Stochastic Processes and Simple Decisions

Review

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{x_1, x_2, x_3, \ldots\}$
- $\{x_t\}_{t=1}^{\infty}$ or just $\{x_t\}$

In a *stationary* stochastic process (all in this class), this relationship does not change with time

Example:

$$x_0 = 0 \qquad x_{t+1} = x_t + v_t$$

$v_t \sim \mathcal{U}(\{0,1\})$ (i.i.d.)

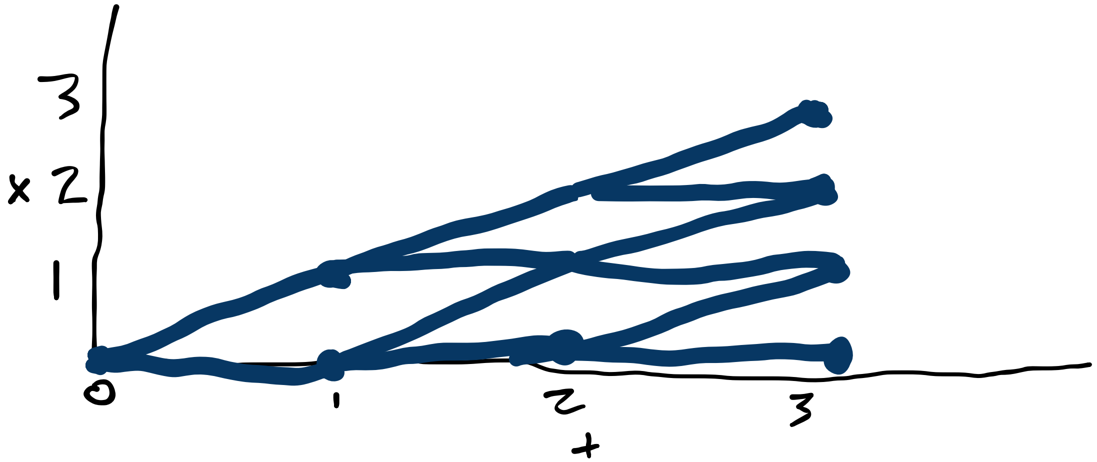
Shorthand:

$$x' = x + v$$

Stochastic Process

$x_0 = 0$

$$x_{t+1} = x_t + v_t$$
$$v_t \sim \mathcal{U}(\{0, 1\}) \text{ (i.i.d.)}$$



$$P(x_{1:n}) = \prod_{t=1}^{n} P(x_t \mid \mathrm{pa}(x_t))$$

For this particular process,

$$P(x_{1:n}) = \prod_{t=1}^{n} P(x_t \mid x_{t-1})$$

Joint

x0	x1	x2	P(x1, x2, x3)
0	0	0	0.25
0	0	1	0.25
0	1	1	0.25
0	1	2	0.25

Marginal

For this particular process, since $\mathrm{pa}(x_t) = x_{t-1}$, if $P(x_{t-1})$ is known,

$$P(x_t) = \sum_{k \in x_{t-1}} P(x_t \mid x_{t-1} = k)\, P(x_{t-1} = k)$$
$$= 0.5\, P(x_{t-1} = x_t - 1) + 0.5\, P(x_{t-1} = x_t)$$

Stochastic Process

Expectation

$$E[x_t] = \sum_{x \in x_t} x P(x_t = x)$$

Expectation of a function
(such as reward)

$$E[f(x_t)] = \sum_{x \in x_t} f(x) P(x_t = x)$$

For this particular process, $x_t = \sum_{i=1}^{t} v_t$, so

$$E[x_t] = E\left[\sum_{i=1}^{t} v_t\right] = \sum_{i=1}^{t} E[v_t] = 0.5t$$

Simulating a Stochastic Process

030-Stochastic-Processes.ipynb

Markov Process

- A stochastic process $\{s_t\}$ is *Markov* if
  $$P(s_{t+1} \mid s_t, s_{t-1}, \dots, s_0) = P(s_{t+1} \mid s_t)$$
- s_t is called the "state" of the process

Break

Suppose you want to create a Markov model that describes how many new COVID cases will start on a particular day. **What information should be in the state of the model?**

Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 14 days after they contract the disease.
- The number of people infected by each person on day d of their illness is roughly $\mathcal{N}(\mu_d, \sigma^2)$

Hidden Markov Model

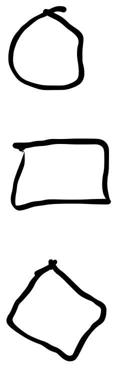
(Often you can't measure the whole state)

Simple Decisions

Value of Information

Decision Networks and MDPs

Decision Network

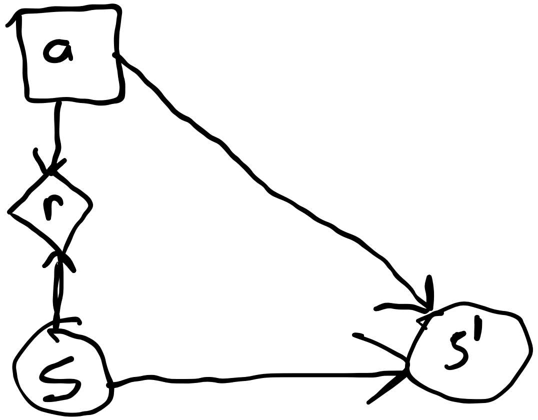


Chance node

Decision node

Utility node

MDP Dynamic Decision Network



MDP Optimization problem

$$\text{maximize} \quad \mathrm{E}\left[\sum_{t=1}^{\infty} r_t\right]$$

Not well formulated!
Infinite

Finite MDP Objectives

1. Finite time

$$\mathbf{E}\left[\sum_{t=0}^{T} r_t\right]$$

2. Average reward

$$\lim_{n\to\infty} \mathbf{E}\left[\sum_{t=0}^{n} r_t\right]$$

3. Discounting

$$\mathbf{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

discount $\gamma \in [0, 1)$

typically 0.9, 0.95, 0.99

if $\underline{r} \leq r_t \leq \bar{r}$

then

$$\frac{\underline{r}}{1-\gamma} \leq \sum_{t=0}^{\infty} \gamma^t r_t \leq \frac{\bar{r}}{1-\gamma}$$

4. Terminal States

Infinite time, but a terminal state (no reward, no leaving) is always reached with probability 1.

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