Continuous Space MDPs

Last Time

- Neural Network Function Approximation

$\hat{y} = f_\theta(x)$

$\{(x_i, y_i)\}_{i=1}^n$

Guiding Questions

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- What tools do we have to solve MDPs with continuous S and A?

Current Tool-Belt

Offline — Value Iteration, Policy Iteration — Curse of Dimensionality, Limited to Discrete Spaces

Online — MCTS — Only computes online, Limited to Discrete Spaces

Today: Four Tools

1. LQR
2. Fitted Value Iteration
3. Sparse UCT/Progressive Widening
4. MPC

Notation: Continuous Random Variables

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support(X) $x \in X$	All the values that X can take	$\{h, t\}$ or $\{0, 1\}$	
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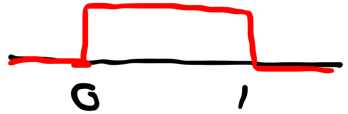
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$ 
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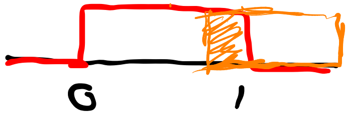
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$  $P(X \in [a,b]) = \int_a^b p(x)dx$
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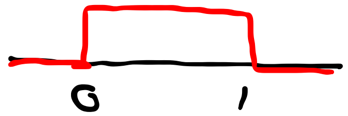
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$  $P(X \in [a,b]) = \int_a^b p(x)dx$ $P(X = 0.5) = 0$
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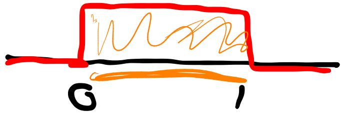
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table (see below)	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$  $P(X \in [a, b]) = \int_a^b p(x)dx$ $P(X = 0.5) = 0$
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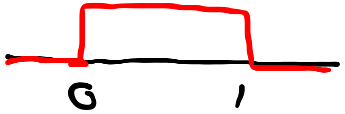
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table <table><tr><th>X</th><th>P(X)</th></tr><tr><td>0</td><td>0.5</td></tr><tr><td>1</td><td>0.5</td></tr></table>	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$  $P(X \in [a,b]) = \int_a^b p(x)dx$ $P(X = 0.5) = 0$
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Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

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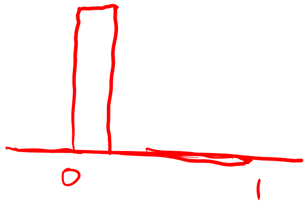

$U([0, 0.1])$

$p(x) = 10\,\mathbb{1}(0 \le x \le 0.1)$

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Multivariate Gaussian Distribution

$$\begin{bmatrix} \boxed{4} & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$x = [x_1, x_2]$

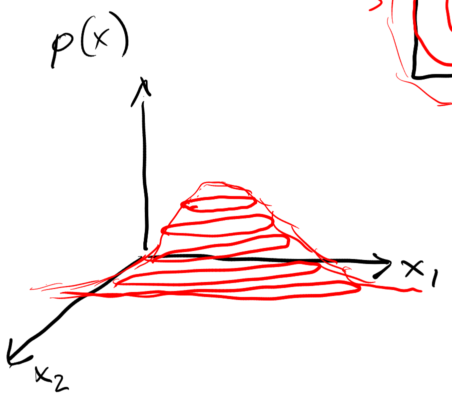
$$\mathcal{N}(\mu, \Sigma)$$

Joint Distribution

$p(x) = \mathcal{N}(\mu, \Sigma)$

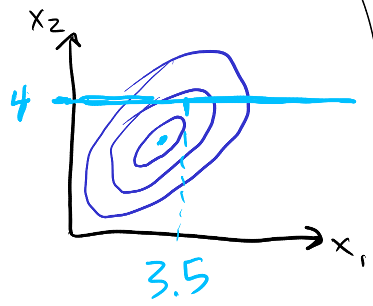
$$p(x) = \frac{\exp\left(\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)}{(2\pi)^{n/2} |\Sigma|^{1/2}}$$

$\mu = [3,3]$ $\quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$



$\mu = [3,3]$

$\Sigma = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$



Conditional Distribution

$p(x_1 | x_2) = \mathcal{N}(\bar{\mu}_1, \bar{\Sigma}_1)$

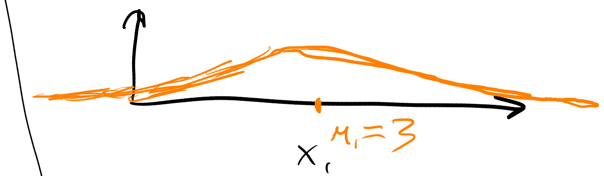
$\bar{\mu}_1 = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2)$

$\bar{\Sigma}_1 = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$

$p(x_1 | x_2 = 4)$

Marginal Distribution

$p(x_1) = \mathcal{N}(\mu_1, \Sigma_{11})$



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The old rules still work!

$$U^*(s) = \max_a \left(R(s,a) + \gamma E[U^*(s')]\right)$$

$$U^\pi(s) = \ldots$$

$$B[U](s) = \underline{\max_a} \left(R(s,a) + \gamma \underline{\mathop{E}_{s' \sim T(s'|s,a)} [U(s')]}\right)$$

hard !!!!

Optimization is harder than integration

$$\int_{s' \in S} T(s'|s,a)\, U(s')\, ds'$$

hard!

Monte Carlo

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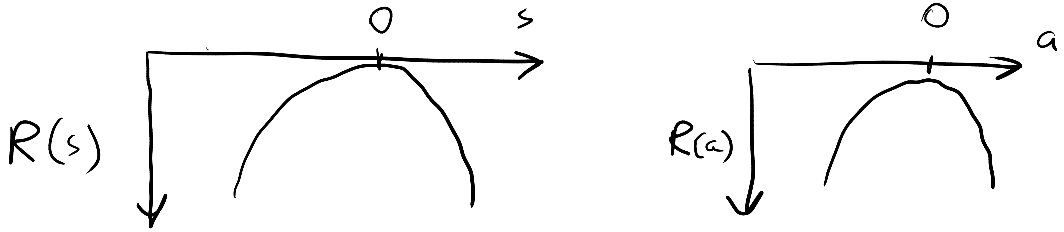
$R(s,a) = R(s) + R(a)$



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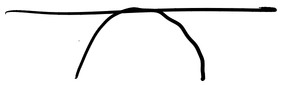
$V_1 = R_s \quad q_1 = 0$

$R(a)$



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Practical Implication: If a continuous problem has roughly linear dynamics, a convex cost function, and roughly zero-mean additive noise, you can use *certainty-equivalent control*, i.e. control as if there is no noise.

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while not converged

$\theta \leftarrow \theta'$

$\hat{V}' \leftarrow B_{\text{approx}}[V_\theta]$

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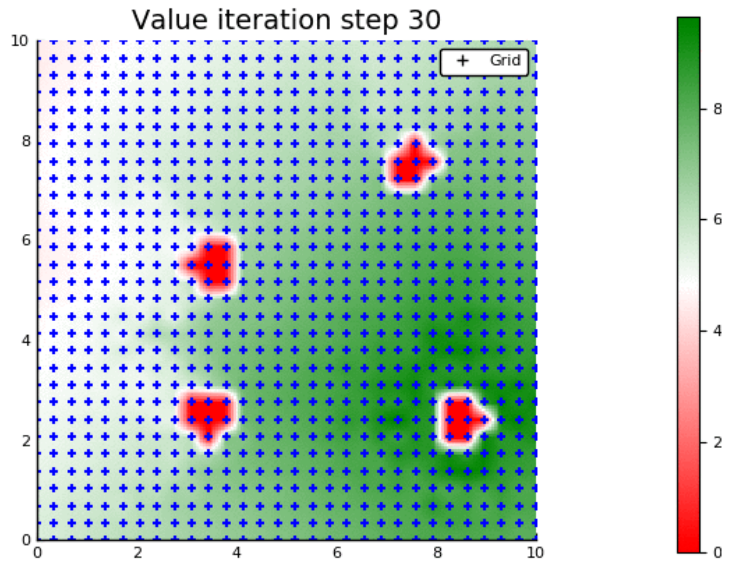

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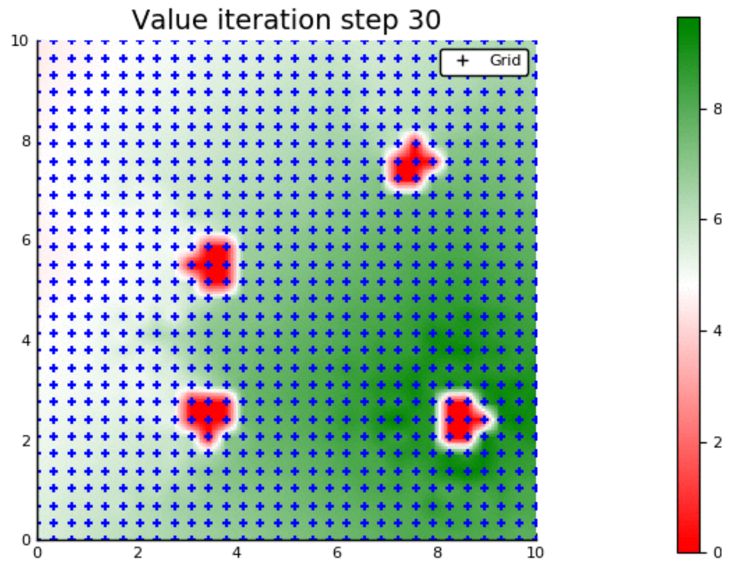

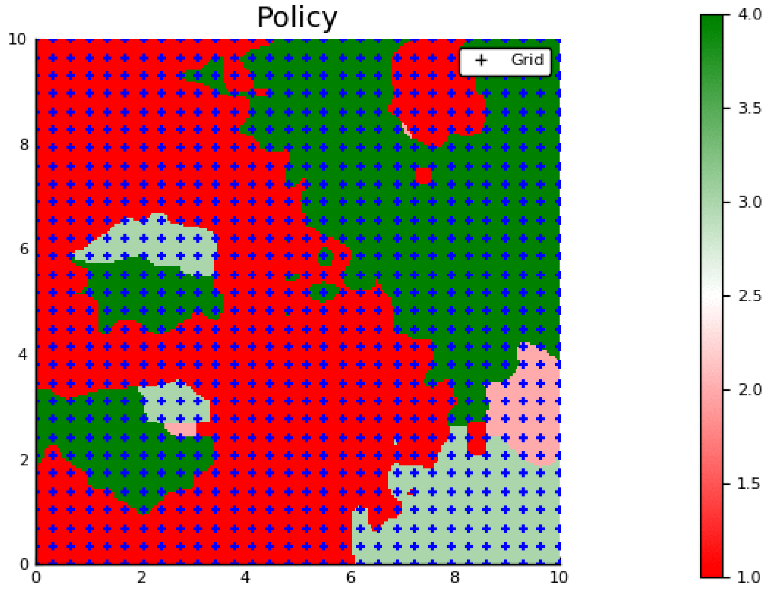

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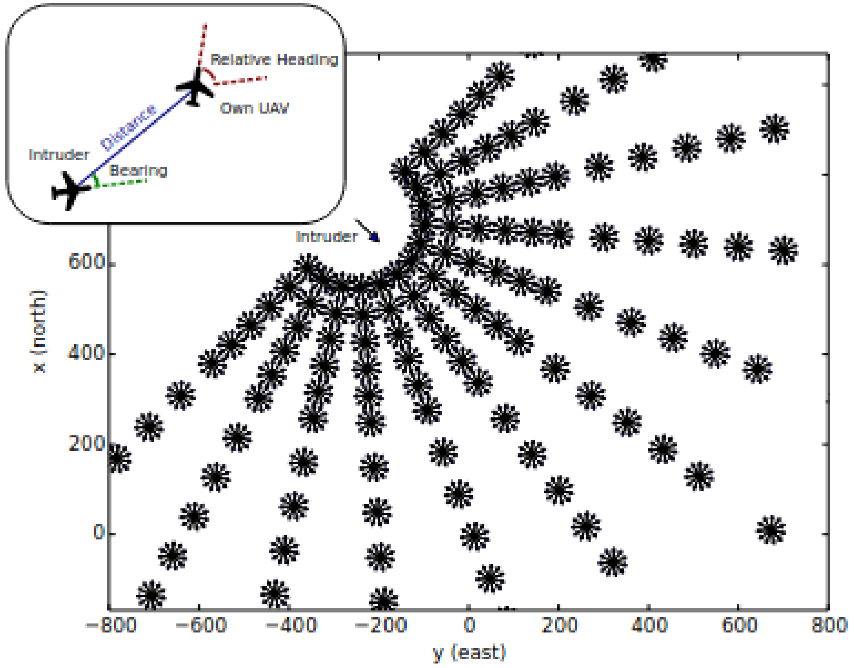
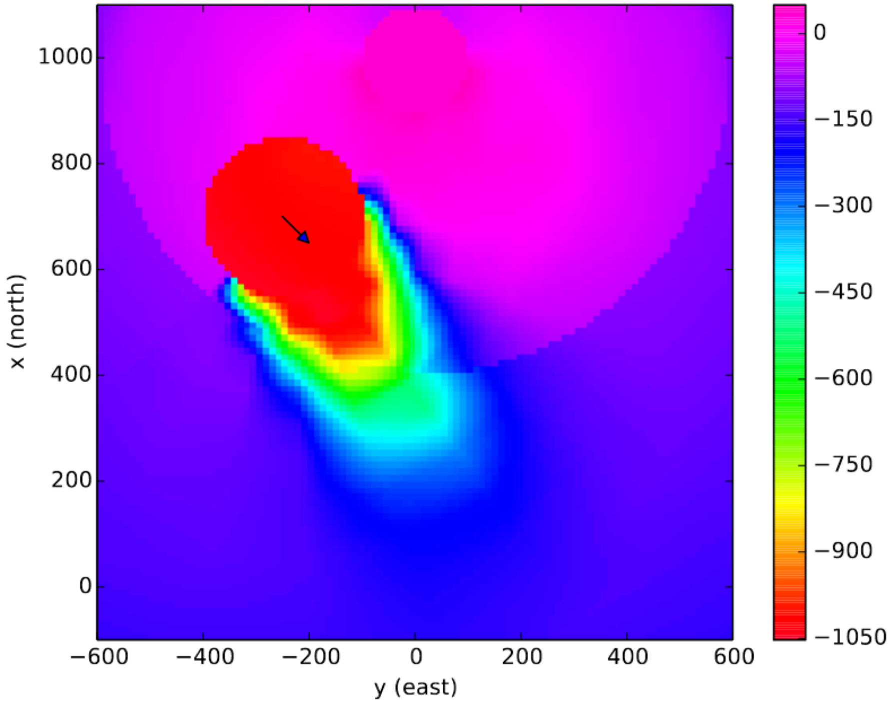
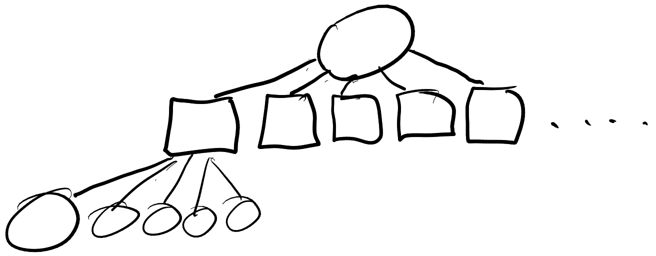




Break

What will a Monte Carlo Tree Search tree look like if run on a problem with continuous spaces?



3. Sparse Tree Search/Progressive Widening

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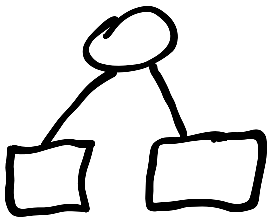

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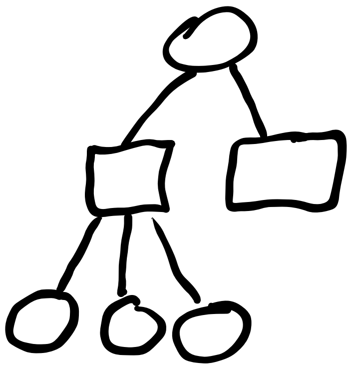

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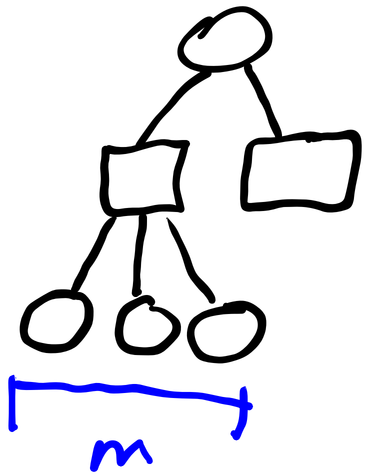

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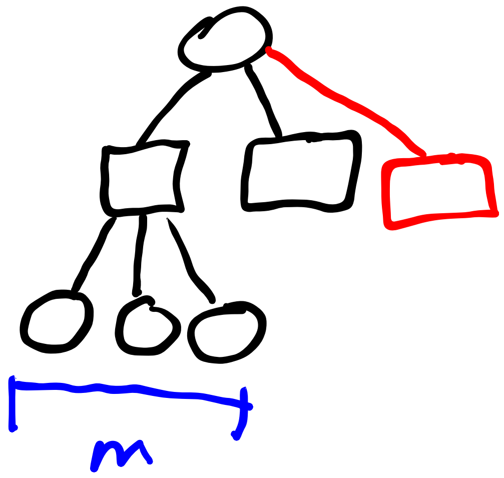

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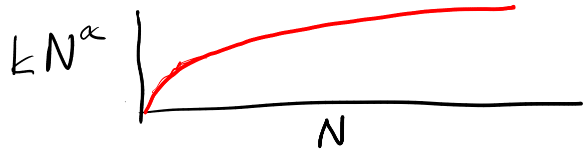
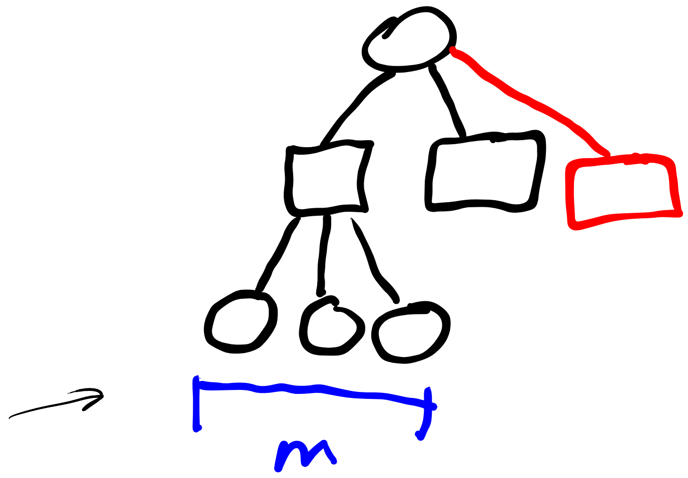


add new branch if $C < kN^\alpha$ $\quad(\alpha < 1)$

number of children



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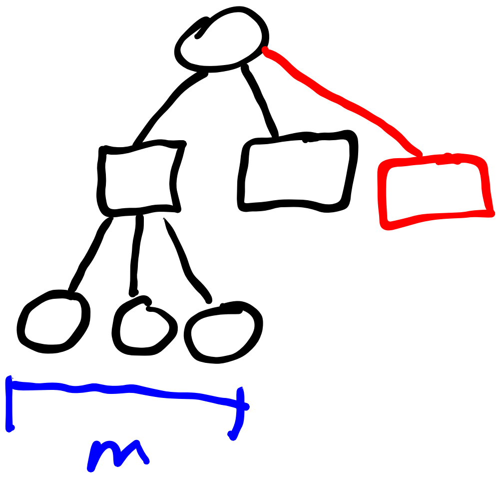

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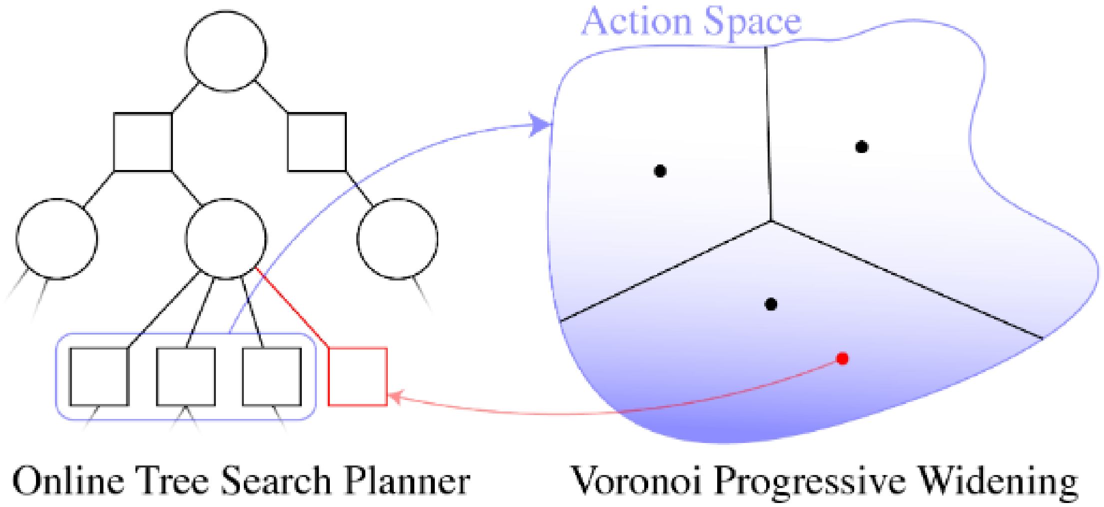

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(Use off-the-shelf optimization software, e.g. Ipopt)

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Hindsight Optimization

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Guiding Questions

- What tools do we have to solve MDPs with continuous S and A?