Stochastic Processes and Simple Decisions

Review

Guiding Question

- What does "Markov" mean in "Markov Decision Process"?
- How do we find an optimal action based on maximizing expected utility?

Stochastic Process

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$$P(X_{t+1} \mid X_{0:t}) = \begin{cases} 0.5 & \text{if } X_{t+1} = X_t \\ 0.5 & \text{if } X_{t+1} = X_t + 1 \\ 0 & \text{otherwise} \end{cases}$$

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Bayes Net

Trajectories

Simulating a Stochastic Process

030-Stochastic-Processes.ipynb

A More Complex Example

Markov Process

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$$P(S_{t+1} \mid S_{0:t}) = P(S_{t+1} \mid S_t)$$
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- S_t is called the "state" of the process

Dynamic Bayesian Networks

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Assume:

- The population mixes thoroughly (i.e. there are no geographic considerations).
- COVID patients may be contagious up to 2 weeks after they contract the disease.
- Researchers have determined a probabilistic model for the number of new cases given the number of people in the first week of the disease and the number of people in the second week of the disease.

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- $U([S_1 : p_1; \dots ; S_n : p_n]) = \sum_{i=1}^{n} p_i U(S_i)$

Decision Networks

Maximizing Expected Utility

Value of Information

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