Value Iteration Convergence

Review

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- How do we reason about the **future consequences** of actions in an MDP?

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- How do we reason about the **future consequences** of actions in an MDP?
- What are the basic **algorithms for solving MDPs**?

Guiding Questions

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- Does value iteration always converge?
- Is the value function unique?
- Can there be multiple optimal policies?
- Is there always a deterministic optimal policy?

Value Iteration: The Bellman Operator

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Algorithm: Value Iteration

while $\|V - V'\|_\infty > \epsilon$

$\quad V \leftarrow V'$

$\quad V' \leftarrow B[V]$

return V'

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$$B[V](s) = \max_{a \in A} \left(R(s,a) + \gamma E\left[V(s')\right]\right)$$

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Theorem 1: Let $\{V_1, \ldots, V_\infty\}$ be a sequence of value functions for a discrete MDP generated by the recurrence $V_{k+1} = B[V_k]$. If $\gamma < 1$, then $\lim_{k\to\infty} V_k = V^*$.

Metrics

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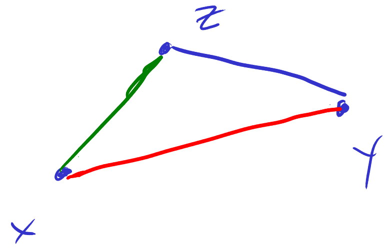

Example: $M = \mathbb{R}^2$ $\quad d(x,y) = \|x - y\|_2$

Contraction Mappings

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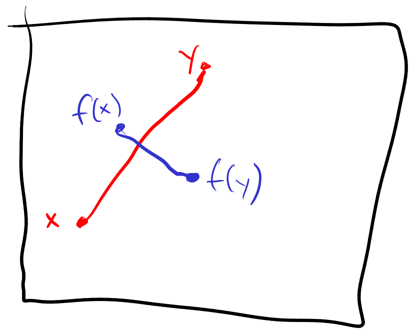
<u>Definition</u>: A *contraction mapping* on metric space (M, d) is a function $f : M \to M$ satisfying

$$d(f(x), f(y)) \leq \alpha\, d(x, y)$$

for some α, $0 \leq \alpha \leq 1$ and all x and y in M.



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$$f(x) = \begin{bmatrix} \frac{x[2]}{2} + 1 \\ \frac{x[1]}{2} + \frac{1}{2} \end{bmatrix}$$

$M = \mathbb{R}^2$

$d(x,y) = \|x - y\|_2$

$$d(f(x), f(y)) = \sqrt{\left(\frac{x[2]}{2} \cancel{+1} - \frac{y[2]}{2} \cancel{-1}\right)^2 + \left(\frac{x[1]}{2} \cancel{+\frac{1}{2}} - \frac{y[1]}{2} \cancel{-\frac{1}{2}}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x[2] - y[2])^2 + (x[1] - y[1])^2}$$

$$= \frac{1}{2} d(x, y)$$

f is a contraction mapping with $\alpha = \frac{1}{2}$

Script: contraction_mapping.jl

Banach's Theorem

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Theorem (Banach): If f is a contraction mapping on metric space (M, d), then

1. f has a single, unique fixed point x^*.
2. If $\{x_k\}$ is a sequence defined by $x_{k+1} = f(x_k)$, then $\lim_{k \to \infty} x_k = x^*$.

Max Norm

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Note: $\|x - y\|_\infty = \max_i |x_i - y_i|$

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<u>Lemma 2</u>: B is a γ contraction mapping on $(\mathbb{R}^{|S|}, \|\cdot\|_\infty)$.

Proof

$$\|B[V_1] - B[V_2]\|_\infty = \max_{s \in S} |B[V_1](s) - B[V_2](s)|$$

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By Banach's theorem (part 1), $\hat{V} = B[\hat{V}]$. Since $\hat{V}$ satisfies Bellman's optimality equation, it is optimal and $\hat{V} = V^*$.

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Proof (sketch):

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Theorem: Policy iteration converges to an optimal policy for a finite MDP in finite time.

Proof (sketch):

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4. By (1), (2), and (3), the policy will improve until it finds the optimal policy, and it will always find the optimal policy.

Properties of optimal MDP solutions

- Does every MDP have a unique optimal value function, V^*? Yes Because of Banach's Theorem
- Does every MDP have a unique optimal policy, π^*? No
- Does every MDP have a *deterministic* optimal policy? Yes $\pi(a|s)$

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Justification

- Suppose that $\tilde{\pi}$ is optimal and, for some s, $\tilde{\pi}(a^1 \mid s) > 0$, $\tilde{\pi}(a^2 \mid s) > 0$, and $\tilde{\pi}(a^1 \mid s) + \tilde{\pi}(a^2 \mid s) = 1$.
- Then $Q^*(s, a^1) = Q^*(s, a^2) = V^*(s)$. If this were not true, then $\tilde{\pi}$ would not be optimal.
- As a consequence, a deterministic policy $\tilde{\pi}'$ with $\tilde{\pi}'(s) = a^1$ is also optimal!

Guiding Questions

Guiding Questions

- Does value iteration always converge?
- Is the value function unique?
- Can there be multiple optimal policies?
- Is there always a deterministic optimal policy?

Break

Conservation MDP