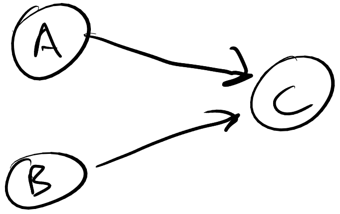
Stochastic Processes and Simple Decisions

Review

Causal

Bayesian Net

1. Structure: D.A.G.
   - ○ node: R.V.
   - → edge: Causal Relationship
2. Parameters
   - Define $P(X \mid Pa(X))$

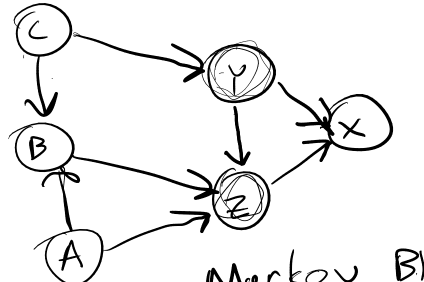


$P(C \mid A, B)$

Chain Rule

$$P(X_{1:n}) = \prod_i P(X_i \mid Pa(X_i))$$

$$P(A,B,C) = P(A)\,P(B)\,P(C \mid A,B)$$



Markov Blanket for <u>X</u> is <u>Y,Z</u>

$$X \perp Y \mid Z$$

$$P(X,Y \mid Z) = P(X \mid Z)\,P(Y \mid Z)$$

$$P(X \mid Y,Z) = P(X \mid Z)$$

D-separation allows you to prove $X \perp Y \mid Z$ based only on the structure of a B.N.

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?
- How do we find an optimal action based on maximizing expected utility?

Stochastic Process

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{X_0, X_1, X_2, \ldots\}$ or $\{X_t\}_{t=0}^{\infty}$ or just $\{X_t\}$

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{X_0, X_1, X_2, \ldots\}$ or $\{X_t\}_{t=0}^{\infty}$ or just $\{X_t\}$

Example: Positive, Uniform Random Walk

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{X_0, X_1, X_2, \ldots\}$ or $\{X_t\}_{t=0}^{\infty}$ or just $\{X_t\}$

Example: Positive, Uniform Random Walk

$X_0 = 0$

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{X_0, X_1, X_2, \ldots\}$ or $\{X_t\}_{t=0}^{\infty}$ or just $\{X_t\}$

Example: Positive, Uniform Random Walk

$X_0 = 0$ $\quad$ $X_{t+1} = X_t + V_t$

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{X_0, X_1, X_2, \ldots\}$ or $\{X_t\}_{t=0}^{\infty}$ or just $\{X_t\}$

Example: Positive, Uniform Random Walk

$X_0 = 0$

$$X_{t+1} = X_t + V_t$$

$$V_t \sim \text{Bernoulli}(0.5) \text{ (i.i.d.)}$$

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{X_0, X_1, X_2, \ldots\}$ or $\{X_t\}_{t=0}^{\infty}$ or just $\{X_t\}$

Example: Positive, Uniform Random Walk

$X_0 = 0$ $\quad\quad$ $X_{t+1} = X_t + V_t$

$V_t \sim \text{Bernoulli}(0.5)$ (i.i.d.)

In a *stationary* stochastic process (all in this class), this relationship does not change with time

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{X_0, X_1, X_2, \ldots\}$ or $\{X_t\}_{t=0}^{\infty}$ or just $\{X_t\}$

Example: Positive, Uniform Random Walk

$X_0 = 0$

$$\underline{X_{t+1} = X_t + V_t}$$

$$V_t \sim \text{Bernoulli}(0.5) \text{ (i.i.d.)}$$

In a *stationary* stochastic process (all in this class), this relationship does not change with time

$$\underline{P(X_{t+1} \mid X_{0:t})} = \begin{cases} 0.5 & \text{if } X_{t+1} = X_t \\ 0.5 & \text{if } X_{t+1} = X_t + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= P(X_{t+1} \mid X_t)$$

Stochastic Process

- A stochastic process is a collection of R.V.s indexed by time.
- $\{X_0, X_1, X_2, \ldots\}$ or $\{X_t\}_{t=0}^{\infty}$ or just $\{X_t\}$

$P(A|B)$

$B=1$

A	$P(A \mid B=1)$
0	0.1
1	0.9

$B=0$

$$P(A|B) = \begin{cases} 0.1 & \text{if } A=0, B=1 \\ 0.9 & \text{if } A=1, B=1 \end{cases}$$

Example: Positive, Uniform Random Walk

P.U.R.W

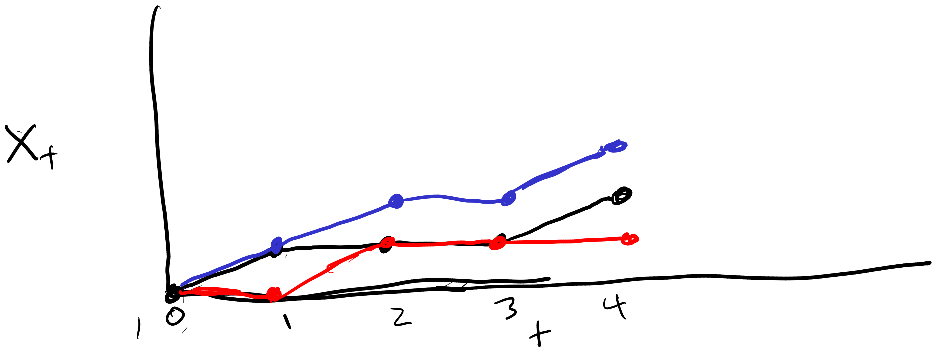
$$X_0 = 0 \qquad X_{t+1} = X_t + V_t \qquad P(X_{t+1} \mid X_{0:t}) = \begin{cases} 0.5 & \text{if } X_{t+1} = X_t \\ 0.5 & \text{if } X_{t+1} = X_t + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$V_t \sim \text{Bernoulli}(0.5) \text{ (i.i.d.)}$$

In a *stationary* stochastic process (all in this class), this relationship does not change with time

Bayes Net

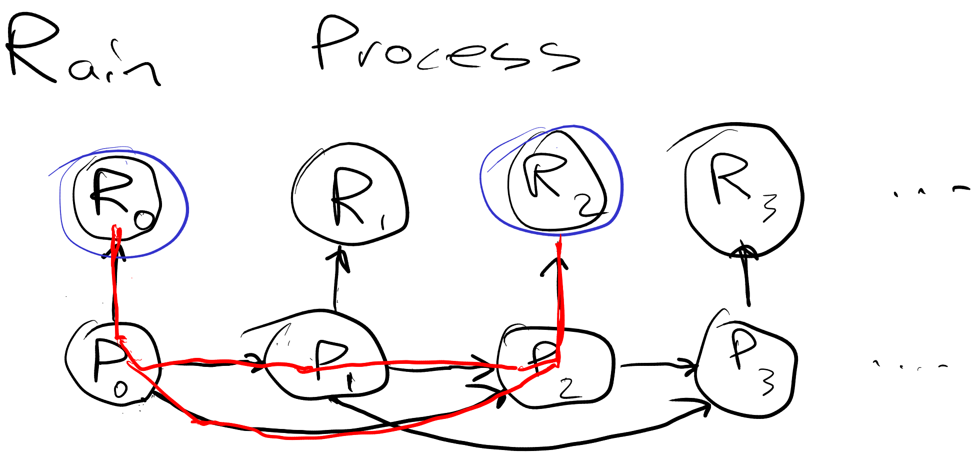
Trajectories



Simulating a Stochastic Process

030-Stochastic-Processes.ipynb

A More Complex Example



$\cancel{P(R_{t+1} \mid R_t)}$

Low-pressure periods last for two days

Want for Markov Process: $R_{t+1} \perp R_{t-e} \mid R_t$?

$R_2 \perp R_0 \mid R_1$? No

Markov Process

Markov Process

- A stochastic process $\{S_t\}$ is *Markov* if

$$P(S_{t+1} \mid S_{0:t}) = P(S_{t+1} \mid S_t)$$

$$S_{t+1} \perp S_{t-\tau} \mid S_t \quad \forall \tau \in 1:t$$

Markov Process

- A stochastic process $\{S_t\}$ is *Markov* if
$$P(S_{t+1} \mid S_{0:t}) = P(S_{t+1} \mid S_t)$$
$$S_{t+1} \perp S_{t-\tau} \mid S_t \quad \forall \tau \in 1:t$$
- S_t is called the "state" of the process

Is the P.U.R.W. $\{X_t\}$ a Markov Process?

Yes, because $P(X_{t+1} \mid X_{0:t}) = P(X_{t+1} \mid X_t)$

Is the Rain Process $\{R_t\}$ a Markov Process?

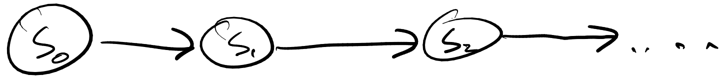
Inconclusive based on structure

No based on information that low pressure lasts for two days.

Dynamic Bayesian Networks

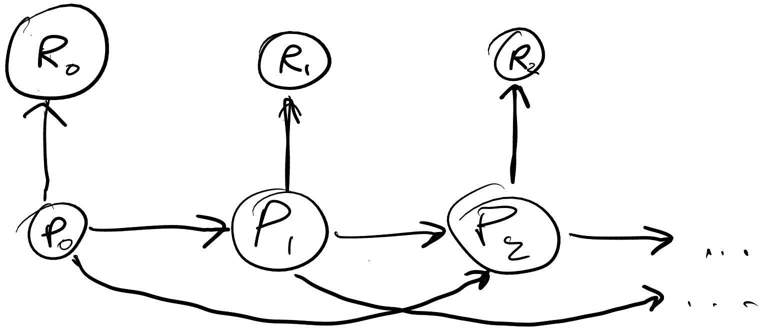
For a Markov Process $\{S_t\}$

Bayes Net



For the Rain Proce

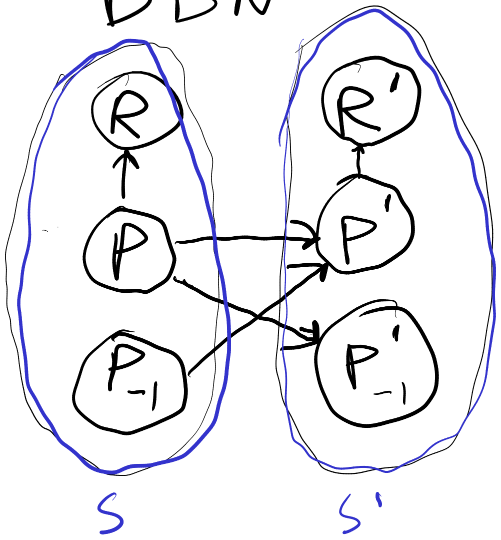
Bayes Net

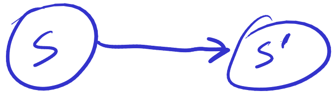


Dynamic Bayes Net

' "prime" means next

DBN





$S = (R, P, P_{-1})$

$\{S_t\}$

is Markov

Break

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start in a particular week. **What information should be in the state of the model?**

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start in a particular week. **What information should be in the state of the model?**

Assume:

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start in a particular week. **What information should be in the state of the model?**

Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).

Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start in a particular week. **What information should be in the state of the model?**

Assume:

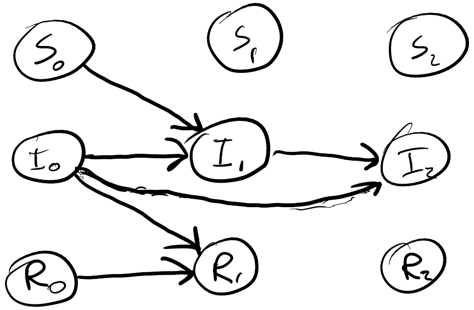
- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 2 weeks after they contract the disease.

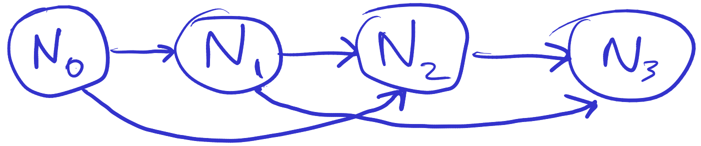
Break

Suppose you want to create a Markov process model that describes how many new COVID cases will start in a particular week. **What information should be in the state of the model?**

Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 2 weeks after they contract the disease.
- Researchers have determined a probabilistic model for the number of new cases given the number of people in the first week of the disease and the number of people in the second week of the disease.





Answer: number infected in current week and previous week

Simple Decisions

Simple Decisions

Simple Decisions

Outcomes

$S_1 \ldots S_n$

Simple Decisions

Outcomes $S_1 \ldots S_n$

Probabilities $p_1 \ldots p_n$

Simple Decisions

Outcomes

$S_1 \ldots S_n$

Probabilities

$p_1 \ldots p_n$

Lottery

$[S_1 : p_1; \ldots; S_n : p_n]$

Simple Decisions

Outcomes
$S_1 \ldots S_n$

Probabilities
$p_1 \ldots p_n$

Lottery
$[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$

Simple Decisions

Outcomes

$S_1 \ldots S_n$

Probabilities

$p_1 \ldots p_n$

Lottery

$[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$

Simple Decisions

Outcomes	Probabilities	Lottery
$S_1 \ldots S_n$	$p_1 \ldots p_n$	$[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$

Simple Decisions

Outcomes	Probabilities	Lottery
$S_1 \ldots S_n$	$p_1 \ldots p_n$	$[S_1 : p_1; \ldots ; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

Simple Decisions

Outcomes
$S_1 \ldots S_n$

Probabilities
$p_1 \ldots p_n$

Lottery
$[S_1 : p_1; \ldots ; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

von Neumann - Morgenstern Axioms

Simple Decisions

Outcomes	Probabilities	Lottery
$S_1 \dots S_n$	$p_1 \dots p_n$	$[S_1 : p_1; \dots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

von Neumann - Morgenstern Axioms

These constraints imply a utility function U with the properties:

Simple Decisions

Outcomes

$S_1 \dots S_n$

Probabilities

$p_1 \dots p_n$

Lottery

$[S_1 : p_1; \dots ; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

von Neumann - Morgenstern Axioms

These constraints imply a utility function U with the properties:

- $U(A) > U(B)$ iff $A \succ B$

Simple Decisions

Outcomes
$S_1 \ldots S_n$

Probabilities
$p_1 \ldots p_n$

Lottery
$[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

von Neumann - Morgenstern Axioms

These constraints imply a utility function U with the properties:

- $U(A) > U(B)$ iff $A \succ B$
- $U(A) = U(B)$ iff $A \sim B$

Simple Decisions

Outcomes $S_1 \ldots S_n$

Probabilities $p_1 \ldots p_n$

Lottery $[S_1 : p_1; \ldots; S_n : p_n]$

- Completeness: Exactly one holds: $A \succ B$, $B \succ A$, $A \sim B$
- Transitivity: If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Continuity: If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$
- Independence: If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$

von Neumann - Morgenstern Axioms

These constraints imply a utility function U with the properties:

- $U(A) > U(B)$ iff $A \succ B$
- $U(A) = U(B)$ iff $A \sim B$
- $U([S_1 : p_1; \ldots; S_n : p_n]) = \sum_{i=1}^{n} p_i U(S_i)$

Decision Networks

Decision

Chance (just like in BN)

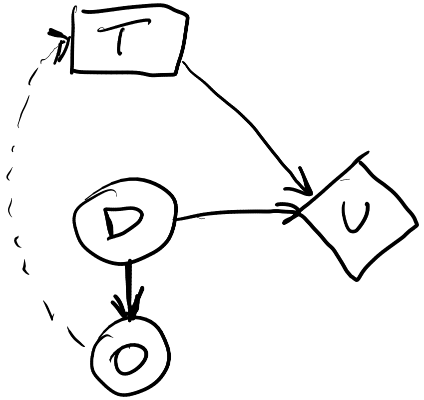
Utility

conditional

informational

what info a decision can be made based on

Example

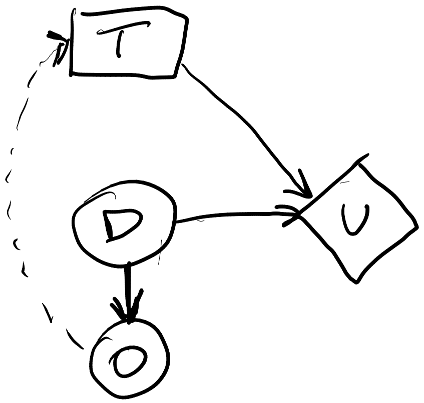


Maximizing Expected Utility

$$EU(a|o) \equiv \sum_{s'} \underline{P(s'|a,o)}\, U(s',a)$$

$$a^* = \underset{a}{\arg\max}\, E(a|o)$$

Example



$P(D=1) = 0.2$

$P(O=1|D=0) = 0.01$

$P(O=1|D=1) = 0.9$

$\rightarrow s' = D$

$P(D|T,O)^{*} = P(D|O)$

$= \dfrac{P(O|D)P(D)}{P(O)}$

Given:

D	T	U
0	0	0
1	0	-10
0	1	-1
1	1	-1

Treat $(T=1)$

$$EU(T=1|O) = \sum_d P(d|1,O)\, \underbrace{U(d,1)}_{-1}$$

$$= \boxed{-1}$$

$$EU(0|O=0) = \sum_d P(d|0,O)\, U(d,O)$$

$$= \frac{1}{\underbrace{P(O=0)}_{0.99 \cdot 0.8 + 0.1 \cdot 0.2}} \Big(P(O=0|D=0)\, P(D=0)\, \underbrace{U(0,0)}_{0} + \underbrace{P(O=0|D=1)}_{0.1}\, \underbrace{P(D=1)}_{0.2}\, \underbrace{U(1,0)}_{-10} \Big)$$

$$= \boxed{-0.24}$$

$$EU(T=0|O=1) = \text{Bad}$$

Value of Information

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?