On One-Off Decisions in Social Networks

By Frank Seidl

Department of Applied Mathematics, University of Colorado Boulder

frse8961@colorado.edu

Summary

I planned to approximate perfect Bayesian agents in a recently-proposed Markov model for decision-making in a social network. The algorithm I intended to use relied on a false assumption, and would not have approximated Bayes' Rule. Therefore, I do not offer experiments of agents playing according to this algorithm. Instead, I rigorously define the model in more generality than the original authors, and offer a simple proof about its dynamics. I proceed to examine the difficulties of simulating Bayesian agents, providing reason to suspect that the exact posteriors are NP-hard to compute. Finally, I explain the subtleties that undermine the false assumption, and propose alternative lines of simulation research.

1. Introduction

Consider a network of agents individually making decisions about their environment under uncertainty. They each receive private observations which contain information about the environment. While they do not share this information explicitly with their counterparts, each agent is able to make inferences about the private observations of others based on how they behave. *The agents have no mutual or competing interests*, so each agent's desired behavior depends only on her beliefs about the stochastic environment, not on the choices of her peers. The economics and psychology literature contain multiple dynamic models for this kind of scenario.

Researchers in the past decade have studied group decision-making in partially observable Markov games (POMGs). (3; 5) In these models, agents repeatedly choose actions from an action set based on private knowledge and the past actions of their neighbors in a network. If actions are chosen *myopically*, with infinite discounting of future rewards, such a POMG reduces to a Markov chain with no strategic complexity. (3) In this case, each agent's action can be reinterpreted as simply announcing (part of) his belief, conditioned on the beliefs his neighbors have announced to him previously. The dynamics of rational agents' beliefs in such situations have been studied at least since the 1970s. (7) Previous authors have shown that their agents' actions and posteriors, after sufficient mutual exchange of social information, converge to agreement with those of their neighbors. (5; 7) Yet, computing the posteriors and actions of each agent before this convergence occurs has been show to be NP-hard in a general social network. (3)

A newer line of research (1; 2) explores models of group decisions in which each agent tries to decide between two exhaustive and mutually exclusive hypotheses about the world, H^+ and $H^- = \neg H^+$. They gather private evidence independently, and each agent makes a decision in favor of one of the hypotheses once her confidence exceeds a threshold. Like the actions of an agent in a myopic POMG or earlier model, this decision is visible to neighboring agents, who may be influenced to decide (or remain undecided) because of her choice. However, unlike previous models, once she decides, she does not continue to communicate with her peers. Therefore in this work, I distinguish such models from their predecessors by calling them "one-off" decision models of social networks. The following sections will define and explore in-depth the particular one-off decision model used in (1).

2. One-Off Group Decision Model

In this section I define the model proposed in (1), from which I borrow notation. In actuality, I define a more general model. However, each example in (1) is consistent with the definition I provide.

What I refer to as a one-off decision model is defined by a directed social graph $\mathcal{G}$ and two probability mass functions f_+, f_- on a finite set Ξ. The agents $1, \ldots, V$ are the vertices of $\mathcal{G}$, and they each begin at time $t = 0$ undecided, with a neutral prior between two opposite hypotheses H^+ and H^-. At each time $t = 1, 2, \ldots$, each undecided i agent makes a private observation $\xi_t^{(i)} \in \Xi$, which is distributed, conditional on the true hypothesis H^+ or H^-, as

$$P(\xi_t^{(i)} = \xi | H^\pm) = f_\pm(\xi). \tag{1}$$

The $\xi_t^{(i)}$ are pairwise independent across i and t. After private observations are made, an "equilibration" process ensues, in which undecided agents update their posteriors, then those with sufficiently high posteriors for either H^+ or H^- make a decision, announcing to their downstream neighbors in the graph $\mathcal{G}$ which hypothesis they believe is likely. The decisions (and lack of decisions) by neighbors cause still-undecided agents to update their posteriors, and potentially decide on $H^\pm$, again and again, resulting in each undecided agent performing a sequence of updates before the next private observations at time $t + 1$.

At this point, more notation can be introduced. Denote by $\mathcal{U}_i$ the set of upstream neighbors of agent i, that is, those j for whom the directed edge $j \longrightarrow i$ is present in $\mathcal{G}$. Define the *decision variable* $d_{t,l}^{(i)} \in \{-1, 0, 1\}$ of agent i at time t during the l^{th} step of equilibration such that $d_{t,l}^{(i)} = \pm 1$ if i has decided on $H^\pm$, or $d_{t,l}^{(i)} = 0$ if i is undecided. Thus all of the information available to an undecided agent i at time t, step l, consists of $\xi_s^{(i)}$ for all $1 \le s \le t$, along with all $d_{s,k}^{(j)}$ for which $j \in \mathcal{U}_i$, $1 \le s \le t$, and $k < l$ if $s = t$. Borrowing a useful concept from (5), I refer to the event described by all of these observations as the *information set* $I_{t,l}^{(i)}$ of agent i at t, l. This allows me to write an equation for the log-likelihood ratio (LLR) of H^+ over H^- according to i's posterior at t, l, provided i is undecided:

$$y_{t,l}^{(i)} = \log\left(\frac{P(H^+ | I_{t,l}^{(i)})}{P(H^- | I_{t,l}^{(i)})}\right). \tag{2}$$

The levels of confidence i requires to decide on $H^\pm$ correspond to thresholds on $y_{t,l}^{(i)}$: he decides H^+ as soon as $y_{t,l}^{(i)} \ge \theta_+$, or H^- as soon as $y_{t,l}^{(i)} \le \theta_-$. Hence

$$d_{t,l}^{(i)} = \begin{cases} -1 & y_{t,l}^{(i)} \le \theta_-, \\ 0 & \theta_- < y_{t,l}^{(i)} < \theta_+, \\ 1 & \theta_+ \le y_{t,l}^{(i)}. \end{cases} \tag{3}$$

Since agents begin as undecided with neutral priors $y_{0,0}^{(i)} = 0$, it is required that $\theta_- < 0 < \theta_+$. Note that decisions are one-off, so once i decides he ceases to make observations, and his LLR $y_{t,l}^{(i)}$ and decision $d_{t,l}^{(i)}$ remain fixed forever.

The model admits a few easy modifications in the assumptions it makes about the thresholds $\theta_\pm$. Special attention has been given to the case in which thresholds are "symmetric," i.e. $\theta_+ = -\theta_- = \theta$, so agents are completely unbiased between H^+ and H^-. (1; 2) Section 5 will discuss how this assumption somewhat simplifies the dynamics. It is also worth noting that (2) found that varying thresholds θ_i between agents resulted in higher accuracy of group decisions, in the case where private evidence is accrued continually rather than in discrete time.

Another easily changeable aspect of the model is the agents' knowledge of each other. Agents are typically assumed to know the thresholds at which not only they, but also their peers, decide. But consensus-biased agents, who have different thresholds yet each assume the others to share their own threshold, have been studied. (2) Finally, it would be consistent with previous work (3) to assume that they know the exact structure of $\mathcal{G}$, although this assumption certainly does not hold for human social networks. Section 4 discusses hardness results that hold even when agents have unrealistically exact knowledge of their social network; it is probably safe to assume that agents with only a probability distribution over possible social graphs, e.g. an Erdos-Renyi prior, (9) face an even more difficult computational challenge.

3. End of Equilibration Process

In this section, I will briefly show that the equilibration at each time t terminates after finitely many steps l, and the LLRs $y_{t,l}^{(i)}$ eventually do not change as l increases. This bodes well for the prospect of simulating the one-off decision model, since it limits the amount of computation that must be performed to accurately simulate each time step.

Lemma 1. *For a fixed time t, there is some finite L such that an undecided agent i already knows $d_{t,l}^{(j)}$ for each j in $\mathcal{U}_i$ at any equilibration step $l \geq L$, meaning that the social information received on step $l+1$ is entirely redundant.*

Proof. Observe that all of i's private observations and her neighbors' decisions up to time t are determined as a function of the $\xi_s^{(j)}$ for $1 \leq s \leq t$ and $1 \leq j \leq V$, which are jointly distributed in the finite space Ξ^{Vt}. Hence the information sets $I_{t,l}^{(i)}$ correspond to subsets of Ξ^{Vt}, and since i does not forget information, they are nested: $I_{t,0}^{(i)} \supseteq I_{t,1}^{(i)} \supseteq \ldots$. A decreasing sequence of nested finite sets is eventually equal to its intersection, so there exists some L for which $I_{t,l}^{(i)} = \cap_l I_{t,l}^{(i)}$ for all $l \geq L$. Consider any $l \geq L$ and $j \in \mathcal{U}_i$. Since $d_{t,l}^{(j)}$ is determined by $I_{t,l+1}^{(i)}$, and $I_{t,l+1}^{(i)} = I_{t,l}^{(i)}$, it follows that i already knows $d_{t,l}^{(j)}$ on step l. □

There are only finitely many agents i and possible realizations of the $\xi_s^{(j)}$ for $1 \leq s \leq t$ and $1 \leq j \leq V$, so some value of L satisfies the assumptions above for every possible agent and information set at a fixed time t. Determining this number, in general, is probably difficult. The next section will examine the (in)tractability of computing the posteriors.

4. Challenges of Computing Bayesian Decisions

Though previous experimental papers have claimed to model "networks" of agents, the one-off decision model has been simulated primarily in the case where $\mathcal{G}$ is complete, so that each agent directly observes each other. It is possible to efficiently compute approximations which converge to the true posteriors for agents in large cliques. (2) But little attention has been given to the incomplete social graph case, with the exception of a few toy examples. (1) Meanwhile, for a similar POMG model from the last decade, it has been shown that simulating rational play, even for myopic agents, is NP-hard for an arbitrary social graph. (3) This hardness result depends on a polynomial time reduction from an NP-hard approximate vertex cover problem, (6) and thus is inapplicable if the social graph is limited to a complete topology. Intuitively, the intractability of the POMG when the social graph is incomplete arises from each agent's uncertainty about the extent to which information from his neighbors represents independent evidence collected by each of them, versus redundant information parroted by all of them. If he cannot witness all interactions of all agents in his vicinity, he cannot necessarily tell whether his neighbors' posteriors are based on novel evidence of their own, or on social information which he has already received. Therefore, rationally weighing their opinions requires him to marginalize over the possible courses taken by social influence through parts of his network which he cannot see.

Agents in one-off decision networks face the same challenge of distinguishing independent and redundant social evidence. For this reason I expect that their rational decisions are also NP-hard to compute. This could be shown formally with a polynomial time reduction from any problem which is known to be NP-hard. (8) However, a reduction from an NP-hard POMG featuring a network of myopic agents would be particularly satisfying, due to the intuition that both problems are hard for the same reason.

5. Two Simulations: one Inaccurate and one Expensive

Due to the aforementioned challenges of modeling perfect Bayesian agents, I began with the goal of simulating *near*-Bayesian agents, in arbitrary networks, under the assumption of symmetric thresholds, so $\theta_+ = -\theta_- = \theta$. In general, an agent i might suspect that an undecided neighbor with asymmetric thresholds has evidence against the hypothesis whose decision threshold is lower. But if all thresholds are symmetric, a basic symmetry argument shows that at times t when all neighbors of an agent i remain undecided, her social evidence is uninformative. (1) That is,

$$y_{t,l}^{(i)} = \log\left(\frac{P(H^+|\xi_{1:t}^{(i)})}{P(H^-|\xi_{1:t}^{(i)})}\right), \tag{4}$$

which is not true if i knows her neighbors to have asymmetric thresholds. With symmetry, if $j \in \mathcal{U}_i$ is the first agent in the network to decide, and does so at time t, he necessarily does so immediately after a private observation, with 0 steps of social information exchange. Due to independence of private observations between agents, i may update her posterior as

$$y_{t,1}^{(i)} = \log\left(\frac{P(H^+|\xi_{1:t}^{(i)})}{P(H^-|\xi_{1:t}^{(i)})}\right) + \log\left(\frac{P(H^+|d_{t,0}^{(j)})}{P(H^-|d_{t,0}^{(j)})}\right). \tag{5}$$

The algorithm I intended to implement relied on a generalization of Equation 5 to simplify computation of an approximate posterior LLR. I assumed that in general, independence guaranteed that i's posterior LLR always breaks into separate components from private and social evidence, i.e.

$$y_{t,l}^{(i)} = \log\left(\frac{P(H^+|\xi_{1:t}^{(i)})}{P(H^-|\xi_{1:t}^{(i)})}\right) + \log\left(\frac{P(H^+|d_{s,k}^{(j)} : j \in \mathcal{U}^i, 1 \le s \le t, s = t \implies k < l)}{P(H^-|d_{s,k}^{(j)} : j \in \mathcal{U}^i, 1 \le s \le t, s = t \implies k < l)}\right). \tag{6}$$

The first term above is easy to compute, since the joint distribution of the $\xi_t^{(i)}$ is known. And the second term can be estimated using empirical probabilities, from simple tabulated data from previous simulations, or more sophisticated learning techniques.

Unfortunately for my project, Equation 6 is not true in general, even with symmetric thresholds, because the symmetry breaks once one agent decides. (1) This is intuitively obvious (in retrospect): for instance, if i's neighbors are overwhelmingly decided towards H^+, but i remains undecided, then her private evidence must support H^-. This shows that $\xi_t^{(i)}$ may depend on $d_{t,l}^{(i)}$ after conditioning on i's knowledge, so the independence assumption underlying Equation 6 typically fails, although it holds in the case of Equation 5. It might still be revealing to study agents who incorrectly update their posteriors by Equation 6, in the tradition of research modeling group decisions of agents who perform Bayesian updates under false simplifying assumptions, e.g. (4). But I chose not to pursue such a project.

Despite the failure of Equation 6, there is still an accurate way to estimate $y_{t,l}^{(i)}$ using simulations. The empirical frequencies of $H^\pm$ within entire information sets $I_{t,l}^{(i)}$ will also converge to true probabilities, and can be used in Equation 2 to estimate LLRs. The problem with such a naive approach is not a lack of convergence, but the *rate* of convergence. The number of information sets $I_{t,l}^{(i)}$ which agents are likely to encounter grows astronomically with the size of Ξ and the size and characteristic degree of $\mathcal{G}$. Obtaining enough samples from every such $I_{t,l}^{(i)}$ to accurately estimate $P(H^\pm|I_{t,l}^{(i)})$ might be prohibitively expensive in many cases.

It may be possible to estimate posteriors more accurately and efficiently than can be achieved with either of the approaches I describe here, even with an arbitrary social graph structure. This area has received little study, and will hopefully be better addressed by subsequent work.

6. Conclusion

The one-off decision model is a recently proposed, rich model for group decision-making in social networks. Although it has been studied and simulated on complete networks (1; 2), the general case of an arbitrary network remains largely unexplored. In this project, I formally defined the one-off decision model in full generality, and proved a promising result about its equilibration. My subsequent discussion outlined two possible avenues for future study. In Section 4, I discussed the likely intractability of computing perfect Bayesian one-off decisions, and suggested seeking a polynomial time reduction from an NP-hard myopic play in partially observable Markov game like that of (3). In Section 5, I described two approaches for simulating the model, each of which could be the subject of future experiments. Finally, I called for further work on efficient approximation of agents' Bayesian posteriors.

7. Contributions and Release

I did this project alone, so all above work which I do not attribute to another source is my own. *I grant permission for this report to be posted publicly.*

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