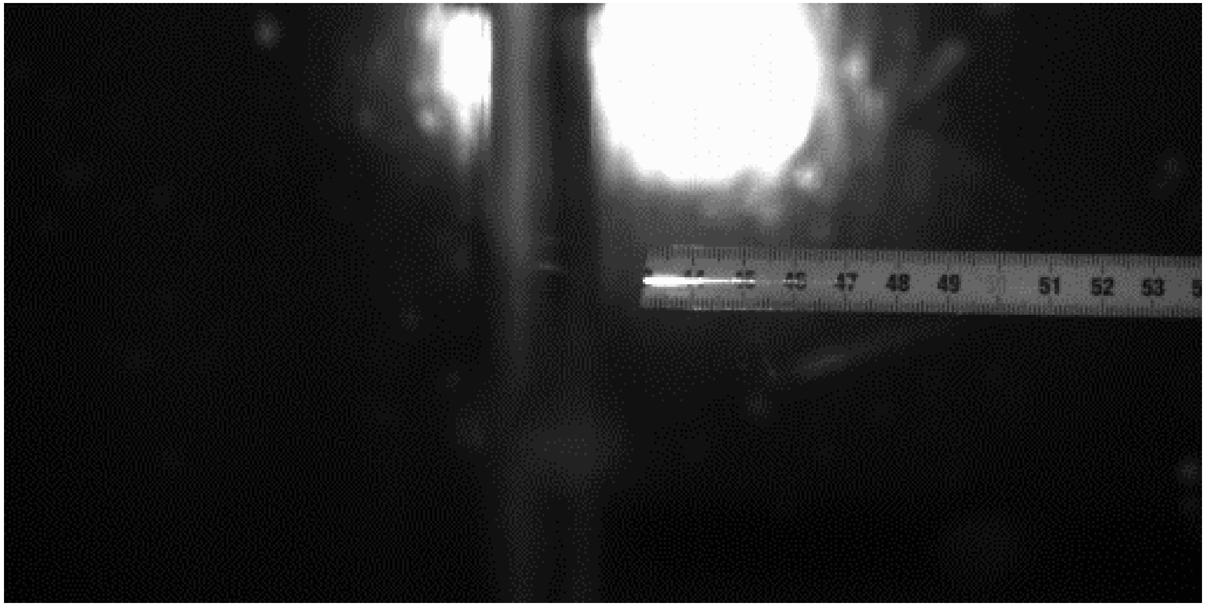
The bigger this energy, the lower the kinetic energy imposed in the colliding objects

Design of ballistic protections and vehicles try to alleviate the reaction forces and velocities enforced to passengers.

Example 1. Ballistic impact

$$E_T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 {v'}_1^2 + \frac{1}{2}m_2 {v'}_2^2 + E_{abs}$$

A ballistic impact consists on a projectile fired at a target



$$v_1 = 320\, m/s$$

$$v_2 = 0\, m/s$$

$$v'_1 > 0\, m/s$$

$$v'_2 > 0\, m/s$$

$$E_{abs} > 0\, J$$

An effective ballistic protection can arrest the fragments

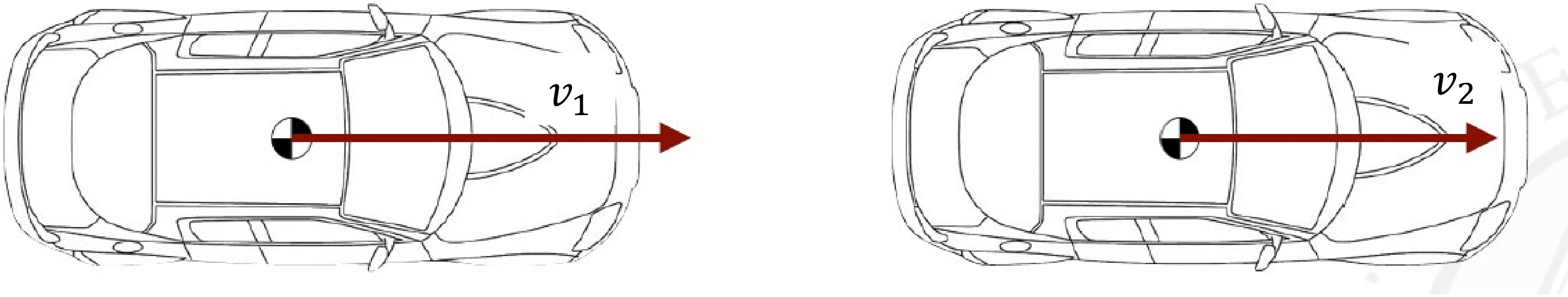
Example. Crashworthiness

Obtain the mutual coefficient of restitution in a two-vehicle collinear collision (rear-end or frontal impact)

Obtain the mutual coefficient of restitution in a two-vehicle collinear collision (rear-end or frontal impact).

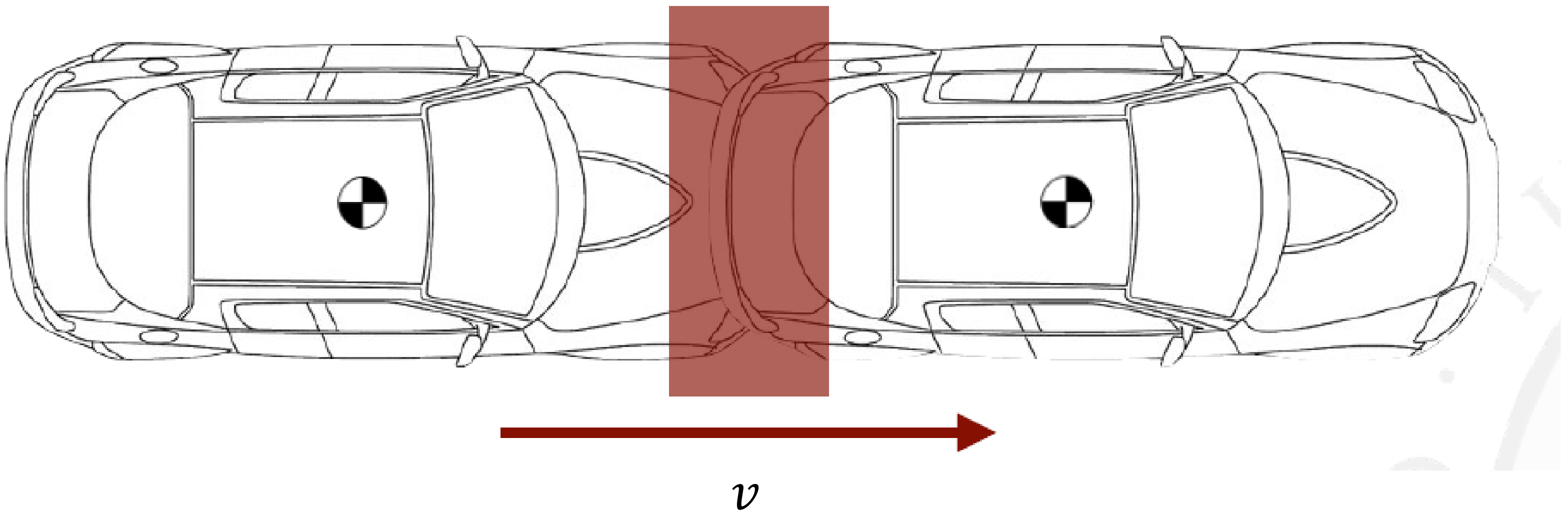
- *This is a property of the impact, not of the individual vehicle, and should be calculated in case any initial conditions changes*

The collision: before the impact (pre-impact velocities)

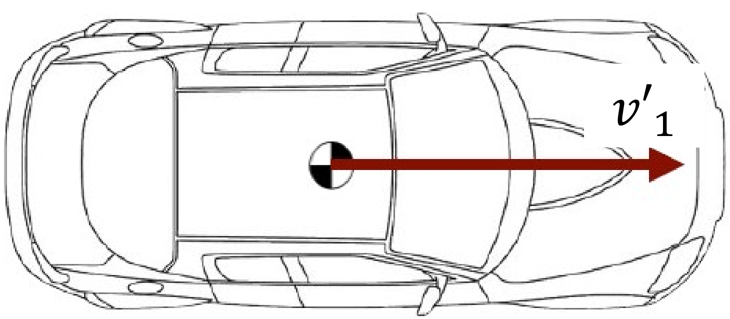


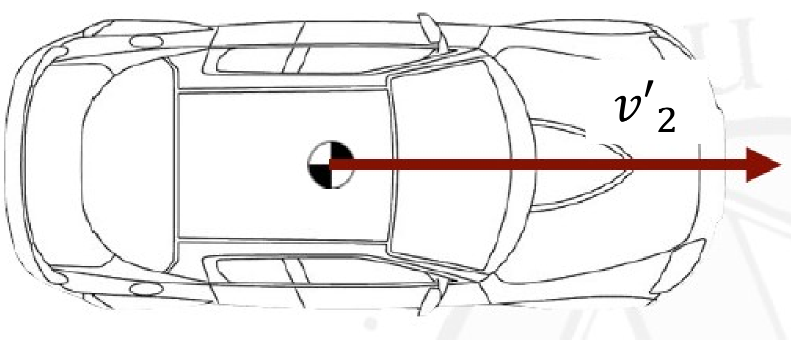
Example 3. Crashworthiness

The collision: during the impact (velocity of the contact point)



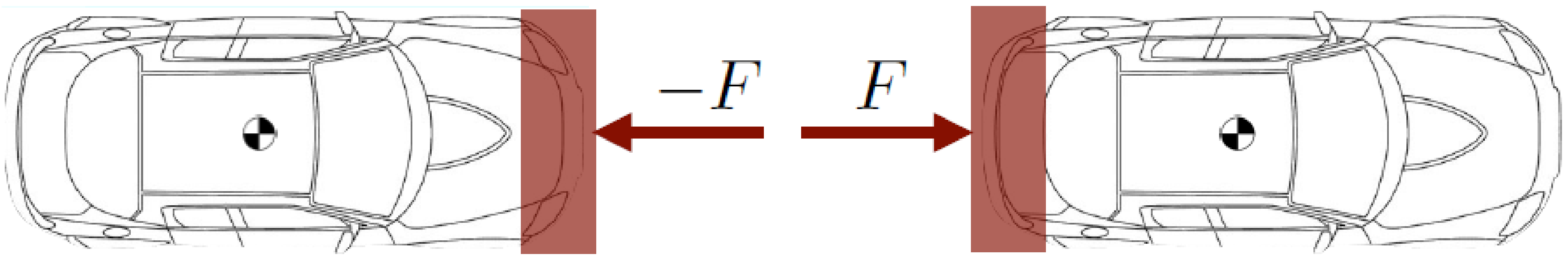
The collision: after the impact (post-impact velocities):





The collision: During the impact

- Newton's 2nd and 3rd laws



$$m_1 a_1 = m_1 \frac{dv_1}{dt} = -F$$

$$m_2 a_2 = m_2 \frac{dv_2}{dt} = F$$

Example 3. Crashworthiness

Velocity after impact

$$v'_1 = v - e\frac{m_2}{m_1 + m_2}\Delta v = v_1 - \frac{(1+e)m_2}{m_1 + m_2}\Delta v$$

$$v'_2 = v + e\frac{m_1}{m_1 + m_2}\Delta v = v_2 + \frac{(1+e)m_1}{m_1 + m_2}\Delta v$$

$$\Delta v = v_1 - v_2$$

Example 3. Crashworthiness

Energy dissipated

$$E_T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v'^2_1 + \frac{1}{2}m_2 v'^2_2 + E_{abs}$$

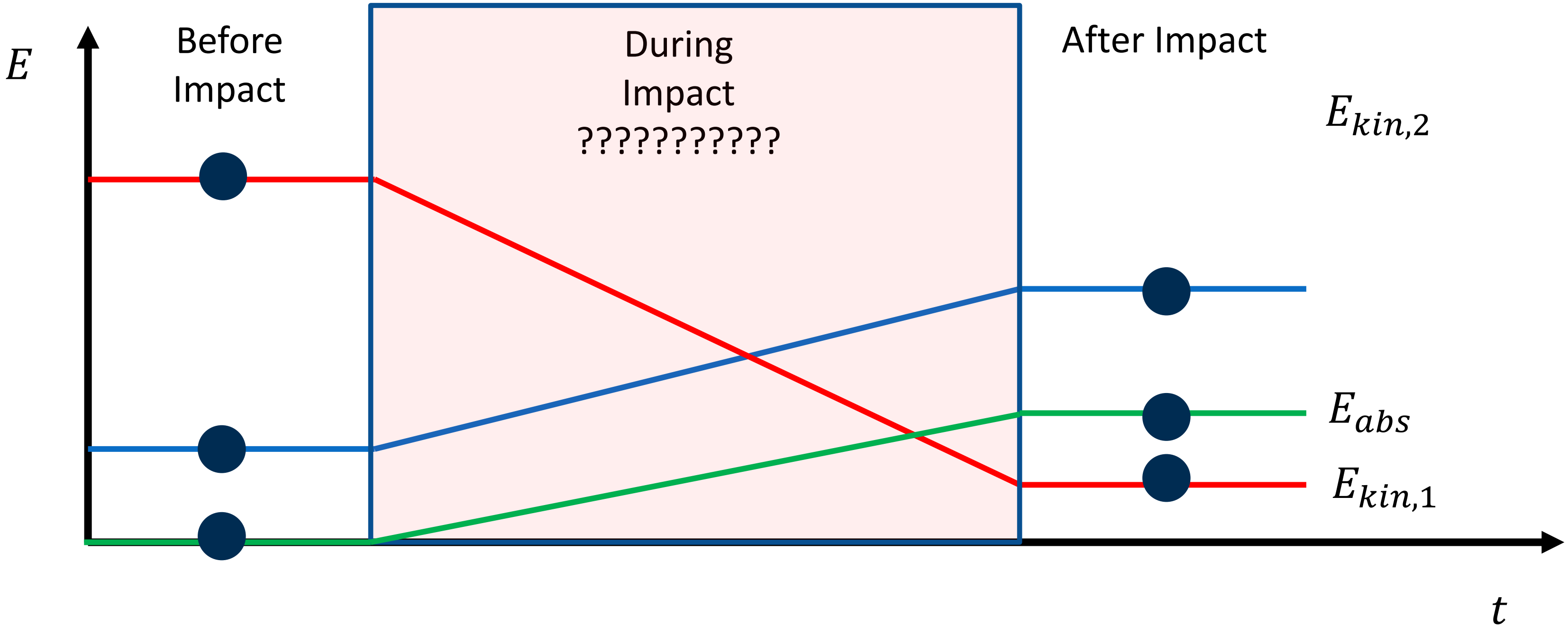
$$E_{abs} = \frac{1}{2}(1-e^2)\frac{m_1 m_2}{m_1+m_2}\Delta v^2 \qquad \Delta v = v_1 - v_2$$

Thermodynamically consistent

$$E_{abs} \geq 0 \rightarrow \quad e \leq 1$$

Limitations of the current approach

Although we have studied what happens before and after the impact …



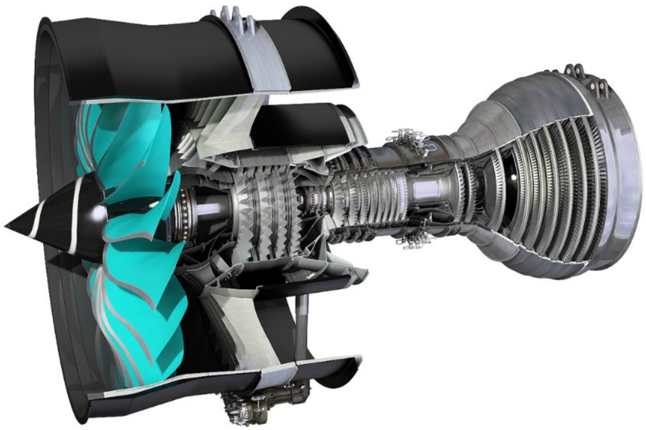
What does really happens during the impact?

This approach only provides discrete results, but can not obtain the energy evolution over time.

Industry requirements

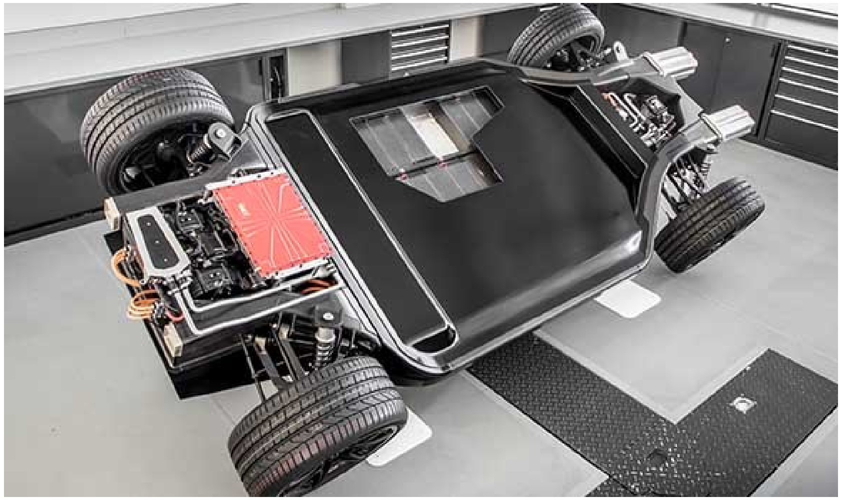
https://app.wooclap.com/events/FEQGIR/0

Applications



Aeronautics

Space



Automotive

Defence

Impact density

Conventional impact problems consist of a projectile which strikes the surface of another object

Kinetic energy $E_{kin} = \frac{1}{2}mv^2$

Impact density $E^*_{kin} = \frac{mv^2}{2A}$

Lead bullet. Nebarnix.

Diablo Crater, Subarcticmike

Impact density

Impact	Mass [kg]	Velocity [m/s]	Energy [J]	Area [m^2]	Energy Density [J/m^2]
Billiard balls					
Stabbing (sharp knife)					
9 mm bullet					
7.62x51 NATO bullet					
Medium size vehicle					
Locomotive					
Meteorite					
Airplane (Boeing 767)					

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Billiard balls	0.2	4	1.6	1×10^{-5}	1.6×10^{5}
Stabbing (sharp knife)	10	0.86	43	2×10^{-7}	215×10^{6}
9 mm bullet	0.008	380	578	63×10^{-6}	9.2×10^{6}
7.62x51 NATO bullet	0.011	900	4455	16×10^{-6}	278×10^{6}
Medium size vehicle	1300	10	65000	2	93×10^{3}
Locomotive	500000	17	72×10^{6}	9	8028×10^{3}
Meteorite	1	20000	200×10^{6}	0.1	200×10^{7}
Airplane (Boeing 767)	180000	153	2106×10^{6}	50	42×10^{6}

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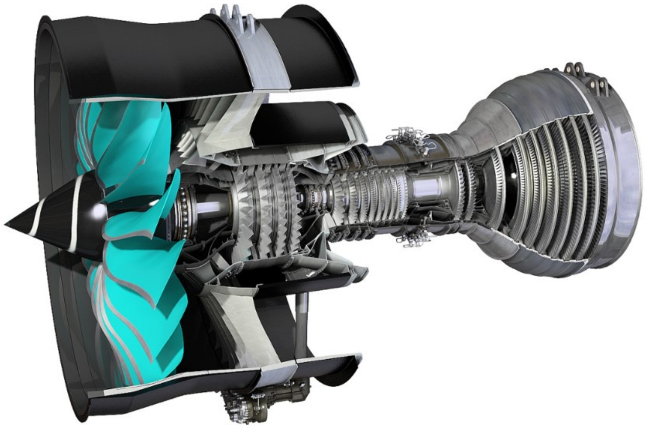


Aeronautics

Lightweight structures

1. To reduce the CO2 emissions, and the operation cost (using less fuel)
2. To create larger systems with improved power efficiency

Protection of critical equipment for a safe flight

How are composite structures designed against impact?

$$E_T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v'^2_1 + \frac{1}{2}m_2 v'^2_2 + \boxed{E_{abs}}$$

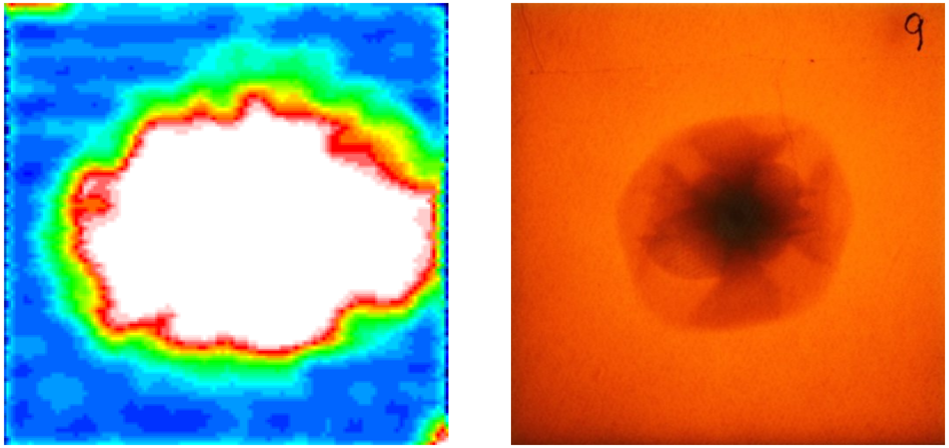
???

Aerospace requirements

Low velocity impact: Ideally e=1 Priority: minimising delamination

Composites show *delamination* (separation between adjacent plies) even at very low impact loads. The compressive strength of the laminate is significantly reduced because the laminate is subdivided into thinner sublaminates with lower buckling load.

Barely visible impact damage can be produced by hail strikes, debris impact, lightning strikes, or even a tool drop during maintenance



Comparison in delamination of a 10 x 10 cm^2 plate against ballistic impact at 45 J for
(a) 5.2 mm thickness laminate of UD CFRP T800S/M21 and
(b) 4.9 mm thickness laminate of Woven GFRP S2G/MTM44

Aerospace requirements

Low velocity impact:

3D-reinforcement

3D WEAVING

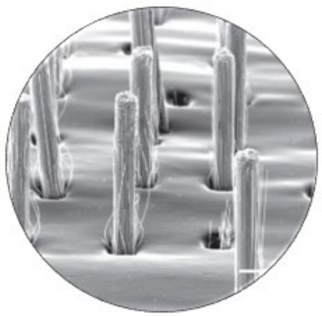
$E = \sim 10\% \downarrow$

$\sigma_u = \sim 20\% \downarrow$

Z-PINNING

$E = \sim 15\% \downarrow$

$\sigma_u = \sim 25\% \downarrow$



[5]

STITCHING

$E = \sim 15\% \downarrow$

$\sigma_u = \sim 25\% \downarrow$

Damage tolerant materials

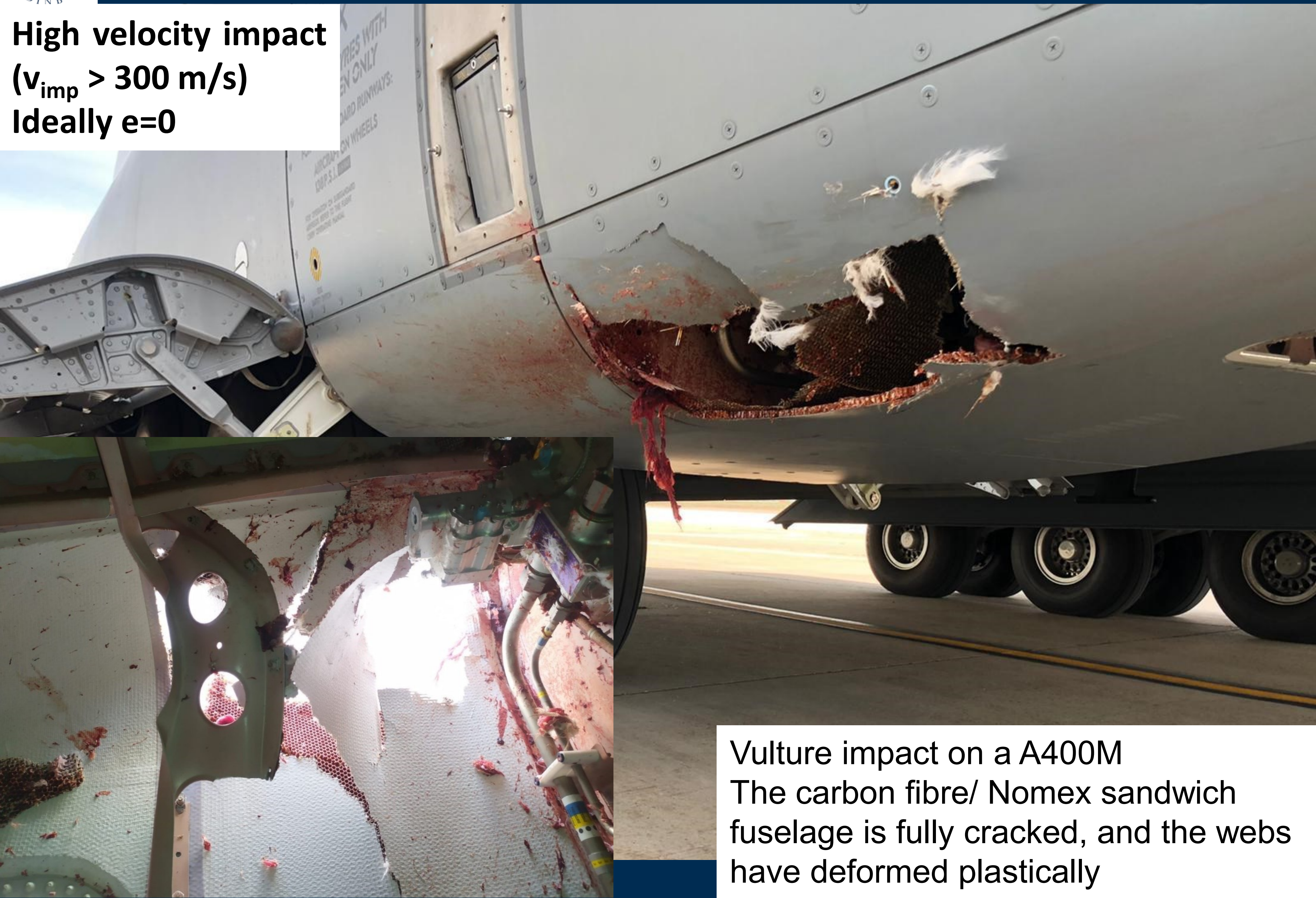
Glass fibre composite

Carbon fibre composite

SMC laminates impacted at 4 m/s, 30 J

Aerospace requirements

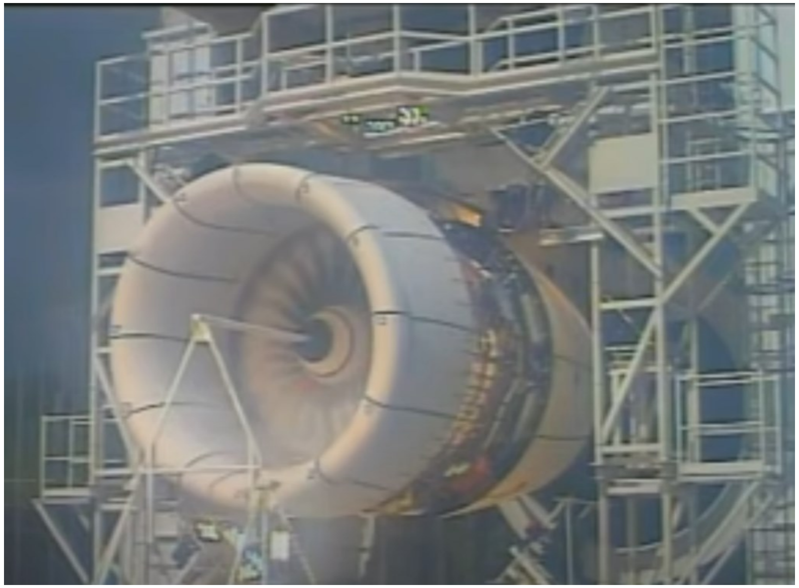
High velocity impact (v_{imp} > 300 m/s)
Ideally e=0

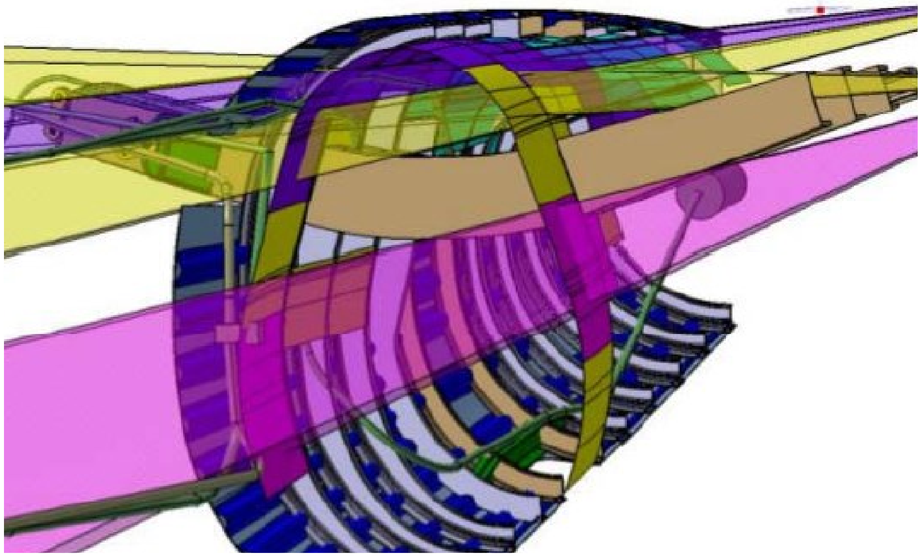


Vulture impact on a A400M
The carbon fibre/ Nomex sandwich fuselage is fully cracked, and the webs have deformed plastically

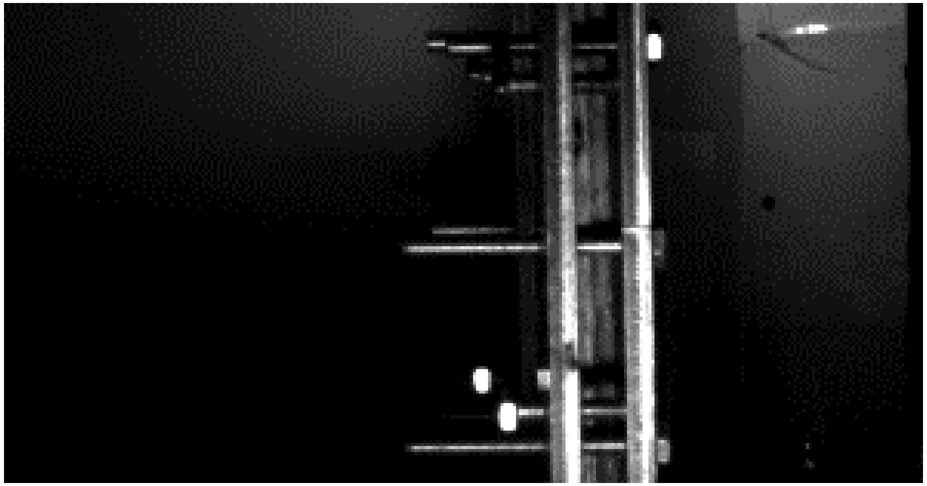
Aerospace requirements

Containment and protective structures





Predicted trajectories for UERF



Successful bird ingest:
https://www.youtube.com/watch?v=0qpD6MDYEGY&t=3s&ab_channel=sbzro

Unsuccessful fan blade off test:
https://www.youtube.com/watch?v=wcALjMJbAvU&ab_channel=RandoWis

Automotive requirements

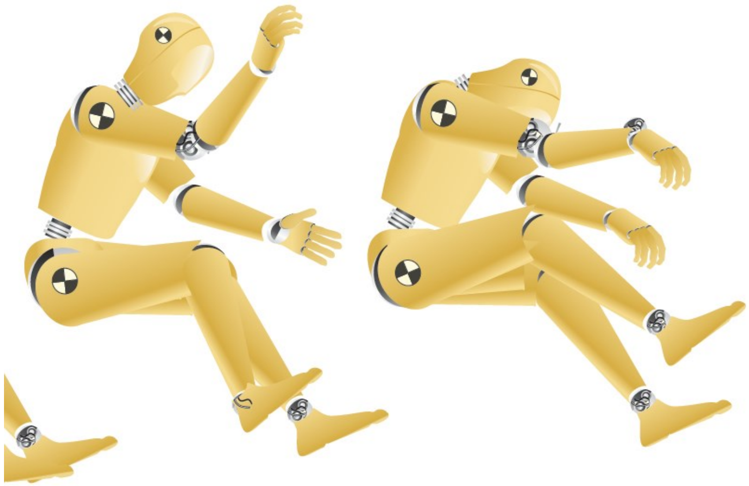
Passenger safety, ideally **e=0**

Minimum safety standards that vehicles must meet to be approved for sale in the EU market are set by the European New Car Assessment Programme (Euro NCAP).

Vehicle crash tests are carried out to understand how the vehicle collision phenomenon takes place.

https://www.youtube.com/watch?v=TikJC0x65X0&ab_channel=4DriveTime

Dummies are instrumented with accelerometers to understand the loads imposed on passengers during impact



Automotive requirements

Crash tubes are used as energy absorbers to reduce the accelerations imposed on passengers and ensure the safety of the occupants

Space Structures

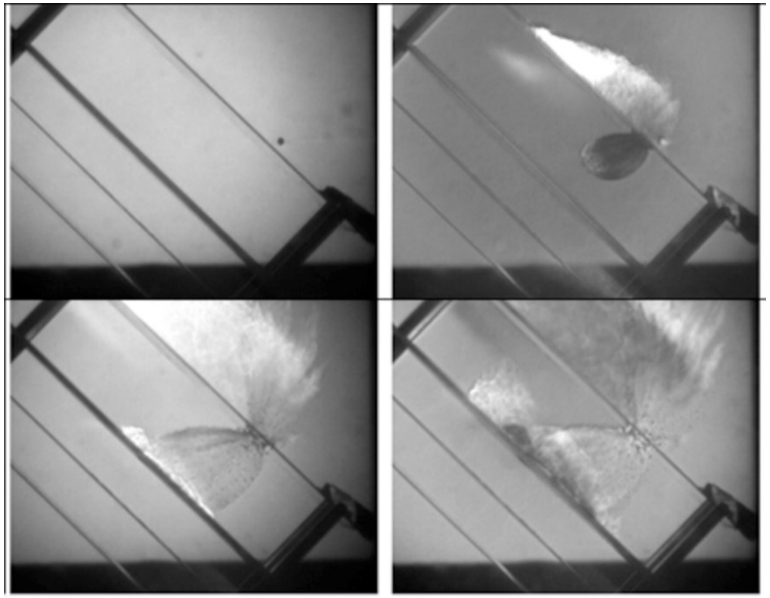
Hypervelocity is categorised as an impact that occurs at a velocity higher than the wave propagation velocity, *c,* of the structure.
(e.g., > 5000 m/s for steel)

Space debris velocity = 15 km/s
Meteoroids velocity = 75 km/s

Passive protection techniques (shields with multiple layers) are used.

- The first layer disintegrates the debris, **e=0.**
- The remaining layers hold the liquified projectile and target material.

Meteoroid scale	Damage
0.001 mm	Surface pits
1 mm	Clear hole penetration
1 cm	Mission critical damage
>10 cm	Catastrophic disintegration



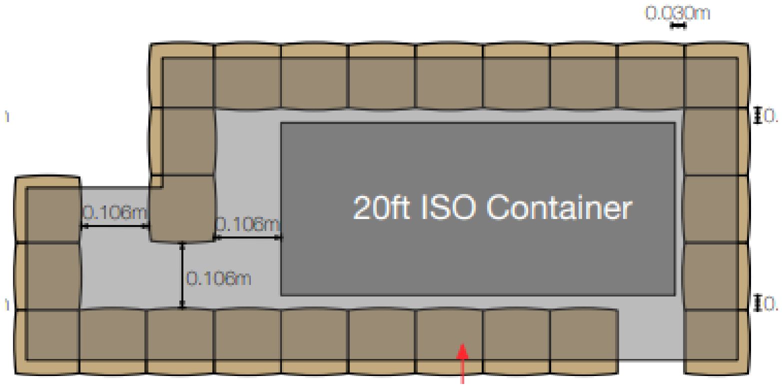
At velocities >4 km/s, the projectile melts

Defence

There are currently no normative to protect civilians buildings from the unlikely event of an explosion.

Overdesigned structures will be extremely heavy, with an increased cost and carbon footprint

It is possible to find approaches to protect from shockwave, mainly in defense sector. Their priority is to vent the shockwave



Hesco-Bastion survival shelter during construction. Steel containment and sand-filled bastions

Impulsive Dynamics

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William Rankine Building, 1.19