

# Stochastic Processes and Simple Decisions

# Review

Causal

Bayesian Net

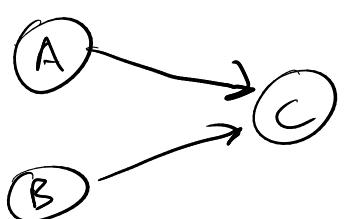
1. Structure: D.A.G.

○ node: R.V.

→ edge: Causal Relationship

2. Parameters

Define  $P(X | P_a(X))$

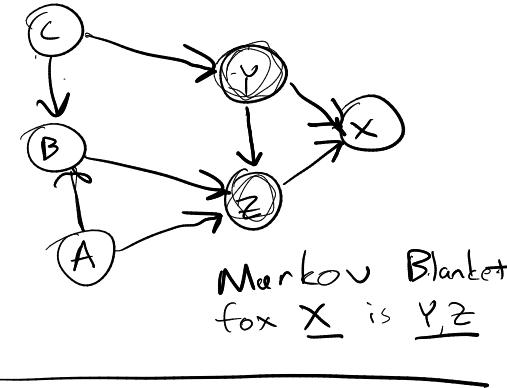


$$P(C | A, B)$$

$$P(A, B, C) = P(A) P(B) P(C | A, B)$$

Chain Rule

$$P(X_{1:n}) = \prod_i P(X_i | P_a(X_i))$$



$$X \perp\!\!\!\perp Y | Z$$

$$P(X, Y | Z) = P(X | Z) P(Y | Z)$$

$$P(X | Y, Z) = P(X | Z)$$

D-separation  
allows you to prove  
 $X \perp\!\!\!\perp Y | Z$  based only  
on the structure of a B.N.

# Guiding Question

- What does "Markov" mean in "Markov Decision Process"?
- How do we find an optimal action based on maximizing expected utility?

# Stochastic Process

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**Example: Positive, Uniform Random Walk**

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$$P(X_{t+1} \mid X_{0:t}) = \underbrace{\begin{cases} 0.5 & \text{if } X_{t+1} = X_t \\ 0.5 & \text{if } X_{t+1} = X_t + 1 \\ 0 & \text{otherwise} \end{cases}}_{\cdot} = P(X_{t+1} \mid X_t)$$

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P.U.R.W

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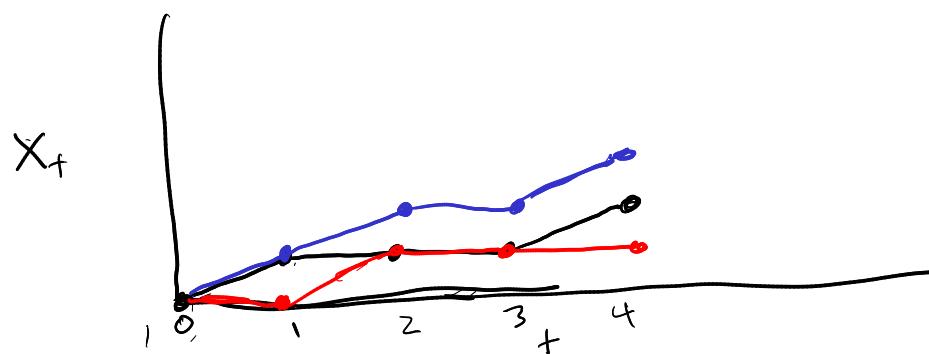
$$P(X_{t+1} | X_{0:t}) = \begin{cases} 0.5 & \text{if } X_{t+1} = X_t \\ 0.5 & \text{if } \underline{X_{t+1}} = \underline{X_t + 1} \\ 0 & \text{otherwise} \end{cases}$$

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Bayes Net



Trajectories



$P(A|B)$

$\therefore B=1$

A	$P(A B=1)$
0	0.1
1	0.9

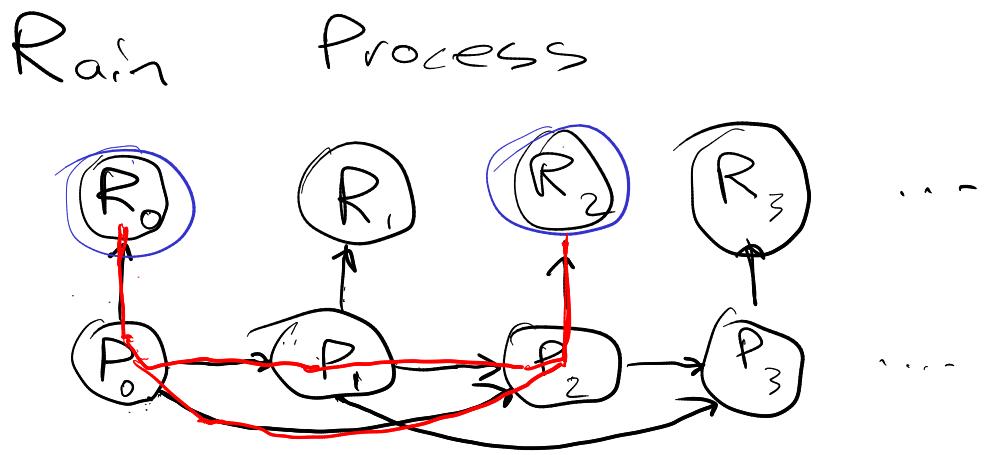
$B=0$

$$P(A|B) = \begin{cases} 0.1 & \text{if } A=0, B=1 \\ 0.9 & \text{if } A=1, B=0 \end{cases}$$

# Simulating a Stochastic Process

030-Stochastic-Processes.ipynb

# A More Complex Example



$$\cancel{P(R_{t+1} | R_t)}$$

Low-pressure periods last for  
two days

Want for Markov Process:  $R_{t+1} \perp R_{t-\epsilon} | R_t ?$

$R_2 \perp R_0 | R_1 ?$  No

# Markov Process

# Markov Process

- A stochastic process  $\{S_t\}$  is *Markov* if

$$P(S_{t+1} \mid S_{0:t}) = P(S_{t+1} \mid S_t)$$
$$S_{t+1} \perp S_{t-\tau} \mid S_t \quad \forall \tau \in 1 : t$$

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$$S_{t+1} \perp S_{t-\tau} | S_t \quad \forall \tau \in 1 : t$$

- $S_t$  is called the "state" of the process

Is the P.U.R.W.  $\{X_t\}$  a Markov Process?

Yes, because  $P(X_{t+1} | X_{0:t}) = P(X_{t+1} | X_t)$

Is the Rain Process  $\{R_t\}$  a Markov Process?

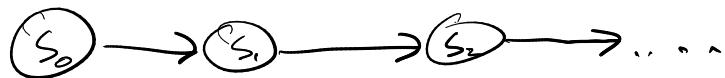
Inconclusive based on structure

No based on information that low pressure lasts for two days.

# Dynamic Bayesian Networks

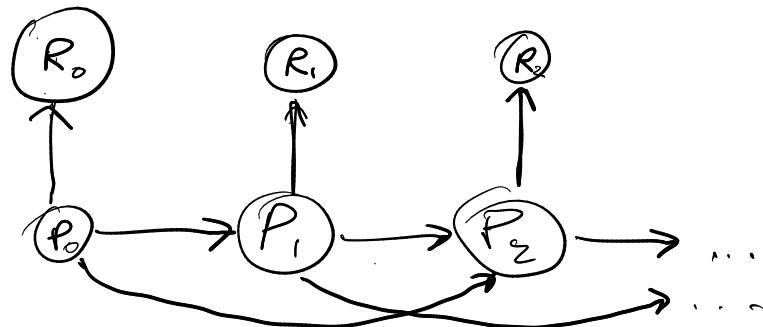
For a Markov Process  $\{S_t\}$

Bayes Net



For the Rain Proce

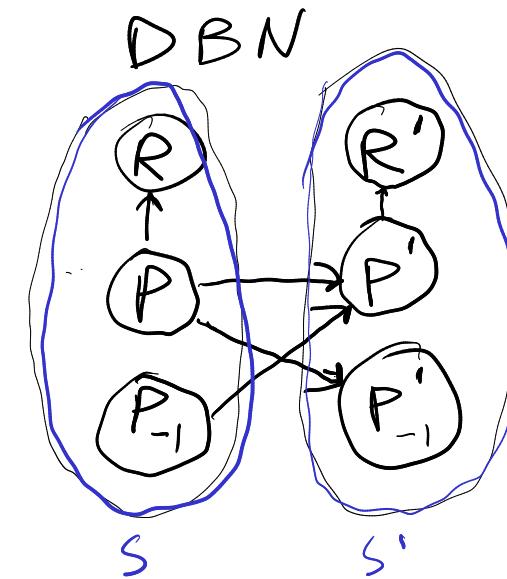
Bayes Net



Dynamic Bayes Net



'prime' means next



$$S = (R, P, P_{-1})$$

$$\{S_t\}$$

is Markov

# **Break**

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Suppose you want to create a Markov process model that describes how many new COVID cases will start in a particular week. **What information should be in the state of the model?**

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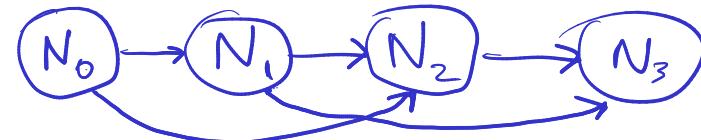
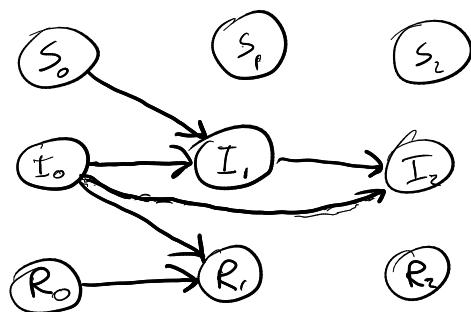
- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 2 weeks after they contract the disease.

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Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 2 weeks after they contract the disease.
- Researchers have determined a probabilistic model for the number of new cases given the number of people in the first week of the disease and the number of people in the second week of the disease.



Answer: number infected in current week  
and previous week

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$S_1 \dots S_n$

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$$[S_1 : p_1; \dots; S_n : p_n]$$

~~Prob~~

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von Neumann - Morgenstern Axioms

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- $U(A) > U(B)$  iff  $A \succ B$

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- $U([S_1 : p_1; \dots; S_n : p_n]) = \sum_{i=1}^n p_i U(S_i)$

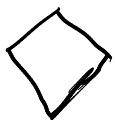
# Decision Networks



Decision



Chance (just like in BN)



Utility



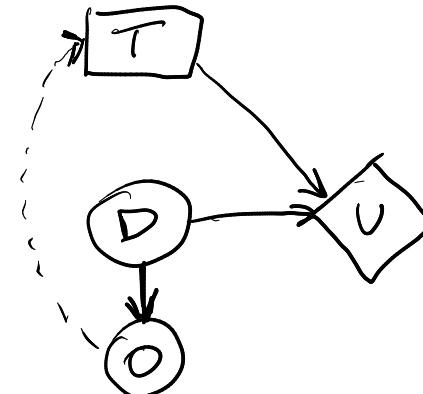
conditional



informational

what info a decision  
can be made based  
on

Example

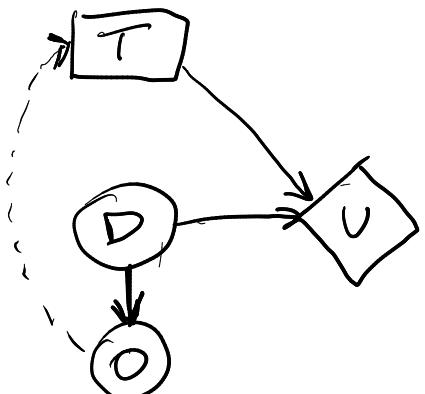


# Maximizing Expected Utility

$$EU(a|o) \equiv \sum_{s'} P(s'|a,o) U(s', a)$$

$$a^* = \underset{a}{\operatorname{argmax}} \, EU(a|o)$$

Example



Given:

		D	T	U
		O	O	0
		1	0	-10
		0	1	-1
		1	1	-1

$$\begin{aligned}P(D=1) &= 0.2 \\P(O=1 | D=0) &= 0.01 \\P(O=1 | D=1) &= 0.9\end{aligned}$$

$$\begin{aligned}\rightarrow s' &= D \\P(D|T,O)^* &= P(D|O) \\&= \frac{P(O|D)P(D)}{P(O)}\end{aligned}$$

Treat ( $T=1$ )

$$\begin{aligned}EU(T=1|o) &= \sum_d P(d|1,o) U(d,1) \\&= -1\end{aligned}$$

$$\begin{aligned}EU(O=0|o=0) &= \sum_d P(d|0,o) U(d,0) \\&= \frac{1}{P(O=0)} \left( P(O=0|D=0) P(D=0) U(0,0) + P(O=0|D=1) P(D=1) U(1,0) \right) \\&= 0.99 \cdot 0.8 + 0.1 \cdot 0.2 \\&= -0.24\end{aligned}$$

$$EU(T=0|o=1) = \text{Bad}$$

# Value of Information

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