Bayesian Network Learning

- Last time:
- Today:

Bayesian Network Learning

- Last time:
  - Conditional independence in Bayesian Networks
- Today:

Bayesian Network Learning

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  - Conditional independence in Bayesian Networks
  - Sampling from Bayesian Networks
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Bayesian Network Learning

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- **Today:**
  - Given a **Bayesian Network** and some **values**, how do we calculate the probability of **other values**?

Bayesian Network Learning

- **Last time:**
  - Conditional independence in Bayesian Networks
  - Sampling from Bayesian Networks
- **Today:**
  - Given a **Bayesian Network** and some **values**, how do we calculate the probability of **other values**?
  - Given **data**, how do we **fit** a Bayesian network?

Bayesian Network

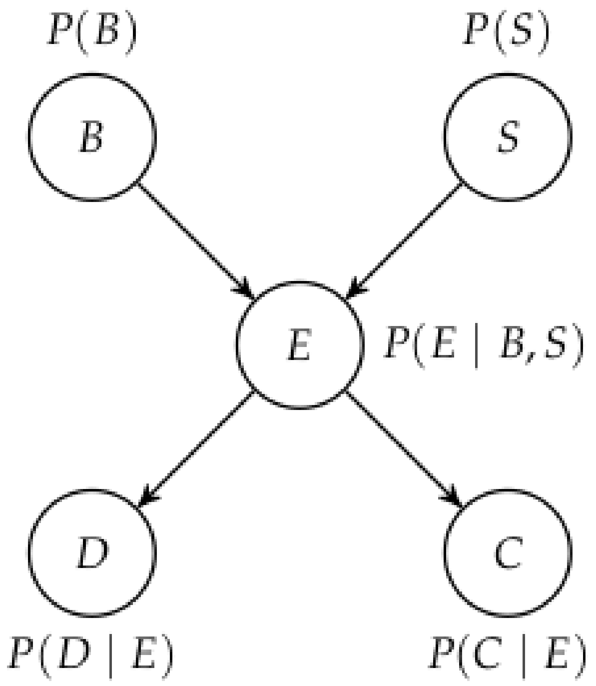
Bayesian Network

Structure

Bayesian Network

Structure **Parameters**

Bayesian Network



B battery failure
S solar panel failure
E electrical system failure
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C communication loss

Structure

Nodes

Edges

Parameters

θ

$P(B=1) \longrightarrow \theta_1$

$P(S=1) \longrightarrow \theta_2$

$P(E=1 \mid S=0, B=0) \rightarrow \theta_3$

$P(E=1 \mid S=1, B=0)$

$\vdots \qquad \vdots$

Inference

Inputs

Outputs

Inference

Inputs

- Bayesian network structure

Outputs

Inference

Inputs

- Bayesian network structure
- Bayesian network parameters

Outputs

Inference

Inputs

- Bayesian network structure
- Bayesian network parameters
- Values of *evidence variables*

Outputs

Inference

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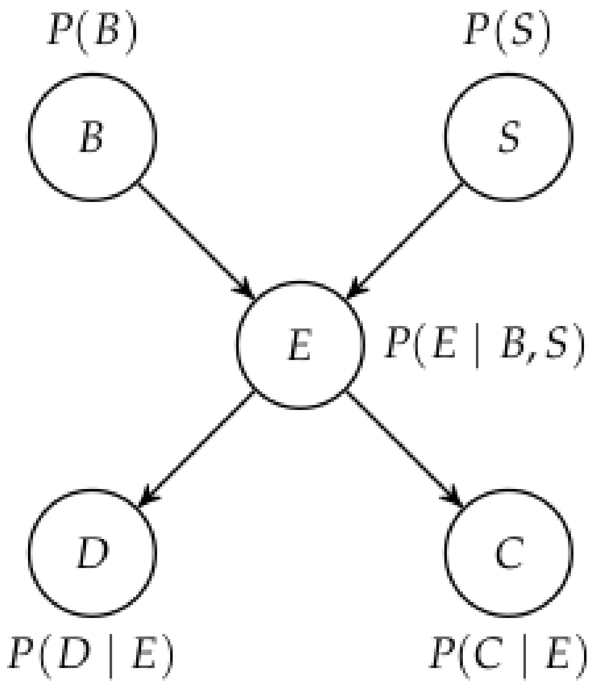
Outputs

- Posterior distribution of *query variables*

Inference



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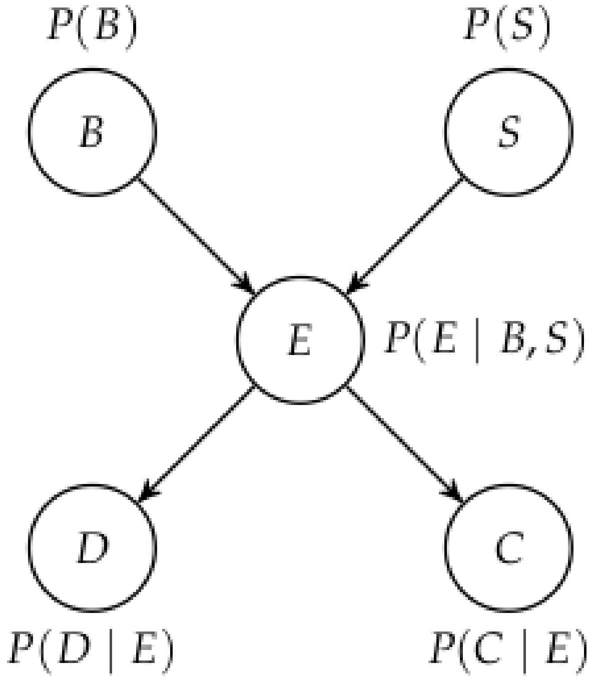

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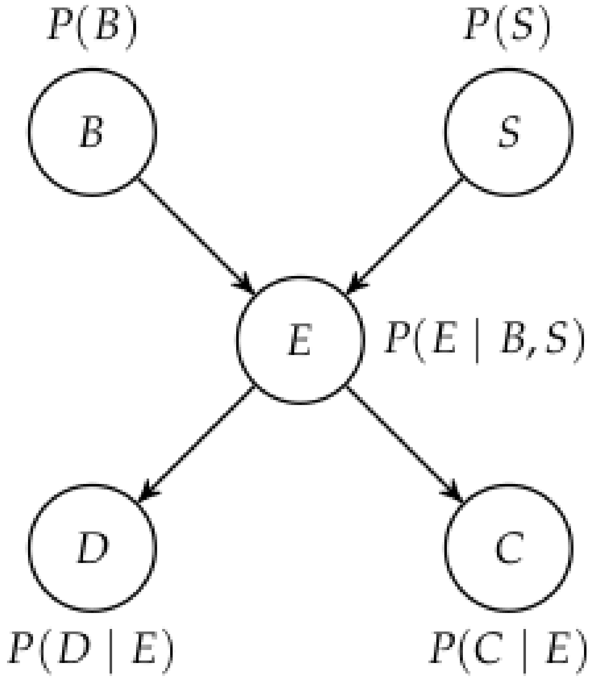
Given that you have detected a trajectory deviation, and the battery has not failed what is the probability of a solar panel failure?

Inference



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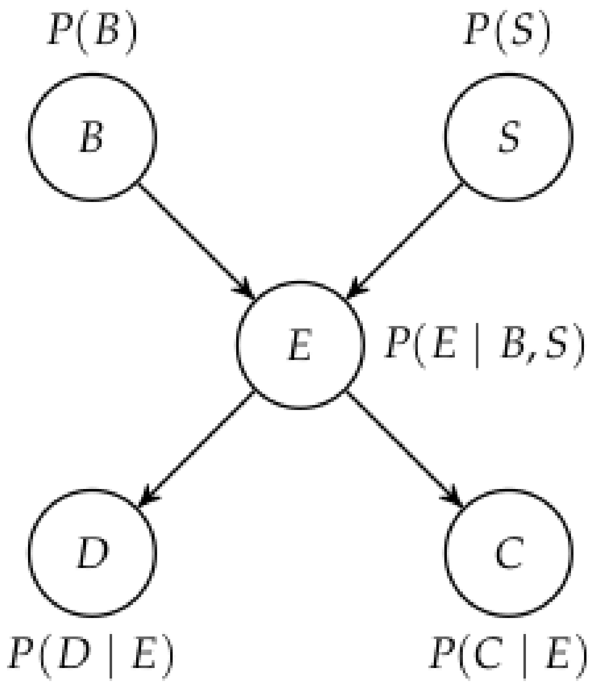
$$P(S = 1 \mid D = 1, B = 0)$$

Inference



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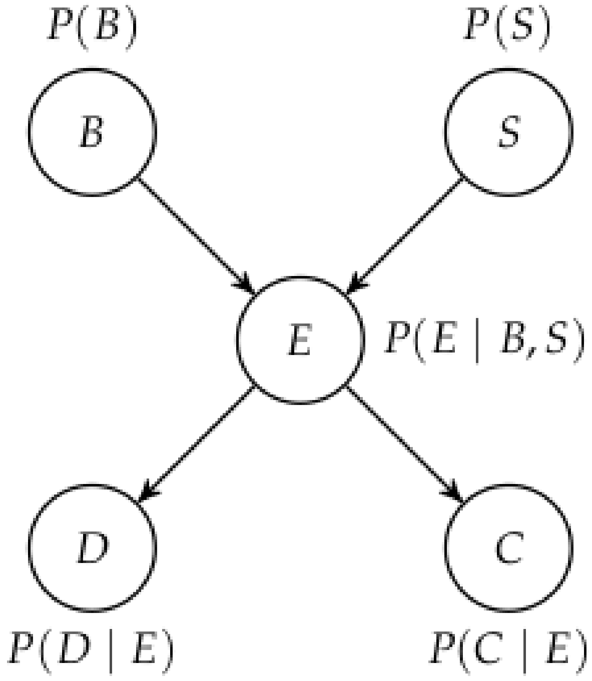
Exact

Inference



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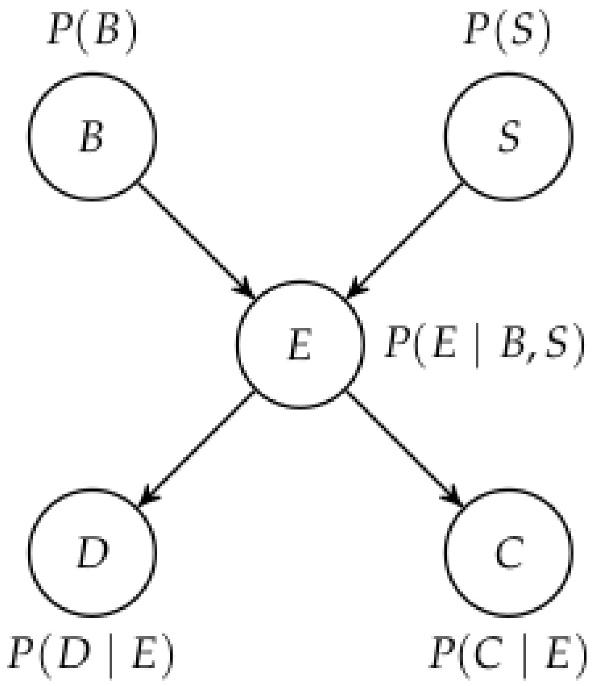
Exact NP-hard

Approximate

Exact Inference

Exact Inference



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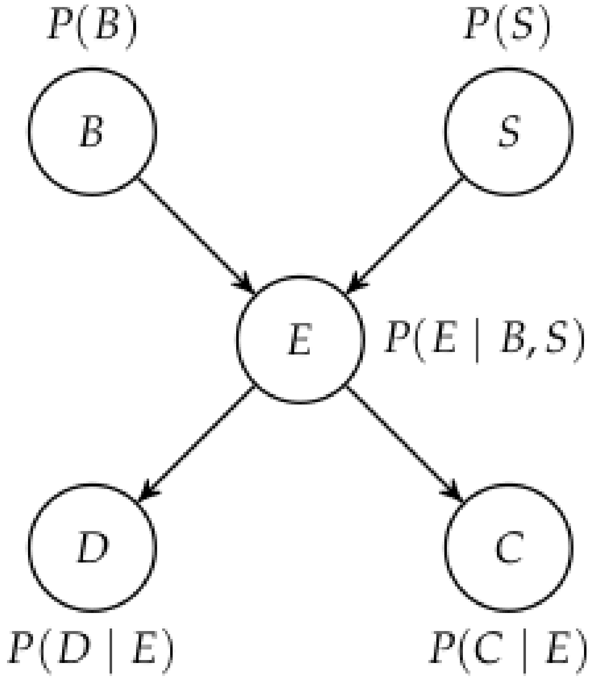

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$$P(S{=}1 \mid D{=}1, B{=}0)$$

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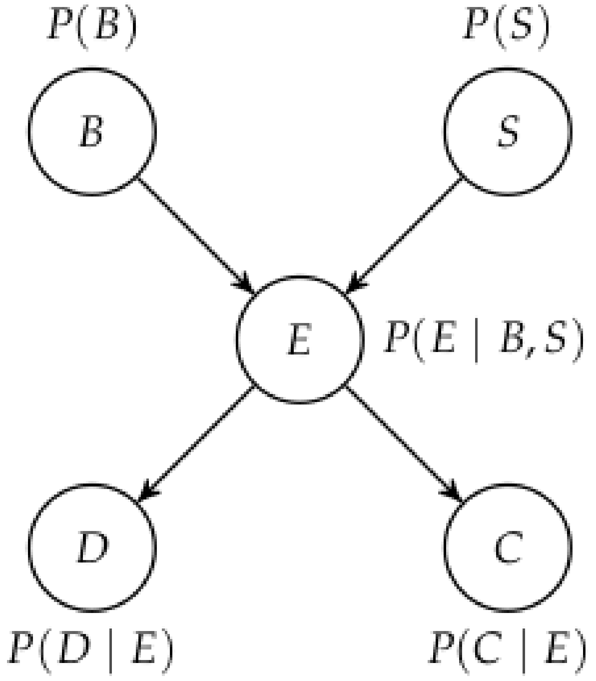

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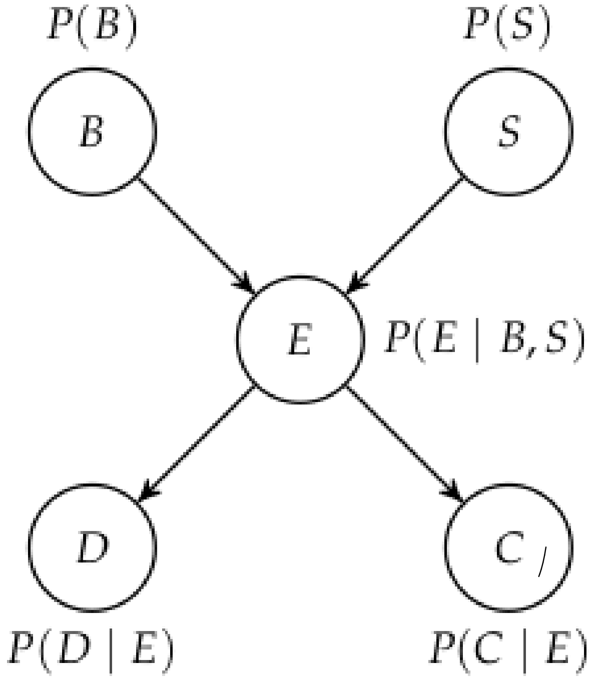
$$P(S{=}1 \mid D{=}1, B{=}0) = \frac{P(S{=}1, D{=}1, B{=}0)}{P(D{=}1, B{=}0)}$$

Exact Inference



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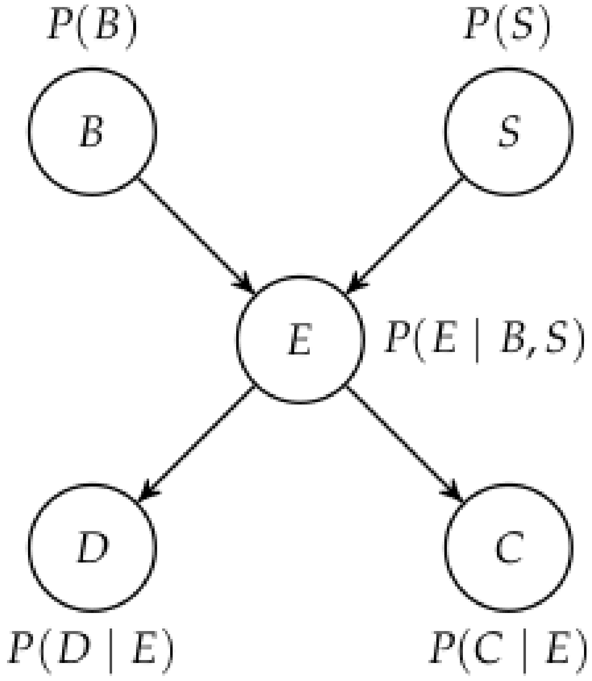
$$P(S{=}1, D{=}1, B{=}0) = \sum_{e,c} P(B{=}0, S{=}1, E{=}e, D{=}1, C{=}c)$$

Exact Inference



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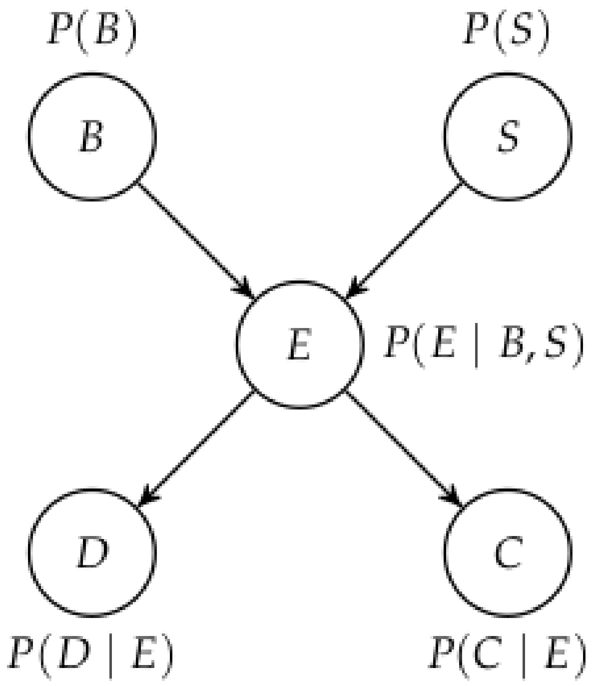

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$$P(B{=}0, S{=}1, E, D{=}1, C)$$
$$= P(B{=}0)\,P(S{=}1)\,P(E \mid B{=}0, S{=}1)\,P(D{=}1 \mid E)\,P(C{=}1 \mid E)$$

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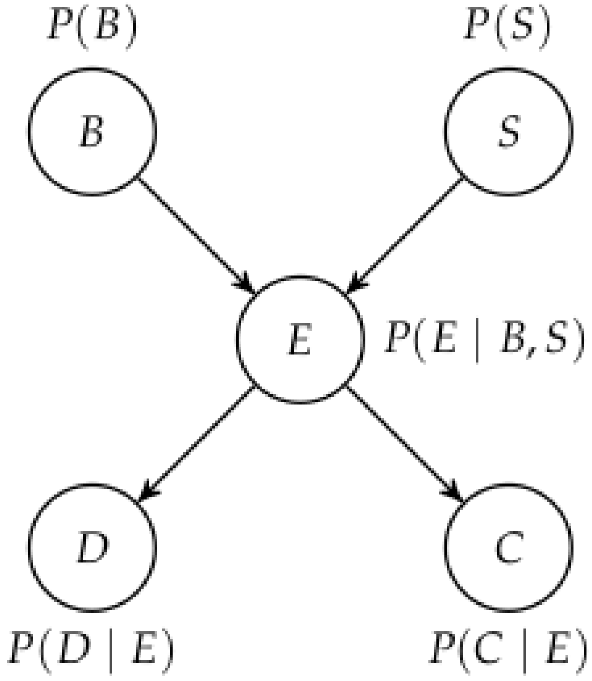

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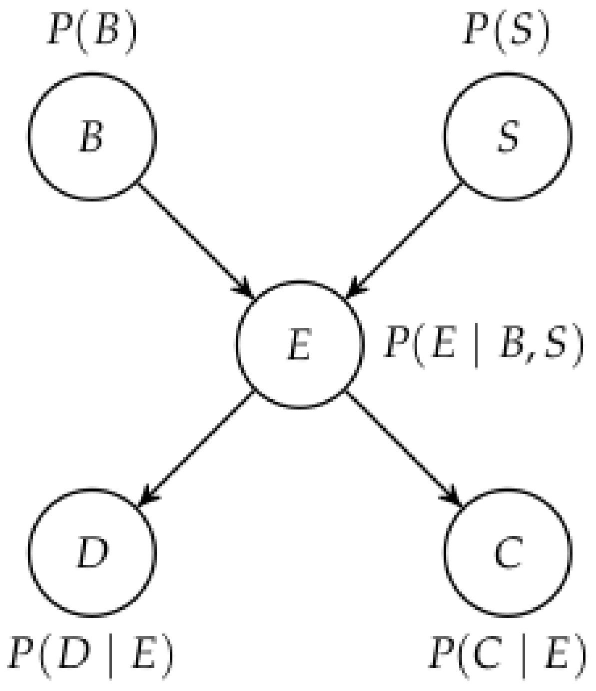
$2^5 = 32$ possible assignments,
but quickly gets too large

Exact Inference

$\phi(A,B,C)$



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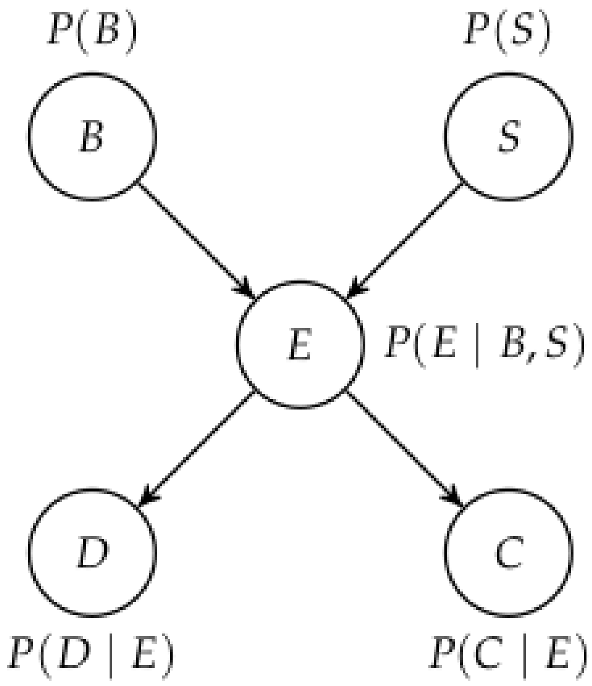

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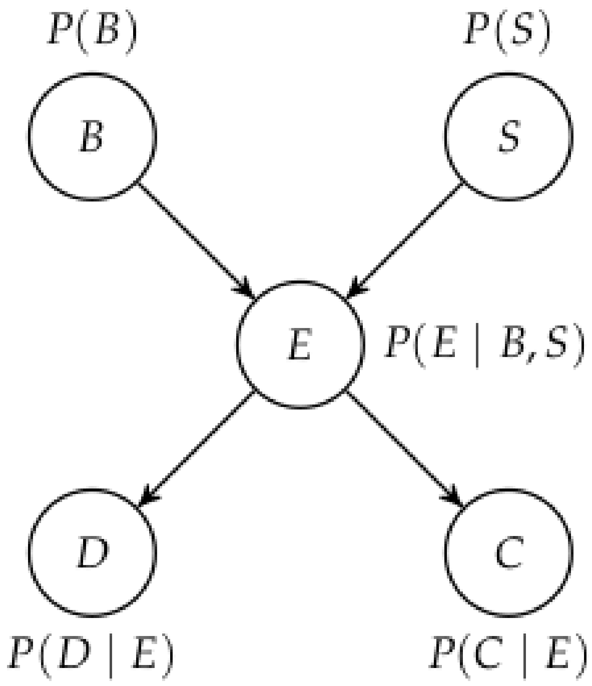
Product

X	Y	$\phi_1(X,Y)$
0	0	0.3
0	1	0.4
1	0	0.2
1	1	0.1

Y	Z	$\phi_2(Y,Z)$
0	0	0.2
0	1	0.0
1	0	0.3
1	1	0.5

X	Y	Z	$\phi_3(X,Y,Z)$
0	0	0	0.06
0	0	1	0.00
0	1	0	0.12
0	1	1	0.20
1	0	0	0.04
1	0	1	0.00
1	1	0	0.03
1	1	1	0.05

Exact Inference



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Condition

X	Y	Z	$\phi(X,Y,Z)$
0	0	0	0.08
0	0	1	0.31
0	1	0	0.09
0	1	1	0.37
1	0	0	0.01
1	0	1	0.05
1	1	0	0.02
1	1	1	0.07

$Y = 1$

X	Z	$\phi(X,Z)$
0	0	0.09
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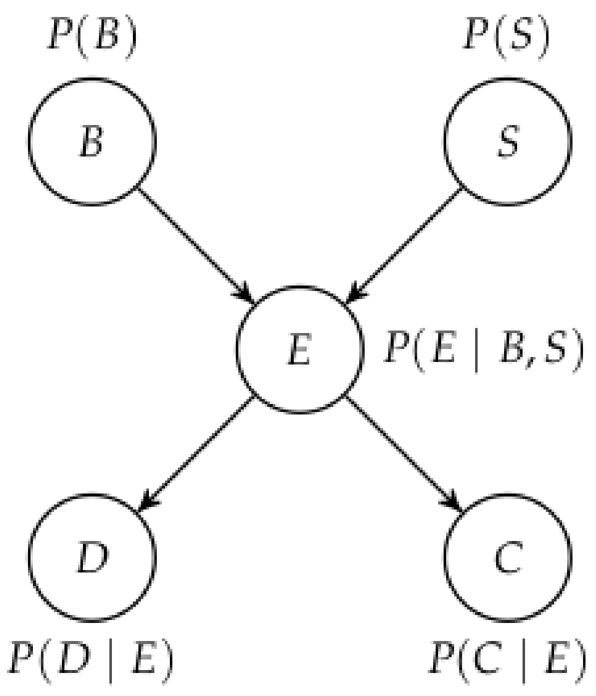
Exact Inference



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Marginalize

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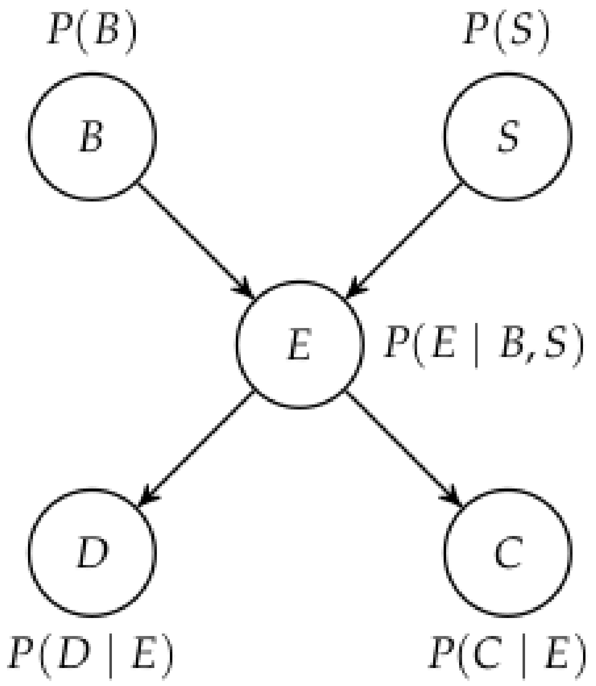
Y

X	Z	$\phi(X,Z)$
0	0	0.17
0	1	0.68
1	0	0.03
1	1	0.12

Exact Inference

$P(S=1 \mid D=1, B=0)$



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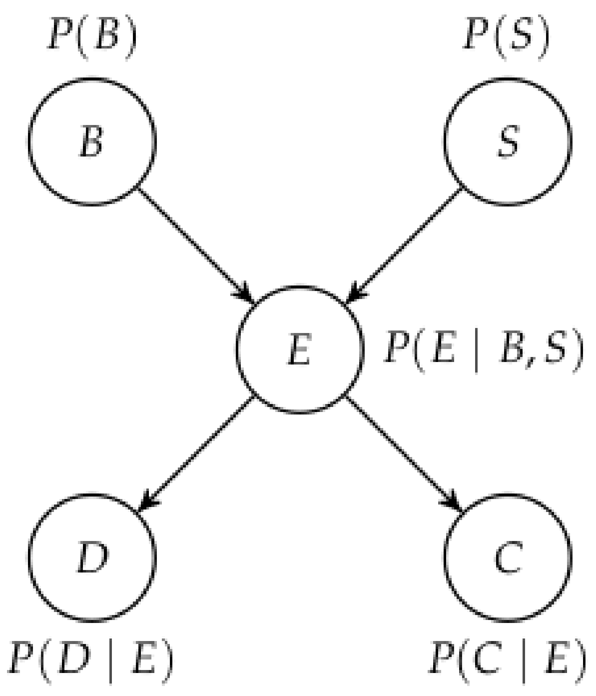
struct ExactInference end

function infer(M::ExactInference, bn, query, evidence)
    ϕ = prod(bn.factors)
    ϕ = condition(ϕ, evidence)
    for name in setdiff(variablenames(ϕ), query)
        ϕ = marginalize(ϕ, name)
    end
    return normalize!(ϕ)
end
```

Exact Inference



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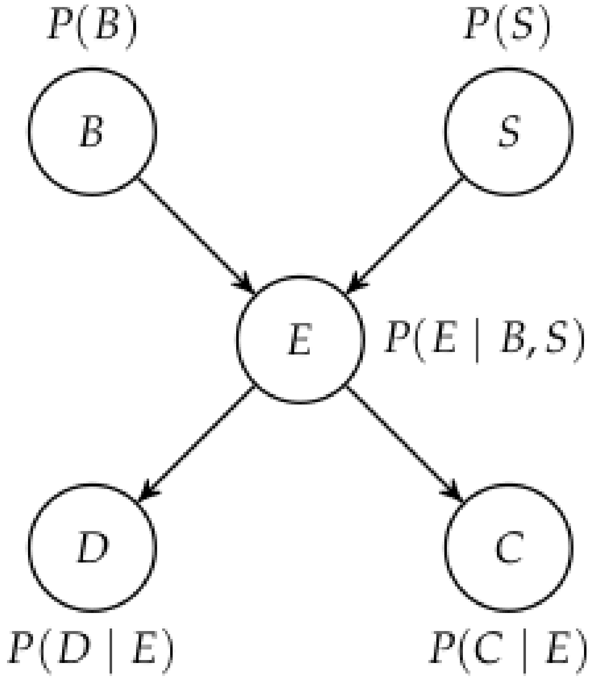
Exact Inference: Variable Elimination



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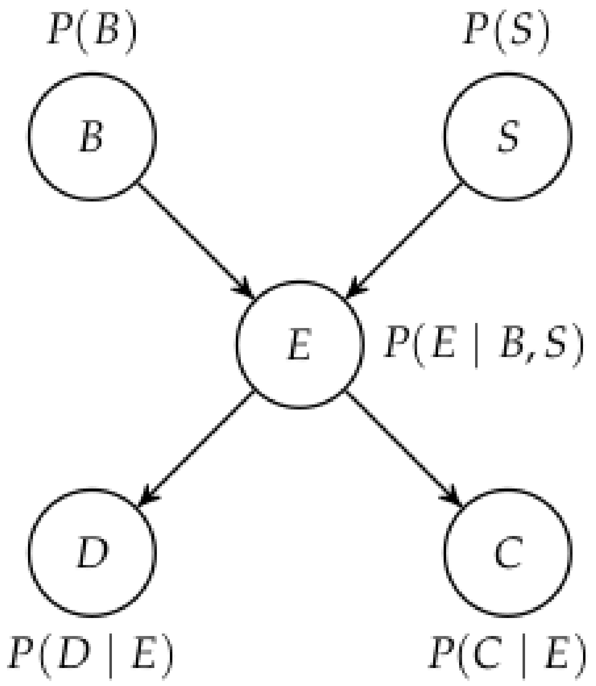

$$P(B \mid d^1, c^1)$$

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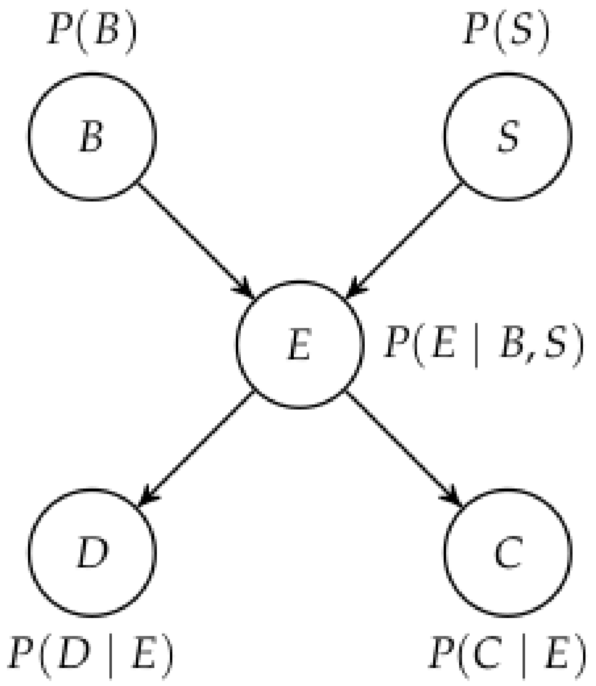

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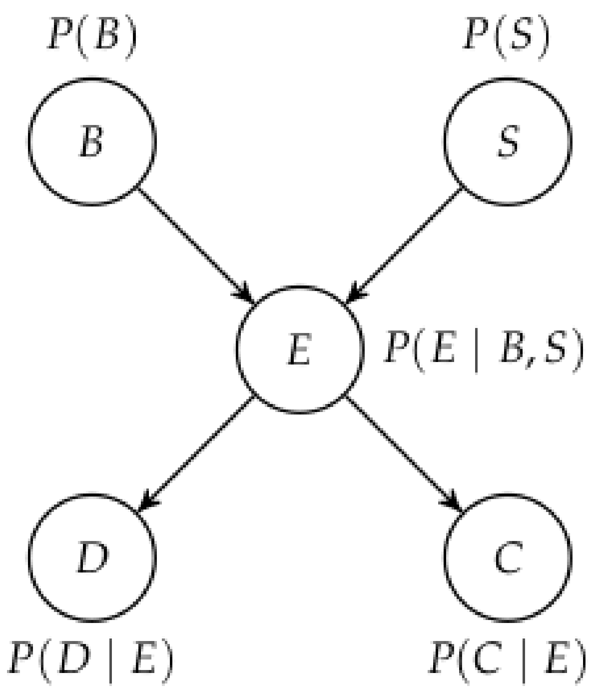
$P(B \mid d^1, c^1)$

Start with

$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Exact Inference: Variable Elimination



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$P(B \mid d^1, c^1)$

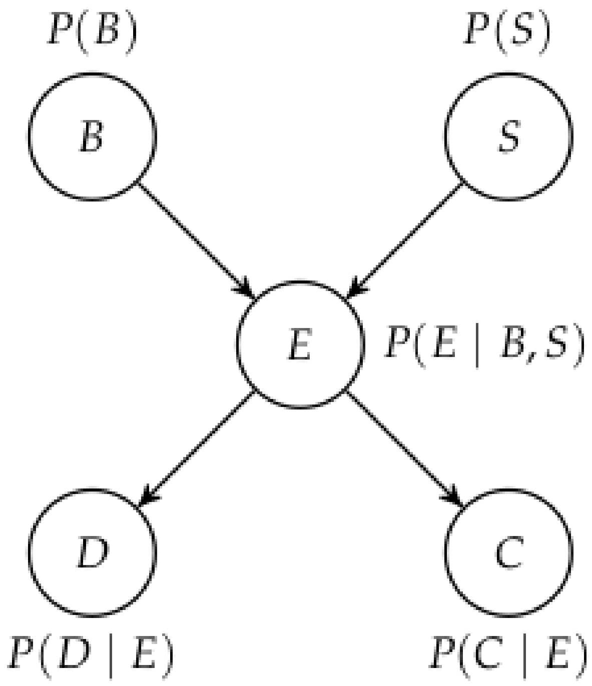
$D=1, C=1$

Start with

$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

Exact Inference: Variable Elimination



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$$P(B \mid d^1, c^1)$$

Start with

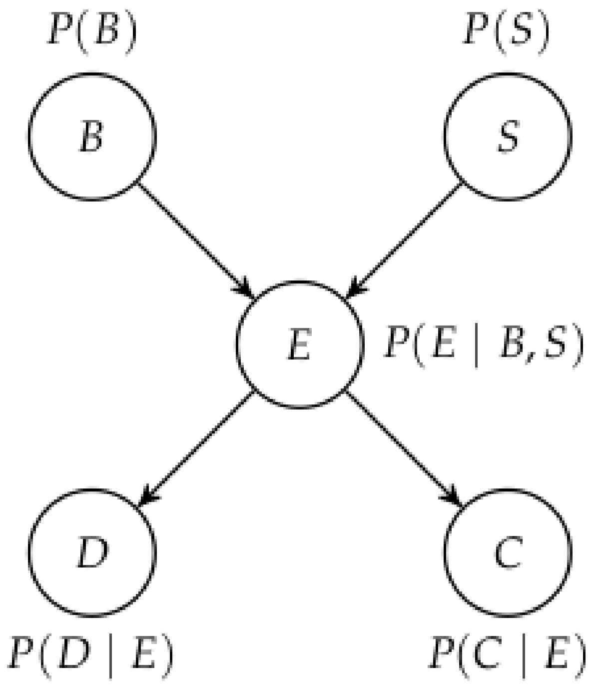
$$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$$

Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

Eliminate E

$$\phi_8(B, S) = \sum_e \phi_3(e, B, S)\phi_6(e)\phi_7(e)$$

Exact Inference: Variable Elimination



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$P(B \mid d^1, c^1)$

Start with

$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

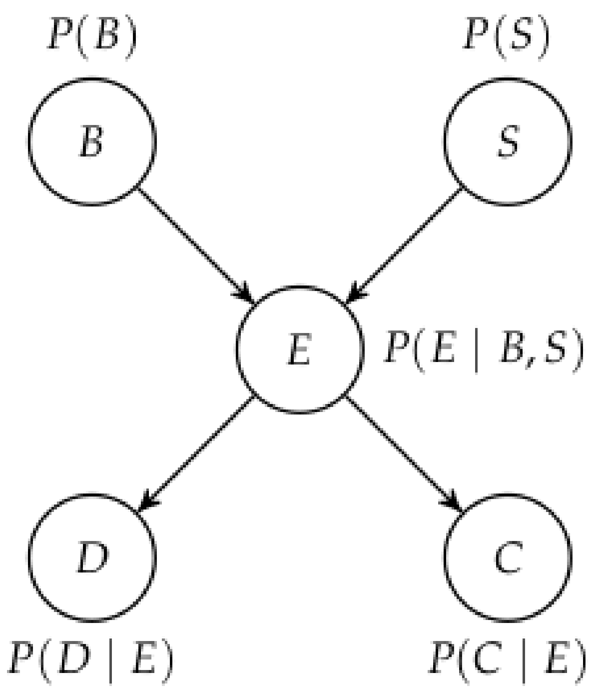
Eliminate E

$$\phi_8(B, S) = \sum_e \phi_3(e, B, S)\phi_6(e)\phi_7(e)$$

Eliminate S

$$\phi_9(B) = \sum_s \phi_2(s)\phi_8(B, s)$$

Exact Inference: Variable Elimination



$P(B \mid d^1, c^1)$

Start with

$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

Eliminate E

$$\phi_8(B, S) = \sum_e \phi_3(e, B, S)\phi_6(e)\phi_7(e)$$

Eliminate S

$$\phi_9(B) = \sum_s \phi_2(s)\phi_8(B, s)$$

B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

Variable Elimination

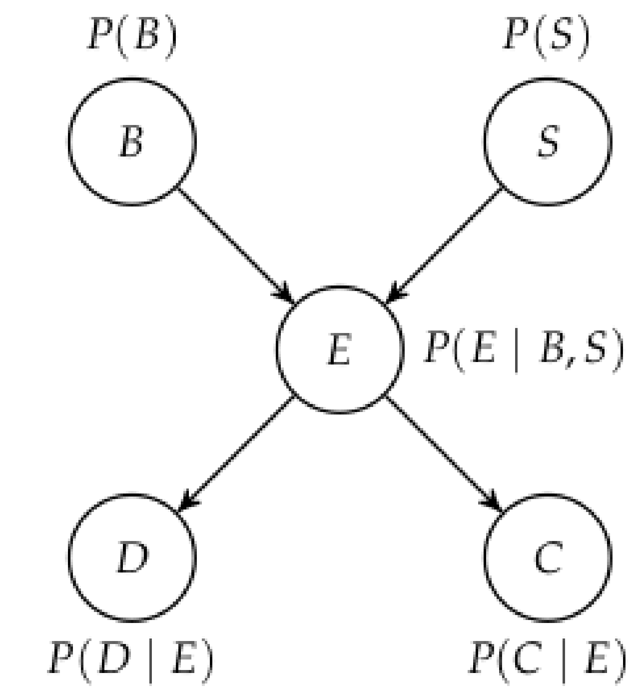
$$P(B \mid d^1, c^1) \propto \phi_1(B) \sum_s \left(\phi_2(s) \sum_e \left(\phi_3(e \mid B, s)\phi_4(d^1 \mid e)\phi_5(c^1 \mid e) \right) \right)$$

vs

Naive Approach

$$P(B \mid d^1, c^1) \propto \sum_s \sum_e \phi_1(B)\phi_2(s)\phi_3(e \mid B, s)\phi_4(d^1 \mid e)\phi_5(c^1 \mid e)$$

Exact Inference: Variable Elimination



B battery failure
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C communication loss

$P(B \mid d^1, c^1)$

Start with

$\phi_1(B), \phi_2(S), \phi_3(E, B, S), \phi_4(D, E), \phi_5(C, E)$

Eliminate D and C (evidence) to get $\phi_6(E)$ and $\phi_7(E)$

Eliminate E

$$\phi_8(B, S) = \sum_e \phi_3(e, B, S)\phi_6(e)\phi_7(e)$$

Eliminate S

$$\phi_9(B) = \sum_s \phi_2(s)\phi_8(B, s)$$

$$P(B \mid d^1, c^1) \propto \phi_1(B) \sum_s \left(\phi_2(s) \sum_e \left(\phi_3(e \mid B, s)\phi_4(d^1 \mid e)\phi_5(c^1 \mid e)\right)\right)$$

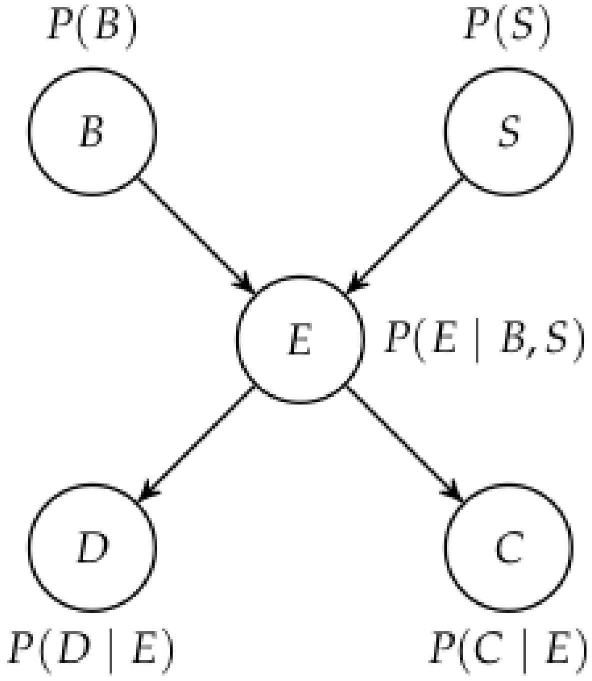
vs

$$P(B \mid d^1, c^1) \propto \sum_s \sum_e \phi_1(B)\phi_2(s)\phi_3(e \mid B, s)\phi_4(d^1 \mid e)\phi_5(c^1 \mid e)$$

Choosing optimal order is NP-hard

Approximate Inference

Approximate Inference: Direct Sampling



B battery failure
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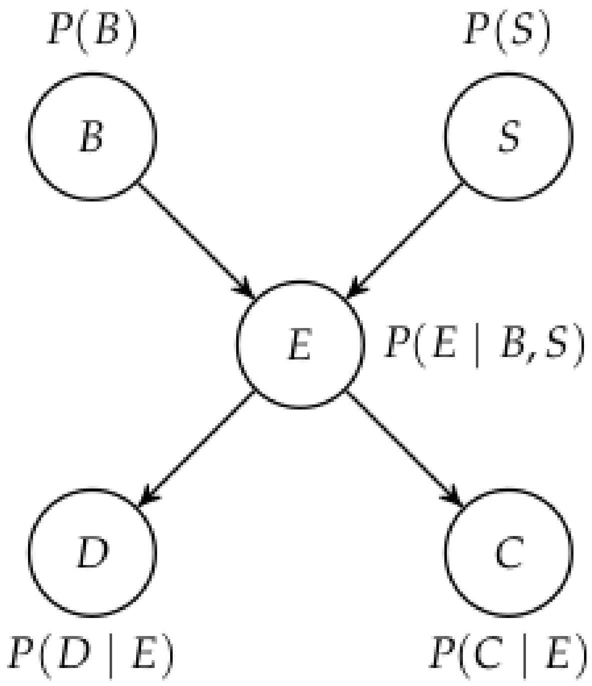
Approximate Inference: Direct Sampling



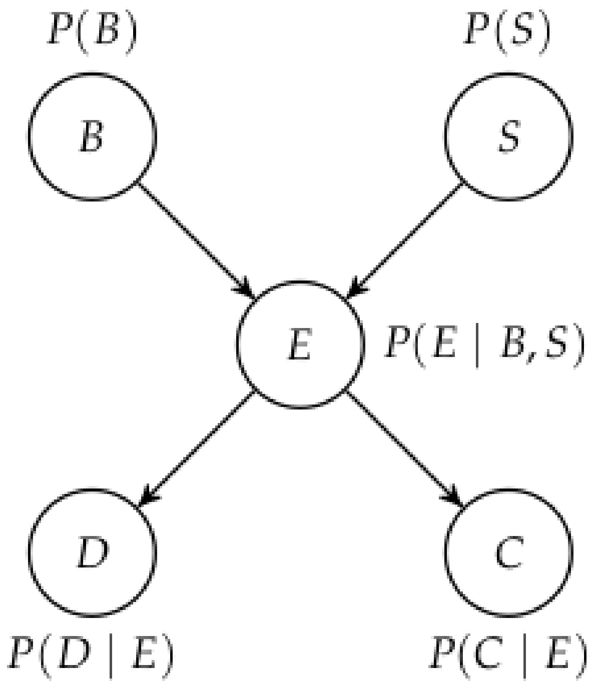
B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i 1(b^{(i)} = 1 \wedge d^{(i)} = 1 \wedge c^{(i)} = 1)}{\sum_i (d^{(i)} = 1 \wedge c^{(i)} = 1)}$$

samples where $B=1, D=1, C=1$

samples where $D=1, C=1$

Approximate Inference: Direct Sampling



B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i (b^{(i)} = 1 \wedge d^{(i)} = 1 \wedge c^{(i)} = 1)}{\sum_i (d^{(i)} = 1 \wedge c^{(i)} = 1)}$$

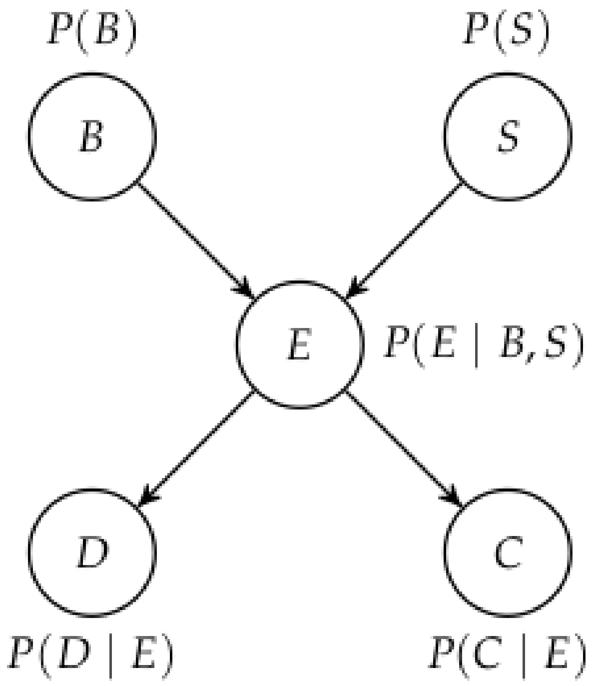
B	S	E	D	C
0	0	1	1	0
0	0	0	0	0
1	0	1	0	0
1	0	1	1	1
0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	1	1	1	1
0	0	0	0	0
0	0	0	1	0

$D=1, C=1$

$P(B=1 \mid C=1, D=1)$
$= \frac{1}{2}$

Unweing

Approximate Inference: Direct Sampling



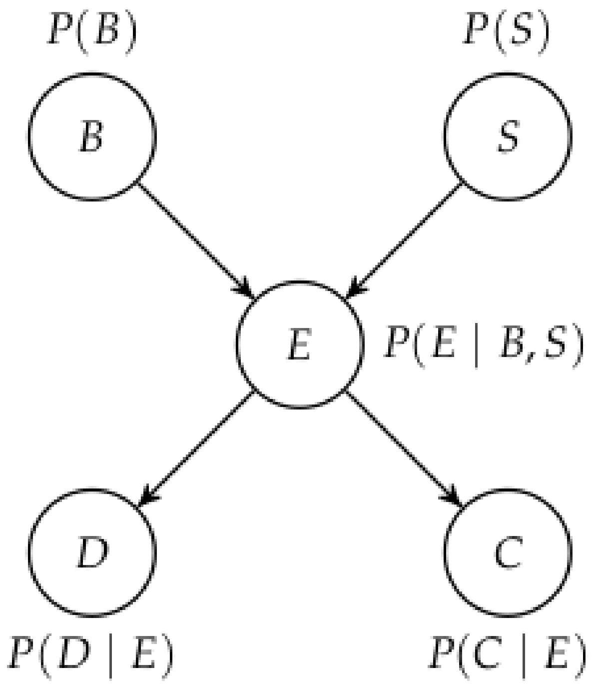
B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i (b^{(i)} = 1 \land d^{(i)} = 1 \land c^{(i)} = 1)}{\sum_i (d^{(i)} = 1 \land c^{(i)} = 1)}$$

B	S	E	D	C
0	0	1	1	0
0	0	0	0	0
1	0	1	0	0
1	0	1	1	1
0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	1	1	1	1
0	0	0	0	0
0	0	0	1	0

Analogous to

Approximate Inference: Direct Sampling



B battery failure
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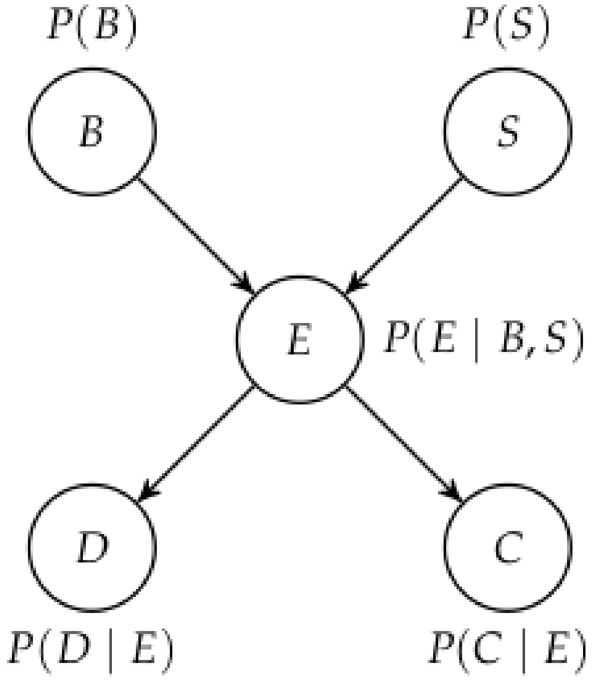
$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i (b^{(i)} = 1 \land d^{(i)} = 1 \land c^{(i)} = 1)}{\sum_i (d^{(i)} = 1 \land c^{(i)} = 1)}$$

B	S	E	D	C
0	0	1	1	0
0	0	0	0	0
1	0	1	0	0
1	0	1	1	1
0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	1	1	1	1
0	0	0	0	0
0	0	0	1	0

Analogous to **unweighted particle filtering**

Approximate Inference: Weighted Sampling



B battery failure
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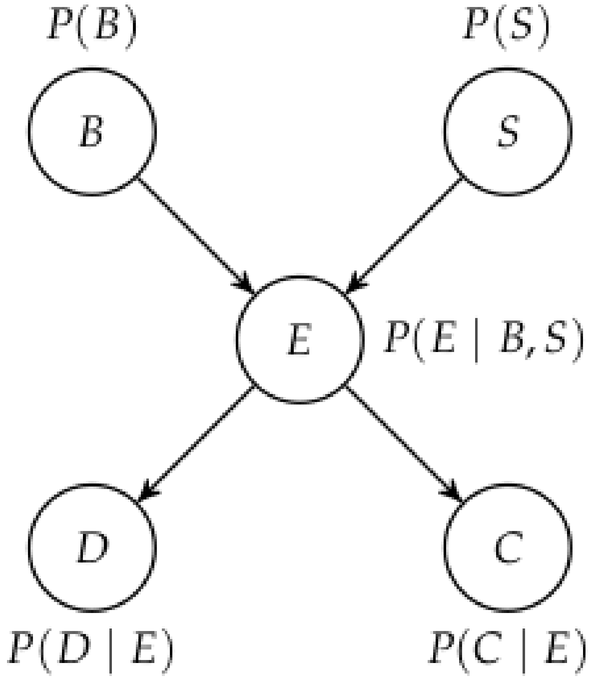
D trajectory deviation
C communication loss

Approximate Inference: Weighted Sampling



B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
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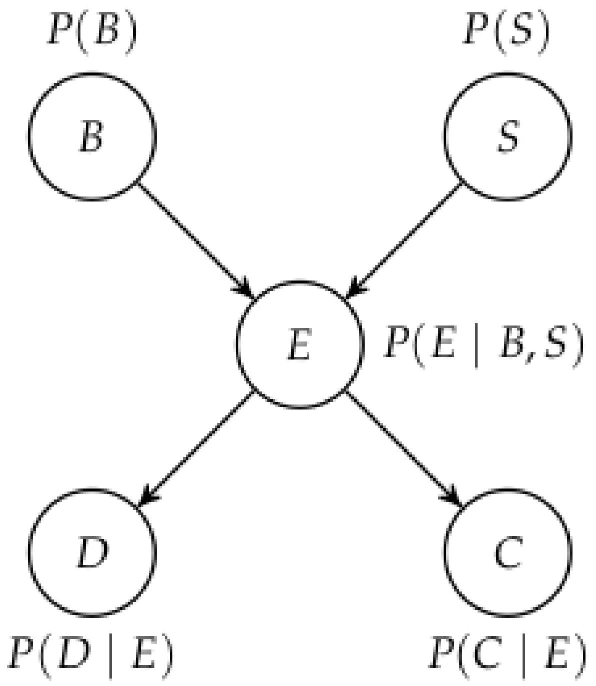
$$
\begin{aligned}
P(b^1 \mid d^1, c^1) &\approx \frac{\sum_i w_i(b^{(i)} = 1 \wedge d^{(i)} = 1 \wedge c^{(i)} = 1)}{\sum_i w_i(d^{(i)} = 1 \wedge c^{(i)} = 1)} \\
&= \frac{\sum_i w_i(b^{(i)} = 1)}{\sum_i w_i}
\end{aligned}
$$

Approximate Inference: Weighted Sampling



B battery failure
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D trajectory deviation
C communication loss

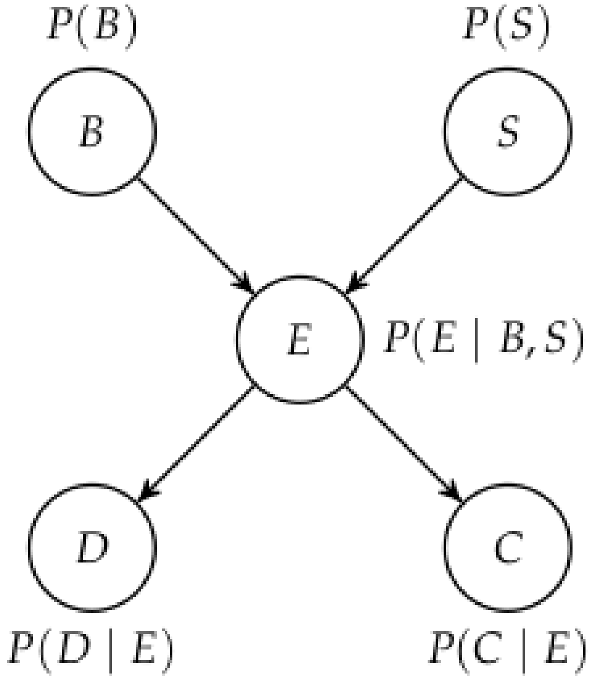
$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i w_i (b^{(i)} = 1 \wedge d^{(i)} = 1 \wedge c^{(i)} = 1)}{\sum_i w_i (d^{(i)} = 1 \wedge c^{(i)} = 1)}$$
$$= \frac{\sum_i w_i (b^{(i)} = 1)}{\sum_i w_i}$$

Query

Evidence

B	S	E	D	C	Weight
1	0	1	1	1	$P(d^1 \mid e^1) P(c^1 \mid e^1)$
0	1	1	1	1	$P(d^1 \mid e^1) P(c^1 \mid e^1)$
0	0	0	1	1	$P(d^1 \mid e^0) P(c^1 \mid e^0)$
0	0	0	1	1	$P(d^1 \mid e^0) P(c^1 \mid e^0)$
0	0	1	1	1	$P(d^1 \mid e^1) P(c^1 \mid e^1)$

Approximate Inference: Weighted Sampling



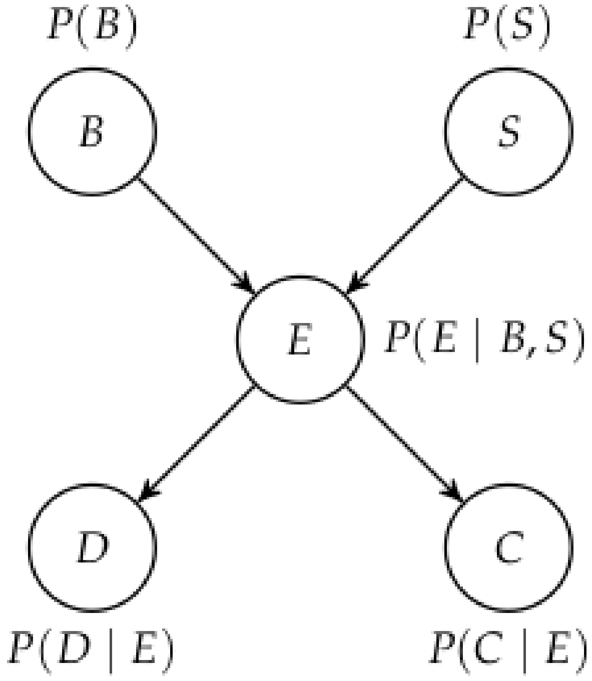
B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i w_i (b^{(i)} = 1 \land d^{(i)} = 1 \land c^{(i)} = 1)}{\sum_i w_i (d^{(i)} = 1 \land c^{(i)} = 1)} = \frac{\sum_i w_i (b^{(i)} = 1)}{\sum_i w_i}$$

B	S	E	D	C	Weight
1	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	1	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$

Analogous to

Approximate Inference: Weighted Sampling



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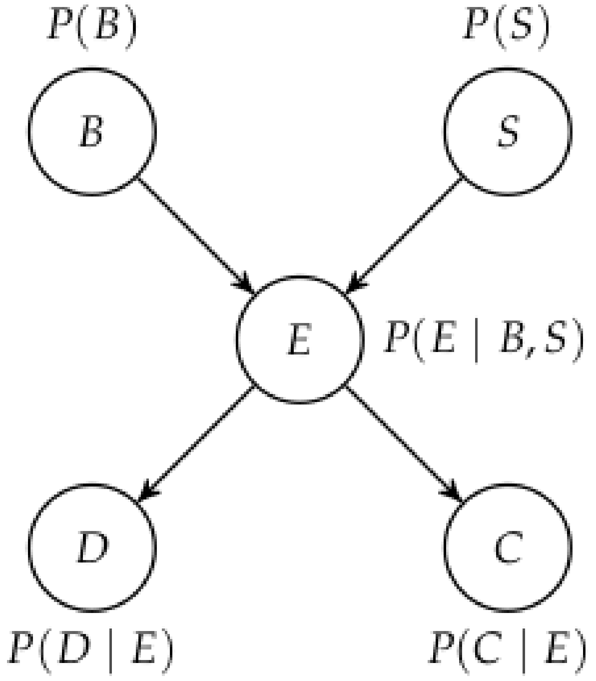
$$P(b^1 \mid d^1, c^1) \approx \frac{\sum_i w_i(b^{(i)} = 1 \land d^{(i)} = 1 \land c^{(i)} = 1)}{\sum_i w_i(d^{(i)} = 1 \land c^{(i)} = 1)}$$
$$= \frac{\sum_i w_i(b^{(i)} = 1)}{\sum_i w_i}$$

B	S	E	D	C	Weight
1	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	1	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	0	1	1	$P(d^1 \mid e^0)P(c^1 \mid e^0)$
0	0	1	1	1	$P(d^1 \mid e^1)P(c^1 \mid e^1)$

Analogous to **weighted particle filtering**

Approximate Inference: Gibbs Sampling



B battery failure
S solar panel failure
E electrical system failure
D trajectory deviation
C communication loss

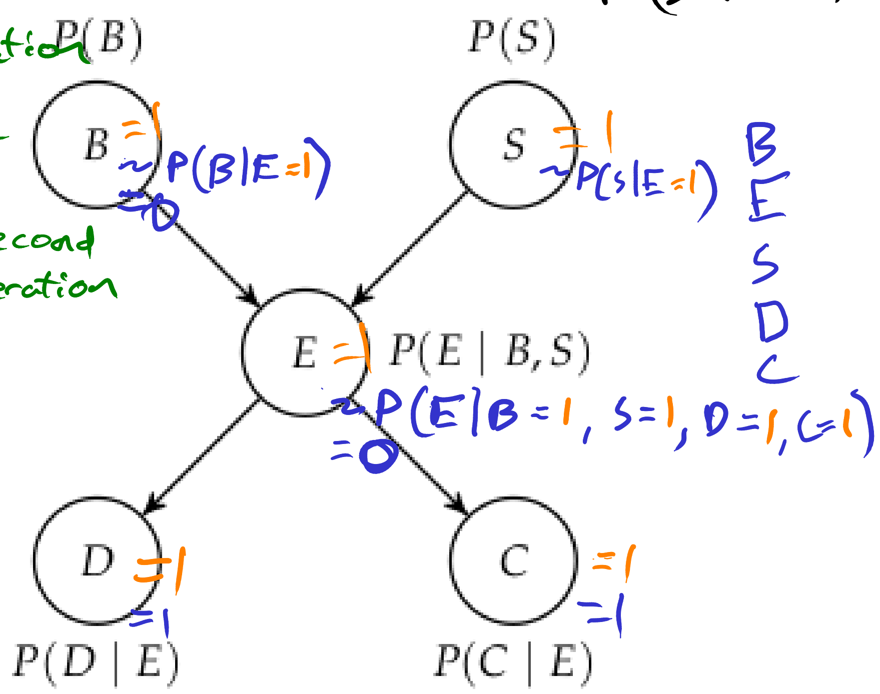
Approximate Inference: Gibbs Sampling

$P(B \mid C=1, D=1)$

2^{nd} iteration

3^{rd} iteration

sample based on second iteration



B battery failure
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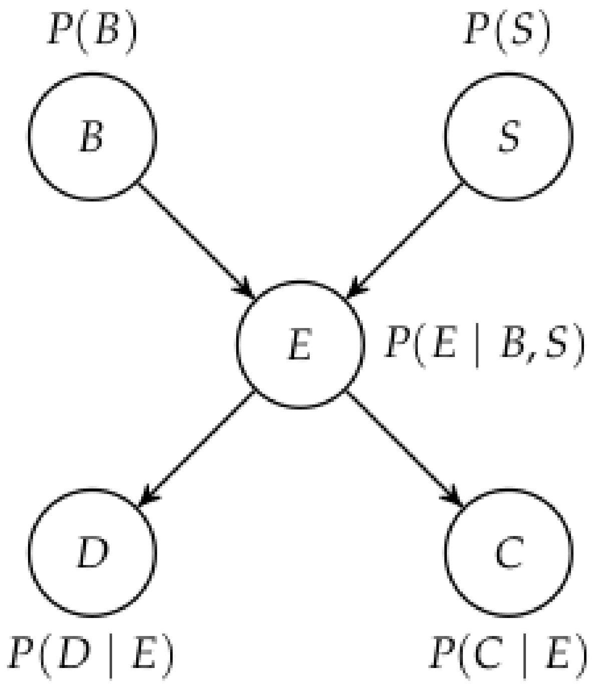
Markov Chain Monte Carlo (MCMC)

1. Choose any order.
2. set evidence variables
3. loop

   sample $x^{(k)}_{\text{non-evidence}}$ based on $x^{(k-1)}$.

Approximate Inference: Gibbs Sampling



B battery failure
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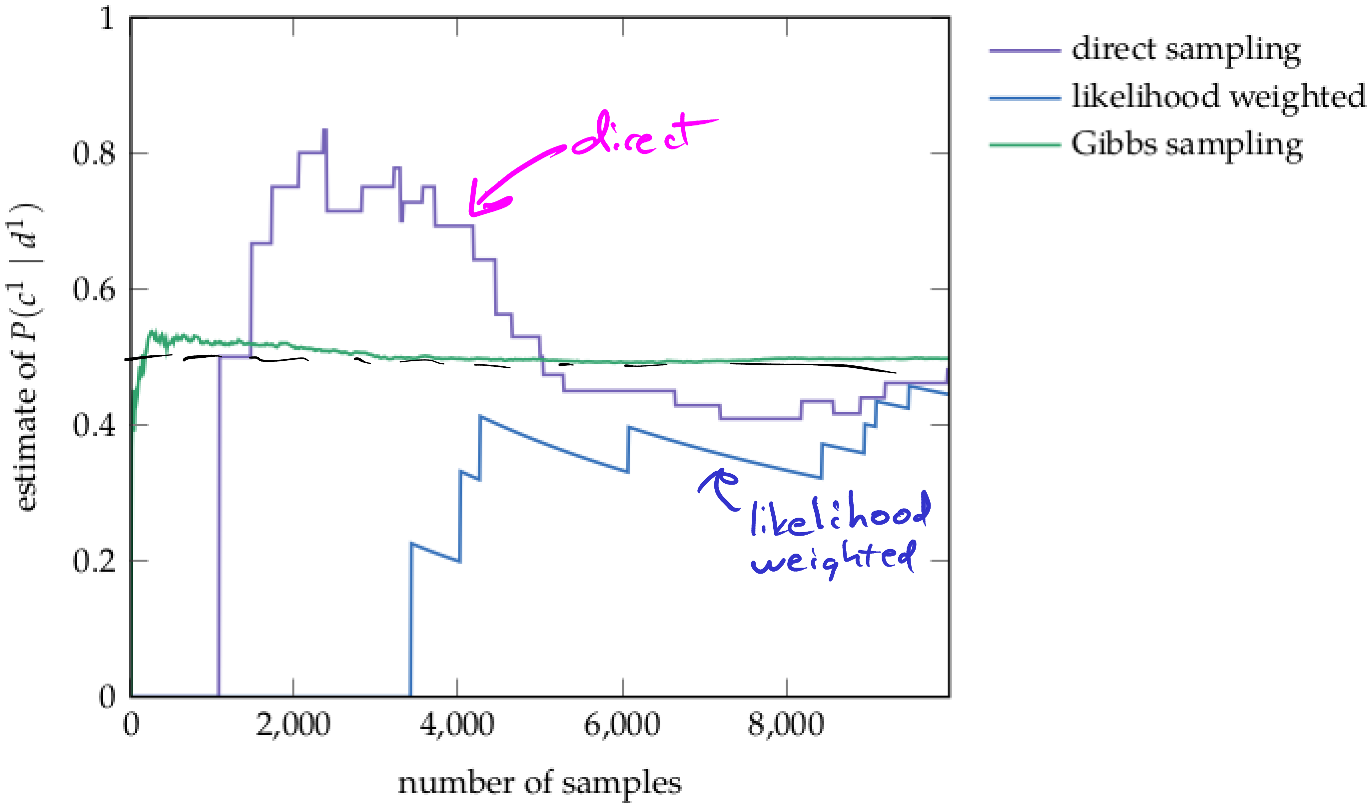
Markov Chain Monte Carlo (MCMC)



Inf

Learning

Bayesian Network Learning

Inputs

Outputs

Bayesian Network Learning

Inputs

- Data, D

Outputs

Bayesian Network Learning

Inputs

- Data, D
- Priors (?)

Outputs

Bayesian Network Learning

Inputs

- Data, D
- Priors (?)

Outputs

- Bayesian network structure, G

Bayesian Network Learning

Inference

BN

Evidence

Prob. of query

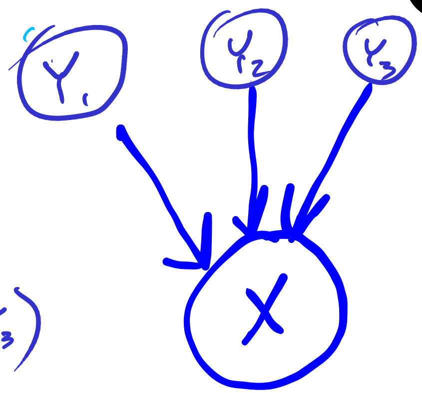
Inputs

- Data, D
- Priors (?)

Outputs

- Bayesian network structure, G
- Bayesian network parameters, θ

Counting Parameters



$P(X \mid Y_1, Y_2, Y_3)$

↑ 8

$P(X=0 \mid Y_1=0, Y_2=0, Y_3=0)$

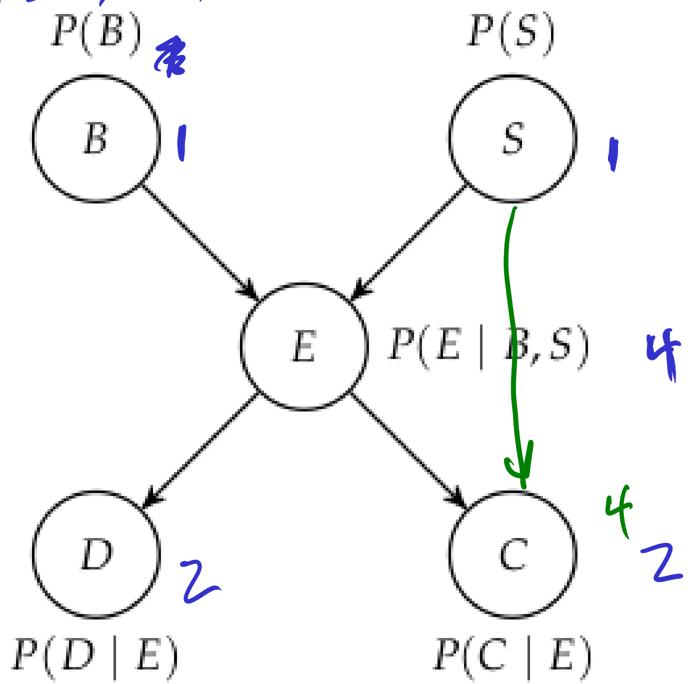
For discrete R.V.s:

$$\dim(\theta_X) = \underbrace{(|\text{support}(X)| - 1)}_{1} \prod_{Y \in Pa(X)} |\text{support}(Y)|$$

"number of params" for X

3   2

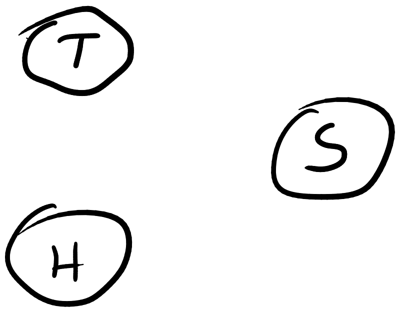
$= 2^3 = 8$

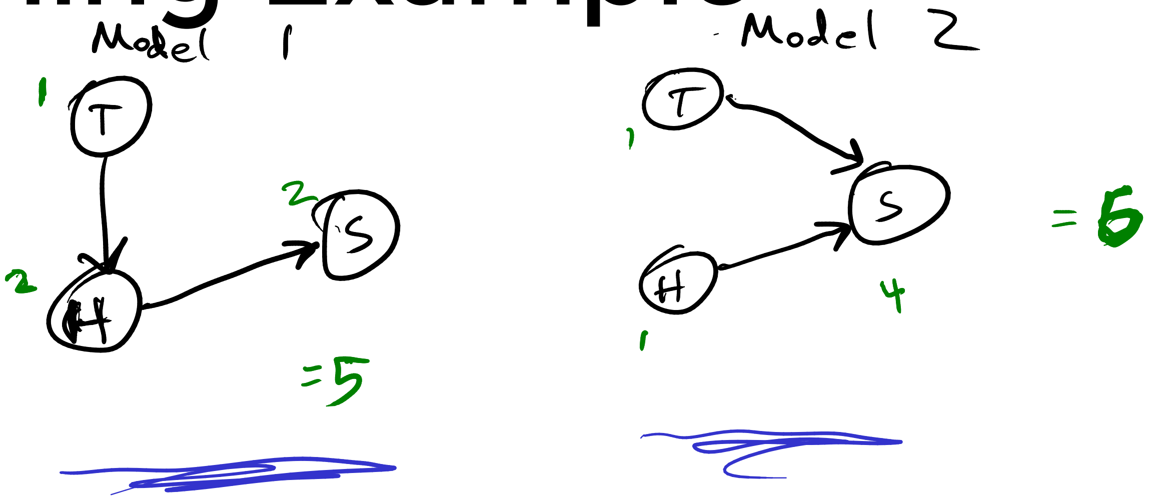


10 parameters total

12 parameters total

Structure Learning Example





Parameter Learning

Parameter Learning

Maximum Likelihood

Parameter Learning

Maximum Likelihood

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_i P(o_i \mid \theta)$$

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_i P(o_i \mid \theta)$$

$$\hat{\theta} = \arg\max_{\theta} \sum_i \log P(o_i \mid \theta)$$

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_i P(o_i \mid \theta)$$

$$\hat{\theta} = \arg\max_{\theta} \sum_i \log P(o_i \mid \theta)$$

Multinomial:

$$\hat{\theta}_i = \frac{n_i}{\sum_{j=1}^{k} n_j}$$

Bayesian

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_i P(o_i \mid \theta)$$

$$\hat{\theta} = \arg\max_{\theta} \sum_i \log P(o_i \mid \theta)$$

Multinomial:

$$\hat{\theta}_i = \frac{n_i}{\sum_{j=1}^{k} n_j}$$

Bayesian

$$\hat{\theta} = \mathbb{E}_{\theta \sim p(\cdot \mid D)}[\theta] = \int \theta p(\theta \mid D) \, \mathrm{d}\theta$$

Parameter Learning

Maximum Likelihood

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$

$$P(D \mid \theta) = \prod_i P(o_i \mid \theta)$$

$$\hat{\theta} = \arg\max_{\theta} \sum_i \log P(o_i \mid \theta)$$

Multinomial:

$$\hat{\theta}_i = \frac{n_i}{\sum_{j=1}^{k} n_j}$$

Bayesian

$$\hat{\theta} = \mathbb{E}_{\theta \sim p(\cdot \mid D)}[\theta] = \int \theta p(\theta \mid D)\, \mathrm{d}\theta$$

Multinomial:

$$p(\theta_{1:n} \mid \alpha_{1:n}, m_{1:n}) = \mathrm{Dir}(\theta_{1:n} \mid \alpha_1 + m_1, \ldots, \alpha_n + m_n)$$

$$\frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j}$$

Structure Learning

Structure Learning

Graph structure

$P(G \mid D)$

Structure Learning

$$P(G \mid D) \propto P(G)P(D \mid G)$$
$$= P(G) \int P(D \mid \theta, G) p(\theta \mid G) \, d\theta$$

Structure Learning

$$P(G \mid D) \propto P(G)P(D \mid G)$$
$$= P(G) \int P(D \mid \theta, G) p(\theta \mid G) \, \mathrm{d}\theta$$

$$P(G \mid D) = P(G) \prod_{i=1}^{n} \prod_{j=1}^{q_i} \frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \prod_{k=1}^{r_i} \frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})}$$

Structure Learning

$$P(G \mid D) \propto P(G)P(D \mid G)$$
$$= P(G) \int P(D \mid \theta, G) p(\theta \mid G)\,\mathrm{d}\theta$$

$$P(G \mid D) = P(G) \prod_{i=1}^{n} \prod_{j=1}^{q_i} \frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \prod_{k=1}^{r_i} \frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})}$$

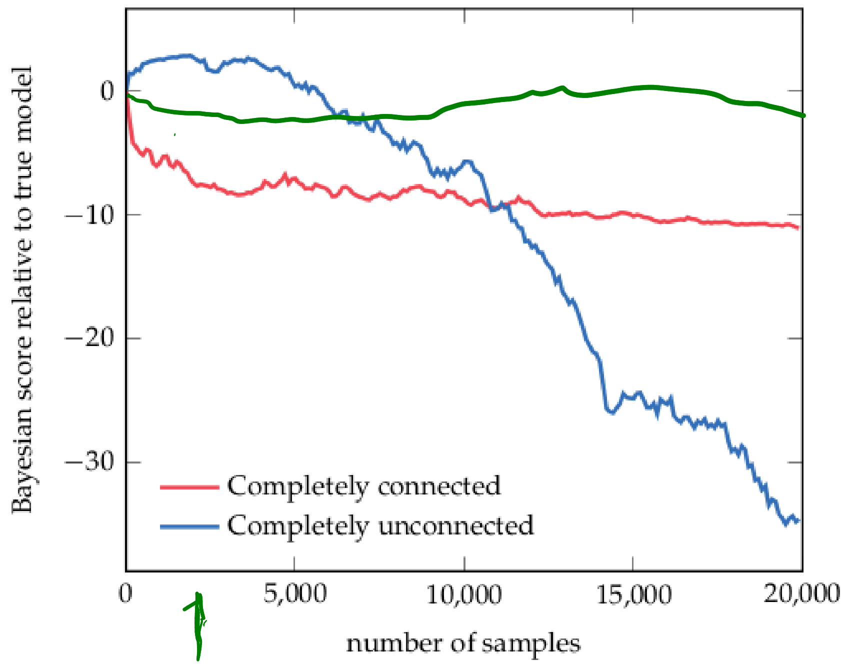
$$\log P(G \mid D)$$
$$= \log P(G) + \sum_{i=1}^{n} \sum_{j=1}^{q_i} \left(\log\left(\frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \right) + \sum_{k=1}^{r_i} \log\left(\frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})} \right) \right)$$

Structure Learning

$$P(G \mid D) \propto P(G)P(D \mid G)$$
$$= P(G) \int P(D \mid \theta, G)p(\theta \mid G)\, d\theta$$

$$P(G \mid D) = P(G) \prod_{i=1}^{n} \prod_{j=1}^{q_i} \frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \prod_{k=1}^{r_i} \frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})}$$

$$\log P(G \mid D)$$
$$= \log P(G) + \sum_{i=1}^{n} \sum_{j=1}^{q_i} \left(\log\left(\frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \right) + \sum_{k=1}^{r_i} \log\left(\frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})} \right) \right)$$



Structure Learning

$$P(G \mid D) \propto P(G)P(D \mid G)$$
$$= P(G) \int P(D \mid \theta, G) p(\theta \mid G) \, d\theta$$

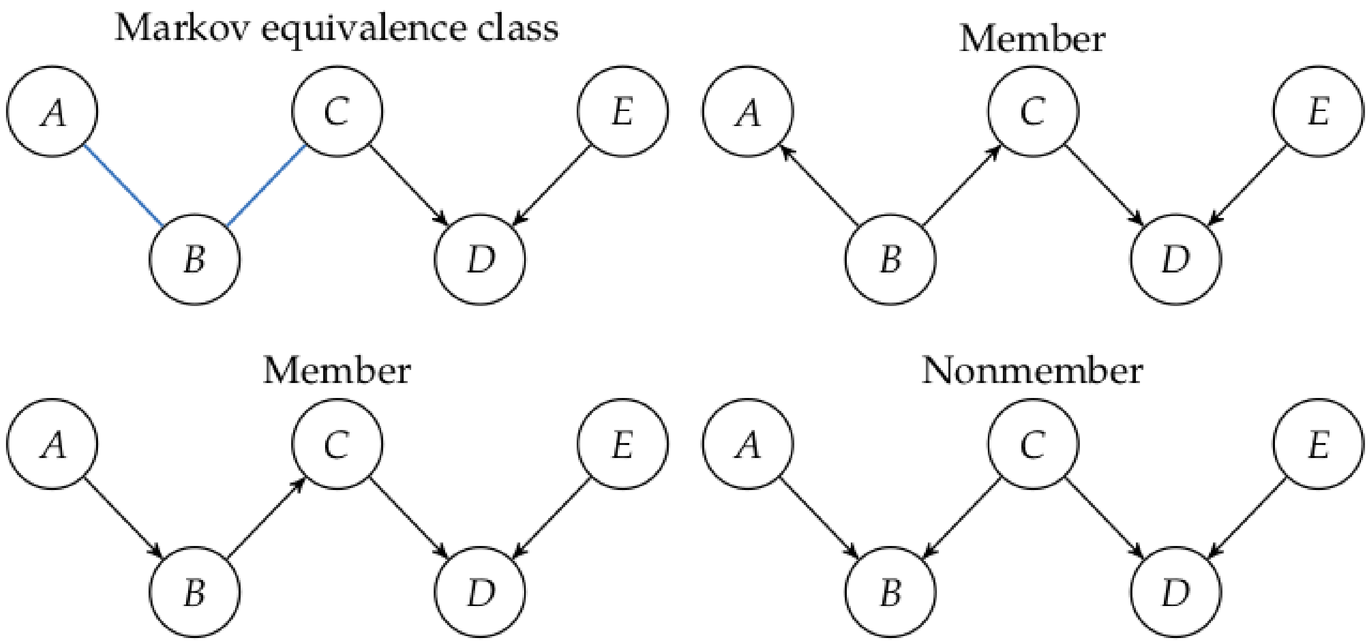
$$P(G \mid D) = P(G) \prod_{i=1}^{n} \prod_{j=1}^{q_i} \frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \prod_{k=1}^{r_i} \frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})}$$

$$\log P(G \mid D)$$
$$= \log P(G) + \sum_{i=1}^{n} \sum_{j=1}^{q_i} \left(\log\left(\frac{\Gamma(\alpha_{ij0})}{\Gamma(\alpha_{ij0} + m_{ij0})} \right) + \sum_{k=1}^{r_i} \log\left(\frac{\Gamma(\alpha_{ijk} + m_{ijk})}{\Gamma(\alpha_{ijk})} \right) \right)$$

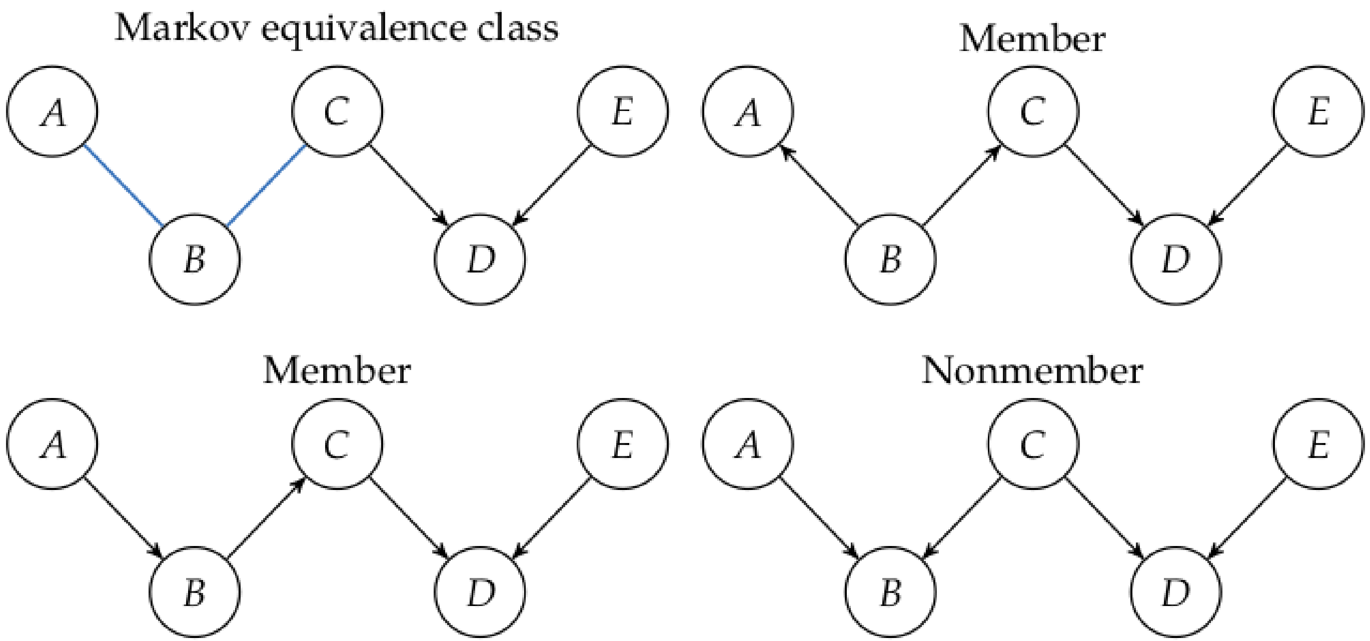
NP-Hard

Markov Equivalence Class

Markov Equivalence Class

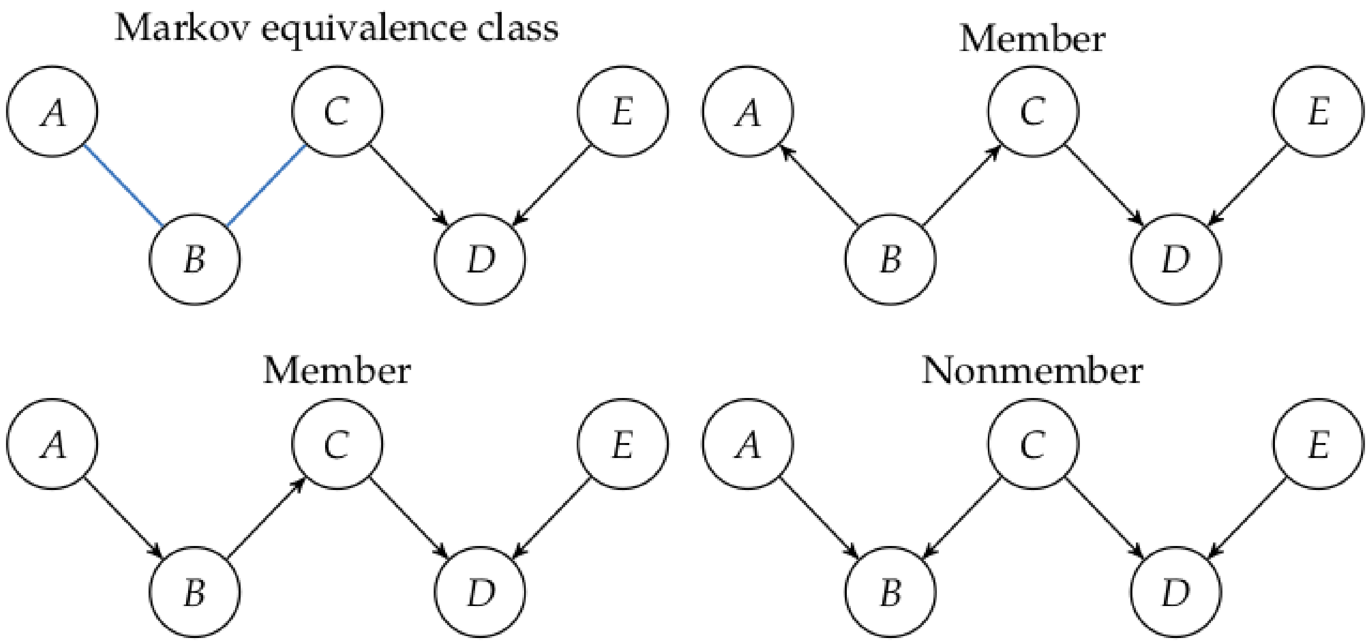


Markov Equivalence Class



Markov Equivalent iff

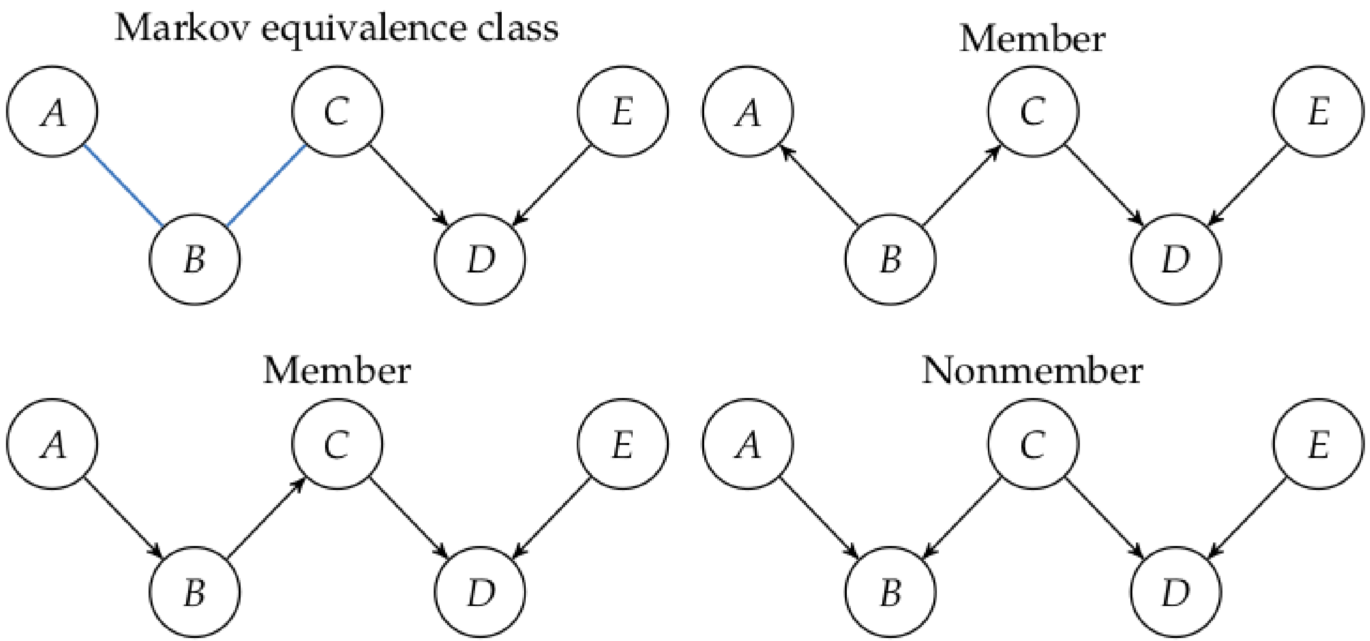
Markov Equivalence Class



Markov Equivalent iff

1. Same undirected edges

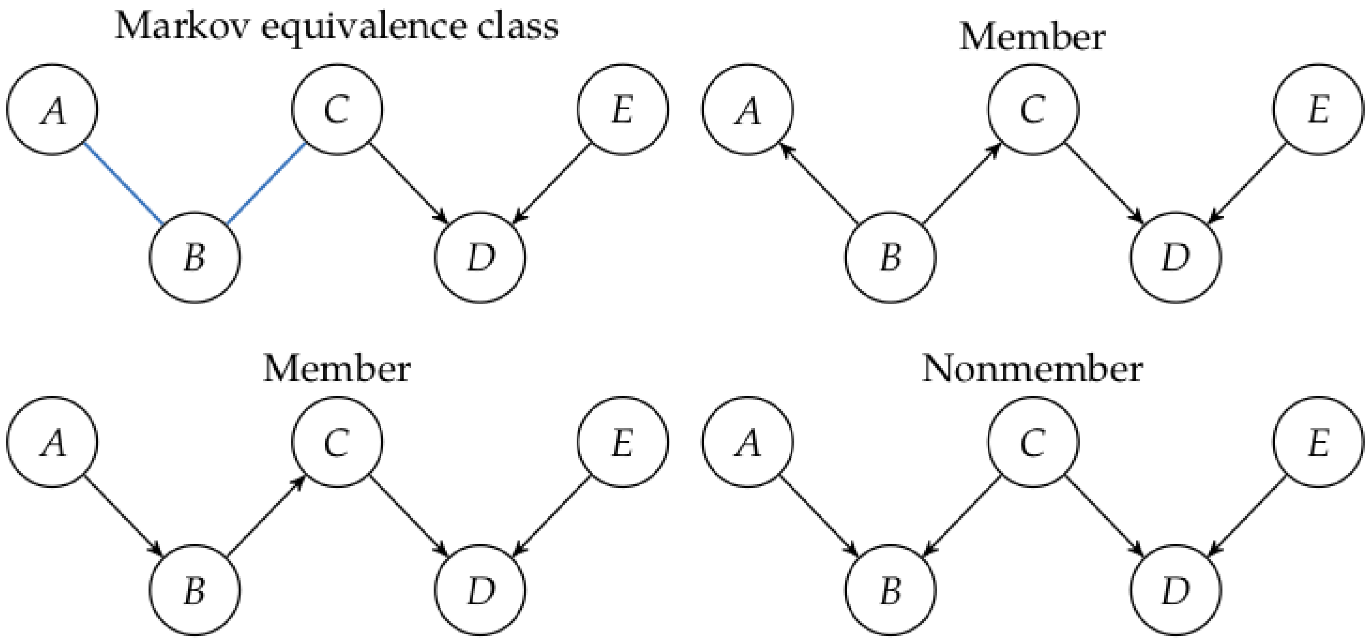
Markov Equivalence Class



Markov Equivalent iff

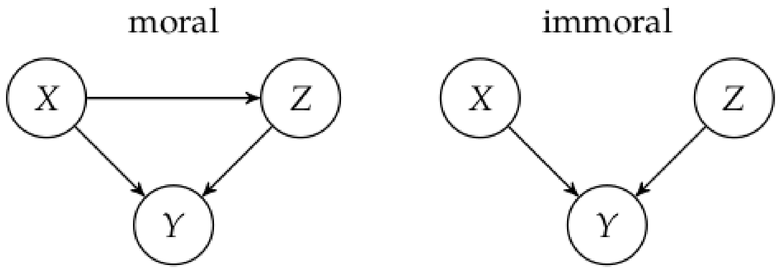
1. Same undirected edges
2. Same set of immoral v-structures

Markov Equivalence Class



Markov Equivalent iff

1. Same undirected edges
2. Same set of immoral v-structures



Recap

Inference

Learning