Continuous Space MDPs

Last Time

- Neural Network Function Approximation

Guiding Questions

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- What tools do we have to solve MDPs with continuous S and A?

Current Tool-Belt

Today: Four Tools

Notation: Continuous Random Variables

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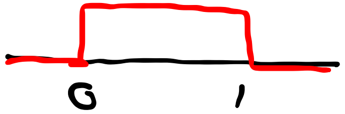 </td></tr>
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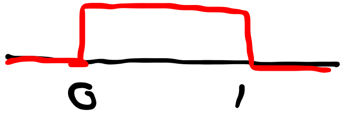
Distribution - Discrete: PMF - Continuous: PDF	Maps each value in the support to a real number indicating its probability	$P(X = 1) = 0.5$ $P(X = 0) = 0.5$ $P(X)$ is a table	$p(x) = \mathbf{1}_{[0,1]}(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ o.w.} \end{cases}$  $P(X \in [a, b]) = \int_a^b p(x)dx$
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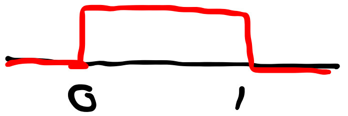

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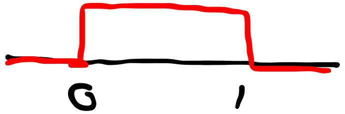
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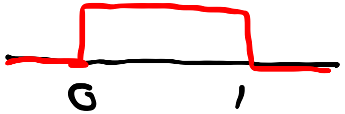
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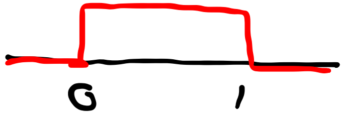

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2) $P(X) = \sum_{y \in Y} P(X, y)$	
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Multivariate Gaussian Distribution

Joint Distribution

Conditional Distribution

Marginal Distribution

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The old rules still work!

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$$\begin{aligned} U_{t+1}^*(s) &= \max_a \left(R(s,a) + \gamma E\left[U_t^*(s)\right]\right) \\ &= \max_a \left(s^\top R_s s + a^\top R_a a + \int p(w) U_t(T_s s + T_a a + w)\, dw\right) \end{aligned}$$

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a^* is where $\nabla_a(\text{max term}) = 0$

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<u>Certainty-Equivalence Principle</u>: For Linear-Quadratic problems, the optimal policy with noise is the same as the optimal policy without noise!

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<u>Certainty-Equivalence Principle</u>: For Linear-Quadratic problems, the optimal policy with noise is the same as the optimal policy without noise!

Practical Implication: If a continuous problem has roughly linear dynamics, a convex cost function, and roughly zero-mean additive noise, you can use *certainty-equivalent control*, i.e. control as if there is no noise.

2. Value Function Approximation

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Fitted Value Iteration

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while not converged
    $\theta \leftarrow \theta'$
    $\hat{V}' \leftarrow B_{\text{approx}}[V_\theta]$
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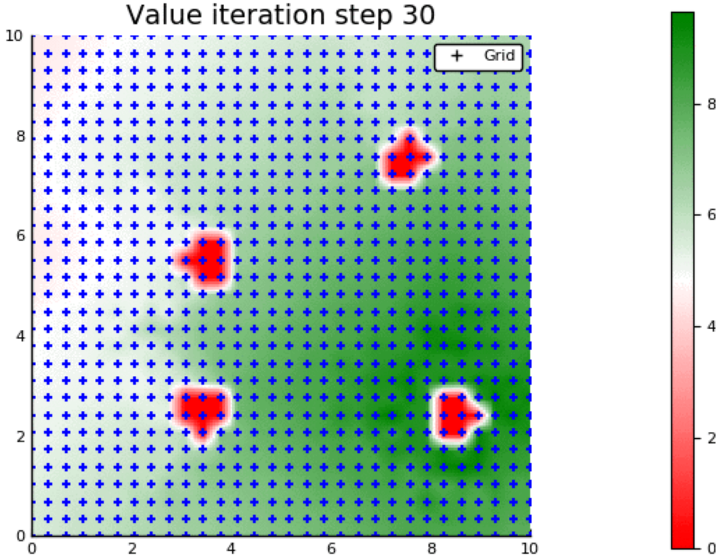

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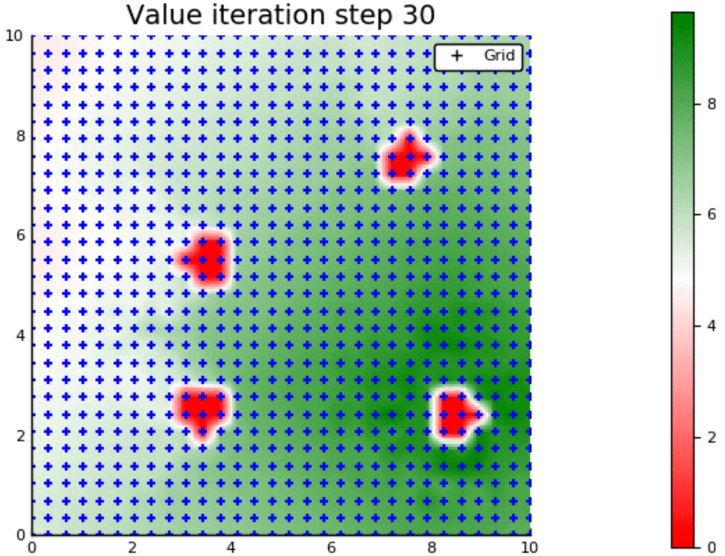

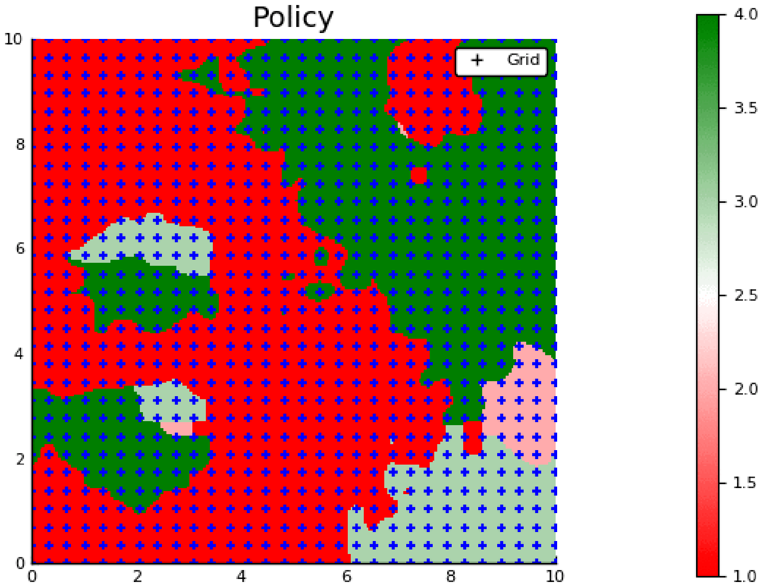

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Function Approximation

Weighting of 2^d points

Weighting of only $d + 1$ points!

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- Global: (e.g. Fourier, neural network)
- Local: (e.g. simplex interpolation)

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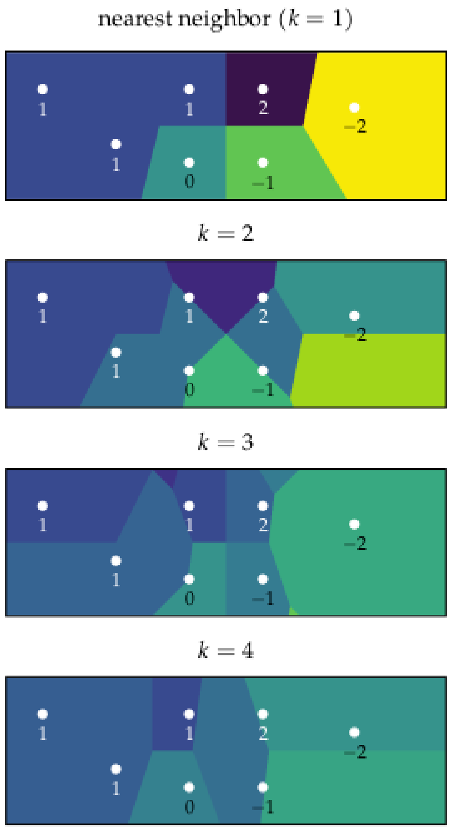
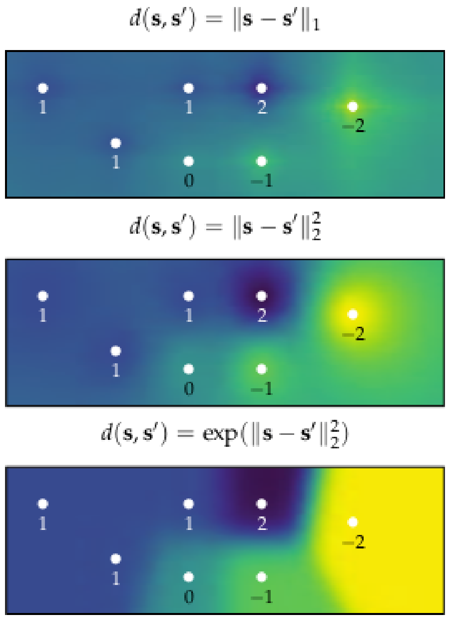


nearest neighbor ($k = 1$)

$k = 2$

$k = 3$

$k = 4$



$d(\mathbf{s}, \mathbf{s}') = \|\mathbf{s} - \mathbf{s}'\|_1$

$d(\mathbf{s}, \mathbf{s}') = \|\mathbf{s} - \mathbf{s}'\|_2^2$

$d(\mathbf{s}, \mathbf{s}') = \exp(\|\mathbf{s} - \mathbf{s}'\|_2^2)$

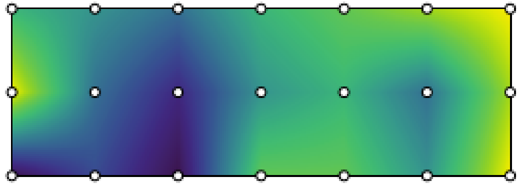


Figure 8.9. Two-dimensional linear interpolation over a 3×7 grid.

Weighting of 2^d points

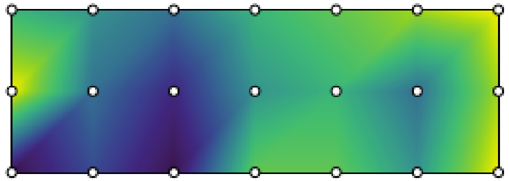
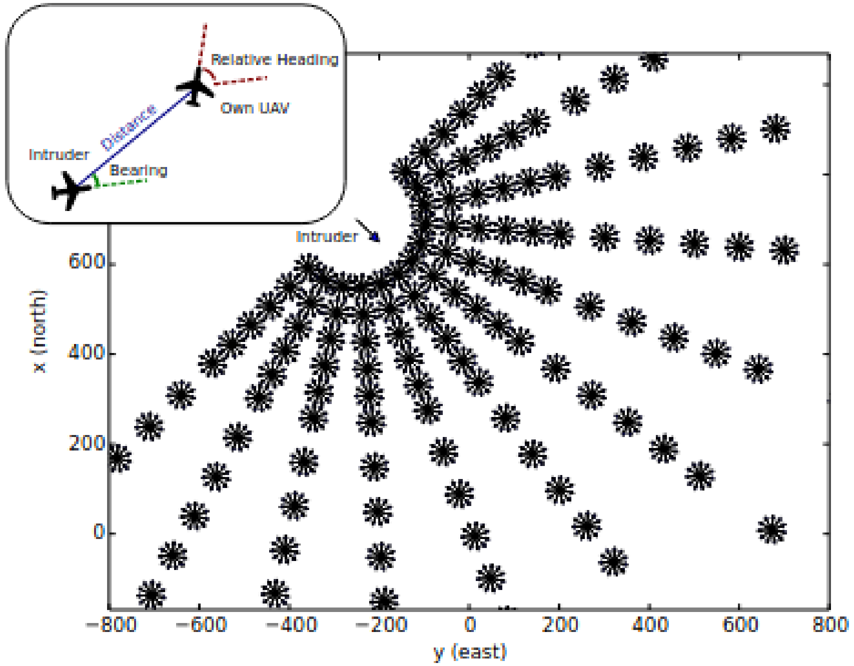
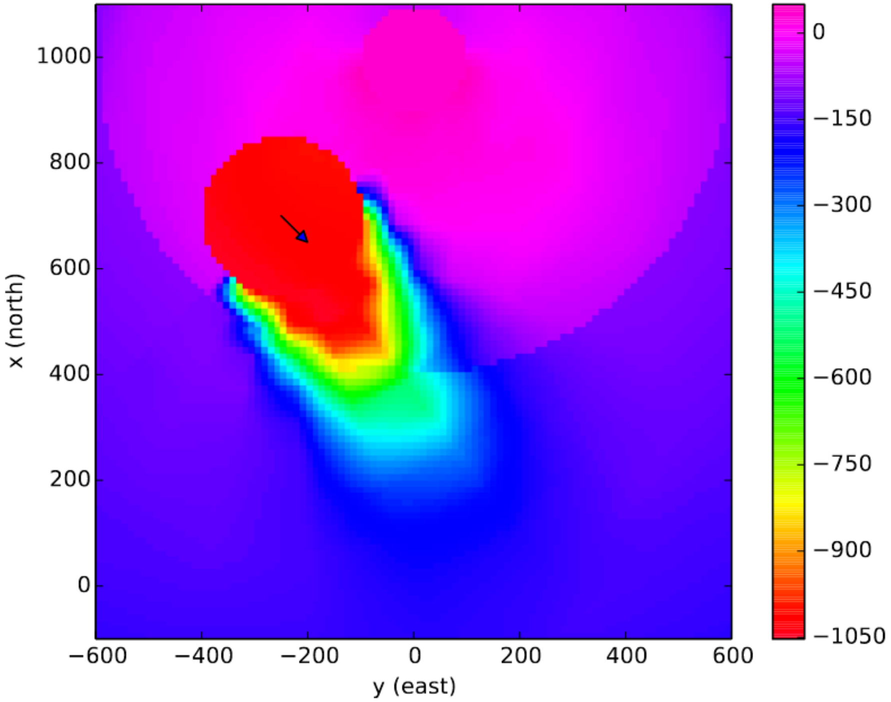


Figure 8.10. Two-dimensional simplex interpolation over a 3×7 grid.

Weighting of only $d + 1$ points!

2. Value Function Approximation





Break

What will a Monte Carlo Tree Search tree look like if run on a problem with continuous spaces?

3. Sparse Tree Search/Progressive Widening

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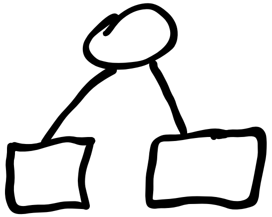

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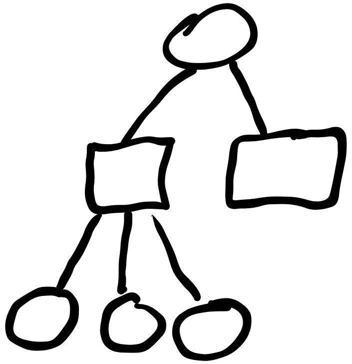

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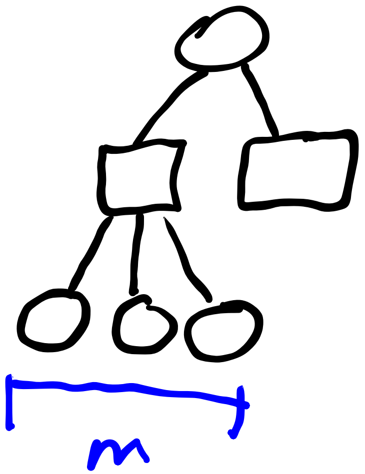

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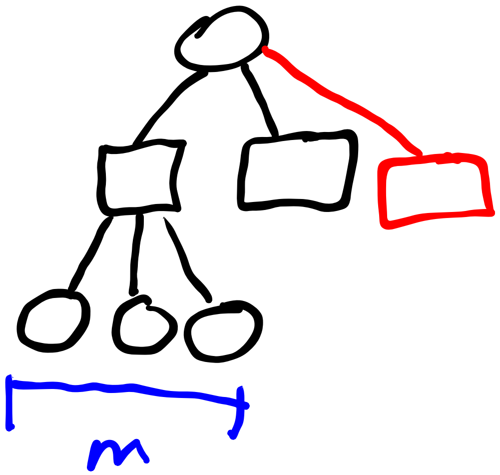

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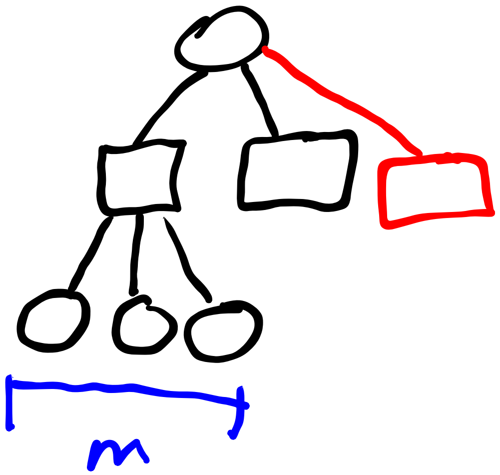

add new branch if $C < kN^{\alpha}$ $\quad (\alpha < 1)$

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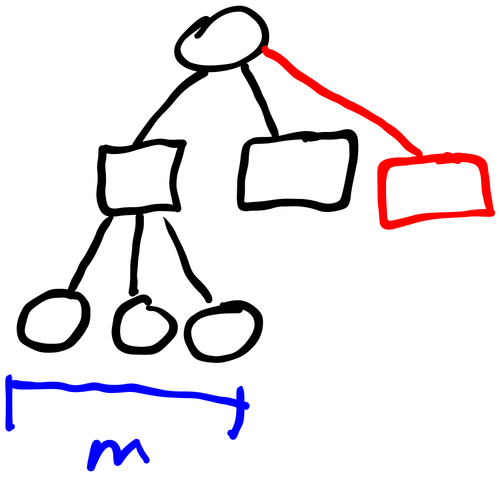

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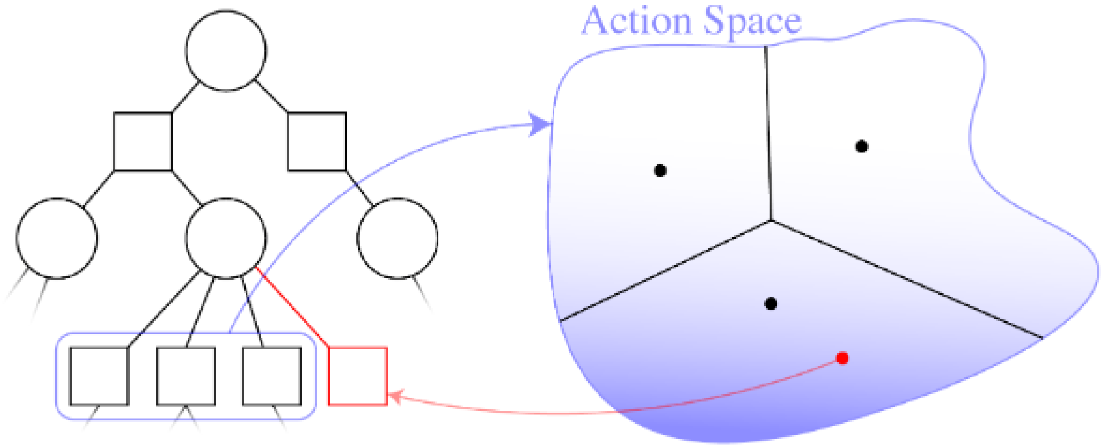

Online Tree Search Planner

Voronoi Progressive Widening

4. Model Predictive Control

(Use off-the-shelf optimization software, e.g. Ipopt)

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Certainty-
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$$\begin{aligned} \underset{a_{1:d}, s_{1:d}}{\text{maximize}} \quad & \sum_{t=1}^{d} \gamma^t R(s_t, a_t) \\ \text{subject to} \quad & s_{t+1} = \mathrm{E}_{s' \sim T(s' \mid s, a)}[s'] \quad \forall t \end{aligned}$$

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Open-Loop

$$\begin{aligned} \underset{a_{1:d}, s_{1:d}^{(1:m)}}{\text{maximize}} \quad & \frac{1}{m} \sum_{i=1}^{m} \sum_{t=1}^{d} \gamma^t R(s_t^{(i)}, a_t) \\ \text{subject to} \quad & s_{t+1} = G(s_t^{(i)}, a_t, w_t^{(i)}) \quad \forall t, i \end{aligned}$$

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Hindsight Optimization

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Guiding Questions

- What tools do we have to solve MDPs with continuous S and A?