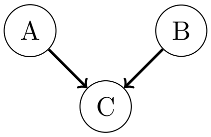
ASEN 5264 Decision Making under Uncertainty
Exam 1: Probabilistic Models and MDPs

Instructions: Clearly indicate your final answers and briefly justify numerical answers with text or mathematical expressions. If you do not understand how to do a problem, skip it and move on so that you have time to attempt all problems. You may consult any static source, but you may NOT communicate with any person except the instructor or TA, and you may not use LLMs such as ChatGPT.

Question 1. (30 pts) Consider the following Bayesian network made up of binary random variables:



a) Is it possible to conclude from the structure alone that A and B are conditionally independent given C? Justify your answer.

Solution: $\boxed{No}$, A and B are not d-separated. The path $A \to C \leftarrow B$ contains an inverted fork at node $C \in \mathcal{C}$.

b) Suppose that the conditional distribution for C is given by the following table:

A	B	$P(C = 1 \mid A, B)$
0	0	0.2
0	1	0.3
1	0	0.4
1	1	0.5

and that $P(A = 1) = 0.6$ and $P(B = 1) = 0.7$. Calculate $P(A = 1 \mid B = 1, C = 1)$.

Solution: By the definition of conditional probability,

$$\begin{aligned} P(A = 1 \mid B = 1, C = 1) &= \frac{P(A = 1, B = 1, C = 1)}{P(B = 1, C = 1)} \\ &= \frac{P(A = 1, B = 1, C = 1)}{\sum_{a \in A} P(A = a, B = 1, C = 1)} \end{aligned}$$

Utilizing the graph structure,

$$P(A = 1, B = 1, C = 1) = P(C = 1 \mid A = 1, B = 1) \cdot P(A = 1) \cdot P(B = 1) = 0.21$$
$$P(A = 0, B = 1, C = 1) = P(C = 1 \mid A = 0, B = 1) \cdot (1 - P(A = 1)) \cdot P(B = 1) = 0.084$$

$$P(A = 1 \mid B = 1, C = 1) = \boxed{0.714}$$

c) Now, suppose that $P(A = 1) = 0.5$, $P(B = 1) = 0.5$, and $P(C = 1 \mid A, B) = 0.5$ for all values of A and B. Are A and B conditionally independent given C? Justify your answer.

Solution:
From problem, $P(A = a) = P(B = b) = P(C = c \mid A = a, B = b) = 0.5 \; \forall \; a \in A, \; b \in B, c \in C$.
By the definition of conditional independence, A and B are conditionally independent given C if and only if $P(A = a \mid B = b, C = c) = P(A = a \mid C = c) \; \forall \; a \in A, b \in B, c \in C$.
Utilizing the graph structure,

$$P(A = a, B = b, C = c) = P(C = c|A = a, B = b) \cdot P(A = a) \cdot P(B = b) = 0.125$$

Marginalizing by A,

$$\begin{aligned} P(B = b, C = c) &= \sum_{a \in A} P(A = a, B = b, C = c) \\ &= P(A = 1, B = b, C = c) + P(A = 0, B = b, C = c) \\ &= 0.25 \end{aligned}$$

By the definition of conditional probability,

$$P(A = a \mid B = b, C = c) = \frac{P(A = a, B = b, C = c)}{P(B = b, C = c)} = 0.5$$

and

$$\begin{aligned} P(A = a \mid C = c) &= \frac{\sum_b P(A = a, B = b, C = c)}{\sum_b P(B = b, C = c)} \\ &= \frac{0.125 + 0.125}{0.25 + 0.25} \\ &= 0.5 \end{aligned}$$

Thus, $P(A = a \mid B = b, C = c) = P(A = a \mid C = c) = 0.5 \; \forall \; a \in A, b \in B, c \in C$ so A and B must be conditionally independent given C.

Question 2. (50 pts) Whiskers the family cat has gotten away again, however this is a fairly common occurrence, so you know that she is either in the alley (A) or at the park (P). You plan to use your camera-equipped drone to quickly search for her, however, since it is late 2024 in New Jersey, you want to avoid being sited and contributing to conspiracy theories.

- You must take off from home (H), and visit both sites (you do not need to account for returning home).
- You seek the policy that minimizes the expected battery usage.
- Due to obstacles and wind the amount of battery used is not proportional to the distance, but instead is given in the table below.
- There is also a probability the drone encounters a person along the route given in the second table below. Since a person may arrive or leave, the presence of a person is independent from time step to time step. If the drone encounters a person, it will return to the location it just left from but will have used the same amount of battery as if it had arrived at the desired location.

Battery usage table

To:		A	P
	H	1	2
From:	A	0	1
	P	1	0

Probability of encountering a person

To:		A	P
	H	0.3	0.1
From:	A	0.0	0.2
	P	0.1	0.0

a) Formulate this as a Markov decision process by writing down the state space, action space, transition probabilities, reward function. You can choose any discount factor 0.9 or above.

Solution: Let F designate the finish state reached when both the park and the alley are searched.

$$S = \{H, A, P, F\}$$

$$A = \{A, P\}$$

$$R(s,a) = \begin{cases} -1 & \text{if } s \in \{H, P\} \text{ and } a = A \\ -1 & \text{if } s = A \text{ and } a = P \\ -2 & \text{if } s = H \text{ and } a = P \\ 0 & \text{o.w.} \end{cases}$$

$$T^A = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^P = \begin{bmatrix} 0.1 & 0 & 0.9 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gamma = 0.9$$

b) Write down the following policies as functions of the states that you defined. Make sure to define an action for every state.

- Policy 1: Go to the alley first, then the park.
- Policy 2: Go to the park first, then the alley.

Solution:

$$\pi_1(s) = \begin{cases} A & \text{if } s = H \\ P & \text{o.w.} \end{cases}$$

$$\pi_2(s) = \begin{cases} P & \text{if } s = H \\ A & \text{o.w.} \end{cases}$$

c) Which of the two policies is better? Justify your answer by showing which policy will accrue the most reward.

Solution: Using the Bellman expectation equation,

$$\vec{U}^{\pi} = (I - \gamma * T^{\pi})^{-1}\vec{R}^{\pi}$$

and the following T and R matrices for each policy,

$$T^{\pi_1} = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^{\pi_1} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$T^{\pi_2} = \begin{bmatrix} 0.1 & 0 & 0.9 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^{\pi_2} = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

we get the following expected values for each policy.

$$U^{\pi_1} = [-2.42, -1.22, 0, 0]^T$$
$$U^{\pi_2} = [-3.18, 0, -1.10, 0]^T$$

Policy 1 has a lower expected fuel cost for state H, so it is the better policy.

Question 3. (10 pts) You are evaluating two MDP policies through simulation. For policy 1, you run 2000 simulations, recording the total reward, $\hat{u}$, obtained in each simulation. The sample standard deviation for this set of simulations is 50. For policy 2, you run 300 simulations, again recording the total reward of each and find that the sample standard deviation is 15. For which policy do you have a better estimate of the expected total reward, $U(\pi)$? Justify your answer.

Solution: Given $N_{\pi_1} = 2000$, $\text{std}(\hat{u}_{\pi_1}) = 50$, $N_{\pi_2} = 300$, and $\text{std}(\hat{u}_{\pi_2}) = 15$.

$$SEM_{\pi_i} = \frac{\text{std}(\hat{u}_{\pi_i})}{\sqrt{N_{\pi_i}}}$$

Calculating the standard error of the mean, $SEM_{\pi_1} \approx 1.12$, $SEM_{\pi_2} \approx 0.866$. Since $SEM_{\pi_1} > SEM_{\pi_2}$, we have a better estimate of $\boxed{U(\pi_2)}$.

Question 4. (10 pts) Consider an MDP with states that are integers. Policy π_1 has value $U^{\pi_1}(1) = 10.0$ and $U^{\pi_1}(2) = 12.0$. Policy π_2 has value $U^{\pi_2}(1) = 11.0$ and $U^{\pi_2}(2) = 10.0$. Is it possible that either one of these policies is an optimal policy? Justify your answer.

Solution: Since the Bellman operator is a contraction mapping, U^* is unique and an optimal policy will have $U^{\pi^*}(s) = U^*(s)$ for all states. $U^*(s) \geq U^\pi(s)$ for all policies π. Since $U^{\pi_1}(1) < U^{\pi_2}(1) \leq U^*(1)$, π_1 is not optimal. Similarly, since $U^{\pi_2}(2) < U^{\pi_1}(2) \leq U^*(2)$, π_2 is not optimal.