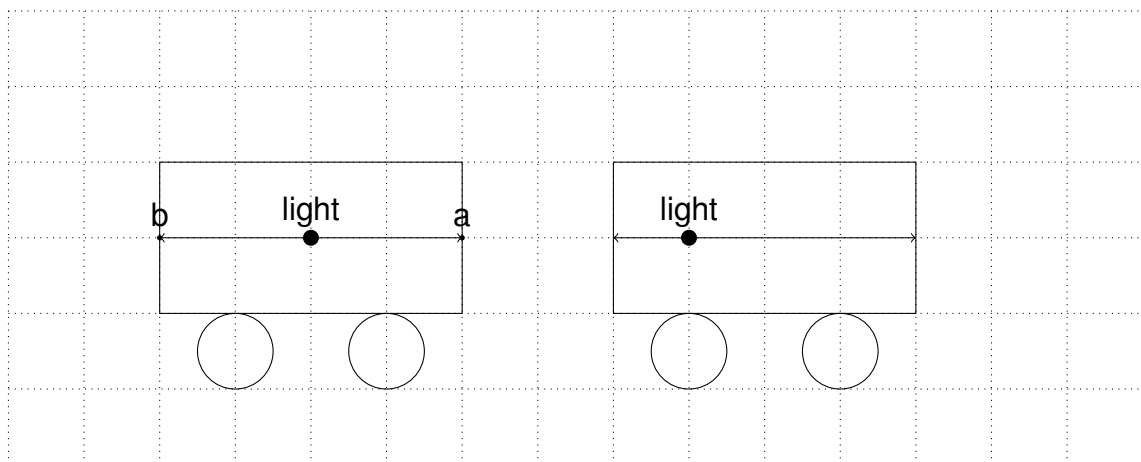


Galileo's velocity addition rule

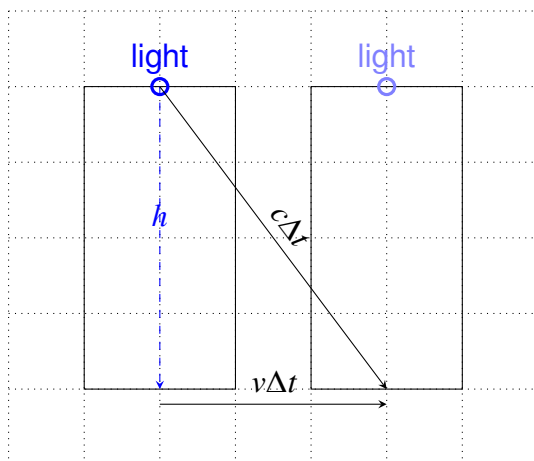
$$u = u' + v$$

Einstein's velocity addition rule

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$



for people at stationary coordinates, b happens first  
for people on the car, a and b happens at the same time.



Moving clocks run slow

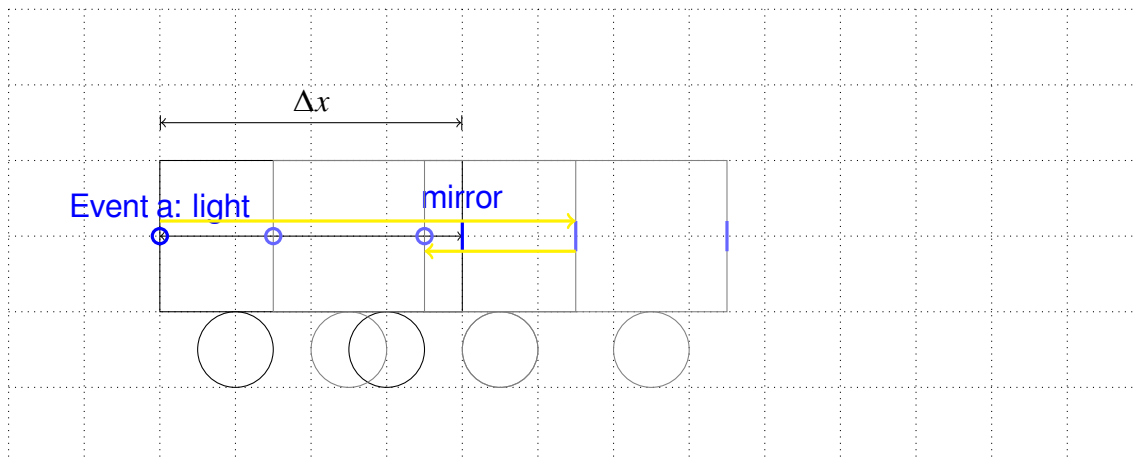
$$(c\Delta t')^2 = h^2 + (v\Delta t)^2$$

$$h^2 = (c^2 - v^2)\Delta t^2$$

$$c^2\Delta t'^2 = (c^2 - v^2)\Delta t^2$$

$$\Delta t' = \sqrt{1 - \frac{v^2}{c^2}}\Delta t$$

$$\Delta t' < \Delta t$$



$$\Delta t' = 2 \frac{\Delta x'}{c}$$

$$\Delta t_1 = \frac{\Delta x + v\Delta t_1}{c} \implies \Delta t_1 = \frac{\Delta x}{c - v}$$

$$\Delta t_2 = \frac{\Delta x - v\Delta t_2}{c} \implies \Delta t_2 = \frac{\Delta x}{c + v}$$

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{2\Delta x}{c} \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\Delta x' = \frac{c}{2} \Delta t'$$

$$= \frac{c}{2} \Delta t \sqrt{1 - \frac{v^2}{c^2}}$$

$$= \frac{c}{2} \frac{2\Delta x}{c} \frac{1}{1 - \frac{v^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}$$

$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta x$$

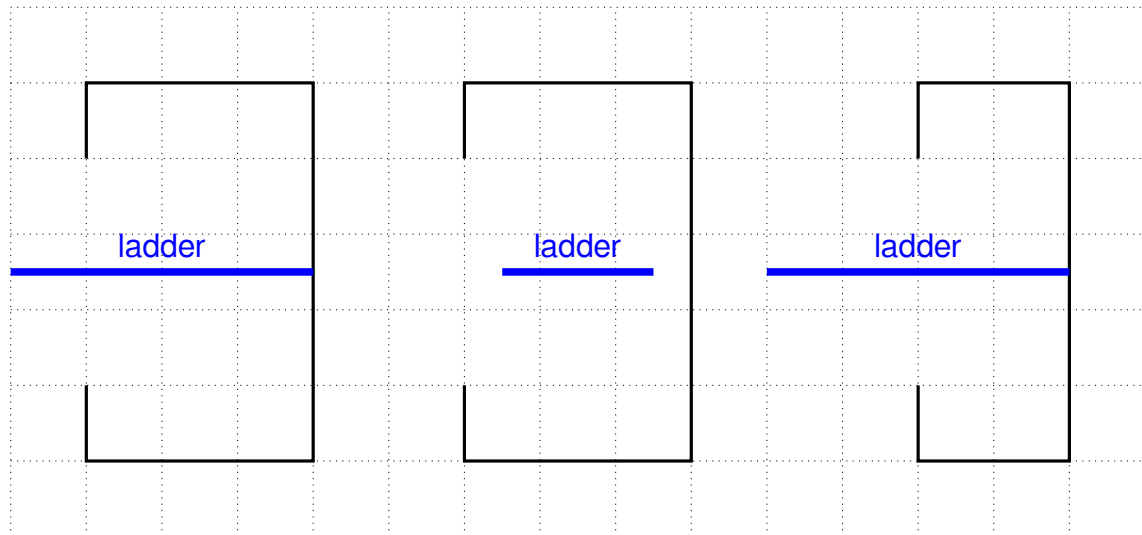
$$\implies \Delta x' > \Delta x$$

## 0.1 The barn and ladder paracby

- (A) Back end of the ladder makes it in the door
- (B) Front end of the ladder hits the wall of the born

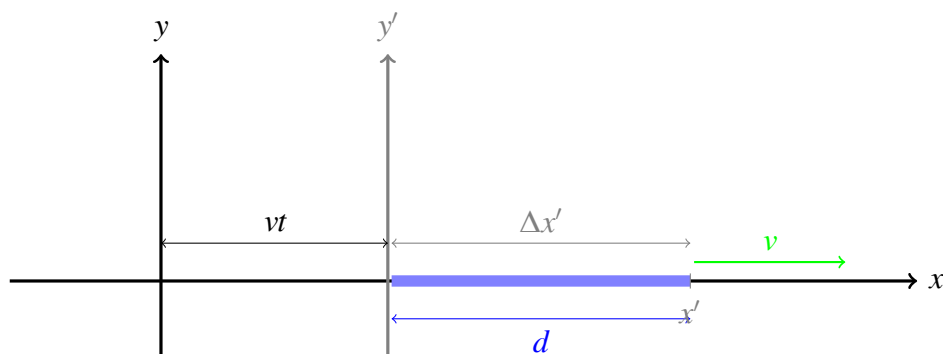
Farmer: (A) before (B)

Son: (B) before (A)



## 0.2 The Lorentz Transformations

- **Event:** Something that take place at a specific location at precise time.
- Knowing  $(x, y, z, t)$ , what is  $(x', y', z', t')$  of the same event
- Let  $x$  be the same position as  $x'$ .  
The stick's length measured in the stationary coordinate be  $d$ .

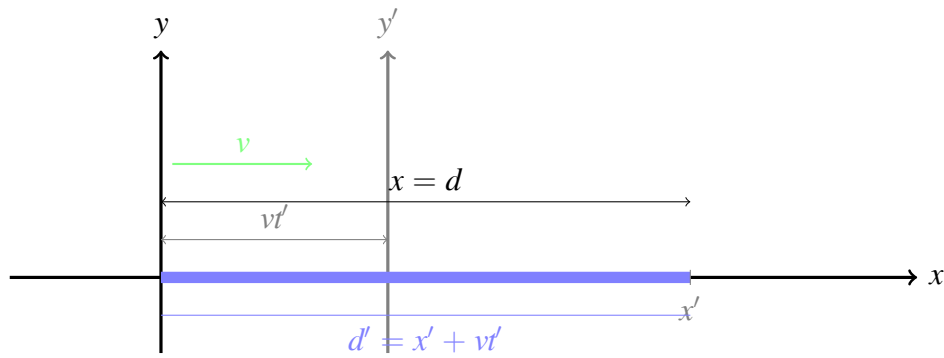


$$d = \sqrt{1 - \frac{v^2}{c^2}} \Delta x'$$

$$(x - vt) = \sqrt{1 - \frac{v^2}{c^2}} x'$$

$$x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt)$$

- Let  $x$  be the same position as  $x'$ .  
The stick's length measured in the stationary coordinate be  $d$ .



$$x' = d' - vt'$$

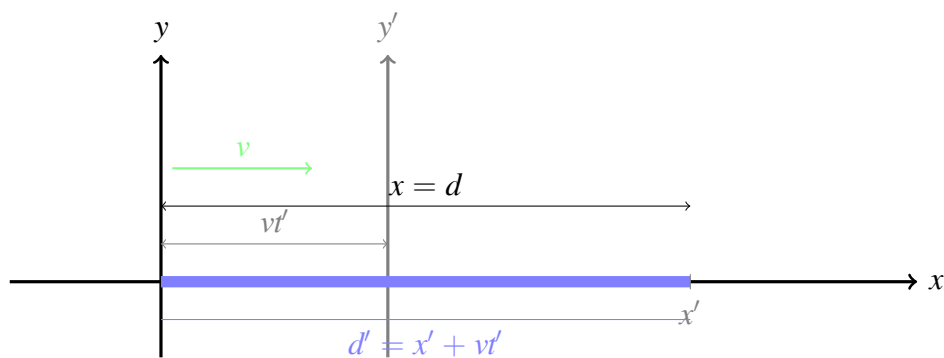
$$d' = \sqrt{1 - \frac{v^2}{c^2}} x$$

$$x' = \sqrt{1 - \frac{v^2}{c^2}} x - vt' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt')$$

$$vt' = \left( \sqrt{1 - \frac{v^2}{c^2}} - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) x - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} vt$$

$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( \frac{v}{c^2} x - vt \right)$$

- Let  $x$  be the same position as  $x'$ .  
The stick's length measured in the stationary coordinate be  $d$ .



$$x' = d' - vt'$$

$$d' = \sqrt{1 - \frac{v^2}{c^2}} x$$

$$x' = \sqrt{1 - \frac{v^2}{c^2}} x - vt' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt)$$

$$vt' = \left( \sqrt{1 - \frac{v^2}{c^2}} - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) x - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} vt$$

$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( vt - \frac{v^2}{c^2} x \right)$$

$$t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( t - \frac{v}{c^2} x \right)$$

- **Check Simultaneously:**

$S$ :

Event A ( $x = 0, t = 0$ )

Event B ( $x = a, t = 0$ )

$S'$ :

Event A ( $x' = 0, t' = 0$ )

Event B ( $x' = \gamma a, t' = -\gamma(\frac{v}{c^2})a$ )

- **Time Dimention:**

$$\Delta x' = 0 = \gamma(\Delta x - v\Delta t) \implies \Delta x = v\Delta t$$

$$\Delta t' = \gamma \left( \Delta t - \frac{v}{c^2} \Delta x \right)$$

$$= \gamma \left( \Delta t - \frac{v^2}{c^2} \Delta t \right)$$

$$= \sqrt{1 - \frac{v^2}{c^2}} \Delta t$$

•

$$\begin{aligned}
 u_x' &= \frac{\Delta x'}{\Delta t'} \\
 &= \frac{\gamma(\Delta x - v\Delta t)}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)} \\
 &= \frac{u_x - v}{1 - \frac{v}{c^2}V_x} \\
 u_y' &= \frac{\Delta y'}{\Delta t'} \\
 &= \frac{\Delta y}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)} \\
 &= \frac{u_y}{\gamma(1 - \frac{v}{c^2})}
 \end{aligned}$$

• **Structure of Spacetime:**

– **Four vectors:**

(Time is the 0th-dimension.)

$$x^0 = ct$$

$$x^1 = x$$

$$x^2 = y$$

$$x^3 = z$$

Lorenze transformation:

$$x^{0'} = \gamma(x^0 - \beta x^1)$$

$$x^{1'} = \gamma(x^1 - \beta x^0)$$

$$x^{2'} = x^2$$

$$x^{3'} = x^3$$

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{aligned}
 (x^\mu)' &= \sum_{\nu=0}^{\delta} (\Lambda_\nu^\mu x^\nu) \\
 \implies (x^\mu)' &= \Lambda_\nu^\mu X^\nu
 \end{aligned}$$

– **Constant in Lorenze Transformation:**

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = a_\mu^\mu \quad (\mu = 0, 1, 2, 3)$$

$$(x^1)^2 + (x^2)^2 + (x^3)^2 \implies \text{length}^2 \text{ of a vector} \implies \text{rotation} \\ \implies \text{inner product}$$

– **The invariant interval:**

Suppose there are two events in frame  $S$ :

- \* event A:  $(x_A^0, x_A^1, x_A^2, x_A^3)$
- \* event B:  $(x_B^0, x_B^1, x_B^2, x_B^3)$
- \* The difference:  $\Delta x^\mu = x_A^\mu - x_B^\mu$  is a four vector.

The interval of  $\Delta x^\mu$  is defined by  $I = \Delta x_\mu \Delta x^\mu = -c^2 \Delta t^2 + d^2$

$$d = \sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}$$

1.  $I < 0$ :

- \* timelike
- \* Causation.
- \* When two events occur at the **same place** ( $d = 0$ ), they are separated only temporally.

2.  $I = 0$ :

- \* lightlike
- \* Causation. ( by light )
- \* Ex: Light and sensor on the bus.
- \* Two events are connected by a signal traveling at the speed of the light.

3.  $I > 0$ :

- \* Spacelike
- \* When two events occur at the **same time** and are separated only spatially.

– **Theorem:** If the interval between two events is timelike, there exists an internal system ( accessible by Lorentze transformation ) in which they occur at the same points likewise, If the interval is spacelike, there exist a system in which the two events occur at the same time.

**Proof:**

Two events A, B occur at  $(t_A, x_A)$  and  $(t_B, x_B)$ .

$$\begin{cases} t'_A = \gamma(t_A - \frac{v}{c^2}x_A) \\ x'_A = \gamma(x_A - vt_A) \end{cases}$$

$$\begin{cases} t'_B = \gamma(t_B - \frac{v}{c^2}x_B) \\ x'_B = \gamma(x_B - vt_B) \end{cases}$$

$$\Delta t = t_A - t_B$$

Timelike: If  $x'_A = x'_B$

$$\implies x_A - vt_A = x_B - vt_B$$

$$\implies \frac{v}{c} = \frac{x_A - x_B}{c(t_A - t_B)} < 1$$

Spacelike: If  $t'_A = t'_B$

$$\implies t_A - \frac{v}{c^2}x_A = t_B - \frac{v}{c^2}x_B$$

$$\implies \frac{v}{c} = \frac{c(t_A - t_B)}{x_A - x_B} < 1$$

### 0.3 Relative Mechanics

- **Proper Time and Proper Velocity:**

- The temporal coordinate of the moving frame.

- 

$$d\tau = \sqrt{1 - \frac{u^2}{c^2}} dt$$

- Example:

$$u = \frac{4}{5}c = \frac{di}{dt}$$

$$\eta = \frac{dl}{d\tau} = \frac{dt}{d\tau} \frac{dl}{dt} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} u$$

- 4-velocity:

$$\eta^\mu = \frac{dx^\mu}{dt}$$

$$dx = c dt$$

$$\eta^0 = \frac{cdt}{d\tau} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Satisfies *Lorenze Transformation*:

$$\begin{cases} \eta^{0'} &= \gamma(\eta^0 - \beta\eta^1) \\ \eta^{1'} &= \gamma(\eta^1 - \beta\eta^0) \\ \eta^{2'} &= \eta^2 \\ \eta^{3'} &= \eta^3 \end{cases}$$

$$(\eta)^\mu = \Lambda^\mu_\nu \eta^\nu$$

- **Exercise:**

Consider a car traveling along the  $\frac{\pi}{4}$  line.

1. Find the componenet  $\mu_x$  and  $\mu_y$  of the ordinary velocity.

$$u_x = \frac{2}{\sqrt{5}} \cos \frac{\pi}{4} = \sqrt{\frac{2}{5}} c = u_y$$



2. Find the component  $\eta_x$  and  $\eta_y$  of the proper velocity.

$$\eta = \frac{\frac{2}{\sqrt{5}}c}{\sqrt{1 - \frac{4}{5}}} = 2c$$

$$\eta_x = \eta_y = \sqrt{2}c$$

3. Find the zeroth component of the 4-velocity.

$$\eta^0 = \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} = \sqrt{5}c$$

A system  $S'$  is moving in the x-direction with ( ordinary ) speed  $\sqrt{\frac{2}{5}}c$ . By using the appropriate transformation laws.

- \* Find the (ordinary) velocity component  $u'_x, u'_y$  in  $S'$ .

$$v = \sqrt{\frac{2}{5}}c$$

$$u'_x = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{\sqrt{\frac{2}{5}}c - \sqrt{\frac{2}{5}}c}{1 - \frac{2}{5}} = 0$$

$$u'_y = \frac{1}{\gamma} \frac{u_y}{\sqrt{1 - u'vc^2}} = \sqrt{1 - \frac{u^2}{c^2}} \frac{u_y}{\sqrt{1 - u'vc^2}} = \sqrt{\frac{2}{5}} \left( \frac{\sqrt{\frac{2}{5}}c}{\frac{3}{5}} \right) = \sqrt{\frac{2}{5}}c$$

- \* Find the proper velocity component  $\eta'_x, \eta'_y$  in  $S'$ .

$$(\eta')^0 = \sqrt{\frac{5}{3}}(\sqrt{5}c - \sqrt{\frac{2}{5}}\sqrt{2}c) = \sqrt{3}c$$

$$(\eta')^1 = \sqrt{\frac{5}{3}}(\sqrt{2}c - \sqrt{\frac{2}{5}}\sqrt{2}c) = 0$$

- \* Check  $\eta' = \frac{u'}{\sqrt{1 - \frac{u'^2}{c^2}}}$ .

$$\eta' = \sqrt{2}c$$

$$u' = \sqrt{\frac{2}{3}}c$$

$$\frac{u'}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{\sqrt{\frac{2}{3}}c}{\sqrt{1 - \frac{2}{3}}} = \sqrt{2}c = \eta'$$

## 0.4 Relativistic Mechanics

– Momentum = mass  $\times$  velocity.

$$\vec{p} = \gamma_u m \vec{u} = m \vec{\eta}$$

$$m_A \eta_A + m_B \eta_B = m_C \eta_C + m_D \eta_D$$

$$\eta = \frac{1}{\gamma} \eta'_A + \beta \eta_A^0 \dots \dots (Loranzze - 2)$$

$$m_A \left( \frac{1}{\gamma} \eta'_A + \beta \eta_A^0 \right) + m_B \left( \frac{1}{\gamma} \eta'_B + \beta \eta_B^0 \right) = m_C \left( \frac{1}{\gamma} \eta'_C + \beta \eta_C^0 \right) + m_D \left( \frac{1}{\gamma} \eta'_D + \beta \eta_D^0 \right)$$

– **Conservation of Mass:**

$$\eta_0 = \frac{c}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$M = \frac{m}{\sqrt{1 - \frac{u^2}{c^2}}} \Rightarrow \text{relativistic mass}$$

$$= \gamma_u m$$

$$\Rightarrow p^0 = m \eta^0 = \gamma_0 m c$$

$$m_A \eta_A^0 + m_B \eta_B^0 = m_C \eta_C^0 + m_D \eta_D^0$$

$$\Rightarrow M_A + M_B = M_C + M_D$$

$$p^0 = \frac{mc}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{E}{c}$$

Einstein defined

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} = mc^2 + \frac{1}{2} mu^2$$

• **Energy-momentum 4-vector:**

$$p^\mu = m \eta^\mu = \left( \frac{E}{c}, p^1, p^2, p^3 \right)$$

$$\vec{p} = \frac{m \vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = mc^2 + kE$$

$|p^\mu|$  is invariant constant

$$\begin{aligned} p^\mu p_\mu &= -(p^0)^2 + (\vec{p} \cdot \vec{p}) = -m^2 c^2 \\ \implies E^2 - p^2 c^2 &= m^2 c^2 \\ \implies E^2 &= p^2 c^2 + m^2 c^2 \end{aligned}$$

As  $m = 0$ : (photon )

$$E = pc$$

- **Example:** Two lumps of clay, each of rest mass  $m$ , collide head-on at  $\frac{3}{5}c$ . They stick together. Question: What is the mass of the composite lump?

$$\begin{aligned} E_i &= E_1 + E_2 \\ &= 2 \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \end{aligned}$$

### • Relativistic: Kinematics

Ex: A pion at rest decays into a muon and a neutrino. Find the momentum of the outgoing muon, in terms of the two mass,  $m_\pi$  and  $m_\mu$ . ( $m_\nu$ )

$$\begin{aligned} E_i &= m_\pi c^2, P_i = 0 \\ E_f &= E_\mu + E_\nu, P_f = 0 = P_\mu + P_\nu \implies \vec{P}_\mu = -\vec{P}_\nu \\ E_\mu^2 &= P_\mu^2 c^2 + m_\mu^2 c^4 \\ E_\nu &= P_\nu c \\ E_i &= E_f \implies m_\pi c^2 = \sqrt{P_\mu^2 c^2 + m_\mu^2 c^4} + P_\mu c \\ P_\mu^2 + m_\mu^2 - 2m_\pi c^2 P_\mu &= (m_\pi c - P_\mu)^2 = P_\mu^2 + m_\mu^2 c^2 \\ \implies P_\mu &= \frac{c(m_\pi^2 - m_\mu^2)}{2m_\pi} \end{aligned}$$

## 0.5 HW note

•

$$f' = f \sqrt{\frac{1 - \beta}{1 + \beta}}$$

•

$$v = \frac{\Delta \lambda}{\lambda} c$$