

Probability and Random Variables

Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables


Utility and Probability

Full story: <https://projecteuclid.org/journals/statistical-science/volume-1/issue-3/The-Axioms-of-Subjective-Probability/10.1214/ss/1177013611.full>

What is a Random Variable?

R.V. X

Vocabulary/Notation

Term	Definition	Bernoulli(0.5) Coinflip Example	$\mathcal{U}(0, 1)$ Uniform Example						
support(X) $x \in X$ $X \in [0, 1]$	All the values that X can take	$\{h, t\}$ or $\{0, 1\}$	$[0, 1]$						
Distribution <ul style="list-style-type: none"> Discrete: PMF Continuous: PDF 	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table <div> <table> <tr> <th>x</th> <th>P(x)</th> </tr> <tr> <td>0</td> <td>0.5</td> </tr> <tr> <td>1</td> <td>0.5</td> </tr> </table> </div>	x	P(x)	0	0.5	1	0.5	 $p(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{o.w.} \end{cases}$ $P(X = 1) = 0$ $P(X \in [a, b]) = \int_a^b p(x) dx$
x	P(x)								
0	0.5								
1	0.5								
Expectation $E[X]$	Single representative value of the random variable, "mean"	$E[X] = \sum_{x \in X} x P(x)$ $= 0.5$	$E[X] = \int_{x \in X} x p(x) dx$ $= 0.5$						

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X, Y, Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Conditional Distribution

$$P(X | Y, Z)$$

(Distribution - valued function)

X	$P(X Y=1, Z=1)$
0	0.84
1	0.16

Marginal Distribution

$$P(X) \quad P(Y) \quad P(Z)$$

X	$P(X)$	Y	$P(Y)$
0	0.85	0	0.45
1	0.15	1	0.55

Z	$P(Z)$
0	0.20
1	0.80

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

$$P(X \mid Y, Z)$$

Marginal Distribution

$$P(X) \ P(Y) \ P(Z)$$

3 Rules (Burrito-level)

- 1) a) $0 \leq P(X \mid Y) \leq 1$
b) $\sum_{x \in X} P(x \mid Y) = 1$

- 2) "Law of total probability"

$$P(X) = \sum_{y \in Y} P(X, y)$$

- 3) Definition of Conditional Probability

$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)}$$

Joint \rightarrow Marginal

Joint + Marginal \rightarrow Conditional

Marginal + Conditional \rightarrow Joint

$$P(X, Y) = P(X \mid Y) P(Y)$$

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

Conditional Distribution

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Marginal Distribution

$$P(X) \ P(Y) \ P(Z)$$

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Joint + Marginal \rightarrow Conditional

Marginal + Conditional \rightarrow Joint

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Break

1) a) $0 \leq P(X | Y) \leq 1$

b) $\sum_{x \in X} P(x | Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

3) $P(X | Y) = \frac{P(X, Y)}{P(Y)}$

$$P(X, Y) = P(X|Y) P(Y)$$

- $P \in \{0, 1\}$: Powder Day
 - $C \in \{0, 1\}$: Pass Clear
 - 1 in 5 days is a powder day
 - The pass is clear 8 in 10 days
 - If it is a powder day, there is a 50% chance the pass is blocked
-
- Write out the joint probability distribution for P and C.
 - What is the probability that the pass is blocked on a non-powder day?

Bayes Rule

- Know: $P(B \mid A), P(A), P(B)$
- Want: $P(A \mid B)$

Independence

Definition: X and Y are *independent* iff $P(X, Y) = P(X) P(Y)$

$$X \perp Y$$

$$P(X|Y) = P(X)$$

Definition: X and Y are *conditionally independent* given Z iff

$$P(X, Y | Z) = P(X | Z) P(Y | Z)$$

$$X \perp Y | Z$$

Rules for Continuous RVs

Discrete

1) a) $0 \leq P(X | Y) \leq 1$

b) $\sum_{x \in X} P(x | Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

3) $P(X | Y) = \frac{P(X, Y)}{P(Y)}$

$$P(X, Y) = P(X | Y) P(Y)$$

Continuous

1) $0 \leq p(X | Y)$

$$\int_X p(x|Y) dx = 1$$

2)

$$p(X) = \int_Y p(X, y) dy$$

3) $p(X | Y) = \frac{p(X, Y)}{p(Y)}$

$$p(X, Y) = p(X | Y) p(Y)$$

Multivariate Gaussian Distribution

Joint Distribution

Conditional Distribution

Marginal Distribution

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