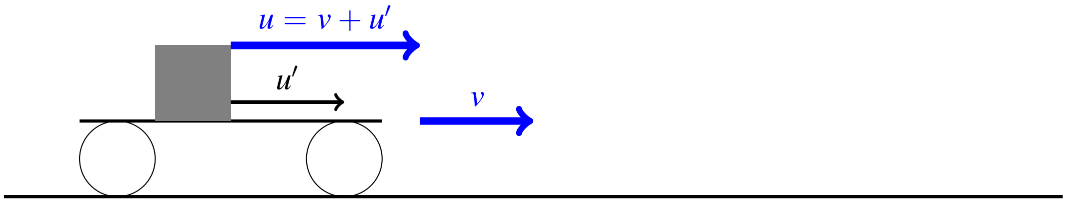
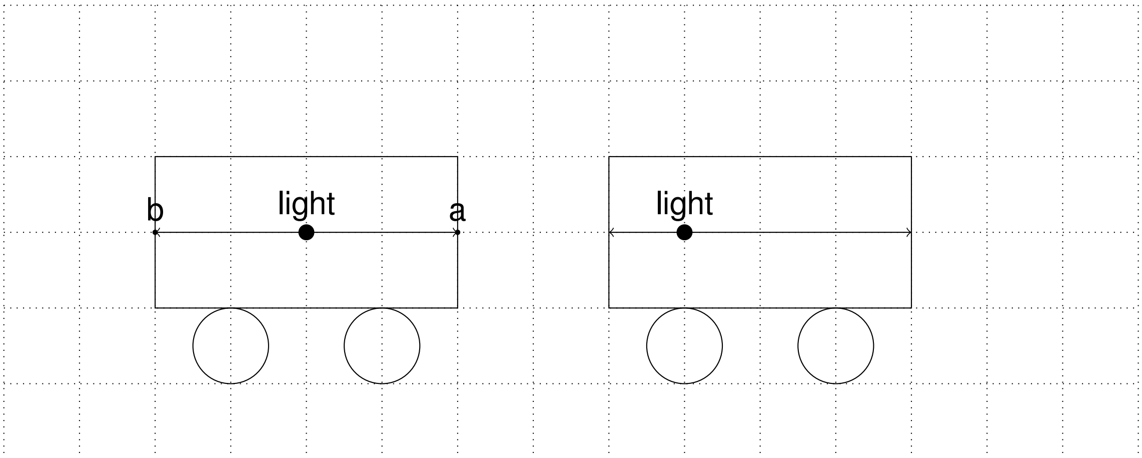


Galileo's velocity addiction rule

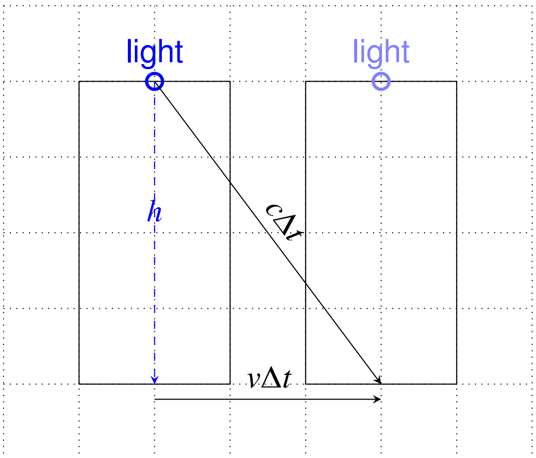
$$u = u' + v$$

Einstein's velocity addiction rule

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

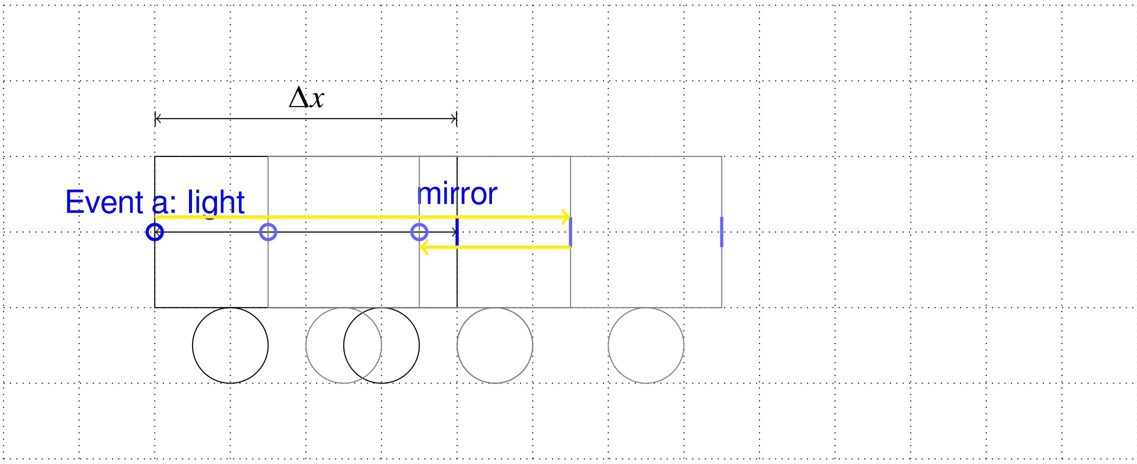


for people at stationary coordination, b happens first
for people on the car, a and b happens at the same time.



$$\begin{aligned}
(c\Delta t)^2 &= h^2 + (v\Delta t)^2 \\
h^2 &= (c^2 - v^2)\Delta t \\
c^2\Delta t'^2 &= (c^2 - v^2)\Delta t \\
\Delta t' &= \sqrt{1 - \frac{v^2}{c^2}}\Delta t \\
\Delta t' &< \Delta t
\end{aligned}$$

Moving clocks run slow

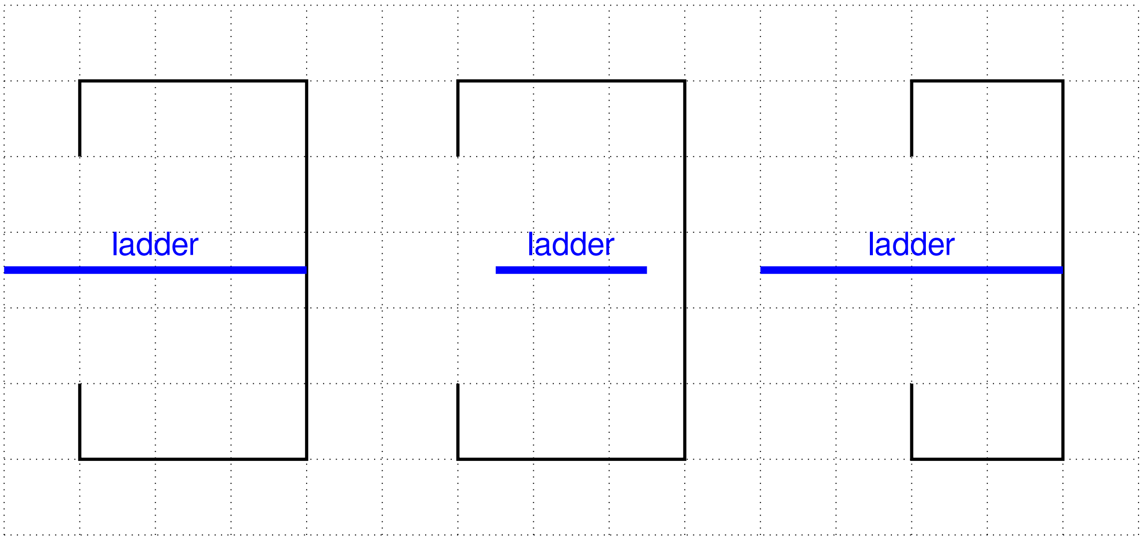


$$
\begin{aligned}
\Delta t' &= 2\frac{\Delta x'}{c} \\
\Delta t_1 &= \frac{\Delta x + v\Delta t_1}{c} \implies \Delta t_1 = \frac{\Delta x}{c-v} \\
\Delta t_2 &= \frac{\Delta x - v\Delta t_2}{c} \implies \Delta t_2 = \frac{\Delta x}{c+v} \\
\Delta t = \Delta t_1 + \Delta t_2 &= \frac{2\Delta x}{c}\frac{1}{1-\frac{v^2}{c^2}} \\
\Delta x' &= \frac{c}{2}\Delta t' \\
&= \frac{c}{2}\Delta t\sqrt{1-\frac{v^2}{c^2}} \\
&= \frac{c}{2}\frac{2\Delta x}{c}\frac{1}{1-\frac{v^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}} \\
&= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Delta x \\
&\implies \Delta x' > \Delta x
\end{aligned}
$$

0.1 The barn and ladder paracby

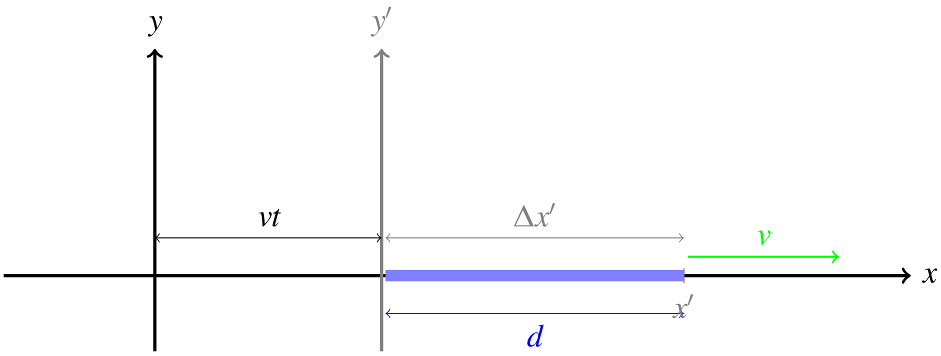
(A) Back end of the ladder makes it in the door
(B) Front end of the ladder hits the wall of the born

Farmer: (A) before (B)
Son: (B) before (A)



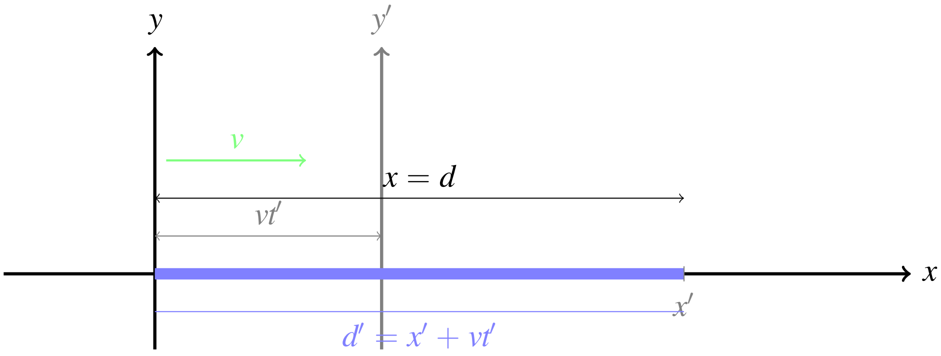
0.2 The Lorentz Transformations

- **Event:** Something that take place at a specific location at precise time.
- Knowing (x, y, z, t), what is (x', y', z', t') of the same event
- Let x be the same position as x'.
  The stick's length measured in the stationary coordiate be d.



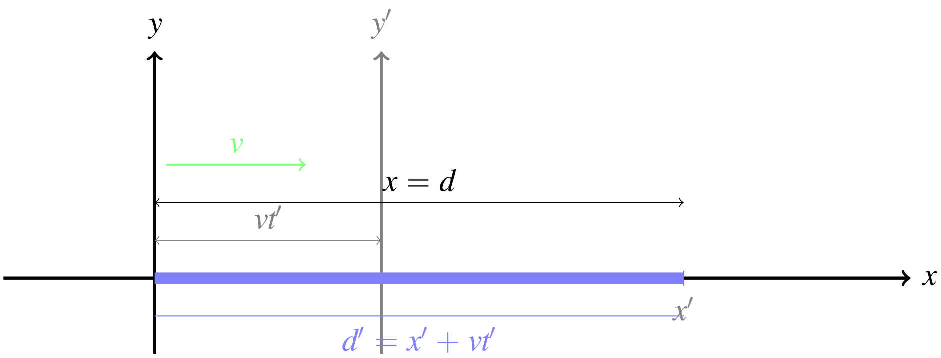
$$
\begin{aligned}
d &= \sqrt{1 - \frac{v^2}{c^2}}\Delta x' \\
(x - vt) &= \sqrt{1 - \frac{v^2}{c^2}}x' \\
x' &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(x - vt)
\end{aligned}
$$

- Let x be the same position as x'.
  The stick's length measured in the stationary coordiate be d.



$$
\begin{aligned}
x' &= d' - vt' \\
d' &= \sqrt{1 - \frac{v^2}{c^2}}x \\
x' &= \sqrt{1 - \frac{v^2}{c^2}}x - vt' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(x - vt) \\
vt' &= (\sqrt{1 - \frac{v^2}{c^2}} - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}})x - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}vt \\
&= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(\frac{v}{c^2}x - vt)
\end{aligned}
$$

- Let x be the same position as x'.
  The stick's length measured in the stationary coordiate be d.



$$
\begin{aligned}
x' &= d' - vt' \\
d' &= \sqrt{1-\frac{v^2}{c^2}}x \\
x' &= \sqrt{1-\frac{v^2}{c^2}}x - vt' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x-vt) \\
vt' &= (\sqrt{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}})x - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}vt \\
&= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(vt - \frac{v^2}{c^2}x) \\
t' &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(t - \frac{v}{c^2}x)
\end{aligned}
$$

- **Check Simutanuously:**
  S:
  Event A $(x=0, t=0)$
  Event B $(x=a, t=0)$
  S':
  Event A $(x'=0, t'=0)$
  Event B $(x'=\gamma a, t'=-\gamma(\frac{v}{c^2})a)$

- **Time Dimention:**
  $\Delta x' = 0 = \gamma(\Delta x - v\Delta t) \implies \Delta x = v\Delta t$

$$
\begin{aligned}
\Delta t' &= \gamma(\Delta t - \frac{v}{c^2}\Delta x) \\
&= \gamma(\Delta t - \frac{v^2}{c^2}\Delta t) \\
&= \sqrt{1-\frac{v^2}{c^2}}\Delta t
\end{aligned}
$$

-
$$
\begin{aligned}
{u_x}' &= \frac{\Delta x'}{\Delta t'} \\
&= \frac{\gamma(\Delta x - v\Delta t)}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)} \\
&= \frac{u_x - v}{1 - \frac{v}{c^2}V_x} \\
{u_y}' &= \frac{\Delta y'}{\Delta t'} \\
&= \frac{\Delta y}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)} \\
&= \frac{u_y}{\gamma(1 - \frac{v}{c^2})}
\end{aligned}
$$

- **Structure of Spacetime:**
  - **Four vectors:**
    (Time is the 0th-dimension.)

$$
\begin{aligned}
x^0 &= ct \\
x^1 &= x \\
x^2 &= y \\
x^3 &= z
\end{aligned}
$$

Lorenze transformation:

$$
\begin{aligned}
x^{0'} &= \gamma(x^0 - \beta x^1) \\
x^{1'} &= \gamma(x^1 - \beta x^0) \\
x^{2'} &= x^2 \\
x^{3'} &= x^3
\end{aligned}
$$

$$
\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}
$$

$$
\begin{aligned}
(x^\mu)' &= \sum_{\nu=0}^{\delta} (\Lambda^\mu_\nu x^\nu) \\
\implies (x^\mu)' &= \Lambda^\mu_\nu X^\nu
\end{aligned}
$$

  - **Constant in Lorenze Transformation:**

$$
-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = a^\mu_\mu \; (\mu = 0, 1, 2, 3)
$$

$$
\begin{aligned}
(x^1)^2 + (x^2)^2 + (x^3)^2 &\implies \textit{length}^2 \textit{of a vector} \implies \textit{rotation} \\
&\implies \textit{inner product}
\end{aligned}
$$

- **The invariant interval:**
  Suppose there are two events in frame S:
  - event A: $(x_A^0, x_A^1, x_A^2, x_A^3)$
  - event B: $(x_B^0, x_B^1, x_B^2, x_B^3)$
  - The difference: $\Delta x^u = x_A^u - x_B^u$ is a four vector.

  The interval of Δx^u is defined by $I = \Delta x_\mu \Delta x^u = -c^2 \Delta t^2 + d^2$

  $$d = \sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}$$

  1. $I < 0$:
     - timelike
     - Causation.
     - When two events occur at the **same place** $(d = 0)$, they are seperated only temporally.
  2. $I = 0$:
     - lightlike
     - Causation. (by light)
     - Ex: Light and sensor on the bus.
     - Two events are connected by a signal traveling at the speed of the light.
  3. $I > 0$:
     - Spacelike
     - When two events occur at the **same time** and are seperated only spatially.

- **Theorem:** If the interval between two events is timelike, there exists an internal system (accessible by Lorenze transformation) in which they occur at the same points likewise, If the interval is spacelike, there exist a system in which the two events occur at the same time.
  Proof:
  Two events A, B occur at (t_A, x_A) and (t_B, x_B).

  $$\begin{cases} t'_A = \gamma(t_A - \frac{v}{c^2} x_A) \\ x'_A = \gamma(x_A - v t_A) \end{cases}$$

  $$\begin{cases} t'_B = \gamma(t_B - \frac{v}{c^2} x_B) \\ x'_B = \gamma(x_B - v t_B) \end{cases}$$

  $$\Delta t = t_A - t_B$$

  Timelike: If $x'_A = x'_B$

  $$
  \begin{aligned}
  &\implies x_A - v t_A = x_B - v t_B \\
  &\implies \frac{v}{c} = \frac{x_A - x_B}{c(t_a - t_b)} < 1
  \end{aligned}
  $$

Spacelike: If $t'_A = t'_B$

$$\implies t_A - \frac{v}{c^2}x_A = t_B - \frac{v}{c^2}x_B$$
$$\implies \frac{v}{c} = \frac{c(t_A - t_B)}{x_A - x_B} < 1$$

0.3 Relative Mechanics

- **Proper Time and Proper Velocity:**
  - The temporal coordinate of the moving frame.
  - $$d\tau = \sqrt{1 - \frac{u^2}{c^2}}dt$$
  - Example:
    $$u = \frac{4}{5}c = \frac{di}{dt}$$
    $$\eta = \frac{dl}{d\tau} = \frac{dt}{d\tau}\frac{dl}{dt} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}u$$
  - 4-velocitty:
    $$\eta^\mu = \frac{dx^\mu}{dt}$$
    $$dx = cdt$$
    $$\eta^0 = \frac{cdt}{dt} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}$$
    Satisfies *Lorenze Transformation*:
    $$\begin{cases} \eta^{0'} &= \gamma(\eta^0 - \beta\eta^1) \\ \eta^{1'} &= \gamma(\eta^1 - \beta\eta^0) \\ \eta^{2'} &= \eta^2 \\ \eta^{3'} &= \eta^3 \end{cases}$$
    $$(\eta)^\mu = \Lambda^\mu_\nu \eta^\nu$$
  - **Exercise:**
    Consider a car traveling along the $\frac{\pi}{4}$ line.

    1. Find the componenet μ_x and μ_y of the ordinary velocity.
       $$u_x = \frac{2}{\sqrt{5}}cos\frac{\pi}{4} = \sqrt{\frac{2}{5}}c = u_y$$

2. Find the compoment η_x and η_y of the proper velocity.

$$\eta = \frac{\frac{2}{\sqrt{5}}c}{\sqrt{1-\frac{4}{5}}} = 2c$$

$$\eta_x = \eta_y = \sqrt{2}c$$

3. Find the zeroth compoment of the 4-velocity.

$$\eta^0 = \frac{u}{\sqrt{1-\frac{u^2}{c^2}}} = \sqrt{5}c$$

A system S' is moving in the x-direction with (ordinary) speed $\sqrt{\frac{2}{5}}c$. By using the qppropriate transformation laws.

* Find the (ordinary) velocity component μ'_x, μ'_y in S'.

$$v = \sqrt{\frac{2}{5}}c$$

$$u'_x = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{\sqrt{\frac{2}{5}}c - \sqrt{\frac{2}{5}}c}{1 - \frac{2}{5}} = 0$$

$$u'_y = \frac{1}{\gamma}\frac{u_1}{\sqrt{1-u'vc^2}} = \sqrt{1-\frac{u^2}{c^2}}\frac{u_y}{\sqrt{1-u'vc^2}} = \sqrt{\frac{2}{5}}\left(\frac{\sqrt{\frac{2}{5}}c}{\frac{3}{5}}\right) = \sqrt{\frac{2}{5}}x$$

* FInd the proper velocity component η'_x, η'_y in S'.

$$(\eta')^0 = \sqrt{\frac{5}{3}}(\sqrt{5}c - \sqrt{\frac{2}{5}}\sqrt{2}c) = \sqrt{3}c$$

$$(\eta')^1 = \sqrt{\frac{5}{3}}(\sqrt{2}c - \sqrt{\frac{2}{5}}\sqrt{2}x) = 0$$

* Check $\eta' = \frac{\mu'}{\sqrt{1-\frac{u^2}{c^2}}}$.

$$\eta' = \sqrt{2}c$$

$$u' = \sqrt{\frac{2}{3}}c$$

$$\frac{u'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{\sqrt{\frac{2}{3}}c}{\sqrt{1-\frac{2}{3}}} = \sqrt{2}c = \eta'$$

0.4 Realativestic Mechanics

- Momentun = mass $\times$ velocity.

$$\overrightarrow{p} = \gamma_u m \overrightarrow{u} = m \overrightarrow{\eta}$$

$$m_A \eta_A + m_B \eta_B = m_C \eta_C + m_D \eta_D$$

$$\eta = \frac{1}{\gamma} \eta'_A + \beta \eta^0_A - - - -(Loranze - 2)$$

$$m_A(\frac{1}{\gamma}\eta'_A + \beta\eta^0_A) + m_B(\frac{1}{\gamma}\eta'_B + \beta\eta^0_C) = m_C(\frac{1}{\gamma}\eta'_C + \beta\eta^0_C) + m_D(\frac{1}{\gamma}\eta'_D + \beta\eta^0_D)$$

- **Conservation of Mass:**

$$\eta_0 = \frac{c}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\begin{aligned} M =& \frac{m}{\sqrt{1 - \frac{u^2}{c^2}}} \implies \textit{relativistic mass} \\ =& \gamma_u m \\ \implies& p^0 = m\eta^0 = \gamma_0 mc \end{aligned}$$

$$m_A \eta^0_A + m_B \eta^0_B = m_C \eta^0_C + m_D \eta^0_D$$

$$\implies M_A + M_B = M_C + M_D$$

$$p^0 = \frac{mc}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{E}{C}$$

Einstein defined

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} = mc^2 + \frac{1}{2}mu^2$$

- **Energy-momentum 4-vector:**

$$\begin{aligned} p^u =& m\eta^u = (\frac{E}{c}, p^1, p^2, p^3) \\ \overrightarrow{p} =& \frac{m\overrightarrow{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \\ E =& \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = mc^2 + kE \end{aligned}$$

$|p^{\mu}|$ is invariant constant

$$\begin{aligned}
& p^{\mu}p_{\mu} = -(p^0)^2 + (\overrightarrow{p} \cdot \overrightarrow{p}) = -m^2c^2 \\
& \implies E^2 - p^2c^2 = m^2c^2 \\
& \implies E^2 = p^2c^2 + m^2c^2
\end{aligned}$$

As $m = 0$: (photon)

$$E = pc$$

- **Example:** Tow lumps of clay, each of rest mass m, collide head-on at $\frac{3}{5}c$. They stick together. Question: What is the mass of the comosite lump?

$$\begin{aligned}
E_i =& E_1 + E_2 \\
=& 2\frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}
\end{aligned}$$

- **Relativistic: Kinematics**
  Ex: A pion at rest decays into a muon and a neutrino. Find the momentum of the outging muon, in terms of the two mass, m_{π} and m_{μ}. (m_{ν})

$$\begin{aligned}
& E_i = m_{\pi}c^2, P_i = 0 \\
& E_f = E_{\mu} + E_{\nu}, P_f = 0 = P_{\mu} + P_{\nu} \implies \overrightarrow{P}_{\mu} = -\overrightarrow{P}_{\nu} \\
& E_{\mu}^2 = P_{\mu}^2c^2 + m_{\mu}^2c^4 \\
& E_{\nu} = P_{\nu}x \\
& E_i = E_f \implies m_{\pi}c^2 = \sqrt{P_{\mu}^2c^2 + m_{\mu}^2c^4} + P_{\mu}c \\
& P_u^2 + m_{\pi}^2 - 2m_{\pi}c^2P_{\mu} = (m_{\pi}c - P_{\mu}) = P_{\mu}^2 + m_{\mu}^2c^2 \\
& \implies P_u = \frac{c(m_{\pi}^2 - m_{\mu}^2)}{2m_{\pi}}
\end{aligned}$$

0.5 HW note

- $$f' = f\sqrt{\frac{1-\beta}{1+\beta}}$$

- $$v = \frac{\Delta\lambda}{\lambda}c$$