Games

Multi-agent interaction

Up to this point:

This week:

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Up to this point:

- Stochastic Uncertainty

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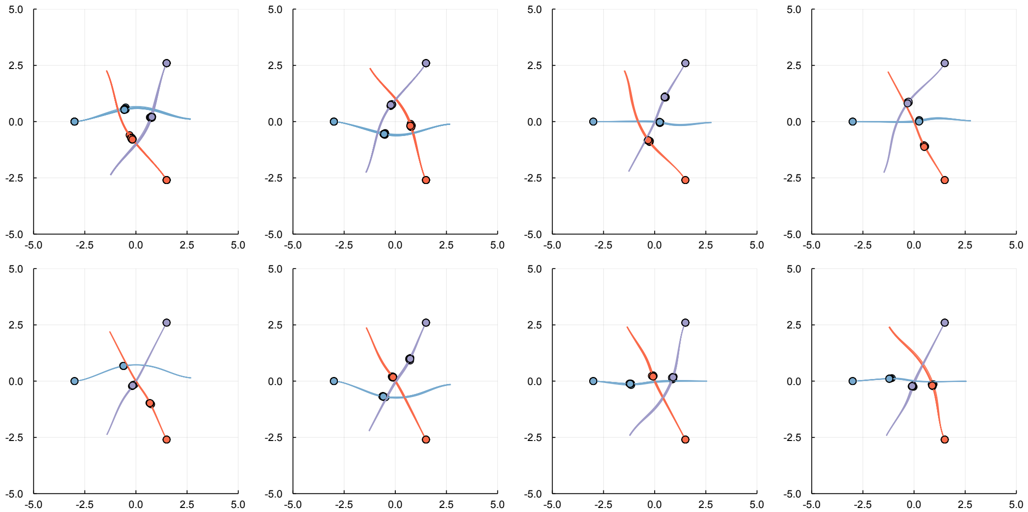
This week:

- Interaction Uncertainty
- Game Solution Concepts



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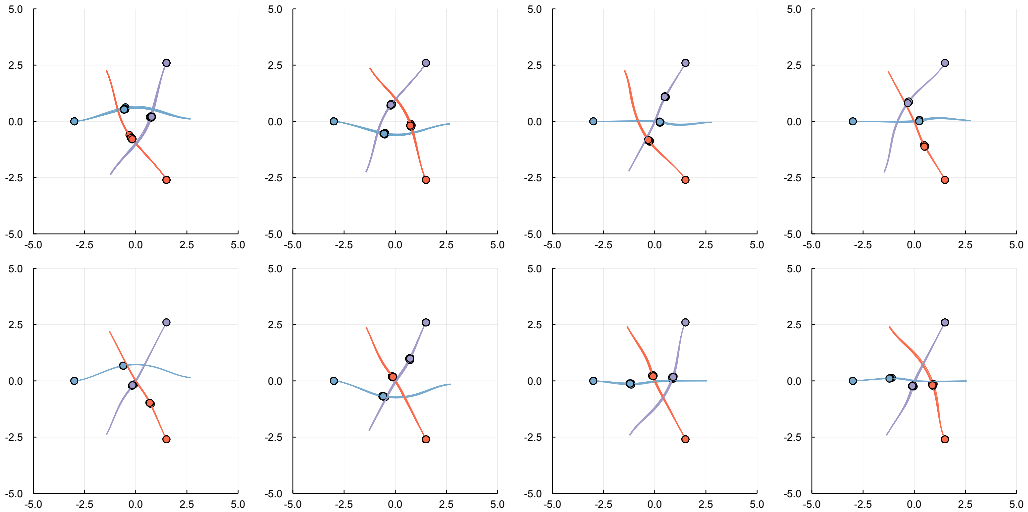

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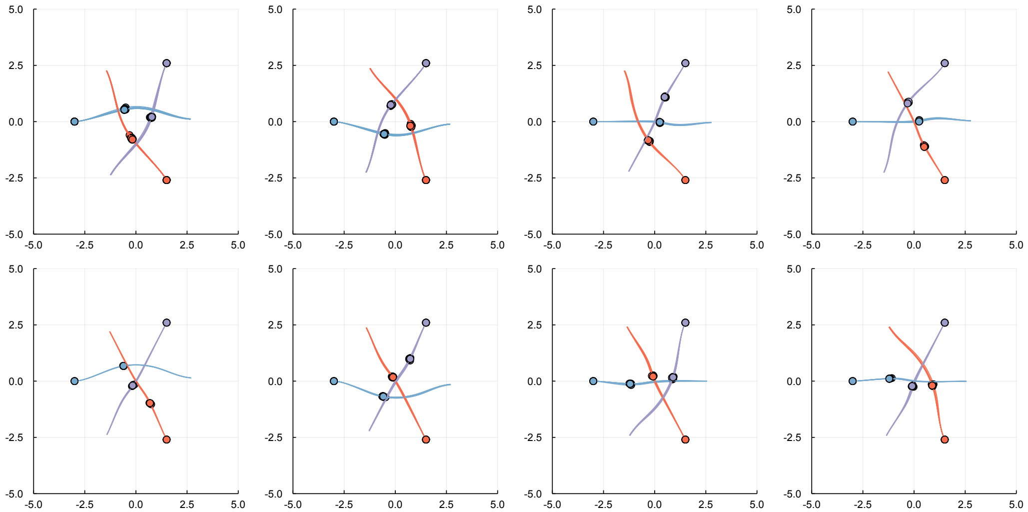

Kriegspiel

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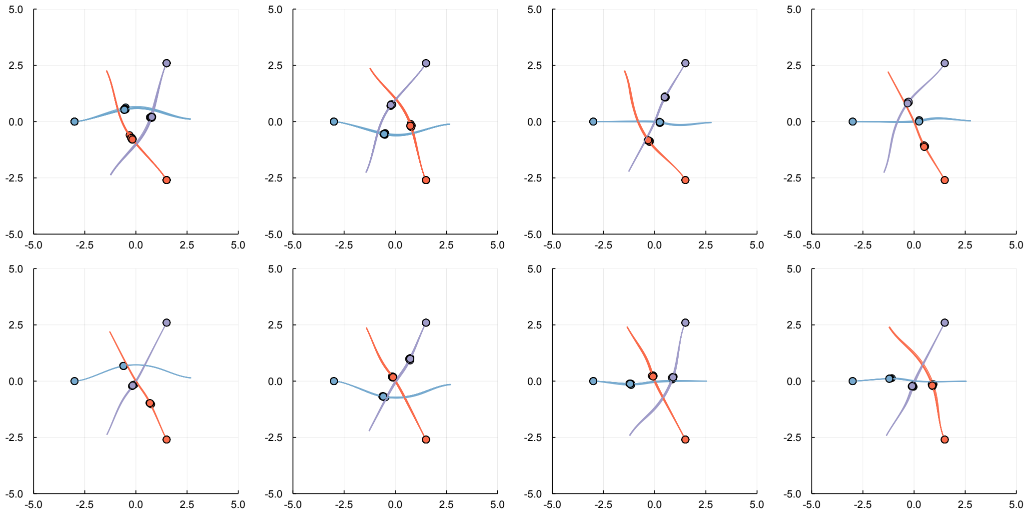

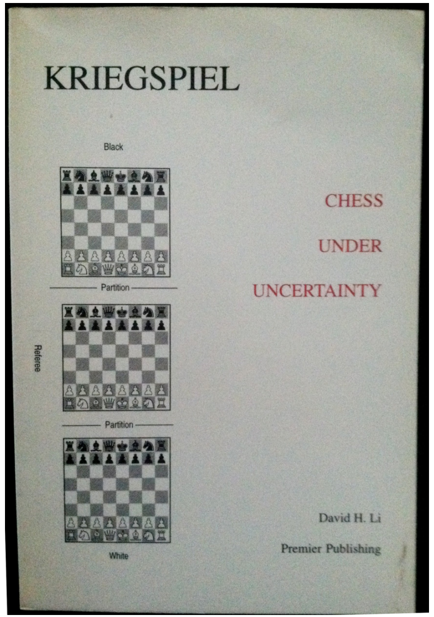

Kriegspiel

Outline

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1. Matrix Games
    1. Linear Programming
    2. Regret Matching

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    1. Minimax Trees
    2. α-β Pruning
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Sources:

- *AI: A Modern Approach* by Russel and Norvig
- "An Introduction to Counterfactual Regret Minimization in Games" by Neller and Lanctot

Game Solution Concepts: Nash and Correlated Equilibria

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Rock Paper Scissors

Game Solution Concepts: Nash and Correlated Equilibria

Rock Paper Scissors

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
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Battle of the Sexes

Gary

Monica	M	G
M	$2,1$	$0,0$
G	$0,0$	$1,2$

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- Correlated Equilibrium: Strategy profile with a shared random source

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$$\sigma_{\text{Gary}}(M) = x$$

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If $x = 1/3$, these are equal, so Monica is indifferent. By symmetry, $\sigma = \{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}\}$ is a Nash Equilibrium.

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$$\begin{aligned} \underset{\boldsymbol{\pi}, U}{\text{minimize}} \quad & \sum_i \left(U^i - U^i(\boldsymbol{\pi}) \right) \\ \text{subject to} \quad & U^i \geq U^i(a^i, \boldsymbol{\pi}^{-i}) \text{ for all } i, a^i \\ & \sum_{a^i} \pi^i(a^i) = 1 \text{ for all } i \\ & \pi^i(a^i) \geq 0 \text{ for all } i, a^i \end{aligned}$$

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Regret Matching

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For action profile $\mathbf{a}$, player i's *regret* for not playing action a' is:

$$R(a_i') = \ U(a_i', a_{-i}) - U(\mathbf{a})$$

that is, the utility missed out on by not playing a'.

Positive regret means that action would have been better.

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Turn-taking Games

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Terminology

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- Max and Min players

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Turn-taking Games

Terminology

- Max and Min players
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- zero-sum
- perfect information

Minimax Trees

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MDP Expectimax Tree

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$$V(s) = \max_{a \in \mathcal{A}} \left(R(s,a) + \mathrm{E}[V(s')] \right)$$

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$$V(s) = \max_{a\in\mathcal{A}} \left(R(s,a) + \mathrm{E}[V(s')]\right)$$

$$V_1(s) = \max_{a\in\mathcal{A}_1} \left(R(s,a) + \min_{a'\in A_2}(R(s',a') + V(s''))\right)$$

Tree Backup Example

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```
function MINIMAX-DECISION(state) returns an action
  return arg max_{a ∈ ACTIONS(s)} MIN-VALUE(RESULT(state, a))
```

```
function MAX-VALUE(state) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← −∞
  for each a in ACTIONS(state) do
    v ← MAX(v, MIN-VALUE(RESULT(s, a)))
  return v
```

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  return v

3 12 8 2 4 6 14 5 2

Why is this harder than an MDP? (think back to sparse sampling)

Alpha-Beta Pruning

```
function ALPHA-BETA-SEARCH(state) returns an action
   v ← MAX-VALUE(state, −∞, +∞)
   return the action in ACTIONS(state) with value v
```

```
function MAX-VALUE(state, α, β) returns a utility value
   if TERMINAL-TEST(state) then return UTILITY(state)
   v ← −∞
   for each a in ACTIONS(state) do
      v ← MAX(v, MIN-VALUE(RESULT(s,a), α, β))
      if v ≥ β then return v
      α ← MAX(α, v)
   return v
```

```
function MIN-VALUE(state, α, β) returns a utility value
   if TERMINAL-TEST(state) then return UTILITY(state)
   v ← +∞
   for each a in ACTIONS(state) do
      v ← MIN(v, MAX-VALUE(RESULT(s,a) , α, β))
      if v ≤ α then return v
      β ← MIN(β, v)
   return v
```

Evaluation Functions

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Examples of evaluation functions:

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- Rollout using default policy

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- Neural Network

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- Rollout using default policy
- Neural Network
- Weighted combination of features (e.g. number of Pawns, Knights, etc.)

Deep Blue

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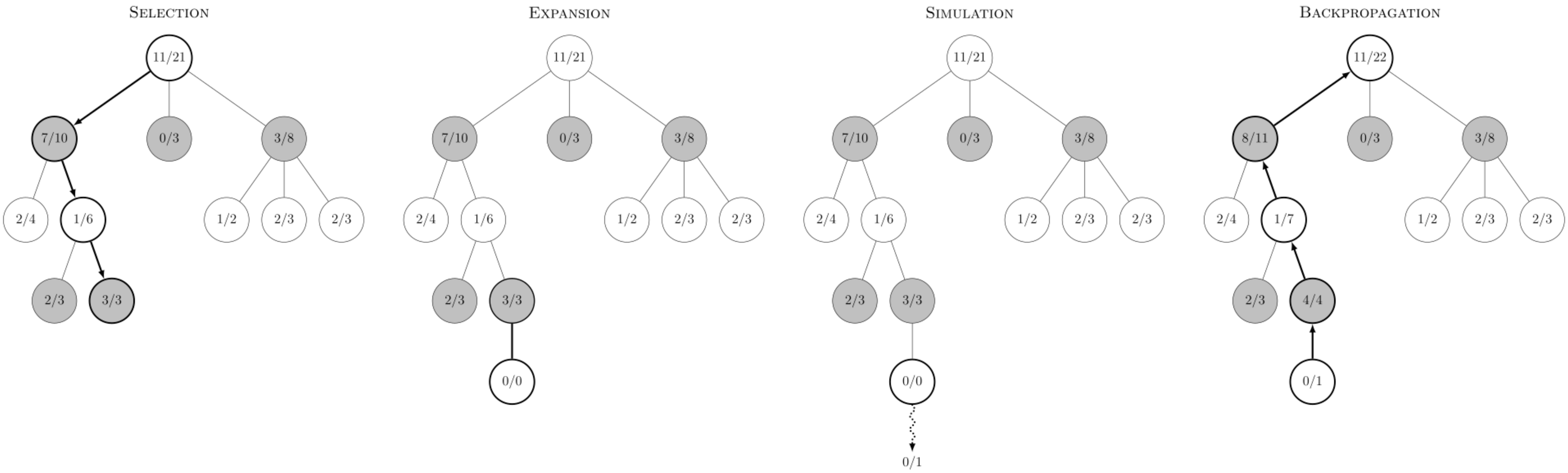
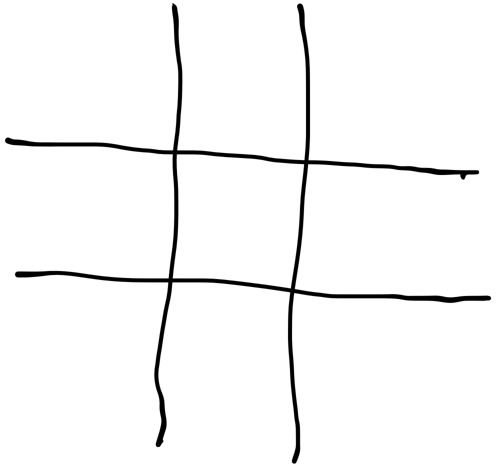
- Database of 700,000 grandmaster games

MCTS for Games



Break Exercise



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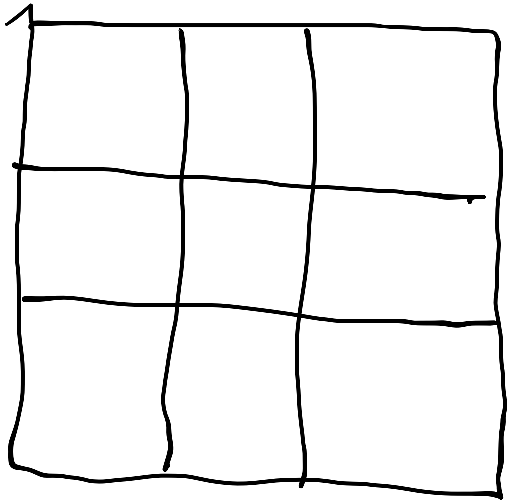

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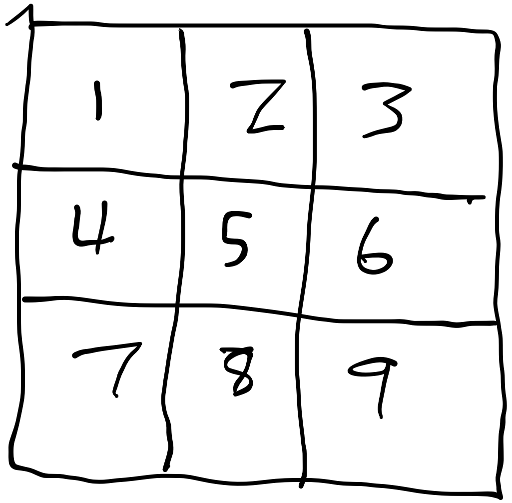

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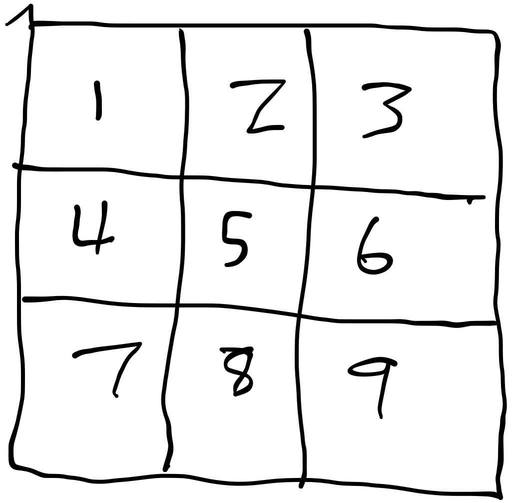
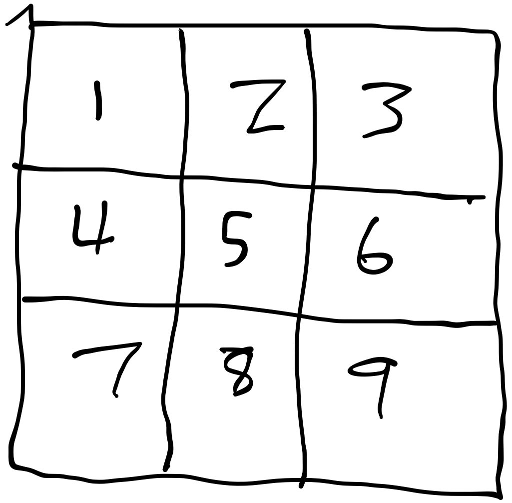


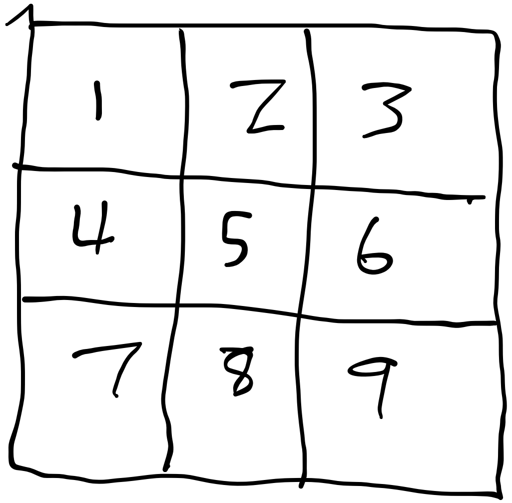
- Start at 5

Break Exercise



- Start at 5
- Max can move up/down

Break Exercise



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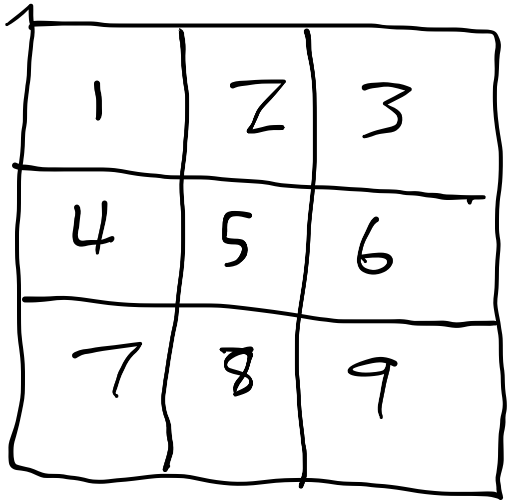

- Start at 5
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- Each player gets 2 turns

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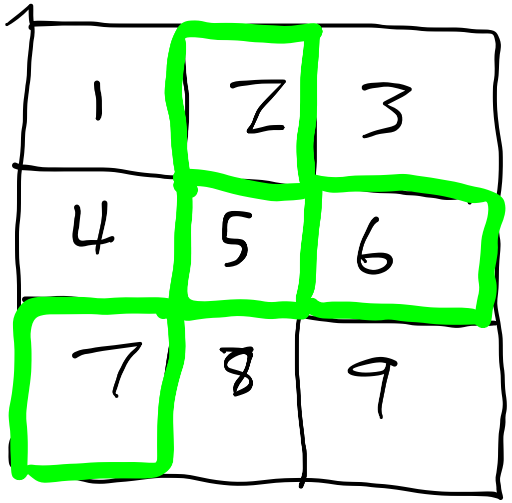

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- On green squares, Max wins

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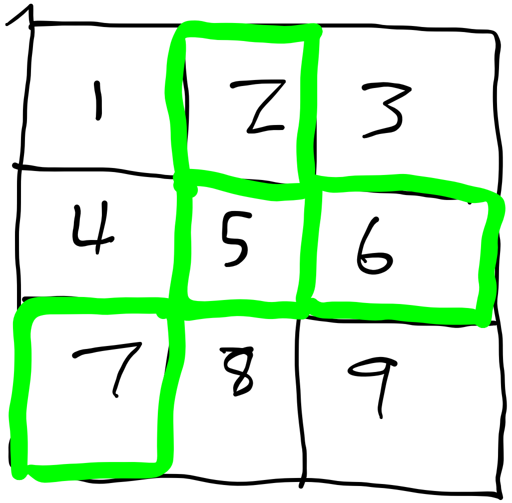

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- Who has advantage and what is first move?

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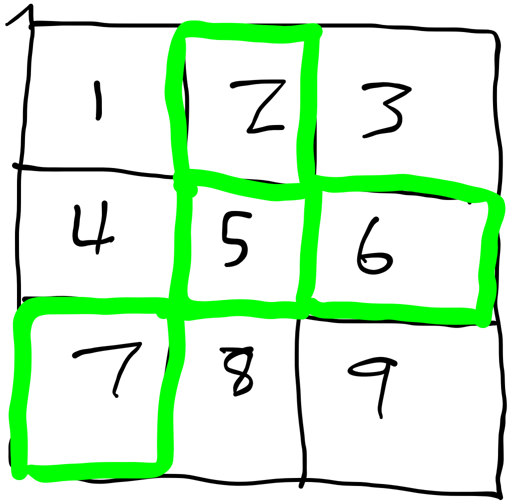

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Bonus: what if 5 was not green?

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- $I(h)$ (information set that each history maps to)
- $U(h)$ (payoff for each leaf node in the game tree)

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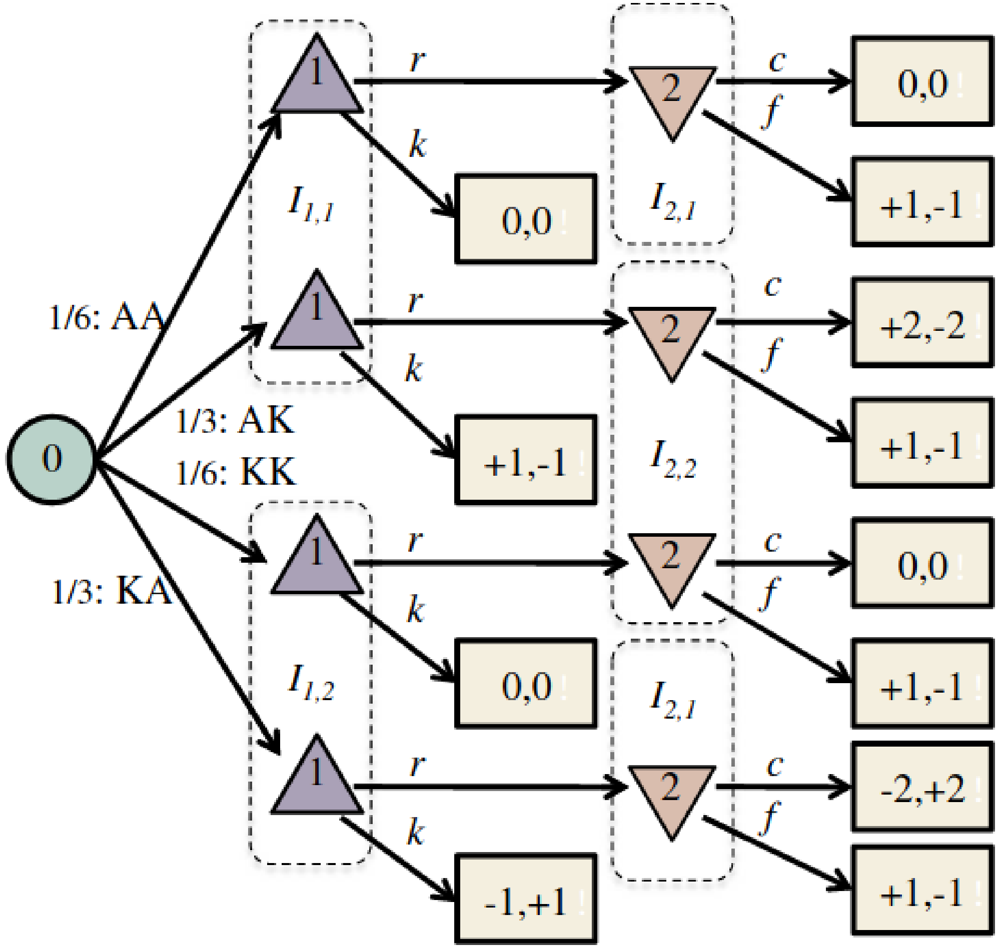
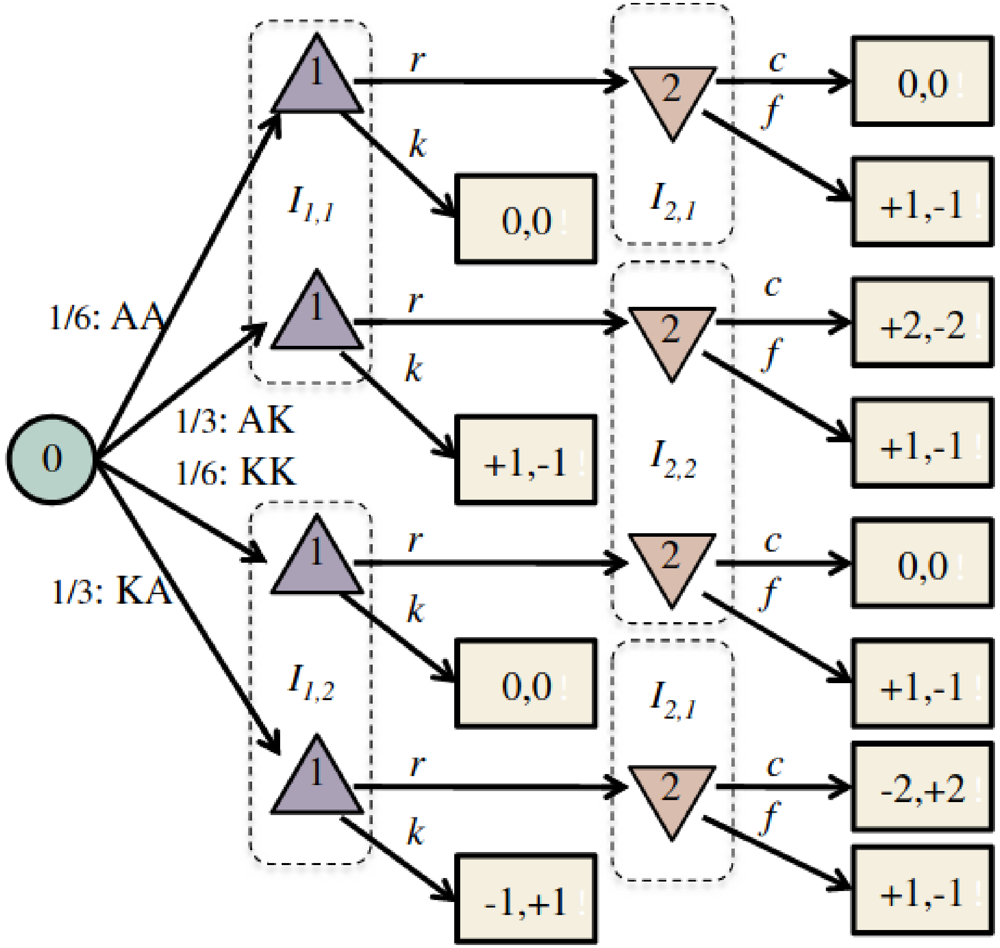
- If P1 raises, P2 can either *call* (c) Player 1's bet, or *fold* (f) the payoff back to 1 point
- The highest card wins



Extensive to Matrix Form



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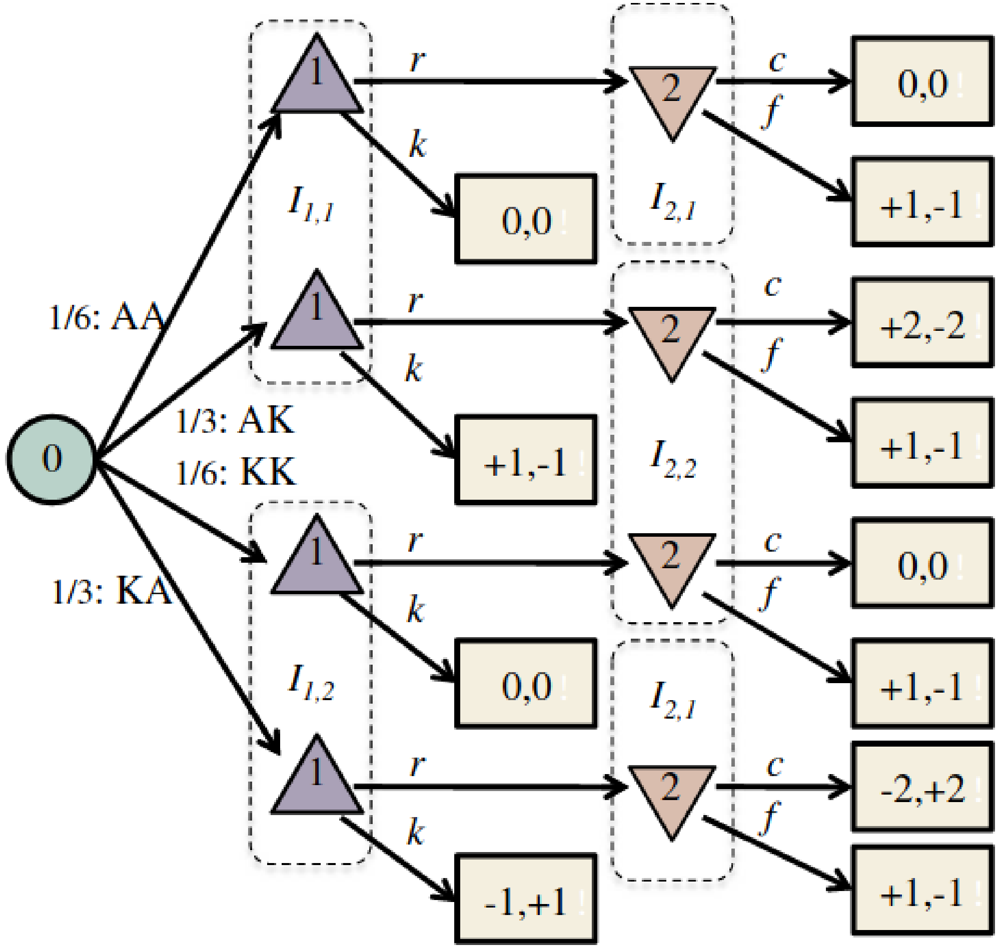
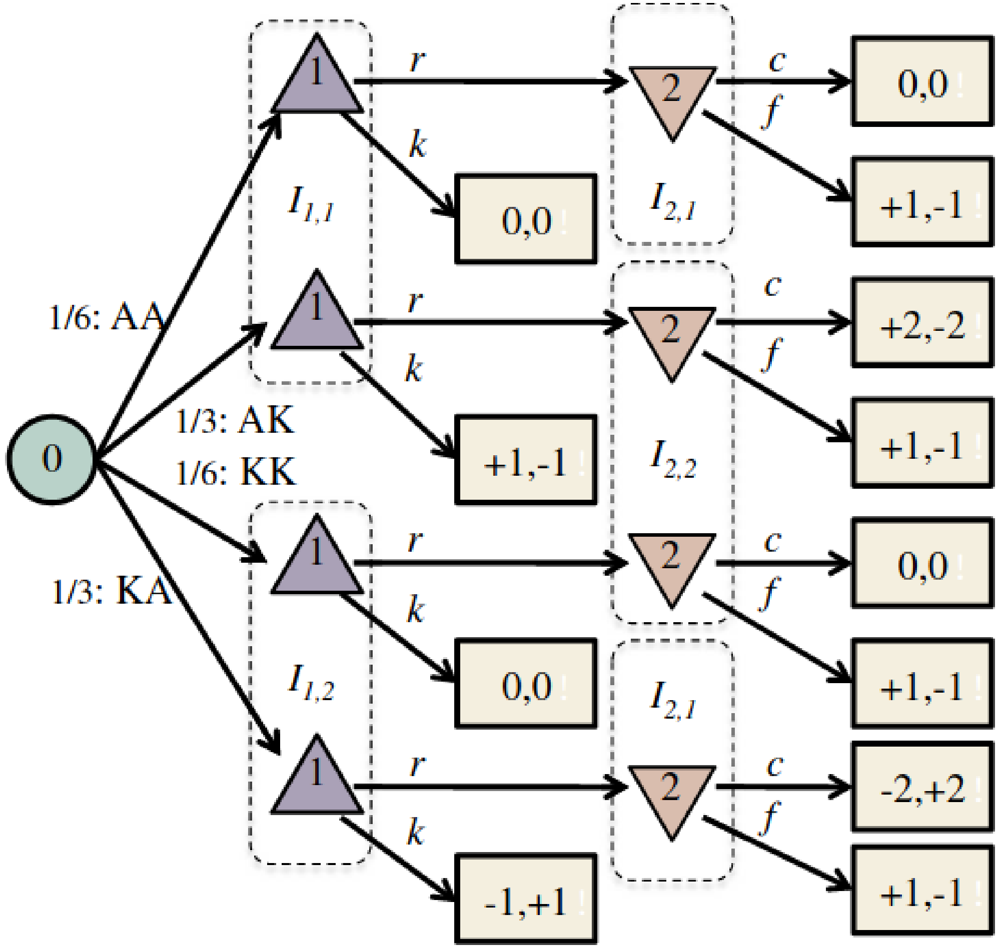


	2:*cc*	2:*cf*	2:*ff*	2:*fc*
1:*rr*	0	-1/6	1	7/6
1:*kr*	-1/3	-1/6	5/6	2/3
1:*rk*	1/3	0	1/6	1/2
1:*kk*	0	0	0	0

Extensive to Matrix Form



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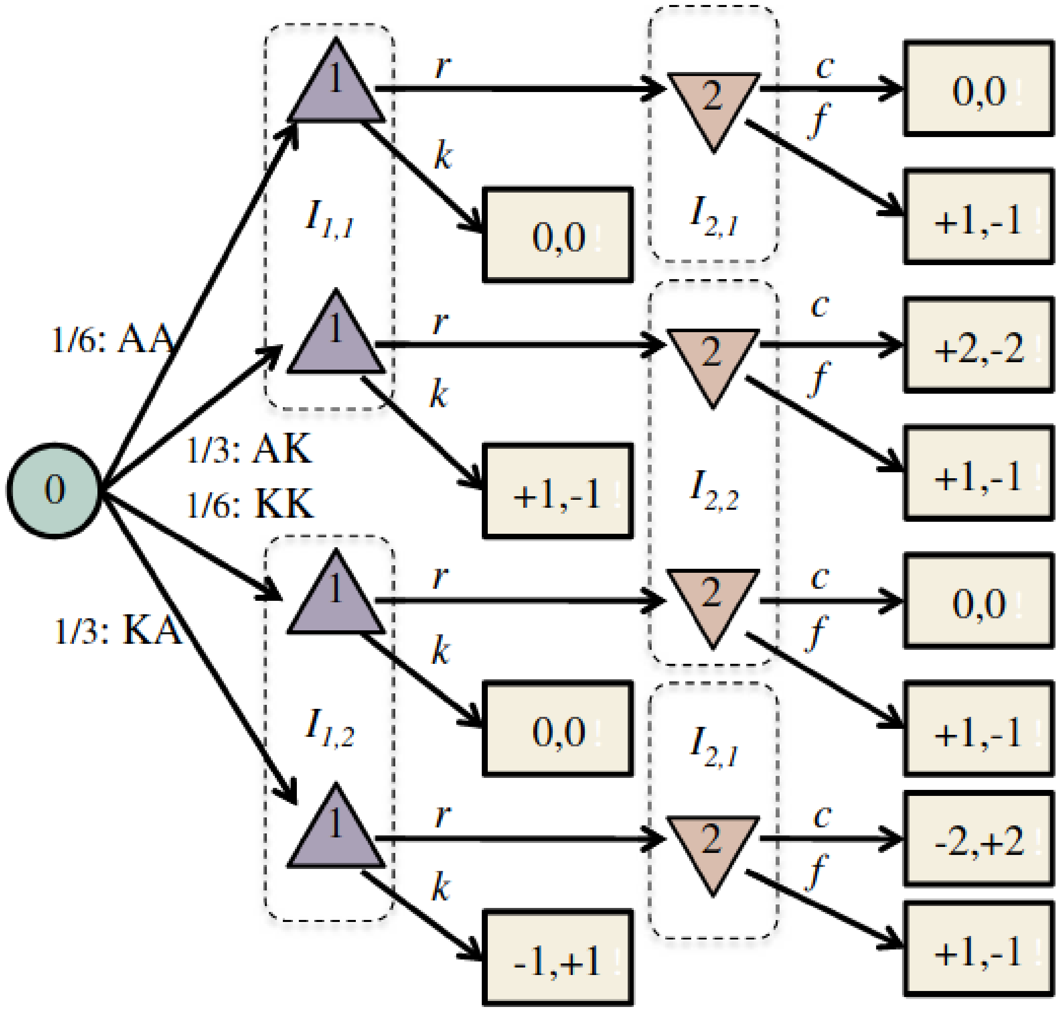
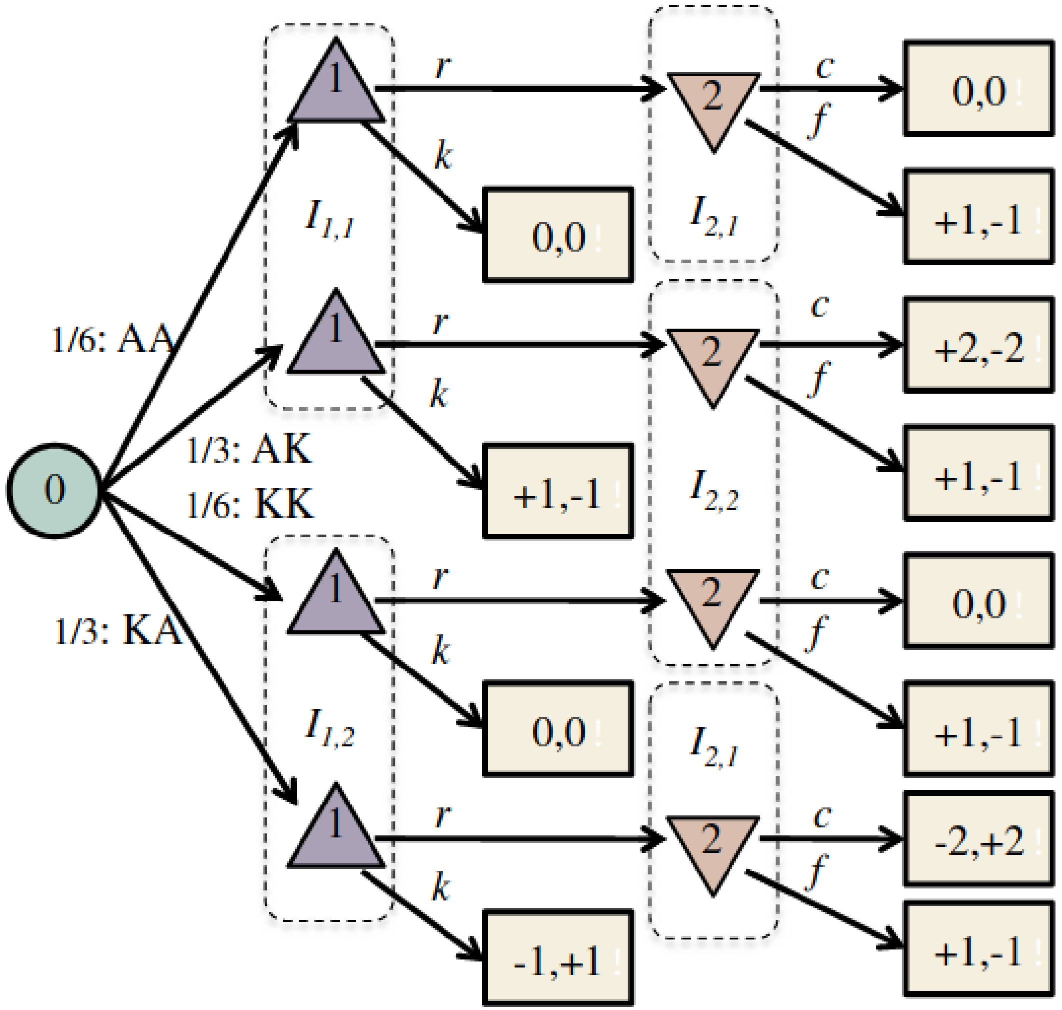
Exponential in number of info states!

Counterfactual Regret Minimization



Counterfactual Regret Minimization

1: Initialize cumulative regret tables: $\forall I, r_I[a] \leftarrow 0$.
2: Initialize cumulative strategy tables: $\forall I, s_I[a] \leftarrow 0$.
3: Initialize initial profile: $\sigma^1(I,a) \leftarrow 1/|A(I)|$



Counterfactual Regret Minimization

```
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```
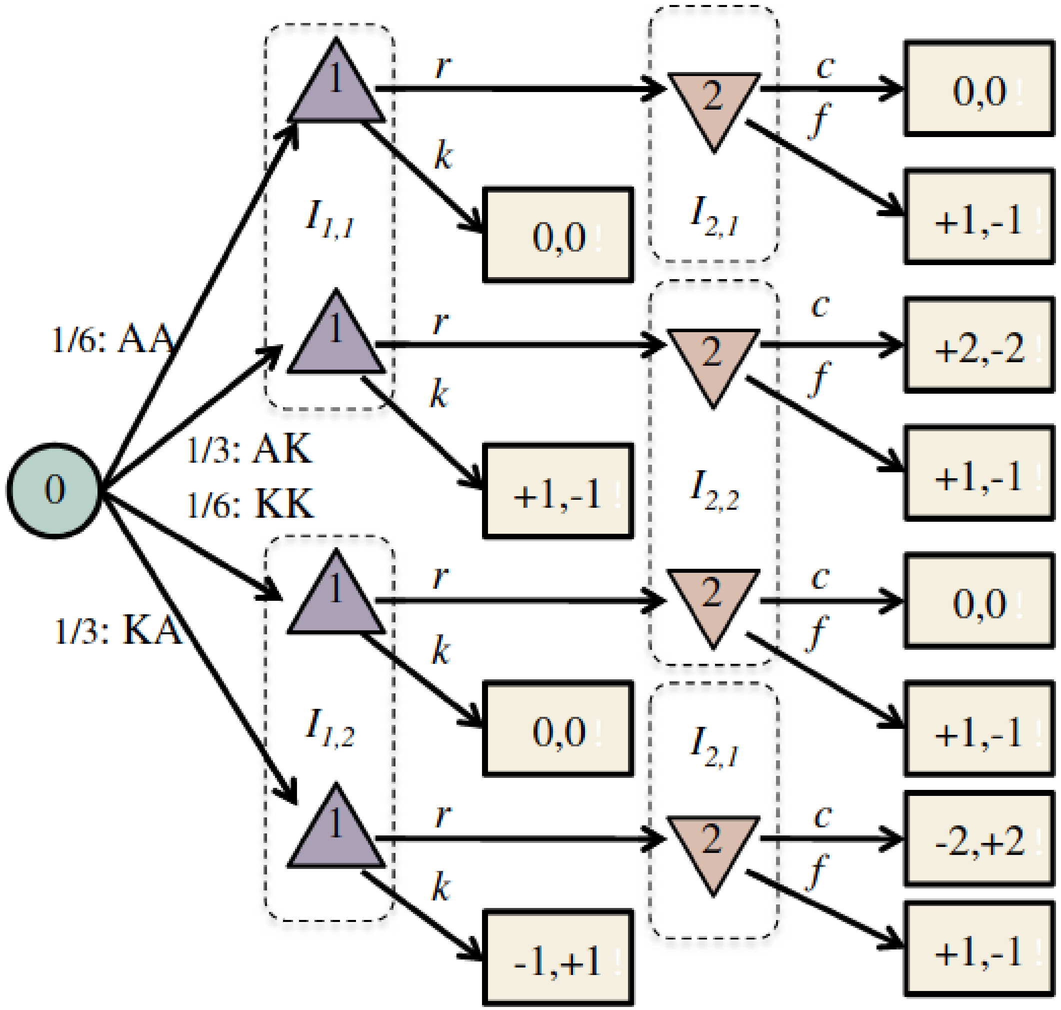

```
function Solve():
for t = {1, 2, 3, ..., T} do
    for i ∈ {1, 2} do
        CFR(∅, i, t, 1, 1)
    end for
end for
```



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Initialize cumulative strategy tables: $\forall I, s_I[a] \leftarrow 0$.
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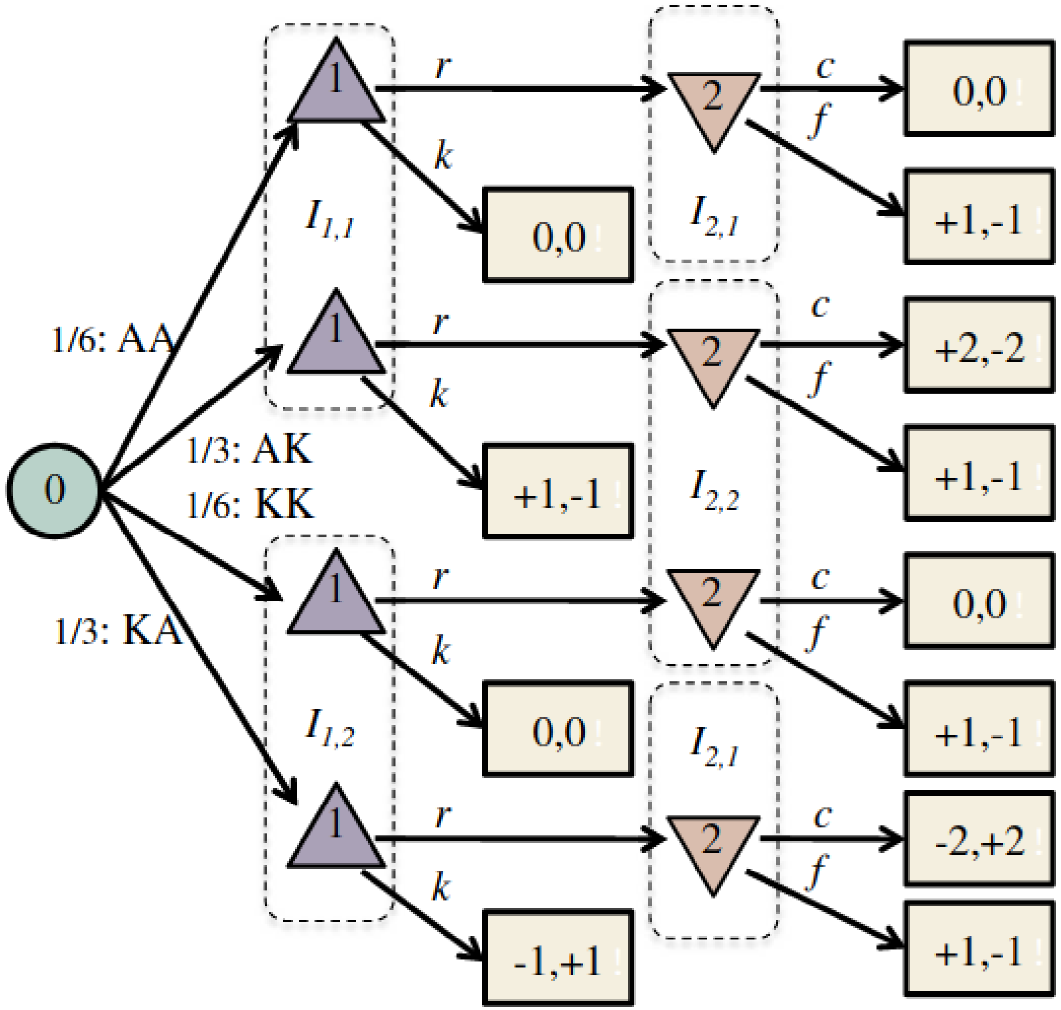
function Solve():
for $t = \{1,2,3,\ldots,T\}$ **do**
  for $i \in \{1,2\}$ **do**
    CFR($\emptyset$, i, t, 1, 1)
  end for
end for

function CFR(h, i, t, π_1, π_2):
if h is terminal **then**
  return $u_i(h)$
else if h is a chance node **then**
  Sample a single outcome $a \sim \sigma_c(h,a)$
  return CFR(ha, i, t, π_1, π_2)
end if
Let I be the information set containing h.
$v_\sigma \leftarrow 0$
$v_{\sigma_{I\rightarrow a}}[a] \leftarrow 0$ for all $a \in A(I)$
for $a \in A(I)$ **do**
  if $P(h) = 1$ **then**
    $v_{\sigma_{I\rightarrow a}}[a] \leftarrow$ CFR(ha, i, t, $\sigma^t(I,a)\cdot\pi_1$, π_2)
  else if $P(h) = 2$ **then**
    $v_{\sigma_{I\rightarrow a}}[a] \leftarrow$ CFR(ha, i, t, π_1, $\sigma^t(I,a)\cdot\pi_2$)
  end if
  $v_\sigma \leftarrow v_\sigma + \sigma^t(I,a)\cdot v_{\sigma_{I\rightarrow a}}[a]$
end for
if $P(h) = i$ **then**
  for $a \in A(I)$ **do**
    $r_I[a] \leftarrow r_I[a] + \pi_{-i}\cdot(v_{\sigma_{I\rightarrow a}}[a] - v_\sigma)$
    $s_I[a] \leftarrow s_I[a] + \pi_i\cdot\sigma^t(I,a)$
  end for
  $\sigma^{t+1}(I) \leftarrow$ regret-matching values
end if
return v_σ



Kuhn Poker Example

3 Cards: 1, 2, and 3

Sequential Actions			Payoff
Player 1	Player 2	Player 1	
pass	pass		+1 to player with higher card
pass	bet	pass	+1 to player 2
pass	bet	bet	+2 to player with higher card
bet	pass		+1 to player 1
bet	bet		+2 to player with higher card

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Solution (found by hand in 1950s):

- Player 1 may pass on a 3 with arbitrary probability γ, but then they must bet on a 1 w.p. $\gamma/3$ and bet on a 2 in the second round w.p. $\gamma/3 + 1/3$
- Player 2 must bet 1/3 of the time when holding a 1 after a pass and bet 1/3 of the time when holding a 2 with a bet

Matrix to Extensive Form

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	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

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- Imperfect information can be represented as an extensive-form game with strategies conditioned on information sets.
- Strategies for imperfect information games often involve stochastic "bluffing"
- Counterfactual regret minimization can be used to solve extensive-form games