Probability and Random Variables

Concepts

1. Utility and Probability
2. Random Variables
3. Relationships between Random Variables

Utility and Probability

Consider events A and B:

Utility indicates preference

$U(A) > U(B)$	Indicates A is *preferable* to B
$U(A) = U(B)$	Indicates *indifference* between A and B

Probability indicates plausibility

$P(A) > P(B)$	Indicates A is *more plausible* (or likely) than B
$P(A) = P(B)$	Indicates A is *equally as plausible* (or equally likely) as B

Full story: https://projecteuclid.org/journals/statistical-science/volume-1/issue-3/The-Axioms-of-Subjective-Probability/10.1214/ss/1177013611.full

What is a Random Variable?

R.V. X

Vocabulary/Notation

Term	Definition	Coinflip Example
support(X) $x \in X$	All the values that X can take	$\{h, t\}$ or $\{0, 1\}$ "Binary random variable"
Distribution	Maps each value in the support to a real number indicating its probability	$\text{Bernoulli}(0.6)$ $P(X = 1) = 0.6$ $P(X = 0) = 0.4$
Expectation $E[X]$	First moment of the random variable, "mean"	$E[X] = \sum_{x \in X} xP(x)$ $= 0.5$

$P(X)$ is a table

X	**P(X)**
0	0.4
1	0.6

Distributions of related R.V.s

Joint Distribution

$$P(X, Y, Z)$$

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	$P(X \mid Y=1, Z=1)$
0	0.84
1	0.16

Marginal Distribution

$$P(X)\ P(Y)\ P(Z)$$

X	$P(X)$
0	0.85
1	0.15

Y	$P(Y)$
0	0.45
1	0.55

Z	$P(Z)$
0	0.20
1	0.80

Distributions of related R.V.s

Joint Distribution

$P(X, Y, Z)$

Conditional Distribution

$P(X \mid Y, Z)$

Marginal Distribution

$P(X)\,P(Y)\,P(Z)$

3 Rules

(Burrito-level)

1) a) $0 \leq P(X \mid Y) \leq 1$

   b) $\sum_{x \in X} P(x \mid Y) = 1$

2) "Law of total probability"

$$P(X) = \sum_{y \in Y} P(X, y)$$

3) Definition of Conditional Probability

$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)}$$

Joint $\rightarrow$ Marginal

Joint + Marginal $\rightarrow$ Conditional

Marginal + Conditional $\rightarrow$ Joint

$$P(X, Y) = P(X|Y)\,P(Y)$$

Distributions of related R.V.s

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Naive Inference

Three Random Variables: A, B, C (Works for any number)

Want to find $P(\underbrace{A=a}_{\text{query}} \mid \underbrace{B=b}_{\text{evidence}})$

C is a "hidden variable"

1. Determine the joint distribution $P(A, B, C)$.
2. Marginalize over hidden and query variables to get

$$P(A=a, B=b) = \sum_{c} P(A=a, B=b, C=c)$$

and

$$P(B=b) = \sum_{a,c} P(A=a, B=b, C=c)$$

3. $P(A=a \mid B=b) = \frac{P(A=a, B=b)}{P(B=b)}$

(Book introduces unnormalized "factors", but process is the same.)

1) a) $0 \leq P(X \mid Y) \leq 1$

b) $\sum_{x \in X} P(x \mid Y) = 1$

2) $P(X) = \sum_{y \in Y} P(X, y)$

3) $P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$

$$P(X, Y) = P(X|Y)\, P(Y)$$

Break

- $P \in \{0, 1\}$: Powder Day
- $C \in \{0, 1\}$: Pass Clear
- 1 in 5 days is a powder day
- The pass is clear 8 in 10 days
- If it is a powder day, there is a 50% chance the pass is blocked

- Write out the joint probability distribution for P and C.
- Suppose it is a non-powder day, what is the probability that the pass is blocked?

Bayes Rule

- Know: $P(B \mid A)$, $P(A)$, $P(B)$
- Want: $P(A \mid B)$

Definitions: Conditional Expectation and Independence

Definition: The conditional expectation of X given Y is

$$E[X \mid Y] = \sum_x x\, P(X = x \mid Y)$$

(function from values of Y to expectations of X)

Definition: X and Y are *independent* iff $P(X, Y) = P(X)\, P(Y)$

$X \perp Y$

$P(X \mid Y) = P(X)$

Definition: X and Y are *conditionally independent* given Z iff $P(X, Y \mid Z) = P(X \mid Z)\, P(Y \mid Z)$

$X \perp Y \mid Z$

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