Continuous Space MDPs

Last Time

- Neural Network Function Approximation

Guiding Questions

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- What tools do we have to solve MDPs with continuous S and A?

Current Tool-Belt

Today: Four Tools

Notation: Continuous Random Variables

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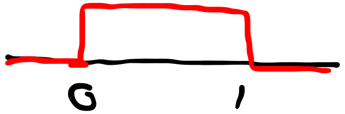 </td></tr>
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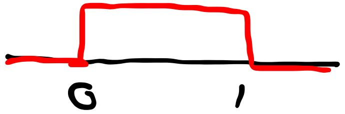
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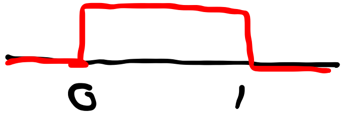
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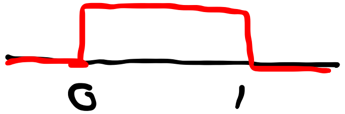
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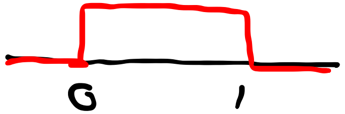
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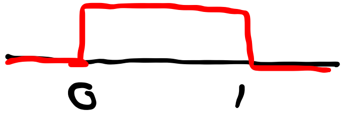
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Multivariate Gaussian Distribution

Joint Distribution

Conditional Distribution

Marginal Distribution

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The old rules still work!

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$$\begin{aligned} U_{t+1}^*(s) &= \max_a \left(R(s,a) + \gamma E\left[U_t^*(s)\right]\right) \\ &= \max_a \left(s^\top R_s s + a^\top R_a a + \int p(w) U_t(T_s s + T_a a + w)\, dw\right) \end{aligned}$$

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Practical Implication: If a continuous problem has roughly linear dynamics, a convex cost function, and roughly zero-mean additive noise, you can use *certainty-equivalent control*, i.e. control as if there is no noise.

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while not converged
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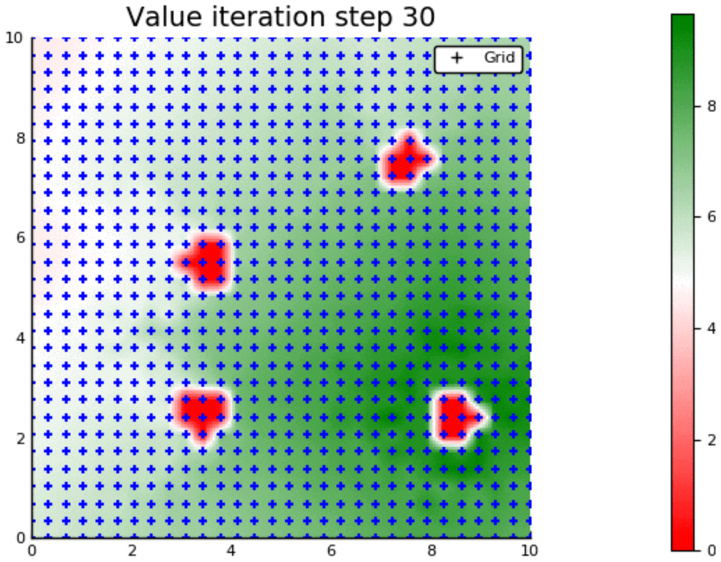
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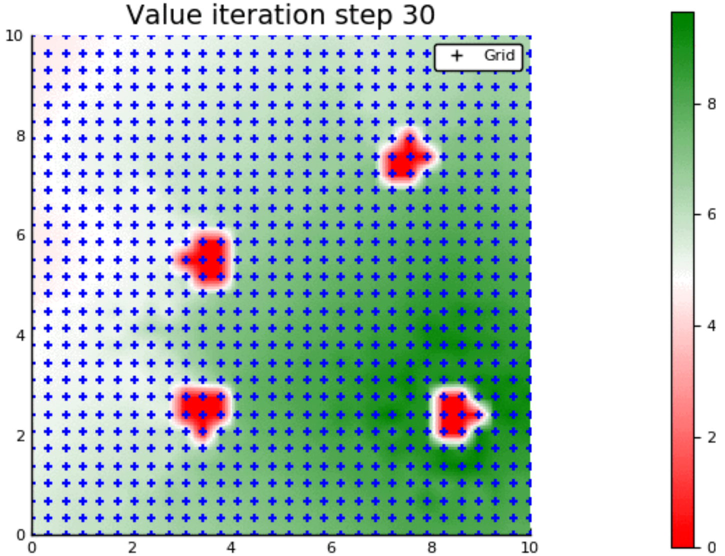
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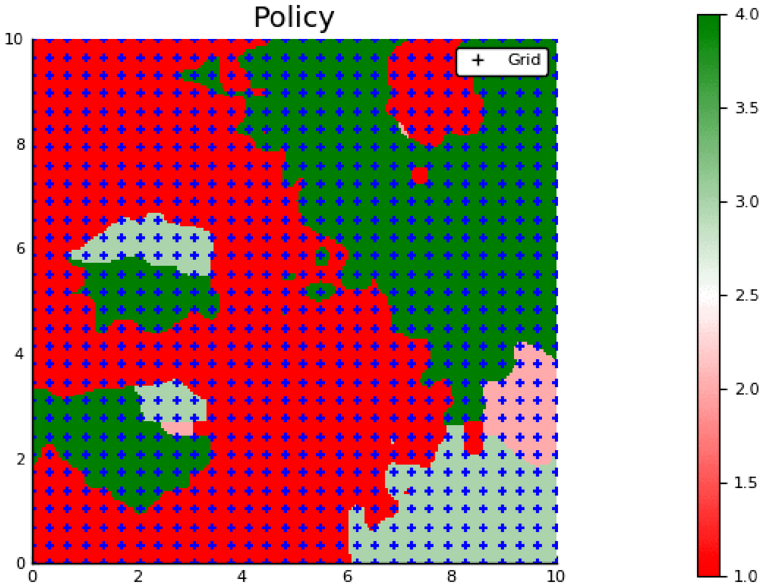
Policy

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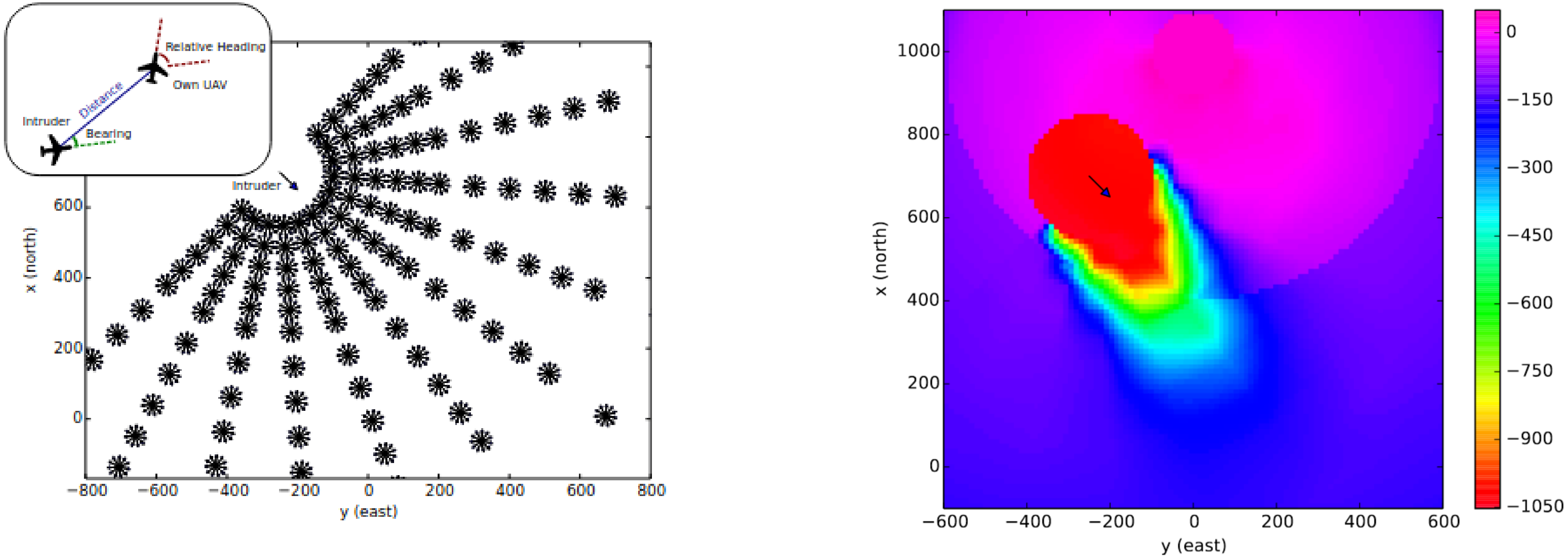

Break

What will a Monte Carlo Tree Search tree look like if run on a problem with continuous spaces?

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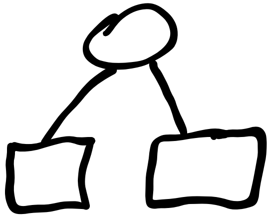

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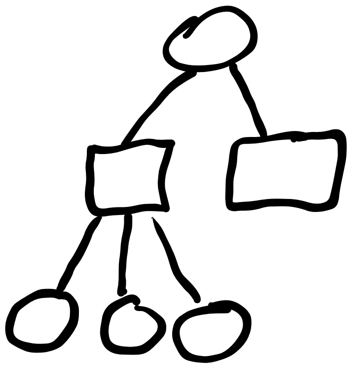

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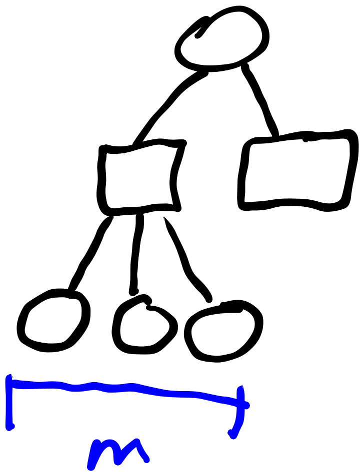

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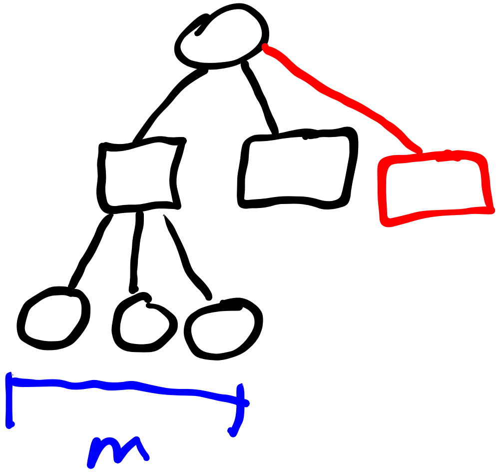

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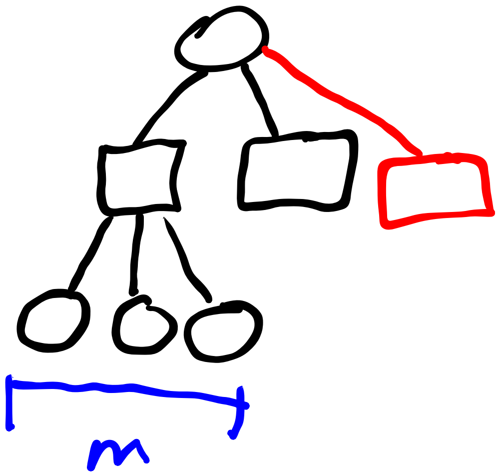

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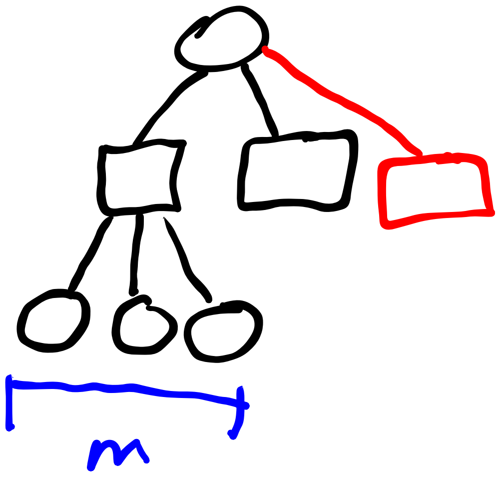

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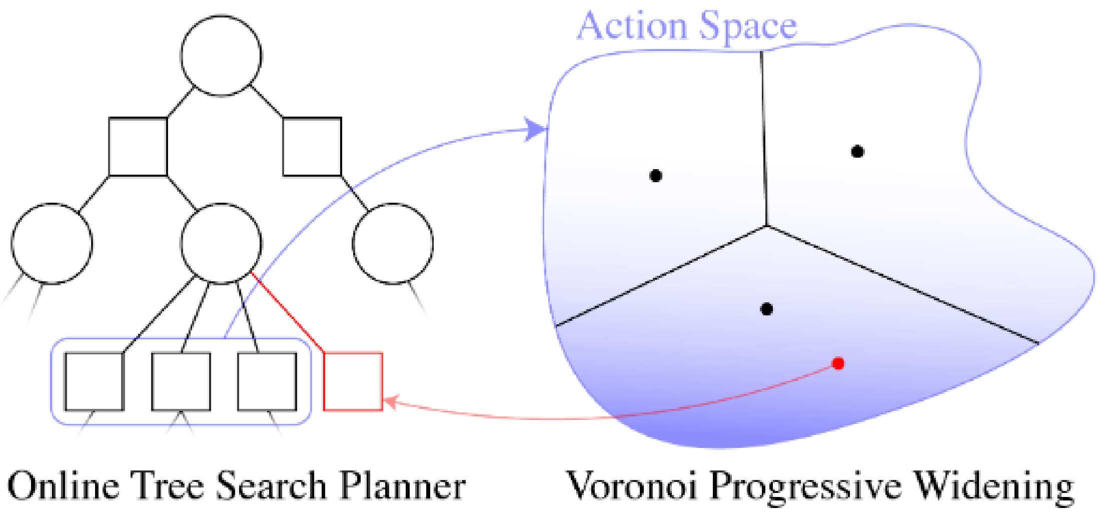

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Hindsight Optimization

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Guiding Questions

- What tools do we have to solve MDPs with continuous S and A?