Markov Decision Processes

Last Time

- What does "Markov" mean in "Markov Process"?

Guiding Questions

Guiding Questions

- What is a **Markov decision process**?

Guiding Questions

- What is a **Markov decision process**?
- What is a **policy**?

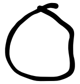
Guiding Questions

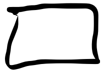
- What is a **Markov decision process**?
- What is a **policy**?
- How do we **evaluate** policies?

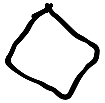
Decision Networks and MDPs

Decision Networks and MDPs

Decision Network

 Chance node

 Decision node

 Utility node

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MDP Decision Network

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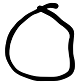
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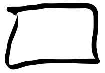
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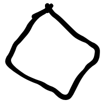
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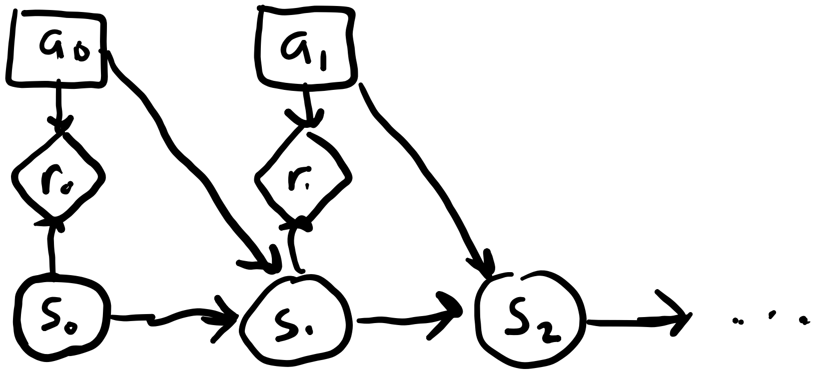

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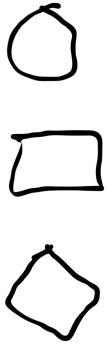

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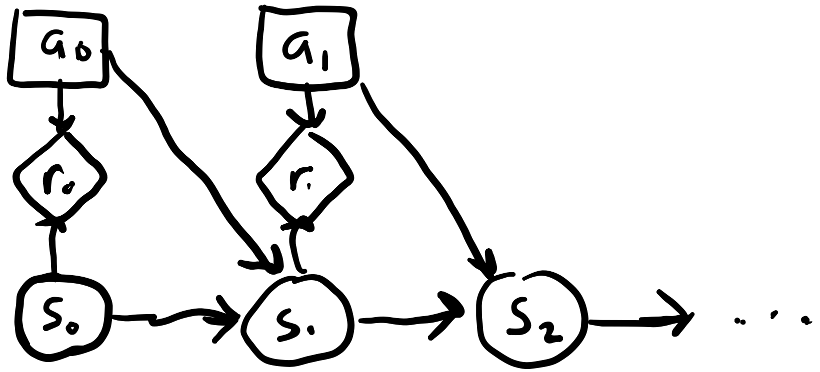

MDP **Dynamic** Decision Network

Decision Networks and MDPs

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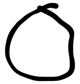
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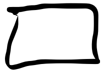
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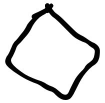
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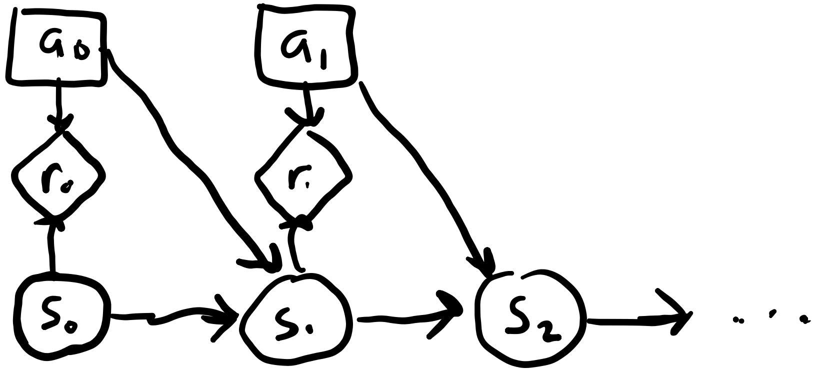

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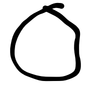
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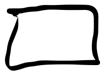
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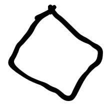
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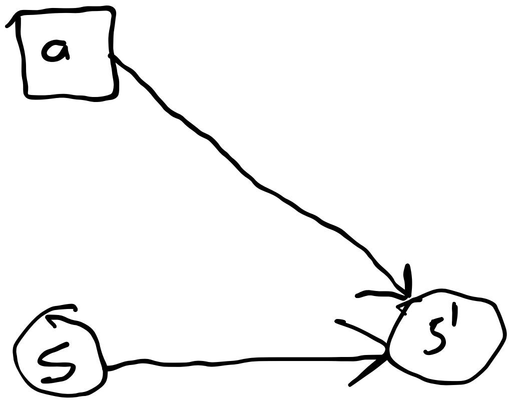

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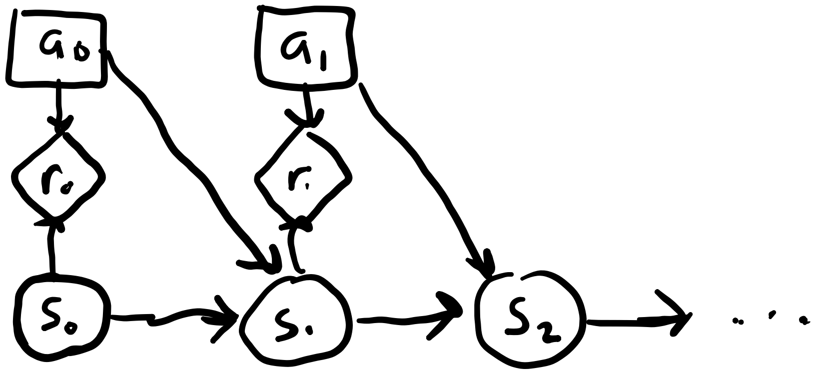

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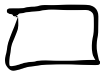
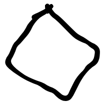
 Decision node

 Utility node

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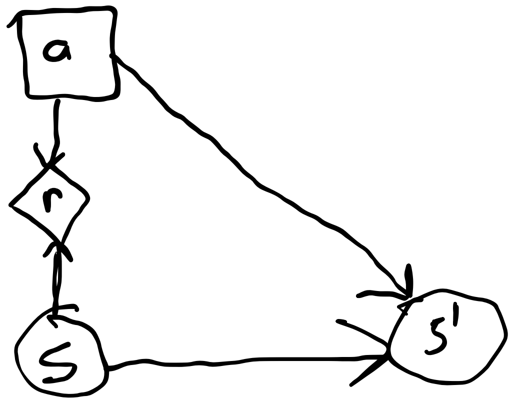

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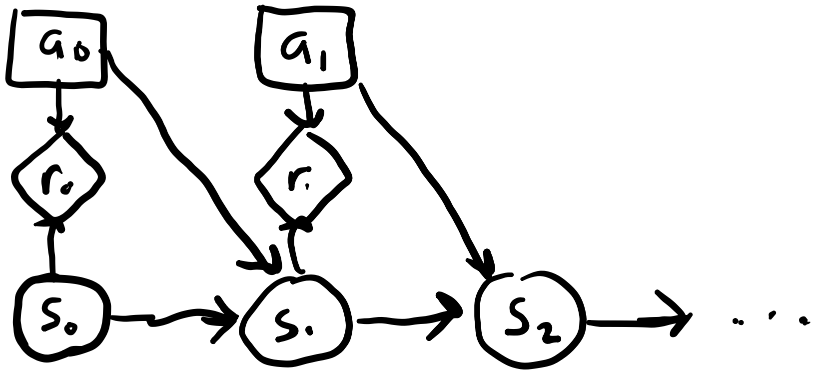

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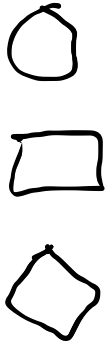

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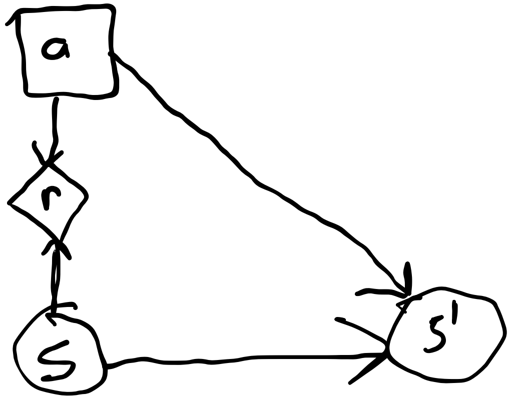

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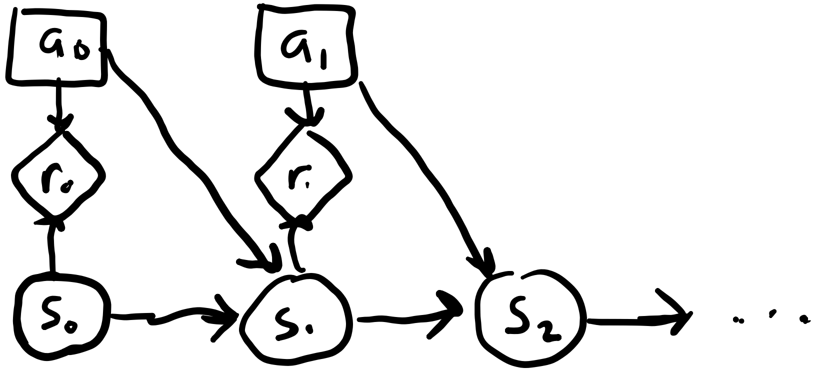

MDP Optimization problem

Decision Networks and MDPs

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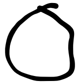
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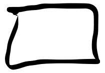
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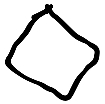
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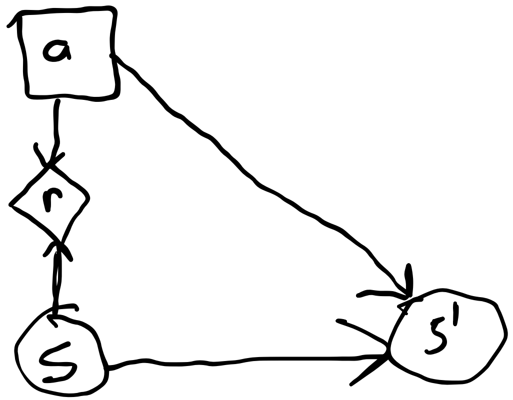

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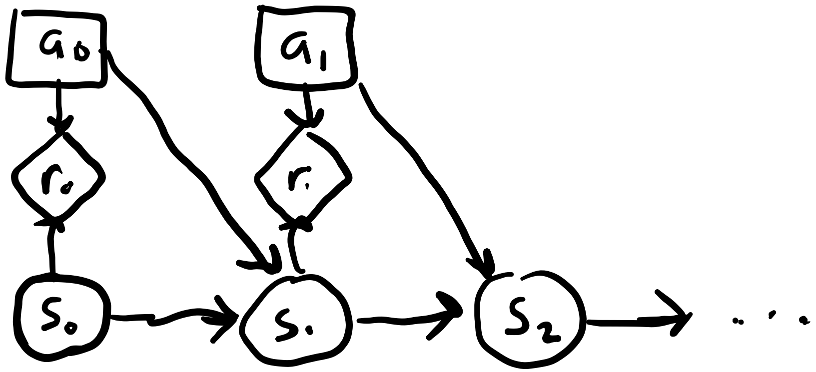

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$$\text{maximize} \quad \mathrm{E}\left[\sum_{t=0}^{\infty} r_t\right]$$

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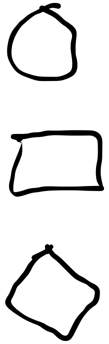

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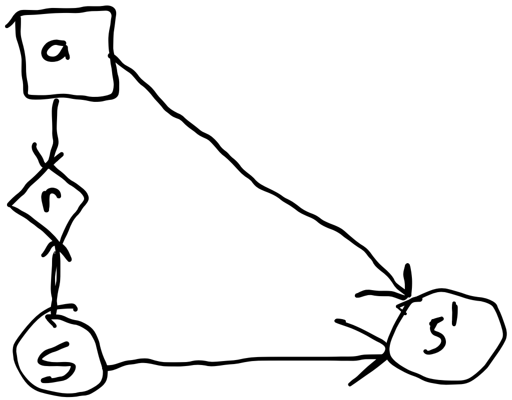

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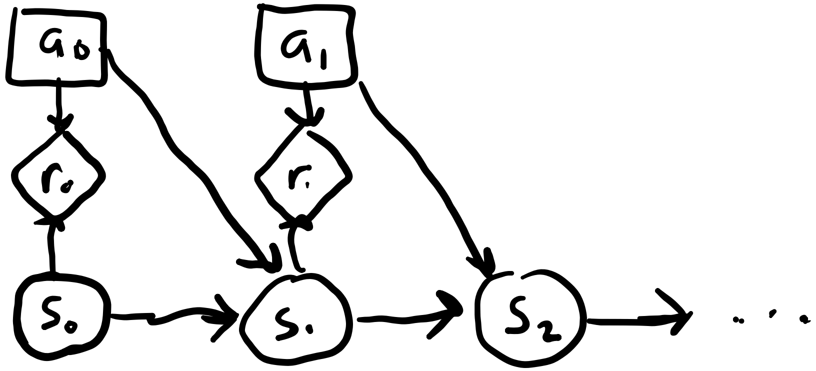

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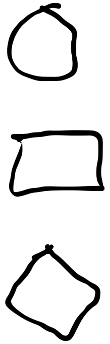

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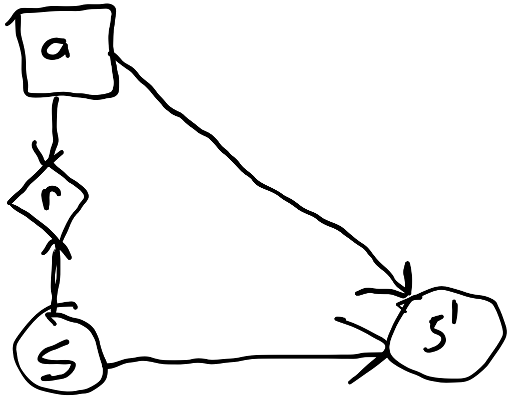

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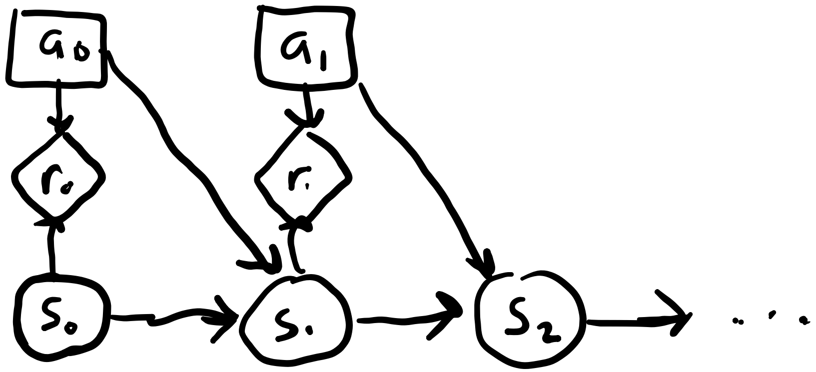

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$$s', r = G(s, a)$$

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- γ: discount factor
- b: initial state distribution
- S_t: set of terminal states

"Generative Model": Alternative to T and R

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MDP Example

Imagine it's a cold day and you're ready to go to work. You have to decide whether to bike or drive.

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Imagine it's a cold day and you're ready to go to work. You have to decide whether to bike or drive.

- If you drive, you will have to pay $15 for parking; biking is free.

MDP Example

Imagine it's a cold day and you're ready to go to work. You have to decide whether to bike or drive.

- If you drive, you will have to pay $15 for parking; biking is free.
- On 1% of cold days, the ground is covered in ice and you will crash if you bike, but you can't discover this until you start riding. After your crash, you limp home with pain equivalent to losing $100.

Policies and Simulation

cf. Alg. 9.1, p. 184

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$$\text{maximize}\, U(\pi) = \mathrm{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t \mid \pi\right]$$

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<u>Algorithm: Rollout Simulation</u>

Inputs: MDP (S, A, R, T, γ, b) (only need generative model, G), Policy π, horizon H

Outputs: Utility estimate $\hat{u}$

$s \leftarrow \text{sample}(b)$

$\hat{u} \leftarrow 0$

for t in $0 \ldots H-1$

    $a \leftarrow \text{sample}(\pi(a \mid s))$

    $s', r \leftarrow G(s, a)$

    $\hat{u} \leftarrow \hat{u} + \gamma^t r$

    $s \leftarrow s'$

return $\hat{u}$

cf. Alg. 9.1, p. 184

Break

- Suggest a policy that you think is optimal for the icy day problem

Naive Policy Evaluation

Naive Policy Evaluation not on Exam

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Naive Policy Evaluation not on Exam

Monte Carlo Policy Evaluation

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- Running a large number of simulations and averaging the accumulated reward is called *Monte Carlo Evaluation*

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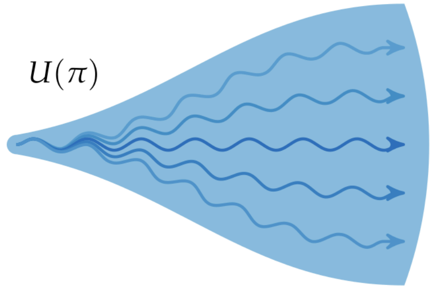

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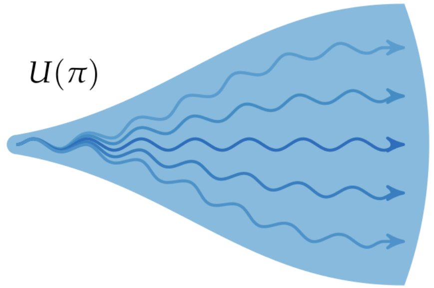

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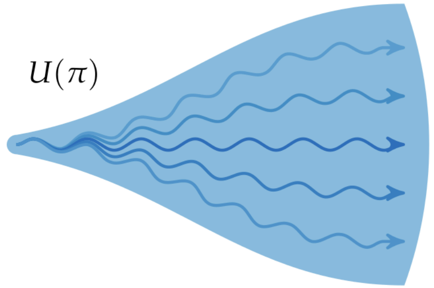

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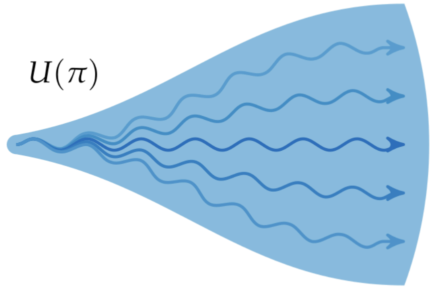

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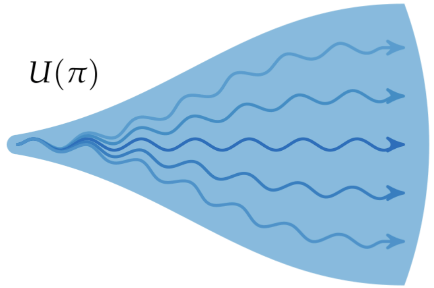

How can we quantify the accuracy of $\bar{u}_m$?

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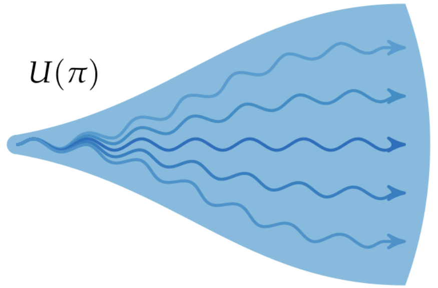

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Value Function-Based Policy Evaluation

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- What is a **Markov decision process**?
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- How do we **evaluate** policies?