

Stochastic Processes and Simple Decisions

Review

Causal

Bayesian Net

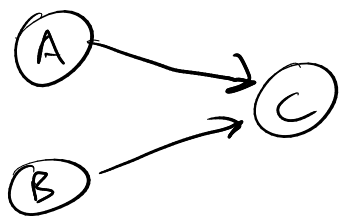
1. Structure: D.A.G.

○ node: R.V.

→ edge: Causal Relationship

2. Parameters

Define $P(X | Pa(X))$

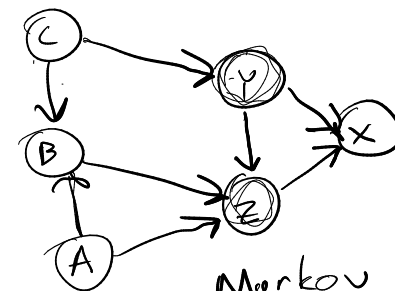


$$P(C | A, B)$$

$$P(A, B, C) = P(A)P(B)P(C | A, B)$$

Chain Rule

$$P(X_{1:n}) = \prod_i P(X_i | Pa(X_i))$$



Markov Blanket
for X is Y, Z

$$X \perp Y | Z$$

$$P(X, Y | Z) = P(X | Z) P(Y | Z)$$

$$P(X | Y, Z) = P(X | Z)$$

D-separation
allows you to prove

$X \perp Y | Z$ based only
on the structure of a B.N.

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?
- How do we find an optimal action based on maximizing expected utility?

Stochastic Process

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$$V_t \sim \text{Bernoulli}(0.5) \text{ (i.i.d.)}$$

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$$\begin{aligned} \overbrace{P(X_{t+1} \mid X_{0:t})} &= \begin{cases} 0.5 & \text{if } X_{t+1} = X_t \\ 0.5 & \text{if } X_{t+1} = X_t + 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \overbrace{P(X_{t+1} \mid X_t)} \end{aligned}$$

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$$P(A|B)$$

B=1		B=0
A	P(A B=1)	
0	0.1	[]
1	0.9	

$$P(A|B) = \begin{cases} 0.1 & \text{if } A=0, B=1 \\ 0.9 & \text{if } A=1, B=1 \\ \vdots & \vdots \end{cases}$$

Example: Positive, Uniform Random Walk

P.U.R.W

$$X_0 = 0$$

$$X_{t+1} = X_t + V_t$$

$$P(X_{t+1} | X_{0:t}) = \begin{cases} 0.5 & \text{if } X_{t+1} = X_t \\ 0.5 & \text{if } X_{t+1} = X_t + 1 \\ 0 & \text{otherwise} \end{cases}$$

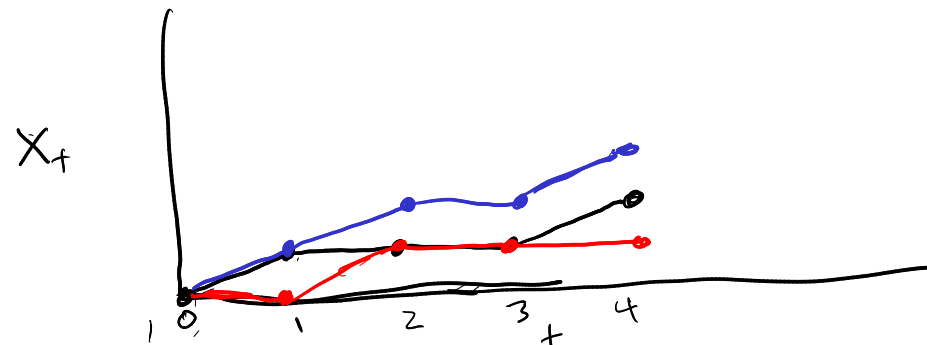
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Bayes Net



Trajectories

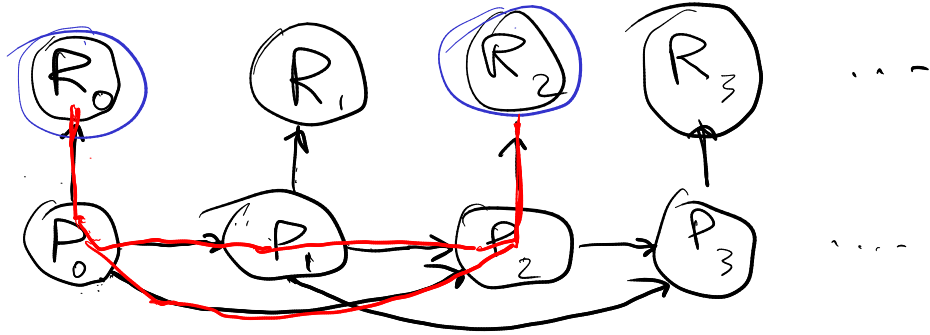


Simulating a Stochastic Process

030-Stochastic-Processes.ipynb

A More Complex Example

Rain Process



$$\cancel{P(R_{t+1} | R_t)}$$

Low-pressure periods last for two days

Want for Markov Process: $R_{t+1} \perp R_{t-1} | R_t$?

$R_2 \perp R_0 | R_1$? **No**

Markov Process

Markov Process

- A stochastic process $\{S_t\}$ is *Markov* if

$$P(S_{t+1} \mid S_{0:t}) = P(S_{t+1} \mid S_t)$$

$$S_{t+1} \perp S_{t-\tau} \mid S_t \quad \forall \tau \in 1 : t$$

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- S_t is called the "state" of the process

Is the P.U.R.W. $\{X_t\}$ a Markov Process?

Yes, because $P(X_{t+1} \mid X_{0:t}) = P(X_{t+1} \mid X_t)$

Is the Rain Process $\{R_t\}$ a Markov Process?

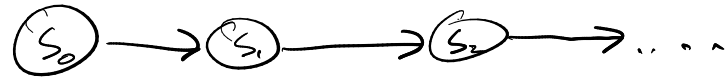
Inconclusive based on structure

No based on information that low pressure lasts for two days.

Dynamic Bayesian Networks

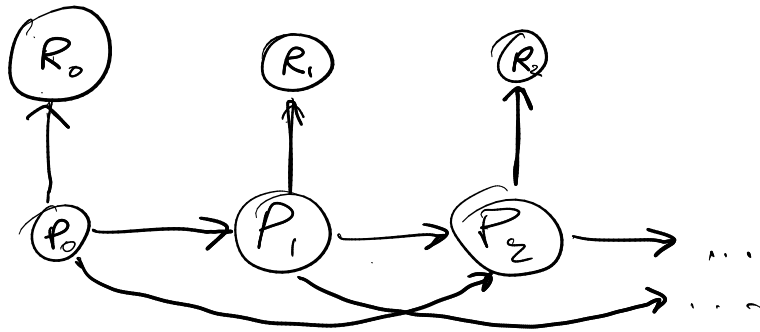
For a Markov Process $\{S_t\}$

Bayes Net

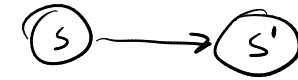


For the Rain Process

Bayes Net

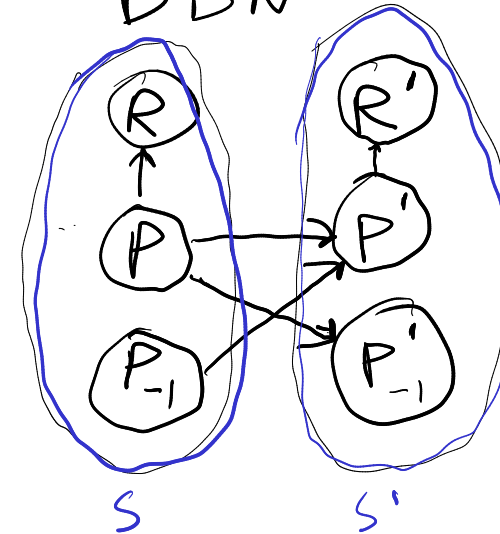


Dynamic Bayes Net



'prime' means next

DBN



$S = (R, P, P_{-1})$

$\{S_t\}$

is Markov

Break

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start in a particular week. **What information should be in the state of the model?**

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Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).

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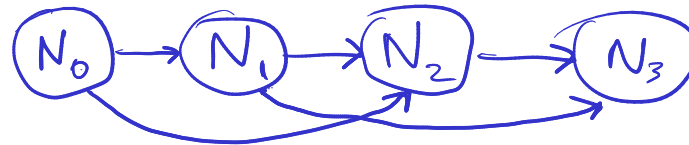
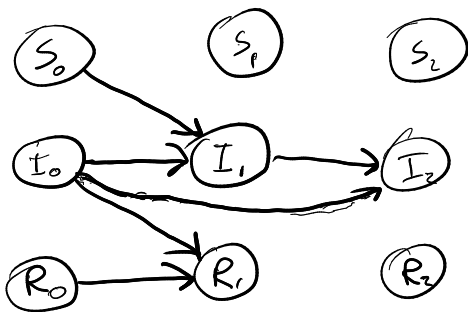
- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 2 weeks after they contract the disease.

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Suppose you want to create a Markov process model that describes how many new COVID cases will start in a particular week. **What information should be in the state of the model?**

Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 2 weeks after they contract the disease.
- Researchers have determined a probabilistic model for the number of new cases given the number of people in the first week of the disease and the number of people in the second week of the disease.



Answer: number infected in current week
and previous week

Simple Decisions

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Outcomes

$s_1 \dots s_n$

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Lottery

$[S_1 : p_1; \dots; S_n : p_n]$

~~p_n~~

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- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$

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These constraints imply a utility function U with the properties:

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- $U(A) > U(B)$ iff $A \succ B$

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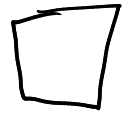
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- $U(A) > U(B)$ iff $A \succ B$
- $U(A) = U(B)$ iff $A \sim B$
- $U([S_1 : p_1; \dots; S_n : p_n]) = \sum_{i=1}^n p_i U(S_i)$

Decision Networks

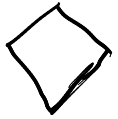


Decision



Chance

(just like in BN)



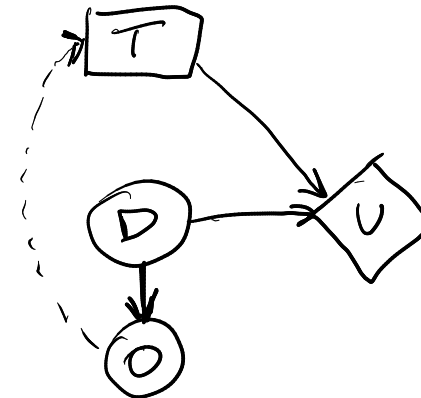
Utility

—————→ conditional

-----→ informational

what info a decision
can be made based
on

Example

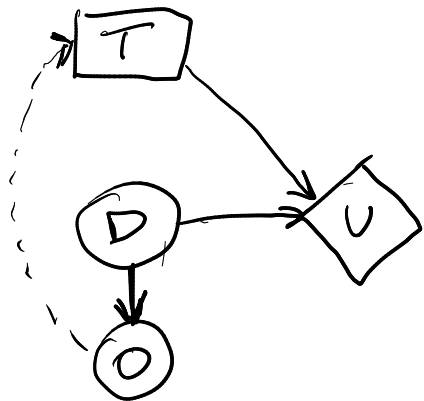


Maximizing Expected Utility

$$EU(a|o) \equiv \sum_{s'} P(s'|a,o) \underline{U(s',a)}$$

$$a^* = \underset{a}{\operatorname{argmax}} E(a|o)$$

Example



Given:

D	T	U
0	0	0
1	0	-10
0	1	-1
1	1	-1

$$P(D=1) = 0.2$$

$$P(O=1|D=0) = 0.01$$

$$P(O=1|D=1) = 0.9$$

$$\rightarrow s' = D$$

$$P(D|T,o)^* = P(D|O) = \frac{P(O|D)P(D)}{P(O)}$$

Treat ($T=1$)

$$EU(T=1|o) = \sum_d P(d|1,o) \underline{U(d,1)}$$

$$= -1$$

$$EU(O=0|o=0) = \sum_d P(d|0,0) U(d,0)$$

$$= \frac{1}{P(O=0)} \left(P(O=0|D=0)P(D=0)U(0,0) + P(O=0|D=1)P(D=1)U(1,0) \right)$$

$$0.99 \cdot 0.8 + 0.1 \cdot 0.2$$

$$= -0.24$$

$$EU(T=0|o=1) = \text{Bad}$$

Value of Information

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