

PMU
- Probabilistic Models
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Probabilistic Models
- MDPs
- Reinforcement Learning
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DMU
    - Probabilistic Models
- MDPs
- Reinforcement Learning
- POMDPs
- Games
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1. 
$$0 \le P(X \mid Y) \le 1$$
  

$$\sum_{x \in X} P(x \mid Y) = 1$$

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  $\sum_{x \in X} P(x \mid Y) = 1$ 

2. 
$$P(X) = \sum_{y \in Y} P(X, y)$$

# P(A) P(A,B) P(AIB)

$$1.~0 \leq P(X \mid Y) \leq 1$$
  $\sum_{x \in X} P(x \mid Y) = 1$ 

2. 
$$P(X) = \sum_{y \in Y} P(X, y)$$

3. 
$$P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$$

### 0 < D(T( | T()

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3 Rules

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$$P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$$

### **Bayes Rule**

$$P(A \mid B) = rac{P(B \mid A)P(A)}{P(B)}$$

#### 3 Rules

1. 
$$0 \le P(X \mid Y) \le 1$$

$$\sum_{x \in X} P(x \mid Y) = 1$$

2. 
$$P(X) = \sum_{y \in Y} P(X, y)$$

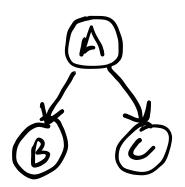
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$$P(X \mid Y) = \frac{P(X,Y)}{P(Y)}$$

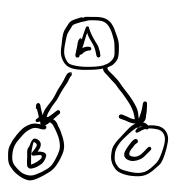
### **Bayes Rule**

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

#### **Independence**

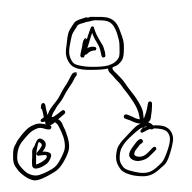
$$A \perp B \iff P(A, B) = P(A)P(B)$$
  
 $A \perp B \mid C \iff P(A, B \mid C) = P(A \mid C)P(B \mid C)$ 





#### **Chain Rule**

$$\underbrace{P(X_{1:n})} = \prod_i \underbrace{P(X_i \mid Pa(X_i))}$$

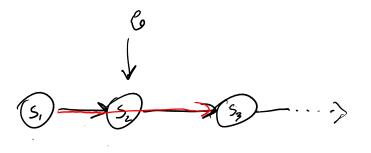


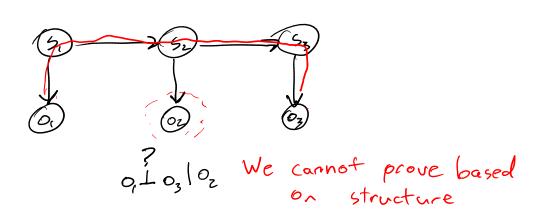
#### **Chain Rule**

$$P(X_{1:n}) = \prod_i P(X_i \mid Pa(X_i))$$

### **Conditional Independence**

 $X \perp Y \mid \mathcal{C}$  if all paths between X and Y are d-separated by  $\mathcal{C}$ 





$$(S, A, R, T, \gamma)$$

$$(S, A, R, T, \gamma)$$

Examples:  $S=\{1,2,3\}$  or  $S=\mathbb{R}^2$ 

 $(S, A, R, T, \gamma)$ 

s e S

Cartesian Product of two sets

$$\{-1,1\} \times \{-1,1\} = \{(-1,-1),(-1,1),(1,-1),(1,1)\}$$

Examples:  $S=\{1,2,3\}$  or  $S=\mathbb{R}^2$ 

$$s=(x,\dot{x})\in S=\mathbb{R}^2$$

$$S = (x', y', x^2, y^2, b)$$
  $S = [-1, 1]^4 \times \{1, 2\}$   
 $b \in \{1, 2\}$ 

A state is usually represented as a vector or tuple of state variables

$$s = (x', y', x^2, y^2)$$

A state space is a set of all possible

$$S = [-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1] = [-1, 1]^4$$
  
 $S = \{-10, ..., 10\}^4$ 

$$(S, A, R, T, \gamma)$$

Examples: 
$$S=\{1,2,3\}$$
 or  $S=\mathbb{R}^2$ 

$$s=(x,\dot{x})\in S=\mathbb{R}^2$$

$$(S, A, R, T, \gamma)$$

Examples: 
$$S=\{1,2,3\}$$
 or  $S=\mathbb{R}^2$ 

$$s=(x,\dot{x})\in S=\mathbb{R}^2$$

$$Q^{\pi}(s,a) = E[\sum_{t=0}^{\infty} r^{t}R(s_{t},a_{t})|s=s,a_{0}=a,a_{t}=\pi(s_{t})]$$

$$(S, A, R, T, \gamma)$$

Examples: 
$$S=\{1,2,3\}$$
 or  $S=\mathbb{R}^2$   $s=(x,\dot{x})\in S=\mathbb{R}^2$ 

$$Q^{\pi}(s,a) = E[\sum_{r=0}^{\infty} r'R(s_{r,a,r}) | s=s, a_{0}=a, a_{r}=\pi(s,r)]$$

$$V^{\pi}(s) = Q^{\pi}(s,\pi(s))$$

$$(S, A, R, T, \gamma)$$

Examples: 
$$S=\{1,2,3\}$$
 or  $S=\mathbb{R}^2$   $s=(x,\dot{x})\in S=\mathbb{R}^2$ 

$$Q^{\pi}(s,a) = E[\sum_{r=0}^{\infty} r^{r}R(s_{r,a+}) | s=s, a_{0}=a, a_{r}=\pi(s_{s+})]$$

$$V^{\pi}(s) = Q^{\pi}(s,\pi(s))$$

$$V^\pi(s) = R(s,a) + \gamma E[V^\pi(s')]$$

$$V^*(s) = \max_a \left\{ R(s,a) + \gamma E[V^*(s')] 
ight\}$$

$$B[V](s) = \max_a \left\{ R(s,a) + \gamma E[V(s')] 
ight\}$$

$$(S, A, R, T, \gamma)$$

Examples: 
$$S=\{1,2,3\}$$
 or  $S=\mathbb{R}^2$ 

$$s=(x,\dot{x})\in S=\mathbb{R}^2$$

$$V^{\pi}(5) = Q^{\pi}(5,\pi(5))$$

$$V^\pi(s) = R(s,a) + \gamma E[V^\pi(s')]$$

**Policy Evaluation** 

$$V^*(s) = \max_a \left\{ R(s,a) + \gamma E[V^*(s')] 
ight\}$$

Bellman's Equation: Certificate of Optimality

$$B[V](s) = \max_a \left\{ R(s,a) + \gamma E[V(s')] 
ight\}$$

Bellman's Operator

### **Policy Iteration**

loop

**Evaluate Policy** 

Improve Policy

### **Policy Iteration**

**Value Iteration** 

loop

**Evaluate Policy** 

Improve Policy

loop

$$V \leftarrow B[V]$$

#### **Policy Iteration**

**Value Iteration** 

loop

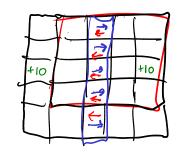
**Evaluate Policy** 

Improve Policy

loop

$$V \leftarrow B[V]$$

Converges because policy always improves and there are a finite number of policies



#### **Policy Iteration**

**Value Iteration** 

loop

**Evaluate Policy** 

Improve Policy

loop

$$V \leftarrow B[V]$$

Converges because policy always improves and there are a finite number of policies

Converges because B is a contraction mapping

**Monte Carlo Tree Search** 

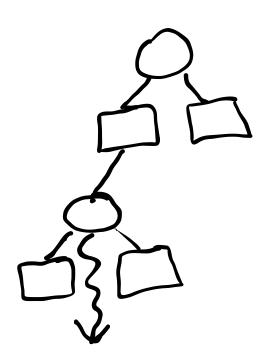
#### **Monte Carlo Tree Search**

Search

Expand

Rollout

Backup



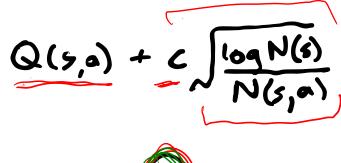
#### **Monte Carlo Tree Search**

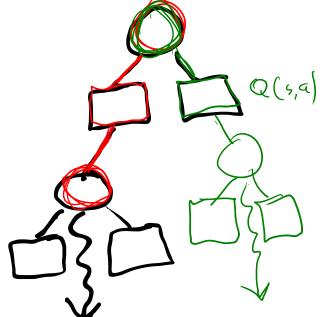


Expand

Rollout

Backup





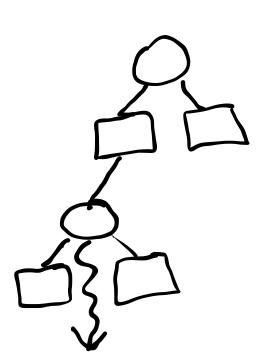
#### **Monte Carlo Tree Search**

Q(5,0) + C \[\log N(s) \\ N(s,0)\]

Expand Rollout

Search

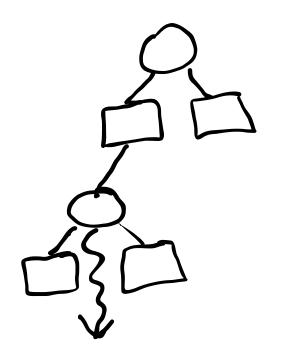
Backup



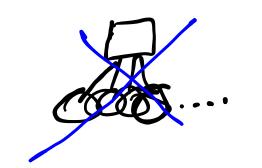
#### **Sparse Sampling**

#### **Monte Carlo Tree Search**

Search Expand Rollout Backup



### **Sparse Sampling**

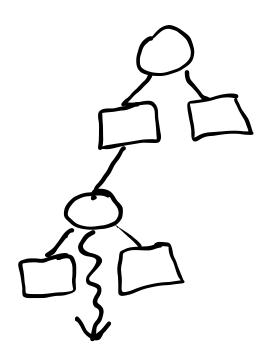




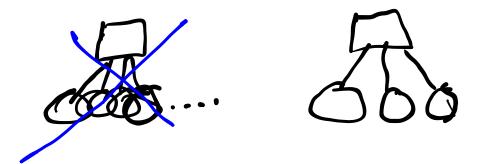
## Online MDP Planning

#### **Monte Carlo Tree Search**

Search Expand Rollout Backup



#### **Sparse Sampling**



Guarantees *independent* of |S|!!

#### LQR

$$\mathbf{s}' = \mathbf{T}_{s}\mathbf{s} + \mathbf{T}_{a}\mathbf{a} + \mathbf{w}$$

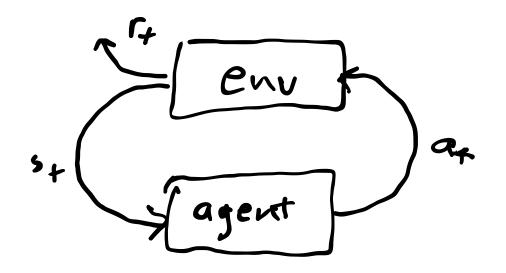
$$R(\mathbf{s},\mathbf{a}) = \mathbf{s}^{\mathsf{T}} \mathbf{R}_{s} \mathbf{s} + \mathbf{a}^{\mathsf{T}} \mathbf{R}_{a} \mathbf{a}$$

$$\pi_{h}(\mathbf{s}) = -\left(\mathbf{T}_{a}^{\top}\mathbf{V}_{h-1}\mathbf{T}_{a} + \mathbf{R}_{a}\right)^{-1}\mathbf{T}_{a}^{\top}\mathbf{V}_{h-1}\mathbf{T}_{s}\mathbf{s}$$

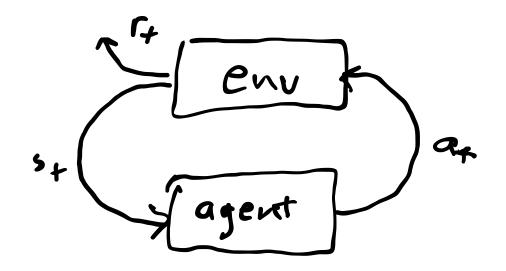
$$\mathbf{V}_{h+1} = \mathbf{R}_{s} + \mathbf{T}_{s}^{\top}\mathbf{V}_{h}^{\top}\mathbf{T}_{s} - \left(\mathbf{T}_{a}^{\top}\mathbf{V}_{h}\mathbf{T}_{s}\right)^{\top}\left(\mathbf{R}_{a} + \mathbf{T}_{a}^{\top}\mathbf{V}_{h}\mathbf{T}_{a}\right)^{-1}\left(\mathbf{T}_{a}^{\top}\mathbf{V}_{h}\mathbf{T}_{s}\right)$$

$$\mathbf{V}_{h} = \mathbf{R}_{s}$$

Challenges:

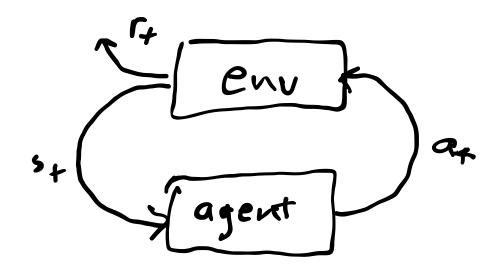


Challenges:



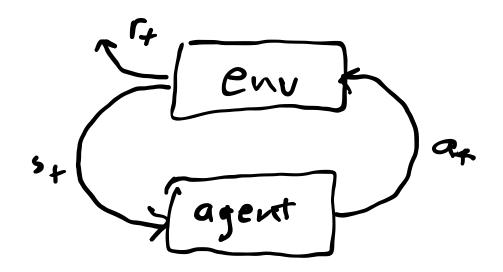
#### Challenges:

1. Exploration and Exploitation



#### Challenges:

- 1. Exploration and Exploitation
- 2. Credit Assignment

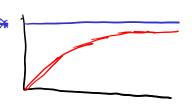


#### Challenges:

- 1. Exploration and Exploitation
- 2. Credit Assignment
- 3. Generalization

#### **Bandits**

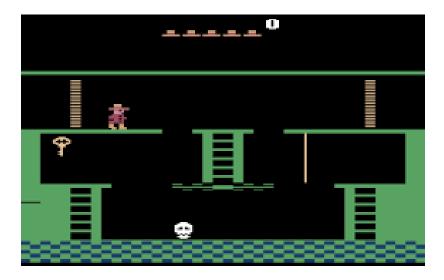
- $\epsilon$ -greedy
- softmax
- UCB
- Thompson Sampling
- Optimal DP Solution (solving a POMDP!)



logarithmic regret

#### **Bandits**

- $\epsilon$ -greedy
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- UCB
- Thompson Sampling
- Optimal DP Solution (solving a POMDP!)

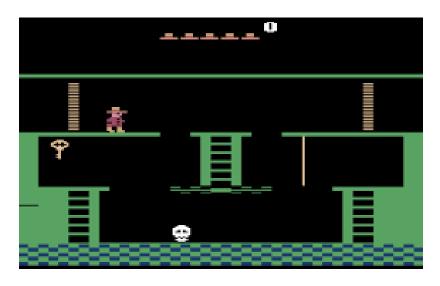


Montezuma's Revenge!

#### **Bandits**

- $\epsilon$ -greedy
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- Thompson Sampling
- Optimal DP Solution (solving a POMDP!)

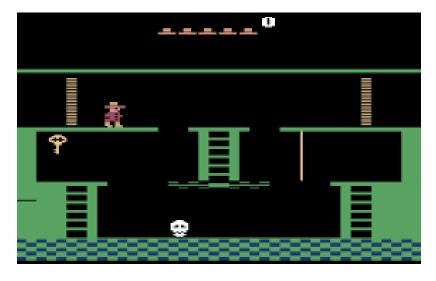
Pseudocounts



Montezuma's Revenge!

#### **Bandits**

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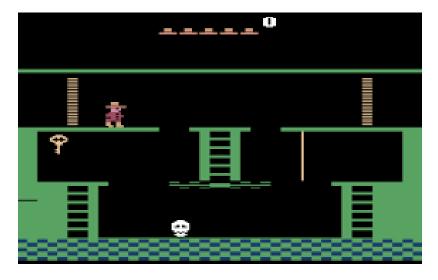


Montezuma's Revenge!

- Pseudocounts
- Curiosity: extra reward for bad prediction

#### **Bandits**

- $\epsilon$ -greedy
- softmax
- UCB
- Thompson Sampling
- Optimal DP Solution (solving a POMDP!)



Montezuma's Revenge!

- Pseudocounts
- Curiosity: extra reward for bad prediction
- Random network distillation

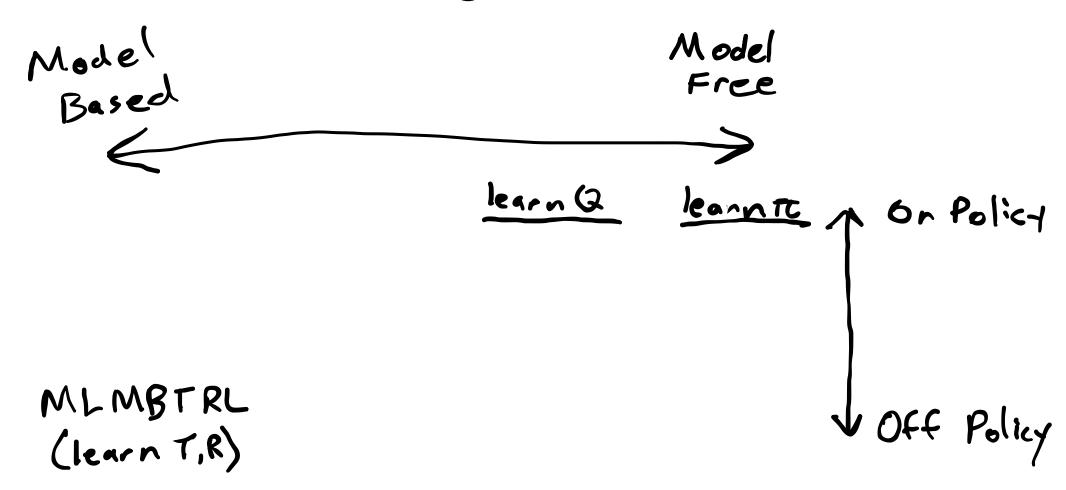


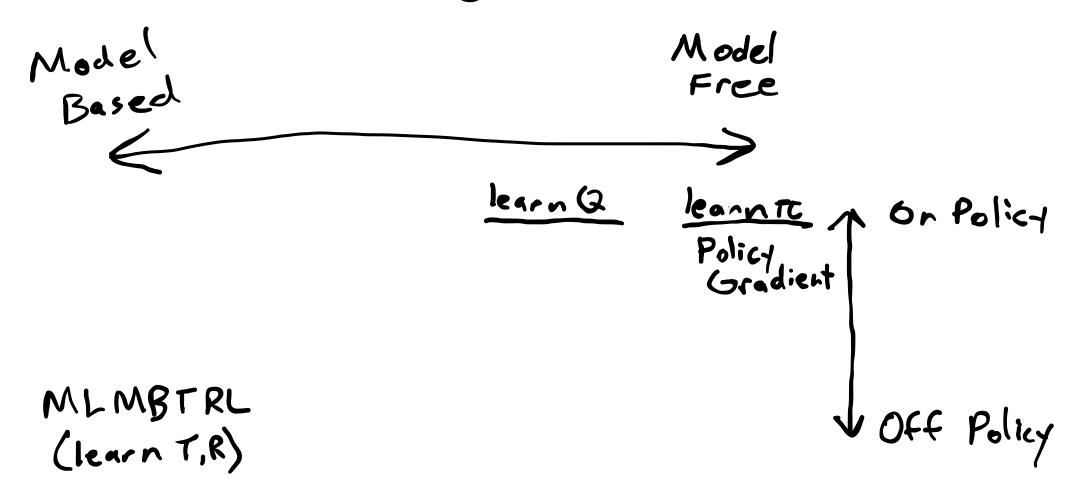


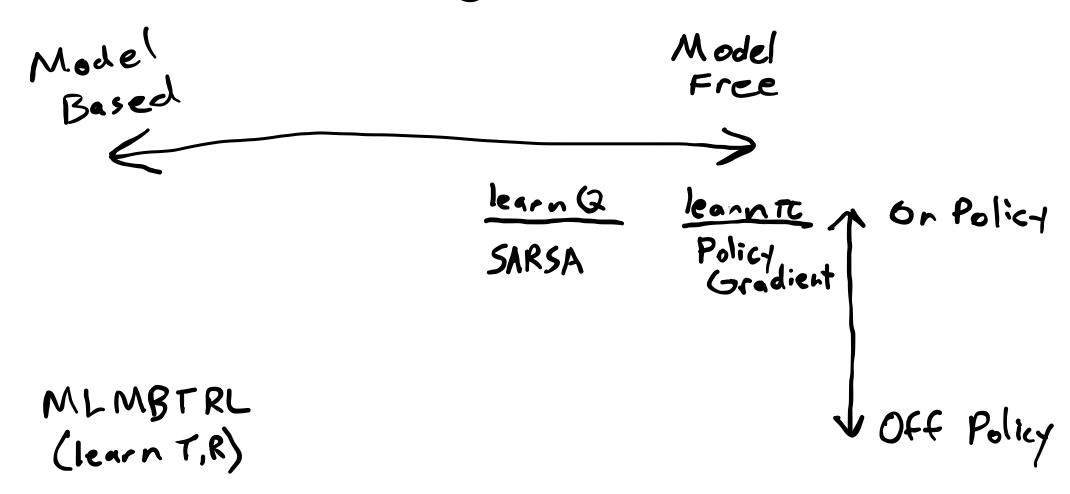
V Off Policy

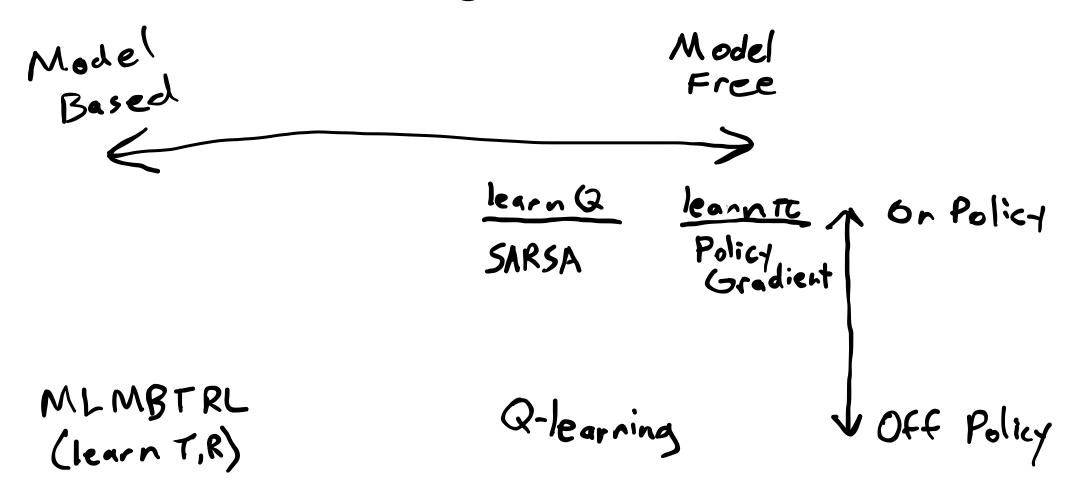


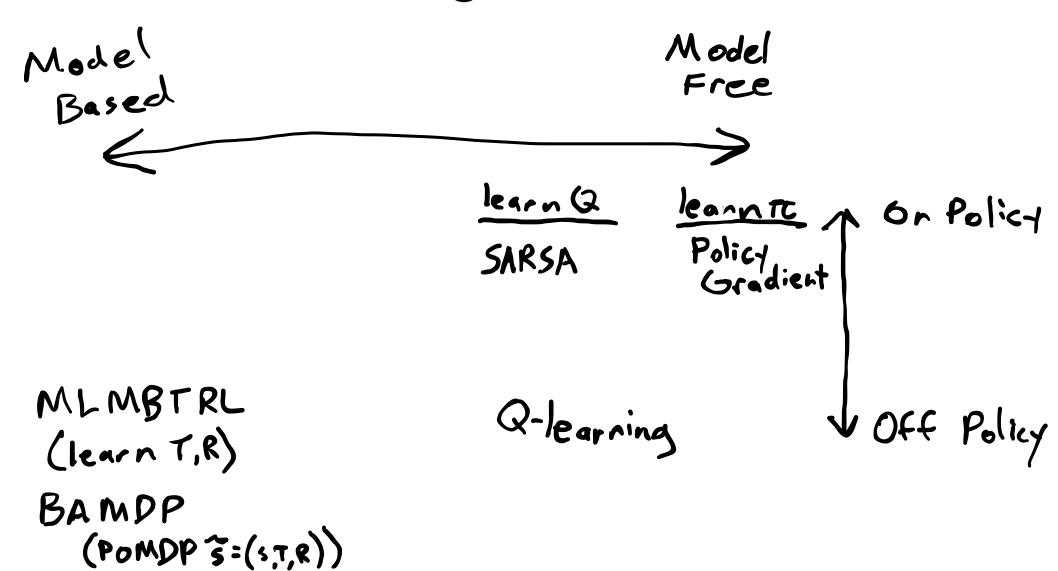
MLMBTRL (learn T,R) V Off Policy











Likelihood ratio trick

$$\nabla_{\theta} p_{\theta}(\tau) = p_{\theta}(\tau) \nabla_{\theta} \log p_{\theta}(\tau)$$

- Likelihood ratio trick
- Causality

$$\nabla_{\theta} p_{\theta}(\tau) = p_{\theta}(\tau) \nabla_{\theta} \log p_{\theta}(\tau)$$

- Likelihood ratio trick
- Causality
- Baseline Subtraction

$$\nabla_{\theta} p_{\theta}(\tau) = p_{\theta}(\tau) \nabla_{\theta} \log p_{\theta}(\tau)$$

Likelihood ratio trick

 $\nabla_{\theta} p_{\theta}(\tau) = p_{\theta}(\tau) \nabla_{\theta} \log p_{\theta}(\tau)$ 

- Causality
- Baseline Subtraction

$$\nabla U(\theta) = \mathbb{E}_{\tau} \left[ \sum_{k=1}^{d} \nabla_{\theta} \log \pi_{\theta}(a^{(k)} \mid s^{(k)}) \gamma^{k-1} \left( r_{\text{to-go}}^{(k)} - r_{\text{base}}(s^{(k)}) \right) \right]$$

Likelihood ratio trick

 $\nabla_{\theta} p_{\theta}(\tau) = p_{\theta}(\tau) \nabla_{\theta} \log p_{\theta}(\tau)$ 

- Causality
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$$\nabla U(\theta) = \mathbb{E}_{\tau} \left[ \sum_{k=1}^{d} \nabla_{\theta} \log \pi_{\theta}(a^{(k)} \mid s^{(k)}) \gamma^{k-1} \left( r_{\text{to-go}}^{(k)} - r_{\text{base}}(s^{(k)}) \right) \right]$$

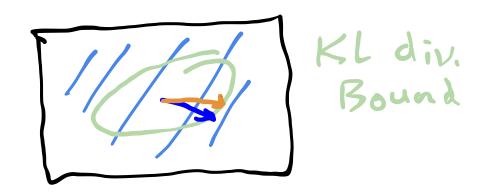
Natural Gradient

- Likelihood ratio trick
- Causality
- Baseline Subtraction

$$\nabla_{\theta} p_{\theta}(\tau) = p_{\theta}(\tau) \nabla_{\theta} \log p_{\theta}(\tau)$$

$$\underbrace{\nabla U(\boldsymbol{\theta})} = \mathbb{E}_{\tau} \left[ \sum_{k=1}^{d} \nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}^{(k)} \mid \boldsymbol{s}^{(k)}) \gamma^{k-1} \Big( r_{\text{to-go}}^{(k)} - r_{\text{base}}(\boldsymbol{s}^{(k)}) \Big) \right]$$

• Natural Gradient



#### **SARSA**

$$Q(s,a) \leftarrow Q(s,a) + lpha(r_t + \gamma Q(s', \underline{a}') - Q(s,a))$$

#### **SARSA**

$$Q(s,a) \leftarrow Q(s,a) + \alpha(r_t + \gamma Q(s',a') - Q(s,a))$$

**Eligibility Traces** 

#### **SARSA**

$$Q(s,a) \leftarrow Q(s,a) + \alpha(r_t + \gamma Q(s',a') - Q(s,a))$$

**Eligibility Traces** 

#### **Q-learning**

$$Q(s,a) \leftarrow Q(s,a) + lpha(r_t + \gamma \max_{a'} Q(s',a') - Q(s,a))$$

#### **SARSA**

$$Q(s,a) \leftarrow Q(s,a) + \alpha(r_t + \gamma Q(s',a') - Q(s,a))$$

**Eligibility Traces** 

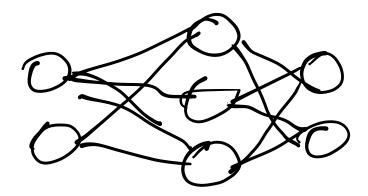
#### **Q-learning**

$$Q(s,a) \leftarrow Q(s,a) + lpha(r_t + \gamma \max_{a'} Q(s',a') - Q(s,a))$$

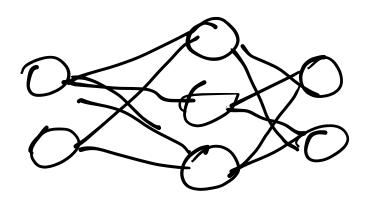
Double Q Learning

## Neural Networks and DQN

### **Neural Networks and DQN**

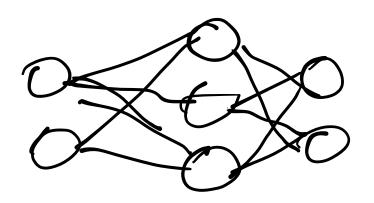


# **Neural Networks and DQN**



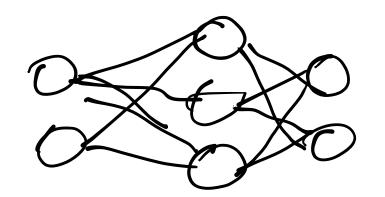
$$f_{ heta}(x) = \sigma(W_2\sigma(W_1x+b_1)+b_2)$$

# **Neural Networks and DQN**



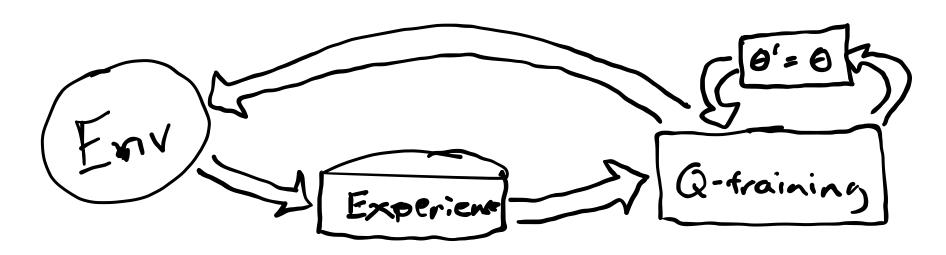
$$f_{ heta}(x) = \sigma(W_2\sigma(W_1x+b_1)+b_2)$$
Backprop

## Neural Networks and DQN



$$f_{ heta}(x) = \sigma(W_2\sigma(W_1x+b_1)+b_2)$$

Backprop



• Actor:  $\pi_{\theta}$ 

• Actor:  $\pi_{\theta}$ 

• Critic:  $Q_{\phi}$ 

• Actor:  $\pi_{\theta}$ 

• Critic:  $Q_{\phi}$ 

**Soft Actor Critic** 

• Actor:  $\pi_{\theta}$ 

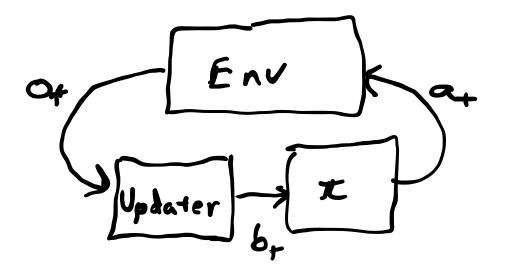
• Critic:  $Q_{\phi}$ 

#### **Soft Actor Critic**

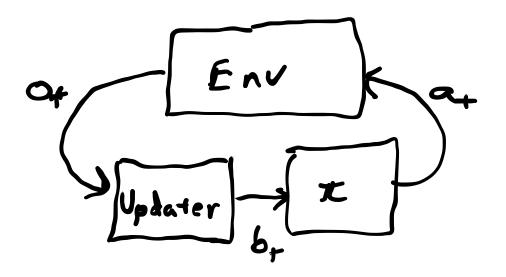
$$J(\pi) = E\left[\sum_{t=0}^{\infty} \gamma^t \left(r_t + lpha \mathcal{H}(\pi(\cdot \mid s_t))
ight)
ight]$$

 $(S, A, T, R, O, Z, \gamma)$ 

 $(S, A, T, R, O, Z, \gamma)$ 



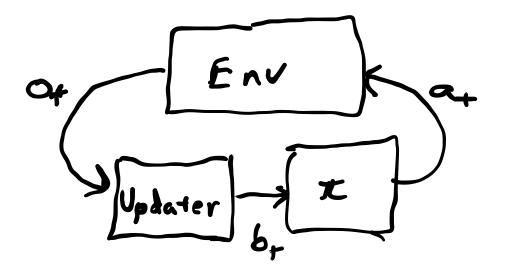
 $(S, A, T, R, O, Z, \gamma)$ 



### **Belief Updates**

- Discrete Bayesian Filter
- Particle Filter

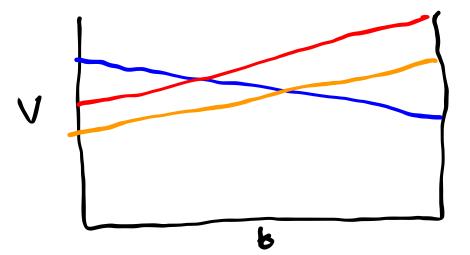
 $(S, A, T, R, O, Z, \gamma)$ 



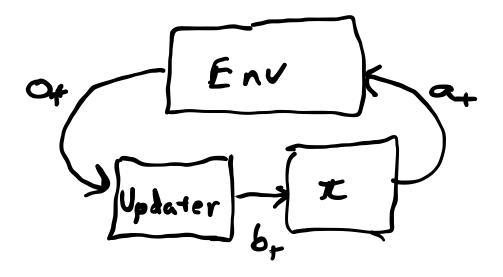
### **Belief Updates**

- Discrete Bayesian Filter
- Particle Filter

### **Alpha Vectors**



 $(S, A, T, R, O, Z, \gamma)$ 

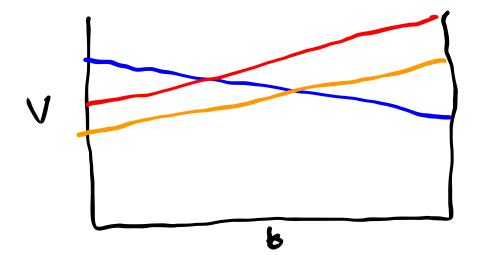


• Each alpha vector corresponds to a conditional plan

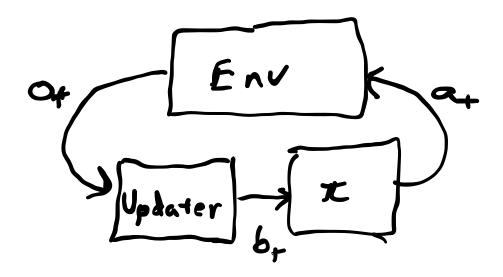
### **Belief Updates**

- Discrete Bayesian Filter
- Particle Filter

### **Alpha Vectors**



 $(S, A, T, R, O, Z, \gamma)$ 

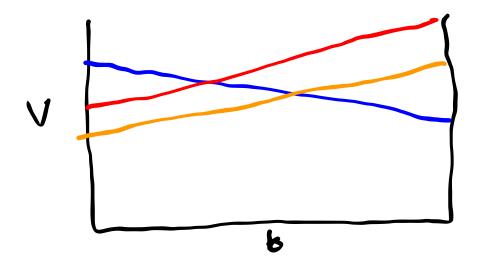


- Each alpha vector corresponds to a conditional plan
- You can prune alpha vectors by solving an LP

### **Belief Updates**

- Discrete Bayesian Filter
- Particle Filter

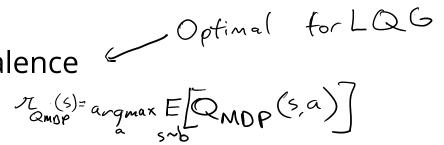
### **Alpha Vectors**



### **Formulation**

• Certainty Equivalence

• QMDP



### **Formulation**

- Certainty Equivalence
- QMDP

**Numerical** 

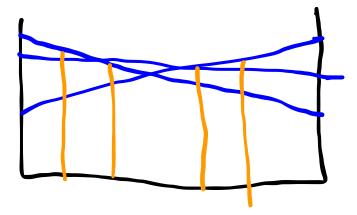
#### **Formulation**

- Certainty Equivalence
- QMDP

#### **Numerical**

#### Offline

- Point-Based Value Iteration
- SARSOP



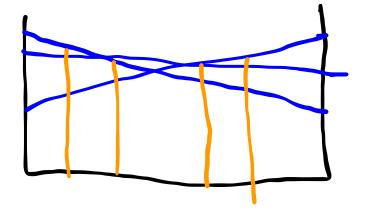
### **Formulation**

- Certainty Equivalence
- QMDP

#### **Numerical**

#### Offline

- Point-Based Value Iteration
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#### **Online**

- POMCP
- DESPOT

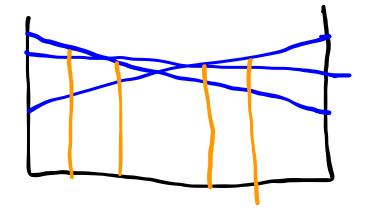
#### **Formulation**

- Certainty Equivalence
- QMDP

#### **Numerical**

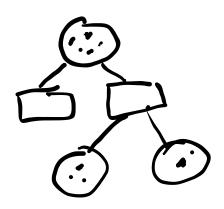
#### Offline

- Point-Based Value Iteration
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#### **Online**

- POMCP
- DESPOT

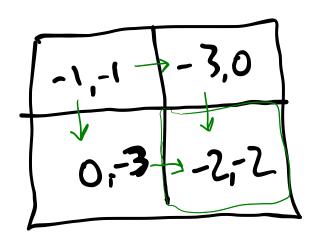


Optimal Solutions

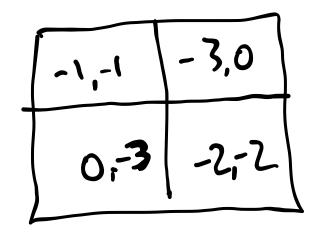
• Optimal Solutions No.

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  Equilibria (e.g. Nash Equilibria)

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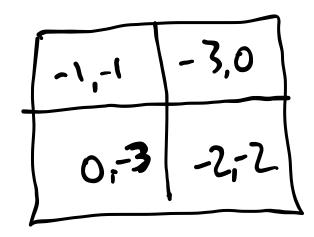


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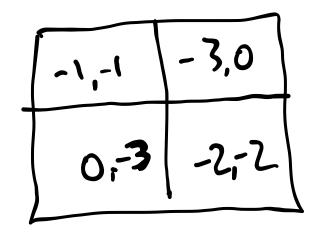
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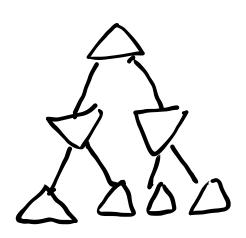


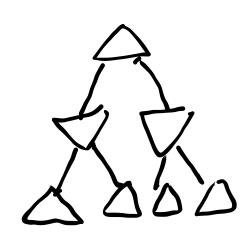
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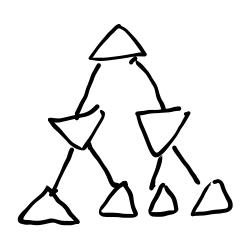


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- Might be pure or mixed
- Algorithms like fictitious play converge in special cases

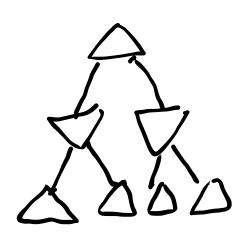




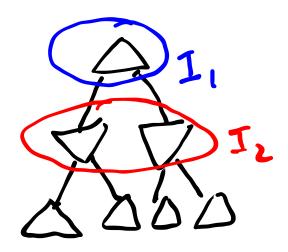
Value Function Backup



- Value Function Backup
- $\alpha\beta$  Pruning



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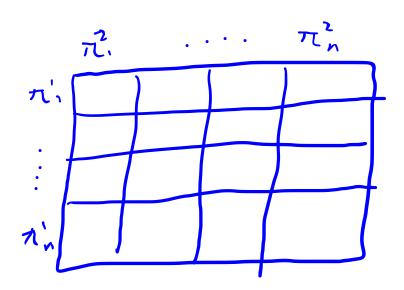
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- All players play simultaneously
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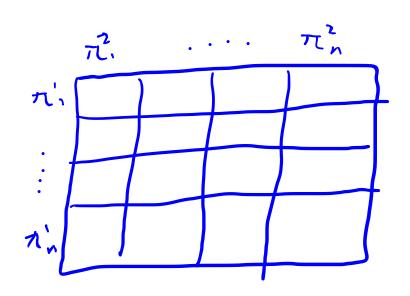


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### **Partially Observable Markov Games**

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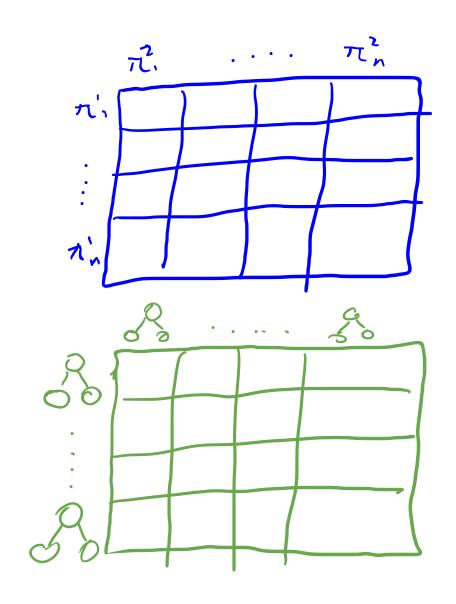


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# Fictitious Play in Markov Games

# After DMU you have basic tools to deal with 4 Big Problems:

1. Immediate and Future Rewards

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- 2. Unknown Models

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- 2. Unknown Models
- 3. Partial Observability

- 1. Immediate and Future Rewards
- 2. Unknown Models
- 3. Partial Observability
- 4. Other Agents

