Probability and Random Variables

Concepts

- 1. Utility and Probability
- 2. Random Variables
- 3. Relationships between Random Variables

Utility and Probability

What is a Random Variable?

R.V. X

Vocabulary/Notation

Term

support(X)

$$x \in X$$

 $X \in [0,1]$

Definition

All the values that X can take

Distribution

- Discrete: PMF
- Continuous: PDF

Maps each value in the support to a real number indicating its probability

Expectation

E[X]

Single representative value of the random variable, "mean"

Bernoulli(0.5)

Coinflip Example

$$\{h, t\}$$
 or $\{0, 1\}$

$$P(X=1)=0.5$$

$$P(X=0)=0.5$$

P(X) is a table

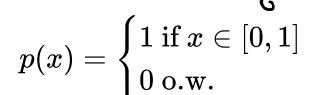
X	P(X)
0	0.5
1	0.5

$$E[X] = \sum_{x \in X} x P(x)$$

$$=0.5$$

$$\mathcal{U}(0,1)$$

Uniform Example



$$P(X=1)=0$$

$$P(X \in [a,b]) = \int_a^b p(x) dx$$

$$E[X] = \int_{x \in X} x p(x) dx$$

$$= 0.5$$

Distributions of related R.V.s

Joint Distribution

\overline{X}	Υ	Z	P(X,Y,Z)
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

Conditional Distribution

$$P(X \mid Y, Z)$$

(Distribution - valued function)

X	<i>P</i> (<i>X</i> <i>Y</i> =1, <i>Z</i> =1)
0	0.84
1	0.16

Marginal Distribution

$$\begin{array}{c|ccccc}
X & P(X) & Y & P(Y) \\
\hline
0 & 0.85 & 0 & 0.45 \\
1 & 0.15 & 1 & 0.55 \\
\hline
\hline
Z & P(Z) \\
\hline
0 & 0.20 \\
1 & 0.80 \\
\hline
\end{array}$$

Distributions of related R.V.s

Joint Distribution

Conditional Distribution

Marginal Distribution

$$P(X \mid Y, Z)$$

3 Rules

(Burrito-level)

- 1) a) $0 \le P(X \mid Y) \le 1$ b) $\sum_{x \in X} P(x \mid Y) = 1$
- 2) "Law of total probability"

$$P(X) = \sum_{y \in Y} P(X,y)$$

3) Definition of Conditional Probability

$$P(X \mid Y) = rac{P(X,Y)}{P(Y)}$$

Joint → Marginal

Joint + Marginal o Conditional Marginal + Conditional o Joint $P(X,Y)=P(X|Y)\,P(Y)$

Distributions of related R.V.s

Joint Distribution

Conditional Distribution

Marginal Distribution

$$P(X \mid Y, Z)$$

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Joint → Marginal

Joint + Marginal → Conditional

Marginal + Conditional → Joint

$$P(X,Y) = P(X|Y) P(Y)$$

1) a)
$$0 \leq P(X \mid Y) \leq 1$$
 b) $\sum_{x \in X} P(x \mid Y) = 1$

Break

2)
$$P(X) = \sum_{y \in Y} P(X, y)$$

3)
$$P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$$
 $P(X,Y) = P(X|Y) P(Y)$

- $P \in \{0,1\}$: Powder Day
- $C \in \{0,1\}$: Pass Clear
- 1 in 5 days is a powder day
- The pass is clear 8 in 10 days
- If it is a powder day, there is a 50% chance the pass is blocked
- Write out the joint probability distribution for P and C.
- What is the probability that the pass is blocked on a non-powder day?

Bayes Rule

- Know: $P(B \mid A)$, P(A), P(B)
- Want: $P(A \mid B)$

Independence

Definition: X and Y are *independent* iff P(X,Y) = P(X) P(Y)

$$P(X|Y) = P(X)$$

Definition: X and Y are conditionally independent given Z iff $P(X,Y\mid Z)=P(X\mid Z)\,P(Y\mid Z)$

Rules for Continuous RVs

Discrete

1) a)
$$0 \leq P(X \mid Y) \leq 1$$
 b) $\sum_{x \in X} P(x \mid Y) = 1$

2)
$$P(X) = \sum_{y \in Y} P(X, y)$$

3)
$$P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$$
 $P(X,Y) = P(X \mid Y) P(Y)$

Continuous

1)
$$0 \leq p(X \mid Y) \ \int_X p(x|Y) \, dx = 1$$

2)
$$p(X) = \int_{Y} p(X, y) dy$$

3)
$$p(X \mid Y) = \frac{p(X,Y)}{p(Y)}$$
 $p(X,Y) = p(X \mid Y) \, p(Y)$

Multivariate Gaussian Distribution

Joint Distribution

Conditional Distribution

Marginal Distribution

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