

Continuous Space MDPs

Last Time

- Neural Network Function Approximation

$$\hat{y} = f_{\theta}(x)$$

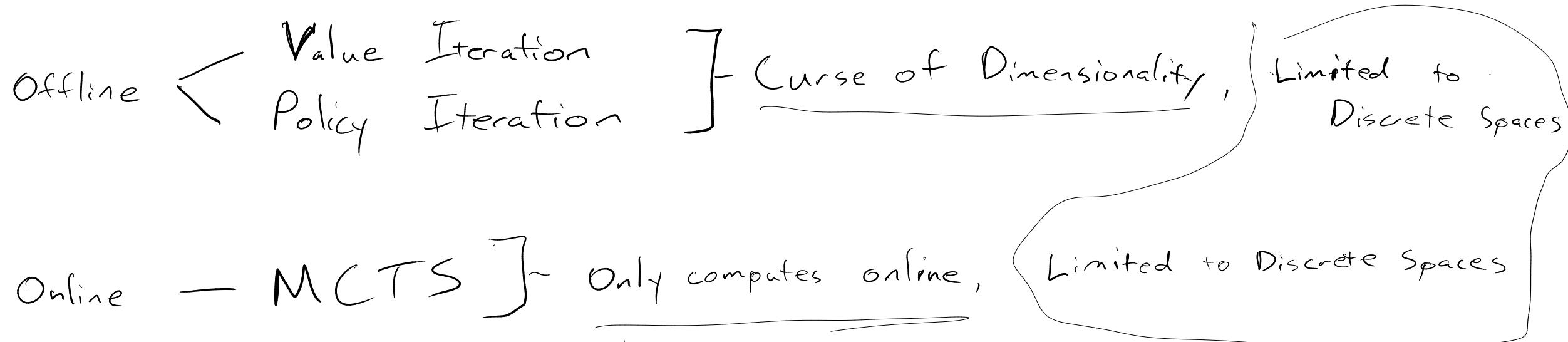
$$\{(x_i, y_i)\}_{i=1}^n$$

Guiding Questions

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- What tools do we have to solve MDPs with continuous S and A ?

Current Tool-Belt



Today: Four Tools

1. LQR
2. Fitted Value Iteration
3. Sparse UCT/Progressive Widening
4. MPC

Notation: Continuous Random Variables

Term	Definition	Coinflip Example	Uniform Example						
$\text{support}(X)$ $x \in X$	All the values that X can take	$\{\text{h}, \text{t}\}$ or $\{0, 1\}$	$\text{Bernoulli}(0.5)$						
Distribution • Discrete: PMF • Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table	<table border="1"><thead><tr><th>x</th><th>P(x)</th></tr></thead><tbody><tr><td>0</td><td>0.5</td></tr><tr><td>1</td><td>0.5</td></tr></tbody></table>	x	P(x)	0	0.5	1	0.5
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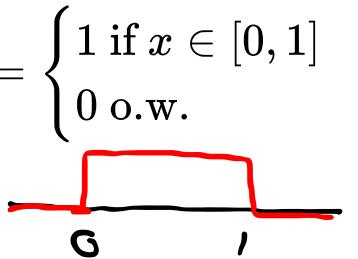
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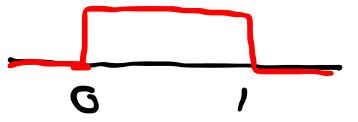
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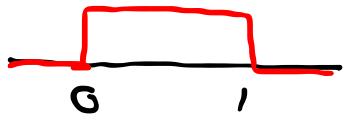
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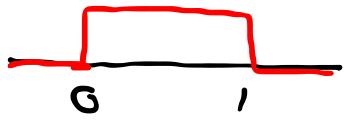
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Rules for Continuous RVs

Discrete

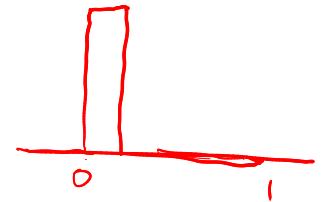
- 1) a) $0 \leq P(X | Y) \leq 1$
- b) $\sum_{x \in X} P(x | Y) = 1$
- 2) $P(X) = \sum_{y \in Y} P(X, y)$

Continuous

- 1)

$$3) P(X | Y) = \frac{P(X, Y)}{P(Y)}$$
$$P(X, Y) = P(X | Y) P(Y)$$

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Continuous

1) $0 \leq p(X | Y)$

$$p(x) = 10 \mathbf{1}(0 \leq x \leq 0.5)$$

3) $P(X | Y) = \frac{P(X, Y)}{P(Y)}$

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$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Multivariate Gaussian Distribution

$$x = [x_1, x_2]$$

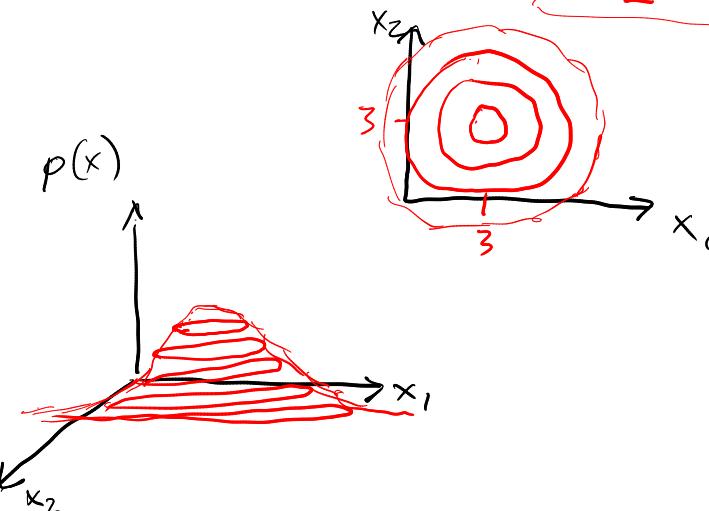
$$\mathcal{N}(\mu, \Sigma)$$

Joint Distribution

$$p(x) = \mathcal{N}(x, \Sigma)$$

$$p(x) = \frac{\exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)}{(2\pi)^{n/2} |\Sigma|^{1/2}}$$

$$\mu = [3, 3] \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$



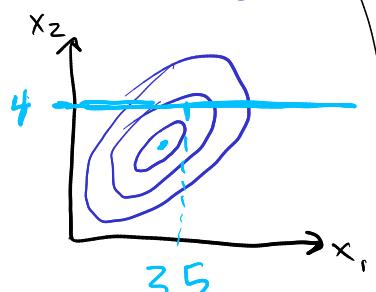
Conditional Distribution

$$p(x_1 | x_2) = \mathcal{N}(\bar{\mu}_1, \bar{\Sigma}_1)$$

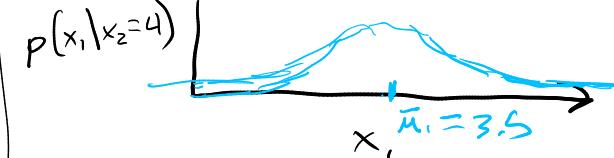
$$\bar{\mu}_1 = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - \mu_2)$$

$$\bar{\Sigma}_1 = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

$$\mu = [3, 3] \quad \Sigma = \begin{bmatrix} 9 & 2 \\ 2 & 4 \end{bmatrix}$$

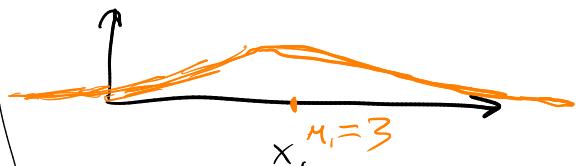


$$p(x_1 | x_2 = 4)$$



Marginal Distribution

$$p(x_1) = \mathcal{N}(\mu_1, \Sigma_{11})$$



Continuous S and A

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e.g. $S \subseteq \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$

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The old rules still work!

$$V^*(s) = \max_a (R(s, a) + \gamma \mathbb{E}[V^*(s')])$$

$$V^\pi(s) = \dots$$

$$B[V](s) = \max_a (R(s, a) + \gamma \mathbb{E}_{\substack{s' \sim T(s'|s, a)}} [V(s')])$$

hard!!!!

Optimization is harder
than integration

$$\int_{S' \in S} T(s'|s, a) V(s') ds'$$

hard!

Monte Carlo

1. Linear Dynamics, Quadratic Reward

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$$s' = T_s s + T_a a + w$$

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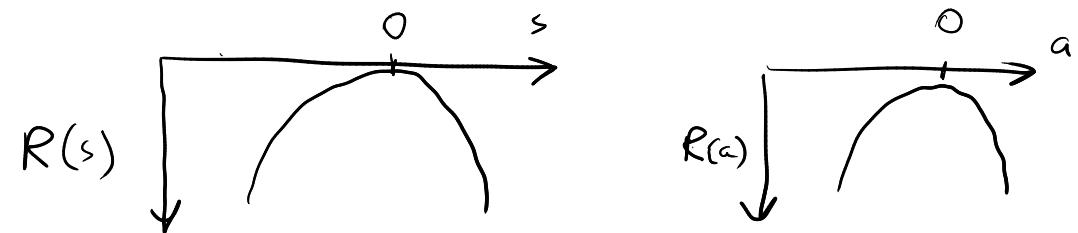
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$$T(s' | s, a) = \mathcal{N}(T_s s + T_a a, \Sigma)$$

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$$s' = T_s s + T_a a + w \quad w \sim \mathcal{N}(0, \Sigma) \quad (\text{Also works with other zero-mean } w.)$$

$$R(s, a) = R(s) + R(a)$$



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Finite Horizon:

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$R(\alpha)$

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$V_1 = R_s$
 $q_1 = 0$



Inductive step: show that if $U_t^* = s^\top V_t s + q_t$, then $U_{t+1}^* = s^\top V_{t+1} s + q_{t+1}$.

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We will show that $U_h^*(s) = s^\top V_h s + q_h$ and $\pi_h^*(s) = -K_h s$ by induction.

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Practical Implication: If a continuous problem has roughly linear dynamics, a convex cost function, and roughly zero-mean additive noise, you can use *certainty-equivalent control*, i.e. control as if there is no noise.

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Fitted Value Iteration

while not converged

$$\theta \leftarrow \theta'$$

$$\hat{V}' \leftarrow B_{\text{approx}}[V_\theta]$$

$$\theta' \leftarrow \text{fit}(\hat{V}')$$

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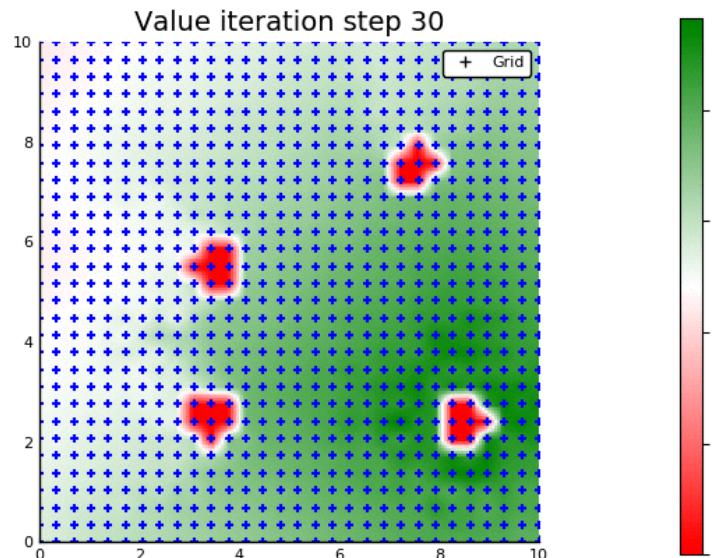
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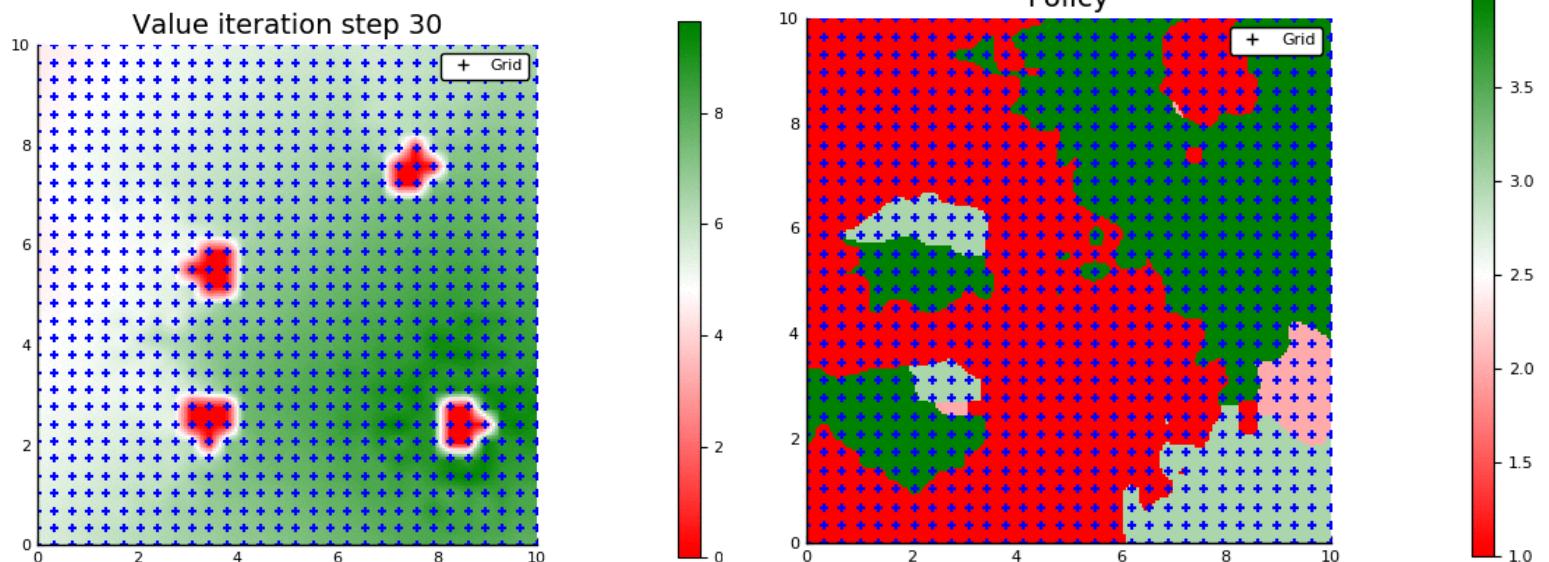
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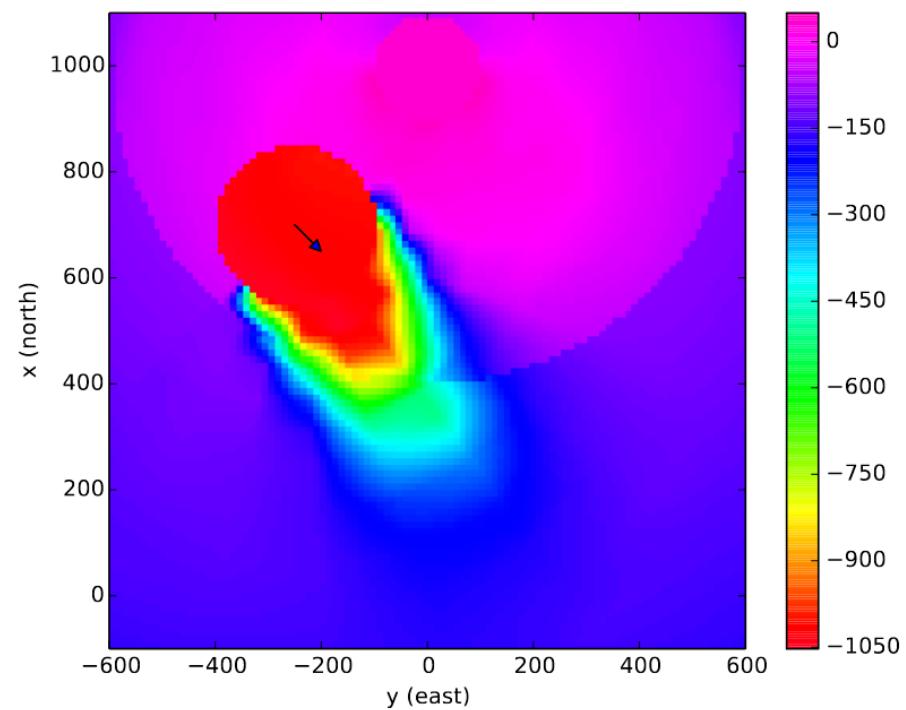
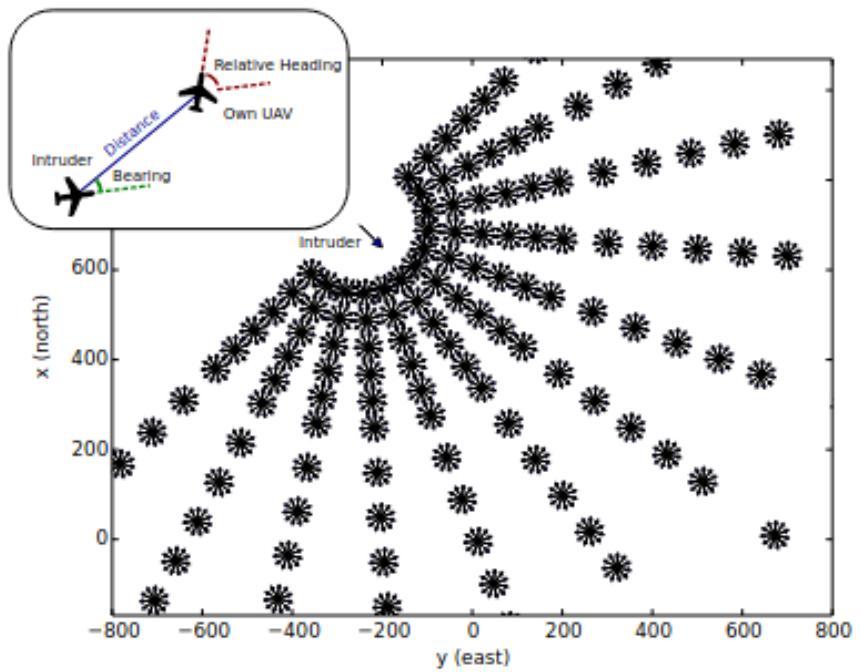
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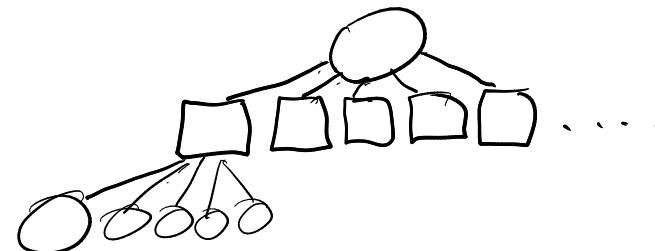
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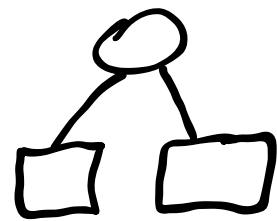
Break

What will a Monte Carlo Tree Search tree look like if run on a problem with continuous spaces?

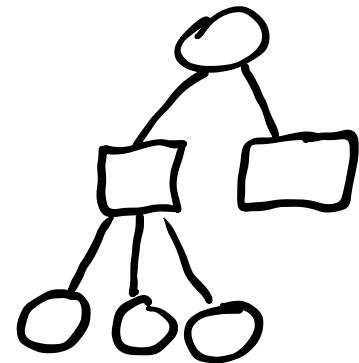


3. Sparse Tree Search/Progressive Widening

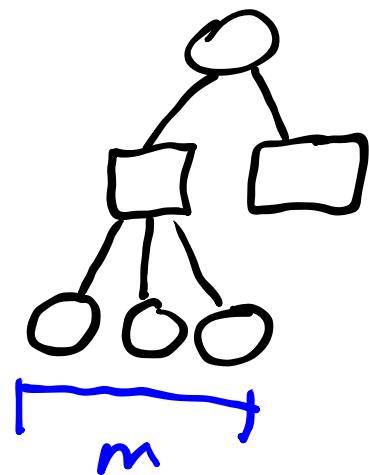
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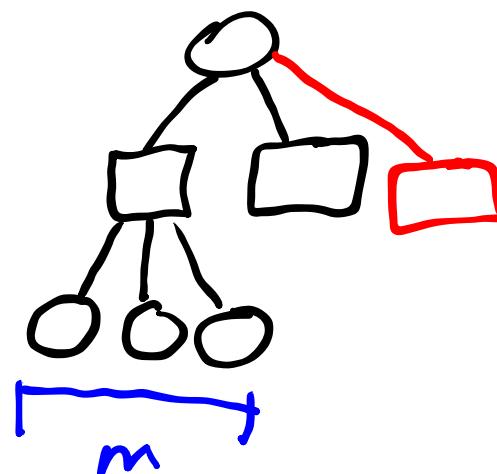
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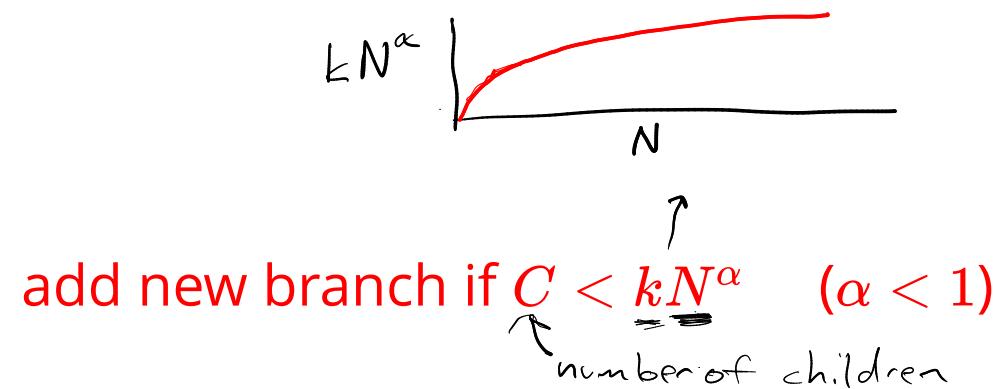
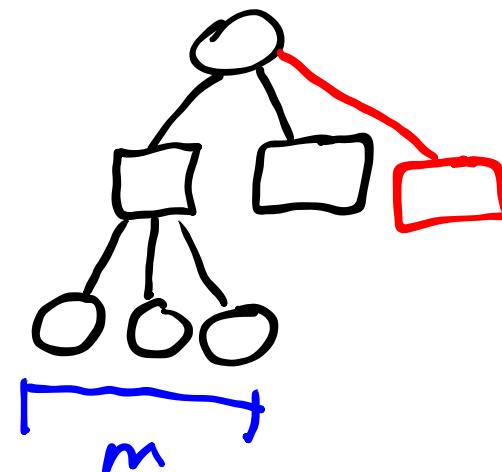
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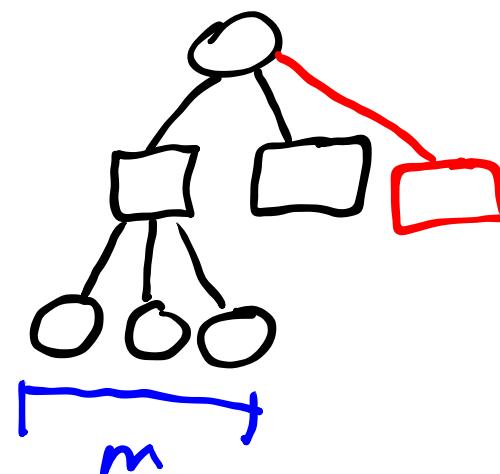


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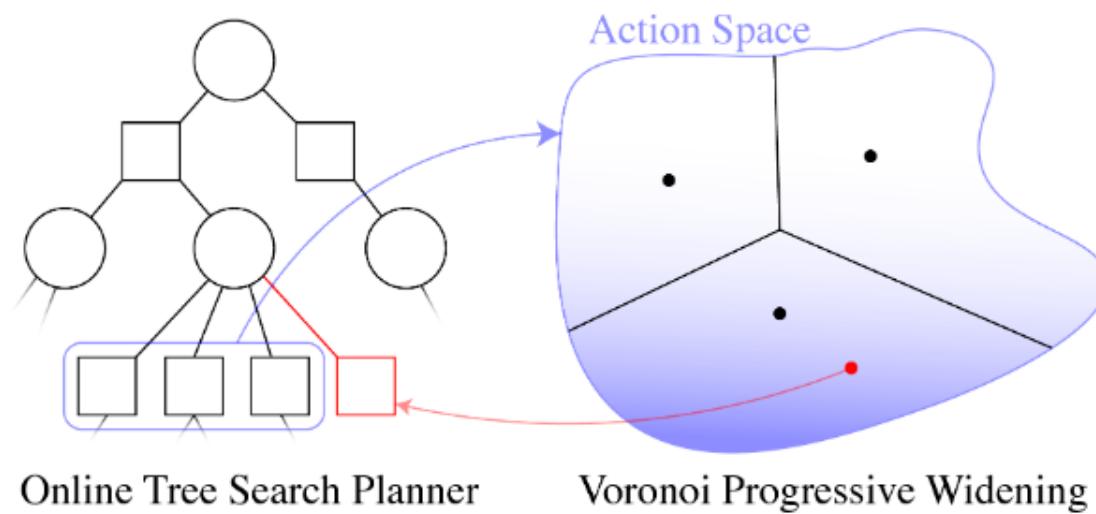


add new branch if $C < \underline{kN^\alpha}$ ($\alpha < 1$)
number of children

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(Use off-the-shelf optimization software, e.g. Ipopt)

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Open-Loop

$$\begin{aligned} & \underset{a_{1:d}, s_{1:d}^{(1:m)}}{\text{maximize}} \quad \frac{1}{m} \sum_{i=1}^m \sum_{t=1}^d \gamma^t R(s_t^{(i)}, a_t) \\ & \text{subject to} \quad s_{t+1} = G(s_t^{(i)}, a_t, w_t^{(i)}) \quad \forall t, i \end{aligned}$$

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Hindsight
Optimization

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Guiding Questions

- What tools do we have to solve MDPs with continuous S and A ?