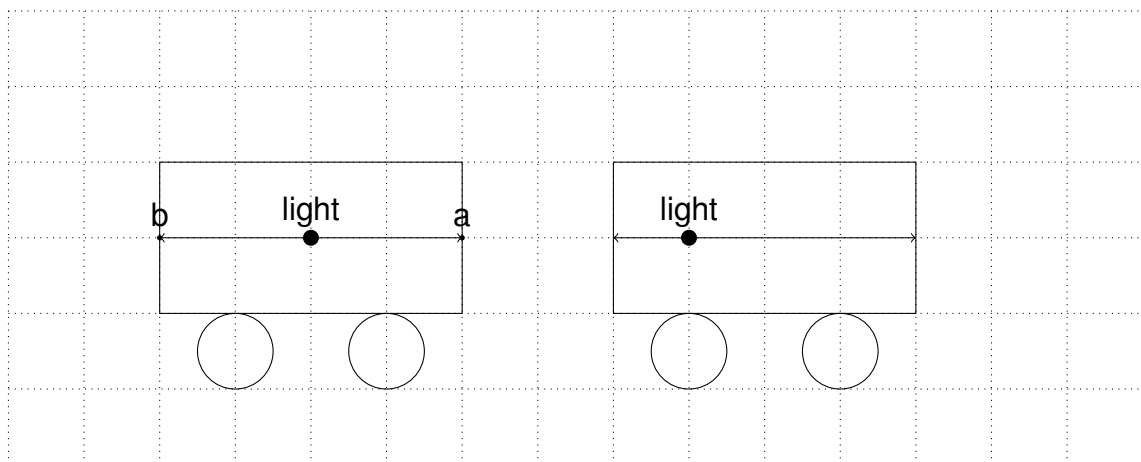


Galileo's velocity addition rule

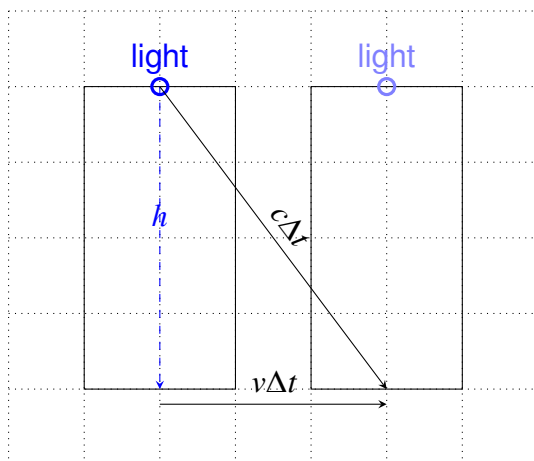
$$u = u' + v$$

Einstein's velocity addition rule

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$



for people at stationary coordinates, b happens first
for people on the car, a and b happens at the same time.



Moving clocks run slow

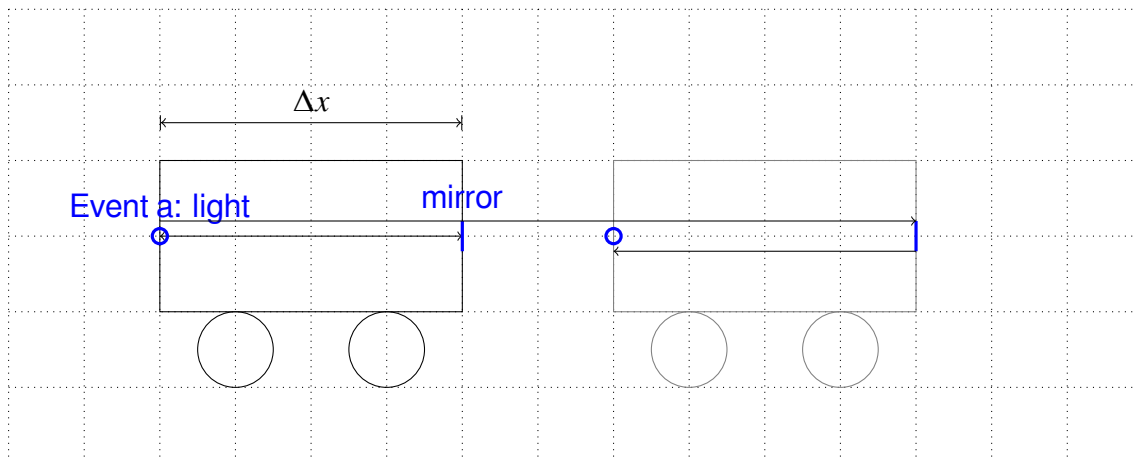
$$(c\Delta t')^2 = h^2 + (v\Delta t)^2$$

$$h^2 = (c^2 - v^2)\Delta t^2$$

$$c^2\Delta t'^2 = (c^2 - v^2)\Delta t^2$$

$$\Delta t' = \sqrt{1 - \frac{v^2}{c^2}}\Delta t$$

$$\Delta t' < \Delta t$$



$$\Delta t' = 2 \frac{\Delta x'}{c}$$

$$\Delta t_1 = \frac{\Delta x + v \Delta t_1}{c} \implies \Delta t_1 = \frac{\Delta x}{c - v}$$

$$\Delta t_2 = \frac{\Delta x - v \Delta t_2}{c} \implies \Delta t_2 = \frac{\Delta x}{c + v}$$

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{2\Delta x}{c} \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\Delta x' = \frac{c}{2} \Delta t'$$

$$= \frac{c}{2} \Delta t \sqrt{1 - \frac{v^2}{c^2}}$$

$$= \frac{c}{2} \frac{2\Delta x}{c} \frac{1}{1 - \frac{v^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}$$

$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta x$$

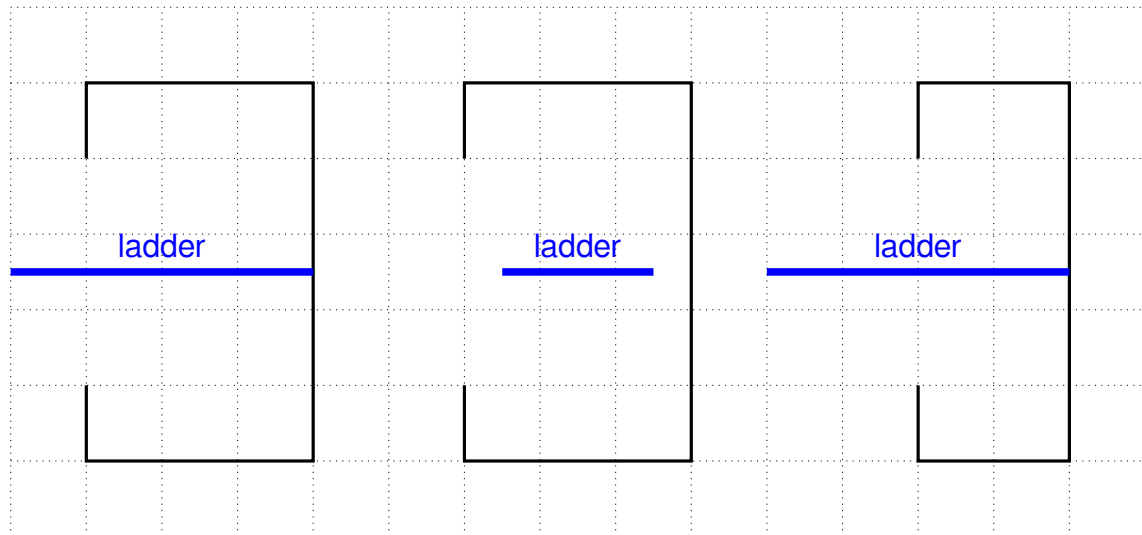
$$\implies \Delta x' > \Delta x$$

0.1 The barn and ladder paracby

- (A) Back end of the ladder makes it in the door
- (B) Front end of the ladder hits the wall of the born

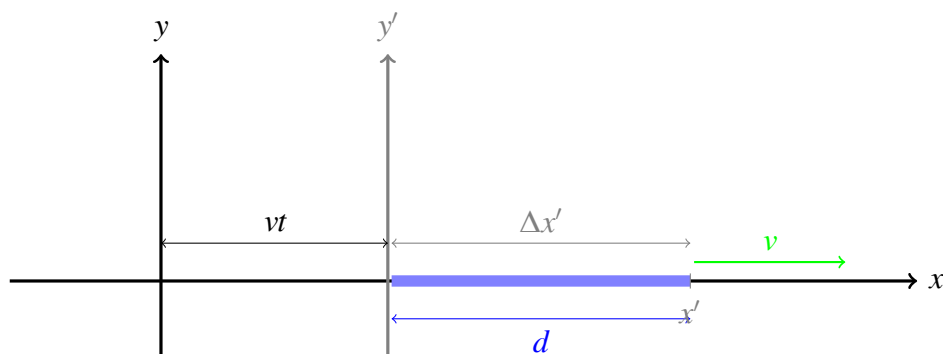
Farmer: (A) before (B)

Son: (B) before (A)



0.2 The Lorentz Transformations

- **Event:** Something that take place at a specific location at precise time.
- Knowing (x, y, z, t) , what is (x', y', z', t') of the same event
- Let x be the same position as x' .
The stick's length measured in the stationary coordinate be d .

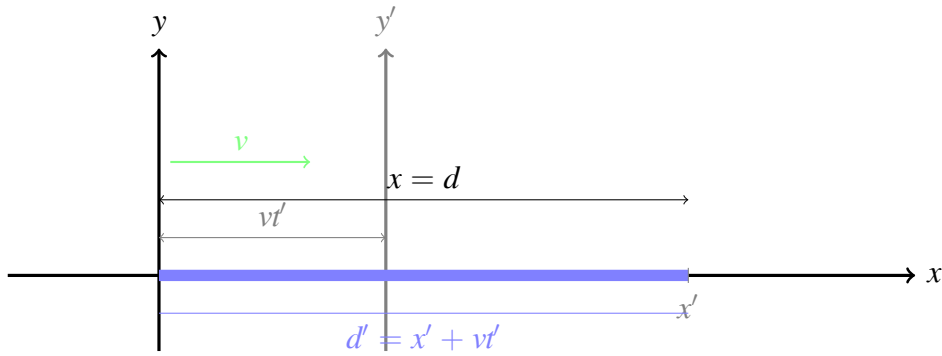


$$d = \sqrt{1 - \frac{v^2}{c^2}} \Delta x'$$

$$(x - vt) = \sqrt{1 - \frac{v^2}{c^2}} x'$$

$$x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt)$$

- Let x be the same position as x' .
The stick's length measured in the stationary coordinate be d .



$$x' = d' - vt'$$

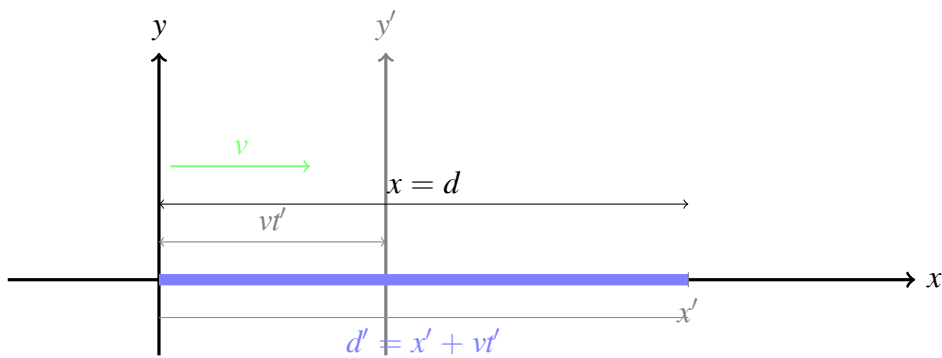
$$d' = \sqrt{1 - \frac{v^2}{c^2}} x$$

$$x' = \sqrt{1 - \frac{v^2}{c^2}} x - vt' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt')$$

$$vt' = \left(\sqrt{1 - \frac{v^2}{c^2}} - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) x - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} vt$$

$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(\frac{v}{c^2} x - vt \right) - vt'$$

- Let x be the same position as x' .
The stick's length measured in the stationary coordinate be d .



$$x' = d' - vt'$$

$$d' = \sqrt{1 - \frac{v^2}{c^2}} x$$

$$x' = \sqrt{1 - \frac{v^2}{c^2}} x - vt' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt)$$

$$vt' = \left(\sqrt{1 - \frac{v^2}{c^2}} - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) x - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} vt$$

$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(vt - \frac{v^2}{c^2} x \right)$$

$$t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(t - \frac{v}{c^2} x \right)$$

- **Check Simultaneously:**

S :

Event A ($x = 0, t = 0$)

Event B ($x = a, t = 0$)

S' :

Event A ($x' = 0, t' = 0$)

Event B ($x' = \gamma a, t' = -\gamma(\frac{v}{c^2})a$)

- **Time Dimention:**

$$\Delta x' = 0 = \gamma(\Delta x - v\Delta t) \implies \Delta x = v\Delta t$$

$$\Delta t' = \gamma \left(\Delta t - \frac{v}{c^2} \Delta x \right)$$

$$= \gamma \left(\Delta t - \frac{v^2}{c^2} \Delta t \right)$$

$$= \sqrt{1 - \frac{v^2}{c^2}} \Delta t$$

•

$$\begin{aligned}
 u_x' &= \frac{\Delta x'}{\Delta t'} \\
 &= \frac{\gamma(\Delta x - v\Delta t)}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)} \\
 &= \frac{u_x - v}{1 - \frac{v}{c^2}V_x} \\
 u_y' &= \frac{\Delta y'}{\Delta t'} \\
 &= \frac{\Delta y}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)} \\
 &= \frac{u_y}{\gamma(1 - \frac{v}{c^2})}
 \end{aligned}$$

• **Structure of Spacetime:**

– **Four vectors:**

(Time is the 0th-dimension.)

$$x^0 = ct$$

$$x^1 = x$$

$$x^2 = y$$

$$x^3 = z$$

Lorenze transformation:

$$x^{0'} = \gamma(x^0 - \beta x^1)$$

$$x^{1'} = \gamma(x^1 - \beta x^0)$$

$$x^{2'} = x^2$$

$$x^{3'} = x^3$$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
 (x^\mu)' &= \sum_{\nu=0}^{\delta} (\Lambda_\nu^\mu x^\nu) \\
 \implies (x^\mu)' &= \Lambda_\nu^\mu X^\nu
 \end{aligned}$$

– **Constant in Lorenze Transformation:**

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = a_\mu^\mu \quad (\mu = 0, 1, 2, 3)$$

$$(x^1)^2 + (x^2)^2 + (x^3)^2 \implies \text{length}^2 \text{ of a vector} \implies \text{rotation} \\ \implies \text{inner product}$$

– **The invariant interval:**

Suppose there are two events in frame S :

- * event A: $(x_A^0, x_A^1, x_A^2, x_A^3)$
- * event B: $(x_B^0, x_B^1, x_B^2, x_B^3)$
- * The difference: $\Delta x^\mu = x_A^\mu - x_B^\mu$ is a four vector.

The interval of Δx^μ is defined by $I = \Delta x_\mu \Delta x^\mu = -c^2 \Delta t^2 + d^2$

$$d = \sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}$$

1. $I < 0$:

- * timelike
- * Causation.
- * When two events occur at the **same place** ($d = 0$), they are separated only temporally.

2. $I = 0$:

- * lightlike
- * Causation. (by light)
- * Ex: Light and sensor on the bus.
- * Two events are connected by a signal traveling at the speed of the light.

3. $I > 0$:

- * Spacelike
- * When two events occur at the **same time** and are separated only spatially.

– **Theorem:** If the interval between two events is timelike, there exists an internal system (accessible by Lorentze transformation) in which they occur at the same points likewise, If the interval is spacelike, there exist a system in which the two events occur at the same time. **Proof:**

Two events A, B occur at (t_A, x_A) and (t_B, x_B) .

$$\begin{cases} t'_A = \gamma(t_A - \frac{v}{c^2}x_A) \\ x'_A = \gamma(x_A - vt_A) \end{cases}$$

$$\begin{cases} t'_B = \gamma(t_B - \frac{v}{c^2}x_B) \\ x'_B = \gamma(x_B - vt_B) \end{cases}$$

$$\Delta t = t_A - t_B$$

Timelike: If $x'_A = x'_B$

$$\implies x_A - vt_A = x_B - vt_B$$

$$\implies \frac{v}{c} = \frac{x_A - x_B}{c(t_A - t_B)} < 1$$

Spacelike: If $t'_A = t'_B$

$$\implies t_A - \frac{v}{c^2}x_A = t_B - \frac{v}{c^2}x_B$$

$$\implies \frac{v}{c} = \frac{c(t_A - t_B)}{x_A - x_B} < 1$$

0.3 Relative Mechanics

- **Proper Time and Proper Velocity:**

- The temporal coordinate of the moving frame.

-

$$d\tau = \sqrt{1 - \frac{u^2}{c^2}} dt$$

- Example:

$$u = \frac{4}{5}c = \frac{di}{dt}$$

$$\eta = \frac{dl}{d\tau} = \frac{dt}{d\tau} \frac{dl}{dt} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} u$$

- 4-velocity:

$$\eta^\mu = \frac{dx^\mu}{dt}$$

$$dx = c dt$$

$$\eta^0 = \frac{cdt}{dt} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Satisfies *Lorenze Transformation*:

$$\begin{cases} \eta^{0'} &= \gamma(\eta^0 - \beta\eta^1) \\ \eta^{1'} &= \gamma(\eta^1 - \beta\eta^0) \\ \eta^{2'} &= \eta^2 \\ \eta^{3'} &= \eta^3 \end{cases}$$

$$(\eta)^\mu = \Lambda^\mu_\nu \eta^\nu$$

- **Exercise:**

Consider a car traveling along the $\frac{\pi}{4}$ line.

1. Find the componenet μ_x and μ_y of the ordinary velocity.

$$u_x = \frac{2}{\sqrt{5}} \cos \frac{\pi}{4} = \sqrt{\frac{2}{5}} c = u_y$$

2. Find the component η_x and η_y of the proper velocity.

$$\eta = \frac{\frac{2}{\sqrt{5}}c}{\sqrt{1 - \frac{4}{5}}} = 2c$$

$$\eta_x = \eta_y = \sqrt{2}c$$

3. Find the zeroth component of the 4-velocity.

$$\eta^0 = \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} = \sqrt{5}c$$

A system S' is moving in the x-direction with (ordinary) speed $\sqrt{\frac{2}{5}}c$. By using the appropriate transformation laws.

* Find the (ordinary) velocity component u'_x, u'_y in S' .

$$v = \sqrt{\frac{2}{5}}c$$

$$u'_x = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{\sqrt{\frac{2}{5}}c - \sqrt{\frac{2}{5}}c}{1 - \frac{2}{5}} = 0$$

$$u'_y = \frac{1}{\gamma} \frac{u_y}{\sqrt{1 - u'vc^2}} = \sqrt{1 - \frac{u^2}{c^2}} \frac{u_y}{\sqrt{1 - u'vc^2}} = \sqrt{\frac{2}{5}} \left(\frac{\sqrt{\frac{2}{5}}c}{\frac{3}{5}} \right) = \sqrt{\frac{2}{5}}c$$

* Find the proper velocity component η'_x, η'_y in S' .

$$(\eta')^0 = \sqrt{\frac{5}{3}}(\sqrt{5}c - \sqrt{\frac{2}{5}}\sqrt{2}c) = \sqrt{3}c$$

$$(\eta')^1 = \sqrt{\frac{5}{3}}(\sqrt{2}c - \sqrt{\frac{2}{5}}\sqrt{2}c) = 0$$

* Check $\eta' = \frac{u'}{\sqrt{1 - \frac{u'^2}{c^2}}}$.

$$\eta' = \sqrt{2}c$$

$$u' = \sqrt{\frac{2}{3}}c$$

$$\frac{u'}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{\sqrt{\frac{2}{3}}c}{\sqrt{1 - \frac{2}{3}}} = \sqrt{2}c = \eta'$$

0.4 Relativistic Mechanics

– Momentum = mass \times velocity.

$$\vec{p} = \gamma_u m \vec{u} = m \vec{\eta}$$

$$m_A \eta_A + m_B \eta_B = m_C \eta_C + m_D \eta_D$$

$$\eta = \frac{1}{\gamma} \eta'_A + \beta \eta_A^0 \dots \dots (Loranzze - 2)$$

$$m_A \left(\frac{1}{\gamma} \eta'_A + \beta \eta_A^0 \right) + m_B \left(\frac{1}{\gamma} \eta'_B + \beta \eta_B^0 \right) = m_C \left(\frac{1}{\gamma} \eta'_C + \beta \eta_C^0 \right) + m_D \left(\frac{1}{\gamma} \eta'_D + \beta \eta_D^0 \right)$$

– **Conservation of Mass:**

$$\eta_0 = \frac{c}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$M = \frac{m}{\sqrt{1 - \frac{u^2}{c^2}}} \Rightarrow \text{relativistic mass}$$

$$= \gamma_u m$$

$$\Rightarrow p^0 = m \eta^0 = \gamma_0 m c$$

$$m_A \eta_A^0 + m_B \eta_B^0 = m_C \eta_C^0 + m_D \eta_D^0$$

$$\Rightarrow M_A + M_B = M_C + M_D$$

$$p^0 = \frac{mc}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{E}{c}$$

Einstein defined

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} = mc^2 + \frac{1}{2} mu^2$$

• **Energy-momentum 4-vector:**

$$p^\mu = m \eta^\mu = \left(\frac{E}{c}, p^1, p^2, p^3 \right)$$

$$\vec{p} = \frac{m \vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = mc^2 + kE$$

$|p^\mu|$ is invariant constant

$$\begin{aligned} p^\mu p_\mu &= -(p^0)^2 + (\vec{p} \cdot \vec{p}) = -m^2 c^2 \\ \implies E^2 - p^2 c^2 &= m^2 c^2 \\ \implies E^2 &= p^2 c^2 + m^2 c^2 \end{aligned}$$

As $m = 0$: (photon)

$$E = pc$$

- **Example:** Two lumps of clay, each of rest mass m , collide head-on at $\frac{3}{5}c$. They stick together. Question: What is the mass of the composite lump?

$$\begin{aligned} E_i &= E_1 + E_2 \\ &= 2 \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \end{aligned}$$

• Relativistic: Kinematics

Ex: A pion at rest decays into a muon and a neutrino. Find the momentum of the outgoing muon, in terms of the two mass, m_π and m_μ . (m_ν)

$$\begin{aligned} E_i &= m_\pi c^2, P_i = 0 \\ E_f &= E_\mu + E_\nu, P_f = 0 = P_\mu + P_\nu \implies \vec{P}_\mu = -\vec{P}_\nu \\ E_\mu^2 &= P_\mu^2 c^2 + m_\mu^2 c^4 \\ E_\nu &= P_\nu c \\ E_i &= E_f \implies m_\pi c^2 = \sqrt{P_\mu^2 c^2 + m_\mu^2 c^4} + P_\mu c \\ P_\mu^2 + m_\mu^2 - 2m_\pi c^2 P_\mu &= (m_\pi c - P_\mu)^2 = P_\mu^2 + m_\mu^2 c^2 \\ \implies P_\mu &= \frac{c(m_\pi^2 - m_\mu^2)}{2m_\pi} \end{aligned}$$

0.5 HW note

•

$$f' = f \sqrt{\frac{1 - \beta}{1 + \beta}}$$

•

$$v = \frac{\Delta \lambda}{\lambda} c$$