Continuous Space MDPs

Last Time

- Neural Network Function Approximation

$\hat{y} = f_\theta(x)$

$\{(x_i, y_i)\}_{i=1}^{n}$

Guiding Questions

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- What tools do we have to solve MDPs with continuous S and A?

Current Tool-Belt

Offline < Value Iteration, Policy Iteration] – Curse of Dimensionality, Limited to Discrete Spaces

Online — MCTS] ~ Only computes online, Limited to Discrete Spaces

Today: Four Tools

1. LQR
2. Fitted Value Iteration
3. Sparse UCT/Progressive Widening
4. MPC

Notation: Continuous Random Variables

Term	**Definition**	$\text{Bernoulli}(0.5)$ **Coinflip Example**	**Uniform Example**
support(X) $x \in X$	All the values that X can take	$\{h, t\}$ or $\{0, 1\}$	
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table	
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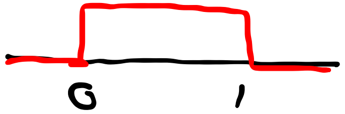
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table (see below)	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$ 
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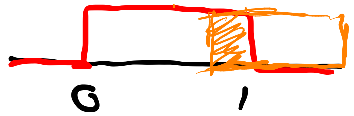

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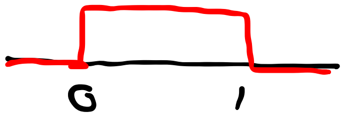
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$  $P(X \in [a,b]) = \int_a^b p(x)dx$ $P(X = 0.5) =$
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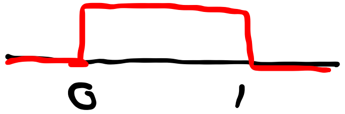
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$  $P(X \in [a, b]) = \int_a^b p(x)dx$ $P(X = 0.5) = 0$
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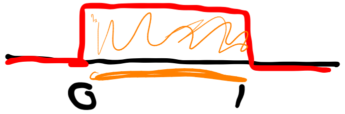
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$  $P(X \in [a,b]) = \int_a^b p(x)dx$ $P(X = 0.5) = 0$
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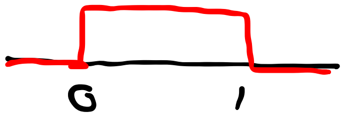
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table <table><tr><th>X</th><th>P(X)</th></tr><tr><td>0</td><td>0.5</td></tr><tr><td>1</td><td>0.5</td></tr></table>	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$  $P(X \in [a,b]) = \int_a^b p(x)dx$ $P(X = 0.5) = 0$
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Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

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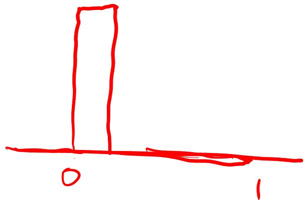
Continuous

1)

Rules for Continuous RVs



$U([0,0.1])$

$p(x) = 10\,\mathbb{1}(0 \le x \le 0.1)$

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$$\begin{bmatrix} \boxed{4} & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Multivariate Gaussian Distribution

$x = [x_1, x_2]$

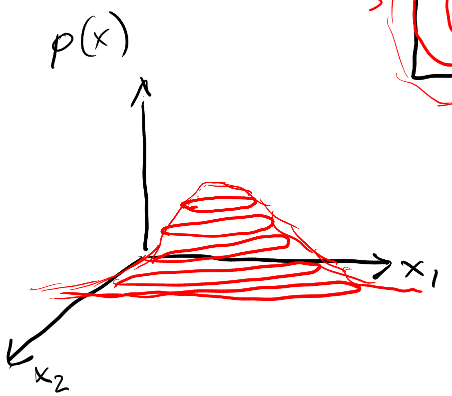
$N(\mu, \Sigma)$

Joint Distribution

$p(x) = N(\mu, \Sigma)$

$$p(x) = \frac{\exp\left(\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)}{(2\pi)^{n/2} |\Sigma|^{1/2}}$$

$\mu = [3, 3]$ $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$



Conditional Distribution

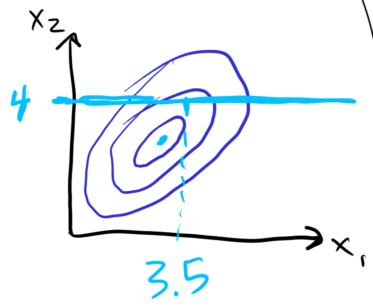
$p(x_1 | x_2) = N(\bar{\mu}_1, \bar{\Sigma}_1)$

$\bar{\mu}_1 = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2)$

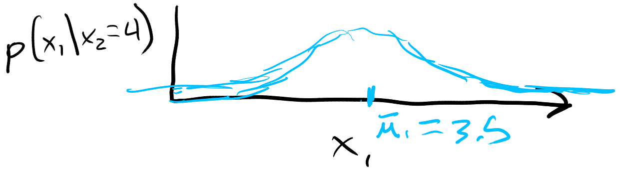
$\bar{\Sigma}_1 = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$

$\mu = [3, 3]$

$\Sigma = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

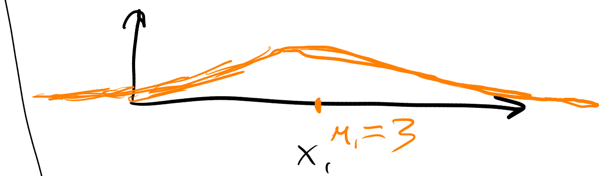


$p(x_1 | x_2 = 4)$



Marginal Distribution

$p(x_1) = N(\mu_1, \Sigma_{11})$



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e.g. $S \subseteq \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$

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The old rules still work!

$$U^*(s) = \max_a \left(R(s,a) + \gamma E\left[U^*(s')\right]\right)$$

$$U^\pi(s) = \ldots$$

$$B[U](s) = \underline{\max_a} \left(R(s,a) + \gamma \underline{\mathop{E}_{s' \sim T(s'|s,a)}\left[U(s')\right]}\right)$$

hard !!!!

$\int_{s' \in S} T(s'|s,a)\, U(s')\, ds'$

Optimization is harder than integration

hard!

Monte Carlo

1. Linear Dynamics, Quadratic Reward

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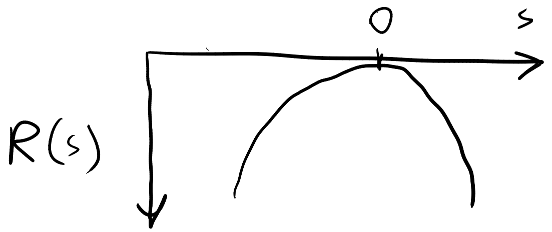
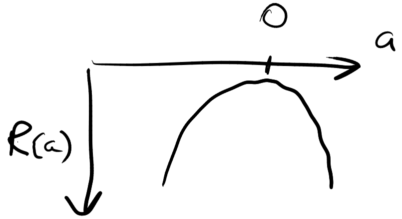
$R(s,a) = R(s) + R(a)$





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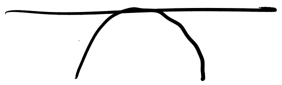
$R(a)$



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$V_1 = R_s$ $\quad q_1 = 0$

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$$\begin{aligned} U_{t+1}^*(s) &= \max_a \left(R(s,a) + \gamma E\left[U_t^*(s)\right]\right) \\ &= \max_a \left(s^\top R_s s + a^\top R_a a + \int p(w) U_t(T_s s + T_a a + w)\, dw\right) \end{aligned}$$

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Inductive step: show that if $U_t^* = s^\top V_t s + q_t$, then $U_{t+1}^* = s^\top V_{t+1} s + q_{t+1}$.

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&= \underline{s^\top R_s s + s^\top T_s^\top V_t T_s s} + \underline{\max_a \left(a^\top R_a a + 2 s^\top T_s^\top V_t T_a a + a^\top T_a^\top V_t T_a a\right)} + \underline{\int p(w) w^\top V_t w\, dw + q_t}
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a^* is where $\nabla_a(\max \text{ term}) = 0$

$$0 = 2R_a a^* + 2T_a^\top V_t T_s s + 2 T_a^\top V_t T_a a^*$$

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$$U_{t+1}^*(s) = s^\top \underbrace{\left(R_s + T_s^\top V_t T_s - (T_a^\top V_t T_s)^\top (R_a + T_a^\top V_t T_a)^{-1} (T_a^\top V_t T_s)\right)}_{V_{t+1}} s + \underbrace{\int p(w) w^\top V_t w \, dw + q_t}_{q_{t+1}}$$

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1. Linear Dynamics, Quadratic Reward

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As $h \to \infty$

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(K_∞ has no dependence on Σ)

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$$\pi^*_\infty(s) = -K_\infty s$$

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<u>Certainty-Equivalence Principle</u>: For Linear-Quadratic problems, the optimal policy with noise is the same as the optimal policy without noise!

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<u>Certainty-Equivalence Principle</u>: For Linear-Quadratic problems, the optimal policy with noise is the same as the optimal policy without noise!

Practical Implication: If a continuous problem has roughly linear dynamics, a convex cost function, and roughly zero-mean additive noise, you can use *certainty-equivalent control*, i.e. control as if there is no noise.

2. Value Function Approximation

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$V_\theta(s) = f_\theta(s)$ (e.g. neural network)

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Fitted Value Iteration

while not converged

$\theta \leftarrow \theta'$

$\hat{V}' \leftarrow B_{\text{approx}}[V_\theta]$

$\theta' \leftarrow \text{fit}(\hat{V}')$

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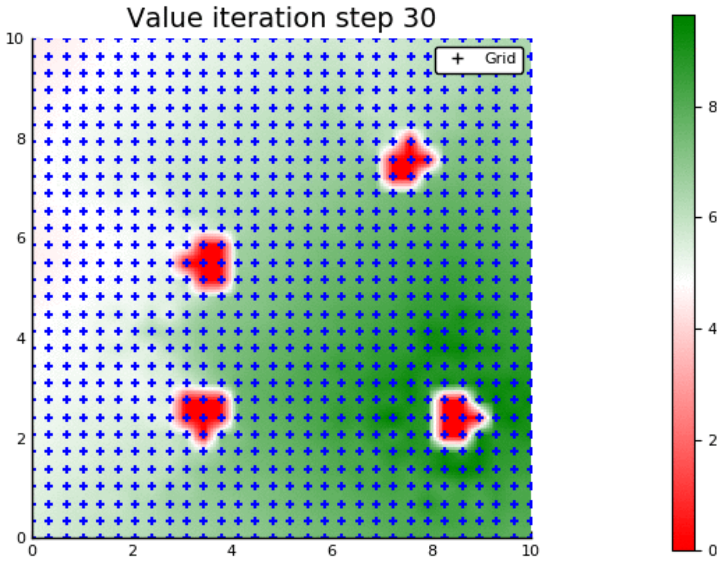

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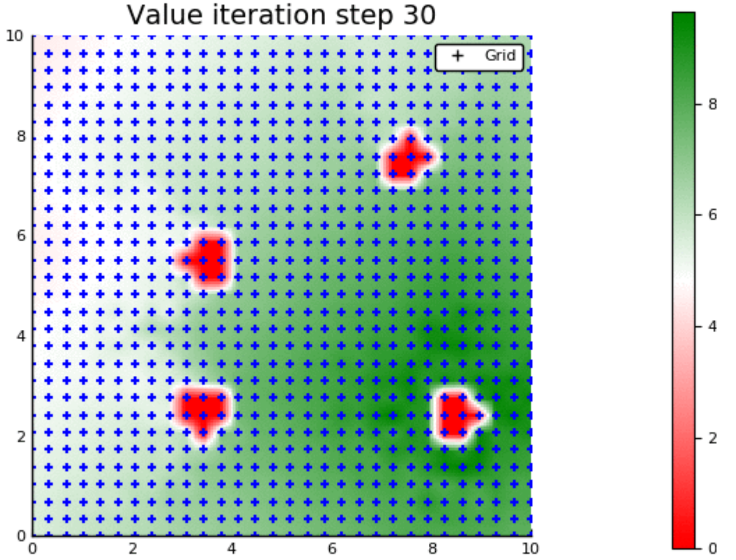

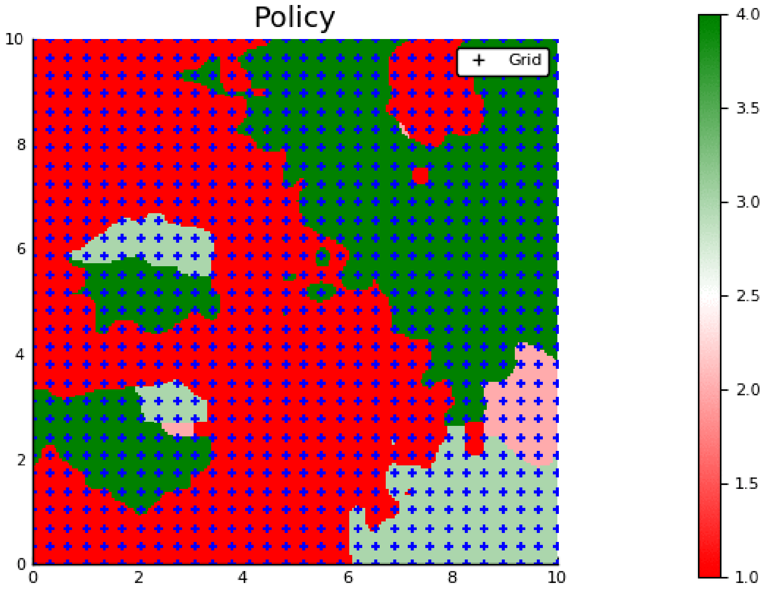

$$B_{\text{MC}(N)}[V_\theta](s) = \max_a \left(R(s,a) + \gamma \frac{1}{N} \sum_{i=1}^{N} V_\theta(G(s,a,w_i)) \right)$$

Function Approximation

Weighting of 2^d points

Weighting of only $d + 1$ points!

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- Global: (e.g. Fourier, neural network)
- Local: (e.g. simplex interpolation)

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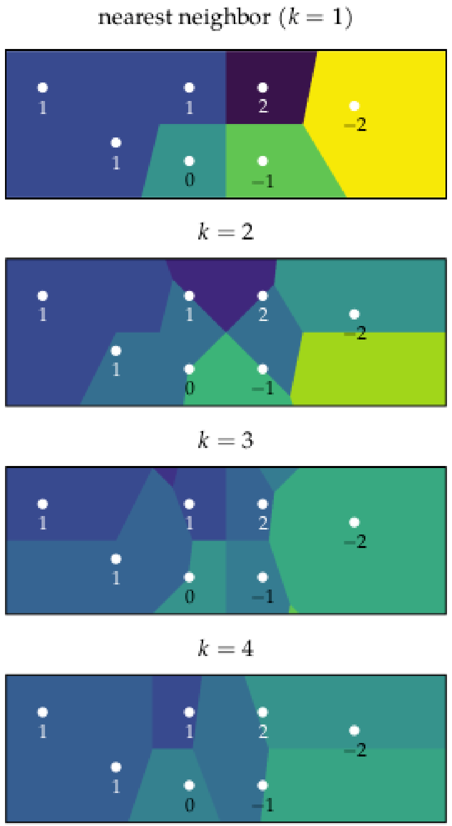
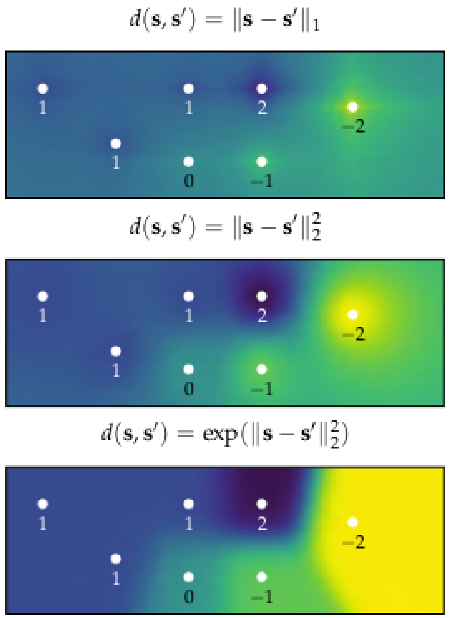




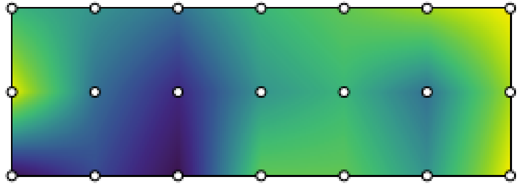


Figure 8.9. Two-dimensional linear interpolation over a 3×7 grid.

Weighting of 2^d points

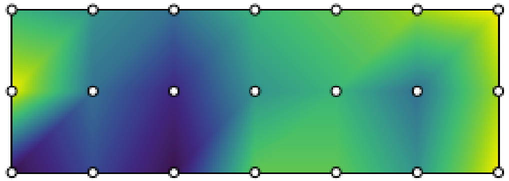
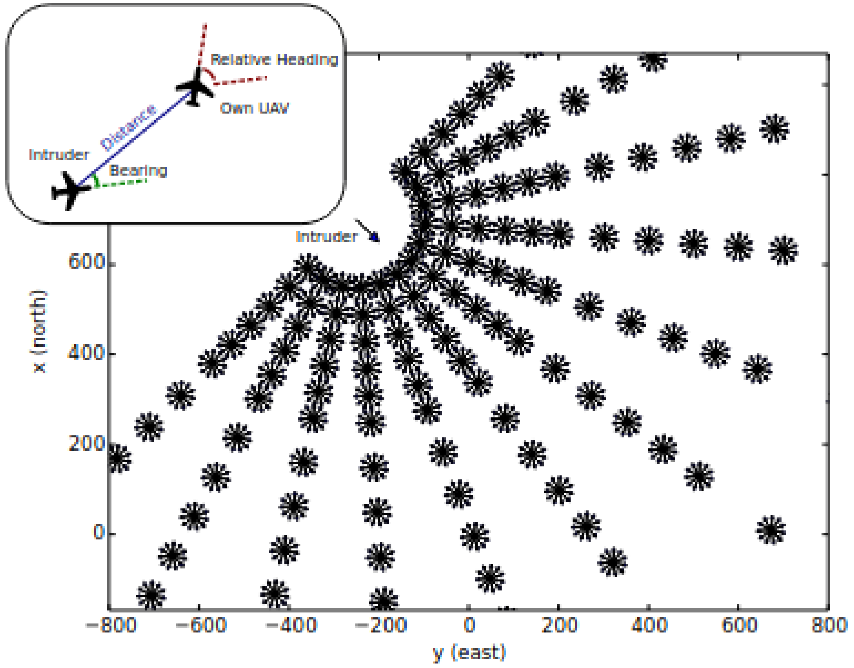
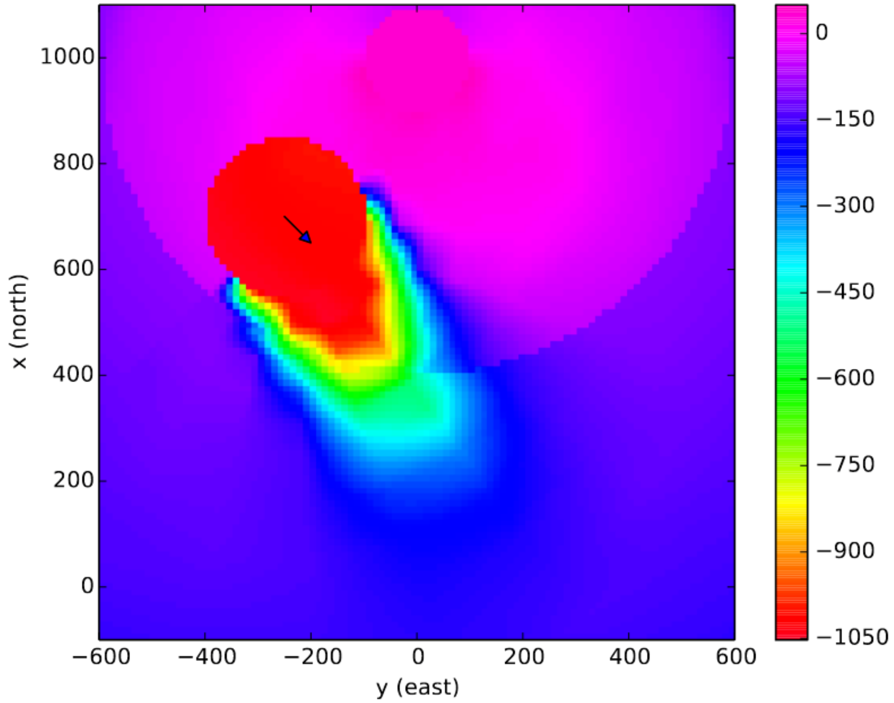


Figure 8.10. Two-dimensional simplex interpolation over a 3×7 grid.

Weighting of only $d + 1$ points!

2. Value Function Approximation





Break

What will a Monte Carlo Tree Search tree look like if run on a problem with continuous spaces?

3. Sparse Tree Search/Progressive Widening

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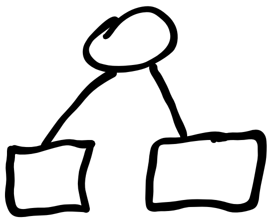

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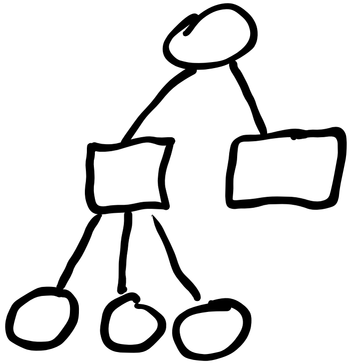
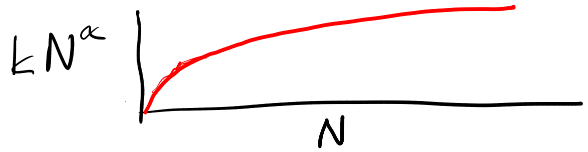
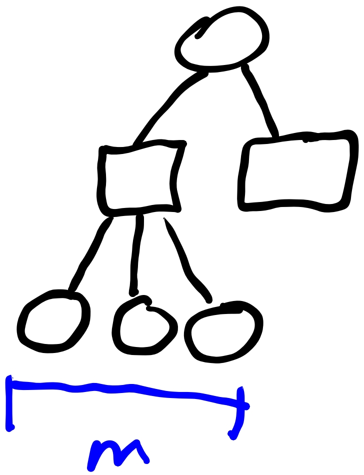




number of children

3. Sparse Tree Search/Progressive Widening



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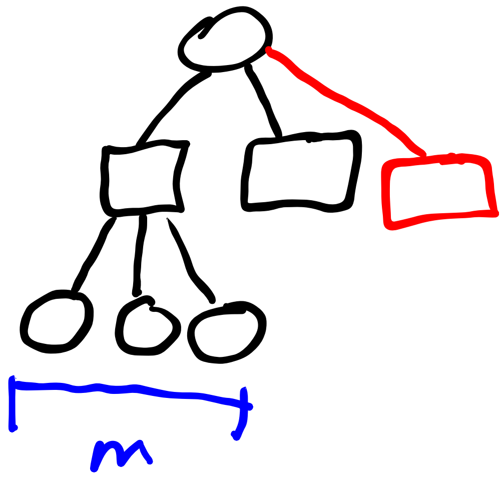

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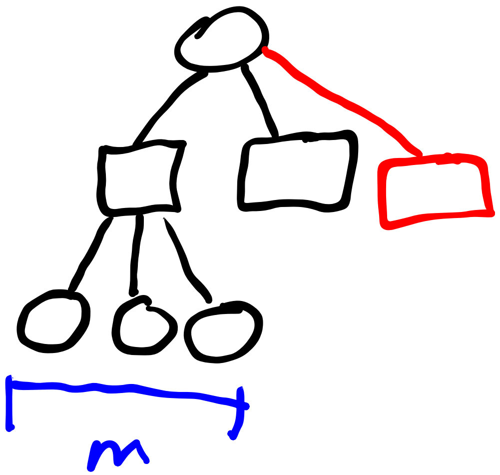

add new branch if $C < kN^{\alpha} \quad (\alpha < 1)$

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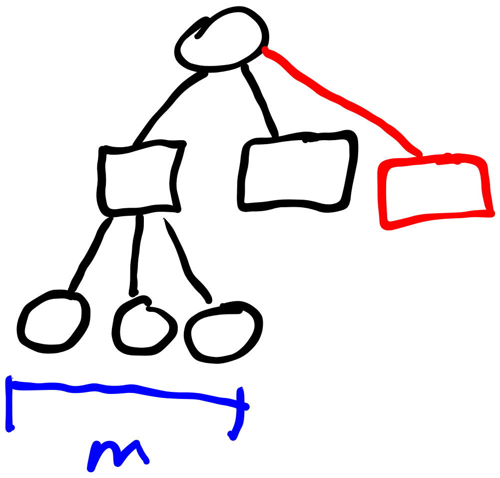

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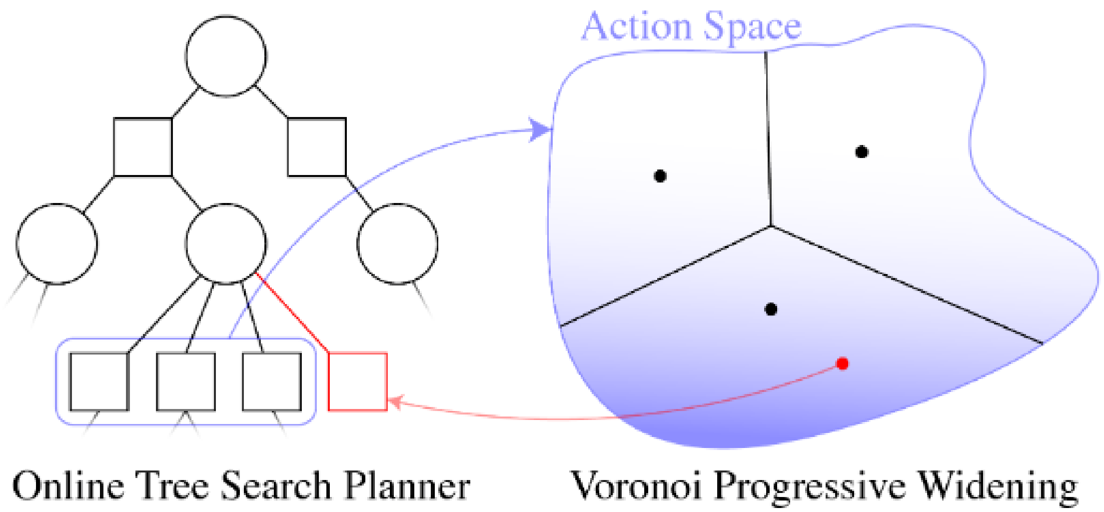

Not on exam

4. Model Predictive Control

(Use off-the-shelf optimization software, e.g. Ipopt)

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Certainty-Equivalent

$$\begin{aligned} \underset{a_{1:d}, s_{1:d}}{\text{maximize}} \quad & \sum_{t=1}^{d} \gamma^t R(s_t, a_t) \\ \text{subject to} \quad & s_{t+1} = \overset{\text{mode}}{\mathrm{E}}_{s' \sim T(s' \mid s, a)}[s'] \quad \forall t \end{aligned}$$

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Open-Loop

$$\begin{aligned} \underset{a_{1:d}, s_{1:d}^{(1:m)}}{\text{maximize}} \quad & \frac{1}{m} \sum_{i=1}^{m} \sum_{t=1}^{d} \gamma^t R(s_t^{(i)}, a_t) \\ \text{subject to} \quad & s_{t+1} = G(s_t^{(i)}, a_t, w_t^{(i)}) \quad \forall t, i \end{aligned}$$

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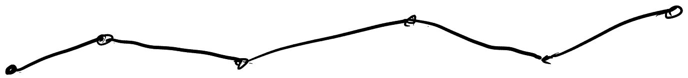

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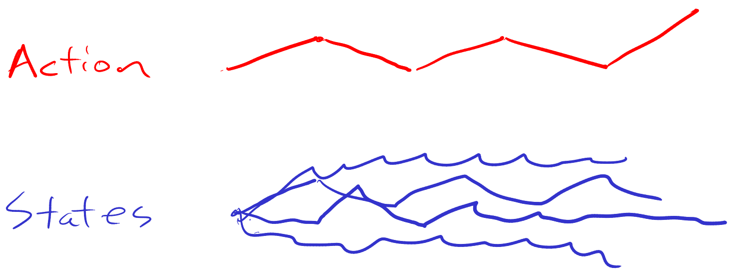

Hindsight Optimization

$$\begin{aligned}\underset{a_{1:d}^{(1:m)}, s_{1:d}^{(1:m)}}{\text{maximize}} \quad & \frac{1}{m}\sum_{i=1}^{m}\sum_{t=1}^{d} \gamma^t R(s_t^{(i)}, a_t^{(i)}) \\ \text{subject to} \quad & s_{t+1} = G(s_t^{(i)}, a_t^{(i)}, w_t^{(i)}) \quad \forall t, i \\ & a_1^{(i)} = a_1^{(j)} \quad \forall i, j\end{aligned}$$