Policy Gradient

Last Time

- Bandits

Guiding Questions

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- What is Policy Optimization?
- What is Policy Gradient?

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- What is Policy Optimization?
- What is Policy Gradient?
- What tricks are needed for it to work effectively?

Map

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Challenges in RL

- Exploration and Exploitation
- Credit Assignment
- Generalization

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Policy Optimization

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$$\underset{\pi}{\text{maximize}} \underset{s \sim b}{E} \left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \mid s_0 = s, a_t = \pi(s_t) \right]$$

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Two approximations:

Policy Optimization

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Two approximations:

1. Parameterized stochastic policies

$$\underset{\theta}{\text{maximize}} \quad U(\pi_\theta) = U(\theta)$$

$$a \sim \pi_\theta(a \mid s)$$

Policy Optimization

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2. Monte Carlo Utility

$$U(\pi) \approx \frac{1}{m} \sum_{i=1}^{m} R(\tau^{(i)})$$

trajectory:

$$\tau = (s_0, a_0, r_0, s_1, a_1, r_1, \ldots s_d, a_d, r_d)$$

Policy Optimization

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Two classes of optimization algorithms:

Policy Optimization

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Two classes of optimization algorithms:

1. Zeroth order (use only $U(\theta)$)

2. First order (use $U(\theta)$ and $\nabla_\theta U(\theta)$)

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Common zeroth-order aproaches:

1. Genetic Algorithms
2. Pattern Search
3. Cross-Entropy

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Cross Entropy:

Initialize d

loop:

population $\leftarrow$ sample(d)

elite $\leftarrow m$ with highest $U(\theta)$

$d \leftarrow$ fit(elite)

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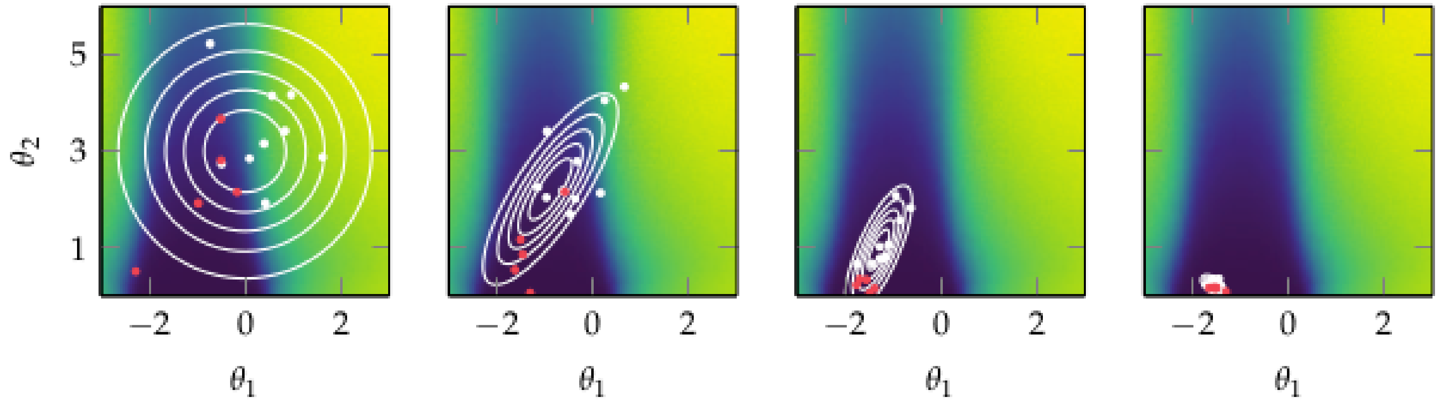
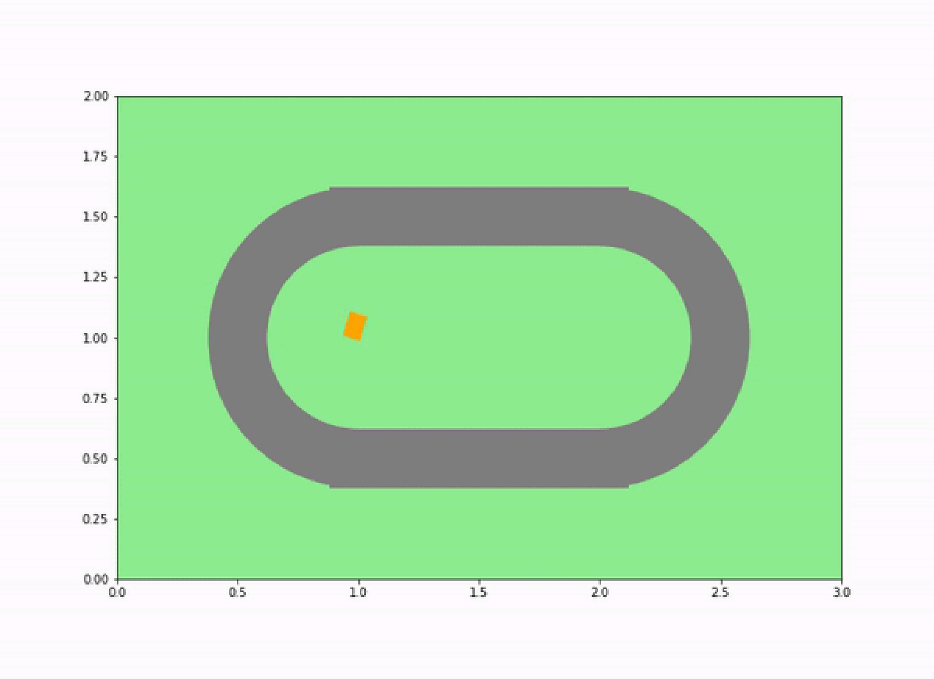


2. First Order Optimization

- Definition of Gradient
- Gradient Ascent
- Stochastic Gradient Ascent

Tricks

Tricks



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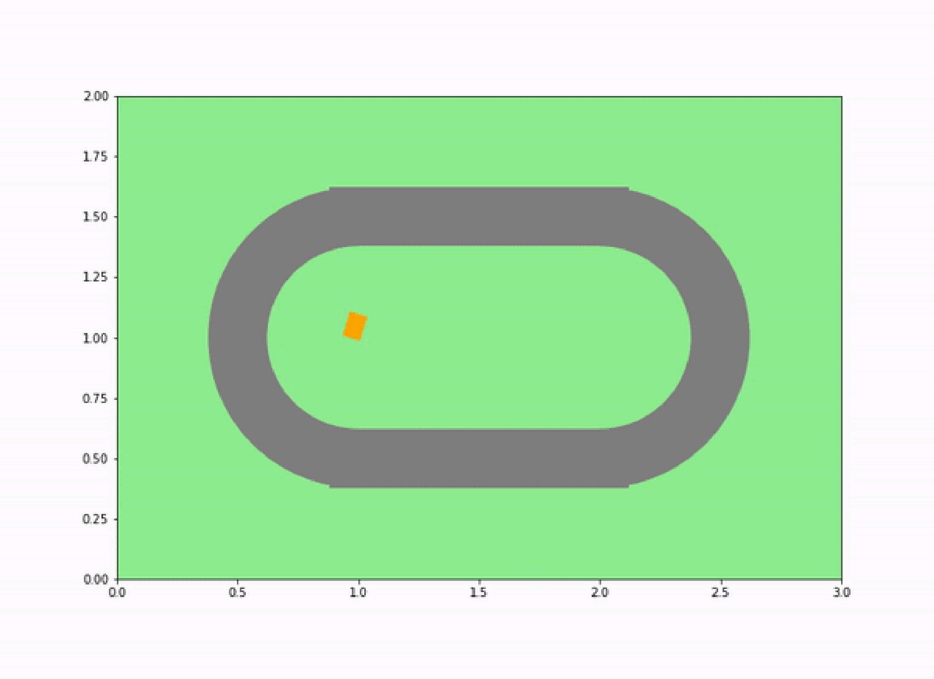


For policy gradient, 3 tricks

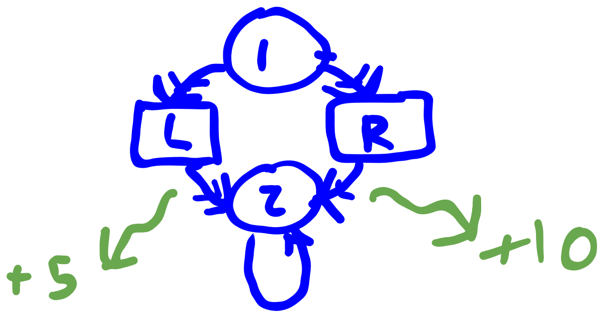
- Likelihood Ratio/Log Derivative
- Reward to go
- Baseline Subtraction

Log Derivative

Trajectory Probability Gradient

Example

$A = \{L, R\}$



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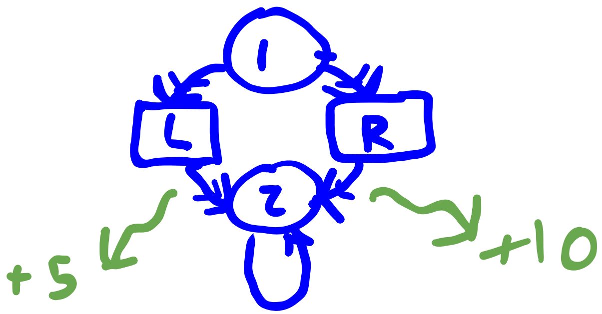

$$\pi_\theta(a = L \mid s = 1) = \text{clamp}(\theta, 0, 1)$$

$$\pi_\theta(a = R \mid s = 1) = \text{clamp}(1 - \theta, 0, 1)$$

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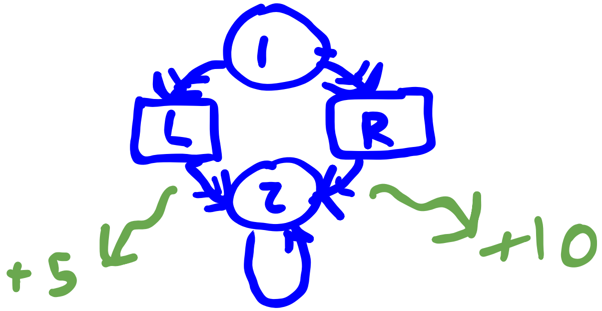

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$$\nabla U(\theta) = \mathrm{E}\left[\sum_{k=0}^{d} \nabla_\theta \log \pi_\theta(a_k \mid s_k) R(\tau)\right]$$

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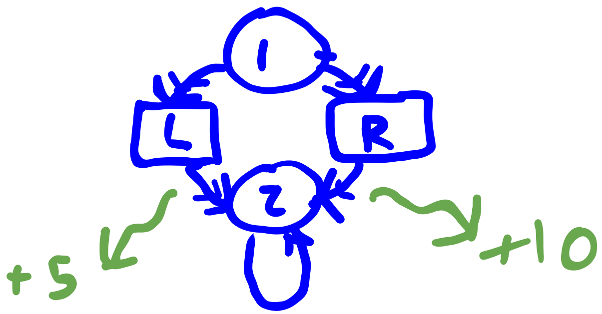

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$$\nabla U(\theta) = \mathrm{E}\left[\sum_{k=0}^{d} \nabla_\theta \log \pi_\theta(a_k \mid s_k) R(\tau)\right]$$

Given $\theta = 0.2$ calculate $\sum_{k=0}^{d} \nabla_\theta \log \pi_\theta(a_k \mid s_k) R(\tau)$ for two cases, (a) where $a_0 = L$ and (b) where $a_0 = R$

Policy Gradient

Policy Gradient

loop

$\tau \leftarrow \text{simulate}(\pi_\theta)$

$\theta \leftarrow \theta + \alpha \sum_{k=0}^{d} \nabla_\theta \log \pi_\theta(a_k \mid s_k) R(\tau)$

Policy Gradient

loop

$\tau \leftarrow \text{simulate}(\pi_\theta)$

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On Policy!

Causality

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$$
\begin{aligned}
\nabla U(\theta) &= \mathrm{E}\left[\sum_{k=0}^{d} \nabla_\theta \log \pi_\theta(a_k \mid s_k) R(\tau)\right] \\
&= \mathrm{E}\left[\left(\sum_{k=0}^{d} \nabla_\theta \log \pi_\theta(a_k \mid s_k)\right)\left(\sum_{k=0}^{d} \gamma^k r_k\right)\right]
\end{aligned}
$$

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&= \mathrm{E}\left[(f_0 + \ldots + f_d)\left(\gamma^0 r_0 + \ldots \gamma^d r_d\right)\right]
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Baseline Subtraction

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does not bias
(proof in book)

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$$r_{\text{base},i} = \frac{\mathbb{E}_{a,s,r_{\text{to-go}},k}\left[\ell_i(a,s,k)^2 r_{\text{to-go}}\right]}{\mathbb{E}_{a,s,k}\left[\ell_i(a,s,k)^2\right]}$$

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$$\ell_i(a,s,k) = \gamma^{k-1} \frac{\partial}{\partial \theta_i} \log \pi_\theta(a \mid s)$$

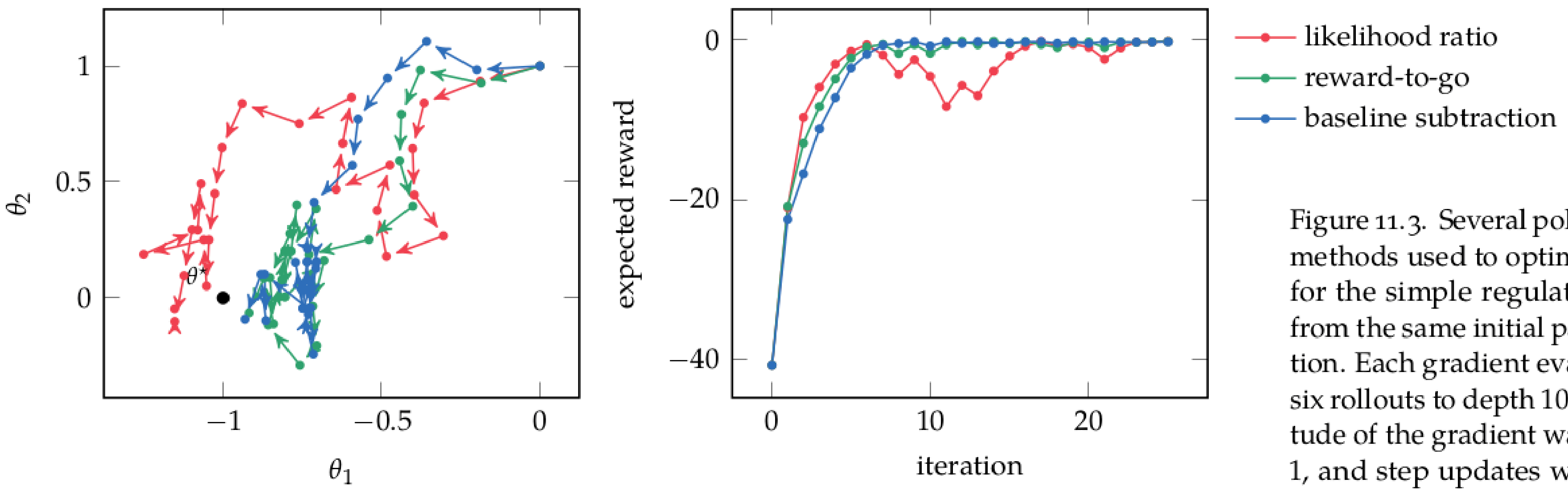

Figure 11.3. Several policy gradient methods used to optimize policies for the simple regulator problem from the same initial parameterization. Each gradient evaluation ran six rollouts to depth 10. The magnitude of the gradient was limited to 1, and step updates were applied with step size 0.2. The optimal policy parameterization is shown in black.

Guiding Questions

- What is Policy Gradient?
- What tricks are needed for it to work effectively?